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Optimal Bidding Strategies In Sponsored Search Advertising Auctions

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A dissertation submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy

University of Washington

2013

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Program Authorized to Offer Degree:
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Abstract

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In this dissertation, I model generalized second price (GSP) auction for keyword search to analyze the optimal bidding strategies of the participating advertisers. The results also apply to a more general setting where multiple goods are being auctioned off. The study in chapter 3 examines the bidding strategies of the advertisers in a complete information static GSP auction. The results show that unlike in standard second price auction, truthful bidding is never a dominant strategy in general second price auction. In chapter 4, I have developed a model of static incomplete information GSP auction. I characterize all possible pure strategy Bayes–Nash equilibrium of the game and show that the consideration of the click through rates ratio plays a key role in determining the equilibrium bidding strategies for the advertisers. Specifically, I find that when the click through rates ratio exceeds a critical value, there will be no pure strategy Bayes-Nash equilibrium. The analysis also reveals that in a game of static incomplete information no asymmetric bidding equilibrium would prevail. The study in chapter 5 analyzes a model of incomplete information dynamic GSP auction. I find that in a dynamic game, the existence of both separating strategy equilibrium and pooling strategy equilibrium would depend upon critical values of click through rates ratio. I also prove that the advertisers with high valuation for a keyword will either reveal their identities at the very beginning or at the very end of

this dynamic game. The results also show that when search engines do not publish the bidding history (i.e. there is ‘minimum disclosure of information’), the advertisers will never try to mimic each other or in other words, there will be no pooling strategy equilibrium.

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ACKNOWLEDGMENTS

First of all, I would like to thank my advisory committee members - Prof. Jeffrey Shulman, Prof. Fabio Caldieraro and Prof. Gary Erickson. Throughout my graduate studenthood, all three of them had guided me, encouraged me to pursue my ideas and provided detailed comments and advice at every stage. Without their extraordinary enthusiasm, this project might not have taken off. My gratitude also goes to Prof. Arvind Tripathi, Prof. Douglas MacLachlan and Prof. Shailendra Pratap Jain. They took special interest in my research and were extremely helpful throughout. I can never thank Prof. Sudipta Sarangi enough for providing me the utmost support during my initial years in the doctoral program. I am also indebted to Prof. Debabrata Dutta and Prof. Krishnendu Ghosh Dastidar for introducing me to the wonderful world of game theory and analytical modeling.

My parents, Prakasendu Chatterjee and Anasuya Chatterjee, and my sister, Debasree Chatterjee had been consistently supportive. This research would not have been possible without their unconditional love and encouragement. My uncles, late Pradyot Kumar Chatterjee and Biswadev Mukhopadhyay, gave selflessly and I can not ever repay what I owe them. Lastly, I want to thank my wife, Chitrani Das, for being there; she has been my primary source of encouragement and without her support I couldn't possibly complete this project. The truth is I can not thank her enough for standing by me in the most demanding times, constantly motivating me throughout the process.

DEDICATION

To Jethu, my beloved uncle

Chapter 1

INTRODUCTION

Sponsored search advertising is a multi-billion dollar industry. In 2012 the search engine industry was expected to grow to 26.5 billion dollars in revenue (Alegria et al. 2012). In this industry, companies pay to have their advertisement appear in a search engine landing page when a consumer searches for a particular keyword. Currently, Google is the largest platform for sponsored search advertising with almost 79 percent market share (Travlos 2012). Google sells its keywords in a generalized second price auction. In a generalized second price auction, advertisers bid on keywords and are allocated positions according to their bid. Positions vary in quality in terms of click through rates. A study conducted by 'First Rate', a Google adwords certified partner showed that in 2010 users clicked on the top link 17 percent of the time while the second spot received 13 percent of the clicks (Sandberg 2012). But another study conducted by Compete.com in 2011 shows that for a different set of keywords, the click through rate (CTR) at the top position was 59 percent while the second position had a CTR of only 15 percent (Andrews 2012). Thus getting a higher position, in most cases, means attracting more viewers and clicks but we can not make a definitive statement regarding the difference in click through rates as it varies by industry or keyword. Additionally, the value of getting the top position itself is unclear. While top position has a higher click through rate meaning more customers, it also comes at a higher price as a consequence.

Practitioners vary in their beliefs about the importance of outbidding other advertiser

for the top position. Gord Hotchkiss (President, Enquiro Search Solutions) advocated for getting the top spot because “Top sponsored ads received 2 to 3 times the click through rates compared to others”¹; whereas Brandt Dainow (CEO, ThinkMetrics) said “You can start saving money by not bidding on the top spot.” (Dainow 2007). This research aims to provide guidance on the conditions under which a company should bid for the top position and when a company should bid below their valuation to get a lower position.

Major search engines like Google, Bing etc. do not provide bidding history to advertisers participating in its generalized second price auction². In contrast, product auction sites such as eBay³ and Sam’s club (Goes et al. 2012) often provide bidding history to current bidders. Advertisers have expressed interest in having bidding history available in keyword auctions. Pardoe and Stone (2010) argued, “*knowing the bids of other advertisers for a specific keyword would have helped an advertiser to predict the ad position and cost per click associated with any click.*” This research shows for which types of advertisers the bidding history will be valuable and for which types of advertisers the bidding history will be irrelevant in decision making.

Prior literature provides guidance to as to how companies should bid in auctions. In a simple second price auction, all bidders should bid truthfully. In other words, all bidders should bid their valuations for the item up for bid (Vickrey 1961). However, in a static complete information generalized second price auction, truthful bidding is not the dominant strategy (Edelman et al. 2007) as the advertisers can maximize their

¹<http://www.webpronews.com/bidding-on-top-ad-spot-a-waste-of-money-2007-01>; accessed on 26th January, 2013

²<http://community.bingads.microsoft.com/ads/en/bingads/f/8/t/74897.aspx>; accessed on 26th January, 2013

³<http://pages.ebay.com/help/buy/bidding-overview.html>; accessed on 26th January, 2013

payoff even without getting the top position. This dissertation aims to show when an advertiser should bid truthfully and when an advertiser should shade the bid in both static and dynamic generalized second price auction with incomplete information.

In practice, most of the online advertisers participate over multiple periods of time and bid on the same keyword or keyphrase in every period. Even though they develop a strategy to mitigate the single period trade-off between getting top position and paying high cost per click, it does not help them in a dynamic setting. There is certainly a trade-off in the dynamic setup as well and most of the advertisers overlook it. Conventional wisdom suggests that as the advertisers rarely change their bids over the time, their bidding behavior remain inefficient. The industry insight reveals *“Given the limited time available to many small business advertisers to test different bid levels, they often lock in on a bid that generates profits and seldom change it. However, never changing bids makes it easy for competitors to push an advertiser into overpaying for clicks.”*¹¹ While the process of learning is certainly important in a dynamic game, my analysis shows that there has to be enough incentives for the advertisers to be engaged in this learning process. In other words, addressing questions like *when to learn* or *whether to learn at all* is as important as developing a dynamic bidding strategy solely based on learning. If the competing advertisers reveal their true identities early in the game, there would be a significant amount of learning at a higher cost as the advertisers would start bidding aggressively. On the other hand, if the advertisers do not reveal their true identities, each advertiser does not get the information to update his belief about competitors’ valuations but has a low expected cost per click. In this dissertation I have developed a dynamic model of incomplete information GSP auction. The model helps us to understand how

¹¹<http://www.searchenginepeople.com/blog/ppc-bid-tweaking.html>

the forward-looking advertisers deal with the trade-off points in dynamic sponsored search advertising auctions.

The theoretical objective of this dissertation is three-fold – (a) to derive the optimal bidding strategies in a static as well as in a dynamic game of incomplete information and (b) to examine the feasibility of asymmetric bidding strategies, (c) to understand the impact of *minimum disclosure of information* (i.e. the search engines do not disclose the bid amounts placed in the previous periods) on advertisers' bidding strategies. The existing literature recognizes that in the keyword search GSP auctions multiple equilibria might exist. My model provides a detailed analytical framework which identifies the conditions under which equilibrium bids would exist. The specific research questions that we address are - (a) under what conditions would the advertisers bid truthfully in static as well as dynamic GSP auction?, (b) can advertisers with similar valuations for the keyword bid differently? and (c) how does the availability of bidding history affect the equilibrium bidding strategies?

For the advertisers, the answer to question (a) demonstrates the incentives for adopting a forward-looking bidding behavior. Similarly, the findings from the research question (b) show that the advertisers with same valuations can maximize their expected payoffs by adopting different bidding paths, they do not need to depend on Google to break a tie. For search engines, the answer to question (c) demonstrates that the decision to withhold bidding information will not affect the bidding behavior of the forward-looking advertisers.

From question (a) the advertisers would understand the incentives for adopting a forward-looking bidding behavior. Answer to the question (b) explains why the bid shading amount of two advertisers (of same type) might differ. The result establishes

the fact that the advertisers with same valuations can have significantly different bidding behavior. Lastly, the question *(c)* answers whether an advertiser needs to have access to the bidding history or not.

Chapter 2 gives an overview of the related literature. In this chapter, I discuss the theoretical literature on standard first price auction, standard second price auction, generalized first price auction and generalized second price auction. I also explain how the existing literature addresses key issues like bid shading, asymmetric bidding, direction of bids over time and bidding under minimum disclosure of information. Lastly, I review the literature on separating and pooling equilibrium.

In chapters 3, 4 and 5, I develop three interrelated game theoretic models of sponsored search advertising auction – *(a)* a static model of complete information GSP auction, *(b)* a static model of incomplete information GSP auction and *(c)* a dynamic model of incomplete information GSP auction. In chapter 6, I discuss the contribution of this dissertation and the future direction of research on sponsored search advertising auctions.

Chapter 2

LITERATURE REVIEW

2.1 Static First Price Auction

In a standard first price auction, bidders submit sealed bids and the bidder who places the highest bid wins the object; winning bidder pays the amount he bids. The basic model of independent private value auctions was first introduced by Vickrey (Vickrey 1961). Typically, in a first price auction, non-truthful bidding is the dominant equilibrium strategy. Vickrey assumed that the bidders are risk neutral. Almost twenty years later, Charles Holt (Holt 1980) showed that the risk averse bidders would bid more aggressively and the equilibrium bid amount would be much higher. Several other papers including Harris and Raviv (1981), Riley and Samuelson (1981), Maskin and Riley (1984) confirmed this result. This also implies that when buyers are risk averse, a seller will strictly prefer a first price auction over a second price auction. The independent private value auction model also confirms that there is a strategic equivalence between a first price auction and a dutch auction (this is also known as descending price auction). Economists have also extensively studied the common value first price auctions. Wilson (1969) first considered the possibility of overbidding, which is known as ‘winner’s curse’, in the common value first price auction. It has also been established that in the first price auction, efficiency can not be ensured i.e. a bidder with low valuation may win the auctioned object under certain parametric conditions.

2.2 Static Second Price Auction

In a second price auction, the winning bidder pays the bid amount of his closest competitor. A second price auction with risk neutral bidders is in fact known as ‘Vickrey auction’. In a second price auction and in an ascending price English auction the optimal strategies are the same when bidders have independent and private valuations (Milgrom and Weber 1982). The authors also proved that in presence of more than two bidders, this notion of strategic equivalence breaks down. Unlike in the first price auction, the dominant strategy in the second price auction is to bid truthfully (Vickrey 1961). Efficiency also prevails in the second price auction as the bidder with the highest valuation wins the auction. Milgrom and Weber (1982) also noted that in presence of a reservation price, the average expected price in second price auction would be higher than the average expected price in the first price auction. However, in general the revenues expected from the second price auction and the first price auction are same – this result is formally known as *revenue equivalence theorem*. While Vickrey (1961) first came up with the notion of revenue equivalence, Riley and Samuelson (1981) proved it in a more general setting. It was also observed (Krishna 2002) that until the internet based auction websites came into business, the Vickrey auction model was more of a theoretical possibility (excluding the stamp auctions hosted since nineteenth century).

2.3 Static Generalized Second Price Auction

Current theoretical literature on keyword search auction mainly deals with the static games of complete information where players know each other’s valuations. Difficulties in identifying the multiple equilibria (as described by the so-called ‘Folk theorem’)

and providing closed-form analytical solutions are always the biggest challenges for researchers working on incomplete information keyword auctions. Nevertheless, these existing analyses yield valuable insights. The first two papers which initiated this new line of research are Edelman, Ostrovsky, and Schwarz (2007) and Varian (2007). Edelman et al. (2007) paper clearly described the rules of the GSP auction. Their model uses these rules (along with some basic assumptions such as risk-neutrality of the advertisers) to build two different models that analyze keyword auctions. First they analyzed a static full information simultaneous GSP auction and then a game of incomplete information *generalized English auction*. The paper assumes that the bidders know each other's per-click valuations in the static full information simultaneous move game. In this static set-up the paper explained why the GSP mechanism leads to the more complicated bidding strategies compared to the traditional auction mechanism known as 'Vickrey-Clarke-Groves' (VCG) mechanism. Essentially, lack of a dominant strategy equilibrium in GSP (compared to the truth-telling equilibrium in VCG) makes the bidding process more challenging. A GSP mechanism even in a static set-up leads to the multiplicity of equilibrium and only one of those equilibria can give the same payment and position offered by the VCG mechanism. The authors named this particular equilibrium as 'Locally envy-free equilibrium' where advertisers do not have incentive to move up or down from their current positions. Edelman et al. (2007) used another framework named *Generalized English auction* which corresponds to GSP. Analyzing 'Generalized English Auction' also helped them to justify the assumption that the bidders reach a long run steady state and become locally envy-free. The analysis of the generalized English auction shows that the equilibria in dominant strategies do not exist and the bid of one advertiser depends on the bid of the other players. However, the analysis of *generalized English auction* does not

fully capture the intricacies of the incomplete information GSP auction. My results explain why analyzing a generalized English auction might not help us to understand the optimal bidding strategies in GSP mechanism. Equilibrium bid functions in generalized English auction do not depend upon bidders' beliefs about each other's types. However, my results show that, unlike in generalized English auctions, the optimal bids in GSP auctions are functions of bidders' beliefs. Later I will show how a full and formal analysis of the incomplete information GSP auction gives us new insights.

In the context of a one period simultaneous game, Varian (2007) found results similar to that of Edelman et al. (2007). Using a more general *position auction* theory (which can be applicable to anything being auctioned off at multiple positions) Varian established the Nash equilibrium bids when the advertisers are fully aware of their competitors' per-click valuations. Varian's (2007) analysis of the incomplete information game was not formal and thus the results are not definitive. The author remarked that a Bayes-Nash equilibrium of a position auction is a straightforward generalization of Bayes-Nash equilibrium of a simple auction. However, in this analysis I find that the Bayes-Nash equilibrium of a position auction is not a straightforward generalization of the Bayes-Nash equilibrium from a simple second price auction – my model suggests that in incomplete information GSP auctions, only low type bidders bid truthfully (whereas in a simple second price auction all bidders, regardless of their valuations, bid truthfully).

Another recent paper Katona and Sarvary (2010) claimed that the equilibrium bidding behavior in sponsored search advertising is significantly influenced by the difference in click through rates. The paper suggests that in a dynamic game set-up, the forward-looking bidding strategy would give the advertisers an edge. The research showed that

in a competition between a forward-looking bidder and a myopic bidder, the forward-looking bidder would have the winning advantage. The paper further observed that the sites with similar valuations might submit significantly different bids. My model shows that in a static game of incomplete information the advertisers with similar valuations might submit significantly different bids.

More recently Gomes and Sweeney (2013) provided a formal analysis of optimal bidding strategies in a static incomplete GSP mechanism. The paper observed that the existing literature is yet to give the readers a complete analysis of the Bayes-Nash equilibrium in a static incomplete information GSP auction. The paper claims that in a specific case of GSP auction with two positions, the existence condition of an efficient equilibrium requires the click through rate at the second position to be sufficiently small. Additionally, the authors also mention that the strictly increasing bid function ensures the existence of a Bayes-Nash equilibrium. Contrary to their claims, I find that if the click through rate at the second position is significantly smaller than the click through rate at the first position then there exists no equilibrium in pure strategies. Furthermore I show that a Bayes-Nash equilibrium may not exist even when the bidding function is monotonically increasing in valuation. The differences in results arise because the bidding function in Gomes and Sweeney (2013) is conditionally monotonic whereas my bidding function is consistently monotonic. The conditional monotonicity arises because in Gomes and Sweeney (2013) the optimization problem is a welfare-maximization problem (i.e. joint payoff maximization problem). As a result when the click through rates are close enough, the welfare-maximization problem does not yield any solution. Mathematically, the bidding function in Gomes and Sweeney (2013) ceases to be monotonic and the equilibrium breaks down. In my model the optimal bidding function is always monotonically increasing in valuation

and the equilibrium breaks down only when the difference between the click through rates is significantly large. This large difference between the click through rates forces the advertisers to place an excessively high bid amount which would eventually give them suboptimal expected payoffs. In this case the advertisers would always be better off by deviating to a value below their previously optimal bids.

2.4 Dynamic Generalized Second Price Auction

The literature on the theoretical aspects of the dynamic GSP auctions is still at a nascent stage (Lambert and Shoham, 2007). As a matter of fact, only very recently researchers across disciplines have started studying general dynamic auctions in details. Bergemann and Said (2010) mentioned that much of the auction literature studies one time static decisions. Often the desirable (e.g., truthful bidding) and the efficient outcomes from the static auctions can not be retained in a dynamic setting. Moreover, an analysis of static auction often ignores more realistic aspects like dynamic bidder population (i.e. continuous entry and exit of the bidders in the game), forward looking behavior of the bidders, role of market transparency etc. In the context of market transparency, Bergemann and Horner (2010) showed that in first price auction with minimum disclosure of information there would always be an efficient separating equilibrium. Unlike Bergemann and Horner (2010), our paper shows that in second price auction mechanism with minimum disclosure of information (when bidding information is not available), inefficient pooling equilibria exist. This is precisely because of the fact that in Bergemann and Horner (2010), bidding information is the only signal for the bidders. In a dynamic GSP auction, in spite of the absence of bidding information, the position and payment information would reveal important information to the participating advertisers. So, the signaling mechanism does not

necessarily go away if the search engines do not disclose the bidding information; the signal only gets weaker. Additionally, we find that only in the short run the ‘minimum disclosure of information’ brings inefficiency in the system. In the long run (i.e. in a multi-period dynamic game), the ‘minimum disclosure of information’ does not have any effect. Our results are in accordance with Athey and Bagwell (2010) which claimed that generally in the context of dynamic auction a separating equilibrium exists when patience is low (i.e. the discount factor is low) and a pooling equilibrium exists when patience is high. Horner and Jamison (2008) showed that higher patience often leads to pooling equilibrium even in a different setting (first price, common value auction). Apart from confirming this result in the context of a dynamic GSP auction we also prove that when patience is really high, a critical value of the click through rates ratio (as a function of the prior belief) would determine whether a pooling or a separating equilibrium would eventually exist.

2.5 Generalized First Price Auction

The only research paper till date which analytically modeled the bidding dynamics in a generalized first price (GFP) keyword auctions is Zhang and Feng (2011). The authors claimed that in a keyword auction the advertisers may engage in cyclical bid adjustments where the bidding prices first follow a price escalating phase and then a price collapsing phase. This is similar to ‘Edgeworth price cycle’ where undercutting continues until every competitor firm offers the reservation price. Once every firm starts offering the reservation price, one firm will deviate and restore the initial price. Every other firm will follow and the cycle repeats in this fashion. Zhang and Feng (2011) showed that this bid adjustment would occur in both first price and second price keyword auctions. However, Zhang and Feng (2011) modeled a keyword auc-

tion where two advertisers with different valuations for the keyword compete for two positions. Once the high valuation advertiser places a bid which is higher than the valuation of the other advertiser, it would be impossible for the low valuation advertiser to engage in further bid escalation to win the top spot. In their model the low valuation player instead places a bid which is incrementally less than the bid of the high valuation advertiser so that the high valuation advertiser incurs a huge cost. If the valuation difference is not large enough then the high valuation advertiser might find it profitable to bid the reserve price to win the second spot. This cyclical bidding war typically takes place because in their model the number of positions is equal to the number of advertisers. This dissertation implicitly shows that as we increase the number of advertisers (compared to the number of positions), this result loses its validity. If we have two high valuation advertisers and one low valuation advertiser then the low valuation advertiser would always get zero payoff and can not force the high type advertisers to engage in bidding war.

2.6 Truthful Bidding Vs. Bid Shading

I show that in a GSP auction, not all advertisers would bid truthfully – some of the advertisers would shade their bid substantially. The existing literature on auction tells us that in a second price static auction (with single unit demand), the advertisers would always bid truthfully (Vickrey 1961). On the other hand in a dynamic second price auction (with single unit demand in every period), the equilibrium strategy would be to bid the true value in the last period and shade in earlier periods (Krishna 2002). The possibility of truthful bidding in a dynamic game has also been supported by Bergemann and Valimki (2003). In a more generalized setting, Albright (1974) explained how dynamic allocations in terms of highest valuations lead to the most

efficient arrangement. In the context of static complete information GSP auction both (Edelman, Ostrovsky, and Schwarz 2007) and (Varian 2007) showed that the truthful bidding is not a dominant strategy. The authors also prove that even in a GSP auction with complete information, multiple equilibria would exist. Existence of multiple equilibria was also supported by Athey and Nekipelov (2010). I however find that in a static GSP auction with incomplete information, truthful bidding would be a dominant strategy for the bidders with lowest valuation; the higher valuation bidders though would not bid truthfully in a static GSP auction. When I extend my analysis to a dynamic setting I find that depending upon parametric conditions, truthful bidding can be a weakly dominating strategy for even the bidders with high valuations.

2.7 Direction of Bids Over Time

In terms of direction of bids over time my results are in accordance with the results from standard first price and second price auctions. Milgrom and Weber (2000) and Krishna (2002) showed that in first price as well as in second price standard auctions, the bid amounts increase over the time. In fact, in a standard second price sequential auction, the last period dominant strategy is to bid truthfully. The research also showed that the forward-looking advertisers place higher bids over the time. However, bidding truthfully in the last period is not a dominant strategy in my case – this result goes hand in hand with the fact that truthful bidding is not a dominant strategy for the bidders with the lowest valuation in a static game of GSP auction.

2.8 Complete and Minimum Disclosure of Information

Recently the researchers working on dynamic auctions have started analyzing the role of market transparency. An important question in this regard is – can the bidding strategies under complete disclosure of information be different from the bidding strategies under minimum disclosure of information? In the GSP context, complete disclosure of information means access to position, payment and bid information whereas minimum disclosure of information means access to only position and payment information. Dufwenberg and Gneezy (2002) showed that in a sequence of standard first price auctions, the bids tend to be higher under minimum disclosure of information (i.e. when bids from the last periods are not revealed to the bidders). Gershkov (2009) showed that sellers would maximize their revenue by revelation of information to all bidders. I find that the bidding strategies of the forward-looking bidders would remain same under both complete disclosure of information and minimum disclosure of information as long as the GSP auctions take place for more than two periods. However, under minimum disclosure of information myopic bidders would place lower bids (compared to the bids they would place under complete disclosure of information) with higher probability.

2.9 Separating and Pooling Equilibrium

Two important equilibrium concepts that I have used in the dynamic setting are separating equilibrium and pooling equilibrium. Separating equilibrium occurs when bidders with different valuations place different bids and pooling equilibrium occurs when bidders with different valuations place the same bid. Bergemann and Horner (2010) showed that in first price auction with minimum disclosure of information there

would always be an efficient separating equilibrium. Unlike their paper, my research shows that in second price auction mechanism with minimum disclosure of information (when bidding information is not available), inefficient pooling equilibria exist. This is precisely because of the fact that in Bergemann and Horner (2010) bidding information is the only signal for the bidders. In a dynamic GSP auction, in spite of the absence of bidding information, the position and payment information would reveal important information to the participating advertisers. So, the signaling mechanism does not necessarily go away if the search engines do not disclose the bidding information; the signal only gets weaker. My results are in accordance with Athey and Bagwell (2010) which shows that generally in the context of dynamic auction a separating equilibrium exists when patience is low (i.e. the discount factor is low) and a pooling equilibrium exists when patience is high. Horner and Jamison (2008) showed that higher patience often leads to pooling equilibrium even in a different setting (first price, common value auction). Apart from confirming this result in the context of a dynamic GSP auction I also prove that when patience is really high, a critical value of the click through rates ratio would determine whether a pooling or a separating equilibrium would eventually exist.

The following table summarizes the contributions of this research in relation to the existing auction (traditional and GSP) literature.

Positioning with respect to the existing auction literature

	Static Game	Dynamic Game
Complete Information	Trivial	Albright (1974), Bergemann and Valimki (2003)
Incomplete Information	Vickrey (1961)	Bergemann and Horner (2010), Gershkov (2009)

Table 2.1: Second Price Auction

	Static Game	Dynamic Game
Complete Information	Edelman, Ostrovsky, and Schwarz (2007), Varian(2007)	
Incomplete Information	Gomes and Sweeney (2013), Athey and Ellison (2011), My research (shows the existence of multiple equilibrium)	My research

Table 2.2: GSP Auction

The above table shows that my research complements some of the existing papers on sponsored search literature as it provides further insights on Bayes–Nash equilibrium in static GSP auction. The results demonstrate that the GSP auction is distinct from a standard second price auction. At the same time, the research shows that truthful bidding might be a weakly dominant strategy for some of the advertisers – this result is quite in accordance with real life experiences of most of the practitioners. Moreover, this dissertation further contributes to the existing literature of sponsored search auction by developing and analyzing a model of dynamic GSP auction.

Chapter 3

STATIC GAME OF COMPLETE INFORMATION

In our model we have three advertisers (X, Y and Z) and two positions (1 and 2) on the search engine landing page. This three bidders–two objects framework has already been used in the traditional auction literature (Bulow and Klemperer 1998). The positions get assigned to the advertisers solely by a generalized second price auction mechanism. The generalized second price mechanism implies that through a simultaneous bidding process the highest bidder will get the first position and pay an amount which is equal to the second highest bidder’s bid amount; similarly the second highest bidder will get the second position and pay an amount equal to the third bidder’s bid amount. In case of a tie, a random draw decides the final position outcome. The following table summarizes all the notations in my model.

Symbols and definitions

$V\{i\}$	Advertiser i ’s valuation for the keyword (it can be either V^H or V^L)
b^i	Advertiser i ’s bid amount
C_j	Click through rates for position j

Table 3.1: Variable Definitions (Static game of complete information)

The advertisers can have either V^H or V^L as their valuation for the keyword ($V^H \geq V^L$). The phrase ‘full information’ here implies that each of the advertisers knows

other advertisers' valuations for the keyword. The respective bids of the advertisers are b^x, b^y and b^z . I also assume that click through rates for two positions are C_1 and C_2 respectively ($C_1 \geq C_2$). Using this framework I analyze the full set of equilibria (i.e. I find out the pure strategy Nash equilibrium bids).

Before I proceed, the following tables describe the ranking and payment rules for some arbitrary b^h, b^l and b^m where $b^h \geq b^m \geq b^l$; where h, m , and l stand for 'high', 'medium' and 'low' respectively.

Ranking rules

b^x	b^y	b^z	R(X)	R(Y)	R(Z)
b^h	b^m	b^l	R_1	R_2	R_3
b^h	b^h	b^m	$0.5(R_1 + R_2)$	$0.5(R_1 + R_2)$	R_3
b^h	b^l	b^l	R_1	$0.5(R_2 + R_3)$	$0.5(R_2 + R_3)$

Table 3.2: Ranking rules in generalized second price auction

Payment rules

b^x	b^y	b^z	P(X)	P(Y)	P(Z)
b^h	b^m	b^l	b^m	b^l	0
b^h	b^h	b^m	b^h	b^m	0
b^h	b^l	b^l	b^l	b^l	0

Table 3.3: Payment rules in generalized second price auction

When all three advertisers place the same bid, each of them can get any position with 0.33 probability; if someone gets the third position, his payoff would be zero. However,

in my analysis I establish tie breaking conditions to ensure a unique position for every bidder. The results show that even when two players have similar valuations, at the equilibrium they can have strictly different positions and thus they can bid differently.

The detailed analysis can be summarized in the following propositions,

Lemma 1. *If all the advertisers have exactly similar valuation for the keyword (either high or low) then the bidding game does not have any equilibrium.*

See appendix A for the complete proof. The analysis shows that if I incorporate the no-tie conditions so that the advertisers can get unique equilibrium positions even when all of them have similar valuations, no equilibrium strategy exists for either of the players. In other words it is not possible for the players to get unique equilibrium positions when all of them have exactly same valuations for the keyword.

Proposition 1. *When all the advertisers do not have exactly similar valuation for the keyword, ‘truth-telling’ is just one of the many equilibrium strategies.*

See appendix A for the complete proof. Regardless of the fact whether most of the advertisers are of high type or low type, truth-telling is just one of the feasible equilibrium strategies. There exist some cases when all the advertisers (irrespective of their type) may choose not to bid truthfully. Further restrictive assumptions like ‘the lowest bidder will always bid truthfully’ may lead to the situation where other advertisers can choose ‘overbidding’ (i.e. bidding more than the valuation) as their equilibrium strategy.

By far I have used a restrictive model as valuations of the keyword for any advertiser can be either V^H or V^L . Let’s analyze a more general situation when the respective valuations of the keyword for the three advertisers are V^x, V^y and V^z . The detailed

analysis in the appendix *A* gives us the following generalized equilibrium strategy profile,

$$b^x \in [V^y - \frac{C_2}{C_1}(V^y - b^z), \infty]$$

$$b^y \in [\max[V^z, b^z], V^x - \frac{C_2}{C_1}(V^x - b^z)]$$

$$b^z \in [0, V^y]$$

These results are in agreement with Varian (2007). The symmetric Nash equilibria analysis in Varian (2007) shows that a player's equilibrium bid is bounded above and below by a convex combination of bid of the player below him and a valuation (either his own or the valuation of the player just above him). In my model I see the same (one minor difference is the top advertiser's bid is not bounded above). Additionally, in my simplified three player model I can show that every advertiser's bid-bounds can be expressed as a function of the bid of the bottom-most advertiser. My results are also in accordance with Edelman et al. (2007) as even in this model I see that in the 'envy-free' situation (i.e. when the advertisers do not want to switch their equilibrium positions) 'truth-telling' is not a dominant strategy and in fact there is no equilibrium in dominant strategies (i.e. there is not a single bid which would dominate all other bids - all three advertisers have multiple equilibrium strategies). However, my model further shows that in a special case (when all the advertisers have same valuations) the GSP action may lead to no equilibrium in bidding strategies.

Lastly, I can get a refined set of equilibria (given below) with the assumption that the low type advertiser would bid truthfully if he is indifferent about all the bids in between 0 and V^y . If the other advertisers can rightly anticipate the value of b^z

(which is V^z in this case), the range of possible equilibrium strategies will be reduced.

$$b^x \in [V^y - \frac{C_2}{C_1}(V^y - V^z), \infty]$$

$$b^y \in (V^z, V^x - \frac{C_2}{C_1}(V^x - V^z)]$$

$$b^z = V^z$$

The static model of complete information game shows that advertisers with same valuations can place significantly different bids i.e. under complete information, advertisers may exhibit asymmetric bidding behavior. The model also proves that as more high type advertisers participate in the GSP auction, the resulting equilibrium bid for the high type advertisers would be higher. When all the advertisers have same valuation for a keyword, no pure strategy equilibrium would exist. While my model confirms the results of Edelman et al. (2007) and Varian (2007), it also contributes to the literature by proving the existence of asymmetric bidding behavior.

Chapter 4

STATIC GAME OF INCOMPLETE INFORMATION

While a full information model gives a number of insights, developing a model of incomplete information keyword auction captures the market reality that the advertisers do not know each other's valuation for a keyword. Typically the participating advertisers in any keyword auction are not sure about their competitors' valuations for the keyword. We model this situation by developing a parsimonious framework where the advertisers can be of high or low type. The modeling assumption allows us to identify the optimal strategies for the bidders. To keep the analysis tractable, we use a model with three players and two positions. However, this specification still captures the economic essence of a GSP auction in a more extended set-up with m players and n positions.

As explained above, in our model the valuations for the keyword are typically private knowledge (all advertisers only know their own valuations) though every advertiser knows that his competitors can either have high valuation (V^H) with probability p or low valuation (V^L) with probability $(1 - p)$. The two parameters, V^H and V^L can in fact be considered as the valuations of the advertisers as well as the different types of advertisement quality. Probability p is a subjective probability but in our model it is the same for every advertiser. We use b^H and b^L to denote the optimal bids of the high type and the low type advertisers. In the general auction context, the specifications as described above were successfully used by previous research such as Zeithammer

(2007), Budish and Takeyama (2001) and Jeitschko (1998). We further assume that the click through rates at the two positions (C_1, C_2) are common knowledge as is also assumed in the incomplete information model of Gomes and Sweeney (2013). The following table summarizes all the notations in our model.

Table 1: Parameter and decision variables

Symbol	Definition
V^i	Advertiser i's valuation for the keyword (it can be either V^H or V^L)
b^i	Advertiser i's bid amount
C_j	Click through rates for position j
p	probability of a given bidder being high type

Table 4.1: Variable Definitions (Static game of incomplete information)

4.1 Symmetric Strategy Equilibrium

As a starting point, I analyze a symmetric game of incomplete information where advertisers of same type place identical bids. In a symmetric game of incomplete information the bidding behavior of any high type (or low type) advertiser can be considered as the bidding behavior of a representative individual for that particular cohort. Consider any symmetric equilibrium in which all high-types bid b^H and all low types bid b^L . I examine whether there is any profitable deviation and identify the values of b^H and b^L that can be sustained in equilibrium. Let p_{jk} represent the probability that the remaining two bidders are of type j and k where $j, k \in \{h, l\}$. A low type bidder's expected utility from adopting b^L given that the high types bid b^H and the low types bid b^L is presented below.

The advertiser's expected payoff from bidding b^L when $V = V^L$ is

$$E(\pi^L | b(L) = b^L) = p_{hh} \cdot 0 + \frac{p_{hl}}{2}(V^L - b^L)C_2 + \frac{p_{ll}}{3}(V^L - b^L)(C_1 + C_2) \quad (4.1)$$

As shown in the appendix A, if $b^L < V^L$ then a low-type bidder can profitably deviate by increasing the bid and consequently the probability of winning the auction. If $b^L > V^L$ then a low-type bidder can profitably deviate by decreasing the bid thereby avoiding the possibility of paying more for a keyword than it is worth to the bidder. Thus, the only potential symmetric bid by low-types is $b^L = V^L$. Note that this result holds true for any $b^H > V^L$. In order to find out the equilibrium bid for the high type advertiser I once again compute the expected payoffs. I proceed in the same fashion as before.

The advertiser's expected payoff from bidding b^H when $V = V^H$ is,

$$\begin{aligned} E(\pi^H | b(H) = b^H) &= \frac{p_{hh}}{3}(V^H - b^H)(C_1 + C_2) + \frac{p_{hl}}{2}(V^H - b^H)C_1 + \frac{p_{hl}}{2}(V^H - b^L)C_2 \\ &+ p_{ll}(V^H - b^L)C_1 \end{aligned} \quad (4.2)$$

For any $b^H > V^H$, a high type advertiser will profitably deviate with a lower bid. However, as shown in appendix B, there exists a range of b^H such that no advertiser can unilaterally deviate to earn higher profit. Thus, there are multiple equilibria. This analysis further shows that the Bayes-Nash equilibrium of GSP auction is not a straightforward generalization of the Bayes-Nash equilibrium of a simple second price auction. We know that in a simple second price auction mechanism advertisers bid truthfully regardless of their type but in case of GSP auction only low type advertisers

bid truthfully. Existing literature also says that the GSP auction is analogous to the generalized English auction. My results further show that unlike in generalized English auction, the bid functions in GSP auction depend on bidders' beliefs. The results are quite important even from a managerial perspective. The search engines would always like to see 'truthful bidding' from every advertiser because that would maximize search engines' revenues. However, after observing the real life practice it has often been speculated that the advertisers generally do not bid truthfully. The question remains whether non-truthful bidding is the optimal strategy for the advertisers irrespective of their valuations for the keyword. My first result throws some light in that direction.

Proposition 2. *In a static incomplete information GSP auction, truthful bidding by a high type advertiser will not occur in equilibrium. A low type advertiser has a unique equilibrium bid, the valuation itself.*

For detailed proof, see appendix *B* – Proof of Proposition 2.

The low type advertiser does not have any other option because deviating downward would give him a zero payoff and deviating upward would lead to negative payoff. However, the high type advertisers have incentive for bid-shading. If the click-through rates at the two positions are close enough, it might be beneficial for a high type advertiser to shade the bid so that his nearest competitor is a low type advertiser. As a result, his gain from a lower cost would outweigh the loss from a lower click through rate. This effect is more prominent when there is a greater likelihood of the competing bidders being low type.

Although Proposition 2 states that truthful bidding by the high type will not occur in pure strategy equilibrium, it is possible that there is no symmetric pure strategy equilibrium. The range of symmetric bids by high types that can be supported in

equilibrium is given by $b^{H1} < b^H < b^{H2}$. However, it is possible that this range does not exist depending on the differences in click-through rates. This result is summarized in the following Proposition.

Proposition 3. *In a symmetric strategy GSP auction with incomplete information the existence of a pure strategy Bayes–Nash equilibrium depends on the click through rates ratio. If the click-through rate of the top spot is less than double the click-through rate of the second spot, then there are multiple pure strategy Bayes–Nash equilibria. If the click-through rate of the second spot is zero, then there is a unique pure strategy Bayes–Nash equilibrium. Otherwise pure strategy Bayes–Nash equilibrium does not exist.*

For detailed proof, please see appendix *C* – Proof of Proposition 3.

The results of the propositions 2 and 3 stem from the interaction of two features of the model: there is variation in click-through rates across positions and consumers are heterogeneous in their appreciation of this difference. To see this, consider the following special cases. If there is no variation in click through rates (i.e., $C_1 = C_2$ or $C_2 = 0$), or no heterogeneity in valuation (i.e., $V^H = V^L$), then the high type advertisers bid truthfully (this is the result from standard second price auction). Thus, the novel predictions of the GSP with incomplete information rely on the presence of both of these realities. To understand the result, consider the literature on new and used durable goods (e.g., Desai and Purohit (1999)) in which there is variation in the appreciation of quality. In both contexts, all players prefer the higher quality offering if at equal prices. However, just as consumers with a lower appreciation of quality buy a used good at a lower price, lower valuation bidders win the lower quality bidding position at a lower price. The lower valuation bidders recognize it will be too

costly to attempt to outbid the high valuation bidders to get the top position. The interesting trade-offs occur for the high valuation bidders.

When the positions offer similar click-through rates, the high-type advertisers view the positions as reasonable substitutes. Bids are thus shaded because if an advertiser loses the bidding for the top position, there is a possibility that the second position will be won. However, if the click-through rates are sufficiently different, there is no pure strategy equilibrium. The reasoning is as follows. If a high-type advertiser undercuts the remaining high types, the equilibrium bid of a high type advertiser increases as benefit of the top position increases. An increase in the equilibrium bid reduces the expected payoffs of a high type advertiser from playing either b^H or $(b^H + \epsilon)$. However, the expected payoff from playing $(b^H - \epsilon)$ is not a function of b^H ; a higher b^H does not adversely affect this expected payoff. When $C_1 > 2C_2$, I observe that within the symmetric equilibrium bid range playing $(b^H - \epsilon)$ as well as $(b^H + \epsilon)$ would always dominate playing b^H . So, within that equilibrium bid range a high type advertiser can always improve his expected payoff by bidding a little less (infinitesimally small) or a little more - as a result, an oscillatory bidding behavior prevails. Consequently, this is not an equilibrium bid in this case. The critical value here is 2 because once the click through rates ratio exceeds this value, the advertisers' perception differences in product quality ceases to exist; every advertiser considers the top spot as the only worthwhile position. As a result the equilibrium payment for a high type advertiser tends to be too high and the expected payoff from the equilibrium bid becomes sub-optimal. The condition $C_1 > 2C_2$ can also be written as $\frac{(C_1 - C_2) + (C_2 - 0)}{2} > \frac{(C_1 + C_2 + 0)}{3}$; the implication is if the average click through rates difference (between two consecutive positions) is greater than the average click through rates across all positions, then the equilibrium does not exist. The following diagrams explain how the high type

advertisers would always perform better by deviating from a potential equilibrium.

Equilibrium in static symmetric GSP auction

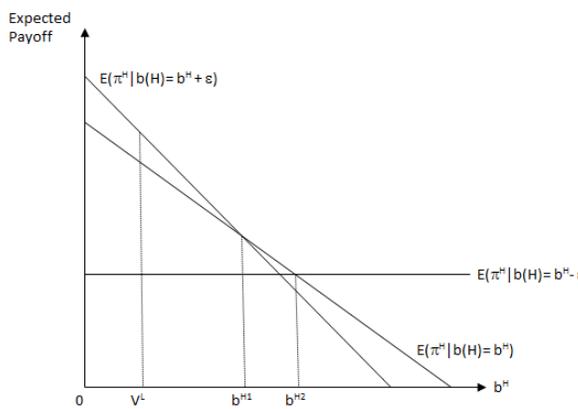


Figure 4.1: Multiple equilibria

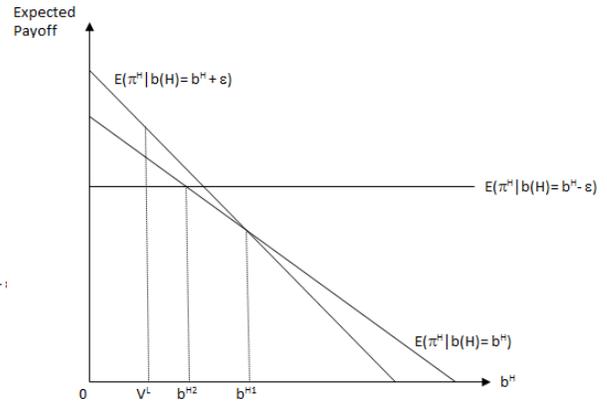


Figure 4.2: No equilibria

In these diagrams I show the expected payoff lines of a high type advertiser from playing three different strategies – (i) bidding b^H , (ii) bidding $(b^H + \epsilon)$ and (iii) bidding $(b^H - \epsilon)$. On the vertical axis I plot the expected payoffs of a high type advertiser and on the horizontal axis I plot a high type advertiser’s bids. When $C_1 > 2C_2$, the expected payoff from playing $(b^H - \epsilon)$ is larger than the expected payoffs from playing either b^H or $(b^H + \epsilon)$ in the equilibrium bid range. In that range a high type advertiser would always gain by bidding $(b^H - \epsilon)$. The stability analysis helps us to clearly identify the cases when GSP auction would not have any pure strategy equilibrium bid for the high type advertisers.

4.2 Can The Advertisers Exhibit Asymmetric Bidding Behavior?

Now, I analyze an asymmetric game of incomplete information where advertisers of same type can adopt different bidding strategies. For instance, one high type advertiser can aim for the top position whereas the second high type can aim for the second position; in such a situation could there ever be an equilibrium? My analysis shows that the advertisers of the same types would adopt identical bidding strategies and the equilibrium outcomes would remain symmetric.

First, I analyze the feasible set of asymmetric strategies for the low type advertisers. Mathematically I can capture all these scenarios just by varying one low type competitor's bids. Let's assume that the initial equilibrium bid for all low type advertisers was b^L . We know that at any point of time one low type advertiser can have at most two low type competitors. Suppose, one of these low type competitors changes his strategy to relatively aggressive bidding and bids b_u^L (where $b_u^L = b^L + \epsilon$); however, the other low type competitor does not deviate from his original bid of b^L . In an incomplete GSP set-up a low type advertiser does not know whether he is competing against the deviating low type or the non-deviating low type or both of them (it is also possible that he is competing against both high type advertisers). In such a situation the low type advertiser has the following five options – (i) do not deviate (bid b^L), (ii) move in-between (bid $b_i^L = b^L + \frac{\epsilon}{2}$), (iii) match (bid $b_u^L = b^L + \epsilon$), (iv) exceed (bid $b_e^L = b^L + 2\epsilon$) and (v) deviate down (bid $b_d^L = b^L - \epsilon$). For each of these strategies of the advertiser i , the strategies of the other two low type competitors are to bid b_u^L and b^L respectively. This explains how the advertiser i can move in between or match or exceed or deviate down in response to the chosen strategies of the other low type advertisers. In order to analytically capture the deviation, I use ϵ which

denotes an infinitesimal change (either increase or decrease) in the bid amount. I also use the notations p_{hh} , p_{hl} and p_{ll} which respectively denote the probabilities of having two high type competitors, one high type and one low type competitor and two low type competitors. Given this information set, I compute the expected payoffs of both types of advertisers.

The strategy ‘do not deviate’ (by bidding b^L) would give the low type advertiser i the following expected payoff ,

$$E(\pi_i^L | b(L_i) = b^L) = p_{hh} \cdot 0 + \frac{p_{hl}}{2} \cdot 0 + \frac{p_{hl}}{4}(V^L - b^L) + p_{ll}(V^L - b^L) \frac{C_2}{2} \quad (4.3)$$

The first term in the expression on the right hand side implies that when the low type advertiser i faces two high type competitors, his expected payoff would be zero. However, when only one of his competitors is low type then with $\frac{1}{2}$ probability the competitor could be a deviating low type or a non-deviating low type advertiser. If the low type competitor is deviating, then advertiser i 's expected payoff would be zero. But if the low type competitor is of non-deviating type then the advertiser i would get the second position with probability $\frac{1}{2}$. Lastly, when both the competitors are low type (and only one of them is deviating), then once again the advertiser i would get the second position with probability $\frac{1}{2}$.

Using similar logic I compute the expected payoffs from other four strategies and compare all five expected payoffs with the baseline expected payoff (when no low type advertiser deviates from b^L); the full expressions are presented in appendix *D*. It can be easily shown that all the six payoffs would be equal only when $b^L = V^L$ (see appendix *D*). This establishes the fact that a low type advertiser would always bid truthfully (i.e. he would bid his own valuation for the keyword).

Similarly, I analyze the feasible set of asymmetric strategies for the high type advertisers. Mathematically I can capture all these scenarios just by varying one high type competitor's bids. Let's assume that the initial equilibrium bid for all high type advertisers was b^H . We know that at any point of time one high type advertiser can have at most two high type competitors. Suppose one of these high type competitors changes his strategy to relatively aggressive bidding and bids b_u^H (where $b_u^H = b^H + \epsilon$); however, the other high type competitor does not deviate from his original bid of b^H . In an incomplete GSP set-up a high type advertiser does not know whether he is competing against the deviating high type or the non-deviating high type or both of them. In such a situation the high type advertiser has the following five options – (i) do not deviate (bid b^H), (ii) move in-between (bid $b_i^H = b^H + \frac{\epsilon}{2}$), (iii) match (bid $b_u^H = b^H + \epsilon$), (iv) exceed (bid $b_e^H = b^H + 2\epsilon$) and (v) deviate down (bid $b_d^H = b^H - \epsilon$).

The strategy ‘do not deviate’ (by bidding b^H) would give the high type advertiser i the following expected payoff,

$$E(\pi_i^H | b(H_i) = b^{H_i}) = p_{hh}(V^H - b^{H_i})\frac{C_2}{2} + p_{hl}((V^H - b^L)\frac{3C_2}{4} + (V^H - b^{H_i})\frac{C_1}{4}) + p_{ll}(V^H - b^L)C_1 \quad (4.4)$$

The first term in the expression on the right side implies that when both the competitors of a high type advertiser (here advertiser i) are also of high type and one of them deviates upward (but the other does not deviate) then the high type advertiser (here advertiser i) would get the second position with probability $\frac{1}{2}$; I already know that the payoff from the second position is $(V^H - b^H)C_2$. However, when only one of his competitors is high type then with probability $\frac{1}{2}$ the competitor could be a deviating high type or a non-deviating high type advertiser. If the high type competitor is

deviating upward, the the advertiser i would certainly get the second position. But if the high type competitor does not deviate then the advertiser i would get the second position with probability $\frac{1}{2}$. Therefore, the total probability with which the advertiser i can get the second position is equal to the sum total of $((\frac{1}{2} \cdot 1) + (\frac{1}{2} \cdot \frac{1}{2}))$, or $\frac{3}{4}$. Lastly, if both the competitors are of low type, then the advertiser i would certainly get the first position.

Using similar logic I compute the expected payoffs from other four strategies and compare all five expected payoffs with the baseline expected payoff (when no high type advertiser deviates from b^H); the full expressions are presented in appendix *D*. However, no single equilibrium b^H can make these six expected payoffs equal. In order to compare these six expected payoffs I use the following graphs which depict the asymmetric bidding strategies of the high type advertisers.

Equilibrium in static asymmetric GSP auction

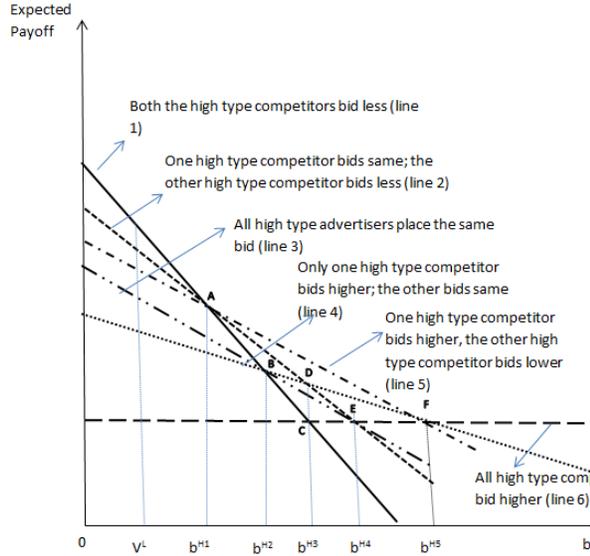


Figure 4.3: Multiple equilibria, all symmetric

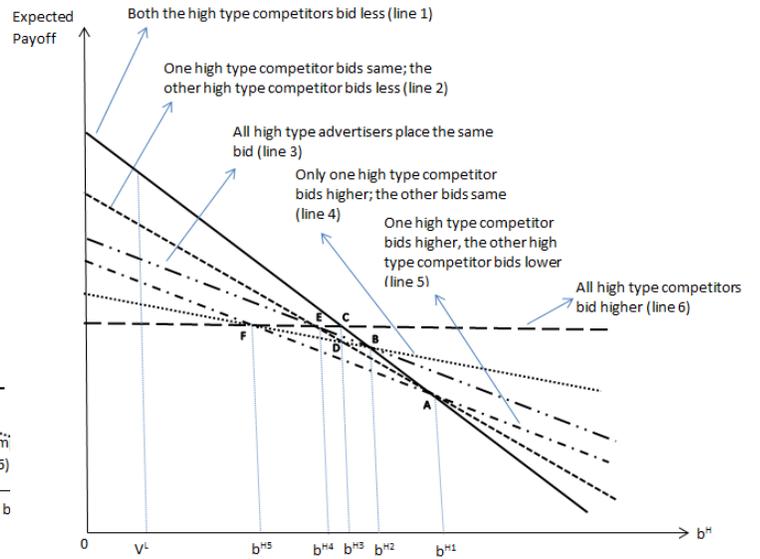


Figure 4.4: No equilibrium

The six reaction functions in the graphs represent the six expected payoffs as mentioned above (for the detailed mathematical properties of these expected payoffs please see the *proof of proposition 4* in appendix *D*). The reaction functions as depicted in these graphs show the relative positions of players in terms of bid. For example, if every player is bidding a certain b^H then the advertisers's expected profits would be derived from reaction function 3; now if one of them deviates upward by bidding b_u^H his expected profit would be determined from the reaction function 1 but the rest two would compute their expected profits using the reaction function 4 (though they are still bidding b^H). If the advertiser deviates downward by bidding b_d^H his expected payoff would be determined from the reaction function 6 but the rest two would

compute their expected payoffs using reaction function 2 (they are still bidding b^H). Whenever one of the advertisers deviate upward or downward, a second advertiser's expected profit would be determined using the reaction function 5. The second advertiser would also be deviating in the same direction but with less magnitude (i.e. in order to enjoy the expected payoff using the reaction function 5, the second advertiser would bid $(b^H + \frac{\epsilon}{2})$ or $(b^H - \frac{\epsilon}{2})$).

The graph in the left panel (figure 4.3) depicts a typical situation when $C_1 < 2C_2$. To see the graph better, I categorize all possible bids in five regions – (a) bids in-between 0 and V^L , (b) bids in between V^L and b^{H1} , (c) bids in between b^{H1} and b^{H2} , (d) bids in between b^{H2} and b^{H4} and (e) bids in between b^{H4} and V^H .

A high type advertiser would not want to bid in between 0 and V^L because he knows with certainty that a low type advertiser would bid V^L . If an advertiser's bid is in between V^L and b^{H1} , he would always want to deviate upward b_u^H as he would get the maximum expected payoff from the reaction function 1. When everybody bids b^{H1} , some advertiser would still deviate upward so that he can still use the reaction function 1. However, his competitors would not want to match his bid. Bidding in the range b^{H1} and b^{H2} would give them maximum expected payoff only when they are using the reaction function 5 (i.e. they would not have any incentive to be either the highest bidder or the lowest bidder). In order to use the reaction function 5 the other two competitors would bid more than b^H but less than $(b^H + \epsilon)$. If the first advertiser deviates by bidding $(b^H + \epsilon)$, the second advertiser would bid $(b^H + \frac{\epsilon}{2})$ and then the third advertiser would retaliate by bidding $(b^H + \frac{2\epsilon}{3})$. While the first advertiser does not have any further incentive for deviation, the second and third advertiser would keep competing until both of them bid $(b^H + \epsilon)$. Once everyone is bidding the same,

one of them would again deviate by bidding ϵ more. This bidding war would continue until everybody bids b^{H2} . Therefore, no bid in between V^L and b^{H1} as well as in between b^{H1} and b^{H2} is an equilibrium bid.

Once all the advertisers are bidding b^{H2} , no player has incentive for further deviation because by further deviating upward the deviating players would have lower expected payoff (compared to the expected payoff at b^{H2}). Also, no player would want to deviate downward, as this would force him to use the reaction function 6 which represents a smaller payoff. This establishes the fact that b^{H2} is indeed an equilibrium bid.

Now let's consider the range in between b^{H4} and V^H . If the advertisers are bidding exactly the same (anywhere in this range) then they would have an incentive for deviating downward (bid b_d^H). Once an advertiser deviates by bidding b_d^H (and uses the reaction function 6), his competitors would want to use the reaction function 5. In order to do so a second advertiser would bid $(b^H - \frac{\epsilon}{2})$, then the third advertiser would retaliate by bidding $(b^H - \frac{2\epsilon}{3})$ and so on. Again, no player has any incentive to be the highest or the lowest bidder. Once everybody bids $(b^H - \epsilon)$, one of the advertiser would further deviate by bidding ϵ less. This bidding war would continue until everybody bids b^{H4} . Therefore, no bid in between b^{H4} and V^H is an equilibrium bid.

Once the advertisers bid b^{H4} , they are indifferent between 'no deviation' and 'deviating downward'. Additionally, if all the advertisers choose an exactly same bid in between b^{H2} and b^{H4} then they would not have any incentive to deviate upward or downward. Thus any bid in this range can be an equilibrium bid. The implication is that even when the advertisers use asymmetric strategies the symmetric equilibria prevail.

The graph in the right panel (figure 4.4) depicts a typical situation when $C_1 > 2C_2$.

The graph shows that this bidding game does not have any equilibrium in pure strategies. For all bids less than b^{H3} the advertisers have incentive to deviate upward. Once an advertiser bids exactly b^{H3} and reach point C on line 1 his competitors would be better off by deviating downward. In fact, even the highest bidder is now indifferent between no deviation and downward deviation. For all bids higher than b^{H3} the advertisers can always do better by deviating downward.

In order to analytically understand the dynamics of this asymmetric GSP auction, I can construct a three player simultaneous move game. In this simultaneous move game all the players are high type advertisers and each one can adopt either of the five strategies discussed earlier – (i) do not deviate (bid b^H), (ii) move in-between (bid $b^H + \frac{\epsilon}{2}$), (iii) match (bid $b^H + \epsilon$), (iv) exceed (bid $b^H + 2\epsilon$) and (v) deviate down (bid $b^H - \epsilon$). An analysis of this three person simultaneous move game (for detailed analysis see *proof of proposition 4*) shows that only pure strategy equilibrium would exist when all the advertisers bid the same which essentially confirms that there would not be any separate asymmetric equilibrium outcome. Whenever at least one of the players use asymmetric bidding strategy, the required profit maximizing condition for the asymmetric strategy player happens to contradict the profit maximizing conditions for the symmetric strategy players. The issue continues to persist even when multiple players use asymmetric bidding strategies. After analyzing the feasible set of position graphs from the three person game I establish the result that when $C_1 < 2C_2$, the low type advertiser would still bid truthfully and the high type advertiser would bid anything in between b^{H2} and b^{H4} (the same bid-bounds from the symmetric strategy game); when $C_1 > 2C_2$, the low type advertiser would continue to bid truthfully but the high type advertisers would not have any equilibrium strategy.

The analysis of the asymmetric strategy incomplete information game leads to the fourth proposition.

Proposition 4. *In an incomplete information GSP auction, both high type and low type advertisers would necessarily use symmetric strategies even when the advertisers with same valuations can place different bids.*

For detailed proof, see appendix *D* – Proof of proposition 4.

4.3 Static Incomplete Information GSP with N Competitors and M Positions

Now I extend this analysis to a more generalized setting. I assume that there are $N + 1$ advertisers (which implies that each advertiser faces N competitors) and M positions. The relationship between N and M has been defined as $j = N - M$. The results are qualitatively similar and we can exactly identify the range of equilibrium b^H . The detailed results have been presented in appendix E. However, the generalized results help us to infer few insightful conclusions as listed below,

Lemma 2. *In a game of static incomplete information the low type advertisers always bid truthfully however the high type advertisers would bid truthfully only under the following conditions,*

(i) when $V^L \rightarrow V^H$, (ii) when $p \rightarrow 1$, (iii) when $C_1 > 0$ but $C_2 = C_3 = \dots = C_M = 0$

The detailed proof has been provided in appendix E. When there is very little difference in the high type's valuation and low type's valuation, every advertiser would bid very aggressively and at the limiting condition truthful bidding would prevail. When the probability of a competitor being high type is 1, the incomplete information game

becomes a complete information game among all high type advertisers; as a result, at the equilibrium everybody would bid truthfully. Lastly, when all positions except the top one give zero click through rate, the GSP auction boils down to a standard second price auction where truthful bidding takes place at the equilibrium. The generalized model also confirms that the existence of equilibrium bidding strategy for the high type advertisers depends upon click through rates and the probability distribution parameters.

Proposition 5. *In a generalized static incomplete information GSP auction, a critical click through rates ratio determines the existence of pure strategy equilibrium. This critical value does not depend on the advertisers' valuations for a keyword and is a function of only the click through rates and the probability distribution parameter.*

For detailed proof, please see appendix *E* – Proof of proposition 5.

In our generalized model the critical click through rates ratio is,

$$C^* = \left(\frac{\sum_{k=j+1}^{N-1} \frac{\binom{N}{N-k} p^{N-k} (1-p)^k C_{(N+1)-k}}{N-k}}{\sum_{k=j+1}^N \frac{\binom{N}{N-k} p^{N-k} (1-p)^k C_{(N+1)-k}}{N+1-k}} \right) \left(\frac{\sum_{i=1}^M C_i \sum_{k=0}^{N-M} \frac{\binom{N}{N-k} p^{N-k} (1-p)^k}{(N+1)-k} + \sum_{k=(N-M)+1}^{N-1} \frac{\binom{N}{N-k} p^{N-k} (1-p)^k \sum_{i=1}^{N-k} C_i}{(N+1)-k}}{\sum_{k=0}^{N-1} \binom{N}{N-k} p^{N-k} (1-p)^k} \right)$$

Whenever the click through rate at the topmost position is lower than this critical value, equilibrium bidding strategies for the high type advertisers exist. When $N = 2, M = 2$ we get $C^* = \frac{2}{3} \sum_{i=1}^2 C_i$ as established earlier.

While analyzing the static GSP auction I observe that as p approaches 1 high type advertisers bid V^H . Varian's (2007) result on GSP auction with complete information becomes a special case of the current model. In general, as the probability of being

a high type advertiser increases, the equilibrium bid of the high type advertisers also increases. I further observe that a positive or negative change in the click through rates do not affect the equilibrium bid of a low type advertiser. However, as the click through rates at both the positions increase so does the equilibrium bid of a high type advertiser. I have already observed that the ratio of the two click through rates has a significant influence on the stability of the equilibrium bids. The comparative static findings show that the ratio $\frac{c_1}{c_2}$ has a monotonic positive relationship with b^H . Lastly, I observe that as a low type advertiser's valuation for a keyword increases both types of advertisers bid more aggressively. More specifically, when a low type advertiser's valuation for the keyword tends to be the same as that for a high type advertiser, then all the advertisers bid truthfully. On the other hand, when the low type advertiser's valuation for the keyword is almost zero, the high type advertisers opt for more bid shading. In this case once again I observe the multiplicity of pure strategy equilibrium. The results are in accordance with Varian (2007). Though Varian did not explicitly mention this result but it can be easily verified that in his model the pure strategy equilibrium bid bounds for any player would become a single equilibrium point (the true valuation) when all the players have same valuation. I also observe that when a high type advertiser's valuation for a keyword increases it does not affect the equilibrium bid of a low type advertiser. However, a high type advertiser in that case places a higher equilibrium bid.

Another interesting result from this section is the non-existence of asymmetric bidding behavior in a static game of incomplete information. The economic intuition also confirms that in the static game of incomplete information, asymmetric bidding strategies would not sustain. For the low type advertiser, bidding higher than other low types would lead to a retaliative bidding behavior yielding negative payoff; on the

other hand, bidding less than the other low types would lead to a zero payoff. For the high type advertiser, bidding higher than other high types might lead to lower payoff if the probability of facing low type competitors is quite high (and the click through rates difference at the two positions is not very high). Bidding less than other high type workers would also not work as the expected payoff would be quite low if the probability of facing high type competitors is very high.

Chapter 5

DYNAMIC GAME OF INCOMPLETE INFORMATION**5.1 Separating and Pooling Equilibrium Strategies**

In this chapter, I analyze the optimal bidding strategies in a T period dynamic game. The number of periods (T) must be greater than or equal to 3 as under minimum disclosure of information, it requires three periods for an incomplete information dynamic game to be transformed into a complete information dynamic game. Initially I assume that the advertisers have *complete disclosure of information* i.e. the advertisers have accessibility to all bidding, payment and position information. I also assume that the advertisers are forward looking and accordingly optimize their aggregate expected payoff. Later I discuss how the results change when I introduce *minimum disclosure of information* in the model.

In a static game all the advertisers bid simultaneously. In a dynamic game the sequence is as follows. In the first stage, all the advertisers simultaneously bid for the keyword. Then the search engine publishes the ranks and the payment information for the first round of auction. Then the advertisers update their beliefs and bid for the second round of auction. The search engine again publishes the ranks and the payment information and the process goes on. The following figure depicts a typical sequence of actions in a dynamic GSP auction.

Sequence of events

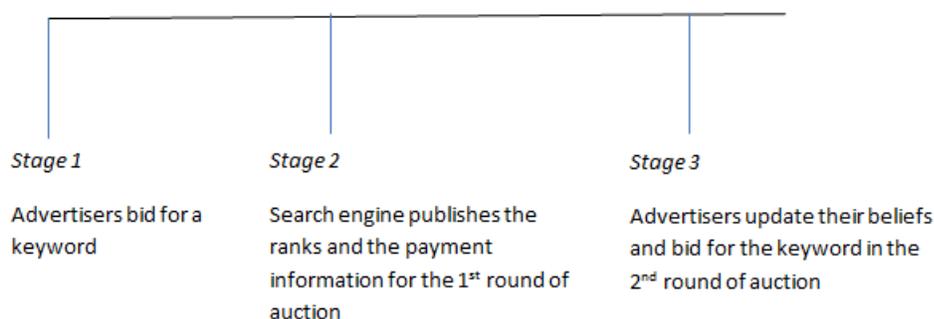


Figure 5.1: Time Sequence of the game

From the static game we know that a low type advertiser would not bid more than V^L as his expected payoff would be negative; bidding less than V^L would also not work as his chance of winning any position would be zero. Therefore a low type advertiser has a unique Nash equilibrium bidding strategy in the single stage game. In any period in the dynamic game, a low type advertiser does not bid less than V^L because there is no other type which bids less than V^L in a static incomplete information game – thus, there is no opportunity of mimicking another type by bidding less than V^L . The low type advertiser does not bid higher than V^L to mimic the high type advertisers because bidding more than V^L will bring negative expected payoff in a given period. Deviation in any given period will never help the low type advertiser to get a positive payoff in future periods. Thus, the dynamic game becomes a sequence of the same single stage game for a low type advertiser. However, the high type advertiser does not bid truthfully in one period incomplete information GSP auction. Additionally, we know that in a finite horizon game, the players would always play the static

Nash equilibrium strategy in the last period as there will not be any opportunity of punishing the others players for defection (Nicholson and Snyder 2011). This leads to the three possible scenarios in a dynamic GSP auction - *(i)* a high type advertiser mimics the low type advertisers in all $(T - 1)$ periods by bidding V^L and then in the T th period plays the equilibrium bidding strategy from the one shot game, *(ii)* a high type advertiser reveals his type in the very first period and updates his prior beliefs at the beginning of each of the following periods to bid accordingly, *(iii)* a high type advertiser reveals his type in the J th period and updates the prior beliefs on other players' valuations at the beginning of the following period.

As we know that no low type advertiser would bid more than V^L (his expected payoff would be negative), any advertiser bidding higher than V^L would be a high type advertiser. Once the high type advertisers' identities are revealed, all the advertisers play a complete information game in the next period. The strategies in the subsequent periods would be independent of the past and future strategies because once the identities have been revealed, the high type advertisers will not bid V^L and bidding anything other than the complete information game bid would bring lower payoff. In other words, there will not be any trade-off point for the high type advertisers in deciding whether to optimally bid or not. Thus, in every period the advertisers would play a complete information game.

At any given period t , high type advertisers may or may not reveal their private information. If no such information is available, then a high type advertiser has two options – he will not reveal his type (bid V^L) or he will reveal his type (bid b^H i.e. the equilibrium bid from the static incomplete information game). However, if such information is available then the high type advertiser will place the optimal bid for

a complete information game (bid b^{HF}). The static incomplete information game analysis shows that b^H is equal to $\frac{C_1(3-p)V^H - C_2((3-2p)V^H - 3(1-p)V^L)}{C_1(3-p) - C_2p}$. The optimal bid for a complete information game, b^{HF} however depends on the number of other high type advertisers. If there are two other high type advertisers, b^{HF} is V^H and if there is just one other high type advertiser then b^{HF} is $\frac{C_1V^H - C_2(V^H - V^L)}{C_1}$. If there is no other high type advertiser (i.e. all the competitors are low type), b^{HF} takes any value higher than V^L .

We find that adopting pooling strategy for the first j periods (and consequently playing separating strategy for the rest of $(T - j)$ periods) is strictly dominated by either pure separating strategy or pure pooling strategy (see appendix F). This implies that a high type advertiser does not reveal his identity in the middle of the dynamic game – either he would reveal his identity at the very beginning or at the very end of the dynamic game. So, we need to compare only two sets of expected payoffs. The following table explains all possible notations used in the expected payoff comparisons.

Various payoffs under complete disclosure of information

Symbol	Definition
π_F^j	Payoff from competing against j number of low type advertisers in a full information game
π_F^i	Payoff from the game when only high type advertiser i has complete information
π_{max}	$(V^H - V^L)C_1$
π_k^j	Payoff from competing against j number of low type advertisers in the k th state; $k \in \{a, b, c, d\}$
p^j	Probability of competing against j number of low type advertisers

Table 5.1: Payoff notations under complete disclosure of information

The index k can take four different values – it is a when the high type advertiser i

as well as the other high type advertisers bid b^H ; when k is b , the advertiser i bids V^L but the other high type advertisers bid b^H ; k takes a value of c when all the high type advertisers bid V^L and a value of d represents that the advertiser i bids b^H but the other high type advertisers bid V^L .

First I will compare the following two expected payoffs related to the separating equilibrium strategy (without deviation and with deviation).

A high type advertiser i 's aggregate expected payoff from bidding b^H in the very first period and bidding b^{HF} in the subsequent periods is (while the other high type advertisers are doing the same in all periods),

$$E(\pi_{S,ND}^i) = \sum_{j=0}^2 p^j \cdot \pi_a^j + (T-1) \sum_{j=0}^2 p^j \cdot \pi_F^j \quad (5.1)$$

A high type advertiser i 's aggregate expected payoff from bidding V^L in the very first period is (while the other high type advertisers are bidding b^H in the very first period),

$$E(\pi_{S,D}^i) = \sum_{j=0}^2 p^j \cdot \pi_b^j + \pi_F^i + (T-2) \sum_{j=0}^2 p^j \cdot \pi_F^j \quad (5.2)$$

Since starting from the third period the game becomes a complete information game, the advertisers will play a complete information game in $(T-2)$ periods.

The expected value comparisons (see appendix *F*) show that $E(\pi_{S,ND}^i) > E(\pi_{S,D}^i)$ when $\frac{C_1}{C_2} < C^a(p)$ (see appendix *F*).

Now I will compare the following two expected payoffs related to the pooling equilibrium strategy (without deviation and with deviation).

A high type advertiser i 's aggregate expected payoff from bidding V^L in the first $(T-1)$ periods and bidding b^H in the last period (while the other high type advertisers are doing the same) is,

$$E(\pi_{P,ND}^i) = (T-1) \sum_{j=0}^2 p^j \cdot \pi_c^j + \sum_{j=0}^2 p^j \cdot \pi_a^j \quad (5.3)$$

A high type advertiser i 's aggregate expected payoff from bidding b^H in the very first period (while the other high type advertisers are bidding V^L in the very first period) is,

$$E(\pi_{P,D}^i) = \pi_{max} + \sum_{j=0}^2 p^j \cdot \pi_d^j + (T-2) \sum_{j=0}^2 p^j \cdot \pi_F^j \quad (5.4)$$

The expected value comparisons show that $E(\pi_{P,ND}^i) > E(\pi_{P,D}^i)$ when $\frac{C_1}{C_2} > C^b(p, T)$ (see appendix F).

However, when p takes a low value, separating equilibrium strategy gives higher payoff for any value of the click through rates ratio. As the probability of competing with low type advertisers increases, a high type advertiser maximizes his payoff by revealing his identity at the very beginning. This ensures the maximum possible payoff in every period. The following proposition summarizes my findings.

Proposition 6. *In a T period dynamic GSP auction with complete disclosure of*

information, pure strategy separating equilibrium takes places if $p \in (0, \underline{p})$. When $p \in (\underline{p}, 1)$, critical click through rates ratio would determine the equilibrium type. $\exists \{C^a, C^b\}$ such that when $p \in (\underline{p}, 1)$, pure strategy separating equilibrium takes place if $\frac{C_1}{C_2} < \min\{C^a, C^b\}$ and pure strategy pooling equilibrium occurs if $\frac{C_1}{C_2} > \max\{C^a, C^b\}$.

For detailed proof, please see appendix *F* – Proof of Proposition 6.

When the probability of a competitor being high type is low (i.e. p takes a relatively low value), a high type advertiser would have higher chances of winning in every period. Thus, a high type advertiser has less incentive to mimic a low type advertiser in this case. As a result, for low values of p , high type advertisers always reveal their true identity at the very beginning of the game. However, for high values of p , the strategic decisions would also depend on the critical click through rates ratio. If the ratio is very high, the value of getting the top position is also very high. As a result the expected payoff of a high type advertiser tends to be very low in any given period. So, the high type advertisers would try to bring down the price of the top position by mimicking the low type.

Proposition 6 confirms the standard result from the second price auction literature – the bid amounts increase over the time. However, the results also show that unlike in the dynamic second price auctions, bidding truthfully in the last period is not a dominant strategy. Also, unlike in the static GSP auction, existence of equilibrium in dynamic GSP auction does not depend on a single critical click through rates ratio.

5.2 Minimum Disclosure of Information

In this subsection I will analyze the dynamic game with minimum disclosure of information. Under minimum disclosure of information the advertisers do not have

accessibility to the bidding history. The search engines do not publish the bidding history and bids placed in previous period auctions typically remain private information. The advertisers can only observe their positions and payments from the last period auction.

Minimum disclosure of information affects the dynamic game in two significant ways. Firstly, the high type advertisers would always find it advantageous to deviate from the pooling strategy equilibrium. In the first period if a high type advertiser i bids anything higher than V^L while all other high type advertisers bid V^L , then advertiser i would get the top position. More importantly, other high type advertisers will not realize that advertiser i has deviated because in the next period they will not be able to observe i 's bid from the first period. So, advertiser i can keep bidding higher than V^L in every period and obtain the top spot (which would also give him the maximum possible payoff in each period). This clearly shows that the symmetric pooling equilibrium will not sustain even for a single period.

Secondly, in case of symmetric pure strategy separating equilibrium, deviating from the equilibrium leads to two possibilities – bidding V^L or bidding $(b^H - \epsilon)$. In a game with complete disclosure of information, bidding $(b^H - \epsilon)$ is not an option for a high type advertiser i because it will give lower payoff in that period and the other high type advertisers will instantaneously recognize his true identity. However, in a game of minimum disclosure of information, if the advertiser i bids $(b^H - \epsilon)$ and gets the third position then the high type advertiser at the top position would not know advertiser i 's true identity (the high type advertiser at the second position would however know i 's identity as that advertiser's payment is advertiser i 's bid).

Once again I compare the expected payoffs from separating strategy equilibrium and

deviation from separating strategy equilibrium. The following table explains all possible notations used in the expected payoff comparisons.

Various payoffs under minimum disclosure of information

Symbol	Definition
Π_F^j	Payoff from competing against j number of low type advertisers in a full information game
Π_F^i	Payoff from the game when only high type advertiser i has complete information
Π_{max}	$(V^H - V^L)C_1$
$\Pi_{(K,t)}^j$	t th period payoff at the k th state facing j low type advertisers; $k \in \{a, b, c\}$
p^j	Probability of competing against j number of low type advertisers

Table 5.2: Payoff notations under minimum disclosure of information

The index K can take three different values – it is a when the high type advertiser i as well as the other high type advertisers bid b^H ; when K is b , the advertiser i bids V^L but the other high type advertisers bid b^H ; K takes a value of c when the advertiser i bids $(b^H - \epsilon)$ and all other high type advertisers bid b^H .

First I will compare the following three expected payoffs related to the separating equilibrium strategy (without deviation and with deviation),

A high type advertiser i 's aggregate expected payoff from bidding b^H in the very first period (while the other high type advertisers are doing the same) is,

$$E(\Pi_{S,ND}^i) = \sum_{j=0}^2 p^j \cdot \Pi^{j(a,1)} + \sum_{j=0}^2 p^j \cdot \Pi_{(a,2)}^j + (T - 2) \sum_{j=0}^2 p^j \cdot \Pi_F^j \quad (5.5)$$

A high type advertiser i 's aggregate expected payoff from bidding V^L in the very first period (while the other high type advertisers are bidding b^H in the very first period

) is,

$$E(\Pi_{S,D1}^i) = \sum_{j=0}^2 p^j \cdot \Pi_{(b,1)}^j + \Pi_F^i + (T-2) \sum_{j=0}^2 p^j \cdot \Pi_F^j \quad (5.6)$$

A high type advertiser i 's aggregate expected payoff from bidding $(b^H - \epsilon)$ in the very first period (while the other high type advertisers are bidding b^H in the very first period) is,

$$E(\Pi_{S,D2}^i) = \sum_{j=0}^2 p^j \cdot \Pi_{(b,1)}^j + \sum_{j=0}^2 p^j \cdot \Pi_{(b,2)}^j + (T-2) \sum_{j=0}^2 p^j \cdot \Pi_F^j \quad (5.7)$$

The expected value comparisons (see appendix G) show that $E(\Pi_{S,ND}^i) > E(\Pi_{S,D1}^i)$ when $\frac{C_1}{C_2} < C^c(p)$ and $E(\Pi_{S,ND}^i) > E(\Pi_{S,D2}^i)$ when $\frac{C_1}{C_2} < C^d(p)$. Region IV in the following graph shows the feasible range of click through rates ratio $(\frac{C_1}{C_2})$ and probability (p) which will lead to a pure strategy symmetric separating equilibrium.

Symmetric separating equilibrium under minimum disclosure of information

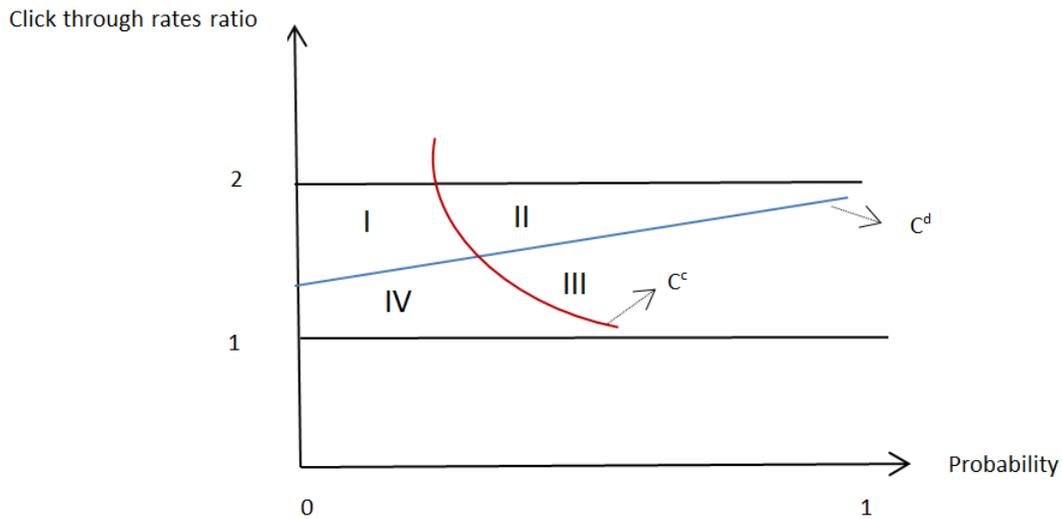


Figure 5.2: Equilibrium under minimum disclosure of information

The above results give us the the next proposition.

Proposition 7. *In a T period dynamic GSP auction with minimum disclosure of information, no pure strategy symmetric pooling equilibrium takes place. A pure strategy symmetric separating equilibrium takes place if $\frac{C_1}{C_2} < \min\{C^c, C^d\}$*

As C^c is strictly greater than C^a , the overall likelihood of pure strategy symmetric separating equilibrium is higher under minimum disclosure of information.

For detailed proof, see appendix G – Proof of Proposition 7.

Overall likelihood is being measured by the area in this two dimensional graph. However, as C^c is strictly greater than C^a (i.e. they do not intersect each other within the relevant click through rates ratio zone), the higher area can just be translated as a higher range in one dimension (i.e. the range in p). I find that in a multi-period dynamic GSP auction with ‘minimum disclosure of information’, all the high type advertisers never mimic the other type at the same time i.e. symmetric pooling equilibrium does not exist. As a result, the equilibrium total payment under minimum disclosure of information is higher than the equilibrium total payment under complete disclosure of information.

Chapter 6

DISCUSSION, CONCLUSION AND DIRECTIONS FOR FUTURE RESEARCH

In this dissertation I analyze the optimal bidding strategies in sponsored search advertising auctions. In chapter 3, I develop a model of complete information GSP auction and show that truthful bidding is only one of the multiple equilibrium strategies. My analysis confirms the results from existing literature on the GSP auction. This is an important result which drives the asymmetric bidding behavior in a dynamic game with incomplete information. In chapter 4, I develop a models of static incomplete information keyword auction and compute all feasible Bayes–Nash equilibria of the game. My analysis and the results show that incomplete information GSP auction can not be treated as a straightforward generalization of a simple second price auction. Unlike some of the existing models in the literature, I show that for a certain type of advertisers (advertisers with low valuations for the keywords) truthful bidding would be the only equilibrium bidding strategy. While I agree with the existing literature on the possibility of having multiple Bayes–Nash equilibria, I also explain why some of those equilibrium bidding strategies would not sustain. From a theoretical perspective, this dissertation has the following contributions – *(i)* in an incomplete information GSP auction, a high type advertiser would bid truthfully only when the difference between the click through rates or the difference between the valuations of a keyword for both types of advertiser is negligible, *(ii)* if the ratio of the click through rates exceeds a critical value, high type advertisers would not have any equilibrium

bidding strategy and *(iii)* no asymmetric bidding behavior would prevail in a static game of incomplete information.

This dissertation particularly contributes to the existing literature on sponsored search advertising auction by developing a model of dynamic incomplete information GSP auction in chapter 4. I analyze the optimal bidding strategies of the advertisers under complete disclosure of information as well as minimum disclosure of information. The results show that – *(i)* the existence of equilibrium depends on multiple critical values of click through rates ratio; a low p does not necessarily guarantee a separating equilibrium or a high p does not necessarily guarantee a pooling equilibrium and *(ii)* under minimum disclosure of information, high type advertisers would never mimic the low type at the equilibrium.

While I contribute to the existing academic literature on GSP auction by providing a complete analysis of the static and the dynamic incomplete information game, I provide additional insights that may help the advertisers and the search engines. In this regard, my results also have some important substantive contributions. The analysis of the incomplete information game in this dissertation shows that a generalized second price auction mechanism does not necessarily motivate the advertisers to bid truthfully. Though the search engines like Google or Yahoo advise the advertisers to bid their true valuations, the existing mechanism does not provide the advertisers a strong incentive to do so. To avoid this problem, the search engines can either tweak the existing mechanism or provide additional incentives in the existing system. For example, they can group the slots on a single page and run separate auctions for different slot-groups. Secondly, the Bayes-Nash equilibrium of my model shows that a low cost equilibrium (which is one of the multiple equilibria) would give the

advertisers a relatively higher payoff. Thirdly, if the click through rates ratio exceeds a critical value, I observe that there does not exist any equilibrium outcome. This has revenue implications for the search engines; perhaps the search engines can do some optimization in terms of number of positions offered per page. Fourthly, it explains why minimum disclosure of information would not affect the bidding behavior of the advertisers in the long run. It has been noted in the literature (Pardoe and Stone (2010)) that the advertisers receive minimal information on the activity of the other advertisers which makes the individual bidding process too complicated. My results show that under such circumstances, the high type advertisers never mimic the low type advertisers. Lastly, some existing literature like Jansen (2011) argue that the ads positioned at the bottom of a search engine landing page might get almost equal number of clicks that the top positions get - this is known as ‘recency effect’. My results show that in presence of such an effect, the high type advertisers would shade their bids by relatively large amounts because the top positions would give same click through rates at higher costs. That means in presence of a ‘recency effect’, high type advertisers are more likely to mimic the low type advertisers. However, as the differences between click through rates widen, the high type advertisers would bid high. In that case, the search engines should try to accommodate as many positions as possible (without making it visibly congested) on the top of the search landing pages ensuring that the ratio of the click through rates does not exceed the critical value.

Some of the topics for future research on sponsored search advertising would include – use of ‘quality score’ in the ranking algorithm, effect of hosting two different GSP auctions instead of one and competition among the search engines.

Some of the search engines have recently started using the *quality score* as another

determinant of rank assignment. *Quality score* determines the relevance of a specific advertisement in the context of a particular keyword search process. Several factors including relevant content, clarity, geographical location can affect an advertiser's *quality score*. I have analyzed the effect of *quality score* on bidding strategies though have not formalized it in this dissertation. The results show that for the advertisers with low quality scores, often the scores do not have any effect on the equilibrium bids. If the low quality score advertisers do not have enough incentives to improve the content and page relevance, the whole objective of incorporating quality score in the payment algorithm might fail. Secondly, although for some of the advertisers (for e.g. 'low valuation-high quality score') high quality score is positively related with high bid, justifying the popular belief, we can not generalize this relationship for every type of advertiser. Another important point to note is that if the low valuation advertisers with high quality scores improve further on the quality scores, they will have incentives to bid low (i.e. getting further away from the truth telling bid strategy). Search engines may want to know whether this trade-off has any potentially important implications for them (the search engines) as the search engines want the advertisers to have high quality score but also expect them to bid truthfully.

The search engines typically place the advertisements on the very top and on the right hand side of the search landing page. Recently Google has started hosting two separate auctions, one for the top positions and the other for the positions on the right hand side. Only ads with particularly high quality can compete in the top-spot auction. It is reasonable to suspect that the bidding strategies of the advertisers participating in two auctions might be significantly different. Also, from the search engine's perspective it would be important to know whether hosting two auctions would necessarily bring higher revenue. This problem can certainly be of some interest

for the analytical modelers.

In reality most of the advertisers participate in multiple GSP auctions hosted by different search engines. For example, an advertiser may want to place her ad on Google search page as well as on Yahoo search page. In doing so, would the advertiser place the same bid in two different auctions? As most of the advertisers are budget-constrained, we can not necessarily argue that the advertiser would exhibit similar bidding behavior across platforms. This is certainly a non-trivial question and using the research insights from this dissertation work, a definite analytical answer can be established.

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Appendix A

PROOF OF PROPOSITION 1

Let's consider the case when majority of the advertisers are of high type i.e. $V(X) = V^H, V(Y) = V^H, V(Z) = V^L$ and the equilibrium positions are assigned as follows $X \rightarrow 1, Y \rightarrow 2, Z \rightarrow 3$. If at the equilibrium X prefers position 1 to position 2 and 3 (position 3 has 0 click through rates which is equivalent to no position), the following conditions must hold,

$$C_1(V^H - b^y) \geq C_2(V^H - b^z) \quad (\text{A.1})$$

$$C_1(V^H - b^y) \geq 0 \quad (\text{A.2})$$

The tie breaking conditions are given by,

$$C_1(V^H - b^y) \geq \frac{1}{3}(C_1(V^H - b^y) + C_2(V^H - b^y) + 0) \quad (\text{A.3})$$

$$C_1(V^H - b^y) \geq \frac{1}{2}(C_1(V^H - b^y) + C_2(V^H - b^z)) \quad (\text{A.4})$$

$$C_1(V^H - b^y) \geq \frac{1}{2}(C_2(V^H - b^z) + 0) \quad (\text{A.5})$$

Binding constraint in this case are equation (A.1) or equation (A.4) (note that these equations are equivalent). Similarly if Y prefers position 2 to position 1 and 3 (at the equilibrium), the following conditions must be satisfied,

$$C_2(V^H - b^z) \geq C_1(V^H - b^x) \quad (\text{A.6})$$

$$C_2(V^H - b^z) \geq 0 \quad (\text{A.7})$$

The tie breaking conditions are given by,

$$C_2(V^H - b^z) \geq \frac{1}{3}(C_1(V^H - b^x) + C_2(V^H - b^x) + 0) \quad (\text{A.8})$$

$$C_2(V^H - b^z) \geq \frac{1}{2}(C_1(V^H - b^x) + C_2(V^H - b^z)) \quad (\text{A.9})$$

$$C_2(V^H - b^z) \geq \frac{1}{2}(C_2(V^H - b^z) + 0) \quad (\text{A.10})$$

Binding constraint in this case are equation (A.6) or (A.9) (note that these equations are equivalent).

Lastly, the analogous conditions for advertiser Z are given by,

$$0 \geq C_1(V^L - b^x) \quad (\text{A.11})$$

$$0 \geq C_2(V^L - b^y) \quad (\text{A.12})$$

The tie breaking conditions are,

$$0 \geq \frac{1}{3}(C_1(V^L - b^x) + C_2(V^L - b^x) + 0) \quad (\text{A.13})$$

$$0 \geq \frac{1}{3}(C_1(V^L - b^y) + C_2(V^L - b^y) + 0) \quad (\text{A.14})$$

$$0 \geq \frac{1}{2}(C_2(V^L - b^x) + 0) \quad (\text{A.15})$$

$$0 \geq \frac{1}{2}(C_2(V^L - b^y) + 0) \quad (\text{A.16})$$

Here (A.11) is binding (which is equivalent to (A.13) and (A.14)). From (A.1) we get, $b^y \leq \frac{(C_1 - C_2)V^H + C_2b^z}{C_1}$ and from (A.6) $b^x \geq \frac{(C_1 - C_2)V^H + C_2b^z}{C_1}$

Therefore, the above conditions lead to the following equilibrium strategy profile,

$$b^x \in [V^H - \frac{C_2}{C_1}(V^H - b^z), \infty]$$

$$b^y \in [\max[V^L, b^z], V^H - \frac{C_2}{C_1}(V^H - b^z)]$$

$$b^z \in [0, V^H]$$

However with the further assumption that the low type advertiser would bid truthfully (as he is indifferent about all possible bids between 0 and V^H) we get a refined set of equilibria,

$$b^x \in [V^H - \frac{C_2}{C_1}(V^H - V^L), \infty]$$

$$b^y \in (V^L, V^H - \frac{C_2}{C_1}(V^H - V^L)]$$

$$b^z = V^L$$

Now consider the case when majority are low type advertisers : $V(X) = V^H$, $V(Y) = V^L$, $V(Z) = V^L$ and $X \rightarrow 1, Y \rightarrow 2, Z \rightarrow 3$. The equilibrium conditions for advertiser X are given by,

$$C_1(V^H - b^y) \geq C_2(V^H - b^z) \tag{A.17}$$

$$C_1(V^H - b^y) \geq 0 \tag{A.18}$$

$$C_1(V^H - b^y) \geq \frac{1}{3}(C_1(V^H - b^y) + C_2(V^H - b^z) + 0) \tag{A.19}$$

$$C_1(V^H - b^y) \geq \frac{1}{2}(C_1(V^H - b^y) + C_2(V^H - b^z)) \tag{A.20}$$

$$C_1(V^H - b^y) \geq \frac{1}{2}(C_2(V^H - b^z) + 0) \tag{A.21}$$

The equilibrium conditions for advertiser Y are given by,

$$C_2(V^L - b^z) \geq C_1(V^L - b^x) \quad (\text{A.22})$$

$$C_2(V^L - b^z) \geq 0 \quad (\text{A.23})$$

$$C_2(V^L - b^z) \geq \frac{1}{3}(C_1(V^L - b^x) + C_2(V^L - b^z) + 0) \quad (\text{A.24})$$

$$C_2(V^L - b^z) \geq \frac{1}{2}(C_1(V^L - b^x) + C_2(V^L - b^z)) \quad (\text{A.25})$$

$$C_2(V^L - b^z) \geq \frac{1}{2}(C_2(V^L - b^z) + 0) \quad (\text{A.26})$$

Lastly, the analogous conditions for advertiser Z are given by,

$$0 \geq C_1(V^L - b^x) \quad (\text{A.27})$$

$$0 \geq C_2(V^L - b^y) \quad (\text{A.28})$$

$$0 \geq \frac{1}{3}(C_1(V^L - b^x) + C_2(V^L - b^y) + 0) \quad (\text{A.29})$$

$$0 \geq \frac{1}{3}(C_1(V^L - b^y) + C_2(V^L - b^x) + 0) \quad (\text{A.30})$$

$$0 \geq \frac{1}{2}(C_2(V^L - b^y) + 0) \quad (\text{A.31})$$

$$0 \geq \frac{1}{2}(C_2(V^L - b^x) + 0) \quad (\text{A.32})$$

The implying conditions are $b^y \leq \frac{(C_1 - C_2)V^H + C_2b^z}{C_1}$, $b^z \leq V^L$ and $b^x \geq \frac{(C_1 - C_2)V^L + C_2b^z}{C_1}$.

Notice that $\frac{(C_1 - C_2)V^H + C_2b^z}{C_1} \geq \frac{(C_1 - C_2)V^L + C_2b^z}{C_1}$. These conditions lead to the following equilibrium strategies,

$$b^x \in \left[\frac{(C_1 - C_2)V^H + C_2b^z}{C_1}, \infty \right]$$

$$b^y \in \left[V^L, \frac{(C_1 - C_2)V^H + C_2b^z}{C_1} \right]$$

$$b^z \in [0, V^L]$$

With further refinement (assuming if advertiser Z is indifferent about all feasible strategies then he will continue to bid truthfully) we get the following equilibrium strategy profiles,

$$b^x \in \left[\frac{(C_1 - C_2)V^H + C_2V^L}{C_1}, \infty \right]$$

$$b^y \in \left(V^L, \frac{(C_1 - C_2)V^H + C_2V^L}{C_1} \right]$$

$$b^z = V^L$$

Now we consider the unrestricted model with V^x , V^y and V^z . The equilibrium positions are assigned as follows $X \rightarrow 1$, $Y \rightarrow 2$ and $Z \rightarrow 3$. If at the equilibrium X prefers position 1 to position 2 and 3 (position 3 has 0 click through rates which is equivalent to no position), the following conditions must hold,

$$C_1(V^x - b^y) \geq C_2(V^x - b^z) \tag{A.33}$$

$$C_1(V^x - b^y) \geq 0 \tag{A.34}$$

The tie breaking conditions are given by,

$$C_1(V^x - b^y) \geq \frac{1}{3}(C_1(V^x - b^y) + C_2(V^x - b^y) + 0) \tag{A.35}$$

$$C_1(V^x - b^y) \geq \frac{1}{2}(C_1(V^x - b^y) + C_2(V^x - b^z)) \tag{A.36}$$

$$C_1(V^x - b^y) \geq \frac{1}{2}(C_2(V^x - b^z) + 0) \tag{A.37}$$

Binding constraint in this case are equation (A.1) or equation (A.4) (note that these equations are equivalent). Similarly if Y prefers position 2 to position 1 and 3 (at the equilibrium), the following conditions must be satisfied,

$$C_2(V^y - b^z) \geq C_1(V^y - b^x) \quad (\text{A.38})$$

$$C_2(V^y - b^z) \geq 0 \quad (\text{A.39})$$

The tie breaking conditions are given by,

$$C_2(V^y - b^z) \geq \frac{1}{3}(C_1(V^y - b^x) + C_2(V^y - b^x) + 0) \quad (\text{A.40})$$

$$C_2(V^y - b^z) \geq \frac{1}{2}(C_1(V^y - b^x) + C_2(V^y - b^z)) \quad (\text{A.41})$$

$$C_2(V^y - b^z) \geq \frac{1}{2}(C_2(V^y - b^z) + 0) \quad (\text{A.42})$$

Binding constraint in this case are equation (A.38) or (A.41) (note that these equations are equivalent).

Lastly, the analogous conditions for advertiser Z are given by,

$$0 \geq C_1(V^z - b^x) \quad (\text{A.43})$$

$$0 \geq C_2(V^z - b^y) \quad (\text{A.44})$$

The tie breaking conditions are,

$$0 \geq \frac{1}{3}(C_1(V^z - b^x) + C_2(V^z - b^x) + 0) \quad (\text{A.45})$$

$$0 \geq \frac{1}{3}(C_1(V^z - b^y) + C_2(V^z - b^y) + 0) \quad (\text{A.46})$$

$$0 \geq \frac{1}{2}(C_2(V^z - b^x) + 0) \quad (\text{A.47})$$

$$0 \geq \frac{1}{2}(C_2(V^z - b^y) + 0) \quad (\text{A.48})$$

Here (A.43) is binding (which is equivalent to (A.45) and (A.46)). From (A.33) we get, $b^y \leq \frac{(C_1 - C_2)V^x + C_2 b^z}{C_1}$ and from (A.38) $b^x \geq \frac{(C_1 - C_2)V^y + C_2 b^z}{C_1}$

Therefore, the above conditions lead to the following equilibrium strategy profile,

$$b^x \in [V^y - \frac{C_2}{C_1}(V^y - b^z), \infty]$$

$$b^y \in [\max[V^z, b^z], V^x - \frac{C_2}{C_1}(V^x - b^z)]$$

$$b^z \in [0, V^y]$$

Appendix B

PROOF OF PROPOSITION 2

To develop a player's belief structure we assume that the valuation of a keyword is a random variable which can take a value of V^H (high valuation) with probability p and a value of V^L (low valuation) with a probability $(1 - p)$. Probability p is a subjective probability but in my model it is same for every advertiser. The two events of an advertiser being either a high type or a low type are both mutually exclusive and mutually exhaustive. Using this individual probability distribution we formulate a joint probability distribution as described below,

Advertiser's belief structure

Joint probability distribution	In terms of p	Competitor type
p_{hh}	$p \cdot p$	Two high type
p_{hl}	$2 \cdot p \cdot (1 - p)$	One high type, one low type
p_{ll}	$(1 - p) \cdot (1 - p)$	Two low type

Table B.1: Joint probability distribution

Expected payoff from bidding b^L when $V = V^L$ is,

$$E(\pi^L | b(L) = b^L) = p_{hh} \cdot 0 + \frac{p_{hl}}{2}(V^L - b^L)C_2 + \frac{p_{ll}}{3}(V^L - b^L)(C_1 + C_2) \quad (\text{B.1})$$

The first term in the expression on the right side implies that when a low type advertiser is facing two high type advertisers, he would certainly not get any position; so, his payoff will be zero. However, if he faces one high type competitor and one low type competitor then with probability $\frac{1}{2}$ he would get the second position. If the low type advertiser gets the second position then his net payoff (per click) would be the difference between his valuation for the keyword and the bid. As the second position gives C_2 number of clicks, his total payoff would be $(V^L - b^L)C_2$. Lastly, if both of his competitors are of low type then with probability $\frac{1}{3}$ each he would get either the first position or the second position or nothing. The third term captures a low type advertiser's expected payoff in this regard.

Using similar logic we can compute the expected payoffs of the advertisers in various situations. Expected payoff from bidding $(b^L + \epsilon)$ when $V = V^L$ (and no other player is deviating) is,

$$E(\pi^L | b(L) = b^L + \epsilon) = p_{hh} \cdot 0 + p_{hl}(V^L - b^L)C_2 + p_{ll}(V^L - b^L)C_1 \quad (\text{B.2})$$

Expected payoff from bidding $(b^L - \epsilon)$ when $V = V^L$ (and no other player is deviating) is 0. Therefore, a low type player will never bid $(b^L - \epsilon)$. Similarly, expected payoff from bidding b^H when $V = V^H$ is,

$$\begin{aligned} E(\pi^H | b(H) = b^H) &= \frac{p_{hh}}{3}(V^H - b^H)(C_1 + C_2) + \frac{p_{hl}}{2}(V^H - b^H)C_1 \\ &+ \frac{p_{ll}}{2}(V^H - b^L)C_2 + p_{lu}(V^H - b^L)C_1 \end{aligned} \quad (\text{B.3})$$

Expected payoff from bidding $(b^H + \epsilon)$ when $V = V^H$ (and no other player is deviating)

is,

$$\begin{aligned} E(\pi^H | b(H) = b^H + \epsilon) &= p_{hh}(V^H - b^H)C_1 + p_{hl}(V^H - b^H)C_1 \\ &+ p_{lu}(V^H - b^L)C_1 \end{aligned} \quad (\text{B.4})$$

Expected payoff from bidding $(b^H - \epsilon)$ when $V = V^H$ (and no other player is deviating) is,

$$E(\pi^H | b(H) = b^H - \epsilon) = p_{hh} \cdot 0 + p_{hl}(V^H - b^L)C_2 + p_{lu}(V^H - b^L)C_1 \quad (\text{B.5})$$

In order to be indifferent between bidding b^L and bidding $(b^L + \epsilon)$ the two expected payoffs must be same. Solving the equation $E[\pi^L | b(L) = b^L] = E[\pi^L | b(L) = b^L + \epsilon]$ we get $b^L = V^L$.

On the other hand, a high type player will prefer to bid b^H over $(b^H + \epsilon)$ if

$$\begin{aligned} \frac{p_{hh}}{3}(V^H - b^H)(C_1 + C_2) + \frac{p_{hl}}{2}(V^H - b^H)C_1 + \frac{p_{hl}}{2}(V^H - b^L)C_2 + p_{lu}(V^H - b^L)C_1 &> \\ p_{hh}(V^H - b^H)C_1 + p_{hl}(V^H - b^H)C_1 + p_{lu}(V^H - b^L)C_1 & \\ \Rightarrow b^H > \frac{C_1(3-p)V^H - C_2((3-2p)V^H - 3(1-p)V^L)}{C_1(3-p) - C_2p} = b^{H1} & \end{aligned}$$

Similarly, a high type player will prefer to bid b^H over $(b^H - \epsilon)$ if

$$\begin{aligned} \frac{p_{hh}}{3}(V^H - b^H)(C_1 + C_2) + \frac{p_{hl}}{2}(V^H - b^H)C_1 + \frac{p_{hl}}{2}(V^H - b^L)C_2 + p_{lu}(V^H - b^L)C_1 &> \\ p_{hh} \cdot 0 + p_{hl}(V^H - b^L)C_2 + p_{lu}(V^H - b^L)C_1 & \\ \Rightarrow \frac{C_1(3-2p)V^H - C_2((3-4p)V^H - 3(1-p)V^L)}{C_1(3-2p) + C_2p} = b^{H2} > b^H & \end{aligned}$$

Therefore, any bid b^H which lies between b^{H1} and b^{H2} can be an equilibrium bid for

the high type advertiser. We observe that,

$$b^{H1} - V^L = \frac{(C_1(3-p) - C_2(3-2p))(V^H - V^L)}{C_1(3-p) - C_2p} > 0, \quad b^{H2} - V^L = \frac{(C_1(3-2p) - C_2(3-4p))(V^H - V^L)}{C_1(3-2p) + C_2p} > 0.$$

Appendix C

PROOF OF PROPOSITION 3

We see that $b^{H1} - b^{H2} = \frac{3(C_1 - 2C_2)C_2(1-p)p(V^H - V^L)}{(C_1(3-p) - C_2p)(C_2p + C_1(3-2p))}$. The result implies that as long as $C_1 < 2C_2$, b^{H1} is less than b^{H2} . However, if $C_1 > 2C_2$ then b^{H2} is greater than b^{H1} .

Now, consider any b^H which satisfies the condition $b^{H1} \leq b^H \leq b^{H2}$.

$$b^{H1} \leq b^H \Rightarrow E(\pi^H | b(H) = b^H + \epsilon) \leq E(\pi^H | b(H) = b^H)$$

$$b^H \leq b^{H2} \Rightarrow E(\pi^H | b(H) = b^H - \epsilon) \leq E(\pi^H | b(H) = b^H)$$

This implies that when $C_1 < 2C_2$, expected payoff from playing b^H will be the highest.

So, a high type advertiser would not deviate from b^H when $C_1 < 2C_2$.

Further, consider any b^H which satisfies the condition $b^{H2} \leq b^H \leq b^{H1}$.

$$b^{H2} \leq b^H \Rightarrow E(\pi^H | b(H) = b^H) \leq E(\pi^H | b(H) = b^H - \epsilon)$$

$$b^H \leq b^{H1} \Rightarrow E(\pi^H | b(H) = b^H) \leq E(\pi^H | b(H) = b^H + \epsilon)$$

This implies that when $C_1 > 2C_2$, expected payoff from playing b^H will be the lowest.

So, a high type advertiser would always deviate from b^H when $C_1 > 2C_2$.

Appendix D

PROOF OF PROPOSITION 4

First, we compute the six possible expected payoffs of the low type advertiser i . Suppose, a low type advertiser i has two low type competitors; one of the low type competitors bids $(b^L + \epsilon)$ and the other low type competitor does not deviate from b^L . Then the low type advertiser i has following five options :

(i) Do not deviate (by bidding b^L) with the following expected payoff,

$$\begin{aligned} E(\pi_i^L | b(L_i) = b^{L_i}) &= p_{hh} \cdot 0 + p_{hl} \left(\left(0 \cdot \frac{1}{2} + (V^L - b^{L_i}) \frac{C_2}{4} \right) \right. \\ &\left. + p_{ul} (V^L - b^L) \frac{C_2}{2} \right) \end{aligned} \quad (\text{D.1})$$

(ii) Move in-between (by bidding $b^L + \frac{\epsilon}{2}$) with the following expected payoff,

$$\begin{aligned} E(\pi_i^L | b(L_i) = b^{L_i} + \frac{\epsilon}{2}) &= p_{hh} \cdot 0 + p_{hl} \left(\left(0 \cdot \frac{1}{2} + (V^L - b^{L_i}) \frac{C_2}{2} \right) \right. \\ &\left. + p_{ul} (V^L - b^L) C_2 \right) \end{aligned} \quad (\text{D.2})$$

(iii) Match (by bidding $b^L + \epsilon$) with the following expected payoff,

$$\begin{aligned} E(\pi_i^L | b(L_i) = b^{L_i} + \epsilon) &= p_{hh} \cdot 0 + p_{hl} \left((V^L - b^{L_i}) \cdot \frac{1}{2} + (V^L - b^{L_i}) \frac{C_2}{2} \right) \\ &+ p_{ul} (V^L - b^L) \frac{(C_1 + C_2)}{2} \end{aligned} \quad (\text{D.3})$$

(iv) Exceed (by bidding $b^L + 2\epsilon$) with the following expected payoff,

$$E(\pi_i^L | b(L_i) = b^{L_i} + 2\epsilon) = p_{hh} \cdot 0 + p_{hl}(V^L - b^{L_i})C_2 + p_{ul}(V^L - b^L)C_1 \quad (\text{D.4})$$

(v) Deviate down (by bidding $b^L - \epsilon$) with the following expected payoff,

$$E(\pi_i^L | b(L_i) = b^{L_i} - \epsilon) = p_{hh} \cdot 0 + p_{hl} \cdot 0 + p_{ul} \cdot 0 = 0 \quad (\text{D.5})$$

Lastly, when all the low type advertisers bid the same they get the following expected payoff,

$$\begin{aligned} E(\pi_i^L | b(L_i) = b(L_i) = b^{L_i}) &= p_{hh} \cdot 0 + \frac{p_{hl}}{2}(V^L - b^{L_i})C_2 \\ &+ \frac{p_{ul}}{3}(V^L - b^{L_i})(C_1 + C_2) \end{aligned} \quad (\text{D.6})$$

Whenever we equate any two of the above six expected payoffs, the resulting equation can be written as $A(V^L - b^{L_i}) = 0$ where A is some constant. So, all the expected payoffs would be equal only when $b^{L_i} = V^L$. This also means that the expected payoff of a low type advertiser will always be 0.

Before we analyze high type player's strategies let's consider the following table.

Position Graph

Bid graph (for player 2)	Position Graph (for player 2)									
Line 1 - Both the high type competitors bid lower	<table style="margin: auto; border: none;"> <tr><td>2</td><td>2</td><td>2</td></tr> <tr><td>1</td><td>3</td><td>1 3</td></tr> <tr><td>3</td><td>1</td><td></td></tr> </table>	2	2	2	1	3	1 3	3	1	
2	2	2								
1	3	1 3								
3	1									
Line 2 - One high type competitor bids same and the other high type bids lower	<table style="margin: auto; border: none;"> <tr><td>1 2</td><td>3 2</td></tr> <tr><td>3</td><td>1</td></tr> </table>	1 2	3 2	3	1					
1 2	3 2									
3	1									
Line 3 - All the high type advertisers bid same	1 2 3									
Line 4 - One high type competitor bids higher and the other high type bids same	<table style="margin: auto; border: none;"> <tr><td>1</td><td>3</td></tr> <tr><td>2 3</td><td>1 2</td></tr> </table>	1	3	2 3	1 2					
1	3									
2 3	1 2									
Line 5 - One high type competitor bids higher and the other high type bids lower.	<table style="margin: auto; border: none;"> <tr><td>1</td><td>3</td></tr> <tr><td>2</td><td>2</td></tr> <tr><td>3</td><td>1</td></tr> </table>	1	3	2	2	3	1			
1	3									
2	2									
3	1									
Line 6 - Both the high type competitors bid higher	<table style="margin: auto; border: none;"> <tr><td>1 3</td><td>1 3</td></tr> <tr><td>2</td><td>3 1</td></tr> <tr><td></td><td>2 2</td></tr> </table>	1 3	1 3	2	3 1		2 2			
1 3	1 3									
2	3 1									
	2 2									

Figure D.1: Feasible positions with asymmetric strategies

As the table shows the high type advertiser may have several feasible strategies for retaliation if one of his high type competitors deviates upward by ϵ . He may match (with ϵ upward deviation), may not deviate at all, may deviate downward (with ϵ downward deviation), may move in between (with $\frac{\epsilon}{2}$ upward deviation) or may exceed (with 2ϵ upward deviation). The position graph table considers all possible cases (including the case when all three bid the same amount) for a high type advertiser (player 2) when one or both of his high type competitors (player 1 and player 3) deviate from b^H .

Mathematically we can capture all these scenarios just by varying one high type competitor's bids. Suppose, a high type advertiser i has two high type competitors; one of the high type competitors bids $(b^H + \epsilon)$ and the other high type competitor does not deviate from b^H . Then the advertiser i has following five options :

(i) Do not deviate (by bidding b^H) with the following expected payoff (represents row 4 in the position table),

$$\begin{aligned}
 E(\pi_i^H | b(H_i) = b^{H_i}) &= p_{hh}(V^H - b^{H_i})\frac{C_2}{2} + p_{hl}((V^H - b^L)\frac{3C_2}{4} + (V^H - b^{H_i})\frac{C_1}{4}) \\
 &+ p_{ll}(V^H - b^L)C_1
 \end{aligned}
 \tag{D.7}$$

The first term in the expression on the right side implies that when both the competitors of a high type advertiser (here advertiser i) are also of high type and one of them deviates upward (but the other does not deviate) then the high type advertiser (here advertiser i) would get the second position with probability $\frac{1}{2}$; we already know that the payoff from the second position is $(V^H - b^H)C_2$. However, when only one of his competitors is high type then with probability $\frac{1}{2}$ he could be a deviating high type or non-deviating high type advertiser. If the high type competitor is deviating upward, then the advertiser i would certainly get the second position. But if the high type competitor does not deviate then the advertiser i would get the second position with probability $\frac{1}{2}$. Therefore, the total probability with which the advertiser i can get the second position is the sum total of $\frac{1}{2} \cdot 1$ and $\frac{1}{2} \cdot \frac{1}{2}$ or $\frac{3}{4}$. Lastly, if both the competitors are of low type, then the advertiser would certainly get the first position.

(ii) Move in-between (by bidding $b^H + \frac{\epsilon}{2}$) with the following expected payoff (represents row 5 in the position table),

$$\begin{aligned}
 E(\pi_i^H | b(H_i) = b^{H_i} + \frac{\epsilon}{2}) &= p_{hh}(V^H - b^{H_i})C_2 + p_{hl}((V^H - b^L)\frac{C_2}{2} + (V^H - b^{H_i})\frac{C_1}{2}) \\
 &+ p_{ll}(V^H - b^L)C_1
 \end{aligned}
 \tag{D.8}$$

(iii) Match (by bidding $b^H + \epsilon$) with the following expected payoff (represents row 2 in the position table),

$$E(\pi_i^H | b(H_i) = b^{H_i} + \epsilon) = p_{hh}(V^H - b^{H_i})\frac{(C_1 + C_2)}{2} + p_{hl}((V^H - b^L)\frac{C_2}{4} + (V^H - b^{H_i})\frac{3C_1}{4}) + p_u(V^H - b^L)C_1 \quad (\text{D.9})$$

(iv) Exceed (by bidding $b^H + 2\epsilon$) with the following expected payoff (represents row 1 in the position table),

$$E(\pi_i^H | b(H_i) = b^{H_i} + 2\epsilon) = p_{hh}(V^H - b^{H_i})C_1 + p_{hl}(V^H - b^{H_i})C_1 + p_u(V^H - b^L)C_1 \quad (\text{D.10})$$

(v) Deviate down (by bidding $b^H - \epsilon$) with the following expected payoff (represents row 6 in the position table),

$$E(\pi_i^H | b(H_i) = b^{H_i} - \epsilon) = p_{hh}(V^H - b^L)0 + p_{hl}(V^H - b^L)C_2 + p_u(V^H - b^L)C_1 \quad (\text{D.11})$$

Lastly, when all the high type advertisers bid the same they get the following expected payoff (represents row 3 in the position table),

$$E(\pi_i^H | b(H_i) = b(H_i) = b^{H_i}) = \frac{p_{hh}}{3}(V^H - b^{H_i})(C_1 + C_2) + \frac{p_{hl}}{2}(V^H - b^{H_i})C_1 + \frac{p_{hl}}{2}(V^H - b^L)C_2 + p_u(V^H - b^L)C_1 \quad (\text{D.12})$$

Before we analyze the asymmetric strategy bidding game, let's first analytically es-

establish the graphs presented in the model (figure 3 and figure 4).

First we observe that when $b^{H_i} = 0$ (regardless of whether $C_1 > 2C_2$ or $C_1 < 2C_2$), $D.10 > D.9$ (line 1's intercept is higher than line 2's intercept), $D.9 > D.8$ (line 2's intercept is higher than line 5's intercept), $D.9 > D.12$ (line 2's intercept is higher than line 3's intercept), $D.8 > D.7$ (line 5's intercept is higher than line 4's intercept), $D.12 > D.7$ (line 3's intercept is higher than line 4's intercept), $D.7 > D.11$ (line 4's intercept is higher than line 6's intercept). However, line 3's intercept would be higher than that of line 5 only when $C_1 > 2C_2$.

Now let's express bids depicted in graph 1 in terms of all the parameters,

$$b^{H1_i} = \frac{C_1 V^H - C_2 (V^H - (1-p)V^L)}{C_1 - C_2 p}; \quad b^{H2_i} = \frac{C_1 (3-p)V^H - C_2 ((3-2p)V^H - 3(1-p)V^L)}{C_1 (3-p) - C_2 p};$$

$$b^{H3_i} = \frac{C_1 (2-p)V^H - 2C_2 (1-p)(V^H - V^L)}{C_1 (2-p)}; \quad b^{H4_i} = \frac{C_1 (3-2p)V^H - C_2 ((3-4p)V^H - 3(1-p)V^L)}{C_1 (3-2p) + C_2 p};$$

$$b^{H5_i} = \frac{C_1 (1-p)V^H - C_2 ((1-2p)V^H - (1-p)V^L)}{C_1 (1-p) - C_2 p}$$

When $C_1 < 2C_2$ we have $b^{H1_i} < b^{H2_i} < b^{H3_i} < b^{H4_i} < b^{H5_i} < V^H$. This establishes graph 1.

When $C_1 > 2C_2$ we have $V^H > b^{H1_i} > b^{H2_i} > b^{H3_i} > b^{H4_i} > b^{H5_i}$. This establishes graph 2.

Now we will show that for every feasible position graph (except position graph in row 3 which represents symmetric bidding), there is no equilibrium in pure strategies (throughout our analysis we assume that $C_1 < 2C_2$).

Let's consider the following bidding game.

Feasible positions in the three players simultaneous move game

Player 2 (column player), player 3 (row player)	b^H	$b^H + \epsilon/2$	$b^H + \epsilon$	$b^H + 2\epsilon$	$b^H - \epsilon$
b^H	1 3 2	1 2 3	1 2 3	2 1 3	1 3 2
$b^H + \epsilon/2$	1 3 2	1 2 3	1 2 3	2 1 3	1 3 2
$b^H + \epsilon$	1 3 2	1 3 2	1 2 3	2 1 3	1 3 2
$b^H + 2\epsilon$	3 1 2	3 1 2	3 1 2	3 2 1	3 1 2
$b^H - \epsilon$	1 2 1	1 2 3	1 2 3	2 1 3	1 2 3

Player 1 is bidding $b^H + \epsilon$

Figure D.2: Bidding game with asymmetric strategies

This bidding game has every feasible position graph (as shown in the position graph table). We will map the position graphs from the position graph table to the payoff cells in the bidding game table. We assume that for all of these position graphs player 1's bid is $(b^H + \epsilon)$.

Position graph in row 1: In the bidding table this graph would exist when the player 2 is bidding $(b^H + 2\epsilon)$ and the player 3 is bidding b^H (though there are other bidding strategies which would give us the same position graph). In order to satisfy the equilibrium conditions, it requires that for the lowest position holder player (in this case player 3) the expected payoff from this cell must be greater than the expected

payoff from the third cell in the same column (D.11 must be greater than D.7) and for the highest position holder player (in this case player 2) the expected payoff from this cell must be greater than the expected payoff from the first cell in the same row (D.12 must be greater than D.7)– which implies that for player 2, $b^H < \frac{C_1(3-p)V^H - C_2((3-2p)V^H - 3(1-p)V^L)}{C_1(3-p) - C_2p}$ but for player 3, $b^H > \frac{C_1(1-p)V^H - C_2((1-2p)V^H - (1-p)V^L)}{C_1(1-p) + C_2p}$. These two conditions are contradictory to each other.

Position graph in row 2 : In the bidding table this graph would exist when the player 2 is bidding $(b^H + \epsilon)$ and the player 3 is bidding b^H . In order to satisfy the equilibrium conditions, it requires that for the lowest position holder player (in this case player 3) the expected payoff from this particular cell must be greater than the expected payoff from the fourth cell in the same column (D.11 must be greater than D.10). This implies $b^H > \frac{C_1(2-p)V^H - 2C_2(1-p)(V^H - V^L)}{C_1(2-p)}$ but for player 2 the expected payoff from this particular cell must be greater than the expected payoff from the first cell in the same row (D.9 must be greater than D.7) which implies that $b^H < \frac{C_1(2-p)V^H - 2C_2(1-p)(V^H - V^L)}{C_1(2-p)}$. The two conditions contradict each other.

Position graph in row 3 : In the bidding table this graph would exist when both player 2 and 3 are bidding $(b^H + \epsilon)$. In order to satisfy the equilibrium conditions, it requires that for all 3 players D.12 must be greater than D.10 and D.11. These two conditions can be simultaneously valid when $\frac{C_1(3-p)V^H - C_2((3-2p)V^H - 3(1-p)V^L)}{C_1(3-p) - C_2p} < b^H < \frac{C_1(3-2p)V^H - C_2((3-4p)V^H - 3(1-p)V^L)}{C_1(3-2p) + C_2p}$.

Position graph in row 4 : In the bidding table this graph would exist when the player 2 is bidding $(b^H + \epsilon)$ and the player 3 is bidding $(b^H + 2\epsilon)$. In order to satisfy the equilibrium conditions, it requires that for the player 2 the expected payoff from this particular cell must be greater than the expected payoff from the fourth cell in the same row

(D.7 must be greater than D.9) which implies that $b^H > \frac{C_1(2-p)V^H - 2C_2(1-p)(V^H - V^L)}{C_1(2-p)}$ whereas for the player 3 the expected payoff from this particular cell must be greater than expected payoff from the fifth cell in the same column (D.10 must be greater than D.11) which implies that $b^H < \frac{C_1(2-p)V^H - 2C_2(1-p)(V^H - V^L)}{C_1(2-p)}$ - the two conditions contradict each other.

Position graph in row 5 : In the bidding table this graph would exist when the player 2 is bidding $(b^H + \frac{\epsilon}{2})$ and the player 3 is bidding b^H . This graph is exactly like the position graph in row 1 though player 1 and player 2 have interchanged their positions. This implies that for player 3, $b^H > \frac{C_1(1-p)V^H - C_2((1-2p)V^H - (1-p)V^L)}{C_1(1-p) + C_2p}$ but for player 1, $b^H < \frac{C_1(3-p)V^H - C_2((3-2p)V^H - 3(1-p)V^L)}{C_1(3-p) - C_2p}$. The two conditions contradict each other.

Position graph in row 6 : In the bidding table this graph would exist when the player 2 is bidding $(b^H - \epsilon)$ and the player 3 is bidding b^H . This graph is exactly like the position graph in row 1 though all three players have interchanged their positions. This implies that for player 2, $b^H > \frac{C_1(1-p)V^H - C_2((1-2p)V^H - (1-p)V^L)}{C_1(1-p) + C_2p}$ but for player 1, $b^H < \frac{C_1(3-p)V^H - C_2((3-2p)V^H - 3(1-p)V^L)}{C_1(3-p) - C_2p}$. The two conditions contradict each other.

The above analysis shows that when player 1 is bidding $(b^H + \epsilon)$, there is no asymmetric equilibrium in pure strategies. We can verify that even when player 1 chooses other strategies we would not have any equilibrium in pure strategies as all the position graphs can be represented by the above 6 categories.

Now let's analyze the graphs when $C_1 > 2C_2$.

Position graph in row 1: In the bidding table this graph would exist when the player 2 is bidding $(b^H + 2\epsilon)$ and the player 3 is bidding b^H (though there are other bidding strategies which would give us the same position graph). In order to satisfy the

equilibrium conditions, it requires that for the lowest position holder player (in this case player 3) the expected payoff from this cell must be greater than all other expected payoffs in the same column (D.11 must be greater than D.7 and D.9) and for the highest position holder player (in this case player 2) the expected payoff from this cell must be greater than all other expected payoffs in the same row (D.10 must be greater than D.7, D.8, D.9, D.11). These conditions can simultaneously be valid when $\frac{C_1(3-2p)V^H - C_2((3-4p)V^H - 3(1-p)V^L)}{C_1(3-2p) + C_2p} < b^H < \frac{C_1(2-p)V^H - 2C_2(1-p)(V^H - V^L)}{C_1(2-p)}$. However, now we need to check the payoff conditions for player 1 as well. For player 1 also, the expected payoff from this cell must be greater than all other expected payoffs in the same row (D.8 must be greater than D.7, D.9, D.10, D.11). But in order to satisfy the conditions we must have $b^H < \frac{C_1(1-p)V^H - C_2((1-2p)V^H - (1-p)V^L)}{C_1(1-p) - C_2p}$. This condition contradicts the earlier condition that $\frac{C_1(3-2p)V^H - C_2((3-4p)V^H - 3(1-p)V^L)}{C_1(3-2p) + C_2p} < b^H$.

Position graph in row 2 : In the bidding table this graph would exist when the player 2 is bidding ($b^H + \epsilon$) and the player 3 is bidding b^H . In order to satisfy the equilibrium conditions, it requires that for the lowest position holder player (in this case player 3) the expected payoff from this particular cell must be greater than the expected payoff from the fourth cell in the same column (D.11 must be greater than D.10). This implies $b^H > \frac{C_1(2-p)V^H - 2C_2(1-p)(V^H - V^L)}{C_1(2-p)}$ but for player 2 the expected payoff from this particular cell must be greater than the expected payoff from the first cell in the same row (D.10 must be greater than D.8) which implies $b^H < \frac{C_1(2-p)V^H - 2C_2(1-p)(V^H - V^L)}{C_1(2-p)}$. The two conditions contradict each other.

Position graph in row 3 : In the bidding table this graph would exist when both player 2 and 3 are bidding ($b^H + \epsilon$). In order to satisfy the equilibrium conditions, it requires that for player 3 the expected payoff from this particular cell must be greater

than the expected payoff from the first cell in the same column (D.12 must be greater than D.11). This implies $b^H < \frac{C_1(3-2p)V^H - C_2((3-4p)V^H - 3(1-p)V^L)}{C_1(3-2p) + C_2p}$ but for player 2 the expected payoff from this particular cell must be greater than the expected payoff from the fourth cell in the same row (D.12 must be greater than D.10) which implies $b^H > \frac{C_1(3-p)V^H - C_2((3-2p)V^H - 3(1-p)V^L)}{C_1(3-p) - C_2p}$. The two conditions contradict each other.

Position graph in row 4 : In the bidding table this graph would exist when the player 2 is bidding $(b^H + \epsilon)$ and the player 3 is bidding $(b^H + 2\epsilon)$. In order to satisfy the equilibrium conditions, it requires that for the player 2 the expected payoff from this particular cell must be greater than the expected payoff from the fourth cell in the same row (D.7 must be greater than D.9) which implies that $b^H > \frac{C_1(2-p)V^H - 2C_2(1-p)(V^H - V^L)}{C_1(2-p)}$ whereas for the player 3 the expected payoff from this particular cell must be greater than expected payoff from the fifth cell in the same column (D.10 must be greater than D.11) which implies that $b^H < \frac{C_1(2-p)V^H - 2C_2(1-p)(V^H - V^L)}{C_1(2-p)}$ - the two conditions contradict each other.

Position graph in row 5 : In the bidding table this graph would exist when the player 2 is bidding $(b^H + \frac{\epsilon}{2})$ and the player 3 is bidding b^H . This graph is exactly like the position graph in row 1 though player 1 and player 2 have interchanged their positions. We already know that the strategies do not give any equilibrium outcome.

Position graph in row 6 : In the bidding table this graph would exist when the player 2 is bidding $(b^H - \epsilon)$ and the player 3 is bidding b^H . This graph is exactly like the position graph in row 1 though all three players have interchanged their positions. We already know that the strategies do not give any equilibrium outcome.

The above analysis shows that when $C_1 > 2C_2$, there is no equilibrium (symmetric or asymmetric) in pure strategies.

Appendix E

PROOF OF PROPOSITION 5

From previous analysis we know that regardless of the number of competitors, a low type advertiser would always bid V^L . A high type advertiser's expected payoff from bidding b^H is,

$$\begin{aligned}
& \left(\binom{N}{N} p^N (1-p)^0 (V^H - b^H) \frac{(C_1 + \dots + C_M)}{N+1} + \dots \right. \\
& + \binom{N}{N-j} p^{N-j} (1-p)^j (V^H - b^H) \frac{(C_1 + \dots + C_M)}{(N-j)+1} \Big) \\
& + \left(\binom{N}{N-j-1} p^{N-j-1} (1-p)^{j+1} ((V^H - b^H) \frac{C_1 + \dots + C_{M-1}}{N-j} \right. \\
& + (V^H - V^L) \frac{C_M}{N-j} \Big) + \dots + \left(\binom{N}{1} p^1 (1-p)^{N-1} ((V^H - b^H) \frac{C_1}{2} + (V^H - V^L) \frac{C_2}{2}) \right) \\
& + \binom{N}{0} p^0 (1-p)^N (V^H - V^L) C_1
\end{aligned} \tag{E.1}$$

The above expression can be written in the following form,

$$\begin{aligned}
& (V^H - b^H) \left(\sum_{i=1}^M C_i \sum_{k=0}^{N-M} \frac{\binom{N}{N-k} p^{N-k} (1-p)^k}{(N+1)-k} \right. \\
& + \left. \sum_{k=(N-M)+1}^{N-1} \frac{\binom{N}{N-k} p^{N-k} (1-p)^k \sum_{i=1}^{N-k} C_i}{(N+1)-k} \right) \\
& + (V^H - V^L) \sum_{k=(N-M)+1}^N \frac{\binom{N}{N-k} p^{N-k} (1-p)^k C_{(N+1)-k}}{(N+1)-k}
\end{aligned} \tag{E.2}$$

Now, a high type advertiser's expected payoff from bidding $b^H + \epsilon$ is,

$$(V^H - b^H) C_1 \sum_{k=0}^{N-1} \binom{N}{N-k} p^{N-k} (1-p)^k + (V^H - V^L) C_1 (1-p)^N \tag{E.3}$$

A high type advertiser's expected payoff from bidding $b^H - \epsilon$ is,

$$(V^H - V^L) \sum_{k=(N-M)+1}^N \binom{N}{N-k} p^{N-k} (1-p)^k C_{(N+1)-k} \tag{E.4}$$

We equate *E.2* to *E.3* and to *E.4* respectively to find out b^{H*} and b^{H**} (as shown below),

$$\begin{aligned}
b^{H*} &= V^H - (V^H - V^L) \cdot \\
& \left((C_1 (1-p)^N - \sum_{k=(N-M)+1}^N \frac{\binom{N}{N-k} p^{N-k} (1-p)^k C_{(N+1)-k}}{(N+1)-k}) \right) \cdot \\
& \left(\frac{1}{\sum_{i=1}^M C_i \sum_{k=0}^{N-M} \frac{\binom{N}{N-k} p^{N-k} (1-p)^k}{(N+1)-k} + \sum_{k=(N-M)+1}^{N-1} \frac{\binom{N}{N-k} p^{N-k} (1-p)^k \sum_{i=1}^{N-k} C_i}{(N+1)-k} - C_1 \sum_{k=0}^{N-1} \binom{N}{N-k} p^{N-k} (1-p)^k} \right)
\end{aligned}$$

$$b^{H**} = V^H - (V^H - V^L).$$

$$\left(\frac{\sum_{k=(N-M)+1}^N \binom{N}{N-k} p^{N-k} (1-p)^k C_{(N+1)-k} - \sum_{k=(N-M)+1}^N \frac{\binom{N}{N-k} p^{N-k} (1-p)^k C_{(N+1)-k}}{(N+1)-k}}{\sum_{i=1}^M C_i \sum_{k=0}^{N-M} \frac{\binom{N}{N-k} p^{N-k} (1-p)^k}{(N+1)-k} + \sum_{k=(N-M)+1}^{N-1} \frac{\binom{N}{N-k} p^{N-k} (1-p)^k \sum_{i=1}^{N-k} C_i}{(N+1)-k}} \right)$$

The above results also give us the bid-shedding amounts $(V^H - V^L)A$ and $(V^H - V^L)B$ where,

$$A =$$

$$\left(\frac{C_1(1-p)^N - \sum_{k=(N-M)+1}^N \frac{\binom{N}{N-k} p^{N-k} (1-p)^k C_{(N+1)-k}}{(N+1)-k}}{\sum_{i=1}^M C_i \sum_{k=0}^{N-M} \frac{\binom{N}{N-k} p^{N-k} (1-p)^k}{(N+1)-k} + \sum_{k=(N-M)+1}^{N-1} \frac{\binom{N}{N-k} p^{N-k} (1-p)^k \sum_{i=1}^{N-k} C_i}{(N+1)-k} - C_1 \sum_{k=0}^{N-1} \binom{N}{N-k} p^{N-k} (1-p)^k} \right)$$

$$=$$

$$\left(\frac{- \sum_{k=(N-M)+1}^{N-1} \frac{\binom{N}{N-k} p^{N-k} (1-p)^k C_{(N+1)-k}}{(N+1)-k}}{\sum_{i=1}^M C_i \sum_{k=0}^{N-M} \frac{\binom{N}{N-k} p^{N-k} (1-p)^k}{(N+1)-k} + \sum_{k=(N-M)+1}^{N-1} \frac{\binom{N}{N-k} p^{N-k} (1-p)^k \sum_{i=1}^{N-k} C_i}{(N+1)-k} - C_1 \sum_{k=0}^{N-1} \binom{N}{N-k} p^{N-k} (1-p)^k} \right)$$

and

$$B = \left(\frac{\sum_{k=(N-M)+1}^N \binom{N}{N-k} p^{N-k} (1-p)^k C_{(N+1)-k} - \sum_{k=(N-M)+1}^N \frac{\binom{N}{N-k} p^{N-k} (1-p)^k C_{(N+1)-k}}{(N+1)-k}}{\sum_{i=1}^M C_i \sum_{k=0}^{N-M} \frac{\binom{N}{N-k} p^{N-k} (1-p)^k}{(N+1)-k} + \sum_{k=(N-M)+1}^{N-1} \frac{\binom{N}{N-k} p^{N-k} (1-p)^k \sum_{i=1}^{N-k} C_i}{(N+1)-k}} \right)$$

$$= \left(\frac{\sum_{k=(N-M)+1}^N \binom{N}{N-k} p^{N-k} (1-p)^k C_{(N+1)-k} \frac{N-k}{(N+1)-k}}{\sum_{i=1}^M C_i \sum_{k=0}^{N-M} \frac{\binom{N}{N-k} p^{N-k} (1-p)^k}{(N+1)-k} + \sum_{k=(N-M)+1}^{N-1} \frac{\binom{N}{N-k} p^{N-k} (1-p)^k \sum_{i=1}^{N-k} C_i}{(N+1)-k}} \right)$$

Now, $(V^H - V^L)A \rightarrow 0$ implies that either $V^L \rightarrow V^H$ or $A \rightarrow 0$. $A \rightarrow 0$ implies that $p \rightarrow 1$. Similarly, $B \rightarrow 0$ implies that either $p \rightarrow 1$ or $C_2 = C_3 = \dots = C_M = 0$ (C_1 does not need to be 0 because the corresponding $\frac{N-k}{(N+1)-k}$ would be equal to 0 as $k \rightarrow N$).

From our previous analysis we know that the requirement for a stable equilibrium is the condition $b^{H^*} < b^{H^{**}}$. Now, the equation $b^{H^*} - b^{H^{**}} = 0$ gives us the following expression for C^* ,

$$\begin{aligned}
b^{H^*} - b^{H^{**}} &= 0 \\
&\Rightarrow \left(\sum_{k=(N-M)+1}^N \binom{N}{N-k} p^{N-k} (1-p)^k C_{(N+1)-k} - C_1 (1-p)^N \right) \\
&\left(\sum_{i=1}^M C_i \sum_{k=0}^{N-M} \frac{\binom{N}{N-k} p^{N-k} (1-p)^k}{(N+1)-k} + \sum_{k=(N-M)+1}^{N-1} \frac{\binom{N}{N-k} p^{N-k} (1-p)^k \sum_{i=1}^{N-k} C_i}{(N+1)-k} \right) \\
&= C_1 \sum_{k=0}^{N-1} \binom{N}{N-k} p^{N-k} (1-p)^k \\
&\left(\sum_{k=(N-M)+1}^N \binom{N}{N-k} p^{N-k} (1-p)^k C_{(N+1)-k} - \right. \\
&\quad \left. \sum_{k=(N-M)+1}^N \frac{\binom{N}{N-k} p^{N-k} (1-p)^k C_{(N+1)-k}}{(N+1)-k} \right) \\
&\Rightarrow C_1 = \left(\frac{\sum_{k=j+1}^{N-1} \frac{\binom{N}{N-k} p^{N-k} (1-p)^k C_{(N+1)-k}}{N-k}}{\sum_{k=j+1}^N \frac{\binom{N}{N-k} p^{N-k} (1-p)^k C_{(N+1)-k}}{N+1-k}} \right) \\
&\left(\frac{\sum_{i=1}^M C_i \sum_{k=0}^{N-M} \frac{\binom{N}{N-k} p^{N-k} (1-p)^k}{(N+1)-k} + \sum_{k=(N-M)+1}^{N-1} \frac{\binom{N}{N-k} p^{N-k} (1-p)^k \sum_{i=1}^{N-k} C_i}{(N+1)-k}}{\sum_{k=0}^{N-1} \binom{N}{N-k} p^{N-k} (1-p)^k} \right) = C^*
\end{aligned}$$

Appendix F

PROOF OF PROPOSITION 6

When a high type advertiser i does not mimic a low type advertiser in the first period, his expected payoff from the first period is,

$$\begin{aligned} \sum_{j=0}^2 p^j \cdot \pi_a^j &= \frac{p_{hh}}{3}(V^H - b_1^H)(C_1 + C_2) + \frac{p_{hl}}{2}(V^H - b_1^H)C_1 + \frac{p_{hl}}{2}(V^H - V^L)C_2 \\ &+ p_{ul}(V^H - V^L)C_1 \end{aligned} \quad (\text{F.1})$$

The next period game becomes a complete information game. The payoff from the complete information game is,

$$\begin{aligned} \sum_{j=0}^2 p^j \cdot \pi_F^j &= \frac{p_{hh}}{3} \cdot 0 + \frac{p_{hl}}{2} \left(V^H - \frac{C_1 V^H - C_2 (V^H - V^L)}{C_1} \right) \frac{C_1}{2} + \frac{p_{hl}}{2} (V^H - V^L) \frac{C_2}{2} \\ &+ p_{ul}(V^H - V^L)C_1 \end{aligned} \quad (\text{F.2})$$

Therefore, the high type advertiser's T periods aggregate expected payoff from playing

the separating strategy is,

$$E(\pi_{S,ND}^i) = \sum_{j=0}^2 p^j \cdot \pi_a^j + (T-1) \sum_{j=0}^2 p^j \cdot \pi_F^j \quad (\text{F.3})$$

When a high type advertiser i deviates from separating equilibrium strategy in the first period and bids V^L , his expected payoff would be,

$$\sum_{j=0}^2 p^j \cdot \pi_b^j = p_{hh} \cdot 0 + \frac{p_{hl}}{2}(V^H - V^L) \frac{C_2}{2} + p_{ul}(V^H - V^L) \frac{(C_1 + C_2)}{3} \quad (\text{F.4})$$

In the second period the high type advertiser has complete information but his other high type competitors do not have complete information. So, the high type advertiser's payoff would be,

$$\pi_F^i = p_{hh}(V^H - \frac{C_1 V^H - C_2(V^H - V^L)}{C_1}) + p_{hl}(V^H - V^L)C_1 + p_{ul}(V^H - V^L)C_1 \quad (\text{F.5})$$

Here we make the 'low cost equilibrium' refinement and assume that when a high type advertiser competes against two low type advertisers, he bids incrementally higher than V^L .

Starting from the third period, the game becomes a complete information game. Thus, the high type advertiser's T periods aggregate expected payoff from deviating from

the separating strategy equilibrium is,

$$E(\pi_{S,D}^i) = \sum_{j=0}^2 p^j \cdot \pi_b^j + \pi_F^i + (T-2) \sum_{j=0}^2 p^j \cdot \pi_F \quad (\text{F.6})$$

Expected payoff comparison shows that,

$$\begin{aligned} E(\pi_{S,ND}^i) &> E(\pi_{S,D}^i) \\ \Rightarrow \frac{C_1}{C_2} &< \\ \frac{-3 + 32p - 43p^2 + 3\sqrt{1 - 24p + 170p^2 - 416p^3 + 441p^4 - 200p^5 + 32p^6}}{2(-6 + 32p - 34p^2 + 8p^3)} &= C^a \end{aligned} \quad (\text{F.7})$$

Comparative static result shows that as p increases, C^a decreases. When p is equal to the critical value \underline{p} , C^a is equal to 2. Solving this equation we get \underline{p} is equal to 0.38.

Now, we will consider the high type advertiser's aggregate expected payoff from the pooling strategy. When the high type advertiser mimics the low type advertiser in the first $(T-1)$ periods, the following would be his expected payoff in the first period,

$$(T-1) \sum_{j=0}^2 p^j \cdot \pi_c^j = (T-1) \frac{(V^H - V^L)(C_1 + C_2)}{3} \quad (\text{F.8})$$

The last period expected payoff would be same as the expected payoff from one period incomplete information GSP auction i.e. $\sum_{j=0}^2 p^j \cdot \pi_a^j$

Thus, the high type advertiser's T period aggregate expected payoff from the pooling strategy is,

$$E(\pi_{P,ND}^i) = (T-1) \sum_{j=0}^2 p^j \cdot \pi_c^j + \sum_{j=0}^2 p^j \cdot \pi_a^j \quad (\text{F.9})$$

When a high type advertiser i deviates from pooling equilibrium strategy in the first period and bids b^H , his expected payoff would be,

$$\pi_{max} = (V^H - V^L)C_1 \quad (\text{F.10})$$

As a result in the second period all high type advertisers except advertiser i would have complete information. In this case, advertiser i 's expected payoff is,

$$\sum_{j=0}^2 p^j \cdot \pi_d^j = p_{hh} \cdot 0 + p_{hl}(V^H - V^L)C_2 + p_{ul}(V^H - V^L)C_1 \quad (\text{F.11})$$

Starting from the third period, all the advertisers will be playing a complete information game and their payoff would be π^F . Thus, the high type advertiser's T period aggregate expected payoff from deviating from the pooling strategy equilibrium is,

$$E(\pi_{P,D}^i) = \pi_{max} + \sum_{j=0}^2 p^j \cdot \pi_d^j + (T-2)\pi_F \quad (\text{F.12})$$

Expected payoff comparison shows that,

$$E(\pi_{P,ND}^i) > E(\pi_{P,D}^i) \Rightarrow \frac{C_1}{C_2} > \frac{-B - \sqrt{B^2 - 4AC}}{2A} = C^b \quad (\text{F.13})$$

where,

$$A = -6 + 38p - 30p^2 + 6p^3 + 6T - 20pT + 15p^2T - 3p^3T \quad (\text{F.14})$$

$$B = 3 - 35p + 39p^2 - 9p^3 - 3T + 17pT - 18p^2T - 3p^3T \quad (\text{F.15})$$

$$C = -p + 6p^2 - 6p^3 + pT - 6p^2T + 6p^3T \quad (\text{F.16})$$

Comparative statics shows that as T increases, C^b decreases and as p increases, C^b increases.

Only symmetric pure strategy separating equilibrium exists when $E(\pi_{S,ND}^i) > E(\pi_{S,D}^i)$ and $E(\pi_{P,ND}^i) < E(\pi_{P,D}^i)$; the resulting conditions are $\frac{C_1}{C_2} < C^a$ and $\frac{C_1}{C_2} < C^b$. This implies that as long as $\frac{C_1}{C_2} < \min\{C^a, C^b\}$, there will be only symmetric pure strategy separating equilibrium. Following similar logic, we can conclude that as long as $\frac{C_1}{C_2} > \max\{C^a, C^b\}$, there will be only pure strategy pooling equilibrium.

We also find that if all the high type advertisers reveal their identities in the J th period (where $J < T$), then each high type advertiser's expected aggregate payoff will be,

$$\begin{aligned}
E(\pi_{SP}^i) &= (J-1)\left\{\frac{(V^H - V^L)(C_1 + C_2)}{3}\right\} + \left\{\frac{p_{hh}}{3}(V^H - b_1^H)(C_1 + C_2) + \right. \\
&\frac{p_{hl}}{2}(V^H - b_1^H)C_1 + \frac{p_{hl}}{2}(V^H - V^L)C_2 + p_u(V^H - V^L)C_1\} + \\
&(T-J)\left\{\left(\frac{p_{hh}}{3} \cdot 1 \cdot 0\right) + \left(\frac{p_{hh}}{3} \cdot 1 \cdot 0 + \frac{p_{hl}}{2}(V^H - \frac{C_1 V^H - C_2(V^H - V^L)}{C_1})\right)\frac{C_1}{2} + \right. \\
&\frac{p_{hl}}{2}(V^H - V^L)\frac{C_2}{2} + \left.\left(\frac{p_{hh}}{3} \cdot 1 \cdot 0 + \frac{p_{hl}}{2}(V^H - \frac{C_1 V^H - C_2(V^H - V^L)}{C_1})\right)\frac{C_1}{2} \right. \\
&\left. + \frac{p_{hl}}{2}(V^H - V^L)\frac{C_2}{2} + p_u(V^H - V^L)C_1\right\}
\end{aligned} \tag{F.17}$$

Comparisons among $E(\pi_{SP}^i)$, $E(\pi_{P,ND}^i)$ and $E(\pi_{S,ND}^i)$ show that,

$$E(\pi_{P,ND}^i) > E(\pi_{SP}^i) > E(\pi_{S,ND}^i) \Rightarrow 2 > \frac{C_1}{C_2} > \frac{6p(1-p) - 1}{1 - 3(1-p)^2} = C^* > 1 \quad (\text{F.18})$$

and,

$$E(\pi_{P,ND}^i) < E(\pi_{SP}^i) < E(\pi_{S,ND}^i) \Rightarrow 1 < \frac{C_1}{C_2} < \frac{6p(1-p) - 1}{1 - 3(1-p)^2} = C^* < 2 \quad (\text{F.19})$$

There is no $\frac{C_1}{C_2}$ in between 1 and 2 which validates the following inequalities : $E(\pi_{SP}^i) > E(\pi_{P,ND}^i) > E(\pi_{S,ND}^i)$ and $E(\pi_{SP}^i) > E(\pi_{S,ND}^i) > E(\pi_{P,ND}^i)$.

Appendix G

PROOF OF PROPOSITION 7

If the high type advertiser gets the top position in the first period auction and the second position holder is also a high type advertiser then the type of advertiser at the third position would not be known to the advertiser at the top position. In that case the high type advertiser would face two high type advertisers with probability p and one high type, one low type with probability $(1-p)$. This is a new game of incomplete information when only one rival advertiser's type is unknown. The parameter b_2^H is the equilibrium bid of a high type advertiser in such a static incomplete information game. An analysis of a one period incomplete information game with the above mentioned belief structure reveals that $b_2^H = \frac{C_1(3+p)V^H - C_2((3-p)V^H - 3(1-p)V^L)}{C_1(3+p) - 2C_2p}$. The low cost equilibrium bid as derived in symmetric static incomplete information game would lead to this expression when $p_{hh} = p$, $p_{hl} = (1-p)$ and $p_{ll} = 0$.

Now, a high type advertiser i 's aggregate expected payoff from bidding b^H in the very first period (while the other high type advertisers are doing the same) is,

$$E(\Pi_{S,ND}^i) = \sum_{j=0}^2 p^j \cdot \Pi_{(a,1)}^j + \sum_{j=0}^2 p^j \cdot \Pi_{(a,2)}^j + (T-2) \sum_{j=0}^2 p^j \cdot \Pi_F^j \quad (\text{G.1})$$

Where,

$$\begin{aligned} \sum_{j=0}^2 p^j \cdot \Pi_{(a,1)}^j &= \frac{p_{hh}}{3}(V^H - b_1^H)(C_1 + C_2) + \frac{p_{hl}}{2}(V^H - b_1^H)C_1 + \frac{p_{hl}}{2}(V^H - V^L)C_2 \\ &+ p_u(V^H - V^L)C_1 \end{aligned} \quad (\text{G.2})$$

$$\begin{aligned} \sum_{j=0}^2 p^j \cdot \Pi_{(a,2)}^j &= \left(\frac{p_{hh}}{3} + \frac{p_{hl}}{2}\right) \left(p(V^H - \frac{C_1(3+p)V^H - C_2((3-p)V^H - 3(1-p)V^L)}{C_1(3+p) - 2C_2p})\right) \\ &\frac{C_1 + C_2}{3} + (1-p) \left(\left(V^H - \frac{C_1(3+p)V^H - C_2((3-p)V^H - 3(1-p)V^L)}{C_1(3+p) - 2C_2p}\right) \frac{C_1}{2} \right. \\ &\left. + (V^H - V^L) \frac{C_2}{2}\right) + p_u(V^H - V^L)C_1 \end{aligned} \quad (\text{G.3})$$

$$\begin{aligned} \sum_{j=0}^2 p^j \cdot \Pi_F^j &= \frac{p_{hh}}{3} \cdot 0 + \frac{p_{hl}}{2} \left(V^H - \frac{C_1V^H - C_2(V^H - V^L)}{C_1}\right) \frac{C_1}{2} + \frac{p_{hl}}{2} (V^H - V^L) \frac{C_2}{2} \\ &+ p_u(V^H - V^L)C_1 \end{aligned} \quad (\text{G.4})$$

A high type advertiser i 's aggregate expected payoff from bidding V^L in the very first period (while the other high type advertisers are bidding b^H in the very first period)

is ,

$$E(\Pi_{S,D1}^i) = \sum_{j=0}^2 p^j \cdot \Pi_{(b,1)}^j + \Pi_F^i + (T-2) \sum_{j=0}^2 p^j \cdot \Pi_F^j \quad (\text{G.5})$$

Where,

$$\sum_{j=0}^2 p^j \cdot \Pi_{(b,1)}^j = p_{hh} \cdot 0 + p_{hl}(V^H - V^L) \frac{C_2}{2} + p_{lu}(V^H - V^L) \frac{C_1 + C_2}{3} \quad (\text{G.6})$$

$$\begin{aligned} \Pi_F^i &= p_{hh} \cdot \left(V^H - \frac{C_1(3+p)V^H - C_2((3-p)V^H - 3(1-p)V^L)}{C_1(3+p) - 2C_2p} \right) C_1 + p_{hl}(V^H - V^L) C_1 \\ &+ p_{lu}(V^H - V^L) C_1 \end{aligned} \quad (\text{G.7})$$

A high type advertiser i 's aggregate expected payoff from bidding $(b^H - \epsilon)$ in the very first period (while the other high type advertisers are bidding b^H in the very first period) is,

$$E(\Pi_{S,D2}^i) = \sum_{j=0}^2 p^j \cdot \Pi_{(b,1)}^j + \sum_{j=0}^2 p^j \cdot \Pi_{(b,2)}^j + (T-2) \sum_{j=0}^2 p^j \cdot \Pi_F^j \quad (\text{G.8})$$

Where,

$$\sum_{j=0}^2 p^j \cdot \Pi_{(b,1)}^j = p_{hh} \cdot 0 + p_{hl}(V^H - V^L)C_2 + p_{ul}(V^H - V^L)C_1 \quad (\text{G.9})$$

$$\begin{aligned} \sum_{j=0}^2 p^j \cdot \Pi_{(b,2)}^j &= p_{hh}(V^H - V^L)\frac{C_1}{2} \\ &+ p_{hh}\left(V^H - \frac{C_1(3+p)V^H - C_2((3-p)V^H - 3(1-p)V^L)}{C_1(3+p) - 2C_2p}\right)\frac{C_2}{2} \\ &+ p_{hl}\left(V^H - \frac{C_1V^H - C_2(V^H - V^L)}{C_1}\right)\frac{C_1}{2} + p_{hl}(V^H - V^L)\frac{C_2}{2} + p_{ul}(V^H - V^L)C_1 \end{aligned} \quad (\text{G.10})$$

The expected value comparisons show that,

$$\begin{aligned} E(\Pi_{S,ND}^i) &> E(\Pi_{S,D1}^i) \\ \Rightarrow \frac{C_1}{C_2} &< \left(\frac{x}{2y} + \frac{(1+i\sqrt{3})w}{(6)X(2^{\frac{2}{3}})X(y(z+\sqrt{4w^3+z^2})^{\frac{1}{3}})} \right. \\ &\left. - \frac{(1-i\sqrt{3})(z+\sqrt{4w^3+z^2})^{\frac{1}{3}}}{(12)X(2^{\frac{1}{3}})X(y)} \right) = C^c \end{aligned}$$

where,

$$x = 3 - 24p - 13p^2 + 4p^3$$

$$y = 9 - 36p - p^2 + 4p^3$$

$$w = -81 + 1782p - 9666p^2 + 7722p^3 - 1809p^4 - 576p^5 + 144p^6$$

$$z = 1458 - 48114p + 523422p^2 - 2049948p^3 + 1759806p^4$$

$$+ 384426p^5 - 904446p^6 + 158436p^7 + 62208p^8 - 15552p^9$$

and,

$$E(\Pi_{S,ND}^i) > E(\Pi_{S,D2}^i) \Rightarrow \frac{C_1}{C_2} < \frac{9 + 9p + \sqrt{81 - 6p + 25p^2}}{4(3 + p)} = C^d.$$

Thus, a pure strategy symmetric separating equilibrium occurs if $\frac{C_1}{C_2} < \min\{C^c, C^d\}$.

VITA

Prabirendra Chatterjee was born in Kolkata (formerly Calcutta), India on 16th of March, 1980. He received his Bachelor of Science degree in Economics from University of Calcutta in 2001 and Master of Arts degree in Economics from Jawaharlal Nehru University in 2003. At the University of Washington, two Master degrees were conferred on him - a Master of Arts in Economics in 2007 and a Master of Science in Business Administration in 2012. In June, 2013 Prabirendra earned a Doctor of Philosophy degree in Business Administration from the Foster School of Business at University of Washington.