ON NON-INCREASE OF BROWNIAN MOTION

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Abstract. A new proof of the non-increase of Brownian paths is given.

David Aldous pointed out in his recent book (Aldous (1989) K7, K13) that none of the published proofs of the non-increase of Brownian motion is totally satisfactory. The original proof of Dvoretzky et al. (1961) is hard, those of Adelman (1985), Karatzas and Shreve (1987) or Knight (1981) do not shed much light on the difference between points of increase and local maxima. Aldous' own argument is only a "Poisson clumping heuristic." This note is an attempt at the impossible i.e., a proof which is at the same time rigorous, short, simple, elementary, and, last but not least, it elucidates the difference between the points of increase and local maxima. I believe that the proof is new, although its ideas may be traced back to Adelman (1985), Aldous (1989) and Davis (1983).

Suppose that $\{B(t), t \geq 0\}$ is a Brownian motion, $B(0) = 0$. Dvoretzky et al. (1961) showed that with probability 1, there are no $t, \varepsilon > 0$ such that $B(s) \leq B(t)$ for $s \in (t-\varepsilon, t)$ and $B(u) \geq B(t)$ for $u \in (t, t+\varepsilon)$. We will prove that $P(A_0) = 0$ where

$$\begin{aligned} A_0 = \{\exists\, t, u > 0 : t < u,\ B(t) \leq 1,\ B(u) \geq B(t) + 2, \\ B(s) \leq B(t) \quad \text{for } s \in [0,t),\ B(s) \geq B(t) \quad \text{for } s \in (t,u]\}. \end{aligned}$$

Standard arguments show that the result of Dvoretzky et al. (1961) follows from $P(A_0) = 0$. Let us start with some inductive definitions. Fix some $\varepsilon \in (0,1)$.

$$\begin{aligned} M_0 &= 0, \quad U_0 = 0, \\ T_k &= \inf\{t > U_k : B(t) = M_k - \varepsilon \ \text{ or } \ B(t) = M_k + 2\}, \quad k \geq 0, \\ M_{k+1} &= \max\{B(t) : t \in [0, T_k]\}, \quad k \geq 0, \\ U_{k+1} &= \inf\{t > T_k : B(t) = M_{k+1}\}, \quad k \geq 0, \\ X_k &= M_k - M_{k-1}, \quad k \geq 1. \end{aligned}$$

Note that

$$P(X_k > x) = \begin{cases} \varepsilon/(x+\varepsilon) & \text{for } \ 0 < x < 2, \\ 0 & \text{for } \ x \geq 2, \end{cases} \tag{1}$$

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and, consequently,

$$EX_k = \varepsilon(\log(2+\varepsilon) - \log\varepsilon) \geq \varepsilon|\log\varepsilon|. \tag{2}$$

If $X_{k+1} = 2$ then we may say that "an approximate point of increase" occured at the level M_k. Let

$$A_\varepsilon = \{\exists\, k : M_k \leq 1,\ X_{k+1} = 2\}.$$

Since $A_0 \subset A_\varepsilon$ for every $\varepsilon \in (0,1)$, it will suffice to show that $\lim_{\varepsilon\to 0} P(A_\varepsilon) = 0$. Let N be the largest number k such that $M_k \leq 1$ (possibly $N = \infty$).

$$\begin{aligned}
P(A_\varepsilon) &= P(\exists\, k : M_k \leq 1,\ X_{k+1} = 2)\\
&= \sum_{k=0}^{\infty} P(M_k \leq 1,\ X_{k+1} = 2) \qquad \text{(disjoint events)}\\
&= \sum_{k=0}^{\infty} P(M_k \leq 1) P(X_{k+1} = 2) \qquad \text{(independence)}\\
&= \sum_{k=0}^{\infty} P(M_k \leq 1) \frac{\varepsilon}{2+\varepsilon} \qquad \text{(use (1))}\\
&= \frac{\varepsilon}{2+\varepsilon} E(N+1).
\end{aligned}$$

It remains to estimate $E(N+1)$. We have $M_k = \sum_{j=1}^{k} X_j$, the random variables X_j are i.i.d. and positive. By Wald's identity, $EM_{N+1} = E(N+1)EX_1$ (see (4.3) and (4.12), Section 5.4 of Karlin and Taylor (1975)). Thus, by (2),

$$E(N+1) = EM_{N+1}/EX_1 \leq 3/(\varepsilon|\log\varepsilon|).$$

Hence

$$P(A_0) \leq \lim_{\varepsilon\to 0} P(A_\varepsilon) \leq \lim_{\varepsilon\to 0} \frac{\varepsilon}{2+\varepsilon} \cdot \frac{3}{\varepsilon|\log\varepsilon|} = 0$$

and the proof is complete.

Let us attempt to prove in a similar way that the local maxima do not exist. We will try to show that $P(\widetilde{A}_0) = 0$ where

$$\widetilde{A}_0 = \{\exists\, t, u > 0 : t < u,\ B(t) \leq 1,\ B(u) \leq B(t) - 2,\ B(s) \leq B(t) \ \text{ for } \ s \in [0,u]\}.$$

Let $\widetilde{M}_0 = 0, \widetilde{U}_0 = 0$,

$$\widetilde{T}_k = \inf\{t > \widetilde{U}_k : B(t) = \widetilde{M}_k + \varepsilon \ \text{ or } \ B(t) = \widetilde{M}_k - 2\},$$

and define $\widetilde{M}_k, \widetilde{U}_k$, etc. in terms of $\widetilde{T}_k$ in the same way as M_k, U_k, etc. were defined relative to T_k.

The proof will fail because $E\widetilde{X}_k \approx \varepsilon$, $E(\widetilde{N}+1) \approx 1/\varepsilon$ and $\lim_{\varepsilon\to 0} P(\widetilde{A}_\varepsilon) \neq 0$.

If an "approximate point of increase" does not occur at the level M_k then there is no such point between levels M_k and M_{k+1}, which are separated by a relatively large distance ($EX_{k+1} \approx \varepsilon|\log\varepsilon|$). In the case of maxima, the non-occurence of the analogous event at the level $\widetilde{M}_k$ does not preclude the occurence of such an event any further than $\widetilde{M}_k + \varepsilon$ (cf. Dvoretzky et al. (1961) Remark 8.2).

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