Rational Points on Conic Bundles over $\mathbb{P}^1$

Sam Roven

A dissertation
submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy

University of Washington

2022

Reading Committee:

Bianca Viray, Chair

Sándor Kovács

Jarod Alper

Program Authorized to Offer Degree:
Mathematics

University of Washington

Abstract

Rational Points on Conic Bundles over $\mathbb{P}^1$

Sam Roven

Chair of the Supervisory Committee:
Professor Bianca Viray
Department of Pure Mathematics

In this paper, we focus on obstructions to the existence of rational points for a special class of algebraic varieties. In particular, we consider the case where $\pi\colon X \to \mathbb{P}^1_k$ is a smooth conic bundle and k is a number field. We show that if X/k has four geometric singular fibers with $X(\mathbb{A}_k) \neq \emptyset$ or X has non-trivial Brauer group, then X satisfies the Hasse principle over any even degree extension L/k. Furthermore for arbitrary conic bundles X we show that, conditional on Schinzels hypothesis, X satisfies the Hasse principle over all but finitely many quadratic extensions of k. We prove these results by showing the Brauer-Manin obstruction vanishes and then apply fibration method results of Colliot-Thélène, following Colliot-Thélène and Sansuc.

TABLE OF CONTENTS

ACKNOWLEDGMENTS

My journey from birth to this PhD has been a wild and unexpected journey. It is hard to sit down and generate a list of everyone that has played a role in getting me here. To the people mentioned on this list, there are no words that accurately convey my gratitude for your existence but this is my best attempt at doing so.

First and foremost, my deepest gratitude goes out to my advisor, Bianca Viray. The constant support that she has provided over the last five years has helped me grow as a mathematician in ways that I could have never imagined. She has always listened and taught with incredible patience, allowing me to evolve in the most organic way possible.

I am also grateful for professors Sándor Kovács and Jarod Alper for all the secrets of algebraic geometry that they have shown me. My work lies on a foundation that they helped me build. Additionally, I also thank Masahiro Nakahara for his willingness to take on a project with me. I have learned a tremendous amount of arithmetic geometry in my time spent working with Masahiro, and his relentless drive and enthusiasm to find answers to questions continues to inspire me in my own pursuits.

Among the many friends I have made throughout my time in Seattle, I am especially and eternally grateful for my beautiful fiancée Molly Baird. Molly, thank you for always supporting me in the lightest and darkest moments. I always knew you existed but I was never certain that I would actually find you in this lifetime. I am however, fairly certain that you did not bargain to learn as much arithmetic geometry as you have over the last few years. No matter the hurdle, you have always been there to dig in and help me learn, whether it be math or simply writing a gramatically correct sentence. Among the infinite list of thanks that I attribute to you, I am especially thankful for the ways in which you help me grow as a human being every day. Your mere presence is grounding and calming and I am not entirely sure if I could have gotten through graduate school in one piece had I not met you.

In addition to Molly, I am grateful for a number of incredible friends that I have made in graduate school:

Josh Southerland, thank you for being an awesome golf partner, beer drinking buddy, and complainant during all the times that I (and we) were completely stuck on a math problem. The word Laplacian will never have the same meaning to me, thanks to you. I cannot wait to see the the path that you continue to carve out for yourself and I will always be a phone call away.

Nico Courts, thank you for being the ultimate "ride or die" grad school friend. I lost count of the number of nights we spent in the library until 4am, trying to understand differential geometry and algebra. Words do not convey how much better graduate school was thanks to you. I am grateful for all that you have shown me and most of all, for our continued friendship.

Tuomas Tajakka , thank you for single-handedly guiding me through my formative years as an arithmetic geometer. Your impeccable attention to detail and constant willingness to always drop your pencil and help me learn has had a massive influence on the mathematician that I have become. All math aside, you have also been an amazing friend and I look forward to dragging you from Finland soon to get some California sunshine.

Sami Davies, thank you for accepting me with open arms months before we even met. Our chats about complex analysis have not been forgotten. I felt instantly welcome and comfortable working together with you through the grueling first year of graduate algebra. I will look forward to the next text message about a random graph theory question and even more so, I look forward to getting you out to California as well! You will always have a place to stay with Molly and I.

Thomas Carr, thank you for being the mathematical brother I never imagined having. You have shown me how to approach problems in ways that I never would have. You have also helped me never feel alone in research and for that I am truly grateful.

I also thank the many others that made my time in Seattle full of rich memories and excellent company. Gabriel Dorfsman-Hopkins, Tejas Devanur, Kristine Hampton, David Clancy, Smart Thep, Sam Fairchild, Max Goering, and Graham Gordon.

In addition to graduate school friends, I will forever be grateful for those who showed me the beauty of math and helped me get where I am today.

Jono Schrode, an expression of my gratitude for your existence is where the english language sees its biggest failure. Thank you for taking me under your wing when I needed it. Thank you for showing me that there was so much more to the world of science than I could have ever known. Thank you for your patience and constant willingness to inspire me to learn and be a better person. Thank you for being my friend. You are and always will be the fire that lit the flame for me.

Paul Zeitz, you are and always will be the bucket of gasoline that was dumped on the flame. When I first set out as a math major, I was filled with indecision and uncertainty, but you saw something in me and showed me all the ways in which I could succeed. Your support throughout my formative years as a mathematician can never be matched and I will always attribute my love for number theory to you and the constant inspiration you provided for me. I will never forget my graduation, seeing you hiding with a little notepad trying to show that the sum of Legendre symbols

$$\left(\frac{a}{p}\right) + \left(\frac{2a}{p}\right) + \cdots + \left(\frac{(p-1)a}{p}\right) = 0$$

We solved every problem in that number theory book except this one...until I recieved your solution the day after graduation. This is an example of just a sliver of your character and one of the many things I have been lucky enough to share with you. Thank you for helping me get here. I look so so forward to our friendship continuing to grow.

None of this could have been possible without my family. Mom, thank you for supporting me and helping me see this journey through to its completion. Thank you for listening to me kvetch every morning about the struggles of research. Thank you for always believing in me and always being there to listen and welcome me back home with open arms. Thank you for giving me the incredible life that I have. Thank you for giving me the space and freedom that has allowed me to become the person I am today. I also thank Talia, Justin, Mia, and Levi for the joy they bring into my life. Lastly, thank you to my brother Daryn, my nephew Jonathan, and my niece Sophia for always being willing to listen to my vector bundle vomit whenever we would get together.

I also want to thank my lifelong best friends whom I grew up with. They shaped who I have become and sparked my love of surfing, which in turn led to my love of weather and mathematics. Jake Gulick, Ben Gulick, Charlie Gulick, Nick Leonard, Morgan Albertoli, Trevor Bloom, Petr Odehnal, Josh Scofield, and Noah Drever.

Lastly, a special thank you goes out to my dog Gia. No matter the day, you have always been there to welcome me home after a long days work, showering me with dog slobber and love. I will buy you a million chuck-it balls when this thesis is complete and I will play fetch to your hearts content.

DEDICATION

To those that I miss the most. My grandfather Max Webb, my father Fred Roven, my brother Jack Baird, and my sister Amy Frischer. This work is in honor of you.

Chapter 1

INTRODUCTION

The roots of arithmetic geometry date back to the 19th century, in particular, to Gauss' observation that one can obtain a rational solution to a homogeneous polynomial equation if and only if an integral solution can be found. In the late 1920s, André Weil drew connections between number theory and algebraic geometry, ultimately leading to the famous Mordell-Weil theorem. Since then, many techniques have been introduced to decide whether a variety over a number field has a rational point. In this paper, we focus on obstructions to the existence of rational points on varieties.

Let X be a smooth geometrically integral variety over a number field k. We say X satisfies the Hasse principle if the set $X(k)$ of k-rational points is non-empty whenever the set of adélic points $X(\mathbb{A}_k)$ is also non-empty. If X fails the Hasse principle, it is natural to ask about the obstructions that account for this failure. In 1970, Manin used the Brauer group of X to define the **Brauer-Manin set** $X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k)$. It was proven [Man71] that $X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}}$ hence this set can obstruct the existence of k-points on X. This obstruction is known as the **Brauer-Manin obstruction**.

A wide area of research investigates the extent with which sets like $X(\mathbb{A}_k)^{\mathrm{Br}}$ give obstructions to the Hasse principle. In particular one can fix numerical invariants, like the dimension of X, and attempt to classify when the Brauer-Manin set explains this failure. In 1971, Iskovskikh [Isk71] constructed an example of a smooth projective surface that failed the Hasse principle. Years later, in the landmark paper [CTSSD87a], Colliot-Thélène, Sansuc, and Swinnerton-Dyer showed that the Brauer-Manin obstruction explains all failures of the Hasse principle for a class of surfaces known as Châtelet surfaces, which are surfaces that contain an affine open subscheme cut out by $y^2 - az^2 = P(\lambda)$ with P a separable degree 4 polynomial. In this paper, we explain how one can use the Brauer group of a Châtelet surface (modulo constant algebras) to give a Brauer-Manin obstruction to the Hasse principle then we go on to prove new results concerning the arithmetic of conic bundles.

1.1 Outline

In Chapter 2, we introduce quaternion algebras and give some results that allow one to determine when they are split. We then generalize to all finite dimensional central simple k-algebras and define the Brauer group of a field. We continue by giving a characterization of Brauer groups of local and global fields before defining the Brauer group of a scheme, as well as the Brauer-Manin set, proving several related and crucial results along the way. We then use the Hochschild-Serre spectral sequence to give an isomorphism that will enable us to find Brauer classes which give a Brauer-Manin obstruction to a Châtelet surface X.

In Chapter 3 we formally define Châtelet surfaces, one of the central objects of interest in this paper. We give an overview of results concerning the classification of ruled surfaces that we then use to prove that Châtelet surfaces are geometrically rational. We finish off this chapter by giving explicit descriptions of the Picard group and intersection theory on a Châtelet surface.

We then state and prove the main results of this paper in Chapter 4 for arbitrary conic bundles.

1.1.1 Notation

Throughout this paper, k will always denote a field of characteristic 0 and all the k-algebras we consider are finite dimensional. We use $\overline{k}$ to denote a fixed algebraic (hence separable) closure of k and let $G_k := \mathrm{Gal}(\overline{k}/k)$ denote the absolute Galois group of k. If k is a global field, we let $\mathbb{A}_k$ denote the adéle ring of k and Ω_k the set of places of k. For a fixed $v \in \Omega_k$, we let k_v denote the completion of k at v, let $\mathcal{O}_v$ denote the valuation ring in k_v, and $\mathbb{F}_v$ the residue field of $\mathcal{O}_v$.

Given a scheme X over k, and an extension L/k, we write $X_L \coloneqq X \times_{\text{Spec k}} \text{Spec } L$ and $\overline{X} \coloneqq X_{\overline{k}}$. We also let $X(k)$ be the k-points of X and $X(\mathbb{A}_k)$ the adélic points of X.

By a k-variety we mean a separated scheme of finite type over k and by nice k-variety we mean a smooth projective geometrically integral k-variety. By surface we mean a smooth projective variety of dimension 2.

Chapter 2

BRAUER GROUPS

2.1 Quaternion Algebras

Definition 2.1.1. (Quaternion Algebra) For any two elements $a, b \in k^\times$ the *(generalized) quaternion algebra* (a, b) is the 4-dimensional k-algebra with basis $1, i, j, ij$, and multiplication being determined by

$$i^2 = a,\ j^2 = b,\ ji = -ij$$

Remark 2.1.2. The isomorphism class of the algebra (a, b) depends only on the classes of a and b in $k^\times/k^{\times 2}$. The substitution $i \mapsto ui, j \mapsto vj$ induces an isomorphism $(a, b) \cong (u^2a, v^2b)$ for all $u, v \in k^\times$.

Taking the map $i \mapsto j, j \mapsto i$ we get

$$(a, b) \cong (b, a)$$

Remark 2.1.3. The assignment

$$i \mapsto I := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ \ j \mapsto J := \begin{bmatrix} 0 & b \\ 1 & 0 \end{bmatrix}$$

defines an isomorphism $(1, b) \cong M_2(k)$, because the matrices

$$\mathrm{Id} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ I = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},\ J = \begin{bmatrix} 0 & b \\ 1 & 0 \end{bmatrix},\ IJ = \begin{bmatrix} 0 & b \\ -1 & 0 \end{bmatrix}$$

generate $M_2(k)$ as a k-vector space and satisfy the relations

$$I^2 = \mathrm{Id},\ \ J^2 = b\,\mathrm{Id},\ \ IJ = -JI$$

Given a quaternion algebra (a, b), we say that (a, b) is **split** if $(a, b) \cong \mathrm{M}_2(k)$. Identfying when a quaternion algebra is split will be of central importance to us and it can be done in a a number of ways. Before giving the main result of this section, we give one more definition

Definition 2.1.4. (**The Associated Conic**) Given a quaternion algebra (a, b), we define the associated conic, $C(a, b)$, to be the projective plane curve defined by the homogeneous equation

$$ax^2 + by^2 = z^2$$

where x, y, z are homogeneous coordinates in $\mathbb{P}^2$. In the case of $(1, 1) \cong M_2(k)$ we get the circle $x^2 + y^2 = z^2$.

The following result will be our primary tool.

Proposition 2.1.5. *For a quaternion algebra (a, b), the following are equivalent.*
(1) (a, b) is split
(2) (a, b) is not a division algebra
(3) The norm map $N : (a, b) \to k$ defined by $N(q) = q\overline{q}$ has a nontrivial zero.
(4) The element b is a norm from the field extension $k(\sqrt{a})/k$.
(5) $C(a, b)$ has a k-rational point

Proof. The implication (1) $\implies$ (2) is obvious. To prove (2) $\implies$ (3), assume that for all $q \neq 0$, $N(q) \neq 0$. Then $\overline{q}/N(q)$ is an inverse to q.

Next, assume (3) and that $a \notin k^{\times 2}$ else the result is obvious. Let $q = x + yi + zj + wij$ so that $N(q) = x^2 - ay^2 - bz^2 + abw^2 = 0$. This implies that $x^2 - ay^2 = b(z^2 - aw^2) = b(z + \sqrt{a}w)(z - \sqrt{a}w)$. Note that since $a \notin k^{\times 2}$, $x^2 - ay^2 \neq 0$ and $z^2 - aw^2 \neq 0$ hence we can set

$$b = \frac{N(x + \sqrt{a}y)}{N(z + \sqrt{a}w)}$$

and so (4) follows from the fact that the norm is multiplicative.

Next we show (4) $\implies$ (1) and then separately, that (4) $\Leftrightarrow$ (5). To deduce (1), we show that $(a,b) \cong (1, 4a^2)$. Assuming again that a is not a square in k, if b is a norm from $k(\sqrt{a})$ then so is b^{-1} thus we can find $x, y \in k$ such that $b^{-1} = x^2 - ay^2$. Setting $u = xj + yij$ we have $u^2 = bx^2 - aby^2 = bb^{-1} = 1$. Furthermore, one can check that $ui = -iu$. Setting $v = (1+a)i + (1-a)ui$ we can check that $uv = (1+a)ui + (1-a)i = -vu$ and $v^2 = (1+a)^2a - (1-a)^2a = 4a^2$. Note that by considering any non-trivial linear combination $z_1 + z_2u + z_3v + z_4uv = 0$, it is easy to show that $z_1 = z_2 = z_3 = z_4 = 0$ using that fact that $\{1, i, j, ij\}$ is a basis. This implies that $\{1, u, v, uv\}$ is a quaternion basis, and by changing to this basis, we obtain the required isomorphism $(a,b) \cong (1, 4a^2)$ for which (a,b) is split by remark 2 above.

Finally, assuming (4) again we have $x, y \in k$ such that $b = x^2 - ay^2$. From here it is immediate that $(y, 1, x)$ is a k-point on $C(a,b)$. For the converse, if there exist (x_0, y_0, z_0) satisfying $a(x_0)^2 + b(y_0)^2 = z_0^2$ we first observe that we can take $y_0 \neq 0$ else we show that a is a square and are done. This means we can multiply by y_0^{-1} and find that $b = (\frac{z_0}{y_0})^2 - a(\frac{x_0}{y_0})^2$, and (4) is satisfied. If $y_0 = 0$ then we must have $x_0 \neq 0$ otherwise b is a square and we are done. Multiplying by x_0^{-1} we find that a is a norm from $k(\sqrt{b})$. $\square$

2.2 *Central Simple Algebras*

Quaternion algebras are a special case of a more general class of algebras known as central simple algebras. We breifly mention some classical results concerning central simple algebras below. For further reading see [GS17, Chapters 1 and 2] and [Voi21, Part I]

Definition 2.2.1. A k-algebra is called *simple* if it has no non-trivial (two-sided) ideals. A k-algebra is *central* if its center equals k. A *central simple algebra* is a k-algebra that is both central and simple.

Theorem 2.2.2. ***(Wedderburn)***[GS17, Theorem 2.1.3]

Let A be a finite-dimensional simple algebra over k. Then there exists an integer $n \geq 1$

and a division algebra $D \supset k$ such that $A \cong \mathrm{M}_n(D)$. Moreover, the division algebra D is uniquely determined up to isomorphism.

If we take our finite-dimensional simple algebra to also be central, we can say more.

Theorem 2.2.3. [GS17, Corollary 2.2.12] *A finite-dimensional k-algebra A is a central simple algebra if and only if there exists an integer $n > 0$ and a finite Galois extension L/k such that $A \otimes_k L$ is isomorphic to the matrix algebra $\mathrm{M}_n(L)$.*

This theorem implies that for any central simple algebra A/k, $\dim_k(A)$ is a square.

Definition 2.2.4. **(Splitting field and degree of a CSA)** A field extension L/k for which $A \otimes_k L \cong M_n(L)$ is called a *splitting field* for A. The integer $\sqrt{\dim_k(A)}$ is called the *degree* of A.

Proposition 2.2.5. [GS17, Prop 2.2.9)] *Let A be a central simple algebra over k. There is a canonical isomorphism $A \otimes_k A^{\mathrm{opp}} \cong \mathrm{End}_k(A) \cong \mathrm{M}_n(k)$, where n is the degree of A.*

We now have all the tools needed to define the Brauer group of a field.

2.3 The Brauer Group of a Field

Since central simple algebras over a field k can be characterized by those algebras A for which there exists a finite Galois extension L/k and an integer $n > 1$ such that $A \otimes_k L \cong M_n(L)$, we can define the following set.

Let $\mathrm{CSA}_L(n)$ denote the set of k- isomorphism classes of central simple k-algebras of degree n split by L. We regard it as a pointed set with the base point being the class of $M_n(k)$. Two central simple k-algebras A and B are *Brauer equivalent* if $A \otimes_k M_m(k) \cong B \otimes_k M_n(k)$ for some $m, n > 0$.

Definition 2.3.1. (The Brauer group of a field) Brauer equivalence defines an equivalence relation on the union of the sets $\mathrm{CSA}_L(n)$. We denote the set of equivalence classes by $\mathrm{Br}(L/k)$ and define the Brauer group, denoted $\mathrm{Br}\,k$, to be

$$\mathrm{Br}\,k \coloneqq \bigcup_{L/k \text{ finite Galois}} \mathrm{Br}(L/k)$$

We note here that for any fixed extension L/k, we will refer to $\mathrm{Br}(L/k)$ as the *relative* Brauer group (of the extension L/k). It can alternatively be defined as the kernel of the homomorphism

$$\mathrm{Br}(L/k) = \ker(\mathrm{Br}\ k \to \mathrm{Br}\ L), \ \ \mathcal{A} \mapsto \mathcal{A} \otimes_k L$$

Proposition 2.3.2. *The set* $\mathrm{Br}\,k$ *forms an abelian group under tensor product.*

Proof. Basic properties of tensor product imply that the binary operation is commutative and associative. Clearly the identity element is the class of $\mathrm{M}_n(k)$. Moreover, Proposition 1 implies that given a class in $\mathrm{Br}(L/k)$ represented by A, the class of the opposite algebra A^{opp} yields an inverse. □

Remark 2.3.3. Equivalently, [Ser91, Chapter X, section 5] one can define the Brauer group via Galois cohomology

$$\mathrm{Br}\ k \coloneqq \mathrm{H}^2(G_k, \overline{k}^{\times})$$

We say that a Brauer class $\mathcal{A} \in \mathrm{Br}\ k$ is *split by* L if $\mathcal{A}$ is contained in the subgroup $\mathrm{Br}(L/k)$. We also note that Br is a covariant functor from the category of fields to the category of abelian groups.

We can see that each non-trivial Brauer class contains (up to isomorphism) a unique division algebra and $\mathrm{Br}(L/k)$ classifies division algebras split by L. To see why, let $\mathcal{A}, \mathcal{A}'$ be central simple k-algebras, so by Theorem 2.2.2

$$\mathcal{A} \cong \mathrm{M}_n(D), \ \ \mathcal{A}' \cong \mathrm{M}_{n'}(D')$$

for some division algebras D and D'. If $\mathcal{A}$ and $\mathcal{A}'$ are Brauer equivalent then

$$\mathcal{A} \otimes \mathrm{M}_m(k) \cong \mathcal{A}' \otimes \mathrm{M}_{m'}(k) \implies \mathrm{M}_{nm}(D) \cong \mathrm{M}_{n'm'}(D')$$

From here, uniqueness of the division algebra implies that $D \cong D'$

It now follows from Theorem 2.2.2 that if $\mathcal{A} \sim_{\mathrm{Br}} \mathcal{A}'$ and $\dim_k(\mathcal{A}) = \dim_k(\mathcal{A}')$, then $\mathcal{A} \cong \mathcal{A}'$, so each Brauer class contains exactly one central simple algebra (up to isomorphism) of fixed degree.

In the case where our ground field is algebraically closed, the Brauer group becomes trivial.

Lemma 2.3.4. *Let $k = \overline{k}$. Then every central simple k-algebra is isomorphic to $M_n(k)$ for some $n \geq 1$, hence* $\mathrm{Br}\, k = 0$.

Proof. To prove this, it is enough to show that the only finite-dimensional division algebra $D \supset k$ is k itself. Take any $d \in D$ and consider the extension $k[d]$. Since D is finite dimensional over k, d is algebraic. Since $k = \overline{k}$, $k[d] = k$. $\square$

Lastly, from the cohomological definition of the Brauer group, we can also obtain a definition of the relative Brauer group of a Galois extension L/k.

Proposition 2.3.5. *Let L/k be a Galois extension, then* $\mathrm{Br}(L/k) \cong H^2(\mathrm{Gal}(L/k), L^\times)$

Proof. We apply the inflation restriction exact sequence to the group G_k with normal subgroup G_L, both acting on the Galois-module $\overline{k}^\times$. We have that $\overline{k} = \overline{L}$, $(\overline{k}^\times)^{G_L} = \overline{L}^\times$, $G_k/G_L = \mathrm{Gal}(L/k)$, and Hilbert's theorem 90 (particularly applied to $H^1(G_L, \overline{L}^\times)$) gives

$$0 \to H^2(\mathrm{Gal}(L/k), L^\times) \to H^2(G_k, \overline{k}^\times) \to H^2(G_L, \overline{L}^\times)$$

Using the cohomological definition of the Brauer group of a field, the above sequence becomes

$$0 \to H^2(\mathrm{Gal}(L/k), L^\times) \to \mathrm{Br}\ k \to \mathrm{Br}\ L$$

hence $\ker\, \mathrm{Br}\ (L/k) \coloneqq (\mathrm{Br}\ k \to \mathrm{Br}\ L) \cong H^2(\mathrm{Gal}(L/k), L^\times)$. $\square$

Proposition 2.3.6. [GS17, Lemma 1.5.2] *Given elements $a, b, b' \in k^\times$ we have an isomorphism*

$$(a, b) \otimes_k (a, b') \cong (a, bb') \otimes M_2(k)$$

Corollary 2.3.7. *For the quaternion algebra (a, b) we have the isomorphism $(a, b) \otimes_k (a, b) \cong M_4(k)$.*

Proof. Applying the above proposition with $b = b'$ we have $(a, b) \otimes (a, b) \cong (a, b^2) \otimes M_2(k) \cong M_2(k) \otimes M_2(k) \cong M_4(k)$ □

Corollary 2.3.7 implies that the Brauer class of any quaternion algebra is 2-torsion in the Brauer group. In fact, it is a theorem of Merkurjev [Voi21, Theorem 8.3.5] that all the 2-torsion in the Brauer group is accounted for by classes of quaternion algebras.

Definition 2.3.8. Let L/k be a cyclic extension of degree n with σ a fixed generator of $\mathrm{Gal}(L/k)$ and let $b \in k^\times$. We define the *cyclic algebra* (σ, b) to be the L-vector space

$$\frac{L \oplus Ly \oplus Ly^2 \oplus \cdots \oplus Ly^{n-1}}{y^n = b,\ \sigma(\alpha)y = y\alpha\ \forall \alpha \in L}$$

Remark 2.3.9. With a bit of work, one can show that (σ, b) is a central simple algebra defined over k and split by L [GS17, section 2.5]. Moreover, this generalizes the construction of quaternion algebras. In general, if k contains the nth roots of unity, then by Kummer theory, any cyclic extension L of degree n is of the form $L = k(\sqrt[n]{a})$ for some $a \in k^\times$. Letting σ be a generator of $\mathrm{Gal}(L/k)$, we denote the cyclic algebra (σ, b) by $(a, b)_n$. For any quadratic extension $k(\sqrt{a})$ with $\sigma(\sqrt{a}) = -\sqrt{a}$, we have $(\sigma, b) \cong (a, b)$ thus recovering the generalized the quaternion algebras defined earlier. This shows that quaternion algebras are indeed central simple algebras.

Given any cyclic extension L/k we also have a nice description of the relative Brauer group

Proposition 2.3.10. [Gui18, Cor 7.19] *We have an explicit isomorphism*

$$\frac{k^\times}{N_{L/k}(L^\times)} \xrightarrow{\sim} \mathrm{Br}(L/k), \quad b \mapsto (\sigma, b)$$

In particular, we can see that (σ, b) is trivial in Br k if and only if b is a norm from $L^\times$.

Corollary 2.3.11. $\operatorname{Br}\mathbb{R} \cong \mathbb{Z}/2\mathbb{Z}$

Proof. We can begin by observing that Br $\mathbb{R}$ = Br $(\mathbb{C}/\mathbb{R})$. Moreover, $N_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times) = \mathbb{R}_{>0}$ so Proposition 2.3.10 implies that

$$\text{Br } \mathbb{R} = \mathbb{R}/\mathbb{R}_{>0} \cong \mathbb{Z}/2\mathbb{Z}$$

To find an explicit representative of the one nontrivial Brauer classe, it suffices to take any quaternion algebra generated by two elements of $\mathbb{R}_{<0}$ since they are not norms from $\mathbb{C}^\times$. Taking the quaternion algebra $(-1, -1)$, otherwise known as Hamilton's quaternions, will do. □

2.3.1 The Brauer Group of a Local Field

Let k_v denote a non-archimedean local field. As defined in [Mil13, section 2.2], there is an isomorphism

$$\operatorname{inv}_v \colon \operatorname{Br} k_v \to \mathbb{Q}/\mathbb{Z}$$

known as the Hasse invariant. The invariant map will be of central importance in computing the Brauer-Manin set. Here we exhibit some nice properties of this map, when applied to cyclic algebras.

Let L/k_v be an unramified (cyclic) extension and let $\sigma \in \operatorname{Gal}(L/k_v)$ be the automorphism which induces the Frobenius map on the residue field, then by [Mil13, Chap IV, Ex. 4.2 and Prop 4.3] we have

$$\operatorname{inv}_v((\sigma, b)) = \frac{v(b)}{[L : k_v]} \in \mathbb{Q}/\mathbb{Z}$$

Remark 2.3.12. If we restrict the domain of the invariant map to Br $(k_v)[2]$, namely the quaternion algebras, then the isomorphism shows that there exists a unique non-trivial 2-torsion class. In other words, $\operatorname{inv}_v|_{\operatorname{Br}(k_v)[2]} \colon \operatorname{Br}(k_v)[2] \xrightarrow{\sim} \frac{1}{2}\mathbb{Z}/\mathbb{Z}$.

2.3.2 The Brauer Group of a Global Field

Let k be a global field and let Ω_k denote the set of places of k. The fundamental exact sequence of global class field theory completely characterizes the Brauer group of any global field.

$$0 \to \mathrm{Br}\ k \to \bigoplus_{v \in \Omega_k} \mathrm{Br}\ \ k_v \xrightarrow{\sum_v \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0 \tag{2.3.1}$$

The Brauer group of k is identified with a subgroup of the direct sum $\bigoplus_{v \in \Omega_k} \mathrm{Br}\, k_v$ and the inclusion map is given by tensoring any central simple algebra over k with each completion. The Brauer group of each completion at nonarchimedian places is isomorphic to $\mathbb{Q}/\mathbb{Z}$ and $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$ at the archimedian place. The Brauer group of k is then the kernel of the sum of local invariants.

It is worth noting that not only is this group infinite, but even the 2-torsion is infinite. In fact, even the relative Brauer group Br $(\mathbb{Q}(\sqrt{2})/\mathbb{Q})$ is infinite!

2.4 The Brauer Group of a Scheme and the Brauer-Manin Obstruction

Throughout this section, let X be a nice variety over a global field k of characteristic zero.

Definition 2.4.1. The Brauer Group of a Scheme Given a scheme X we define

$$\mathrm{Br}\, X \coloneqq H^2_{\text{ét}}(X, \mathbb{G}_m)$$

By functoriality of cohomology, Br is a functor from the category of schemes to the category of abelian groups. However, in contrast to fields, Br is **contra**variant for schemes.

Remark 2.4.2. Note that this definition generalizes the notion of the Brauer group of a field, and given a field k, we have $\mathrm{Br}\, k = H^2_{\text{ét}}(\mathrm{Spec}\ k, \mathbb{G}_m)$. Furthermore, when X is nice, Br X is torsion [Mil80, Example III.2.22], as is the case for fields.

Theorem 2.4.3. [Gro68b, Corollarie 7.5] *If X is a smooth projective surface, then* $\mathrm{Br}\, X$ *depends only on the birational class of X.*

Corollary 2.4.4. *Let X be a nice geometrically rational surface over a field k. Then* $\operatorname{Br}\overline{X} = 0$

Proof. From Theorem 2.4.3 , we have an isomorphism $\operatorname{Br}\overline{X} \cong \operatorname{Br}\mathbb{P}^n_{\overline{k}}$. Moreover, the induced map $\operatorname{Br} k \to \operatorname{Br}\mathbb{P}^n_k$ coming from the structure morphism $\mathbb{P}^n_k \to \operatorname{Spec} k$ is an isomorphism, see [Poo17, Proposition 6.9.9]. Combining this with Lemma 2.3.4, we get the string of isomorphisms $\operatorname{Br}\overline{X} \cong \operatorname{Br}\mathbb{P}^n_{\overline{k}} \cong \operatorname{Br}\overline{k} = 0$. ☐

Given a morphism $X \to \operatorname{Spec} k$ we obtain a map on Brauer classes $\operatorname{Br} k \to \operatorname{Br} X$ by pulling back. Let X be a variety over a field and define

$$\operatorname{Br}_0\ X \coloneqq \operatorname{im}(\operatorname{Br} k \to \operatorname{Br} X) \quad \text{and} \quad \operatorname{Br}_1\ X \coloneqq \ker(\operatorname{Br} X \to \operatorname{Br}\overline{X})$$

One can show the inclusion $\operatorname{Br}_0 X \subset \operatorname{Br}_1 X \subset \operatorname{Br} X$. Elements of $\operatorname{Br}_0 X$ are called *constant* and elements of $\operatorname{Br}_1 X$ are called *algebraic*.

Proposition 2.4.5. *If X is geometrically rational then* $\operatorname{Br}_1 X = \operatorname{Br} X$. *If X is over a global field k and $X(\mathbb{A}_k) \neq \emptyset$ then* $\operatorname{Br} k = \operatorname{Br}_0 X$.

Proof. If $\overline{X}$ is rational, then by Corollary 2.4.4, $\operatorname{Br}_1 X = \operatorname{Br} X$. To show that the natural map $\operatorname{Br} k \to \operatorname{Br} X$ is injective, the existence of an adélic point gives a map $P_v : \operatorname{Spec} k_v \to X_{k_v}$ as in the following diagram

$$\begin{tikzcd} X_{k_v} \arrow[r] \arrow[d, "\pi_v"] & X \arrow[d] \\ \operatorname{Spec} k_v \arrow[r] \arrow[u, bend left, "P_v"] & \operatorname{Spec} k \end{tikzcd}$$

By functoriality of Brauer groups we have

$$\begin{tikzcd} \operatorname{Br} X \arrow[r] & \operatorname{Br} X_{k_v} \arrow[d, bend left, "P_v^*"] \\ \operatorname{Br} k \arrow[r] \arrow[u] & \operatorname{Br} k_v \arrow[u, "\pi_v^*"] \end{tikzcd}$$

We can see that P_v splits the base change of the structure morphism of X for every $v \in \Omega_k$ hence the natural maps $\pi_v^* : \operatorname{Br} k_v \to \operatorname{Br} X_{k_v}$ split for every $v \in \Omega_k$. Combining this with (2.3.1), it follows that the induced map $\operatorname{Br} k \to \operatorname{Br} X$ coming from the structure map of X is injective. □

Our considerations up to this point mainly involve Brauer groups of fields and varieties over fields but in some cases, it will be useful to consider Brauer groups of ring. Given a commutative ring R, we can define $\operatorname{Br} R \coloneqq \operatorname{Br}(\operatorname{Spec} R)$.

Lemma 2.4.6. [Mil80, III.3.11(a)] *Let R be a non-archimedian local ring with residue field $\mathbb{F}$. The quotient map $R \to k$ induces an isomorphism $\operatorname{Br} R \xrightarrow{\sim} \operatorname{Br} \mathbb{F}$.*

Corollary 2.4.7. *Let k be a complete valued field with valuation denoted by v and valuation ring $\mathcal{O}_v$. Then $\operatorname{Br} \mathcal{O}_v = 0$.*

Proof. By Lemma 2.4.6, $\mathcal{O}_v \to \mathbb{F}_v$ induces the isomorphism $\operatorname{Br} \mathbb{F}_v \cong \operatorname{Br} \mathcal{O}_v$. For any non-archimedian local field, $\mathbb{F}_v$ is finite. Since every finite division algebra is a field, the only central simple algebras over $\mathbb{F}_v$ are matrix algebras. Hence $\operatorname{Br} \mathbb{F}_v = 0$. □

2.4.1 The Brauer-Manin Set

Let Ω_k denote the set of places of k. If $v \in \Omega_k$, write k_v for the completion of k at v and $\mathcal{O}_v$ for the valuation ring in k_v. Let $\mathbb{A}_k$ be the adéle ring of k, that is, $\mathbb{A}_k$ is the restricted product $\mathbb{A}_k = \prod'_{v \in \Omega_k}(k_v, \mathcal{O}_v)$. This is a subring of the product $\prod_{v \in \Omega_k} k_v$ containing tuples (P_v) for which there exists a finite set of places S such that $P_v \in \mathcal{O}_v$ for all $v \notin S$.

Let $X(\mathbb{A}_k)$ denote the set of adélic points of X. A priori, $X(\mathbb{A}_k)$ is a subset of $X(\prod_{v \in \Omega_k} k_v)$ but one can show ([Poo17], Exercise 3.4) that if X is proper, then $X(\mathbb{A}_k) = \prod_{v \in \Omega_k} X(k_v)$.

For any $P_v \in X(k_v)$ and any $\mathcal{A} \in \operatorname{Br} X$, we can pullback $\mathcal{A}$ along $P_v \colon \operatorname{Spec} k_v \to X$ and obtain an element of Br k_v. We denote this element $\mathcal{A}(P_v)$ and regard it as the image of P_v under the map $\mathrm{ev}_{\mathcal{A}} \colon X(k_v) \to \operatorname{Br} k_v$, which we call the evaluation map. A priori, this gives a map $\mathrm{ev}_{\mathcal{A}} \colon X(\mathbb{A}_k) \to \prod_{v \in \Omega_k} \operatorname{Br} k_v$.

Given a point $(P_v) \in X(\mathbb{A}_k)$, for some finite set of places $S \subset \Omega_k$, there exists a finite type $\mathcal{O}_{k,S}$ scheme $\mathcal{X}$ equipped with a map $X \hookrightarrow \mathcal{X}$, such that for fixed $\mathcal{A} \in \operatorname{Br} X$, one can find $\widetilde{\mathcal{A}} \in \operatorname{Br} \mathcal{X}$ with $\widetilde{\mathcal{A}}$ pulling back to $\mathcal{A}$ under $\operatorname{Br} \mathcal{X} \to \operatorname{Br} X$, [Poo17, Corollary 6.6.11]. In particular, for a k_v point P_v, we have $P_v(\operatorname{Spec} k_v) \in \operatorname{Spec} A \subset X$ and since X is finite type, $A = k_v[x_1, ..., x_n]/(f_1, ..., f_r)$. We take S to be the (finite) set of valuations that are negative when applied to the finitely many coefficients of the f_i. For all $v \notin S$, we have $P_v \in X(\mathcal{O}_v)$. This implies that for all but finitely many v, $\operatorname{ev}_{\mathcal{A}}(P_v) \in \operatorname{Br} \mathcal{O}_v$. By Corollary 2.4.7, $\operatorname{Br} \mathcal{O}_v = 0$ so $\operatorname{ev}_{\mathcal{A}}(P_v) = 0$ for almost all v. Therefore, $\operatorname{ev}_{\mathcal{A}}$ gives a map $X(\mathbb{A}_k) \to \oplus_{v \in \Omega_k} \operatorname{Br} k_v$.

Composing this with the sum of local invariants, $\operatorname{inv}_v : \operatorname{Br} k_v \to \mathbb{Q}/\mathbb{Z}$, we obtain a well-defined map $X(\mathbb{A}_k) \to \mathbb{Q}/\mathbb{Z}$ given by

$$(P_v) \mapsto \sum_{v \in \Omega_k} \operatorname{inv}_v(\operatorname{ev}_{\mathcal{A}}(P_v))$$

Before defining the Brauer-Manin set, we cite one lemma that will be of use in the coming results.

Lemma 2.4.8. [Vir10, Lemma 3.3.2] *Let k_v be a local field. For any $\mathcal{A} \in \operatorname{Br} X$*

$$\operatorname{ev}_{\mathcal{A}} \colon X(k_v) \to \mathbb{Q}/\mathbb{Z}$$

is continuous for the discrete topology on $\mathbb{Q}/\mathbb{Z}$

Definition 2.4.9. (**The Brauer-Manin Set**) Given $\mathcal{A} \in \operatorname{Br} X$ let

$$X(\mathbb{A}_k)^{\mathcal{A}} = \left\{ (P_v) \in X(\mathbb{A}_k) \colon \sum_{v \in \Omega_k} \operatorname{inv}_v(\operatorname{ev}_{\mathcal{A}}(P_v)) = 0 \right\}$$

We call

$$X(\mathbb{A}_k)^{\operatorname{Br}} = \bigcap_{\mathcal{A} \in \operatorname{Br} X} X(\mathbb{A}_k)^{\mathcal{A}}$$

the *Brauer Manin set.*

Lemma 2.4.8 shows that $X(\mathbb{A}_k)^{\mathcal{A}}$ is a closed subset of $X(\mathbb{A}_k)$.

Proposition 2.4.10. $X(k) \subset X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k)$

Proof. For $\mathcal{A} \in \operatorname{Br} X$, we obtain a commuting diagram

$$\begin{tikzcd}
 & X(k) \arrow[r] \arrow[d, "\mathrm{ev}_{\mathcal{A}}"] & X(\mathbb{A}_k) \arrow[d, "\mathrm{ev}_{\mathcal{A}}"] & & \\
0 \arrow[r] & \operatorname{Br} k \arrow[r, "\phi"] & \bigoplus_{v \in \Omega_k} \operatorname{Br} k_v \arrow[r, "\sum_v \mathrm{inv}_v"] & \mathbb{Q}/\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

where the bottom row is the usual exact sequence from global class field theory. It is immediate from the definition of $X(\mathbb{A}_k)^{\mathrm{Br}}$ that $X(\mathbb{A}_k)^{\mathrm{Br}} \subset X(\mathbb{A}_k)$. To see the other inclusion, take $P \in X(k)$ and its image $(P_v) \in X(\mathbb{A}_k)$. Commutativity of the diagram implies that $\phi(\mathrm{ev}_{\mathcal{A}}(P)) = \mathrm{ev}_{\mathcal{A}}(P_v)$. Moreover, exactness of the bottom row then gives $\sum_v \mathrm{inv}_v(\phi(\mathrm{ev}_{\mathcal{A}}(P))) = \sum_v \mathrm{inv}_v(\mathrm{ev}_{\mathcal{A}}(P_v)) = 0$ hence $P \in X(\mathbb{A}_k)^{\mathcal{A}}$. ☐

We say there is a **Brauer-Manin obstruction to the Hasse principle** if $X(\mathbb{A}_k) \neq \emptyset$ and $X(\mathbb{A}_k)^{\mathrm{Br}} = \emptyset$. The Brauer-Manin set and it's usefulness in finding obstructions to the Hasse principle cannot be emphasized enough. The arithmetic of many nice varieties is largely controlled by their Brauer groups ands serves as ample motivation to find Brauer classes that give a Brauer-Manin obstruction. In searching for such algebras, and computing the Brauer-Manin set, several helpful reductions can be made.

Corollary 2.4.11. *For any* $\mathcal{A} \in \operatorname{Br}_0 X$, $X(\mathbb{A}_k)^{\mathcal{A}} = X(\mathbb{A}_k)$.

Proof. From Proposition 2.4.5 and the exactness of the bottom row of the diagram in Proposition 2.4.10, we have that for any $\mathcal{A} \in \operatorname{Br} k$

$$\mathrm{inv}_v(\mathrm{ev}_{\mathcal{A}}(P_v)) = 0$$

for every v. ☐

Corollary 2.4.12. *To compute* $X(\mathbb{A}_k)^{\mathrm{Br}}$*, it is enough to compute the intersection over a set of representatives of* $\operatorname{Br} X / \operatorname{Br}_0 X$.

Proof. This is immediate from the above Corollary. ☐

2.5 Computing Brauer Groups: The Hochschild-Serre Spectral Sequence in Étale Cohomology

Let X be a nice variety over a global field k that is everywhere locally solvable. If X is geometrically rational, then the Hochschild-Serre spectral sequence in étale cohomology gives a tool for computing the group $\operatorname{Br} X / \operatorname{Br}_0 X$.

Proposition 2.5.1. *Let k be a global field and X/k a nice geometrically rational variety with $X(\mathbb{A}_k) \neq \emptyset$. Then we obtain an exact sequence*

$$0 \to \operatorname{Pic} X \to (\operatorname{Pic} \overline{X})^{G_k} \to \operatorname{Br} k \to \operatorname{Br} X \to H^1(G_k, \operatorname{Pic} \overline{X}) \to H^3(k, \mathbb{G}_m)$$

Since k is a global field, $H^3(k, \mathbb{G}_m) = 0$ and we have an isomorphism

$$\frac{\operatorname{Br} X}{\operatorname{Br}_0 X} \cong H^1(G_k, \operatorname{Pic} \overline{X})$$

Proof. Let L/k be a finite extension with $G = \operatorname{Gal}(L/k)$. The Hochschild-Serre sprectral sequence

$$E_2^{p,q} := H^p(G, H^q_{\text{ét}}(X_L, \mathbb{G}_m)) \implies L^{p+q} := H^{p+q}_{\text{ét}}(X, \mathbb{G}_m)$$

gives rise to the usual seven term exact sequence

$$0 \to E_2^{1,0} \to L^1 \to E_2^{0,1} \to E_2^{2,0} \to \ker(L^2 \to E_2^{0,2}) \to E_2^{1,1} \to E_2^{3,0}$$

In our case, we get

$$0 \to H^1(G, H^0_{\text{ét}}(X_L, \mathbb{G}_m)) \to H^1_{\text{ét}}(X, \mathbb{G}_m) \to H^0(G, H^1_{\text{ét}}(X_L, \mathbb{G}_m)) \to H^2(G, H^0_{\text{ét}}(X_L, \mathbb{G}_m))$$

$$\to \ker(H^2_{\text{ét}}(X, \mathbb{G}_m) \to H^0(G, H^2_{\text{ét}}(X_L, \mathbb{G}_m))) \to H^1(G, H^1_{\text{ét}}(X_L, \mathbb{G}_m)) \to H^3(G, H^0_{\text{ét}}(X_L, \mathbb{G}_m))$$

One can show (see e.g. [Poo17, Proposition 6.6.1]) that

$$H^0_{\text{ét}}(X_L, \mathbb{G}_m)) \cong L^\times, \quad \text{and} \quad H^1_{\text{ét}}(X_L, \mathbb{G}_m)) \cong \operatorname{Pic} X_L$$

which gives

$$0 \to \operatorname{Pic} X \to (\operatorname{Pic} X)^G \to H^2(G, L^\times) \to \ker(\operatorname{Br} X \to \operatorname{Br} X_L) \to H^1(G, \operatorname{Pic} X_L) \to H^3(G, L^\times).$$

In particular, for the extension $\overline{k}/k$ with Galois group G_k we get

$$0 \to \operatorname{Pic} X \to (\operatorname{Pic} \overline{X})^{G_k} \to \operatorname{Br} k \to \ker(\operatorname{Br} X \to \operatorname{Br} \overline{X}) \to H^1(G_k, \operatorname{Pic} \overline{X}) \to H^3(G_k, \overline{k}^\times)$$

Furthermore, if k is a global field, then $H^3(G_k, \overline{k}^\times) = 0$, which is a result due to Tate, see [NSW08, 8.3.11(iv), 8.3.17]. Proposition 2.4.5 then gives rise to the short exact sequence

$$0 \to \operatorname{Br} k \to \operatorname{Br} X \to H^1(G_k, \operatorname{Pic} \overline{X}) \to 0$$

yeilding the desired isomorphism □

Remark 2.5.2. When $X(\mathbb{A}_k) \neq \emptyset$, if H is a subgroup of G_k, then by looking at the first two terms of the above sequence, injectivity of the natural map $\operatorname{Br} k \to \operatorname{Br} X$ gives the isomorphism

$$\operatorname{Pic} X_{\overline{k}^H} \cong (\operatorname{Pic} \overline{X})^H$$

where $\overline{k}^H$ denotes the fixed field of H.

2.5.1 Galois action on the Picard Group

The Galois group $\operatorname{Gal}(\overline{k}/k)$ acts on $\operatorname{Pic} \overline{X}$ as follows. For $\sigma \in \operatorname{Gal}(\overline{k}/k)$ let $\tilde{\sigma} \in \operatorname{Aut}(\operatorname{Spec} \overline{k})$ be the corresponding morphism. By considering the base change of X to $\overline{k}$, the pullback of the morphism $\operatorname{id}_X \times \tilde{\sigma} \colon \overline{X} \to \overline{X}$ induces an automorphism $(\operatorname{id}_X \times \tilde{\sigma})^*$ of $\operatorname{Pic} \overline{X}$. This gives a group homomorphism

$$\operatorname{Gal}(\overline{k}/k) \to \operatorname{Aut}(\operatorname{Pic} \overline{X}), \quad \sigma \mapsto (\operatorname{id}_X \times \tilde{\sigma})^*$$

This action preserves the intersection pairing and thus takes exceptional curves to exceptional curves [Man74, Theorem 23.8].

If X has torsion free Picard group, we define the **splitting field** of X to be the smallest extension L of k in $\overline{k}$ for which the action of $\mathrm{Gal}(\overline{k}/L)$ on $\operatorname{Pic}\overline{X}$ is trivial.

Proposition 2.5.3. *Let X be a nice surface with torsion free geometric Picard group. Let L be the splitting field of X and let $\operatorname{Pic}\overline{X} \cong \mathbb{Z}^r$ for some $r > 0$. Then the inflation map $\inf\colon H^1(\mathrm{Gal}(L/k), (\operatorname{Pic}\overline{X})^H) \to H^1(G_k, \operatorname{Pic}\overline{X})$ is an isomorphism.*

Proof. Applying the inflation restriction exact sequence to the subgroup $H = \mathrm{Gal}(\overline{k}/L)$, we have $G_k/H = \mathrm{Gal}(L/k)$ and the first three terms of inflation restriction are

$$0 \to H^1(\mathrm{Gal}(L/k), (\operatorname{Pic}\overline{X})^H) \xrightarrow{\inf} H^1(G_k, \operatorname{Pic}\overline{X}) \xrightarrow{\mathrm{res}} H^1(\mathrm{Gal}(\overline{k}/L), \operatorname{Pic}\overline{X})^{\mathrm{Gal}(L/k)}$$

Since $\mathrm{Gal}(\overline{k}/L)$ is a limit of finite Galois groups, each of which acts trivially on the free abelian group $\mathbb{Z}^r$, it follows that $H^1(\mathrm{Gal}\,\overline{k}/L, \operatorname{Pic}\overline{X}) = 0$ because it is a limit of the groups $\mathrm{Hom}(H, \mathbb{Z}^r)$ which are all trivial, since H is finite. The result now follows. □

If we further assume that X has adélic points, then Remark 2.5.2 gives

$$H^1(\mathrm{Gal}(L/k), \operatorname{Pic} X_L) \cong H^1(G_k, \operatorname{Pic}\overline{X})$$

Moreover, if we have irreducible curves C on X cut out by equations with coefficients in L, then an element $\sigma \in \mathrm{Gal}(L/k)$ acts on C by applying σ to each coefficient.

Chapter 3

THE GEOMETRY OF CHÂTELET SURFACES

The goal of this section is to define Châtelet surfaces and then prove that over $\overline{k}$, a Châtelet surface is the blow-up of a Hirzebruch surface at four points. In other words, that Châtelet surfaces are geometrically rational. We begin with some background on ruled surfaces.

3.1 Background

Let X be a nice surface, let C, D be divisors on X that intersect properly and take $C.D = \#(C \cap D) = \deg_C(\mathcal{O}_X(D) \otimes \mathcal{O}_C)$ to be the usual intersection pairing on Div X × Div X [Har77, V.3]. For a nice variety, the Picard group $\operatorname{Pic} X$ coincides with the group of Weil divisors modulo linear equivalence.

Definition 3.1.1. (**Ruled Surface**) A ruled surface is a surface X, together with a surjective morphism $\pi\colon X \to C$ to a nonsingular curve C, such that

1) for every point $y \in C$, the fiber X_y is isomorphic to $\mathbb{P}^1$

2) π admits a section, $\sigma\colon C \to X$.

Proposition 3.1.2. [Har77, Proposition V.2.2]

If $\pi\colon X \to C$ is a ruled surface, then there exists a locally free sheaf $\mathcal{E}$ of rank 2 on C such that $X \cong \mathbb{P}(\mathcal{E})$ over C. Conversely, every such $\mathbb{P}(\mathcal{E})$ is a ruled surface over C. Additionally, $\mathbb{P}(\mathcal{E}) \cong \mathbb{P}(\mathcal{E}')$ if and only if there exists a line bundle $\mathcal{L}$ on C such that $\mathcal{E} \cong \mathcal{E}' \otimes \mathcal{L}$.

Definition 3.1.3. (Hirzebruch Surface) A Hirzebruch surface $\mathbb{F}_n$ is a ruled surface associated to the locally free sheaf $\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(-n)$ for $n \geq 0$.

Definition 3.1.4. (Blowing up a variety at a point) Let X be a variety and P a closed point of X. The blow-up of X at P is a variety $\widetilde{X}$ equipped with a morphism $\pi\colon \widetilde{X} \to X$ such that π induces an isomorphism of $\widetilde{X} - \pi^{-1}(P)$ to $X - P$. The preimage of P under π is called the *exceptional divisor* of the blow-up, and we denote it by E. Moreover, given any divisor D (containing P) on X, we define the *strict transform* of D, denoted $\widetilde{D}$, to be the closure of $\pi^{-1}(D - P)$.

Proposition 3.1.5. [Har77, Proposition V.3.2] *The intersection theory on $\widetilde{X}$ is defined by the following rules:*

- *If $C, D \in \text{Pic } X$, then $(\pi^*C).(\pi^*D) = C.D$*
- *If $C \in \text{Pic } X$, then $(\pi^*C).E = 0$*
- $E^2 = -1$

We refer to any curve on X of self-intersection $-n$ as a $(-n)$-curve.

Proposition 3.1.6. [Har77, Proposition V.3.6] *Let C be an effective divisor on X, let P be a point of multiplicty r on C, and let $\pi : \widetilde{X} \to X$ be the blow-up of X at P. Then*

$$\pi^*C = \widetilde{C} + rE$$

Theorem 3.1.7. ***(Castelnuovo)***[Har77, Theorem V.5.7] *If Y is a (-1)-curve on a smooth projective surface X, with $Y \cong \mathbb{P}^1$, then there exists a nonsingular projective surface X_0, a point $P \in X_0$, and a commutative diagram*

$$\begin{tikzcd}
X \arrow[r, "\sim"] \arrow[d, "f"] & \mathrm{Bl}_P X_0 \arrow[dl] \\
X_0 &
\end{tikzcd}$$

This result allows us to blow-down all (-1)-curves on X and obtain the "simplest" birational model of X. As a result of Theorem 2.4.3, we will be interested in determining when a given surface is rational. Due to the work of Castelnuovo, there is an especially nice criterion to determine this. Define $q(S) := h^1(S, \mathcal{O}_S)$ and for $n \geq 1$, $P_n(S) := h^0(S, K_S^{\otimes n})$ where K_S denotes the canonical divisor on S. We then have

Theorem 3.1.8. ***(Castelnuovo's Rationality Criterion)***[Bea78, Theorem V.1] *Let S be a surface. If $q(S) = P_2(S) = 0$ then S is rational.*

Proposition 3.1.9. [Bea78, Proposition III.21] *Let S be a ruled surface over C and let g be the genus of C.Then*

$$q(S) = g \quad \textit{and} \quad P_n(S) = 0 \quad \forall n \geq 2$$

Corollary 3.1.10. *$\mathbb{F}_n$ is rational.*

For more on rational surfaces see [Bea78].

We finish out our preliminary results with two important facts concerning Picard groups.

Proposition 3.1.11. ***(Picard Group of a ruled surface)***[Har77, Prop V.2.3] *Let $\pi : X \to C$ be a ruled surface, let $\sigma(C) \cong C_0 \subset X$ be a section, and F be a fiber. Then*

$$\text{Pic } X \cong \mathbb{Z} \oplus \pi^* \text{Pic } C$$

with $\mathbb{Z}$ generated by C_0. Additionally, C_0 and F satisfy $C_0.F = 1, F^2 = 0$.

Proposition 3.1.12. ***(Picard Group of a blow up)***[Har77, Prop V.3.2] *Given the natural map $\pi : \widetilde{X} \to X$, we have maps $\pi^* : Pic\ X \to Pic\ \widetilde{X}$ and $\mathbb{Z} \to Pic\ \widetilde{X}$ defined by $1 \mapsto 1 \cdot E$ which give rise to the isomorphism*

$$Pic\ \widetilde{X} \cong Pic\ X \oplus \mathbb{Z}$$

One of the main objects of interest in this paper will be Châtelet surfaces over $\mathbb{P}^1$. In attempting to study their arithmetic, we will need to have a good understanding of their geometry. In particular, geometric data will enable us to compute the Brauer group of a Châtelet surface. By Proposition 2.5.1, we must begin with a careful examination of the codimension one behavior on such a surface, i.e. the Picard group. We will now define a Châtelet surface and use results of the previous section to compute its Picard group.

3.2 The Picard Group of a Châtelet Surface

Let $P(\lambda) \in k[\lambda]$ be a separable polynomial of degree 4 and let $a \in k^{\times}$. Take

$$X_1 \coloneqq \operatorname{Proj} \frac{k[\lambda][y,z,t]}{(y^2 - az^2 - P(\lambda)t^2)} \subset \mathbb{P}^2_{\mathbb{A}^1_k}$$

with coordinates $(y : z : t, \lambda)$ and

$$X_2 \coloneqq \operatorname{Proj} \frac{k[\mu][Y,Z,T]}{(Y^2 - aZ^2 - Q(\mu)T^2)} \subset \mathbb{P}^2_{\mathbb{A}^1_k}$$

with coordinates $(Y : Z : T, \mu)$ and

$$Q(\mu) = \mu^4 P\Big(\frac{1}{\mu}\Big)$$

Let X be the surface obtained by gluing X_1 and X_2 via the isomorphism

$$X_1 - \{\lambda = 0\} \xrightarrow{\sim} X_2 - \{\mu = 0\}$$

$$(y : z : t, \lambda) \mapsto (Y : Z : \mu^2 T, 1/\mu)$$

which is indeed an isomorphism because

$$(y^2 - az^2 - P(\lambda)t^2) \mapsto (Y^2 - aZ^2 - P(1/\mu)\mu^4 T^2) = (Y^2 - aZ^2 - Q(\mu)T^2)$$

Definition 3.2.1. We call X the **Châtelet surface** given by $y^2 - az^2 = P(\lambda)$.

Observe that X comes with a morphism to $\pi\colon X \to \mathbb{P}^1_k$ obtained by gluing the projections $X_1 \to \mathbb{A}^1_k$ and $X_2 \to \mathbb{A}^1_k$ given by $(y : z : t, \lambda) \mapsto \lambda$ and $(Y : Z : T, \mu) \mapsto \mu$, respectively.

We note here that the original surfaces studied by Châtelet [Châ59] took $P(\lambda)$ to be degree 3 or 4 but we can obtain the degree 4 case from the degree 3 case by homogenizing, and if necessary, making a linear change of variables that shifts a root away from the point at infinity.

Proposition 3.2.2. *X is smooth, projective, and geometrically integral.*

Proof. To show smoothness, we can use the Jacobian criterion on an open cover of X. We can take as our open cover the 6 open affines given by $y \neq 0, z \neq 0$, and $t \neq 0$ on the $\operatorname{Spec} k[\lambda]$ patch, and $Y \neq 0, Z \neq 0$, and $T \neq 0$ on the $\operatorname{Spec} k[\mu]$ patch. It is easy to see that the Jacobian has rank 1 on each patch, and the only non-trivial computation happens where $t \neq 0$. The Jacobian here is given by

$$\begin{bmatrix} 2y & -2az & \frac{\partial P}{\partial \lambda} \end{bmatrix}$$

This has rank 1 at all points of X because separability of P implies that $\frac{\partial P}{\partial \lambda}$ and P are coprime.

To see that X is projective, we show it is proper. Since properness is local on the base, it is clear that π is a projective morphism on the standard affine cover of $\mathbb{P}^1_k$, hence π is proper. Since smooth proper **surfaces** are projective, X is projective. Note, Hironaka's example shows that smooth proper schemes are not projective in general, but when we restrict to the case of surfaces, it is in fact true [Har70, II.4.2].

Finally, to see that it is geometrically integral, note that smoothness implies regularity and regular local rings are always reduced, so it suffices to check that $\overline{X}$ is irreducible. It is enough to check it on an open cover. We first observe that the 6 standard affines intersect non-trivially, and on the open affine $t \neq 0, \mu \neq 0$, we consider the ring $\frac{k[\lambda][y,z,t]}{(y^2-az^2-P(\lambda))}$. Since a is a square in $\overline{k}$, consider the prime ideal $\mathfrak{p} = (y - \sqrt{a}z)$. By generalized Eisenstein, $y^2 - az^2 - P(\lambda)$ is irreducible so long as $P(\lambda)$ is not identically zero. This implies that $\frac{\overline{k}[\lambda][y,z,t]}{(y^2-az^2-P(\lambda))}$ is an integral domain. Verifying irreducibility on the other open affines follows similarly. □

As the ground field changes so does the Galois action on $\operatorname{Pic} \overline{X}$. Let $\sqrt{a}$ denote a fixed square root of a. When we base change X to $k(\sqrt{a})$ we obtain two sections C_0 and $\widetilde{C}_0$ which extend the sections $\lambda \mapsto (\sqrt{a} : 1 : 0, \lambda)$ and $\lambda \mapsto (-\sqrt{a} : 1 : 0, \lambda)$ respectively of the projection $X_{1_{k(\sqrt{a})}} \to \mathbb{A}^1_{k(\sqrt{a})}$

Let $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ be the roots of P in $\overline{k}$ so that we have a factorization of P given by

$$P(\lambda) = c\prod_{i=1}^{4}(\lambda - \lambda_i)$$

Denote the base change of our structure morphism by $\overline{\pi} : \overline{X} \to \mathbb{P}^1_{\overline{k}}$. All fibers above points of $\mathbb{P}^1_{\overline{k}}$ are smooth conics except for the 4 fibers above the roots λ_i on the open set $\overline{X}_1$, which degenerate to the union of two intersecting lines. We denote these subschemes of $\overline{X}_1$ by

$$E_i = \{\lambda = \lambda_i, y - \sqrt{a}z\} \quad \text{and} \quad \widetilde{E}_i = \{\lambda = \lambda_i, y + \sqrt{a}z\}$$

and both E_i and $\widetilde{E}_i$ are isomorphic to $\mathbb{P}^1_{\overline{k}}$. Finally, we let F denote the smooth conic over ∞ ($\mu = 0$).

Proposition 3.2.3.

$$\mathrm{Div}_{\overline{X}}(\lambda - \lambda_i) = E_i + \widetilde{E}_i - F \tag{3.2.1}$$

$$\mathrm{Div}_{\overline{X}}\left(\frac{y - \sqrt{a}z}{t}\right) = E_1 + E_2 + E_3 + E_4 + C_0 - \widetilde{C}_0 - 2F \tag{3.2.2}$$

Proof. To compute $\mathrm{Div}_{\overline{X}}(\lambda - \lambda_i)$ we observe that over $\operatorname{Spec} k[\lambda]$ it suffices to compute $V(\lambda - \lambda_i)$ to find the zeroes, this gives $E_i + \widetilde{E_i}$. Over $\operatorname{Spec} k[\mu]$ we apply the transition function $\lambda \mapsto \frac{1}{\mu}$ to obtain $\frac{1-\mu\lambda_i}{\mu}$. This has a pole at $\mu = 0$ which gives the smooth fiber F hence

$$\mathrm{Div}_{\overline{X}}(\lambda - \lambda_i) = E_i + \widetilde{E}_i - F$$

To carry out the remaining computation, we must compute valuations of the rational function $\frac{y-\sqrt{a}z}{t}$ over each affine patch, of which there are 6 . To do this we must find the codimension 1 primes **containing** $y - \sqrt{a}z$ or t. By localizing the appropriate ring at these primes, we obtain a DVR that has non-trivial valuation when applied to the functions in question. When localizing at the codimension 1 primes not containing the above functions, $y - \sqrt{a}z$ and t will become units in the localization, hence we safely ignore those.

To find these primes, we first let $A = \frac{\overline{k}[\lambda][y,z,t]}{(y^2-az^2-P(\lambda)t^2)}$ and then use the fact that codimension 1 primes containing $y-\sqrt{a}z$ or t, respectively correspond to codimension 0 (minimal) primes in $A/(y-\sqrt{a}z)$ and A/t, respectively. After finding these primes $\mathfrak{p}$, we find uniformizers in $A_{\mathfrak{p}}$ by using the relation $y^2-az^2 = P(\lambda)t^2$, and then compute valuations. We give divisors according to the equations that cut them out:

- $E_i = \{\lambda = \lambda_i, y - \sqrt{a}z = 0\}$
- $\widetilde{E}_i = \{\lambda = \lambda_i, y + \sqrt{a}z = 0\}$
- $C_0 = \{y - \sqrt{a}z = 0, t = 0\}$
- $\widetilde{C}_0 = \{y + \sqrt{a}z = 0, t = 0\}$
- $F = \{\mu = 0\}$

We can now begin the computation:

- $(z \neq 0, \mu \neq 0)$. We first find codimension 1 primes containing (t), hence look for minimal primes in $A/(t)$.

  $$A/(t) = \frac{k[\lambda][y]}{(y^2-a)} = \frac{\overline{k}[\lambda,y]}{(y+\sqrt{a})} \times \frac{\overline{k}[\lambda,y]}{(y-\sqrt{a})} = \overline{k}[\lambda]^2$$

  The minimal primes are clearly $(y \pm \sqrt{a})$ and the corresponding codimension 1 primes in A are simply obtained by lifting (i.e. adding t as a generator) thus yielding the primes $\mathfrak{p}_1 = (y+\sqrt{a}, t)$ and $\mathfrak{p}_2 = (y-\sqrt{a}, t)$

  In the local ring $A_{\mathfrak{p}_1}$, we can write the relation $y^2-az^2 = P(\lambda)t^2$ as

  $$y+\sqrt{a} = t^2\Big(\frac{P(\lambda)}{y-\sqrt{a}}\Big)$$

  Since $\frac{P(\lambda)}{y-\sqrt{a}}$ is a unit we have $y+\sqrt{a} \subset (t)$, hence t generates $\mathfrak{p}_1$ and $v_{\mathfrak{p}_1}(t) = 1$. An almost identical calculation shows that t is also a uniformizer in $A_{\mathfrak{p}_2}$. As a result, we get t vanishing to order 1.

Finding the codimension 1 primes containing $y-\sqrt{a}$ on the same patch, we look at

$$A/(y-\sqrt{a}) = \frac{\overline{k}[\lambda][y,t]}{(y^2-a-P(\lambda)t^2, y-\sqrt{a})} = \frac{\overline{k}[\lambda,t]}{(P(\lambda)t^2)} = \prod_{i=1}^{4}\frac{\overline{k}[\lambda,t]}{(\lambda-\lambda_i)} \times \frac{\overline{k}[\lambda,t]}{(t^2)}$$

and after lifting, we get minimal primes $\mathfrak{q}_i = (\lambda-\lambda_i, y-\sqrt{a})$ and $\mathfrak{p}_2 = (t, y-\sqrt{a})$.

In the local ring $A_{\mathfrak{q}_i}$, the relation

$$(y-\sqrt{a})(y+\sqrt{a}) = \Big(\prod_{j=1}^{4}(\lambda-\lambda_i)\Big)t^2$$

gives us the two expressions

$$y-\sqrt{a} = \frac{\Big(\prod_{j=1}^{4}(\lambda-\lambda_i)\Big)t^2}{(y+\sqrt{a})} \quad \text{and} \quad \lambda-\lambda_i = (y-\sqrt{a})\Big(\frac{y+\sqrt{a}}{\prod_{i\neq j}(\lambda-\lambda_i)t^2}\Big)$$

Thus both $\lambda-\lambda_i$ and $y-\sqrt{a}$ are uniformizers and at all codimension 1 points $\mathfrak{p}_i$, $\lambda-\lambda_i$ and $y-\sqrt{a}$ vanish to order 1. Since $V(\mathfrak{q}_i)$ is precisely E_i, we know we will have $\sum E_i$ in our computation.

Localizing at $\mathfrak{q} = (y-\sqrt{a}, t)$ we have $y-\sqrt{a} = (\text{unit})t^2$ thus t is a uniformizer and by applying $v_{\mathfrak{q}}$ to the previous equation we get $v_{\mathfrak{q}}(y-\sqrt{a}) = v_{\mathfrak{q}}(t^2) = 2$. We complete the computation on the $z\neq 0$ patch by observing that at the prime $\mathfrak{q}$, the divisor corresponding to $(y-\sqrt{a}, t)$ vanishes to order 2. This divisor is C_0. Directly computing the valuation of the rational function $\frac{y-\sqrt{a}}{t}$ in these localizations, we obtain

$$\mathrm{Div}_{\overline{X}_{z\neq 0}}\Big(\frac{y-\sqrt{a}}{t}\Big) = E_1+E_2+E_3+E_4+2C_0-(C_0+\widetilde{C}_0) = E_1+E_2+E_3+E_4+C_0-\widetilde{C}_0$$

- ($y \neq 0, \mu \neq 0$) We make a simplification by noting that the transition function from the $z \neq 0$ patch to the $y \neq 0$ patch is multiplication by $\frac{z}{y}$. On the $z \neq 0$ patch, we are given the function $\frac{\frac{y}{z}-\sqrt{a}}{t}$ and moving to the patch where $y \neq 0$ we get

$$\frac{\frac{z}{y}(\frac{y}{z}-\sqrt{a})}{t} = \frac{1-z\sqrt{a}}{t}$$

By observing how we determined the order of vanishing of t in the local rings $A_{\mathfrak{p}_1}$ and $A_{\mathfrak{p}_2}$ and the order of vanishing of $y-\sqrt{a}$ in $A_{\mathfrak{p}_i}$ and $A_{\mathfrak{q}}$, we see that applying the transition function, while changing the ideals, does not give new divisors in our computation. As a result, any divisor coming from the computation where $y \neq 0$ will be double counted, hence there is nothing more to compute.

- ($t \neq 0, \mu \neq 0$) This patch is defined by $\operatorname{Spec}(A_t)_0$ so t is a unit and there are no (codimension 1) primes containing t. Moreover, to find primes containing $y-\sqrt{a}z$ we look at $A/(y-\sqrt{a}z)$ which is

$$\frac{\overline{k}[\lambda][y,z,t]}{(y^2-az^2-P(\lambda), y-\sqrt{a}z)} = \frac{\overline{k}[\lambda]}{P(\lambda)} = \prod \frac{\overline{k}[\lambda]}{(\lambda-\lambda_i)} = \overline{k}^{\times 4}$$

There are 4 minimal primes, namely $(\lambda-\lambda_i)$ but this gives no new information.

- We now move to the other standard open affine of $\mathbb{P}^1_{\overline{k}}$. Write $Q(\mu) = \sum_{i=1}^4 a_i\mu^i$ and observe that under the map $(y:z:t,\lambda) \mapsto (Y:Z:\mu^2T, 1/\mu)$ our rational function is sent to

$$\frac{y-\sqrt{a}z}{t} \mapsto \frac{Y-\sqrt{a}Z}{\mu^2T}$$

When $Y, Z \neq 0$ the divisor computation is identical to what we did when $\mu \neq 0$. Moreover, we have looked at all possibilities when $\mu \neq 0$ so it remains to check what happens when $\mu = 0$ on either of the patches $Y \neq 0$ or $Z \neq 0$.

- ($Y \neq 0, \lambda \neq 0$) Let $B = \frac{\overline{k}[\mu][Z,T]}{(1-aZ^2-Q(\mu)T^2)}$ and let a_0 the constant term of $Q(\mu)$. Note that

a_0 is the leading term of $P(\lambda)$ so must be nonzero. On the locus where $\mu = 0$ we get

$$B/\mu = \frac{\overline{k}[\mu][Z,T]}{(1 - aZ^2 - Q(\mu)T^2, \mu)} = \frac{\overline{k}[Z,T]}{(1 - aZ^2 - a_0T^2)}$$

To see that B/μ is an integral domain, it suffices to show that $1 - aZ^2 - a_0T^2$ is irreducible. By taking $\mathfrak{p} = (z + \sqrt{a})$ as our prime ideal, its easy to see that $\mathfrak{p} | Z^2 - a$ and $\mathfrak{p}^2 \not| Z^2 - a$, hence by generalized Eisenstein, $1 - aZ^2 - a_0T^2$ is irreducible. This then implies that the only codimension 1 prime where μ vanishes is $\mathfrak{q} = (1 - aZ^2 - a_0T^2, \mu)$.

In the local ring $B_{\mathfrak{q}}$, the relation $1 - aZ^2 - Q(\mu)T^2$ can be written as

$$1 - aZ^2 - a_0T^2 - \mu(\mu^3 + a_3\mu^2 + a_2\mu + a_1)T^2$$

Without loss of generality, we are assuming $a_1 \neq 0$. If it were zero, we would take the first non-zero coefficient and factor out the lowest power of μ attached to it. The existence of this constant term ensures that right-most term in the above equation is a unit times μT^2. Furthermore, T is a unit as well, hence $1 - aZ^2 - a_0T^2 \in (\mu)$ and μ is a uniformizer. Considering valuations, we can see that $1 - \sqrt{a}z \notin (1 - aZ^2 - a_0T^2)$ for if it were we would have $Z = \frac{1}{\sqrt{a}}$ in B/μ, which in turn would imply that B/μ is zero dimensional. It now remains to compute $v_{\mathfrak{q}}(\mu^2 T)$.
Using the fact that μ is a uniformizer, we get

$$v_{\mathfrak{q}}(\mu^2 T) = 2v_{\mathfrak{q}}(\mu) = 2$$

The vanishing of μ gives the smooth fiber F . We can then say that

$$\mathrm{Div}_{\overline{X}_{y\neq 0}}\left(\frac{y - \sqrt{a}z}{\mu^2 T}\right) = -2F$$

By the argument we made for $\mu \neq 0$, the patch where $T \neq 0$ gives no new information, thus

$$\mathrm{Div}_{\overline{X}_{z\neq 0}}\left(\frac{y - \sqrt{a}}{t}\right) = E_1 + E_2 + E_3 + E_4 + C_0 - \widetilde{C}_0 - 2F$$

From this we can conclude that $2F = E_1 + E_2 + E_3 + E_4 + C_0 - \widetilde{C}_0$ in $\mathrm{Pic}\,\overline{X}$.

□

Proposition 3.2.4. *Let X be a Châtelet surface defined over k. Then $\overline{X}$ is a blow-up of a Hirzebruch surface $\mathbb{F}_n$ at 4 points.*

Proof. First, observe that all vertical fibers (preimages of $\overline{\pi}$ over closed points of $\mathbb{P}^1_{\overline{k}}$) are linearly equivalent. Let $F = \overline{\pi}^{-1}(P)$ and $F' = \overline{\pi}^{-1}(P')$, then $F^2 = F.F' = 0$. From Proposition 3.2.3, equation (3.2.1), we know that $E_i + \widetilde{E}_i - F$ is a principal divisor hence, in the Picard group, F and $E_i + \widetilde{E}_i$ are in the same class. We can now deduce that

$$0 = F^2 = F.(E_i + \widetilde{E}_i) = F.E_i + F.\widetilde{E}_i$$

Moreover, since the Galois action preserves the intersection pairing, we can conclude that $F.E_i = F.\widetilde{E}_i = 0$. Now, since intersecting with a principal divisor is 0 we have

$$0 = E_i.(E_i + \widetilde{E}_i - F) = E_i.E_i + E_i.\widetilde{E}_i + E_i.F = E_i^2 + 1$$

so $E_i^2 = -1$, for $i = 1, 2, 3, 4$. Intersecting with $\widetilde{E}_i$ we also see that $\widetilde{E}_i^2 = -1$. Now, we can take the four skew lines, E_i, and blow them down. Let $P_i \in E_i \cap \widetilde{E}_i$ in $\overline{X}$ and by abusing notation, let P_i denote the image of P_i under f. We have that P_i is a smooth (multiplicity one) point on $\widetilde{E}_i$, thus $\widetilde{E}_i \cong \mathbb{P}^1_{\overline{k}}$. We are left with a $\mathbb{P}^1_{\overline{k}}$-bundle over $\mathbb{P}^1_{\overline{k}}$ which is none other than $\mathbb{F}_n$ for some n. □

Corollary 3.2.5. *Let $\overline{X}$ be a Châtelet surface defined over $\overline{k}$. Then*

$$\text{Pic } X = \mathbb{Z}F \oplus \mathbb{Z}C_0 \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2 \oplus \mathbb{Z}E_3 \oplus \mathbb{Z}E_4$$

with intersection theory given by

$$C_0.F = \widetilde{C}_0.F = 1 \tag{3.2.3}$$

$$C_0.E_i = \widetilde{C}_0.\widetilde{E}_i = 1 \tag{3.2.4}$$

$$C_0.\widetilde{E}_i = \widetilde{C}_0.E_i = 0 \tag{3.2.5}$$

$$E_i.\widetilde{E}_i = 1 \tag{3.2.6}$$

$$C_0.\widetilde{C}_0 = 0 \tag{3.2.7}$$

Proof. By realizing $\overline{X}$ as the blow-up of a Hirzebruch surface, we know that $C_0.F = 1$. Using Proposition 3.2.3, equation (3.2.2)), we can conclude that $F.\widetilde{C}_0 = 1$. Since $F \sim E_i + \widetilde{E}_i$ we can see that

$$1 = C_0.F = C_0.(E_i + \widetilde{E}_i) = C_0.E_i + C_0.\widetilde{E}_i$$

Looking at the equations that cut out C_0 and E_i we can see that $C_0.E_i = 1$. Equation (3.2.4) now follows as does (3.2.5) by a similar argument. Equation (3.2.6) is clear, and to deduce (3.2.7) observe that $\{y - \sqrt{a}z = 0, t = 0\}$ and $\{y + \sqrt{a}z = 0, t = 0\}$ cut out C_0 and $\widetilde{C}_0$ respectively. If the two curves were to intersect, their equations would have a common solution, but this solution must be $y = z = t = 0$. Since y, z, t are homogeneous coordinates, at least one of them must be non-zero. □

3.3 The Brauer Group of a Châtelet Surface

In this section, we compute the Brauer group of a Châtelet surface, and in doing so, will realize that classes of Châtelet surfaces that can fail the Hasse Principle are those in which the factorization of $P(\lambda)$ has a certain form [CTSSD87a], greatly reducing a complex problem to a simpler one.

Theorem 3.3.1. *Let L denote the splitting field of P so that $L(\sqrt{a})$ is the splitting field of X. Assume $a \notin L^{\times 2}$. The Brauer group of X depends on the factorization of P with Brauer groups given by*

$$H^1(G_k, Pic(\overline{X})) = \begin{cases} \left(\mathbb{Z}/2\mathbb{Z}\right)^2 & P(\lambda) \textit{ has four rational roots} \\ \mathbb{Z}/2\mathbb{Z} & P(\lambda) \textit{ has at least one irredicible quadratic factor} \\ \{0\} & \textit{otherwise} \end{cases}$$

The proof of Theorem 3.3.1 is explicit but requires several important reductions.

Lemma 3.3.2. *The surface $X_{k(\sqrt{a})}$ is rational. In particular, if $a \in k^{\times 2}$, then X is rational.*

Proof. Let $P(\lambda) = \prod g_j(\lambda)$ and let Q_{g_j} denote the closed point of degree $\deg(g_j)$ corresponding to g_j. The fiber over Q_{g_j} degenerates to the union of two lines over $k(\sqrt{a})$ which we label E_i and $\widetilde{E}_i$. Letting $\lambda = \alpha$ denote a smooth fiber F we have

$$\mathrm{Div}_{X_{k(\sqrt{a})}}\left(\frac{g_j(\lambda)}{(\lambda-\alpha)^{\deg(g_j)}}\right) = E_j + \widetilde{E}_j - (\deg(g_j)F)$$

Since $E_j.\widetilde{E}_j = \deg(g_j)$ and intersecting with a principal divisor is always zero we have

$$E_j.(E_j + \widetilde{E}_j - (\deg(g_j)F)) = E_j^2 - E_j.\widetilde{E}_j - (\deg(g_j))E_j.F = E_j^2 + \deg(g_j) = 0$$

thus E_j is a $(-\deg(g_j))$-curve. Moreover, each component of E_j is a (-1)-curve
For each factor of $P(\lambda)$ over $k(\sqrt{a})$ we have such a collection of curves. As in the proof of Proposition 3.2.4, by blowing down these skew groups of curves we obtain a Hirzebruch surface, which is rational by Corollary 3.1.10. □

One can see that if $a \in k^{\times 2}$, then X is rational hence has no Brauer-Manin obstruction.

Lemma 3.3.3. *The Brauer group of X modulo constant algebras is 2-torsion. That is*

$$H^1(G_k, \operatorname{Pic}\overline{X})[2] = H^1(G_k, \operatorname{Pic}\overline{X})$$

Proof. First, if $a \in k^{\times 2}$ then $\operatorname{Br} X = 0$, hence $H^1(G_k, \operatorname{Pic}\overline{X}) = 0$. If a is not a square in k then consider the subgroup $H = \operatorname{Gal}(\overline{k}/k(\sqrt{a}))$ of G_k, with $G_k/H \cong \operatorname{Gal}(k(\sqrt{a})/k) \cong \mathbb{Z}/2\mathbb{Z}$.

By Lemma 3.3.2
$$\frac{\operatorname{Br} X_{k(\sqrt{a})}}{\operatorname{Br} k(\sqrt{a})} \cong H^1(\operatorname{Gal}(\overline{k}/k(\sqrt{a})), \operatorname{Pic} \overline{X}) = 0$$
Restriction-corestriction implies that $Cor \circ Res = [2]$ and furthermore, this is the zero map on $H^1(G_k, \operatorname{Pic} \overline{X})$. ☐

Lemma 3.3.4.
$$H^1(G_k, \operatorname{Pic} \overline{X}) \cong \frac{\left(\frac{\operatorname{Pic} \overline{X}}{2 \operatorname{Pic} \overline{X}}\right)^{G_k}}{\left(\operatorname{im}(\operatorname{Pic} \overline{X})^{G_k} \to \left(\frac{\operatorname{Pic} \overline{X}}{2 \operatorname{Pic} \overline{X}}\right)^{G_k}\right)}$$

Proof. Taking Galois cohomology with respect to the exact sequence
$$0 \to \operatorname{Pic} \overline{X} \xrightarrow{\times 2} \operatorname{Pic} \overline{X} \to \frac{\operatorname{Pic} \overline{X}}{2 \operatorname{Pic} \overline{X}} \to 0$$
we obtain the long exact sequence
$$0 \to (\operatorname{Pic} \overline{X})^{G_k} \xrightarrow{\times 2} (\operatorname{Pic} \overline{X})^{G_k} \xrightarrow{\psi} \left(\frac{\operatorname{Pic} \overline{X}}{2 \operatorname{Pic} \overline{X}}\right)^{G_k} \xrightarrow{\delta} H^1(G_k, \operatorname{Pic} \overline{X}) \xrightarrow{[2]} 0$$
Lemma 3.3.3 implies that $H^1(G_k, \operatorname{Pic} \overline{X})[2] = \mathrm{H}^1(G_k, \operatorname{Pic} \overline{X})$ by exactness so we obtain the desired isomorphism. ☐

Lemma 3.3.5. *A basis for* $(\operatorname{Pic} \overline{X})^{G_k}$ *is given by* $\{F\}$

Proof. Let L be the splitting field of $P(\lambda)$, so then $L(\sqrt{a})$ is the splitting field of X. Let $G = \operatorname{Gal}(L\sqrt{a})/k)$. It is easy to see that
$$(\operatorname{Pic} X_{L(\sqrt{a})})^G = \bigcap_{\sigma \in G} \ker(\sigma - \mathrm{id})$$
so in order to find a basis for $(\operatorname{Pic} \overline{X})^{G_k}$, we intersect bases for all the above eigenspaces.

We can easily see that $\sigma(F) = F$ for all $\sigma \in G$. To find a basis for $\bigcap_{\sigma \in G} \ker(\sigma - \mathrm{id})$, will show that $\ker(\sigma_{\sqrt{a}} - \mathrm{id}) = \operatorname{Span}\{F\}$, where $\sigma_{\sqrt{a}}$ denotes the involution coming from $\operatorname{Gal}(k(\sqrt{a})/k)$ and so then $(\operatorname{Pic} \overline{X})^{G_k} = \operatorname{Span}\{F\}$. Recalling that

- $E_i = \{\lambda = \lambda_i, y - \sqrt{a}z = 0\}$

- $\widetilde{E}_i = \{\lambda = \lambda_i, y + \sqrt{a}z = 0\}$
- $C_0 = \{y - \sqrt{a}z = 0, t = 0\}$
- $\widetilde{C}_0 = \{y + \sqrt{a}z = 0, t = 0\}$
- $F = \{\mu = 0\}$

we now fix the basis $\{F, C_0, E_1, E_2, E_3, E_4\}$ for $\operatorname{Pic} \overline{X}$ and represent the linear map $\sigma_{\sqrt{a}} - \mathrm{id}$ via a matrix in this basis.

$$\sigma_{\sqrt{a}} - id = \begin{bmatrix} 0 & -2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 & -2 \end{bmatrix}$$

From here it is immediate that this matrix has rank 5 with kernel spanned by F. This computation holds independent of the factorization of $P(\lambda)$ hence $(\operatorname{Pic} \overline{X})^{G_k} = \operatorname{Span}\{F\}$, and so $\operatorname{im}\left((\operatorname{Pic} \overline{X})^{G_k} \to \left(\frac{\operatorname{Pic} \overline{X}}{2 \operatorname{Pic} \overline{X}}\right)^{G_k}\right) = \operatorname{Span}\{F\}$. ☐

Proposition 3.3.6. *A basis for $\left(\frac{\operatorname{Pic} \overline{X}}{2 \operatorname{Pic} \overline{X}}\right)^{G_k}$ depends on the factorization of $P(\lambda)$.*

$$\left(\frac{\operatorname{Pic} \overline{X}}{2 \operatorname{Pic} \overline{X}}\right)^{G_k} = \begin{cases} \operatorname{Span}\{F, E_1 + E_2, E_2 + E_3\} & P(\lambda) \textit{has four rational roots} \\ \operatorname{Span}\{F, E_1 + E_2\} & P(\lambda) \textit{has one irredicible quadratic factor} \\ \operatorname{Span}\{F\} & \textit{otherwise} \end{cases}$$

Proof. Since $\operatorname{Pic} \overline{X} \cong \mathbb{Z}^6$ we obtain a basis for $\left(\frac{\operatorname{Pic} \overline{X}}{2 \operatorname{Pic} \overline{X}}\right)^{G_k}$ by reducing the entries of the matrices $\sigma - \mathrm{id}$ modulo 2, finding bases for their respective kernels, and finally, computing a basis for their intersection. Each of the following cases can be easily computed by associating, to each $\sigma \in G$, its corresponding element of S_4. We note that $\sigma_{\sqrt{a}}(C_0) = \widetilde{C}_0 \neq C_0$ hence to determine the basis for each eigenspace, we need only look at how the E_i are permuted. Moreover, the permutations of these roots are given by $\mathrm{Gal}(L/k)$, hence it is enough to compute bases for $\ker(\sigma - \mathrm{id})$ where $\sigma \in \mathrm{Gal}(L/k)$. Given an automorphism τ, we denote its eigenspace of eigenvalue 1 by S_τ. A basis for S_τ is then given by the span of the following vectors

- F
- Sums of exceptional curves $\sum_{i \in I} E_i$ where $I = \{i | \tau(i) \neq i\}$
- Exceptional curves E_j where $\tau(j) = j$.

We note that for any $\tau \in \mathrm{Gal}(L/k(\sqrt{a}))$, $\tau - \mathrm{id}$ is represented as a matrix, and one can easily see that the rank of this matrix (over k) is the same as its rank mod 2, so the above method does in fact compute bases modulo $(2\,\mathrm{Pic}\,\overline{X})^{G_k}$.

We begin by noting that since the action of $\sigma_{\sqrt{a}}$ does not depend in the factorization of P, the basis for $\ker(\sigma_{\sqrt{a}} - \mathrm{id})$ modulo $(2\,\mathrm{Pic}\,\overline{X})^{G_k}$ will be the same for each case. Furthermore, we can observe that $\sigma_{\sqrt{a}}$ fixes any sum of the form $E_i + E_j$ because from equation (3.2.1)

$$\sigma_{\sqrt{a}}(E_i + E_j) = \widetilde{E}_i + \widetilde{E}_j = F - E_i + F - E_j = 2F + E_i + E_j = E_i + E_j \in \left(\frac{\mathrm{Pic}\,\overline{X}}{2\,\mathrm{Pic}\,\overline{X}}\right)^{G_k}$$

Reducing the matrix $(\sigma_{\sqrt{a}} - \mathrm{id})$ modulo 2 we obtain a linear map whose kernel is $\mathrm{Span}\{F, E_1 + E_2, E_2 + E_3, E_3 + E_4\}$. Furthermore, by equation (3.2.2) we know that $E_1 + E_2 + E_3 + E_4$ is linearly equivalent to $2F$ so $E_1 + E_2 + E_3 + E_4 = 0$ in $\left(\frac{\mathrm{Pic}\,\overline{X}}{2\,\mathrm{Pic}\,\overline{X}}\right)^{G_k}$ (C_0 is not fixed by any element of G). We can now conclude that $E_1 + E_2$ and $E_3 + E_4$ represent the same class in

$$\frac{\left(\frac{\mathrm{Pic}\,\overline{X}}{2\,\mathrm{Pic}\,\overline{X}}\right)^{G_k}}{\left(\mathrm{im}(\mathrm{Pic}\,\overline{X})^{G_k} \to \left(\frac{\mathrm{Pic}\,\overline{X}}{2\,\mathrm{Pic}\,\overline{X}}\right)^{G_k}\right)}$$

We now proceed by considering the different possible factorizations of $P(\lambda)$.

Case 1: $P(\lambda) = \underbrace{(\lambda^2 - b)}_{\textbf{irreducible}} \underbrace{(\lambda^2 - c)}_{\textbf{irreducible}}$

In this case, $\mathrm{Gal}(L/k) \cong (\mathbb{Z}/2\mathbb{Z})^2$ and up to relabeling, is generated by the premutations (12) and (34) thus

$$S_{(12)} = \mathrm{Span}\{F, E_1 + E_2, E_3, E_4\} \quad \text{and} \quad S_{(34)} = \mathrm{Span}\{F, E_1, E_2, E_3 + E_4\}$$

hence

$$\Big(\frac{\mathrm{Pic}\,\overline{X}}{2\,\mathrm{Pic}\,\overline{X}}\Big)^{G_k} = \mathrm{Span}\{F, E_1 + E_2, E_3 + E_4\}$$

We conclude that

$$\frac{\big(\frac{\mathrm{Pic}(\overline{X})}{2\,\mathrm{Pic}(\overline{X})}\big)^{G_k}}{\Big(\mathrm{im}(\mathrm{Pic}(\overline{X})^{G_k} \to \big(\frac{\mathrm{Pic}(\overline{X})}{2\,\mathrm{Pic}(\overline{X})}\big)^{G_k}\Big)} = \frac{\mathrm{Span}\{F, E_1 + E_2, E_3 + E_4\}}{\mathrm{Span}\{F\}} = \mathrm{Span}\{E_1 + E_2\} \cong \mathbb{Z}/2\mathbb{Z}$$

so

$$H^1(G_k, \mathrm{Pic}(\overline{X})) \cong \mathbb{Z}/2\mathbb{Z}$$

Case 2: $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \underbrace{(\lambda^2 - b)}_{\textbf{irreducible}}$

In this case, $\mathrm{Gal}(L/k) \cong \mathbb{Z}/2\mathbb{Z}$ and is generated by (34), so $S_{(34)} = \mathrm{Span}\{F, E_1, E_2, E_3 + E_4\}$ and $\Big(\frac{\mathrm{Pic}\,\overline{X}}{2\,\mathrm{Pic}\,\overline{X}}\Big)^{G_k} = \mathrm{Span}\{F, E_1 + E_2, E_3 + E_4\}$. By the same argument as the previous case, we conclude that

$$H^1(G_k, \mathrm{Pic}(\overline{X})) \cong \mathbb{Z}/2\mathbb{Z}$$

and is generated by the class of $E_1 + E_2$.

Case 3: $P(\lambda) = \prod_{i=1}^{4}(\lambda - \lambda_i)$

In this case, $\sigma_{\sqrt{a}}$ is the only non-trivial automorphism. This implies that

$$\Big(\frac{\mathrm{Pic}\,\overline{X}}{2\,\mathrm{Pic}\,\overline{X}}\Big)^{G_k} = \ker(\sigma_{\sqrt{a}} - \mathrm{id}) = \mathrm{Span}\{F, E_1 + E_2, E_2 + E_3, E_3 + E_4\}$$

It now follows that

$$H^1(G_k, \mathrm{Pic}(\overline{X})) \cong (\mathbb{Z}/2\mathbb{Z})^2$$

and is generated by classes of $E_1 + E_2$ and $E_2 + E_3$.

Case 4: $P(\lambda) = (\lambda - \lambda_1)f(\lambda)$**, with** f **irreducible**

(a) $\underline{\mathrm{disc}(f) \in k^{\times 2}}$

Let $\tau = (123)$ be the generator of $\mathrm{Gal}(L/k) = A_3$. This means that $\ker(\tau - \mathrm{id}) = \mathrm{Span}\{F, E_1 + E_2 + E_3\}$ so intersecting with $\ker(\sigma_{\sqrt{a}} - \mathrm{id})$ we get a subspace of dimension 1, containing F hence

$$\Big(\frac{\mathrm{Pic}\,\overline{X}}{2\,\mathrm{Pic}\,\overline{X}}\Big)^{G_k} = \mathrm{Span}\{F\} \quad \text{and} \quad H^1(G_k, \mathrm{Pic}(\overline{X})) = \{0\}$$

(b) $\underline{\mathrm{disc}(f) \notin k^{\times 2}}$

Here, $\mathrm{Gal}(L/k) = \langle \rho = (12), \tau = (123) \rangle$. We get

$$\ker(\tau - \mathrm{id}) = \mathrm{Span}\{F, E_1 + E_2 + E_3\}, \quad \ker(\rho - \mathrm{id}) = \mathrm{Span}\{F, E_1, E_2 + E_3\}$$

so intersecting again with $\ker(\sigma_{\sqrt{a}} - id)$ we get a subspace of dimension 1 that must contain F, giving the same result.

Case 5: $P(\lambda)$ **irreducible**

Recall that the possibilities for $\mathrm{Gal}(L/k)$ are the transitive subgroups of S_4. These are the subgroups $S_4, A_4, D_4, \mathbb{Z}/4\mathbb{Z}$, and the unique normal subgroup that is isomorphic to $\Big(\mathbb{Z}/2\mathbb{Z}\Big)^2$.

Any transitive subgroup containing a 4-cycle will have eigenspace generated by F and $E_1 + E_2 + E_3 + E_4$. No other sum of exceptional divisors (hence no individual exceptional divisor) is contained in this eigenspace, therefore no exceptional divisor is contained in $\cap_\sigma \ker(\sigma - id)$.

From this we can conclude that

$$H^1(G_k, \mathrm{Pic}(\overline{X})) \cong \{0\}$$

whenever $\mathrm{Gal}(L/k) \cong S_4, D_4$, or $\mathbb{Z}/4\mathbb{Z}$.

If $\mathrm{Gal}(L/k) \cong \left(\mathbb{Z}/2\mathbb{Z}\right)^2 = \{(1), (12)(34), (13)(24), (14)(23)\}$, the non-trivial elements give eigenspaces $\mathrm{Span}\{F, E_1 + E_2, E_3 + E_4\}, \mathrm{Span}\{F, E_1 + E_3, E_2 + E_4\}$, and $\mathrm{Span}\{F, E_1 + E_4, E_2 + E_3\}$ respectively. Intersecting these three with $\ker(\sigma_{\sqrt{a}} - \mathrm{id}) = \mathrm{span}\{F, E_1 + E_2, E_2 + E_3, E_3 + E_4\}$ we see that only multiples of F and the element $E_1 + E_2 + E_3 + E_4$ lie in all 4. This shows that $H^1(G_k, \mathrm{Pic}(\overline{X}))$ is also trivial in this case.

Lastly, if $\mathrm{Gal}(L/k) \cong A_4$, then the non-trivial automorphisms correspond to 8 cycles of type $(1, 3)$ and 3 cycles of type $(2, 2)$. These correspond to eigenspaces of the form $\mathrm{Span}\{F, E_i + E_j + E_k\}$ for $i, j, k \in \{1, 2, 3, 4\}$. Assume that an element of the form $E_i + E_j + E_k \in \mathrm{Span}\{F, E_1 + E_2, E_2 + E_3, E_3 + E_4\} = \ker(\sigma_{\sqrt{a}} - \mathrm{id})$, then in particular,

$$(E_i + E_j + E_k) + (E_i + E_j) = 2E_i + 2E_j + E_k = E_k \in \ker(\sigma_{\sqrt{a}} - \mathrm{id})$$

but $\sigma_{\sqrt{a}}(E_k) = \widetilde{E}_k \neq E_k$, hence $\bigcap_{\sigma \in G} \ker(\sigma - \mathrm{id}) = \mathrm{Span}\{F\}$ and we can conclude that $H^1(G_k, \mathrm{Pic}(\overline{X})) = \{0\}$ whenever P is irreducible. □

We can now conclude the result of Theorem 3.3.1 by using the isomorphism given in Lemma 3.3.4. In particular, we compute a basis for the quotient by considering the bases given in Lemma 3.3.5 and Proposition 3.3.6.

3.4 *Iskovskikh's Example*

In 1971, Iskovskikh provided an example of a Châtelet surface that failed the Hasse principle [Isk71]. This occured when the Brauer-Manin set was first discovered as a tool to show that certain varieties can fail the Hasse principle. In fact, Iskovskikh's original example used methods of quadratic reciprocity to show that the given Châtelet surface failed the Hasse principle. It was only later that this example was explained via the Brauer-Manin obstruction. We now review the construction of Iskovskikh's Châtelet surface that fails the Hasse priciple, following the exposition made in [Poo17, Section 8.2.5]. In particular, we show that there is a Brauer-Manin obstruction to the Hasse principle.

Let X be the Châtelet surface given by

$$y^2 + z^2 = (3 - \lambda^2)(\lambda^2 - 2)$$

over $\mathbb{Q}$. Given any regular, integral, Noetherian scheme X, we have an injection $\operatorname{Br} X \hookrightarrow \operatorname{Br} K(X)$ [Gro68a, Corollaire 1.10], where $K(X)$ denotes the function field of X. Given any two elements $a, b \in K(X)^\times$, one can define a quaternion algebra $(a, b) \in (\operatorname{Br} K(X))[2]$. Considering the quaternion algebra $\mathcal{A} = (3 - \lambda^2, -1) \in \operatorname{Br} K(X)$, one can show that $\mathcal{A}$ lies in the subgroup $\operatorname{Br} X$ using residue homomorphisms. In particular, given an open cover $\{U_i\}$ of X, we have a sequence of injections $\operatorname{Br} X \hookrightarrow \operatorname{Br} U_i \hookrightarrow \operatorname{Br} K(X)$. Furthermore, given Brauer classes $\mathcal{B}_i \in \operatorname{Br} U_i$ whose images agree in $\operatorname{Br} K(X)$, one can conclude that they come from $\operatorname{Br} X$. This is a non-trivial fact, proven in [Poo17, Theorem 6.8.3].

Given any $g \in K(X)^\times$, the class of $(g, -1)$ is unaffected by multiplying g by a square or a norm from $k(\sqrt{-1})$. Let

$$\mathcal{B} = (\lambda^2 - 2, -1) \quad \text{and} \quad \mathcal{C} = (3/\lambda^2 - 1, -1)$$

From Proposition 2.3.6, we know that

$$[\mathcal{A}] + [\mathcal{B}] = [(y^2 + z^2, -1)] = 0 \quad \textit{and} \quad [\mathcal{A}] + [\mathcal{C}] = [((\frac{\lambda^2 - 3}{\lambda^2})^2, -1)] = 0$$

Since $2[\mathcal{A}] = 0$ we conclude that

$$[\mathcal{A}] = [\mathcal{B}] = [\mathcal{C}]$$

One can show [Poo17, Proposition 8.2.14] that there exists an open cover $\{U_{\mathcal{A}}, U_{\mathcal{B}}, U_{\mathcal{B}}\}$ of X on which $\mathcal{A}, \mathcal{B}, \mathcal{C}$ represent respective Brauer classes on each open set. It then follows that

$[\mathcal{A}] \in \operatorname{Br} X$.

Each of these three representatives will be used to compute the set $X(\mathbb{A}_{\mathbb{Q}})^{\operatorname{Br}}$. To evaluate $\mathcal{A}$ at a point $P \in X(k)$ for any $k \supset \mathbb{Q}$, choose any of $\mathcal{A}, \mathcal{B}$, or $\mathcal{C}$ such that the rational function of λ is defined and non-zero at P, and replace the rational function by its value at P. For example, if P is defined and non-zero at $3-\lambda^2$ then $\mathcal{A}(P) = (3-\lambda(P)^2, -1) \in \operatorname{Br} k[2]$. More generaly if k_v is a local field and X is any variety, then for any fixed algebra $\mathcal{A} \in \operatorname{Br} K(X)$ we have an *evaluation map*, $\mathrm{ev}_{\mathcal{A}} \colon X(k_v) \to \operatorname{Br} k_v$ given by $P \mapsto \mathcal{A}(P)$. This map, as we have seen already in Proposition 2.4.10, will be of central importance throughout the rest of this paper.

Since $\frac{\operatorname{Br} X}{\operatorname{Br} \mathbb{Q}} \cong \mathbb{Z}/2\mathbb{Z}$, we need only compute $X(\mathbb{A}_{\mathbb{Q}})^{\mathcal{A}}$ with $\mathcal{A}$, a generator of $\frac{\operatorname{Br} X}{\operatorname{Br} \mathbb{Q}}$. We will show that $\mathcal{A}$ gives a Brauer-Manin obstruction to the Hasse principle, that is, we show that $X(\mathbb{A}_{\mathbb{Q}})^{\mathcal{A}} = \emptyset$. Since, in this case, $X(\mathbb{A}_{\mathbb{Q}})^{\mathcal{A}} = X(\mathbb{A}_{\mathbb{Q}})^{\operatorname{Br}}$, emptiness of $X(\mathbb{A}_{\mathbb{Q}})^{\mathcal{A}}$ implies emptiness of $X(\mathbb{A}_{\mathbb{Q}})^{\operatorname{Br}}$, which in turn implies that $X(\mathbb{Q}) = \emptyset$ by [CTSSD87a]. We begin by noting that X has a $\mathbb{Q}_p$ point for every $p \leq \infty$, in other words

Proposition 3.4.1. $X(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$

Before we prove the proposition we recall the notion of a Hilbert symbol along with a basic property. For $v \in \Omega_k$ and $t, u \in k_v^{\times}$ we define the **Hilbert symbol** $(t, u)_v \in \{\pm 1\}$ by the rule $(t, u)_v = 1$ if and only if $x^2 - ty^2 = uz^2$ has a solution $(x, y, z) \neq (0, 0, 0) \in k_v^{\times 3}$. The main property of Hilbert symbols we will use is the following

Lemma 3.4.2. [Ser73, Chapter III] *Suppose that v is odd and that $v(t) = 0$. Then $(t, u)_v = -1$ if and only is $v(u)$ is odd and the image of t in $\mathbb{F}_v$ is a non-square.*

Proof. The fact that $X(\mathbb{R}) \neq \emptyset$ is obvious so we begin by assuming that v is 2-adic. By considering the univariate polynomial $f(\lambda) = (\lambda^2 - 2)(3 - \lambda^2)$ obtained by setting $y = z = 0$ we can see that

$$v(f(0)) > 2v(f'(0))$$

Hence the strong version of Hensel's Lemma [Lan94, Chapter II, Proposition 2.2] implies that there exists a root in $\mathbb{Q}_2$.

If v is odd and $v \neq v_3$, we aim to prove that $(-1,(x^2-2)(3-x^2))_v = 1$ for $x \in \mathbb{Q}_p$. Note that this would imply the existence of a p-adic solution to $y^2+z^2 = (\lambda^2-2)(3-\lambda^2)$ for odd p not equal to 3. By Lemma 3.4.2, it is enough to find $x \in \mathbb{Q}_p$ such that $v_p((x^2-2)(3-x^2))$ is even. Pick any $x \in \mathbb{Q}_p$ such that $v_p(x) < 0$, it follows that

$$v_p((x^2-2)(3-x^2)) = v_p(x^2-2) + v_p(3-x^2) = 2v_p(x^2) = 4v_p(x)$$

So X has a $\mathbb{Q}_p$ point for all $p \neq 3$.

When $p = 3$, the proof follows similarly. For $x \in \mathbb{Q}_3$ such that $v_3(x) < 0$ we have

$$v_3((x^2-2)(3-x^2)) = 4v_3(x)$$

Hence X has a $\mathbb{Q}_3$ point. □

It now remains to compute $\operatorname{inv}_{v_p}(\mathcal{A}(P))$ and by implicit function theorem, it is enough to compute this for $P \in U(\mathbb{Q}_p)$ for any Zariski-dense open set $U \subset X$.

Lemma 3.4.3. *Let k be a local field and let X be smooth of dimension n over k. Let $U \subset X$ be a nonempty Zariski open set of X. Then $U(k)$ is analytically dense in $X(k)$.*

Proof. Take $P \in X(k)$. We show that we can find a sequence of points $P_i \in U(k)$ that converge to P. Let $\pi : X \to \operatorname{Spec} k$ be the structure morphism of X, by [Poo17, Prop 3.5.48], there exists a Zariski open neighborhood V of P and an étale morphism $\varphi : V \to \mathbb{A}^n_k$ such that

$$\begin{tikzcd}
V \arrow[r, "\varphi"] \arrow[d, "\pi"] & \mathbb{A}^n_k \arrow[dl] \\
\operatorname{Spec} k &
\end{tikzcd}$$

The étale map φ satisfies the hypothesis of the implicit function theorem, thus there exists *analytically* open neighborhoods $O_1 \ni P$ and O_2 of $V(k)$ and k^n respectively, such that φ

induces a homeomorphism $\theta : O_1 \to O_2$.

Let $X \setminus U$ denote the Zariski complement of U and consider $(X \setminus U) \cap V$, which contains both P and O_1. Consider the closed set $C = \overline{\varphi(X \setminus U) \cap V)}$ of $\mathbb{A}^n_k$. Since C doesn't contain the image of U under φ, it is not all of $\mathbb{A}^n_k$ hence $\dim C < n$. Note that for any Zariski closed set $C = V(f_1, ..., f_n)$ we can construct a sequence of points P_i converging to $\varphi(P)$, by openness of O_2. Moreover, we can ensure that all $P_i \in O_2 \setminus C$. To see why, observe that the polynomials f_i are all continuous in the analytic toplogy. Since $V(f_i) = f_i^{-1}(0)$, we conclude that C is also closed in the analytic topology, hence $O_2 \setminus C$ is analytically open. It now follows that $\theta^{-1}(P_i) \in U(k)$ and by continuity of θ we have

$$\theta^{-1}(\lim P_i) = \lim \theta^{-1}(P_i) = P$$

□

Proposition 3.4.4. *Fix a place p of $\mathbb{Q}$. Then for any $P \in X(\mathbb{Q}_p)$,*

$$\mathrm{inv}_{v_p}(\mathcal{A}(P)) = \begin{cases} 0 & p \neq 2 \\ \frac{1}{2} & p = 2 \end{cases}$$

Proof. Let X_0 be the affine surface in $\mathbb{A}^3$ given by $y^2+z^2 = (3-\lambda^2)(\lambda^2-2)$. Since X is smooth, Lemma 3.4.3 shows that $X_0(\mathbb{Q}_p)$ is p-adically dense in $X(\mathbb{Q}_p)$. By Lemma 2.4.8, $\mathrm{inv}_{v_p} \circ \mathrm{ev}_{\mathcal{A}}$ is a continuous function on $X(\mathbb{Q}_p)$, hence it suffices to prove the result for $P \in X_0(\mathbb{Q}_p)$.

For each $P \in X_0(\mathbb{Q}_p)$, $\mathcal{A}(P) = (3 - \lambda(P)^2, -1)$ defines a quaternion algebra in $\mathrm{Br}(\mathbb{Q}_p)$. In particular, $\mathcal{A}$ can be viewed as a cyclic algebra where $j^2 = -1$ and $i^2 = 3 - \lambda^2$. Furthermore, we can compute the image of $\mathcal{A}(P)$ under the invariant map since $\mathrm{inv}_{v_p}(\mathcal{A}(P)) = 0$ if and only if $3 - \lambda(P) \in N_{\mathbb{Q}_p(i)/\mathbb{Q}_p}(\mathbb{Q}_p(i)^\times)$. In other words,

$$\mathrm{inv}_{v_p}(\mathcal{A}(P)) = \begin{cases} 0 & \mathcal{A}(P) \text{is split} \\ \frac{1}{2} & \mathcal{A}(P) \text{is non-split} \end{cases}$$

Case 1: $p \notin \{2, \infty\}$

If $v_p(\lambda) < 0$ then $v_p(3/\lambda^2 - 1) = \min\{v_p(3) - 2v_p(\lambda), 0\}$. Note that $v_p(3) = 0$ or 1 but

$-2v_p(\lambda) > 0$ hence $v_p(3) - 2v_p(\lambda) > 0$ and $v_p(3/\lambda^2 - 1) = 0$. This implies that $3/\lambda^2 - 1 \in \mathbb{Z}_p^\times$. If $v_p(\lambda) \geq 0$ then

$$0 = v_p(1) = v_p(3 - \lambda^2 + \lambda^2 - 2) \geq \min\{v_p(3 - \lambda^2), v_p(\lambda^2 - 2)\}$$

which implies that

$$\min\{v_p(3), v_p(\lambda^2)\} \leq v_p(3 - \lambda^2) \leq 0 \quad \text{or} \quad \min\{v_p(2), v_p(\lambda^2)\} \leq v_p(\lambda^2 - 2) \leq 0$$

hence one of $3 - \lambda^2$ or $\lambda^2 - 2$ are in $\mathbb{Z}_p^\times$.

Now, we can recognize $\mathcal{A}(P)$ as an element of $\mathrm{Br}(\mathbb{Z}_p)$ via the definition of an Azumaya algebra over the ring $\mathbb{Z}_p$. It is given by a $\mathbb{Z}_p$-algebra A_p that is free and of finite rank as a $\mathbb{Z}_p$-module, such that $A_p \otimes_{\mathbb{Z}_p} \mathbf{k}(x)$ is a central simple algebra over $\mathbf{k}$, where $\mathbf{k}$ denotes the residue field of $x \in \mathbb{Z}_p$. We can now see that $\mathrm{ev}_{\mathcal{A}}(P) = \mathcal{A}(P) = (u, -1)$ with $u \in \mathbb{Z}_p^\times$. By considering the Azumaya algebra $A_p := \mathbb{Z}_p \otimes \mathbb{Z}_p i \otimes \mathbb{Z}_p j \otimes \mathbb{Z}_p ij$, with multiplication defined as for $\mathcal{A}(P)$, we have that $A_p \in \mathrm{Br}\,\mathbb{Z}_p$. Moreover, $A_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathcal{A}(P)$ thus A_p maps to $\mathcal{A}(P)$ under $\mathrm{Br}\,\mathbb{Z}_p \to \mathrm{Br}\,\mathbb{Q}_p$ and $\mathrm{Br}\,\mathbb{Z}_p = 0$ by Corollary 2.4.7. This implies that $\mathcal{A}(P)$ is trivial in $\mathrm{Br}\,\mathbb{Q}_p$ hence $\mathrm{inv}_{v_p}(\mathcal{A}(P)) = 0$.

Case 2: $p = \infty$

Any $P \in X_0(\mathbb{R})$ satisfies

$$(3 - \lambda(P)^2)(\lambda(P)^2 - 2) = y^2 + z^2 > 0$$

hence either

$$(3 - \lambda(P)^2) \geq 0 \quad \text{and} \quad (\lambda(P)^2 - 2) \geq 0$$

or

$$(3 - \lambda(P)^2) \leq 0 \quad \text{and} \quad (\lambda(P)^2 - 2) \leq 0$$

The latter isn't possible for if it were we could have $3 \leq \lambda(P)^2 \leq 2$. Since the former holds then

$$(3 - \lambda(P)^2), (\lambda(P)^2 - 2) \in \mathbb{R}_{>0} = \mathrm{N}_{\mathbb{C}/\mathbb{R}}(\mathbb{C}^\times)$$

implying that $\mathcal{A}(P) = 0 \in \operatorname{Br}\mathbb{R}$, thus $\operatorname{inv}_{v_\infty}(\operatorname{ev}_{\mathcal{A}}(P)) = 0$.

Case 3: $p = 2$

Let $P \in X(\mathbb{Q}_2)$ and recall that an element $x \in \mathbb{Z}_2$ is not of the form $a^2 + b^2$ if $x \equiv -1 \pmod 4$. We consider three possibilities.

If $v_2(\lambda(P)) > 0$ then $\lambda(P)^2 \equiv 0 \pmod 4$ hence $3 - \lambda(P)^2 \equiv 3 \equiv -1 \pmod 4$

If $v_2(\lambda(P)) = 0$ then $\lambda(P)^2 \equiv 1 \pmod 4$ hence $\lambda(P)^2 - 2 \equiv -1 \pmod 4$

If $v_2(\lambda(P)) < 0$ then $\frac{1}{\lambda(P)^2} \equiv 0 \pmod 4$ hence $3/\lambda(P)^2 - 1 \equiv -1 \pmod 4$

We can see that in all three cases, $\mathcal{A}(P) = (x, -1)$ where x is not a norm from $\mathbb{Q}_2(i)/\mathbb{Q}_2$, hence $\mathcal{A}(P)$ is non-split and $\operatorname{inv}_{v_2}(\mathcal{A}(P)) = \frac{1}{2}$.

$\square$

Corollary 3.4.5.

$$X(\mathbb{A}_k)^{\mathcal{A}} = \emptyset$$

Proof. From Proposition 3.4.4, it follows that $\sum_{v \in \Omega_k} \operatorname{inv}_v(\operatorname{ev}_{\mathcal{A}}(P_v)) = \frac{1}{2}$ so

$$X(\mathbb{A}_k)^{\mathcal{A}} = \{(P_v) \in X(\mathbb{A}_k) \colon \sum_{v \in \Omega_k} \operatorname{inv}_v(\operatorname{ev}_{\mathcal{A}}(P_v)) = 0\} = \emptyset$$

By [CTSSD87a, Theorem A], $X(\mathbb{Q}) = \emptyset$ if and only if $X(\mathbb{A}_{\mathbb{Q}})^{\operatorname{Br}} = \emptyset$, thus we can conclude that $X(\mathbb{Q}) = \emptyset$ and X fails the Hasse principle over $\mathbb{Q}$. $\square$

3.5 Potential Hasse Principle Failures for Châtelet Surfaces

Since the discovery of Iskovskikh's example of a Châtelet surface which fails the Hasse principle, Poonen constructed Châtelet surfaces which, over fixed number fields, also fail the Hasse principle [Poo09]. Given the existence of these examples, it is natural to wonder what positive results can be obtained within a larger framework. The questions of primary interest to us throughout the remainder of this paper are two-fold:

Question 1. What can be said about the arithmetic of larger classes of surfaces, in particular, ones which contains the class of Châtelet surfaces?

To this end, much work has been done over the past several decades and the natural class of surfaces to consider has been smooth conic bundles $X \to \mathbb{P}^1_k$. Over local and global fields, the arithmetic of such conic bundles has been well studied. By the Hasse-Minkowski theorem, conic bundles over $\mathbb{Q}$ with two geometric singular fibers always satisfy the Hasse principle and work of Iskovskikh shows that conic bundles over $\mathbb{Q}$ with three geometric singular fibers always have rational points [Isk96]. For the case of conic bundles with four geometric singular fibers, the example of Section 3.4 shows that the arithmetic is not as well-behaved. Thankfully, further investgation of this case is a tractable problem due to the landmark result of Colliot-Thélène, Sansuc, and Swinnerton-Dyer [CTSSD87a] which shows that for Châtelet surfaces, the Brauer-Manin obstruction to the Hasse principle is the only one. These results were extended far more generally to the class of conic bundles with five or fewer singular fibers, thanks to a paper of Colliot-Thélène and Swinnerton-Dyer [CTSD94]. In the same paper, it was shown that if one assumes Schinzel's hypothesis (a wide reaching generalization of Dirichlet's theorem on primes in arithmetic progressions) the same conclusion can be made for conic bundles with any number of geometric singular fibers. In other words the arithmetic of an arbitrary conic bundle is largely controlled by its Brauer group. This idea is the central tool which enables us to investigate the second question of interest.

Question 2. Given a number field k and a Châtelet surface or, more generally, a smooth conic bundle $X \to \mathbb{P}^1_k$ with five or fewer geometric singular fibers, what extensions L/k guarantee the existence of L-rational points?

With the current body of knowledge surrounding the arithmetic of such conic bundles, this question can naturally be broken down into two scenarios.

1. If $X(\mathbb{A}_k) \neq \emptyset$ and $X(k) = \emptyset$, that is, X fails the Hasse principle over k.

2. If $X(\mathbb{A}_k) = \emptyset$, that is, X trivially satisfies the Hasse principle over k.

The scenario in case 1 is the one which we consider first. In particular, since the Brauer-Manin obstruction to the Hasse principle is the only one, we consider when points of $X(\mathbb{A}_k)$ can be chosen to lie in $X(\mathbb{A}_L)^{\operatorname{Br} X_L}$. The existence of such a point would imply that $X(\mathbb{A}_L)^{\operatorname{Br} X_L} \neq \emptyset$ and hence by [CTSD94], $X(L) \neq \emptyset$.

Case 2, on the other hand, can require a bit more work. In particular, since computing $X(\mathbb{A}_L)^{\operatorname{Br} X_L}$ relies on knowing the Brauer group of X_L, one can encounter instances where Brauer classes of $\operatorname{Br} X_L$, which are not defined over k, can appear upon base extension and *potentially* give a Brauer-Manin obstruction to the Hasse principle over L. Such an example would require the existence of a variety X/k with $X(\mathbb{A}_k) = \emptyset$ *and* a fixed extension L/k such that $X(\mathbb{A}_L) \neq \emptyset$ but $X(L) = \emptyset$. To obtain the latter condition it is of course enough to show that $X(\mathbb{A}_L)^{\operatorname{Br} X_L} = \emptyset$.

Examples of varieties failing the Hasse principle over extensions of the ground field has only recently been given any consideration.

Definition 3.5.1. Given a nice variety X/k such that $X(k) = \emptyset$ we say X is a **potential Hasse principle failure** (PHPF) if there exists an extension L/k such that $X(L) = \emptyset$ and $X(\mathbb{A}_L) \neq \emptyset$.

In 2009, Pete Clark coined the notion of a potential Hasse principle failure for smooth, geometrically irreducible varieties over number fields. Clark showed the existence of infinitely many curves which were PHPFs. It was also conjectured that for any positive genus curve C/k with no k-rational points, C is a PHPF, see [Cla09, Theorem 1, Conjecture 4]. In particular, he conjectured the following

Conjecture 3.5.1. Every curve C/k of genus ≥ 2 with $X(k) = \emptyset$ is a potential Hasse principle failure.

This conjecture still remains an open question but the case for surfaces was seldom studied until a few years ago. In 2019, Bianca Viray and Brendan Creutz investigated a related question and obtained a positive result for nice varieties over global fields satisfying a series of necessary conditions on their Brauer groups. [CV22, Corollary 2.4].

Building upon these ideas in the next chapter, we provide a complete classification of the behavior of rational points on conic bundles over even degree extensions of the base field, answering Question 2 in its entirety.

Chapter 4

ON THE HASSE PRINCIPLE FOR CONIC BUNDLES OVER EVEN DEGREE EXTENSIONS

Since conics have numerous quadratic points, for any conic bundle there are numerous quadratic extensions over which they satisfy the Hasse principle. However, we show that a much stronger statement holds, namely that for general conic bundles with 4 geometric singular fibers, the Hasse principle holds over **every** even degree extension.

Theorem 4.0.1. *Let k be a number field, let $X \to \mathbb{P}^1_k$ be a conic bundle with four geometric singular fibers, and assume that one of the following holds:*

1. $\frac{\operatorname{Br} X}{\operatorname{Br} k} \neq 0$
2. $X(\mathbb{A}_k) \neq \emptyset$
3. *The singular fibers lie over an A_4 or S_4 extension of k*

Then, if L/k is an even degree extension, we have

$$X(L) \neq \emptyset \Leftrightarrow X(\mathbb{A}_L) \neq \emptyset. \tag{4.0.1}$$

Recalling the notion of a potential Hasse principle failure from Definition 3.5.1, Theorem 4.0.1 tells us that most conic bundles with four bad fibers are not PHPFs. However, there do exist some cases where new Brauer classes can give a Brauer-Manin obstruction over L.

If the conditions of Theorem 4.0.1 fail, then one can prove that the singular fibers of $X \to \mathbb{P}^1_k$ lie over a single closed point of $\mathbb{P}^1_k$ and there exists a quadratic extension L/k such that the singular fibers of $X_L \to \mathbb{P}^1_L$ lie over two closed points of degree 2. For quadratic extensions like L/k, the Hasse principle can fail.

Theorem 4.0.2. *The Châtelet surface $X/\mathbb{Q}$ given by*

$$y^2 - 5z^2 = \frac{3}{5}(5\lambda^4 + 7\lambda^2 + 1)$$

has no $\mathbb{A}_{\mathbb{Q}}$-points and fails the Hasse principle over $L = \mathbb{Q}(\sqrt{29})$, i.e. X is a potential Hasse principle failure.

Indeed, we show that quadratic extensions which exhibit this behavior are the only even degree extensions over which the Hasse principle can fail (see Corollary 4.2.2). In addition, we trace the failure of the Hasse principle to a parity condition on the number of ramification places of a conic that splits in a fixed quadratic extension (see Theorem 4.4.1).

For arbitrary conic bundles we do not have such unconditional results, however, we can extend our prior results to conic bundles with more geometric singular fibers at the expense of restricting to quadratic extensions.

Theorem 4.0.3. *Let k be a number field, let $X \to \mathbb{P}^1_k$ be a conic bundle, let S denote the set of closed points on $\mathbb{P}^1_k$ corresponding to the singular fibers, and for $P \in S$, let $\mathbf{k}(P)$ denote its residue field. Assume Schinzel's hypothesis. If L/k is a quadratic extension linearly disjoint from $\mathbf{k}(P)$ for all $P \in S$, then*

$$X(L) \neq \emptyset \Leftrightarrow X(\mathbb{A}_L) \neq \emptyset.$$

4.0.1 Outline of the proof of Theorems 4.0.1 and 4.0.3

The statements of both Theorems follow from results about Brauer-Manin obstructions. We prove these results by first showing (in section 4.2) that

$$X(\mathbb{A}_L)^{\operatorname{Res}_{L/k}(\operatorname{Br}(X_k))} \neq \emptyset \Leftrightarrow X(\mathbb{A}_L) \neq \emptyset. \tag{4.0.2}$$

In using this result to deduce the main theorems, a careful examination of the map $\operatorname{Res}_{L/k} \colon \frac{\operatorname{Br} X_k}{\operatorname{Br} k} \to \frac{\operatorname{Br} X_L}{\operatorname{Br} L}$ is needed. We then show that the same assumptions on X imply

$$X(\mathbb{A}_L)^{\operatorname{Br}(X_L)} \neq \emptyset \Leftrightarrow X(\mathbb{A}_L) \neq \emptyset. \tag{4.0.3}$$

A case by case analysis shows that when $\mathrm{Res}_{L/k}$ is not surjective, statement (4.0.3) still holds for all even degree extensions L/k except (at most) three.

When $X \to \mathbb{P}^1_k$ is a Châtelet surface, Theorem 4.0.1 follows from statement (4.0.3) by a landmark result of Colliot-Thélène, Sansuc, and Swinnerton-Dyer [CTSSD87a] which states that for Châtelet surfaces, the Brauer-Manin obstruction to the Hasse principle is the only one. More generally, Theorem 4.0.1 follows from (4.0.3) by work of Colliot-Thélène and Coray combined with results of Colliot-Thélène and Swinnerton-Dyer. Colliot-Thélène and Coray show that if X/k is a conic bundle with five or fewer geometric singular fibers, then $X(k) \neq \emptyset$ if and only if there exists a zero cycle of degree one [CTC79]. Furthermore Colliot-Thélène and Swinnerton-Dyer, building upon work of Salberger [Sal88, Sal90], have results which, when applied to conic bundles, state that if there is no Brauer-Manin obstruction to the existence of a 0-cycle of degree one then there exists a 0-cycle of degree one [CTSD94, Theorem 5.1]. In particular, no Brauer-Manin obstruction to the existence of rational points implies no Brauer-Manin obstruction to the existence of a zero cycle of degree one, hence we can conclude Theorem 4.0.1.

When X has arbitrarily many geometric singular fibers, Theorem 4.0.3 follows from statement (4.0.3) by different means. In 1982, Colliot-Thélène and Sansuc pioneered the method of using Schinzel's hypothesis and the fibration method to prove that certain varieties have rational points [CTS82]. In particular, a theorem of Colliot-Thélène and Swinnerton-Dyer shows that if k is a number field, X is a conic bundle, and one assumes Schinzel's hypothesis, then $X(\mathbb{A}_k)^{\mathrm{Br}(X_k)} \neq \emptyset \Leftrightarrow X(k) \neq \emptyset$ [CTSD94]. Consequently, the conditional statement of Theorem 4.0.3 follows from this work.

4.1 The Brauer group of conics and conic bundles

In this section, we collect some general results concerning Brauer groups of conic bundles. For a background on the Brauer-Manin obstruction see [Poo17, §8.2]. We note that many of these results are well known but we include their proofs for completeness.

Lemma 4.1.1. *Let k be a local field and let C/k be a smooth conic. If L/k is an even degree extension then $C(L) \neq \emptyset$.*

Proof. From local class field theory, the restriction map factors as

$$\mathrm{Res}_{L/k}\colon\ \mathrm{Br}\,k \cong \mathbb{Q}/\mathbb{Z} \xrightarrow{\cdot[L\colon k]} \mathbb{Q}/\mathbb{Z} \cong \mathrm{Br}\,L$$

Using the correspondence between smooth conics and quaternion algebras, we can apply this to any element of $\mathrm{Br}\,k[2]$ and obtain the result. □

4.1.1 The Brauer group of a conic bundle

Throughout, let k denote an infinite field of characteristic not equal to 2 and let $X \to \mathbb{P}^1_k$ be a conic bundle. After a suitable change of coordinates on $\mathbb{P}^1_k$, we can and will assume that the fiber over the point at infinity, X_∞, is a smooth conic. Let t be the coordinate on $\mathbb{A}^1_k = \mathbb{P}^1_k \smallsetminus \{\infty\}$ and let S denote the finite set of closed points on $\mathbb{A}^1_k$ with geometric singular fiber. Let $|S|$ denote the number of such closed points, not counting their degree, i.e. if X has four geometric singular fibers and $|S| = 1$ then S is irreducible and $\deg(S) = 4$. For any such point $P \in \mathbb{A}^1_k$, let $\mathbf{k}(P)$ denote its residue field. The fiber X_P degenerates to the union of two lines, ℓ_P and ℓ'_P, defined over $\mathbf{k}(P)(\sqrt{a_P})$ for some $a_P \in \mathbf{k}(P)^*$. Let $\tau_P \in \mathbf{k}(P)$ be the image of t in $\mathbf{k}(P)$ and let $\varepsilon = (\varepsilon_P) \in \mathbb{F}_2^{|S|}$ be any vector satisfying

$$\prod_{P \in S} N_{\mathbf{k}(P)/k}(a_P)^{\varepsilon_P} \in k^{\times 2}$$

where $N_{\mathbf{k}(P)/k}$ denotes the usual norm map. With this setup, we obtain an especially useful generating set for $\frac{\mathrm{Br}\,X}{\mathrm{Br}\,k}$ which comes from the following fact.

Lemma 4.1.2. [CTS21, Corollary 10.2.4] *Let* $\pi\colon X \to \mathbb{P}^1_k$ *be a conic bundle. For any* $P \in S$ *define* $\mathcal{A}_p$ *to be the quaternion algebra* $(t - \tau_P, a_P)$ *in* $\operatorname{Br} \boldsymbol{k}(P)(t)$*. Then* $\frac{\operatorname{Br} X}{\operatorname{Br} k}$ *is generated by elements of the form* $\pi^*(A_\varepsilon)$ *where*

$$A_\varepsilon = \sum_{P \in S} \varepsilon_P \operatorname{Cor}_{\boldsymbol{k}(P)/k} \mathcal{A}_P \tag{4.1.1}$$

Corollary 4.1.3. *The subgroup* $\ker(\operatorname{Br} X[2] \to \operatorname{Br} X_\infty)$ *generates* $\frac{\operatorname{Br} X}{\operatorname{Br} k}$.

Proof. By the description given in equation (4.1.1), we can see that $\mathcal{A}_P(\infty) = (1, a_P) = 0 \in \operatorname{Br} \mathbf{k}(P)$. Since $\operatorname{Cor}_{\mathbf{k}(P)/k}$ is a group homomorphism, $A_\varepsilon \in \ker(\operatorname{Br} \mathbf{k}(\mathbb{P}^1)[2] \xrightarrow{\infty^*} \operatorname{Br} k[2])$ and so $\pi^*(A_\varepsilon) \in \ker(\operatorname{Br} X[2] \to \operatorname{Br} X_\infty)$. □

From Lemma 4.1.2, we can deduce the following.

Corollary 4.1.4. *Let* k *be a global field of characteristic not equal to* 2 *and let* X/k *be a conic bundle with four geometric singular fibers. Then*

$$\frac{\operatorname{Br} X}{\operatorname{Br} k} = \begin{cases} \left(\mathbb{Z}/2\mathbb{Z}\right)^2 & \textit{if } |S| = 4, \\ \mathbb{Z}/2\mathbb{Z} & \textit{if } S \text{ has at least one irreducible degree two point}, \\ \{0\} & \text{otherwise} \end{cases}$$

Moreover, if $X(k) = \emptyset$ *and* $\frac{\operatorname{Br} X}{\operatorname{Br} k} \neq 0$ *then there exists a Galois-invariant decomposition* $S = S_1 \cup S_2$ *with both* S_i *irreducible of degree* 2.

Proof. The computations for $\frac{\operatorname{Br} X}{\operatorname{Br} k}$ follow from a case-by-case analysis, see [Sko15, §2.2]. In particular, following the notation of Lemma 4.1.2, any Brauer class of the form A_ε for $\varepsilon = (1, 1, \dots, 1) \in \mathbb{F}_2^{|S|}$ is trivial, hence $|S| = 1$ implies that $\frac{\operatorname{Br} X}{\operatorname{Br} k} = 0$.

If any of the singular fibers of X lie over a degree 1 point $P \in \mathbb{A}^1_k$, then we can obtain a k-rational point on X by taking the intersection point of the lines ℓ_P and ℓ'_P for such $P \in S$. Therefore, if $X(k) = \emptyset$ and $\frac{\operatorname{Br} X}{\operatorname{Br} k} \neq 0$, it must be the case that $|S| = 2$ and the singular fibers both lie over degree 2 points of k, hence we obtain the Galois-invariant decomposition $S_1 \cup S_2$. □

Corollary 4.1.5. *Let k be a global field of characteristic not equal to 2 and let $X \to \mathbb{P}^1_k$ be a conic bundle with four geometric singular fibers. If there exists an even degree extension L/k such that $X(L) = \emptyset$ and $\operatorname{Res}_{L/k} \colon \frac{\operatorname{Br} X}{\operatorname{Br} k} \to \frac{\operatorname{Br} X_L}{\operatorname{Br} L}$ is not surjective, then S is irreducible over k and the singular fibers of $X_L \to \mathbb{P}^1_L$ lie over two degree 2 closed points that are interchanged by $\operatorname{Gal}(L/k)$.*

Proof. From Corollary 4.1.4, we have that $\operatorname{Res}_{L/k}$ is not surjective when $\frac{\operatorname{Br} X}{\operatorname{Br} k} = \{0\}$ and $\frac{\operatorname{Br} X_L}{\operatorname{Br} L} \neq \{0\}$ or $\frac{\operatorname{Br} X}{\operatorname{Br} k} = \mathbb{Z}/2\mathbb{Z}$ and $\frac{\operatorname{Br} X_L}{\operatorname{Br} L} \neq (\mathbb{Z}/2\mathbb{Z})^2$. If $\frac{\operatorname{Br} X}{\operatorname{Br} k} = \mathbb{Z}/2\mathbb{Z}$ and $\frac{\operatorname{Br} X_L}{\operatorname{Br} L} = (\mathbb{Z}/2\mathbb{Z})^2$, then X_L will always have an L-rational point. If $\frac{\operatorname{Br} X}{\operatorname{Br} k} = \{0\}$ and $\frac{\operatorname{Br} X_L}{\operatorname{Br} L} \neq \{0\}$, then the only case where X_L may not have an L-rational point is when all geometric singular fibers of X/k lie over a single closed point of $\mathbb{A}^1_k$ and over L, there exists a Galois-invariant decomposition $S = S_1 \cup S_2$ with both S_i of degree 2. $\square$

Remark 4.1.6. *Corollary 4.1.5 implies that the degree 4 closed point of $\mathbb{A}^1_k$ must correspond to a polynomial of the form $N_{k_0/k}(g(\lambda))$, where $g(\lambda) \in k_0[\lambda]$ is an irreducible quadratic, and k_0/k is a quadratic extension. In fact, finite extensions such as these are the only possible extensions of k over which the Hasse principle may fail.*

4.2 Brauer-Manin Obstructions over Extensions

In this section, we prove some general results relating the Brauer-Manin obstruction on a conic bundle to the Brauer-Manin obstruction over certain extensions. These results will play a critical role in the proofs of the main theorems.

Theorem 4.2.1. *Let k be a global field of characteristic not equal to 2 and let $X \to \mathbb{P}^1_k$ be a conic bundle with four geometric singular fibers, then for any even degree extension L/k*

$$X(\mathbb{A}_L)^{\operatorname{Res}_{L/k}(\operatorname{Br} X_k)} \neq \emptyset \Leftrightarrow X(\mathbb{A}_L) \neq \emptyset.$$

In particular, if $\operatorname{Res}_{L/k} \colon \frac{\operatorname{Br} X}{\operatorname{Br} k} \to \frac{\operatorname{Br} X_L}{\operatorname{Br} L}$ is surjective, then $X(\mathbb{A}_L)^{\operatorname{Br}(X_L)} \neq \emptyset \Leftrightarrow X(\mathbb{A}_L) \neq \emptyset$.

Proof. We begin by observing that one direction is immediate since $X(\mathbb{A}_L) = \emptyset$ implies that $X(\mathbb{A}_L)^{\operatorname{Res}_{L/k}(\operatorname{Br} X_k)} = \emptyset$. Now assume that $X(\mathbb{A}_L) \neq \emptyset$ and let Ω_k denote the set of places of

k. We will show that for all $v \in \Omega_k$ and all $w|v$, there exists a point $(P_w) \in X(L \otimes_k k_v)$ such that $\sum_{w|v} \operatorname{inv}_w(\operatorname{ev}_{\mathcal{A}}(P_w)) = 0$ for all $\mathcal{A} \in \ker(\operatorname{Br} X[2] \to \operatorname{Br} X_\infty)$, and so

$$\sum_{v\in\Omega_k} \sum_{w|v} \operatorname{inv}_w(\operatorname{ev}_{\mathcal{A}}(P_w)) = 0.$$

Since by Corollary 4.1.3, $\ker(\operatorname{Br} X[2] \to \operatorname{Br} X_\infty)$ generates $\frac{\operatorname{Br} X}{\operatorname{Br} k}$, this will imply that $X(\mathbb{A}_L)^{\operatorname{Res}_{L/k}(\operatorname{Br} X_k)} \neq \emptyset$.

We first consider the case when $X(k_v) \neq \emptyset$. Choose a point $P_v \in X(k_v)$, and for all $w|v$, set $P_w = P_v$. Then

$$\sum_{w|v} \operatorname{inv}_w(\operatorname{ev}_{\mathcal{A}}(P_w)) = \sum_{w|v} \operatorname{inv}_w(\operatorname{Res}_{L_w/k_v}(\operatorname{ev}_{\mathcal{A}}(P_v))) = \sum_{w|v}[L_w \colon k_v] \operatorname{inv}_v(\operatorname{ev}_{\mathcal{A}}(P_v))$$

$$= \operatorname{inv}_v(\operatorname{ev}_{\mathcal{A}}(P_v)) \sum_{w|v}[L_w \colon k_v] = \operatorname{inv}_v(\operatorname{ev}_{\mathcal{A}}(P_v)) \cdot [L \colon k] = 0 \in \mathbb{Q}/\mathbb{Z}.$$

Now consider the case where $X(k_v) = \emptyset$. By [CTC79], $X(k_v) \neq \emptyset$ if and only if there exists a zero cycle of degree 1, hence $X(k_v) = \emptyset$ implies that $X(L_w) = \emptyset$ for all odd degree extensions L_w/k_v. Since $X(L \otimes_k k_v) \neq \emptyset$, L_w/k_v is an even degree extension for all $w|v$. Thus, by Lemma 4.1.1, there exists a point $P_w \in X_\infty(L_w)$ satisfying $\operatorname{inv}_w(\operatorname{ev}_{\mathcal{A}}(P_w)) = 0$. □

Corollary 4.2.2. *Let k be a global field of characteristic not equal to 2, let $X \to \mathbb{P}^1_k$ be a conic bundle with four geometric singular fibers. There exist at most three quadratic extensions k_i/k such that for all even degree extensions L/k that do not contain any k_i, we have*

$$X(\mathbb{A}_L)^{\operatorname{Br}(X_L)} \neq \emptyset \Leftrightarrow X(\mathbb{A}_L) \neq \emptyset.$$

Proof. If $X(k) \neq \emptyset$ then $X(\mathbb{A}_L)^{\operatorname{Br}(X_L)}$ and $X(\mathbb{A}_L)$ are non-empty for all extensions L/k, hence we assume that $X(k) = \emptyset$. By Theorem 4.2.1, it remains to show that there exist quadratic extensions $k_i \supseteq k$ such that $\operatorname{Res}_{L/k} \colon \frac{\operatorname{Br} X}{\operatorname{Br} k} \to \frac{\operatorname{Br} X_L}{\operatorname{Br} L}$ is surjective whenever $k_i \not\subseteq L$ for any i. If $\operatorname{Res}_{L/k}$ is not surjective for all L/k, then by Corollary 4.1.5, it remains to consider the case when $|S| = 1$. Here, the problematic extensions k_i/k are precisely the extensions for which the singular fibers of $X_{k_i} \to \mathbb{P}^1_{k_i}$ lie over two degree 2 closed points that are interchanged by

$\mathrm{Gal}(k_i/k)$.

By Remark 4.1.6, we have at most three such extensions, arising from the case where S corresponds to a polynomial of the form $N_{F/k}(\ell(\lambda))$, where F is a bi-quadratic extension of k and $\ell(\lambda) \in k[\lambda]$ is a linear polynomial. In this case, the extensions k_i/k correspond to the three quadratic subextensions of F/k. If L does not contain any such k_i, then this case never occurs, and the result follows. ☐

We can now extend Theorem 4.2.1 to arbitrary conic bundles at the expense of restricting to quadratic extensions.

Theorem 4.2.3. *Let k be a global field of characteristic not equal to 2 and let $X \to \mathbb{P}^1_k$ be a conic bundle, then for any quadratic extension L/k*

$$X(\mathbb{A}_L)^{\mathrm{Res}_{L/k}(\mathrm{Br}\, X_k)} \neq \emptyset \Leftrightarrow X(\mathbb{A}_L) \neq \emptyset.$$

Proof. Observe that one direction is again immediate, hence we assume that $X(\mathbb{A}_L) \neq \emptyset$. In a similar fashion to the proof of Theorem 4.2.1, we will show that for all $v \in \Omega_k$ there exists a point $(P_w) \in X(L \otimes_k k_v)$ such that $\sum_{w|v} \mathrm{inv}_w(\mathrm{ev}_{\mathcal{A}}(P_w)) = 0$ for all $\mathcal{A} \in \ker(\mathrm{Br}\, X[2] \to \mathrm{Br}\, X_\infty)$, and thus $X(\mathbb{A}_L)^{\mathrm{Res}_{L/k}(\mathrm{Br}\, X_k)} \neq \emptyset$.

If $X(k_v) \neq \emptyset$ then by an identical argument to the proof of Theorem 4.2.1, there exists a point $(P_w) \in X(\mathbb{A}_L)$ such that $\sum_{w|v} \mathrm{inv}_w(\mathrm{ev}_{\mathcal{A}}(P_w)) = 0$ for all $\mathcal{A} \in \ker(\mathrm{Br}\, X[2] \to \mathrm{Br}\, X_\infty)$.

If $X(k_v) = \emptyset$ and $X(L \otimes_k k_v) \neq \emptyset$, then v is non-split and so L_w/k_v is quadratic. Thus, Lemma 4.1.1 implies the existence of a point $P_w \in X_\infty(L_w)$ which satisfies $\mathrm{inv}_w(\mathrm{ev}_{\mathcal{A}}(P_w)) = 0$ for all $\mathcal{A} \in \ker(\mathrm{Br}\, X[2] \to \mathrm{Br}\, X_\infty)$. ☐

When considering extensions analogous to the extensions k_i of Corollary 4.2.2 for arbitrary conic bundles there are more issues that can arise, but nonetheless, there are still only finitely many. These problematic extensions arise in a similar manner to those of Corollary 4.2.2. Namely, if $S \subset \mathbb{P}^1_k$ is the finite set of closed points with geometric singular fiber, then the problematic extensions, k_i/k, are those which coincide with residue fields, $\mathbf{k}(P)$, for $P \in S$. Over these extensions, $\mathrm{Res}_{k_i/k}\colon \frac{\mathrm{Br}\, X}{\mathrm{Br}\, k} \to \frac{\mathrm{Br}\, X_{k_i}}{\mathrm{Br}\, k_i}$ could fail to be surjective and new Brauer classes in $\frac{\operatorname{Br} X_{k_i}}{\operatorname{Br} k_i}$ could give a Brauer-Manin obstruction to X_{k_i}. For all extensions L/k over which the points in S remain unchanged, $\operatorname{Res}_{L/k}$ is surjective and indeed $X(\mathbb{A}_L)^{\operatorname{Br}(X_L)} \neq \emptyset \Leftrightarrow X(\mathbb{A}_L) \neq \emptyset$. Since there are only finitely many geometric singular fibers, there are only finitely many problematic extensions of this form, hence we have the following corollary.

Corollary 4.2.4. *Let k be a global field of characteristic not equal to 2, let $X \to \mathbb{P}^1_k$ be a conic bundle, and let L/k be a quadratic extension. If $L \not\subseteq \mathbf{k}(P)$ for all $P \in S$, then*

$$X(\mathbb{A}_L)^{\operatorname{Br}(X_L)} \neq \emptyset \Leftrightarrow X(\mathbb{A}_L) \neq \emptyset.$$

Remark 4.2.5. *Quite generally, if k is a number field and X/k is a smooth, projective, and geometrically rational variety, then Pic$(\overline{X})$ is a Galois lattice split by a finite Galois extension K/k. If E/k is a finite field extension which is linearly disjoint from K/k, then the map* $\operatorname{Res}_{E/k} \colon \frac{\operatorname{Br}(X)}{\operatorname{Br}_0(X)} \to \frac{\operatorname{Br}(X_E)}{\operatorname{Br}_0(X_E)}$ *is an isomorphism, and in particular, is surjective. Corollary 4.2.4 is a slight refinement of this result for the case of conic bundles.*

4.3 Proofs Of The Main Theorems

4.3.1 Proof of Theorem 4.0.1

If $\frac{\operatorname{Br} X}{\operatorname{Br} k} \neq 0$ then, by Corollary 4.1.4, $X(k) \neq \emptyset$ or there exists a Galois-invariant decomposition $S = S_1 \cup S_2$. The result trivially follows if $X(k) \neq \emptyset$. If the latter case holds, then $\frac{\operatorname{Br} X}{\operatorname{Br} k} \cong \mathbb{Z}/2\mathbb{Z}$ and for any even degree extension L/k we have

$$\frac{\operatorname{Br} X_L}{\operatorname{Br} L} \cong \mathbb{Z}/2\mathbb{Z} \quad \text{or} \quad \frac{\operatorname{Br} X_L}{\operatorname{Br} L} \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

If $\frac{\operatorname{Br} X_L}{\operatorname{Br} L} \cong \mathbb{Z}/2\mathbb{Z}$, then $\operatorname{Res}_{L/k}$ is surjective and the result follows from Theorem 4.2.1. If $\frac{\operatorname{Br} X_L}{\operatorname{Br} L} \cong (\mathbb{Z}/2\mathbb{Z})^2$, then $X(L) \neq \emptyset$ by Corollary 4.1.5.

It remains to prove the theorem when $\frac{\operatorname{Br} X}{\operatorname{Br} k} = 0$ and $X(\mathbb{A}_k) \neq \emptyset$. If $\frac{\operatorname{Br} X}{\operatorname{Br} k} = 0$ then

$$X(\mathbb{A}_k)^{\operatorname{Br}(X_k)} = X(\mathbb{A}_k) \neq \emptyset.$$

By [CTSD94, Theorem 5.1], $X(\mathbb{A}_k)^{\operatorname{Br}(X_k)} \neq \emptyset \Leftrightarrow X(k) \neq \emptyset$ hence

$$X(\mathbb{A}_k)^{\operatorname{Br}(X_k)} \neq \emptyset \implies X(k) \neq \emptyset \implies X(L) \neq \emptyset$$

completing the proof. □

4.3.2 Proof of Theorem 4.0.2

Let $X/\mathbb{Q}$ be the Châtelet surface given by

$$y^2 - 5z^2 = \frac{3}{5}(5\lambda^4 + 7\lambda^2 + 1).$$

One can check that $5\lambda^4 + 7\lambda^2 + 1$ is irreducible hence $\frac{\operatorname{Br} X}{\operatorname{Br} \mathbb{Q}} = 0$ by Corollary 4.1.4.

We will begin by showing $X(\mathbb{Q}_p) \neq \emptyset$ for all $p \neq 3$ and that $X(\mathbb{Q}_3) = \emptyset$. First, observe that for all $p \neq 3, 5$ we have $X_\infty(\mathbb{Q}_p) \neq \emptyset$ because X_∞ is the conic $y^2 - 5z^2 = 3t^2$, which has $\mathbb{Q}_p$ points for all $p \neq 3, 5$. Furthermore, the fiber X_2 is the conic $y^2 - 5z^2 = \frac{1239}{5}t^2$ which has $\mathbb{Q}_5$ points since $1239 \in \mathbb{Q}_5^{\times 2}$. For the case when $p = 3$, observe that $5 \notin \mathbb{Q}_3^{\times 2}$ which implies that $y^2 - 5z^2$ is a norm from the unramified extension $\mathbb{Q}_3(\sqrt{5})$. It remains to see that $5\lambda^4 + 7\lambda^2 + 1$ always has even 3-adic valuation and this follows from the fact that $5\lambda^4 + 7\lambda^2 + 1$ is irreducible over the residue field $\mathbb{F}_3$. We can now conclude that $\frac{3}{5}(5\lambda^4 + 7\lambda^2 + 1)$ always has odd valuation hence is never of the form $y^2 - 5z^2$, so $X(\mathbb{Q}_3) = \emptyset$. This shows that $X(\mathbb{A}_\mathbb{Q}) = \emptyset$.

Now, consider the quadratic extension $L = \mathbb{Q}(\sqrt{29})$. One can see that $[\mathbb{Q}_3(\sqrt{29}) \colon \mathbb{Q}_3] = 2$, hence $X(\mathbb{Q}_3(\sqrt{29})) \neq \emptyset$ from Lemma 4.1.1. This shows that $X(\mathbb{A}_L) \neq \emptyset$. Furthermore, a computation shows that $X/\mathbb{Q}(\sqrt{29})$ is the Châtelet surface given by

$$y^2 - 5z^2 = 3\left(\lambda^2 + \frac{1}{10}(7 + \sqrt{29})\right)\left(\lambda^2 + \frac{1}{10}(7 - \sqrt{29})\right)$$

hence by Corollary 4.1.4 we have

$$\frac{\operatorname{Br} X_{\mathbb{Q}(\sqrt{29})}}{\operatorname{Br} \mathbb{Q}(\sqrt{29})} \cong \mathbb{Z}/2\mathbb{Z} = \langle \mathcal{A} \rangle$$

where $\mathcal{A}$ denotes the quaternion algebra $\left(5, \lambda^2 + \frac{1}{10}(7 + \sqrt{29})\right)$.

It remains to show that $X(\mathbb{A}_{\mathbb{Q}(\sqrt{29})})^{\operatorname{Br}(X_{\mathbb{Q}(\sqrt{29})})} = \emptyset$ and we do so by first showing that $\operatorname{ev}_{\mathcal{A}}\colon X(L_w) \to \operatorname{Br} L_w$ is identically zero for all $w \not| 5$. Begin by observing that for all primes $w|p$ where $p \neq 3, 5$, we know that $X_\infty(L_w) \neq \emptyset$, hence $\operatorname{ev}_{\mathcal{A}}\colon X(L_w) \to \operatorname{Br} L_w$ takes the value 0 at such primes. Since the evaluation map is constant at all primes w of good reduction [CTS13, Theorem 3.1], it remains to check $\operatorname{ev}_{\mathcal{A}}(X(L_w))$ for primes w lying over $p = 2, 3, 5$, and 29. Now observe that, for $w|2, 3, 29$, $5 \in L_w^{\times 2}$ hence the algebra $\mathcal{A} = \left(5, \lambda^2 + \frac{1}{10}(7 + \sqrt{29})\right)$ is identically zero and $\operatorname{ev}_{\mathcal{A}}(X(L_w)) = 0$. Thus for any $(P_w) \in X(\mathbb{A}_L)$ we have

$$\sum_{w \in \Omega_L} \operatorname{inv}_w(\operatorname{ev}_{\mathcal{A}}(P_w)) = \sum_{w|5} \operatorname{inv}_w(\operatorname{ev}_{\mathcal{A}}(P_w)).$$

Let w_1 and w_2 denote the places lying over 5 corresponding to the embeddings in which $\sqrt{29} \equiv 2 \pmod 5$ and $\sqrt{29} \equiv 3 \pmod 5$ respectively and let $P_i = (\lambda_i, y_i, z_i) \in X(L_{w_i})$. We now show that

$$\operatorname{inv}_{w_i}(\mathcal{A}(P_i)) = \begin{cases} \frac{1}{2} & \text{if } i = 1 \\ 0 & \text{if } i = 2 \end{cases}$$

Let $\alpha = \frac{1}{10}(7 + \sqrt{29})$ and $\overline{\alpha} = \frac{1}{10}(7 - \sqrt{29})$. Over $\mathbb{Q}_5$, we have

$$X/\mathbb{Q}_5\colon y^2 - 5z^2 = 3(\lambda^2 + \alpha)(\lambda^2 + \overline{\alpha})$$

hence $\frac{\operatorname{Br} X_{\mathbb{Q}_5}}{\operatorname{Br} \mathbb{Q}_5}$ is generated by the quaternion algebra $\mathcal{A} = (5, \lambda^2 + \alpha)$. Note that in $\operatorname{Br} X$, the quaternion algebra $\mathcal{A}$ is equivalent to the algebra $\mathcal{B} = (5, 3(\lambda^2 + \overline{\alpha}))$. This will be an important tool in the final part of our proof.

First we consider the place w_1 and observe that since $\sqrt{29} \equiv 2 \pmod 5$, we have $w_1(\alpha) = -1$. Moreover, since $\alpha\overline{\alpha} = \frac{1}{5}$, it follows that $w_1(\overline{\alpha}) = 0$. Take $P_1 = (\lambda_1, y_1, z_1)$ and assume $w_1(\lambda_1) \geq 0$. Then, by the strong triangle inequality, $w_1(\lambda_1^2 + \alpha) = w_1(\alpha)$ and since $5\alpha \equiv \frac{1}{2}(7 + \sqrt{29}) \equiv 2 \pmod 5$, it follows that $10(\lambda_1^2 + \alpha) \in \mathbb{Q}_5^{\times 2}$, meaning $\operatorname{ev}_{\mathcal{A}}(P_1) = (5, 10)$ which is a non-split quaternion algebra in $\operatorname{Br} \mathbb{Q}_5$. We can now see that $\operatorname{inv}_{w_1}(\operatorname{ev}_{\mathcal{A}}(P_1)) = \frac{1}{2}$.

Further, assume that $w_1(\lambda_1) < 0$ and consider the polynomial $P(\lambda) = \frac{3}{5}(5\lambda^4 + 7\lambda^2 + 1)$. We have that $5^{-4w_1(\lambda_1)}P(\lambda) \equiv 3 \pmod 5$ hence $P(\lambda)$ is never a norm from $\mathbb{Q}_5(\sqrt{5})$ so all $P_1 \in X(L_{w_1})$ must satisfy $w_1(\lambda_1) \geq 0$. Therefore, for all $P_1 \in X(L_{w_1})$ we have $\operatorname{inv}_{w_1}(\operatorname{ev}_{\mathcal{A}}(P_1)) = \frac{1}{2}$.

Lastly, we consider the case of w_2 and recall that in this case, $\sqrt{29} \equiv 3 \pmod 5$, hence $w_2(\alpha) = 0$ and $w_2(\overline{\alpha}) = -1$. Take $P_2 = (\lambda_2, y_2, z_2)$ and observe that just as in the previous case, if $P_2 \in X(L_{w_2})$ we must have $w_2(\lambda_2) \geq 0$. If $w_2(\lambda_2) > 0$ then by the same argument for when $w_1(\lambda_1) \geq 0$, we have that $\lambda_2^2 + \alpha$ is always a square in $\mathbb{Q}_5$, hence $\mathrm{ev}_{\mathcal{A}}(P_2) = 0 \in \operatorname{Br}\mathbb{Q}_5$. It remains to consider the case when $w_2(\lambda_2) = 0$ and for this, we consider the algebra $\mathcal{B} = (5, 3(\lambda^2 + \overline{\alpha}))$ and show that $\mathrm{ev}_{\mathcal{B}}(P_2)$ is identically zero in $\operatorname{Br}\mathbb{Q}_5$. For such values of λ_2 we have $w_2\left(3(\lambda^2 + \overline{\alpha})\right) = w_2(\overline{\alpha})$ hence $5\left(3(\lambda^2 + \overline{\alpha})\right) \equiv 3\overline{\alpha} \pmod 5$, which by Hensel's lemma, is a square in $\mathbb{Q}_5$. It now follows that for all $P_2 \in X(L_{w_2})$ we have $\mathrm{inv}_{w_2}(\mathrm{ev}_{\mathcal{A}}(P_2)) = 0$.

We can now conclude that for any $P_w \in X(\mathbb{A}_L)$ we have

$$\sum_{w \in \Omega_L} \mathrm{inv}_w \left(\mathrm{ev}_{\mathcal{A}}(P_w)\right) = \frac{1}{2}$$

and X fails the Hasse principle over L. ☐

4.3.3 Proof of Theorem 4.0.3

This result follows immediately from Corollary 4.2.4. ☐

4.4 A Partial Converse to Theorem 4.0.2

In our consideration of even degree extensions over which X may fail the Hasse principle, we saw that the extensions which intersect with $\mathbf{k}(S)$ in a quadratic extension F/k open the possibility for a potential Hasse principle failure. In the case of Châtelet surfaces, this can occur for surfaces X/k of the form

$$y^2 - az^2 = cN_{F/k}(g(\lambda))$$

where $a, c \notin k^{\times 2}$, $g(\lambda) \in F[\lambda]$ is a monic, irreducible polynomial, and $N_{F/k}$ denotes the usual norm map. In contrast to Theorem 4.0.2 not all such extensions F/k are guaranteed to produce a Brauer-Manin obstruction over F. In order for such an obstruction to exist, the fiber X_∞, the places for which it has no local points, and the places of bad reduction, must satisfy several necessary conditions.

Theorem 4.4.1. *Let k be a number field, let F/k be a quadratic extension, and let X/k be the Châtelet surface given by*

$$y^2 - az^2 = cN_{F/k}(g(\lambda))$$

where $a, c \notin k^{\times 2}$ and $g(\lambda) \in F[\lambda]$ is monic and irreducible. Let Ω_k denote the set of places of k, let Ω_F denote the set of places of F, and assume $X(\mathbb{A}_k) = \emptyset$.

If

$$\sum_{\substack{v \in \Omega_k \\ v \text{ splits in } F}} \operatorname{inv}_v(a, c) = 0 \in \mathbb{Q}/\mathbb{Z}$$

then for all even degree extensions L/k, we have $X(L) \neq \emptyset \Leftrightarrow X(\mathbb{A}_L) \neq \emptyset$.

Proof. Let $v \in \Omega_k$, let $w \in \Omega_F$ such that $w|v$. Let $\mathcal{A} = (a, g(\lambda))$ be a generator of $\frac{\operatorname{Br} X_F}{\operatorname{Br} F}$ and let σ denote the generator of $\operatorname{Gal}\left(F/k\right)$. If $[F_w \colon k_v] = 2$ then there exists a point $P_w \in X_\infty(F_w)$ such that $\operatorname{inv}_w(\mathcal{A}(P_w)) = 0$ by Lemma 4.1.1. Now assume that $F_w = k_v$, then there exists a unique place w' such that $w' \neq w$ and $w'|v$, hence $F_w = F_{w'} = k_v$. Take $P_v \in X(k_v)$ and set $P_v = P_w = P_{w'} = (\lambda_v, y_v, z_v)$. We then have

$$\operatorname{inv}_w(\mathcal{A}(P_v)) + \operatorname{inv}_{w'}(\mathcal{A}(P_v)) = \operatorname{inv}_w\Big((a, g(\lambda_v)) + (a, \sigma(g)(\lambda_v))\Big) = \operatorname{inv}_w\Big(a, g(\lambda_v)\sigma(g)(\lambda_v)\Big) = \operatorname{inv}_v((a, c))$$

Now, picking $(P_w) \in X(\mathbb{A}_F)$ as above, we have

$$\sum_{w \in \Omega_F} \operatorname{inv}_w\Big(\mathcal{A}(P_w)\Big) = \sum_{\substack{w \in \Omega_F \\ [F_w \colon k_v] = 2}} 0 + \sum_{\substack{w \in \Omega_F \\ [F_w \colon k_v] = 1}} \operatorname{inv}_w\Big(\mathcal{A}(P_w)\Big) = \sum_{\substack{v \in \Omega_k \\ v \text{ splits in } F}} \operatorname{inv}_v(a, c)$$

Now, if

$$\sum_{\substack{v \in \Omega_k \\ v \text{ splits in } F}} \operatorname{inv}_v(a, c) = 0 \in \mathbb{Q}/\mathbb{Z}$$

then $X(\mathbb{A}_F)^{\operatorname{Br}(X_F)} \neq \emptyset$ and it follows that $X(F) \neq \emptyset$. Since F/k is the only even degree extension over which the set S admits a Galois-invariant decomposition $S = S_1 \cup S_2$, it follows from Corollary 4.1.4 and Corollary 4.1.5 that if L/k is any even degree extension, then $X(L) \neq \emptyset \Leftrightarrow X(\mathbb{A}_L) \neq \emptyset$. □

Chapter 5

WEAK APPROXIMATION ON CHÂTELET SURFACES

While much of our previous works concerns the *existence* of rational points, if one is presented with a situation in which rational points are known to exist, it is reasonable to wonder if we can actually find explicit rational points. For general varieties, this notion is known as weak approximation. More generally, given a nice variety X if one knows p-adic solutions for all but finitely many primes p, then X satisfies weak approximation if a global solution can be obtained by means of the known local solutions. For example, the projective line $\mathbb{P}^1$ satisfies weak approximation and specifically, weak approximation on $\mathbb{P}^1$ is the chinese remainder theorem.

In this chapter, we investigate weak approximation for Châtelet surfaces over number fields when all singular fibers are defined over rational points. We also consider Châtelet surfaces which satisfy weak approximation over every finite extension of the ground field. We prove many of these results by showing that the Brauer-Manin obstruction vanishes, then apply results of Colliot-Thélène, Sansuc, and Swinnerton-Dyer.

As was mentioned earlier, the arithmetic of these surfaces have been studied extensively by Colliot-Thélène, Sansuc, and Swinnerton-Dyer in [CTSSD87a, CTSSD87b], where it was proven that over a number field, rational points on these surfaces are controlled by the Brauer–Manin obstruction. More precisely, if X is a Châtelet surface over a number field k, then $X(k)$ is dense in the Brauer–Manin set $X(\mathbb{A})^{\mathrm{Br}}$.

In this chapter, we focus on weak approximation on Châtelet surfaces. In *loc. cit.*, it is shown that Châtelet surfaces where $P(x)$ is irreducible or is a product of a linear and irreducible cubic, always satisfy weak approximation [CTSSD87a, Theorem B]. Our first result shows what happens in the opposite scenario.

Theorem 5.0.1. *Let X be a Châtelet surface over a number field k with $X(k) \neq \emptyset$. Assume that $P(x)$ splits into linear factors. Then X fails weak approximation if and only if either*

1. *X has a place v of bad reduction with $a \notin k_v^{\times 2}$ or,*

2. *k has a real embedding and $a < 0$.*

We also collect some partial results and show that the situation is not as simple in the remaining case when $P(x)$ is a product of two irreducible quadratics, see Examples 1, 2.

Our next result investigates 'potential' type questions in regards to weak approximation. For Châtelet surfaces, behavior of the Hasse principle or weak approximation under finite extensions have been studied in [Lia18, Wu22, Rov22]. To this end, we coin the notion of a surface satisfying perpetual weak approximation, that is, X/k satisfies **perpetual weak approximation** if X_L satisfies weak approximation for every finite extension L/k. In [Lia18, Wu22], the authors give a Châtelet surface over any arbitrary number field k, which satisfies weak approximation over k but fails over some finite extension. Using Theorem 5.0.1 we show that the same holds true for almost all Châtelet surfaces.

Corollary 5.0.2. *Let X be a Châtelet surface over a number field k. Let L be the splitting field of $P(x)$. Assume that X has a place v of bad reduction with $a \notin (k_vL)^{\times 2}$. Then X_L fails weak approximation.*

Conversely, we show that without the assumptions of the previous Corollary, there exist Châtelet surfaces that satisfy perpetual weak approximation.

Theorem 5.0.3. *Let X be a Châtelet surface over a number field k. Then* $\operatorname{Br} X_L / \operatorname{Br}_0 X_L = 0$ *for every finite extension L/k if and only if the splitting field of $P(x)$ is a $\mathbb{Z}/4\mathbb{Z}$-extension containing $k(\sqrt{a})$. In particular, X satisfies perpetual weak approximation.*

5.1 Cyclic algebras

In this section let k be a nonarchimedian local field containing an nth roots of unity ζ. Let $\mathcal{O}$ be its ring of integers with uniformizer $\pi \in \mathcal{O}$ and $\mathbb{F}_q = \mathcal{O}/(\pi)$. Recall that the invariant map gives an isomorphism $\operatorname{inv}\colon \operatorname{Br} k \to \mathbb{Q}/\mathbb{Z}$.

For $a, b \in k^\times$, let $(a,b)_\zeta = (a,b)_\zeta \in \operatorname{Br} k$ be the class of the cyclic algebra as defined in [GS17, §2.5]. When $n = 2$, we will simply write $(a,b) = (a,b)_{-1}$.

Lemma 5.1.1. *If $k(\sqrt[n]{a})/k$ is unramified, then for any $b \in k^\times$,* $\operatorname{inv}(a,b)_\zeta = v_\pi(b)/n$.

Proof. [Ser86, §2.5 Proposition 2] ☐

Recall the invariant map from local class field theory gives an isomorphism $\operatorname{inv}\colon \operatorname{Br} k \to \mathbb{Q}/\mathbb{Z}$.

Lemma 5.1.2. *Assume the residue characterisitc of $\mathbb{F}_q$ is even and $n = 2$. If $k(\sqrt[n]{a})/k$ is ramified, then there exists $u \in \mathcal{O}$ such that* $\operatorname{inv}(a, 1 - \pi u) = 1/2$.

Proof. Since a is not a square in $\mathcal{O}$, there exists some $b \in \mathcal{O}$ such that $\operatorname{inv}(a,b) = 1/2$. Moreoever, as $k(\sqrt{a})/k$ is ramified, there exists some $c \in \mathcal{O}$ such that $v(c) = 1$ and $(a,c) = 0$. Hence, we have $\operatorname{inv}(a, c^n b) = 1/2$ for all $n \in \mathbb{Z}$ so that we may assume $v(b) = 0$. Since the squaring map $x \mapsto x^2$ is an isomorphism on $\mathbb{F}_q^\times$, there exists $d \in \mathcal{O}^\times$ such that $b - d^2 \equiv 0 \bmod \pi$. Write $b - d^2 = -\pi e$ for some $e \in \mathcal{O}$. Then,

$$(a,b) = (a, d^2 + b - d^2) = (a, d^2 - \pi e) = (a, 1 - \pi(e/d^2)).$$

Setting $u = e/d^2 \in \mathcal{O}$ gives our result. ☐

5.2 *Generators for the Brauer group*

Throughout the rest of this paper, X will denote a Châtelet surface over a number field k with affine model given by the equation

$$y^2 - az^2 = P(x)$$

where $a \in k^\times \setminus k^{\times 2}$ and $P(x)$ is a separable polynomial of degree 3 or 4. The morphism $X \to \mathbb{P}^1$ given by the x-coordinate gives X the structure of a conic bundle. The singular fibers are precisely the roots of $P(x)$ together with $\infty \in \mathbb{P}^1$ if $\deg(P(x)) = 3$.

Let $\operatorname{Br} X = \mathrm{H}^2_{\text{et}}(X, \mathbb{G}_m)$ denote the cohomological Brauer group of X. Let $\operatorname{Br}_0 X$ be the image of the natural map $\operatorname{Br} k \to \operatorname{Br} X$. The Brauer group of Châtelet surfaces, or more generally conic bundles, have been extensively studied, see [CTS21, §11.3] for a detailed exposition. We summarize the necessary results for our purposes here.

Proposition 5.2.1. *The Brauer group of X depends on the factorization of $P(x)$ over k. Let L/k be the splitting field of $P(x)$. Then*

$$\frac{\operatorname{Br} X}{\operatorname{Br}_0 X} = \begin{cases} \left(\mathbb{Z}/2\mathbb{Z}\right)^2 & \text{if } P(x) \text{ splits completely,} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } P(x) \text{ has at least one irreducible quadratic factor and } \sqrt{a} \notin L, \\ \{0\} & \text{otherwise.} \end{cases}$$

The points on $\mathbb{P}^1_k$ corresponding to the singular fibers determine the generators for the quotient $\operatorname{Br} X / \operatorname{Br}_0 X$. In particular, if $P(x)$ splits completely, we can map three of the roots of $P(x)$ to $0, 1$, and $\infty \in \mathbb{P}^1$ via an automorphism of $\mathbb{P}^1$. In doing so, we may assume that $P(x) = cx(x-1)(x-\lambda)$ for some $\lambda \in k \smallsetminus \{0, 1\}$ and squarefree c.

The main proofs of this paper work with fixed generators for the Brauer groups, and for the cases above where $\operatorname{Br} X / \operatorname{Br}_0 X \neq 0$, our generating algebras are as follows:

- If $P(x)$ splits completely, then $\operatorname{Br} X / \operatorname{Br}_0 X$ is generated by quaternion algebras of the form

$$\mathcal{A} = (a, x(x-1)) \quad \text{and} \quad \mathcal{B} = (a, x(x-\lambda))$$

- If $P(x) = cf(x)g(x)$ where both f and g are monic and at least one of $f(x)$ or $g(x)$ of degree 2, then $\operatorname{Br} X / \operatorname{Br}_0 X$ is generated by the quaternion algebra

$$\mathcal{C} = (a, f(x)) = (a, cg(x))$$

We now prove Theorem 5.0.3 from the introduction.

Theorem 5.2.2. *Let X be a Châtelet surface over a number field k. Then $\operatorname{Br} X_K / \operatorname{Br}_0 X_K = 0$ for every finite extension K/k if and only if the splitting field of $P(x)$ is a $\mathbb{Z}/4\mathbb{Z}$-extension containing $k(\sqrt{a})$.*

Proof. Let L/k be the splitting field of $P(x)$. Assume that L is $\mathbb{Z}/4\mathbb{Z}$ with $k(\sqrt{a})$ as the unique quadratic subfield. Let K/k be a finite extension. If $P(x)$ is irreducible over K, then $\operatorname{Br} X_K / \operatorname{Br}_0 X_K = 0$ by above. Otherwise, $P(x)$ factors, and by assumption on L, this means that $\sqrt{a} \in K$. In this case, X_K is K-rational and so $\operatorname{Br} X_K / \operatorname{Br}_0 X = 0$.

Conversely assume $\operatorname{Br} X_K / \operatorname{Br}_0 X_K = 0$ for every finite extension K/k. If $P(x)$ is reducible, then $\operatorname{Br} X / \operatorname{Br}_0 X$ being trivial implies it contains an irreducible factor of degree 3. Let α be a root of the cubic factor. Then over $K = k(\alpha)$, the polynomial $P(x)$ either splits or has an irreducible quadratic factor. In either case, $\sqrt{a} \notin K$ so $\operatorname{Br} X_K / \operatorname{Br}_0 X_K \neq 0$ by Proposition 5.2.1 which is a contradiction. Hence, $P(x)$ must be irreducible over k. Taking $K = L$ shows that $\sqrt{a}$ must be in L by Proposition 5.2.1. Therefore, there exists a normal index 2 subgroup $H \subset \operatorname{Gal}(L/k)$ corresponding to $k(\sqrt{a})$. We break into cases based on the isomorphism class of $\operatorname{Gal}(L/k)$.

(S_4) There is a unique quadratic extension contained in L/k corresponding to $A_4 \subset S_4$ which must be $k(\sqrt{a})$. Now take any subgroup $J \subset S_4$ of order 2 not contained in A_4, and let K/k be the corresponding extension. Since $J \not\subset A_4$, it follows that $\sqrt{a} \notin K$. Moreover, $P(x)$ must factorizes over K into two linear and one quadratic factor. It follows from Proposition 5.2.1 that $\operatorname{Br} X_K / \operatorname{Br}_0 X_K \cong \mathbb{Z}/2\mathbb{Z}$, which is a contradiction.

(A_4) There are no index 2 subgroups so this case is impossible.

(D_8) For every subgroup of index 2 in D_8, there exists a subgroup $J \subset D_8$ of order 2 that is not contained in it. The argument then follows the same way as in the S_4 case.

(K_4) Here L/k is a biquadratic extension, and taking K/k to be the quadratic extension not equal to $k(\sqrt{a})$ gives a contradiction by Proposition 5.2.1 again.

This leaves us with the last remaining case which is a $\mathbb{Z}/4\mathbb{Z}$-extension containing $k(\sqrt{a})$. ☐

Corollary 5.2.3. *Let $P(x)$ be a degree 4 irreducible polynomial with splitting field a $\mathbb{Z}/4\mathbb{Z}$-extension containing $k(\sqrt{a})$. Then the Châtelet surface*

$$y^2 - az^2 = P(x)$$

satisfies weak approximation over all finite extensions K/k.

Proof. By Theorem 5.2.2, there is no Brauer–Manin obstruction to weak approximation over all such extensions K/k. So by [CTSSD87a, Theorem B(ii)(a)] they satisfy weak approximation. ☐

5.3 *Failure of weak approximation when $P(x)$ is split*

In this section, we consider the case when $P(x)$ factors into linear factors. Our goal is to prove Theorem 5.0.1. As discussed in §5.2, after moving one of the singular fibers to $\infty \in \mathbb{P}^1$, we may assume

$$P(x) = cx(x-1)(x-\lambda) \tag{5.3.1}$$

for some $\lambda \in k$ not equal to $0, 1$ and c squarefree.

Let v be a finite place of k where $a \notin k_v^{\times 2}$, and $\pi \in \mathcal{O}_k$ be a uniformizer. Suppose the Châtelet surface X has bad reduction over v. Our goal in this section is to analyze the surjectivity of the evaluation map $\mathrm{ev}_{\mathcal{A}}\colon X(k_v) \to \operatorname{Br} k_v[2]$ for the algebra $\mathcal{A} \in \operatorname{Br}(X)$ listed in §5.2. Bad reduction means either one of $v(c), v(\lambda), v(\lambda-1)$ is nonzero or that $k(\sqrt{a})/k$ is ramified at v. We first consider the unramified case, and then deal with the ramified case separately for odd and even primes.

5.3.1 Unramified Case

In this section, we assume that $k_v(\sqrt{a})/k_v$ is unramified. For $x \in k_v$, there exists a point in $X(k_v)$ lying over $x \in \mathbb{P}^1$ if $v(P(x))$ is even. By Lemma 5.1.1, for any point $Q = (x, y, z) \in X(k_v)$,

$$\operatorname{inv}_v(\mathcal{A}(Q)) = \operatorname{inv}_v(a, x(x-1)) = v(x(x-1))/2 \in \mathbb{Q}/\mathbb{Z}.$$

The map $\operatorname{ev}_{\mathcal{A}}\colon X(k_v) \to \operatorname{Br} k_v[2]$ is surjective if and only if it is nonconstant, and for $\mathcal{A}(Q)$ to be nontrivial it is equivalent to show that $v(x(x-1))$ is odd.

Proposition 5.3.1. *Assume that* $v(\lambda) = v(\lambda-1) = 0$. *Then the evaluation map* $\operatorname{ev}_{\mathcal{A}}\colon X(k_v) \to \operatorname{Br} k_v[2]$ *is surjective.*

Proof. The assumptions imply we must have $v(c) = 1$. Take $x = 1/\pi$, then $v(cP(x)) = -2$, so that there exists $Q \in X(k_v)$ lying over x. Then $\mathcal{A}(Q) = 0$ sinc $v((1/\pi)(1/\pi - 1)) = -2$ is even. On the other hand, if we take $x = \pi$, then $v(cP(x)) = 2$ so again there exists $Q \in X(k_v)$ lying over x. Then $\mathcal{A}(Q) \neq 0$ since $v((\pi)(\pi-1)) = 1$ is odd. Hence, the proposition follows. □

Proposition 5.3.2. *Assume that* $v(\lambda)$ *or* $v(\lambda-1) \neq 0$. *Then the evaluation map* $\operatorname{ev}_{\mathcal{A}}\colon X(k_v) \to \operatorname{Br} k_v[2]$ *is surjective.*

Proof. By considering the six cross-ratios on $\mathbb{P}^1$, we may assume that $v(\lambda) > 0$. We now consider two cases, namely when $v(c) = 0$ and $v(c) = 1$. Recall that

$$v(P(x)) = v(c) + v(x) + v(x-1) + v(x-\lambda).$$

If $v(c) = 0$ then by picking $x = \frac{1}{\pi}$ we can conclude that $v(P(x)) = -6$. Hence there exists a point $Q = (x, y, z) \in X(k_v)$. Moreover, $v(x(x-1)) = -4$ hence $\mathcal{A}(Q) = 0$. Now to show that $\operatorname{ev}_{\mathcal{A}}$ is surjective, it suffices to find a different choice of x with $\mathcal{A}(Q) \neq 0 \in \operatorname{Br} k_v$. Pick x such that

$$0 < v(x) \leq v(\lambda), \quad v(x) \equiv 1 \bmod 2, \quad \text{and} \quad v(x-\lambda) = v(x).$$

Then we have $v(P(x)) = 2v(x)$ hence $Q \in X(k_v)$ and $v(x(x-1)) = v(x)$. Since $v(x) \equiv 1 \bmod 2$, we have $\mathcal{A}(Q) \neq 0$ as desired.

If $v(c) = 1$, then first set $x = \frac{1}{\pi}$. For this choice of x, $v(P(x)) = -2$ hence $Q \in X(k_v)$ and $v(x(x-1)) = -2$ so $\mathcal{A}(Q) = 0$. Furthermore, if we set $x = 1 + \pi$, then $v(P(x)) = 2$ and $v(x(x-1)) = 1$ thus $\mathcal{A}(Q) \neq 0$ as desired.

□

5.3.2 Ramified case odd

If v is an odd place and $k_v(\sqrt{a})/k_v$ is ramified, then $v(a) = 1$.

Proposition 5.3.3. *Assume that $v(a) = 1$ and $P(x)$ has the form (5.3.1). Then the evaluation map $\mathrm{ev}_\alpha \colon X(k_v) \to \operatorname{Br} k_v[2]$ is surjective for some $\alpha \in \{\mathcal{A}, \mathcal{B}\}$.*

Proof. By applying an automorphism on the base $\mathbb{P}^1$ that shuffles the points $0, 1, \infty$, we may assume $v(\lambda) \geq 0$. Dividing the equation (**??**) for X by a power of $-a$ and exchanging y, z if necessary, we can also assume that $v(c) = 0$. Let $R \in X(k)$ be the unique rational point over $\infty \in \mathbb{P}^1$. By using the following alternative formula for $\mathcal{A}, \mathcal{B}$ defined at R,

$$\mathcal{A} = (a, w(1-w)), \mathcal{B} = (a, w(1-\lambda w)),$$

where $w = 1/x$, we see that $\mathrm{ev}_{\mathcal{A}}(R) = \mathrm{ev}_{\mathcal{B}}(R) = 0$. To finish the proof, it suffices to exhibit a point where the evaluation is nontrivial.

Assume first that $\lambda \not\equiv 0, 1 \bmod \pi$. Let $E \subset X$ be the closed subset given by $z = 0$. Then E is an elliptic curve given by

$$y^2 = cx(x-1)(x-\lambda).$$

For $Q = (x, y) \in E(k_v)$, the quantities

$$cx, \ cx - c, \ cx - c\lambda$$

are all squares in k_v if and only if $Q \in 2E(k_v)$ (use [Hus04, §1.4 Theorem 4.1] after suitable coordinate change).

Lemma 5.3.4. *$E(k_v) \setminus 2E(k_v)$ is nonempty*

Proof. Since $v \nmid 2$, E has good reduction at v and $E(k_v)[2^\infty] = \widetilde{E}(\mathbb{F}_v)[2^\infty]$, where $\widetilde{E}$ is the reduction modulo v. In particular, this means $E(k_v)[2^\infty]$ is finite and moreover nontrivial as $E(k_v)[2] \cong (\mathbb{Z}/2\mathbb{Z})^2$. It follows then that $E(k_v) \setminus 2E(k_v)$ is nonempty. □

Remark 5.3.5. Although we will not need it, the previous lemma is true for $v \mid 2$ as well. Then there is a subgroup $E^* \subset E(k_v)$ of finite index such that $\mathcal{O}_v \cong E^*$. Take any $u \in \mathcal{O}_v$ with $v(u) = 0$. Then u is not divisible by 2 in E^*, and iteratively dividing by 2 in $E(k_v)$ would produce infinitely many points in the quotient $E(k_v)/E^*$, a contradiction. Hence u is not 2-divisible in $E(k_v)$.

Let $Q \in E(k_v) \setminus 2E(k_v)$ so then exactly two of $cx, cx - c, cx - c\lambda$ are not squares. This means in particular that $v(x) \geq 0$, and so at most one of $v(x), v(x-1), v(x-\lambda)$ can be nonzero and they must all be even since $v(x(x-1)(x-\lambda))$ is even. We also obtain that at least one of $x(x-1), x(x-\lambda)$ is not a square. Hence, at least on of those products is not a square with even valuation which implies it is not a norm from $k_v(\sqrt{a})/k_v$. Hence, $\operatorname{inv}_v \alpha(Q) = 1/2$ for some $\alpha \in \{\mathcal{A}, \mathcal{B}\}$.

Now assume that $\lambda \equiv 0 \bmod \pi$. Choose $\overline{x} \in \mathbb{F}_v^\times \setminus \mathbb{F}_v^{\times 2}$ such that $\overline{x} - 1 \in \mathbb{F}_v^{\times 2}$. Lift $\overline{x}$ to $x \in \mathcal{O}_v$. Then $(a, x) = (a, x - \lambda) \neq 0$ but $(a, x-1) = 0$. Hence, there exists some point $Q \in X(k_v)$ lying over $x \in \mathbb{P}^1$ where

$$\operatorname{inv}_v \mathcal{A}(Q) = (a, x(x-1)) = 1/2.$$

The case $\lambda \equiv 1 \bmod \pi$ is very similar. □

5.3.3 Ramified case even

Proposition 5.3.6. *Let v be a place lying over 2. Assume that $v(a) = 1$ and $P(x)$ has the form (5.3.1). Then the evaluation map $\operatorname{ev}_{\mathcal{A}} \colon X(k_v) \to \operatorname{Br} k_v[2]$ is surjective.*

We give the proof of this result after establishing some basic facts on the distribution of norms inside $\mathcal{O}_v$. For the remainder of this section, v will denote a place lying over 2 where

$k_v(\sqrt{a})/k_v$ is ramified. Let w be the place lying over v and $L_w := k_v(\sqrt{a})$. Let $\mathrm{N}\colon L_w \to k_v$ denote the norm map.

Equidistribution of norms among residues

The subgroup of norms $\{x \in \mathcal{O}_v^\times \mid x \in \mathrm{N}(\mathcal{O}_w^\times)\}$ has index 2 inside $\mathcal{O}_v^\times$. For any subset $H \subset k_v$, let $H \bmod \pi^n$ denote the set of equivalence classes $H/\sim$ where $h_1 \sim h_2$ if $h_1 - h_2 \in \pi^n \mathcal{O}_v$.

Lemma 5.3.7. *Let* $r \in \mathcal{O}_v$. *Then*

$$\lim_{n\to\infty} \frac{\#\{x \in \mathcal{O}_v \mid x \in \mathrm{N}(\mathcal{O}_w), x \equiv r \bmod \pi\} \bmod \pi^n}{\#\{x \in \mathcal{O}_v \mid x \equiv r \bmod \pi\} \bmod \pi^n} = \frac{1}{2}.$$

Proof. Let $\mathcal{O}_v^{(r)} := \{x \in \mathcal{O}_v \mid x \equiv r \bmod \pi\}$. We first prove the case when $r = 1$. Note that $\mathrm{N}(\mathcal{O}_w) \cap \mathcal{O}_v^{(1)} \subset \mathcal{O}_v^{(1)}$ is a subgroup of index at most 2 under the multiplicative structure. By Lemma 5.1.2, there exists $u \in \mathcal{O}_v$ such that $1 - \pi u \notin \mathrm{N}(\mathcal{O}_w)$, and so $1 - \pi u \in \mathcal{O}_v^{(1)} \setminus (\mathrm{N}(\mathcal{O}_w) \cap \mathcal{O}_v^{(1)})$. Hence it follows $\mathrm{N}(\mathcal{O}_w) \cap \mathcal{O}_v^{(1)}$ has index 2. Consider the quotient map

$$q_n\colon \mathcal{O}_v^{(1)} \to \mathcal{O}_v^{(1)}/\pi^n.$$

The image $q_n(\mathrm{N}(\mathcal{O}_w) \cap \mathcal{O}_v^{(1)})$ has either index 1 or 2. Since norms of finite index are open in the v-adic topology, it follows that for n large enough, this image has index 2. The statement about the limit follows immediately.

Now assume $r \not\equiv 0 \bmod \pi$. Take any $x \in \mathcal{O}_v$ such that $x \equiv r \bmod \pi$. Multiplication by $1/x$ gives a bijection $\mathcal{O}_v^{(r)} \to \mathcal{O}_v^{(1)}$. Depending on x, this map sends $\mathrm{N}(\mathcal{O}_w) \cap \mathcal{O}_v^{(r)}$ to either $\mathrm{N}(\mathcal{O}_w) \cap \mathcal{O}_v^{(1)}$ or $\mathcal{O}_v^{(1)} \setminus \mathrm{N}(\mathcal{O}_w) \cap \mathcal{O}_v^{(1)}$. Hence, the limit then follows from what we proved for $\mathcal{O}_v^{(1)}$ above.

Finally assume $r = 0$. Then noting that $-\pi \in \mathrm{N}(\mathcal{O}_w)$ and setting $x' = x/(-\pi)$ gives

$$\{x \in \mathcal{O}_v \mid x \in \mathrm{N}(\mathcal{O}_w), x \equiv 0 \bmod \pi\} = -\pi\{x' \in \mathcal{O}_v \mid x' \in \mathrm{N}(\mathcal{O}_w)\}.$$

Hence the limit in question is

$$\lim_{n\to\infty} \frac{\#\{x' \in \mathcal{O}_v \mid x' \in \mathrm{N}(\mathcal{O}_w)\} \bmod \pi^{n-1}}{\#\mathcal{O}_v \bmod \pi^{n-1}}.$$

Now, we can divide this limit according to the image of x' in $\mathcal{O}_v/\pi\mathcal{O}_v$. We can apply our previous result for $x' \not\equiv 0 \bmod \pi$ and argue inductively for those $x' \equiv 0 \bmod \pi$. Hence, we obtain that the above limit is $1/2$. □

The following limit follows immediately from Lemma 5.3.7.

$$\lim_{n\to\infty} \frac{\#\{x \in \mathcal{O}_v^\times \mid x \in \mathrm{N}(\mathcal{O}_w)\} \bmod \pi^n}{\#\mathcal{O}_v^\times \bmod \pi^n} = \frac{1}{2}.$$

Remark 5.3.8. The above lemma fails when v does not lie over 2. Indeed whether a unit x is a norm or not can be determined by looking modulo π.

Let $\mathcal{O}_v^* = \mathcal{O}_v \setminus \{0\}$ be the nonzero elements (not to be confused with $\mathcal{O}_v^\times$, the nonzero unites). Define the sets

$$A = \{x \in \mathcal{O}_v \mid x(x-1) \in \mathrm{N}(\mathcal{O}_w^*)\}, \quad B = \{x \in \mathcal{O}_v \mid x - \lambda \in \mathrm{N}(\mathcal{O}_w^*)\}.$$

Since $x \mapsto x(x-1)$ and $x \mapsto x - \lambda$ are both continuous endomorphisms on $\mathcal{O}_v$, both A, B, being inverse images of the open and closed subset $\mathrm{N}(\mathcal{O}_w^*)$, are open and closed inside $\mathcal{O}_v$. Our first goal is to establish the following.

Proposition 5.3.9.

$$\lim_{n\to\infty} \frac{\#A \bmod \pi^n}{\#\mathcal{O}_v \bmod \pi^n} = \lim_{n\to\infty} \frac{\#B \bmod \pi^n}{\#\mathcal{O}_v \bmod \pi^n} = \frac{1}{2}$$

Proof. Since $\{x - \lambda \mid x \in \mathcal{O}_v\} = \mathcal{O}_v$, the limit for B follows immediately in view of Lemma 5.3.7. To prove the limit for A, we first divide $\mathcal{O}_v$ in the following way

$$\mathcal{O}_v^{(0)} = \{x \in \mathcal{O}_v \mid x \equiv 0 \bmod \pi\}, \mathcal{O}_v^{(1)} = \{x \in \mathcal{O}_v \mid x \equiv 1 \bmod \pi\}, \mathcal{O}_v' = \mathcal{O}_v \setminus (\mathcal{O}_v^{(0)} \cup \mathcal{O}_v^{(1)}).$$

We divide the set A in the analogous way

$$A^{(0)} = \{x \in A \mid x \equiv 0 \bmod \pi\}, A^{(1)} = \{x \in A \mid x \equiv 1 \bmod \pi\}, A' = A \setminus (A^{(0)} \cup A^{(1)}).$$

It suffices to show each of the following limits,

$$\lim_{n\to\infty} \frac{\#A^{(0)} \bmod \pi^n}{\#\mathcal{O}_v^{(0)} \bmod \pi^n} = \lim_{n\to\infty} \frac{\#A^{(1)} \bmod \pi^n}{\#\mathcal{O}_v^{(1)} \bmod \pi^n} = \lim_{n\to\infty} \frac{\#A' \bmod \pi^n}{\#\mathcal{O}_v' \bmod \pi^n} = \frac{1}{2}.$$

Define the map $f\colon k_v^\times \to k_v \setminus \{1\}$ given by $f(x) = 1 - 1/x$. Observe that f is a bijection. Moreover, for any $x \in \mathcal{O}_v^* \setminus \{1\}$, $x(x-1)/f(x) = x^2$. Hence, $x(x-1) \in \mathrm{N}(\mathcal{O}_w^*)$ if and only if $f(x) \in \mathrm{N}(L_w^\times)$. In particular, this means for $x \in A$ if and only if $f(x) \in \mathrm{N}(L_w^\times)$.

Let n be a positive integer and $x \in \mathcal{O}_v^\times$. Then

$$f(x + \pi^n\mathcal{O}_v) = 1 - 1/x + \pi^n\mathcal{O}_v = y + \pi^n\mathcal{O}_v$$

where $y = 1 - 1/x \in \mathcal{O}_v$. Observe that $y \not\equiv 1 \bmod \pi$. Hence, f induces a bijection between the following two sets,

$$\{x + \pi^n\mathcal{O}_v \mid x \in \mathcal{O}_v^\times\} \overset{f}{\longleftrightarrow} \{y + \pi^n\mathcal{O}_v \mid y \in \mathcal{O}_v, y \not\equiv 1 \bmod \pi\}.$$

We may further decompose into the following bijections

$$\{x + \pi^n\mathcal{O}_v \mid x \in \mathcal{O}_v, x \not\equiv 0, 1 \bmod \pi\} \overset{f}{\longleftrightarrow} \{y + \pi^n\mathcal{O}_v \mid y \in \mathcal{O}_v, y \not\equiv 0, 1 \bmod \pi\},$$

$$\{x + \pi^n\mathcal{O}_v \mid x \in \mathcal{O}_v, x \equiv 1 \bmod \pi\} \overset{f}{\longleftrightarrow} \{y + \pi^n\mathcal{O}_v \mid y \in \mathcal{O}_v, y \equiv 0 \bmod \pi\}.$$

Written another way, f induces bijections

$$\mathcal{O}_v' \bmod \pi^n \overset{f}{\longleftrightarrow} \mathcal{O}_v' \bmod \pi^n,$$

$$\mathcal{O}_v^{(1)} \bmod \pi^n \overset{f}{\longleftrightarrow} \mathcal{O}_v^{(0)} \bmod \pi^n.$$

Moreover, under this bijection, $A \bmod \pi^n$ maps to

$$A' \bmod \pi^n \overset{f}{\longleftrightarrow} \{y \in \mathcal{O}_v \mid y \in \mathrm{N}(\mathcal{O}_w), y \not\equiv 0, 1 \bmod \pi\} \bmod \pi^n,$$

$$A^{(1)} \bmod \pi^n \overset{f}{\longleftrightarrow} \{y \in \mathcal{O}_v \mid y \in \mathrm{N}(\mathcal{O}_w), y \equiv 0 \bmod \pi\} \bmod \pi^n.$$

Therefore,

$$\lim_{n\to\infty} \frac{\#A' \bmod \pi^n}{\#\mathcal{O}_v' \bmod \pi^n} = \lim_{n\to\infty} \frac{\#\{y \in \mathcal{O}_v \mid y \in \mathrm{N}(\mathcal{O}_w), y \not\equiv 0, 1 \bmod \pi\} \bmod \pi^n}{\#\mathcal{O}_v' \bmod \pi^n} = \frac{1}{2},$$

$$\lim_{n\to\infty} \frac{\#A^{(1)} \bmod \pi^n}{\#\mathcal{O}_v^{(1)} \bmod \pi^n} = \lim_{n\to\infty} \frac{\#\{y \in \mathcal{O}_v \mid y \in \mathrm{N}(\mathcal{O}_w), y \equiv 0 \bmod \pi\} \bmod \pi^n}{\#\mathcal{O}_v^{(0)} \bmod \pi^n} = \frac{1}{2}$$

by Lemma 5.3.7. It remains to prove the limit for $A^{(0)}$. For this, consider the map $g\colon \mathcal{O}_v \to \mathcal{O}_v$ given by $g(x) = 1 - x$. This is clearly a bijection and sends $\mathcal{O}_v^{(0)}$ to $\mathcal{O}_v^{(1)}$. Moreover, since $x(x-1) = (g(x))(g(x)-1)$, g sends $A^{(0)}$ bijectively to $A^{(1)}$. Hence,

$$\lim_{n\to\infty} \frac{\#A^{(0)} \bmod \pi^n}{\#\mathcal{O}_v^{(0)} \bmod \pi^n} = \lim_{n\to\infty} \frac{\#A^{(1)} \bmod \pi^n}{\#\mathcal{O}_v^{(1)} \bmod \pi^n} = \frac{1}{2}. \qquad \square$$

Applying the equidistribution results

Finally, we return to the proof of Proposition 5.3.6

Proof of Proposition 5.3.6. We first consider the case when $(a,c) = 0$. This means in particular that the fiber over $\infty \in \mathbb{P}^1$ has a point $Q \in X_\infty(k_v)$ and $\operatorname{inv}_v \mathcal{A}(Q) = \operatorname{inv}_v \mathcal{B}(Q) = 0$. So it suffices to find another point with invariant $1/2$.

Lemma 5.3.10.

$$\lim_{n\to\infty} \frac{\#\mathcal{O}_v \setminus (A\cup B) \bmod \pi^n}{\#\mathcal{O}_v \bmod \pi^n} > 0.$$

Proof. Since $X(k_v) \neq \emptyset$ by assumption, there exists $x \in \mathcal{O}_v$ with $x \neq 0, 1, \lambda$ such that $x(x-1)(x-\lambda) \in \mathrm{N}(\mathcal{O}_w)$. This means either $x \in \mathcal{O}_v \setminus (A\cup B)$ or $x \in A\cap B$. In the former case, the lemma then follows since $A\cap B$ is open in $\mathcal{O}_v$. In the latter case $\lim_{n\to\infty}(\#A\cap B)/(\#\mathcal{O}_v \bmod \pi^n) > 0$, and so combining with Proposition 5.3.9 gives the desired result. $\square$

Let $x \in \mathcal{O}_v \setminus (A\cup B)$ and $x \neq 0, 1, \lambda$. This means $x(x-1)(x-\lambda) \in \mathrm{N}(L_w)$ but $x(x-1) \notin \mathrm{N}(L_w)$. Let $Q \in X(k_v)$ be a point with x-coordinate is x. Then

$$\operatorname{inv}_v \mathcal{A}(Q) = (a, x(x-1)) = 1/2.$$

For the case $(a,c) \neq 0$, we have the following

Lemma 5.3.11.

$$\lim_{n\to\infty} \frac{\#(A\setminus B) \bmod \pi^n}{\#\mathcal{O}_v \bmod \pi^n} > 0, \quad \lim_{n\to\infty} \frac{\#(B\setminus A) \bmod \pi^n}{\#\mathcal{O}_v \bmod \pi^n} > 0$$

Proof. By Proposition 5.3.9, it suffices to show at least one of the limit is positive since that will imply the other is positive as well. Since $X(k_v) \neq \emptyset$ by assumption, there exists $x \in \mathcal{O}_v$ with $x \neq 0, 1, \lambda$ such that one of the following two cases happen

Case 1, $x(x-1) \in \mathrm{N}(\mathcal{O}_w)$ *and* $x - \lambda \notin \mathrm{N}(\mathcal{O}_w)$ OR

Case 2, $x(x-1) \notin \mathrm{N}(\mathcal{O}_w)$ *and* $x - \lambda \in \mathrm{N}(\mathcal{O}_w)$.

Either case, at least one of $A \setminus B$ or $B \setminus A$ is nonempty, and the limit must also be positive since both are open in $\mathcal{O}_v$. ☐

To finish the proof, we choose $x_1 \in A \setminus B$ and $x_2 \in B \setminus A$. Then there exists $Q_1, Q_2 \in X(k_v)$ with x coordinate corresponding to x_1, x_2 respectively. It follows that

$$\operatorname{inv}_v \mathcal{A}(Q_1) = 0, \quad \operatorname{inv}_v \mathcal{A}(Q_2) = 1/2. \qquad \square$$

5.3.4 Proof of Theorem 5.0.1

Proof of Theorem 5.0.1. Let X be a Châtelet surface where $P(x)$ has the form (5.3.1). First assume that for every nonarchimedian place v, either X has good reduction or $\sqrt{a} \in k_v$. In the first case, by [CTS13, Lemma 2.2] the evaluation map $\operatorname{ev}_\alpha \colon X(k_v) \to \operatorname{Br} k_v[2]$ must be constant for any $\alpha \in \operatorname{Br} X$. In the latter case, ev_α is also constant since the Brauer classes listed in §3 are trivial over k_v. Moreover, if $a > 0$ or k does not have a real embedding, then for any archimedian place v, $\sqrt{a} \in k_v$, so the evaluation map is constant again. Since $X(k)$ is clearly nonempty, it follows X satisfies weak approximation.

Conversely, assume either X has a place v of bad reduction with $\sqrt{a} \notin k_v$ or v is a real place and $a < 0$. To show failure of weak approximation, it suffices to show that there is a Brauer–Manin obstruction given by the surjectivity of the evaluation map $\operatorname{ev}_{\mathcal{A}} \colon X(k_v) \to \operatorname{Br} k_v[2]$.

If $a < 0$, then the evaluation map is surjective at v since taking x such that exactly two of $x, x-1, x-\lambda$ is negative gives rise to a real point Q where either $\operatorname{ev}_{\mathcal{A}}(Q)$ or $\operatorname{ev}_{\mathcal{B}}(Q)$ is nontrivial. On the other hand, taking x so that all $x, x-1, x-\lambda$ are positive gives rise to a real point Q such that $\operatorname{ev}_{\mathcal{A}}(Q) = \operatorname{ev}_{\mathcal{B}}(Q) = 0$.

Now assume v is a place of bad reduction. If $k(\sqrt{a})/k$ is unramified at v, then one of $v(c), v(\lambda), v(\lambda - 1)$ must be nonzero. Then Propositions 5.3.1 and 5.3.2 imply the result. If $k(\sqrt{a})/k$ is ramified then Proposition 5.3.3 for odd v or Proposition 5.3.6 for even v gives the desired result. □

Proof of Corollary 5.0.2. Let L be the splitting field of $P(x)$ as stated in the theorem. If v is a place of bad reduction as given in the hypothesis, then there exists a place w of L lying over v such that X_L has bad reduction at w and $a \notin L_w^{\times 2}$. Theorem 5.0.1 then implies that X_L fails weak approximation. □

5.4 *Weak approximation in the quadratic case*

In this section, we consider the case when $P(x)$ factors as

$$y^2 - az^2 = cP_1(x)P_2(x)$$

where P_1, P_2 are irreducible monic quadratic polynomials. By §2, the Brauer group modulo $\operatorname{Br}_0 X$ is generated by the quaternion algebra

$$\mathcal{C} = (a, P_1(x)) = (a, cP_2(x)).$$

If the above Brauer class is constant (meaning it comes from $\operatorname{Br} k$), then X satisfies weak approximation. Hence, for the rest of this section, we assume that the class above is nonconstant. This is equivalent to the fact that $\sqrt{a}$ is not in the splitting field of $P_1(x)$ or $P_2(x)$. Moreover, after a change of coordinates, we assume the coefficients of $P_1(x), P_2(x)$ are in $\mathcal{O}_k$.

Let v be a nonarchimedian place of k and $\pi \in \mathcal{O}_k$ a uniformizer.

Lemma 5.4.1. *Let $R(x) \in \mathbb{F}_v[x]$ be a monic irreducible quadratic polynomial. Then for exactly $(q-1)/2$ many of the values $x \in \mathbb{F}_v$, $R(x)$ is a square in $\mathbb{F}_v$.*

Proof. It suffices to show that $R(x) = y^2$ has $q - 1$ solutions in $(x, y) \in \mathbb{F}_v^2$ (since $R(x)$ is irreducible, y is never 0). We may homogenize to define a smooth conic in $\mathbb{P}^2_{\mathbb{F}_v}$. Since this conic has two points at infinity, it is isomorphic to $\mathbb{P}^1_{\mathbb{F}_v}$ and thus has $q + 1$ points. Removing the two points at infinity gives $q - 1$ solutions to the original equation. □

Proposition 5.4.2. *Assume that $v(a) = 1$. If $P_1(x), P_2(x)$ are irreducible modulo π, then there is an obstruction to weak approximation.*

Proof. Write

$$P_i(x) = x^2 + d_i x + r_i$$

where $d_i, r_i \in \mathcal{O}_v$. Since $P_i(x)$ is irreducible modulo π, we must have $r_i \in \mathcal{O}_v^\times$. Suppose X has a k_v point on a smooth fiber $x = x_0 \in k_v$. If $v(x_0) < 0$, then the fiber over $\infty \in \mathbb{P}^1$ also has a k_v point. Applying the automorphism $x \mapsto 1/x$ on $\mathbb{P}^1$, we may rewrite the equation for X as

$$y^2 - az^2 = cr_1r_2(x^2 + d_1x/r_1 + 1/r_1)(x^2 + d_2x/r_2 + 1/r_2)$$

which has a k_v point over the smooth fiber $x = 0$. Hence, we may reduce to the case where there exists a point $Q_0 = [y_0, z_0, x_0] \in X(k_v)$ on a smooth fiber where $v(x_0) \geq 0$.

It suffices to find a point $Q_1 \in X(k_v)$ such that $\operatorname{inv}_v \mathcal{A}(Q_0) \neq \operatorname{inv}_v \mathcal{A}(Q_1)$. Let $\alpha = x_0 \bmod \pi \in \mathbb{F}_v$. By Lemma 5.4.1, there must be another $\beta \in P^1(\mathbb{F}_v)$ such that

1. $P_1(\alpha)$ is a square if and only if $P_1(\beta)$ is a nonsquare,

2. $P_2(\alpha)$ is a square if and only if $P_2(\beta)$ is a nonsquare.

Here we take the convention that $P_1(\infty) = P_2(\infty) = 1$ are squares. This is only needed if both $P_1(\alpha), P_2(\alpha)$ are nonsquares. Then $P_1(\beta)P_2(\beta)$ is nonzero and in the same square class as $P_1(\alpha)P_2(\alpha)$. Therefore, we may use Hensel's lemma to lift to a point $Q_1 = (y_1, z_1, x_1) \in X(k_v)$ where $x_1 \equiv \beta \bmod \pi$. But then

$$\operatorname{inv}_v \mathcal{C}(Q_0) = (a, x_0) \neq (a, x_1) = \operatorname{inv}_v \mathcal{C}(Q_1).$$

The inequality of the two Hilbert symbols is due to (1) and (2) above. □

If the hypothesis of Proposition 5.4.2 does not hold, the existence of an obstruction to weak approximation is much more intrinsic to the surface in question. In particular, one cannot expect a uniform result similar to the case when $P(x)$ splits completely or is irreducible. We illustrate this subtlety by considering two Châtelet surfaces whose defining equation differs by one coefficient.

Example 1. Let $X/\mathbb{Q}$ be the Châtelet surface given by

$$y^2 - 17z^2 = 3(x^2-7)(17x^2-43).$$

By Proposition 5.2.1, $\operatorname{Br} X/\operatorname{Br}_0 X$ is generated by $\mathcal{C} = (17, x^2-7) = (17, 3(17x^2-43))$. We show that X satisfies weak approximation.

We begin by showing $X(\mathbb{Q}_p) \neq \emptyset$ for all primes p. First, observe that for $p \neq 3, 17$ we have $X_\infty(\mathbb{Q}_p) \neq \emptyset$ because X_∞ is the conic $y^2 - 17z^2 = 51w^2$, which has $\mathbb{Q}_p$ points for $p \neq 3, 17$. Next, we observe that the fiber X_1 is the conic $y^2 - 17z^2 = 468w^2$ which has $\mathbb{Q}_3$ and $\mathbb{Q}_{17}$-points since 468 is a square in $\mathbb{Q}_3$ and $\mathbb{Q}_{17}$.

We show that the map $\mathrm{ev}_{\mathcal{C}}\colon X(\mathbb{Q}_p) \to \operatorname{Br}\mathbb{Q}_p$ is constant at all primes p. Since the evaluation map is constant at all primes of good reduction [CTS13, Theorem 3.1], it remains to check $\mathrm{ev}_{\mathcal{C}}\left(X(\mathbb{Q}_p)\right)$ for the primes $p = 2, 3, 7, 17$ and 43. Let $(x_0, y_0, z_0) \in X(\mathbb{Q}_p)$ and $P(x) = 3(x^2-7)(17x^2-43)$.

($p=2$) Note that $\mathrm{inv}_2(\mathrm{ev}_C\left(X(\mathbb{Q}_2)\right)) = 0$ because $17 \in \mathbb{Q}_2^{\times 2}$.

($p=17$) Assume $v_{17}(x_0) < 0$ so then $v_{17}\left(P(x_0)\right) = 4v_{17}(x_0) + 1$. Furthermore, we can see that

$$17^{-(4v_{17}(x_0)+1)}P(x_0) \equiv 3 \pmod{17}$$

which is not a square in $\mathbb{Q}_{17}$. We can now conclude that in this case, $P(x)$ is never a norm from the ramified extension $\mathbb{Q}_{17}(\sqrt{17})$. Hence we must have $v_{17}(x_0) \geq 0$. For these values of x_0, $3(17x_0^2-43) \equiv 7 \pmod{17}$ is not a square so $\mathrm{inv}_{17}(\mathrm{ev}_{\mathcal{C}}(X(\mathbb{Q}_{17}))) = 1/2$.

($p=7$) Since $3(17x^2-43)$ is irreducible over $\mathbb{Q}_7$, $v_7(3(17x_0^2-43))$ must be even. Hence $\mathrm{inv}_7(\mathrm{ev}_{\mathcal{C}}\left(X(\mathbb{Q}_7)\right)) = 0$.

($p=43$) Since x^2-7 is irreducible over $\mathbb{Q}_{43}$, we have $v_{43}(x_0^2-7)$ must be even. Hence $\mathrm{inv}_{43}(\mathrm{ev}_{\mathcal{C}}\left(X(\mathbb{Q}_{43})\right)) = 0$.

($p=3$) Since $17x^2-43$ is irreducible over $\mathbb{Q}_3$, $v_3(17x_0^2-43)$ must be even. Hence $v_3(3(17x_0^2-43))$ is odd and so $\mathrm{inv}_3(\mathrm{ev}_{\mathcal{C}}(X(\mathbb{Q}_3))) = 1/2$.

Combining the above calculations, we obtain that the sum of invariants is always 0, which means weak approximation holds.

After a minor adjustments to the surface of Example 1, we obtain another Châtelet surface which fails weak approximation.

Example 2. Let $X/\mathbb{Q}$ be the Châtelet surface given by

$$y^2 - 17z^2 = 3(x^2 - 7)(17x^2 - 7 \cdot 43).$$

We first show that $X(\mathbb{A}_{\mathbb{Q}}) \neq \emptyset$. In a similar fashion to Example 1 we have $X(\mathbb{Q}_p) \neq \emptyset$ for all $p \neq 3, 17$ hence only non-emptiness of $X(\mathbb{Q}_3)$ and $X(\mathbb{Q}_{17})$ must be checked directly. One can compute the fibers X_1 and X_3 and see that the conic X_1 has $\mathbb{Q}_3$-points and the conic X_3 has $\mathbb{Q}_{17}$-points.

We claim that X fails weak approximation and to prove this, it suffices to verify that the map $\mathrm{ev}_{\mathcal{C}} \colon X(\mathbb{Q}_7) \to \operatorname{Br} \mathbb{Q}_7[2]$ is surjective. To see this, first note that $X_\infty(\mathbb{Q}_7) \neq \emptyset$ and $\mathrm{ev}_{\mathcal{C}}$ is trivial over such a point. On the other hand, as $P(0) = 3 \cdot 7^2 \cdot 43$ has even valuation, there exists $Q \in X_0(\mathbb{Q}_7)$ lying over $0 \in \mathbb{P}^1(\mathbb{Q}_7)$ where

$$\mathrm{ev}_{\mathcal{C}}(Q) = (17, -7) \neq 0.$$

This proves our claim.

BIBLIOGRAPHY

[Bea78] Arnaud Beauville, *Complex Algebraic Surfaces*, Press Syndicate of the University of Cambridge, Cambridge, 1978. ↑22

[Châ59] F. Châtelet, *Points rationnels sur certaines courbes et surfaces cubiques*, L'Enseignement Mathématique. Revue Internationale. IIe Serie (1959). ↑23

[Cla09] Pete L. Clark, *On the Hasse principle for Shimura curves*, Israel J. Math. **171** (2009), 349–365. ↑46

[CTC79] J.L. Colliot-Thélène and Daniel Coray, *L'équivalence rationnelle sur les points fermés des surfaces rationnelles fibrées en coniques*, Compositio Math. **39** (1979), no. 3, 301–332. ↑50, 54

[CTS82] Jean-Louis Colliot-Thélène and Jean-Jacques Sansuc, *Sur le principe de Hasse et l'approximation faible, et sur une hypothèse de Schinzel*, Acta Arith. **41** (1982), no. 1, 33–53 (French). ↑50

[CTS21] J.L. Colliot-Thélène and A. Skorobogatov, *The Brauer-Grothendeick Group*, available at https://wwwf.imperial.ac.uk/ anskor/brauer.pdf, 2021. ↑52, 64

[CTS13] Jean-Louis Colliot-Thélène and Alexei N. Skorobogatov, *Good reduction of the Brauer-Manin obstruction*, Trans. Amer. Math. Soc. **365** (2013), no. 2, 579–590. ↑58, 74, 77

[CTSSD87a] J.L. Colliot-Thélène, J.J. Sansuc, and H.P.F. Swinnnerton-Dyer, *Intersection of two quadrics and Châtelet surfaces*, J. Reine Angew. Math **374** (1987). ↑1, 31, 40, 44, 45, 50, 61, 62, 66

[CTSSD87b] ______, *Intersection of two quadrics and Châtelet surfaces II*, J. Reine Angew. Math **374** (1987), 72-168. ↑61

[CTSD94] Jean-Louis Colliot-Thélène and Peter Swinnerton-Dyer, *Hasse principle and weak approximation for pencils of Severi-Brauer and similar varieties*, J. Reine Angew. Math. **453** (1994), 49–112. ↑45, 46, 50, 57

[CV22] Brendan Creutz and Bianca Viray, *Quadratic points on intersections of two quadrics* (2022). available at https://arxiv.org/pdf/2106.08560.pdf. ↑47

[Gro68a] Alexander Grothendieck, *Le groupe de Brauer. II. Théorie cohomologique*, Dix Exposés sur la Cohomologie des Schémas, North-Holland, 1968. ↑39

[Gro68b] ———, *Le groupe de Brauer. III. Théorie cohomologique*, Dix Exposés sur la Cohomologie des Schémas, North-Holland, 1968. ↑12

[GS17] Philippe Gille and Tamás Szamuely, *Central Simple Algebras and Galois Cohomology*, Cambidge University Press, Cambridge, 2017. ↑6, 7, 10, 63

[Gui18] Pierre Guillot, *A Gentle Course in Local Class Field Theory*, Cambridge University Press, Cambridge, 2018. ↑10

[Har70] Robin Hartshorne, *Ample Subvarieties of Algebraic Varieties*, Springer-Verlag, New York-Heidelberg, 1970. ↑24

[Har77] ———, *Algebraic Geometry*, Springer-Verlag, New York-Heidelberg, 1977. ↑20, 21, 22

[Hus04] Dale Husemller, *Elliptic Curves*, 2nd ed., Graduate Texts in Mathematics, vol. 111, Springer, New York, 2004. ↑68

[Lia18] Yongqi Liang, *Non-invariance of weak approximation properties under extension of the ground field*, `arxiv:1805.08851`, 2018. preprint. ↑62

[Isk71] V. A. Iskovskih, *A counterexample to the Hasse principle for systems of two quadratic forms in five variables*, Mat. Zametki **10** (1971), 253–257 (Russian). ↑1, 38

[Isk96] V.A. Iskovskih, *A rationality criterion for conic bundles*, Matematicheskii Sbornik **187, Number 7** (1996). ↑45

[Lan94] Serge Lang, *Algebraic Number Theory*, Springer-Verlag, New York, 1994. ↑41

[Man71] Yu. I Manin, *Le groupe de Brauer-Grothendieck en geometrie diophantienne*, Actes du Congres International des Mathematiciens (Nice, 1970) (1971). ↑1

[Man74] Yuri Manin, *Cubic forms: algebra, geometry, arithmetic*, North- Holland Publishing Co., Amsterdam, 1974. ↑18

[Mil80] J.S. Milne, *Étale cohomology*, Princeton Mathematical Series **33** (1980). ↑12, 14

[Mil13] ______, *Class Field Theory (v4.02)*, 2013. available at www.jmilne.org/math/. ↑11

[NSW08] J Neukirch, A Schmidt, and K Wingberg, *Cohomology of number fields*, Springer-Verlag, Berlin, 2008. ↑18

[Poo09] Bjorn Poonen, *Existence of rational points on smooth projective varieties*, J. Eur. Math. Soc. (JEMS) **11** (2009). ↑44

[Poo17] ______, *Rational points on varieties*, Graduate Studies in Mathematics, vol. 186, American Mathematical Society, Providence, RI, 2017. ↑13, 14, 15, 17, 39, 41, 51

[Rov22] Sam Roven, *On the Hasse Principle for conic bundles over even degree extensions*, `arxiv:1804.08763`, 2022. preprint. ↑62

[Sal88] P. Salberger, *Zero-cycles on rational surfaces over number fields*, Invent. Math. **91** (1988), no. 3, 505–524. ↑50

[Sal90] ______, *Some new Hasse principles for conic bundle surfaces*, Séminaire de Théorie des Nombres, Paris 1987–88, Progr. Math., vol. 81, Birkhäuser Boston, Boston, MA, 1990, pp. 283–305. ↑50

[Sko15] A. Skorobogatov, *Rational Points on Higher-Dimensional Varieties, Hay-On-Wye* (2015). ↑52

[Ser73] Jean-Pierre Serre, *A Course in Arithmetic*, Springer-Verlag, New York, 1973. ↑40

[Ser91] ______, *Local Fields*, Springer-verlag, New York, 1991. ↑8

[Ser86] J.P. Serre, *Local class field theory*, Algebraic number theory: Proceedings of the instructional conference held at the University of Sussex, Brighton, September 1–17, 1965, 1986, pp. 129–161. Reprint of the 1967 original. ↑63

[Vir10] Bianca Viray, *The algebraic Brauer-Manin obstruction on Chatelet surfaces, degree 4 del Pezzo surfaces, and Enriques surfaces* (2010). ↑15

[Voi21] John Voight, *Quaternion algebras*, Graduate Texts in Mathematics, vol. 288, Springer, Cham, 2021. ↑6, 10

[Wu22] Han Wu, *Arithmetic of Chatelet surfaces under extensions of base fields*, `arxiv:2203.09156`, 2022. preprint. ↑62