THE HEAT EQUATION AND REFLECTED BROWNIAN MOTION IN TIME-DEPENDENT DOMAINS

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Abstract. The paper is concerned with reflecting Brownian motion (RBM) in domains with deterministic moving boundaries, also known as "non-cylindrical domains," and its connections with partial differential equations. Construction is given for reflecting Brownian motion in C^3-smooth time-dependent domains in the n-dimensional Euclidean space $\mathbf{R}^n$. Various sample path properties of RBM, its two-sided transition density functions estimate and the probabilistic representation of solutions for the corresponding partial differential equations are obtained. Furthermore, the one-dimensional case is throughly studied, with the assumptions on the smoothness of the boundary drastically relaxed.

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1. Introduction

This is the first part of a two-paper series on the heat equation and reflecting Brownian motion in time-dependent domains. The paper is concerned with reflecting Brownian motion in domains with deterministic moving boundaries, also known as "non-cylindrical domains," and its connections with partial differential equations. A related paper, Burdzy, Chen and Sylvester (2002a), studies the existence and uniqueness of solutions to the heat equation in this context from the analytic point of view. Some of the results of this paper, for example, a Feynman-Kac type formula, are the basis for several effective quantitative and qualitative arguments in the second paper in this series, Burdzy, Chen and Sylvester (2002b).

The analytic literature on the heat equation and related problems is enormous and we would rather let the reader search the library than provide an exceedingly imperfect review. Crank (1984) provides an excellent review of various problems related to free and moving boundaries. Although one can see obvious general similarities between our problem and the classical Stefan's problem, it remains to be seen if there exist any connections at the technical level. For an analytic approach to the same model as in our paper, see Hofmann and Lewis (1996) and Lewis and Murray (1995).

Brownian motion in time-dependent domains belongs to "classical" subjects in probability. The model appears in the context of a problem often referred to as "boundary crossing." The literature on the problem is huge; we suggest Anderson and Pitt (1997) and Durbin (1992) as starting points. The boundary crossing problem was mainly motivated by statistical questions but the estimates derived in this area have been also applied to study Brownian path properties, see, e.g., Bass and Burdzy (1996) or Greenwood and Perkins (1983). In the context of our article, this classical model may be described as a Brownian motion killed on the boundary of a time-dependent domain. The corresponding analytic problem may be called the heat equation in time-dependent domain with Dirichlet boundary conditions.

Our article is devoted to Brownian motion *reflected* on rather than killed at the boundary of a time-dependent domain. The analytic counterpart of the model is a heat equation with Neumann rather than Dirichlet boundary conditions. We are not aware of any article devoted to a systematic study of such a process but this stochastic process appeared in literature in several unrelated contexts; see Bass and Burdzy (1999), Cranston and Le Jan (1989), El Karoui and Karatzas (1991a,b), Knight (1999) or Soucaliuc, Toth and Werner (1999).

There exists an extensive literature devoted to the relationship of Brownian motion and the heat equation. We suggest four books as possible starting points: Bass (1997), Doob (1984), Durrett (1984) and Port and Stone (1978). But, to the authors' best knowledge, the interplay between the reflected Brownian motion and the heat equation in time-dependent domains has not been investigated before.

One of the strongest assertions about existence and uniqueness of reflecting Brownian motion (RBM) in a smooth *time-independent* domain has the following form (Lions and Sznitman (1984)). Suppose B_t is a Brownian motion in $\mathbf{R}^n$. For any bounded C^2-smooth domain $D \subset \mathbf{R}^n$, there exists a unique solution X_t (reflected Brownian motion) to the following Skorohod equation,

$$X_t = X_0 + B_t + \int_0^t \mathbf{n}(X_s) dL_s, \quad t \geq 0, \tag{1.1}$$

where $\mathbf{n}$ is the inward normal vector on the boundary ∂D and L is a continuous nondecreasing process with $L_0 = 0$ which increases only when X_t is on the boundary ∂D, that is,

$$L_t = \int_0^t 1_{\{X_s \in \partial D\}} dL_s. \tag{1.2}$$

Recently strong existence and pathwise uniqueness have been established by Bass, Burdzy and Chen (2002) for reflecting Brownian motions in bounded planar lip domains.

When the domain D in $\mathbf{R}^n$ is C^3-smooth, there are a number of ways of constructing a reflecting Brownian motion in D—all these methods yield the same continuous strong Markov process on $\overline{D}$. Reflecting Brownian motion can be constructed using Dirichlet form methods (Bass and Hsu (1990,1991), Fukushima (1967)). It can be obtained by solving the deterministic Skorohod problem (1.1)-(1.2) or by solving the corresponding stochastic differential equation (Costantini (1992), Dupuis and Ishii (1993), Lions and Sznitman (1984), Saisho (1987), Tanaka (1979)). It can also be constructed by solving a submartingale problem (Stroock and Varadhan (1971)), or using an analytic method starting by solving the heat equation for the transition density function for reflecting Brownian motion (Hsu (1984), Sato and Ueno (1965)). See the introduction to Williams and Zheng (1990) for more information.

When the Dirichlet form method is applied, the smoothness assumption on the boundary of D can be dramatically relaxed. One can construct reflecting Brownian motion on an arbitrary domain and effectively study its various properties (see Burdzy and Chen (1998), Burdzy and Khoshnevisan (1998), Chen (1993,1996), Chen, Fitzsimmons and Williams (1993), Fukushima (1967), Fukushima and Tomisaki (1996), Williams and Zheng (1990)). RBM constructed in this way is unique in the sense of distribution. In some non-smooth domains, RBM is a semimartingale (see Chen, Fitzsimmons and Williams (1993), Fukushima (1999), Fukushima and Tomisaki (1996), Williams and Zheng (1990)). This holds when, intuitively speaking, the boundary of the domain has locally finite "surface area". For such domains, a generalized definition of the normal vector $\mathbf{n}$ for D can be given and one can find a Brownian motion B such that a Skorohod decomposition similar to (1.1) holds for the reflected Brownian motion X. All results mentioned in the last three paragraphs hold for reflecting Brownian motions in *time-independent* domains. Motivated by a preliminary version of our paper, Oshima (2001) recently constructed reflecting diffusions in certain time-dependent domains by using time-dependent Dirichlet form approach.

At the other extreme, parts of the theory of reflecting Brownian motion and the corresponding heat equation are known to hold only in domains with Lipschitz or Hölder continuous boundaries (cf. Bass and Hsu (1991) and the references therein).

We would like to point out that in the one-dimensional case the strongest existence results for RBM and solutions to the heat equation are obtained via the deterministic Skorohod equation (see Section 3 below).

The remaining of the paper is organized as follows. In section 2, we give the construction of reflecting Brownian motion in C^3-smooth time-dependent domains in the n-dimensional Euclidean space $\mathbf{R}^n$ and derive an upper bound estimate for its transition density functions, also called the heat kernels. We prove the existence of boundary local time for the reflecting Brownian motion and derive its Skorohod decomposition. We then focus on the probabilistic representation of solutions for the corresponding partial differential equations. For this, exponential integrability of the boundary local time is established. Several results will elucidate the relationship between "forward" and "backward" equations and the time reversal transformation of the reflected Brownian motion. We would like to point out Corollary 2.12, which contains Feynman-Kac formulas in terms of reflecting Brownian motions in a space-time domain as well as in its time-reversed domain. This formula is one of the main technical tools in Burdzy, Chen and Sylvester (2002b) to study the detailed properties of the heat equation solutions, including the existence of heat atoms and singularities.

Section 3 of this paper is devoted to the one-dimensional case. A deterministic version of the Skorohod equation allows us to drastically relax the assumptions on the smoothness of the boundary. Various properties concerning the heat kernels or the marginal distributions of the reflecting Brownian motion are studied using probabilistic means. For example, it is shown for a one-dimensional time-dependent domain that, as long as the boundary is continuous, the marginal distribution of the reflecting Brownian motion in the interior is absolutely continuous with respect to the Lebesgue measure and its density function satisfies the heat equation. When the boundary of a one-dimensional time-dependent domain is given by a continuous function $g(t)$ whose distributional derivative $g'(t)$ is locally square integrable then transforming it into a time-independent domain can result in a useful representation of the heat equation solution—see Theorem 3.9 for a precise statement. This implies, in particular, that under these assumptions the reflecting Brownian motion has transition density functions up to the boundary. It should be noted that if the local square integrability of $g'(t)$ is not satisfied, then the distribution of the reflecting Brownian motion at the boundary point can be singular with respect to the Lebesgue measure—this is one of the main topics of the second paper in this series, Burdzy, Chen and Sylvester (2002b).

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2. Multidimensional reflecting Brownian motions in time-dependent domains

Let $\dot{D}$ be a subset of $\mathbf{R}_+ \times \mathbf{R}^n$ such that the projection of $\dot{D}$ onto the time axis is $[0,T)$ with $0 < T \leq \infty$, and that for each $0 \leq t < T$, $D(t) = \{x \in \mathbf{R}^n : (t,x) \in D\}$ is a bounded connected open set in $\mathbf{R}^n$. In this section we will assume that $\partial \dot{D} \cap (0,T) \times \mathbf{R}^n$ is C^3-smooth. Let $\mathbf{n}(t,x)$ be the unit inward normal of $D(t)$ at a boundary point x. Sometimes we will identify $\mathbf{n}(t,x)$ with a vector in $\mathbf{R}_+ \times \mathbf{R}^n$ in an obvious way. Let $\overrightarrow{\gamma}$ be the unit inward normal vector field on $\partial \dot{D}$.

Theorem 2.1. *Suppose that $\overrightarrow{\gamma} \cdot \mathbf{n} \geq c_0$ on $\partial \dot{D} \cap (0,T) \times \mathbf{R}^n$ for some positive constant $c_0 > 0$. Suppose that B is a Brownian motion in $\mathbf{R}^n$ with $B(0) = \mathbf{0}$. Then for each $(s,x) \in \overline{\dot{D}}$ with $s < T$, there is a unique pair of continuous processes $(X^{s,x}, L^{s,x})$ adapted to the minimal admissible filtration of B such that* *(i)* $(t, X_t^{s,x}) \in \overline{\dot{D}}$ *for* $t \in [s, T)$ *with* $X_s^{s,x} = x$,

(ii) $\{L_t^{s,x},\ t \in [s, T)\}$ *is a nondecreasing process with* $L_s^{s,x} = 0$ *such that* $t \to L_t^{s,x}$ *increases only when the process* (t, X_t) *is on the boundary of* $\dot{D}$, *i.e.,* $L_t^{s,x} = \int_s^t \mathbf{1}_{\partial \dot{D}}((r, X_r)) dL_r^{s,x}$ *for* $s \le t < T$,

(iii) $X_t^{s,x} = x + (B_t - B_s) + \int_s^t \mathbf{n}(r, X_r^{s,x}) dL_r^{s,x}$ *for* $s \le t < T$.

Proof. Theorem 4.3 of Lions and Sznitman (1984) may be applied to construct from the space-time Brownian motion (t, B_t) a new process, a diffusion $X^{s,x}$ in $\dot{D}$ with oblique reflection vector field $\mathbf{n}$. All assertions of Theorem 2.1 follow immediately from that result. □

Let $\mathbf{P}^{(s,x)}$ denote the law of $X^{(s,x)}$ induced on $C[0, \infty)$, the space of continuous functions equipped with uniform topology on each compact time interval. Let X be the canonical map on $C[0, \infty)$. The uniqueness of $X^{s,x}$ implies that $X = (X, \mathbf{P}^{(s,x)}, (s, x) \in \overline{\dot{D}})$ is a time-inhomogeneous strong Markov process. It is in fact a continuous Feller process as we will see in Theorem 2.5.

We will now prove the existence of the transition density for X and find some estimates for it, using a parametric method from the theory of partial differential equations (see, e.g. Itô (1957) or Hsu (1987)).

From now on, we will work with "the half Laplacian" operator $\frac{1}{2}\Delta$ rather then the standard Laplacian Δ because the standard Brownian motion is related to $\frac{1}{2}\Delta$. One can pass from one normalization to the other by a trivial change of variable.

Theorem 2.2. *There exists a fundamental solution* $p(s, x; t, y)$, (s, x), $(t, y) \in \dot{D}$, $s < t < T$, *for the following differential equation:*

$$\begin{cases} \frac{\partial p}{\partial s} + \frac{1}{2}\Delta_x p = 0 & \text{for } (s,x) \in \dot{D} \text{ with } s < t, \\ \frac{\partial p}{\partial \mathbf{n}} = 0 & \text{for } (s,x) \in \partial\dot{D} \text{ with } s < t, \\ \lim_{s \uparrow t} p(s, x; t, y) dx = \delta_{\{y\}}(dx) & \text{for } (t, y) \in \dot{D}. \end{cases} \tag{2.1}$$

The function $p(s, x; t, y)$ *is continuous on* $\overline{\dot{D}} \times \overline{\dot{D}}$ *with* $s < t < T$, *continuously differentiable in* $s \in (0, t)$ *and of class* $C^2(D(s)) \cap C^1(\overline{D(s)})$ *as a function of* x.

Proof. We will use $|\,.\,|$ to denote the Euclidean norm and d to denote the Euclidean distance. Let $\Gamma(s, x; t, y)$ be the fundamental solution for the heat equation $\frac{\partial u}{\partial s} + \frac{1}{2}\Delta_x u = 0$ in $\mathbf{R}^n$; that is, $\Gamma(s, x; t, y) = (2\pi(t - s))^{-\frac{n}{2}} \exp\left(-\frac{|x-y|^2}{2(t-s)}\right)$. For $(s, x) \in \dot{D}$, let $x_0 \in \partial D(s)$ be such that $|x - x_0| = d(x, \partial D(s))$ and let $x_s^* = 2x_0 - x$, the point symmetric to x with respect to x_0. Note that since $D(s)$ is C^3, x_0 and x_s^* are uniquely determined by (s, x) and are C^2-smooth in (s, x) provided (s, x) is sufficiently close to the boundary $\partial\dot{D}$. For each fixed $T_0 < T$, let $\phi \in C_c^\infty(\mathbf{R}_+ \times \mathbf{R}^n)$ (the space of infinitely differentiable functions with compact support) with $0 \le \phi \le 1$ and such that for $s \le T_0$,

$$\phi(s, x) = \begin{cases} 1 & \text{if } d((s,x),\, \partial\dot{D}) \le \varepsilon_0/2, \\ 0, & \text{if } d((s,x),\, \partial\dot{D}) \ge \varepsilon_0, \end{cases}$$

where ε_0 is a fixed small constant and $d((s,x), \partial\dot{D})$ is the Euclidean distance between (s,x) and the boundary of $\dot{D}$ in $\mathbf{R}_+ \times \mathbf{R}^n$. As a first approximation of p, set

$$p_0(s,x;t,y) = \Gamma(s,x;t,y) + \phi(s,x)\Gamma(s,x_s^*;t,y).$$

This function satisfies the boundary and terminal conditions in (2.1). The idea of the remaining part of the argument is to find a suitable function $f(s,x;t,y)$ so that if

$$p_1(s,x;t,y) = \int_s^t \left(\int_{D(r)} p_0(s,x;r,z) f(r,z;t,y) dz \right) dr$$

then

$$p(s,x;t,y) = p_0(s,x;t,y) + p_1(s,x;t,y), \qquad s < t \le T_0, \tag{2.2}$$

is the fundamental solution for (2.1). Note that p defined in (2.2) satisfies the boundary and terminal condition in (2.1). We would like the function p defined in (2.2) to satisfy the heat equation, i.e.,

$$\left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta_x\right) p_0 + \left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta_x\right) \int_s^t \left(\int_{D(r)} f(r,z;t,y) p_0(s,x;r,z) dz \right) dr = 0.$$

This is equivalent to

$$\begin{aligned} f(s,x;t,y) = &\left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta_x\right) p_0(s,x;t,y) \\ &+ \int_s^t \left(\int_{D(r)} f(r,z;t,y) \left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta_x\right) p_0(s,x;t,y) dz \right) dr. \end{aligned} \tag{2.3}$$

It remains to solve (2.3) for f. This is an integral equation of Volterra type, which can be solved by the method of iteration. Let

$$\begin{aligned} f_0(s,x;t,y) &= \left(\frac{\partial}{\partial s} + \frac{1}{2}\Delta_x\right) p_0(s,x;t,y), \\ f_k(s,x;t,y) &= \int_s^t \left(\int_{D(r)} f_0(s,x;r,z) f_{k-1}(r,z;t,y) dz \right) dr, \quad k \ge 1, \\ f(s,x;t,y) &= \sum_{k=0}^{\infty} f_k(s,x;t,y). \end{aligned} \tag{2.4}$$

We will show below that $\sum_{k=0}^{\infty} f_k(s,x;t,y)$ is absolutely convergent and solves (2.3).

Using induction, we can show that (cf. p.375 of Hsu (1987)) for each fixed $l < T$, there are constants K_1, K_2 and C such that for all $(s,x),(t,y) \in \overline{\dot{D}}$ with $s < t \le l$,

$$|f_k(s,x;t,y)| \le K_1 K_2^k \Gamma\left(\frac{k+1}{2}\right)^{-1} (t-s)^{(k-1-n)/2} \exp\left(-\frac{c|x-y|^2}{(t-s)}\right), \quad k \ge 1. \tag{2.5}$$

Here $\Gamma(\lambda)$ is the Gamma function defined by $\Gamma(\lambda) = \int_0^\infty t^{\lambda-1}e^{-t}dt$ for $\lambda > 0$. Thus $f(s,x;t,y) = \sum_{k=0}^\infty f_k(s,x;t,y)$ is well defined and continuous. It is easy to deduce from (2.4) that it satisfies equation (2.3). □

For a fixed $t < T$, and a bounded continuous function ϕ on $\overline{D}(t)$, we see from Theorem 2.2 that

$$u(s,x) = \int_{D(t)} p(s,x;t,y)\phi(y)dy$$

is a solution of the following equation:

$$\begin{cases} \frac{\partial u}{\partial s} + \frac{1}{2}\Delta_x u = 0 & \text{for } (s,x) \in \dot{D} \text{ with } s < t \;, \\ \frac{\partial u}{\partial \mathbf{n}} = 0 & \text{for } (s,x) \in \partial\dot{D} \text{ with } s < t, \\ \lim_{s\uparrow t} u(s,x) = \phi(x). \end{cases} \tag{2.6}$$

The following theorem is a special case of the uniqueness result in Friedman (1964, Theorem 15 in Chapter 2).

Theorem 2.3. *For fixed $(t,x) \in \dot{D}$, the solution of the heat equation (2.6) is unique.*

Theorem 2.4. *The function $p(s,x;t,y)$ in Theorem 2.2 has the following properties:*

(i) $p(s,x;t,y)$ is strictly positive, and C^2-smooth on $\{(s,x,t,y) \in \overline{\dot{D}} \times \overline{\dot{D}} : s < t < T\}$;

(ii) For $s < t < T$, $(s,x) \in \overline{\dot{D}}$,

$$\int_{D(t)} p(s,x;t,y)dy = 1;$$

(iii) The Chapman-Kolmogorov equations hold: for any $0 \le s < r < t < T$ and any $(s,x),(t,y) \in \overline{\dot{D}}$,

$$p(s,x;t,y) = \int_{D(r)} p(s,x;r,z)p(r,z;t,y)dz;$$

(iv) For each fixed $0 < l < T$, there exist constants $K_l > 0$ and $C_l < \infty$ such that

$$p(s,x;t,y) \le C_l(t-s)^{-n/2}\exp\left(\frac{-K_l|x-y|^2}{(t-s)}\right)$$

for $s < t < l$ and $(s,x),(t,y) \in \overline{\dot{D}}$;

(v) Let $\dot{D}_\varepsilon = \{(t,x) \in \dot{D} : d(x,\partial D(t)) < \varepsilon\}$. For each fixed $0 < l < T$, there are constants $\varepsilon_l > 0$ and $C_l > 0$ such that for $0 < \varepsilon < \varepsilon_l$, $0 < s < t \le l$ and $(s,x) \in \dot{D}$,

$$\frac{1}{\varepsilon}\int_{\dot{D}_\varepsilon} p(s,x;t,y)dy \le C_l/\sqrt{t-s}.$$

Proof. (i) The positivity of $p(s,x;t,y)$ is a consequence of a strong version of the maximum principle (see Theorem 1 in Chapter 2 of Friedman (1964)), while the C^2 smoothness follows from equations (2.2) and (2.3). Assertions (ii) and (iii) follow from Theorem 2.3. Claim (iv) follows from the estimate (2.5) and equation (2.2). Finally, (v) follows from (iv). □

Theorem 2.5. *The function $p(s,x;t,y)$ in Theorem 2.2 is the transition density of the time-inhomogeneous reflecting Brownian motion X on $\dot{D}$ defined in Theorem 2.1. Therefore X is a continuous Feller process and hence a strong Markov process.*

Proof. For any fixed $t < T$, and a bounded continuous function ϕ on $\overline{D}(t)$, let

$$u(s,x) = \int_{D(t)} p(s,x;t,y)\phi(y)dy.$$

The function $u(s,x)$ is a C^2-smooth solution to equation (2.6). For $(s,x) \in \overline{\dot{D}}$, applying the Itô formula to $u(r, X_r^{s,x})$, we have

$$\begin{aligned} du(r, X_r^{s,x}) &= u_s(r, X_r^{s,x})dr + \frac{1}{2}\Delta u(r, X_r^{s,x})dr + \nabla u(r, X_r^{s,x})dB_r + \frac{\partial u}{\partial \mathbf{n}}(r, X_r^{s,x})dL_r^{s,x} \\ &= \nabla u(r, X_r^{s,x})dB_r. \end{aligned}$$

Hence

$$u(s,x) = \mathbf{E}[u(t, X_t^{s,x})] = \mathbf{E}[\phi(X_t^{s,x})].$$

This shows that the distribution of $X_t^{s,x}$ is absolutely continuous with respect to the Lebesgue measure and its density function is $p(s,x;t,y)$. From the continuity of p, we see that for bounded measurable function ϕ on $\overline{D}(t)$, $u(s,x) = \mathbf{E}[\phi(X_t^{s,x})]$ is a continuous function in $\overline{\dot{D}} \cap [0,t) \times \mathbf{R}^n$. This means that X is a Feller process. The Feller property together with the continuity of the sample paths implies that X is a strong Markov process. Note that an alternative way of proving the strong Markov property has been indicated in the proof of Theorem 2.1 and the remark following it. □

For $\varepsilon > 0$, let $\dot{D}_\varepsilon = \{(s,x) \in \dot{D} : \, d(x, \partial D(s)) < \varepsilon\}$ and let σ_r denote the surface area measure on $\partial D(r)$.

Theorem 2.6. *For $(s,x) \in \overline{\dot{D}}$ and $s < t < T$,*

$$L_t^{s,x} = \lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int_s^t \mathbf{1}_{\dot{D}_\varepsilon}(r, X_r^{s,x})dr, \tag{2.7}$$

in L^2 and a.s., uniformly on relatively compact sets of t. For each fixed $0 < l < T$, there is a constant $c_l > 0$ such that for $(s,x) \in \overline{\dot{D}}$ and $s < t \le l$,

$$\mathbf{E}[L_t^{s,x}] = \frac{1}{2}\int_s^t \left(\int_{\partial D(r)} p(s,x;r,z)\sigma_r(dz) \right) ds \le c_l \sqrt{t-s}. \tag{2.8}$$

Proof. For each fixed small constant $\varepsilon > 0$, define $\psi_\varepsilon(\delta) = (\varepsilon - \delta)^2/2$ if $0 \le \delta \le \varepsilon$ and 0 if $\delta > \varepsilon$. Let

$$f_\varepsilon(s,x) = \psi_\varepsilon(d(x, D(s)^c)).$$

Since $\dot{D}$ is a C^3-smooth domain, f_ε is twice differentiable with bounded second derivative on $\{(t,x) \in \dot{D} : t \leq l\}$ for each fixed $l < T$. Note that $0 \leq f_\varepsilon \leq \varepsilon^2$, $\frac{\partial f_\varepsilon}{\partial s} \leq c_l \varepsilon$, $|\nabla_x f_\varepsilon| \leq c_l \varepsilon$, $\nabla_x f_\varepsilon(s,x) = -\varepsilon \mathbf{n}(s,x)$ for $(s,x) \in \partial \dot{D}$, and $\Delta_x f_\varepsilon = (1 + O(\varepsilon)) \mathbf{1}_{\dot{D}_\varepsilon}$. By the Itô formula,

$$f_\varepsilon(t, X_t^{s,x}) = f_\varepsilon(s,x) + \int_s^t \nabla f_\varepsilon(r, X_r^{s,x})\, dB_r + \int_s^t \frac{\partial f_\varepsilon}{\partial \mathbf{n}}(r, X_r^{s,x})\, dL_r^{s,x} + \frac{1}{2} \int_s^t \Delta f_\varepsilon(r, X_r^{s,x})\, dr.$$

The second spatial derivative of the function f_ε is not continuous so the usual Itô formula does not apply to f_ε. We will sketch a standard approximation argument justifying the last formula. Let $\phi > 0$ be a smooth function on $\mathbf{R}^n$ with compact support and such that $\int_{\mathbf{R}^n} \phi(x) dx = 1$. Let $\phi_n(x) = n\phi(nx)$ and $f_{\varepsilon,n}(s,x) = \int f_\varepsilon(s, x-y)\phi_n(y) dy$. We can apply the Itô formula to $f_{\varepsilon,n}(t, X_t^{s,x})$, and then pass to the limit with $n \to \infty$, using Theorem 2.4 (iv).

Dividing both sides of the last formula by ε, we obtain,

$$L_t^{s,x} - \frac{1}{2\varepsilon} \int_s^t \mathbf{1}_{\dot{D}_\varepsilon}(r, X_r^{s,x}) dr = \frac{1}{\varepsilon} \int_s^t \nabla f_\varepsilon(r, X_r^{s,x})\, dB_r + O(\varepsilon) + O(\varepsilon) \int_s^t \mathbf{1}_{\dot{D}_\varepsilon}(r, X_r^{s,x}) dr. \tag{2.9}$$

By Doob's maximal inequality and Theorem 2.4 (v),

$$\begin{aligned} \mathbf{E}\left[\sup_{s \leq t \leq l} \left| \frac{1}{\varepsilon} \int_s^t \nabla f_\varepsilon(r, X_r^{s,x})\, dB_r \right|^2 \right] &\leq \frac{4}{\varepsilon^2} \mathbf{E}\left[\int_s^l |\nabla f_\varepsilon(r, X_r^{s,x})|^2 dr \right] \\ &\leq c_l \mathbf{E}\left[\int_s^l |\mathbf{1}_{\dot{D}_\varepsilon}(r, X_r^{s,x}) dr \right] \\ &= c_l \int_s^l \left(\int_{\dot{D}_\varepsilon} p(s,x;r,y) dy \right) dr \\ &\leq C\varepsilon \int_s^l (r-s)^{-1/2} dr = C\, \varepsilon \sqrt{l-s}. \end{aligned} \tag{2.10}$$

This and (2.9) imply that

$$\lim_{\varepsilon \downarrow 0} \mathbf{E}\left[\sup_{s \leq t \leq l} \left| L_t^{s,x} - \frac{1}{2\varepsilon} \int_s^t \mathbf{1}_{\dot{D}_\varepsilon}(r, X_r^{s,x}) dr \right|^2 \right] = 0,$$

i.e., (2.7) holds in L^2-sense. From (2.10) and Chebyshev's inequality we see that

$$\sum_{k=1}^\infty \mathbf{P}\left(\sup_{s \leq t \leq l} \left| k^4 \int_s^t \nabla f_{1/k^4}(r, X_r^{s,x})\, dB_r \right| \geq \frac{1}{k} \right) \leq C \sum_{k=1}^\infty \frac{1}{k^2} < \infty.$$

By the Borel-Cantelli Lemma, with probability 1,

$$\lim_{k \to \infty} \sup_{s \leq t \leq l} k^4 \left| \int_s^t \nabla f_{1/k^4}(r, X_r^{s,x})\, dB_r \right| = 0.$$

This implies that, a.s.,

$$\lim_{k \to \infty} \sup_{s \leq t \leq l} \left| L_t^{s,x} - \frac{k^4}{2} \int_s^t \mathbf{1}_{\dot{D}_{1/k^4}}(r, X_r^{s,x}) dr \right| = 0. \tag{2.11}$$

For $0 < \varepsilon < 1$, let $k_\varepsilon \geq 1$ be the integer such that $1/k_\varepsilon^4 < \varepsilon \leq 1/(k_\varepsilon - 1)^4$. Since $\dot{D}_{1/k_\varepsilon^4} \subset \dot{D}_\varepsilon \subset \dot{D}_{1/(k_\varepsilon-1)^4}$, we have

$$\frac{(k_\varepsilon - 1)^4}{2} \int_s^t \mathbf{1}_{\dot{D}_{1/k_\varepsilon^4}}(r, X_r^{s,x}) dr \leq \frac{1}{2\varepsilon} \int_s^t \mathbf{1}_{\dot{D}_\varepsilon}(r, X_r^{s,x}) dr \leq \frac{k_\varepsilon^4}{2} \int_s^t \mathbf{1}_{\dot{D}_{1/(k_\varepsilon-1)^4}}(r, X_r^{s,x}) dr.$$

This, together with (2.11), implies that, a.s.,

$$\lim_{\varepsilon \downarrow 0} \sup_{s \leq t \leq l} \left| L_t^{s,x} - \frac{1}{2\varepsilon} \int_s^t \mathbf{1}_{\dot{D}_\varepsilon}(r, X_r^{s,x}) dr \right| = 0.$$

Inequality (2.8) follows from (2.7) and Theorem 2.4 (v). □

The following result on exponential integrability of the local time is needed for the probabilistic representation of solutions to the heat equation given in Theorem 2.8.

Lemma 2.7. *For each fixed $\alpha < \infty$ and $0 < l < T$,*

$$\sup_{\substack{(s,x) \in \overline{\dot{D}} \\ s < t \leq l}} \mathbf{E}[\exp(\alpha L_t^{s,x})] < \infty.$$

Proof. It follows from Theorem 2.6 there is a $\delta > 0$ such that

$$\sup_{\substack{(s,x) \in \dot{D},\, s < r \leq l \\ |s-r| < \delta}} \mathbf{E}[\alpha L_r^{s,x}] < \frac{1}{2},$$

and therefore by Khasminskii's inequality (see, e.g., p.231 of Durrett (1984)),

$$\sup_{\substack{(s,x) \in \dot{D},\, s < r \leq l \\ |s-r| < \delta}} \mathbf{E}[\exp(\alpha L_r^{s,x})] \leq 2.$$

Let $k \geq 1$ be such that $l/k < \delta$. Then by the Markov property of X and the additivity of local time L, we have for any $0 \leq s < t \leq l$ and $(s,x) \in \overline{\dot{D}}$

$$\mathbf{E}[\exp(\alpha L_t^{s,x})] \leq \left(\sup_{\substack{(s,x) \in \dot{D},\, s < r < l \\ |s-r| < \delta}} \mathbf{E}[\exp(\alpha L_r^{s,x})] \right)^k \leq 2^k.$$

□

Theorem 2.8. *Fix some $t > 0$. Let $f(s,x)$ be a bounded function defined on $\partial \dot{D}$ and ϕ be a continuous function on $\overline{\dot{D}(t)}$. Suppose $u(s,x) \in C^2(\dot{D}) \cap C^1(\overline{\dot{D}})$ is a C^2-smooth solution for*

$$\begin{cases} \frac{\partial u}{\partial s} + \frac{1}{2}\Delta_x u = 0 & \text{for } (s,x) \in \dot{D} \text{ with } s \leq t \;, \\ \frac{\partial u}{\partial \mathbf{n}} + f(s,x) u = 0 & \text{for } (s,x) \in \partial \dot{D} \text{ with } s < t, \\ \lim_{s \uparrow t} u(s,x) = \phi(x). \end{cases} \tag{2.12}$$

Then for $(s,x) \in \overline{\dot{D}}$ *with* $s<t$,

$$u(s,x) = \mathbf{E}\left[\exp\left(\int_s^t f(r, X_r^{s,x}) dL_r^{s,x}\right) \phi\left(X_t^{s,x}\right)\right]. \tag{2.13}$$

Conversely, if $f(s,x)$ *is a bounded continuous function on* $\partial\dot{D}$*, then the function* $u(s,x)$ *defined by (2.13) is continuous on* $\overline{\dot{D}}$ *for* $s \le t$*, it is continuously differentiable in* $s \in (0,t)$*, it belongs to class* $C^2(D(s)) \cap C^1(\overline{D(s)})$ *as a function of* x*, and it solves the equation (2.12).*

Proof. Assume that $u(s,x)$ solves (2.12). By the Itô formula,

$$\begin{aligned}
& d\left(\exp\left(\int_s^r f(v, X_v^{s,x}) dL_v^{s,x}\right) u(r, X_r^{s,x})\right) \\
&= \exp\left(\int_s^r f(v, X_v^{s,x}) dL_v^{s,x}\right)\left(u(r, X_r^{s,x}) f(r, X_r^{s,x}) dL_r^{s,x} + \left(\frac{\partial}{\partial r} + \frac{1}{2}\Delta_x\right) u(r, X_r^{s,x})\, dr\right. \\
&\qquad \left. + \nabla u(r, X_r^{s,x})\, dB_r + \frac{\partial u}{\partial \mathbf{n}}(r, X_r^{s,x}) dL_r^{s,x}\right) \\
&= \exp\left(\int_s^r f(v, X_v^{s,x}) dL_v^{s,x}\right) \nabla u(r, X_r^{s,x})\, dB_r.
\end{aligned}$$

Hence $\{\exp\left(\int_s^r f(v, X_v^{s,x}) dL_v^{s,x}\right) u(r, X^{s,x}),\ s \le r \le t\}$ is a local martingale. By Lemma 2.7, it is in fact a martingale since u and f are bounded. This implies

$$\begin{aligned}
u(s,x) &= \mathbf{E}\left[\exp\left(\int_s^t f(r, X_r^{s,x}) dL_r^{s,x}\right) u(t, X_t^{s,x})\right] \\
&= \mathbf{E}\left[\exp\left(\int_s^t f(v, X_r^{s,x}) dL_r^{s,x}\right) \phi(X_t^{s,x})\right],
\end{aligned}$$

and hence proves (2.13).

Now suppose that $f(s,x)$ is a bounded continuous function on $\partial\dot{D}$ and u is a function defined by (2.13). Clearly $\lim_{s\uparrow t} u(s,x) = \phi(x)$. We have

$$\begin{aligned}
u(s,x) &= \mathbf{E}[\phi(X_t^{s,x})] + \mathbf{E}\left[\left(\exp\left(\int_s^t f(r, X_r^{s,x}) dL_r^{s,x}\right) - 1\right) \phi(X_t^{s,x})\right] \\
&= \mathbf{E}[\phi(X_t^{s,x})] - \mathbf{E}\left[\int_s^t f(r, X_r^{s,x}) \exp\left(\int_r^t f(v, X_v^{s,x}) dL_v^{s,x}\right) \phi(X_t^{s,x}) dL_r^{s,x}\right] \\
&= \mathbf{E}[\phi(X_t^{s,x})] - \mathbf{E}\left[\int_s^t f(r, X_r^{s,x}) u(r, X_r^{s,x}) dL_r^{s,x}\right] \\
&= \int_{\dot{D}(t)} p(s,x;t,y)\phi(y) dy - \frac{1}{2}\int_s^t \left(\int_{\partial D(r)} p(s,x;r,z) f(r,z) u(r,z) \sigma_r(dz)\right) dr.
\end{aligned} \tag{2.14}$$

From (2.14), we see that $u \in C^2(\dot{D}) \cap C^1(\overline{\dot{D}})$ for $s<t$. By Theorem 2.2, u satisfies $\frac{\partial u}{\partial s} + \frac{1}{2}\Delta_x u = 0$ in $\dot{D}$. To show that u satisfies the boundary conditions in (2.12), we adapt an approach from Hsu (1987), Proposition 3.2. Applying Itô's formula, we have for $s<r<t$,

$$u(r, X_r^{s,x}) - u(s,x) = \int_s^l \nabla u(v, X_v^{s,x}) dB_v + \int_s^l \frac{\partial u}{\partial \mathbf{n}}(v, X_v^{s,x}) dL_v^{s,x}. \tag{2.15}$$

On the other hand,

$$
\begin{aligned}
u(r, X_r^{s,x}) &= \mathbf{E}\left[\exp\left(\int_r^t f(v, X_v^{r,X_r^{s,x}})dL_v^{r,X_r^{s,x}}\right)\phi(X_t^{r,X_r^{s,x}})\big|\mathcal{F}_{s,r}\right] \\
&= \exp\left(-\int_s^r f(v, X_v^{s,x})dL_v^{s,x}\right)\mathbf{E}\left[\exp\left(\int_s^t f(v, X_v^{s,x})dL_v^{s,x}\right)\phi(X_t^{s,x})\big|\mathcal{F}_{s,r}\right],
\end{aligned}
$$

where $\mathcal{F}_{s,r}$ is the σ-field generated by $X_v^{s,x}$ for $v \in [s, r]$. Let

$$
M_r = \mathbf{E}\left[\exp\left(\int_s^t f(v, X_v^{s,x})dL_v^{s,x}\right)\phi(X_t^{s,x})\big|\mathcal{F}_{s,r}\right].
$$

In view of Lemma 2.7, M_r is a martingale so by the Itô formula,

$$
\begin{aligned}
&u(r, X_r^{s,x}) - u(s,x) \\
&= \int_s^r \exp\left(-\int_s^\theta f(v, X_v^{s,x})dL_v^{s,x}\right)dM_\theta \\
&\qquad - \int_s^r f(\theta, X_\theta^{s,x})\exp\left(-\int_s^\theta f(v, X_v^{s,x})dL_v^{s,x}\right)M_\theta dL_\theta^{s,x} \\
&= \int_s^r \exp\left(-\int_s^\theta f(v, X_v^{s,x})dL_v^{s,x}\right)dM_\theta - \int_s^r f(\theta, X_\theta^{s,x})u(\theta, X_\theta^{s,x})dL_\theta^{s,x}. \qquad (2.16)
\end{aligned}
$$

From (2.15) and (2.16), we see that the bounded variation process

$$
\int_s^r \left(\frac{\partial u}{\partial \mathbf{n}}(v, X_v^{s,x}) + f(v, X_v^{s,x})u(v, X_v^{s,x})\right)dL_v^{s,x}
$$

is a continuous martingale and therefore it must be identically zero. Were $\frac{\partial u}{\partial \mathbf{n}} \neq -fu$ on $\partial\dot{D}$, say $\frac{\partial u}{\partial \mathbf{n}}(s,x) + f(s,x)u(s,x) > 0$ for some $(s,x) \in \partial\dot{D}$, there would be a neighborhood U of (s,x) such that $\frac{\partial u}{\partial \mathbf{n}}(s,x) + f(s,x)u(s,x) \geq \varepsilon_0 > 0$ on $U \cap \partial\dot{D}$. Let $\tau = \inf\{r \geq s : (r, X_r^{s,x}) \in \partial\dot{D} \setminus U\}$. Clearly $\tau > 0$ almost surely and therefore there is $t_0 > 0$ such that $P^{s,x}(\tau > t_0) > 0$. Then on $\{\tau > t_0\}$,

$$
0 = \int_s^{t_0}\left(\frac{\partial u}{\partial \mathbf{n}}(v, X_v^{s,x}) + f(v, X_v^{s,x})u(v, X_v^{s,x})\right)dL_v^{s,x} \geq \varepsilon_0 dL_{t_0}^{s,x}.
$$

This is impossible as (s,x) is a regular point of $\dot{D}$ for the space-time Brownian motion because $\dot{D}$ is C^3-smooth and therefore $L_{t_0}^{s,x} > 0$, $P^{s,x}$-almost surely. Therefore $\frac{\partial u}{\partial \mathbf{n}} = fu$ on $\partial\dot{D}$. □

Remark 2.11. Uniqueness of C^2-smooth solutions to (2.12) is a by-product of the probabilistic representation (2.13).

The equation in (2.1) is the "backward partial differential equation" for the transition density function $p(s,x;t,y)$ of X, in variables s and x. Our next result is concerned with $p(s,x;t,y)$ as a function of t and y. If we view $L = \frac{\partial}{\partial s} + \frac{1}{2}\Delta_x$ as an operator in $\dot{D}$ together with its zero Neumann boundary condition given in (2.1), and we let L^* be its formal adjoint operator in $L^2(\dot{D})$, then the function $(t,y) \to p(s,x;t,y)$ is in the domain $\mathcal{D}(L^*)$ of L^* and it satisfies the differential equation $L^*p = 0$ (cf. Stroock and Varadhan (1979), pages 2-3). The following result is an application of the divergence formula in $\mathbf{R}^{n+1}$. Recall that $\overrightarrow{\gamma} = (\gamma_1, \gamma_2, \cdots, \gamma_{n+1})$ denotes the unit inward normal vector on the boundary of $\dot{D}$.

Theorem 2.9. *The function $p(s,x;t,y)$ satisfies the following forward differential equation in (t,y) for each fixed $(s,x) \in \overline{\dot{D}}$:*

$$\begin{cases} \frac{\partial p}{\partial t} - \frac{1}{2}\Delta_y p = 0 & \text{for } (t,y) \in \dot{D} \text{ with } s < t, \\ \frac{\partial p}{\partial \mathbf{n}} - \frac{2\gamma_1}{\overrightarrow{\gamma}\cdot\mathbf{n}} p = 0 & \text{for } (t,y) \in \partial\dot{D} \text{ with } s < t, \\ \lim_{t\downarrow s} p(s,x;t,y)dy = \delta_{\{x\}}(dy) & \text{for } (s,x) \in \dot{D}. \end{cases} \tag{2.17}$$

Proof. A function ψ belongs to $\mathcal{D}(L^*) \subset L^2(\dot{D})$ if and only if there is $\phi \in L^2(\dot{D})$ such that

$$\int_{\dot{D}} \psi L u \, dtdx = \int_{\dot{D}} \phi u \, dtdx$$

for any $u \in \mathcal{D}(L)$, and in this case $L^*\psi = \phi$. In view of the remarks about the function $(t,y) \to p(s,x;t,y)$ preceding the theorem, it will suffice to show that $\psi \in \mathcal{D}(L^*)$ if and only if $L^*\psi = \left(-\frac{\partial}{\partial t} + \frac{1}{2}\Delta\right)(\psi)$ and $\frac{\partial p}{\partial \mathbf{n}} - \frac{2\gamma_1}{\overrightarrow{\gamma}\cdot\mathbf{n}}\psi = 0$.

For any test function $\psi \in C_c^\infty((0,T) \times \mathbf{R}^n)$ and $u \in \mathcal{D}(L)$,

$$\begin{aligned}
&\int_{\dot{D}} \psi \left(\frac{\partial}{\partial t} + \frac{1}{2}\Delta\right) u dtdx \\
&= \int_{\dot{D}} \left(\frac{\partial(u\psi)}{\partial t} - u\frac{\partial \psi}{\partial t}\right) dtdx + \frac{1}{2}\int_0^T \left(\int_{\dot{D}(t)} \psi \Delta u dx\right) dt \\
&= \int_{\dot{D}} \left(\frac{\partial(u\psi)}{\partial t} - u\frac{\partial \psi}{\partial t}\right) dtdx + \frac{1}{2}\int_0^T \left(\int_{\dot{D}(t)} u \Delta \psi dx\right) dt + \frac{1}{2}\int_0^T \left(\int_{\partial\dot{D}(t)} u\frac{\partial \psi}{\partial \mathbf{n}} \sigma_t(dx)\right) dt \\
&= \int_{\dot{D}} \left(\frac{\partial(u\psi)}{\partial t} - u\frac{\partial \psi}{\partial t}\right) dtdx + \frac{1}{2}\int_0^T \left(\int_{\dot{D}(t)} u \Delta \psi dx\right) dt \\
&\quad + \frac{1}{2}\int_0^T \left(\int_{\dot{D}(t)} \operatorname{div}_{\mathbf{R}^n}(-u\nabla\psi) dx\right) dt \\
&= \int_{\dot{D}} u\left(-\frac{\partial}{\partial t} + \frac{1}{2}\Delta\right)\psi \, dt \, dx + \int_{\dot{D}} \operatorname{div}_{\mathbf{R}^{n+1}}\left(u\psi, -\frac{1}{2}u\nabla_x\psi\right) dtdx \\
&= \int_{\dot{D}} u\left(-\frac{\partial}{\partial t} + \frac{1}{2}\Delta\right)\psi \, dt \, dx + \int_{\partial\dot{D}} u \overrightarrow{\gamma} \cdot \left(-\psi, \frac{1}{2}\nabla_x\psi\right) d\sigma.
\end{aligned}$$

Therefore $\psi \in D(L^*)$ if and only if $\overrightarrow{\gamma} \cdot \left(-\psi, \frac{1}{2}\nabla_x\psi\right) = 0$ on $\partial\dot{D}$ and $L^*\psi = \left(-\frac{\partial}{\partial t} + \frac{1}{2}\Delta\right)\psi$. This is equivalent to $L^*\psi = \left(-\frac{\partial}{\partial t} + \frac{1}{2}\Delta\right)\psi$ and $\frac{\partial p}{\partial \mathbf{n}} - \frac{2\gamma_1}{\overrightarrow{\gamma}\cdot\mathbf{n}}\psi = 0$. The proof is complete. □

Remark. The differential equation (2.17) above is equivalent to the differential equation (2.25) in Burdzy, Chen and Sylvester (2000a) by a straightforward change of variable.

For fixed $l < T$, let $\widetilde{D}_l = \{(t,x) : (l-t,x) \in \dot{D}\}$. For $(t,x) \in \widetilde{D}_l$, let $Y^{t,x}$ be the reflecting Brownian motion in $\widetilde{D}_l$ constructed via Theorem 1, with $Y^{t,x}_t = x$ and local time $\widetilde{L}^{t,x}$.

Theorem 2.10. *Let $f(t,x)$ be a bounded function defined on $\partial\dot{D}$ and ϕ be a continuous function on $\overline{\dot{D}(s)}$. Suppose that v is a C^2-smooth solution for*

$$\begin{cases} \frac{\partial v}{\partial t} - \frac{1}{2}\Delta v = 0 & \text{for } (t,x) \in \dot{D} \text{ with } s < t < l \ , \\ \frac{\partial v}{\partial \mathbf{n}} + f(t,x)v = 0 & \text{for } (t,x) \in \partial\dot{D} \text{ with } s \le t \le l, \\ v(s,x) = \phi(x). \end{cases} \tag{2.18}$$

Then for $(t,x) \in \overline{\dot{D}}$ with $s < t < l$,

$$v(t,x) = \mathbf{E}\left[\exp\left(\int_{l-t}^{l-s} f(l-r, Y^{l-t,x}_r)d\widetilde{L}^{l-t,x}_r\right)\phi\left(Y^{l-t,x}_{l-s}\right)\right]. \tag{2.19}$$

Conversely, if $f(t,x)$ is a bounded continuous function on $\partial\dot{D}$, then the function $v(t,x)$ defined by (2.19) is continuous on $\overline{\dot{D}} \times \overline{\dot{D}}$ for $s \le t < T$, continuously differentiable in $t \in (s,T)$, it belongs to class $C^2(D(t)) \cap C^1(\overline{D(t)})$ as a function of x, and solves equation (2.18).

Proof. Let $u(t,x) = v(l-t,x)$. Then u solves

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{2}\Delta u = 0 & \text{for } (t,x) \in \widetilde{D}_l \text{ with } 0 < t < l-s \ , \\ \frac{\partial u}{\partial \mathbf{n}} + f(l-t,x)u = 0 & \text{for } (t,x) \in \partial\widetilde{D}_l \text{ with } 0 < t < l-s, \\ \lim_{t\uparrow l-s} u(t,x) = \phi(x) & \text{for } x \in \dot{D}(s) = \widetilde{D}_l(l-s). \end{cases}$$

Hence by Theorem 2.8,

$$v(l-t,x) = u(t,x) = \mathbf{E}\left[\exp\left(\int_{t}^{l-s} f(l-r, Y^{t,x}_r)d\widetilde{L}^{t,x}_r\right)\phi\left(Y^{t,x}_{l-s}\right)\right].$$

This proves the theorem. □

Remark 2.11. Uniqueness of C^2-smooth solutions to (2.18) is a by-product of the probabilistic representation (2.19), just like uniqueness of solutions to (2.12) follows from (2.13), as noted above.

We will use $\widetilde{p}$ to denote the transition density function for process Y in $\widetilde{D}_l$ and use $\widetilde{\mathbf{E}}^{(l-t,x)}_{(l-s,y)}$ to denote the expectation under the law for the process Y conditioned by $\{Y^{l-t,x}_{l-t} = x\}$ and $\{Y^{l-t,x}_{l-s} = y\}$.

Letting $f(s,x) = -\frac{2\gamma_1}{\overrightarrow{\gamma}\cdot\mathbf{n}}$ on $\partial\dot{D}$, we obtain the following result from Theorem 2.10.

Corollary 2.12. *The function*

$$\begin{aligned} u(t,x) &= \mathbf{E}\left[\exp\left(\int_{l-t}^{l-s} \frac{-2\gamma_1}{\overrightarrow{\gamma}\cdot\mathbf{n}}(l-r, Y^{l-t,x}_r)d\widetilde{L}^{l-t,x}_r\right)\phi\left(Y^{l-t,x}_{l-s}\right)\right] \\ &= \int_{\dot{D}(s)} \phi(y)\widetilde{p}(l-t,x;l-s,y)\widetilde{\mathbf{E}}^{(l-t,x)}_{(l-s,y)}\left[\exp\left(\int_{l-t}^{l-s} \frac{-2\gamma_1}{\overrightarrow{\gamma}\cdot\mathbf{n}}(l-r, Y^{l-t,x}_r)d\widetilde{L}^{l-t,x}_r\right)\right] dy \end{aligned}$$

solves

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{1}{2}\Delta u = 0 & \text{for } (t,y) \in \dot{D} \text{ with } s < t < l, \\ \frac{\partial u}{\partial \mathbf{n}} - \frac{2\gamma_1}{\overrightarrow{\gamma}\cdot\mathbf{n}} u = 0 & \text{for } (t,y) \in \partial\dot{D} \text{ with } s < t < l, \\ u(s,x) = \phi(x). \end{cases} \tag{2.20}$$

By Theorem 2.9, the function $u(t,x)$ solving (2.20) satisfies

$$u(t,x) = \int_{\dot{D}(s)} p(s,y;t,x)\phi(y)dy.$$

Therefore,

$$p(s,y;t,x) = \widetilde{p}(l-t,x;l-s,y)\widetilde{\mathbf{E}}^{(l-t,x)}_{(l-s,y)}\left[\exp\left(-\int_{l-t}^{l-s} \frac{2\gamma_1}{\overrightarrow{\gamma}\cdot\mathbf{n}}(r, Y_r^{l-t,x})d\widetilde{L}_r^{l-t,x}\right)\right]$$

for $0 \le s < t < l$. It follows that for $s < t < l$,

$$\frac{p(s,y;t,z)p(t,z;l,x)}{p(s,y;l,x)} = \frac{\widetilde{p}(s,y;t,z)\widetilde{p}(t,z;l,x)}{\widetilde{p}(s,y;l,x)} \times \tag{2.21}$$
$$\frac{\widetilde{\mathbf{E}}^{(l-t,z)}_{(l-s,y)}\left[\exp\left(-\int_{l-t}^{l-s}\frac{2\gamma_1}{\overrightarrow{\gamma}\cdot\mathbf{n}}(r,Y_r^{l-t,z})d\widetilde{L}_r^{l-t,z}\right)\right]\widetilde{\mathbf{E}}^{(0,x)}_{(l-t,z)}\left[\exp\left(-\int_0^{l-t}\frac{2\gamma_1}{\overrightarrow{\gamma}\cdot\mathbf{n}}(r,Y_r^{0,x})d\widetilde{L}_r^{0,x}\right)\right]}{\mathbf{E}^{(0,x)}_{(l-s,y)}\left[\exp\left(-\int_0^{l-s}\frac{2\gamma_1}{\overrightarrow{\gamma}\cdot\mathbf{n}}(r,Y_r^{0,x})d\widetilde{L}_r^{0,x}\right)\right]}.$$

For $0 \le s < l < T$, let $\widehat{\mathbf{P}}^{0,x}_{l-s,y}$ denote the law of $\{X^{s,y}_{l-v}, v \in [0, l-s]\}$ conditioned by $\{X^{s,y}_l = x\}$, and let $\widetilde{\mathbf{P}}^{0,x}_{l-s,y}$ be the law of $Y^{0,x}$ conditioned by $\{Y^{0,x}_{l-s} = y\}$.

Theorem 2.13.

$$\frac{d\widehat{\mathbf{P}}^{0,x}_{l-s,y}}{d\widetilde{\mathbf{P}}^{0,x}_{l-s,y}} = \frac{\exp\left(-\int_0^{l-s}\frac{2\gamma_1}{\overrightarrow{\gamma}\cdot\mathbf{n}}(r,Y_r^{0,x})d\widetilde{L}_r^{0,x}\right)}{\mathbf{E}^{(0,x)}_{(l-s,y)}\left[\exp\left(-\int_0^{l-s}\frac{2\gamma_1}{\overrightarrow{\gamma}\cdot\mathbf{n}}(r,Y_r^{0,x})d\widetilde{L}_r^{0,x}\right)\right]}.$$

Proof. Clearly by (2.21), the above assertion is true on cylindrical sets. A standard measure theoretical argument shows the Theorem 2.13 is true for general measurable sets as well. □

3. One-dimensional reflecting Brownian motion in a time-dependent domain

In the one-dimensional case, the existence of a reflecting Brownian motion in $\dot{D}$ can be proved under dramatically relaxed assumptions on the smoothness of the boundary of $\dot{D}$. We will show that such a process can be constructed on any space-time domain lying between the graphs of measurable functions. Almost all domains discussed in Burdzy, Chen and Sylvester (2000b) will have continuous boundaries. We need the existence result for domains with measurable boundaries mainly for technical reasons but some interesting theoretical questions arise in this context as well—we defer their discussion to a separate article.

Recall the notation $v^+ = \max\{v, 0\}$ and $v^- = \max\{-v, 0\}$.

The following lemma is a variation of the famous Skorohod decomposition. The result is deterministic. Its proof is a modification of that of Lemma 3.6.14 in Karatzas and Shreve (1994). Soucaliuc, Toth and Werner (1999) pointed out that reflecting a continuous function on another continuous function is quite easy.

Lemma 3.1. *Suppose that g is a locally bounded measurable function from $\mathbf{R}_+$ to $\mathbf{R}$. Let $\widehat{g}(t) = \max\left(g(t), \limsup_{s\downarrow t} g(s)\right)$. For every continuous function $b(t)$, $t \geq 0$, there is a unique pair of functions $(x(t), l(t))$, $t \geq 0$, such that*

(i) $x(t) \stackrel{\text{df}}{=} b(t) + l(t) \geq \widehat{g}(t)$ for $t \geq 0$,

(ii) $l(t)$ is a nondecreasing right-continuous function with $l(0) = (b(0) - \widehat{g}(0))^-$,

(iii) if $x(t) > \widehat{g}(t)$ for $t \in [s_1, s_2]$ then $l(s_1) = l(s_2)$, and

(iv) if $l(t)$ has a jump at $t = t_1$, i.e., $\lim_{t\uparrow t_1} l(t) < l(t_1)$ then $x(t_1) = \widehat{g}(t_1)$.

Moreover,

$$l(t) = \sup_{0\leq s\leq t} (b(s) - \widehat{g}(s))^- . \tag{3.1}$$

Proof. We first prove uniqueness. Suppose that $(x(t), l(t))$ and $(\widetilde{x}(t), \widetilde{l}(t))$ have properties (i)-(iv) and that for some t_* we have $l(t_*) - \widetilde{l}(t_*) > 0$. By the right-continuity of l and $\widetilde{l}$, there exist $t_1 > t_*$ and $a > 0$ such that $l(t_1) - \widetilde{l}(t_1) = a$ and $l(t) - \widetilde{l}(t) > 0$ for all $t \in [t_*, t_1]$. Let $t_0 = \inf\{t \leq t_1 : x(s) > \widetilde{x}(s)\ \forall s \in [t, t_1]\}$. For every $t \in (t_0, t_1]$ we have $x(t) > \widetilde{x}(t) \geq \widehat{g}(t)$ so (iii) implies that $l(t) = l(t_1)$. This implies that, for $t \in (t_0, t_1]$,

$$0 < a = x(t_1) - \widetilde{x}(t_1) = l(t_1) - \widetilde{l}(t_1) \leq l(t) - \widetilde{l}(t) = x(t) - \widetilde{x}(t). \tag{3.2}$$

Since b, l and $\widetilde{l}$ are right-continuous, so are x and $\widetilde{x}$ and so (3.2) in fact holds for all $t \in [t_0, t_1]$. In particular, $x(t_0) - \widetilde{x}(t_0) \geq a > 0$, which implies, in view of (ii), that $t_0 > 0$. The definition of t_0 and the fact that $x(t_0) - \widetilde{x}(t_0) \geq a > 0$ require that $\liminf_{t\uparrow t_0} x(t) - \widetilde{x}(t) \leq 0$. Another consequence of (3.2) is that $l(t_0) - \widetilde{l}(t_0) \geq a$. We obtain

$$\begin{aligned}\left(\liminf_{t\uparrow t_0} l(t)\right) - (l(t_0) - a) &\leq \left(\liminf_{t\uparrow t_0} l(t)\right) - \widetilde{l}(t_0) \leq \liminf_{t\uparrow t_0}(l(t) - \widetilde{l}(t)) \\ &= \liminf_{t\uparrow t_0}(x(t) - \widetilde{x}(t)) \leq 0.\end{aligned}$$

We see that $\lim_{t\uparrow t_0} l(t) < l(t_0)$ and so, according to (iv), $x(t_0) = \widehat{g}(t_0)$. However, $\widetilde{x}(t_0) \le x(t_0) - a = \widehat{g}(t_0) - a$. This implies that $\widetilde{x}(t) < \widehat{g}(t)$ for some t, which is a contradiction. The proof of uniqueness is complete.

We will finish the proof by showing that $l(t)$ defined in (3.1), together with $x(t) = b(t) + l(t)$, satisfy (i)-(iv). Property (i) is evident and so is the fact that $l(t)$ is non-decreasing. It is easy to see that $\widehat{g}(t) \ge \limsup_{s\downarrow t} \widehat{g}(s)$. Right-continuity of $l(t)$ easily follows from this observation.

To prove (iv), note that by the continuity of $b(t)$ and right-continuity of $l(t)$, $x(t)$ is right-continuous and so $\lim_{t\downarrow t_1} x(t)$ exists and is equal to $x(t_1)$. Let

$$t_* = \sup\{s : \sup_{0\le r\le s} (b(s) - \widehat{g}(s))^- = 0\}.$$

Then $(x(t), l(t)) = (b(t), 0)$ is the unique solution to the Skorohod problem on $[0, t_*)$ and so $l(t)$ has no jumps on this interval. Suppose now that $t_1 \ge t_*$ is a jump time for l. Let $b(t_1) = c$. Then $l(t_1) = -c + \widehat{g}(t_1)$ because of continuity of $b(t)$ and the fact that $l(t)$ has a jump at $t = t_1$. We conclude that

$$x(t_1) = b(t_1) + l(t_1) = c - c + \widehat{g}(t) = \widehat{g}(t),$$

which proves (iv).

It remains to prove (iii). Suppose that $x(t) > \widehat{g}(t)$ for $t \in [s_1, s_2]$. Then $l(t) > \widehat{g}(t) - b(t)$ on the same interval. Since

$$l(s_2) = \max\left\{ \sup_{0\le s\le s_1} (b(s) - \widehat{g}(s))^-, \sup_{s_1\le s\le s_2} (b(s) - \widehat{g}(s))^- \right\},$$

we must have $l(s_2) = l(s_1)$. □

Remarks 3.2. (i) If the function $g(t)$ is continuous then $\widehat{g}(t) = g(t)$. Then formula (3.1) shows that $l(t)$ and, consequently, $x(t)$ are continuous.

(ii) If $g_1(t)$ and $g_2(t)$ are locally bounded measurable and there exist continuous functions $g_3(t)$ and $g_4(t)$, such that $g_1(t) < g_3(t) < g_4(t) < g_2(t)$, then for any continuous function $b(t)$ one can construct a function $x(t)$ which satisfies $g_1(t) \le x(t) \le g_2(t)$ and is a sum of $b(t)$ and a "local time" $l(t)$ which does not change when $g_1(t) < x(t) < g_2(t)$. The proof of this generalization of Lemma 3.1 is somewhat tedious but completely elementary. See Burdzy and Toby (1995) for a similar version of the Skorohod lemma.

The first two of the following results follow immediately from formula (3.1).

Corollary 3.3. *In the setting of Lemma 3.1, consider the Skorohod problem for a fixed function $b(t)$, relative to two different measurable functions $g_1(t)$ and $g_2(t)$. Let $(x_1(t), l_1(t))$ and $(x_2(t), l_2(t))$ denote the corresponding solutions of the Skorohod problem. If $g_1(t) \le g_2(t)$ for all t then $x_1(t) \le x_2(t)$ for all t. If $|g_1(t) - g_2(t)| \le \varepsilon$ for all t then $|x_1(t) - x_2(t)| \le \varepsilon$ for all t.*

Corollary 3.4. *Suppose that we have a family of continuous functions $g_\alpha(t)$, where α is an index in some metric space and assume that the mapping $\alpha \to g_\alpha(\,.\,)$ is continuous in the uniform* *topology. Let $(x_\alpha(t), l_\alpha(t))$ denote solutions of the Skorohod problem for a fixed $b(t)$ (same for all α), relative to $g_\alpha(t)$. Then the mapping $\alpha \to x_\alpha(t)$ is continuous in the uniform topology.*

Corollary 3.5. *Suppose that we have a family of measurable functions $g_a(t)$, $a \in \mathbf{R}$. Assume that $a \to g_a(t)$ is non-decreasing for each t. Let $(x_a(t), l_a(t))$ denote solutions of the Skorohod problem for a fixed $b(t)$ (same for all a), relative to $g_a(t)$. If $\lim_{a\uparrow a_0} g_a(t) = g_{a_0}(t)$ for every t then $\lim_{a\uparrow a_0} x_a(t) = x_{a_0}(t)$.*

Proof. It is not hard to verify the convergence in (3.1). We note, however, that it is not necessarily true that $\lim_{a\downarrow a_0} g_a(t) = g_{a_0}(t)$ implies $\lim_{a\downarrow a_0} x_a(t) = x_{a_0}(t)$. □

Fix for a moment $s \geq 0$ and let B be a one-dimensional Brownian motion starting from 0 at time s, that is, $B_s = 0$. Suppose that $g(t), t \geq s$, is a measurable function and let $\dot{D} = \{(t,x) : t \geq s, x > g(t)\}$. For any $x \geq g(s)$, let $(X^{s,x}, L^{s,x})$ be the solution of the Skorohod problem defined in Lemma 3.1 (i), with $b(t) = x + B(t)$. If $g(t)$ is C^3-smooth then clearly $X^{s,x}$ is the reflecting Brownian motion in $\dot{D}$ in the sense of Theorem 2.1.

We will use $\mathbf{P}^x$ to denote the probability law on the canonical sample space $C([0,\infty), \mathbf{R})$ induced by $X^{0,x}$. The σ-field generated by $\omega(r)$ for $0 \leq r \leq t$ will be denoted by $\mathcal{F}_t$.

Let Y be the reflecting Brownian motion on $[0,\infty)$ with $Y_0 = y \geq 0$, defined on the canonical sample space $C([0,\,\infty),\,\mathbf{R})$. Then Y_t can be represented as

$$Y_t = y + W_t + L_t, \quad t \geq 0,$$

where W is a Brownian motion on $\mathbf{R}$ with $W_0 = 0$ and L_t is the local time of Y_t at 0, satisfying $L_t = \int_0^t \mathbf{1}_{\{Y_r = 0\}} dL_r$. Set $Z_t = Y_t + g(t),\ t \geq 0$. Then the process (t, Z_t) takes values in $\overline{\dot{D}}$ and satisfies

$$Z_t = z + (W_t + g(t) - g(0)) + L_t, \qquad t \geq 0,$$

and

$$L_t = \lim_{\varepsilon\downarrow 0} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{\{|Z_s - g(s)| < \varepsilon\}} ds,$$

where $z = y + g(0)$. Note that

$$L_t = \int_0^t \mathbf{1}_{\{Z_r = g(r)\}} dL_r.$$

Denote the distribution of Z by $\widetilde{\mathbf{P}}^x$.

Theorem 3.6. *The measure $\mathbf{P}^x$ is absolutely continuous with respect to $\widetilde{\mathbf{P}}^x$ on $\mathcal{F}_l$ if and only if $g \in \mathrm{H}^1[0,l]$ (that is, $\int_0^l |g'(t)|^2 dt < \infty$)). If $g \in \mathrm{H}^1[0,l]$, then*

$$\frac{d\mathbf{P}^x}{d\widetilde{\mathbf{P}}^x} = \exp\left(-\int_0^l g'(t) dW_t - \frac{1}{2}\int_0^l |g'(t)|^2 dt \right) \quad \text{on } \mathcal{F}_l. \tag{3.3}$$

Proof. The "*if*" part and (3.3) follow from Lemma 3.1 and Girsanov's theorem. For the "*only if*" part, let $M_l = \frac{d\mathbf{P}^x}{d\widetilde{\mathbf{P}}^x}\Big|_{\mathcal{F}_l}$, and $M_t = E(M_l \mid \mathcal{F}_t)$ for $0 \le t \le l$. Then M_t is a continuous martingale. Define $N_t = \int_0^t M_s^{-1}\mathbf{1}_{\{M_s>0\}}dM_s$. Clearly $(N_t, 0 \le t \le l)$, is a continuous local martingale with respect to the filtration $\{\mathcal{F}_t\}_{0\le t\le l}$. By the martingale representation theorem (cf. Theorem 3.4.2 of Karatzas and Shreve (1994)), there exists an adapted process H_s with $\int_0^l H_s^2 ds < \infty$ a.s., such that $N_t = \int_0^t H_s dW_s$. According to the Girsanov transform, under the measure $d\mathbf{P}^x = M_l d\widetilde{\mathbf{P}}^x$ on $\mathcal{F}_l$, the process

$$\widetilde{W}_t = W_t - \int_0^t H_s ds, \quad 0 \le t \le l,$$

is a Brownian motion and Z can be rewritten as

$$Z_t = x + \widetilde{W}_t + \int_0^t (H_s + g'(s))ds + L_t, \quad 0 \le t \le l.$$

Since under $\mathbf{P}^x$, Z_t is a reflecting Brownian motion in $\dot{D}$, by the uniqueness of the Skorohod decomposition, $\int_0^t (H_s + g'(s))ds + L_t$ must be the boundary local time of Z and hence $H_s + g'(s) = 0$ for a.e. $0 \le s \le l$. This implies that $g' \in L^2[0,l]$ and therefore $g \in \mathrm{H}^1[0,l]$. □ .

The following result gives the exponential integrability of the boundary local time for reflecting Brownian motion, which will be used to give an probabilistic representation for solutions of the corresponding heat equation.

Lemma 3.7. *Suppose that the domains $\dot{D}_k = \{(s,x) : s \ge 0, x > g_k(s)\}$ have smooth boundaries, the functions $g_k(t)$ converge to $g(t)$ uniformly on compact subsets of positive half-line and $\dot{D} = \{(s,x) : s \ge 0, x > g(s)\}$. Let $X^{s,x,k}$'s be the reflecting Brownian motions in $\dot{D}_k$'s driven by a common Brownian motion B, as in Lemma 3.1. Then, a.s., $X^{s,x,k}$'s converge uniformly to the reflected Brownian motion $X^{s,x}$ in $\dot{D}$, driven by the same Brownian motion B.*

Proof. The result follows from the second assertion of Corollary 3.3. □

Now we show that reflecting Brownian motion in $\dot{D} = \{(s,x) : s \ge 0,\ x > g(s)\}$, where g is a continuous function, always has transition density function in the interior of the domain.

Theorem 3.8. *Suppose that $g(s)$ is a continuous function and let $\dot{D} = \{(s,x) : s \ge 0,\ x > g(s)\}$. For all $t > s \ge 0$ and $x \ge g(s)$, there is a positive function $p(s,x;t,y)$ such that*

$$\mathbf{P}(X_t^{s,x} \in A) = \int_A p(s,x;t,y)dy,$$

for all Borel subsets A of the interior of $D(t)$. The function $p(s,x;t,y)$ is (locally) Hölder continuous on $\dot{D} \times \dot{D}$. Moreover, $p(s,x;t,y)$ satisfies

$$\frac{\partial p}{\partial s} + \frac{1}{2}\Delta_x p = 0,$$

for $(s,x)\in\dot D$ *with* $s<t$, *and*

$$\frac{\partial p}{\partial t}-\frac{1}{2}\Delta_y p=0,$$

for $(t,y)\in\dot D$ *with* $s<t$.

Proof. For each fixed $l>t$, there is a sequence of smooth functions $\{g_k\}_{k\ge1}$ on $[0,l]$ such that $\lim_{k\to\infty}\sup_{0\le r\le l}|g_k(r)-g(r)|=0$. Define $\dot D_k=\{(r,z):r\ge0,\,z>g_k(r)\}$ and denote the reflecting Brownian motion in $\dot D_k$ by X^k, and its transition density function by $p_k(s,x;t,y)$. For $(t,y)\in\dot D$ with $s<t<l$, there is a constant $0<\varepsilon<\min\{l-t,t-s\}/4$ such that $[t-3\varepsilon,\,t+3\varepsilon]\times[y-3\varepsilon,y+3\varepsilon]\subset\dot D$. Without loss of generality, we may assume that $[t-3\varepsilon,\,t+3\varepsilon]\times[y-3\varepsilon,y+3\varepsilon]\subset\dot D_k$ for every $k\ge1$. By Moser's Harnack inequality (see Theorem 2 of Moser (1964)), there is a constant $c_1=c_1(n,\varepsilon)>$ such that

$$\log\frac{p_k(s,x;t_1,y_1)}{p_k(s,x;t_2,y_2)}\le c\left(\frac{|y_1-y_2|^2}{t_2-t_1}+\frac{t_2-t_1}{\varepsilon^2}+1\right)$$

for $t-2\varepsilon\le t_1<t_2\le t+2\varepsilon$, $y_1,y_2\in[y-2\varepsilon,y+2\varepsilon]$ and $k\ge1$. Therefore there is a constant $c_2=c_2(\varepsilon)>0$ such that

$$p_k(s,x;t_1,y_1)\le c_2p_k(s,x;t+2\varepsilon,y_2)$$

for $t_1\in[t-\varepsilon,\,t+\varepsilon]$, $y_1,y_2\in[y-2\varepsilon,y+2\varepsilon]$ and $k\ge1$. Thus

$$p_k(s,x;t_1,y_1)\le\frac{c_2}{4\varepsilon}\int_{[y-2\varepsilon,y+2\varepsilon]}p_k(s,x;t+2\varepsilon,y_2)dy_2\le\frac{c_2}{4\varepsilon}$$

for $(t_1,y_1)\in[t-\varepsilon,\,t+\varepsilon]\times[y-2\varepsilon,y+2\varepsilon]$ and $k\ge1$. Now by Nash's Hölder continuity result for solutions to heat equation (see (2.4) of Moser (1964)), there are constants $0<\alpha<1$ and $c>0$ that depend only on n and ε such that

$$|p_k(s,x;t_1,y_1)-p_k(s,x;t_2,y_2)|\le c\left(|y_1-y_2|^\alpha+|t_1-t_2|^{\alpha/2}\right)$$

for $k\ge1$ and $(t_i,y_i)\in[t-\varepsilon/2,\,t+\varepsilon/2]\times[y-\varepsilon,y+\varepsilon]$ with $i=1,2$. So there is a subsequence of $p_k(s,x;r,z)$ that converges to some function $p(s,x;r,z)$ uniformly in $(r,z)\in[t-\varepsilon/2,\,t+\varepsilon/2]\times[y-\varepsilon,y+\varepsilon]$. Clearly, $p(s,x;r,z)$ is Hölder continuous in (r,z) on $[t-\varepsilon/2,\,t+\varepsilon/2]\times[y-\varepsilon,y+\varepsilon]$ and by Lemma 3.7,

$$\mathbf{P}(X_t^{s,x}\in A)=\int_A p(s,x;t,y)dy\qquad\text{for }A\in\mathcal{B}((y-\varepsilon,\,y+\varepsilon)).$$

Since y is an arbitrary point in $\dot D(t)$, it follows that $p(s,x;t,y)$ is the density function for $X_t^{s,x}$ inside $\dot D(t)$. Nash's inequality again implies that $p(s,x;t,y)$ is locally Hölder continuous in $(s,x)\in\dot D$ so p is locally Hölder continuous in $\dot D\times\dot D$.

The function $p(s,x;t,y)$ satisfies the forward and backward heat equations because the functions $p_k(s,x;t,y)$ do and they converge to $p(s,x;t,y)$ uniformly on balls in $\dot D$. □

When the boundary of the domain is sufficiently smooth, transforming it into a time-independent domain can result in a useful representation of the heat equation solution. A similar general idea underlies the arguments in Burdzy, Chen and Sylvester (2000b) but the following result is completely different at the technical level. The representation (3.4) is somewhat similar to that in Corollary 2.12 but the crucial difference is that the local time in (3.4) corresponds to the reflected Brownian motion on a half-line. The distribution of this process is well known—we use it as an essential ingredient of the proof of Theorem 3.9. The smoothness assumptions on the boundary of the domain are significantly weaker in Theorem 3.9 than in Corollary 2.12.

Theorem 3.9. *Suppose that g is a continuous function on $\mathbf{R}_+$ and that $\dot{D} = \{(t,y) : t \geq 0, y < g(t)\}$. Let X be the reflecting Brownian motion in $\dot{D}$ with initial distribution X_0 being the Lebesgue measure on $(-\infty, g(0))$. Let B_t be the standard Brownian motion and $Y_t = Y_0 + B_t - L_t$ be the reflecting Brownian on $(-\infty, 0]$, with L_t the local time of Y at 0. Let $\mathbf{P}^x$ denote the law of Y with $Y_0 = x$. Assume that $\int_0^1 |g'(s)|^2 ds < \infty$ and let*

$$N_t = \exp\left(\int_0^t g'(t-s)dB_s - \frac{1}{2}\int_0^t |g'(t-s)|^2 ds - 2\int_0^t g'(t-s)dL_s\right), \quad 0 \leq t \leq 1. \tag{3.4}$$

Then for each $t \in [0, 1]$, $\mathbf{E}_x[N_t]$ is bounded on compact intervals of $(-\infty, 0)$ and the distribution of X_t is absolutely continuous with respect to the Lebesgue measure on $(-\infty, g(t)]$ with density function u given by $u(t, g(t)+x) = \mathbf{E}_x[N_t]$.

Proof. The absolute continuity of the distribution of X_t is a consequence of Theorem 3.6.

Step 1. We first assume that g is C^3-smooth. By Theorem 2.9, the density $u(t,x)$ for the distribution of X_t is a C^2 smooth function and satisfies the following heat equation:

$$\begin{cases} u_t = \frac{1}{2}u_{xx} & \text{for } (t,x) \in \dot{D}\ , \\ u_x + 2g'(t)u = 0 & \text{for } (t,x) \in \partial\dot{D}, \\ u(0,x) = 1 & \text{for } x \leq g(0). \end{cases}$$

Let $v(t,x) = u(t, g(t)+x)$ for $t \geq 0$ and $x \leq 0$. Clearly $v_x = u_x$ and $v_t = u_t + g'(t)u_x$ and so v satisfies the partial differential equation

$$\begin{cases} v_t = \frac{1}{2}v_{xx} + g'(t)v_x & \text{for } x > 0, \\ v_x + 2g'(t)v = 0 & \text{for } x = 0, \\ v(0,x) = 1 & \text{for } x \leq 0. \end{cases} \tag{3.5}$$

Fix some $T \in (0, 1]$ and for $t \in [0,T]$ let

$$M_t = \exp\left(\int_0^t g'(T-s)dB_s - \frac{1}{2}\int_0^t |g'(T-s)|^2 ds - 2\int_0^t g'(T-s)dL_s\right).$$

Note that $M_T = N_T$. By Itô's formula, using (3.5),

$$\begin{aligned} & d\left(v(T-t, Y_t)M_t\right) \\ &= M_t\left(-v_t(T-t,Y_t)dt + v_x(T-t,Y_t)dB_t - v_x(T-t,Y_t)dL_t + \frac{1}{2}v_{xx}(T-t,Y_t)dt\right) \\ &\quad + v(T-t,Y_t)M_t\left(g'(T-t)dB_t - \frac{1}{2}|g'(T-t)|^2 dt - 2g'(T-t)dL_t + \frac{1}{2}|g'(T-t)|^2 dt\right) \\ &\quad + v_x(T-t,Y_t)M_t g'(T-t)dt \\ &= M_t(v_x(T-t,Y_t) + v(T-t,Y_t)g'(T-t))dB_t. \end{aligned}$$

This shows that $t \to v(T-t, Y_t)M_t$ is a local martingale for $t \in [0,T]$. We will prove that this process is in fact a martingale. It will suffice to show that $E(v(T-t,Y_t)M_t)^2 < \infty$ for $t \in [0,T]$. We first note that $v(t,x)$ is bounded on $\dot{D} \cap [0,\, T] \times \mathbf{R}^n$ in view of Theorem 2.4 (iv). It remains to estimate EN_t^2. In view of the boundedness of the first three derivatives of g, the process

$$t \to \exp\left(\int_0^t 4g'(T-s)dB_s - \frac{1}{2}\int_0^t |4g'(T-s)|^2 ds \right)$$

is a martingale so its expectation is equal to 1 for every t. We obtain for any $t \in [0,T]$,

$$\begin{aligned}
&\mathbf{E}_x[M_t^2] \\
&= \exp\left(-\int_0^t |g'(T-s)|^2 ds\right) \mathbf{E}_x\left[\exp\left(2\int_0^t g'(T-s)dB_s - 4\int_0^t g'(T-s)dL_s\right)\right] \\
&= \exp\left(2\int_0^t |g'(T-s)|^2 ds\right) \\
&\qquad \times \mathbf{E}_x\left[\exp\left(2\int_0^t g'(T-s)dB_s - 2\sqrt{2}\int_0^t |g'(T-s)|^2 ds - 4\int_0^t g'(T-s)dL_s\right)\right] \\
&\le \exp\left(2\int_0^t |g'(T-s)|^2 ds\right) \left(\mathbf{E}_x\left[\exp\left(-8\int_0^t g'(T-s)dL_s\right)\right]\right)^{1/2} \\
&\qquad \times \left(\mathbf{E}_x\left[\exp\left(\int_0^t 4g'(T-s)dB_s - \frac{1}{2}\int_0^t |4g'(T-s)|^2 ds\right)\right]\right)^{1/2} \\
&\le \exp\left(2\int_0^T |g'(s)|^2 ds\right) \left(\mathbf{E}_x\left[\exp\left(8\|g'\|_{L^\infty[0,T]}\, L_t\right)\right]\right)^{1/2} \\
&\le c,
\end{aligned} \tag{3.6}$$

for some positive constant $c < \infty$ independent of $x < 0$. The first factor in the second to last line is bounded because g' is bounded on $[0,T]$, while the second one is bounded due to Lemma 2.7. This shows that $\mathbf{E}_x M_t^2 < \infty$ and thus completes the proof of the fact that $t \to v(T-t, Y_t)M_t$ is a martingale. Thus

$$v(T,x) = \mathbf{E}_x\left[v(T-0,x)M_0\right] = \mathbf{E}_x\left[v(T-T,x)M_T\right] = \mathbf{E}_x\left[N_T\right],$$

and the theorem follows for C^3-smooth g.

Step 2. For the general case, let g_n be a sequence of smooth functions with $g_n(0) = g(0)$ such that g_n' converge to g' in $L^2[0,\,1]$. Then g_n converge to g uniformly on $[0,\,1]$. We will prove the theorem only for $t = 1$ as the argument is analogous for $t < 1$. Let X^n be the reflecting Brownian motion in the domain $\dot{D}_n = \{(t,x) : t \ge 0,\, x < g_n(t)\}$ with the density of the initial distribution equal to 1 on $(-\infty,\, g_n(0)]$ and let $u_n(t,x)$ be the density of the distribution of X_t^n. Let N^n be defined by (3.4) with g_n in place of g. If we let $v_n(t,x) = u_n(t, g(t)+x)$, we have from Step 1, $v_n(1,x) = \mathbf{E}_x[N_1^n]$ for $x < 0$. Note that

$$\mathbf{E}_x|N_1 - N_1^n|$$

$$= \mathbf{E}_x \left[N_1^n \left| \exp\left(\frac{1}{2}\int_0^1 (|g_n'(s)|^2 - |g'(s)|^2)ds\right) \right.\right.$$
$$\left.\left. \times \exp\left(\int_0^1 (g'-g_n')(1-s)dB_s - 2\int_0^1 (g'-g_n')(1-s)dL_s\right) - 1 \right| \right]$$
$$\le \left(\mathbf{E}_x[(N_1^n)^2]\right)^{1/2} \left(\mathbf{E}_x\left(\exp\left(\frac{1}{2}\int_0^1 (|g_n'(s)|^2 - |g'(s)|^2)ds\right)\right.\right. \tag{3.7}$$
$$\left.\left. \times \exp\left(\int_0^1 (g'-g_n')(1-s)dB_s - 2\int_0^1 (g'-g_n')(1-s)dL_s\right) - 1\right)^2\right)^{1/2}.$$

We now estimate the second factor on the right hand side of (3.7). First of all, for $k > 0$,

$$\exp\left(k\int_0^1 (|g'(s)|^2 - |g_n'(s)|^2)ds\right) \to 1$$

because $g_n' \to g'$ in L^2.

Recall that for a fixed $s > 0$, the density of the distribution of L_s is equal to $\frac{2}{\sqrt{2\pi}} s^{-1/2} e^{-a^2/(2s)}$ (see, e.g., P.211 in Katzas and Shreve (1991)). Hence, for any $a < 0$,

$$\lim_{n\to\infty} \sup_{x\le a} \mathbf{E}_x \left[\int_0^1 |(g_n' - g')(1-s)| dL_s\right]$$
$$= \lim_{n\to\infty} \frac{2}{\sqrt{2\pi}} \int_0^1 |(g_n' - g')(1-s)|\, s^{-1/2} e^{-a^2/(2s)} ds = 0 \tag{3.8}$$

and therefore by Khasminskii's inequality

$$\lim_{n\to\infty} \sup_{x\le a} \mathbf{E}_x \exp\left(k\int_0^1 |(g_n'-g')(1-s)| dL_s\right) = 1. \tag{3.9}$$

Since

$$t \to \exp\left(\int_0^t k(g_n' - g')(1-s)dB_s - \frac{1}{2}\int_0^t |k(g_n'-g')(1-s)|^2 ds\right)$$

is a martingale, the expectation of its value at $t = 1$ is the same as at $t = 0$, i.e., it is equal to 1. We use this observation in the following computation,

$$\mathbf{E}_x\left[\exp\left(\int_0^1 4(g'-g_n')(1-s)dB_s\right)\right]$$
$$= \exp\left(\int_0^1 8|g'-g_n'|^2(s)ds\right)$$
$$\times \mathbf{E}_x \exp\left(\int_0^1 4(g'-g_n')(1-s)dB_s - \frac{1}{2}\int_0^1 |4(g'-g_n')(1-s)|^2 ds\right)$$
$$= \exp\left(\int_0^1 8|g'-g_n'|^2(s)ds\right). \tag{3.10}$$

The last expression converges to 1 so using (3.9),

$$\begin{aligned}
&\lim_{n\to\infty}\sup_{x\le a}\mathbf{E}_x\left[\exp\left(\int_0^1 2(g'-g_n')(1-s)dB_s - 4\int_0^1 (g'-g_n')(1-s)dL_s\right)\right]\\
&\le \lim_{n\to\infty}\sup_{x\le a}\left(\mathbf{E}_x\left[\exp\left(\int_0^1 4(g'-g_n')(1-s)dB_s\right)\right]\right.\\
&\qquad\left.\times\,\mathbf{E}_x\left[\exp\left(8\int_0^1 |(g'-g_n')(1-s)|dL_s\right)\right]\right)^{1/2}\\
&\le\ 1.
\end{aligned}$$

On the other hand, by Jensen's inequality and (3.8)

$$\begin{aligned}
&\lim_{n\to\infty}\inf_{x\le a}\mathbf{E}_x\left[\exp\left(\int_0^1 (g'-g_n')(1-s)dB_s - 2\int_0^1 (g'-g_n')(1-s)dL_s\right)\right]\\
&\ge \exp\left(-2\sup_{x\le a}\mathbf{E}_x\left[\int_0^1 |(g'-g_n')(1-s)|dL_s\right]\right)\\
&= 1.
\end{aligned}$$

This, together with (3.9) and (3.10), shows that

$$\begin{aligned}
&\lim_{n\to\infty}\sup_{x\le a}\mathbf{E}_x\left(\exp\left(\frac{1}{2}\int_0^1(|g'(s)|^2-|g_n'(s)|^2)ds\right)\right.\\
&\qquad\left.\times\ \exp\left(\int_0^1 (g'-g_n')(1-s)dB_s - 2\int_0^1 (g'-g_n')(1-s)dL_s\right) - 1\right)^2\Bigg)^{1/2}\\
&= 0. \qquad (3.11)
\end{aligned}$$

Since g_n is C^3-smooth, we see from (3.6) with $T=t=1$ that $\mathbf{E}_x[(N_1^n)^2]$ is bounded. Hence $\mathbf{E}_x[|N_1-N_1^n|]$ is locally bounded on $(-\infty,0)$. Therefore N_1 is $\mathbf{P}^x$-integrable and $v(1,x)=\mathbf{E}_x[N_1]$ is locally bounded on $(-\infty,0)$. A similar argument as above shows that $\mathbf{E}_x[(N_1)^2]$ is locally bounded on $(-\infty,0)$. Note that

$$\begin{aligned}
&|v(1,x)-v_n(1,x)| = |\mathbf{E}_x[N_1-N_1^n]|\\
&\le \mathbf{E}_x\left[N_1\left|1-\exp\left(\frac{1}{2}\int_0^1(|g'(s)|^2-|g_n'(s)|^2)ds\right)\right.\right.\\
&\qquad\left.\left.\times\ \exp\left(\int_0^1 (g_n'-g')(1-s)dB_s - 2\int_0^1 (g_n'-g')(1-s)dL_s\right)\right|\right]\\
&\le \left(\mathbf{E}_x[(N_1)^2]\right)^{1/2}\left(\mathbf{E}_x\left(1-\exp\left(\frac{1}{2}\int_0^1(|g'(s)|^2-|g_n'(s)|^2)ds\right)\right.\right.\\
&\qquad\left.\left.\times\ \exp\left(\int_0^1 (g_n'-g')(1-s)dB_s - 2\int_0^1 (g_n'-g')(1-s)dL_s\right) - 1\right)^2\right)^{1/2}.
\end{aligned}$$

By a proof completely analogous to that of (3.11), the second factor in last display goes to zero uniformly on compact intervals of $(-\infty,0)$. Thus $\lim_{n\to\infty} v_n(1,x)=v(1,x)$ uniformly on compact intervals in $(-\infty, 0)$. As g_n converge to g uniformly on [0, 1], by Corollary 3.4, X_1^n converge to X_1 uniformly. Since $v_n(1, x - g_n(1))$ is the density function for X_1^n, we see that $v(1, x - g(1))$ is the density function for X_1. □

Remark 3.10. It is shown in Burdzy, Chen and Sylvester (2002b) that the distribution of X_t can be singular with respect to the Lebesgue measure at a boundary point $x = g(t)$ if the L^2-integrability of g' is not satisfied.

REFERENCES

1. J. Anderson and L. Pitt (1997), Large time asymptotics for Brownian hitting densities of transient concave curves *J. Theoret. Probab.* **10**, 921–934.
2. R.F. Bass (1997), *Diffusions and Elliptic Operators*, Springer, New York.
3. R. Bass and K. Burdzy (1996), A critical case for Brownian slow points *Probab. Th. Rel. Fields* **105**, 85–108.
4. R. Bass and K. Burdzy (1999), Stochastic bifurcation models *Ann. Probab.* **27**, 50–108.
5. R. Bass, K. Burdzy and Z.-Q. Chen (2002), Uniqueness for reflecting Brownian motion in lip domains. Preprint, 2002.
6. R.F. Bass and P. Hsu (1990), The semimartingale structure of reflecting Brownian motion, Proc. A.M.S. **108**, 1007-1010.
7. R.F. Bass and P. Hsu (1991), Some potential theory for reflecting Brownian motion in Hölder and Lipschitz domains, Ann. Probab. **19**, 486-508.
8. K. Burdzy and Z.-Q. Chen (1998), Weak convergence of reflecting Brownian motions. *Electronic Communications in Probability*, **3**, 29-33.
9. K. Burdzy, Z.-Q. Chen and J. Sylvester (2002a) The heat equation in time dependent domains with insulated boundaries. To appear in *J. Math. Anal. Appl.*
10. K. Burdzy, Z.-Q. Chen and J. Sylvester (2002b) The heat equation and reflected Brownian motion in time-dependent domains II: Singularities of solutions. To appear in *J. Funct. Anal.*
11. K. Burdzy and W. Kendall (2000), Efficient Markovian couplings, examples and counterexamples *Ann. Appl. Probab.* (to appear).
12. K. Burdzy and D. Khoshnevisan (1998), Brownian motion in a Brownian crack *Ann. Appl. Probab.* **8**, 708–748
13. K. Burdzy and E. Toby (1995), A Skorohod-type lemma and a decomposition of reflected Brownian motion *Ann. Probab.* **23**, 586–604.
14. Z. Q. Chen (1993), On reflecting diffusion processes and Skorokhod decompositions. *Probab. Theor. Rel. Fields*, **94**, 281-316.
15. Z. Q. Chen (1996), Reflecting Brownian motions and a deletion result for Sobolev spaces of order $(1, 2)$. *Potential Analysis*, **5**. 383-401.
16. Z. Q. Chen, P. J. Fitzsimmons, and R. J. Williams, Reflecting Brownian motions: quasimartingales and strong Caccioppoli sets, *Potential Analysis* **2** (1993), 219-243
17. C. Costantini (1992), The Skorohod oblique reflection principle in domains with corners and applications to stochastic differential equations. *Probab. Theor. Rel. Fields,* **91**, 43-70.

18. J. Crank (1984), *Free and Moving Boundary Problems.* Clarendon Press, Oxford.
19. M. Cranston and Y. Le Jan (1989), Simultaneous boundary hitting for a two point reflecting Brownian motion. Sminaire de Probabilités, XXIII, 234–238, Lecture Notes in Math., 1372, Springer, Berlin.
20. J.L. Doob (1984), *Classical Potential Theory and Its Probabilistic Counterpart*, Springer, New York.
21. P. Dupuis and H. Ishii (1993), SDEs with oblique reflection on nonsmooth domains. *Ann. Probab.* **21**, 554–580.
22. J. Durbin (1992), The first-passage density of the Brownian morion process to a curved boundary. With an appendix by D. Williams. *J. Appl. Probab.* **29**, 291–304.
23. R. Durrett (1984), *Brownian Motion and Martingales in Analysis*, Wadsworth, Belmont, California.
24. N. El Karoui and I. Karatzas (1991a), A new approach to the Skorohod problem, and its applications. *Stochastics Stochastics Rep.* **34**, 57–82.
25. N. El Karoui and I. Karatzas (1991b), Correction: "A new approach to the Skorohod problem, and its applications". *Stochastics Stochastics Rep.* **36**, 265.
26. A. Friedman (1964), *Partial Differential Equations of Parabolic Type.* Prentice-Hall, Inc, 1964.
27. M. Fukushima (1967), A construction of reflecting barrier Brownian motions for bounded domains. *Osaka J. Math.* **4** (1967), 183-215.
28. M. Fukushima (1999), On semimartingale characterization of functionals of symmetric Markov processes. *Elect. J. Probab.* **4**, 1-31.
29. M. Fukushima and M. Tomisaki (1996), Construction and decomposition of reflecting diffusions on Lipschitz domains with Hölder cusps. *Probab. Theory Relat. Fields*, **106**, 521-557.
30. P. Greenwood and E. Perkins (1983), A conditioned limit theorem for random walk and Brownian local time on square root boundaries *Ann. Probab.* **11**, 227–261.
31. S. Hofmann and J.L. Lewis (1996) L^2 solvability and representation by caloric layer potentials in time-varying domains. *Ann. Math.* **144**, 349–420.
32. P. Hsu (1984), Reflecting Brownian, Boundary local time, and the Neumann boundary value problem. Ph.D. thesis, Stanford University.
33. P. Hsu (1987), On the Poisson kernel for the Neumann problem of Schrödinger operators. *J. London Math. Soc. (2)* **36**, 370-384.
34. N. Ikeda and S. Watanabe (1981), *Stochastic Differential Equations and Diffusion Processes.* North-Holland.
35. S. Itô (1957), Fundamental solutions of parabolic differential equations and boundary value problems. *Japan J. Math.* **27**, 55-102.
36. I. Karatzas and S. Shreve (1994) *Brownian Motion and Stochastic Calculus, Second Edition.* Springer-Verlag, New York.
37. S. Karlin and H.M. Taylor (1981) *A Second Course in Stochastic Processes.* Academic Press, New York.
38. F. Knight (1981) *Essentials of Brownian Motion and Diffusion.* Mathematical Surveys 18, American Mathematical Society, Providence, RI

39. F. Knight (1999) On the path of an inert object impinged on one side by a Brownian particle (preprint)
40. J.L. Lewis and M.A.M. Murray (1995) The method of layer potentials for the heat equation in time-varying domains. *Memoir AMS 545*, vol. 114.
41. P. L. Lions and A. S. Sznitman (1984), Stochastic differential equations with reflecting boundary conditions. *Comm. Pure Appl. Math.* **37**, 511-537.
42. J. Moser (1964), A Harnack inequality for parabolic differential equations. *Comm. Pure Appl. Math.* **17**, 101-134.
43. Y. Oshima (2001), On a construction of diffusion processes on moving domains. Preprint.
44. S. Port and C. Stone (1978) *Brownian Motion and Classical Potential Theory*, Academic Press, New York.
45. G. Roberts (1991) A comparison theorem for conditioned Markov processes *J. Appl. Prob.* **28**, 74–83.
46. Y. Saisho (1987), Stochastic differential equations for multi-dimensional domain with reflecting boundary. *Probab. Theor. Rel. Fields* **74**, 455-477.
47. K. Sato and T. Ueno (1965), Multi-dimensional diffusion and the Markov process on the boundary. *J. Math. Kyoto Univ.* **4-3**, 529-605.
48. M. Shimura (1985) Excursions in a cone for two-dimensional Brownian motion. *J. Math. Kyoto Univ.* **25**, 433–443.
49. F. Soucaliuc, B. Toth and W. Werner (2000), Reflection and coalescence between independent one-dimensional Brownian paths. *Ann. Inst. H. Poincaré Probab. Statist,* **36**, 509–545.
50. D.W. Stroock and S.R.S. Varadhan (1971), Diffusion processes with boundary conditions. Comm. Pure Appl. Math. **24**, 147-225.
51. D. W. Stroock and S. R. S. Varadhan (1979), *Multidimensional Diffusion Processes.* Springer-Verlag, Berlin Heidelber New York.
52. M. Tanaka (1979), Stochastic differential equations with reflecting boundary conditions in convex regions. Hiroshima Math. J. **9** (1979), 163-177.
53. R. J. Williams and W. A. Zheng (1990), On reflecting Brownian motion—a weak convergence approach. *Ann. Inst. Henri Poincaré* **26**, 461-488.

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