Pattern Avoidance Criteria for Smoothness of Positroid Varieties Via Decorated Permutations, Spirographs, and Johnson Graphs

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Abstract

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Chair of the Supervisory Committee:
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Mathematics

Positroids are certain representable matroids originally studied by Postnikov in connection with the totally nonnegative Grassmannian and now used widely in algebraic combinatorics. The positroids give rise to determinantal equations defining positroid varieties as subvarieties of the Grassmannian variety. Rietsch, Knutson-Lam-Speyer, and Pawlowski studied geometric and cohomological properties of these varieties. In this thesis, we continue the study of the geometric properties of positroid varieties by establishing several equivalent conditions characterizing smooth positroid varieties using a variation of pattern avoidance defined on decorated permutations, which are in bijection with positroids. This allows us to give two formulas for counting the number of smooth positroids along with two q-analogs. We also include results of Christian Krattenthaler, which give additional formulas for counting smooth positroids and the coefficients of our q-analogs as well as an asymptotic growth formula for the number of smooth positroids. Furthermore, we give a combinatorial method for determining the dimension of the tangent space of a positroid variety at key points using an induced subgraph of the Johnson graph. We also give a Bruhat interval characterization of positroids. The results and much of the text of this thesis appear in joint work with Sara Billey [BW22a; BW22b]. The enumerative results due to Krattenthaler presented here were

inspired by conjectures and results announced in [BW22a]. His results will be included as an appendix to [BW22b].

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ACKNOWLEDGMENTS

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DEDICATION

to my family

Chapter 1

INTRODUCTION

The use of permutation patterns emerged about a century ago, and this topic has, in the intervening time, become a prominent field of study. The work of Donald Knuth in the 1960s brought into light applications of permutation pattern avoidance to computer science [Knu73]. Widespread study in this area developed in the 1990s, and 2003 saw the establishment of the international Permutation Patterns conference, devoted solely to results in this area of study. Over the last several decades, several thousand papers related to permutation patterns have emerged, touching many different facets of the field.

Problems in enumerative geometry can be traced back over two thousand years, but a more rigorous treatment of the subject began in the 1800s. In the late 1800s, Hermann Schubert brought great developments to the field through the introduction of Schubert calculus, whose key objects are *Schubert cells* [Sch86; Sch79]. Hilbert's fifteenth problem called for a rigorous foundation to Schubert's enumerative calculus, which influenced a strong emphasis on these topics in the field of intersection theory within algebraic geometry throughout the twentieth century. In particular, *Schubert varieties*, which arise as the closure of Schubert cells in both the Grassmannian variety and the flag variety, have been thoroughly studied. A nonexhaustive list of major contributions in the last 100 years includes work produced by Ehresmann [Ehr34], Borel [Bor53], Weil [Wei62], Bernstein-Gelfand-Gelfand [BGG73], Lascoux-Schützenberger [LS82], Macdonald [Mac91], Chevalley [Che94], Billey-Haiman [BH95], Fulton [Ful92], Fomin-Gelfand-Postnikov [FGP97], Goresky-Kottwitz-MacPherson [GKM98], and Lam-Lapointe-Morse-Shimozono [Lam+14].

One reason that Schubert varieties have undergone such comprehensive examination is that they provide ready examples of algebraic varieties which are often singular. Many nice properties of algebraic varieties apply only to varieties that are smooth, while singular varieties may require more careful and specialized treatment. See [Har77, Chapter 1] for more on smooth varieties. Therefore, in the study of algebraic geometry, it is helpful to have examples of varieties, and knowledge of whether a particular variety is smooth or singular is essential.

In [LS90], Lakshmibai and Sandyha characterized exactly when a Schubert variety in the complete flag variety, $\mathcal{F}\ell(n)$, is smooth. As Schubert varieties in $\mathcal{F}\ell(n)$ can be indexed by permutations in S_n, their characterization is given via a permutation pattern avoidance criterion in the corresponding permutations. In particular, a Schubert variety X_w corresponding to a permutation w is smooth if and only if w avoids 1324 and 2143 as permutation patterns. We note that in much of the literature Schubert varieties and opposite Schubert varieties are indexed differently than how we index them here. We will spell out our notation more carefully in Chapter 2. In presenting this result, Lakshmibai and Sandyha united these prominent fields of permutation patterns and the study of Schubert varieties. Their result provided the first application of permutation patterns to the study of the geometry of Schubert varieties. Related work was also done by Ryan [Rya87], Wolper [Wol89], and Haiman [Hai92].

There have since been many results in the same spirit of the result of Lakshmibai and Sandyha. In [BP05, Thm 2.4], Billey and Postnikov extend this result by giving a pattern avoidance criterion for smoothness of Schubert varieties for all Weyl groups. Given a singular Schubert variety, X_w, [BW03, Thm 1], [KLR03, Thm 1.3], [Man01, Thm 2], and [Cor03] developed criteria based on permutation patterns to determine the singular points of X_w. See [BL00] for additional details on singularities of Schubert varieties and [AB16] for a nice survey on patterns in relation to Schubert calculus and geometry.

Positroids are an important family of realizable matroids originally defined by Postnikov in [Pos06] in the context of the totally nonnegative part of the Grassmannian variety. These matroids and the totally positive part of the Grassmannian variety have played a critical role in the theory of cluster algebras and soliton solutions to the KP equations and have connections to statistical physics, integrable systems, and scattering amplitudes [AH+16; BGY06; FWZ22; Lus98; Rie06; Wil07; Wil]. Positroids are closed under restriction, contraction, duality, and cyclic shift of the ground set, and furthermore they have particularly elegant matroid polytopes [ARW16].

Positroid varieties were studied by Knutson, Lam, and Speyer in [KLS13], building on the work of Lusztig, Postnikov and Rietsch [Lus98; Pos06; Rie98; Rie06]. They are homogeneous subvarieties of the complex Grassmannian variety $Gr(k,n)$ which are defined by determinantal equations determined by the nonbases of a positroid. They can also be described as projections of Richardson varieties in the complete flag manifold to $Gr(k,n)$. These varieties have beautiful geometric, representation theory, and combinatorial connections [KLS14; Paw18]. See the background section for notation and further background.

In [Pos06], Postnikov shows that positroids are in bijection with *decorated permutations*, which generalize ordinary permutations. Hence, in analogy with the indexing of Schubert varieties by permutations, positroid varieties can be indexed by decorated permutations. When studying the partially asymmetric exclusion process and its surprising connection to the Grassmannian, Sylvie Corteel posed the idea of considering patterns in decorated permutations [Cor07]. This suggestion was the foundation for the present work. Specifically, we present pattern avoidance criteria for smoothness of positroid varieties which parallel the pattern avoidance criterion for smoothness of Schubert varieties. We also extend our criterion for decorated permutations to criteria on Grassmann intervals, matroid Johnson graphs, and positroids, all of which are used as indexing sets for positroid varieties. The full list of criteria is given in Theorem 1.2.5.

The majority of this thesis is devoted to our results on positroid varieties. In particular, our pattern avoidance criteria for positroid varieties are achieved by considering patterns as arrangements of arcs in chord diagrams of the associated decorated permutations. As a separate consideration of patterns in decorated permutations, we provide an alternative one-line representation of decorated permutations. Using this representation, we describe a version of pattern avoidance for decorated permutations that more closely resembles classical permutation pattern avoidance. We provide enumerative results for this version of decorated permutation pattern avoidance in Chapter 7.

1.1 Background on Positroid Varieties

Positroids and positroid varieties can be bijectively associated with many different combinatorial objects [Oh11; Pos06]. For the purposes of this thesis, we will need to use the bijections associating each of the following types of objects with each other:

1. positroids $\mathcal{M}$ of rank k on a ground set of size n,
2. decorated permutations w^{Ω} on n elements with k anti-exceedances,
3. Grassmann necklaces $\mathcal{N} = (I_1, \dots, I_n) \in \binom{[n]}{k}^n$, and
4. Grassmann intervals $[u, v]$ in $Gi(k, n)$.

Here, a decorated permutation w^{Ω} on n elements is a permutation $w \in S_n$ together with an orientation clockwise or counterclockwise, denoted $\overrightarrow{i}$ or $\overleftarrow{i}$ respectively, assigned to each fixed point of w. A *Grassmann interval* $[u, v] \in Gi(k, n)$ is an interval in Bruhat order on permutations in S_n such that v has at most one descent, specifically in position k. In Chapter 2, we will sketch the relevant bijections and remaining terminology. In addition to these, there are bijections to juggling sequences, Ⅎ-diagrams, equivalence classes of plabic graphs, and bounded affine permutations [ARW16; KLS13; Pos06].

Many of the properties of positroid varieties can be "read off" from one or more of these bijectively equivalent definitions. Thus, we will index a positroid variety

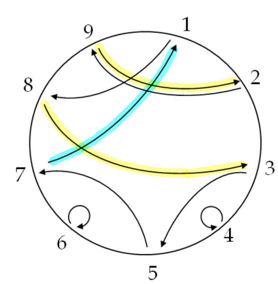
$$\Pi_{\mathcal{M}} = \Pi_{w^{\Omega}} = \Pi_{\mathcal{N}} = \Pi_{[u,v]} \tag{1.1.1}$$

using any of the associated objects, depending on the relevant context. For example, the codimension is easy to read off from the decorated permutation using the notions of the chord diagram and its alignments.

Let $S^{\circlearrowleft}_{n,k}$ be the set of decorated permutations on n elements with k anti-exceedances. The *chord diagram* $D(w^{\circlearrowleft})$ of $w^{\circlearrowleft} \in S^{\circlearrowleft}_{n,k}$ is constructed by placing the numbers $1, 2, \ldots, n$ on n vertices around a circle in clockwise order, and then, for each i, drawing a directed arc from i to $w(i)$ with a minimal number of crossings between distinct arcs while staying completely inside the circle. The arcs beginning at fixed points should be drawn clockwise or counterclockwise according to their orientation in $w^{\circlearrowleft}$.

An *alignment* in $D(w^{\circlearrowleft})$ is a pair of directed edges $(i \mapsto w(i), j \mapsto w(j))$ which can be drawn as distinct noncrossing arcs oriented in the same direction. A pair of directed edges $(i \mapsto w(i), j \mapsto w(j))$ which can be drawn as distinct noncrossing arcs oriented in opposite directions is called a *misalignment*. A pair of directed edges which must cross if both are drawn inside the cycle is called a *crossing* [Pos06, Sect. 5]. Let *Alignments*$(w^{\circlearrowleft})$ denote the set of alignments of $w^{\circlearrowleft}$.

Example 1.1.1. Let $w^{\circlearrowleft} = 895\overleftarrow{4}7\overrightarrow{6}132$ be the decorated permutation with a counterclockwise fixed point at 4 and a clockwise fixed point at 6. The chord diagram for $w^{\circlearrowleft}$ is the following.



Here for example, $(9 \mapsto 2, 8 \mapsto 3)$ highlighted in yellow is an alignment, $(2 \mapsto 9, 8 \mapsto 3)$ is a misalignment, and both $(7 \mapsto 1, 8 \mapsto 3)$ and $(7 \mapsto 1, 5 \mapsto 7)$ are crossings. Note, $(7 \mapsto 1, 6 \mapsto 6)$ is an alignment and $(7 \mapsto 1, 4 \mapsto 4)$ is a misalignment.

Theorem 1.1.2. *[KLS13; Pos06] For any decorated permutation $w^{\circlearrowleft} \in S^{\circlearrowleft}_{n,k}$ and associated Grassmann interval $[u, v]$, the codimension of $\Pi_{w^{\circlearrowleft}}$ in $Gr(k, n)$ is*

$$\operatorname{codim}(\Pi_{w^{\circlearrowleft}}) = \#\mathit{Alignments}(w^{\circlearrowleft}) = k(n-k) - [\ell(v) - \ell(u)]. \tag{1.1.2}$$

We use the explicit equations defining a positroid variety in $Gr(k,n)$ to determine whether the variety is smooth or singular. In general, a variety X defined by polynomials $f_1,\ldots,f_s$ is *singular* if there exists a point $x \in X$ such that the Jacobian matrix, Jac, of partial derivatives of the f_i satisfies $\operatorname{rank}(Jac|_x) < \operatorname{codim} X$. It is *smooth* if no such point exists. The value $\operatorname{rank}(Jac|_x)$ is the codimension of the tangent space to X at the point x. Thus, $\operatorname{rank}(Jac|_x) < \operatorname{codim} X$ implies the dimension of the tangent space to the variety X at x is strictly larger than the dimension of the variety X, hence x is a singularity like a cusp on a curve. In the case of a positroid variety $\Pi_{w^{\circlearrowleft}}$, Theorem 1.1.2 implies that a point $x \in \Pi_{w^{\circlearrowleft}}$ is a singularity of $\Pi_{w^{\circlearrowleft}}$ if

$$\operatorname{rank}(Jac|_x) \;<\; \operatorname{codim} \Pi_{w^{\circlearrowleft}} \;=\; \#Alignments(w^{\circlearrowleft}). \tag{1.1.3}$$

1.2 Main Results

Our first main theorem reduces the problem of finding singular points in a positroid variety to checking the rank of the Jacobian only at a finite number of T-fixed points. For any $J = \{j_1,\ldots,j_k\} \subseteq [n]$, let A_J be the element in $Gr(k,n)$ spanned by the elementary row vectors e_i with $i \in J$, or equivalently the subspace represented by a $k \times n$ matrix with a 1 in cell (i, j_i) for each $j_i \in J$ and zeros everywhere else. These are the T-fixed points of $Gr(k,n)$, where T is the set of invertible diagonal matrices over $\mathbb{C}$. The reduction follows from the decomposition of $\Pi_{[u,v]}$ as a projected Richardson variety. Every point $A \in \Pi_{[u,v]}$ lies in the projection of some intersection of a Schubert cell with an opposite Schubert variety $C_y \cap X^v$ for $y \in [u,v]$. In particular, if $y = y_1 y_2 \cdots y_n \in [u,v]$ in one-line notation and we define $y[k] := \{y_1, y_2, \ldots, y_k\}$ to be an *initial set* of y, then $A_{y[k]}$ is in the projection of $C_y \cap X^v$.

Theorem 1.2.1. *Assume $A \in \Pi_{[u,v]}$ is the image of a point in $C_y \cap X^v$ projected to $Gr(k,n)$ for some $y \in [u,v]$. Then the codimension of the tangent space to $\Pi_{[u,v]}$ at A is bounded below by* $\operatorname{rank}(Jac|_{A_{y[k]}})$.

Theorem 1.2.1 indicates that the T-fixed points of the form $A_{y[k]}$ such that $y \in [u,v]$ are key to understanding the singularities of $\Pi_{[u,v]}$. In fact, the equations determining $\Pi_{[u,v]}$ and the bases of the positroid $\mathcal{M}$ associated to the interval $[u,v]$ can be determined from the permutations in the interval by the following theorem. Our proof of the following theorem depends on Knutson-Lam-Speyer's Theorem/Definition 2.4.7 of a positroid variety as the projection of a Richardson variety and Theorem 2.1.6. It also follows from [KW15, Lemma 3.11]. Both groups, Knutson-Lam-Speyer and Kodama-Williams, were aware of this result in the context of positroid varieties for Grassmannians, but it does not appear to be in the literature in the form we needed, hence we prove the result in Chapter 3.

Theorem 1.2.2. Let $w^{\Omega} \in S^{\Omega}_{n,k}$ with associated Grassmann interval $[u,v]$ and positroid $\mathcal{M}$. Then $\mathcal{M}$ is exactly the collection of initial sets of permutations in the Grassmann interval $[u,v]$,

$$\mathcal{M} = \{y[k] \, : \, y \in [u,v]\}.$$

Our next theorem provides a method to compute the rank of the Jacobian of $\Pi_{[u,v]}$ explicitly at the T-fixed points. Therefore, we can also compute the dimension of the tangent space of a positroid variety at those points. Comparing that with the number of alignments gives a test for singularity of points in positroid varieties by (1.1.3).

Theorem 1.2.3. *Let $w^{\Omega} \in S^{\Omega}_{n,k}$ with associated Grassmann interval $[u,v]$ and positroid $\mathcal{M}$. For $y \in [u,v]$, the codimension of the tangent space to $\Pi_{[u,v]} \subseteq Gr(k,n)$ at $A_{y[k]}$ is*

$$\mathrm{rank}(Jac|_{A_{y[k]}}) \;=\; \#\left\{I \in \binom{[n]}{k} \setminus \mathcal{M} \, : \, |I \cap y[k]| = k-1\right\}. \tag{1.2.1}$$

The formula in (1.2.1) is reminiscent of the *Johnson graph* $J(k,n)$ with vertices given by the k-subsets of $[n]$ such that two k-subsets I,J are connected by an edge precisely if $|I \cap J| = k-1$. For a positroid $\mathcal{M} \subseteq \binom{[n]}{k}$, let $J(\mathcal{M})$ denote the induced subgraph of the Johnson graph on the vertices in $\mathcal{M}$. We call $J(\mathcal{M})$ the *Johnson graph of* $\mathcal{M}$. Note, the Johnson graph is closely related to bases of matroids by the Basis Exchange Property. Theorem 1.2.3 implies $J(\mathcal{M})$ encodes aspects of the geometry of the positroid varieties like the Bruhat graph in the theory of Schubert varieties [Car94].

To state our main theorem characterizing smoothness of positroid varieties, we need to define two types of patterns that may occur in a chord diagram. First, given an alignment $(i \mapsto w(i), j \mapsto w(j))$ in $D(w^{\Omega})$, if there exists a third arc $(h \mapsto w(h))$ which forms a crossing with both $(i \mapsto w(i))$ and $(j \mapsto w(j))$, we say $(i \mapsto w(i), j \mapsto w(j))$ is a *crossed alignment* of w^{Ω}. In the example above, $(9 \mapsto 2, 8 \mapsto 3)$ is a crossed alignment; this alignment is crossed for instance by $(7 \mapsto 1)$, highlighted in blue.

The second type of pattern is related to the Spirograph™ toy, designed by Denys Fisher and trademarked by Hasbro to draw a variety of curves inside a circle which meet the circle in a finite number of discrete points. See Figure 1.1. Once oriented and vertices are added, such curves each determine a chord diagram from a special class, which we will call spirographs. We think of alignments, crossings, crossed alignments, and spirographs as *subgraph patterns* for decorated permutations.

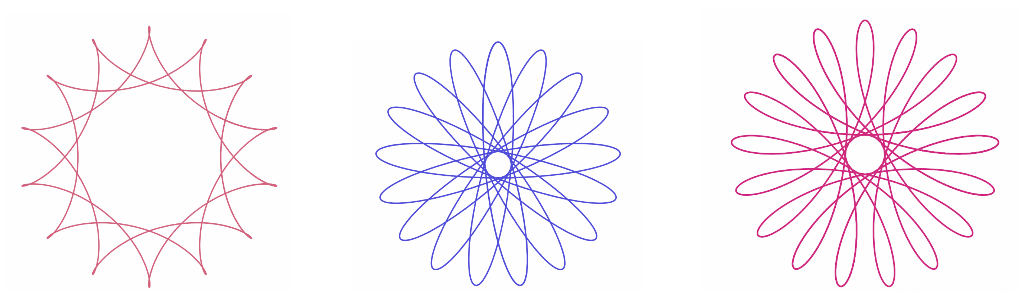


Figure 1.1: Spirographs made by the Spirograph Maker app for the iphone.

Definition 1.2.4. A decorated permutation $w^{\Omega} = (w, \mathrm{co}) \in S_{n,k}^{\Omega}$ is a *spirograph permutation* if there exists a positive integer m such that $w(i) = i + m \pmod n$ for all i and w^{Ω} has at most one fixed point. The chord diagram of a spirograph permutation will be called a *spirograph.*

Theorem 1.2.5. *Let $w^{\Omega} \in S_{n,k}^{\Omega}$ with associated Grassmann interval $[u, v]$ and positroid $\mathcal{M} = \{y[k] : y \in [u, v]\}$. Then, the following are equivalent.*

1. *The positroid variety $\Pi_{w^{\Omega}} = \Pi_{[u,v]} = \Pi_{\mathcal{M}}$ is smooth.*

2. *For every* $y \in [u,v]$, $\#\{I \in \mathcal{M} : |I \cap y[k]| = k-1\} = \ell(v) - \ell(u)$.

3. *For every* $J \in \mathcal{M}$, $\#\{I \in \mathcal{M} : |I \cap J| = k-1\} = k(n-k) - \#Alignments(w^{\Omega})$.

4. *The graph* $J(\mathcal{M})$ *is regular, and each vertex has degree* $\ell(v) - \ell(u)$.

5. *The decorated permutation* w^{Ω} *has no crossed alignments.*

6. *The chord diagram* $D(w^{\Omega})$ *is a disjoint union of spirographs.*

7. *The positroid* $\mathcal{M}$ *is a direct sum of uniform matroids.*

In Chapter 2, we provide background material from the literature and define our notation. In Chapter 3, we study initial sets for Grassmann intervals and prove Theorem 1.2.2 as well as the equalities (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) in Theorem 1.2.5.

In Chapter 4, we prove several reduction steps that enable us to focus our study on a particular class of decorated permutations and associated positroid varieties. This is done first by proving Theorem 1.2.1 and Theorem 1.2.3 in Section 4.1. Then, Theorem 1.2.1 and Theorem 1.2.3 together yield the equality (1) $\Leftrightarrow$ (2) in Theorem 1.2.5. In Section 4.2, we consider several rigid transformations of chord diagrams and the associated transformations of the related objects. Furthermore, we show in Lemma 4.2.6 that applying these transformations to the objects indexing positroid varieties preserves the property of being smooth. In Section 4.3, we utilize a decomposition of $\mathcal{M}$ into connected components on a non-crossing partition and show that a positroid variety is smooth if and only if all of the components of the decomposition correspond to smooth positroid varieties.

In Chapter 5, we connect crossed alignments and spirographs in chord diagrams to the study of singular and smooth positroid varieties. In Section 5.1, we identify the special properties of positroid varieties indexed by spirograph permutations, which leads to the implication (6) $\Rightarrow$ (1) and the equalities (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) in Theorem 1.2.5. In Section 5.2, we complete the proof of Theorem 1.2.5 by showing that (1) $\Rightarrow$ (5). This implication is accomplished by constructing an injective map from the *anti-exchange pairs for* a particular set $J \in \mathcal{M}$ to $Alignments(w^{\circlearrowleft})$.

The last two chapters are devoted to enumerative results. In Chapter 6, we provide some enumerative results for smooth positroid varieties. This includes relevant results due to Christian Krattenthaler, which are found in Section 6.3. We conclude in Chapter 7 by presenting an alternate version of pattern avoidance for decorated permutations and corresponding enumerations.

Chapter 2
BACKGROUND

We begin by giving notation and some background on several combinatorial objects and theorems from the literature. These objects will be used to index the varieties discussed throughout the thesis. We will then introduce notation for several geometrical objects, including Grassmannian varieties, flag varieties, Schubert varieties, Richardson varieties, and positroid varieties.

2.1 *Combinatorial objects*

2.1.1 Subsets and Partitions

For integers $i \leq j$, let $[i,j]$ denote the set $\{i, i+1, \ldots, j\}$, and write $[n] := [1,n]$ for a positive integer n. Let $\binom{[n]}{k}$ be the set of size k subsets of $[n]$ for $k \in [0,n]$. Call $J \in \binom{[n]}{k}$ a k-subset of $[n]$.

Define the *Gale partial order*, $\leq$, on k-subsets of $[n]$ as follows. Let $I = \{i_1 < \cdots < i_k\}$ and $J = \{j_1 < \cdots < j_k\}$. Then $I \leq J$ if and only if $i_h \leq j_h$ for all $h \in [k]$. This partial order is known by many other names; we are following [ARW16] for consistency. Gale studied this partial order in the context of matroids in the 1960s [Gal68].

For any $k \times n$ matrix A and any set $J \in \binom{[n]}{k}$, define $\Delta_J(A)$ to be the determinant of the $k \times k$ submatrix of A lying in column set J. The minors $\Delta_I(A)$ for $I \in \binom{[n]}{k}$ are called the *Plücker coordinate* of A. We think of Δ_J as a polynomial function on the set of all $k \times n$ matrices over a chosen field using variables of the form x_{ij} indexed by row $i \in [k]$ and column $j \in J$.

Definition 2.1.1. Let S be a partition $[n] = B_1 \sqcup \cdots \sqcup B_t$ of $[n]$ into pairwise disjoint, nonempty subsets. We say that S is a *non-crossing partition* if there are no distinct a, b, c, d

in cyclic order such that $a, c \in B_i$ and $b, d \in B_j$ for some $i \neq j$. Equivalently, place the numbers $1, 2, \ldots, n$ on n vertices around a circle in clockwise order, and then for each B_i, draw a polygon on the corresponding vertices. If no two of these polygons intersect, then S is a non-crossing partition of $[n]$.

2.1.2 Matroids

A *matroid* of rank k on $[n]$, defined by its bases, is a nonempty subset $\mathcal{M} \subseteq \binom{[n]}{k}$ satisfying the following Basis Exchange Property: if $I, J \in \mathcal{M}$ such that $I \neq J$ and $a \in I \setminus J$, then there exists some $b \in J \setminus I$ such that $(I \setminus \{a\}) \cup \{b\} \in \mathcal{M}$. Compare the notion of matroid basis exchange to basis exchange in linear algebra. We call the sets in $\binom{[n]}{k} \setminus \mathcal{M}$ the *nonbases* of $\mathcal{M}$, and we denote this collection of sets by $\mathcal{Q}(\mathcal{M})$. The set $\binom{[n]}{k}$ is the *uniform matroid* of rank k on $[n]$. For more on matroids, see [Ard15; Oxl11].

Example 2.1.2. A notable family of matroids called *representable matroids* comes from matrices. Let A be a full rank $k \times n$ matrix. The *matroid of* A is the set

$$\mathcal{M}_A := \left\{ J \in \binom{[n]}{k} \;:\; \Delta_J(A) \neq 0 \right\}.$$

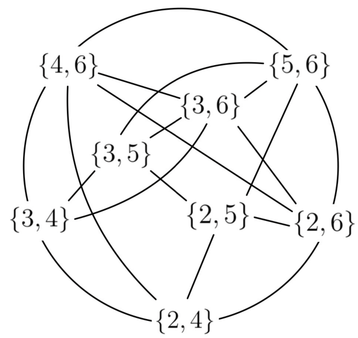
The matroid of

$$A = \begin{bmatrix} 0 & 3 & 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{bmatrix}$$

is

$$\mathcal{M}_A = \{\{2,4\}, \{2,5\}, \{2,6\}, \{3,4\}, \{3,5\}, \{3,6\}, \{4,6\}, \{5,6\}\} \subseteq \binom{[6]}{2}.$$

Recall, the *Johnson graph* $J(k,n)$ with vertices given by the k-subsets of $[n]$ such that two k-subsets I, J are connected by an edge precisely if $|I \cap J| = k - 1$. For a matroid $\mathcal{M} \subseteq \binom{[n]}{k}$, $J(\mathcal{M})$ is the induced subgraph of the Johnson graph on the vertices in $\mathcal{M}$. The Basis Exchange Property for $\mathcal{M}$ implies that $J(\mathcal{M})$ is connected, and furthermore, between any two vertices I, J in $J(\mathcal{M})$, there exists a path in $J(\mathcal{M})$ which is a minimal length path between I and J in $J(k,n)$. The Johnson graph $J(\mathcal{M}_A)$ from Example 2.1.2 is



The direct sum of two matroids on disjoint ground sets, denoted $M_1 \oplus M_2$, is the matroid with bases given by $\{I \cup J \,:\, I \in M_1, J \in M_2\}$ on the ground set which is the disjoint union of the ground sets of the matroids M_1 and M_2. A matroid M on ground set $[n]$ is *connected* if M cannot be expressed as the direct sum of two matroids. Every matroid can be decomposed into the direct sum of its connected components.

Given a matroid $\mathcal{M}$ on ground set $[n]$, the *dual matroid* M^* is the matroid on $[n]$ with bases $\{[n] \setminus I \subseteq [n] : I \in \mathcal{M}\}$. If $\mathcal{M}$ has rank k, then its dual matroid has rank $n-k$.

2.1.3 Permutations

Let S_n be the set of permutations of $[n]$, where we think of a permutation as a bijection from a set to itself. For $w \in S_n$, let $w_i = w(i)$, and write w in *one-line notation* as $w = w_1 w_2 \cdots w_n$. For an interval of indices $[i,j]$, we denote the image of $[i,j]$ under w by $w[i,j] = \{w_i, w_{i+1}, \ldots, w_j\}$. The permutation matrix M_w of w is the $n \times n$ matrix that has a 1 in cell (i, w_i) for each $i \in [n]$ and zeros elsewhere. The *length of* $w \in S_n$ is

$$\ell(w) := \#\{(i,j) \,:\, i < j \text{ and } w(i) > w(j)\}.$$

The permutation in S_n of maximal length is $w_0 := n(n-1)\cdots 21$.

Example 2.1.3. For $w = 3124$, the length of $w = 3124$ is $\ell(w) = 2$, and M_w is the matrix

$$M_{3124} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that the permutation matrix of w^{-1} is M_w^T. Furthermore, permutation multiplication is given by function composition so that if $wv = u$, then $w(v(i)) = u(i)$. By this definition, $M_w^T M_v^T = M_u^T$.

Definition 2.1.4. [BB05a, Chapter 2] For u, v in S_n, $u \leq v$ in *Bruhat order* if $u[i] \leq v[i]$ for all $i \in [n]$. Equivalently, Bruhat order is the ranked poset defined as the transitive closure of the relation $u < ut_{ij}$ whenever $u_i < u_j$, where t_{ij} is the permutation transposing i and j and fixing all other values. For each $u \leq v$ in Bruhat order, the *interval* $[u, v]$ is defined to be

$$[u, v] := \{y \in S_n \ : \ u \leq y \leq v\}. \tag{2.1.1}$$

For $0 \leq k \leq n$, write $S_k \times S_{n-k}$ for the subgroup of S_n consisting of permutations that send $[k]$ to $[k]$ and $[k+1, n]$ to $[k+1, n]$. If $k = 0$, consider $[k]$ to be the empty set. A permutation $w \in S_n$ is k-*Grassmannian* if $w_1 < \cdots < w_k$ and $w_{k+1} < \cdots < w_n$. This is equivalent to saying that w is the minimal length element of its coset $w \cdot (S_k \times S_{n-k})$. For example, the permutation $w = 3124$ is 1-Grassmannian. The set of *Grassmannian permutations* in S_n is the union over $k \in [0, n]$ of all k-Grassmannian permutations.

Definition 2.1.5. Assume $u \leq v$ in Bruhat order on S_n. Then, the interval $[u, v]$ is a *Grassmann interval* provided v is a k-Grassmannian permutation for some $k \in [0, n]$. Denote by $Gi(k, n)$ the set of all Grassmann intervals $[u, v]$ in S_n, where v is k-Grassmannian.

The Grassmann intervals $[u, v]$ are key objects for this thesis. Note that u need not be a Grassmannian permutation. In the case where v is Grassmannian, there is a simpler criterion for Bruhat order that follows closely from work of Bergeron-Sottile [BS98, Theorem A]. This criterion also appears in [Pos06, Lemma 20.2]. We will use this criterion extensively.

Theorem 2.1.6. *Let $u, v \in S_n$, where v is k-Grassmannian. Then $u \leq v$ if and only if*

(i) for every $1 \leq j \leq k$, we have $u(j) \leq v(j)$, and

(ii) for every $k < m \leq n$, we have $u(m) \geq v(m)$.

2.1.4 Classical Permutation Pattern Avoidance and Sequences

The notion of permutation patterns has been a well studied topic since the 1960s and has been hugely popular since the 1990s, [Bón16; Bón17]. Of particular importance has been pattern containment and avoidance.

Let $L = \{l_1 < l_2 < \cdots < l_m\} \subseteq [n]$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n$. Denote by x_L the restriction of x to the coordinates indexed by L, $x_L := (x_{l_1}, x_{l_2}, \ldots, x_{l_m}) \in \mathbb{C}^m$. Identify a permutation $w = w_1 w_2 \cdots w_n \in S_n$ with the n-tuple $(w_1, w_2, \ldots, w_n)$ so that $w_L = (w_{l_1}, w_{l_2}, \ldots, w_{l_m})$. For a sequence of distinct integers $x = (x_1, x_2, \ldots, x_m)$, let $\mathrm{fl}(x)$ be the unique permutation in S_m whose entries occur in the same relative order as those of x. We will refer to this as *flattening w on the set L.*

Example 2.1.7. Let $w = 31824756$ and $L = \{1, 3, 4, 7\}$. Then $w_L = (3, 8, 2, 5)$, and the corresponding permutation in S_4 is $\mathrm{fl}(w_L) = \mathrm{fl}(3, 8, 2, 5) = 2413$.

For $v \in S_m$ and $w \in S_n$, where $m \leq n$, we say that w *contains* v if there is some m-subset L of $[n]$ such that $\mathrm{fl}(w_L) = v$. If there is no such subset L, we say that w *avoids* v. Denote the set of all $w \in S_n$ that avoid v by $S_n(v)$. The set S_m may be partitioned into *Wilf equivalence classes* according to the enumerations for $\#S_n(v)$ for $v \in S_m$. Here u and v in S_m are in the same class if $\#S_n(u) = \#S_n(v)$ for all n.

Consider, for example, the case $m = 3$. A well known fact in pattern avoidance is that there is only one Wilf equivalence class of permutations in S_3. In particular, Knuth showed that for any $v \in S_3$ and any $n \geq 0$, $\#S_n(v) = C_n$ [Knu73], where C_n is the nth *Catalan number*. The sequence of Catalan numbers is defined by setting $C_0 = 1$ and letting

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$$

for $n > 0$. More on Catalan numbers and the many objects they count can be found in [Sta15].

Since Knuth's enumeration for $S_n(v)$ for $v \in S_3$ was given, enumerations and related results for $v \in S_m$ for $m \geq 4$ have proliferated. For $m = 4$, there are three Wilf equivalence classes of permutations. Enumerations for $\#S_n(v)$ have been found for two of these equivalence classes. The enumeration for the third class is an open problem.

For a sequence of natural numbers, it is of interest to study the asymptotic growth of the sequence. In particular, for running an algorithm which is iterated over a sequence, the asymptotic growth of the sequence is vital to determining the time complexity of the algorithm. In Knuth-Theta, Knuth discusses several notations for expressing asymptotic growth. One of these is the Θ notation, as defined below. See [Wik20] for more information on related notation.

Definition 2.1.8. Let g and f be functions from the set of natural numbers to itself. The function f is said to be $\Theta(g)$, if there are constants $c_1, c_2 > 0$ and a natural number n_0 such that $c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$ for all $n \geq n_0$.

2.2 Grassmannian, Flag, and Richardson Varieties

Schubert varieties in both the flag variety and the Grassmannian are a well studied class of varieties that have found importance in algebraic geometry, representation theory, and combinatorics. Singularities of Schubert varieties, in particular, have been investigated thoroughly. See [BL00; Ful97; Kum02] for further background on these varieties.

For $0 \leq k \leq n$, the points in the *Grassmannian variety*, $Gr(k,n)$, are the k-dimensional subspaces of $\mathbb{C}^n$. Up to left multiplication by a matrix in GL_k, we may represent $V \in Gr(k,n)$ by a full rank $k \times n$ matrix A_V such that V is the row span of A_V. Let Mat_{kn} be the set of full rank $k \times n$ matrices. The points in $Gr(k,n)$ can be bijectively identified with the cosets $\mathrm{GL}_k \setminus \mathrm{Mat}_{kn}$. The Grassmannian varieties are smooth manifolds via the Plücker coordinate embedding of $Gr(k,n)$ into projective space. This includes the case when $k = n = 0$, in which case $Gr(k,n)$ consists of one point, which is the 0-dimensional vector space in $\mathbb{C}^0$.

Let $\mathcal{F}\ell(n)$ be the *complete flag variety* of nested subspaces of $\mathbb{C}^n$. A complete flag $V_\bullet = (0 \subset V_1 \subset \cdots \subset V_n)$ can be represented as an invertible $n \times n$ matrix where the row span of the first k rows is the subspace V_k in the flag. Throughout the thesis, we will often identify a full rank matrix with the point it represents in $Gr(k,n)$ or $\mathcal{F}\ell(n)$. For a subset $J \subseteq [n]$, let $\mathrm{Proj}_J : \mathbb{C}^n \to \mathbb{C}^{|J|}$ be the projection map onto the indices specified by J. Then, for every permutation $w \in S_n$, there is a *Schubert cell* C_w and an *opposite Schubert cell* C^w in $\mathcal{F}\ell(n)$ defined by

$$C_w = \{V_\bullet \in \mathcal{F}\ell(n) \;:\; \dim(\mathrm{Proj}_{[j]}(V_i)) = |w[i] \cap [j]| \text{ for all } i,j\},$$
$$C^w = \{V_\bullet \in \mathcal{F}\ell(n) \;:\; \dim(\mathrm{Proj}_{[n-j+1,n]}(V_i)) = |w[i] \cap [n-j+1,n]| \text{ for all } i,j\}.$$

By row elimination and rescaling, we can find a *canonical matrix representative* $A_{V_\bullet} := (a_{i,j})$ for each $V_\bullet \in C_w$ such that $a_{i,w_i} = 1$ for $i \in [n]$, $a_{i,j} = 0$ for all $1 \le j < w_i$, and $a_{h,w_i} = 0$ for $h > i$. Canonical matrices for C^w can be found similarly, but so that every leading 1 has all zeros to the right instead of the left. For example, the canonical matrices for C_{3124} have the form below, where entries labeled $*$ can be replaced by any element of $\mathbb{C}$.

$$\begin{bmatrix} 0 & 0 & 1 & * \\ 1 & * & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The *Schubert variety* X_w is the closure of C_w in the Zariski topology on $\mathcal{F}\ell(n)$, and similarly, the *opposite Schubert variety* X^w is the closure of C^w. Specifically, X_w and X^w can be defined by

$$X_w = \{V_\bullet \in \mathcal{F}\ell(n) \;:\; \dim(\mathrm{Proj}_{[j]}(V_i)) \le |w[i] \cap [j]| \text{ for all } i,j\},$$
$$X^w = \{V_\bullet \in \mathcal{F}\ell(n) \;:\; \dim(\mathrm{Proj}_{[n-j+1,n]}(V_i)) \le |w[i] \cap [n-j+1,n]| \text{ for all } i,j\}.$$

Bruhat order determines which Schubert cells are in a Schubert variety,

$$X_w = \bigsqcup_{y \ge w} C_y \quad \text{and} \quad X^w = \bigsqcup_{v \le w} C^v. \tag{2.2.1}$$

Schubert varieties in the Grassmannian manifold $Gr(k,n)$ are indexed by the sets in $\binom{[n]}{k}$ and can be defined as the projections of Schubert varieties in $\mathcal{F}\ell(n)$. Let

$$\pi_k : \mathcal{F}\ell(n) \to Gr(k,n) \tag{2.2.2}$$

be the projection map which sends a flag $V_\bullet = (0 \subset V_1 \subset \cdots \subset V_n)$ to the k-dimensional subspace V_k. Identifying a full rank $n \times n$ matrix M with the point it represents in $\mathcal{F}\ell(n)$, then $\pi_k(M)$ denotes the span of the top k rows of M. For each $J \in \binom{[n]}{k}$, there exists a k-Grassmannian permutation v such that $J = v[k] \coloneqq \{v_1, v_2, \ldots, v_k\}$. The *Grassmannian Schubert variety* indexed by J is the projection $\pi_k(X_v)$.

For permutations u and v in S_n, with $u \leq v$, the *Richardson variety* is a nonempty variety in $\mathcal{F}\ell(n)$ and is defined as the intersection $X_u^v \coloneqq X_u \cap X^v$. Then $\dim X_u^v = \ell(v) - \ell(u)$. The decompositions of X_u and X^v into Schubert cells and opposite Schubert cells in (2.2.1) yield

$$X_u^v = \bigsqcup_{u \leq y \leq v} (C_y \cap X^v) = \Big(\bigsqcup_{y \geq u} C_y\Big) \cap \Big(\bigsqcup_{t \leq v} C^t\Big).$$

By Equation (2.2.1), one can observe that the permutation matrix $M_y \in (C_y \cap X^v) \subset X_u^v$ for each $y \in [u,v]$.

In [LS90], Lakshmibai and Sandhya prove that X_w is smooth if and only if w avoids 1324 and 2143 as permutation patterns. Related work was also done by Ryan [Rya87], Wolper [Wol89], and Haiman [Hai92]. In [BP05, Thm 2.4], Billey and Postnikov extend this result by giving a pattern avoidance criterion for smoothness of Schubert varieties for all Weyl groups. Given a singular Schubert variety, X_w, [BW03, Thm 1], [KLR03, Thm 1.3], [Man01, Thm 2], and [Cor03] developed criteria based on permutation patterns to determine the singular points of X_w. These theorems characterizing smooth versus singular points in Schubert varieties using permutation patterns motivated this work.

Singularities in Richardson varieties and their projections have also been studied in the literature [Bri05; BC12; KLS14; KL04]. The characterizations of smooth versus singular Richardson varieties described there are not based on pattern avoidance but rely on computations in the associated cohomology rings.

2.3 Decorated Permutations

A decorated permutation $w^{\circlearrowleft}$ is defined by a permutation w together with a circular orientation map called co from the fixed points of w to the set of clockwise or counterclockwise orientations, denoted by $\{\circlearrowright, \circlearrowleft\}$. Therefore, we will sometimes describe a decorated permutation as a pair (w, co).

Postnikov made the following definitions in [Pos06, Sect 16]. Given a decorated permutation $w^{\circlearrowleft}$, call $i \in [n]$ an *anti-exceedance* of $w^{\circlearrowleft} = (w, \mathrm{co})$ if $i < w^{-1}(i)$ or if $w^{\circlearrowleft}(i) = \overrightarrow{i}$ is a clockwise fixed point. If $i \in [n]$ is not an anti-exceedance, it is an *exceedance*. Let $I_1(w^{\circlearrowleft}) = I_1$ be the *set of anti-exceedances* of $w^{\circlearrowleft}$. For an arc $(i \mapsto w(i))$ in the chord diagram of $w^{\circlearrowleft}$, we say the arc is an *anti-exceedance arc* or *exceedance arc* depending on whether $w(i)$ is in $I_1(w^{\circlearrowleft})$ or not. Let $k(w^{\circlearrowleft}) := |I_1(w^{\circlearrowleft})|$. Recall that $S^{\circlearrowleft}_{n,k}$ is the set of decorated permutations with anti-exceedance set of size k.

More generally, let $<_r$ be the *shifted linear order* on $[n]$ given by $r <_r (r+1) <_r \cdots <_r n <_r 1 <_r \cdots <_r (r-1)$ for $r \in [n]$. The *shifted anti-exceedance set* $I_r(w^{\circlearrowleft})$ of $w^{\circlearrowleft}$ is the anti-exceedance set of $w^{\circlearrowleft}$ with respect to the shifted linear order $<_r$ on $[n]$,

$$I_r(w^{\circlearrowleft}) = \{i \in [n] \,:\, i <_r w^{-1}(i) \text{ or } w^{\circlearrowleft}(i) = \overrightarrow{i}\}.$$

An element $i \in I_r(w^{\circlearrowleft})$ is called an r*-anti-exceedance*, and an element $i \notin I_r(w^{\circlearrowleft})$ is called an r*-exceedance*. Note from the construction that either $I_{r+1}(w^{\circlearrowleft}) = I_r(w^{\circlearrowleft})$ or $I_{r+1}(w^{\circlearrowleft}) = I_r(w^{\circlearrowleft}) \setminus \{r\} \cup \{w(r)\}$, so $|I_1(w^{\circlearrowleft})| = \cdots = |I_n(w^{\circlearrowleft})| = k(w^{\circlearrowleft})$. Furthermore, $I_{r+1}(w^{\circlearrowleft}) = I_r(w^{\circlearrowleft})$ if and only if r is a fixed point of w, so clockwise fixed points will be in all of the shifted anti-exceedance sets and counterclockwise fixed points will be in none. Therefore, $w^{\circlearrowleft}$ is easily recovered from $(I_1(w^{\circlearrowleft}), \ldots, I_n(w^{\circlearrowleft}))$. The sequence of $k(w^{\circlearrowleft})$-subsets $(I_1(w^{\circlearrowleft}), \ldots, I_n(w^{\circlearrowleft}))$ is called the *Grassmann necklace* associated to $w^{\circlearrowleft}$.

Given a decorated permutation $w^{\circlearrowleft} = (w, \mathrm{co}) \in S^{\circlearrowleft}_{n,k}$ and its anti-exceedance set, we can also easily identify the Grassmann interval associated to it. The k-Grassmannian permutation v is determined by $v[k] = w^{-1}(I_1(w^{\circlearrowleft}))$, and then u is determined by $u = wv$, and hence $I_1(w^{\circlearrowleft}) = u[k]$ by this construction. Let $u(w^{\circlearrowleft}) = u$ and $v(w^{\circlearrowleft}) = v$. The interval $[u, v] \in Gi(k, n)$ is the *Grassmann interval* associated to $w^{\circlearrowleft} \in S_{n,k}^{\circlearrowleft}$. To identify a decorated permutation from a Grassmann interval $[u, v] \in Gi(k, n)$, simply let

$$w = uv^{-1} \quad \text{with} \quad co(j) := \begin{cases} \circlearrowleft & \text{if } j \in u[k] \\ \circlearrowright & \text{if } j \notin u[k] \end{cases}. \tag{2.3.1}$$

We visualize the bijection from decorated permutations to Grassmann intervals as a shuffling algorithm.

1. Write $w^{\circlearrowleft}$ in two-line notation with the numbers $1, 2, \ldots, n$ on the first row and $w(1)$, $w(2)$,..., $w(n)$ on the second row, including orientations on fixed points.

2. Highlight the columns $i_1 < i_2 < \cdots < i_k$ such that $i_j > w(i_j)$ or $w^{\circlearrowleft}(i_j) = \overrightarrow{i_j}$, corresponding with anti-exceedances.

3. Keeping the columns intact, reorder the columns so that the highlighted columns $i_1 < i_2 < \cdots < i_k$ come first followed by the non-highlighted columns maintaining their relative order within the highlighted and non-highlighted blocks. Then, drop any orientation arrows to obtain a $2 \times n$ array of positive integers $\left[\begin{smallmatrix} v \\ u \end{smallmatrix}\right]$ with the one-line notation for u determined by the lower row and the one-line notation for v determined by the upper row.

From this shuffling algorithm, note that v is a k-Grassmannian permutation by construction and $v[k] = w^{-1} \cdot I_1(w^{\circlearrowleft})$. Furthermore, $u \in S_n$ and its initial set $u[k] = I_1(w^{\circlearrowleft})$. Also observe that $u_i \leq v_i$ for all $1 \leq i \leq k$, and $u_i \geq v_i$ for all $k + 1 \leq i \leq n$. Hence, $u \leq v$ in Bruhat order by Theorem 2.1.6. Thus, $[u, v]$ is the Grassmann interval associated to $w^{\circlearrowleft}$, since the shuffling algorithm above is equivalent to the permutation multiplication $wv = u$.

Conversely, to go from $[u, v]$ to $w^{\circlearrowleft} = (w, \text{co})$, simply take the two line array $\left[\begin{smallmatrix} v \\ u \end{smallmatrix}\right]$, sort the columns by the top row to obtain w. Observe that j is a fixed point of w if and only if $u^{-1}(j) = v^{-1}(j)$. If j is a fixed point of w, then $\text{co}(j) = \circlearrowleft$ if $j \in u[k]$, and $co(j) = \circlearrowright$ otherwise.

Remark 2.3.1. Observe that the chord diagram is just as easily obtained from $[u,v]$ as it is from of w^{Ω}. The chord diagram arcs $\{(i \mapsto w(i)) : i \in [n]\}$ are determined by the columns in the two line notation of w^{Ω}, so $\{(i \mapsto w(i)) : i \in [n]\} = \{(v_j \mapsto u_j) : j \in [n]\}$ with fixed points oriented appropriately. Therefore, the graphical patterns determining properties for decorated permutations also determine patterns for Grassmann intervals.

Remark 2.3.2. Note that the map above between decorated permutations in $S^{\Omega}_{n,k}$ and Grassmann intervals in $Gi(k,n)$ follows the work of [KLS13]. In [Pos06, Sect. 20], a slightly different map is given. In particular, in Postnikov's work, the decorated permutation w^{Ω} obtained from a Grassmann interval $[u,v]$ is computed as $w = w_0 v u^{-1} w_0$, with a fixed point i assigned a clockwise orientation if $n+1-i$ is in $u[k]$. Here $w_0 = n \dots 21 \in S_n$. We will return to this involution on decorated permutations and associated objects in Remark 4.2.5. Our reason for prioritizing the Knutson-Lam-Speyer bijection is the direct connection to the corresponding positroid given in Theorem 1.2.2.

Example 2.3.3. For the decorated permutation $w^{\Omega} = 54127\overrightarrow{6}9\overleftarrow{8}3$, the anti-exceedance set is $I_1(w^{\Omega}) = \{1,2,6,3\}$. These values occur in positions $\{3,4,6,9\}$ in w^{Ω}. The Grassmann necklace for w^{Ω} is

$$(I_1, \dots, I_9) = (\{1236\}, \{2356\}, \{3456\}, \{1456\}, \{1256\}, \{1267\}, \{1267\}, \{1269\}, \{1269\}).$$

Write the two-line notation, highlight the columns corresponding to anti-exceedances, and shuffle the anti-exceedances to the front to identify the associated Grassmann interval $[u,v]$ $= [126354798, 346912578]$,

$$w^{\Omega} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 5 & 4 & 1 & 2 & 7 & \overrightarrow{6} & 9 & \overleftarrow{8} & 3 \end{bmatrix} \implies \begin{bmatrix} v \\ u \end{bmatrix} = \begin{bmatrix} 3 & 4 & 6 & 9 & 1 & 2 & 5 & 7 & 8 \\ 1 & 2 & 6 & 3 & 5 & 4 & 7 & 9 & 8 \end{bmatrix}.$$

Using Postnikov's map, the decorated permutation corresponding to the interval $[u,v] =$ $[126354798, 346912578]$ is $3\overleftarrow{2}5\overrightarrow{4}98167$. Postnikov's inverse map would associate the Grassmann interval $[416732598, 478912356]$ to the original $w^{\Omega} = 54127\overrightarrow{6}9\overleftarrow{8}3$.

A key artifact of a decorated permutation in our work is its set of alignments. To formally define alignments, we first establish the following notation for a cyclic interval of elements in $[n]$ for a fixed integer n.

Definition 2.3.4. Let $a, b \in [n]$. Then

$$[a,b]^{cyc} := \begin{cases} [a,b] & \text{if } a \leq b \\ [a,n] \cup [1,b] & \text{if } a > b \end{cases}. \tag{2.3.2}$$

Definition 2.3.5. An *alignment* of $w^{\circ} = (w, co) \in S_n^{\circ}$ is a pair of arcs $(p \mapsto w(p))$ and $(s \mapsto w(s))$ in $D(w^{\circ})$ which can be drawn as distinct noncrossing arcs such that

1. both $w(p) \in [p, w(s)-1]^{cyc}$ and $w(s) \in [w(p)+1, s]^{cyc}$,
2. if $w(s) = s$, then $co(s) = \circlearrowright$, and
3. if $w(p) = p$, then $co(p) = \circlearrowleft$.

In this case, we denote the alignment by $(p \mapsto w(p), s \mapsto w(s))$ or $A(p,s)$ if w° is understood from context. We say the arc $(p \mapsto w(p))$ is the *port side* and the arc $(s \mapsto w(s))$ is the *starboard side* of the alignment. See Figure 2.1. We use $Alignments(w^{\circ})$ to denote the set of all alignments of w°.

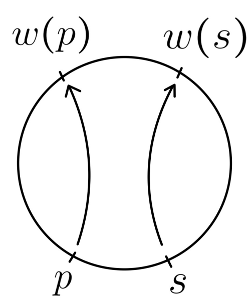


Figure 2.1: Alignment with port side $(p \mapsto w(p))$ and starboard side $(s \mapsto w(s))$.

Example 2.3.6. Recall the chord diagram in Example 1.1.1 for $w^{\circlearrowleft} = 895\overleftarrow{4}7\overrightarrow{6}132$, the alignment $(9 \mapsto 2, 8 \mapsto 3)$ highlighted in yellow has port side $(9 \mapsto 2)$ and starboard side $(8 \mapsto 3)$ as if they were two sides of a boat with its bow pointing to the right. Furthermore, $w^{\circlearrowleft}$ has 13 alignments,

$$\begin{aligned} Alignments(w^{\circlearrowleft}) = \{&A(3,1), A(3,2), A(3,6), A(4,1), A(4,2), A(4,3), A(4,6), \\ &A(5,1), A(5,2), A(7,6), A(8,6), A(9,6), A(9,8)\}. \end{aligned}$$

Definition 2.3.7. A *crossed alignment* $A(p,s,x)$ consist of an alignment $A(p,s)$ of $w^{\circlearrowleft}$ and an additional arc $(x \mapsto w(x))$ crossing both $(p \mapsto w(p))$ and $(s \mapsto w(s))$. We partition the set of crossed alignments according to which side of the alignment the crossing arc intersects first as it passes from x to $w(x)$.

1. If $x \in [w(s), s]^{cyc}$ and $w(x) \in [p, w(p)]^{cyc}$, we say $A(p,s,x)$ is a *starboard tacking crossed alignment.*

2. If $x \in [p, w(p)]^{cyc}$ and $w(x) \in [w(s), s]^{cyc}$, we say $A(p,s,x)$ is a *port tacking crossed alignment.*

Note that if $A(p,s,x)$ is a crossed alignment, then p and s cannot be fixed points. The highlighted crossed alignment in Example 1.1.1 $A(9,8,7)$ is an example of a starboard tacking crossed alignment, while $A(9,8,1)$ is a port tacking crossed alignment.

The following family of (decorated) permutations are key to this work. As a subset of S_n, these permutations have nice enumerative properties [Cal04], [OEIS, A075834].

Definition 2.3.8. [Cal04] A permutation $w \in S_n$ is *stabilized-interval-free* provided no proper nonempty interval $[a,b] \subset [n]$ exists such that $w[a,b] = [a,b]$.

Note, the only stabilized-interval-free permutation with a fixed point is the identity permutation in S_1. Thus, the definition of stabilized-interval-free permutations extends easily to *decorated stabilized-interval-free permutations.* Both decorated permutations in $S_1^{\circ\bullet}$ are stabilized-interval-free, and for $n \geq 2$ the SIF permutations and the decorated SIF permutations are the same.

2.4 Positroids and Positroid Varieties

Postnikov and Rietsch considered an important cell decomposition of the totally nonnegative Grassmannian [Pos06; Rie06]. The term *positroid* does not appear in either paper, but has become the name for the matroids that index the nonempty matroid strata in that cell decomposition. They also individually considered the closures of those cells, which determines an analog of Bruhat order. The cohomology classes for these cell closures was investigated by Knutson, Lam, and Speyer [KLS13; KLS14] and Pawlowski [Paw18]. Further geometrical properties of positroid varieties connected to Hodge structure and cluster algebras can be found in [GL21; Lam19].

Definition 2.4.1. A real valued $k \times n$ matrix A is *totally nonnegative* (tnn) if each maximal minor $\Delta_I(A)$ satisfies $\Delta_I(A) \geq 0$ for $I \in \binom{[n]}{k}$. Let $Gr(k,n)^{tnn}$ be the points in $Gr(k,n)$ that can be represented by totally nonnegative matrices.

Definition 2.4.2. A *positroid* of rank k on the ground set $[n]$ is a matroid of the form $\mathcal{M}_A$ for some matrix $A \in Gr(k,n)^{tnn}$. More generally, a positroid can be defined on any ordered ground set $B = \{b_1 < \cdots < b_n\}$.

Positroids are an especially nice class of realizable matroids. For example, they are closed under the matroid operations of restriction, contraction, and duality, as well as a cyclic shift of the ground set [ARW16]. The following theorem is nice connection between positroids and non-crossing partitions as explored by Ardila, Rincón and Williams. See also [For15] for one direction of this theorem.

Theorem 2.4.3. *[ARW16, Thm. 7.6] The connected components of a positroid on ground set $[n]$ give rise to a non-crossing partition of $[n]$. Conversely, each positroid $\mathcal{M}$ on $[n]$ can be uniquely constructed by choosing a non-crossing partition $B_1 \sqcup \cdots \sqcup B_t$ of $[n]$, and then putting the structure of a connected positroid $\mathcal{M}_i$ on each block B_i, so $\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_t$.*

The connected components appearing in the decomposition of $\mathcal{M}$ above each correspond to decorated SIF permutations. This relationship is given in the following corollary.

Corollary 2.4.4. *The decomposition of a positroid $\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_t$ into connected components gives rise to a noncrossing partition $[n] = B_1 \sqcup \cdots \sqcup B_t$ and a decomposition of the associated decorated permutation $w^{\circ} = w^{(1)} \oplus \cdots \oplus w^{(t)}$, where each $w^{(i)}$ is a decorated stabilized-interval-free permutation on B_i, and the converse holds as well.*

As mentioned in the introduction, positroids are in bijection with decorated permutations. The following bijection goes by way of the associated Grassmann necklace. The bijection depends on the *shifted Gale order* $\prec_r$ on $\binom{[n]}{k}$ using the shifted linear order $<_r$ on $[n]$. Specifically, if I and J are k-sets that can be written under $<_r$ as $I = \{i_1 <_r \cdots <_r i_k\}$ and $J = \{j_1 <_r \cdots <_r j_k\}$, then $I \preceq_r J$ if $i_h \leq_r j_h$ for all $h \in [k]$.

Theorem 2.4.5. *[Oh11; Pos06] For $w^{\circ} \in S^{\circ}_{n,k}$, the set*

$$\mathcal{M}(w^{\circ}) := \left\{ I \in \binom{[n]}{k} \,:\, I_r(w^{\circ}) \preceq_r I \text{ for all } r \in [n] \right\} \tag{2.4.1}$$

is a positroid of rank k on ground set $[n]$. Conversely, for every positroid $\mathcal{M}$ of rank k on ground set $[n]$, there exists a unique decorated permutation $w^{\circ} \in S^{\circ}_{n,k}$ such that the sequence of minimal elements in the shifted Gale order on the subsets in $\mathcal{M}$ is the Grassmann necklace of w°.

Corollary 2.4.6. *For $w^{\circ} = (w, \mathrm{co}) \in S^{\circ}_{n,k}$ with associated positroid $\mathcal{M}(w^{\circ})$, every shifted anti-exceedance set $I_r(w^{\circ})$ is in $\mathcal{M}(w^{\circ})$.*

Theorem/Definition 2.4.7. *[KLS13, Thm 5.1] Given a decorated permutation $w^{\circ} \in S^{\circ}_{n,k}$ along with its associated Grassmann interval $[u,v]$ and positroid $\mathcal{M} = \mathcal{M}(w^{\circ}) \subseteq \binom{[n]}{k}$, the following are equivalent definitions of the positroid variety $\Pi_{w^{\circ}} = \Pi_{[u,v]} = \Pi_{\mathcal{M}}$.*

1. *The positroid variety $\Pi_{\mathcal{M}}$ is the homogeneous subvariety of $Gr(k,n)$ whose vanishing ideal is generated by the Plücker coordinates indexed by the nonbases of $\mathcal{M}$, $\{\Delta_I : I \in \mathcal{Q}(\mathcal{M})\}$.*

2. *The positroid variety* $\Pi_{[u,v]}$ *is the projection of the Richardson variety* $X_u^v \subseteq \mathcal{F}\ell(n)$ *to* $Gr(k,n)$*, so* $\Pi_{[u,v]} = \pi_k(X_u^v)$.

Example 2.4.8. In Example 2.1.2, the matrix A has all nonnegative 2×2 minors, so the associated matroid is a positroid. The minimal elements in shifted Gale order are $(\{24\},\{24\},\{34\},\{46\},\{56\}, \{26\})$, which is the Grassmann necklace for the decorated permutation $\overleftarrow{1}36524$. The associated Grassmann interval is $[241365, 561234]$. The set of nonbases of $\mathcal{M}$ is

$$\mathcal{Q} = \{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{2,3\},\{4,5\}\}.$$

Therefore, the points in the positroid variety $\Pi_{w^{\mathcal{Q}}}$ are represented by the full rank complex matrices of the form

$$\begin{bmatrix} 0 & a_{12} & ca_{12} & a_{14} & da_{14} & a_{16} \\ 0 & a_{22} & ca_{22} & a_{24} & da_{24} & a_{26} \end{bmatrix}.$$

As mentioned in the introduction, there are many other objects in bijection with positroids and decorated permutations. We refer the reader to [ARW16] for a nice survey of many other explicit bijections.

Chapter 3

POSITROID CHARACTERIZATION USING INITIAL SETS

The definition of positroid varieties in Theorem/Definition 2.4.7 is given in terms of positroids/positroid complements and the Grassmann intervals. In this section, we show how the initial sets of permutations in Grassmann intervals determine the positroids, proving Theorem 1.2.2. This theorem completes the commutative diagram below.

$$\begin{tikzcd}
S_{n,k}^{\circlearrowleft} \arrow[rr, "shuffle"] \arrow[d, "necklace"] & & Gi(k,n) \arrow[d, "initial"] \\
\text{Grassmann necklaces in } \binom{[n]}{k}^n \arrow[rr, "shiftedGale"] & & \text{Positroids in } \binom{[n]}{k}
\end{tikzcd}$$

3.1 *Canonical Representatives*

For a k-set $I = \{i_1 < i_2 < \cdots < i_k\}$ such that $i_j \leq v_j$ for all $1 \leq j \leq k$, we will show there exists a unique maximal element below v in each $S_k \times S_{n-k}$ coset with initial set I. Let $u(I,v) \leq v$ be this maximal element when it exists. We think of $u(I,v)$ as the *canonical representative* of I in $[id, v]$.

Lemma 3.1.1. Let $u \leq v$ be two permutations in S_n, and assume v is k-Grassmannian. Then there exists a unique permutation $u' \in [u,v]$ such that u' is maximal in Bruhat order among all permutations in the interval $[u,v]$ with $[u'_1, \dots, u'_k] = [u_1, \dots, u_k]$ fixed as an ordered list. Similarly, there exists a unique permutation $u' \in [u,v]$ such that u' is maximal in Bruhat order among all permutations in the interval $[u,v]$ with $[u'_{k+1}, \dots, u'_n] = [u_{k+1}, \dots, u_n]$ fixed.

Proof. Let $u' = u'_1, \dots, u'_n$ be defined by $u'_a := u_a$ for $a \leq k$ and for $b = n, n-1, \dots, k+1$ by

$$u'(b) = \min\, [v(b), n] \smallsetminus (u'[k] \cup u'[b+1, n]). \tag{3.1.1}$$

We claim that u' is well defined. To see this, recall that $u \leq v$ implies $u(b) \geq v(b)$ for all $b > k$ by Theorem 2.1.6, since v is k-Grassmannian. Hence, for all b' such that $k < b \leq b' \leq n$, we have $u(b') \geq v(b') \geq v(b)$, so $|[v(b), n] \cap u[k+1, n]| \geq n - b + 1$. Then, at each step, the set $[v(b), n] \smallsetminus (u'[k] \cup u'[b+1, n]) = ([v(b), n] \cap u[k+1, n]) \smallsetminus u'[b+1, n]$ is nonempty.

By construction, $u'(a) = u(a) \leq v(a)$ for all $a \leq k$ and $u'(b) \geq v(b)$ for all $b > k$, so again by Theorem 2.1.6, we know $u' \leq v$. To see that $u \leq u'$, observe that we can obtain u' from u by applying a sequence of transpositions which increase length at each step. Since $u'(n) = u(b)$ for some $b \in [k+1, n]$, then $u(b) \leq u(n)$ by (3.1.1). Therefore, $u < ut_{bn}$ by Definition 2.1.4. Next, $u'(n-1) = ut_{bn}(c)$ for some $c \in [k+1, n-1]$ such that $ut_{bn}(c) \leq ut_{bn}(n-1)$ by (3.1.1). Hence $u < ut_{bn} < ut_{bn}t_{c(n-1)}$, etc. for $b = n-2, \ldots, k+1$.

Consider the permutation matrix $M_{u'}$ below row k. For each $k < b \leq n$, we see from the construction of u' that the rectangle northeast of $(b, v(b))$ and southwest of $(k+1, n)$ contains a decreasing sequence of 1's ending at $u'(b) \geq v(b)$. Therefore, there is no $a < b$ such that $u' < u't_{ab} \leq v$ unless $a \leq k$ which would change the first k values of u'. So, u' is maximal among all elements below v with prefix $[u_1, \ldots, u_k]$.

It remains to show that u' is the unique maximal element below v with prefix $[u_1, \ldots, u_k]$. Say $x \in S_n$ is another element such that $u \leq x \leq v$ and $u_i = x_i$ for all $1 \leq i \leq k$. If $u' \neq x$, then there exists a maximal j such that $u'(j) \neq x(j)$. Since $x \leq v$, we know $x(j) \in [v(j), n]$ by Theorem 2.1.6 again. By construction $u'(j)$ is the minimal element in

$$[v(j), n] \smallsetminus (u'[k] \cup u'[j+1, n]\}) = [v(j), n] \smallsetminus (u'[k] \cup x[j+1, n]),$$

so $u'(j) < x(j)$. Since u, u', and x are bijections, there exists some $k < i < j$ such that $x(i) = u'(j)$. Therefore, $x < xt_{ij}$, $x(i) \geq v(j)$, and $x(j) \geq v(i)$, so $xt_{ij} \leq v$ by Theorem 2.1.6. Hence, x is not a maximal element below v with initial values $[u(1), \ldots, u(k)]$.

The proof for the second claim is very similar. We leave the details to the reader. □

Definition 3.1.2. Given a k-Grassmannian permutation $v \in S_n$ and a k-subset $I = \{i_1 < i_2 < \cdots < i_k\} \in \binom{[n]}{k}$ such that $i_j \leq v_j$ for all $1 \leq j \leq k$, let $u' = u(I, v)$ be the permutation obtained as follows.

- Loop for j from 1 to k, assuming $u'(1), u'(2), \dots, u'(j-1)$ are defined. Let

$$u'(j) = \max \{i \in I : i \leq v_j\} \setminus u'[j-1]. \tag{3.1.2}$$

- Loop for j from n down to $k+1$, assuming $u'(n), u'(n-1), \dots, u'(j+1)$ are defined. Let

$$u'(j) = \min [v(j), n] \setminus (u'[k] \cup u'[j+1, n]). \tag{3.1.3}$$

Corollary 3.1.3. *Let v be a k-Grassmannian permutation, and let $u \leq v$. If $I = \{u_1, \dots, u_k\}$ then $u \leq u(I, v) \leq v$, and $u(I, v)$ is the unique maximal element in the $S_k \times S_{n-k}$ right coset containing u.*

Proof. Use Lemma 3.1.1 to find the maximal permutation $u \leq u' \leq v$ with the prefix determined by u. Then use Lemma 3.1.1 again to find the maximal permutation below v with suffix determined by u' to obtain $u(I, v)$. □

Remark 3.1.4. In preparation of this manuscript, we have learned that the restriction to k-Grassmannian permutations is unnecessary in Corollary 3.1.3. In fact, Oh and Richmond have proved a substantially more general statement. Their results imply that for any elements $u \leq v$ in any Coxeter group W and any parabolic subgroup W_J, there exists a unique maximal element in $[u, v] \cap uW_J$, see [OR22, Thm. 2.1].

3.2 Intervals and Positroids

For $I \subseteq [n]$, define $w_0 \cdot I := \{n + 1 - i : i \in I\}$. For any integer s, we also establish the notation

$$I^{+s} := \{i + s \ : \ i \in I\},$$
$$I^{-s} := \{i - s \ : \ i \in I\},$$

where all values are taken mod n in the range $[n]$.

Lemma 3.2.1. *Let $w^{\circlearrowleft} = (w, co)$ be a decorated permutation in $S^{\circlearrowleft}_{n,k}$. Let $\mathcal{M}$ and $[u,v]$ be the positroid and Grassmann interval corresponding to $w^{\circlearrowleft}$. Then every $I \in \mathcal{M}$ satisfies*

$$I_1(w^{\circlearrowleft}) = u[k] \leq I \leq v[k] = w^{-1}(I_1(w^{\circlearrowleft})).$$

Furthermore, for any $r \in [n]$, I satisfies

$$I_r(w^{\circlearrowleft}) \leq_r I \leq_r w^{-1}(I_r(w^{\circlearrowleft})).$$

Proof. Recall the equalities $u[k] = I_1(w^{\circlearrowleft})$ and $v[k] = w^{-1}(I_1(w^{\circlearrowleft}))$ from the bijection between decorated permutations and Grassmann intervals given in Section 2.3. To see the lower bound, use the characterization of $\mathcal{M}$ from Theorem 2.4.5. Any $I \in \mathcal{M}$ must satisfy $I_r(w^{\circlearrowleft}) \leq_r I$ for every shifted anti-exceedance set $I_r(w^{\circlearrowleft})$. In particular, for $r = 1$, I must satisfy $u[k] = I_1(w^{\circlearrowleft}) \leq_1 I$.

To prove the upper bound $I \leq v[k]$, let $I = \{i_1 < \cdots < i_k\}$, and suppose that $I \nleq v[k] = \{v_1 < \cdots < v_k\}$. Then there is some maximal index r for which $i_r > v_r$. In particular, we have $|v[k] \cap [i_r, n]| < k - r + 1$. Consider the positroid variety $\Pi_{w^{\circlearrowleft}} = \pi_k(X_u \cap X^v)$. Recall that the opposite Schubert variety X^v is defined by

$$X^v = \{V_\bullet \in \mathcal{F}\ell(n) \,:\, \dim(\mathrm{Proj}_{[n-j+1,n]}(V_i)) \leq |v[i] \cap [n-j+1, n]| \text{ for all } i, j\}.$$

Any point in $\Pi_{w^{\circlearrowleft}}$ is the projection of a point in X^v. Thus, for any matrix A representing a point in $\Pi_{w^{\circlearrowleft}}$, the rank of columns $[i_r, n]$ of A must be strictly less than $k-r+1$. In particular, the rank of A in the column set $\{i_r, \ldots, i_k\}$ is strictly less than $k - r + 1$ so $\mathrm{rank}(A_I) < k$, which implies $\Delta_I(M) = 0$. Since this argument holds for every matrix representing a point in $\Pi_{w^{\circlearrowleft}}$, then by Theorem/Definition 2.4.7 it follows that $I \notin \mathcal{M}$. Thus, $I \in \mathcal{M}$ must also satisfy $I \leq v[k]$.

For general r, we have already seen above from the definition of $\mathcal{M}(w^{\circlearrowleft})$ that I must satisfy $I_r(w^{\circlearrowleft}) \leq_r I$. Thus again, it remains to show that $I \leq_r w^{-1}(I_r(w^{\circlearrowleft}))$. Consider the action of rotating $D(w^{\circlearrowleft})$ in the counterclockwise direction by $r-1$ positions. As discussed further in Section 4.2, this transformation of $D(w^{\circlearrowleft})$ has the action of cyclically shifting the anti-exceedance sets of $w^{\circlearrowleft}$ as well as the ground set of $\mathcal{M}(w^{\circlearrowleft})$. Let $z^{\circlearrowleft} = (z, co')$ be the decorated permutation whose chord diagram is obtained by applying this rotation to $D(w^{\circlearrowleft})$. Then $I_1(z^{\circlearrowleft}) = I_r(w^{\circlearrowleft})^{-(r-1)}$, $z^{-1}(I_1(z^{\circlearrowleft})) = w^{-1}(I_r(w^{\circlearrowleft}))^{-(r-1)}$, and $\mathcal{M}(z^{\circlearrowleft}) = \mathcal{M}(w^{\circlearrowleft})^{-(r-1)}$. Applying the result above yields that $I^{-(r-1)} \leq_r z^{-1}(I_1(z^{\circlearrowleft})) = w^{-1}(I_r(w^{\circlearrowleft}))^{-(r-1)}$. Cyclically shifting the values back up by $r-1$ positions, it follows that $I = (I^{-(r-1)})^{+(r-1)} \leq_r (w^{-1}(I_r(w^{\circlearrowleft}))^{-(r-1)})^{+(r-1)} = w^{-1}(I_r(w^{\circlearrowleft}))$, as desired. $\square$

The result of Lemma 3.2.1 gives a constraint on the elements in a positroid. From this constraint, we prove the characterization of a positroid as a collection of initial sets of the corresponding Grassmann interval.

Proof of Theorem 1.2.2. Let $\mathcal{S} = \{y[k] : y \in [u,v]\}$ be the collection of initial sets for $[u,v]$. For $y \in [u,v]$, the point represented by the permutation matrix M_y is in the Richardson variety X_u^v. Let $A_y = A_{y[k]}$ be the submatrix of M_y given by the top k rows. Then A_y represents a point in the positroid variety $\pi_k(X_u^v) = \Pi_{[u,v]}$, by Part (1) of Theorem/Definition 2.4.7. The restriction of A_y to the columns indexed by $I = y[k]$ is a $k \times k$ permutation matrix. Therefore, Δ_I does not vanish at $A_y \in \Pi_{[u,v]}$. Since Δ_I is not in the vanishing ideal of $\Pi_{[u,v]}$, we must have $I \in \mathcal{M}$ by Part (2) of Theorem/Definition 2.4.7. Thus, we have the inclusion $\mathcal{S} \subseteq \mathcal{M}$.

For the reverse inclusion, let $I \in \mathcal{M}$. By Lemma 3.2.1, I must satisfy $u[k] \leq I \leq v[k]$. In particular, since $I \leq v[k]$, we may use Definition 3.1.2 to define a permutation $y = u(I,v)$ with initial set $y[k] = I$. By Corollary 3.1.3, $y \leq v$ under Bruhat order, so to prove $I \in \mathcal{S}$ it remains only to show that this y satisfies $u \leq y$.

By [BB05b, Ex. 8, Ch 2], $u \leq y$ if and only if $u[j] \leq y[j]$ for all $j \in [k]$ and $u[n-j+1,n] \geq y[n-j+1,n]$ for all $j \in [n-k]$. This exercise follows from Definition 2.1.4. We will prove that $u[j] \leq y[j]$ for $j \in [k]$ by induction on j.

Let $w^{\circlearrowleft}$ be the decorated permutation associated to $[u,v]$. Since $I \in \mathcal{M}$, we know by Theorem 2.4.5 that $I_r(w^{\circlearrowleft}) \leq_r I$ for all $r \in [n]$ where $I_r(w^{\circlearrowleft})$ is the shifted anti-exceedance set for $w^{\circlearrowleft}$. By the shuffling algorithm described in Chapter 2 mapping $w^{\circlearrowleft}$ to $[u,v]$, one can observe for all $1 \le j \le k$ that $u_j \in I_r(w^{\Omega})$ unless $u_j < r \le v_j$. In particular, $u[j] \subset I_r(w^{\Omega})$ for all $r > v_j$ since $v_1 < \cdots < v_j$ and $u \le v$. Therefore,

$$|u[j] \cap [i, v_j]| \le |I_r(w^{\Omega}) \cap [i, v_j]| \;\forall\; r > v_j. \tag{3.2.1}$$

For the case $j = 1$, fix $r = v_1 + 1$ (or if $v_1 = n$ fix $r = 1$) so v_1 is maximal under the order $<_r$. As observed above, $u_1 \in I_r(w^{\Omega})$. Since $I_r(w^{\Omega}) \cap [u_1, v_1]$ is nonempty, $I_r(w^{\Omega}) \le_r I$ implies $I \cap [u_1, v_1]$ must also be nonempty. By construction of $y = u(I, v)$ and Definition 3.1.2, y_1 is the maximal element of $I \cap [u_1, v_1]$. Thus $u_1 \le y_1$.

For the inductive step, fix $j \in [2, k-1]$, and assume that

$$u[j-1] = \{a_1 < \ldots < a_{j-1}\} \;\le\; y[j-1] = \{b_1 < \ldots < b_{j-1}\}. \tag{3.2.2}$$

We note that all $a_i, b_i \in [v_j]$ since v is k-Grassmannian and $u, y \le v$ in Bruhat order by Theorem 2.1.6. If $u_j \le y_j$, then no matter where u_j and y_j fit in among the increasing sequence of a_i's and b_i's, we will have $u[j] \le y[j]$ by (3.2.2). Therefore, assume $1 \le y_j < u_j \le v_j$. By the induction hypothesis, $1 \le a_i \le b_i \le v_i$ for each $1 \le i < j$, so if m is the largest index such that $b_m < y_j$, then

$$u[j] \cap [1, a_m] \le y[j] \cap [1, b_m].$$

So, to prove $u[j] \le v[j]$, it suffices to show that for all $i \in [y_j, v_j]$

$$|u[j] \cap [i, v_j]| \le |y[j] \cap [i, v_j]|. \tag{3.2.3}$$

Fix $r = v_j + 1$ modulo n so v_j is maximal in the shifted $\le_r$ order. By definition of $y = u(I, v)$, we know y_j is maximal in $(I \setminus y[j-1]) \cap [v_j]$ under $\le_r$ and the usual order on the integers. Therefore, for all $i \in [y_j, v_j]$ we have $y[j] \cap [i, v_j] = I \cap [i, v_j]$. Hence, since $I_r(w^{\Omega}) \le_r I$, we have

$$|I_r(w^{\Omega}) \cap [i, v_j]| \le |I \cap [i, v_j]| = |y[j] \cap [i, v_j]|. \tag{3.2.4}$$

By (3.2.1),

$$|u[j] \cap [i, v_j]| \le |I_r(w^{\Omega}) \cap [i, v_j]|, \tag{3.2.5}$$

so (3.2.4) and (3.2.5) together imply (3.2.3) for all $i \in [y_j, v_j]$. Hence, $u[j] \leq y[j]$ for all $j \in [k]$.

Observe that the map on permutations in S_n sending $x \to x' = w_0 x w_0$ rotates the permutation matrices 180 degrees. Using Theorem 2.1.6 we have $[u, v] \in Gi(k, n)$ if and only if $[w_0 u w_0, w_0 v w_0] \in Gi(n-k, n)$. By the symmetry in Definition 3.1.2, we observe that $u(y[k+1, n], w_0 v w_0) = w_0 y w_0$. Therefore, by applying the argument above to $y' \in [u', v']$ with $I' = w_0 \cdot y[k+1, n]$, we have $u'[j] \leq y'[j]$ for all $j \in [n-k]$ which implies $u[n-j+1, n] \geq y[n-j+1, n]$ for $j \in [n-k]$ as needed to complete the proof. □

Proof of Theorem 1.2.5, (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). By Theorem 1.2.2, we have the equality $\mathcal{M} = \{y[k] : y \in [u, v]\}$. By Theorem 1.1.2, we have $\#Alignments(w^{\circlearrowleft}) = k(n-k) - [\ell(v) - \ell(u)]$. Hence, $\ell(v) - \ell(u) = k(n-k) - \#Alignments(w^{\circlearrowleft})$. Together, these facts yield (2) $\Leftrightarrow$ (3).

The equality (3) $\Leftrightarrow$ (4) comes from the definition of the matroid Johnson graph $J(\mathcal{M})$. For a set $J \in \mathcal{M}$, the vertices adjacent to J in $J(\mathcal{M})$ are exactly the sets $I \in \mathcal{M}$ such that $|I \cap J| = k - 1$. So again, the equality $\ell(v) - \ell(u) = k(n-k) - \#Alignments(w^{\circlearrowleft})$ from Theorem 1.1.2 implies (3) $\Leftrightarrow$ (4). □

Chapter 4

REDUCTIONS

In this chapter, we prove several results which enable us to reduce our study to a particular case of decorated permutations. We use these reductions in this chapter to prove equality (1) $\Leftrightarrow$ (2) in Theorem 1.2.5 and in Chapter 5.

In Section 4.1, we reduce the study of positroid varieties to analysis at a few key points. To determine whether a variety is singular, in general one would have to test every point in the variety. In this section, we show that for a positroid variety $\Pi_{[u,v]}$, one need only check whether $\Pi_{[u,v]}$ is singular at the T-fixed points $A_{y[k]}$ for $y \in [u,v]$. In particular, we will complete the proofs of Theorem 1.2.1 and Theorem 1.2.3. Together these two theorems along with Theorem 1.2.2 prove the equality between (1) and (2) in Theorem 1.2.5.

In Section 4.2, we consider a set of rigid transformations of a chordal diagram and the corresponding action on the associated objects. In particular, we show in Lemma 4.2.6 that applying any of these transformations preserves the property of being smooth or singular in the associated positroid variety.

In Section 4.3, we enable a reduction to connected positroids. Specifically, we will show in Corollary 4.3.5 that a positroid variety is smooth if and only if the positroid varieties corresponding with each connected component are smooth.

4.1 Reduction to T-fixed points

To prove the ability to reduce to T-fixed points, we first recall some specifics for the definition of the Jacobian matrix of a positroid variety. By Theorem/Definition 2.4.7 Part (1), the polynomials generating the vanishing ideal of $\Pi_{[u,v]}$ are exactly the determinants Δ_I for

$I \in \mathcal{Q}_{[u,v]}$ where

$$\mathcal{Q}_{[u,v]} \;:=\; \binom{[n]}{k} \setminus \mathcal{M} \;=\; \left\{ I \in \binom{[n]}{k} : I \neq y[k] \;\forall y \in [u,v] \right\} \tag{4.1.1}$$

is the set of nonbases of the positroid $\mathcal{M}$ corresponding to $[u,v] \in Gi(k,n)$. The second equality in (4.1.1) holds by Theorem 1.2.2. Each determinant Δ_I is a polynomial function using variables of the form x_{ij} indexed by row $i \in [k]$ and column $j \in [n]$. Let $Jac_{[u,v]}$, or just Jac if $[u,v]$ is understood, be the Jacobian of $\Pi_{[u,v]}$. Similarly, we will suppress the subscript $\mathcal{Q} = \mathcal{Q}_{[u,v]}$. Then, the rows of Jac are indexed by the sets $I \in \mathcal{Q}$, the columns are indexed by variables x_{ij}, and the (I, x_{ij}) entry of Jac is $\frac{\partial \Delta_I}{\partial x_{ij}}$. If A is a $k \times n$ complex matrix representing a point in $\Pi_{[u,v]} \subseteq Gr(k,n)$, then $Jac|_A$ is the matrix with entries in the complex numbers obtained from Jac by evaluating each entry at A,

$$Jac|_A(I, x_{ij}) = \frac{\partial \Delta_I}{\partial x_{ij}}(A). \tag{4.1.2}$$

By Theorem/Definition 2.4.7 Part (2), we can write $\Pi_{[u,v]}$ as the union

$$\Pi_{[u,v]} \;=\; \pi_k(X_u^v) = \pi_k\Big(\bigsqcup_{u \le y \le v} (C_y \cap X^v) \Big) \;=\; \bigcup_{u \le y \le v} \pi_k(C_y \cap X^v). \tag{4.1.3}$$

For any $y \in [u,v]$, recall $A_{y[k]}$ is the projection of the permutation matrix M_y to the top k rows.

Lemma 4.1.1. *For $[u,v] \in Gi(k,n)$ and $y \in [u,v]$, the matrix $Jac_{[u,v]}|_{A_{y[k]}}$ is a partial permutation matrix, up to the signs of the entries. Furthermore, the nonzero entries occur exactly in the entries (I, x_{st}), where $I \in \mathcal{Q}_{[u,v]}$ satisfies $|I \cap y[k]| = k-1$, the value $s \in [k]$ is determined by $y[k] \setminus I = \{y_s\}$, and $t \in [n]$ is determined by $I \setminus y[k] = \{t\}$.*

Proof. Write $A_{y[k]}|_I$ for the restriction of $A_{y[k]}$ to column set I. By expanding Δ_I along row $i \in [k]$ of the partial permutation matrix $A_{y[k]}$, observe from (4.1.2) that entry (I, x_{ij}) of $Jac|_{A_{y[k]}}$ is

(i) $(j \in I)$ up to sign, the $(k-1) \times (k-1)$ minor of $A_{y[k]}|_I$ in column set $I \setminus \{j\}$ and row set $[k] \setminus \{i\}$, or

(ii) $(j \notin I)$ 0, since Δ_I does not contain x_{ij}.

Consider any $I \in \mathcal{Q}$. By Theorem 1.2.2, $y[k] \in \mathcal{M}$, so $I \neq y[k]$. Since I and $y[k]$ are distinct k-sets, then $|I \cap y[k]|$ is at most $k-1$. If $|I \cap y[k]| \leq k-2$, then $A_{y[k]}$ has at most $k-2$ ones in $A_{y[k]}|_I$, and the remaining entries of $A_{y[k]}|_I$ are zeros. In that case, every $(k-1)\times(k-1)$ minor of $A_{y[k]}|_I$ is zero. Therefore, row I of $Jac|_{A_{y[k]}}$ contains only zeros.

Otherwise, $I \in \mathcal{Q}$ satisfies $|I \cap y[k]| = k-1$. Say $I \setminus y[k] = \{t\}$ and $y[k] \setminus I = \{y_s\}$ for some $s \in [k]$. Then column t of $A_{y[k]}$ contains all zeros, row s of $A_{y[k]}|_I$ contains all zeros, and the submatrix of $A_{y[k]}|_I$ obtained by removing row s and column t is a permutation matrix. Therefore, by cofactor expansion of the determinant

$$Jac\Big|_{A_{y[k]}}(I, x_{st}) = \frac{\partial \Delta_I}{\partial x_{st}}(A_{y[k]}) = \pm 1,$$

and x_{st} is the unique variable such that $\frac{\partial \Delta_I}{\partial x_{ij}}(A_{y[k]})$ is nonzero.

Conversely, we claim column x_{st} contains a unique nonzero entry in row I. If $I' \in \mathcal{Q}$ is another set such that $|I' \cap y[k]| = k-1$, let $I' \setminus y[k] = \{t'\}$ and $y[k] \setminus I' = \{y_{s'}\}$. Then, the only nonzero entry of $Jac|_{A_{y[k]}}$ in row I' occurs in column $x_{s't'}$ by the same reasoning as above. If $t \neq t'$, then $x_{st} \neq x_{s't'}$. If $t = t'$, either $s = s'$ and $I' = I$ or else $y_s \in I'$ so $s' \neq s$ since y is a bijection, in which case $x_{st} \neq x_{s't'}$. Therefore, $Jac|_{A_{y[k]}}$ is a partial permutation matrix up to signs as stated. □

Proof of Theorem 1.2.3. From Lemma 4.1.1, $Jac|_{A_{y[k]}}$ is a partial permutation matrix, up to the signs of its entries. Thus, the rank of $Jac|_{A_{y[k]}}$ is exactly the number of its nonzero entries, which are exactly in the rows indexed by the sets $I \in \mathcal{Q}$ satisfying $|I \cap y[k]| = k-1$. Therefore, the rank of $Jac|_{A_{y[k]}}$ is equal to the number of these sets,

$$\operatorname{rank}(Jac|_{A_{y[k]}}) = \#\left\{I \in \binom{[n]}{k} \setminus \mathcal{M} \;:\; |I \cap y[k]| = k-1\right\}.$$

□

Next, consider a flag $V_\bullet \in C_y \cap X^v$ and its canonical matrix representative $A_{V_\bullet}$ as defined in Chapter 2. Let A be the projection of $A_{V_\bullet}$ to $Gr(k,n)$. We will see that the rank of the Jacobian matrix evaluated at A is at least the rank of the Jacobian evaluated at $A_{y[k]}$.

Proof of Theorem 1.2.1. Without loss of generality, we can assume $A \in \pi_k(C_y \cap X^v)$ is a canonical matrix with leading ones in exactly the same entries as $A_{y[k]}$. Therefore, permuting the columns of A and $A_{y[k]}$ in the same way, we can assume the first k columns of both have the form of an upper triangular matrix with ones on the diagonal. If we apply the same permutation to the values in the k-sets in $\mathcal{Q}$ and the variables x_{ij}, then such a rearrangement of columns will not change the rank of $Jac|_{A_{y[k]}}$ or $Jac|_A$. After permuting, we can then assume y is the identity permutation, so $A_{y[k]} = A_{[k]}$. Let $N = \operatorname{rank}(Jac|_{A_{[k]}})$. By Theorem 1.2.3, we must show that

$$\operatorname{rank}(Jac|_A) \ \geq \ N = \ \#\{I \in \mathcal{Q} \, : \, |I \cap [k]| = k-1\}.$$

Let $R = \{I_1, \ldots, I_N\}$ and $C = \{(s_1, t_1), \ldots, (s_N, t_N)\}$ be the unique row and column sets such that the $N \times N$ minor of $Jac|_{A_{[k]}}$ in rows R and columns C is nonzero, as determined in Lemma 4.1.1. Then, for each $I_j \in R$, we know $I_j = ([k] \smallsetminus \{s_j\}) \cup \{t_j\}$, where $s \in [k]$ and $t_j > k$ after permuting values. Since the first k columns of A form an upper triangular matrix with ones along the diagonal, we observe that for all $j \in [N]$,

$$Jac|_A(I_j, x_{s_j t_j}) = \frac{\partial \Delta_{I_j}}{\partial x_{s_j t_j}}(A) = \pm 1. \tag{4.1.4}$$

Fix $j \in [N]$. Assume that the sets in R are ordered lexicographically as sorted lists, and let $1 \leq h < j \leq N$. Either $t_h \neq t_j$ or $t_h = t_j$ and $s_h > s_j$ by lex order. If $t_h \neq t_j$, then I_h does not contain t_j, by construction of I_h. Hence, Δ_{I_h} does not depend on the variable $x_{s_j t_j}$, so $\frac{\partial \Delta_{I_h}}{\partial x_{s_j t_j}}(A) = 0$. Otherwise, $s_h > s_j$, and $\frac{\partial \Delta_{I_h}}{\partial x_{s_j t_j}}(A)$ is the determinant of an upper triangular matrix with a zero on its diagonal. Hence, in either case

$$Jac|_A(I_h, x_{s_j t_j}) = \frac{\partial \Delta_{I_h}}{\partial x_{s_j t_j}}(A) = 0. \tag{4.1.5}$$

Arrange the rows and columns of Jac so that $I_1, \ldots, I_N$ are the top N rows listed in order and $x_{s_1 t_1}, x_{s_2 t_2}, \ldots, x_{s_N k t_N}$ are the first N columns listed in order. Then, the $N \times N$ upper left submatrix of $Jac|_A$ is lower triangular with plus or minus one entries along the diagonal. Thus $Jac|_A$ contains a rank N submatrix, so $\operatorname{rank}(Jac|_A) \geq N$. $\square$

Example 4.1.2. Rearranging the columns of the matrix in Example 2.1.2 so that $y[k] = [k]$, consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 & 2 & 1 \end{bmatrix}.$$

The nonbases of $\mathcal{M}_A$ are

$$\mathcal{Q} = \{\{1,3\},\{1,4\},\{2,4\},\{2,5\},\{3,4\},\{4,5\},\{4,6\}\} \subseteq \binom{[6]}{2}.$$

With $k = 2$, the sets $I \in \mathcal{Q}$ satisfying $|[k] \cap I| = k-1$, ordered lexicographically, are $I_1 = \{1,3\}$, $I_2 = \{1,4\}$, $I_3 = \{2,4\}$, and $I_4 = \{2,5\}$. The corresponding x_{st} so that $I_j = ([k] \smallsetminus \{s_j\}) \cup \{t_j\}$ are x_{23}, x_{24}, x_{14}, and x_{15}, respectively. Arranging the rows and columns of $Jac|_A$ as described in the proof above, the upper left submatrix of $Jac|_A$ is a lower triangular matrix with plus or minus one entries on the diagonal, as shown below.

$$\begin{array}{c c} & \begin{array}{cccc} x_{23} & x_{24} & x_{14} & x_{15} \end{array} \\ \begin{array}{c} \{1,3\} \\ \{1,4\} \\ \{2,4\} \\ \{2,5\} \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Corollary 4.1.3. *Let $[u,v] \in Gi(k,n)$ and $y \in [u,v]$. Then $\pi_k(C_y \cap X^v) \subseteq \Pi_{[u,v]}$ contains a singular point of $\Pi_{[u,v]}$ if and only if $\Pi_{[u,v]}$ is singular at $A_{y[k]}$. Furthermore, $A_{y[k]}$ is a singularity of $\Pi_{[u,v]}$ if and only if*

$$\#\{I \in \mathcal{Q}_{[u,v]} \; ; \; |I \cap y[k]| = k-1\} \; < \; k(n-k) - [\ell(v) - \ell(u)].$$

Proof. Recall from (1.1.3) that $A \in \Pi_{[u,v]}$ is a singular point of $\Pi_{[u,v]}$ if and only if $\mathrm{rank}(Jac|_A)$ $< \mathrm{codim}\,\Pi_{[u,v]}$. By Theorem 1.2.1, for any $A \in \pi_k(C_y \cap X^v)$, $\mathrm{rank}(Jac|_{A_{y[k]}}) \leq \mathrm{rank}(Jac|_A)$. Therefore, if there exists some $A \in \pi_k(C_y \cap X^v)$ that is a singular point of $\Pi_{[u,v]}$, then

$$\mathrm{rank}(Jac|_{A_{y[k]}}) \; \leq \; \mathrm{rank}(Jac|_A) \; < \; \mathrm{codim}\,\Pi_{[u,v]}$$

implies that $A_{y[k]}$ is also a singular point of $\Pi_{[u,v]}$. Conversely, note that since the permutation matrix M_y is in $C_y \cap X^v$, then $A_{y[k]} = \pi_k(M_y) \in \pi_k(C_y \cap X^v)$. Thus, if $A_{y[k]}$ is a singular point of $\Pi_{[u,v]}$, then $A_{y[k]}$ is already a singular point of $\Pi_{[u,v]}$ in $\pi_k(C_y \cap X^v)$. This proves the first statement.

For the second statement, recall from Theorem 1.1.2 that

$$\text{codim } \Pi_{[u,v]} = k(n-k) - [\ell(v) - \ell(u)].$$

By Theorem 1.2.3, $\text{rank}(Jac|_{A_{y[k]}}) = \#\{I \in \mathcal{Q} : |I \cap y[k]| = k-1\}$. Therefore, $A_{y[k]}$ is a singular point of $\Pi_{[u,v]}$ if and only if

$$\#\{I \in \mathcal{Q} : |I \cap y[k]| = k-1\} = \text{rank}(Jac|_{A_{y[k]}}) < \text{codim}\,\Pi_{[u,v]} = k(n-k) - [\ell(v) - \ell(u)].$$

$\square$

Proof of Theorem 1.2.5, (1) $\Leftrightarrow$ (2). From the definition of a smooth variety and the decomposition

$$\Pi_{[u,v]} = \pi_k(X_u^v) = \pi_k\Big(\bigsqcup_{u \le y \le v} (C_y \cap X^v)\Big) = \bigcup_{u \le y \le v} \pi_k(C_y \cap X^v),$$

it follows that $\Pi_{[u,v]}$ is smooth if and only if, for every $y \in [u,v]$, every point in $\pi_k(C_y \cap X^v)$ is a smooth point of $\Pi_{[u,v]}$. By Corollary 4.1.3, every point of $\pi_k(C_y \cap X^v)$ is a smooth point of $\Pi_{[u,v]}$ if and only if $A_{y[k]}$ is a smooth point of $\Pi_{[u,v]}$, which occurs if and only if

$$\#\{I \in \mathcal{Q} : |I \cap y[k]| = k-1\} \ \ge\ k(n-k) - [\ell(v) - \ell(u)]. \tag{4.1.6}$$

Since $\text{rank}(Jac|_{A_{y[k]}})$ is the codimension of the tangent space to $\Pi_{[u,v]}$ at $A_{y[k]}$, it follows that $\text{rank}(Jac|_{A_{y[k]}})$ is bounded above by $\text{codim}\,\Pi_{[u,v]}$. Thus, the inequality in (4.1.6) can never be strict. Therefore, $A_{y[k]}$ is a smooth point of $\Pi_{[u,v]}$ if and only if

$$\#\{I \in \mathcal{Q} : |I \cap y[k]| = k-1\} = \text{rank}(Jac|_{A_{y[k]}}) = \text{codim}\,\Pi_{[u,v]} = k(n-k) - [\ell(v) - \ell(u)]. \tag{4.1.7}$$

We can compute the left side of (4.1.7) as follows. For any $j \in y[k]$ and any element $i \in [n] \backslash y[k]$, observe that there are $k(n-k)$ sets $I \in \binom{[n]}{k}$ of the form $I = (y[k] \backslash \{j\}) \cup \{i\}$ so that $|I \cap y[k]| = k-1$. Since $\mathcal{Q} = \binom{[n]}{k} \setminus \mathcal{M}$, then

$$\#\{I \in \mathcal{M} : |I \cap y[k]| = k-1\} \;+\; \#\{I \in \mathcal{Q} : |I \cap y[k]| = k-1\} \;=\; k(n-k). \tag{4.1.8}$$

Substituting the result of (4.1.8) into (4.1.7) yields that $A_{y[k]}$ is a smooth point of $\Pi_{[u,v]}$ if and only if

$$\#\{I \in \mathcal{M} : |I \cap y[k]| = k-1\} \;=\; \ell(v) - \ell(u). \tag{4.1.9}$$

Thus, $\Pi_{[u,v]}$ contains only smooth points if and only if every $y \in [u,v]$ satisfies (4.1.9). □

Finally in this section, we relate the vertex degree conditions in the Johnson graph to the sets of nonbases of positroids similar to Corollary 4.1.3. If a positroid $\mathcal{M}$ corresponds with the decorated permutation w°, let $Alignments(\mathcal{M}) = Alignments(w^{\circ})$.

Corollary 4.1.4. *Given a rank k positroid $\mathcal{M}$ on ground set $[n]$, let $\mathcal{Q} := \binom{[n]}{k} \setminus \mathcal{M}$ be the corresponding set of nonbases. For any $J \in \mathcal{M}$, the codimension of the tangent space to $\Pi_{\mathcal{M}}$ at A_J is*

$$\#\{I \in \mathcal{Q} \,:\, |I \cap J| = k-1\} \;\leq\; \#Alignments(\mathcal{M}) = \operatorname{codim}(\Pi_{\mathcal{M}}). \tag{4.1.10}$$

Furthermore, A_J is a singular point in $\Pi_{\mathcal{M}}$ if and only if

$$\#\{I \in \mathcal{Q} \,:\, |I \cap J| = k-1\} \;<\; \#Alignments(\mathcal{M}). \tag{4.1.11}$$

Proof. Equation (4.1.10) follows from Equation (1.1.3), Theorem 1.2.2, and Theorem 1.2.3. The second claim now follows by Corollary 4.1.3 and Theorem 1.2.2. □

4.2 Rigid Transformations

The authors of [ARW16] show that the set of positroids is closed under restriction, contraction, duality, and a cyclic shift of the ground set. We add to this list reversal of the ground set. The fact that positroids are closed under duality, cyclic shift of the ground set, and reversal of the ground set can be obtained by considering rigid transformations of the chord diagram of the associated decorated permutation. We consider three types of rigid transformations on chord diagrams: arc reversal, reflection, and rotation. We associate these transformations with the symmetric group operations of taking the inverse, conjugation by w_0, and conjugation by a cycle. Because these rigid transformations are bijections on the set of chord diagrams, they generate a group of transformations on Grassmann intervals, Grassmann necklaces, and positroids. The results are collected in Proposition 4.2.2.

Throughout this section, fix $w^{\circlearrowleft\!\!\!\circlearrowright} = (w, co) \in S^{\circlearrowleft\!\!\!\circlearrowright}_{n,k}$, and let $z^{\circlearrowleft\!\!\!\circlearrowright} = (z, co')$ be the decorated permutation whose chord diagram $D(z^{\circlearrowleft\!\!\!\circlearrowright})$ is obtained from $D(w^{\circlearrowleft\!\!\!\circlearrowright})$ by a rigid transformation. Denote by $F(w^{\circlearrowleft\!\!\!\circlearrowright})$ the set of fixed points of $w^{\circlearrowleft\!\!\!\circlearrowright}$. Define the map $flip : \{\circlearrowright, \circlearrowleft\} \to \{\circlearrowleft, \circlearrowright\}$ to be the involution on $\{\circlearrowright, \circlearrowleft\}$ that reverses the orientation. Let $\chi \in S_n$ be the cycle with $\chi(i) = i + 1 \bmod n$.

To give the maps on decorated permutations corresponding to the chord diagram transformations, we will consider the transformations on the arcs of the chord diagrams. The transformations of arcs then lead to transformations of the two-line notation for decorated permutations.

First, consider *arc reversal*, where where $D(z^{\circlearrowleft\!\!\!\circlearrowright})$ is obtained from $D(w^{\circlearrowleft\!\!\!\circlearrowright})$ by reversing all the arcs in $D(w^{\circlearrowleft\!\!\!\circlearrowright})$. So, an arc $i \mapsto w(i)$ becomes $w(i) \mapsto i$. When $i \in F(w^{\circlearrowleft\!\!\!\circlearrowright})$, then i becomes a fixed point in $z^{\circlearrowleft\!\!\!\circlearrowright}$ with opposite orientation of i. Then $z = w^{-1}$, and $F(z^{\circlearrowleft\!\!\!\circlearrowright}) = F(w^{\circlearrowleft\!\!\!\circlearrowright})$ with all fixed point orientations reversed. Therefore, $co' = flip \circ co$. With these observations in mind, we define

$$(w^{\circlearrowleft\!\!\!\circlearrowright})^{-1} := (w^{-1}, flip \circ co). \tag{4.2.1}$$

The two-line notation for $(w^{\circlearrowleft\!\!\!\circlearrowright})^{-1}$ is obtained from the two-line notation of $w^{\circlearrowleft\!\!\!\circlearrowright}$ by swapping the rows, reversing arrows labeling fixed points, and reordering the columns so that the entries of the top line appear in increasing order.

Next, consider *reflection*, where $D(z^{\circlearrowleft\!\!\!\circlearrowright})$ is obtained from $D(w^{\circlearrowleft\!\!\!\circlearrowright})$ by reflecting $D(w^{\circlearrowleft\!\!\!\circlearrowright})$ across the vertical axis. So, an arc $i \mapsto w(i)$ becomes $w_0(i) \mapsto w_0(w(i))$. When $i \in F(w^{\circlearrowleft\!\!\!\circlearrowright})$, then $w_0(i)$ becomes a fixed point of $z^{\circlearrowleft\!\!\!\circlearrowright}$ with opposite orientation of i. Then $z = w_0 w w_0$, and $F(z^{\circlearrowleft\!\!\!\circlearrowright}) = w_0 \cdot F(w^{\circlearrowleft\!\!\!\circlearrowright})$ with all fixed point orientations reversed after applying w_0 to the value of a fixed point in $F(z^{\circlearrowright})$. Therefore, $co' = flip \circ co \circ w_0$, and we define

$$w_0 \cdot w^{\circlearrowright} := (w_0 w w_0, flip \circ co \circ w_0). \tag{4.2.2}$$

The two-line notation for $z^{\circlearrowright}$ is obtained from the two-line notation of $w^{\circlearrowright}$ by replacing i with $w_0(i)$ in both lines, reversing all arrows labeling fixed points, and reversing the order of the columns.

Finally, consider *rotation*, where $D(z^{\circlearrowright})$ is obtained from $D(w^{\circlearrowright})$ by rotating $D(w^{\circlearrowright})$ by s units in the clockwise direction. So, an arc $i \mapsto w(i)$ becomes $i+s \mapsto w(i)+s$, taken modulo n. When $i \in F(w^{\circlearrowright})$, then $i+s$ becomes a fixed point of $z^{\circlearrowright}$ with the same orientation as i. Then $z = \chi^s w \chi^{-s}$, and $F(z^{\circlearrowright}) = F(w^{\circlearrowright})^{+s}$ with all fixed point labels preserved after applying χ^{-s} to the value of a fixed point in $F(z^{\circlearrowright})$. Therefore, $co' = co \circ \chi^{-s}$, and we define

$$\chi^s \cdot w^{\circlearrowright} := (\chi^s w \chi^{-s}, co \circ \chi^{-s}). \tag{4.2.3}$$

The two-line notation for $z^{\circlearrowright}$ is obtained from the two-line notation of $w^{\circlearrowright}$ by replacing i with $i+s \bmod n$ in both lines and cyclically shifting all columns s units to the right.

Associated with the maps listed above are maps on Grassmann intervals. Let $[u,v] \in Gi(k,n)$ be the Grassmann interval corresponding to $w^{\circlearrowright}$, and let $[u',v']$ be the Grassmann interval corresponding to $z^{\circlearrowright}$. As we know from the shuffling algorithm described in Section 2.3, u' and v' are easily extracted from the two-line notation for $z^{\circlearrowright}$ by reordering the columns so that the highlighted columns, corresponding to the anti-exceedances of $z^{\circlearrowright}$, appear on the left and the top line has the form of a $k(z^{\circlearrowright})$-Grassmannian permutation. This new array is the two-line array $\left[\begin{smallmatrix} v' \\ u' \end{smallmatrix}\right]$.

We have already described how the two-line notation for $z^{\circlearrowright}$ is obtained from the two-line notation of $w^{\circlearrowright}$ under the three rigid transformations of $D(w^{\circlearrowright})$. From these descriptions, we obtain the following maps $\left[\begin{smallmatrix} v \\ u \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} v' \\ u' \end{smallmatrix}\right]$.

(1) For the arc reversal map, $\left[\begin{smallmatrix} v' \\ u' \end{smallmatrix}\right]$ is obtained from $\left[\begin{smallmatrix} v \\ u \end{smallmatrix}\right]$ by swapping the rows, swapping the left k-column block with the right $(n-k)$-column block, and reordering the columns within the two blocks so that the top row gives an $(n-k)$-Grassmannian permutation.

Define $[u,v]^{-1} \in Gi(n-k,n)$ to be the Grassmann interval whose two-line array is obtained in this way from $\left[\begin{smallmatrix} v \\ u \end{smallmatrix}\right]$.

(2) For the reflection map, $\left[\begin{smallmatrix} v' \\ u' \end{smallmatrix}\right]$ is obtained from $\left[\begin{smallmatrix} v \\ u \end{smallmatrix}\right]$ by replacing every i with $w_0(i)$ in both rows and reversing all columns. It follows that $u' = w_0uw_0$ and $v' = w_0vw_0$. Define $w_0 \cdot [u,v] \coloneqq [w_0uw_0, w_0vw_0] \in Gi(n-k,n)$. Note that this operation on $[u,v]$ of conjugation by w_0 was used at the end of the proof of Theorem 1.2.2.

(3) For the rotation map, $\left[\begin{smallmatrix} v' \\ u' \end{smallmatrix}\right]$ is obtained from $\left[\begin{smallmatrix} v \\ u \end{smallmatrix}\right]$ by replacing every i with $i+s$, highlighting all columns corresponding to anti-exceedances, then reordering the columns into a highlighted and an unhighlighted block so that the elements of the top row are increasing within each block. Define $\chi^s \cdot [u,v] \in Gi(k,n)$ to be the Grassmann interval obtained from $[u,v]$ in this way.

There are also naturally associated maps on the positroids corresponding to w^Ω and z^Ω. These maps on positroids, listed in Proposition 4.2.2, can be seen from the following facts.

(1) The dual map on positroids corresponds to arc reversal, as shown in the following lemma.

(2) The map on Grassmann intervals which corresponds to the map $w^\Omega \mapsto w_0w^\Omega w_0$ on decorated permutations is $[u,v] \mapsto [w_0uw_0, w_0vw_0] \in Gi(n-k,n)$, and $[w_0uw_0, w_0vw_0] = \{w_0yw_0 : y \in [u,v]\}$. In particular, the map $y \mapsto w_0yw_0$ is an interval isomorphism between $[u,v]$ and $[w_0uw_0, w_0vw_0]$. The related map on positroids then follows from Theorem 1.2.2.

(3) Every r-anti-exceedance of w^Ω translates to an $(r+s)$-anti-exceedance of z^Ω so that $I_r(w^\Omega)^{+s} = I_{r+s}(z^\Omega)$. Also, for sets $I, J \in \binom{[n]}{k}$, $I \leq_r J$ if and only if $I^{+s} \leq_{r+s} J^{+s}$.

Lemma 4.2.1. *Let $w^\circ \in S^\circ_{n,k}$ have associated positroid $\mathcal{M}(w^\circ) \subseteq \binom{[n]}{k}$, and let $z^\circ = (w^\circ)^{-1} \in S^\circ_{n,n-k}$. The positroid associated with z° is the dual of $\mathcal{M}(w^\circ)$,*

$$\mathcal{M}(z^\circ) = \{[n] \setminus I : I \in \mathcal{M}(w^\circ)\} \subseteq \binom{[n]}{n-k}. \tag{4.2.4}$$

Proof. Since arc reversal is an involution, to show that $\mathcal{M}(z^\circ)$ is the dual of $\mathcal{M}(w^\circ)$, as in (4.2.4), it suffices to show that $I \in \mathcal{M}(w^\circ)$ implies that $[n] \setminus I \in \mathcal{M}(z^\circ)$. In particular, we will show that $I \in \mathcal{M}(w^\circ)$ implies that $I_r(z^\circ) \leq_r [n] \setminus I$ for all $r \in [n]$.

For $r \in [n]$, let $J_r(w^\circ)$ be the set of r-exceedances of w°, $J_r(w^\circ) = [n] \setminus I_r(w^\circ)$. Fix $r \in [n]$ and $I \in \mathcal{M}(w^\circ)$. By definition, $D(z^\circ)$ is obtained from $D(w^\circ)$ by reversing every arc. Then every arc $a \mapsto w(a)$ corresponding an r-anti-exceedance $w(a) \in I_r(w^\circ)$ yields an arc $w(a) \mapsto a$ in $D(z^\circ)$, which corresponds to an r-exceedance of z°, so that $a \in J_r(z^\circ)$. Similarly, r-exceedances $w(a) \in J_r(w^\circ)$ yield r-anti-exceedances $a \in I_r(z^\circ)$. Therefore, we have

$$\begin{aligned} J_r(z^\circ) &= w^{-1}(I_r(w^\circ)) \\ I_r(z^\circ) &= w^{-1}(J_r(w^\circ)) = [n] \setminus w^{-1}(I_r(w^\circ)). \end{aligned}$$

Since $I \in \mathcal{M}$, then by Lemma 3.2.1, I must satisfy $I_r(w^\circ) \leq_r I \leq_r w^{-1}(I_r(w^\circ))$. In particular, $I \leq_r w^{-1}(I_r(w^\circ))$ implies that $[n] \setminus I \geq_r [n] \setminus w^{-1}(I_r(w^\circ)) = I_r(z^\circ)$, as desired. Therefore, $[n] \setminus I$ is in $\mathcal{M}(z^\circ)$ by Theorem 2.4.5. $\square$

The transformations of arc reversal, reflection, and rotation can now be applied to any of the bijectively equivalent objects. These transformations are summarized in the following proposition.

Proposition 4.2.2. *Fix a decorated permutation $w^\circ = (w, co) \in S^\circ_{n,k}$ with associated Grassmann interval $[u, v]$, Grassmann necklace $(I_1(w^\circ), \dots, I_n(w^\circ))$, and positroid $\mathcal{M}(w^\circ)$. Let z° be a decorated permutation whose chord diagram is obtained from $D(w^\circ)$ by (1) arc reversal, (2) reflection, or (3) rotation. Let $[u', v']$ be the Grassmann interval associated with z°. The table below describes z° and its associated objects and values.*

Trans \ Obj	z°	$[u', v']$	$I_r(z^{\circ})$	$\mathcal{M}(w^{\circ}) \mapsto \mathcal{M}(z^{\circ})$	$k(z^{\circ})$
arc reversal	$(w^{\circ})^{-1}$	$[u,v]^{-1}$	$[n] \setminus w^{-1}(I_r(w^{\circ}))$	$I \mapsto [n] \setminus I$	$n-k$
reflection	$w_0 \cdot w^{\circ}$	$w_0 \cdot [u,v]$	$w_0 \cdot ([n] \setminus I_{w_0(r)}(w^{\circ}))$	$I \mapsto w_0 \cdot ([n] \setminus I)$	$n-k$
rotation	$\chi^s \cdot w^{\circ}$	$\chi^s \cdot [u,v]$	$I_{r-s}(w^{\circ})^{+s}$	$I \mapsto I^{+s}$	k

Corollary 4.2.3. *The set of positroids is closed under reversal of the ground set* $[n]$*.*

Proof. Let $\mathcal{M} \subseteq \binom{[n]}{k}$ be a positroid, and let w° be the decorated permutation associated with $\mathcal{M}$ so that $\mathcal{M} = \mathcal{M}(w^{\circ})$. Let $w_0 \cdot \mathcal{M} = \{w_0 \cdot I : I \in \mathcal{M}\}$ be the matroid obtained from $\mathcal{M}$ by reversal of the ground set $[n]$. We show that $w_0 \cdot \mathcal{M}$ is a positroid.

Let z° be the decorated permutation whose chord diagram is obtained from $D(w^{\circ})$ by arc reversal followed by a reflection across the vertical axis. By Proposition 4.2.2, $z^{\circ} = w_0 \cdot (w^{\circ})^{-1}$, and in particular, $\mathcal{M}(z^{\circ})$ is obtained from $\mathcal{M}(w^{\circ})$ via the sequence of maps corresponding with arc reversal followed by reflection on the positroid,

$$I \quad \mapsto \quad [n] \setminus I \quad \mapsto \quad w_0 \cdot ([n] \setminus ([n] \setminus I)) = w_0 \cdot I.$$

Therefore, $\mathcal{M}(z^{\circ})$ is exactly $w_0 \cdot \mathcal{M}(w^{\circ})$, so $w_0 \cdot \mathcal{M}(w^{\circ}) = w_0 \cdot \mathcal{M}$ is a positroid by Theorem 2.4.5. □

Remark 4.2.4. The closure of the set of positroids under reversal of the ground set can also be obtained using the fact that positroids are the matroids of totally nonnegative matrices. In particular, suppose A is a totally nonnegative $k \times n$ matrix with positroid $\mathcal{M}(A)$. Let A' be obtained by reversing the columns of A and multiplying the bottom row by $(-1)^{\binom{k}{2}}$. Then A' is totally nonnegative, and $\mathcal{M}(A') = \{w_0 \cdot I : I \in \mathcal{M}(A)\}$.

Remark 4.2.5. Using the notation of rigid transformations, Postnikov's bijection from decorated permutations to Grassmann intervals mentioned in Remark 2.3.2 maps w° to $w_0 \cdot [u,v]^{-1}$, which preserves the size of the anti-exceedance set. This map is similar to the reversal of ground set involution on positroids.

Observe that each of the three transformations in Proposition 4.2.2 induces a bijection from $Alignments(w^{\Omega})$ to $Alignments(z^{\Omega})$. Then $\#Alignments(w^{\Omega}) = \#Alignments(z^{\Omega})$, so it follows from Theorem 1.1.2 that $\operatorname{codim}\Pi_{w^{\Omega}} = \operatorname{codim}\Pi_{z^{\Omega}}$. Furthermore, as we have seen from the results of Section 4.1, one may determine whether a positroid variety $\Pi_{\mathcal{M}}$, corresponding to a positroid $\mathcal{M}$, is smooth or singular by performing certain computations involving the sets in $\mathcal{M}$. This fact and the maps given in Proposition 4.2.2 imply the following relationship between $\Pi_{w^{\Omega}}$ and $\Pi_{z^{\Omega}}$.

Lemma 4.2.6. *Let $w \in S^{\Omega}_{n,k}$, and let z^{Ω} be a decorated permutation whose chord diagram is obtained from $D(w^{\Omega})$ by (1) arc reversal, (2) reflection, or (3) rotation. Then, in any of these three cases, $\Pi_{w^{\Omega}}$ is smooth if and only if $\Pi_{z^{\Omega}}$ is smooth.*

Proof. We know from the equivalence of Parts (1) and (3) of Theorem 1.2.5 that $\Pi_{w^{\Omega}} \subseteq Gr(k,n)$ is smooth if and only if, for every $J \in \mathcal{M}(w^{\Omega})$, J satisfies

$$\#\{I \in \mathcal{M}(w^{\Omega}) : |I \cap J| = k-1\} = k(n-k) - \#Alignments(w^{\Omega}). \tag{4.2.5}$$

As noted above, in all three cases, $\#Alignments(z^{\Omega}) = \#Alignments(w^{\Omega})$. Observe that $\Pi_{z^{\Omega}} \subseteq Gr(k(z^{\Omega}),n)$, where $k(z^{\Omega}) = n-k$ in the cases of arc reversal and reflection, and $k(z^{\Omega}) = k$ in the case of rotation. Therefore, $\Pi_{z^{\Omega}}$ is smooth if and only if every $J \in \mathcal{M}(z^{\Omega})$ satisfies

$$\#\{I \in \mathcal{M}(z^{\Omega}) : |I \cap J| = k(z^{\Omega}) - 1\} = k(n-k) - \#Alignments(w^{\Omega}). \tag{4.2.6}$$

For the cases of arc reversal and reflection, recall from Proposition 4.2.2 that the positroids $\mathcal{M}(z^{\Omega})$ and their nonbases in these cases are obtained by the maps $I \mapsto [n] \smallsetminus I$ and $I \mapsto w_0 \cdot ([n] \smallsetminus I)$, respectively. Let $I, J \in \binom{[n]}{k}$. Then $|I \cap J| = k-1$ if and only if $|([n] \smallsetminus I) \cap ([n] \smallsetminus J)| = (n-k)-1$ if and only if $|w_0 \cdot ([n] \smallsetminus I) \cap w_0 \cdot ([n] \smallsetminus J)| = (n-k)-1$. Hence, $I \in \mathcal{M}(w^{\Omega})$ contributes to the set in (4.2.5) if and only if $[n] \smallsetminus I \in \mathcal{M}((w^{\Omega})^{-1})$ contributes to the set in (4.2.6) for the case of arc reversal if and only if $w_0 \cdot ([n] \smallsetminus I) \in \mathcal{M}(w_0 \cdot w^{\Omega})$ contributes to the set in (4.2.6) for the case of reflection. Therefore, the result follows in these cases.

For case of rotation, recall from Proposition 4.2.2 that $\mathcal{M}(z^{\Omega}) = \mathcal{M}(w)^{+s}$. For $I, J \in \binom{[n]}{k}$, $|I \cap J| = k-1$ if and only if $|I^{+s} \cap J^{+s}| = k-1$. Therefore $I \in \mathcal{M}(w^{\Omega})$ contributes to the set in (4.2.5) if and only if $I^{+s} \in \mathcal{M}(z^{\Omega})$ contributes to the set in (4.2.6), so the result also follows for this case. □

Remark 4.2.7. In [Pos06], Postnikov defined a partial order on decorated permutations called circular Bruhat order. This order determines the containment relations on positroid varieties just as Bruhat order determines the containment relation on Schubert varieties. The covering relations in circular Bruhat order on $S^{\Omega}_{n,k}$ are determined by exchanging a simple crossing with a simple alignment by [Pos06, Thm. 17.8]. Since the rigid transformations preserve simple crossings and simple alignments, we see these operations are order preserving under circular Bruhat order.

4.3 Reduction to Connected Positroids

Recall that by Theorem 2.4.3, each positroid $\mathcal{M}$ on $[n]$ can be uniquely constructed by choosing a non-crossing partition $B_1 \sqcup \cdots \sqcup B_t$ of $[n]$, and then putting the structure of a connected positroid $\mathcal{M}_i$ on each block B_i, so $\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_t$. The noncrossing partition also determines a decomposition of the chord diagram of the associated decorated permutation into connected components as a union of directed arcs inscribed in the plane. We will show that a positroid variety is smooth if and only if the positroid varieties corresponding with each connected component are smooth. By direct analysis of the Jacobian matrix at a T-fixed point, we show that this matrix can be decomposed into a block diagonal matrix corresponding with the connected components of the associated positroid. We will need a slight refinement of Corollary 2.4.4.

Lemma 4.3.1. *Let $\mathcal{M}$ be a positroid on ground set $[n]$. If $\mathcal{M}$ is not connected, then up to a possible cyclic shift, it has a decomposition of the form*

$$\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2^{+n_1} = \{I \cup J^{+n_1} : I \in M_1, J \in M_2\}.$$

where $n = n_1 + n_2$, $\mathcal{M}_1$ and $\mathcal{M}_2$ are positroids on ground sets $[n_1]$ and $[n_2]$ respectively.

Proof. By [ARW16, Prop 7.4], if $\mathcal{M}$ is not connected then we can assume it is the direct sum of two positroids $\mathcal{M}_1$ and $\mathcal{M}_2$ on disjoint cyclic intervals. Using Lemma 4.2.6 and the fact that positroids are closed under the transformation of rotating the set $[n]$, we can assume that $\mathcal{M}_1$ has ground set $[1, n_1]$ and $\mathcal{M}_2$ has ground set $[n_1 + 1, n]$. □

As discussed in Section 4.1, for a set $I \in \mathcal{M}$, one may classify the point $A_I \in \Pi_{\mathcal{M}}$ as a smooth or singular point by computing the rank of the Jacobian matrix, $Jac(\mathcal{M})$, for $\Pi_{\mathcal{M}}$ evaluated at A_I. In the regime where $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2^{+n_1}$, as in Lemma 4.3.1, $Jac(\mathcal{M})|_{A_I}$ can be written as a block diagonal matrix as follows.

Lemma 4.3.2. *Let $\mathcal{M}_i \subseteq \binom{[n_i]}{k_i}$ be positroids for $i \in \{1,2\}$, and let $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2^{+n_1}$. Let $I_i \in \mathcal{M}_i$ for $i \in \{1,2\}$, and let $I = I_1 \sqcup I_2^{+n_1} \in \mathcal{M}$. Then the Jacobian matrix $Jac(\mathcal{M})|_{A_I}$ can be written as a block diagonal matrix whose first and second blocks are, up to signs of entries, the matrices $Jac(\mathcal{M}_i)|_{A_{I_i}}$ for $i \in \{1,2\}$, and whose third block has rank $k_1(n_2 - k_2) + k_2(n_1 - k_1)$.*

Proof. Recall that for a positroid $\mathcal{M} \subset \binom{[n]}{k}$, the rows of $Jac(\mathcal{M})$ are indexed by the nonbases in $\mathcal{Q}(\mathcal{M}) = \binom{[n]}{k} \smallsetminus \mathcal{M}$, and the columns of $Jac(\mathcal{M})$ are indexed by variables x_{ij}, where $i \in [k]$ and $j \in [n]$. Consider $I \in \mathcal{M}$. By Theorem 1.2.2, if $[u, v] \in Gi(k, n)$ is the Grassmann interval corresponding to $\mathcal{M}$, then there is some $y \in [u, v]$ such that $I = y[k]$. By Lemma 4.1.1, $Jac(\mathcal{M})|_{A_I}$ is, up to the signs of the entries, a partial permutation matrix whose nonzero entries occur exactly in the cells (J, x_{st}), where $J \in \mathcal{Q}(\mathcal{M})$ satisfies $|I \cap J| = k - 1$, $I \smallsetminus J = \{y_s\}$, and $J \smallsetminus I = \{t\}$.

Set $n = n_1 + n_2$, $k = k_1 + k_2$, $Jac = Jac(\mathcal{M})$, $Jac_i = Jac(\mathcal{M}_i)|_{A_{I_i}}$, and $\mathcal{Q}_i = \mathcal{Q}(\mathcal{M}_i) \subset \binom{[n_i]}{k_i}$ for $i \in \{1,2\}$. Let $[u^{(i)}, v^{(i)}]$ be the Grassmann interval corresponding to $\mathcal{M}_i$, and let $y^{(i)} = u(I_i, v) \in [u^{(i)}, v^{(i)}]$, as in Definition 3.1.2. By considering the decorated permutations corresponding to $\mathcal{M}$ and the $\mathcal{M}_i$ and the associated decomposition, as in Corollary 2.4.4, $v_j = v_j^{(1)}$ for $j \in [k_1]$ and $v_{j+k_1} = v_j^{(2)} + n_1$ for $j \in [k_2]$. Since every element of $I_2^{+n_1}$ is greater than every element of $v[k_1]$, then the construction of $y = u(I, v)$ replicates the constructions of the $y^{(i)}$ so that $y_j = y_j^{(1)}$ for $j \in [k_1]$ and $y_{j+k_1} = y_j^{(2)} + n_1$ for $j \in [k_2]$.

By definition, $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2^{+n_1}$. Therefore, the collection of nonbases can be partitioned as $\mathcal{Q} = B \sqcup C \sqcup D \sqcup E \sqcup F$, where the sets in the partitioned are defined as

$$B = \mathcal{Q}_1 \oplus \{I_2^{+n_1}\}, \qquad C = \{I_1\} \oplus \mathcal{Q}_2^{+n_1},$$
$$D = \left\{J \in \binom{[n]}{k} : |J \cap [n_1]| > k_1 \text{ or } |J \cap [n_1+1, n]| > k_2\right\},$$
$$E = \mathcal{Q}_1 \oplus \left(\binom{[n_1+1, n]}{k_2} \smallsetminus \{I_2^{+n_1}\}\right), \text{ and } \quad F = (\mathcal{M}_1 \smallsetminus \{I_1\}) \oplus \mathcal{Q}_2^{+n_1}.$$

We also partition the set $[k] \times [n] =$ **(I)** $\sqcup$ **(II)** $\sqcup$ **(III)** in the following way.

- **(I)** $= [k_1] \times [n_1]$
- **(II)** $= [k_1+1, k] \times [n_1+1, n]$
- **(III)** $= ([k_1+k, k] \times [n_1]) \cup ([k_1] \times [n_1+1, n])$

The partitions above yield partitions of the row and column sets of $Jac|_{A_I}$. From these partitions, $Jac|_{A_I}$ can be decomposed as a block diagonal matrix in the following way.

Block 1: ($J \in B$) The rows of the first block will be indexed by the sets $J \in B$, and the columns of this block will be indexed by x_{rc} with pairs (r, c) in **(I)**. The maps $J_1 \mapsto J_1 \sqcup I_2^{+n_1}$ for $J_1 \in \mathcal{Q}_1$ and $(r, c) \mapsto (r, c)$ for $(r, c) \in [k_1] \times [n_1]$ together give a bijection between the cells in Jac_1 and the cells in this upper left block of $Jac|_{A_I}$.

Since $I = I_1 \sqcup I_2^{+n_1}$, then a set $J = J_1 \sqcup I_2^{+n_1} \in \mathcal{Q}_1 \oplus \{I_2^{+n_1}\}$ satisfies $|I \cap J| = k-1$ if and only if $|I_1 \cap J_1| = k_1 - 1$. By Lemma 4.1.1 and Theorem 1.2.2, for any such $J_1 \in \mathcal{Q}_1$, the unique nonzero entry of Jac_1 in row J_1 is in column x_{st}, where $I_1 \smallsetminus J_1 = \{y_s^{(1)}\}$ and $J_1 \smallsetminus I_1 = \{t\}$. Since $I = y[k] = y^{(1)}[k_1] \cup y^{(2)}[k_2]^{+n_1}$, then $I \smallsetminus J = \{y_s\}$ and $J \smallsetminus I = \{t\}$. Hence, for this same pair (s, t), entry (J, x_{st}) of $Jac|_{A_I}$ will be nonzero. Note also that this unique pair is in **(I)**.

Therefore, this upper left block with rows indexed by B and columns indexed by **(I)** looks, up to sign, like Jac_1. Furthermore, all entries in the rows indexed by B, but outside of the columns indexed by **(I)**, will be zeros.

<u>Block 2</u>: $(J \in C)$ This case is similar to the previous case. The rows of the second block will be indexed by the sets $J \in C$, and the columns of this block will be indexed by x_{rc} with (r,c) in **(II)**. The maps $J_2 \mapsto I_1 \sqcup J_2^{+n_1}$ for $J_2 \in \mathcal{Q}_2$ and $(r,c) \mapsto (r+k_1, c+n_1)$ for $(r,c) \in [k_2] \times [n_2]$ together give a bijection between the cells in Jac_2 and the cells in the second block of $Jac|_{A_I}$. A similar argument to that of the Block 1 case with elements and indices shifted by these maps shows that this second block looks, up to sign, like Jac_2 and that all entries in the rows indexed by C, but outside of the columns indexed by **(II)**, will be zeros.

<u>Block 3</u>: $(J \in D \sqcup E \sqcup F)$ The rows of the third block will be indexed by the sets $J \in D \sqcup E \sqcup F$, and the columns of this block will be indexed by pairs (r,c) in set **(III)**. First, consider $J = J_1 \cup J_2 \in D$, where $J_1 = J \cap [n_1]$ and $J_2 = J \cap [n_1+1, n]$. Then J satisfies $|I \cap J| = k-1$ if and only if either

(i) $J_1 = I_1 \setminus \{y_s\}$ for some $y_s \in I_1$ and $J_2 = I_2^{+n_1} \cup \{t\} \in \binom{[n_1+1,n]}{k_2+1}$, or

(ii) $J_1 = I_1 \cup \{t\} \in \binom{[n_1]}{k_1+1}$ and $J_2 = I_2^{+n_1} \backslash \{y_s\}$ for some $y_s \in I_2^{+n_1}$.

In case (i), the column pair (s,t) satisfying Lemma 4.1.1 has $s \in [k_1]$ since $y_s \in [n_1]$ and $t \in [n_1+1, n]$. There are k_1 choices for s and $n_2 - k_2$ choices for t. In case (ii), the pair (s,t) satisfying Lemma 4.1.1 has $s \in [k_1+1, k]$ since $y_s \in [n_1+1, n]$ and $t \in [n_1]$. There are $n_1 - k_1$ choices for t and k_2 choices for s. Thus, the rank of this block restricted to the rows in D is $k_1(n_2-k_2) + k_2(n_1-k_1)$. Furthermore, all nonzero entries in the rows indexed by D occur in the columns in **(III)** by arguments similar to the proof of Lemma 4.1.1.

We claim that for $J \in E \sqcup F$, $|I \cap J| < k-1$. Hence, $Jac|_{A_I}$ has all zeros in row J by Lemma 4.1.1. To prove the claim, consider $J = J_1 \sqcup J_2 \in E$, where $J_1 \in \mathcal{Q}_1$ and $J_2 \in \binom{[n_1+1,n]}{k_2} \setminus \{I_2^{+n_1}\}$. Since J_1 is in $\mathcal{Q}_1$, so $J_1 \neq I_1$, then $|I_1 \cap J_1| \leq k_1 - 1$. Similarly, since $J_2 \neq I_2^{+n_1}$, then $|I_2^{+n_1} \cap J_2| \leq k_2 - 1$. Therefore, $|I \cap J| \leq (k_1-1) + (k_2-1) = k-2$. Next, consider $J = J_1 \sqcup J_2^{+n_1} \in F$, where $J_1 \in \mathcal{M}_1 \setminus \{I_1\}$ and $J_2 \in \mathcal{Q}_2$. Since $J_1 \neq I_1$, then $|I_1 \cap J_1| \leq k_1 - 1$. Similarly, since J_2 is in $\mathcal{Q}_2$, and therefore is not I_2, then $|I_2 \cap J_2| = |I_2^{+n_1} \cap J_2^{+n_1}| \leq k_2 - 1$. Therefore, $|I \cap J| \leq (k_1-1) + (k_2-1) = k-2$. $\square$

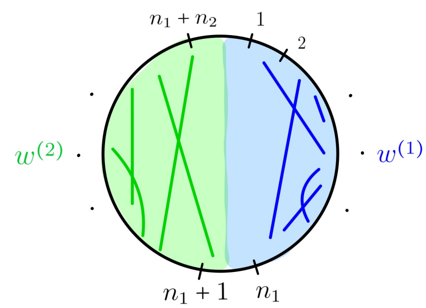


Figure 4.1: Chord diagram for $w^{(1)} \oplus w^{(2)}$.

Lemma 4.3.3. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be positroids on grounds sets $[n_1]$ and $[n_2]$. Let $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2^{+n_1}$ and let $w^{\circlearrowleft} = w^{(1)} \oplus w^{(2)}$ be the associated decorated permutation. Let k_i be the number of anti-exceedances of $w^{(i)}$. Then,*

$$\#Alignments(w^{\circlearrowleft}) \;=\; \#Alignments(w^{(1)}) + \#Alignments(w^{(2)}) + k_1(n_2 - k_2) + k_2(n_1 - k_1).$$

Proof. Consider the chord diagram for $w^{\circlearrowleft}$ partitioned by a line separating the connected components of the noncrossing partition corresponding with $w^{(1)}$ and $w^{(2)}$. See Figure 4.1. Every alignment of $w^{(i)}$ remains an alignment of $w^{\circlearrowleft}$. In addition, every anti-exceedance arc of $w^{(i)}$ forms an alignment with every exceedance arc of $w^{(i')}$, where $i' = \frac{3+(-1)^{i-1}}{2}$. Since $w^{(i)}$ has k_i anti-exceedances and $n_i - k_i$ exceedances, the result follows. □

Lemma 4.3.4. *Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be positroids on grounds sets $[n_1]$ and $[n_2]$ of rank k_1 and k_2, respectively. Let $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2^{+n_1}$. For any $I = I_1 \sqcup I_2^{+n_1} \in \mathcal{M}$, A_I is a smooth point of $\Pi_{\mathcal{M}}$ if and only if the A_{I_1} and A_{I_2} are both smooth points of $\Pi_{\mathcal{M}_1}$ and $\Pi_{\mathcal{M}_2}$, respectively.*

Proof. By Theorem 2.4.3 and the construction, $\mathcal{M}$ is a positroid of rank $k = k_1 + k_2$ on ground set $[n]$ for $n = n_1 + n_2$. Let $w^{\circlearrowleft} = w^{(1)} \oplus w^{(2)}$ be the corresponding decomposition of the associated decorated permutation, as in Lemma 4.3.3. Let $Jac = Jac(\mathcal{M})|_{A_I}$ and $Jac_i = Jac(\mathcal{M}_i)|_{A_{I_i}}$ for $i \in \{1, 2\}$. Since $\operatorname{rank}(Jac)$ is bounded above by $\#Alignments(w^{\circlearrowleft})$, then by (1.1.3), A_I is a smooth point of $\Pi_{\mathcal{M}}$ if and only if $\operatorname{rank}(Jac) = \#Alignments(w^{\circlearrowleft})$. Similarly, A_{I_i} is a smooth point of $\Pi_{\mathcal{M}_i}$ if and only if $\operatorname{rank}(Jac_i) = \#Alignments(w^{(i)})$.

By Lemma 4.3.2, $\operatorname{rank}(Jac) = \operatorname{rank}(Jac_1) + \operatorname{rank}(Jac_2) + k_1(n_2 - k_2) + k_2(n_1 - k_1)$. It follows from Lemma 4.3.3 that A_I is a smooth point of $\Pi_{\mathcal{M}}$ if and only if

$$\operatorname{rank}(Jac_1) + \operatorname{rank}(Jac_2) = \#Alignments(w^{(1)}) + \#Alignments(w^{(2)}). \tag{4.3.1}$$

Again, since $\operatorname{rank}(Jac_i) \leq \#Alignments(w^{(i)})$, then (4.3.1) holds if and only if $\operatorname{rank}(Jac_i) = \#Alignments(w^{(i)})$ for both $i = 1, 2$, which holds if and only if A_{I_i} is a smooth point of $\Pi_{\mathcal{M}_i}$ for both $i = 1, 2$. □

Corollary 4.3.5. *Let $\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_t$ be a positroid decomposed into its connected components. Then $\Pi_{\mathcal{M}}$ is smooth if and only if $\Pi_{\mathcal{M}_i}$ is smooth for each $i \in [t]$.*

Proof. If $\mathcal{M}$ is connected the statement holds, so assume $\mathcal{M}$ is not connected. We know that $\Pi_{\mathcal{M}}$ is smooth if and only if all cyclic rotations of $\mathcal{M}$ correspond with smooth positroid varieties by Lemma 4.2.6. Therefore, by Lemma 4.3.1, we can assume it has a decomposition of the form $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2^{+n_1}$ where $n = n_1 + n_2$ for some $1 \leq n_1, n_2 < n$, and $\mathcal{M}_1$ and $\mathcal{M}_2$ are positroids on ground sets $[n_1]$ and $[n_2]$, respectively. By Lemma 4.3.4, $\Pi_{\mathcal{M}}$ has a singular T-fixed point A_I for $I \in \mathcal{M}$ if and only if either $\Pi_{\mathcal{M}_1}$ or $\Pi_{\mathcal{M}_2}$ has a singular T-fixed point. Therefore, the result holds by Theorem 1.2.2, Corollary 4.1.3, and induction on t. □

Remark 4.3.6. By Corollary 2.4.4, the connected components of positroids are associated with decorated SIF permutations. In order to complete the proof of Theorem 1.2.5, it remains to characterize the decorated SIF permutations indexing smooth positroid varieties by Corollary 4.3.5. We will show that these are in bijection with the spirograph permutations, discussed further in the next section.

Chapter 5

DECORATED PERMUTATION PATTERNS AND POSITROID VARIETIES

This chapter is concerned with connecting two of our central decorated permutation patterns to singular and smooth positroid varieties, namely crossed alignments and spirographs. Recall crossed alignments and the class of spirograph permutations introduced in Chapter 1.

In Section 5.1, we show that spirograph permutations always index smooth positroid varieties. We will also show in this section that direct sums of spirograph permutations on a noncrossing partition are exactly the same as decorated permutations that have no crossed alignments. In particular, we prove the equivalence of Parts (5), (6), and (7) in Theorem 1.2.5.

In Section 5.2, we relate decorated permutations with crossed alignments to singular positroids varieties as determined by the vertex degrees of the matroid Johnson graphs in Parts (1)-(4) in Theorem 1.2.5. By the end of Section 5.1, we will have proved Parts (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) and (6) $\Rightarrow$ (1) in Theorem 1.2.5. To finish the proof of Theorem 1.2.5, it suffices to prove (1) $\Rightarrow$ (5), namely if $\Pi_{w^{\Omega}}$ is smooth, then w^{Ω} has no crossed alignments.

5.1 Connecting Crossed Alignments and Spirographs

By Definition 1.2.4, the spirograph permutations are a subset of the decorated SIF permutations. The chord diagram and the positroid associated to a spirograph permutation is always connected. The two decorated permutations in $S_1^{\circ\bullet}$ are spirographs, and for $n > 1$, there are $n-1$ distinct spirograph permutations corresponding with $m \in [n-1]$. Therefore, the generating function for spirograph permutations in S_n for $n \geq 1$ is

$$S(x) = 2x + x^2 + 2x^3 + 3x^4 + \ldots = \frac{2x - 3x^2 + 2x^3}{(1-x)^2}. \tag{5.1.1}$$

Remark 5.1.1. Each $S_{n,k}^{\circlearrowleft}$ has a unique spirograph permutation, denoted $\pi_{n,k}$. This spirograph permutation is defined so that $\pi_{n,k}(i) = i + k$ for all i and is the unique maximal element in circular Bruhat order in $S_{n,k}^{\circlearrowleft}$ for $n > 1$. For $k = n = 1$, $\pi_{1,1}$ is the unique decorated permutation in $S_{n,k}^{\circlearrowleft}$, which consists of a single clockwise fixed point. For $k = 0, n = 1$, $\pi_{1,0}$ is the unique decorated permutation in $S_{n,k}^{\circlearrowleft}$, which has a counterclockwise fixed point.

Lemma 5.1.2. If $w^{\circlearrowleft}$ is the unique the spirograph permutation in $S_{n,k}^{\circlearrowleft}$, then $\Pi_{w^{\circlearrowleft}}$ is $Gr(k,n)$, which is a smooth variety.

Proof. Observe that the chord diagram of a spirograph permutation $w^{\circlearrowleft}$ has no alignments, so the codimension of $\Pi_{w^{\circlearrowleft}}$ as a subvariety of $Gr(k,n)$ is zero by Theorem 1.1.2. Hence, $\Pi_{w^{\circlearrowleft}} = Gr(k,n)$. □

Proof of Theorem 1.2.5, $(6) \Rightarrow (1)$. Let $D(w^{\circlearrowleft})$ be a disjoint union of spirographs corresponding to the decomposition $w^{\circlearrowleft} = w^{(1)} \oplus \cdots \oplus w^{(t)}$ of $w^{\circlearrowleft}$ into decorated SIF permutations with $\mathcal{M}(w^{\circlearrowleft}) = \mathcal{M}(w^{(1)}) \oplus \cdots \oplus \mathcal{M}(w^{(t)})$ using Corollary 2.4.4. By Lemma 5.1.2, the positroid varieties $\Pi_{w^{(i)}}$ are all smooth. Hence, $\Pi_{w^{\circlearrowleft}} = \Pi_{\mathcal{M}(w^{\circlearrowleft})}$ is smooth by Corollary 4.3.5. □

Proof of Theorem 1.2.5, $(5) \Leftrightarrow (6)$. By definition of a chord diagram $D(w^{\circlearrowleft})$, arcs from distinct components of the associated noncrossing partition must be drawn so that they do not intersect. It follows that, if $w^{\circlearrowleft}$ contains a crossed alignment, then all arcs involved in the crossed alignment must be contained in the same connected component of $D(w^{\circlearrowleft})$.

A spirograph permutation has no crossed alignments since it has no alignments. By the observation above, if every connected component of $D(w^{\circlearrowleft})$ is a spirograph, then $w^{\circlearrowleft}$ has no crossed alignments.

For the converse, observe that the property of containing a crossed alignment is invariant under rotation of the chord diagram. Hence, by Corollary 2.4.4 and the fact that a crossed alignment is contained a single connected component of $D(w^{\circlearrowleft})$, the argument may be completed by assuming that $w^{\circlearrowleft}$ is a decorated SIF permutation that is not a spirograph and showing that $w^{\circlearrowleft}$ has a crossed alignment. For $n < 4$, $w^{\circlearrowleft}$ is either a spirograph permutation

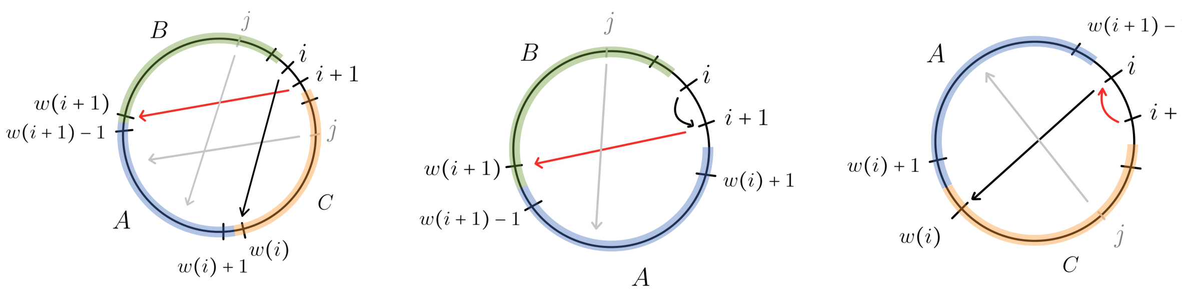


Figure 5.1: Chord diagrams with $(i \mapsto w(i))$ and $(i+1 \mapsto w(i+1))$ crossing.

or has more than one connected component. Therefore, we must have $n \geq 4$, in which case we will denote w^{Ω} simply by w since it has no fixed points.

Since w has no fixed points, $m_i := w(i) - i \pmod n \in [n-1]$ for each $i \in [n]$. Since w is not a spirograph permutation, the m_i are not all equal. Therefore, taking indices mod n, there must exist some $i \in [n]$ such that $m_i < m_{i+1}$. Observe from the chord diagram that $m_i < m_{i+1}$ implies the arcs $(i \mapsto w(i))$ and $(i+1 \mapsto w(i+1))$ form a crossing. Use the crossing arcs to create three disjoint cyclic intervals. Let $A = [w(i)+1, w(i+1)-1]^{cyc}$. The fact that $m_i < m_{i+1}$ implies that A is nonempty. Let $B = [w(i+1), i-1]^{cyc} \setminus [i, w(i+1)-1]^{cyc}$ and $C = [i+2, w(i)]^{cyc} \setminus [w(i)+1, i+1]^{cyc}$. See Figure 5.1.

Since w is a SIF permutation, the arcs incident to elements of A cannot form an isolated connected component. Thus, there must be an arc mapping some $j \in [n] \setminus (A \cup \{i, i+1\}) = B \cup C$ to $w(j) \in A$. If $j \in B$, then $(i \mapsto w(i), j \mapsto w(j))$ is an alignment crossed by $(i+1 \mapsto w(i+1))$. If $j \in C$, then $(j \mapsto w(j), i+1 \mapsto w(i+1))$ is an alignment crossed by $(i \mapsto w(i))$. Therefore, w contains a crossed alignment. □

Proof of 1.2.5, (6) ⇔ (7). The unique spirograph permutation $\pi_{n,k}$ in $S^{\Omega}_{n,k}$, mentioned in Remark 5.1.1, has shifted anti-exceedance sets $I_r(\pi_{n,k}) = [k]^{+(r-1)}$ for all $r \in [n]$. Since $[k]^{+(r-1)}$ is the minimal element of $\binom{[n]}{k}$ under $\prec_r$, it follows from Theorem 2.4.5 that $\mathcal{M}(\pi_{n,k}) = \binom{[n]}{k}$. Hence, spirograph permutations correspond exactly to uniform matroids. By the correspondence in Corollary 2.4.4, a positroid $\mathcal{M} = \mathcal{M}(w^{\Omega})$ is a direct sum of uniform matroids if and only if w° is a direct sum of spirograph permutations. $\square$

5.2 Anti-Exchange Pairs and Crossed Alignments

Recall from the beginning of this chapter that to finish the proof of Theorem 1.2.5, it suffices to prove (1) $\Rightarrow$ (5), namely if $\Pi_{w^{\circ}}$ is smooth, then w° has no crossed alignments. This final step will be proved using the characterization of smoothness from Corollary 4.1.4 utilizing the nonbases in $\mathcal{Q}(w^{\circ})$. Specifically, for any $w^{\circ} \in S^{\circ}_{n,k}$, $\Pi_{w^{\circ}}$ is singular if and only if there exists a $J \in \mathcal{M}(w^{\circ})$ such that

$$\#\{I \in \mathcal{Q}(w^{\circ}) \,:\, |I \cap J| = k-1\} \;<\; \#Alignments(w). \tag{5.2.1}$$

Thus, we now carefully study the sets I that occur on the left side of this equation. The following vocabulary refers back to the Basis Exchange Property of matroids from Chapter 2.

Definition 5.2.1. For a set J in a positroid $\mathcal{M} \subseteq \binom{[n]}{k}$, let $a \in J$ and $b \in [n] \setminus J$. If $(J \setminus \{a\}) \cup \{b\} \in \mathcal{M}$, we say the pair (a,b) is an *exchange pair for* J and that the values a and b are *exchangeable*. Otherwise, $(J \setminus \{a\}) \cup \{b\} \in \mathcal{Q} = \binom{[n]}{k} \setminus \mathcal{M}$, in which case we say (a,b) is an *anti-exchange pair for* J and that the values a and b are *not exchangeable*.

For a set $J \in \mathcal{M} \subseteq \binom{[n]}{k}$, pairs (a,b) with $a \in J$ and $b \notin J$ are in bijection with the vertices in the full Johnson graph $J(k,n)$ adjacent to J. Exchange pairs for J are in bijection with the vertices adjacent to J in the matroid Johnson graph $J(\mathcal{M})$. Anti-exchange pairs for J are in bijection with the vertices adjacent to J in $J(k,n)$ that do not appear in $J(\mathcal{M})$.

5.2.1 Characterizing Exchange Pairs in the Johnson Graph

For a positroid $\mathcal{M}(w^{\circ})$, we will focus on the set $J = I_1(w^{\circ})$, which is in $\mathcal{M}(w^{\circ})$ by Corollary 2.4.6. The following lemma characterizes exchange pairs for $I_1(w^{\circ})$.

Lemma 5.2.2. For $w^{\circ} = (w, \mathrm{co}) \in S^{\circ}_{n,k}$, suppose that $a \in I_1(w^{\circ})$ and $b \in [n] \setminus I_1(w^{\circ})$. Then $I = (I_1(w^{\circ}) \setminus \{a\}) \cup \{b\}$ is in $\mathcal{M}(w^{\circ})$ if and only if $a < b$ and for every $r \in [a+1, b]$, both of the following conditions hold:

(1) there exists $x \in [a, r-1]$ such that $w^{-1}(x) \in [r, n]$, and

(2) there exists $y \in [r, b]$ such that $w^{-1}(y) \in [1, r-1]$.

Proof. Let $I_r = I_r(w^{\Omega})$. Recall from Theorem 2.4.5 that I is in $\mathcal{M} = \mathcal{M}(w^{\Omega})$ if and only if $I_r \preceq_r I$ for all $r \in [n]$. We use this characterization of $\mathcal{M}$ in terms of the elements of the Grassmann necklace to derive the conditions of the lemma.

Consider the case when $a > b$. Then I is obtained by replacing an element of I_1 with a smaller element. By definition of the Gale order, I cannot satisfy $I_1 \preceq I$, so $I \notin \mathcal{M}$.

For the remainder of the proof, assume $a < b$. For $r \in [b+1, a]^{cyc}$, the inequality $a <_r b$ holds, which implies that I is obtained by replacing an element of I_1 with an element that is larger under $<_r$. Thus, $I_1 \prec_r I$. By Corollary 2.4.6, I_1 is in $\mathcal{M}$, so by Theorem 2.4.5, I_1 must satisfy $I_r \preceq_r I_1$. Hence, the sequence of inequalities $I_r \preceq_r I_1 \prec_r I$ holds for every $r \in [b+1, a]^{cyc}$. Therefore, $I \in \mathcal{M}$ if and only if $I_r \preceq_r I$ for every $r \in [a+1, b]$.

Fix some $r \in [a+1, b]$, so $a >_r b$. An arc $(i \mapsto j)$ with $i \in [r, n]$ and $j \in [a, r-1]$ is an anti-exceedance arc, but not an r-anti-exceedance arc, so $j \in I_1 \setminus I_r$. Similarly, an arc $(i \mapsto j)$ with $i \in [1, r-1]$ and $j \in [r, b]$ is an r-anti-exceedance arc, but not an anti-exceedance arc, so $j \in I_r \setminus I_1$. Therefore, the following statements hold.

$$\text{For } x \in [a, r-1],\ w^{-1}(x) \in [r, n] \Leftrightarrow x \in I_1 \setminus I_r. \tag{5.2.2}$$

$$\text{For } y \in [r, b],\ w^{-1}(y) \in [1, r-1] \Leftrightarrow y \in I_r \setminus I_1. \tag{5.2.3}$$

See Figure 5.2 for these two cases. From these observations, for the fixed $r \in [a+1, b]$, Conditions (1) and (2) in the statement of the lemma are equivalent to the following:

(1′) $(I_1 \setminus I_r) \cap [a, r-1] \neq \varnothing$, and

(2′) $(I_r \setminus I_1) \cap [r, b] \neq \varnothing$.

We must now compare the sets I_r and I under the shifted Gale order $\prec_r$ for $r \in [a+1, b]$. Recall that I is obtained from I_1 by exchange of the elements a and b, where $a \in I_1$ and

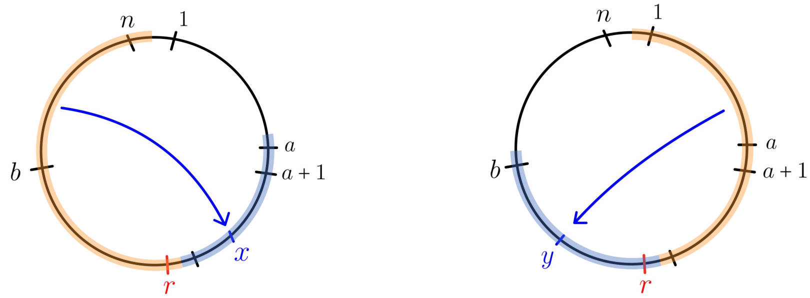


Figure 5.2: Two conditions of Lemma 5.2.2

$b \notin I_1$. Thus we begin by writing $I_1 = \{j_1 <_r \cdots <_r j_k\}$ increasing under the $<_r$ order. Setting $f = |I_1 \cap [r, b]| + 1$, $g = |I \cap [r, n]| + 1$, and $h = |I \cap [r, a]^{cyc}|$, we refine the expression of I_1 in the following way,

$$I_1 = \{\underbrace{j_1 <_r \cdots <_r j_{f-1}}_{\in [r,b]} <_r \underbrace{j_f <_r \cdots <_r j_{g-1}}_{\in [b+1,n]} <_r \underbrace{j_g <_r \cdots <_r j_{h-1} <_r a}_{\in [1,a]} <_r \underbrace{j_{h+1} <_r \cdots <_r j_k}_{\in [a+1,r-1]}\}.$$

Here, $a = j_h$. Note that since $b \notin I_1$, then $j_{f-1} <_r b$. Write I_r and I under $<_r$ as

$$\begin{aligned} I_r &= \{i_1 <_r \cdots <_r i_{f-1} <_r i_f <_r i_{f+1} <_r \cdots <_r i_g <_r i_{g+1} <_r \cdots <_r i_h <_r i_{h+1} <_r \cdots <_r i_k\} \\ I &= \{j_1 <_r \cdots <_r j_{f-1} <_r b <_r j_f <_r \cdots <_r j_{g-1} <_r j_g <_r \cdots <_r j_{h-1} <_r j_{h+1} <_r \cdots <_r j_k\}. \end{aligned}$$

Then $I_r \leq_r I$ if and only if all three of the following hold:

(i) $i_\ell \leq_r j_\ell$ for all $\ell \in [1, f-1] \cup [h+1, k]$;

(ii) $i_f \leq_r b$ and $i_{\ell+1} \leq_r j_\ell$ for all $\ell \in [f, g-1]$; and

(iii) $i_{\ell+1} \leq_r j_\ell$ for all $\ell \in [g, h-1]$.

For $\ell \in [1, f-1] \cup [h+1, k]$, $I_r \leq_r I_1$ already implies that $i_\ell \leq_r j_\ell$, so (i) is always satisfied. Thus, $I_r \leq_r I$ if and only if (ii) and (iii) are satisfied.

Consider Condition (ii) above. We will show that (ii) is satisfied if and only if $(I_r \setminus I_1) \cap [r, b]$ is nonempty, as in Condition $(2')$.

We require the following observations about the elements $j_1,\ldots,j_{g-1}$. By construction, $\{j_1,\ldots,j_{g-1}\} = I_1 \cap [r,n]$. Each of these j_ℓ has $w^{-1}(j_\ell) \in [j_\ell, n]$, as in Figure 5.3a. But then j_ℓ and $w^{-1}(j_\ell)$ also satisfy $r \leq_r j_\ell \leq_r w^{-1}(j_\ell) \leq_r n$. In particular, this implies that every such j_ℓ is also an element of I_r so that $\{j_1,\ldots,j_{g-1}\} \subseteq I_r$. Furthermore, $j_1 <_r \cdots <_r j_{g-1}$ appear in I_r in the same relative order as they do in I_1, but there may be additional elements of I_r that are interspersed among them. Therefore, for every $\ell \in [1, g-1]$, there is some $\ell' \geq \ell$ such that $i_{\ell'} = j_\ell$.

For the only if direction, (ii) implies (2′), assume $(I_r \setminus I_1) \cap [r,b]$ is empty. By construction, $I_1 \cap [r,b] = \{j_1,\ldots,j_{f-1}\}$. Since $\{j_1,\ldots,j_{f-1}\} \subset I_r$ by the previous paragraph and $(I_r \setminus I_1) \cap [r,b] = \varnothing$, then $I_r \cap [r,b] = \{j_1,\ldots,j_{f-1}\} = \{i_1,\ldots,i_{f-1}\}$. In particular, $i_f \notin [r,b]$, so $b <_r i_f$, violating Condition (ii).

Conversely, suppose $(I_r \setminus I_1) \cap [r,b]$ is nonempty. Then $|I_r \cap [r,b]| \geq |I_1 \cap [r,b]| + 1 = f$. In particular, i_f must be in $[r,b]$, so $i_f \leq_r b$. Furthermore, this extra element in $I_r \cap [r,b]$ shifts the elements $j_f,\ldots,j_{g-1}$ to the right in I_r. Specifically, for $\ell \in [f, g-1]$, $i_{\ell'} = j_\ell$ for some $\ell' > \ell$. Therefore, for $\ell \in [f, g-1]$, we have the inequalities $i_{\ell+1} \leq_r i_{\ell'} = j_\ell$. Thus, $(I_r \setminus I_1) \cap [r,b] \neq \varnothing$ implies that Condition (ii) is satisfied.

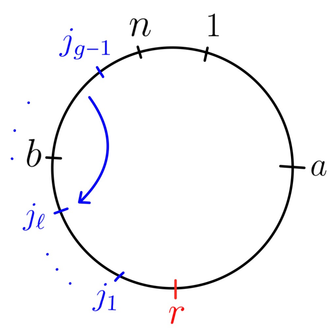
Now, consider Condition (iii) above. We will show that (iii) is satisfied if and only if Condition (1′) holds. The argument is symmetric to the argument above for Condition (ii).

Consider any $i_\ell \in I_r \cap [1, r-1]$, as in Figure 5.3b. Since $i_\ell \in I_r$, then $w^{-1}(i_\ell)$ must be in $[i_\ell, r-1]$ so that $1 \leq_r i_\ell \leq_r w^{-1}(i_\ell) \leq_r r-1$. Hence, i_ℓ is also in I_1, so $I_r \cap [1, r-1] \subseteq I_1 \cap [1, r-1]$. Thus, these $i_\ell \in I_r \cap [1, r-1]$ appear among the elements $j_g,\ldots,j_k$, with possibly some additional elements. So, each of these i_ℓ has some $j_{\ell'}$ with $\ell' \leq \ell$ for which $i_\ell = j_{\ell'}$. After possibly deleting a, all of these elements are in I.

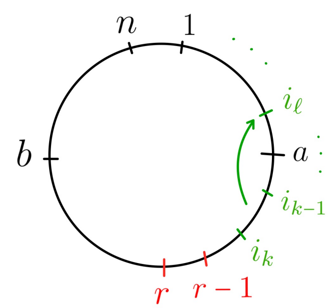
For the only if direction in this case, suppose $[a, r-1] \cap (I_1 \setminus I_r)$ is empty. By construction, $I_1 \cap [a, r-1] = \{j_h,\ldots,j_k\}$. Then $I_1 \cap [a, r-1] = I_r \cap [a, r-1]$, so $i_\ell = j_\ell$ for every $\ell \in [h,k]$. In particular, $i_h = j_h = a$. But then we have $i_h = a >_r j_{h-1}$, which violates Condition (iii) for $\ell = h-1$.

Conversely, suppose $(I_1 \setminus I_r) \cap [a, r-1]$ is nonempty. Then $|I_1 \cap [a, r-1]| > |I_r \cap [a, r-1]|$.

Since $I_1 \cap [a, r-1] = \{j_h, \dots, j_k\}$, then i_h cannot be in $[a, r-1]$, so $i_h <_r a$, and thus $i_\ell <_r a$ for all $\ell \in [g+1, h]$. Furthermore, the existence of at least one extra element in $I_1 \cap [a, r-1]$ implies that the elements of $I_r \cap [1, a-1]$ all appear shifted to the left in I_1. Specifically, for $i_\ell \in \{i_{g+1}, \dots, i_h\} \cap [1, a-1]$, there is some $\ell' < \ell$ for which $i_\ell = j_{\ell'}$. Then, for these i_ℓ, we have $i_\ell = j_{\ell'} \leq_r j_{\ell-1}$. The remaining $i_\ell \in \{i_{g+1}, \dots, i_h\}$ are in $[r, n]$, so $i_\ell <_r j_{\ell-1} \in [1, a-1]$. Hence, $i_\ell \leq_r j_{\ell-1}$ for all $\ell \in [g+1, h]$, which is equivalent to Condition (iii).



(a) $j_\ell \in I_1 \cap [r, n] \Rightarrow j_\ell \in I_r$.



(b) $i_\ell \in I_r \cap [1, r-1] \Rightarrow i_\ell \in I_1$.

Figure 5.3: Elements in both I_1 and I_r.

□

5.2.2 *Mapping Anti-Exchange Pairs to Alignments*

For a fixed $w^{\Omega} \in S_n^{\Omega}$ whose chord diagram contains a crossed alignment, we will show that (5.2.1) holds for $J = I_1(w^{\Omega})$. This will be achieved using an injective map $\Psi_{w^{\Omega}}$, which maps anti-exchange pairs for $I_1(w^{\Omega})$ to alignments in $D(w^{\Omega})$ and is defined below for any $w^{\Omega} \in S_n^{\Omega}$ using Lemma 5.2.2.

Given $w^{\Omega} \in S_n^{\Omega}$, let $\mathrm{AE}(w^{\Omega})$ be the set of all anti-exchange pairs (a, b) for $I_1(w^{\Omega})$, so $a \in I_1(w^{\Omega})$, $b \notin I_1(w^{\Omega})$, and $(I_1(w^{\Omega}) \setminus \{a\}) \cup \{b\} \notin \mathcal{M}(w^{\Omega})$. For each $(a, b) \in \mathrm{AE}(w^{\Omega})$, either $a > b$ or $a < b$ and there must exist some $r \in [a+1, b]$ for which Condition (1) or (2) of Lemma 5.2.2 fails. In the $a < b$ case, we say r is a *witness* for (a, b) to be in $\mathrm{AE}(w^{\Omega})$. Let

$\mathrm{AE}_>(w^{\circlearrowleft})$ be all the anti-exchange pairs with $a > b$. Let $\mathrm{AE}_1(w^{\circlearrowleft})$ be all the anti-exchange pairs with $a < b$ such that Condition (1) fails for some $r \in [a+1, b]$. Let $\mathrm{AE}_2(w^{\circlearrowleft})$ be the anti-exchange pairs with $a < b$ such that Condition (1) holds for all $r \in [a+1, b]$. Lemma 5.2.2 then implies for $(a, b) \in \mathrm{AE}_2(w^{\circlearrowleft})$ that Condition (2) fails for some $r \in [a+1, b]$. These three sets form a partition of the anti-exchange pairs for $I_1(w^{\circlearrowleft})$,

$$\mathrm{AE}(w^{\circlearrowleft}) = \mathrm{AE}_>(w^{\circlearrowleft}) \sqcup \mathrm{AE}_1(w^{\circlearrowleft}) \sqcup \mathrm{AE}_2(w^{\circlearrowleft}). \tag{5.2.4}$$

Lemma 5.2.3. *Fix $w^{\circlearrowleft} = (w, co) \in S_n^{\circlearrowleft}$ and $a \in I_1(w^{\circlearrowleft})$. Let $\{b_1 < \cdots < b_s\}$ be the set of all elements in $[a+1, n] \smallsetminus I_1(w^{\circlearrowleft})$ such that for each $1 \le i \le s$, there exists some minimal $r_i \in [a+1, b_i]$ for which Condition (1) of Lemma 5.2.2 fails, so $(a, b_i) \in \mathrm{AE}_1(w^{\circlearrowleft})$. If $\{b_1 < \cdots < b_s\}$ is nonempty, then defining $\bar{r}(a, w^{\circlearrowleft}) := r_1$ we have*

$$\{b_1 < \cdots < b_s\} = [\bar{r}(a, w^{\circlearrowleft}), n] \smallsetminus I_1,$$

and $\bar{r}(a, w^{\circlearrowleft}) \in [a+1, b_i]$ is a witness for Condition (1) failing for (a, b_i) for each i.

Proof. Assume $\{b_1 < \cdots < b_s\}$ is nonempty. By construction, $r_1 \in [a+1, b_1] \subseteq [a+1, b_i]$ for all i. Furthermore, r_1 is chosen so that Condition (1) of Lemma 5.2.2 fails for the pair (a, b_1), and hence there is no $x \in [a, r_1 - 1]$ such that $w^{-1}(x) \in [r_1, n]$. This last condition only depends on r_1, so Condition (1) still fails for r_1 when determining whether or not a is exchangable with any given $b \in [r_1, n] \smallsetminus I_1$, according to Lemma 5.2.2. Hence, $\{b_1 < \cdots < b_s\} = [r_1, n] \smallsetminus I_1$. □

Corollary 5.2.4. *If $\{b_1 < \cdots < b_s\}$ from Lemma 5.2.3 is nonempty, then $b < \bar{r}(a, w^{\circlearrowleft})$ for any $(a, b) \in \mathrm{AE}_2(w^{\circlearrowleft})$.*

Proof. For $(a, b) \in \mathrm{AE}_2(w^{\circlearrowleft})$, Condition (1) of Lemma 5.2.2 is satisfied for every $r \in [a+1, b]$. In particular, b is not one of the b_is as in Lemma 5.2.3, so b is not in $[\bar{r}(a, w^{\circlearrowleft}), n] \smallsetminus I_1$. Since $b \notin I_1$ by definition of an anti-exchange pair, b must be in $[1, \bar{r}(a, w^{\circlearrowleft}) - 1]$. □

Lemma 5.2.5. *Fix $w^{\circlearrowleft} = (w, co) \in S_n^{\circlearrowleft}$ and $b \notin I_1(w^{\circlearrowleft})$. Let $\{a_1 < \cdots < a_s\}$ be the elements of $[b] \cap I_1(w^{\circlearrowleft})$ such that there exists some maximal $r_i \in [a_i + 1, b]$ for which Condition (2)* *of Lemma 5.2.2 fails, so* $(a_i, b) \in \mathrm{AE}(w^{\circlearrowright})$. *If* $\{a_1 < \cdots < a_s\}$ *is not empty, then defining* $\underline{r}(b, w^{\circlearrowright}) := r_s$ *we have*

$$\{a_1 < \cdots < a_s\} \;=\; [1, \underline{r}(b, w^{\circlearrowright}) - 1] \cap I_1(w^{\circlearrowright}),$$

and $\underline{r}(b, w^{\circlearrowright}) \in [a_i + 1, b]$ *is a witness for Condition (2) failing for* (a, b_i) *for each* i.

Proof. The proof is similar to the proof of Lemma 5.2.3 by reflecting $D(w^{\circlearrowright})$ across the vertical axis. □

For each $w^{\circlearrowright} \in S_n^{\circlearrowright}$, we construct a map $\Psi_{w^{\circlearrowright}}$ from the anti-exchange pairs for $I_1(w^{\circlearrowright})$ partitioned according to (5.2.4) to ordered pairs of the form $(p \mapsto w(p), s \mapsto w(s))$ for $p \neq s$. We will show in the lemma that follows that the range of $\Psi_{w^{\circlearrowright}}$ is contained in the set of alignments for $w^{\circlearrowright}$, as defined in Definition 2.3.5.

Definition 5.2.6. For $w^{\circlearrowright} = (w, co) \in S_n^{\circlearrowright}$, let $\Psi_{w^{\circlearrowright}}$ be defined by the following algorithm.

Input : $(a, b) \in \mathrm{AE}(w^{\circlearrowright}) = \mathrm{AE}_{>}(w^{\circlearrowright}) \sqcup \mathrm{AE}_1(w^{\circlearrowright}) \sqcup \mathrm{AE}_2(w^{\circlearrowright})$

Output: $(p \mapsto w(p), s \mapsto w(s)) \in \mathit{Alignments}(w^{\circlearrowright})$

set $p \leftarrow w^{-1}(b)$ and $s \leftarrow w^{-1}(a)$.
if $(a, b) \in \mathrm{AE}_1(w)$ **then**
  while $p \notin [\bar{r}(a, w^{\circlearrowright}), n]$, **do**
    update $p \leftarrow w^{-1}(p)$.
  end
end
if $(a, b) \in \mathrm{AE}_2(w)$ **then**
  while $s \notin [1, \underline{r}(b, w^{\circlearrowright}) - 1]$, **do**
    update $s \leftarrow w^{-1}(s)$.
  end
end
return $(p \mapsto w(p), s \mapsto w(s))$.

Lemma 5.2.7. *For $w^{\circlearrowleft} \in S_n^{\circlearrowleft}$ and $(a,b) \in \mathrm{AE}(w^{\circlearrowleft})$, the algorithm defined in Definition 5.2.6 terminates in finitely many steps, and the image $\Psi_{w^{\circlearrowleft}}(a,b)$ is in Alignments$(w^{\circlearrowleft})$.*

Proof. First, consider the case when $(a,b) \in \mathrm{AE}_{>}(w^{\circlearrowleft})$, so $a > b$. Since $a \in I_1(w^{\circlearrowleft})$ and $b \notin I_1(w^{\circlearrowleft})$, we have the inequalities $w^{-1}(b) \le b < a \le w^{-1}(a)$, as indicated in Figure 5.4. Thus, $\Psi_{w^{\circlearrowleft}}(a,b) = (w^{-1}(b) \mapsto b, w^{-1}(a) \mapsto a)$ is an alignment. Note that a and b could be fixed points with $co(a) = \circlearrowright$ and $co(b) = \circlearrowleft$, according to the definition of anti-exceedances from Section 2.3.

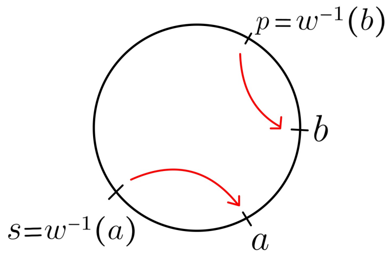
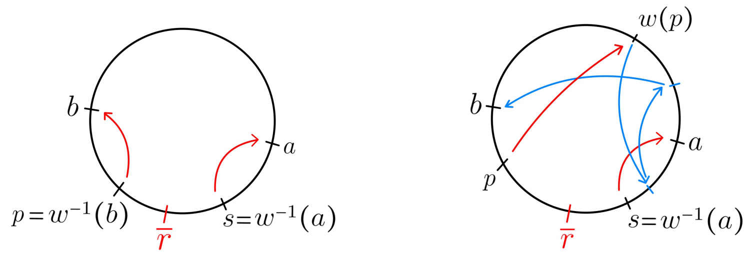


Figure 5.4: The image of (a,b) when $a > b$ is $(w^{-1}(a) \mapsto a, w^{-1}(b) \mapsto b)$.

Next, consider the case when $(a,b) \in \mathrm{AE}_1(w^{\circlearrowleft})$, and let $\bar{r} = \bar{r}(a, w^{\circlearrowleft})$. Then $\Psi_{w^{\circlearrowleft}}(a,b) = (p \mapsto w(p), w^{-1}(a) \mapsto a)$, where p is in $[\bar{r}, n]$. By definition of $\mathrm{AE}_1(w^{\circlearrowleft})$, Condition (1) of Lemma 5.2.2 fails for some $r \in [a+1, b]$. It follows from Lemma 5.2.3 that $b \ge \bar{r}$, and furthermore, that Condition (1) fails for this $\bar{r}$. In particular, there is no $x \in [a, \bar{r}-1]$ with preimage in $[\bar{r}, n]$. Since $a \in I_1(w^{\circlearrowleft})$, then $w^{-1}(a) \in [a, n]$. But $a \in [a, \bar{r}-1]$ implies that $w^{-1}(a) \notin [\bar{r}, n]$. Hence, $w^{-1}(a) \in [a, \bar{r}-1]$, as drawn in Figure 5.5.

The termination of the while loop in Line 3 of Definition 5.2.6 comes from the fact that p is obtained by tracing in reverse the cycle containing b. By Lemma 5.2.3, $b \in [\bar{r}, n]$. If $w^{-1}(b) \in [\bar{r}, n]$, then $p = w^{-1}(b)$, and the loop ends immediately. In this case, since b is an exceedance by the assumption that $(a,b) \in \mathrm{AE}_1(w^{\circlearrowleft})$, then $w^{-1}(b) \in [\bar{r}, b]$. See Figure 5.5a. However, even if $w^{-1}(b)$ is not in $[\bar{r}, n]$ so that the reverse cycle immediately leaves the interval $[\bar{r}, n]$, it must eventually return to $[\bar{r}, n]$, since $b \in [\bar{r}, n]$. In this case, the first element at which the reverse cycle returns to $[\bar{r}, n]$ is p, which implies that $w(p) \in [1, \bar{r}-1]$. However, the fact that Condition (1) is not satisfied for $\bar{r}$ then implies that $w(p) \notin [a, \bar{r}-1]$, and thus $w(p) \in [1, a-1]$. See Figure 5.5b. In both cases, $(p \mapsto w(p), w^{-1}(a) \mapsto a)$ is an alignment with port side $(p \mapsto w(p))$ and starboard side $(w^{-1}(a) \mapsto a)$.



(a) $w^{-1}(b) \geq \bar{r}$ **(b)** $w^{-1}(b) < \bar{r}$

Figure 5.5: The red alignment is the image of $(a, b) \in \mathrm{AE}_1$.

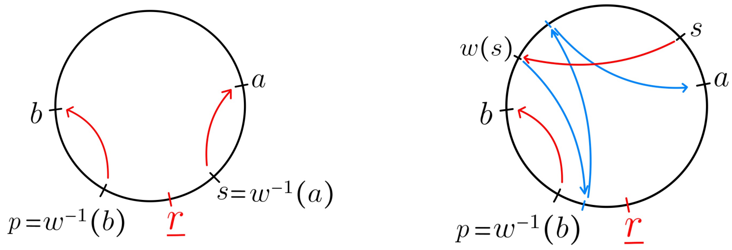
Finally, consider the case where $(a, b) \in \mathrm{AE}_2(w^{\Omega})$, and let $\underline{r} = \underline{r}(b, w^{\Omega})$. Again, this case is symmetric to the $\mathrm{AE}_1(w^{\Omega})$ case above. From the algorithm, $\Psi_{w^{\Omega}}(a, b) = (w^{-1}(b) \mapsto b, s \mapsto w(s))$, where $s \in [1, \underline{r} - 1]$. By Lemma 5.2.5, $a < \underline{r}$, and Condition (2) is not satisfied for $\underline{r}$. Thus, no $y \in [\underline{r}, b]$ has preimage in $[1, \underline{r} - 1]$. Since $b \notin I_1(w^{\Omega})$, we must have $w^{-1}(b) \in [\underline{r}, b]$, as drawn in Figure 5.6.

The termination of the while loop in Line 8 of Definition 5.2.6 is again seen to terminate by tracing the cycle containing a in reverse. Since $a \in I_1(w^{\Omega})$, then $a \leq w^{-1}(a)$. If $w^{-1}(a) \in [a, \underline{r} - 1]$, then $s = w^{-1}(a)$, as in Figure 5.6a. If $w^{-1}(a) \in [\underline{r}, n]$, then s is the first element at which the reverse cycle returns to the interval $[1, \underline{r} - 1]$, as in Figure 5.6b. Thus $w(s) \in [\underline{r}, n]$, and the fact that Condition (2) of Lemma 5.2.2 fails for $\underline{r}$ implies that $w(s) \in [b + 1, n]$. So, in both cases, $(w^{-1}(b) \mapsto b, s \mapsto w(s))$ is an alignment with port side $(w^{-1}(b) \mapsto b)$ and starboard side $(s \mapsto w(s))$.

□

Lemma 5.2.8. *Fix $w^{\Omega} = (w, co) \in S_n^{\Omega}$. The map $\Psi_{w^{\Omega}} : \mathrm{AE}(w^{\Omega}) \longrightarrow Alignments(w^{\Omega})$ is injective.*

Proof. Let (a_1, b_1) and (a_2, b_2) be two distinct anti-exchange pairs for $I_1(w^{\Omega})$. We need to



(a) $w^{-1}(a) < \underline{r}$ **(b)** $w^{-1}(a) \geq \underline{r}$

Figure 5.6: The red alignment is the image of $(a, b) \in \mathrm{AE}_2$.

show $\Psi_{w^{\Omega}}(a_1, b_1) \neq \Psi_{w^{\Omega}}(a_2, b_2)$. We again utilize the partition of $\mathrm{AE}(w^{\Omega})$ from (5.2.4) and consider several cases.

Case 1: Assume at least one of (a_1, b_1) and (a_2, b_2) is in $\mathrm{AE}_{>}(w^{\Omega})$. Observe from Definition 5.2.6 that anti-exchange pairs $(a, b) \in \mathrm{AE}_{>}(w^{\Omega})$ are the only pairs assigned alignments $\Psi_{w^{\Omega}}(a, b) = (p \mapsto w(p), s \mapsto w(s))$ with $p \leq w(p) < s \leq w(s)$. Thus, $\Psi_{w^{\Omega}}(a_1, b_1) \neq \Psi_{w^{\Omega}}(a_2, b_2)$ unless both (a_1, b_1) and (a_2, b_2) are in $\mathrm{AE}_{>}(w^{\Omega})$. In this case, $\Psi_{w^{\Omega}}(a_i, b_i) = (w^{-1}(b_i) \mapsto b_i, w^{-1}(a_i) \mapsto a_i)$ for both i by definition. Therefore, since (a_1, b_1) and (a_2, b_2) are distinct, we must have that $\Psi_{w^{\Omega}}(a_1, b_1) \neq \Psi_{w^{\Omega}}(a_2, b_2)$.

Case 2: Assume both (a_1, b_1) and (a_2, b_2) are in $\mathrm{AE}_1(w^{\Omega})$ and $\Psi_{w^{\Omega}}(a_i, b_i) = (p_i \mapsto w(p_i), w^{-1}(a_i) \mapsto a_i)$. If $a_1 \neq a_2$, then the two alignments have distinct starboard sides. If $a_1 = a_2 = a$, then $b_1 \neq b_2$. Since the p_i are determined uniquely by tracing the cycle containing b_i backwards and finding the first element in the interval $[\bar{r}(a, w), n]$, then $b_1 \neq b_2$ implies that $p_1 \neq p_2$, so the alignments have distinct port sides. Either way, we have $\Psi_{w^{\Omega}}(a_1, b_1) \neq \Psi_{w^{\Omega}}(a_2, b_2)$.

Case 3: Assume both (a_1, b_1) and (a_2, b_2) are in $\mathrm{AE}_2(w^{\Omega})$ and $\Psi_{w^{\Omega}}(a_i, b_i) = (w^{-1}(b_i) \mapsto b_i, s_i \mapsto w(s_i))$. If $b_1 \neq b_2$, then the two alignments have distinct port sides. If $b_1 = b_2 = b$, then $a_1 \neq a_2$. Since the s_i are determined uniquely by tracing the cycle containing a_i backwards and finding the first element in the interval $[1, \underline{r}(b, w^{\Omega}) - 1]$, then $a_1 \neq a_2$ implies $s_1 \neq s_2$.

Either way, $\Psi_{w^{\circlearrowright}}(a_1, b_1) \neq \Psi_{w^{\circlearrowright}}(a_2, b_2)$.

Case 4: Assume one of the (a_i, b_i) is in $\mathrm{AE}_1(w^{\circlearrowright})$ and the other anti-exchange pair is in $\mathrm{AE}_2(w^{\circlearrowright})$. Without loss of generality, we may assume that $(a_1, b_1) \in \mathrm{AE}_1(w^{\circlearrowright})$ with $\bar{r}(a_1, w^{\circlearrowright}) \in [a_1 + 1, b_1]$ and $(a_2, b_2) \in \mathrm{AE}_2(w^{\circlearrowright})$ with $\underline{r}(b_2, w^{\circlearrowright}) \in [a_2 + 1, b_2]$. Then $\Psi_{w^{\circlearrowright}}(a_1, b_1) = (p \mapsto w(p), w^{-1}(a_1) \mapsto a_1)$ for some $p \geq \bar{r}(a_1, w^{\circlearrowright})$, and $\Psi_{w^{\circlearrowright}}(a_2, b_2) = (w^{-1}(b_2) \mapsto b_2, s \mapsto w(s))$ for some $s < \underline{r}(b_2, w^{\circlearrowright})$.

If $a_1 = a_2 = a$, then Corollary 5.2.4 shows that $b_2 < \bar{r}(a, w^{\circlearrowright})$ since $(a, b_2) \in \mathrm{AE}_2(w^{\circlearrowright})$ and $(a, b_1) \in \mathrm{AE}_1(w^{\circlearrowright})$. Since $b_2 \notin I_1(w^{\circlearrowright})$, $w^{-1}(b_2) \leq b_2 < \bar{r}(a, w^{\circlearrowright}) \leq p$. Therefore, once again, the two alignments cannot have the same port sides, so $\Psi_{w^{\circlearrowright}}(a_1, b_1) \neq \Psi_{w^{\circlearrowright}}(a_2, b_2)$.

If $a_1 \neq a_2$, observe from Definition 5.2.6 that $(s \mapsto w(s))$ is either the arc $(w^{-1}(a_2) \mapsto a_2)$, or it is an arc with $s < \underline{r}(b_2, w)$ and $w(s) \geq \underline{r}(b_2, w^{\circlearrowright})$, in which case it is an exceedance arc. See Figure 5.6 for these two cases. In either case, $(s \mapsto w(s))$ is not the anti-exceedance arc $(w^{-1}(a_1) \mapsto a_1)$. Hence, $\Psi_{w^{\circlearrowright}}(a_1, b_1) \neq \Psi_{w^{\circlearrowright}}(a_2, b_2)$. □

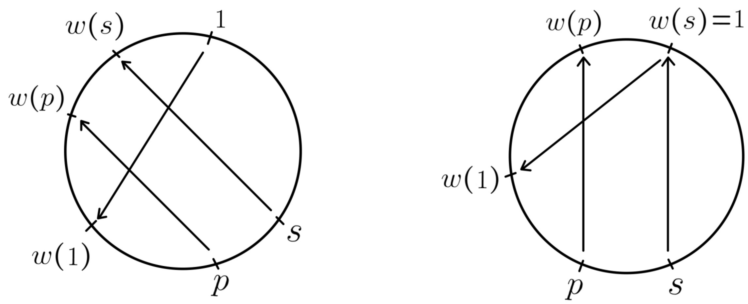
5.2.3 The Case of Crossed Alignments

Recall the definition of a starboard tacking crossed alignment from Definition 2.3.7. Figure 5.7 depicts two examples of a starboard tacking crossed alignment where the tail of the crossing arc is 1.

Lemma 5.2.9. *If $w^{\circlearrowright} = (w, co) \in S_n^{\circlearrowright}$ has a starboard tacking crossed alignment $(p \mapsto w(p), s \mapsto w(s))$, where 1 is the tail of the crossing arc, then there is no element of $\mathrm{AE}(w^{\circlearrowright})$ mapping to $(p \mapsto w(p), s \mapsto w(s))$ under $\Psi_{w^{\circlearrowright}}$. Hence, $\Psi_{w^{\circlearrowright}}$ is not surjective.*

Proof. Assume by way of contradiction that there is some anti-exchange pair $(a, b) \in \mathrm{AE}(w^{\circlearrowright})$ such that $\Psi_{w^{\circlearrowright}}(a, b) = (p \mapsto w(p), s \mapsto w(s))$. Since fixed point loops cannot cross other arcs, then none of the arcs involved in a crossed alignment are loops. Therefore, none of 1, p, or s is a fixed point. There are two cases to consider, depending on whether $w(s) = 1$.

If $w(s) \neq 1$, as in Figure 5.7a, then $w(p)$ and $w(s)$ are both exceedances. By considering the various cases in the algorithm defined in Definition 5.2.6, $w(p)$ and $w(s)$ can both be



(a) $d \neq 1$ **(b)** $d = 1$

Figure 5.7: Two cases for starboard tacking crossing arc with tail at 1.

exceedances only if $a < b$ and $(a,b) \in \mathrm{AE}_2(w^{\circlearrowleft})$, as in Figure 5.6b. In this case, we must have that $w(p) = b$. Hence, the arc $(1 \mapsto w(1))$ satisfies Condition (2) of Lemma 5.2.2 for all $r \in [2, w(1)]$, and the arc $(p \mapsto w(p) = b)$ satisfies Condition (2) for $r \in [p+1, w(p)]$. Since the alignment is crossed by $(1 \mapsto w(1))$, then $w(1) \geq p$. It follows that Condition (2) is satisfied for all $r \in [2, w(p)]$ and thus for all r in $[a+1, w(p)]$. This contradicts that $(a,b) \in \mathrm{AE}_2(w^{\circlearrowleft})$.

If $w(s) = 1$, as in Figure 5.7b, then the arrangement of the arcs in $(p \mapsto w(p), s \mapsto w(s))$ implies that $a < b$. Thus, $(a,b) \in \mathrm{AE}_1(w^{\circlearrowleft}) \sqcup \mathrm{AE}_2(w^{\circlearrowleft})$. If $(a,b) \in \mathrm{AE}_2(w^{\circlearrowleft})$, then we must have $w(p) = b$. Again, looking at the arcs $(1 \mapsto w(1))$ and $(p \mapsto w(p))$ contradicts the assumption that Condition (2) fails for some $r \in [a+1, w(p)]$. Otherwise, $(a,b) \in \mathrm{AE}_1(w^{\circlearrowleft})$, and the starboard side being $(s \mapsto w(s))$ implies that $a = 1$. The port side being an exceedance implies that $w(p) = b$, as in Figure 5.5a. Thus, Condition (1) fails for some $r \in [2, w(p)]$. If $r \in [2, w(1)]$, let $x = 1$, and if $r \in [w(1)+1, w(p)]$, let $x = p$. In either case, the arc $(x \mapsto w(x))$ maps from $[1, r-1]$ to $[r, n]$. The cycle containing the arc $(x \mapsto w(x))$ must then also contain an arc $(y \mapsto w(y))$ from $[r, n]$ to $[1, r-1]$. The existence of the arc $(y \mapsto w(y))$ contradicts the fact that Condition (1) fails for r. Therefore, $(a,b) \notin \mathrm{AE}_1(w^{\circlearrowleft})$.

For these two cases of $w(s) \neq 1$ and $w(s) = 1$, we have obtained contradictions that (a,b) is an anti-exchange pair. Therefore, there can be no such anti-exchange pair mapping to

$(p \mapsto w(p), s \mapsto w(s))$ under $\Psi_{w^{\circlearrowleft}}$. $\square$

Proof of Theorem 1.2.5, $(1) \Rightarrow (5)$. We prove the implication $(1) \Rightarrow (5)$ using the contrapositive. Thus, assume that $w^{\circlearrowleft}$ is a decorated permutation whose chord diagram contains a crossed alignment. The chord diagram $D(w^{\circlearrowleft})$ may be rotated and reflected as necessary to produce a new chord diagram with a starboard tacking crossed alignment whose crossing arc has its tail at 1. By Lemma 4.2.6, any such operations on $D(w^{\circlearrowleft})$ preserve the property of being smooth or singular. Therefore, we may assume that $D(w^{\circlearrowleft})$ has a starboard tacking crossed alignment whose crossing arc has its tail at 1. It follows from Lemma 5.2.9 that $\#\,\mathrm{AE}(w^{\circlearrowleft}) < \#\mathit{Alignments}(w^{\circlearrowleft})$. Since $I_1(w) \in \mathcal{M}$, then Corollary 4.1.4 and Definition 5.2.1 of anti-exchange pairs imply that $\Pi_{w^{\circlearrowleft}}$ is singular. $\square$

Chapter 6

ENUMERATION OF SMOOTH POSITROIDS

Let $w^{\circlearrowleft}$ be a decorated permutation in $S_n^{\circlearrowleft}$ with associated positroid $\mathcal{M}$ of rank k on ground set $[n]$. We say $w^{\circlearrowleft}$ is a *smooth decorated permutation* and $\mathcal{M}$ is a *smooth positroid* if $\Pi_{w^{\circlearrowleft}} = \Pi_{\mathcal{M}}$ is a smooth positroid variety.

Definition 6.0.1. Let $s(n)$ be the number of smooth positroids on ground set $[n]$, and let $s_c(n)$ be the number of connected smooth positroids on ground set $[n]$.

By Theorem 1.2.5, each connected smooth positroid can be bijectively associated with a spirograph permutation. Thus, by (5.1.1), we have

$$s_c(n) = \begin{cases} 2 & n = 1 \\ n-1 & n > 1 \end{cases}. \tag{6.0.1}$$

In this chapter, we give enumerative results for smooth positroids. Our enumerations rely on the characterization of the corresponding decorated permutations as a union of spirographs on a non-crossing partition of $[n]$, found in Theorem 1.2.5. From the sum over non-crossing partitions, we utilize result of Speicher [Spe94] to obtain $s(n)$ as the coefficient of a power of a polynomial, Theorem 6.1.1. By a formula due to Faà di Bruno, which utilizes partial Bell polynomials, this expression for $s(n)$ leads to a formula for $s(n)$ as a sum over set partitions of $[n]$. We are then able to refine the sequence $(s(n))_{\geq 0}$ according to three different statistics in Section 6.2 and show that two of the refinements coincide.

In Section 6.3, we list further enumerative results due to Christian Krattenthaler. In this work, Krattenthaler expanded on our enumerations to provide elegant closed formulas for $s(n)$ as well as the refinements that we define in Section 6.2. In addition, Krattenthaler provides an exact formula for the asymptotic growth of $s(n)$ in Theorem 6.3.5.

6.1 Enumeration with Bell polynomials

In analogy with the enumeration of smooth Schubert varieties studied by Haiman, Bona, Bousquet-Mélou and Butler [B9́8; BMB07; Hai92], the enumeration of smooth positroids on $[n]$ gives rise to several interesting sequences. See [OEIS, A349413, A349456, A349457, A349458, A353131, A353132]. The following formula for enumerating smooth positroids is very similar to the results in [ARW16, Thm. 10.2]. In particular, we use the results of Beissinger [Bei85] and Speicher [Spe94] for counting structures induced on the blocks of non-crossing partitions and the Lagrange inversion formula for formal power series [Sta99, Sect. 5.4]. The sequence begins $2, 5, 16, 61, 256, 1132, 5174, 24229, 115654$ for $n = 1, \ldots, 10$ [OEIS, A349458]. If G is a polynomial or power series in x, then $\langle x^n \rangle\, G(x)$ denotes the coefficient of x^n in $G(x)$.

Theorem 6.1.1. *The number of smooth positroids on ground set $[n]$ is the coefficient*

$$s(n) = \langle x^n \rangle \frac{1}{n+1} \left(1 + 2x + \sum_{i=2}^{\infty} (i-1) x^i \right)^{n+1} = \langle x^n \rangle \frac{1}{n+1} \left(1 + 2x + \sum_{i=2}^{n} (i-1) x^i \right)^{n+1}. \tag{6.1.1}$$

Proof. Let $\mathcal{M} = \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_t$ be a positroid decomposed into its connected components on a non-crossing partition. Then, $\Pi_{\mathcal{M}}$ is smooth if and only if $\Pi_{\mathcal{M}_i}$ is smooth for each $i \in [t]$ by Corollary 4.3.5. Each connected smooth positroid can be bijectively associated with a spirograph permutation by Theorem 1.2.5. Therefore, every smooth positroid on $[n]$ can be uniquely determined by a non-crossing partition of $[n]$ along with a spirograph permutation on each block. Hence, the number of smooth positroids on ground set $[n]$ is given by

$$s(n) = \sum s_c(\#B_1)\, s_c(\#B_2) \cdots s_c(\#B_t) \tag{6.1.2}$$

where the sum is over all non-crossing partitions $B_1 \sqcup \cdots \sqcup B_t$ of $[n]$. Equation (6.1.1) now follows directly from [Spe94, Corollary 0] and (6.0.1). □

The *partial Bell polynomial*, $B_{n,k}(x_1, \ldots, x_{n-k+1})$ introduced in [Bel27], is defined as

$$B_{n,k}(x_1, \ldots, x_{n-k+1}) = \sum_{B_1 \sqcup \cdots \sqcup B_k} \prod_{i=1}^{k} x_{|B_i|}, \tag{6.1.3}$$

where the sum is taken over all set partitions of $[n]$ into k blocks. The following formula of Faà di Bruno expresses the nth derivative of a composition of functions in terms of the partial Bell polynomials [Joh02]

$$\frac{d^n}{dx^n} f(g(x)) = \sum_{k=1}^{n} f^{(k)}(g(x)) \cdot B_{n,k}(g'(x), g''(x), \ldots, g^{(n+1-k)}(x)). \tag{6.1.4}$$

Define the triangle of numbers $b_{n,k}$ using the Bell polynomials evaluated at $x_i = s_c(i) \cdot i!$ for all i, so

$$b_{n,k} = B_{n,k}(2 \cdot 1!, 1 \cdot 2!, 2 \cdot 3!, \ldots, (n-k) \cdot (n-k+1)!) = \sum_{i=1}^{n-k+1} \binom{n-i}{i-1} \cdot s_c(i) \cdot i! \cdot b_{n-i,k-1} \tag{6.1.5}$$

for $1 \le k \le n$ along with initial conditions $b_{0,0} = 1$, and $b_{0,k} = b_{n,0} = 0$ if $n > 0$ or $k > 0$, see [OEIS, A353131]. Also, let $(n)_k$ denote the falling factorial,

$$(n)_k \;:=\; n\,(n-1)\cdots(n-k+1).$$

Corollary 6.1.2. *The number of smooth positroids on ground set* $[n]$ *is*

$$s(n) = \frac{1}{(n+1)!} \sum_{k=1}^{n} (n+1)_k \cdot b_{n,k} = \sum_{k=1}^{n} \frac{b_{n,k}}{(n-k+1)!}. \tag{6.1.6}$$

Proof. Set $f(x) = x^{n+1}$ and $g(x) = 1 + 2x + \sum_{i=2}^{n} (i-1)x^i$. By Theorem 6.1.1, $s(n) = \frac{1}{n+1} \cdot \langle x^n \rangle\, f(g(x))$. Furthermore, the coefficient of x^n in the composition $f(g(x))$ can be computed as $\langle x^n \rangle\, f(g(x)) = \frac{1}{n!} \frac{d^n}{dx^n} f(g(x))\Big|_{x=0}$. Then, by Faà di Bruno's formula (6.1.4),

$$\begin{aligned} s(n) &= \frac{1}{n+1} \cdot \frac{1}{n!} \cdot \frac{d^n}{dx^n} f(g(x))\Big|_{x=0} & (6.1.7)\\ &= \frac{1}{(n+1)!} \sum_{k=1}^{n} f^{(k)}(g(x)) \cdot B_{n,k}(g'(x), g''(x), \ldots, g^{(n+1-k)}(x))\Big|_{x=0}. & (6.1.8) \end{aligned}$$

The kth derivative of $f(x) = x^{n+1}$ is $f^{(k)}(x) = (n+1)_k \cdot x^{n-k+1}$, so

$$f^{(k)}(g(x))\Big|_{x=0} = (n+1)_k \Big(1 + 2x + \sum_{i=1}^{n} (i-1)x^i\Big)^{n-k+1}\Big|_{x=0} = (n+1)_k.$$

The derivatives of g are $g'(x) = 2 + \sum_{i=1}^{n-1} i(i+1)x^i$ and $g^{(j)}(x) = \sum_{i=0}^{n-j} (i+j-1) \cdot (i+j)_j x^i$ for $j > 1$. Hence, $g^{(j)}(x)|_{x=0} = s_c(j) \cdot j!$ for all $j \ge 1$. Therefore, the formula in (6.1.6) follows from (6.1.5) and (6.1.8). $\square$

It is interesting to consider the asymptotic growth function for the number $s(n)$ of smooth positroids on $[n]$. We use the second formula in (6.1.1) to obtain the following data as examples. Observe from this data the speed of growth of the number of smooth positroids:

$$\begin{aligned} s(51)/s(50) &\approx 5.4489775, \\ s(101)/s(100) &\approx 5.528236, \\ s(151)/s(150) &\approx 5.555362, \\ s(201)/s(200) &\approx 5.569062, \\ s(251)/s(250) &\approx 5.5773263. \end{aligned}$$

Based on this data, we conjectured in [BW22a] that the growth function is of the order $O(c^n)$ for some constant $c < 6$. The exact formula for growth was determined by Christian Krattenthaler and is stated in Theorem 6.3.5. The proof is given in Section 6.3. The result was determined separately by Omer Angel and Sergi Elizalde.

6.2 Equality of q-analogs.

Note that the value k in the Corollary 6.1.2 above does not represent the same value $k = k(w^{\circ})$ used elsewhere in the thesis. These distinct values may be used to refine the enumeration of $s(n)$. To this end, we define

$$s_1(n,k) := \#\{\text{smooth positroid varieties in } Gr(k,n)\} \quad \text{for } 0 \le k \le n \tag{6.2.1}$$

$$= \#\{\text{smooth decorated permutations in } S^{\circ}_{n,k}\}, \tag{6.2.2}$$

$$s_2(n,k) := \#\left\{\begin{matrix}\text{smooth decorated permutations in } S^{\circ}_{n} \text{ with} \\ \text{exactly } k \text{ components in its SIF decomposition}\end{matrix}\right\} \text{for } 1 \le k \le n, \text{ and} \tag{6.2.3}$$

$$s_3(n,k) := \frac{b_{n,k}}{(n-k+1)!} \quad \text{for } 1 \le k \le n. \tag{6.2.4}$$

The terms $s_i(n,k)$ are displayed in Figure 6.3 and Figure 6.4 for $1 \le n \le 10$.

For each $i \in \{1,2,3\}$, we may define a q-analog of $s(n)$ as

$$s^{(i)}(n;q) = \sum_{k\leq n} s_i(n,k)q^k. \tag{6.2.5}$$

In fact, the q-analogs for $i = 2$ and $i = 3$ coincide. This equality leads to the following theorem.

Theorem 6.2.1. *For any $1 \leq k \leq n$, the number of smooth decorated permutations in S_n° with exactly k components in its SIF decomposition is $\frac{b_{n,k}}{(n-k+1)!}$.*

Using the notation from the proof of Corollary 6.1.2 so that $g(x) = 1 + \sum_{n\geq 1} s_c(n)x^n$, we know from Theorem 6.1.1 that $s(n)$ is the coefficient of x^n in $\frac{1}{n+1}(g(x))^{n+1}$. Now, let $g(x,t) = 1 + t\sum_{n\geq 1} s_c(n)x^n$. We will show that $s_2(n,k)$ and $s_3(n,k)$ are both the coefficient $\langle x^n t^k\rangle \frac{1}{n+1}(g(x,t))^{n+1}$, and hence are equal. Expanding the product $(g(x,t))^{n+1}$ yields a sum over weak compositions of n into $n+1$ parts. With (6.1.2) in mind, which expresses $s(n)$ via a sum over noncrossing partitions of $[n]$, we now relate certain weak compositions to noncrossing partitions of $[n]$.

Let $\mathcal{C} = (d_1,\ldots,d_{n+1})$ be a weak composition of n into $n+1$ parts, and suppose that $\mathcal{C}$ has k nonzero elements, say $d_{m_1},\ldots,d_{m_k}$. We will associate with $\mathcal{C}$ a partition $\mathcal{B} = B_1 \sqcup \cdots \sqcup B_k$ of a set $B \in \binom{[n+1]}{n}$ with k nonempty blocks. We think of the elements of $[n+1]$ as points sitting around a circle in the clockwise order. Then, a set of elements is assigned to a block by drawing an arc between every pair of consecutive elements in that set. Sets of elements are assigned to a block using the following algorithm. All indices m_i and elements defining cyclic intervals should be considered mod $n+1$ in the set $[n+1]$, and all subindices i on the d_{m_i} should be considered mod k in $[k]$.

Definition 6.2.2. For a weak composition $\mathcal{C} = (d_1,\ldots,d_{n+1})$ of n into $n+1$ parts, let $\mathcal{B}(\mathcal{C}) = \{B_1,\ldots,B_k\}$ be defined by the following algorithm.

Input : $\mathcal{C} = (d_1, \dots, d_{n+1})$

Output: $\mathcal{B}(\mathcal{C}) = \{B_1, \dots, B_k\}$

set $\mathcal{B} = \varnothing$.

while $\|\mathcal{C}\|_0 > 0$ **do**

  set $D = \{i \in [k] : d_{m_i} \leq |[m_i, m_{i+1} - 1]^{cyc}|\}$

  for $i \in D$ **do**

    set $j = \#\mathcal{B}$

    set $B_{j+1} = [m_i, m_i + (d_{m_i} - 1)]^{cyc}$

    update $\mathcal{B} = \mathcal{B} \cup \{B_{j+1}\}$.

    Delete d_{m_i} and the $d_{m_i} - 1$ zeros cyclically to the right of d_{m_i} from $\mathcal{C}$.

  end

  set $n' = n - \sum_{i \in D} d_{m_i}$

  Relabel the remaining $n' + 1$ elements with the consecutive integers in $[n' + 1]$.

  set $n \leftarrow n'$

  set $k \leftarrow k - |D|$

end

return $\mathcal{B} = \{B_1, \dots, B_k\}$.

Example 6.2.3. Consider the weak composition $\mathcal{C} = (0, 0, 3, 0, 3, 0, 0, 0, 0, 0, 0, 3, 2, 0, 0, 5, 0)$ of 16. In the first iteration of the while loop, $(d_{m_1}, \dots, d_{m_k}) = (3, 3, 3, 2, 5)$, $(m_1, \dots, m_k) = (3, 5, 12, 13, 16)$, and $D = \{2, 4\}$. Hence, we assign the sets $\{5, 6, 7\}$ and $\{13, 14\}$ to blocks, as in the image on the left below. The weak compositions and the corresponding block assignments for each iteration of the loop are shown below.

(1) $(0, 0, 3, 0, 3, 0, 0, 0, 0, 0, 0, 3, 2, 0, 0, 5, 0)$

(2) $(0, 0, 3, 0, \cancel{3, 0, 0}, 0, 0, 0, 0, 3, \cancel{2, 0}, 0, 5, 0)$

(3) $(0, 0, \cancel{3, 0, 3, 0, 0, 0}, 0, 0, 0, 3, \cancel{2, 0}, 0, 5, 0)$

(4) $(\cancel{0, 0, 3, 0, 3, 0, 0, 0, 0}, 0, 0, 3, \cancel{2, 0}, 0, \cancel{5, 0})$

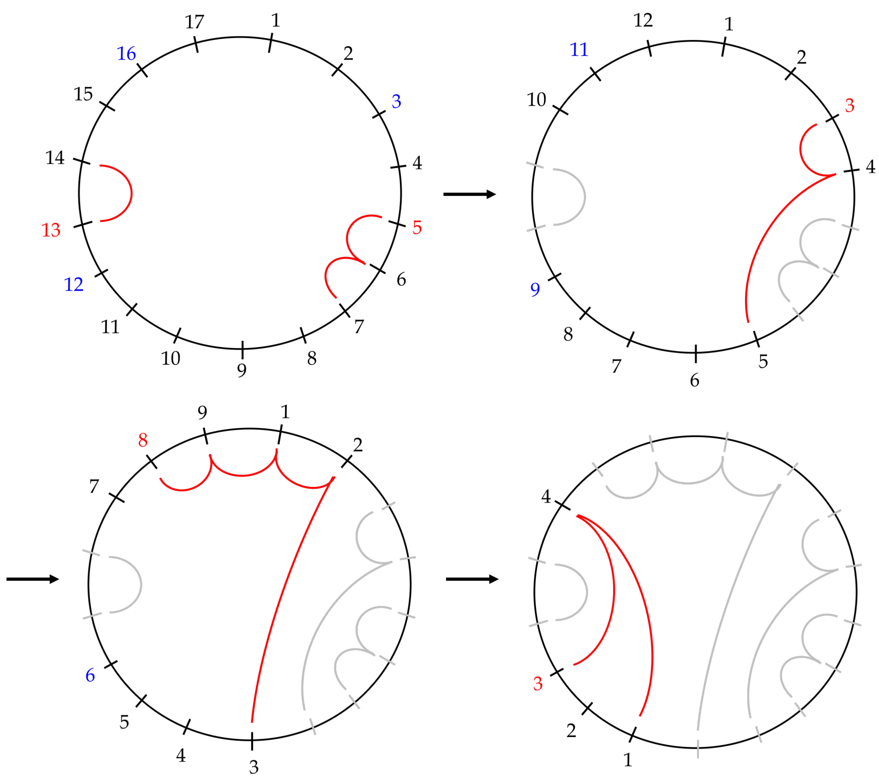


Figure 6.1: Iterations of algorithm.

Relabeling the elements of $[n+1] = 17$ yields the figure below, corresponding to the partition of $B = [17] \setminus \{11\}$ so that 11 is the unique element not contained in any block of $\mathcal{B}$,

$$\mathcal{B} = \{1,2,9,16,17\} \sqcup \{3,4,8\} \sqcup \{5,6,7\} \sqcup \{10,12,15\} \sqcup \{13,14\}$$

For a weak composition of n into $n+1$ parts, the set D must always be nonempty, for otherwise the sum of the elements would be at least $n+1$. The algorithm begins with a composition of n into $n+1$ parts. At each step, we reduce the sum of the composition by some element d_{m_i} and simultaneously delete d_{m_i} terms from the composition. Therefore, at each iteration, we always have a weak composition $\mathcal{C}'$ of some n' into $n'+1$ parts, and by the previous observation, the set D will always be nonempty for this $\mathcal{C}'$. Hence, the loop will terminate in finitely many steps.

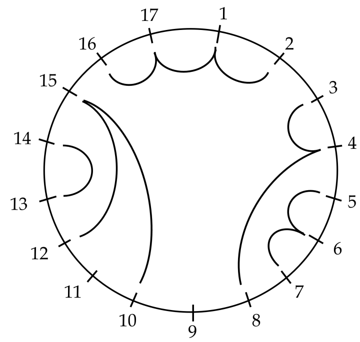


Figure 6.2: Resulting partition.

By selecting intervals of the form $[m_i, m_i + (d_{m_i} - 1)]^{cyc}$ to be blocks, then there will be a block of size d_{m_i} for each $i \in [k]$. Since $\mathcal{C}$ is a weak composition of n, then there will be exactly one element $x \in [n+1]$ which is not assigned to any element. Thus, we consider the partition $\mathcal{B}$ together with the element x, where $\mathcal{B}$ is a partition of some n-subset $B \subset [n+1]$ and $[n+1] \setminus B = \{x\}$.

Lemma 6.2.4. *For a weak composition $\mathcal{C} = (d_1, \ldots, d_{n+1})$ of n into $n+1$ parts, all $n+1$ cyclic rotations $(d_i, d_{i+1}, \ldots, d_{n+1}, d_1, \ldots, d_{i-1})$ are distinct.*

Proof. Let $\mathcal{B}$ be the noncrossing partition associated with $\mathcal{C}$. Then $\mathcal{B}$ has a unique associated element $x \in [n+1]$ such that x is not contained in any of the blocks of $\mathcal{B}$. Since the elements and indices in the algorithm are considered mod $n+1$ and k, respectively, then a cyclic rotation of $\mathcal{C}$ simply corresponds to a cyclic rotation of $\mathcal{B}$. Since each rotation $\mathcal{B}'$ of $\mathcal{B}$ has the unique element x' which is not contained in any of the blocks of $\mathcal{B}'$, then all $n+1$ rotations of $\mathcal{B}$ yield distinct partitions. Therefore, the weak compositions from which these partitions came must also all be distinct. □

From the arguments above, each cyclic rotation equivalence class of weak compositions of n into $n+1$ parts yields exactly one noncrossing partition such that $n+1$ is not contained in any block of the partition. Hence, this is the unique partition of $[n]$ for this rotation class. We will consider the weak composition which yields this partition of $[n]$ to be the representative of the cyclic rotation class and its partition to be the partition associated with the equivalence class.

Lemma 6.2.5. *Let $\mathcal{C}_1, \ldots, \mathcal{C}_s$ be the representative elements of the cyclic rotation equivalence classes of weak compositions of n into $n+1$ parts. Then the associated noncrossing partitions $\mathcal{B}_1, \ldots, \mathcal{B}_s$ of $[n]$ are all distinct.*

Proof. For any of the $\mathcal{C}_j$ and the corresponding partition $\mathcal{B}_j$, recall that $n+1$ is not assigned to any of the blocks of $\mathcal{B}_j$. The element $n+1$ can only be excluded from every block if for every assignment of an interval of the form $I = [m_i, m_i + (d_{m_i} - 1)]^{cyc}$ to a block, I is in fact an ordinary interval with $I \subseteq [n]$. Then, m_i will be the minimal element of the block in which it is contained. Hence, for all of the nonzero elements $d_{m_1}, \ldots, d_{m_k}$ of $\mathcal{C}_j$, the block containing m_i is a block of size d_{m_i} with minimal element m_i. Since $\mathcal{B}_j$ has exactly k blocks, for any $i \notin \{m_1, \ldots, m_k, n+1\}$, corresponding to a zero of $\mathcal{C}_j$, if $i \in B_\ell \in \mathcal{B}_j$, then i is not the minimal element of B_ℓ.

Let $\mathcal{C}_1 = (c_1, \ldots, c_{n+1})$ and $\mathcal{C}_2 = (d_1, \ldots, d_{n+1})$ be distinct rotation class representatives with associated partitions $\mathcal{B}_1$ and $\mathcal{B}_2$. Let i be the minimal index in $[n+1]$ at which $\mathcal{C}_1$ and $\mathcal{C}_2$ differ. If, without loss of generality, $c_i = 0$, so $d_i \neq 0$, then i is not minimal element of its block in $\mathcal{B}_1$, but i is the minimal element of its block in $\mathcal{B}_2$. If c_i and d_i are both nonzero, but distinct, then the blocks containing i in $\mathcal{B}_1$ and $\mathcal{B}_2$ have different sizes. In either case, $\mathcal{B}_1 \neq \mathcal{B}_2$. $\square$

Lemma 6.2.6. *The coefficient $\langle x^n t^k \rangle \frac{1}{n+1}(g(x,t))^{n+1}$ is equal to $s_2(n,k)$.*

Proof. Consider the expansion

$$\begin{aligned}
\frac{1}{n+1}(g(x,t))^{n+1} &= \frac{1}{n+1}\Big(1 + t\sum_{i\geq 1} s_c(i)x^i\Big) \\
&= \frac{1}{n+1}\left(1 + \sum_{j\geq 1}\Big(\sum_{\substack{d_1,\ldots,d_{n+1}\in[0,j] \\ d_1+\cdots+d_{n+1}=j}} s_c(d_1)\cdots s_c(d_n)t^{\|(d_1,\ldots,d_n)\|_0}\Big)x^j\right),
\end{aligned}$$

where the inner sum is over all weak compositions of j into $n+1$ parts and $\|(d_1,\dots,d_n)\|_0$ is the number of nonzero elements in the $(n+1)$-tuple $(d_1,\dots,d_n)$. Thus,

$$\langle x^n t^k\rangle \frac{1}{n+1}(g(x,t))^{n+1} = \sum_{\substack{d_1,\dots,d_{n+1}\in[0,n]\\ d_1+\cdots+d_{n+1}=n}} s_c(d_1)\cdots s_c(d_n),$$

where the sum is over only compositions containing exactly k nonzero elements.

For each equivalence class of weak compositions under cyclic rotation, recall that one may choose a representative $\mathcal{C}=(d_1,\dots,d_{n+1})$ such that the associated noncrossing partition $\mathcal{B}$ is a partition of $[n]$. As above, let $d_{m_1},\dots,d_{m_k}$ be the nonzero elements of $\mathcal{C}$. Writing $\mathcal{B}=B_1\sqcup\cdots\sqcup B_k$ so that $d_{m_i}=|B_i|$, then the number of smooth decorated permutations whose underlying set partition is $\mathcal{B}$ is exactly $s_c(|B_1|)\cdots s_c(|B_k|)=s_c(d_{m_1})\cdots s_c(d_{m_k})$. Since each cyclic rotation of $\mathcal{C}$ corresponds to a cyclic rotation of $\mathcal{B}$, then each cyclic rotation of $\mathcal{C}$ also contributes the term $s_c(d_{m_1})\cdots s_c(d_{m_k})$ to the sum above. By Lemma 6.2.4, there are $n+1$ distinct cyclic rotations of $\mathcal{C}$. Hence, we get the term $s_c(d_{m_1})\cdots s_c(d_{m_k})$ for each cyclic rotation equivalence class.

By Lemma 6.2.5, the noncrossing partitions associated to these equivalence classes are all unique. Since there are $\binom{2n}{n}$ weak compositions of n into $n+1$ parts, and each cyclic rotation equivalence class has size $n+1$, then there are $C_n=\frac{1}{n+1}\binom{2n}{n}$ equivalence classes contributing to the sum.

Noncrossing partitions are a Catalan object. Hence, there are also C_n noncrossing partitions of $[n]$. Therefore, the map $\mathcal{C}\mapsto\mathcal{B}$ gives a bijection between the equivalence classes and noncrossing set partitions. In particular, every noncrossing partition is accounted for in the sum above exactly once. It follows that $\frac{1}{n+1}\langle x^n\rangle (g(x,1))^{n+1}$ recovers exactly (6.1.2). Furthermore, the compositions with $k=\|(d_1,\dots,d_n)\|_0$ are exactly those whose equivalence class is associated with a noncrossing partition of $[n]$ into k blocks. Hence $\langle x^n t^k\rangle \frac{1}{n+1}(g(x,t))^{n+1}$ is exactly $s_2(n,k)$. □

Lemma 6.2.7. *The coefficient* $\langle x^n t^k\rangle \frac{1}{n+1}(g(x,t))^{n+1}$ *is equal to* $s_3(n,k)$.

Proof. Observe that for $n\ge 1$, $\frac{1}{n+1}\langle x^n t^k\rangle (g(x,t))^{n+1} = \frac{1}{(n+1)!}\langle x^0 t^k\rangle \frac{d^n}{dx^n}(g(x,t))^{n+1}$. By applying Faà di Bruno's formula with $f(x) = x^{n+1}$, we have

$$\begin{aligned}\frac{d^n}{dx^n} f(g(x,t)) &= \sum_{j=1}^{n} f^{(j)}(g(x,t)) \cdot B_{n,j}(g'(x,t), g''(x,t), \dots, g^{(n+1-j)}(x,t)) \\ &= \sum_{j=1}^{n} f^{(j)}(g(x,t)) \sum_{B_1 \sqcup \cdots \sqcup B_j} \prod_{i=1}^{j} g^{(|B_i|)}(x,t),\end{aligned}$$

where all derivatives are with respect to x and the inner sum is over all partitions of $[n]$ into j parts. Since every block of every such partition is nonempty, then all derivatives $g^{(|B_i|)}(x,t)$ are of order at least one. Then $g^{(|B_i|)}(x,t) = tg^{(|B_i|)}(x)$, so t can be factored out of $g^{(|B_i|)}(x,t)$ for each block of the partition. Therefore,

$$\begin{aligned}\frac{1}{(n+1)!} \left\langle x^0 t^k \right\rangle \frac{d^n}{dx^n} (g(x,t))^{n+1} &= \frac{1}{(n+1)!} \left\langle x^0 t^k \right\rangle \sum_{j=1}^{n} f^{(j)}(g(x,t)) \cdot t^j \sum_{B_1 \sqcup \cdots \sqcup B_j} \prod_{i=1}^{j} g^{(|B_i|)}(x) \\ &= \frac{1}{(n+1)!} \left\langle x^0 \right\rangle f^{(k)}(g(x,t)) \cdot \sum_{B_1 \sqcup \cdots \sqcup B_j} \prod_{i=1}^{j} \left\langle x^0 \right\rangle g^{(|B_i|)}(x) \\ &= \frac{1}{(n+1)!} (n)_k \cdot \sum_{B_1 \sqcup \cdots \sqcup B_j} \prod_{i=1}^{j} h(|B_i|) \\ &= s_3(n,k).\end{aligned}$$

□

Proof of Theorem 6.2.1. From Lemma 6.2.6 and Lemma 6.2.7, the coefficient of $x^n t^k$ in $\frac{1}{n+1}(g(x,t))^{n+1}$ is equal to both $s_2(n,k)$ and $s_3(n,k)$. Hence, $s_2(n,k) = s_3(n,k)$. □

Corollary 6.2.8. *The numbers $\frac{b_{n,k}}{(n-k+1)!}$ are integers for any $1 \le k \le n$.*

n \ k	0	1	2	3	4	5	6	7	8	9	10
1	1	1									
2	1	3	1								
3	1	7	7	1							
4	1	15	29	15	1						
5	1	31	96	96	31	1					
6	1	63	282	440	282	63	1				
7	1	127	771	1688	1688	771	127	1			
8	1	255	2011	5803	8089	5803	2011	255	1		
9	1	511	5074	18520	33721	33721	18520	5074	511	1	
10	1	1023	12488	55998	127698	166325	127698	55998	12488	1023	1

Figure 6.3: Table of $s_1(n,k)$.

n \ k	1	2	3	4	5	6	7	8	9	10
1	2									
2	1	4								
3	2	6	8							
4	3	18	24	16						
5	4	40	100	80	32					
6	5	78	305	440	240	64				
7	6	140	798	1750	1680	672	128			
8	7	236	1876	5838	8400	5824	1792	256		
9	8	378	4056	17136	34524	35616	18816	4608	512	
10	9	580	8190	45480	122682	175896	137760	57600	11520	1024

Figure 6.4: Table of $s_2(n,k) = s_3(n,k)$.

6.3 Further enumerative results

All results in this section are due to Christian Krattenthaler[1]. These results are also given in [BW22b, Appendix]. The constant given in Theorem 6.3.5 was also found separately by Omer Angel and Sergi Elizalde.

In this appendix, we derive explicit formulas in terms of single respectively double sums for the number $s(n)$ of smooth positroids on the ground set $[n]$ (see Theorem 6.3.1), for the number $s_2(n,k)$ of smooth decorated permutations in $S^{\circlearrowleft}_{n,k}$ with exactly k components in its SIF decomposition (and thus, by Theorem 6.2.1, for $s_3(n,k)$; see Theorem 6.3.3), and for the number $s_1(n,k)$ of smooth positroid varieties in $Gr(k,n)$ (or, by (6.2.1), equivalently, for the number of smooth decorated permutations in $S^{\circlearrowleft}_{n,k}$; see Theorem 6.3.4). Furthermore, we find a linear recurrence of order 4 with polynomial coefficients for the numbers $s(n)$ (see Theorem 6.3.2), which allows for a quick and efficient computation of these numbers. Finally, we confirm the conjecture from the end of Chapter 6 on the asymptotic growth of the numbers $s(n)$ and determine the growth constant exactly (see Theorem 6.3.5).

Theorem 6.3.1. *For all non-negative integers n, the number of smooth positroids on the ground set $[n]$ is given by*

$$s(n) = \frac{1}{n+1} \sum_{r=0}^{\lfloor (n+1)/2 \rfloor} (-1)^r 2^r \binom{n+1}{r} \binom{3n-3r+1}{n-2r}. \tag{6.3.1}$$

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Proof. Starting from (6.1.1), we compute

$$\begin{aligned} s(n) &= \frac{1}{n+1}\langle x^n\rangle \left(1+2x+\sum_{i=2}^{\infty}(i-1)x^i\right)^{n+1} \\ &= \frac{1}{n+1}\langle x^n\rangle \left(\frac{1}{(1-x)^2}-2\frac{x^2}{1-x}\right)^{n+1} \qquad (6.3.2)\\ &= \frac{1}{n+1}\langle x^n\rangle \sum_{r=0}^{n+1}(-1)^r 2^r\binom{n+1}{r}x^{2r}(1-x)^{-2(n+1-r)-r} \\ &= \frac{1}{n+1}\langle x^n\rangle \sum_{r=0}^{n+1}(-1)^r 2^r\binom{n+1}{r}x^{2r}\sum_{s\geq 0}\binom{(2n-r+2)+s-1}{s}x^s \\ &= \frac{1}{n+1}\sum_{r=0}^{\lfloor (n+1)/2\rfloor}(-1)^r 2^r\binom{n+1}{r}\binom{3n-3r+1}{n-2r}. \end{aligned}$$

□

Theorem 6.3.2. *The sequence* $(s(n))_{n\geq 0}$ *satisfies the recurrence relation*

$$\begin{aligned} &2(n-2)(n+1)(2n+1)(26n-33)s(n) \\ &\quad -(1118n^4-3343n^3+2092n^2+367n-66)s(n-1) \\ &\quad +2(n-1)(2002n^3-6181n^2+4435n+18)s(n-2) \\ &\quad -4(n-2)(n-1)(1586n^2-3547n+555)s(n-3) \\ &\quad +152(n-3)(n-2)(n-1)(26n-7)s(n-4)=0, \qquad (6.3.3) \end{aligned}$$

with initial conditions $s(0)=1$, $s(1)=2$, $s(2)=5$, *and* $s(3)=16$.

Proof. In (6.3.1), we may (artificially) extend the sum up to $r=n$,

$$s(n)=\frac{1}{n+1}\sum_{r=0}^{n}(-1)^r 2^r\binom{n+1}{r}\binom{3n-3r+1}{n-2r},$$

by adopting the convention that $\binom{m}{j}=0$ for $m\geq 0$ and $j<0$. In this form, the sum may be fed into the Gosper–Zeilberger algorithm [PWZ96; Zei90; Zei91] to find a recurrence relation for the sequence $s(n)$. (We used the *Mathematica* implementation by Paule and Schorn [PS95].) It finds the recurrence in (6.3.3). □

Theorem 6.3.3. *For all integers* n *and* k *with* $n\geq k>0$, *the number of smooth decorated permutations in* $S_{n,k}^{\circlearrowleft}$ *with exactly* k *components in its SIF decomposition is given by*

$$s_2(n,k)=s_3(n,k)=\frac{1}{n+1}\binom{n+1}{k}\sum_{l=0}^{k}2^{k-l}\binom{k}{l}\binom{n-k+l-1}{n-k-l}. \qquad (6.3.4)$$

Proof. Here our starting point is Theorem 6.2.1, which says that

$$s_2(n,k) = s_3(n,k) = \frac{1}{(n-k+1)!} B_{n,k}(2\cdot 1!, 1\cdot 2!, 2\cdot 3!, \dots, (n-k)\cdot(n-k+1)!) \qquad (6.3.5)$$

(cf. (6.1.5)). The Bell polynomials $B_{n,k}$ have the following generating function (cf. [Com74, Sec. 3.3]):

$$\sum_{n\ge k\ge 0} B_{n,k}(x_1,\dots,x_{n-k+1}) \frac{t^n}{n!} u^k = \exp\left(u \sum_{j\ge 1} x_j \frac{t^j}{j!}\right).$$

By substituting the values of the x_i's from (6.3.5), we get

$$\begin{aligned}\sum_{n\ge k\ge 0} B_{n,k}(2\cdot 1!, 1\cdot 2!, 2\cdot 3!, \dots, (n-k)\cdot(n-k+1)!) \frac{t^n}{n!} u^k &= \exp\left(u\left(2t + \sum_{j\ge 2}(j-1)t^j\right)\right) \\ &= \exp\left(u\left(2t + \frac{t^2}{(1-t)^2}\right)\right) \\ &= \sum_{k\ge 0} \frac{u^k}{k!}\left(2t + \frac{t^2}{(1-t)^2}\right)^k \\ &= \sum_{k\ge 0} \frac{u^k}{k!} \sum_{l=0}^{k} \binom{k}{l} 2^{k-l} t^{k+l} (1-t)^{-2l} \\ &= \sum_{k\ge 0} \frac{u^k}{k!} \sum_{l=0}^{k} \binom{k}{l} 2^{k-l} t^{k+l} \sum_{i\ge 0} \binom{2l+i-1}{i} t^i.\end{aligned}$$

By extracting the coefficient of $\frac{t^n}{n!}u^k$ on both sides, we obtain

$$B_{n,k}(2\cdot 1!, 1\cdot 2!, 2\cdot 3!, \dots, (n-k)\cdot(n-k+1)!) = \frac{n!}{k!} \sum_{l=0}^{k} 2^{k-l} \binom{k}{l} \binom{n-k+l-1}{n-k-l}.$$

The claim now follows by dividing this expression by $(n-k+1)!$. □

Theorem 6.3.4. *For all integers n and k with $n \ge k \ge 0$, the number of smooth positroid varieties in $Gr(k,n)$ is given by*

$$s_1(n,k) = \frac{1}{n+1} \sum_{k_1,k_2\ge 0} (-1)^{k_1+k_2} \binom{n+1}{k_1,k_2} \binom{2n-k-2k_1-k_2}{n-k-2k_1} \binom{n+k-k_1-2k_2}{k-2k_2}, \qquad (6.3.6)$$

where the multinomial coefficient is defined by $\binom{n+1}{k_1,k_2} := \frac{(n+1)!}{k_1!\,k_2!\,(n+1-k_1-k_2)!}$.

Proof. We have to count non-crossing partitions of $\{1,2,\dots,n\}$ in which each block carries the structure of a spirograph permutation, and where the arising "compound" permutation has exactly k anti-exceedances. We do this by computing the generating function $\sum_\sigma t^{\#I_1(\sigma)}$, where $\#I_1(\sigma)$ denotes the number of anti-exceedances of σ, and where the sum is over all these objects σ.

To prepare for this, we first consider a weighted generating function for non-crossing partitions of $\{1,2,\dots,n\}$. We define the weight $w(\pi)$ of a noncrossing partition π by

$$w(\pi) := \prod_{i=1}^{n} x_i^{\#(\text{blocks of } \pi \text{ of size } i)}.$$

Then it is not difficult to see by standard generating function calculus and the use of Lagrange inversion (alternatively, one may use [Kre72, Théorème 4]) that

$$\sum_\pi w(\pi) = \langle x^n \rangle \frac{1}{n+1} \left(1 + \sum_{i\ge1} x_i x^i\right)^{n+1}.$$

We must replace x_i by the generating function $\sum_\rho t^{\#I_1(\rho)}$ of spirograph permutations ρ of i elements. Now, it is straightforward that this generating function equals $1+t$ for $i=1$ and otherwise

$$(t + t^2 + \cdots + t^{i-1}) = \frac{1}{1-t}(t - t^i).$$

Hence, the generating function $\sum_\sigma t^{\# I_1(\sigma)}$ for our compound permutations equals

$$\begin{aligned}
&\langle x^n\rangle \frac{1}{n+1}\left(1+(1+t)x+\sum_{i\ge 2}\frac{1}{1-t}(t-t^i)x^i\right)^{n+1}\\
&\qquad = \langle x^n\rangle \frac{1}{n+1}\left(1+(1+t)x+\frac{tx^2}{(1-x)(1-tx)}\right)^{n+1}\\
&\qquad = \frac{1}{n+1}\langle x^n\rangle\left(\frac{1}{(1-x)(1-tx)}-\frac{x^2}{1-x}-\frac{t^2x^2}{1-tx}\right)^{n+1}\\
&\qquad = \frac{1}{n+1}\langle x^n\rangle\sum_{k_1,k_2\ge 0}(-1)^{k_1+k_2}\binom{n+1}{k_1,k_2}\frac{t^{2k_2}x^{2(k_1+k_2)}}{(1-x)^{n+1-k_2}(1-tx)^{n+1-k_1}}\\
&\qquad = \frac{1}{n+1}\langle x^n\rangle\sum_{k_1,k_2\ge 0}(-1)^{k_1+k_2}\binom{n+1}{k_1,k_2}t^{2k_2}x^{2(k_1+k_2)}\\
&\qquad\qquad\qquad \cdot\sum_{l_1,l_2\ge 0}\binom{n-k_2+l_2}{l_2}\binom{n-k_1+l_1}{l_1}t^{l_1}x^{l_1+l_2}\\
&\qquad = \frac{1}{n+1}\sum_{k_1,k_2\ge 0}(-1)^{k_1+k_2}\binom{n+1}{k_1,k_2}\\
&\qquad\qquad\qquad \cdot\sum_{l_1\ge 0}\binom{2n-2k_1-3k_2-l_1}{n-2k_1-2k_2-l_1}\binom{n-k_1+l_1}{l_1}t^{2k_2+l_1}.
\end{aligned}$$

If we now extract the coefficient of t^k on both sides, then we obtain (6.3.6). □

Theorem 6.3.5. *The asymptotic growth of the number of smooth positroids on the ground set $[n]$ is given by*

$$s(n)\sim\frac{\rho^{n+1}}{n^{3/2}\sqrt{2\pi\xi}}=\frac{(5.61071\dots)^{n+1}}{(8.74042\dots)\cdot n^{3/2}},\quad \textit{as } n\to\infty, \tag{6.3.7}$$

where $\rho = 5.61071\dots$ is the real root of

$$4x^3-35x^2+84x-76=0,$$

and $\xi = 12.15864\dots$ is the real root of

$$38x^3-425x^2-416x-416=0.$$

Proof. We use the formula for $s(n)$ in (6.3.2) together with the large powers theorem (cf. [FS09, Theorem VIII.8]):

$$\langle z^N\rangle A(z)B^n(z)\sim A(\zeta)\frac{B^n(\zeta)}{\zeta^{N+1}\sqrt{2\pi n\xi}},\quad \text{as } n\to\infty,$$

where $A(z)$ and $B(z)$ are power series satisfying certain properties (given as $\mathbf{L}_1$–$\mathbf{L}_3$ in [FS09, p. 586]), $\lambda = N/n$, ζ is the unique real root of $zB'(z)/B(z) = \lambda$, and

$$\xi = \frac{d^2}{dz^2}\Big(\log B(z) - \lambda \log z\Big)\bigg|_{z=\zeta}.$$

For our purposes, we choose $\lambda = 1$ and $A(z) = B(z) = \frac{1}{(1-z)^2} - 2\frac{z^2}{1-z}$, which satisfies $\mathbf{L}_1$–$\mathbf{L}_3$. Thus, $\zeta = 0.39660\ldots$ is the real root of

$$2z^3 - 2z^2 + 3z - 1 = 0,$$

$\rho := B(\zeta)/\zeta = 5.61071\ldots$ (the growth rate) is the real root of

$$4z^3 - 35z^2 + 84z - 76 = 0,$$

and $\xi = 12.15864\ldots$ is the real root of

$$38z^3 - 425z^2 - 416z - 416 = 0.$$

Altogether, we obtain the claimed result. $\square$

Chapter 7

MORE ON PATTERN AVOIDANCE IN DECORATED PERMUTATIONS

So far, we have described all decorated permutations using the two-line notation $w^{\circ} = (w, co)$ and the one-line notation which includes orientation decorations for all fixed points, e.g. $w^{\circ} = 895\overleftarrow{4}\,7\,\overrightarrow{6}\,132$. We now define the *flattened one-line notation* for w° as a length n string which incorporates the information from both the permutation w and the co function. We learned of this technique from Brendon Rhoades, and this notation for decorated permutations first appeared in [BRT21]. To write decorated permutations using this notation, we think of fixed points as decorated ($\circlearrowright$) or undecorated ($\circlearrowleft$). In the flattened one-line notation for w, we first replace all decorated fixed points with zeros to mark their locations. Then, if the remaining nonzero entries occur in positions $L = \{\ell_1 < \cdots < \ell_j\}$, we replace them with the entries of the permutation $\mathrm{fl}(w_L) \in S_j$, using the notation for a flattened permutation defined in Section 2.1.4. For an example of converting a decorated permutation from the (w, co) notation to flattened one-line notation, see Example 7.0.1 below. For the rest of this section, assume that all decorated permutations are given in flattened one-line notation.

We note that writing decorated permutations in flattened one-line notation leads to an easy way to count $\#S_n^{\circ}$. We partition S_n° according to the number of nonzero elements in each decorated permutation. A decorated permutation in S_n° with j nonzero elements is determined by the unique set in $\binom{[n]}{j}$ giving the positions of the nonzero elements and the unique permutation in S_j occurring in the nonzero positions. Therefore,

$$\#S_n^{\circ} = \sum_{j=0}^{n} \binom{n}{j} j! = \sum_{j=0}^{n} \frac{n!}{j!}.$$

From this enumeration, observe that

$$\lim_{n\to\infty} \frac{\#S_n^{\circlearrowleft}}{n!} = \lim_{n\to\infty} \sum_{j=0}^{n} \frac{1}{j!} = e.$$

Thus the sequence $\#S_n^{\circlearrowleft}$ is in $\Theta(n!)$. See Section 2.1.4 for the definition of Θ notation. Recall from Theorem 6.3.5 the asymptotic growth of the number of smooth decorated permutation in $S_n^{\circlearrowleft}$, $s(n) \sim \frac{(5.61071...)^{n+1}}{(8.74042...)\cdot n^{3/2}}$. In particular, the set of smooth decorated permutations is asymptotically a very small fraction of the set of all decorated permutations.

In general, given a subset $L \subseteq [n]$, define $\mathrm{fl}(w_L^{\circlearrowleft})$ to be the decorated permutation in $S_{|L|}^{\circ\bullet}$ obtained from $w_L^{\circlearrowleft}$ by preserving all zeros and flattening the set of nonzero entries. Then, for $m \leq n$ and a decorated permutation $v^{\circlearrowleft} \in S_m^{\circlearrowleft}$, say that $w^{\circlearrowleft}$ *contains the pattern* $v^{\circlearrowleft}$ if $\mathrm{fl}(w_L^{\circlearrowleft}) = v^{\circlearrowleft}$ for some subset $L \subseteq [n]$. If there is no such L, then $w^{\circlearrowleft}$ *avoids* $v^{\circlearrowleft}$.

Example 7.0.1. Let $w^{\circlearrowleft} = (w, co)$ be defined by

$$w = 5231764, \quad co : \begin{cases} 2 \mapsto \circlearrowright \\ 3 \mapsto \circlearrowleft \\ 6 \mapsto \circlearrowright \end{cases} .$$

The two fixed points with clockwise orientation are 2 and 6, so we first rewrite w as 5031704. Then $\mathrm{fl}(53174) = 42153 \in S_5$. Thus, in flattened one-line notation, $w^{\circlearrowleft} = 4021503$. Then $w^{\circlearrowleft}$ contains the pattern $v^{\circlearrowleft} = 1020$ since for $L = \{1, 2, 5, 6\}$, we have $\mathrm{fl}(w_L^{\circlearrowleft}) = \mathrm{fl}(4050) = 1020$.

For $v^{\circlearrowleft} \in S_m^{\circlearrowleft}$ with $m \leq n$, let $S_n^{\circlearrowleft}(v^{\circlearrowleft})$ be the set of decorated permutations in $S_n^{\circlearrowleft}$ that avoid the pattern $v^{\circlearrowleft}$. We consider S_n to be the subset of decorated permutations in $S_n^{\circlearrowleft}$ with no zeros. Therefore, if $v^{\circlearrowleft} \in S_m$, then $S_n(v^{\circlearrowleft}) = S_n^{\circlearrowleft}(v^{\circlearrowleft}) \cap S_n$. We have considered avoidance of certain patterns of small length and corresponding enumerations.

7.1 Avoiding permutations in S_n

Let $v^{\circlearrowleft} \in S_m$, and set $a_j(v^{\circlearrowleft}) = \#S_j(v^{\circlearrowleft})$. As with counting $\#S_n^{\circlearrowleft}$, we partition the set $S_n^{\circlearrowleft}(v^{\circlearrowleft})$ according to the number of zeros contained in each decorated permutation. Any $w^{\circlearrowleft} \in S_n^{\circlearrowleft}(v^{\circlearrowleft})$

with j nonzero elements is determined uniquely by the locations of the j nonzero elements and the permutation $u \in S_j(v^{\circ})$ occupying the nonzero entries of w°. Since the nonzero elements may be placed in $\binom{n}{j}$ ways, then $\#S_n^{\circ}(v^{\circ})$ is the binomial transform of the $a_j(v^{\circ})$ sequence.

Proposition 7.1.1. The number of decorated permutations in S_n° avoiding $v^{\circ} \in S_m$ is

$$\#S_n^{\circ}(v^{\circ}) = \sum_{j=0}^{n} \binom{n}{j} a_j(v^{\circ}).$$

7.1.1 Permutations of length 3

For $v^{\circ} \in S_3$, recall that $a_j(v^{\circ}) = \#S_j(v^{\circ}) = C_j$, where C_j is the jth Catalan number. The enumeration $a_j(v^{\circ}) = C_j$ provides the following enumeration for decorated permutations avoiding v°.

Corollary 7.1.2. The number of decorated permutations in S_n° avoiding any $v^{\circ} \in S_3$ is

$$\#S_n^{\circ}(v^{\circ}) = \sum_{j=0}^{n} \binom{n}{j} C_j.$$

The binomial transform of the Catalan numbers can be found in the OEIS at [OEIS, A007317]. Other objects counted by $\sum_{j=0}^{n} \binom{n}{j} C_j$ include the following examples.

- Rooted plane trees with nodes that have positive integer weights and whose total weight is $n+1$, where total weight is the sum of the node weights. Noted by Brad Jones.

- Complete rooted binary trees of weight $n+1$, where the leaves have positive integer weights. Noted by Michael Somos.

- Symmetric hex trees with $2n$ edges. A *hex tree* is an ordered, rooted tree where each vertex has 0, 1, or 2 children, and when only one child is present, it is a left, median, or right child. See [HR70] and [KS16]. Noted by Emeric Deutsch.

- 321-avoiding set partitions of $[n+1]$. Given a partition $\mathcal{P}$ of $[n]$, write the entries of each block in increasing order, and then arrange the blocks of $\mathcal{P}$ in increasing order of their first entries. Then $\mathrm{fl}(\mathcal{P})$ is the permutation in S_n obtained by removing the divisors between blocks. We say that $\mathcal{P}$ is *v-avoiding* if $\mathrm{fl}(\mathcal{P})$ is v-avoiding. See [Cal09].

7.1.2 Permutations of length 4

There are three Wilf equivalence classes for permutations in S_4. The enumerations found for two of the classes give the following corollaries.

Corollary 7.1.3. [Ges90] [OEIS, A005802] The number of decorated permutations in S_n° avoiding 1234 is

$$\#S_n^{\circ}(1234) = \sum_{j=0}^{n} \binom{n}{j} a_j(1234),$$

where

$$a_j(1234) = 2\sum_{k=0}^{j} \binom{2k}{k} \binom{j}{k}^2 \frac{3k^2 + 2k + 1 - j - 2kn}{(k+1)^2(k+2)(j-k+1)}.$$

Corollary 7.1.4. [Bón97] [OEIS, A022558] The number of decorated permutations in S_n° avoiding 1342 is

$$\#S_n^{\circ}(1342) = \sum_{j=0}^{n} \binom{n}{j} a_j(1342),$$

where

$$a_j(1342) = \frac{(7j^2 - 3j - 2)}{2}(-1)^{j-1} + 3\sum_{k=2}^{j} 2^{k+1} \frac{(2k-4)!}{k!(k-2)!} \binom{j-k+2}{2} (-1)^{j-k}.$$

The sequence $\#S_j(1324)$ can be found at [OEIS, A061552]. The exact formula for this sequence is still an open problem. A recursive formula is given in [MR03].

7.2 More enumerations for decorated permutations

Proposition 7.2.1. We have the following enumerations for $\#S_n^{\circ}(v^{\circ})$ for the given decorated permutations v°.

(i) $v^{\circlearrowleft} = 0^m$ [OEIS, A334156]:

$$\#S_n^{\circlearrowleft}(0^m) = n! \sum_{j=0}^{m-1} \frac{1}{j!}$$

(ii) $v^{\circlearrowleft} = 1$:

$$\#S_n^{\circlearrowleft}(1) = 1$$

(iii) $v^{\circlearrowleft} = 01$ or 10 [OEIS, A003422]:

$$\#S_n^{\circlearrowleft}(01) = \#S_n^{\circlearrowleft}(10) = \sum_{j=0}^{n} j!$$

(iv) $v^{\circlearrowleft} = 12$ or 21 [OEIS, A000079]:

$$\#S_n^{\circlearrowleft}(12) = \#S_n^{\circlearrowleft}(21) = \sum_{j=0}^{n} \binom{n}{j} = 2^n$$

(v) $v^{\circlearrowleft} = 001$ or 100 or 010 [OEIS, A334155]:

$$\#S_n^{\circlearrowleft}(001) = \#S_n^{\circlearrowleft}(100) = n! + \sum_{j=0}^{n-1} (j+1)!$$

(vi) $v^{\circlearrowleft} = 012$ or 210 [OEIS, A334154]:

$$\#S_n^{\circlearrowleft}(012) = n! + \sum_{j=1}^{n} \sum_{\ell=1}^{n-j+1} \binom{n-\ell}{j-1} \binom{n-j}{\ell-1} (\ell-1)!.$$

(vii) $v^{\circlearrowleft} = 102$ or 201 [OEIS, A051295]:

$$\#S_n^{\circlearrowleft}(102) = n! + \sum_{j=1}^{n} (j-1)! \cdot (\#S_{n-j}^{\circ\bullet}(102)).$$

From the OEIS entry, this sequence also counts

- $w \in S_{n+1}$ that contain a 132 pattern only as part of a 4132 pattern.
- $w \in S_{n+1}$ such that the elements of each cycle of w form an interval. Noted by Michael Albert.

Proof.

(i) Observe that $w^{\circlearrowleft} \in S_n^{\circlearrowleft}(0^m) \Leftrightarrow w^{\circlearrowleft}$ has at most $m-1$ zeros. Therefore,

$$\#S_n^{\circlearrowleft}(0^m) = \sum_{j=0}^{m-1} \binom{n}{j}(n-j)! = \sum_{j=0}^{m-1} \frac{n!}{j!} = n! \sum_{j=0}^{m-1} \frac{1}{j!}.$$

(ii) Observe that $w^{\circlearrowleft} \in S_n^{\circlearrowleft}(1) \Leftrightarrow w^{\circlearrowleft} = 00\cdots 0$.

(iii) Observe that $w^{\circlearrowleft} \in S_n^{\circlearrowleft}(01) \Leftrightarrow w^{\circlearrowleft} = u0\cdots 0$ for some $u \in S_m$, $m \le n$. Then $w^{\circlearrowleft}$ is completely determined by the permutation u.

(iv) Observe that $w^{\circlearrowleft} \in S_n^{\circlearrowleft}(12) \Leftrightarrow$ the nonzero entries of $w^{\circlearrowleft}$ are decreasing. Therefore, $w^{\circlearrowleft}$ is completely determined by the positions of its zeros.

(v) Partition $S_n^{\circlearrowleft}$ according to the number of nonzero entries appearing in a decorated permutation. Consider $w^{\circlearrowleft} \in S_n^{\circlearrowleft}(001)$ with j nonzero entries. If $j = n$, then $w^{\circlearrowleft} \in S_n$. If $j \in [0, n-1]$, then $w^{\circlearrowleft}$ has the form $w^{\circlearrowleft} = u_1 0 u_2 0 \cdots 0$, where the concatenation of the words u_1 and u_2 in flattened one-line notation, $u = u_1 u_2$, is in S_j. There are $j!$ choices of the permutation u, and there are $j+1$ ways to partition u into u_1 and u_2. Summing over all j,

$$\#S_n^{\circlearrowleft}(001) = n! + \sum_{j=0}^{n-1} (j+1)!.$$

Now consider decorated permutation pattern 010. Note that $w^{\circlearrowleft} \in S_n^{\circlearrowleft}(010) \Leftrightarrow w^{\circlearrowleft}$ does not have two nonadjacent strings of zeros. Again, partition $S_n^{\circlearrowleft}$ according to the number of nonzero entries appearing in a decorated permutation. Assume $w^{\circlearrowleft}$ has j nonzero entries. If $j = n$, then $w^{\circlearrowleft} \in S_n$. For $j \in [0, n-1]$, $w^{\circlearrowleft}$ has the form $w^{\circlearrowleft} = w_1 \cdots w_i 0 \cdots 0 w_{i+1} \cdots w_j$, where $w = w_1 \cdots w_j \in S_j$ and $0 \le i \le j$. The following map gives a bijection between decorated permutations in $S_n^{\circlearrowleft}(010)$ with $n-j$ zeros and decorated permutations in $S_n^{\circlearrowleft}(001)$ with $n-j$ zeros:

$$w^{\circlearrowleft} = w_1 \cdots w_i \underbrace{0 \cdots 0}_{n-j} w_{i+1} \cdots w_j \quad \mapsto \quad \sigma^{\circ\bullet} = w_1 \cdots w_i 0 w_{i+1} \cdots w_j \underbrace{0 \cdots 0}_{n-j-1}.$$

Therefore $\# S_n^{\circlearrowleft}(010) = \# S_n^{\circlearrowleft}(001)$.

(vi) Partition $S_n^{\circlearrowleft}$ according to the number of zeros appearing in a decorated permutation. Consider $w^{\circlearrowleft} \in S_n^{\circlearrowleft}(012)$ with j zeros. If $j = 0$, then $w^{\circlearrowleft} \in S_n$, and $S_n(012) = S_n$. For $j > 0$, let ℓ be the index of the first zero. If the first zero occurs at index $\ell > n-j+1$, then by the pigeonhole principle, $w^{\circlearrowleft}$ must have fewer than j zeros. Therefore ℓ must be in $[1, n-j+1]$. Choose the positions of the remaining zeros in $\binom{n-\ell}{j-1}$ ways. The first $\ell - 1$ entries of w may be chosen in $\binom{n-j}{\ell-1}$ ways and arranged in any order. All nonzero entries after the first zero must be decreasing, so they can only be arranged in one way. Therefore, there are $\binom{n-\ell}{j-1}\binom{n-j}{\ell-1}(\ell-1)!$ decorated permutations in $S_n^{\circlearrowleft}(012)$ containing $j > 0$ zeros whose first zero occurs at index ℓ. Summing over all $j \in [0, n]$ all $\ell \in [1, n-j+1]$ yields

$$\# S_n^{\circlearrowleft}(012) = n! + \sum_{j=1}^{n} \sum_{\ell=1}^{n-j+1} \binom{n-\ell}{j-1}\binom{n-j}{\ell-1}(\ell-1)!.$$

(vii) Partition $S_n^{\circlearrowleft}$ according to the number of zeros appearing in a decorated permutation. Let $w^{\circlearrowleft} \in S_n^{\circlearrowleft}(102)$ contain j zeros. If $j = 0$, then $w^{\circlearrowleft} \in S_n$, and $S_n(102) = S_n$. For $j > 0$, assume that the first zero of $w^{\circlearrowleft}$ occurs at index $n-j+1$. Then we can write $w^{\circlearrowleft}$ as $w^{\circlearrowleft} = u_1 0 u_2$, where the elements of u_1 are all greater than the elements of u_2, and $u_2 \in S_{n-j}^{\circ\bullet}(102)$. The elements of u_1 can be arranged in any order since there are no zeros among the entries of u_1. Summing over all j yields the recursive formula

$$\# S_n^{\circlearrowleft}(102) = n! + \sum_{j=1}^{n} (j-1)! \cdot (\# S_{n-j}^{\circ\bullet}(102)).$$

Similar logic can be used for the other two objects listed to show that they also satisfy this recursive formula.

$\square$

The sequences A334154, A334155, and A334156 are new to the OEIS and were added by the author. We note that all of the sequences given in Section 7.1 and Section 7.2 are distinct sequences.

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