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# QUALITATIVE STABILITY PROPERTIES OF MATRICES

by

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Abstract

QUALITATIVE STABILITY PROPERTIES OF MATRICES

by Terrence Bone

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A matrix A is sign stable if some matrix with the same sign pattern of positive, negative, and zero entries has all eigenvalues with negative real parts. It is potentially stable if it is not sign unstable. The system $A x + b = 0$ is positively sign solvable if the sign pattern of the solution vector x contains all positive entries and is determined by the sign patterns of A and b.

The characterization of potentially stable matrices remains unresolved. Counterexamples are presented refuting some previous results of Quirk and Campbell. New classes of potentially stable matrices are defined by means of recursive constructions, and a new necessary condition for potential stability is given.

A characterization is presented of systems $A x + b = 0$ for which A is sign stable and the system is positively sign solvable. This leads to a recognition algorithm of time complexity which is linear in the order of A and the number of nonzero entries of A. The relation between this characterization and previous results of Manber is explored.

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# CHAPTER 1.

# INTRODUCTION

## 1.1 Origin of qualitative matrix problems

Qualitative matrix problems arise in such applications as economics, ecology, chemistry, mechanics, and energy planning, whenever we would like to draw conclusions about the stability characteristics of dynamical systems whose parameters we are not prepared to specify.

The origin of such problems is usually attributed to questions raised by Paul Samuelson in his Foundations of Economic Analysis. [31] Samuelson said that it would be highly desirable to be able to infer properties of economic systems from qualitative information alone, both because numerical values of coefficients are often not available, and because the computations involved are distasteful even when such values are available.

As an example of the kind of problems Samuelson was dealing with, consider an economy with n commodities with prices $p_1,\dots,p_n$, and at least one other parameter q (e.g., national income). We suppose there exist excess demand functions $E_i(p_1,\dots,p_n,q)$ and some equilibrium $(\hat{p},\hat{q})$ at which all $E_i(\hat{p},\hat{q})=0$. Let q be perturbed from the equilibrium value $\hat{q}$. There are then two approaches. We may postulate that equilibrium is quickly restored, so that the equations $E_i(p,q)=0$ give p as an implicit function of q. We then seek to solve for p explicitly in a neighborhood of $\hat{p}$. The second approach is to suppose that the system evolves according to a differential equation that relates the rate of change of each price to the corresponding excess demand. In this case we ask whether, in fact, equilibrium is restored after each perturbation.

Samuelson's challenge to carry out such analysis using only qualitative information was soon taken up by other economists. The first approach led to the sign solvability problem; the second to the sign stability problem. A related problem that was investigated was to determine which systems must return to equilibrium regardless of the rates at which prices adjust to non-zero excess demand. This became known as the D-stability problem. [22, 23, 27]

As work proceeded, some economists became skeptical regarding the applicability of purely qualitative results to most economic systems. [29] But the work they had done was pointed to by R. H. May as a tool in the study of ecological communities of several species. [20] Here the variables are the populations of the species, given either in terms of individuals or in aggregate mass or energy units. Qualitative information consists of the classification of the interactions into such categories as commensalism, amensalism, mutualism, competition, and predation, and is conveniently represented by a signed directed graph.

Another area in which such problems have been examined is that of energy planning. F. S. Roberts has encouraged what he calls "geometric methodologies" because of the complexity of the systems being analyzed and the difficulty of assigning magnitudes to all of the variables that must be considered in any public policy context. [30]

A final example of such problems is furnished by mechanical vibration theory. In his book <u>Stability of Motion,</u> Wolfgang Hahn asks when must a system be unstable as a result of the signs of the interconnections, regardless of the magnitudes of the coefficients. He refers to this property as "structural instability". [8]

In each of these examples we have a dynamical system and an associated interaction matrix A (such as the Jacobian of a differential equation). Either the magnitudes of the entries of A are unknown or uncertain, or we choose to ignore them in order to simplify the analysis or to obtain insight into the system. We seek to determine stability or solvability properties of the system that depend only upon the sign pattern of A.

## 1.2 Definitions and required theorems

Let A be a real $n \times m$ matrix. We define the qualitative matrix Q(A) as the set of all matrices with the same sign pattern as A of positive, negative, and zero entries. That is,

$$Q(A)=\{B\in R^{n\times m}: \text{sgn } b_{ij} = \text{sgn } a_{ij} \text{ for } 1\leq i\leq n \text{ and } 1\leq j\leq m\}.$$

Alternatively, we say that $A \sim B$ if $A\in Q(B)$.

The matrix A is stable if the system $dx/dt = Ax$ is asymptotically stable at $x = 0$. Equivalently, A is stable if all its eigenvalues have negative real part. A is sign stable if $B \in Q(A)$ implies B is stable. A is potentially stable if some $B\in Q(A)$ is stable. The pair (A,b) is sign solvable if the linear equation $Ax+b=0$ has a unique solution and if $(C, d) \sim (A, b)$ implies C is nonsingular and $-C^{-1}d\in Q(x)$. If no $x_i = 0$ the pair is said to be strongly sign solvable.

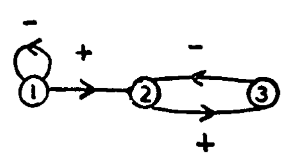
Sign stability and solvability are often discussed in terms of graphs.

Let A be a square matrix. Define S(A), the signed digraph of A, as the directed graph with vertex set $V=\{1,2,...,n\}$ and edge set $E= P \cup N$ with positive edges $P= \{(i,j):a_{ji}>0\}$ and negative edges $N= \{(i,j):a_{ji}<0\}$ . We say distinct vertices i and j are neighbors if both (i, j) and $(j, i) \in E$. We will use the term path or k-path to refer both to the sequence of k consecutive edges $(i_1,i_2)(i_2,i_3)...(i_k,i_{k+1})$ in S(A) with no vertex repeated, and to the corresponding product of matrix entries $a_{i_2 i_1} a_{i_3 i_2} \cdots a_{i_{k+1} i_k}$. We say the path is positive or negative depending on the sign of this product. We abbreviate the notation as $(i_1, i_2, \cdots, i_{k+1})$ or $(i_1 \to i_{k+1})$ for a k-path in S(A) and as $a(i_1, i_2, \cdots, i_{k+1})$ or $a(i_1 \to i_{k+1})$ for the corresponding product. Similarly, we use the term cycle or k–cycle to refer to the sequence of k edges $(i_1, i_2, \cdots, i_k)(i_k, i_1)$, denoted as $[i_1, i_2, \cdots, i_k]$ or $[i_1 \to i_k]$ when referring to the graph, and as $a[i_1, i_2, \cdots, i_k]$ or $a[i_1 \to i_k]$ for the corresponding product. Whenever $a_{ii} \neq 0$, we also refer to one–cycles [i] corresponding to loops $(i, i) \in E$. If a one-cycle is negative, we say the corresponding vertex is self–limiting. We define the distance between two vertices as $\mathrm{dist}(i,j) = \min \{k\text{: there exists a k-path } (i \to j)\}$.

We illustrate these notions by showing the signed digraph S(A) for a $3 \times 3$ matrix A with the indicated sign pattern. The signs of the edges are indicated by (+,–) labels. The graph has vertices 1, 2, and 3, positive edges (1,2) and (2,3) corresponding to $a_{21} > 0$ and $a_{32} > 0$, and negative edges (1,1) and (3,2) corresponding to $a_{11} < 0$ and $a_{23} < 0$. This gives rise to paths $(1 \to 2)$, $(1 \to 3)$, and $(3 \to 2)$ (positive, positive, and negative), and cycles [1] and [2,3], both negative. Vertex 1 is self-limiting, and dist (1,3) = 2.

$$\begin{bmatrix} - & 0 & 0 \\ + & 0 & - \\ 0 & + & 0 \end{bmatrix}$$

Q(A)




S(A)

There are three main theorems to which we will refer in subsequent sections. The first is the Routh-Hurwitz criterion.

Consider the polynomial $p(x) = x^n + c_1 x^{n-1} + \cdots + c_n$, and let $H_2, \ldots, H_{n-1}$

denote the Hurwitz determinants. That is, $H_k$ is the determinant of the matrix

$$\begin{bmatrix} c_1 & c_3 & c_5 & \cdots & 0 \\ 1 & c_2 & c_4 & & \cdot \\ 0 & c_1 & c_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot\ \cdot & \cdot \\ \cdot & & & \cdot\ \cdot & \cdot \\ \cdot & & & \cdot & c_{k+1} \\ 0 & \cdot & \cdot & \cdots & c_k \end{bmatrix}$$

with the convention $c_j = 0$ if $j>n$. The following theorem provides us with a method of testing the stability of a matrix in terms of the coefficients of its characteristic polynomial. [18]

Theorem 1.1 (Routh-Hurwitz). All zeros of $p(x)$ have negative real parts iff $c_i > 0$, for $1 \leqq i \leqq n$, and $H_j > 0$, for $2 \leqq j \leqq n-1$.

The next result we cite is the Quirk-Ruppert-Jeffries characterization of sign stability. First, we define two ways of coloring the vertices of S(A). In each case one colors each vertex of S(A) either black or white according to the following rules.

Im-coloring:

(i) Every self-limiting vertex is black.

(ii) No black vertex has exactly one white neighbor.

(iii) Each white vertex has at least one white neighbor.

0-coloring:

(i) and (ii) same as above.

(iii) No white vertex has a white neighbor.

(In the literature these are known as $\delta$-colorings and $\varepsilon$-colorings, respectively. We use the above terminology here for mnemonic reasons.) We then have the following theorem. [11,12]

Theorem 1.2 (Quirk-Ruppert-Jeffries). Matrix A is sign stable if and only if it satisfies the following conditions:

(a) All one-cycles are nonpositive.

(b) All two-cycles are nonpositive.

(c) No k-cycle exists for $k \geqq 3$ .

(d) In every Im-coloring all vertices are black.

(e) In every 0-coloring all vertices are black.

The algorithms of Klee and van den Driessche [17] for recognizing sign solvable matrices use an alternate condition equivalent to (e). Define the graph $G = (V, E_u)$ with $E_u=\{(i,j): a[i,j]\neq 0\}$. A Z–complete matching for any $Z \subset V$ is a subset $M \subset E_u$ consisting of vertex disjoint edges such that every vertex of Z is incident to at least one edge in M. Let vertex subset $R_A$ consist of all self-limiting vertices. We then have the following condition.

(e') The graph G admits a $(V - R_A)$- complete matching.

Conditions (a), (b), and (c), due to Quirk and Ruppert [28], are sufficient to guarantee that each $B \in Q(A)$ has no eigenvalues with positive real parts. In such a case we refer to A and S(A) as semistable. Conditions (d) and (e), due to Jeffries [9,10], detect eigenvalues with zero real part. Let S(A) be semistable. If it has any Im–coloring in which some vertex is white, then A or some $C \in Q(A)$ has a purely imaginary eigenvalue and the system $dx/dt=Cx$ has a periodic solution which does not converge to the origin. Similarly, white vertices of a 0-coloring of S(A) correspond to nonzero entries of a vector in the null space of some $C \in Q(A)$. Zero eigenvalues are also indicated when there is no matching that covers all vertices of $(V - R_A)$.

Finally, we cite Manber's graph theoretic criterion for sign solvability. [19,15] A system $Ax + b = 0$ is said to be in standard form if $a_{ii} < 0$, $b_i \geqq 0$, for $1 \leqq i \leqq n$. Given any system $Ax + B = 0$, where B is an $n \times n$ matrix, define the augmented graph $S(A,B)=(V', P', N')$ as follows. Let $V'=V \cup W$, where we create additional vertices $W=\{w_{ij}:b_{ij}\neq 0\}$, and define the augmented edge sets by $P'=P\cup\{(w_{ij},i):b_{ij}>0\}$, and $N'=N\cup\{(w_{ij},i):b_{ij}<0\}$. That is, the augmented graph is formed by adding appropriately signed edges into each vertex i for which $b_{ij}\neq 0$. In the usual case where B is a column vector we denote the augmented graph as $S(A,b)$. We may formulate Manber's criterion in terms of the augmented graph of a system as follows.

Theorem 1.3 (Manber). Let (A, b) be in standard form. Then the system is sign solvable if and only if S(A, b) satisfies the following:

(a) No cycle in S(A, b) is positive.

(b) Every path of the form $(w_i \rightarrow v)$ is positive, where $w_i \in W$ and $v \in V$.

Moreover, (A, b) is strongly sign solvable if and only if it is sign solvable and satisfies the condition:

(c) For every $v \in V$, there exists a path $(w_i \rightarrow v)$ from some $w_i \in W$.

## 1.3 Overview

Chapter 2 examines potential stability. This is equivalent to asking which matrices are sign unstable. Counterexamples to some previous theorems are displayed. Compound cycles of even and odd parity are defined, and shown to be closely related to the coefficients of the characteristic polynomial. Sufficient Hurwitz criteria are then used to define some classes of potentially stable matrices. Section 2.5 uses a different approach to obtain a necessary condition for potential stability. Concluding sections discuss related results and prominent open questions. [2]

Chapter 3 gives an exposition of some discoveries of Jeffries characterizing the structure of sign stable systems which are also positively sign solvable. The time complexity of recognizing such systems is shown to be bounded above by a linear function of the number of vertices and edges in the associated augmented digraph. Algorithms are given for some stages of this recognition procedure. The last section provides an alternate derivation of the characterization from the results of Manber. For the extension of these results to the nonlinear case, the reader is referred to a forthcoming paper. [3]

## CHAPTER 2.

## POTENTIAL STABILITY

### 2.1 Introduction

It may be said that there has been negative progress on the potential stability problem in that we now no longer know some things of which we were once quite certain. Accordingly, we begin with some cautionary examples.

Potential stability was first defined by Quirk in [26]. We cite Proposition 2 from the same paper.

Proposition. Given $A \in R^{n \times n}$, define B by

$$b_{ij} = \begin{cases} 0 & \text{if } i=j \text{ and } a_{ii}>0 \\ a_{ij} & \text{otherwise} \end{cases}$$

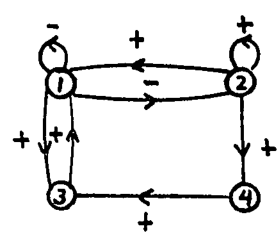
Then A is potentially stable if and only if B is potentially stable.

Counterexample

That A is potentially stable does not imply that B is. Let A be a matrix with sign pattern and digraph shown below. Then A is potentially stable, but B is not.

$$\begin{bmatrix} - & + & + & 0 \\ - & + & 0 & 0 \\ + & 0 & 0 & + \\ 0 & + & 0 & 0 \end{bmatrix}$$

Q(A)



S(A)

Proof

We apply the Routh-Hurwitz criterion, which states that A is stable if and only if the coefficients $c_1,\dots,c_n$ and the Hurwitz determinants $H_2, \dots, H_{n-1}$ are all positive. Let a, b, c, d, e > 0 with $a_{11}=-a$, $a_{22}=b$, $a_{12}a_{21}=-c$, $a_{13}a_{31}=d$, and $a_{13}a_{34}a_{42}a_{21}=-e$. Then the characteristic polynomial coefficients are $c_1=a-b$, $c_2=-ab+c-d$, $c_3=bd$, and $c_4=e$, while the Hurwitz determinants are $H_2=cc_1+a(b^2-ab-d)$ and $H_3=c_3H_2-c_1^2c_4$. These can all be made positive if a > b, c is large enough, and e is small enough. Thus A is potentially stable.

Now suppose we obtain the matrix B from A by changing the positive one-cycle $a_{22}>0$ to $b_{22}=0$ and leaving all other entries unchanged. Then for Q(B) we will have $c_3=0$ and $H_3=-c_1^2c_4<0$, so all matrices of this sign pattern will have an eigenvalue with nonnegative (in fact, positive) real part and are thus unstable. (Note that making $b_{22}<0$ does not produce potential stability either.)

□

The opposite implication of Quirk's proposition is correct, as we shall see in section 2.4. This direction fails because he does not distinguish between setting positive diagonal entries of A to zero, and defining B = A − aI, for some a > 0.

Next we restate Theorem 2.5 of Campbell's work on potentially stable matrices. [4] Let $\sigma$ be a k-cycle $\sigma=[i_1,i_2,\ldots,i_k]$. Define the <u>vertex set</u> of $\sigma$ as $V(\sigma)=\{i_1,i_2,\ldots,i_k\}$. Let <u>J</u> be the class of all real, indecomposable n × n matrices in which, for each s with $1\leq s\leq n$, there is a negative k-cycle with vertex set $\{i_{s-k+1},\ldots,i_s\}$, for some $k\leq s$.

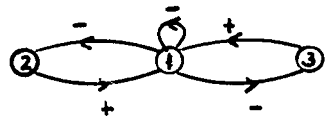
<u>Theorem.</u> If A ∈ <u>J</u>, then A is potentially stable.

<u>Counterexample</u>

Consider any matrix A with the following sign pattern.

$$\begin{bmatrix} - & + & + \\ - & 0 & 0 \\ - & 0 & 0 \end{bmatrix}$$

Q(A)



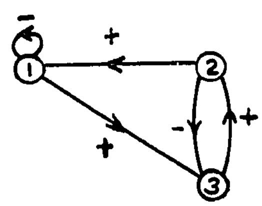

S(A)

A is indecomposable and we have negative cycles [1], [1,2], and [1,3] for s=1, 2, and 3, respectively. Yet whenever $B \in Q(A)$, we have det $B = 0$, automatically rendering B unstable. Campbell errs by allowing the negative cycles to overlap (that is, allowing $k<s$) instead of requiring them to be nested, thereby allowing sign patterns in which one or more of the characteristic polynomial coefficients $c_i=0$.

We present one more cautionary example, also noted by both Quirk and Campbell in the cited papers. Consider matrix A with the indicated sign pattern.

$$\begin{bmatrix} - & + & 0 \\ 0 & 0 & + \\ + & - & 0 \end{bmatrix}$$

Q(A)




S(A)

Matrix A is potentially stable. But if its positive 3-cycle is made negative, A is forced to be unstable.

For let $a_{11}=-a$, $a_{23}a_{32}=-b$, and $a_{12}a_{23}a_{31}=c$. The Routh-Hurwitz criterion shows A is stable when $ab>c$ (and all are positive). But if we reverse the sign of the 3-cycle so that $c<0$, the resulting matrix is not stable.

2.2 Compound cycles

The preceding examples may serve to qualify our intuition that negative feedback is good for stability, while positive feedback undermines it. To better understand what is going on, we must go beyond looking at the signs of simple cycles.

Let $\sigma_1,\sigma_2,\ldots,\sigma_r$ be simple cycles which are vertex disjoint. That is, $i\neq j$ implies $V(\sigma_i)\cap V(\sigma_j)=\phi$. We define a compound k−cycle $\Sigma=\sigma_1\sigma_2\cdots\sigma_r$, where k is the cardinality of the vertex set $V(\Sigma)=V(\sigma_1)\cup\cdots\cup V(\sigma_r)$. We write $\Sigma$ as a product to remind us of the corresponding product of matrix entries. We say the compound cycle has even parity if all but an even number of its factor cycles are negative. Otherwise, it has odd parity.

As an example consider the digraph in the last example. It contains two compound 3-cycles, $\Sigma_1$ = [1][2,3] and $\Sigma_2$ =[1,3,2]. $\Sigma_1$ is even parity, since it contains an even number of positive cycles (zero), while $\Sigma_2$ is odd parity, since it contains exactly one positive cycle.

Note that the terms of any k × k principal minor of A are all compound k-cycles which appear with sign $(-1)^k$ if they are of even parity, $(-1)^{k+1}$ if of odd parity. Since characteristic polynomial coefficient

$$c_k=(-1)^k\sum(\text{ all } k\times k \text{ principal minors}),$$

the following necessary condition is immediate.

Lemma 2.1. If A is potentially stable, then S(A) contains an even parity compound cycle of every order k for $1\leq k\leq n$.

Obviously, an even parity 1-cycle is just a negative 1-cycle. It is not hard to see that every stable matrix must also contain a negative 2-cycle or a second negative 1-cycle. But for $k \geqq 2$, it is not necessary that the compound k-cycles consist of negative cycles. Indeed, we will exhibit arbitrarily large stable matrices which contain only two negative cycles (Construction 2 below).

In the following sections we investigate sufficient conditions involving even parity compound cycles.

## 2.3 Hurwitz polynomials

We first consider further conditions on the coefficients of a polynomial (e.g., the characteristic polynomial of a matrix) which are related to the property that the polynomial $p(x)=x^n+c_1x^{n-1}+\cdots+c_n$ with real coefficients $c_i$ is Hurwitz (all roots have negative real parts).

The set of necessary and sufficient conditions given by the Routh-Hurwitz criterion is rather unwieldy. For instance, if $p(x)$ is the characteristic polynomial of a full $4 \times 4$ matrix, then the expansion of $H_{n-1}$ in terms of matrix entries has 684 terms. It is not surprising that weaker conditions have been formulated to apply to specific cases.

The following approach is due to Strelitz [32]. Factor $p(x)=\prod_{i=1}^{n}(x-\lambda_i)$ over the complex numbers, let m be an integer with $1 < m < n$, and let $N=\binom{n}{m}$. Define $\rho_1, \rho_2, \ldots, \rho_N$ as the elements of the set $\{\lambda_{i_1}+\lambda_{i_2}+\cdots+\lambda_{i_m}: 1 \leqq i_1 < i_2 < \cdots < i_m \leqq n\}$. That is, the $\rho_i$ are all possible sums of the original roots taken m at a time. Let $q(x)$ be the polynomial with the $\rho_i$ as roots.

$$q(x) = \prod_{i=1}^{N}(x-\rho_i) = C_0x^N+C_1x^{N-1}+\cdots+C_N.$$

The coefficients $C_i$ must be real. If $p(x)$ is Hurwitz, then the $C_i$ must be positive. Strelitz shows that for $m=2$ the converse holds: all $C_i>0$ (and all $c_i>0$) implies $p(x)$ is Hurwitz.

In general, explicit formulas for the $C_i$ in terms of the original $c_i$ are long and complicated. An exception is the case when we choose $m=n-1$, but whenever $m>2$, the resulting criterion is no longer sufficient for $p(x)$ to be Hurwitz. We obtain the following.

Lemma 2.2. Polynomial $p(x)$ with positive coefficients $c_i$ is Hurwitz only if coefficients $C_1,\ldots,C_n$ are positive, where

$$C_1=(n-1)c_1$$
$$C_k=\binom{n-1}{k}c_1^k+\sum_{r=2}^{k}(-1)^r\binom{n-r}{k-r}c_1^{k-r}c_r,\ \text{if } 1<k<n$$
$$C_n=\sum_{r=2}^{n}(-1)^r c_1^{n-r}c_r$$

Proof

Let $\Lambda=\sum_{i=1}^{n}\lambda_i=-c_1$. Consider subindices of the set $I_k$ of k-indices $I_k=\{(i_1,i_2,\ldots,i_k):1\leqq i_1<i_2<\cdots<i_k\leqq n\}$. Observe that any subindex, such as $(1,2,\ldots,r)$, appears in exactly $\binom{n-r}{k-r}$ of the indices of $I_k$. Thus, for $1<k<n$, we obtain

$$C_k=(-1)^k\sum_I(\Lambda-\lambda_{i_1})\cdots(\Lambda-\lambda_{i_k}).$$

$$=(-1)^k\left[\binom{n}{k}\Lambda^k+\binom{n-1}{k-1}\Lambda^{k-1}c_1+\binom{n-2}{k-2}\Lambda^{k-2}c_2+\cdots+(n-k+1)\Lambda c_{k-1}+c_k\right]$$

After simplifying, we obtain the stated expressions.

$\square$

A set of simple sufficient conditions are of the form

$$c_k^2 > m\, c_{k-1}\, c_{k+1}, \quad \text{for } 1 \leqq k \leqq n-1, \tag{1}$$

where $c_0 \equiv 1$. One such formulation appears in an article by C.S. Ballantine [1], who in turn attributes the result to "folklore". He states that when the conditions (1) hold for m=4, then p(x) is Hurwitz, and all roots of p(x) are real and simple. This is proven by finding numbers $0=r_0>r_1> \cdots >r_n$ such that $\operatorname{sgn} p(r_k)=(-1)^k$.

The following apparently similar result is established quite differently.

Theorem 2.3. Let $k=\lfloor\frac{n}{2}\rfloor$ and let m be a positive number satisfying $\sum_{j=1}^{k} m^{-j^2}<0.5$. Then p(x) with positive coefficients $c_j>0$ is Hurwitz if (1) holds. (In this case p(x) need not have real simple roots.)

Proof

We will show that each of the following matrices, for $2 \leqq j \leqq n-1$, is quasi-diagonal dominant. That is, there are positive numbers by which we can multiply each row [or each column] so that the diagonal entries of the resulting matrix have absolute value exceeding the sum of the absolute values of all other entries in their columns [respectively, rows].

$$\begin{bmatrix} c_1 & c_3 & & & \cdot \\ c_0 & c_2 & & & \cdot \\ 0 & c_1 & \cdot & & \cdot \\ \cdot & c_0 & & \cdot & \cdot \\ \cdot & 0 & & \cdot & c_{j+1} \\ \cdot & \cdot & & & c_j \end{bmatrix}$$

As usual, we define $c_k=0$ when $k > n$.

It is then the case that all eigenvalues of the above matrices must have positive real part. (See McKenzie [24]). Thus, the corresponding determinants $H_2,\ldots,H_{n-1}$ are positive, and the Routh-Hurwitz criterion may be applied.

It suffices to show that the largest such matrix (with j=n-1) is quasi-diagonal dominant. To demonstrate this, multiply the rth row by the positive number $c_r$ for $1 \leqq r \leqq n-1$. We then obtain a matrix whose kth column has the following form.

$$\begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ c_{k-2}c_{k+2} \\ c_{k-1}c_{k+1} \\ c_k^2 \\ c_{k+1}c_{k-1} \\ c_{k+2}c_{k-2} \\ \cdot \\ \cdot \\ \cdot \end{bmatrix}$$

It is claimed that the diagonal element $c_k^2$ exceeds the sum of all other elements in the kth column.

To establish this claim we first show by induction that $c_k^2 > m^{j^2} c_{k-j}c_{k+j}$ for $1 \leqq j \leqq \min\{k, n-k\}$.

The case with $j=1$ is the hypothesis of the theorem. Assume the inequality holds for $1 \leqq r < j$. Since $\frac{c_k}{c_{k-1}} > m\frac{c_{k+1}}{c_k}$, we have $\frac{c_{k-j+1}}{c_{k-j}} > m^{2j-1}\frac{c_{k+j}}{c_{k+j-1}}$ or $c_{k-j+1}c_{k+j-1} > m^{2j-1}c_{k-j}c_{k+j}$. By inductive hypothesis $c_k^2 > m^{(j-1)^2}c_{k-j+1}c_{k+j-1}$. Thus, $c_k^2 > m^{(j-1)^2}m^{2j-1}c_{k-j}c_{k+j} = c^{j^2}c_{k-j}c_{k+j}$.

We apply this by observing that

$$2\sum_{j=1}^{\min\{k,\, n-k\}} c_{k-j}c_{k+j} < 2\sum_{j=1}^{\min\{k,\, n-k\}} m^{-j^2}c_{k^2} < c_{k^2}$$

since min{k, n-k} $\leqq \lfloor n/2 \rfloor$. Thus, the diagonal elements dominate each column as claimed.

□

The lower bound for the value of m will vary with n, but, for example, values in excess of 2.198 are always permissible.

In general, of course, for no positive value of m is it necessary that $c_k^2 > mc_{k-1}c_{k+1}$. In the special case where all roots of p(x) are real and negative, however, it is then the case that $c_{k-r}c_{k+r} > c_{k-r-1}c_{k+r+1}$, for $1 \leqq r \leqq k \leqq n-r-1$, and when r=0, then $c_k^2 \geqq 2c_{k-1}c_{k+1}$, as is readily established by set inclusion arguments on the index sets. If the roots are simple, the latter inequality becomes strict.

Suppose we have functions $c_i$:D→R for $0 \leqq i \leqq n$, and let m be a positive real number. We will refer to $c_0,\ldots,c_n$ as <u>downward independent</u> with respect to D and m, if there exists u ∈ D such that for $1 \leqq i \leqq n-1$, $0 < c_{i+1}(u) < \frac{c_i^2(u)}{mc_{i-1}(u)}$. Then we may rephrase Theorem 2.3 as saying that the matrix A is potentially stable if the coefficients of its characteristic polynomial are downward independent with respect to Q(A) and some positive number m satisfying $\sum_{j=1}^{k} m^{-j^2} < 0.5$.

## 2.4 Sufficient conditions

Given an $n \times n$ matrix A, we define a skeleton of A as any $n \times n$ matrix B for which S(B) is a subgraph of S(A). That is, we obtain Q(B) from Q(A) by setting an arbitrary number of entries equal to zero.

Lemma 2.4. If any skeleton of A is potentially stable, then A is potentially stable.

Proof.

The function $F: R^{n\times n} \to R^{2n-n}$ by $F(A) = (c_1,\ldots,c_n,H_2,\ldots,H_{n-1})$ is continuous, and therefore $F^{-1}(]0,\infty]^{2n-2})$ is an open set. By the Routh-Hurwitz criterion this is precisely the set of stable $n \times n$ matrices. If B is a skeleton of A, and $B_0 \in Q(B)$ is stable, then some neighborhood $N(B_0)$ is stable, and $N(B_0) \cap Q(A) \neq \phi$. Therefore, there exists stable $A_0 \in Q(A)$.

□

Lemma 2.4 is based upon the fact that stability is an open property, in the sense that any stable matrix is an interior point in the set of all such matrices. It includes the correct direction of Quirk's Proposition 2, the converse of which was shown false in section 2.1.

The following construction may be regarded as a procedure which starts with one potentially stable matrix (initially $1 \times 1$) and borders it (adds one row and one column) to produce another, repeating the process until a potentially stable matrix of the desired dimension is obtained.

Construction 1. We construct an $n \times n$ matrix A as follows:

(i) for $k = 1,2,\ldots,n$, A contains an even parity compound k-cycle $\Sigma_k$;

(ii) the cycles are nested, that is, $V(\Sigma_1) \subset V(\Sigma_2) \subset \ldots \subset V(\Sigma_n)$;

(iii) for $k=2,\dots,n$, whenever $\Sigma_k$ contains a factor $\Sigma'$ and $m<k$ is the smallest integer such that $\Sigma_1 \cdots \Sigma_m$ contains a (simple or compound cycle) factor $\Sigma''$ with $V(\Sigma') = V(\Sigma'')$, then $\Sigma' = \Sigma''$.

Condition (iii) may be paraphrased as follows: no two distinct factors of $\Sigma_1, \Sigma_2, \dots, \Sigma_n$ have the same vertex set. If the $\Sigma_k$ consist entirely of negative cycles, then any matrix constructed according to conditions (i) and (ii) is necessarily constructed according to condition (iii) as well. On the other hand, if some $\Sigma_k$ contains a positive factor, then condition (iii) forbids that any other cycle $\Sigma_j$ contain a negative factor covering the same vertices, for in this case making $\Sigma_k$ dominant in $c_k$ would conflict with making $\Sigma_j$ dominant in $c_j$.

Theorem 2.5. The matrix A produced by Construction 1 is potentially stable.

Proof

One checks that the coefficients of the characteristic polynomial of a constructed matrix A are downward independent with respect to Q(A) and any $m>0$.

Let $A_r$ be a matrix in the closure $\overline{Q(A)}$ of Q(A) with all entries zero except those entries which appear in one or more of the compound cycles $\Sigma_1, \dots, \Sigma_r$. Suppose (1) holds for the coefficients $c_i$ of $\det(xI - A_r)$ for $1 \leq i \leq r-1$. Then we can obtain $A_{r+1} \in \overline{Q(A)}$ from $A_r$ by creating additional nonzero entries so small that (1) still holds for the coefficients of $A_{r+1}$ for $1 \leq i \leq r-1$, and so that the magnitude of $a[\Sigma_{r+1}]$ is less than $\frac{c_r{}^2}{mc_{r-1}}$. Conditions (ii) and (iii) of Construction 1 guarantee that $a[\Sigma_{r+1}]$ is the only term in the expansion of $c_{r+1}$ for $A_{r+1}$, and condition (i) guarantees that it is positive.

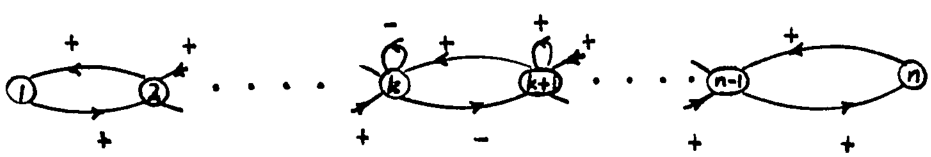
Proceeding inductively, we obtain potentially stable skeleton $A_n \in \overline{Q(A)}$, making A potentially stable by Theorem 2.4.

□

The same result could be deduced as a corollary of the Fisher-Fuller theorem on the diagonal stabilization of matrices with a nested sequence of nonzero principal minors. [6,5] Conversely, an argument parallel to the above proof may be applied to prove the Fisher-Fuller theorem. [1]

We now use Construction 1 to demonstrate the existence of potentially stable $n \times n$ matrices of each order $n \geqq 2$ with all but two cycles positive.

<u>Construction 2.</u> Let $k = \lfloor n/2 \rfloor$. That is, n is either 2k or 2k-1. We construct an $n \times n$ matrix A as follows. A is arbitrary except that we require $a_{kk}<0$, $a_{k+1,k+1}>0$, $a_{k,k+1}a_{k+1,k} < 0$, and $a_{i,i+1}a_{i+1,i} > 0$ for all $i \neq k$, $i=1,\ldots,n-1$. For example, if n is even, A is tridiagonal, and all $a_{ii}=0$ if $k \neq i \neq k+1$, then S(A) as follows.




<u>Lemma 2.6.</u> Construction 2 always produces potentially stable matrices.

<u>Proof</u>

Let $\Sigma_1=a_{kk}<0$ and $\Sigma_2=a_{k,k+1}a_{k+1,k}<0$. The resulting $2 \times 2$ submatrix is potentially stable by Theorem 2.5, and by Lemma 2.4 it is still potentially stable when we add the positive 1-cycle $a_{k+1,k+1}>0$. Since $V(\Sigma_2) = V(\Sigma_1 a_{k+1,k+1})$, condition (iii) of the construction requires that $\Sigma_2$ be used in all subsequent cycles with a factor in vertices k and k+1. Let $\Sigma_3=(a_{k-1,k}a_{k,k-1})(a_{k+1,k+1})$ and $\Sigma_4=(a_{k-1,k}a_{k,k-1})(a_{k+1,k+2}a_{k+2,k+1})$. Each is a product of two positive cycles and therefore of even parity. For r=5, 6,..., n, let $V(\Sigma_r)=\{k-\lfloor\frac{r-1}{2}\rfloor,\ldots,k+\lfloor\frac{r}{2}\rfloor\}$. Notice that the extreme vertices of $V(\Sigma_r)$ together with condition (iii) unambiguously determine $\Sigma_r$ as the product of the two extreme positive 2-cycles and $\Sigma_{r-4}$. Thus, by the construction the resulting matrix is potentially stable.

□

The following construction allows us to create potentially stable matrices which violate the nesting condition of Construction 1 and the Fisher-Fuller Theorem. This generalizes the last example of section 2.1.

Construction 3. We construct $n \times n$ matrix A as follows:

(i) for k = 1,2,...,n, A contains a compound k-cycle $\Sigma_k$;

(ii) $\Sigma_1$ and $\Sigma_2$ are even parity cycles;

(iii) for k=1,...,n-2, if $\Sigma_{k+2}$ is even parity then $V(\Sigma_k)\subset V(\Sigma_{k+1})$;

(iv) for k=1,...,n-2, if $\Sigma_k$ and $\Sigma_{k+1}$ are not nested, then $\Sigma_{k+2}$ is odd parity, $\Sigma_{k+1}=\Sigma_{k-1}\sigma_{k+1}$ for some negative two-cycle $\sigma_{k+1}$, and $V(\Sigma_k)\cup V(\Sigma_{k+1})=V(\Sigma_{k+2})$;

(v) no two distinct factors of $\Sigma_1,\ldots,\Sigma_n$ have the same vertex set.

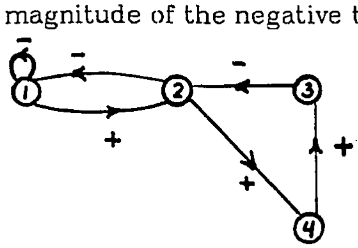
Theorem 2.7. Any matrix A produced by Construction 3 is potentially stable.

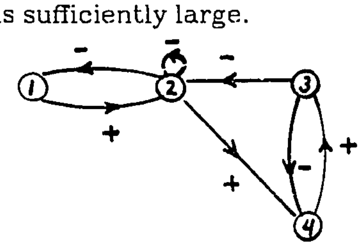
Proof

Again, we show downward independence of the $c_i$ with respect to Q(A) and arbitrary m. Inductively assume downward independence holds for $c_1,\ldots,c_r$ coefficients of the characteristic polynomial of $A_r$, which has all entries zero except those appearing in cycles $\Sigma_1,\ldots,\Sigma_r$. Create additional nonzero entries corresponding to $\Sigma_{r+1}$, and consider coefficients $\tilde{c}_1,\ldots,\tilde{c}_{r+1}$ for $A_{r+1}$. If $\Sigma_r$ and $\Sigma_{r+1}$ are nested, even parity compound cycles, proceed as in the proof of Theorem 2.5. If they are not nested, it is nevertheless true that $\sigma_{r+1}$ is a factor of every term of $\tilde{c}_{r+1}>0$, which can thus be made arbitrarily small. Finally, if $\Sigma_r$, $\Sigma_{r+1}$ are nested, but $\Sigma_{r+1}$ is odd parity, then $\tilde{c}_{r+1}=c_{r-1}\,|a[\Sigma_{r+1}]|$, which can be made positive by making the newly created entries for $\Sigma_{r+1}$ small enough, and which can be made arbitrarily small by making the entries for $\sigma_r$ small enough.

□

Although Construction 3 gives us more potentially stable matrices than before, it still does not give us all of them. As examples, consider the 4 × 4 matrices whose digraphs appear below. Each matrix will be stable when the magnitude of the negative two-cycle [1,2] is sufficiently large.




In such a case we can still apply parts of Constructions 1 and 3 to construct larger potentially stable matrices by bordering existing ones. We have the following.

Construction 4. Given an (n-1) × (n-1) matrix $A^-$, we construct an n × n matrix A as follows:

(i) let $A^-$ appear as a principal submatrix of A;

(iii) A contains a negative n-cycle.

Construction 5. Given an (n-2) × (n-2) matrix $A^-$, we construct an n × n matrix A as follows:

(i) let $A^-$ appear as a principal submatrix of A;

(ii) the 2 × 2 principal submatrix complementary to $A^-$ contains a negative 2-cycle;

(iii) A contains a positive n-cycle.

Theorem 2.8. If matrix $A^-$ is potentially stable, then the matrix A produced by either Construction 4 or Construction 5 is also potentially stable.

Proof

First, suppose A results from Construction 5.

For convenience suppose $A^-=A_0^- \in Q(A^-)$ is itself stable. Let t be the value of the positive n-cycle, and first set t=0. When the n-cycle is not present, the eigenvalues of A are just the eigenvalues of $A^-$ together with those of the complementary 2 × 2 submatrix. By continuity there exists $\varepsilon > 0$ such that, when $0 < t < \varepsilon$, we may continuously factor $p_A(x) = (x^{n-2}+a_1x^{n-3}+\cdots+a_{n-2})(x^2+b_1x+b_2)$ so that the roots of the first factor remain stable. When t=0, the first factor is $p_{A^-}(x)$, $b_1$ is 0, and $b_2$ is the absolute value of the negative 2-cycle. As t increases from 0 the conjugate imaginary roots of the second factor become stable if $b_1$ becomes positive. At the same time the growing positive n-cycle decreases $c_n$. Thus, it suffices to show $\frac{\partial b_1}{\partial c_n}\big|_{\{t=0\}}<0$.

Some computation yields the following expression.

$$\frac{\partial(c_1, \ldots, c_n)}{\partial(a_1, \ldots, a_{n-2}; b_1, b_2)}\Big|_{\{t=0\}} = \begin{bmatrix} 1 & 0 & \cdots & & 0 & 1 & 0 \\ 0 & 1 & \cdot & & & a_1 & 1 \\ b_2 & 0 & \cdot\ \cdot & & & a_2 & a_1 \\ 0 & b_2 & \cdot\ \cdot\ \cdot & & & \cdot & \cdot \\ \cdot & & \cdot\ \cdot\ \cdot & & 0 & \cdot & \cdot \\ \cdot & & \cdot\ \cdot & & 1 & \cdot & \cdot \\ \cdot & & \cdot & & 0 & a_{n-2} & \cdot \\ 0 & \cdot & \cdot\ \cdot & 0 & b_2 & 0 & a_{n-2} \end{bmatrix}$$

Note that $\partial b_1/\partial c_n|_{\{t=0\}}$ is the (n-1,n) entry of the inverse matrix. Assuming the magnitude of the negative 2-cycle is sufficiently small, we can ignore $b_2$ and obtain $\operatorname{sgn} \frac{\partial b_1}{\partial c_n}\Big|_{\{t=0\}} = \operatorname{sgn} \frac{-a_{n-3}}{a_{n-2}^2} = -1$, as claimed.

In the case of Construction 4 a similar computation yields the desired result.

□

The above proof also shows that when a negative two-cycle is joined to a stable digraph on $n-2$ vertices by means of a small negative n-cycle, the resulting matrix is unstable. This provides another instance in which a positive cycle is preferred to a negative cycle for purposes of stability.

## 2.5 Necessary conditions

We might define a solvability property corresponding to potential stability as follows. Let us say that the system $Ax + b = 0$ is <u>potentially solvable</u> if there exists $C \in Q(A)$, $d \in Q(B)$, such that $Cx + d = 0$. This property is easily recognized. As Klee and Ladner note in [15] (and we repeat in section 3.2), (A,b) is potentially solvable iff each of the sets $\{a_{i1}x_1, \ldots, a_{in}x_n, b_i\}$ is <u>balanced</u> (either all zero elements or at least one positive and one negative element). In this section we explore the relationship between stability properties and the solvability of the matrix equation $AX+XA^T=-I$.

The following theorem is from Taussky [34]. Let $\Pi_n$ represent the set of positive definite $n \times n$ hermitian matrices. That is, $H \in \Pi_n$ if $h_{ij}=h_{ji}$ and $x \in R^n-\{0\}$ implies that $x^T Hx>0$.

Theorem 2.9. (Taussky) The following conditions are equivalent:

(a) A is a stable matrix;

(b) there exists $H \in \Pi_n$ such that $AH+HA^T=-I$;

(c) there exist $G, H \in \Pi_n$ such that $AH+HA^T=-G$;

(d) for every $G \in \Pi_n$, there exists $H \in \Pi_n$ such that $AH+HA^T=-G$.

This suggests that we should be interested in the solvability of $AX+XA^T+I=0$, where X is positive definite. We may ask how great a restriction is imposed by the requirement that X be positive definite. To answer that we have the following theorem. Let X be a real symmetric $n \times n$ matrix.

Theorem 2.10. $X \in \Pi_n$ iff X is quasi-diagonal dominant with negative diagonal.

Proof

The "if" direction follows from Gerschgorin's theorem and the fact that, since symmetric matrices are orthogonally diagonalizable, a symmetric matrix with positive eigenvalues must be positive definite.

To prove the "only if" implication, we proceed as follows. Obviously, all $x_{ii}<0$. To prove quasi-diagonal dominance, first note that $i \neq j$ implies $x_{ij}^2 < x_{ii}x_{jj}$. For the vector u with all entries zero except $u_i = \text{sgn}\,(x_{ij})/\sqrt{x_{ii}}$, $u_j = -1/\sqrt{x_{jj}}$, has $u^T X\, u = 2\,(1 - |x_{ij}|/\sqrt{x_{ii}x_{jj}}\,) > 0$. It follows that the maximal compound r-cycle consists of 1-cycles. That is, if $(i \rightarrow j) = (i_1, i_2, \ldots, i_r)$, then

$\left| \dfrac{x_{ij}\, x(i_1,i_2,\ldots,i_r)}{x[i_1]x[i_2]\cdots x[i_r]} \right| < 1$, as is apparent when both sides of the inequality are squared.

Without loss of generality, suppose X is irreducible, and that $x_{ii}$ is the maximal diagonal element. Then $x_{ii}$ already dominates the off-diagonal elements of row i. Define $d_i = 1$. For $j \neq i$, let $P(i,j)$ be the set of all paths $(i \to j)$, and define

$$d_j = \max_{(i_0,i_1,\ldots,i_r) \in P(i,j)} \left\{ \left| \frac{x(i_0,i_1,\ldots,i_r)}{x[i_1]\cdots x[i_r]} \right| \right\}.$$

Then DA is diagonally dominant, where $D = \text{diag}\,\{d_1,\ldots,d_n\}$. For let j and k be any distinct vertices and let $(i_0,i_1,\ldots,i_r)$ be the path $(i \to k)$ for which the maximum value $d_k$ is achieved. If this path does not pass through vertex j, then

$$d_k |x_{jk}| = \left| \frac{x(i_0,i_1,\ldots,i_r)x(k,j)}{x[i_1]\cdots x[i_r]x[j]} \right| x_{jj} \leq d_j x_{jj}.$$

If the path does pass through j, then we have

$$\begin{aligned} d_k |x_{jk}| &= \left| \frac{x(i_0 \to j)x(j \to k)}{x[i_1]\cdots x[j]\cdots x[k]} \right| |x_{jk}| \\ &= \left| \frac{x(i_0 \to j)x[j]}{x[i_1]\cdots x[j]} \right| \left| \frac{x_{jk}x(j \to k)}{x[j]\cdots x[k]} \right| \\ &< \left| \frac{x(i_0 \to j)x[j]}{x[i_1]\cdots x[j]} \right| \leq d_j x_{jj}. \end{aligned}$$

□

We may apply this to derive the following necessary condition for potential stability.

<u>Theorem 2.11.</u> A is potentially stable only if there exists $Y \in \{-1,0,1\}^{n \times n}$ such that each of the following $n^2+n$ sets is balanced:

(a) $\{y_{ii}, -1\}$, for $1 \leqq i \leqq n$,

(b) $\{y_{ij}, -y_{ji}\}$, for $1 \leqq i < j \leqq n$,

(c) $\{a_{i1}y_{i1}, \ldots, a_{in}y_{in}, 1\}$, for $1 \leqq i \leqq n$,

(d) $\{a_{i1}y_{j1}, \ldots, a_{in}y_{jn}, a_{j1}y_{i1}, \ldots, a_{jn}y_{in}\}$, for $1 \leqq i < j \leqq n$.

<u>Proof</u>

The condition that the sets in (a) and (b) are balanced is just the requirement that Y be symmetric with positive diagonal, while the condition that the sets in (c) and (d) are balanced is necessary for the solvability of $AY+YA^T+D=0$, for some $D \in Q(I) \subset \Pi_n$.

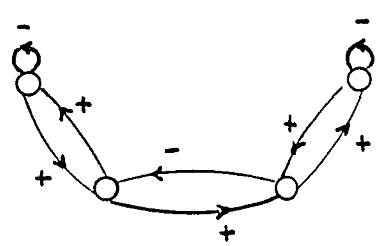
□

Define $\varphi(A)=A \otimes I + I \otimes A$ and define $E \in R^{n^2}$ by $E_{n(i-1)+j}=\delta_{ij}$. Then the above theorem gives necessary and sufficient conditions for potential solvability of $\varphi(A)Y+E=0$. Unfortunately, $\tilde{B} \in Q(\varphi(A))$ does not imply $\tilde{B}=\varphi(B)$ with $B \in Q(A)$, so the conditions are not sufficient for potential solvability of $AY+YA^T=-I$.

We illustrate this by means of the following example. Let Q(A) and S(A) be as follows.

$$\begin{bmatrix} - & 0 & 0 & + \\ 0 & - & + & 0 \\ 0 & + & 0 & - \\ + & 0 & - & 0 \end{bmatrix}$$

Q(A)



S(A)

One may verify that there exist B, C $\in$ Q(A) such that $BY+YC+I=0$ when $y_{ii}=2$, $y_{ij}=1$. But A cannot be potentially stable since S(A) contains no even parity compound 3-cycle.

## 2.6 On some theorems of Campbell

In this section we explore the relationship between our results and the earlier work of Campbell. [4]

Campbell also makes use of the Fisher-Fuller theorem to derive sufficient conditions for potential stability. He proves that the following classes of real $n \times n$ matrices are potentially stable:

(a) Matrices in which vertices $i_1,\dots,i_t$ are self-limiting and, for each s with $t+1 \leqq s \leqq n$, there is a negative s-cycle with vertex set $\{i_1,\dots,i_s\}$.

(b) The class J of section 2.1.

(c) The class K of matrices, which he defines as irreducible, combinatorially symmetric matrices with all one-cycles and even cycles nonpositive, some one-cycles negative, and some nonzero term in the expansion of the determinant of A consisting of one-cycles and even cycles.

The first class is included in the second. Unfortunately, the proof that class J is potentially stable is erroneous, as we have noted in section 2.1.

The theorem that class K is potentially stable first appears with an incomplete proof in a paper of Maybee. [21] Campbell furnishes a complete proof via a series of six lemmas. A shorter proof is the following.

Theorem 2.12. Let A be an irreducible matrix that satisfies the following conditions:

(a) A is combinatorially symmetric, that is, $a_{ij}=0$ iff $a_{ji}=0$.

(b) All $a_{ii} \leqq 0$ and there exists k such that $a_{kk}<0$.

(c) All even cycles of A are nonpositive.

(d) There is a term in det A consisting of 1-cycles and even cycles.

Then A is potentially stable.

Proof

First, note that irreducibility, (a), and (c) imply that all the vertices of A are connected via negative two-cycles. Consider the undirected graph $G = (V,E_u)$ where $E_u=\{(i,j) : a[i,j]\neq 0\}$. Given any even cycle $\sigma$, by combinatorial symmetry there exists a compound cycle $\Sigma$ consisting of two-cycles with $V(\Sigma)=V(\sigma)$. Thus, we may interpret hypothesis (d) as saying that G admits a $(V-R_A)$-complete matching, where $R_A$ is the set of self- limiting vertices. So A contains a skeleton which is sign semistable and nonsingular. Since the subset of nontrivially Im-colorable semistable matrices which exhibit a pure imaginary eigenvalue contains no open set, it follows that A is potentially stable.

□

We note that the paper by Klee and van den Driessche contains matching algorithms which answer the question posed by Campbell of efficiently recognizing class K. [17]

Campbell uses what we have referred to as even parity compound cycles when he considers necessary conditions for potential stability. He proves that the class of real $n \times n$ potentially stable matrices is contained in the following classes (as a proper subset for $n\geq 3$):

(a) The class of potentially pre-stable matrices, which he defines as all matrices A for which there exists $B \in Q(A)$ with all positive coefficients of its characteristic polynomial.

(b) The class N which he defines as matrices in which either each vertex i is contained in a negative cycle or there exists a disjoint pair of positive cycles.

(c) More generally, the class of matrices in which there is an even parity compound k-cycle for $1\leq k\leq n$.

Campbell shows that the classes of parts (a) and (c) are the same for $n \leqq 3$, or when the even parity compound cycles contain no positive cycles. In general, however, potentially pre-stable matrices are a proper subset of those containing an even parity compound k-cycle. Campbell illustrates this for $n=4$ and generalizes his example to a theorem which we may restate as follows.

Theorem 2.13. (Campbell) Suppose $S(A)$ contains a positive s-cycle $\sigma$ and there exists r such that

(a) for every even parity compound r-cycle $\Sigma$, $V(\sigma) \cap V(\Sigma) = \phi$,

(b) $\sigma$ is a factor of every even parity compound (s+r)-cycle.

Then A is not potentially pre-stable.

Proof (after Campbell)

Let $c_0, \ldots, c_n$ be the coefficients of det $(Ix-B)$ for any $B \in Q(A)$. Then every term of $c_{r+s}$ which does not involve $b[\sigma]$ is negative. Thus, $c_{r+s} \leqq (-1)^s b[\sigma] c_r$. Since $(-1)^s b[\sigma] < 0$, this implies that either $c_{r+s} \leqq 0$ or $c_r \leqq 0$.

□

## 2.7 Open questions

The characterization of potentially stable matrices remains unresolved. Within this general problem there are some smaller problems of interest.

The first problem is to determine which matrices of some restricted class are potentially stable. For instance, a matrix is defined to be an acyclic-k matrix if its digraph contains no r-cycles for any $r \geqq k$. Thus, condition (c) of Theorem 1.2 states that a matrix which is sign stable must be acyclic-3. We conjecture that every potentially stable acyclic-3 matrix may be obtained by Construction 1. If one could show that every such matrix of order n contains a potentially stable principal submatrix of order n−1, then the conjecture would be verified. Note that the class of acyclic-3 matrices includes the subclass of tri-diagonal matrices.

A related problem is the simplification of the Routh-Hurwitz criteria for acyclic-k matrices. For example, if matrix A is acyclic-3, then the condition that $H_3>0$ requires that S(A) contain a 1-cycle and a compound 2-cycle of the same parity and with overlapping vertex sets.

Another open question concerns the computational complexity of recognizing potentially stable matrices. Tarski's decision procedure for elementary algebra [33] together with the Routh-Hurwitz criterion provides a finite algorithm of a very high order of complexity. It is hoped that the approach of Theorem 2.11 might lead to a reduction of the problem to that of recognizing singularity of certain matrices constructed from the original matrix. In this case it is conjectured that the complexity would be similar to that of recognizing L-matrices (see [17]).

## CHAPTER 3.

## SOLVABILITY OF SIGN STABLE SYSTEMS

### 3.1 Introduction

We will say that the system $Ax+b=0$ is positively sign solvable if $(C,d)\sim(A,b)$ implies C is nonsingular and $-C^{-1}d>0$ in all entries. That is, (A, b) is strongly sign solvable and sgn $x_i = 1$. We are interested in systems (A, b) for which A is sign stable and (A, b) is positively sign solvable. We will refer to such systems as viable.

As noted in section 1.2, criteria exist for recognizing sign stability and for recognizing sign solvability of systems in standard form. Unfortunately, the operations used to bring a system into standard form preserve sign solvability, but not sign stability. Thus, although we can separately test sign solvability and sign stability of any given system, the two separate characterizations do not immediately tell us what structure typifies systems with both of these properties.

### 3.2 Assignments

The following considerations are often useful for determining positive sign solvability. Consider (A,B) where B is a real n×m matrix.

An assignment is a vector $y\in\{-1,0,1\}^n$. Given a vertex $i\in V$, we say an assignment y is admissible at i if the set $\{a_{i1}y_1,\ldots,a_{in}y_n,b_{i1},\ldots,b_{im}\}$ is all zero or contains at least one positive and one negative member. If y is admissible at i for all $i\in V$, we say that y is admissible.

Clearly, if B is a column vector and $Ax+B=0$ or, more generally, if each $\sum_{j=1}^{n}(a_{ij}x_j+b_{ij})=0$, then sgn x must be an admissible assignment. It is not hard to see that if y is any admissible assignment for (A,B), we can find (C,D)~(A,B) such that each $\sum_{j=1}^{n}(c_{ij}x_j+d_{ij})=0$. In fact, (C,D) can be chosen so that the augmented matrices (A;B) and (C;D) differ in at most one entry per row. Thus, a system is positively sign solvable iff its only admissible assignment is the vector y with $y_i=1$ for all $i \in V$.

The following allows us to apply our test for sign solvability to one strong component of S(A) at a time. We say i is <u>upstream</u> of j (and j is <u>downstream</u> of i) if there is a path from i to j, but not from j to i. This notion generalizes naturally to strong components of S(A).

<u>Lemma 3.1.</u> Suppose $S_G$ is a strong component of S(A) and y is an assignment with $y_i \neq 1$ for some vertex of $S_G$. If y is admissible for all vertices of $S_G$ and upstream components, then (A,B) is not positively sign solvable.

<u>Proof</u>

Suppose (A,B) is positively sign solvable. Relabel the vertices of V so that $i<j$ whenever i is upstream from j. Then for any (C,D)~(A,B), the matrix of C in block form is lower triangular, and every principal minor of this block matrix must be nonsingular. Thus, every assignment admissible for all vertices of $S_G$ and upstream components has the uniquely determined restriction $y_i=1$ for each of those vertices.

□

In what follows we assume that S(A)=(V, P, N) is semistable. If S(A,B)=(V', P', N') is the augmented graph associated with the system Ax + B = 0, note that there are no edges to vertices in V'−V, so that an assignment is admissible for V' iff it is admissible for V.

## 3.3 Elimination of branches

Define a branch based at v as a strongly connected vertex induced subgraph $S_B=(V_B,P_B,N_B)$ of S(A,b) with base $v \in V_B$ such that there is no edge in S(A,b) from $V'-V_B$ to $V_B-\{v\}$. A vertex of S(A,b) is an end vertex of a subgraph if it has exactly one neighbor in the subgraph.

Lemma 3.2. If (A,b) is viable, then all end vertices of a branch other than the base vertex are self-limiting.

Proof

Let i be the end vertex and x any solution of $Ax+b=0$. We cannot have exactly one element of the set $\{b_i, a_{i1}x_1, \ldots, a_{in}x_n\}$ nonzero. Since $b_i=0$ for branch vertices other than the base vertex, but the element corresponding to i's neighbor is nonzero, we must have $a_{ii}x_i \neq 0$.

□

Lemma 3.3. If (A,b) is viable, then every branch is outward positive. That is, $(v \to i)$ is positive for all vertices $i \in V_B-\{v\}$ of a branch based at v.

Proof

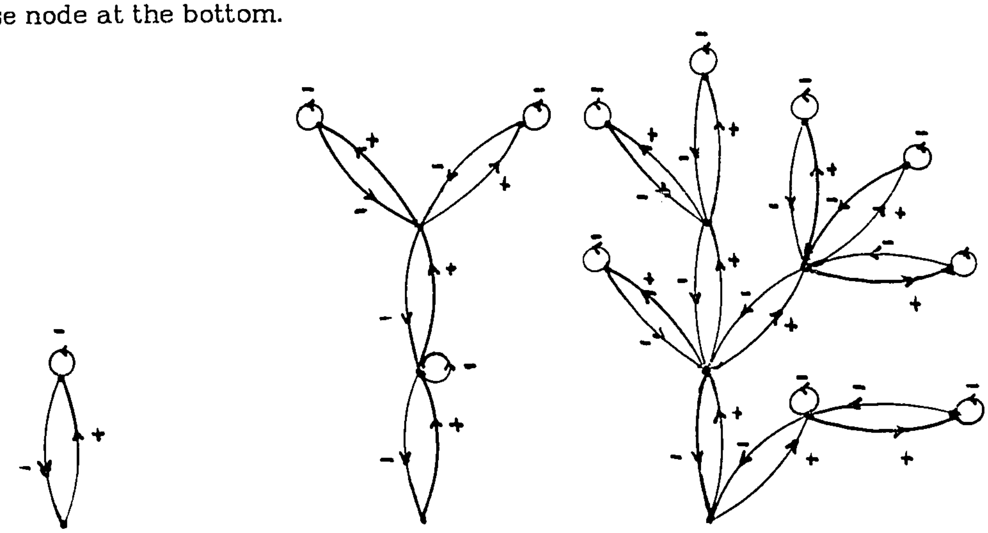
This is equivalent to showing that if $i,j \in V_B$ are neighbors and dist(i,v)> dist(j,v), then a(i,j)>0 (recall that $a(i,j)=a_{ji}$ ). If i is an end vertex this follows from admissibility at i, since $a_{ii}x_i \leqq 0$ when (A,b) is viable. Suppose we have established the result for all vertices of $V_B$ whose distance from the base exceeds k=dist(v,i), and suppose i is not an end vertex. Since all two-cycles are negative, it follows that $a_{im}<0$ whenever m is a neighbor of i with dist(v,m)>k. Hence, all products $a_{im}x_m$ are negative, so admissibility requires that $a_{ij}x_j>0$ for some neighbor j with dist(v,j)<k. But i has only one neighbor closer to v than itself, so the result is established.

□

Accordingly, define a <u>same sign branch</u> as an outward positive branch in which all end vertices other than the base are self-limiting.

<u>Example</u>

Below are three typical same sign branches. Each branch is drawn with the base node at the bottom.



The next two lemmas are easily checked on the example.

<u>Lemma 3.4.</u> There are no nontrivial 0- or Im- colorings of a same sign branch.

<u>Proof</u>

For k ranging from the diameter of $S_B$ down to 0, let $i \in V_B$ be any vertex with dist(v,i)=k. If i is an end vertex, then i is black by coloring condition (i).

If i is not an end vertex, then it has a neighbor j further out. By induction all of j's neighbors other than i are black, so i must also be black by coloring condition (ii).

□

Lemma 3.5. If $S_B=(V_B,P_B,N_B)$ is a same sign branch of S(A,b) with base v, and if Ax+b=0, then sgn $x_i$= sgn $x_v$ for all $i \in V_B$ .

Proof

We let k range from the diameter of $S_B$ down to 1 and establish that whenever $i \in V_B$ with dist(i,v)=k has neighbor $j \in V_B$ with dist(j,v)=k−1, then sgn $x_i$= sgn $x_j$. It follows from admissibility at i that $a_{ij}x_j$ must have sign opposite to that of any nonzero element of $\{a_{ii}x_i\} \cup \{a_{im}x_m\text{:dist}(m,v)=k+1\}$. Since $a_{ij}$ is of opposite sign to any nonzero element of the corresponding set of coefficients $\{a_{ii}\} \cup \{a_{im}\text{:dist}(m,v)=k+1\}$, it follows that $x_j$ must be of the same sign as sgn $x_i$= sgn $x_m$.

□

In the rest of this section it is shown how we can replace the same sign branches of a graph by self-limiting vertices while preserving the sign solvability and sign stability properties of the graph.

Define the reduced system $(\tilde{A},\tilde{B})$ of a pair (A,b) as follows. Let $\{i_1,\ldots,i_k\} \subset V$ consist of those vertices which do not appear in same sign branches of S(A,b) together with the base vertices of all such branches. That is,

$$\{i_1,\ldots,i_k\} = V - (\ (V_{B_1}-\{v_1\}) \cup \cdots \cup (V_{B_r}-\{v_r\})\ ),$$

where $S_{B_1}, \cdots ,S_{B_r}$ are the same sign branches of S(A,b). Let $\{j_1,\ldots,j_{n-k}\}$ denote the remaining vertices of V. For $1 \leqq r \leqq k$ and for $1 \leqq s \leqq n-k$ the entries of the k×k matrix $\tilde{A}$ and the k×(n−k+1) matrix $\tilde{B}$ are given by the following.

$$\tilde{a}_{rs} = \begin{cases} a_{i_r i_s} & \text{if } i_r \neq i_s \text{ or } i_s \text{ is not the base vertex of a same sign branch} \\ -1 & \text{if } i_r = i_s \text{ is such a base vertex} \end{cases}$$

$$\tilde{b}_{rs} = \begin{cases} a_{j_r i_s} & \text{if } i_s \text{ is not the base vertex of a same sign branch that includes } j_r \\ 0 & \text{if } i_s \text{ is such a base vertex} \end{cases}$$

$$\tilde{b}_{r,n-k+1} = b_{i_r}$$

Thus, the reduced system is formed by replacing all same sign branches with self-limiting vertices, and by reclassifying edges from the deleted vertices of each branch to any downstream vertex as edges in $E'-E$.

Theorem 3.6. (A,b) is viable iff the reduced system $(\tilde{A},\tilde{B})$ is viable.

Proof

Neither adding nor removing a same sign branch creates positive cycles or k-cycles for $k \geq 3$ if none exist before. In either a 0-coloring or an Im-coloring both the same sign branch and the self-limiting vertex which replaces it constitute black blocks with the same set of neighbors. Thus A is sign stable iff $\tilde{A}$ is.

To establish positive sign solvability we check that y is an admissible assignment for (A,b) iff $\tilde{y}$ is an admissible assignment for $(\tilde{A},\tilde{B})$, where we obtain $\tilde{y}$ from y by setting $\tilde{y}_r = y_{i_r}$, and we extend $\tilde{y}$ to y by requiring that $y_j = \tilde{y}_r$ if j is part of a same sign branch based at $i_r$. Note that if either (A,b) or $(\tilde{A},\tilde{B})$ is positively sign solvable, all entries of both y and $\tilde{y}$ must be 1. Thus, mutual admissibility follows from the equality of the two sets of coefficients $\{a_{i_r 1}, \cdots, a_{i_r n}, b_{i_r}\} = \{\tilde{a}_{r1}, \cdots, \tilde{a}_{rk}, \tilde{b}_{r1}, \cdots, \tilde{b}_{r,n-k+1}\}$.

□

Checking the branches of a graph and reducing it may be done by efficient algorithms. Let us suppose that the graph is presented in terms of adjacency lists that distinguish between neighbors and other vertices. Thus, for every $v \in V$ we have the sets (or lists) NEIGHBOR(v), the neighbors of v and INPUT(v) = $\{w \in V' : (w,v) \in E', \text{ but } (v,w) \notin E'\}$. (If we begin with adjacency lists indexed by the initial vertex of each edge, we preprocess the graph in time proportional to the number of edges to obtain the sets NEIGHBOR(v) and INPUT(v).) A standard search algorithm can be adapted to find all branches and base vertices of S(A,b) in time proportional to the number of two-way edges. [25] Given any base vertex, we can then eliminate the attached branch while checking it for outward positivity and self-limiting end vertices in time proportional to the number of vertices in the branch. The entire procedure of finding, checking, and eliminating branches can be performed in time O(n) for semistable graphs.

An even simpler approach is used by the following algorithm. Instead of finding branches, it just finds successive end vertices. In addition to the arrays already mentioned, the algorithm makes use of SELF(v) to identify self-limiting vertices, V(v) to identify vertices of V (as opposed to V'), and NBRDEGREE(v) to keep count of the number of neighbors of each vertex. These are updated as branch vertices are moved from V to V'−V and base vertices are made self-limiting. Current end vertices are listed in the first-in, first-out queue ENDVERTQ. It is also assumed that the sign of every edge is stored so as to permit access in constant time. One may verify that the time required for the algorithm remains linear.

Algorithm REDUCE

Input: Graph S(A, b) with edges given by the lists NEIGHBOR(v), INPUT(v), and SELF(v)
Output: Edges and vertex set of the reduced system, or FALSE if any branch is not a same sign branch

```
begin
        initialize;
        while  ENDVERTQ ≠ φ  do
        begin
                remove v from the front of ENDVERTQ;
                let  w∈NEIGHBOR(v) such that V(w) is TRUE;
                V(w) := FALSE;
                if  (v,w) ∈ P then return  (FALSE);
                SELF(w) := TRUE;
                NBRDEGREE(w):=NBRDEGREE(w) - 1;
                if  NBRDEGREE(w)=1 and  INPUT(w)=φ then
                    add w to the back of ENDVERTQ;
        end
end

procedure  initialize

begin
        for all  v∈V do  V(v) := TRUE;
        for all  v∈V do  NBRDEGREE(v):=|NEIGHBOR(v)|;
        ENDVERTQ:=φ;
        for all  v∈V do
                if  NBRDEGREE(v)=1 and  INPUT(v)=φ then
                    if  SELF(v) then  add v to the back of ENDVERTQ
                    else return  (FALSE)
end
```

Thus we have reduced the question of viability of (A,b) to determining the viability of a system $(\tilde{A},\tilde{B})$ whose graph is branchless. We will refer to the strong components of a branchless semistable graph S(A) as cores. In succeeding sections we characterize the structure of cores of viable systems.

## 3.4 Singleton cores and straight cores

We first consider the case of a strong component of S(A) consisting of a single vertex.

Lemma 3.7. If (A,b) is viable, then every singleton core is self-limiting and receives at least one positive and no negative edges from upstream components in S(A,b).

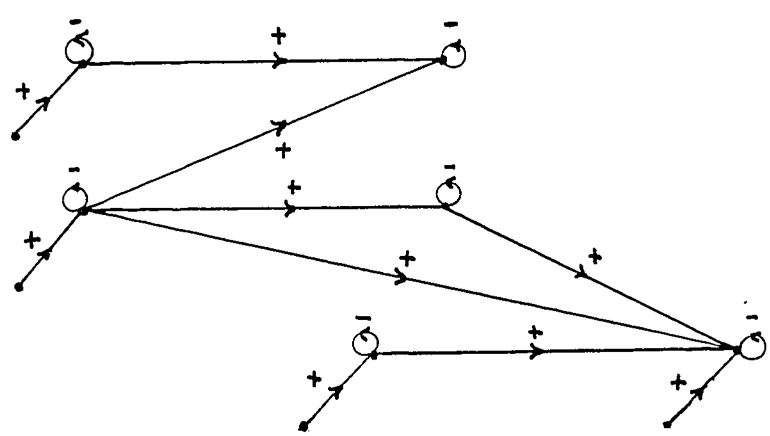
Proof

By part (i) of the coloring conditions the vertex of the core must be colored black iff it is self-limiting. For the other conditions let v be the self-limiting vertex of a singleton core and suppose admissible assignment y has $y_i=1$ for all i upstream of v. Then by Lemma 3.1 admissibility at v must determine $y_v=1$, which is the case only if the core receives positive but no negative edges from upstream components.

□

Example

Below is a viable system with all singleton components.

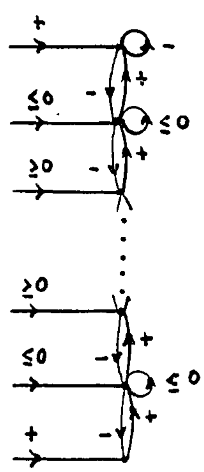


Now suppose we have a core that is not a singleton. The simplest case is a core that is all on one path.

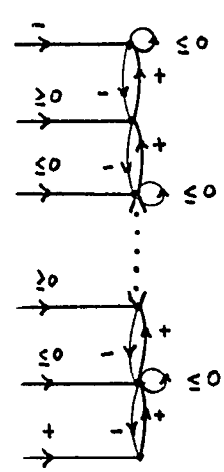
A <u>straight chain with bottom v and top w</u> is a strongly connected vertex induced subgraph $S_C=(V_C, P_C, N_C)$ of $S(A,b)$ with at least two vertices $v,w \in V_C$, such that $V_C$ is the set of vertices of the path $(v \to w)$, there exist edges from $V'-V_C$ to v and w, and for all $i \in V_C-\{v\}$ the path $(v \to i)$ is positive. We refer to a strong component consisting of a straight chain as a <u>straight core.</u> Note that straight chains differ from branches in that there are <u>external edges</u> (edges from $V'-V_C$ to $V_C$) to both top and bottom, as well as, perhaps, to <u>intermediate vertices</u> (vertices in $V_C-\{v,w\}$).

<u>Example</u>

Below are the two straight chain subgraphs which occur in cores of viable systems. In the even case the top vertex must be self-limiting.




dist(v,w) is even




dist(v,w) is odd

The following two lemmas are immediate consequences of the admissibility criteria applied to straight chains.

Lemma 3.8. Suppose $S_C=(V_C, P_C, N_C)$ is a straight chain with bottom v and top w, and y is an assignment with $y_i=1$, for $i \in V_C$. Then y is admissible for $V_C$ iff

(1) there is a positive external edge to v, and

(2) w is self-limiting or there is a negative external edge to w.

Proof

Let the vertices of $V_C=\{i_0, i_1, \cdots, i_m\}$ where $dist(v, i_k)=k$. (Thus $i_0=v, i_m=w$.) Then y is always admissible at intermediate vertices $i_1, \cdots, i_{m-1}$ because $a_{i_k i_{k-1}} y_{i_{k-1}} > 0$ and $a_{i_k i_{k+1}} y_{i_{k+1}} < 0$. For v and w one inequality will hold. Conditions (1) and (2) are equivalent to the other inequality needed for v and w.

□

Lemma 3.9. Suppose S(A,b) contains a straight chain $S_C=(V_C, P_C, N_C)$ with bottom v and top w. Let S'(A,b) be obtained from S(A,b) by deleting one or more one-cycles or external edges to $V_C$, and suppose y is an admissible assignment for S'(A,b) with $y_i \neq 0$, $i \in V_C$. Then y is admissible for S(A,b) also.

Proof

Each of the sets $\{b_i, a_{i1}y_1, \cdots, a_{in}y_n\}$, $i \in V_C$, contains a nonzero element. Since y is admissible for S'(A,b), each set must contain at least one positive and one negative member - a property retained when deleted edges are added back to the graph.

□

The remainder of this section is devoted to establishing necessary and sufficient conditions for straight cores of viable systems. The conditions of the following lemmas may be clarified by checking them against the example above.

Lemma 3.10. Suppose S(A,b) contains a straight core $S_C=(V_C, P_C, N_C)$ with bottom v and top w satisfying the conditions of Lemma 3.8 as well as the following.

(1) If $i \in V_C$ and dist(i,v) is odd, there is no positive external edge to i.

(2) If $i \in V_C$ and dist(i,v) is even, there is no negative external edge to i.

(3) If $i \in V_C-\{w\}$ and dist(i,v) is even, i is not self-limiting.

If y is an admissible assignment for S(A,b) with $y_i=1$ for all i upstream of $S_C$, then $y_i=1$ for all $i \in V_C$ as well.

Proof

Conditions (2) and (3) imply that the only possible negative product in the set $\{b_{i_k}, a_{i_k 1}y_1, \cdots, a_{i_k n}y_n\}$ when k is even is $a_{i_k i_{k+1}}y_{i_{k+1}}$. Thus, admissibility at each succeeding vertex $i_k$ with k even requires $y_{i_{k+1}}=1$. If dist(v,w)=m is odd, then admissibility at the odd vertices $i_m, i_{m-2}, i_{m-4}, \ldots, i_1$ similarly requires $y_{i_{m-1}}=y_{i_{m-3}}= \cdots =y_{i_0}=1$ by condition (1). If m is even, admissibility at the top vertex $i_m$ requires $y_{i_m}=1$. It is then the case that admissibility at the odd vertices $i_{m-1}, i_{m-3}, \cdots, i_1$ requires $y_{i_{m-2}}=y_{i_{m-4}}= \cdots =y_{i_0}=1$ as before.

□

Note that conditions (1)-(3) do not guarantee that the straight core will be sign stable. If a straight core satisfies them, then in every 0- coloring all its vertices must be black. But if dist(v,w) is odd, neither w nor i is self-limiting, and the distance between any two self-limiting vertices of the core always exceeds two, then a nontrivial Im-coloring exists.

The necessity of (1)-(3) is established in the following.

Lemma 3.11. Suppose $S(A,b)$ contains a straight core $S_C=(V_C, P_C, N_C)$ with bottom v and top w satisfying the conditions of Lemma 3.8 and $S(A,b)$ is viable. Then $S_C$ must satisfy conditions (1)-(3) of Lemma 3.10 above.

Proof

Suppose one of the conditions (1)-(3) is violated. Then in each case we find an assignment y with some $y_i=-1$ which is admissible for $S_C$ and upstream components, contradicting viability by Lemma 3.1. In each case we may first use Lemma 3.9 to simplify $S_C$ before finding the assignment.

(0) Special case. Suppose $\text{dist}(v,w)=m$ is even and there is a negative external edge to w. Remove all other external edges to $V_C$ except for one positive edge to v, and remove all negative one-cycles in $V_C$. We assume $y_i=1$ if i is upstream from V. Let $y_{i_r}=(-1)^{1+r}$. This assignment is admissible on $V_C$ and thus extends to an admissible assignment on V. Since it is nowhere-zero, the conditions of Lemma 3.9 are met and the assignment is also admissible for the unsimplified graph, contradicting viability. In what follows, therefore, we may always assume that if m is even, then the top vertex is self-limiting and receives no negative external edge.

(1) Simplify $S_C$ to remove all external edges except one to v, one to w, and one offending positive external edge to $i_k$, for some odd k. Also remove all negative one-cycles except at w. Let

$$y_{i_r}=\begin{cases}(-1)^{1+r} & \text{if } k<r\leqq m\\ +1 & \text{otherwise.}\end{cases}$$

One may check that this assignment is admissible on $V_C$.

(2) Simplify $S_C$ as in (1) except that the offending external edge is now negative and k is even. In this case the admissible assignment is

$$y_{i_r}=\begin{cases}(-1)^r & \text{if } k<r\leq m\\ +1 & \text{otherwise.}\end{cases}$$

(3) Simplify $S_C$ as in (2) except that the vertex $i_k$ has a negative one-cycle instead of a negative external edge. The same assignment as in (2) is admissible.

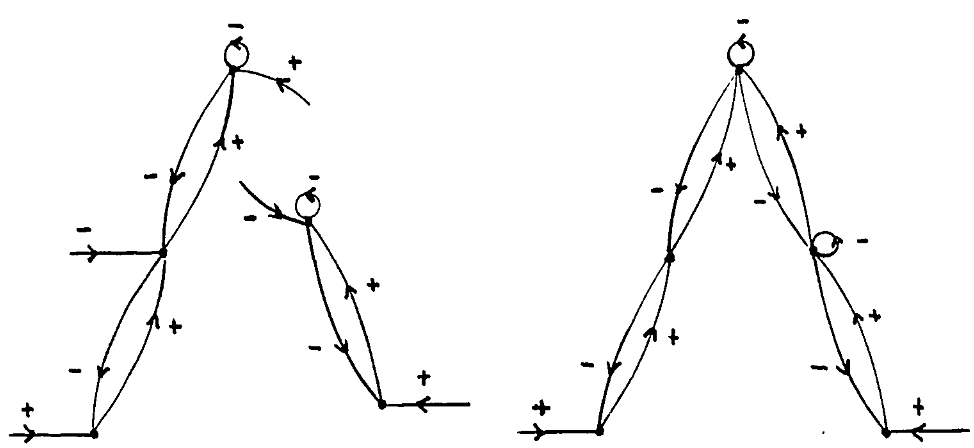
□

## 3.5 General cores

The following example shows how to create viable cores by fusing straight chains.

### Example

The straight chains on the left are fused to obtain the core on the right. The external edges required by the two top vertices are supplied by a negative 2-cycle between the top vertices.



A general core $S_G=(V_G, P_G, N_G)$ may be regarded as the result of fusing straight chains and singletons whenever its vertex set is the disjoint union of the vertex sets of the straight chains and singletons. In such a case the sufficient conditions of section 4 carry over to the fused core. These conditions are restated here for clarity.

We say that a general core $S_G$ satisfies the general solvability conditions if its vertex set $V_G=V_1\bigcup V_2\bigcup \cdots \bigcup V_r$, where $V_i\bigcap V_j=0$ if $i\neq j$, and each $V_k$ satisfies the following.

(a) If $V_k$ contains a single vertex $v_k$, then $v_k$ is self-limiting and receives at least one positive and no negative edges.

(b) If $V_k$ contains more than one vertex, then $V_k$ is the vertex set of a straight chain with top $w_k$ and bottom $v_k$ such that

(1) $v_k$ receives at least one positive and no negative edges;

(2) if $dist(v_k,w_k)$ is even, $w_k$ is self-limiting;

(3) if $i\in V_k$ and $dist(i,v_k)$ is odd, i receives no positive external edge;

(4) if $i\in V_k$ and $dist(i,v_k)$ is even, i receives no negative external edge;

(5) if $i\in V_k-\{w_k\}$ and $dist(i,v_k)$ is even, i is not self-limiting.

Lemma 3.12. Suppose every core of S(A,b) satisfies the general solvability conditions. Then the system is positively sign solvable.

Proof

We have $V_G=V_1\bigcup V_2\bigcup \cdots \bigcup V_r$ as above. Let y be any admissible assignment and consider the set $U=\{i\in V: y_i\neq 1\}$. Given any $i_0\in U$ we find an $i_1\in U$ and a path $(i_1\rightarrow i_0)$ as follows. Let $i_0\in V_k$. If $V_k=\{i_0\}$, then by admissibility for $i_0$, $y_{i_0}\neq 1$ implies that some $y_{i_1}\neq 1$ where $(i_1,i_0)$ is positive. On the other hand, if $V_k$

is a straight chain and dist$(i_0, v_k)$ is odd, we must have $y_{i_1} \neq 1$ where $i_1$ is the self-limiting neighbor below $i_0$ in the straight chain - in which case $(i_1, i_0)$ is positive - or where $i_1$ is another neighbor of this neighbor. If dist$(i_0, v_k)$ is even and $i_0 = w_k$, then $i_1$ is a neighbor of $i_0$ and $(i_1, i_0)$ is again positive. Finally, if dist$(i_0, v_k)$ is even and $i_0$ is not the top, then $i_1$ is a neighbor of the neighbor vertex above $i_0$. In each case dist$(i_1, i_0)=2$ or $(i_1, i_0)$ is positive.

Similarly, given $i_1 \in U$ we find $i_1 \in U$ with path $(i_2 \rightarrow i_1)$, and so on. Since $U \subset V'$ we must eventually obtain $i_s = i_r$, $r<s$. This gives rise to a cycle $[i_r \rightarrow i_s]$ which is either positive or of length greater than two, violating the semistability of S(A,b). Hence, U=0.

The admissibility of the all-one assignment is a consequence of Lemmas 3.7 and 3.8.

□

The process opposite to fusing straight chains and singletons is to disconnect a general core into them. A procedure for doing this is given by the following algorithm.

## Algorithm DISCONNECT

Input: A general core $S_G=(V_G, P_G, N_G)$ with edges given by adjacency lists
Output: A partition of $V_G$ into $V_1, V_2, \ldots, V_r$ each of which is a singleton or the vertex set of a straight chain

```
begin
        k:=0;
        while  there exists unmarked v∈V_G do
        begin
                increment k;
                let v∈V_G be an unmarked vertex such that k=1 or
                    dist(v,⋃{V_j:j<k})=1;
                findchain (v,k);
                mark V_k
        end
end
```

procedure findchain (v,k)
( comment: Finds vertex set $V_k$ of maximal straight chain through v with top $w_k$ and bottom $v_k$)

```
begin
        V_k:={v};
        w_k:=v;
        while  there exists unmarked w such that (w_k,w)∈P_G
            do  w_k:=w,V_k:=V_k⋃{w};
        v_k:=v;
        while  there exists unmarked w such that (w,v_k)∈P_G
            do  v_k:=w,V_k:=V_k⋃{w};
end
```

<u>Lemma 3.13.</u> The algorithm DISCONNECT applied to a general core $S_G=(V_G, P_G, N_G)$ partitions $V_G$ into $V_1, V_2, \ldots, V_r$ in time $O(|V_G|)$ such that

(i) each $V_k$ is maximal in the sense that if $j \neq k$, then $V_k \cup V_j$ is not the vertex set of a straight chain;

(ii) each $V_k$ is either the vertex set of a straight chain or a singleton containing an end vertex of $S_G$.

<u>Proof</u>

First, note that the algorithm certainly partitions $V_G$ into maximal $V_1, \ldots, V_r$. Suppose that $V_k=\{i\}$ is a singleton. If i is not an end vertex, then there exist distinct $v, w \in V_G$ such that $\text{dist}(i,v)=\text{dist}(i,w)=1$. Since $V_k$ is maximal, both v and w must be marked vertices when $V_k$ is found. But if v and w are elements of straight chains found before $V_k$, semistability is violated. So i must be an end vertex.

The time complexity of the algorithm is established as follows. At worst, the procedure <u>findchain</u> examines each neighbor of each vertex in $V_k$. Thus the total time used by this part of the algorithm is $O(|P_G|+|N_G|)$. Marking each vertex of $V_1, \ldots, V_r$ requires $O(|V_G|)$ steps. Finally, the algorithm can be implemented so that the unmarked vertex v can be found in constant time by using a doubly linked list to keep track of eligible vertices. This list is updated to add unmarked neighbors of new chains and remove marked vertices. Since $|P_G|=|N_G|=|V_G|-1$, we have $O(|V_G|)$ time for the whole algorithm.

□

Note that the vertex set usually can be partitioned in more than one way. It is expected that any implementation of the algorithm would result in a partition that depends upon the labelling of vertices.

The desired necessary condition for general cores is given in the following lemma.

Lemma 3.14. If (A,b) is viable and $S_G$ is any core of S(A,b), then $S_G$ satisfies the general solvability conditions.

Proof

Apply DISCONNECT to $V_G$ to obtain $V_1, V_2, \cdots, V_r$. Using Lemmas 3.8 and 3.9 then simplify $S_G$ to $S_{G'}$ so that for each $V_k$

(i) every bottom $v_k$ receives only one positive external edge;

(ii) every top $w_k$ receives only one external edge;

(iii) no vertices are self-limiting other than tops $w_k$ and singletons $w_k=v_k$.

We will refer to the components of $S_{G'}$ as subcomponents to distinguish them from components of the original graph.

First, let us dispose of the singleton subcomponents. By Lemma 3.13(ii) each singleton subcomponent $V_k=\{i\}$ will have exactly one edge to i from some other subcomponent and no edges from i to any other subcomponent. Thus the reasoning of Lemma 3.7 applies, establishing general solvability condition (a).

The remaining subcomponents of $S_{G'}$ are all straight chains. Thus the assignments used in the proof of Lemma 3.11 again show that edges to the top of each straight chain must satisfy part (b) of the general solvability conditions. In arguing that each of these assignments for the subcomponents is also admissible for the original general core, we must take care that each nowhere-zero assignment for a given subcomponent extends to a nowhere-zero assignment on downstream subcomponents, allowing us to invoke Lemma 3.9. This follows from the fact that in $S_{G'}$ each $y_i=y_j$, where i is an odd

[even] vertex of a straight chain in $S_{G'}$ and j is the vertex such that there is an external edge from j to the bottom [respectively, top] of the straight chain containing i.

Now suppose $S_G$ contains an offending edge (j,i) where vertex i is not a singleton subcomponent or the top of a straight chain. Add (j,i) to $S_{G'}$. If $S_{G'}$ does not contain (i,j), then the reasoning of the previous paragraph applies, and we may again use the assignments of the proof of Lemma 3.11 to obtain a contradiction. On the other hand if (j,i) fuses two straight chains $S_1$ and $S_2$, where i is a vertex of $S_1$ and j of $S_2$, we may apply these assignments to $S_1$ and the all-one assignment to $S_2$ because each of these assignments leaves $y_i=+1$. In either case we obtain a nowhere-zero assignment on $S_{G'}$ other than the all-one assignment, leading to a contradiction.

□

The general solvability conditions only require the existence of a single disconnection into straight chains and singletons satisfying (a) and (b)(1)-(5). This may be strengthened as follows.

Lemma 3.15. Let $S_G$ be a core of viable S(A,b) and let $U_1,\ldots,U_s$ be any partition of $V_G$ into singletons and straight chains such that each $U_k$ is maximal, in the sense that $j \neq k$ implies that $U_j \cup U_k$ is not the vertex set of a straight chain. Then $U_1,\ldots,U_s$ satisfy conditions (a) and (b)(1)-(5).

Proof

Let $V_1,\ldots,V_r$ be the disconnection of the previous lemma. We know that $V_1,\ldots,V_r$ satisfy conditions (a) and (b)(1)-(5). Let $v_k$ [$u_k$] denote the bottom vertex or only vertex of $V_k$ [respectively, $U_k$]. Call v an even distance vertex for $V_1,\ldots,V_r$ if $v \in V_k$ implies $\mathrm{dist}(v, v_k)$ is even. The following argument establishes that the even distance vertices for $V_1,...,V_r$ are the same as the even distance vertices for $U_1,...,U_s$.

Suppose $U_i$ is a straight chain which intersects $V_k$ nontrivially and that $V_k$ does not contain the top vertex of $U_i$. Let $v_f$ be the final vertex of the straight subchain $U_i \cap V_k$, that is, the vertex maximizing $\text{dist}(v_k,v_f)$. Then $v_f$ receives a negative edge from outside of $V_k$, so it must be an odd distance vertex for $V_1,...,V_r$. Now consider $u_j \in V_k$. If $u_j = v_k$, then $\text{dist}(v_k,u_j)$ is even. If $u_j \neq v_k$, let $v_p$ be the vertex preceding $u_j$ in $V_k$. Clearly, $v_p$ is the final vertex of some $U_i \cap V_k$. Since $U_i \cup U_j$ cannot be the vertex set of a straight chain, $v_p$ is not the top vertex of $U_i$, hence $\text{dist}(v_k,v_p)$ is odd, so $\text{dist}(v_k,u_j)$ is even. Either way, $u_j$ must be an even distance vertex for $V_1,...,V_r$. Finally, if $u_j \notin V_k$ but $U_i \cap V_k \neq \phi$, then the initial vertex $v_i$ of $U_i \cap V_k$ receives a positive edge from outside of $V_k$, so it must be an even distance vertex for $V_1,...,V_r$.

Thus, conditions (b)(1), (3) and (4) must hold for $U_1,...,U_s$, since they hold for $V_1,...,V_r$. By maximality of the $U_k$, each singleton subcomponent vertex and each even distance top of a straight chain for $U_1,...,U_r$ must be either a singleton or an even distance top vertex for $V_1,...,V_r$. Thus, (a) and (b)(2) and (5) also hold for $U_1,...,U_r$.

□

This assures us that every reasonable disconnection of a viable system exhibits the desired properties. Algorithm DISCONNECT merely provides a convenient way to obtain such a disconnection. On the other hand, if the maximality provision is not observed, one may easily find disconnections which violate the general solvability conditions, such as disconnection of a straight core into two straight chains.

## 3.6 Viable systems

We may summarize the results of the preceding sections as follows.

Theorem 3.16. A system (A,b) is viable if and only if

(i) the system is sign stable;

(ii) every branch of the augmented graph S(A,b) is a same sign branch;

(iii) every core in the graph $S(\widetilde{A},\widetilde{B})$ of the corresponding reduced system satisfies the general solvability conditions.

□

To determine whether a given system (A,b) is viable, we follow a procedure with similar steps:

(i) Test for sign stability using the algorithm in [17], which can be carried out in time $O(|V|+|E|)$ for suitably presented systems.

(ii) Check the branches of S(A,b) while forming the reduced system $S(\widetilde{A},\widetilde{B})$ as outlined in section 3 above - an $O(|V|)$ procedure.

(iii) Finally, use the standard algorithms to partition $S(\widetilde{A},\widetilde{B})$ into strong components, then DISCONNECT each of these cores, and check the sign of every edge to every vertex of every subcomponent, in times $O(|V|)$, $O(|V|)$, and $O(|E|)$, respectively.

Thus, the entire recognition procedure can be carried out in $O(|V|+|E|)$ steps.

## 3.7 On Manber's criterion

An alternative to steps (ii) and (iii) of the procedure for recognizing viable systems given in the preceding section is to transform the augmented matrix into a standard form with all diagonal entries negative, and then apply

Manber's criterion, cited above as Theorem 1.3. To transform a matrix into standard form one must use operations that preserve sign solvability. Manber lists several of these including interchanging columns and multiplying rows by -1. She shows that transforming a system, then applying the criteria for positive sign solvability, can be carried out in $O(|V||E|)$ steps. [19]

One may transform a sign stable system into standard form as follows. By Theorem 1.2 (e') there is a matching M which covers all the vertices of S(A) except some which are already in standard form ($a_{vv}<0$ already). Without loss of generality we shall suppose that the two-cycle [i,j] is represented in M by its positive edge. For every $(i,j) \in M$ define an $n \times n$ matrix T(i,j) such that left multiplication by T(i,j) has the effect of multiplying row j by -1 and interchanging columns i and j. Multiplying row j by -1 corresponds to reversing the signs of all edges to vertex j. This converts [i,j] into a $(-,-)$ cycle. Interchanging columns i and j corresponds to replacing each edge (i,k) by an edge (j,k) of the same sign, and each edge (j,k) by an edge (i,k) of the same sign. So this converts [i,j] into [i] [j] with both one-cycles negative. Thus we obtain $(\hat{A};\hat{b}) = \prod_{(i,j)\in M} T(i,j)\,(A;b)$ with all diagonal entries negative. Note that the T(i,j) commute since they interchange disjoint pairs of columns.

For the following lemma consider a matching M with respect to a particular disconnection of S(A,b) into maximal subcomponents so that if $i_0,i_1,\ldots,i_k$ are vertices in a straight chain subcomponent, listed in order from bottom to top, they are paired as $(i_0,i_1),(i_2,i_3),(i_4,i_5),\ldots$, etc. If k is even and $i_k$ is self-limiting, then $i_k$ is not covered by M. If k is even and $i_k$ is not self-limiting, we pair $i_k$ with a vertex of the branch for which it constitutes the base.

Lemma 3.17. Let (A,b) satisfy the hypotheses of Theorem 3.16 and let M be a matching meeting the above provisions with corresponding transformed matrix $(\widehat{A};\widehat{b})$. Then $S(\widehat{A},\widehat{b})$ satisfies conditions (a), (b) and (c) of Theorem 1.3.

Proof

The proof is by induction. We consider two separate cases. Consult the following example for moral support.

Case 1. Consider vertices of same sign branches including base vertices which are not self-limiting and which appear as singletons or tops of straight chains with an even number of vertices. Beginning with vertices closest to the base of the same sign branch, we perform the transformation T(i,j). We verify that at each stage

(a) there is a positive path from base v to all other vertices of the branch;

(b) no positive cycles are created.

Consider a single T(i,j) and for convenience let a(.) indicate the state of affairs before applying T(i,j) and $\tilde{a}(.)$ the resulting situation afterwards. Consider vertex k of the branch. By induction there is some path $a(v\to k) > 0$. If this path does not go through i or if k=i, then $\tilde{a}(v\to k) = a(v\to k)$. Otherwise, either j is an end vertex, making $\operatorname{sgn} \tilde{a}(i,j) = -\operatorname{sgn} a[j] > 0$, or there is a vertex p beyond j such that $a(j,p) > 0$, in which case $\operatorname{sgn} \tilde{a}(i\to j) = \operatorname{sgn} \tilde{a}(i\to p)\, \tilde{a}(p,j) = -\operatorname{sgn} a(j,p)\, a(p,j) > 0$. Either way we have

$$\operatorname{sgn} \tilde{a}(v\to k) = \operatorname{sgn} \tilde{a}(v\to i)\, \tilde{a}(i\to j)\, \tilde{a}(j\to k) = \operatorname{sgn} a(v\to k)\, \tilde{a}(i\to j) > 0.$$

Thus, the positive path persists as diagonal entries are made negative.

Now consider the creation of cycles. Note that $\tilde{a}[k\to i] \neq 0$ implies that $a[k\to i] \neq 0$. For $\tilde{a}[k\to i]$ comes from $a(k\to i)\, a(j\to k)$, and the latter, together with $a(i,j)$, is a cycle, unless j is on the path $(k\to i)$, in which case $\tilde{a}[k\to i]$ is not a cycle. Similarly for $\tilde{a}[k\to j]$. So we only need to guard against the possibility that a negative cycle $a[k \to i]$ or $a[k \to j]$ gives rise to positive $\tilde{a}[k \to i]$ or $\tilde{a}[k \to j]$.

First, note that $\text{sgn}\, \tilde{a}[i,j] = -\text{sgn}\, a[i]\, a[j] \leqq 0$. Next, note that $\text{sgn}\, \tilde{a}(i,j) = -\text{sgn}\, a[j] \geqq 0$, so $\tilde{a}[k \to i] \neq 0$ implies that $\tilde{a}(i,j) > 0$ and $\text{sgn}\, \tilde{a}[k \to i] = \text{sgn}\, \tilde{a}(k \to i)\, \tilde{a}(i,j)\, \tilde{a}(j \to k) = \text{sgn}\, a(k \to i)\, a(i \to k) = \text{sgn}\, a[k \to i]$. Finally, $\text{sgn}\, \tilde{a}(j,i) = \text{sgn}\, a[i] \leqq 0$, and $\tilde{a}[k \to j] \neq 0$ implies that $\text{sgn}\, \tilde{a}[k \to j] = \text{sgn}\, \tilde{a}(k \to j)\, \tilde{a}(j,i)\, \tilde{a}(i \to k) = -\text{sgn}\, \tilde{a}(k \to j)\, \tilde{a}(i \to k) = \text{sgn}\, a(k \to j)\, a(j \to k) = \text{sgn}\, a[k \to j]$. This establishes that no new positive cycle is created.

Case 2. Consider the remaining vertices. These are matched up (if at all) in consecutive pairs from straight chain subcomponents. Here we have the following before and after situation. To even distance vertex i we have edges with $a(j,i)<0$, $a(k,i) \geqq 0$ if $k \neq j$, and $a[i]=0$. To odd distance vertex i are edges $a(i,j)<0$, $a(k,j) \leqq 0$ if $k \neq i$, and $a[j] \leqq 0$. Afterwards, this becomes $\tilde{a}[i]<0$, $\tilde{a}(k,i) \geqq 0$ if $k \neq j$, $\tilde{a}(j,i)=0$, $\tilde{a}[j]<0$, $\tilde{a}(k,j) \geqq 0$ if $k \neq i$, and $\tilde{a}(i,j) \geqq 0$.

After all the vertices of the straight chain have been transformed, there is a positive path from the bottom up through all the even distance vertices, and another positive path (possibly an extension of the first path) from the top back down through all the odd distance vertices. There is no path from an odd distance vertex back to any even distance vertex within the straight chain. In fact, once the transformation is carried out for all vertices of a strong component, there will be no edge from any odd distance vertex of any straight chain within that component to any even distance vertex of any straight chain (or singleton) within the component. This precludes any cycles of length greater than two among such vertices, in particular any positive cycles.

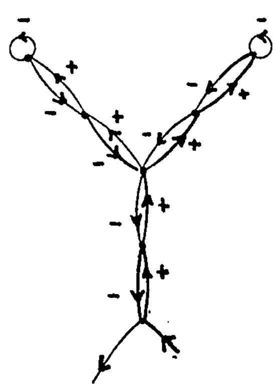
Thus, there are no positive cycles in $S(\hat{A},\hat{b})$. If j is a singleton or even distance top vertex, then in $S(A,b)$ already every external edge to j is positive.

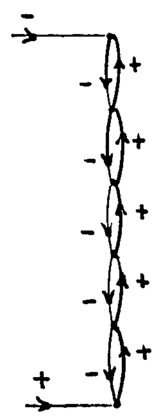
If j is a branch vertex other than the base there are no external edges to j. As we have seen, the transformed system has all external edges to remaining vertices positive. This implies that every path $(w_i \rightarrow j)$ is positive in $S(\hat{A},\hat{b})$. And such a path exists to every straight chain vertex through the bottom or top vertices of its chain, to every singleton, and to every branch vertex through the base vertex. Hence, the conditions of Theorem 1.3 are satisfied.

□

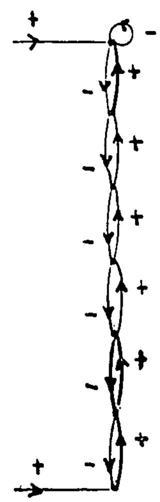
Example

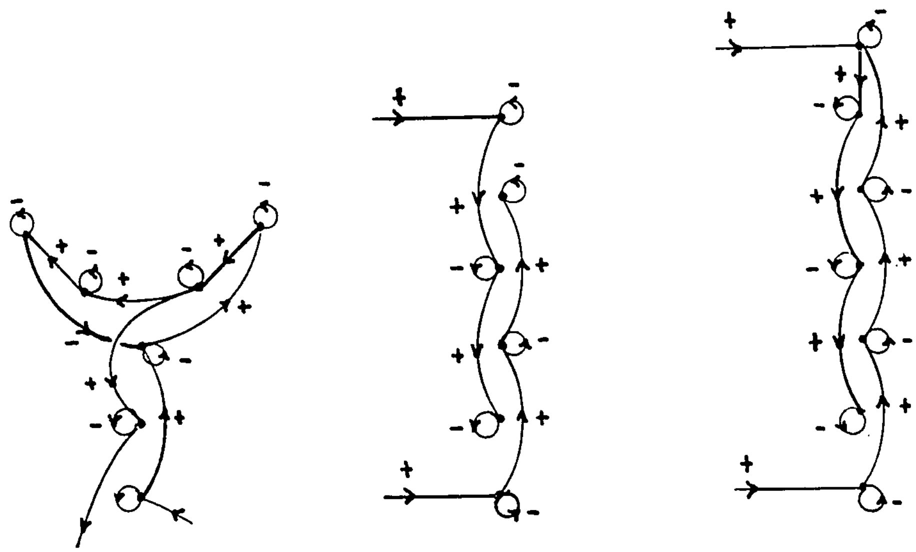
The first row of digraphs show three typical structures of a viable system. The second row shows the digraphs that result when the matrix is converted to standard form.





S(A,b)





$S(\hat{A},\hat{b})$

For the converse we consider the same matching M with respect to the same disconnection of S(A,b), and verify that all subcomponents must satisfy the general solvability conditions.

<u>Lemma 3.18.</u> Given S(A,b) with A sign stable, let M be the matching discussed above and $(\hat{A};\hat{b})$ the corresponding transformed matrix. If $S(\hat{A},\hat{b})$ satisfies conditions (a), (b) and (c) of Theorem 1.3, then every branch of S(A,b) is a same sign branch, and all straight chain and singleton subcomponents satisfy the general solvability conditions.

<u>Proof</u>

If end vertex j is not self-limiting and (i,j) is the unique edge to j, the only way to make $\tilde{a}_{jj}\neq 0$ is to interchange columns i and j. But this makes vertex j isolated, violating condition (c) of Theorem 1.3. In any branch based at v, let i be the branch vertex nearest v for which $(v\rightarrow j)=(v\rightarrow i)(i,j)$ with (i,j) negative. Suppose $(i,j)\in M$. Then T(i,j) makes $\tilde{a}[i]>0$, so we must reverse the signs of all edges to i, making $\tilde{a}(v\rightarrow i)<0$ in violation of Theorem 1.3(b) If $(i,j)\not\in M$, but $(j,k)\in M$, then $T(j,k)$ makes $\tilde{a}[j]>0$, so we must reverse signs of edges to j, giving $\tilde{a}(v\rightarrow j)<0$. If $(i,j)\not\in M$, and no $(j,k)\in M$, then again $\tilde{a}(v\rightarrow j)<0$. Thus branches must be outward positive with self-limiting end vertices - what we have defined as same sign branches.

Similarly, conditions (b) and (c) require that external edges to singleton subcomponents and bottoms of straight chains exist and are all positive, and that singletons and even distance top vertices which are not bases of same sign branches are self-limiting. Suppose any even distance vertex i of a straight chain receives a negative external edge. Then either i is unmatched or $(i,j)\in M$. In either case the sign of the edge is unchanged, so there will be a negative path from some $(w_k\rightarrow i)$ in $S(\hat{A},\hat{b})$. Similarly, any odd distance vertex j is covered by some $(i,j)\in M$ so that $T(i,j)$ will reverse the sign of the positive edge to j, creating negative path $(w_k\rightarrow j)$ in $S(\hat{A},\hat{b})$. Finally, if any even distance vertex i other than the top is self-limiting, $T(i,j)$ transforms $a[i]<0$ into $\tilde{a}(j,i)<0$, creating negative path $\tilde{a}(w_k\rightarrow i) = \tilde{a}(w_k\rightarrow w)\ \tilde{a}(w\rightarrow j)\ \tilde{a}(j,i)$ where w is the top vertex of the straight chain. Thus all subcomponents satisfy the general solvability conditions.

$\square$

It is clear from the above that the conditions for positive sign solvability of a sign stable system also apply to sign nonsingular sign semistable systems, that is, systems which may admit an Im-coloring in which some vertices are white, but which admit no such 0-coloring. The proofs of the above lemmas suggest that a related characterization might apply for nonnegative sign solvability of such systems.

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# VITA

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