

**STOCHASTIC DIFFERENTIAL EQUATIONS  
DRIVEN BY STABLE PROCESSES FOR  
WHICH PATHWISE UNIQUENESS FAILS <sup>1</sup>**

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**Abstract.** Let  $Z_t$  be a one-dimensional symmetric stable process of order  $\alpha$  with  $\alpha \in (0, 2)$  and consider the stochastic differential equation

$$dX_t = \phi(X_{t-})dZ_t.$$

For  $\beta < \frac{1}{\alpha} \wedge 1$ , we show there exists a function  $\phi$  that is bounded above and below by positive constants and which is Hölder continuous of order  $\beta$  but for which pathwise uniqueness of the stochastic differential equation does not hold. This result is sharp.

**AMS 2000 Mathematics Subject Classification:** Primary 60H10, Secondary 60G52.

**Key Words:** Stable processes, pathwise uniqueness, stochastic differential equations, time change, crossing estimates

**Short Title:** SDEs driven by stable processes

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<sup>1</sup>Research partially supported by NSF grants DMS-9988496 and DMS-0071486.

# 1 Introduction

Let  $Z_t$  be a one-dimensional symmetric stable process of order  $\alpha$  with  $\alpha \in (0, 2)$ . In this paper we are concerned with whether or not pathwise uniqueness holds for the stochastic differential equation

$$dX_t = \phi(X_{t-})dZ_t. \quad (1.1)$$

In integrated form this can be written as

$$X_t = X_0 + \int_0^t \phi(X_{s-})dZ_s. \quad (1.2)$$

For details concerning the stochastic calculus of processes with jumps, see [M].

It is relatively straightforward, using Picard iteration, to show that if  $\phi$  is Lipschitz, then the solution to (1.1) exists and is pathwise unique. If  $\alpha > 1$ , it was shown in [Bs2] that if  $\phi$  is bounded, has modulus of continuity  $\rho$ , and  $\rho$  satisfies

$$\int_{0+} \frac{1}{\rho(x)^\alpha} dx = \infty, \quad (1.3)$$

then (1.1) admits a strong solution and the solution is pathwise unique. As an example, if  $\phi$  is Hölder continuous of order  $1/\alpha$ , then (1.3) holds. Condition (1.3) is the exact analogue of the Yamada-Watanabe condition for stochastic differential equations driven by a Brownian motion.

Just as in the Brownian case, one can show that condition (1.3) is sharp. That is, if the integral is finite, one can find a continuous function  $\phi$  having  $\rho$  as its modulus of continuity for which pathwise uniqueness for (1.1) does not hold; see [Bs2]. However, just as in the Brownian case, the examples in [Bs2] showing sharpness are a bit unsatisfying:  $\phi$  degenerates to 0 and not only does pathwise uniqueness fail, but one does not have uniqueness in law either. In [Br], for each  $\beta < \frac{1}{2}$ , Barlow constructed examples of nondegenerate (i.e., bounded away from 0 and infinity) functions  $\phi$  that were Hölder continuous of order  $\beta$ , but for which pathwise uniqueness did not hold for the equation

$$dX_t = \phi(X_t)dB_t,$$

driven by a one-dimensional Brownian motion  $B_t$ .

Our main result in this paper is the extension of Barlow's theorem to the stable case. We prove

**Theorem 1.1.** *Let  $\alpha_0 = \frac{1}{\alpha} \wedge 1$ . If  $\beta < \alpha_0$ , there exists  $\phi$  that is bounded above and bounded below by strictly positive finite constants and such that  $\phi$  is Hölder continuous of order  $\beta$ , but for which two distinct solutions to (1.2) exist.*

We see from (1.3) that the result in Theorem 1.1 is sharp as far as Hölder exponents go.

See [Br] for definitions of weak, strict, and strong solutions of SDEs, weak uniqueness and pathwise uniqueness, and for information about the implications between the existence of weak solutions, strong solutions, weak uniqueness and pathwise uniqueness. We just mention here that weak uniqueness and the existence of a strong solution imply pathwise uniqueness. It is well known that when  $\phi$  is bounded between two strictly positive constants, a weak solution to (1.2) exists and its law is unique (cf. Proposition 3.3 of Bass [Bs2]). So Theorem 1.1 implies that no strong solution to (1.2) exists for the  $\phi$  in Theorem 1.1. We do not pursue this here and refer the reader to [Bs2] for further information.

In Section 3 of [Bs2] it is asserted that if  $\alpha < 1$ , there is pathwise uniqueness for (1.2) if  $\phi$  is bounded above and below by positive constants and  $\phi$  is continuous. There is an error in the proof of Proposition 3.2 there – the argument that the strong solution constructed there is adapted is faulty. In fact, in view of Theorem 1.1 of the present paper,  $\phi$  being bounded between two positive constants and only continuous is not sufficient for pathwise uniqueness.

A recent paper by Williams [W] is also concerned with pathwise solutions for SDEs driven by Lévy processes. The paper [W], however, involves the Stratonovich stochastic integral rather than the Itô integral considered here.

Our method owes a great deal to Barlow’s paper [Br], but because we are working with jump processes, there are also significant differences. We give a brief outline of our proof.

For  $\varepsilon > 0$ , we let  $X_t(\varepsilon), Y_t(\varepsilon), Z_t(\varepsilon), Z'_t(\varepsilon)$  be processes such that  $Z(\varepsilon)$  and  $Z'(\varepsilon)$  are independent symmetric stable processes of order  $\alpha$  and

$$\begin{aligned} dX_t(\varepsilon) &= \phi(X_{t-}(\varepsilon))dZ_t(\varepsilon), & X_0(\varepsilon) &= x_0, \\ dY_t(\varepsilon) &= [\phi(X_{t-}(\varepsilon) + Y_{t-}(\varepsilon)) - \phi(X_{t-}(\varepsilon))]dZ_t(\varepsilon) + \varepsilon dZ'_t(\varepsilon), & Y_0(\varepsilon) &= 0. \end{aligned} \tag{1.4}$$

Suppose we can show that as  $\varepsilon \downarrow 0$ , the joint law of  $(X_t(\varepsilon), Y_t(\varepsilon), Z_t(\varepsilon), Z'_t(\varepsilon))$

has a weak limit  $(X_t, Y_t, Z_t, Z'_t)$  where  $Y_t$  is not identically zero. Then

$$\begin{aligned} X_t &= x_0 + \int_0^t \phi(X_{s-}) dZ_s, \\ Y_t &= \int_0^t [\phi(X_{s-} + Y_{s-}) - \phi(X_{s-})] dZ_s, \end{aligned}$$

and so

$$X_t + Y_t = x_0 + \int_0^t \phi(X_{s-} + Y_{s-}) dZ_s.$$

Hence  $X_t$  and  $X_t + Y_t$  are distinct solutions to (1.1) and we have pathwise nonuniqueness.

Let  $T_b^\varepsilon = \inf\{t : |Y_t(\varepsilon)| \geq b\}$ . The main goal is to show that for some  $b \leq 1/2$  the quantity  $\mathbb{E} T_b^\varepsilon$  is bounded uniformly in  $\varepsilon$ . Once we have that, we can argue as in the first part of Barlow's paper [Br] to show that  $Y_t(\varepsilon)$  has a nonzero limit.

For notational convenience we will omit the  $\varepsilon$  from  $Y_t(\varepsilon)$  and  $T_b^\varepsilon$ . Let  $I_k = [2^{-k}, 2^{-k+1}]$  and  $I_k^* = [2^{-k-1}, 2^{-k+2}]$ . Roughly speaking, for some  $b \leq 1/2$ , the strong Markov property tells us that the amount of time  $Y_t$  spends in  $(0, b)$  up to time  $T_b$  is bounded by

$$\begin{aligned} &\sum_k [\text{expected number of crossings by } Y_t \text{ from } I_k \text{ to } (I_k^*)^c \text{ before } T_b] \\ &\quad \times [\text{maximum expected time to exit } I_k^* \text{ from } I_k]. \end{aligned}$$

(See section 5 for details.)

The proof is now reduced to finding estimates for the terms in the last sum. Assuming this is done, we can obtain a similar estimate for the time spent in  $(-b, 0)$ , then we argue that no time is spent at 0, and thus we obtain a uniform bound on  $\mathbb{E} T_b$ . We will now give a few more details of this strategy.

To estimate the expected number of crossings from  $I_k$  to  $(I_k^*)^c$  by  $Y_t$ , we observe that  $Y_t$  is a time change of a symmetric stable process, so this is the same as the expected number of crossings from  $I_k$  to  $(I_k^*)^c$  by a symmetric stable process before time  $T_b$ . We estimate this using a bound for the Green function of a symmetric stable process on an interval.

The expected time for a symmetric stable process  $Z_t$  to exit  $I_k^*$  starting from a point in  $I_k$  is of order  $(2^{-k})^\alpha$ , by scaling. For a constant  $h$ , the expected length of time for  $hZ_t$  to exit  $I_k^*$  starting from  $I_k$  is the same as

the expected length of time for  $Z_t$  to leave  $\frac{1}{h}I_k^*$  starting from  $\frac{1}{h}I_k$ , which is of order  $(2^{-k}/h)^\alpha$ . For  $h$  we want to take  $h = |\phi(x+y) - \phi(x)|$ , because

$$dY_t = [\phi(X_{t-} + Y_{t-}) - \phi(X_{t-})]dZ_t.$$

To complete the argument, we would like to apply the above estimates with large  $h$ , but we cannot construct  $\phi$  so that  $|\phi(x+y) - \phi(x)|$  is large for all  $x$  and  $y$ . We can, however, construct it so this expression is large enough for many  $x$ 's, and that turns out to be good enough.

In Section 2 we construct  $\phi$ , while in Section 3 we estimate the number of crossings from the set  $I_k$  to the complement of  $I_k^*$ . Section 4 is where the estimate on the expected time for  $Y_t$  to leave an interval is given, and all the parts of the proof are put together in Section 5.

We use the letter  $c$  with subscripts to denote strictly positive finite constants whose exact value is unimportant. For a process  $V_t$  that is right continuous with left limits, we denote the left limit at  $t$  by  $V_{t-}$  and the jump at time  $t$  by  $\Delta V_t$ .

## 2 Constructing $\phi$

Fix any  $\gamma \in (0, 1)$ . Let  $\bar{\psi}$  be the piecewise linear function on  $[0, 1]$  such that  $\bar{\psi}(0) = \bar{\psi}(1) = 0$  and  $\bar{\psi}(\frac{1}{2}) = 1$ ; that is,

$$\bar{\psi}(x) = \begin{cases} 2x & 0 \leq x \leq \frac{1}{2}, \\ 2 - 2x & \frac{1}{2} \leq x \leq 1. \end{cases}$$

Define  $\psi_0 : \mathbb{R} \rightarrow [0, 1]$  by  $\psi_0(x) = \bar{\psi}(x - [x])$ , where  $[x]$  is the integer part of  $x$ . Note that  $\psi_0$  is periodic with period 1 and agrees with  $\bar{\psi}$  on  $[0, 1]$ .

Set

$$\psi_n(x) = \psi_0(2^n x) \quad \text{and} \quad \phi(x) = 1 + \sum_{n=0}^{\infty} 2^{-\gamma n} \psi_n(x).$$

Note that the function  $\phi$  is bounded and bounded away from 0 because  $\psi_n(x) \geq 0$  and  $\sum_{n=0}^{\infty} 2^{-\gamma n} < \infty$ .

The family of functions which are Hölder continuous with exponent  $\eta$  will be denoted  $C^\eta$ . We will first show that  $\phi \notin C^{\gamma+\varepsilon}$  for any  $\varepsilon > 0$ . We have

$$\phi(2^{-n-1}) - \phi(0) \geq 2^{-\gamma n} [\psi_n(2^{-n-1}) - \psi_n(0)] = 2^{-n\gamma}. \quad (2.1)$$

Then

$$\frac{|\phi(2^{-n-1}) - \phi(0)|}{(2^{-n-1})^{\gamma+\varepsilon}} \geq 2^{\gamma+\varepsilon+n\varepsilon},$$

which will surpass any positive constant if  $n$  is large enough.

**Proposition 2.1.** *If  $0 < \zeta < \gamma$ , then  $\phi \in C^\zeta$ .*

**Proof.** Since  $\sum_{n=1}^{\infty} 2^{-\gamma n}$  is summable,  $\phi$  is bounded. It is easy to see that  $|\psi_n(x) - \psi_n(y)|/|x - y|^\zeta$  cannot surpass the maximum value of  $\psi_n(z)/z^\zeta$  for  $z \in (0, 2^{-n-1}]$ . Since  $\psi'_n(z)$  equals  $2^{n+1}$  for such  $z$ ,

$$\frac{\psi_n(z)}{z^\zeta} = 2^{n+1} z^{1-\zeta} \leq 2^{(n+1)\zeta}.$$

Therefore

$$|\psi_n(x) - \psi_n(y)| \leq 2^{(n+1)\zeta} |x - y|^\zeta,$$

and then

$$|\phi(x) - \phi(y)| \leq \sum_{n=0}^{\infty} 2^{-\gamma n} 2^{(n+1)\zeta} |x - y|^\zeta \leq c_1 |x - y|^\zeta,$$

since  $\gamma > \zeta$ . □

Let  $|A|$  denote the Lebesgue measure of a Borel set  $A$  in  $\mathbb{R}$ . Let  $I_k^* = [2^{-k-1}, 2^{-k+2}]$  and

$$A_k(\theta) = \{x : |\phi(x+y) - \phi(x)| > \theta 2^{-k\gamma} \text{ for all } y \in I_k^*\}.$$

**Proposition 2.2.** *There are positive constants  $k_0, \theta, L$ , and  $\delta$  such that if  $J$  is an interval of length larger than  $L2^{-k}$ , then  $|J \cap A_k(\theta)| \geq \delta|J|$  for all  $k \geq k_0$ .*

**Proof.** Let  $k \geq 5$ ,  $r = 2^{-k}$ ,  $n = k - 5$ , and  $j_0$  a positive integer to be fixed later on. Since  $\psi_n$  has slope  $2^{n+1}$  on  $[0, 2^{-n-1}]$ , we have for  $y \in I_k^*$  and  $0 \leq x \leq r/16$

$$2^{-\gamma n} \psi_n(x+y) - 2^{-\gamma n} \psi_n(x) = 2^{-\gamma n} 2^{n+1} y \geq 2^{-\gamma n} 2^{n+1} 2^{-k-1} = c_1 r^\gamma, \quad (2.2)$$

where  $c_1 = 2^{-5(1-\gamma)}$ . If  $0 \leq j < n$ , the slope of  $\psi_j$  is positive on  $[0, 2^{-n}]$ , so

$$\psi_j(x+y) - \psi_j(x) \geq 0 \quad (2.3)$$

if  $0 \leq x \leq r/16$  and  $y \in I_k^*$ . Next we see that

$$\left| \sum_{l=n+j_0}^{\infty} 2^{-\gamma l} \psi_l(x) \right| \leq \sum_{l=n+j_0}^{\infty} 2^{-l\gamma} = \frac{2^{-\gamma(n+j_0)}}{1-2^{-\gamma}} = \frac{2^{-\gamma(j_0-5)}}{1-2^{-\gamma}} r^\gamma.$$

Provided  $j_0$  is chosen large enough,

$$\left| \sum_{l=n+j_0}^{\infty} 2^{-\gamma l} \psi_l(x) \right| \leq c_1 r^\gamma / 4. \quad (2.4)$$

The derivative of  $\psi_l$  is bounded by  $2^{l+1}$ , so if  $y \in I_k^*$ ,

$$\begin{aligned} |2^{-\gamma l} \psi_l(x+y) - 2^{-\gamma l} \psi_l(x)| &\leq 2^{-\gamma l+1} |y| \\ &\leq 2^{(1-\gamma)l+3} r^{1-\gamma} r^\gamma \leq 2^{l(1-\gamma)+3} 2^{-k(1-\gamma)} r^\gamma. \end{aligned} \quad (2.5)$$

So if  $j_0$  is chosen to be sufficiently large and  $n > j_0$ ,

$$\begin{aligned} \sum_{l=1}^{n-j_0-1} |2^{-\gamma l} \psi_l(x+y) - 2^{-\gamma l} \psi_l(x)| &\leq \frac{2^{(n-j_0-1)(1-\gamma)+3}}{2^{1-\gamma} - 1} 2^{-k(1-\gamma)} r^\gamma \\ &\leq c_1 r^\gamma / 4. \end{aligned} \quad (2.6)$$

If  $0 \leq \varepsilon \leq \frac{1}{16}$  and  $0 \leq x \leq \varepsilon r$ , then since  $|\psi'_l|$  is bounded by  $2^{l+1}$ ,

$$\begin{aligned} \left| \sum_{l=n+1}^{n+j_0-1} 2^{-\gamma l} \psi_l(x) \right| &\leq \sum_{l=n+1}^{n+j_0-1} 2^{l(1-\gamma)+1} x \\ &= \frac{2^{(n+j_0)(1-\gamma)+1} - 2^{(n+1)(1-\gamma)+1}}{2^{1-\gamma} - 1} x \\ &\leq (2^{1-\gamma} - 1)^{-1} \varepsilon 2^{n(1-\gamma)} 2^{j_0(1-\gamma)+1} r^\gamma r^{1-\gamma} \\ &\leq c_2 \varepsilon 2^{j_0(1-\gamma)} r^\gamma. \end{aligned} \quad (2.7)$$

Choose  $j_0$  so that (2.4) and (2.6) hold, and then choose  $\varepsilon < 1/16$  small so that (2.7) implies

$$\left| \sum_{l=n+1}^{n+j_0-1} 2^{-\gamma l} \psi_l(x) \right| \leq c_1 r^\gamma / 4. \quad (2.8)$$

Since  $2^{-\gamma l} \psi_l(x+y) \geq 0$  for all  $l$ , combining (2.4) and (2.8),

$$\sum_{l=n+1}^{\infty} [2^{-\gamma l} \psi_l(x+y) - 2^{-\gamma l} \psi_l(x)] \geq -c_1 r^\gamma / 2 \quad (2.9)$$

if  $x \in (0, \varepsilon r]$  and  $y \in I_k^*$ . Let

$$\tilde{\phi}(x) = \sum_{l=n-j_0}^{\infty} 2^{-\gamma l} \psi_l(x).$$

We obtain from (2.2), (2.3) and (2.9),

$$\tilde{\phi}(x+y) - \tilde{\phi}(x) \geq c_1 r^\gamma / 2 \quad (2.10)$$

if  $x \in (0, \varepsilon r]$  and  $y \in I_k^*$ .

The function  $\tilde{\phi}$  is periodic with period  $2^{-(n-j_0)}$ . So if  $J$  is an interval of length at least  $2^{-(n-j_0)+1}$ , then

$$|\{x \in J : \tilde{\phi}(x+y) - \tilde{\phi}(x) \geq c_1 r^\gamma / 2 \text{ for all } y \in I_k^*\}| \geq \varepsilon 2^{-j_0-5} |J|. \quad (2.11)$$

Using (2.6), if  $|J| \geq 2^{-(n-j_0)+1}$ , then

$$|\{x \in J : \phi(x+y) - \phi(x) \geq c_1 r^\gamma / 4 \text{ for all } y \in I_k^*\}| \geq \varepsilon 2^{-j_0-5} |J|.$$

This implies the proposition with  $k_0 = j_0 + 6$ ,  $\theta = c_1/4$ ,  $L = 2^{j_0+6}$ , and  $\delta = \varepsilon 2^{-j_0-5}$ .  $\square$



### 3 Expected number of crossings

On a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ , a real-valued stochastic process  $X_t$  is said to be an  $\{\mathcal{F}_t\}$ -adapted 1-dimensional symmetric stable process of order  $\alpha \in (0, 2)$  if for every  $\lambda \in \mathbb{R}$ ,  $t > 0$  and  $s > 0$ ,

$$\mathbb{E} [e^{i\lambda(X_{t+s} - X_s)} \mid \mathcal{F}_s] = e^{-t|\lambda|^\alpha}.$$

In other words, for every  $s > 0$ , process  $t \mapsto X_{t+s} - X_s$  is independent of  $\mathcal{F}_s$  and is a symmetric  $\alpha$ -stable process starting from the origin.

In this section,  $\phi$  is a continuous function on  $\mathbb{R}$  that is bounded between two strictly positive constants.

**Proposition 3.1.** *For each  $\varepsilon > 0$ ,  $x_0, y_0 \in \mathbb{R}$ , there exists a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$  with processes  $X_t, Y_t, Z_t, Z'_t$ , such that  $Z_t$  and  $Z'_t$  are independent 1-dimensional  $\{\mathcal{F}_t\}$ -adapted symmetric stable processes of order  $\alpha$ ,*

$$X_t = x_0 + \int_0^t \phi(X_{s-}) dZ_s, \quad (3.1)$$

and

$$Y_t = y_0 + \int_0^t [\phi(X_{s-} + Y_{s-}) - \phi(X_{s-})] dZ_s + \varepsilon Z'_t. \quad (3.2)$$

**Proof.** Using the substitution  $K_t = X_t + Y_t$ , it is easy to see that the equations (3.1)-(3.2) are equivalent to the following two equations

$$\begin{aligned} X_t &= x_0 + \int_0^t \phi(X_{s-}) dZ_s, & (3.3) \\ K_t &= x_0 + y_0 + \int_0^t \phi(K_{s-}) dZ_s + \varepsilon Z'_t. \end{aligned}$$

The idea of the proof of weak existence for (3.3)-(3.4) is standard; cf. [Bs1], Section 3. We take smooth  $\phi_n$  which converge uniformly to  $\phi$  on compact intervals and find (unique) solutions to

$$\begin{aligned} dX_t^n &= \phi_n(X_{t-}^n) d\tilde{Z}_t, & X_0^n &= x_0, \\ dK_t^n &= \phi_n(K_{t-}^n) d\tilde{Z}_t + \varepsilon d\tilde{Z}'_t, & K_0^n &= x_0 + y_0, \end{aligned}$$

where  $\tilde{Z}_t$  and  $\tilde{Z}'_t$  are independent 1-dimensional symmetric  $\alpha$ -stable processes. It is routine to show tightness and also routine to show that a

weak subsequential limit  $(X_t, K_t, Z, Z')$  of  $(X_t^n, K_t^n, \tilde{Z}, \tilde{Z}')$  satisfies (3.3)-(3.4), where  $\tilde{Z}, \tilde{Z}'$  are independent symmetric  $\alpha$ -stable processes. Then if we take  $Y_t = K_t - X_t$ , we see that  $(X_t, Y_t, Z, Z')$  solves (3.1)-(3.2).  $\square$

**Proposition 3.2.** *Let  $(X_t, Y_t)$  be a weak solution of (3.1)-(3.2). Define*

$$A_t = \int_0^t (|\phi(X_{s-} + Y_{s-}) - \phi(X_{s-})|^\alpha + \varepsilon^\alpha) ds$$

and  $\sigma_t = \inf\{s \geq 0 : A_s > t\}$  for  $t \geq 0$ . Then  $W_t = Y_{\sigma_t}$  is a symmetric  $\alpha$ -stable process starting from  $y_0$ .

**Proof.** The proof is a straightforward modification of arguments used in Proposition 3.1 and Theorem 3.1 of [RW].  $\square$

Recall that

$$I_k = [2^{-k}, 2^{-k+1}], \quad I_k^* = [2^{-k-1}, 2^{-k+2}].$$

Let  $R_1^Y = \inf\{t : Y_t \in I_k\}$ ,

$S_i^Y = \inf\{t > R_i^Y : Y_t \notin I_k^*\}$  and  $R_{i+1}^Y = \inf\{t > S_i^Y : Y_t \in I_k\}$  for  $i \geq 1$ .

Let

$$N_k^Y(t) = \sup\{j : R_j^Y \leq t\},$$

the number of crossings from  $I_k$  to  $(I_k^*)^c$ . Let  $T_b^Y = \inf\{t : |Y_t| \geq b\}$ . Recall that  $W_t = Y_{\sigma_t}$  and define  $R_i^W, S_i^W, N_k^W$ , and  $T_b^W$  analogously, but in terms of  $W$  instead of  $Y$ .

**Proposition 3.3.** *For  $b > 0$ , there exists  $c_1 = c_1(b) > 0$  such that*

$$\mathbb{E} N_k^W(T_b^W) \leq \begin{cases} c_1 2^{k(\alpha-1)} & \alpha > 1, \\ c_1 k & \alpha = 1, \\ c_1 & \alpha < 1. \end{cases} \quad (3.4)$$

**Proof.** We drop the superscripts  $W$  from the notation. Let  $\tau_k$  be the first exit from  $I_k^*$  by  $W_t$ . Since  $W_t \in I_k^*$  when  $R_i < t < S_i$ , by the strong Markov property,

$$\begin{aligned} \mathbb{E} \int_0^{T_b} 1_{I_k^*}(W_s) ds &\geq \sum_{i=1}^{\infty} \mathbb{E}(S_i \wedge T_b - R_i \wedge T_b) \\ &= \sum_{i=1}^{\infty} \mathbb{E} \left[ \mathbb{E}^{W_{R_i \wedge T_b}} S_1; R_i < T_b \right] \\ &\geq \left[ \mathbb{E}(N_k(T_b) - 1) \right] \left[ \inf_{x \in I_k} \mathbb{E}^x \tau_k \right]. \end{aligned} \quad (3.5)$$

Let  $U_k = \inf\{t : |W_t - W_0| \geq 2^{-k-3}\}$ , the time for  $W_t$  to move a distance at least  $2^{-k-3}$ . If  $x \in I_k$ ,  $\mathbb{E}^x \tau_k \geq \mathbb{E}^x U_k = \mathbb{E}^0 U_k$ . By scaling,

$$\mathbb{E}^0 U_k = c_2(2^{-k})^\alpha \mathbb{E}^0 U_0 = c_3 2^{-k\alpha}.$$

Combining with (3.6) we have

$$\mathbb{E} N_k(T_b) \leq 1 + c_3 2^{k\alpha} \mathbb{E} \int_0^{T_b} 1_{I_k^*}(W_s) ds. \quad (3.6)$$

Suppose  $\alpha > 1$ . The Green function for  $W_t$  killed on exiting  $[-b, b]$  is bounded (see Corollary 4 of [BGR]), so

$$\mathbb{E} \int_0^{T_b} 1_{I_k^*}(W_s) ds \leq c_4 |I_k^*| \leq c_5 2^{-k}.$$

If  $\alpha = 1$ , the Green function is bounded by  $c_6 \log(1/|x|)$  (again see [BGR]), and then

$$\begin{aligned} \mathbb{E} \int_0^{T_b} 1_{I_k^*}(W_s) ds &\leq c_6 \int_{I_k^*} \log(1/|x|) dx \\ &= c_6 \int_{2^{-k-1}}^{2^{-k+2}} \log(1/|x|) dx \leq c_7 k 2^{-k}. \end{aligned}$$

Finally, if  $\alpha < 1$ , the Green function is bounded by  $c_8 |x|^{\alpha-1}$ ; see [BGR]. In this case

$$\mathbb{E} \int_0^{T_b} 1_{I_k^*}(W_s) ds \leq c_9 \int_{I_k^*} |x|^{\alpha-1} dx \leq c_{10} 2^{-k\alpha}.$$

If we substitute the appropriate estimate for  $\mathbb{E} \int_0^{T_b} 1_{I_k^*}(W_s) ds$  into (3.7), we obtain the proposition.  $\square$

**Corollary 3.4.** *For  $b > 0$ , there exists  $c_1 = c_1(b) > 0$  such that*

$$\mathbb{E} N_k^Y(T_b^Y) \leq \begin{cases} c_1 2^{k(\alpha-1)} & \alpha > 1, \\ c_1 k & \alpha = 1, \\ c_1 & \alpha < 1. \end{cases} \quad (3.7)$$

**Proof.** This follows from Proposition 3.3 and the fact that  $Y$  is a nondegenerate time change of  $W$  (see Proposition 3.2).  $\square$

## 4 Expected time to leave an interval

Let  $(X_t, Y_t)$  be a weak solution of (3.1)-(3.2). We want an estimate on  $\mathbb{E} \tau_k$ , where  $\tau_k = \inf\{t : Y_t \notin I_k^*\}$  (note that here  $\tau_k$  is defined in terms of  $Y_t$ ). Let  $\alpha_0 = \frac{1}{\alpha} \wedge 1$ , choose any  $\beta < \alpha_0$ , and then fix any  $\gamma \in (\beta, \alpha_0)$ . Construct  $\phi$  as in Section 2, and let  $k_0, \theta, L$ , and  $\delta$  be as in the statement of Proposition 2.2.

Fix  $k \geq k_0$ . For simplicity write  $r$  for  $2^{-k}$  and set  $t_0 = r^{\alpha(1-\gamma)}$ . Recall the definition of  $A_k(\theta)$  from Section 2. Let

$$C_t = \sum_{s \leq t} 1_{\{|\Delta Z_s| > 8\theta^{-1}r^{1-\gamma}\}} 1_{\{X_{s-} \in A_k(\theta)\}}. \quad (4.1)$$

**Lemma 4.1.** *There is a constant  $c_1 > 0$ , independent of  $k \geq k_0$  and such that  $\mathbb{E} C_{t_0} \geq c_1$ .*

**Proof.** Recall that the symmetric  $\alpha$ -stable process  $Z$  has Lévy kernel  $\frac{c(\alpha)}{|z|^{1+\alpha}}$  for some  $c(\alpha) > 0$ ; see [Be], p. 13. The process

$$V_t = \sum_{s \leq t} 1_{\{|\Delta Z_s| > 8\theta^{-1}r^{1-\gamma}\}} \quad (4.2)$$

is a Poisson process with parameter  $c(\alpha)\alpha^{-1}(8\theta^{-1}r^{1-\gamma})^{-\alpha}$  (cf. [Be]). Since  $Z_t$  is an  $\{\mathcal{F}_t\}$ -adapted symmetric  $\alpha$ -stable process, it follows that  $M_t = V_t - c(\alpha)\alpha^{-1}(8\theta^{-1}r^{1-\gamma})^{-\alpha}t$  is a purely discontinuous square integrable martingale

with respect to  $\{\mathcal{F}_t\}$  (note that this filtration is larger than the natural filtration generated by  $M_t$ ). Hence the stochastic integral  $\int_0^t 1_{A_k(\theta)}(X_{s-})dM_s$  is also a square integrable martingale with respect to  $\{\mathcal{F}_t\}$ . It follows that

$$\begin{aligned}\mathbb{E} C_t &= \mathbb{E} \int_0^t 1_{A_k(\theta)}(X_{s-})dV_s \\ &= c_2 \mathbb{E} \int_0^t 1_{A_k(\theta)}(X_{s-})r^{-(1-\gamma)\alpha}ds \\ &= c_2 r^{-\alpha(1-\gamma)} \mathbb{E} \int_0^t 1_{A_k(\theta)}(X_s)ds.\end{aligned}\tag{4.3}$$

In the last equality we used the fact that  $X_{s-} = X_s$  for all but countably many  $s$ 's.

Since  $X_t = x_0 + \int_0^t \phi(X_{s-})dZ_s$  and  $\phi$  is bounded between two positive numbers, by Theorem 3.1 of [RW],  $W_t = X_{\sigma_t}$  is a symmetric  $\alpha$ -stable process starting from  $x_0$ , where

$$\sigma_t = \inf \left\{ s \geq 0 : \int_0^s \phi(X_{u-})^\alpha du > t \right\}.$$

Note that  $d\sigma_t/dt$  is bounded between two positive constants since  $\phi$  is. Therefore, for some  $c_3$  and  $c_4$ ,

$$\begin{aligned}\mathbb{E} \int_0^{t_0} 1_{A_k(\theta)}(X_s)ds &= \mathbb{E} \int_0^{\sigma^{-1}(t_0)} 1_{A_k(\theta)}(W_t) \frac{d\sigma_t}{dt} dt \\ &\geq c_3 \mathbb{E} \int_0^{c_4 t_0} 1_{A_k(\theta)}(W_t) dt \\ &\geq c_3 \mathbb{E} \int_{c_4 t_0/2}^{c_4 t_0} 1_{A_k(\theta)}(W_t) dt,\end{aligned}\tag{4.4}$$

where we used the change of variables  $s = \sigma_t$  in the first line. If  $p_s(x, y)$  is the transition density for a symmetric stable process of order  $\alpha$ , then there exists (see Proposition 3.1 of [K])  $c_5 > 0$  such that

$$p_s(x, y) \geq c_5 t_0^{-1/\alpha} \quad \text{for } s \in [c_4 t_0/2, c_4 t_0] \text{ and } |y-x| \in [-Lt_0^{1/\alpha}, Lt_0^{1/\alpha}].\tag{4.5}$$

Let  $J = [x_0 - Lt_0^{1/\alpha}, x_0 + Lt_0^{1/\alpha}]$ . Putting (4.4) and (4.5) together and using

Proposition 2.2,

$$\begin{aligned}\mathbb{E} \int_0^{t_0} 1_{A_k(\theta)}(X_s) ds &\geq c_6 t_0^{1-\frac{1}{\alpha}} |A_k(\theta) \cap J| \\ &\geq c_7 t_0^{1-\frac{1}{\alpha}} \delta |J| = 2c_7 t_0^{1-\frac{1}{\alpha}} \delta L t_0^{\frac{1}{\alpha}} \\ &\geq c_8 t_0.\end{aligned}$$

Therefore, using (4.3),

$$\mathbb{E} C_{t_0} \geq c_9 t_0 r^{-(1-\gamma)\alpha} = c_9.$$

□

**Proposition 4.2.** *There exists  $c_1 \geq 0$  not depending on  $x_0, y_0 \in \mathbb{R}$ ,  $k \geq k_0$  and  $\varepsilon \in (0, 1)$  such that  $\mathbb{P}(C_{t_0} \geq 1) \geq c_1$ .*

**Proof.** With  $V_t$  defined as in (4.2), we have  $C_t \leq V_t$ , and as  $V_t$  is a Poisson process with parameter  $c_2 r^{-\alpha(1-\gamma)}$ ,

$$\mathbb{E} V_{t_0}^2 = (c_2 t_0 r^{-\alpha(1-\gamma)})^2 + c_2 t_0 r^{-\alpha(1-\gamma)} = c_2^2 + c_2 = c_3.$$

By Lemma 4.1,

$$\begin{aligned}c_4 &\leq \mathbb{E} C_{t_0} = \mathbb{E} [C_{t_0}; C_{t_0} \geq 1] \leq \mathbb{E} [V_{t_0}; C_{t_0} \geq 1] \\ &\leq \left( \mathbb{E} V_{t_0}^2 \right)^{1/2} \left( \mathbb{P}(C_{t_0} \geq 1) \right)^{1/2} = c_3^{1/2} \left( \mathbb{P}(C_{t_0} \geq 1) \right)^{1/2}.\end{aligned}$$

Rearranging yields the result. □

Recall that  $k \geq k_0$  and  $\tau_k = \inf\{t : Y_t \notin I_k^*\}$ .

**Proposition 4.3.** *There exists  $c_1 \geq 0$ , not depending on  $k$  and  $\varepsilon \in (0, 1)$ , such that for every starting point  $(x_0, y_0)$  for  $(X_t, Y_t)$  in (3.1)-(3.2),  $\mathbb{E} \tau_k \leq c_1 (2^{-k})^{\alpha(1-\gamma)}$ .*

**Proof.** Let  $U_s = \inf\{t > s : C_t \geq 1\}$ . Then

$$\begin{aligned} \mathbb{P}(U_0 > mt_0) &\leq \mathbb{P}(U_{(m-1)t_0} > mt_0, U_0 > (m-1)t_0) \\ &= \mathbb{E} [\mathbb{P}(U_{(m-1)t_0} > mt_0 \mid \mathcal{F}_{(m-1)t_0}); U_0 > (m-1)t_0]. \end{aligned} \quad (4.6)$$

The conditional law of  $(X_{t+(m-1)t_0}, Y_{t+(m-1)t_0})$  given  $\mathcal{F}_{(m-1)t_0}$  solves an SDE of the same form as (3.1) and (3.2); cf. [Bs1], proof of Proposition 3.2. This and Proposition 4.2 give

$$\mathbb{P}(C_{mt_0} - C_{(m-1)t_0} \geq 1 \mid \mathcal{F}_{(m-1)t_0}) \geq c_2. \quad (4.7)$$

The inequality (4.7) implies

$$\mathbb{P}(U_{(m-1)t_0} > mt_0 \mid \mathcal{F}_{(m-1)t_0}) \leq 1 - c_2.$$

Substituting this in (4.6),

$$\mathbb{P}(U_0 > mt_0) \leq (1 - c_2)\mathbb{P}(U_0 > (m-1)t_0).$$

Using induction,  $\mathbb{P}(U_0 > mt_0) \leq (1 - c_2)^m$ , and from this it follows that

$$\mathbb{E} U_0 \leq c_3 t_0 = c_4 (2^{-k})^{\alpha(1-\gamma)}. \quad (4.8)$$

Recall that the probability that  $Z$  and  $Z'$  jump at the same time is 0. At time  $U_0$  the process  $C$  has a jump so  $\Delta Z_{U_0} > 8\theta^{-1}r^{1-\gamma}$  and  $X_{U_0-} \in A_k(\theta)$ . Had  $Y_t$  not exited  $I_k^*$  by that time, then  $\phi(X_{U_0-} + Y_{U_0-}) - \phi(X_{U_0-}) > \theta r^\gamma$  (by the definition of  $A_k(\theta)$ ), and therefore  $Y$  would have had a jump of size at least  $(8\theta^{-1}r^{1-\gamma})(\theta r^\gamma) = 8r$ . This would have meant that  $Y_{U_0} \notin I_k^*$ . We have thus shown that  $\tau_k \leq U_0$ . This combined with (4.7) completes the proof.  $\square$

## 5 Pathwise nonuniqueness

It follows from Proposition 3.1 that for each  $i \geq 1$ , there exists a filtered probability space  $(\Omega^{(i)}, \mathcal{F}^{(i)}, \{\mathcal{F}_t^{(i)}\}, \mathbb{P}^{(i)})$  and processes  $X_t^{(i)}, Y_t^{(i)}, Z_t^{(i)}, Z_t^{\prime(i)}$  such that  $Z_t^{(i)}$  and  $Z_t^{\prime(i)}$  are independent  $\{\mathcal{F}_t^{(i)}\}$ -adapted symmetric stable processes of order  $\alpha$ ,

$$X_t^{(i)} = x_0 + \int_0^t \phi(X_{s-}^{(i)}) dZ_s^{(i)}, \quad (5.1)$$

and

$$Y_t^{(i)} = \int_0^t [\phi(X_{s-}^{(i)} + Y_{s-}^{(i)}) - \phi(X_{s-}^{(i)})] dZ_s^{(i)} + \frac{1}{i} Z_t^{(i)}. \quad (5.2)$$

Let  $T_b^i = \inf\{t : |Y_t^{(i)}| \geq b\}$  and define  $N_k^{Y^{(i)}}(t)$  analogously to  $N_k^Y(t)$  in Section 3.

**Proposition 5.1.** *Let  $k_0$  be as in Proposition 2.2 and  $b = 2^{-k_0}$ . If  $k \geq k_0$ , then*

$$\mathbb{E} \int_0^{T_b^i} 1_{I_k}(Y_s^{(i)}) ds \leq c_1 (2^{-k})^{\alpha(1-\gamma)} \mathbb{E} N_k^{Y^{(i)}}(T_b^i), \quad (5.3)$$

where  $c_1$  is independent of  $k$ .

**Proof.** We drop the  $(i)$ 's from the notation. Suppose  $R$  is any finite stopping time. The conditional law of  $(X_t, Y_t)$  given  $\mathcal{F}_R$  is again a solution to

$$\begin{aligned} dX_t &= \phi(X_{t-}) dZ_t, \\ dY_t &= [\phi(X_{t-} + Y_{t-}) - \phi(X_{t-})] dZ_t + \frac{1}{i} dZ_t', \end{aligned} \quad (5.4)$$

starting from  $(X_R, Y_R)$ . So the argument of Section 4 shows that the expected amount of time for  $Y_t$  to leave  $I_k^*$  after time  $R$  is again bounded by  $c_2 (2^{-k})^{\alpha(1-\gamma)}$  (see Proposition 4.3). Let  $R_j = R_j^Y, S_j = S_j^Y$  be defined as in Section 3. Then

$$\begin{aligned} \mathbb{E} \int_0^{T_b} 1_{I_k}(Y_s) ds &\leq \sum_{j=1}^{\infty} \mathbb{E} (S_j \wedge T_b - R_j \wedge T_b) \\ &= \sum_{j=1}^{\infty} \mathbb{E} \left[ \mathbb{E} [(S_j \wedge T_b - R_j \wedge T_b) \mid \mathcal{F}_{R_j \wedge T_b}]; R_j < T_b \right] \\ &\leq \sum_{j=1}^{\infty} c_2 (2^{-k})^{\alpha(1-\gamma)} \mathbb{E} 1_{(R_j < T_b)} \\ &= c_2 (2^{-k})^{\alpha(1-\gamma)} \mathbb{E} \sum_{j=1}^{\infty} 1_{(R_j < T_b)} \\ &= c_2 (2^{-k})^{\alpha(1-\gamma)} \mathbb{E} N_k^Y(T_b). \end{aligned}$$

□



Recall that  $\alpha_0 = \frac{1}{\alpha} \wedge 1$ ,  $\beta < \alpha_0$ ,  $\gamma \in (\beta, \alpha_0)$ , and  $b = 2^{-k_0}$ , where  $k_0$  is given in Proposition 2.2.

**Theorem 5.2.** *There exists  $c_1$  such that  $\mathbb{E} T_b^i \leq c_1$  for all  $i \geq 1$ .*

**Proof.** By Proposition 5.1,

$$\begin{aligned} \mathbb{E} \int_0^{T_b^i} 1_{(0,b)}(Y_s^{(i)}) ds &\leq \sum_{k=1}^{\infty} \mathbb{E} \int_0^{T_b^i} 1_{I_k}(Y_s^{(i)}) ds \\ &\leq \sum_{k=1}^{\infty} c_2 (2^{-k})^{\alpha(1-\gamma)} \mathbb{E} N_k^{Y^{(i)}}(T_b^i). \end{aligned} \quad (5.5)$$

If  $\alpha > 1$ , then by Corollary 3.4, the right hand side is bounded by

$$\sum_{k=1}^{\infty} c_2 (2^{-k})^{\alpha(1-\gamma)} c_3 (2^{-k})^{1-\alpha}.$$

As  $\alpha(1-\gamma) + (1-\alpha) = 1 - \alpha\gamma > 0$ , this is summable, and we have

$$\mathbb{E} \int_0^{T_b^i} 1_{(0,b)}(Y_s^{(i)}) ds \leq c_4.$$

If  $\alpha \leq 1$ , then by Corollary 3.4 the right hand side of (5.5) is bounded by

$$\sum_{k=1}^{\infty} c_2 (2^{-k})^{\alpha(1-\gamma)} c_5 k \quad \text{or} \quad \sum_{k=1}^{\infty} c_2 (2^{-k})^{\alpha(1-\gamma)} c_5.$$

In either case, as  $\gamma < 1$ , we have  $\alpha(1-\gamma) > 0$ , and both series are summable.

The same arguments with only cosmetic changes imply that

$$\mathbb{E} \int_0^{T_b^i} 1_{(-b,0)}(Y_s^{(i)}) ds \leq c_4$$

with  $c_4$  independent of  $i$ . Since the expected amount of time a symmetric stable process of order  $\alpha$  spends at 0 is 0 and  $Y_t^{(i)}$  is a nondegenerate time change of a symmetric stable process, then  $Y_t^{(i)}$  spends 0 time at 0. That is,

$$\mathbb{E} \int_0^{T_b^i} 1_{\{0\}}(Y_s^{(i)}) ds = 0.$$

Combining, we have our theorem.  $\square$

**Proof of Theorem 1.1.** It is routine ([Bs1], Section 3) to see that the quadruples of processes  $(X_t^{(i)}, Y_t^{(i)}, Z_t^{(i)}, Z_t^{\prime(i)})$  are tight and any subsequential limit point  $(X_t, Y_t, Z_t, Z_t')$  under weak convergence will satisfy (3.1)-(3.2) with  $y_0 = 0$  and  $\varepsilon = 0$  there. By Theorem 5.2,  $\mathbb{E}T_b^i \leq c_1$ . We have that  $X_t$  satisfies (1.2) and so does  $X_t + Y_t$ . We have for  $t_1 > 0$ ,

$$\begin{aligned} \mathbb{P}\left(\sup_{s \leq t_1} |Y_s| \leq b\right) &\leq \limsup_i \mathbb{P}\left(\sup_{s \leq t_1} |Y_s^{(i)}| \leq b\right) \\ &\leq \limsup_i \mathbb{P}(T_b^i \geq t_1) \leq \limsup_i \frac{\mathbb{E}T_b^i}{t_1}. \end{aligned}$$

If we set  $t_1 = 2c_1$ , then the right side is less than  $\frac{1}{2}$ , which proves that with probability at least  $\frac{1}{2}$ , we have  $\sup_{s \leq t_1} |Y_s| \geq b$ . Therefore our two solutions  $X_t$  and  $X_t + Y_t$  are not identically equal and pathwise uniqueness fails.  $\square$

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