# NON-INTERSECTION EXPONENTS FOR BROWNIAN PATHS PART II. ESTIMATES AND APPLICATIONS TO A RANDOM FRACTAL

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Abstract. Let $X$ and $Y$ be independent 2-dimensional Brownian motions, $X(0) = (0,0), Y(0) = (\varepsilon, 0)$, and let $p(\varepsilon) = P(X[0,1] \cap Y[0,1] = \emptyset)$, $q(\varepsilon) = \{Y[0,1]$ does not contain a closed loop around $0\}$. Asymptotic estimates (when $\varepsilon \to 0$) of $p(\varepsilon), q(\varepsilon)$, and some related probabilities, are given. Let $F$ be the boundary of the unbounded connected component of $\mathbb{R}^2 \backslash Z[0,1]$, where $Z(t) = X(t) - tX(1)$ for $t \in [0,1]$. Then $F$ is a closed Jordan arc and the Hausdorff dimension of $F$ is less or equal to $3/2 - 1/(4\pi^2)$.

**1. Introduction and main results.** Let $n = 2$ or 3 and let $X_1, X_2, \dots, X_k$, $Y_1, Y_2, \dots, Y_m$ be independent $n$-dimensional Brownian motions,

$$X_1(0) = X_2(0) = \ldots = X_k(0) = 0,$$

$$Y_1(0) = Y_2(0) = \ldots = Y_m(0) = (\varepsilon, 0) \text{ or } (\varepsilon, 0, 0), \quad \varepsilon \in (0,1).$$

For $Z = X_j$ or $Y_j$, let

$$T_Z = \inf\{t > 0 : |Z(t)| \geq 1\},$$

$$p_{n,k,m}(\varepsilon) = P(\bigcup_{j=1}^{k} X_j[0, T_{X_j}] \cap \bigcup_{j=1}^{m} Y_j[0, T_{Y_j}] = \emptyset).$$

By Theorem 1.1 of part I of this article (Burdzy and Lawler (1988)), the following "non-intersection exponents" are well defined.

$$\xi(n,k,m) = \lim_{\varepsilon \to 0} \log p_{n,k,m}(\varepsilon) / \log \varepsilon.$$

It was proved in part I that the analogous non-intersection exponents for random walks exist as well and are equal to those for Brownian motion. Moreover, they are exactly 2 times larger than

$$\zeta(n,k,m) \overset{\text{df}}{=} \lim_{\varepsilon \to 0} \log P(\bigcup_{j=1}^{k} X_j[0,\varepsilon] \cap \bigcup_{j=1}^{m} Y_j[0,\varepsilon] \neq \emptyset) / \log \varepsilon$$

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or the analogous exponent for random walks. These results enable us to translate some known results about random walks into the present context. For example, the results of Lawler (1988) imply that

$$\xi(2,2,1) = 2 \tag{1.1}$$

and

$$\xi(3,2,1) = 1. \tag{1.2}$$

Burdzy, Lawler and Polaski (1988) proved that $\xi(2,1,1) \in (1,3/2]$. Duplantier and Kwon (1988) have a number of conjectures for $n = 2$ including $\xi(2,1,1) = 5/4$. Computer simulations (Duplantier and Kwon (1988), Burdzy, Lawler and Polaski (1988)) seem to support this conjecture. The paper starts with an improvement of the estimate for $\xi(2,1,1)$:

**Theorem 1.1.**

$$\xi(2,1,1) \in [1 + 1/(2\pi^2), 3/2). \tag{1.3}$$

The methods to prove the lower bound in (1.3) could in fact be improved to show that $\xi(2,1,1) \in [1+1/(4\pi), 3/2)$; however, since this is still far from the conjectured value, we will only derive (1.3). The upper bound is part of a more general result:

**Theorem 1.2.** *For all* $k, m \geq 1$,
*(i)* $\xi(2,k,m) < \xi(2,k,m+1) - 1/2$,
*(ii)* $\xi(3,k,m) < \xi(3,k,m+1)$.

Theorem 1.2 (i) combined with (1.1) gives the upper bound in (1.3). Theorem 1.2 (ii) and (1.2) give $\xi(3,1,1) < 1$ which is a slight improvement over the best previous bounds $\xi(3,1,1) \in [1/2, 1]$. Computer simulations (Burdzy, Lawler and Polaski (1988)) suggest $\xi(3,1,1) \approx .57$. There are reasons to believe that the inequalities in Theorem 1.2 are best possible among those that hold uniformly for all $k, m \geq 1$. The methods of part I of the present paper or other elementary arguments may be used to prove for $n = 2, 3,\ k, m \geq 1$,

$$\xi(n,k,m) = \xi(n,m,k),$$

so Theorem 1.2 is slightly more general than stated.

The proofs of Theorem 1.1 and 1.2 (i) use estimates on the probability that Brownian motion makes a loop about the origin. Suppose that $n = 2$ and let $A_m(\varepsilon)$ denote the event that (0,0) and (2,0) belong to the same connected component of $\mathbb{R}^2 \setminus \bigcup_{j=1}^m Y_j[0, T_{Y_j}]$. In other words, $A_m^c(\varepsilon)$ holds iff $\bigcup_{j=1}^m Y_j[0, T_{Y_j}]$ contains a closed loop around $(0,0)$.

**Theorem 1.3.** *(i)* $P(A_1(\varepsilon)) \leq \varepsilon^{\pi^{-2}}$ *for* $\varepsilon \in (0,1)$.
*(ii)* $\liminf_{\varepsilon \to 0} \log P(A_2(\varepsilon)) / \log \varepsilon \geq 1/2 + 1/(4\pi^2)$.

Theorem 1.3 (i) will be used as a lemma in the proof of Theorem 1.1. On the other hand, Theorem 1.3 (ii) is a corollary of Theorem 1.1 and will be used to derive Theorem 1.5 (i) below.

Bertrand Duplantier (private communication) has conjectured that

$$\lim_{\varepsilon\to 0} \log P(A_1(\varepsilon))/\log\varepsilon = 1/4.$$

Some computer simulations and conjectures related to "self-avoiding Brownian motion" (for a definition, see below or Mandelbrot (1982)) suggest that the limit in Theorem 1.3 (ii) is equal to 2/3.

Now we will present some applications of our estimates of non-intersection exponents to the geometric structure of Brownian paths. Let $X$ be an $n$-dimensional Brownian motion, $n \geq 1$. A point $x \in \mathbb{R}^n$ will be called a cut point if there exists $t \in (0,1)$ such that $X(t) = x$ and $X[0,t) \cap X(t,1] = \emptyset$. A point $x \in \mathbb{R}^n$ will be called a double cut point if there exist $s,t \in (0,1)$ with $0 < s < t < 1$, $X(s) = X(t) = x$, and

$$X([0,s) \cup (t,1]) \cap X((s,t)) = \emptyset.$$

For $n = 1$, cut points would be points of increase and Dvoretzky et al. (1961) proved that such points do not exist. Brownian paths in four or more dimensions do not intersect (Dvoretzky et al. (1950)) so they contain cut points but no double cut points. For $n = 2,3$, for every fixed $t$, $X(t)$ is not a cut point; however, Burdzy (1987a) proved that, with probability one, a Brownian path contains cut points. Here we prove that none of these cut points can be a double cut point.

**Theorem 1.4.** *For $n = 2,3$, Brownian paths have no double cut points with probability 1.*

Let $Z$ be a 2-dimensional Brownian motion conditioned to return to its starting point at time 1. Formally speaking, let $Z(t) = X(t) - tX(1)$ for $t \in [0,1]$, where $X$ is a
2-dimensional Brownian motion, $X(0) = (0,0)$. Let $F$ denote the boundary of the unbounded connected component of $\mathbb{R}^2 \setminus Z[0,1]$. Mandelbrot (1982) calls $F$ a "self-avoiding Brownian motion". Of course, this name must be taken with a grain of salt since $F$ is a set and not a stochastic process. (See Westwater (1985) for a review of results on other models of self-avoiding Brownian motion.)

There are two questions concerning the set $F$ which we will address here. The first one involves the Hausdorff dimension of $F$. The question is important since Mandelbrot (1982) gave the name "self-avoiding Brownian motion" to the set $F$ because the computer simulations indicated that its Hausdorff dimension is 4/3 and is the same as that of a more natural candidate for this name (see Mandelbrot (1982) for more details).

The second question related to $F$ is best expressed as a

**Problem.** *Is "self-avoiding Brownian motion" self-avoiding?*

The question is more delicate than it may seem at the first sight. Let $\widetilde{F}$ be the boundary of the unbounded connected component of $\mathbb{R}^2 \setminus X[0,1]$, where $X$ is a standard Brownian motion. It follows easily from Theorem 2.1 of Burdzy (1987a) that, with positive probability, the set $\widetilde{F}$ is not a closed Jordan arc, i.e. it is not homeomorphic to a circle. We will see that a seemingly unimportant technical assumption (i.e. conditioning by $\{X(0) = X(1)\}$) makes a lot of difference.

**Theorem 1.5.** *(i) The Hausdorff dimension of $F$ is less or equal to $3/2 - 1/(4\pi^2)$ a.s.*

*(ii) The set $F$ is a closed Jordan arc a.s.*

**2. Preliminaries.** This section is devoted to notation and a brief review of some useful results. One may find more information in the following books and articles.

(i) Ahlfors (1973)–harmonic measure, extremal distance,
(ii) Doob (1984), Port and Stone (1978)–Brownian motion, $h$-processes, potential theory,
(iii) Maisonneuve (1975), Burdzy (1987b)–exit systems,
(iv) Itô and McKean (1974), Durrett (1984)–conformal invariance of Brownian motion,
(v) Revuz (1970)–continuous additive functionals.

The sets of real, complex, integer and rational numbers will be denoted $\mathbb{R}, \mathbb{C}, \mathbb{Z}$ and $\mathbb{Q}$, resp. We will identify $\mathbb{R}^2$ and $\mathbb{C}$.

$$S(x,r) = \{z \in \mathbb{R}^n : |z-x| = r\}.$$

For a set $A \subset \mathbb{R}^n$, its boundary, closure and complement $\mathbb{R}^n \backslash A$ will be denoted by $\partial A, \overline{A}$ and $A^c$.

The Hausdorff dimension of a set $A \subset \mathbb{R}^n$ is defined by proclaiming that it is less or equal to $\alpha$ if and only if for every $\beta > \alpha$ and every $\varepsilon > 0$ one can find a sequence $\{B_k\}_{k\geq 1}$ of balls with radii $r_k$ such that $A \subset \bigcup_{k\geq 1} B_k$ and $\sum_{k\geq 1}(r_k)^\beta < \varepsilon$.

The underlying probability structure will be irrelevant most of the time. For definiteness, we will now describe the canonical space of paths. Let $\Omega$ be the set of all functions $\omega : [0,\infty) \to \mathbb{R}^n \cup \{\delta\}$ which are continuous on $[0,R)$ and equal to $\delta$ otherwise. The "lifetime" $R$ may be infinite. The "coffin state" $\delta$ is an isolated trap in $\mathbb{R}^n \cup \{\delta\}$.

The canonical process is defined by $X(t) = X(\omega,t) = \omega(t)$ for all $\omega$ and $t$. We will use various other names: $Y$, $X_1$, $Y_j$, etc. for canonical and other processes. Quite often, we will consider several processes simultaneously, for example, several canonical processes on the product space $\Omega^j$. We will also need another canonical space of paths $\widetilde{\Omega}$ which differs from $\Omega$ only in that the paths in $\widetilde{\Omega}$ are defined on $(0,\infty)$ rather than $[0,\infty)$.

The trace of a process will be denoted $X[s,t) = X([s,t))$; the symbols $X(s,t), Y(s,t]$, etc. will have the analogous meaning.

We will use many different measures on $\Omega$ and $\widetilde{\Omega}$; analogous measures on $\Omega$ and $\widetilde{\Omega}$ will be denoted by the same symbol. The distribution of the standard Brownian motion starting from $x$ will be denoted $P^x$. The distribution of Brownian motion in $D$ (i.e. Brownian motion killed at the hitting time of $D^c$) starting at $x$ will be denoted $P^x_D$. For a Greenian domain $D \subset \mathbb{R}^n$ and a superharmonic function $h$ in $D$, the symbol $P^x_h$ will stand for the distribution of an $h$-process in $D$ starting at $x$.

If $X$ and $Y$ are independent and have distributions $P^x_{h_1}$ and $P^y_{h_2}$ then their joint distribution will be denoted $P^{x,y}_{h_1,h_2}$. For measures $\sigma$ and $\lambda$ on $\mathbb{R}^n$, the distributions $P^{\sigma,\lambda}_{h_1,h_2}$, $P^\sigma_h$, etc. will be the usual mixtures of measures.

Sometimes we will ignore the above notation concerning the probability measures. If need arises, we will describe a process in words (say, "$X$ is a Brownian motion in $D$, $X(0) = x$") and then we will use the generic symbol $P$ for probability.

**Lemma 2.1.** *(Brownian scaling) Suppose that $h$ is a superharmonic and positive function in a Greenian domain $D \subset \mathbb{R}^n$, $x \in \overline{D}$ and $A \subset \Omega$. Denote*

$D_c = \{y \in \mathbb{R}^n : \exists z \in D \text{ such that } cz = y\},$

$h_c(z) = h(z/c)$,
$A_c = \{\omega \in \Omega : \exists \omega_1 \in A \; \forall t \quad c\omega_1(t) = \omega(c^2 t)\}$.
*Then* $P^{cx}_{h_c}(A_c) = P^x_h(A)$.

*Proof.* The Lemma follows from the scaling properties of Brownian motion and superharmonic functions, and the definition of an $h$-process (Doob (1984)). □

For a process $Z$ and a set $M \subset \mathbb{R}^n$ we will write

$$T(M) = T_Z(M) = \inf\{t > 0 : \lim_{s \uparrow\uparrow t} Z(t) \in M\}.$$

The harmonic measure of $M \subset \partial D$ at $x \in D$ with respect to a region $D$ will be denoted $\mu(x, D, M)$. Probabilistic significance of $\mu$ is explicated by

$$\mu(x, D, M) = P^x(T(M) \le T(D^c)).$$

If $X$ is a 2-dimensional Brownian motion, $X(0) \in D$, and $f : D \to \mathbb{C}$ is analytic then $f(X)$, after a suitable time change, is also a Brownian motion. A similar statement is also true for $h$-processes in $D$.

Now we are going to present an exit system formula. Suppose that $D \subset \mathbb{R}^n$ is open and $\partial D$ is non-polar. For $t > 0$ such that $X(t) \in \partial D$ define excursions $\{e_t(s), s > 0\} \in \widetilde{\Omega}$ of $X$ in $D$ as follows.

$$e_t(s) = \begin{cases} X(t+s) & \text{if} \quad \inf\{u > t : X(u) \in D^c\} > t + s, \\ \delta & \text{otherwise.} \end{cases}$$

Let $L_t$ denote the local time of the process $X$ under $P^{\bullet}$ (i.e. Brownian motion) on $\partial D$. A $\sigma$-finite measure $H^x$ on $\widetilde{\Omega}$ will be called a standard (Brownian) excursion law in $D$ if

$$H^x(\lim_{t \downarrow\downarrow 0} X(t) \ne x) = 0,$$

$H^x$ is strong Markov for the $P^{\bullet}_D$-transition probabilities, and for every compact non-polar set $K \subset D$ we have $0 < H^x(T_X(K) < \infty) < \infty$. Let $E^{\bullet}$ be the expectation corresponding to $P^{\bullet}$.

**Theorem 2.1.** *(Maisonneuve (1975), Burdzy (1987b)). There exists a family* $\{H^x\}_{x \in \mathbb{R}^n}$ *of $\sigma$-finite measures such that*

$$E^{\bullet}\Big(\sum_{0<u<\infty} f \circ e_u\Big) = E^{\bullet}\Big(\int_0^{\infty} H^{X(s)}(f) dL_s\Big) \tag{2.1}$$

*for all universally measurable non-negative $f$ on $\widetilde{\Omega}$ which vanish on constant excursions equal to $\delta$.*

*The measures $H^x$ may be chosen so that $H^x \equiv 0$ for $x \notin \partial D$, and for every $x \in \mathbb{R}^n$ either $H^x \equiv 0$ or $H^x$ is a standard excursion law in $D$.*

Here is a short review of some useful facts about $h$-processes. The proofs may be found in Doob (1984) and Meyer, Smythe and Walsh (1972).

Let $D \subset \mathbb{R}^n$ be a Greenian domain and $h$ be a positive superharmonic function in $D$. Let $p_t^D(x,y)$ be the transition density for Brownian motion killed at $T(D^c)$ and

$$p_t^h(x,y) = p_t^D(x,y)h(y)/h(x).$$

Any process with the $p_t^h$-transition densities will be called an $h$-process (conditioned Brownian motion).

Suppose that $M$ is a closed subset of $D$ and let

$$L = \sup\{t < R : X(t) \in M\}$$

be the last exit time from $M$. Denote

$Y_1(t) = X(t), \quad t \in (0, T_X(M)),$
$Y_2(t) = X(T_X(M) + t), \quad t \in (0, R - T_X(M)),$
$Y_3(t) = X(t), \quad t \in (0, L),$
$Y_4(t) = X(L + t), \quad t \in (0, R - L),$
$Y_5(t) = X(R - t), \quad t \in (0, R).$

Under $P_h^x$, each process $Y_k$ is an $h_k$-process in a domain $D_k$.

$$D_1 = D_4 = D \backslash M, \quad D_2 = D_3 = D_5 = D.$$

$h_1 = h_2 = h.$

$h_3$ is a potential supported by $\partial M$.

$h_4$ has the boundary values 0 on $\partial M$ and the same boundary values as $h$ on $\partial D \backslash \partial M$.

$h_5$ is the Green function $G_D(x, \cdot)$ if $x \in D$ or a harmonic function with a pole at $x$ if $x \in \partial D$.

If $\lambda(dy)$ is the $P^x$-distribution of

$$X(\inf\{t < T_X(D^c) : X(t) \in M\})$$

then the $P_h^x$-distribution of this random variable is $\lambda(dy)h(y)/h(x)$.

Let $D \subset \mathbb{C}$ be an open set, $M_1, M_2 \subset \partial D$, and let $\Gamma$ be the family of all arcs in $D$ joining $M_1$ and $M_2$. Let $z = x + iy$. The extremal distance of $M_1$ and $M_2$ in $D$ is defined by

$$d_D(M_1, M_2) = \sup_\rho \frac{\inf_{\gamma \in \Gamma} \int_\gamma \rho |dz|}{\iint\limits_D \rho^2 \, dxdy}$$

where the supremum is taken over all non-negative Borel measurable $\rho$ subject to the condition $0 < \iint\limits_D \rho^2 \, dxdy < \infty$.

**3. Closed loops around** 0 **(one Brownian path).** In this section, we will consider 2-dimensional processes. Recall that $X$ under $P^0$ is a Brownian motion starting from 0. Let

$$M_t = \sup\{\Re X_s : s \le t\}.$$

For each $t \ge 0$ such that $M_t = \Re X_t$, define

$$R_t^f = \inf\{s > 0 : M_{t+s} = \Re X_{t+s}\},$$

$$f_t(s) = \begin{cases} X(t+s) & \text{for } s \in (0, R_t^f), \\ \delta & \text{otherwise.} \end{cases}$$

Roughly speaking, $f$'s are excursions of $X$ to the left of the maximum of $\Re X$. Some excursions $f_t$ are null, i.e. $f_t \equiv \delta$.

**Lemma 3.1.** *For $a > 0$ we have*

$$P^0\left(\exists t \geq 0 : M_t < a, \quad |f_t(0+) - f_t(R_t^f -)| > 2\pi\right) = 1 - \exp(-a/\pi^2).$$

*Proof.* Denote $K = \{z \in \mathbb{C} : \Re z = 0\}$ and let $g_t$ be excursions of $X$ from $K$, i.e. for $t > 0$ such that $X_t \in K$ let

$$R_t^g = \inf\{s > 0 : X_{t+s} \in K\},$$

$$g_t(s) = \begin{cases} X(t+s) & \text{for } s \in (0, R_t^g), \\ \delta & \text{otherwise.} \end{cases}$$

Let $L_t$ be the local time of $X_t$ on $K$, under $P^\bullet$; it may be identified with the local time of $\Re X_t$ at 0. The local time $L_t$ will be normalized so that it has the same distribution as $M_t$ under $P^0$ (Williams (1979)).

Now we will describe an exit system $(dL, H)$ of $X$ from $K$. The process $L$ has just been defined. For each $x \notin K$, let $H^x \equiv 0$.

Let $H_*^0$ be the standard excursion law in $D_* \stackrel{\text{df}}{=} \{z \in \mathbb{C} : \Re z > 0\}$, and let $H_-^0$ be the distribution of $-\Re X_t + i\Im X_t$ under $H_*^0$. Define $H^0 = H_*^0 + H_-^0$ and normalize $H^0$ so that

$$H^0(\sup_{t<R} |\Re X(t)| \geq 1) = 1. \tag{3.1}$$

We will discuss this normalization at the end of the proof. For $x \in K$, let $H^x$ be the distribution of $x + X_t$ under $H^0$.

Let us show that $(dL, H)$ described above is an exit system from $K$. For any exit system $(dL, H)$ from $K$, all excursion laws $H^x$ must be translates of $H^0$, since Brownian motion is translation invariant. Similarly, the symmetry of Brownian motion forces $H^0$ to be the sum of two symmetric excursion laws on both sides of $K$. Finally, there is only one (up to a multiplicative constant) standard excursion law $H_*^0$ in $D_*$ (Burdzy (1987b)).

Denote

$K_1 = \{z \in \mathbb{C} : \Re z = 1\}$,

$K_2 = \{z \in K : |z| > 2\pi\}$,

$A_j = \{T_X(K_j) < \infty\}, \quad j = 1, 2.$

By Theorem 4.1 of Burdzy (1987b) we have

$$H_*^0(A_2) = H_*^0(A_1) \lim_{\substack{x \to 0 \\ x \in D_*}} P_{D_*}^x(A_2)/P_{D_*}^x(A_1). \tag{3.2}$$

By (3.1),

$$H_*^0(A_1) = 1/2. \tag{3.3}$$

We have

$$P_{D_*}^x(A_1) = \Re x \quad \text{for } x \in D_*, |x| < 1, \tag{3.4}$$

since this probability is equal to the chance that the 1-dimensional Brownian motion $\Re X$ starting from $\Re x$ will hit 1 before 0.

As for $P^x_{D_*}(A_2)$, recall that the distribution of $X(R-)$ under $P^x_{D_*}$ is Cauchy with the density

$$k_x(y) = \frac{1}{\pi \Re x \left(1 + \left(\frac{y - \Im x}{\Re x}\right)^2\right)}$$

for $y \in K$. Thus,

$$P^x_{D_*}(A_2) = \int\limits_{\substack{y \in K \\ |\Im y - \Im x| > 2\pi}} k_x(y) dy.$$

This, (3.4) and some elementary calculations imply that

$$\lim_{\substack{x \to 0 \\ x \in D_*}} P^x_{D_*}(A_2)/P^x_{D_*}(A_1) = \left( \int\limits_{-\infty}^{-2\pi} + \int\limits_{2\pi}^{\infty} \right) (\pi y^2)^{-1} dy = \pi^{-2}.$$

Then, by (3.2) and (3.3), $H^0_*(A_2) = (2\pi^2)^{-1}$ and, consequently, $H^0(A_2) = \pi^{-2}$. It follows that

$$H^x(|X(0+) - X(R-)| > 2\pi) = \pi^{-2} \quad \text{for } x \in K. \tag{3.5}$$

Let $N_t$ be the number of excursions $g_s$ such that $L_s < t$ and

$$|g_s(0+) - g(R^g_s -)| > 2\pi.$$

Then the exit system formula (2.1) and (3.5) imply that $t \to N_t$ is a Poisson process with intensity $\pi^{-2}$. Hence, for $a > 0$,

$$P(N_a = 0) = \exp(-a/\pi^2).$$

The processes $(L_t, |\Re X_t| + i\Im X_t)$ and $(M_t, M_t - \Re X_t + i\Im X_t)$ have the same distribution under $P^0$ (Williams (1979)). Observe that this means that excursions $f$ and $g$ correspond to each other. It follows that the $P^0$-chance that there are no excursions $f_t$ with $M_t < a$ and

$$|f_t(0+) - f_t(R^f_t -)| > 2\pi$$

is equal to $\exp(-a/\pi^2)$.

The proof is complete but we would like to make an important comment. Several normalizations of the local time $L_t$ and excursion laws $H^x$ may be found in literature and, therefore, it is easy to make a mistake. The choice of the right normalization is crucial to our estimate so we would like to indicate briefly how one can check whether our normalization is correct.

By (3.1) and Proposition 5.1 of Burdzy (1987b) we have for $b > 0$ and $x \in K$

$$H^x(\sup_{t<R} |\Re X_t| \geq b) = b^{-1}. \tag{3.6}$$

Let $\widetilde{N}_t$ be the number of excursions $g_s$ of $X$ from $K$ such that $L_s < t$ and

$$\sup_{u<R_s^g} |\Re g_s(u)| > L_s + 1.$$

Then, by (3.6) and the exit system formula (2.1), $\widetilde{N}_t$ is a Poisson process with intensity $(1+t)^{-1}$. Thus

$$P(\widetilde{N}_1 = 0) = \exp(-\int_0^1 (1+t)^{-1} dt) = 1/2.$$

In terms of $M_t$ and excursions $f$, it means that the $P^0$-chance that there are no excursions $f_s$ with $M_s < 1$ and

$$\sup_{u<R_s^f} |\Re f_s(0+) - \Re f_s(u)| > M_s + 1$$

is equal to $1/2$. The last event may be described equivalently by saying that $\Re X$ hits 1 before $-1$. By symmetry, its chance is $1/2$. Since our computation produced the same value, we conclude that our normalization of $L_t$ and $H^x$ is correct. □

*Proof of Theorem 1.3 (i).* Recall $X, f_t$, $K$ etc. from the last proof. Choose any $\varepsilon \in (0,1)$ and let $a = \log \varepsilon$. By the translation invariance of Brownian motion, we obtain from Lemma 3.1

$$P^{(a,0)}(\exists t \in [0, T_X(K)) : |f_t(0+) - f_t(R_t^f -)| > 2\pi) = 1 - \exp(a/\pi^2). \tag{3.7}$$

Let $Y$ be a time-changed version of $\exp(X)$ so that $Y$ is a Brownian motion. The process $Y$ has the distribution $P^{(\varepsilon,0)}$. Denote

$$D_t = \{z \in \mathbb{C} : |z| < \exp(\Re X(t))\}.$$

Each excursion $f_t$ of $X$ with

$$|f_t(0+) - f_t(R_t^f -)| > 2\pi$$

corresponds to an excursion of $Y$ inside $D_t$ which contains a closed loop around $(0,0)$ and, therefore, cuts off $(0,0)$ from $(2,0)$.

This and (3.7) imply that the chance that the path of $Y$ does not cut off $(0,0)$ from $(2,0)$ is less or equal to

$$\exp(a/\pi^2) = \exp(\log \varepsilon/\pi^2) = \varepsilon^{\pi^{-2}}$$

which completes the proof. □

**4. Inequalities between non-intersection exponents.** Let $X$ and $Y$ be independent 2-dimensional Brownian motions, $X(0) = (0,0)$, $Y(0) = (\varepsilon, 0)$ and let $Q_\varepsilon$ be the conditional probability of

$$\{T_Y(S(0,1)) < T_Y(X[0, T_X(S(0,1))])\}$$

given $X$.

**Lemma 4.1.** *For every $\beta < \infty$ there are $\alpha > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have*

$$P(Q_\varepsilon > \varepsilon^{1/2+\alpha}) \le \varepsilon^\beta.$$

*Proof. Step 1.* Suppose that $Z$ is a 2-dimensional Brownian motion with $\Re Z(0) = \log \varepsilon$ a.s. Denote

$T = \inf\{t > 0 : \Re Z(t) = 0\}$,

$x = (\log \varepsilon, 0)$.

Suppose that $\Gamma = \{\Gamma(t), t \ge 0\}$ is a continuous, possibly random, curve such that $\Gamma(0) = Z(0)$,

$$\Re\Gamma(t) \le \log \varepsilon \quad \text{for } t \ge 0,$$

and

$$\lim_{t\to\infty} \Re\Gamma(t) = -\infty.$$

Denote

$$K = \{z \in \mathbb{C} : \Re z = 0\} \cup \bigcup_{k\in\mathbb{Z}} (\Gamma + k \cdot 2\pi i) \cup \bigcup_{k\in\mathbb{Z}} (Z[0,T] + k \cdot 2\pi i).$$

Let $D$ be the connected component of $\mathbb{C}\backslash K$ which contains $x$, provided $x \notin K$, and

$$M_1 = \{z \in \partial D : \Re z = 0\}.$$

We will assume that $x \notin K$ and $M_1 \notin \emptyset$; the remaining cases will be discussed at the end of the proof (they are trivial).

Note that if $M_1 \ne \emptyset$ then $M_1$ is a line segment of length $2\pi$. Let $M_2$ be the connected component (line segment) of $\{z \in D : \Re z = \Re x\}$ which contains $x$. Let $D_1$ be the connected component of $D\backslash M_2$ which contains $M_1$ and $M_2$ in its boundary.

Let $\eta(b)$ be the total length of intervals comprising $\{z \in D_1 : \Re z = b\}$. For $z \in D_1$, denote $\rho(z) = 1/\eta(\Re z)$. Every path joining $M_1$ and $M_2$ in $D_1$ has length greater or equal to $\int_{\log\varepsilon}^0 db/\eta(b)$ in the metric $\rho(z)|dz|$. This integral is also the $\rho$-area of $D_1$. Thus, the extremal distance $d_{D_1}(M_1, M_2)$ of $M_1$ and $M_2$ in $D_1$ satisfies

$$d_{D_1}(M_1, M_2) \ge \int_{\log\varepsilon}^{0} \frac{db}{\eta(b)}. \tag{4.1}$$

See Section 4-5 of Ahlfors (1973) for more details.

*Step 2.* For every $p < 1$ one can find $r > 0$ such that Brownian motion starting from 0 makes a closed loop around $S(0, r)$ before hitting $S(0, 1)$ with probability greater than $p$. This may be easily proved using the scaling property of Brownian motion and the 0-1 law; see also Section 7.16 on "Spinning" in Itô and McKean (1974).

Let $m$ be the largest integer not greater than $-\log\varepsilon - 1$, and for $k \in (0, m], k \in \mathbb{Z}$, let

$$T_k = \inf\{t > 0 : \Re Z(t) = -k\}.$$

Let $A_k$ denote the event that the process $\{Z(T_k + t), t \geq 0\}$ makes a closed loop around $S(Z(T_k), r)$ before hitting $S(Z(T_k), 1)$. By the strong Markov property applied at $T_k$'s, the events $A_k$ are independent and each one has probability greater than $p$. Let $N$ be the number of events $A_k$, $1 \leq k \leq m$, which occured.

In order to estimate the tail of the distribution of $N$, we will use the normal approximation and the following elementary inequality. For $a \leq -1$,

$$\int_{-\infty}^{a} \exp(-x^2/2)\, dx \leq \int_{-\infty}^{a} -x \exp(-x^2/2)\, dx = \exp(-a^2/2).$$

Let $q = 1 - p$. For large $m$ (i.e. small $\varepsilon$) we have

$$\begin{aligned} P(N < mp/3) &< \int_{-\infty}^{mp/2} \frac{1}{\sqrt{2\pi mpq}} \exp\left[-\frac{(u-mp)^2}{2mpq}\right] du \\ &= \int_{-\infty}^{\frac{mp/2-mp}{\sqrt{mpq}}} \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dv \\ &\leq \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{mp/2-mp}{\sqrt{mpq}}\right)^2\right] \\ &= \frac{1}{\sqrt{2\pi}} (e^{-m})^{p/8q}. \end{aligned}$$

Choose sufficiently large $p < 1$ (and, consequently, small $r$) so that $p/8q \geq \beta + 1$. Since $|m + \log \varepsilon| < 2$, we have for small $\varepsilon > 0$ (i.e. large $m$)

$$\begin{aligned} P(N < mp/3) &\leq \frac{1}{\sqrt{2\pi}} (e^{-m})^{p/8q} \\ &\leq \frac{1}{\sqrt{2\pi}} (e^{-m})^{\beta+1} \\ &\leq \varepsilon^{\beta}. \end{aligned} \tag{4.2}$$

*Step 3.* If the event $A_k$ holds then a square with center $Z(T_k)$, side length $r$ and sides parallel to the axes, is contained in $S(Z(T_k), r)$ and, therefore, lies totally outside $D_1$, since $S(Z(T_k), r)$ is enclosed in a loop of $Z$. Then

$$\eta(b) \leq 2\pi - r \qquad \text{for } b \in [-k - r/2, -k + r/2].$$

For all $b$, $\eta(b) \leq 2\pi$.

Suppose that $N \geq mp/3$. Then $\eta(b) \leq 2\pi - r$ for $b$ in a subset of $(\log \varepsilon, 0)$ of measure greater or equal to $mpr/3$. This and (4.1) imply that, for sufficiently small

$\varepsilon$,

$$
\begin{aligned}
(4.3) \qquad d_{D_1}(M_1, M_2) &\geq \int_{\log \varepsilon}^{0} \frac{db}{\eta(b)} \\
&\geq \frac{1}{2\pi}(-\log\varepsilon - mpr/3) + \frac{1}{2\pi - r} mpr/3 \\
&\geq \frac{1}{2\pi}(m - mpr/3) + \frac{1}{2\pi - r} mpr/3 \\
&= m\frac{1}{2\pi}\left(1 - pr/3 + \frac{2\pi}{2\pi - r} pr/3\right) \\
&\overset{\text{df}}{=} m\frac{1}{2\pi}(1+2a) \\
&\geq -(1+a)\frac{1}{2\pi}\log\varepsilon.
\end{aligned}
$$

Notice that $a > 0$. By (4.2),

$$
(4.4) \qquad P\left(d_{D_1}(M_1, M_2) \geq -(1+a)\frac{1}{2\pi}\log\varepsilon\right) \geq 1 - \varepsilon^{\beta}.
$$

*Step 4.* We will now evaluate the harmonic measure $\mu(x, D, M_1)$ under the assumption that inequality (4.3) holds.

First, map $D$ conformally onto the strip

$$
D_2 \overset{\text{df}}{=} \{z \in \mathbb{C} : \Re z < 0, -\pi < \Im z < \pi\}
$$

in such a way that the endpoints of $M_1$ are mapped onto $-\pi i$ and $\pi i$ and $M_2$ is mapped onto a curve $M_3$ joining $\{z \in \mathbb{C} : \Im z = -\pi\}$ and $\{z \in \mathbb{C} : \Im z = \pi\}$.

Inequality (4-23) of Ahlfors (1973) and our inequality (4.3) imply that, for small $\varepsilon$, the point $x$ is mapped onto a point $y$ with

$$
(4.5) \qquad \Im y < (1+a)\log\varepsilon + 2\log 32.
$$

Note that our estimate of $\Im y$ differs by a factor of $2\pi$ from the one given in (4-23) of Ahlfors (1973) because we work with strips of width $2\pi$ rather than 1.

For small $\varepsilon$, (4.5) yields

$$
(4.6) \qquad \Im y < (1 + a/2)\log\varepsilon.
$$

Let

$D_3 = \{z \in \mathbb{C} : |z| < 1\} \setminus \{z \in \mathbb{C} : \Im z = 0, \Re z \leq 0\}$,

$M_4 = S(0, 1)$.

It is easy to check that, for some $c < \infty$ and all small $\varepsilon$,

$$
\mu((\varepsilon, 0), D_3, M_4) \leq c\varepsilon^{1/2}
$$

and by the Buerling theorem (Theorem 3.6 of Ahlfors (1973))

$$
(4.7) \qquad \mu(z, D_3, M_4) \leq c\varepsilon^{1/2}
$$

for all $z \in D_3$ with $|z| = \varepsilon$. The last inequality may be obtained by other, more elementary means as well. The function $z \to e^z$ maps $D_2$ onto $D_3$ and $M_3$ onto $M_4$. By the conformal invariance of harmonic measure, (4.6) and (4.7), we see that

$$\begin{aligned} \mu(y, D_2, M_3) &\le c(\exp((1 + a/2)\log \varepsilon))^{1/2} \\ &= c\varepsilon^{1/2+a/4}. \end{aligned} \tag{4.8}$$

Choose $\alpha \in (0, a/4)$ and apply the conformal invariance of harmonic measure again, together with (4.4) and (4.8) to conclude that, for small $\varepsilon$,

$$P(\mu(x, D, M_1) \le \varepsilon^{1/2+\alpha}) \ge 1 - \varepsilon^\beta. \tag{4.9}$$

*Step 5.* Recall that $X$ and $Y$ are independent 2-dimensional Brownian motions, $X(0) = (0, 0)$, $Y(0) = (\varepsilon, 0)$. Let

$$\widetilde{T} = \inf\{t > 0 : |X(t)| = \varepsilon\}.$$

The (multivalued) function $z \to \log z$ maps $\{X(\widetilde{T} + t), t \ge 0\}$ onto a time-changed Brownian motion which we may identify with $Z$. Similarly, $\{X(t), t \in (0, \widetilde{T}]\}$ is mapped onto a curve $\Gamma$. By the conformal invariance of Brownian motion and harmonic measure, $Q_\varepsilon$ is equal to $\mu(x, D, M_1)$, and, in view of (4.9),

$$P(Q_\varepsilon > \varepsilon^{1/2+\alpha}) \le \varepsilon^\beta$$

for small $\varepsilon$.

Finally, we come back to our assumption made in Step 1 that $x \notin K$ and $M_1 \ne \emptyset$. If any one of them is violated then $Q_\varepsilon = 0$. □

Now we will prove an analogous lemma for 3-dimensional independent Brownian motions $X$ and $Y$, $X(0) = (0, 0, 0)$, $Y(0) = (\varepsilon, 0, 0)$. As before, $Q_\varepsilon$ will denote the conditional probability of

$$\{T_Y(S(0, 1)) < T_Y(X[0, T_X(S(0, 1))])\}$$

given $X$.

**Lemma 4.2.** *For every $\beta < \infty$ there exist $\alpha > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ we have*

$$P(Q_\varepsilon > \varepsilon^\alpha) \le \varepsilon^\beta.$$

*Proof.* The distributions of $X$ and $Y$ will be denoted $P^x_X$ and $P^y_Y$, resp., provided $(X, Y)$ has the distribution $P^{x,y}$. We will write $a = (a, 0, 0)$. Denote

$$A = \left\{P^{3/4,1/2}\left(T_Y(S(0, 1)) > T_Y[X(0, T_X(S(3/4, 1/8))]) \mid X\right) > \eta\right\}.$$

Three-dimensional Brownian paths intersect with positive probability (Dvoretzky et al. (1950)). Therefore, for each $p < 1$ there exists $\eta > 0$ such that

$$P^{3/4}_X(A) > p.$$

By the Harnack principle applied in $\{z \in \mathbb{R}^3 : |z| < 9/16\}$, we have for some $c > 0$,

$$P_X^{3/4}\Big(\forall |x| = 1/2 \quad P^{3/4,x}\left(T_Y(S(0,1)) > T_Y(X[0,T_X(S(3/4,1/8))]) \mid X\right) > \eta c\Big) > p.$$

Denote for $k \geq 1$,

$$\begin{aligned} T_k &= \inf\{t > 0 : |X(t)| = (3/4)2^{-k}\}, \\ U_k &= \inf\{t > T_k : X(t) \in S(X(T_k), (1/8)2^{-k})\}, \\ A_k &= \left\{\forall |x| = (1/2)2^{-k} \quad P^{0,x}\left(T_Y(S(0,2^{-k})) > T_Y(X[T_k,U_k]) \mid X\right) > \eta c\right\}. \end{aligned}$$

By Brownian scaling, rotation invariance of Brownian motion and the strong Markov property applied at $T_k$, the events $A_k, k = 1, 2, \dots$ are independent under $P_X^0$ and each one has probability greater than $p$. Let $N$ be the number of events $A_k, k = 1, 2, \dots, m$, which hold. As in Step 2 of Lemma 4.1, we obtain for a sufficiently large $p < 1$ and all large $m$

$$\begin{aligned} P_X^0(N < mp/3) &\leq \frac{1}{\sqrt{2\pi}}(e^{-m})^{p/(8(1-p))} \\ &= \frac{1}{\sqrt{2\pi}}(2^{-m})^{p/(8(1-p)\log 2)} \\ &\leq (2^{-m-1})^{\beta}. \end{aligned} \tag{4.10}$$

If $A_k$ holds then the $P_Y^\varepsilon$-chance that the paths of $Y$ and $X$ do not intersect is less than $1-\eta c$, by the strong Markov property of $Y$ applied at $T_Y(S(0,(1/2)2^{-k}))$, assuming that $\varepsilon < (1/2)2^{-k}$. Suppose that $N \geq mp/3$. A similar argument to the one given above shows that, for $\varepsilon \in (2^{-m-1}, 2^{-m}]$, given $X$ and $\{N \geq mp/3\}$, the $P_Y^\varepsilon$-conditional probability of

$$\{X[0,T_X(S(0,1))] \cap Y[0,T_Y(S(0,1))] = \emptyset\}$$

is less than

$$(1-\eta c)^{mp/3} \leq (2^{-m-1})^{\alpha} \leq \varepsilon^{\alpha}$$

where $\alpha = -[p\log(1-\eta c)]/(6\log 2) > 0$. In view of (4.10),

$$P^{0,\varepsilon}(Q_\varepsilon > \varepsilon^\alpha) \leq (2^{-m-1})^{\beta} \leq \varepsilon^{\beta},$$

for small $\varepsilon > 0, \varepsilon \in (2^{-m-1}, 2^{-m}]$. □

*Proof of Theorem 3.1.* We will discuss only the inequality $\xi(2,1,1) < \xi(2,1,2) - 1/2$. The remaining inequalities may be proved in a similar way.

Let $X_1$, $Y_1$ and $Y_2$ be independent 2-dimensional Brownian motions $X_1(0) = (0,0)$, $Y_1(0) = Y_2(0) = (\varepsilon, 0)$. Recall the definition of $p_{n,k,m}(\varepsilon)$ from the introduction.

Let $\beta > \xi(2,1,2)$ and choose $\alpha$ according to Lemma 4.1. Let $\alpha_1 \in (0,\alpha)$.

The paths of $X_1$ and $Y_1$ do not intersect with probability $p_{2,1,1}(\varepsilon)$. According to Lemma 4.1, given $X_1$, the process $Y_2$ has less than $\varepsilon^{1/2+\alpha}$ chance of not intersecting $X_1$, except for a set of $X_1$-paths of probability $\varepsilon^\beta$. In symbols, we have

$$p_{2,1,2}(\varepsilon) \leq p_{2,1,1}(\varepsilon)\varepsilon^{1/2+\alpha} + \varepsilon^{\beta}$$

and

$$\begin{aligned} p_{2,1,1}(\varepsilon) &\geq p_{2,1,2}(\varepsilon)\varepsilon^{-1/2-\alpha} - \varepsilon^{\beta}\varepsilon^{-1/2-\alpha} \\ &\geq p_{2,1,2}(\varepsilon)\varepsilon^{-1/2-\alpha}/2 \\ &\geq p_{2,1,2}(\varepsilon)\varepsilon^{-1/2-\alpha_1} \end{aligned}$$

for small $\varepsilon$. Thus, $\xi(2,1,1) \leq \xi(2,1,2) - 1/2 - \alpha_1$.

The proof of Theorem 3.1 (ii) uses Lemma 4.2 rather than Lemma 4.1. □

## 5. Estimate of $\xi(2,1,1)$.

*Proof of Theorem 1.1.* Theorem 3.1 (i) and (1.1) imply that $\xi(2,1,1) < 3/2$.

Now we will prove the lower bound in (1.3). Let $X_1, X_2$ and $Y_1$ be independent 2-dimensional Brownian motions, $X_1(0) = X_2(0) = (0,0)$, $Y_1(0) = (\varepsilon, 0)$. Recall the definitions of $T_{X_1}$, $p_{n,k,m}$, $A_1(\varepsilon)$, etc. from the introduction.

Let $Q = Q(\varepsilon)$ be the conditional probability of

$$\{X_1[0,T_{X_1}] \cap Y_1[0,T_{Y_1}] = \emptyset\}$$

given $Y_1$. Since $X_1, X_2$ and $Y_1$ are independent and $(X_1, Y_1)$ and $(X_2, Y_1)$ have identical distributions, the conditional probability of

$$\{(X_1[0,T_{X_1}] \cup X_2[0,T_{X_2}]) \cap Y_1[0,T_{Y_1}] = \emptyset\}$$

given $Y_1$ is equal to $Q^2$. Observe that

$$E(Q \mid A_1^c(\varepsilon)) = 0.$$

For every $\xi_0 < \xi(2,2,1)$ and small $\varepsilon$, we obtain, by the Schwartz inequality,

$$\begin{aligned} \varepsilon^{-\xi_0} &\geq p_{2,2,1}(\varepsilon) \\ &= EQ^2 \\ &\geq [E(Q\mathbf{1}_{A_1(\varepsilon)})]^2[E(\mathbf{1}_{A_1(\varepsilon)})^2]^{-1} \\ &= [EQ]^2[P(A_1(\varepsilon))]^{-1}. \end{aligned}$$

This and Theorem 1.3 (i) imply that

$$\begin{aligned} p_{2,1,1}(\varepsilon) = EQ &\leq \varepsilon^{-\xi_0/2}[P(A_1(\varepsilon))]^{1/2} \\ &\leq \varepsilon^{-\xi_0/2-1/(2\pi^2)} \end{aligned}$$

for small $\varepsilon$. Hence,

$$\xi(2,1,1) \geq \xi_0/2 + 1/(2\pi^2)$$

and, because $\xi_0$ is an arbitrary number less than $\xi(2,2,1) = 2$ (see (1.1)), we have

$$\xi(2,1,1) \geq 1 + 1/(2\pi^2). \qquad \square$$

**6. Closed loops around 0 (two Brownian paths).**

*Proof of Theorem 1.3 (ii).* Let $D = \{z \in \mathbb{C} : |z| < 1\}$ and let $X_1$ and $X_2$ be independent 2-dimensional Brownian motions, $X_1(0) = (-\varepsilon, 0)$, $X_2(0) = (\varepsilon, 0)$, $\varepsilon \in (0,1)$. For a process $V$, we will write $T_V = T_V(S(0,1))$.

Let $f$ be a one-to-one conformal mapping of $D$ onto itself, such that

$$f((-\varepsilon,0)) = (0,0), \quad f((\varepsilon,0)) = (\varepsilon_1, 0), \qquad \varepsilon_1 > 0.$$

For small $\varepsilon$, we have $\varepsilon_1 < 3\varepsilon$.

Let $Z_1$ and $Z_2$ be Brownian motions obtained from $f(X_1)$ and $f(X_2)$ by a suitable time-change. Denote

$$B = \{X_1[0, T_{X_1}] \cap X_2[0, T_{X_2}] = \emptyset\},$$

$$B_1 = \{Z_1[0, T_{Z_1}] \cap Z_2[0, T_{Z_2}] = \emptyset\}.$$

We have $B = B_1$, and, by Theorem 1.1, for an arbitrary $\xi_0 < 1 + 1/(2\pi^2)$ and small $\varepsilon$,

$$P(B) = P(B_1) \le \varepsilon_1^{\xi_0} \le (3\varepsilon)^{\xi_0}. \tag{6.1}$$

Let $g(z) = z^2$ and let $Y_1$ and $Y_2$ be time-changed processes $g(X_1)$ and $g(X_2)$ so that $Y_1$ and $Y_2$ are Brownian motions. Both processes $Y_1$ and $Y_2$ start from $(\varepsilon^2, 0)$.

Easy geometry shows that if $B$ does not hold then $Y_1[0, T_{Y_1}] \cup Y_2[0, T_{Y_2}]$ contains a closed loop around $(0,0)$. In view of (6.1), and using the notation of Theorem 1.3, this may be expressed as

$$P(A_2(\varepsilon^2)) \le P(B) \le (3\varepsilon)^{\xi_0},$$

for small $\varepsilon > 0$. Thus,

$$P(A_2(\varepsilon)) \le 3^{\xi_0} \varepsilon^{\xi_0/2}$$

for small $\varepsilon$ and, since $\xi_0$ is an arbitrary number less than $1 + 1/(2\pi^2)$, the theorem follows. □

**7. Double cut points.** We will offer two proofs of Theorem 1.4 (ii). The first one is based on our estimates of non-intersection exponents. The idea of the second one will be outlined afterwards.

In this section, $\sigma_r$ will denote the uniform probability measure on $S(0,r)$.

**Lemma 7.1.** *Let $n = 2$ or $3$, $\varepsilon \in (0,1)$, $D = D(\varepsilon) = \{z \in \mathbb{R}^n : |z| \in (\varepsilon, 1)\}$ and let $h = h_\varepsilon$ be harmonic in $D$ with boundary values 1 on $S(0,1)$ and 0 otherwise. Suppose that $X_1, X_2, \dots, X_k, Y_1, Y_2, \dots, Y_m$ are independent processes and each one has the distribution $P_h^{\sigma_\varepsilon}$. Denote $T_Z = T_Z(S(0,1))$ and*

$$\widetilde{p}_{n,k,m}(\varepsilon) = P\Big(\bigcup_{j=1}^{k} X_j[0, T_{X_j}] \cap \bigcup_{j=1}^{m} Y_j[0, T_{Y_j}] = \emptyset\Big).$$

*Then*

$$\lim_{\varepsilon \to 0} \log \widetilde{p}_{n,k,m}(\varepsilon) / \log \varepsilon = \xi(n,k,m).$$

*Proof.* We will sketch the proof for $n = 3, k = m = 1$ only.

It has been shown in Lemma 3.3 of Burdzy and Lawler (1988) that $P^{\sigma,\sigma}_{h,h}(A(r)) \geq p_{2r}/c_1$ which translates into our present notation as

$$\widetilde{p}_{n,k,m}(\varepsilon) \geq p_{n,k,m}(\varepsilon/2)/c_1$$

for some constant $c_1 < \infty$. This implies that

$$\limsup_{\varepsilon\to 0} \log \widetilde{p}_{n,k,m}(\varepsilon)/\log \varepsilon \leq \xi(n,k,m).$$

Now suppose that the conclusion of Lemma 3.3 of Burdzy and Lawler (1988) holds, i.e.

$$\liminf_{\varepsilon\to 0} \log \widetilde{p}_{n,k,m}(\varepsilon)/\log \varepsilon \leq \xi_0.$$

The rest of the proof of (3.1) in Burdzy and Lawler (1988), including Lemmas 3.4–3.6, shows that $\xi(n,k,m) \leq \xi_0$, so

$$\xi(n,k,m) \leq \liminf_{\varepsilon\to 0} \log \widetilde{p}_{n,k,m}(\varepsilon)/\log \varepsilon. \quad \square$$

*First proof of Theorem 1.4.* We will consider the 3-dimensional case first. We will work with the canonical process $X$ which becomes a 3-dimensional Brownian motion under $P^x$. Fix some $y \in \mathbb{R}^3$, $|y| \in [3,4]$, and denote

$S_r = S(y,r)$,
$D = \{x \in \mathbb{R}^3 : |x-y| \in (2\varepsilon, 1)\}$.

For $\varepsilon \in (0,1/2)$ define

$T_1 = \inf\{t > 0 : X(t) \in S_{2\varepsilon}\}$,
$L_1 = \sup\{t < T_1 : X(t) \in S_1\}$,
$T_2 = \inf\{t > 0 : X(t) \in S_\varepsilon\}$,
$T_3 = \inf\{t > T_2 : X(t) \in S_1\}$,
$L_2 = \sup\{t < T_3 : X(t) \in S_{2\varepsilon}\}$,
$T_4 = \inf\{t > T_3 : X(t) \in S_2\}$,
$T_5 = \inf\{t > T_4 : X(t) \in S_{2\varepsilon}\}$,
$L_3 = \sup\{t < T_5 : X(t) \in S_1\}$,
$T_6 = \inf\{t > T_5 : X(t) \in S_\varepsilon\}$,
$T_7 = \inf\{t > T_6 : X(t) \in S_1\}$,
$L_4 = \sup\{t < T_7 : X(t) \in S_{2\varepsilon}\}$,
$\widetilde{Z}_1(t) = X(L_1 + t), \quad t \in (0, T_1 - L_1)$,
$Z_1(t) = \widetilde{Z}_1(T_1 - L_1 - t), \quad t \in (0, T_1 - L_1)$,
$Z_2(t) = X(L_2 + t), \quad t \in (0, T_3 - L_2)$,
$\widetilde{Z}_3(t) = X(L_3 + t), \quad t \in (0, T_5 - L_3)$,
$Z_3(t) = \widetilde{Z}_3(T_5 - L_3 - t), \quad t \in (0, T_5 - L_3)$,
$Z_4(t) = X(L_4 + t), \quad t \in (0, T_7 - L_4)$,
$T_8 = \inf\{t > 0 : X(t) \in S_{12}\}$,
$A = \{T_7 < T_8\}$.

We will now analyse the conditional distributions of the processes $Z_k$ given $A$. The joint conditional distribution of $(Z_1, Z_2)$ under $P^0$ given $A$ will be denoted $P_{Z_1,Z_2}$, and $P_{\widetilde{Z}_1}, P_{Z_1,Z_2,Z_3,Z_4}$, etc. will have an analogous meaning.

Let $h$ be harmonic in $D$ with boundary values 1 on $S_1$ and 0 otherwise. For $x \in \mathbb{R}^3$, $|x-y| > \varepsilon$, let $h_1(x) = P^x(A)$. Let $h_2$ be a harmonic function in $D$, such that $h_2 = h_1$ on $S_1$ and $h_2$ has boundary values 0 otherwise.

The process $\{X(t), t \in (0, T_1)\}$ under $P^0$, conditioned by $A$, is an $h_1$-process in $\{x \in S_{12} : |x-y| > 2\varepsilon\}$ and, consequently, $\{\widetilde{Z}_1(t), t \in (0, T_1 - L_1)\}$ is an $h_2$-process in $D$. Denote
$\eta_1(dx) = P^0(\widetilde{Z}_1(0+) \in dx \mid A)$,
$\eta_2(dx) = P^0(\widetilde{Z}_1(T_{\widetilde{Z}_1}(S_{2\varepsilon})) \in dx \mid A)$.
By the Harnack principle applied in $\{x \in \mathbb{R}^3 : |x-y| \in (\varepsilon, 3\varepsilon)\}$, we have

$$h_1(x)/h_1(z) \in (c^{-1}, c)$$

for $x, z \in S_{2\varepsilon}$, and some constant $c < \infty$ (independent of $x, z$ and $\varepsilon$). This and formula (2.1) 2.X.2 of Doob (1984) imply that $d\eta_2/d\sigma_{2\varepsilon} > c_1 > 0$. For similar reasons, $d\eta_1/d\sigma_1 > c_2 > 0$. It follows that

$$dP_{\widetilde{Z}_1}/dP^{\sigma_1}_{1-h} > c_1c_2 = c_3 > 0$$

and, by the time reversal,

$$dP_{Z_1}/dP^{\sigma_{2\varepsilon}}_h > c_3.$$

By the strong Markov property of $X$ under $P^0$ applied at $T_2$ and an argument similar to that given in the case of $Z_1$, the conditional distribution $\widetilde{P}$ of $Z_2$ given $Z_1$, $X(T_2)$ and $A$ satisfies

$$d\widetilde{P}/dP^{\sigma_{2\varepsilon}}_h > c_4 > 0.$$

Integrate over the distribution of $Z_1$ and $X(T_2)$ to obtain

$$dP_{Z_1,Z_2}/dP^{\sigma_{2\varepsilon},\sigma_{2\varepsilon}}_{h,h} > c_3c_4 > 0.$$

A similar reasoning gives

$$dP_{Z_1,Z_2,Z_3,Z_4}/d(P^{\sigma_{2\varepsilon}}_h \times P^{\sigma_{2\varepsilon}}_h \times P^{\sigma_{2\varepsilon}}_h \times P^{\sigma_{2\varepsilon}}_h) > c_5 > 0. \tag{7.1}$$

Denote

$$B = \{[Z_1(0, T_1 - L_1) \cup Z_4(0, T_7 - L_4)] \cap [Z_2(0, T_3 - L_2) \cup Z_3(0, T_5 - L_3)] = \emptyset\}.$$

By (1.2) and Theorem 1.2 (ii) we have $\xi(3,2,2) > 1$ and $\alpha \stackrel{\text{df}}{=} (\xi(3,2,2) - 1)/2 > 0$. Lemma 7.1 and (7.1) imply that, for small $\varepsilon$,

$$P^0(B \mid A) < c_5\varepsilon^{1+\alpha}. \tag{7.2}$$

The probability of hitting a sphere $S(x,r)$ by a 3-dimensional Brownian motion starting from $z$ is equal to $r/|z-x|$ for $|x-z| > r$. Thus,

$$P^0(T_2 < \infty) \le \varepsilon/3$$

and, by the strong Markov property applied at $T_4$,

$$P^0(A) \le P^0(T_2 < \infty, T_6 < \infty) \le (\varepsilon/3)(\varepsilon/2) = \varepsilon^2/6. \tag{7.3}$$

We combine (7.2) and (7.3) to obtain

$$P^0(A \cap B) = P^0(B \mid A) P^0(A) \le c_6 \varepsilon^{3+\alpha}, \tag{7.4}$$

for small $\varepsilon$.

Let $\{y_k\}_{k=1}^N$, $N = N(\varepsilon)$, be the sequence of all points of the set

$$\{x \in \mathbb{R}^3 : (2/\varepsilon)x \in \mathbb{Z}^3, |x| \in [3,4]\}.$$

A crude estimate gives

$$N(\varepsilon) \le (16/\varepsilon)^3. \tag{7.5}$$

Let $C_k$ be the event $A \cap B$ defined relative to $y_k$ rather than $y$. Then (7.4) and (7.5) yield, for small $\varepsilon$,

$$\begin{aligned} P^0(\bigcup_{k=1}^N C_k) &\le c_6 \varepsilon^{3+\alpha} (16/\varepsilon)^3 \\ &\le c_7 \varepsilon^{\alpha}. \end{aligned} \tag{7.6}$$

Denote

$$B_1(\varepsilon) = \bigcup_{k=1}^N C_k,$$

$$\begin{aligned} B_2(a,b,c,d) = \{\exists s,t,u \quad &\text{such that} \quad 0 < s < u < t < 1, \quad X(s) = X(t) = y, \\ &|y| \in [a,b], \quad |X(u) - y| > c, \quad T_X(S(0,d)) > 1, \\ &(X[0,s) \cup X(t,1]) \cap X(s,t) = \emptyset\}. \end{aligned}$$

In view of (7.6),

$$\lim_{\varepsilon \to 0} P^0(B_1(\varepsilon)) = 0.$$

Observe that

$$B_2(3,4,1,7) \in \bigcap_{\substack{\varepsilon > 0 \\ \varepsilon \in \mathbb{Q}}} B_1(\varepsilon)$$

so $P^0(B_2(3,4,1,7)) = 0$. By analogy, $P^0(B_2(a,b,c,d)) = 0$ simultaneously for all rational $a,b,c,d$, such that $a,b,c,d > 0$, $c < a < b$, $d < b + c$. If a double cut point exists (as described in the introduction) then the event $\bigcup_{a,b,c,d \in \mathbb{Q}} B_2(a,b,c,d)$ holds. We conclude that, with $P^0$-probability 1, double cut points do not exist.

The proof in the 2-dimensional case is completely analogous. By (1.1) and Theorem 1.2 (i) we have

$$\xi(2,2,2) > 2 + \alpha \qquad \text{for some } \alpha > 1/2.$$

Therefore, in the 2-dimensional case, our estimates (7.2)–(7.6) are replaced by

$$\begin{aligned} P^0(B \mid A) &< c_5 \varepsilon^{2+\alpha}, \\ P^0(A) &\le 1, \\ P^0(A \cap B) &\le c_6 \varepsilon^{2+\alpha}, \\ N(\varepsilon) &\le (16/\varepsilon)^2, \\ P(\bigcup_{k=1}^N C_k) &\le c_6 \varepsilon^{2+\alpha} (16/\varepsilon)^2 \le c_7 \varepsilon^{\alpha}. \quad \square \end{aligned}$$

The main idea of the second proof of Theorem 1.4 is to give a correspondence between paths with double cut points and paths with isolated intersection points. It was shown in Lemma 3.9 of Burdzy and Lawler (1988) that with probability one isolated intersection points do not exist. The correspondence roughly goes as follows: suppose that $X$ and $Y$ are independent Brownian motions and $X[0,1]$ and $Y[0,1]$ intersect at a single point $x \in \mathbb{R}^n$. Cut both paths at $x$ and reassemble them by joining the initial part of $X$ to the terminal part of $Y$ and vice versa. Then the isolated intersection point $x$ becomes a "local double cut point" (or, better, a cut point for each of the two new processes). The above idea contains two potential pitfalls: (i) the described transformation is not one-to-one and (ii) path to path transformations do not necessarily preserve measure; in other words, even if the two new processes can be defined rigorously, it is not clear at all whether they are independent Brownian motions.

*Second proof of Theorem 1.4.* Suppose that $n = 2$ or 3 and that $X$ and $Y$ are canonical processes on the product space $\Omega^2$ equipped with the probability measure $P^{0,1}$ so that $X$ and $Y$ are independent Brownian motions starting from 0 and 1, where $1 = (1,0)$ or $(1,0,0)$.

Suppose that $y \in \mathbb{R}^n, |y| > 3$, and denote $S_r = S(y,r)$. For $Z = X$ or $Y$ and an integer $m > 1$ define

$$\begin{aligned}
T_Z^1 &= \inf\{t > 0 : Z(t) \in S_{2^{-m}}\}, \\
L_Z &= \sup\{t < T_Z^1 : Z(t) \in S_{2\cdot 2^{-m}}\}, \\
T_Z^2 &= \inf\{t > T_Z^1 : Z(t) \in S_{2\cdot 2^{-m}}\}, \\
T_Z^3 &= \inf\{t > T_Z^2 : Z(t) \in S_1\}, \\
A(y) &= \{(X[0,L_X] \cup X[T_X^2,T_X^3]) \cap (Y[0,L_Y] \cup Y[T_Y^2,T_Y^3]) = \emptyset, \\
&\qquad X[T_X^1,T_X^2] \cap Y[T_Y^1,T_Y^2] \neq \emptyset\}.
\end{aligned}$$

Let $\{y_m^k\}_{k=1}^\infty$ be the sequence of all elements of the set

$$\{x \in \mathbb{R}^n : 2^{m+1}x \in \mathbb{Z}^n, |x| > 3\}.$$

Denote $B(m) = \bigcup_{k=1}^\infty A(y_m^k)$. Observe that $B(m+1) \subset B(m)$ for all $m > 1$. Let $B = \bigcap_{m=2}^\infty B(m)$. If $B$ holds then $X$ and $Y$ have an isolated intersection point and, therefore, by Lemma 3.9 of Burdzy and Lawler (1988), $P(B) = 0$. Since the sequence of events $\{B(m)\}_{m\ge 2}$ is monotone, we have

$$\lim_{m\to\infty} P(B(m)) = 0. \tag{7.7}$$

Note that a sample path may belong to only a finite number $N$ of events $A(y_m^k)$, $k \ge 1$, and a crude estimate of $N$ is $N \le 19^3$. This implies that

$$\sum_{k=1}^\infty P(A(y_m^k)) \le NP(\bigcup_{k=1}^\infty A(y_m^k)) = NP(B(m)). \tag{7.8}$$

Now define events

$$\begin{aligned}
\widetilde{A}(y) &= \{(X[0,L_X] \cup Y[T_Y^2,T_Y^3]) \cap (Y[0,L_Y] \cup X[T_X^2,T_X^3]) = \emptyset, \\
&\qquad X[T_X^1,T_X^2] \cap Y[T_Y^1,T_Y^2] \neq \emptyset\}. \\
\widetilde{B}(m) &= \bigcup_{k=1}^\infty \widetilde{A}(y_m^k).
\end{aligned}$$

The event $\widetilde{A}(y)$ is obtained from $A(y)$ be exchanging the roles of $X[T_X^2, T_X^3]$ and $Y[T_Y^2, T_Y^3]$.

Given $\{L_X < \infty, L_Y < \infty\}$, $X[0, L_X]$, $Y[0, L_Y]$, $X(L_X) = x_1$, $Y(L_Y) = y_1$, the $P^{0,1}$-distribution of

$$\{(X(T_X^1 + t), Y(T_Y^1 + t)), t \geq 0\}$$

is $P^{\sigma_{x_1}, \sigma_{y_1}}$, for some $\sigma_{x_1}$ and $\sigma_{y_1}$, by the strong Markov property applied at $T_X^1$ and $T_Y^1$. It is not hard to see that

$$\sigma_{x_1}(dx) \times \sigma_{y_1}(dy) \leq c\,\sigma_{x_1}(dy) \times \sigma_{y_1}(dx)$$

where $c < \infty$ is independent of $x_1, y_1$, $x$ and $y$. Thus, given $X[0, L_X]$ and $Y[0, L_Y]$, for every event $C$, the conditional $P^{0,1}$-probability of $\{(Y(T_Y^1 + \cdot), X(T_X^1 + \cdot)) \in C\}$ is less or equal to $c$ times the conditional $P^{0,1}$-probability of $\{(X(T_X^1 + \cdot), Y(T_Y^1 + \cdot)) \in C\}$. By integrating over the distributions of $X[0, L_X]$ and $Y[0, L_Y]$, we obtain (for suitable $C$),

$$P(\widetilde{A}(y)) \leq cP(A(y)).$$

This and (7.8) imply that

$$P(\widetilde{B}(m)) \leq \sum_{k=1}^{\infty} P(\widetilde{A}(y_m^k)) \leq c \sum_{k=1}^{\infty} P(A(y_m^k)) \leq cNP(B(m)).$$

By (7.7),

$$\lim_{m \to \infty} P(\widetilde{B}(m)) = 0. \tag{7.9}$$

Denote

$$\begin{aligned} C_1(a, b) =& \{\exists s, t, u_1, u_2 > 0 \text{ such that } X(s) = Y(t), \\ & |X(s) - X(0)| > a, \quad |Y(t) - Y(0)| > a, \\ & (X[0, s) \cup Y(t, t + u_1]) \cap (Y[0, t) \cup X(s, s + u_2]) = \emptyset, \\ & |X(s + u_2) - X(s)| > b, \quad |Y(t + u_1) - Y(t)| > b\}, \\ C_2 =& \{\exists s, t, u > 0 \text{ such that } X(s) = Y(t), \\ & (X[0, s) \cup Y(t, t + u]) \cap (Y[0, t) \cup X(s, s + u]) = \emptyset\}. \end{aligned}$$

We have $C_1(4, 1) \subset \widetilde{B}(m)$ for all $m \geq 2$, so, by (7.9), $P^{0,1}(C_1(4, 1)) = 0$. For similar reasons, $P^{0,1}(C_1(a, b)) = 0$ for all rational $a, b > 0$ simultaneously and, therefore, $P^{0,1}(C_2) = 0$. By analogy,

$$P^{x,y}(C_2) = 0 \tag{7.10}$$

for all $x$ and $y$.

Suppose that $Z$ is a Brownian motion and $0 < t_1 < t_2 < 1$. The joint distribution of $\{Z(t), t \in [0, t_1]\}$ and $\{Z(t_2 + t), t \in [0, 1 - t_2]\}$ is mutually absolutely continuous with $P^{\sigma_1, \sigma_2}$ on $[0, \min(t_1, 1 - t_2)]$, for suitable $\sigma_1$ and $\sigma_2$. Then (7.10) implies that the probability of

$$\{\exists s, t \text{ such that } s \in (0, t_1), t \in (t_2, 1), Z(s) = Z(t),$$

$$(Z[0,s) \cup Z(t,1]) \cap (Z(s,t_1] \cup Z[t_2,t)) = \emptyset\}$$

is zero. This holds for all rational $t_1, t_2$, $0 < t_1 < t_2 < 1$, simultaneously, so double cut points do not exist, with probability 1. □

*Proof of Theorem 1.5 (ii).* We will only sketch the proof. It uses a version of Theorem 1.4 and otherwise it is an elementary exercise in the theory of the Carathéodory prime ends boundary. Readers are referred to Section 9.2 of Pommerenke (1975) for the definitions of prime ends, their impressions, null-chains etc.

First consider an arbitrary continuous curve $\Gamma = \{\Gamma(t), t \in [0,1]\} \subset \mathbb{C}$ and let $D$ be the unbounded connected component of $\mathbb{C} \setminus \Gamma$. We will show that for every prime end $K$ in $D$, its impression consists of a single point. Let $\{C_k\}_{k\geq 1}$ be a null-chain corresponding to $K$ and let $x_k$ and $y_k$ be the endpoints of $C_k$. By compactness, some subsequences of $\{x_k\}$ and $\{y_k\}$ converge and without loss of generality we assume that $x_k \to x$ and $y_k \to x$; both sequences must converge to the same point because $\operatorname{diam}(C_k) \to 0$. Choose $s_k, t_k \in [0,1]$ so that $\Gamma(s_k) = x_k$ and $\Gamma(t_k) = y_k$. By compactness, we may assume that $s_k \to s$ and $t_k \to t$. The continuity of $\Gamma$ implies that $\Gamma(s) = \Gamma(t) = x$. In order to simplify the notation, let us pretend that $s_k \leq s$ and $t_k \geq t$ although it is irrelevant. By the continuity of $\Gamma$, $\operatorname{diam} \Gamma[s_k, s] \to 0$ and $\operatorname{diam} \Gamma[t, t_k] \to 0$. Thus

$$\operatorname{diam}(C_k \cup \Gamma[s_k, s] \cup \Gamma[t, t_k]) \to 0$$

and it follows that $\operatorname{diam} \overline{\operatorname{Int} C_k} \to 0$ (see Pommerenke (1975) for the definition of $\operatorname{Int} C_k$). This immediately implies that the impression of $K$ consists of a single point. It follows from Corollary 9.3 of Pommerenke (1975) that if $f$ is a one-to-one conformal mapping of $D_1 \overset{\text{df}}{=} \{z \in \mathbb{C} : |z| > 1\}$ onto $D$ then $f$ has a continuous extension to $\overline{D_1}$.

Now we will prove that $f : \overline{D_1} \to D$ is one-to-one under suitable additional assumptions about $\Gamma$. First, assume that $\Gamma(0) = \Gamma(1)$. Now suppose that $f(x_1) = f(x_2)$ for some $x_1, x_2 \in \partial D_1$, $x_1 \neq x_2$. Let $C$ be an arc in $D_1$ with endpoints $x_1$ and $x_2$. Then $C_0 \overset{\text{df}}{=} f(C)$ is a closed Jordan arc. It is easy to see that $\Gamma$ cannot lie totally inside or outside $C_0$. Denote $y = \Gamma \cap \overline{C_0}$. We may assume without loss of generality that $\Gamma(0) \neq y$. Find $t$ so that $\Gamma(0)$ and $\Gamma(t)$ belong to distinct components of $\mathbb{C} \setminus C_0$. Then there exist $s, t_1, t_2$ and $u$ such that

$$0 < s \leq t_1 < t < t_2 \leq u < 1,$$

$$\Gamma(s) = \Gamma(t_1) = \Gamma(t_2) = \Gamma(u) = y,$$

$$(\Gamma[0,s) \cup \Gamma(u,1]) \cap \Gamma(t_1,t_2) = \emptyset.$$

If $\Gamma$ does not satisfy the last property then $f$ is a continuous and one-to-one function on the closure of $D_1$ and, consequently, $\partial D$ is a closed Jordan arc. With probability 1, the paths of Brownian bridge have the properties of the curve $\Gamma$ and they do not satisfy the last property. This may be proved in the way completely analogous to the first proof of Theorem 1.4. We conclude that $F$ is a closed Jordan arc. □

**8. Hausdorff dimension of "self-avoiding Brownian motion".** First we will prove a result (Lemma 8.3) similar to Theorem 1.3 (ii) but considerably stronger. The proof of Theorem 1.3 (ii) was based on the fact that the two Brownian paths started from the same point. We will get rid of this assumption.

**Lemma 8.1.** *Let $D = \{z \in \mathbb{C} : |z| \in (\varepsilon, 1)\}$ and let $h = h_\varepsilon$ be harmonic in $D$ with boundary values* 1 *on $S(0,1)$ and* 0 *otherwise. Suppose that $x, y \in S(0, \varepsilon)$, $x = x(\varepsilon)$, $y = y(\varepsilon)$. Denote*

$$A_1 = \{X[0,R) \cap Y[0,R) = \emptyset\}.$$

*Then*

$$\liminf_{\varepsilon \to 0} \log P^{x,y}_{h,h}(A_1)/\log \varepsilon \geq \xi(2,1,1).$$

*Proof.* This lemma is completely analogous to Lemma 7.1 and may be obtained from the latter by the last exit decomposition of $P^{x,y}_{D,D}$-processes at $\sup\{t < R : Z(t) \in S(0, 2\varepsilon)\}$, for $Z = X$ and $Y$. □

**Lemma 8.2.** *Let $D, h, X$ and $Y$ be as in Lemma 8.1. Suppose that $X(0) = x$, $Y(0) = y$, $x = \varepsilon e^{i\alpha}$, $y = \varepsilon e^{i\beta}$, $\alpha, \beta \in [0, 2\pi)$. Let $A_2$ denote the event that $X[0,R) \cup Y[0,R)$ contains a (continuous) curve $\Gamma = \{\Gamma(u), u \in [0,1]\}$ with the following properties.*

$$\Gamma(0) = x, \quad \Gamma(1) = y, \quad \arg\Gamma(0) = \alpha,$$

$$\arg\Gamma(1) = \beta \quad \text{or} \quad \arg\Gamma(1) = \beta + 4\pi.$$

*Here and elsewhere in this section we assume that the suitable version of* arg *is chosen so that it is continuous along continuous curves.*

*We have*

$$\liminf_{\varepsilon \to 0} \log P^{x,y}_{h,h}(A_2^c)/\log \varepsilon \geq \xi(2,1,1)/2.$$

*Proof.* By the rotation invariance of Brownian motion, we may assume that $\beta = 0$. We will discuss only the case $\alpha \in (0, \pi)$, the other cases being capable of similar treatment.

Let $D_1 = \{z \in \mathbb{C} : |z| \in (\sqrt{\varepsilon}, 1)\}$ and let $h_1$ be harmonic in $D_1$ with boundary values 1 on $S(0,1)$ and 0 otherwise. Denote $x_1 = \sqrt{\varepsilon} e^{i\alpha/2}, y_1 = (\sqrt{\varepsilon}, 0)$, and suppose that $(X_1, Y_1)$ has the distribution $P^{x_1,y_1}_{h_1,h_1}$.

If the event

$$A_3 \stackrel{\text{df}}{=} \{X_1[0,R) \cap Y_1[0,R) \neq \emptyset\}$$

holds then $X_1[0,R) \cup Y_1[0,R)$ contains a continuous curve $\Gamma_1 = \{\Gamma_1(u), u \in [0,1]\}$ with $\Gamma_1(0) = x_1, \Gamma_1(1) = y_1$, $\arg\Gamma_1(0) = \alpha/2$, $\arg\Gamma_1(1) = 0$ or $2\pi$.

The mapping $f(z) = z^2$, and a suitable time change, transform $X_1, Y_1$ and $\Gamma_1$ onto processes and a curve with the properties of $X$, $Y$ and $\Gamma$ (under $P^{x,y}_{h,h}$). Thus

$$P^{x,y}_{h,h}(A_2^c) \leq P^{x_1,y_1}_{h_1,h_1}(A_3)$$

and, by Lemma 8.1,

$$\liminf_{\varepsilon \to 0} \log P^{x,y}_{h,h}(A_2^c)/\log \varepsilon \geq \xi(2,1,1)/2. \quad \square$$

**Lemma 8.3.** *Let $D, h, X$ and $Y$ be as in Lemma 8.1. Let $A_4$ denote the event that $X[0,R) \cup Y[0,R)$ contains a closed loop around 0. Then*

$$\liminf_{\varepsilon \to 0} \log P^{x,y}_{h,h}(A_4^c)/\log \varepsilon \geq \xi(2,1,1)/2.$$

*Proof.* Fix some $\xi_0 < \xi(2,1,1)$. For $r \in (0,1)$ and $Z = X$ or $Y$ denote
$T^r_Z = T_Z(S(0, \varepsilon^{1-r}))$,
$L^r_Z = \sup\{t > 0 : |Z(t)| = \varepsilon^{1-r}\}$,
$X_1(t) = X(t), \quad t \in [0, T^r_X)$,
$Y_1(t) = Y(t), \quad t \in [0, T^r_Y)$,
$X_2(t) = X(L^r_X + t), \quad t \in [0, R - L^r_X)$,
$Y_2(t) = Y(L^r_Y + t), \quad t \in [0, R - L^r_Y)$.
Note that, up to Brownian scaling, the processes $(X_1, Y_1)$ and $(X_2, Y_2)$ satisfy the assumptions of Lemmas 8.1 and 8.2. Let

$$A_5(r) = \{X_1[0, T^r_X) \cap Y_1[0, T^r_Y) = \emptyset\}$$

and let $A_6(r)$ denote the event that $X_2[0, R - L^r_X) \cup Y_2[0, R - L^r_Y)$ contains a continuous curve $\Gamma = \{\Gamma(u), u \in [0,1]\}$ such that $\Gamma(0) = X_2(0), \Gamma(1) = Y_2(0)$,
$\arg \Gamma(0) = \arg X_2(0) \in [0, 2\pi)$,
$\arg Y_2(0) \in [0, 2\pi)$,
$\arg \Gamma(1) = \arg Y_2(0)$ or $\arg \Gamma(1) = \arg Y_2(0) + 4\pi$.

We will write $P = P^{x,y}_{h,h}$. By Lemma 8.1,

$$P(A_5(r)) < (\varepsilon^r)^{\xi_0}, \tag{8.1}$$

if $\varepsilon^r$ is small. Lemma 8.2 implies that

$$P(A_6^c(r) \mid X_2(0), Y_2(0)) < \varepsilon^{(1-r)\xi_0/2} \tag{8.2}$$

provided $\varepsilon^{(1-r)}$ is small.

Suppose that $A_5(r)$ does not hold. Then $X[0, L^r_X] \cup Y[0, L^r_Y]$ contains a continuous curve $\Gamma_1 = \{\Gamma_1(u), u \in [0,1]\}$ with $\Gamma_1(0) = X(L^r_X), \Gamma_1(1) = Y(L^r_Y)$. Assume that
$\arg \Gamma_1(0) = \arg X(L^r_X) \in [0, 2\pi)$,
$\arg Y(L^r_Y) \in [0, 2\pi)$,
$\arg \Gamma_1(1) = \arg Y(L^r_Y) + 2\pi$.
The only other possibility i.e. $\arg \Gamma_1(1) = \arg Y(L^r_Y)$, may be handled in a similar way.

If, in addition to $A_5^c(r)$, the event $A_6(r)$ holds then $\Gamma \cup \Gamma_1$ forms a closed loop around 0.

In view of (8.2), we have by the last exit decomposition at $L^r_X$ and $L^r_Y$,
(8.3)

$$P(A_4^c \mid A_5^c(r), X[0, L^r_X], Y[0, L^r_Y]) \leq P(A_6^c \mid A_5^c(r), X[0, L^r_X], Y[0, L^r_Y]) \leq \varepsilon^{(1-r)\xi_0/2}$$

if $\varepsilon^{1-r}$ is small.

Define a random variable $V$ by declaring that $\{V > r\} = A_5(r)$ for all $r \in (0,1)$. Choose $\varepsilon_0 > 0$ so that, according to Lemma 8.1 and (8.1), $P(A_5(r)) < (\varepsilon^r)^{\xi_0}$

whenever $\varepsilon^r < \varepsilon_0$. Let $r_0 = \log \varepsilon_0 / \log \varepsilon$ so that $\varepsilon^{r_0} = \varepsilon_0$. We have for small $\varepsilon$, by (8.3),

$$\begin{aligned} P(A_4^c \cap \{V \le r_0\}) &= P(A_4^c \mid V \le r_0) P(V \le r_0) \\ &\le P(A_4^c \mid V \le r_0) \\ &= P(A_4^c \mid A_5^c(r_0)) \\ &\le \varepsilon^{(1-r_0)\xi_0/2} \\ &= \varepsilon^{\xi_0/2} \varepsilon_0^{\xi_0/2}. \end{aligned} \tag{8.4}$$

We obtain from (8.1), for small $\varepsilon$,

$$\begin{aligned} P(A_4^c \cap \{V \ge 1/2\}) &\le P(V \ge 1/2) \\ &= P(A_5(1/2)) \\ &\le \varepsilon^{\xi_0/2}. \end{aligned} \tag{8.5}$$

By (8.3), $P(A_4^c \mid V) \le \varepsilon^{(1-r)\xi_0/2}$ on $\{V < r\}$. Since $P(V > r) < \varepsilon^{r\xi_0}$ for $r \ge r_0$ and the function $r \to \varepsilon^{(1-r)\xi_0/2}$ is increasing, we have

$$\begin{aligned} P(A_4^c \cap \{V \in (r_0, 1/2)\}) &\le \int_{r_0}^{1/2} \varepsilon^{(1-r)\xi_0/2} P(V \in dr) \\ &\le \int_{r_0}^{1/2} \varepsilon^{(1-r)\xi_0/2} d(-\varepsilon^{r\xi_0}) \\ &= \int_{r_0}^{1/2} \varepsilon^{(1-r)\xi_0/2} \xi_0 |\log \varepsilon| \varepsilon^{r\xi_0} dr \\ &\le \xi_0 |\log \varepsilon| \varepsilon^{\xi_0/2} \int_0^{1/2} \varepsilon^{r\xi_0/2} dr \\ &= \xi_0 |\log \varepsilon| \varepsilon^{\xi_0/2} \left[ \frac{2}{\xi_0 \log \varepsilon} \varepsilon^{r\xi_0/2} \right] \Big|_0^{1/2} \\ &\le 2\varepsilon^{\xi_0/2}(1 - \varepsilon^{\xi_0/4}). \end{aligned} \tag{8.6}$$

For small $\varepsilon$, we obtain from (8.4)–(8.6)

$$P(A_4^c) \le c\varepsilon^{\xi_0/2}$$

which proves the lemma. □

*Proof of Theorem 1.5 (i).* Let $D = \{z \in \mathbb{C} : |z| < 4\}$ and suppose that $X$ has the distribution $P_D^{x_0}$, where $x_0 = (3, 0)$. Fix some $y, |y| \le 1$, and denote
$D_1 = \{z \in \mathbb{C} : |z - y| \in (\varepsilon, 1)\}$,
$D_2 = \{z \in D : |z - y| > \varepsilon\}$,

$T_1 = T_X(S(y,\varepsilon))$,
$L_1 = \sup\{t < T_1 : X(t) \in S(y,1)\}$,
$T_2 = \inf\{t > T_1 : X(t) \in S(y,1)\}$,
$L_2 = \sup\{t < T_2 : X(t) \in S(y,\varepsilon)\}$,
$X_1(t) = X(T_1 - t), \quad t \in (0, T_1 - L_1)$,
$X_2(t) = X(L_2 + t), \quad t \in (0, T_2 - L_2)$.

Let $h$ and $g$ be harmonic in $D_1$ with boundary values 0 on $S(y,\varepsilon)$ and $h(x) = 1$, $g(x) = G_{D_2}(x, x_0)$ for $x \in S(y,1)$. Here, $G_{D_2}$ stands for the Green function.

Conditional on $\{T_1 < T_X(D^c)\}$, the process $(X_1, X_2)$ has the distribution $P^{\sigma_1,\sigma_2}_{g,h}$ for some $\sigma_1$ and $\sigma_2$. By the Harnack principle, $g$ is bounded away from 0 and $\infty$ on $S(y,1)$, so

$$dP^{\sigma_1,\sigma_2}_{g,h} / dP^{\sigma_1,\sigma_2}_{h,h} < c$$

for some $c < \infty$, independent of $y, |y| \le 1$.

Choose a $\xi_0 < \xi(2,1,1)$ and assume that $T_1 < T_X(D^c)$. By Lemma 8.3, the union of paths of $X_1$ and $X_2$ contains a closed loop around 0 with probability greater than $1 - c\varepsilon^{\xi_0/2}$, for small $\varepsilon$. It follows that the $P^{x_0}_D$-chance that the path of $X$ intersects $S(y,\varepsilon)$ and does not contain a closed loop around $S(y,\varepsilon)$ is less or equal to $c\varepsilon^{\xi_0/2}$, for small $\varepsilon$.

Let $\{S_k\}_{k=1}^N$ be the sequence of all discs $S_k = \{z \in \mathbb{C} : |z - y_k| \le \varepsilon\}$, where $|y_k| \le 1$ and $(2/\varepsilon) y_k \in \mathbb{Z}^2$. Then $N = N(\varepsilon) \le c_1 \varepsilon^{-2}$ for some $c_1 < \infty$ and all $\varepsilon$. Let $N_1(\varepsilon)$ be the number of discs $S_k$ which intersect the path of $X$ but are not encircled by any closed loop contained in this path. Then, for small $\varepsilon$,

$$EN_1(\varepsilon) \le c_1 \varepsilon^{-2} \cdot c\varepsilon^{\xi_0/2} = c_2 \varepsilon^{-2+\xi_0/2},$$

and, for $\gamma > 0$,

$$P(N_1(\varepsilon) > c_2 \varepsilon^{-2+\xi_0/2-\gamma}) \le c_2 \varepsilon^{-2+\xi_0/2} / c_2 \varepsilon^{-2+\xi_0/2-\gamma} = \varepsilon^\gamma.$$

Denote

$$A_7(m) = \{N_1(2^{-m}) > c_2 (2^{-m})^{-2+\xi_0/2-\gamma}\}.$$

Then

$$\sum_{m=1}^\infty P(A_7(m)) \le \sum_{m=1}^\infty (2^{-m})^\gamma < \infty,$$

so only a finite number of events $A_7(m)$ hold a.s.

Let $F_1$ be the boundary of the unbounded connected component of $\mathbb{C} \backslash X[0,R)$. Let $\{S^1_k\}_{k=1}^{N_2}$ be the subsequence of $\{S_k\}_{k=1}^N$ which consists of all discs $S_k$ which intersect $F_1$. Note that if $S_k \cap F_1 \ne \emptyset$ then $S_k$ intersects the path of $X$ but this path does not contain a closed loop around $S_k$. Hence, $N_2 = N_2(\varepsilon) \le N_1(\varepsilon)$.

It follows that, with probability 1, for all $m$ greater than some random $m_0$,

$$N_2(2^{-m}) \le c_2 (2^{-m})^{-2+\xi_0/2-\gamma}.$$

For every $\beta > 0$

$$\sum_{k=1}^{N_2(2^{-m})} (2^{-m})^{-(-2+\xi_0/2-\gamma)+\beta} \le c_2 (2^{-m})^\beta \xrightarrow[m\to\infty]{} 0.$$

This implies that the Hausdorff dimension of $F_2 \stackrel{\text{df}}{=} F_1 \cap \{z \in \mathbb{C} : |z| \leq 1\}$ is less or equal to $2 - \xi_0/2 + \gamma + \beta$, where $\xi_0 < \xi(2,1,1)$, $\gamma > 0$, $\beta > 0$, but otherwise $\xi_0$, $\gamma$ and $\beta$ are arbitrary. Thus, the Hausdorff dimension of $F_2$ is less or equal to $2 - \xi(2,1,1)$ which, by Theorem 1.1, is less or equal to $3/2 - 1/(4\pi^2)$.

Standard arguments may be used to extend this result to $F_1$ and $F$. □

**References.**

[1] L.V. Ahlfors (1973) *Conformal Invariants. Topics in Geometric Function Theory.* McGraw-Hill, New York.

[2] K. Burdzy (1987a) Cut points on Brownian paths. *Ann. Probab.* (to appear).

[3] K. Burdzy (1987b) *Multidimensional Brownian Excursions and Potential Theory.* Longman, Harlow, Essex.

[4] K. Burdzy and G.F. Lawler (1988) Non-intersection exponents for Brownian paths. Part I. Existence and an invariance principle. (preprint).

[5] K. Burdzy, G.F. Lawler and T. Polaski (1988) On the critical exponent for random walk intersections. (preprint).

[6] J.L. Doob (1984) *Classical Potential Theory and Its Probabilistic Counterpart.* Springer, New York.

[7] B. Duplantier and K.-H. Kwon (1988) Conformal invariance and intersections of random walks. *Physical Review Letters* **61**, 2514-2517.

[8] R. Durrett (1984) *Brownian Motion and Martingales in Analysis.* Wadsworth, Belmont, Ca.

[9] A. Dvoretzky, P. Erdös and S. Kakutani (1950) Double points of paths of Brownian motion in $n$-space. *Acta Sci. Math. (Szeged)* **12**, 75–81.

[10] A. Dvoretzky, P. Erdös and S. Kakutani (1961) Nonincrease everywhere of the Brownian motion process. *Proc. 4-th Berkeley Symp. Math. Stat. Prob.* vol. II, 103–116, Univ. of California Press, Berkeley, CA.

[11] K. Itô and H.P. McKean (1974) *Diffusion Processes and Their Sample Paths*, 2nd printing. Springer, New York.

[12] G.F. Lawler (1988) Intersections of random walks with random sets. *Israel J. Math.* (to appear).

[13] B. Maisonneuve (1975) Exit systems. *Ann. Probab.* **3**, 399–411.

[14] B.B. Mandelbrot (1982) *The Fractal Geometry of Nature.* Freeman and Co., New York.

[15] P.A. Meyer, R.T. Smythe and J.B. Walsh (1972) Birth and death of Markov processes. *Proc. 6-th Berkeley Symp. Math. Stat. Prob.* vol. III, 295–305, Univ. of California Press, Berkeley, CA.

[16] Ch. Pommerenke (1975) *Univalent Functions.* Vandenhoeck and Ruprecht, Göttingen.

[17] S.C. Port and C.J. Stone (1978) *Brownian Motion and Classical Potential Theory.* Academic Press, New York.

[18] D. Revuz (1970) Mesures associees aux fonctionelles additives de Markov. I. *Trans. Amer. Math. Soc.* **148**, 501–531.

[19] J. Westwater (1985) On Edwards' model for polymer chains. pp. 385-404, in S. Albeverio and Ph. Blanchard (eds.) *Bielefeld Encounters in Mathematics and Physics IV and V.* World Scientific Publishing Co.

[20] D. Williams (1979) *Diffusions, Markov Processes and Martingales.* vol. I. Wiley, New York.

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