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Graded group schemes

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Abstract

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We define graded group schemes and graded group varieties and develop their theory. We give a generalization of the result that connected graded bialgebras are graded Hopf algebra. Our result is given for a broader class of graded Hopf algebras: the coordinate rings of graded group varieties. We also give a classification for graded group algebras and graded group varieties. We proceed to using tools of representation theory to get a better understanding of the cohomology of graded group schemes. For that, we focus our attention on the case in which the base field is of characteristic $p > 0$. Using as inspiration the work on [SFB97b], [SFB97a] and [FP05], we define graded p -points and build the theory of graded 1-parameter subgroups denoted by $V_r^*(G)$. We give a natural homomorphism of bigraded \mathbf{k} -algebras $\psi : H^{*,*}(G, \mathbf{k}) \rightarrow \mathbf{k}[V_r^*(G)]$, where $\mathbf{k}[V_r^*(G)]$ is the bigraded coordinate ring for $V_r^*(G)$, and we show that ψ is an F -monomorphism.

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DEDICATION

a Alberto, Beatrice y a la trulla

INTRODUCTION

Group schemes, and in particular infinitesimal group schemes, are very well studied algebro-geometric objects. Group schemes are a generalization of algebraic groups, and infinitesimal group schemes are an important class of group schemes as they take the role of Lie groups. This work explores the theory of group schemes in the graded realm, that is, when the coordinate rings are graded Hopf algebras and the underlying category is that of graded commutative algebras. The concepts of graded group schemes and graded group varieties will take the place of group schemes and infinitesimal group schemes, respectively. The purpose of this work is to define these concepts and study their properties, in particular their cohomology.

One of the main challenges of this work comes from the fact that the idea of adding a grading is not as innocent as it may seem. Subtleties arise from the fact not all graded Hopf algebras come from ungraded Hopf algebras. Hence, we cannot just work on the ungraded category by forgetting the grading and directly use known results. Also, in the classical theory of graded Hopf algebra it is usually assumed that the algebras are algebraically connected, that is, their degree zero part is the base field. We do not make this assumption, and, with some work, we refine some results from ungraded group schemes (ungraded Hopf algebras) and graded, algebraically connected, Hopf algebras.

This work is divided into five chapters in which we study graded group schemes and their cohomology using different tools, some from homological algebra, some from representation theory, others from algebraic geometry.

We start chapter 1 with the basics; we define graded group schemes, provide some simple examples, and set up the notation. In 1.1 we define the graded connected and graded separable components of a graded group scheme and show that connectivity of graded group schemes relate to the connectivity of the graded spectrum. Then, in 1.3 we use the

constructions from 1.1 in order to give a decomposition of finite graded group schemes in terms of connected and étale components and we give examples of such decompositions.

The motivation of this work is to find a representation theoretic interpretation of the cohomology of graded group schemes. In chapter ?? we define graded p -points, (inspired by the ungraded p -points in [FP05]) as a way of understanding the cohomology of graded group schemes. In [FP05] it is shown that there is a homeomorphism between the projectivization of the cohomology variety, $\text{Proj}|G|$, and space of p -points, $P(G)$. To do this, they rely (among other things) on the following.

- Let G be a finite group scheme and M a finite dimensional G -module, then $H^*(G, \mathbf{k})$ is a finitely generated \mathbf{k} algebra and $H^*(G, M)$ is a finite $H^*(G, \mathbf{k})$ -module.
- Let G be an infinitesimal group scheme of height $\leq r$. They use the 1-parameter subgroups of G , $V_r(G)$ and in particular the F -isomorphism $\psi : H^{ev}(G, \mathbf{k}) \rightarrow \mathbf{k}[V_r(G)]$.

We develop the list above in our context of graded group schemes. In chapter 2 we introduce graded group varieties. In the ungraded setting, for instance in [SFB97b] and [SFB97a], they focus on infinitesimal group schemes. In the graded setting works like that of [Wil81] focus on algebraically connected, graded Hopf algebras. Graded group varieties, in particular finite ones, are graded group schemes with the desired properties of their ungraded and graded counterparts. In 2.1 we give a generalization of the result that connected graded bialgebras are graded Hopf algebra in the context of graded group varieties. Our result is the algebraic analogue of a well known geometric result regarding projective group schemes. As we mentioned before, the coordinate rings of graded group schemes are not necessarily algebraically connected, making our work more difficult. To circumvent this issue, in 2.2, we construct the algebraic connectivization of a graded group scheme as an algebraically connected graded group scheme that is closely related to the original one. The algebraic connectivization proves to be a very useful construction as it allows us to describe the structure of graded group varieties. That is, we provide a description of the graded algebra structure of the coordinate ring of a graded group variety. It also helps us relate, via a

spectral sequence, the cohomology of graded group schemes with that of group schemes and algebraically connected graded Hopf algebras.

We proceed to chapter 3, where we use tools of representation theory to get a better understanding of the cohomology of graded group schemes. For that, we focus our attention on the case in which the base field is of characteristic $p > 0$. Using the work of [SFB97b] and [SFB97a] as inspiration, we build the theory of graded 1-parameter subgroups denoted by $V_r^*(G)$. Ungraded 1-parameter subgroups play the same role that elementary abelian p -subgroups or shifted subgroups play when studying finite groups. An analogue of Quillen's result holds: the cohomology classes of infinitesimal group schemes can be detected, up to nilpotents, by 1-parameter subgroups. Wilkerson explored this question for graded cocommutative Hopf algebras in [Wil81]. In his work, elementary sub-Hopf algebras take the place of elementary abelian p -subgroups. He shows that, in this context, Quillen's result no longer holds. More precisely, he shows that for each prime p , there exists a graded cocommutative Hopf algebra and a nonnilpotent cohomology class which restricts to zero on every abelian sub-Hopf algebra. We merge both concepts of 1-parameter subgroups and elementary Hopf algebras and provide evidence that with some modification a result like that of Quillen could exist for graded group schemes. As part of the exploration of this idea we give a natural homomorphism of bigraded \mathbf{k} -algebras $\psi : H^{*,*}(G, \mathbf{k}) \rightarrow \mathbf{k}[V_r^*(G)]$, where $\mathbf{k}[V_r^*(G)]$ is the bigraded coordinate ring for $V_r^*(G)$. In the ungraded case ψ is an F -isomorphism, allowing us to compute the cohomology of G (up to nilpotents). We show that the graded ψ is an F -monomorphism and compute some examples that suggest that ψ is an F -isomorphism.

In chapter 4, we define graded p -points and set up the concepts needed to develop their theory.

Finally, we discuss further work in chapter 5, where we explain the current obstacles we need to overcome to get a complete Quillen-type result, the F -isomorphism of ψ , and other works we wish to explore in the future.

Chapter 1

GRADED GROUP SCHEMES

This chapter, and most of the next can be found on [AR14]. Throughout this work, our underlying category will be \mathcal{GR} : the category of finitely generated graded commutative \mathbf{k} -algebras, where \mathbf{k} is a field. A graded algebra is graded commutative if, for a, b homogeneous elements in A , we have that $ab = (-1)^{|a||b|}ba$. Note that this way the multiplication map from $m_A : A \otimes A \rightarrow A$ is a graded algebra map where the multiplication of $A \otimes A$ is given by $(A \otimes A) \otimes (A \otimes A) \xrightarrow{A \otimes \tau \otimes A} (A \otimes A) \otimes (A \otimes A) \xrightarrow{m_A \otimes m_A} A \otimes A$, where $\tau : A \otimes A \rightarrow A \otimes A$ is given by $\tau(a \otimes b) = (-1)^{|a||b|}b \otimes a$.

Definition 1.0.1. A representable functor $G : \mathcal{GR} \rightarrow (\text{groups})$ is called an *affine graded group scheme*. We will call them *gr-group schemes* for short. The graded algebra representing G is denoted by $\mathbf{k}[G]$ and is called the *coordinate algebra or coordinate ring* of G . We will drop the word affine from now on, as all our gr-schemes will be assumed to be affine.

As in the ungraded case, by Yoneda's Lemma, there is an equivalence of categories between gr-group schemes and graded commutative Hopf algebras.

Definition 1.0.2. If a graded Hopf algebra is over a field of characteristic $p = 2$ or if it is generated by elements of even degree in the case of $p > 2$, we will call it an *evenly graded Hopf algebra*.

Definition 1.0.3. If G is a gr-group scheme such that $\mathbf{k}[G]$ is evenly graded we say that G is an *evenly gr-group scheme*.

Remark 1.0.4. A graded Hopf algebra which is evenly graded can always be made into an ungraded Hopf algebra, since the sign convention does not change the product and coproduct.

There are graded Hopf algebras over \mathbf{k} of characteristics $p > 2$, with elements in odd degrees, that cannot be made into ungraded Hopf algebras. The simplest example is $A = \mathbf{F}_3[u]/(u^2)$ with $|u| = 1$, where $\Delta(u) = u \otimes 1 + 1 \otimes u$ and $\varepsilon(u) = 1$. Note that the coproduct is well defined: $\Delta(u^2) = \Delta(u)\Delta(u) = (u \otimes 1 + 1 \otimes u)(u \otimes 1 + 1 \otimes u) = u^2 \otimes 1 + u \otimes u + (-1)^{|u|}u \otimes u + 1 \otimes u^2 = 0$. However, if we drop the grading, we get that $\Delta(u^2) = 2u \otimes u \neq 0$, hence A cannot be made into a ungraded Hopf algebra.

Definition 1.0.5. We denote $\mathbf{k}[x_1, \dots, x_n]^{gr}$ to be the *graded polynomial ring* over \mathbf{k} in n -variables, where $x_i x_j = (-1)^{|x_i||x_j|} x_j x_i$. Note that if $\text{char}(\mathbf{k}) \neq 2$, then $x_i^2 = 0$ if $|x_i|$ is odd.

Remark 1.0.6. When $\text{char}(\mathbf{k}) = 2$, $\mathbf{k}[x_1, \dots, x_n]^{gr}$ is just the (ungraded) polynomial ring where the x_i 's are graded. For $\text{char}(\mathbf{k}) \neq 2$ a standard notation for $\mathbf{k}[x_1, \dots, x_n]^{gr}$ is $\mathbf{k}[y_1, \dots, y_m] \otimes \Lambda(z_1, \dots, z_k)$, where the y_i 's are evenly graded and the z_i 's are oddly graded. The y_i 's are in the polynomial part (in the traditional sense) and the z_i 's are in the exterior part. We choose not to use this standard notation for the following reason. The exterior part of the graded polynomial rings that arise in the ungraded setting is usually ignored. For example, when studying the cohomology of (ungraded) Hopf algebras or group schemes, people are usually interested in working with a strictly commutative ring. This is not what we want to do in our setting. Our objects are graded to begin with, and the odd degree part is as important as the even degree part. With the notation as defined above, we want to convey to the reader that each element in $\mathbf{k}[x_1, \dots, x_n]^{gr}$ plays an important role. We think of $\mathbf{k}[x_1, \dots, x_n]^{gr}$ as the graded version of the polynomial ring and the fact that x_i^2 may be zero is just a consequence of the graded commutativity.

Definition 1.0.7. We say that a gr-group scheme G is a *finite gr-group scheme* if $\mathbf{k}[G]$ is finite dimensional. In that case we can define $\mathbf{k}G$ as the graded dual of $\mathbf{k}[G]$; $\mathbf{k}G$ is called the *group algebra* for G .

Definition 1.0.8. We say that a gr-group scheme G is a *positive gr-group scheme* if $\mathbf{k}[G]$ is positively graded. That is $\mathbf{k}[G] = \bigoplus_{i \geq 0} (\mathbf{k}[G])_i$.

Definition 1.0.9. We say that a positive gr-group scheme is *algebraically connected* if the

zero degree part is \mathbf{k} , that is, $(\mathbf{k}[G])_0 = \mathbf{k}$. This is the same as saying that $\mathbf{k}[G]$ is a connected graded Hopf algebra.

Example 1.0.10. Consider $\mathbf{k}[t]^{gr}$ where $|t| = i$ and $\Delta(t) = t \otimes 1 + 1 \otimes t$. Then $\text{Hom}_{\mathfrak{GR}}(\mathbf{k}[t]^{gr}, R) = (R_i, +)$.

Example 1.0.11. Let R be a commutative graded ring. Let $\text{nil}_i^m(R) = \{x \in R_i \mid x^m = 0\}$. Let $\text{char}(\mathbf{k}) = p > 0$ and consider $\mathbf{k}[t]^{gr}/(t^{p^r})$ with $|t| = i$ even, if $p > 2$, and any degree, if $p = 2$. Then $\mathbf{k}[t]^{gr}/(t^{p^r})$ is a graded commutative Hopf algebra with $\Delta(t) = t \otimes 1 + 1 \otimes t$ and $\text{Hom}_{\mathfrak{GR}}(\mathbf{k}[t]^{gr}/(t^{p^r}), R) = \text{nil}_i^{p^r}(R)$ is an additive group under sums. We can see that this group structure arises from the comultiplication of the algebra, since if $x, y \in \text{nil}_i^{p^r}(R)$, where $x(t) = x$ and $y(t) = y$, then the convolution product gives that $(x * y)(t) = m(x \otimes y)(\Delta(t)) = m(x * y)(t \otimes 1 + 1 \otimes t) = x(t)y(1) + x(1)y(t) = x + y$, where we see x and y as functions in $\text{Hom}_{\mathfrak{GR}}(\mathbf{k}[t]^{gr}/(t^{p^r}), R)$.

Example 1.0.12 (Dual Steenrod subalgebra $A(1)$). Consider $A(1) = \mathbf{F}_2[\xi_1, \xi_2]^{gr}/(\xi_1^4, \xi_2^2)$, where $|\xi_1| = 1$ and $|\xi_2| = 3$. The Hopf algebra structure on $A(1)$ is given by $\Delta(\xi_1) = \xi_1 \otimes 1 + 1 \otimes \xi_1$, and $\Delta(\xi_2) = \xi_2 \otimes 1 + \xi_1^2 \otimes \xi_1 + 1 \otimes \xi_2$.

As sets $\text{Hom}_{\mathfrak{GR}}(A(1), R) = \text{nil}_1^4(R) \times \text{nil}_3^2(R)$, with product given by $(x, u) * (y, v) = (x + y, u + v + x^2y)$ and inverse given by $(x, u)^{-1} = (x, u + x^3)$. We denote the gr-group scheme represented by $A(1)$ by \mathbf{S}_1 .

1.1 Graded connected and graded separable components

Given a positive, gr-group scheme G , we define the gr-connected component G^0 , and the gr-group of connected components $\pi_0 G$ associated to it.

Definition 1.1.1. Let $A = \mathbf{k}[G]$, we define $\pi_0 A$ to be the largest gr-separable subalgebra of A (c.f A.3.1).

The gr-group scheme corresponding to $\pi_0 A$ is denoted by $\pi_0 G$ and called the *gr-group of connected components*.

By A.3.4, if A is a positively graded \mathbf{k} algebra (\mathbf{k} not graded), then $\pi_0 A = \pi_0 A_0$ where A_0 is the degree zero part of A . Hence

$$\pi_0 G(R) = \text{Hom}_{\text{gr}}(\pi_0 A, R) = \text{Hom}_{\text{alg}}(\pi_0 A_0, R) = \text{Hom}_{\text{alg}}(\pi_0 A_0, R_0).$$

In fact, gr-separable elements over an ungraded field are necessarily trivially graded.

Definition 1.1.2. The inclusion $\pi_0 A \subset A$ gives a map $G \rightarrow \pi_0 G$. Let G^0 be the gr-group scheme corresponding to the kernel of this map: G^0 is the *gr-connected component* of G .

Theorem 1.1.3. *Let G be a positive, gr-group scheme with coordinate ring $A = \mathbf{k}[G]$. If A_0 is finite dimensional, then $\pi_0 A = \prod_i \mathbf{k}_i^s$, where \mathbf{k}_i^s denotes the separable closure of \mathbf{k} in \mathbf{k}_i , where the \mathbf{k}_i 's are the residue fields in the decomposition of A_0 as a product of local rings.*

Proof. Note that since A_0 is finite dimensional $A_0 = \prod_i A_0^i$ where each A_0^i is a local ring. Moreover, $\pi_0 A = \pi_0 A_0$, then $\pi_0 A_0 = \prod_i \pi_0 A_0^i$.

To see that, let $s \in A_0^i$ separable over \mathbf{k} ; then $(0, \dots, 0, s, 0, \dots)$ is in A_0 and separable over \mathbf{k} ; therefore $\prod_i \pi_0 A_0^i \subseteq \pi_0 A_0$.

Let $s = (s_i) \in \pi_0 A_0$; then $s(0, \dots, 0, 1, 0, \dots, 0) = s_i \in \pi_0 A_0$ since s and $(0, \dots, 0, 1, 0, \dots, 0)$ are in $\pi_0 A_0$. Hence the claim is true.

So now it is enough to show that if (A_0, \mathfrak{m}) is a finite local ring with residue field $\tilde{\mathbf{k}}$ then $\pi_0 A_0 = \tilde{\mathbf{k}}^s$ where $\tilde{\mathbf{k}}^s$ denotes the separable closure of \mathbf{k} in $\tilde{\mathbf{k}}$. By definition $\pi_0 A_0$ is a separable subalgebra of A_0 . Hence by the classification of separable algebras, $\pi_0 A_0$ is a product of matrix rings over division rings whose centers are finite dimensional separable field extensions of \mathbf{k} . Since A_0 is commutative it follows that $\pi_0 A_0$ is a product of separable field extensions of \mathbf{k} . Since A_0 is local and $\pi_0 A_0 \subset A_0$, then it follows that $\pi_0 A_0$ is exactly a separable field extension of \mathbf{k} . Every element in $\pi_0 A_0 - \{0\}$ is invertible hence it survives under the quotient $A_0/\mathfrak{m} \cong \tilde{\mathbf{k}}$; hence $\pi_0 A_0$ is a separable field extension of \mathbf{k} contained in $\tilde{\mathbf{k}}$. Thus $\pi_0 A_0 \subseteq \tilde{\mathbf{k}}^s$.

Now by [Wat79, 6.8] we have that if $(A_0, \mathfrak{m}, \tilde{\mathbf{k}})$ is finite dimensional local, then $\pi_0 A = \pi_0 A_0 \cong \pi_0(A_0/\mathfrak{m}) \cong \pi_0(\tilde{\mathbf{k}}) = \tilde{\mathbf{k}}^s$, hence $\pi_0 A = \tilde{\mathbf{k}}^s$. \square

When G be a positive, finite, gr-group scheme, we have a more explicit description for G_0 . Let $A = \mathbf{k}[G]$; by A.2.7 we write $A = \prod_{i=1}^n A^i$ where (A^i, \mathfrak{m}_i) are gr-local rings with residue gr-field \mathbf{k}_i . Let $\mathbf{k} \subset \mathbf{k}_i^s \subset \mathbf{k}_i$, where \mathbf{k}_i^s is the gr-subfield of \mathbf{k}_i whose homogeneous elements are separable over \mathbf{k} .

For a finite gr-group scheme G and $A = \prod_i A^i$ as above, let e_i be the identity of A^i . The counit map $\varepsilon : A \rightarrow \mathbf{k}$ sends all e_i but one to 0, say e_0 . To see this, notice that $\varepsilon(1_A) = 1$ and $1_A = \sum_i e_i$. Let $\varepsilon(e_i) = \lambda_i \in \mathbf{k}$; since the e_i 's are idempotent it follows that $\varepsilon(e_i) = \varepsilon(e_i^2)$ thus $\lambda_i = \lambda_i^2$, which in the case of a field implies that λ_i is either zero or one. Now since $\varepsilon(1_A) = \varepsilon(\sum_i e_i) = \sum_i \lambda_i = 1$ and $\lambda_i \lambda_j = \varepsilon(e_i e_j) = 0$ for $i \neq j$, then exactly one of the λ_i 's, say λ_0 must be nonzero and hence equal to one. Then ε factors through A^0 , $\varepsilon : A \rightarrow A^0 \rightarrow \mathbf{k}$.

Note that since $\varepsilon : A \rightarrow \mathbf{k}$ factors through A^0 and ε is surjective then $\varepsilon : A^0 \rightarrow \mathbf{k}$ is surjective. We know that $\mathbf{k} \subset \mathbf{k}_0$ and since $\hat{\varepsilon} : A^0/\mathfrak{m}_0 = \mathbf{k}_0 \rightarrow \mathbf{k}$ is surjective it follows by Schur's Lemma that $\mathbf{k}_0 = \mathbf{k}$. Therefore A^0 is a gr-local algebra with residue gr-field equal to \mathbf{k} .

Theorem 1.1.4. *Let G be a positive, finite, gr-group scheme. Then the coordinate ring for G^0 is A^0 as above.*

Proof. By 1.1.3 $\pi_0 A = \prod_i \mathbf{k}_i^s$, the map $G \rightarrow \pi_0 G$ is given by the inclusion $\pi_0 A = \prod_i \mathbf{k}_i^s \subset A$. By [Wat79, 2.1] the kernel of $G \rightarrow \pi_0 G$ is represented by $A \otimes_{\pi_0 A} \mathbf{k} = A/(I \cap \pi_0 A)A$ where $I = \ker \varepsilon$ is the augmentation ideal. We have that $\prod_{i \neq 0} \mathbf{k}_i^s \subset I$. We then get that $I \cap \pi_0 A = \prod_{i \neq 0} \mathbf{k}_i^s$. Therefore $A/(I \cap \pi_0 A)A = A/(\prod_{i \neq 0} \mathbf{k}_i^s A) = (\prod_i A^i)/(\prod_{i \neq 0} \mathbf{k}_i^s A^i) = \prod_i A^i / \prod_{i \neq 0} A^i \cong A^0$, which is the algebra that corresponds to G^0 . Thus, A^0 is the coordinate ring for G^0 . \square

1.2 The graded spectrum and connectivity

The graded (prime) spectrum, denoted $\text{Spec}^{gr}(R)$, for a graded ring R , is defined and studied in A.1. We give results regarding the graded spectrum of the coordinate ring of a positive, gr-group scheme.

Proposition 1.2.1. *Let G be a positive, gr-group scheme with $A = \mathbf{k}[G]$ and A_0 finitely generated. Then $\pi_0 G$ is trivial if and only if $\text{Spec}^{gr}(A)$ is connected.*

Proof. Since G is positive, $\pi_0 A = \pi_0 A_0$. Then by [Wat79, 6.6], $\pi_0 G$ is trivial if and only if $\text{Spec}(A_0)$ is connected, and by A.1.18, $\text{Spec}(A_0)$ is connected if and only if $\text{Spec}^{gr}(A)$ is connected. \square

Definition 1.2.2. A gr-group scheme G is *connected* if $\pi_0 G$ is trivial.

Remark 1.2.3. Note that if G is algebraically connected, then G is connected. By A.1.17 G is connected if and only if $\mathbf{k}[G]$ contains no nontrivial idempotents. If G is algebraically connected, that is, the zero degree part of $\mathbf{k}[G]$ is \mathbf{k} , G is connected since $\mathbf{k}[G]$ contains no nontrivial idempotents.

Definition 1.2.4. A finite gr-group G is *étale* if $\mathbf{k}[G]$ is gr-separable.

By A.3.4, if G is positive and étale then $\mathbf{k}[G]$ must be trivially graded.

1.3 Classification of finite graded group schemes

For the next result we can follow the proof of [Wat79, 6.8] to get the graded version.

Proposition 1.3.1. *(From [Wat79]) Let G be a finite, positive, gr-group scheme over a perfect field. Then G is the semi-direct product of G^0 and $\pi_0 G$.*

Proof. Let $A = \mathbf{k}[G]$. Since A is a product of gr-local rings $A = \prod_i A^i$, the nilradical of A is $N = \prod_i \mathfrak{m}_i$ and each $\mathfrak{m}_i = (A^i)^+ \oplus (\mathfrak{m}_i)_0$, that is, each \mathfrak{m}_i is the irrelevant ideal of A^i plus the zeroth part of \mathfrak{m}_i . Then $A/N = A/(N_0 \oplus A^+) = A_0/N_0$, and since A/N is separable (since \mathbf{k} is perfect), then by [Wat79, 6.8] $\pi_0(A) = \pi_0(A_0) \cong A_0/N_0 = A/N$. Hence, A/N defines a gr-subgroup scheme of G which is isomorphic to $\pi_0 G$.

We want the last map in the exact sequence

$$0 \longrightarrow G^0 \longrightarrow G \longrightarrow \pi_0 G$$

to be surjective. If we look at the zero part of A , A_0 we have that for a graded algebra R ,

$$\mathrm{Hom}_{\mathrm{alg}}(A_0, R_0) \twoheadrightarrow \mathrm{Hom}_{\mathrm{alg}}(\pi_0 A_0, R_0) = \mathrm{Hom}_{\mathfrak{GR}}(\pi_0 A, R)$$

is surjective. Let $f \in \pi_0 G(R)$: then by the surjectivity there is a map corresponding to f , say $g \in \mathrm{Hom}_{\mathrm{alg}}(A_0, R_0)$. Then we can define $\hat{g} \in G(R)$ so that it is g in the zero component and zero elsewhere, hence $G(R) = \mathrm{Hom}_{\mathfrak{GR}}(A, R) \twoheadrightarrow \pi_0 G(R) = \mathrm{Hom}_{\mathfrak{GR}}(\pi_0 A, R)$ is surjective.

Notice that the map

$$\pi_0 G^{\mathrm{c}} \longrightarrow G \twoheadrightarrow \pi_0 G$$

corresponds to the composition of maps

$$\prod \mathbf{k}_i^s \longrightarrow A \longrightarrow A/N \cong \prod \mathbf{k}_i^s$$

which is the identity map. □

Theorem 1.3.2. *Let G be an abelian, finite, positive, gr-group scheme over a perfect field, then G splits canonically into four factors of the following types:*

1. *étale with étale dual,*
2. *étale with connected dual,*
3. *connected with étale dual,*
4. *connected with connected dual.*

Proof. Since G is abelian we have that $G = G^0 \times \pi_0 G$ and when taking duals we have the following decomposition,

$$G \cong G^{\#\#} \cong ((G^{\#})^0)^{\#} \times (\pi_0(G^{\#}))^{\#}.$$

Applying this decomposition to both G^0 and $\pi_0 G$ we get that

$$G^0 \cong ((G^0)^{\#})^0)^{\#} \times (\pi_0((G^0)^{\#}))^{\#}$$

which is a product of connected with connected dual and connected with étale dual. Similarly

$$\pi_0 G \cong ((\pi_0(G^\#))^0)^\# \times (\pi_0(\pi_0(G^\#)))^\#$$

which is a product of étale with connected dual and étale with étale dual. \square

1.3.1 Examples of decomposition

We provide some examples of the decomposition of $G = G^0 \times \pi_0 G$ for finite abelian (gr-)group schemes.

Example 1.3.3 (Ungraded case; characteristic zero). Consider $A = \mathbf{R}[x]/(x^3 - 1)$ where $|x| = 0$, $\varepsilon(x) = 1$, $\Delta(x) = x \otimes x$ and $S(x) = x^2$. By the Chinese Remainder Theorem we can write A as a product of local rings,

$$A \cong \mathbf{R}[x]/(x - 1) \times \mathbf{R}[x]/(x^2 + x + 1) \cong \mathbf{R} \times \mathbf{C}.$$

We follow the isomorphism

$$A \cong \mathbf{R}[x]/(x^3 - 1) \cong \mathbf{R}[x]/(x - 1) \times \mathbf{R}[x]/(x^2 + x + 1)$$

by finding nontrivial idempotents of A . We find that two nontrivial idempotents are $e = 1/3(x^2 + x + 1)$ and $(1 - e) = -1/3(x^2 + x - 2)$; note that $\varepsilon(e) = 1$ and $\varepsilon(1 - e) = 0$. Then $Ae \cong \mathbf{R}[x]/(x - 1)$, $A(1 - e) \cong \mathbf{R}[x]/(x^2 + x + 1)$ and, $A^0 \cong \mathbf{R}[x]/(x - 1) \cong \mathbf{R}$ where A^0 is as in 1.1.4. Hence A^0 is a sub-Hopf algebra with group scheme $G^0(R) = \{e\}$ the trivial group. Now $A \otimes \mathbf{C} = \mathbf{C} \times \mathbf{C} \times \mathbf{C}$, hence A is separable and $\pi_0 A = A$, then $\pi_0 G(R) = \mu_3(R) = \{r \in R \mid r^3 = 1\}$ and $G(R) = \mu_3(R) = (G^0 \times \pi_0 G)(R)$.

Example 1.3.4 (Ungraded case; finite characteristic). Consider $B = \mathbf{F}_2[x]/(x^3 - 1)$ where $|x| = 0$, $\varepsilon(x) = 1$, $\Delta(x) = x \otimes x$ and $S(x) = x^2$. By the Chinese Remainder Theorem

$$B \cong \mathbf{F}_2[x]/(x - 1) \times \mathbf{F}_2[x]/(x^2 + x + 1) \cong \mathbf{F}_2 \times \mathbf{F}_2(\zeta)$$

where ζ is a primitive 3rd root of unity. We follow the isomorphism

$$B \cong \mathbf{F}_2[x]/(x^3 - 1) \cong \mathbf{F}_2[x]/(x - 1) \times \mathbf{F}_2[x]/(x^2 + x + 1)$$

by finding nontrivial idempotents of B . We find that two nontrivial idempotents are $e = (x^2 + x + 1)$ and $(1 - e) = (x^2 + x)$; note that $\varepsilon(e) = 1$ and $\varepsilon(1 - e) = 0$. Then $Be \cong \mathbf{F}_2[x]/(x-1)$, $B(1-e) \cong \mathbf{F}_2[x]/(x^2+x+1)$ and, $B^0 \cong \mathbf{F}_2[x]/(x-1) \cong \mathbf{F}_2$. Hence B^0 is a sub-Hopf algebra with group scheme $G^0(R) = \{e\}$ the trivial group. Now $B \otimes \overline{\mathbf{F}}_2 = \overline{\mathbf{F}}_2 \times \overline{\mathbf{F}}_2 \times \overline{\mathbf{F}}_2$, hence B is separable and $\pi_0 B = B$, then $\pi_0 G(R) = \mu_3(R) = \{r \in R \mid r^3 = 1\}$ and $G(R) = \mu_3(R) = (G^0 \times \pi_0 G)(R)$.

Example 1.3.5 (Graded case; finite characteristic). Let $C = \mathbf{F}_2[x, y]^{gr}/(x^3 - 1, y^2)$ where $|x| = 0$, $|y| = 1$, $\varepsilon(x) = 1$, $\varepsilon(y) = 0$, $\Delta(x) = x \otimes x$ and $\Delta(y) = y \otimes 1 + 1 \otimes y$. Since tensor commutes with products we have that

$$\begin{aligned} C &\cong (\mathbf{F}_2[x]^{gr}/(x-1) \times \mathbf{F}_2[x]^{gr}/(x^2+x+1)) \otimes \mathbf{F}_2[y]^{gr}/(y^2) \\ &\cong \mathbf{F}_2[x, y]^{gr}/(x-1, y^2) \times \mathbf{F}_2[x, y]^{gr}/(x^2+x+1, y^2). \end{aligned}$$

This isomorphism is given by two nontrivial idempotents of C . By A.1.17 the idempotents of C are homogeneous of degree zero, therefore the only trivial idempotents of C are, like in the previous example, $e = (x^2 + x + 1)$ and $(1 - e) = (x^2 + x)$, $Ce \cong \mathbf{F}_2[x, y]^{gr}/(x-1, y^2)$ and $C(1-e) \cong \mathbf{F}_2[x, y]^{gr}/(x^2+x+1, y^2)$. Now $\pi_0 C \cong \mathbf{F}_2[x]^{gr}/(x^3-1)$ and $C^0 = \mathbf{F}_2[x, y]^{gr}/(x-1, y^2)$, also $\pi_0 G(R) = \mu_3(R_0)$ where R_0 is the degree zero part of a commutative graded ring R . Now $\mathbf{F}_2[x]^{gr}/(x-1) \cong \mathbf{F}_2$ and $\mathbf{F}_2[x]^{gr}/(x^2+x+1) \cong \mathbf{F}_2(\zeta)$, where ζ is a 3rd primitive root, are local rings. Hence $C \cong (\mathbf{F}_2 \times \mathbf{F}_2(\zeta)) \otimes \mathbf{F}_2[y]^{gr}/(y^2) \cong \mathbf{F}_2[y]^{gr}/(y^2) \times \mathbf{F}_2(\zeta)[y]^{gr}/(y^2)$ where each term is gr-local, both with unique homogeneous maximal ideal (y) . Then $G^0(R) = \text{nil}_1^2(R) = \{r \in R_1 \mid r^2 = 0\}$ with group structure given by sum and $G(R) = (G^0 \times \pi_0 G)(R)$. Let $f, g \in G(R)$; then f and g are determined by what they send x and y to, say $f(x) = r_1, g(x) = r_2 \in \mu_3(R_0)$ and $f(y) = s_1, g(y) = s_2 \in \text{nil}_1^2(R)$. Then $(f * g)(x) = r_1 r_2$ and $(f * g)(y) = s_1 + s_2$, hence $G(R) = \text{nil}_1^2(R) \times \mu_3(R_0)$ where the multiplication is given by $(s_1, r_1) \cdot (s_2, r_2) = (s_1 + s_2, r_1 r_2)$ and $(s, r)^{-1} = (s, r^2)$ which corresponds to the antipode map $S : C \rightarrow C$ where $S(x) = x^2$ and $S(y) = -y$.

Example 1.3.6 (Dual Steenrod subalgebra $A(1)$). For $A(1) = \mathbf{F}_2[\xi_1, \xi_2]^{gr}/(\xi_1^4, \xi_2^2)$, $\pi_0 A(1) = \mathbf{F}_2$, hence $\pi_0 G = \{e\}$. In fact $A(1)$ is a gr-local algebra with graded maximal ideal (ξ_1, ξ_2) and $A^0 = A(1)$ therefore $G \cong G^0$.

Chapter 2

GRADED GROUP VARIETIES

The structure of infinitesimal group schemes is well known, that is, for an infinitesimal group scheme, there is a description of its coordinate ring as algebras (c.f. [Wat79, 14.4]). Similarly, in the graded case, the structure of positively graded, algebraically connected Hopf algebras of finite type (each degree finite dimensional) is also known (c.f. [MM65, 7.8]). In this section we describe the structure of a class of gr-group schemes which we define and call *graded group varieties*. A graded group variety is a graded group scheme that somehow carries the characteristics of infinitesimal group schemes and algebraically connected Hopf algebra of finite type. Its coordinate ring is gr-local, just like for infinitesimal group schemes the coordinate ring is local. Also, the coordinate ring is of finite type, like the graded Hopf algebras in [MM65].

Definition 2.0.7. Let G be a gr-group scheme, and let $A = \mathbf{k}[G]$. If A is gr-local, positively graded, of finite type we say that G is a *graded group variety* (*gr-group variety*).

Note that by A.1.10 if A is positively graded then, A is gr-local is equivalent to A_0 local.

Remark 2.0.8 (About the choice of name: gr-group variety). In algebraic geometry, a variety is a scheme which is, in particular, connected and of finite type (in the geometric sense). Given a gr-group scheme G , we can associate the geometric object $\text{Spec}^{gr}(A)$, where $A = \mathbf{k}[G]$. If G is a gr-group variety, A_0 local implies that $\text{Spec}^{gr}(A)$ is connected (c.f. 1.2.1). Therefore in this sense, gr-group varieties are connected and of finite type.

Note that for a gr-group variety G , it is not required for $\mathbf{k}[G]$ to be finite dimensional, but only of finite type. For this reason, the class of gr-group varieties is broader than what you would obtain with a *verbatim* generalization of infinitesimal group schemes.

2.1 Antipode for graded group varieties

As part of our quest to classify gr-group varieties, a natural question would be: How much information do we need in order to understand the graded Hopf algebra structure of a gr-group variety?

In our attempt to answer that question, we give a generalization of the well known result that states that, if A is an algebraically connected, positively graded bialgebra then there exists an antipode S constructed from the relations on ε and Δ , making A into a graded Hopf algebra (c.f. [MM65, 8.3]).

Theorem 2.1.1. *Let A be a gr-commutative, positively graded, gr-local bialgebra of finite type, there exists an antipode map S making A into a graded Hopf algebra, that is, A is the coordinate ring of a gr-group variety.*

Proof. Let (A_0, \mathfrak{m}) be a local ring. Note that ε restricts to a surjective map on A_0 , hence, $\mathfrak{m} = \ker(\varepsilon) \cap A_0$ and $A_0/\mathfrak{m} \cong \mathbf{k}$. If $a \in A_0$ we can write $a = \lambda + m$ where $\lambda \in \mathbf{k}$ and $m \in \mathfrak{m}$.

We will describe $\Delta(a)$ for $a \in A$. First, let $a \in A_0$, then $\Delta(a) = \sum a_1 \otimes a_2$ where $|a_1| = |a_2| = 0$. The counit diagram gives us that $a = \sum \varepsilon(a_1)a_2 = \sum a_1\varepsilon(a_2)$.

Write $a_1 = \lambda_1 + m_1$ and $a_2 = \lambda_2 + m_2$ where $\lambda_i \in \mathbf{k}$ and $m_i \in \mathfrak{m}$, and note that $\varepsilon(\lambda_i + m_i) = \varepsilon(\lambda_i) = \lambda_i$. So we can write $a = \sum \lambda_1 a_2 = \sum a_1 \lambda_2$. Therefore, for $a \in A_0$

$$\Delta(a) = a \otimes 1 + 1 \otimes a + \sum a_1 \otimes m_2 + \sum m_1 \otimes a_2.$$

If $m \in \mathfrak{m}$, we can write $\Delta(m) = m \otimes 1 + 1 \otimes m + \sum a_1 \otimes a_2$ and rewriting $a_i = \lambda_i + m_i$ by the counit map we get that $\sum a_1 \lambda_2 = \sum \lambda_1 a_2 = 0$. Hence we can write

$$\Delta(m) = m \otimes 1 + 1 \otimes m + \sum m_1 \otimes m_2,$$

where $m_i \in \mathfrak{m}$.

If a has nonzero degree then with similar arguments we can write

$$\Delta(a) = a \otimes 1 + 1 \otimes a + \sum a_1 \otimes m_2 + \sum m_1 \otimes a_2 + \sum b_1 \otimes b_2,$$

where $m_i \in \mathfrak{m}$, $|a_i| = |a|$ and $|b_i| < |a|$.

We can derive S using that $(S * id)(a) = \varepsilon(a)1_A$, where $*$ is the convolution product for $\text{Hom}_{\mathfrak{R}}(A, A)$.

Without loss of generality, a basis for A_0 is of the form $\{1, m_1, \dots, m_k\}$ where $m_i \in \mathfrak{m}$ for each i ; then S must satisfy

$$S(m_l) + m_l + \sum_{1 \leq i, j \leq k} \alpha_{lij} S(m_i) m_j = 0,$$

where $\alpha_{lij} \in \mathbf{k}$.

We can rewrite the equation as

$$S(m_l) [1 + n_l] + \sum_{\substack{i \neq l \\ j}} \alpha_{lij} S(m_i) m_j = -m_l,$$

where $n_l = \sum_j \alpha_{lij} m_j \in \mathfrak{m}$.

We can think of the above equation as a $(k \times k)$ system in A where $S(m_1), \dots, S(m_k)$ are the unknowns. The system corresponds to the matrix

$$\begin{bmatrix} 1 + n_1 & r_{12} & \cdots & r_{1n} \\ r_{21} & 1 + n_2 & \cdots & r_{2n} \\ \vdots & & \ddots & \vdots \\ r_{n1} & \cdots & & 1 + n_k \end{bmatrix},$$

where $r_{li} = \sum_j \alpha_{lij} m_j$. By inspection, this matrix's determinant is in $1 + \mathfrak{m}$, hence, it is invertible and the system has a unique solution as desired.

Let $\{a_1, \dots, a_n\}$ be a basis for A_k for $k > 0$, then S must satisfy

$$S(a_l) + a_l + \sum_{i=1}^n [S(a_i) m_{li} + S(\hat{m}_{li}) a_i] + \sum S(b_i) b_i = 0,$$

where $m_{li}, \hat{m}_{li} \in \mathfrak{m}$ and b_i are of lower degree.

We can rewrite as

$$S(a_l)(1 + m_{ll}) + \sum_{i \neq l} S(a_i)m_{li} = - \left[\sum_{i=1}^n S(\hat{m}_{li})a_i + \sum S(b_1)b_2 + a_l \right].$$

Note that all the terms in the right hand side are known by induction. Think of the above equation as an $(n \times n)$ system in A where $S(a_1), \dots, S(a_n)$ are the unknowns. The system corresponds to the matrix

$$\begin{bmatrix} 1 + m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & 1 + m_{22} & \cdots & m_{2n} \\ \vdots & & \ddots & \vdots \\ m_{n1} & \cdots & & 1 + m_{nn} \end{bmatrix}.$$

Just as before, the system has a unique solution as desired. \square

As an interesting fact, theorem 2.1.1 is an algebraic analogue to the following theorem in algebraic geometry, regarding abelian varieties.

Theorem 2.1.2. (From [Mum74, Appendix 4]) *Let X be a complete variety, $e \in X$ a point, and*

$$m : X \times X \rightarrow X$$

a morphism such that $m(x, e) = m(e, x) = x$ for all $x \in X$. Then X is an abelian variety with group laws and identity e .

Rephrasing, this theorem says that if X is a projective variety with multiplication and identity, then there exists an inverse making X into a group scheme.

2.2 Algebraic connectivization

Most of the literature regarding graded Hopf algebras assume these algebras to be algebraically connected. This is not the case in our work. The next construction is a bridge between the classical setting and our broader class of graded Hopf algebras.

Definition 2.2.1. Let A be a graded Hopf algebra. Let $\kappa(A) = A \otimes_{A_0} \mathbf{k}$. We call $\kappa(A)$ the *algebraic connectivization* of A .

Theorem 2.2.2. Let A be a positively graded Hopf algebra. The algebraic connectivization of A , $\kappa(A)$, is an algebraically connected graded Hopf algebra.

Proof. The counit and antipode maps for $\kappa(A)$ are given by $\kappa(\varepsilon) = \varepsilon \otimes_{A_0} \mathbf{k}$ and $\kappa(S) = S \otimes_{A_0} \mathbf{k}$. The comultiplication is given by the map $\kappa(\Delta)$ that makes the following diagram commute.

$$\begin{array}{ccc} \kappa(A) & \xrightarrow{\Delta \otimes_{A_0} \mathbf{k}} & A \otimes (A \otimes_{A_0} \mathbf{k}) \\ & \searrow \kappa(\Delta) & \downarrow \\ & & \kappa(A) \otimes \kappa(A) \end{array}$$

More explicitly if $a \in A$ we write

$$\Delta(a) = \sum_{|a_1|=|a|} a_1 \otimes a_2 + \sum_{|b_2|=|a|} b_1 \otimes b_2 + \sum_{|c_1|, |c_2| \neq |a|} c_1 \otimes c_2$$

where $|a_1| + |a_2| = |b_1| + |b_2| = |c_1| + |c_2| = |a|$ then

$$\kappa(\Delta)(a \otimes 1) = \sum a_1 \otimes \varepsilon(a_2) + \sum \varepsilon(b_1) \otimes b_2 + \sum c_1 \otimes c_2.$$

With the counit, antipode and comultiplication as above, a quick chasing of diagrams shows that $\kappa(A)$ is a graded Hopf algebra over \mathbf{k} . Note that its zeroth part is $\kappa(A)_0 = (A \otimes_{A_0} \mathbf{k})_0 = A_0 \otimes_{A_0} \mathbf{k} \cong \mathbf{k}$. Hence $\kappa(A)$ algebraically connected. \square

Lemma 2.2.3. Let C be a commutative local ring (C, \mathfrak{m}) with algebra map $\Delta : C \rightarrow C \otimes C$ of the form $\Delta(a) = a \otimes 1 + 1 \otimes a + \sum a_1 \otimes a_2$, where $a_1, a_2 \in \mathfrak{m}$. If M is a finitely generated C -module with some map $\Delta_M : M \rightarrow M \otimes C$ such that for $a \in C$ and $x \in M$

- $\Delta_M(ax) = \Delta(a)\Delta_M(x)$, where Δ is the comultiplication on C , and
- $\Delta_M(x) = x \otimes 1 + x \otimes b + \sum b_1 x \otimes b_2$, where $b, b_1, b_2 \in \mathfrak{m}$

then M is a free C -module.

Proof. Note that the first condition on the statement is saying that $M \otimes C$ is a C -module via Δ and that $\Delta_M : M \rightarrow M \otimes C$ is a C -module morphism. The proof is by induction on the number of generators of M .

Let $M = Cx$. If M is not free, then there exists $a \in \mathfrak{m}$ such that $ax = 0$. For

$$\Delta_M(ax) = x \otimes a + x \otimes ab + \sum b_1 x \otimes ab_2 + \sum a_1 x \otimes a_2 + \sum a_1 x \otimes a_2 b + \sum a_1 b_1 x \otimes a_2 b_2$$

to be zero either the term $x \otimes -$ or $- \otimes a$ must be zero. The former is $a + ab$. If $a + ab = 0$, then $a = 0$, since $(1 + b)$ is invertible. The latter is

$$x + \sum_{\text{if } a_2=a} a_1 x + \sum_{\text{if } a_2 b=a} a_1 x + \sum_{\text{if } a_2 b_2=a} a_1 b_1 x.$$

In any case, this is of the form $x(1 + m)$ where $m \in \mathfrak{m}$, which is zero if and only if $x = 0$ and this contradicts the fact that x generates M over B .

Let $\{x_1, \dots, x_k\}$ be a minimal set of generators for M over C . If M is not free over C , there must exist a relation of the form

$$a_1 x_1 + \dots + a_k x_k = 0.$$

By minimality, $a_i \in \mathfrak{m}$. Let $M' = M/(x_1)$; then $a_2 x_2 + \dots + a_k x_k = 0$ in M' . Since $\Delta_M(x) = (x \otimes 1)(1 \otimes 1 + 1 \otimes b + \sum b_1 \otimes b_2)$, the map Δ_M restricts to M' and has the same form as before, therefore, by induction, $a_2 = \dots = a_k = 0$. Similarly, we can deduce $a_1 = 0$ by looking at $M/(x_2)$. Therefore M is free over C as desired. \square

Lemma 2.2.4. *Let A be the coordinate ring of a gr-group variety. Then A is a graded free A_0 -module.*

Proof. Let (A_0, \mathfrak{m}) be local. Note that since A is a positively graded Hopf algebra, each A_n is an A_0 -module. Also, since A is a positively graded Hopf algebra, A_0 is a sub-Hopf algebra of A .

Consider A_n ; as computed in the proof of 2.1.1, for $a \in A_0$ and $x \in A_n$ we have that,

$$\Delta(a) = a \otimes 1 + 1 \otimes a + \sum a_1 \otimes a_2,$$

where $a_1, a_2 \in \mathfrak{m}$. Similarly

$$\Delta(x) = x \otimes 1 + x \otimes b + \sum b_1 x \otimes b_2 + \sum c_1 \otimes c_2,$$

where $|c_i| < n$ and $b, b_1, b_2 \in \mathfrak{m}$.

When restricting Δ to $\Delta_{A_n} : A_n \rightarrow A_n \otimes A_0$, A_n is an A_0 -module satisfying the conditions in 2.2.3. Therefore A is a free A_0 -module as desired. \square

Theorem 2.2.5. *Let A be a coordinate ring of a gr-group variety, then*

$$A \cong \kappa(A) \otimes A_0$$

as graded algebras.

Proof. The theorem follows from Lemma 2.2.4. \square

2.3 Structure of graded group varieties

We now classify the graded algebra structure of the coordinate rings of gr-group varieties over perfect fields of characteristic $p > 0$. For this we need to recall some known results from [MM65] and [Wat79].

Proposition 2.3.1. *(From [MM65, 7.8])*

Let A be a gr-commutative Hopf algebra over a field \mathbf{k} of characteristic $p > 0$, generated by one positively graded element x , then $A \cong \mathbf{k}[x]^{gr}$, or $A \cong \frac{\mathbf{k}[x]^{gr}}{(x^{p^t})}$, with x primitive.

Proposition 2.3.2. *(From [Wat79, 14.4]) Let G be an infinitesimal group scheme over a perfect field \mathbf{k} of characteristic $p > 0$, then its coordinate ring is isomorphic, as an algebra, to an algebra of the form:*

$$\mathbf{k}[G] \cong \frac{\mathbf{k}[t_1, \dots, t_n]}{(t_1^{p^{r_1}}, \dots, t_n^{p^{r_n}})}.$$

Proposition 2.3.3. (From [MM65, 7.11]) *Let G be a positive, algebraically connected, gr -group scheme over a perfect field of characteristic p . If its coordinate ring is of finite type, then it is isomorphic, as an algebra, to an algebra of the form below.*

1. *If $p > 2$ then,*

$$\mathbf{k}[G] \cong \frac{\mathbf{k}[x_1, \dots, x_n, y_1, \dots, y_m]^{gr}}{(x_1^{p^{l_1}}, \dots, x_n^{p^{l_n}})},$$

where $|x_i|$ are even, $|y_i|$ are even or odd (of nonzero degrees).

2. *If $p = 2$ then,*

$$\mathbf{k}[G] \cong \frac{\mathbf{k}[x_1, \dots, x_n, y_1, \dots, y_m]^{gr}}{(x_1^{2^{l_1}}, \dots, x_n^{2^{l_n}})},$$

where $|x_i|$ and $|y_i|$ are nonzero, and even or odd.

Theorem 2.3.4. *Let G be gr -group variety over a perfect field of characteristic p . Then its coordinate ring, is isomorphic as an algebra to an algebra of the form below.*

1. *If $p > 2$, then*

$$\mathbf{k}[G] \cong \frac{\mathbf{k}[x_1, \dots, x_n, y_1, \dots, y_m]^{gr}}{(x_1^{p^{l_1}}, \dots, x_n^{p^{l_n}})},$$

where $|x_i|$ are even (including degree zero), $|y_i|$ are even or odd.

2. *If $p = 2$, then*

$$\mathbf{k}[G] \cong \frac{\mathbf{k}[x_1, \dots, x_n, y_1, \dots, y_m]^{gr}}{(x_1^{2^{l_1}}, \dots, x_n^{2^{l_n}})},$$

where $|x_i|$ is odd or even (including degree zero), $|y_i|$ is even or odd and nonzero.

Proof. Given G as above then $A = \mathbf{k}[G]$ contains no nontrivial idempotent hence the degree zero part A_0 is a connected Hopf algebra in the ungraded sense, hence A_0 represents an infinitesimal group scheme. Then A_0 is as in 2.3.2. Consider $\kappa(A) = A \otimes_{A_0} \mathbf{k}$ as defined in 2.2.1; then $\kappa(A)$ is as in 2.3.3. By 2.2.5, $A \cong \kappa(A) \otimes_{\mathbf{k}} A_0$ as graded algebras, hence the result follows. \square

2.4 The cohomology of graded group schemes

Recall as defined in 2.2 that for gr-group scheme G , the *algebraic connectivization* of $A = \mathbf{k}[G]$ is the algebraically connected graded Hopf algebra $\kappa(A) = A \otimes_{A_0} \mathbf{k}$. Using the definitions from [Pal01], it can be easily checked that A_0 is the cotensor product $A \square_{\kappa(A)} \mathbf{k}$ and that $\kappa(A)$ is a conormal quotient of A . We then get the following extension:

$$\mathbf{k} \rightarrow A_0 \rightarrow A \rightarrow \kappa(A) \rightarrow \mathbf{k}$$

This extension satisfies the conditions of section 1.4 on [Pal01], so we get the following theorems. For reference $\underline{\text{Ext}}$ is defined and studied on A.4.

Theorem 2.4.1. *(From [Pal01, 1.4.10]) Let G be a gr-group scheme with $A = \mathbf{k}[G]$ let $\kappa(A)$ be the algebraic connectivization of A . Then for any graded A -comodules M_1, M_2, M_3 there is a spectral sequence with*

$$E_2^{p,q,v} = \underline{\text{Ext}}_{A_0}^{p,v}(M_1, \underline{\text{Ext}}_{\kappa(A)}^{q,*}(M_2, M_3)) \Rightarrow \underline{\text{Ext}}_A^{p+q,v}(M_1 \otimes M_2, M_3).$$

The next theorem relates the cohomology of the gr-group scheme G with that of the group scheme G_0 and $\kappa(G)$.

Theorem 2.4.2. *Let G be a gr-group scheme with $A = \mathbf{k}[G]$. Let $\kappa(G)$ and G_0 denote the gr-group schemes corresponding to $\kappa(A)$ and A_0 respectively. Then there is a spectral sequence with E_2 -term $H^{*,*}(G_0, H^{*,*}(\kappa(G), \mathbf{k}))$ abutting to $H^{*,*}(G, \mathbf{k})$.*

Chapter 3

GRADED 1-PARAMETER SUBGROUPS

In the ungraded setting, the additive group scheme \mathbf{G}_a plays an important role. More precisely its r th Frobenius kernel $\mathbf{G}_{a(r)}$ is of main importance in representation theory. Given a group scheme G , the infinitesimal 1-parameter subgroups of height $\leq r$ in G are defined as group homomorphism from $\mathbf{G}_{a(r)} \rightarrow G$ and denoted by $V_r(G)$. In the following sections we define the additive gr-group scheme \mathbf{G}_a^* and its r th Frobenius kernel, $\mathbf{G}_{a(r)}^*$. To do so, we first give precise definitions of the graded Frobenius map and twist. We define the graded general linear group, \mathbf{GL}_I . As in the ungraded case, \mathbf{GL}_I proves useful when computing $V_r^*(G)$.

3.1 Graded Frobenius

Definition 3.1.1. Let A be a gr-commutative ring. We define the *Frobenius p th power map* $F : A \rightarrow A$ such that $F(a) = a^p$.

Definition 3.1.2. Let A be a gr-commutative ring and M a gr-module over A . For any $r \geq 0$, the *r -th Frobenius twist* of M is the A -module $M^{(r)} = M \otimes_{F^r} A$. We grade $M^{(r)}$ as follows: $(M^{(r)})_k = \bigoplus_{p^r i + j = k} M_i \otimes_{F^r} A_j$.

The degree is well-defined and on $M^{(r)}$ a homogeneous element $m \otimes 1$ has degree $p^r |m|$. Moreover on $M^{(r)}$, $|m \otimes a^{p^r}| = p^r |m| + p^r |a| = p^r (|m| + |a|)$ and also $m \otimes a^{p^r} = am \otimes 1$ which is of degree $p^r (|m| + |a|)$.

Proposition 3.1.3. Let A be a gr-commutative ring and M and N be gr-modules over A . For $r \geq 0$, the graded r -th Frobenius twist $I^r : M \mapsto M^{(r)}$ is an additive functor from the

category of graded A modules to itself. Moreover for any M, N

$$(M \otimes_A N)^{(r)} = M^{(r)} \otimes_A N^{(r)}; (M^\#)^{(r)} = (M^{(r)})^\#.$$

Proof. Let $f : M \rightarrow N$ be a graded map, then $I^{(r)}(f) : (m \otimes 1 \mapsto f(m) \otimes 1)$. Now $|m \otimes 1| = p^r |m| \mapsto p^r |f(m)| = p^r |f| + p |m|$. Hence $I^{(r)}(f)$ is a graded map of degree $p^r |f|$ and it easily follows that $I^{(r)}$ is an additive functor. The equalities in the statement are true in the ungraded case. For the graded case they follow the same way. We just need to check that the gradings are compatible, which they are by the computations below.

$$\begin{aligned} (M \otimes N)_k^{(r)} &= \bigoplus_{p^r i + j = k} \left(\bigoplus_{l+m=i} M_l \otimes N_m \right) \otimes_{F^r} A_j \\ &= \bigoplus_{\substack{p^r i + j = k \\ l+m=i \\ u+v=j}} (M_l \otimes_{F^r} A_u) \otimes (N_m \otimes_{F^r} A_v) \\ (M^{(r)} \otimes N^{(r)})_k &= \bigoplus_{l+m=k} M_l^{(r)} \otimes N_m^{(r)} \\ &= \bigoplus_{l+m=k} \left(\bigoplus_{p^r i + j = l} M_i \otimes_{F^r} A_j \right) \otimes \left(\bigoplus_{p^r u + v = m} N_u \otimes_{F^r} A_v \right) \\ &= \bigoplus_{\substack{l+m=k \\ p^r i + j = l \\ p^r u + v = m}} (M_i \otimes_{F^r} A_j) \otimes (N_u \otimes_{F^r} A_v) \end{aligned}$$

A homogeneous element in $(M^\#)^{(r)}$ is of the form $f \otimes_{F^r} 1$, where $f : M \rightarrow A$ is a graded map and $|f \otimes_{F^r} 1| = p^r |f|$. This element corresponds to $\hat{f} \in (M^{(r)})^\#$ where $\hat{f}(m \otimes_{F^r} 1) = f(m) \otimes 1$ with $f : M \rightarrow A$ and $|\hat{f}| = p^r |f|$. \square

Definition 3.1.4. Given a gr-group scheme G over a gr-commutative ring A , since $I^{(r)}$ is a functor, $A[G^{(r)}] = A[G] \otimes_{F^r} A$ is a gr-Hopf algebra. Then $F^r : A \rightarrow A$ ($a \mapsto a^{p^r}$) gives a map $F^r : G \rightarrow G^{(r)}$; we denote by $G_{(r)} = G \times_{G^{(r)}} \{e\}$ the gr-group scheme which is the kernel of F^r . It has coordinate algebra $A[G]/(\{x^{p^r} \mid x \in \ker(\varepsilon)\}A[G])$. At the level of gr-Hopf algebras $F^r : A[G^{(r)}] \rightarrow A[G]$ is the map that sends $a \otimes \alpha \mapsto a^{p^r} \otimes \alpha$. We say that $G_{(r)}$ is the *graded r th Frobenius kernel* of G .

Definition 3.1.5. A gr-group scheme G is of *height r* , if r is the smallest positive integer such that $a^{p^r} = 0$ for every element a in the augmentation ideal of $\mathbf{k}[G]$, where the

augmentation ideal of $\mathbf{k}[G]$ is the kernel of the counit map $\varepsilon : \mathbf{k}[G] \rightarrow \mathbf{k}$.

3.2 Additive graded group scheme

For $p > 2$, let \mathbf{G}_a^* be the gr-group scheme represented by $\mathbf{k}[t, s]^{gr}$ where $|t|$ is even and $|s|$ is odd and both t and s are primitive. We call \mathbf{G}_a^* *additive gr-group scheme*. We would like to point out the difference between the additive gr-group scheme \mathbf{G}_a^* , and the ungraded additive group scheme \mathbf{G}_a . Note that we do not fix the degrees of t nor s . They are left as placeholders.

In the ungraded setting $\mathbf{k}[\mathbf{G}_a] = \mathbf{k}[t]$ and $\mathbf{G}_a(R) = (R, +)$ for any commutative algebra R , hence \mathbf{G}_a is the group scheme that describes the additive group structure for any algebra R . In the graded case, for any graded algebra R , $\mathbf{G}_a^*(R) = (R_{|t|} \times R_{|s|}, +)$. We do not get the additive group structure of the whole ring R , we only get the additive structure of the parts of degree equal to t and s . In order to get the full additive structure of R we would need to consider the gr-group scheme represented by $\bigotimes_{i \in \mathbb{Z}} \mathbf{k}[x_i]^{gr}$ where $|x_i| = i$ and x_i is primitive. This gr-Hopf algebra is too big (not noetherian). It turns out that considering \mathbf{G}_a^* is enough in our setting. In particular, the r th Frobenius kernel of \mathbf{G}_a^* , denoted by $\mathbf{G}_{a(r)}^*$ and represented by $\mathbf{k}[t, s]^{gr}/(t^{p^r})$ would be our main protagonist in the development of graded 1-parameter subgroups and in the further understanding of the cohomology of a gr-group scheme.

For $p = 2$, our definition of \mathbf{G}_a^* is slightly different. This is because this case ‘behaves’ like the ungraded case; our algebras are commutative, not just graded commutative. It is enough to let $\mathbf{k}[\mathbf{G}_a^*] = \mathbf{k}[t]^{gr}$ where t can be even or odd, and in that case, $\mathbf{k}[\mathbf{G}_{a(r)}^*] = \mathbf{k}[t]^{gr}/(t^{2^r}) = \mathbf{k}[t]/(t^{2^r})$.

3.3 Graded general linear group

Definition 3.3.1. Let $I = (I_1, \dots, I_n) \in \mathbb{Z}^n$, then for any graded commutative algebra A we define the gr-group scheme

$$\mathbf{GL}_I(A) = \{(a_{ij}) \in \mathbf{GL}_n(A) \mid |a_{ij}| = I_j - I_i\}.$$

The coordinate algebra for \mathbf{GL}_I is $\mathbf{k}[\mathbf{GL}_I] = \mathbf{k}[x_{ij}, t]_{1 \leq i, j \leq n}^{gr} / (\det(x_{ij})t - 1)$, where $\Delta(x_{ij}) = \sum x_{ik} \otimes x_{kj}$, $\varepsilon(x_{ij}) = \delta_{ij}$, and $|x_{ij}| = I_j - I_i$. The gr-group scheme \mathbf{GL}_I is called the *graded general linear group indexed by I* .

Definition 3.3.2. Let $I = (I_1, \dots, I_n) \in \mathbf{Z}^n$, then for any graded commutative algebra A we define the graded affine scheme

$$\mathbf{M}_I(A) = \{(a_{ij}) \in \mathbf{M}_n(A) \mid |a_{ij}| = I_j - I_i\}.$$

The coordinate algebra for \mathbf{M}_I is $\mathbf{k}[\mathbf{M}_I] = \mathbf{k}[x_{ij}]_{1 \leq i, j \leq n}^{gr}$, where $\Delta(x_{ij}) = \sum x_{ik} \otimes x_{kj}$, $\varepsilon(x_{ij}) = \delta_{ij}$, and $|x_{ij}| = I_j - I_i$. We call it, the graded affine scheme \mathbf{M}_I of *matrices indexed by I* .

Remark 3.3.3. Note that \mathbf{GL}_I is a gr-group scheme as Δ and ε gives an antipode S (using Cramer's rule). Instead, \mathbf{M}_I is just a graded affine scheme, that is, \mathbf{M}_I is represented by $\mathbf{k}[\mathbf{M}_I]$, but for a gr-commutative algebra A , $\mathbf{M}_I(A) = \text{Hom}_{\text{gr}}(\mathbf{k}[\mathbf{M}_I], A)$, is not a group.

The quotient map $\mathbf{k}[\mathbf{M}_I] \rightarrow \mathbf{k}[\mathbf{GL}_I]$ gives an inclusion $\mathbf{GL}_I \hookrightarrow \mathbf{M}_I$ and any gr-group scheme mapping to \mathbf{M}_I factors through \mathbf{GL}_I .

Proposition 3.3.4. *Let G be a gr-group scheme. If M is a finite dimensional graded $\mathbf{k}[G]$ -comodule of dimension n , then the coaction corresponds to a map $G \rightarrow \mathbf{GL}_I$ for some $I = (I_1, \dots, I_n)$.*

Proof. Let M be a graded finite dimensional $\mathbf{k}[G]$ right-comodule with comodule map $\rho : M \rightarrow M \otimes \mathbf{k}[G]$. Let $\{v_1, \dots, v_n\}$ be a homogeneous basis for M . Let $I = (I_1, \dots, I_n) = (|v_1|, \dots, |v_n|)$. Then $\rho(v_j) = \sum v_i \otimes a_{ij}$ where $|a_{ij}| = I_j - I_i$. Since $v_j = \sum v_i \varepsilon(a_{ij})$ and the v_i 's are linearly independent we have that $\varepsilon(a_{ij}) = \delta_{ij}$.

The map $\phi : \mathbf{k}[\mathbf{M}_I] \rightarrow \mathbf{k}[G]$ given by $x_{ij} \mapsto a_{ij}$ is a gr-bialgebra morphism. To see that, $\phi(\varepsilon(x_{ij})) = \phi(\delta_{ij}) = \delta_{ij} = \varepsilon(a_{ij}) = \varepsilon(\phi(x_{ij}))$. Also since $(\rho \otimes id) \circ \rho = (id \otimes \Delta) \circ \rho$

we get that $\sum_i v_i \otimes \Delta(a_{ij}) = \sum_i v_i \otimes \sum_k a_{ik} \otimes a_{kj}$, hence $\Delta(a_{ij}) = \sum_k a_{ik} \otimes a_{kj}$. Hence $\Delta(\phi(x_{ij})) = \phi(\Delta(x_{ij}))$.

Then the map ϕ yields a gr-algebra map from $\phi^* : G(R) \rightarrow \mathbf{M}_I(R)$ given by $\phi^*(f)(x_{ij}) = f(a_{ij})$ for $f \in G(R)$. Since $G(R)$ is actually a group then ϕ^* must factor through $\mathbf{GL}_I(R)$ as desired. \square

Proposition 3.3.5. *Let G be a finite gr-group scheme, then there is a closed gr-subgroup embedding $G \hookrightarrow \mathbf{GL}_I$ for some index I .*

Proof. Since $\mathbf{k}[G]$ is assumed to be finitely dimensional, let f_1, \dots, f_n be a homogeneous basis for $\mathbf{k}[G]$ and $I = (|f_1|, \dots, |f_n|)$. Since $\mathbf{k}[G]$ is a finite dimensional comodule over itself via the comultiplication map, write $\Delta(f_j) = \sum f_i \otimes g_{ij}$.

We claim that $(g_{ij}) \in \mathbf{GL}_I(\mathbf{k}[G])$. Consider the map $\phi : \mathbf{k}[\mathbf{M}_I] \rightarrow \mathbf{k}[G]$, where $x_{ij} \mapsto g_{ij}$. We can check that ϕ is a gr-bialgebra map. Note that since $f_j = \sum \varepsilon(f_i)g_{ij}$ and $(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta$ we get that $\sum_i f_i \otimes \Delta(g_{ij}) = \sum_i f_i \otimes \sum_k g_{ik} \otimes g_{kj}$. Since the f_i 's form a minimal set we get that $\Delta(g_{ij}) = \sum_k g_{ik} \otimes g_{kj}$. Hence $\Delta(\phi(x_{ij})) = \phi(\Delta(x_{ij}))$. Since $(id \otimes \varepsilon) \circ \Delta = id$, we get that $f_j = \sum_i f_i \varepsilon(g_{ij})$, which gives that $\varepsilon(g_{ij}) = \delta_{ij}$, hence $\varepsilon(\phi(x_{ij})) = \phi(\varepsilon(x_{ij}))$. Since $(\varepsilon \otimes id) \circ \Delta = id$ we get that $f_j = \sum_i \varepsilon(f_i)g_{ij}$ and it follows that $\{g_{ij}\}_{1 \leq i, j \leq n}$ is a generating set for $\mathbf{k}[G]$. Therefore ϕ is a surjective gr-bialgebra map which yields a map $\phi^* : G(R) \rightarrow \mathbf{M}_I(R)$. Since $G(R)$ is a group then ϕ^* must factor through $\mathbf{GL}_I(R)$ as desired. \square

Proposition 3.3.6. *(From [Wat79, 3.3]) Let A be a (graded) finitely generated Hopf algebra. Let V be a (graded) comodule for A . Then any (homogeneous) $v \in V$ is contained in a finite dimensional (graded) subcomodule.*

Proof. Let $\{a_i\}$ be a (homogeneous) basis for A then $\rho(v) = \sum v_i \otimes a_i$ where all but finitely many v_i 's are zero (they are homogeneous). Write $\Delta(a_i) = \sum r_{ijk} a_j \otimes a_k$. Then

$$\sum \rho(v_i) \otimes a_i = (\rho \otimes id)\rho(v) = (id \otimes \Delta)\rho(v) = \sum v_i \otimes r_{ijk} a_j \otimes a_k.$$

We get that $\rho(v_k) = \sum v_i \otimes r_{ijk} a_j$. Hence the space spanned by v and the v_i 's is a finite dimensional (graded) subcomodule of V . \square

Remark 3.3.7. We now prove the same results in the case of G any gr-group scheme not necessarily finite.

Proposition 3.3.8. *Let G be a gr-group scheme, then there is a closed gr-subgroup embedding $G \hookrightarrow \mathbf{GL}_I$.*

Proof. Let V be a graded finite dimensional subcomodule of $\mathbf{k}[G]$ containing a finite set of generators for $\mathbf{k}[G]$. Let $\{v_1, \dots, v_n\}$ be a homogeneous basis for V and $I = (|v_1|, \dots, |v_n|)$. Note that $\{v_i\}$ is a generating set for $\mathbf{k}[G]$. Then $\Delta(v_j) = \sum v_i \otimes a_{ij}$ and $v_j = (\varepsilon \otimes id)\Delta(v_j) = \sum \varepsilon(v_i) a_{ij}$, therefore $\{a_{ij}\}$ also generate $\mathbf{k}[G]$. Again consider the map $\mathbf{k}[\mathbf{M}_I] \rightarrow \mathbf{k}[G]$ which sends $x_{ij} \mapsto a_{ij}$ then the map is surjective since the a_{ij} generate $\mathbf{k}[G]$ and is a gr-bialgebra map by the same computations as in 3.3.5 and it gives a closed embedding of G in \mathbf{GL}_I . \square

Example 3.3.9. Recall the Dual Steenrod subalgebra $A(1)$ from 1.0.12, $A(1) = \mathbf{F}_2[\xi_1, \xi_2]^{gr}/(\xi_1^4, \xi_2^2)$, then the generators ξ_1, ξ_2 are contained in the finite dimensional comodule $V = \langle 1, \xi_1, \xi_1^2, \xi_2 \rangle$; a matrix corresponding to the comodule action on V is given by

$$\phi = \begin{bmatrix} 1 & \xi_1 & \xi_1^2 & \xi_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \xi_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now $\det(\phi) = 1$, which is invertible; then the map $\phi^* : \mathbf{k}[\mathbf{GL}_I] \rightarrow \mathbf{k}[\mathbf{S}_1]$ which sends $x_{ij} \mapsto \phi_{ij}$ yields the embedding $\phi : \mathbf{S}_1 \hookrightarrow \mathbf{GL}_I$, where $I = (0, |\xi_1|, |\xi_1^2|, |\xi_2|) = (0, 1, 2, 3)$, as a gr-group scheme $\mathbf{S}_1 \hookrightarrow \mathbf{GL}_I$.

3.4 Graded 1-parameter subgroups

We recall the definition of 1-parameter subgroups as given in [SFB97b].

Definition 3.4.1. Let G be a group scheme over \mathbf{k} and $r > 0$ be a positive integer. We define the functor

$$V_r(G) : (\text{comm } \mathbf{k}\text{-alg}) \rightarrow (\text{sets})$$

by

$$V_r(G)(A) = \text{Hom}_{\text{grp}/A}(\mathbf{G}_{\mathbf{a}(r)} \otimes A, G \otimes A).$$

That is, all the A group scheme homomorphisms $\mathbf{G}_{\mathbf{a}(r)} \otimes A \rightarrow G \otimes A$.

By [SFB97b] for any commutative algebra A , we have the natural identification of

$$V_r(\mathbf{GL}_n)(A) = \{(a_0, \dots, a_{r-1}) \mid a_i \in \mathbf{M}_n(A), a_i^p = 0 = [a_j, a_l] \text{ for all } 0 \leq i, j, l < r\}.$$

We modify the definition of the scheme $V_r(G)$ in the graded case.

Definition 3.4.2. Let G be a gr-group scheme, and $r > 0$ a positive integer, we define the functor

$$V_r^*(G) : \mathcal{GR} \rightarrow (\text{sets})$$

by setting

$$V_r^*(G)(A) = \text{Hom}_{\text{gr-grp}/A}(\mathbf{G}_{\mathbf{a}(r)}^* \otimes A, G \otimes A).$$

We call $V_r^*(G)$ the *graded 1-parameter subgroups* for G .

Let $G = \mathbf{GL}_I$, we see how $V_r^*(\mathbf{GL}_I)$ compares to the ungraded version defined above. We discuss the $p > 2$, the $p = 2$ follows similarly and it is easier to compute. Recall that in this case $\mathbf{G}_{\mathbf{a}(r)}^*$ is represented by $\mathbf{k}[t, s]^{gr}/(t^p)$.

For a graded commutative algebra A , an element $f \in V_r^*(\mathbf{GL}_I)(A)$ is a map of graded A -schemes

$$f : \mathbf{G}_{\mathbf{a}(r)}^* \otimes A \rightarrow \mathbf{GL}_I \otimes A$$

and it corresponds to a $\mathbf{G}_{\mathbf{a}(r)}^* \otimes A$ -module ($A[\mathbf{G}_{\mathbf{a}(r)}^*]$ -comodule) structure on $\Sigma^I A := \bigoplus_{i=1}^n (\Sigma^{I_i} A)$ where $I = (I_1, \dots, I_n)$ and $\Sigma^{I_i} A$ corresponds to A shifted by I_i . Hence it corresponds to a map

$$\rho_f : \Sigma^I A \rightarrow A[\mathbf{G}_{\mathbf{a}(r)}^*] \otimes_A \Sigma^I A.$$

For $v \in \Sigma^I A$ we can write

$$\rho_f(v) = \sum_{k=0}^{p^r-1} (t^k \otimes \beta_k(v) + t^k s \otimes \sigma_k(v)),$$

where $\beta_k, \sigma_k \in \mathbf{M}_n(A)$.

From the counit diagram for ρ_f we get that β_0 is the identity map and from the coassociativity diagram we get the following relations:

for $i + j \leq p^r - 1$,

$$\beta_j \beta_i = \binom{i+j}{j} \beta_{i+j} = \beta_i \beta_j, \quad (3.1)$$

and

$$\beta_j \sigma_i = \beta_i \sigma_j = \sigma_i \beta_j = \sigma_j \beta_i = \binom{i+j}{j} \sigma_{i+j}, \quad (3.2)$$

and for $i + j > p^r - 1$ all the above values are zero. We also get that for any i and j ,

$$\sigma_i \sigma_j = 0. \quad (3.3)$$

Proposition 3.4.3. *Let $\alpha_i = \beta_{p^i}$ for $i = 1, \dots, r-1$ then any β_j and σ_j for $j \neq 0$ can be written in terms of the α_i 's and σ_0 .*

Proof. First note that by formula (3.2) any $\sigma_i = \beta_i \sigma_0$. Now for some β_j not of the form α_i , we can write $j = p^l q$ where p^l is maximal with respect to dividing j and $q \neq 1$. Let us assume by induction that $\beta_{p^l(q-1)}$ can be written in terms of the α_i 's. Then $j = p^l + p^l(q-1)$ and then $\binom{p^l + p^l(q-1)}{p^l} \beta_j = \beta_{p^l(q-1)} \beta_{p^l}$. It can be shown that $p \nmid \binom{p^l + p^l(q-1)}{p^l}$ therefore β_j can be described in terms of the α_i 's. \square

Remark 3.4.4. More precisely, as in [SFB97b, 1.2] we get that for any $j = \sum_{l=0}^{r-1} j_l p^l$ with $0 \leq j_l < p$,

$$\beta_j = \frac{\alpha_0^{j_0} \cdots \alpha_{r-1}^{j_{r-1}}}{(j_0!) \cdots (j_{r-1}!)},$$

and

$$\sigma_i = \frac{\alpha_0^{j_0} \cdots \alpha_{r-1}^{j_{r-1}} \sigma_0}{(j_0!) \cdots (j_{r-1}!)}.$$

The proposition above tells us that in order to describe a map ρ_f it is enough to have maps α_i for $i = 0, \dots, r-1$ and σ_0 . From now on we will relabel $\sigma = \sigma_0$ for simplicity.

Proposition 3.4.5. *Let A be a graded commutative algebra and let $I = (I_1, \dots, I_n)$, then*

$$V_r^*(\mathbf{GL}_I)(A) = \begin{cases} \{(\alpha_0, \dots, \alpha_{r-1}) \in \mathbf{M}_n(A)^r\} & \text{if } p = 2, \\ \{(\alpha_0, \dots, \alpha_{r-1}, \sigma) \in \mathbf{M}_n(A)^{r+1}\} & \text{if } p > 2. \end{cases}$$

Where

- $\alpha_k^p = \sigma^2 = [\alpha_k, \alpha_l] = [\alpha_k, \sigma] = 0$ for all $0 \leq k, l < r$,
- $|(\alpha_k)_{ij}| = I_j - I_i - p^k |t|$, and
- $|\sigma_{ij}| = I_j - I_i - |s|$.

Proof. These relations come formulas (3.1) and (3.2). □

Proposition 3.4.6. *Let A be a graded commutative algebra and let $H \in \mathbf{GL}_I(A[\mathbf{G}_{\mathbf{a}(r)}^*])$ such that H yields a graded comodule structure for $\Sigma^I A$. Then*

$$H = \sum_{k=0}^{p^r-1} \beta_k t^k + \sigma_k t^k s$$

Proof. A homogeneous basis for $\Sigma^I A$ over A must be given by $e_1 = 1_{\Sigma^{I_1} A}, e_2 = 1_{\Sigma^{I_2} A}, \dots, e_n = 1_{\Sigma^{I_n} A}$ where then $|e_i| = I_i$ then $\rho_f(e_j) = \sum g_{ij} \otimes e_i$, where $(g_{ij}) \in \mathbf{GL}_I(A[\mathbf{G}_{\mathbf{a}(r)}^*])$.

Given $v \in \Sigma^I A$ we can write $v = a_1 e_1 + \cdots + a_n e_n$ and

$$\rho_f(v) = \sum_{k=0}^{p^r-1} t^k \otimes \beta_k(v) + t^k s \otimes \sigma_k(v).$$

Note that if instead we write the coaction in terms of the basis elements we get that

$$\rho_f(e_j) = \sum_{i=1}^n g_{ij} \otimes e_i.$$

Comparing these expressions we obtain

$$\begin{aligned} \sum_{i=1}^n g_{ij} \otimes e_i &= \sum_{k=0}^{p^r-1} t^k \otimes \beta_k(e_j) + t^k s \otimes \sigma_k(e_j) \\ &= \sum_{k=0}^{p^r-1} t^k \otimes \sum_{i=1}^n (\beta_k)_{ij} e_i + t^k s \otimes \sum_{i=1}^n (\sigma_k)_{ij} e_i \\ &= \sum_{i=1}^n \sum_{k=0}^{p^r-1} [(\beta_k)_{ij} t^k + (\sigma_k)_{ij} t^k s] \otimes e_i. \end{aligned}$$

We have this Taylor polynomial type relation:

$$g_{ij} = \sum_{k=0}^{p^r-1} (\beta_k)_{ij} t^k + (\sigma_k)_{ij} t^k s.$$

□

Remark 3.4.7. For $p > 2$, given such a map $f : \mathbf{G}_{\mathbf{a}(r)}^* \otimes A \rightarrow \mathbf{GL}_I \otimes A$ as above, we get an $r + 1$ -tuple $(\underline{\alpha}, \sigma) = (\alpha_0, \dots, \alpha_{r-1}, \sigma)$.

Conversely, given $(\underline{\alpha}, \sigma)$, we can construct the map $f : \mathbf{G}_{\mathbf{a}(r)}^* \otimes A \rightarrow \mathbf{GL}_I \otimes A$ as $\exp(\underline{\alpha}, \sigma)$, defined the following way. Let R be a graded A -algebra, we define $\exp(\alpha)$ for any p -nilpotent $\alpha \in \mathbf{M}_n(R)$ as

$$\exp(\alpha) = I + \alpha + \frac{\alpha^2}{2} + \dots + \frac{\alpha^{p-1}}{(p-1)!} \in \mathbf{GL}_I(R).$$

There is a correspondence between $u \in R_{|t|}$ (p^r -nilpotent), $v \in R_{|s|}$, and $g \in (\mathbf{G}_{\mathbf{a}(r)}^* \otimes A)(R)$ given by $g(t) = u$ and $g(s) = v$. We define $\exp(\underline{\alpha}, \sigma)$ as follows: for any $u \in R_{|t|}$ and $v \in R_{|s|}$,

$$\exp(\underline{\alpha}, \sigma)(u, v) = \exp(u\alpha_0) \exp(u^p \alpha_1) \cdots \exp(u^{p^{r-1}} \alpha_{r-1}) \exp(v\sigma) \in \mathbf{GL}_I(R),$$

and from the relations in 3.4.4 we can check that $f = \exp(\underline{\alpha}, \sigma)$.

Beware that even though α_i and σ commute, $(\alpha_i t)(\sigma s)$ may not be equal to $(\sigma s)(\alpha_i t)$ since s is of odd degree, so we need to be careful when computing $\exp(\underline{\alpha}, \sigma)(u, v)$.

Proposition 3.4.8. *Given a gr-group scheme G , $V_r^*(G)$ is a graded affine scheme, that is, $V_r^*(G)$ is a representable functor from graded commutative algebras to sets. Moreover V_r^* is a functor from gr-group schemes to graded affine schemes.*

Proof. By 3.4.5 $V_r^*(\mathbf{GL}_I)$ has as a coordinate ring;

$$\frac{\mathbf{k}[X_{ij}^l]_{0 \leq l \leq r-1, 1 \leq i, j \leq n}^{gr}}{((X^l)^p, [X^l, X^k])},$$

for $p = 2$ and,

$$\frac{\mathbf{k}[X_{ij}^l, Y_{ij}]_{0 \leq l \leq r-1, 1 \leq i, j \leq n}^{gr}}{((X^l)^p, (Y)^2, [X^l, X^k], [X^l, Y])},$$

for $p > 2$.

Where X^l and Y are the $n \times n$ matrices with ij th entry being the variable X_{ij}^l and Y_{ij} respectively. These are graded by $|X_{ij}^l| = (I_j - I_i - p^l |t|)$ and $|Y_{ij}| = (I_j - I_i - |s|)$.

Let G be a gr-group scheme; by 3.3.8 there exists an embedding $\phi : G \hookrightarrow \mathbf{GL}_I$ for some I . Given a graded algebra A , elements in $V_r^*(\mathbf{GL}_I)(A)$ corresponds to an $r + 1$ -tuple $(\underline{\alpha}, \sigma) \in \mathbf{M}_n(A)^{r+1}$ (satisfying the conditions in 3.4.5) via $\exp(\underline{\alpha}, \sigma)(t, s)$. Then given $g \in V_r^*(G)(A)$ there exists an $(\underline{\alpha}, \sigma) \in V_r^*(\mathbf{GL}_I)$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathbf{G}_{\mathbf{a}(r)}^* \otimes A & \xrightarrow{\exp(\underline{\alpha}, \sigma)(t, s)} & \mathbf{GL}_I \otimes A \\ & \searrow g & \uparrow \phi \otimes A \\ & & G \otimes A \end{array}$$

The above diagram says that any element in $V_r^*(G)$ can be describe by the defining equations of the embedding on an exponential map to $\mathbf{GL}_I \otimes A$. More precisely, the embedding

$\phi : G \hookrightarrow \mathbf{GL}_I$ corresponds to a surjective map $\phi^* : \mathbf{k}[\mathbf{GL}_I] \rightarrow \mathbf{k}[G]$ and has defining equations F_1, \dots, F_m describing the kernel of ϕ^* . Then the coordinate algebra of $\mathbf{k}[V_r^*(G)]$ is precisely the quotient of $\mathbf{k}[V_r^*(\mathbf{GL}_I)]$ defined by $F_i(\exp(X_{ij}^l, Y)(t, s)) = 0$.

Note that since the coordinate ring for $V_r^*(G)$ uniquely describes $V_r^*(G)$ as a graded affine scheme then as a graded algebra $\mathbf{k}[V_r^*(G)]$ is independent of the embedding.

We now show that $V_r^*(-)$ is a functor. Let $\varphi : G \rightarrow H$ be gr-group scheme homomorphism, A a gr-commutative algebra, and $V_r^*(\varphi) : V_r^*(G) \rightarrow V_r^*(H)$ is given by the composition $\mathbf{G}_{\mathbf{a}(r)}^* \otimes A \xrightarrow{g} G \otimes A \xrightarrow{\varphi \otimes A} H \otimes A$ hence $V_r^*(\varphi)(g) = (\varphi \otimes A) \circ g \in V_r^*(H)$. It is then clear that V_r^* is a covariant functor from gr-group schemes to affine graded schemes. \square

Remark 3.4.9 (Bigraded rings; notation for bidegree and total degree). There is no standard notation for the bidegree and the total degree of a bigraded ring. We describe the notation we will use. Let R be a bigraded ring. We will use $\|x\| = (i, j)$ to denote the bidegree of a bihomogeneous element $x \in R$. We denote $\|x\|_1 = i$ and $\|x\|_2 = j$, if we wish to refer to the first or second degree. The bigraded ring R can be made into a graded ring via the first or second degree and also by the total degree, we denote the total degree of x by $\text{tot}(x) = i + j$.

We can make $\mathbf{k}[V_r^*(G)]$ into a bigraded ring with some external degrees. By convention these degrees do not introduce any Koszul sign convention. For instance, when we write

$$\mathbf{k}[V_r^*(\mathbf{GL}_I)] = \frac{\mathbf{k}[X_{ij}^l, Y_{ij}]_{0 \leq l \leq r-1, 1 \leq i, j \leq n}^{gr}}{((X^l)^p, (Y)^2, [X^l, X^k], [X^l, Y])}$$

the gr refers to $\mathbf{k}[V_r^*(\mathbf{GL}_I)]$ as a graded polynomial ring with respect to the internal degree that was computed in 3.4.8.

We bigrade $\mathbf{k}[V_r^*(G)]$ the following way

$$\begin{aligned} \|X_{ij}^l\| &= \begin{cases} (p^l, I_j - I_i - |t|p^l) & \text{for } p = 2, \\ (2p^l, I_j - I_i - |t|p^l) & \text{for } p > 2, \text{ and} \end{cases} \\ \|Y_{ij}\| &= (p^r, I_j - I_i - |s|), \text{ for } p > 2. \end{aligned}$$

Proposition 3.4.10. (From [Wil81, 2.2]) For $\mathbf{G}_{\mathbf{a}(r)}^*$ the graded cohomology is

$$H^{*,*}(\mathbf{G}_{\mathbf{a}(r)}^*, \mathbf{k}) = \mathbf{k}[\lambda_1, \dots, \lambda_r]^{gr},$$

where $\|\lambda_i\| = (1, |t|p^{i-1})$ for $p = 2$, and

$$H^{*,*}(\mathbf{G}_{\mathbf{a}(r)}^*, \mathbf{k}) = \mathbf{k}[x_1, \dots, x_r, y, \lambda_1, \dots, \lambda_r]^{gr},$$

where $\|x_i\| = (2, |t|p^i)$, $\|y\| = (1, |s|)$, and $\|\lambda_i\| = (1, |t|p^{i-1})$ for $p > 2$.

The cohomology is bigraded with the first degree corresponding to the cohomological degree and the second degree correspond to the internal degree of $\mathbf{G}_{\mathbf{a}(r)}^*$. As an algebra $H^{*,*}(\mathbf{G}_{\mathbf{a}(r)}^*, \mathbf{k})$ is gr -commutative with respect to the total degree (c.f. A.4.27), that is, the sum of the cohomological and internal degree.

Proposition 3.4.11. *The external grading on $V_r^*(G)$ corresponds to an action of \mathbf{A}^1 on $V_r^*(G)$.*

Proof. Since we can embed G into some \mathbf{GL}_I , it is enough to check this in the case of $V_r^*(\mathbf{GL}_I)$. In that case given $\gamma \in \mathbf{A}^1$ we have that

$$V_r^*(\mathbf{GL}_I) \times \mathbf{A}^1 \rightarrow V_r^*(\mathbf{GL}_I)$$

where

$$\langle \underline{\alpha}, \gamma \rangle \mapsto (\gamma \alpha_0, \gamma^p \alpha_1, \dots, \gamma^{p^{r-1}} \alpha_{r-1})$$

for $p = 2$, and

$$\langle (\underline{\alpha}, \sigma), \gamma \rangle \mapsto (\gamma^2 \alpha_0, \gamma^{2p} \alpha_1, \dots, \gamma^{2p^{r-1}} \alpha_{r-1}, \gamma^{p^r} \sigma)$$

for $p > 2$.

□

Remark 3.4.12. The action of \mathbf{A}^1 above is compatible with the action of \mathbf{A}^1 on $\mathbf{G}_{\mathbf{a}(r)}^*$ where $\langle t, \gamma \rangle \in \mathbf{G}_{\mathbf{a}(r)}^* \times \mathbf{A}^1$ maps to γt for $p = 2$ and $\langle (t, s), \gamma \rangle \in \mathbf{G}_{\mathbf{a}(r)}^* \times \mathbf{A}^1$ maps to $(\gamma^2 t, \gamma^{p^r} s)$ for $p > 2$.

This action does the following on $H^{*,*}(\mathbf{G}_{\mathbf{a}(r)}^*, \mathbf{k})$:

$$\begin{aligned} \gamma^*(\lambda_i) &= \gamma^{2^{i-1}} \lambda_i && \text{for } p = 2, \\ \gamma^*(x_i) &= \gamma^{2^{p^i}} x_i, \gamma^*(y) = \gamma^{p^r} y, \text{ and } \gamma^*(\lambda_i) = \gamma^{2^{p^{i-1}}} \lambda_i && \text{for } p > 2. \end{aligned}$$

3.4.1 Computing $\mathbf{k}[V_r^*(G)]$ for some gr-group schemes

To compute $\mathbf{k}[V_r^*(G)]$ we embed our gr-group schemes into some \mathbf{GL}_I as described in 3.3.8. The defining equations of the embedding $\phi : G \hookrightarrow \mathbf{GL}_I$ corresponds to a surjective map $\phi^* : \mathbf{k}[\mathbf{GL}_I] \rightarrow \mathbf{k}[G]$ and has defining equations F_1, \dots, F_m describing the kernel of ϕ^* . Then the coordinate algebra of $\mathbf{k}[V_r^*(G)]$ is precisely the quotient of $\mathbf{k}[V_r^*(\mathbf{GL}_I)]$ defined by $F_i(\exp(X_{ij}^l, Y)(t, s)) = 0$ for $p > 2$ and by $F_i(\exp(X_{ij}^l)(t)) = 0$ for $p = 2$.

Example 3.4.13. Recall the gr-group scheme \mathbf{S}_1 from 1.0.12, \mathbf{S}_1 is represented by $A(1) = \mathbf{F}_2[\xi_1, \xi_2]^{gr}/((\xi_1^4, \xi_2^2))$. From 3.3.9 the embedding $\phi : \mathbf{S}_1 \rightarrow \mathbf{GL}_I$

$$\phi = \begin{bmatrix} 1 & \xi_1 & \xi_1^2 & \xi_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \xi_1 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where $I = (0, |\xi_1|, |\xi_1^2|, |\xi_2|) = (0, 1, 2, 3)$.

The gr-group scheme \mathbf{S}_1 has height 2, so we compute $\mathbf{F}_2[V_2^*(\mathbf{S}_1)]$. We have that

$$\mathbf{F}_2[V_2^*(\mathbf{GL}_I)] = \frac{\mathbf{k}[X_{ij}^0, X_{ij}^1]_{1 \leq i, j \leq 4}^{gr}}{((X^0)^2, (X^1)^2, [X^0, X^1])}.$$

Let $Z = \exp(X^0, X^1)(t) = I + X^0 t + X^1 t^2 + X^0 X^1 t^3$ be the 4×4 matrix with entries in $\mathbf{F}_2[V_2^*(\mathbf{GL}_I)]$. From ϕ we get that the defining equations are

- $Z_{ii} = 1$,
- $Z_{21} = Z_{23} = Z_{24} = Z_{31} = Z_{32} = Z_{41} = Z_{42} = Z_{43} = 0$,

- $Z_{12} = Z_{34}$,
- $Z_{12}^2 = Z_{13}$, and
- $Z_{12}^4 = 0, Z_{14}^2 = 0$.

We get some relations on the entries of X^0 and X^1 . To begin with, the only possibly nonzero entries are $X_{12}^l, X_{13}^l, X_{14}^l, X_{34}^l$ for $l = 0, 1$. Also, from the definition of $\mathbf{F}_2[V_2^*(\mathbf{GL}_I)]$, we also have that $(X^0)^2 = (X^1)^2 = 0$ and $[X^0, X_1] = 0$. Putting all of these together we get that,

- $X_{12}^0 = X_{34}^0, X_{12}^1 = X_{34}^1$,
- $(X_{12}^0)^2 = X_{13}^1, X_{13}^0 = 0$,
- $(X_{14}^0)^2 = 0$, and
- $(X_{13}^1)^2 = 0, X_{13}^1 X_{12}^1 = 0$.

Therefore

$$\mathbf{F}_2[V_2^*(\mathbf{S}_1)] = \frac{\mathbf{F}_2[X_{14}^0, X_{12}^1, X_{13}^1, X_{14}^1]^{gr}}{((X_{14}^0)^2, (X_{13}^1)^2, (X_{13}^1 X_{12}^1))},$$

which is F -isomorphic to $\mathbf{F}_2[X_{12}^1, X_{14}^1]^{gr}$ where $\|X_{12}^1\| = (2, 1 - 2|t|)$ and $\|X_{14}^1\| = (2, 3 - 2|t|)$.

Example 3.4.14. (Wilkerson's counterexample [Wil81, 6.3])

Consider the quotient of the dual of the Steenrod algebra given by

$$\mathbf{F}_p[W_1] = \frac{\mathbf{F}_p[\xi_1, \xi_2, \xi_3]^{gr}}{(\xi_1^p, \xi_2^{p^2}, \xi_3^p)}.$$

We can embed $\phi : W_1 \rightarrow \mathbf{GL}_I$ where

$$\phi = \begin{bmatrix} 1 & \xi_1 & \xi_2 & \xi_2^p & \xi_3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \xi_1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We compute $\mathbf{F}_p[V_r^*(W_1)]$ for $r = 2$ which is the height of $\mathbf{F}_p[W_1]$.

- For $p = 2$, let $I = (0, 1, 3, 6, 7)$ and

$$\mathbf{F}_2[V_2^*(\mathbf{GL}_I)] = \frac{\mathbf{F}_2[X_{ij}^0, X_{ij}^1]_{1 \leq i, j \leq 5}^{gr}}{((X^0)^2, (X^1)^2, [X^0, X^1])}.$$

Then $\exp(X^0, X^1)(t) = I + X^0 t + X^1 t^2 + X^0 X^1 t^3$. The defining equations for the embedding give that $X_{ij}^0 = X_{ij}^1 = 0$ for $ij \neq 12, 13, 14, 15, 45$. This also implies that $(X^0 X^1)_{ij} = 0$ except for $ij = 15$ and in that case $(X^0 X^1)_{15} = X_{14}^0 X_{45}^1 = X_{14}^1 X_{45}^0$. Among other relations, we get that, $\mathbf{F}_2[V_2^*(W_1)]$ is F -isomorphic to

$$\frac{\mathbf{F}_2[X_{13}^0, X_{12}^1, X_{13}^1, X_{15}^1]^{gr}}{((X_{13}^0)^2 X_{12}^1)},$$

where

- $\|X_{13}^0\| = (1, 3 - |t|)$,
- $\|X_{12}^1\| = (2, 1 - 2|t|)$,
- $\|X_{13}^1\| = (2, 3 - 2|t|)$, and
- $\|X_{15}^1\| = (2, 7 - 2|t|)$.

- For $p > 2$ we have that $I = (0, 2(p-1), 2(p^2-1), 2p(p^2-1), 2(p^3-1))$, and

$$\mathbf{F}_p[V_2^*(\mathbf{GL}_I)] = \frac{\mathbf{F}_p[X_{ij}^0, X_{ij}^1, Y_{ij}]_{1 \leq i, j \leq 5}^{gr}}{((X^0)^p, (X^1)^p, (Y_{ij})^2, [X^0, X^1], [X^0, Y], [X^1, Y])}.$$

Then doing computations as in the $p = 2$ case, we find that $\mathbf{F}_p[V_2^*(W_1)]$ is F -isomorphic to

$$\mathbf{F}_p[X_{13}^0, X_{12}^1, X_{13}^1, X_{15}^1]^{gr},$$

where

- $\|X_{13}^0\| = (2, 2(p^2-1) - |t|)$,
- $\|X_{12}^1\| = (2p, 2(p-1) - p|t|)$,

$$\begin{aligned} - \|X_{13}^1\| &= (2p, 2(p^2 - 1) - p|t|), \\ - \|X_{15}^1\| &= (2p, 2(p^3 - 1) - p|t|), \end{aligned}$$

Example 3.4.15. (Wilkerson's counterexample [Wil81, 6.5]) Let

$$\mathbf{F}_2[W_2] = \frac{\mathbf{F}_2[x_i]_{1 \leq i \leq 5}^{gr}}{(x_i^2)},$$

and x_i is primitive for $i < 5$ and $\Delta(x_5) = x_5 \otimes 1 + x_1 \otimes x_4 + x_2 \otimes x_3 + 1 \otimes x_5$. We can embed $\phi : W_2 \rightarrow \mathbf{GL}_I$ where

$$\phi = \begin{bmatrix} 1 & x_1 & x_2 & x_3 & x_4 & x_5 \\ 0 & 1 & 0 & 0 & 0 & x_4 \\ 0 & 0 & 1 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and $I = (0, 1, 2, 3, 4, 5)$.

The height of $\mathbf{F}_2[W_2]$ is $r = 1$, then

$$\mathbf{F}_2[V_1^*(\mathbf{GL}_I)] = \frac{\mathbf{F}_2[X_{ij}]_{1 \leq i, j \leq 6}^{gr}}{(X^2)},$$

where $\|X_{ij}\| = (1, j - i - |t|)$.

By the defining equation of the embedding we get that

$$\mathbf{F}_2[V_1^*(W_2)] = \frac{\mathbf{F}_2[X_{12}, X_{13}, X_{14}, X_{15}, X_{16}]^{gr}}{(X_{12}X_{15} + X_{13}X_{14})}.$$

Example 3.4.16. Consider the quotient of the dual of the Steenrod algebra given by

$$\mathbf{F}_p[G] = \frac{\mathbf{F}_p[\xi_1, \tau_0, \tau_1]^{gr}}{(\xi_1^p)},$$

for $p > 2$ where $|\xi_1| = 2(p - 1)$, $|\tau_0| = 1$ and $|\tau_1| = 2p - 1$, ξ_1 and τ_0 are primitive and $\Delta(\tau_1) = \tau_1 \otimes 1 + \xi_1 \otimes \tau_0 + 1 \otimes \tau_1$.

We can embed $\phi : G \hookrightarrow \mathbf{GL}_I$ where $I = (0, 2(p - 1), 1, 2p - 1)$ and

$$\phi = \begin{bmatrix} 1 & \xi_1 & \tau_0 & \tau_1 \\ 0 & 1 & 0 & \tau_0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then the height is $r = 1$ and

$$\mathbf{F}_p[V_1^*(\mathbf{GL}_I)] = \frac{\mathbf{F}_p[X_{ij}, Y_{ij}]_{1 \leq i, j \leq 4}^{gr}}{(X^p, Y^2, [X, Y])}.$$

For a defining equation F_i for ϕ , the equation $F_i(\exp(X, Y)(t, s)) = 0$ gives us that

- $X_{ij} = Y_{ij} = 0$ for $ij \neq 12, 13, 14, 24$,
- $(X^2)_{ij} = 0$ except for $(X^2)_{14} = X_{12}X_{24}$, and
- $(XY)_{ij} = 0$ except $(XY)_{14} = X_{12}Y_{24} = Y_{12}X_{24}$,
- $X^3 = 0$ and $XY^2 = 0$,
- $(X_{13})^2 = (X_{14})^2 = 0$,

among other relations. Then $\mathbf{F}_p[V_1^*(G)]$ is F -isomorphic to

$$\frac{\mathbf{F}_p[X_{12}, Y_{12}, Y_{13}, Y_{14}]^{gr}}{(Y_{12}Y_{13}, X_{12}Y_{13} - Y_{12}X_{13})},$$

with bidegrees

- $\|X_{12}\| = (2, 2(p-1) - |t|)$,
- $\|Y_{12}\| = (p, 2(p-1) - |s|)$,
- $\|Y_{13}\| = (p, 1 - |s|)$, and
- $\|Y_{14}\| = (p, 2p-1 - |s|)$.

3.5 The algebra map $\psi : H^{*,*}(G, \mathbf{k}) \rightarrow \mathbf{k}[V_r^*(G)]$

For a gr-group scheme G we give an algebra map from $H^{*,*}(G, \mathbf{k})$ to $\mathbf{k}[V_r^*(G)]$. For reference we state the results in the ungraded case.

Theorem 3.5.1. ([SFB97b, 1.14]) *For any affine group scheme G , there is a natural homomorphism of graded commutative algebras $\psi : H^{ev}(G, \mathbf{k}) \rightarrow \mathbf{k}[V_r(G)]$ which multiplies degrees by $p^r/2$.*

Theorem 3.5.2. ([SFB97b, 1.14.1]) *Assume that $p = 2$. Then for any affine group scheme G , there is a natural homomorphism of graded commutative algebras $\psi : H^*(G, \mathbf{k}) \rightarrow \mathbf{k}[V_r(G)]$ which multiplies degrees by 2^{r-1} .*

For a gr-group variety we get the following graded version of ψ .

Theorem 3.5.3. *Let G be a gr-group scheme. There is an algebra map*

$$\psi : H^{*,*}(G, \mathbf{k}) \rightarrow \mathbf{k}[V_r^*(G)].$$

If $z \in H^{n,m}(G, \mathbf{k})$, then

- for $p > 2$, $\psi(z)$ is a sum of bihomogeneous pieces of bidegree $(np^r, m - |t|p^r l - k|s|)$ where $2l + k = n$;
- for $p = 2$, $\psi(z)$ is bihomogeneous of bidegree $(n2^{r-1}, m - n|t|2^{r-1})$.

Proof. Consider the identity map $1 \in V_r^*(G)(\mathbf{k}[V_r^*(G)])$; this map corresponds to some $u : \mathbf{G}_{\mathbf{a}(r)}^* \times V_r^*(G) \rightarrow G \times V_r^*(G)$.

For any graded algebra A , an element in $(\mathbf{G}_{\mathbf{a}(r)}^* \times V_r^*(G))(A)$ corresponds to $\langle (a, b), f \rangle$, where (a, b) is the map in $\mathbf{G}_{\mathbf{a}(r)}^*(A)$ such that $t \mapsto a$ and $s \mapsto b$ and $f : \mathbf{G}_{\mathbf{a}(r)}^* \otimes A \rightarrow G \otimes A$.

The identity map $1 \in V_r^*(G)(\mathbf{k}[V_r^*(G)])$ corresponds to $u : \mathbf{G}_{\mathbf{a}(r)}^* \times V_r^*(G) \rightarrow (G \times V_r^*(G))$ where for $\langle (a, b), f \rangle \in (\mathbf{G}_{\mathbf{a}(r)}^* \times V_r^*(G))(A)$, $u(\langle (a, b), f \rangle) = \langle f(a, b), f \rangle$.

To see this we focus on the case of \mathbf{GL}_I . It can be checked that

$$1 \in V_r^*(\mathbf{GL}_I)(\mathbf{k}[V_r^*(\mathbf{GL}_I)])$$

corresponds to

$$u = \exp(\underline{X}, Y)(t, s)$$

where $\underline{X} = (X^0, \dots, X^{r-1})$. Note also that f is given by an $r + 1$ -tuple $(\underline{\alpha}, \sigma)$ such that $\exp(\underline{\alpha}, \sigma)(t, s) = f$. Then $u(\langle (a, b), f \rangle) = \langle \exp(\underline{\alpha}, \sigma)(a, b), \exp(\underline{\alpha}, \sigma)(t, s) \rangle$. This can be unraveled by saying that u sends $\langle (a, b), f \rangle \mapsto \langle f(a, b), f \rangle \in (\mathbf{GL}_I \times V_r^*(\mathbf{GL}_I))(A)$.

For a gr-group scheme G and embedding $\phi : G \hookrightarrow \mathbf{GL}_I$ the following diagram commutes:

$$\begin{array}{ccc} \mathbf{G}_{\mathbf{a}(r)}^* \times V_r^*(G) & \hookrightarrow & \mathbf{G}_{\mathbf{a}(r)}^* \times V_r^*(\mathbf{GL}_I) \\ u \downarrow & & \downarrow u \\ G \times V_r^*(G) & \hookrightarrow & \mathbf{GL}_I \times V_r^*(\mathbf{GL}_I), \end{array}$$

which gives that u is as described above.

We now define ψ ; its definition will depend on whether $p = 2$ or $p > 2$, recall $H^{*,*}(\mathbf{G}_{\mathbf{a}(r)}^*, \mathbf{k})$ from 3.4.10.

- For $p = 2$ and $z \in H^{n,m}(G, \mathbf{k})$,

$$u^*(z) = \sum \lambda^j \otimes f_j(z) \in H^{*,*}(\mathbf{G}_{\mathbf{a}(r)}^*, \mathbf{k}) \otimes \mathbf{k}[V_r^*(G)],$$

where $\lambda^j = \lambda_1^{j_1} \dots \lambda_r^{j_r}$. We define $\psi(z)$ as the coefficient for λ_r^n in $\mathbf{k}[V_r^*(G)]$.

- For $p > 2$ and $z \in H^{n,m}(G, \mathbf{k})$,

$$u^*(z) = \sum \lambda^j x^i y^k \otimes f_{ijk}(z) \in H^{*,*}(\mathbf{G}_{\mathbf{a}(r)}^*, \mathbf{k}) \otimes \mathbf{k}[V_r^*(G)],$$

where $\lambda^j x^i y^k = \lambda_1^{j_1} \dots \lambda_r^{j_r} x_1^{i_1} \dots x_r^{i_r} y^k$. We define $\psi(z)$ as the sum of all the coefficients for $x_r^l y^k$ in $\mathbf{k}[V_r^*(G)]$ such that $n = 2l + k$.

For any $\gamma \in \mathbf{A}^1$, the following diagram commutes, where γ is acting on $\mathbf{G}_{\mathbf{a}(r)}^*$ and $V_r^*(G)$ as described in 3.4.11.

$$\begin{array}{ccccc} \mathbf{G}_{\mathbf{a}(r)}^* \times V_r^*(G) & \xrightarrow{1 \times \gamma} & \mathbf{G}_{\mathbf{a}(r)}^* \times V_r^*(G) & \xrightarrow{u} & G \times V_r^*(G) \\ \gamma \times 1 \downarrow & & & & \downarrow \pi_1 \\ \mathbf{G}_{\mathbf{a}(r)}^* \times V_r^*(G) & \xrightarrow{u} & G \times V_r^*(G) & \xrightarrow{\pi_1} & G. \end{array}$$

To figure out the grading of $\psi(z)$ we pullback on the diagram above in the two possible ways and compare them.

For $p = 2$ we get that

$$\sum \gamma^{j_1 + 2j_2 + \dots + 2^{r-1}j_r} \lambda^j \otimes f_j(z),$$

which implies that $\|\psi(z)\| = (n2^{r-1}, m - n|t|2^{r-1})$.

For $p > 2$ we have that

$$\sum \gamma^{2j_1 + 2pj_2 + \dots + 2p^{r-1}j_r + 2pi_1 + \dots + 2p^r i_r + p^r k} \lambda^j x^i y^k \otimes f_{ijk}(z).$$

In this case $\psi(z)$ is only homogeneous with respect to the cohomological degree and not bihomogeneous, but the bihomogeneous pieces of $\psi(z)$ have bidegree $(np^r, m - |t|p^r l - k|s|)$ where $2l + k = n$.

It can be easily checked that ψ is indeed an algebra map. □

3.6 *F-injectivity of ψ*

From [SFB97b] and [SFB97a] we have that for any infinitesimal group scheme of height $\leq r$, the ungraded version of ψ is an F -isomorphism. A result of this type allows us to compute the cohomology of a group scheme G (up to nilpotents) in a fairly straightforward way. We can do this by embedding G into some \mathbf{GL}_n and using the defining equations of this embedding to describe $\mathbf{k}[V_r(G)]$ as a quotient of $\mathbf{k}[V_r(\mathbf{GL}_n)]$.

Given the usefulness of such a result, it is a natural question to ask if we get a similar result in our case. In our quest to answer this question we compute some examples of the

map ψ and we show that they are in fact F -isomorphisms. Then, we define some detection properties of the cohomology of a gr-group scheme and study their relation. Finally, we (partially) answer the question for a class of gr-group schemes.

Recall that a map between \mathbf{k} -algebras is F -isomorphism if its kernel consists of nilpotent elements, and some p th power of every element in the target is actually in the image. We also recall from A.4.27 that the cohomology of a finite gr-group scheme is graded commutative with respect to the total degree.

3.6.1 Examples of ψ for some gr-group schemes

Example 3.6.1. For $\mathbf{G}_{\mathbf{a}(r)}^*$ and $p > 2$:

- $H^{*,*}(\mathbf{G}_{\mathbf{a}(r)}^*, \mathbf{k})$ is F -isomorphic to $\mathbf{k}[x_1, \dots, x_r, y]^{gr}$ where $\|x_i\| = (2, |u|p^i), |y| = (1, |v|)$.
- $\mathbf{k}[V_r^*(\mathbf{G}_{\mathbf{a}(r)}^*)]$ is F -isomorphic to $\mathbf{k}[x^0, \dots, x^{r-1}, y^0]^{gr}$ where $\|x^i\| = (2p^i, |u| - |t|p^i)$, and $\|y^0\| = (p^r, |v| - |s|)$.

By comparing bidegrees, $\psi(x_i) = (x^{r-i})^{p^i}, \psi(y) = y^0$.

Example 3.6.2. Recall the gr-group scheme \mathbf{S}_1 from 1.0.12 which has coordinate algebra $\mathbf{k}[\mathbf{S}_1] = \frac{\mathbf{F}_2[\xi_1, \xi_2]^{gr}}{(\xi_1^4, \xi_2^2)}$. We have that $H^{*,*}(\mathbf{S}_1, \mathbf{F}_2)$ is F -isomorphic to $\mathbf{F}_2[h_{10}, h_{20}]$, where $\|h_{10}\| = (1, 1)$, and $\|h_{20}\| = (1, 3)$.

We compute on 3.4.13 that $\mathbf{F}_2[V_2^*(\mathbf{S}_1)]$ is F -isomorphic to $\mathbf{F}_2[x, y]^{gr}$ where $\|x\| = (2, 1 - 2|t|)$ and $\|y\| = (2, 3 - 2|t|)$. By comparing bidegrees, $\psi(h_{10}) = x$ and $\psi(h_{20}) = y$.

Example 3.6.3. Recall Wilkerson's counterexample from 3.4.14, $\mathbf{F}_p[W_1]$. For $p > 2$, we have that $\mathbf{F}_p[V_2^*(W_1)]$ is F -isomorphic to $\mathbf{F}_p[X_{13}^0, X_{12}^1, X_{13}^1, X_{15}^1]^{gr}$. From [Wil81, 6.3] we have that $H^{*,*}(W_1, \mathbf{k})$ is F -isomorphic to

$$\mathbf{F}_p[b_{10}, b_{20}, b_{21}, b_{30}]^{gr}$$

where

- $\|b_{10}\| = (2, 2p(p-1))$,
- $\|b_{20}\| = (2, 2p(p^2-1))$,
- $\|b_{21}\| = (2, 2p^2(p^2-1))$, and
- $\|b_{30}\| = (2, 2p(p^3-1))$.

By comparing bidegrees, ψ corresponds to the map that sends $b_{10} \mapsto (X_{12}^1)^p$, $b_{20} \mapsto (X_{13}^1)^p$, $b_{21} \mapsto (X_{13}^0)^{p^2}$, and $b_{30} \mapsto (X_{15}^1)^p$.

Example 3.6.4. Recall Wilkerson's counterexample from 3.4.15: $\mathbf{F}_2[W_2]$, $\mathbf{F}_2[V_1^*(W_2)]$ is F -isomorphic to $\mathbf{F}_2[x_1, x_2, x_3, x_4, x_5]/(x_1x_4 + x_2x_3)$, where $\|x_i\| = (1, i - |t|)$. From [Wil81, 6.5], $H^{*,*}(W_2, \mathbf{k})$ is F -isomorphic to

$$\frac{\mathbf{F}_2[z_1, z_2, z_3, z_4, z_5^2]}{(z_1z_4 + z_2z_3)},$$

where $\|z_i\| = (1, i)$. We then have that $\psi(z_i) = x_i$ for $i < 5$ and $\psi(z_5^2) = x_5^2$.

3.6.2 Detection properties

In [SFB97a], they show that the ungraded ψ is an F -isomorphism. To show that ψ is an F -monomorphism they first show that the cohomology of infinitesimal group schemes of height $\leq r$ satisfies a detection property. By a *detection property* we mean some sort of Quillen-type result in which the cohomology of our object can be detected (up to nilpotent) by understanding the cohomology of some sub-class or restriction of this object.

For infinitesimal group schemes, the detection property is one that detects the cohomology up to nilpotents by restricting to 1-parameter subgroups. More precisely, in [SFB97a, 4.3] they show that $z \in H^n(G, \mathbf{k})$ is nilpotent if and only if for every field extension K/\mathbf{k} and every group scheme homomorphism over K , $\nu : \mathbf{G}_{a(r)} \otimes K \rightarrow G \otimes K$, the cohomology class $\nu^*(z_K) \in H^n(\mathbf{G}_{a(r)} \otimes K, K)$ is nilpotent.

For gr-Hopf algebras, in [Wil81], a detection property is defined in terms of elementary sub-Hopf algebras. A graded Hopf algebra as in [Wil81] satisfies the detection property if

each nonnilpotent cohomology class has a nonzero restriction to at least one elementary sub-Hopf algebra. It is important to note that not all gr-Hopf algebras as in [Wil81] satisfy this detection property. Hence we do not get a Quillen-type result as we do in the ungraded case.

Wilkerson's counterexamples in [Wil81] are example 3.4.14 (for the $p > 2$ case) and 3.4.15 above. He showed that they do not satisfy his detection property. Although we computed that for these gr-group schemes ψ is an F -isomorphism, what happens is that even though they do not satisfy this detection property they satisfy some detection property (one like that of [SFB97a]), that suffices to guarantee that ψ is an F -monomorphism. We define such detection property below and compare it to Wilkerson's. We conclude by proving that if a gr-group variety satisfies such detection property, then ψ is an F -monomorphism.

Definition 3.6.5. (From [Pal01, 2.1.6]) A gr-group scheme G_E is said to be an *elementary gr-group scheme* if its coordinate ring E is isomorphic (as a graded Hopf algebra) to a tensor product of graded Hopf algebras of the forms

- $\mathbf{k}[t]^{gr}/(t^{2^n})$ for $p = 2$, and
- $\mathbf{k}[t]^{gr}/(t^{p^n})$ and $\mathbf{k}[s]^{gr}$ where $|t|$ is even and $|s|$ is odd for $p > 2$,

where t and s are primitive elements.

Remark 3.6.6. Note that as usual, our definition includes the case in which t is possibly of degree zero, while the definition in [Pal01] and [Wil81] is only for algebraically connected gr-group schemes.

We now give the definition of two detection properties; one based on that of [SFB97a] the other on [Wil81].

Definition 3.6.7. Let G be a gr-group scheme, we say that G has the *W-detection property* if $z \in H^{n,m}(G, \mathbf{k})$ is nilpotent if and only if for each elementary gr-subgroup scheme of G , G_E , its restriction to $H^{n,m}(G_E, \mathbf{k})$ is nilpotent.

Given a field K , we can construct the *graded field* $K[X, X^{-1}]$ ($K[X^\pm]$ for short), where $|X| = 1$ if $p = 2$, and $|X| = 2$ if $p > 2$. Constructing this graded field is a useful tool that allows us to ‘grade’ scalars on our field K . For instance, if we want to view a scalar $\lambda \in K$ as an element of degree $l \in \mathbf{Z}$, then we will identify λ with $\lambda X^l \in K[X^\pm]$. Note that for $p = 2$, $|\lambda X^l| = l$, while for $p > 2$, $|\lambda X^l| = 2l$.

Definition 3.6.8. Let G be a gr-group scheme of height $\leq r$, we say that G has the *SFB-detection property* if $z \in H^{n,m}(G, \mathbf{k})$ is nilpotent if and only if for every field extension K of \mathbf{k} and every gr-group scheme homomorphism over $K[X^\pm]$, $\nu : \mathbf{G}_{\mathbf{a}(r)}^* \otimes K[X^\pm] \rightarrow G \otimes K[X^\pm]$ the restriction of z to $H^{n,m}(\mathbf{G}_{\mathbf{a}(r)}^* \otimes K[X^\pm], K[X^\pm])$ is nilpotent.

Note that while the detection property in [Wil81] is for algebraically connected cocommutative graded Hopf algebras and [SFB97a] detection property is for (ungraded) infinitesimal group schemes, our detection properties are constructed for gr-group schemes.

One word on why we choose $K[X^\pm]$ for the SFB-detection property: for a field \mathbf{k} , a graded field extension of \mathbf{k} may be one of the following; K where K is a field extension in the ungraded sense, or the graded field $K[X^\pm]$ where K is a field extension in the ungraded sense.

We state the result saying that all infinitesimal group schemes satisfy an ungraded detection property and we compare it to the SFB-detection property defined above. We will refer to this result as the ungraded SFB-detection property.

Theorem 3.6.9. (From [SFB97a, 4.3]) *Let G be an infinitesimal group scheme of height $\leq r$ over \mathbf{k} . Let $z \in H^n(G, \mathbf{k})$, then z is nilpotent if and only if for every field extension K/\mathbf{k} and every group scheme homomorphism over K , $\nu : \mathbf{G}_{\mathbf{a}(r)} \otimes K \rightarrow G \otimes K$, the cohomology class $\nu^*(z_K) \in H^n(\mathbf{G}_{\mathbf{a}(r)} \otimes K, K)$ is nilpotent.*

Lemma 3.6.10. *Let G be an evenly gr-group scheme and K an ungraded field, then $\text{Hom}(\mathbf{k}[G], K) = \text{Hom}_{\mathfrak{gr}}(\mathbf{k}[G], K[X^\pm])$.*

Proof. We have the following identification, $f \in \text{Hom}(\mathbf{k}[G], K) \leftrightarrow \hat{f} \in \text{Hom}_{\mathfrak{gr}}(\mathbf{k}[G], K[X^\pm])$, where for g homogeneous in $\mathbf{k}[G]$, $f(g) = \lambda \leftrightarrow \hat{f}(g) = \lambda X^{|g|}$. \square

Proposition 3.6.11. *Let G be an evenly gr-group scheme and let K be an ungraded field, then $V_r(G)$ and $V_r^*(G)$ have the same coordinate ring and $V_r(G)(K) = V_r^*(G)(K[X^\pm])$.*

Proof. Since G is evenly graded we can embed G into a \mathbf{GL}_I which is also evenly graded. In this case $\mathbf{k}[V_r^*(\mathbf{GL}_I)]$ is commutative (not just graded commutative) hence it is equal to $\mathbf{k}[V_r(\mathbf{GL}_n)]$, by just forgetting the grading, where n is the length of I . Similarly, $\mathbf{k}[V_r^*(G)] = \mathbf{k}[V_r(G)]$ if we forget the grading. Hence $V_r(G)$ and $V_r^*(G)$ have the same coordinate ring and $V_r(G)(K) = V_r^*(G)(K[X^\pm])$ by 3.6.10 \square

Note that $V_r(G)$ is not equal to $V_r^*(G)$, since they are representable functors on different categories even though they have the same coordinate ring. That is, $V_r(G)(A) = \text{Hom}_{\mathfrak{gR}}(\mathbf{k}[V_r(G)], A)$ where A is a commutative algebra, while $V_r^*(G)(B) = \text{Hom}(\mathbf{k}[V_r(G)], B)$ where B is a graded commutative algebra.

Proposition 3.6.12. *Let G be an evenly gr-group scheme. Then the ungraded SFB-detection property (see 3.6.9) implies the SFB-property (see 3.6.8). Moreover, G has the SFB-detection property.*

Proof. First note that by A.4.8 $H^*(G, \mathbf{k}) = H^{*,*}(G, \mathbf{k})$ if we forget the internal grading. Let $z \in H^{n,m}(G, \mathbf{k})$ be nilpotent, then since G has the ungraded SFB-detection property we get that for every $\nu : \mathbf{G}_{\mathbf{a}(r)}^* \otimes K[X^\pm] \rightarrow G \otimes K[X^\pm]$, $\nu^*(z) \in H^{n,m}(\mathbf{G}_{\mathbf{a}(r)}^* \otimes K[X^\pm], K[X^\pm])$ is nilpotent (by tensoring by $\mathbf{k}[X^\pm]$).

Let $\nu^*(z) \in H^{n,m}(\mathbf{G}_{\mathbf{a}(r)}^* \otimes K[X^\pm], K[X^\pm])$ be nilpotent for every $\nu : \mathbf{G}_{\mathbf{a}(r)}^* \otimes K[X^\pm] \rightarrow G \otimes K[X^\pm]$. By 3.6.11 it follows that z is nilpotent. \square

Proposition 3.6.13. *Let G be a finite gr-group scheme, the W -detection property implies the SFB-detection property. In particular, elementary gr-group schemes have the SFB-detection property.*

Proof. Without loss of generality, we can assume that $G = G_E$ is an elementary gr-group scheme of height $\leq r$. We will prove that G_E has SFB-detection property. For simplicity

let $p > 2$; the $p = 2$ case follows in the same way. The coordinate ring of G_E is

$$E = \frac{\mathbf{k}[t_1, \dots, t_n, s_1, \dots, s_m]^{gr}}{(t_1^{p^{r_1}}, \dots, t_n^{p^{r_n}})},$$

where the t_i 's and s_i 's are primitive, $|t_i|$ is even, and $|s_i|$ is odd.

Let $A = K[X^\pm]$, consider the map from $A[G_E] \rightarrow A[\mathbf{G}_{\mathbf{a}(r)}^*]$ given by $t_i \mapsto t^{p^{r-r_i}} X^{k_i}$ and $s_i \mapsto s X^{m_i}$ where $k_i = (|t_i| - p^{r-r_i}|t|)/2$ and $m_i = (|s_i| - |s|)/2$; which are integers. This yields the map $\nu_E : \mathbf{G}_{\mathbf{a}(r)}^* \otimes A \rightarrow G_E \otimes A$. From 3.4.10, the cohomology of G_E and $\mathbf{G}_{\mathbf{a}(r)}^*$ are:

$$H^{*,*}(G_E, \mathbf{k}) = \bigotimes_{i=1}^n (\mathbf{k}[x_{i1}, \dots, x_{ir_i}, y_i, \lambda_{i1}, \dots, \lambda_{ir_i}])^{gr}, \text{ and}$$

$$H^{*,*}(\mathbf{G}_{\mathbf{a}(r)}^*, \mathbf{k}) = \mathbf{k}[x_1, \dots, x_r, y, \lambda_1, \dots, \lambda_r]^{gr}.$$

Then $\nu_E^* : H^{*,*}(G_E \otimes A, A) \rightarrow H^{*,*}(\mathbf{G}_{\mathbf{a}(r)}^* \otimes A, A)$ is given by $x_{ij} \mapsto x_{r-r_i+j} X^{p^j k_i}$, $y_i \mapsto y X^{m_i}$, and $\lambda_{ij} \mapsto \lambda_j X^{l_i}$ where $l_i = (|t_i| - |t|)/2$. Then ν_E^* sends nonnilpotent elements to nonnilpotent elements, hence G_E has the SFB-detection property. \square

Remark 3.6.14. In [Wil81] Wilkerson showed that examples $\mathbf{F}_p[W_1]$ and $\mathbf{F}_2[W_2]$ in 3.4.14 and 3.4.15, respectively, do not satisfy the W-detection property. However, since W_1 and W_2 are evenly gr-group schemes, by 3.6.9 and 3.6.12 they satisfy the SFB-detection property.

Lemma 3.6.15. *Let G be a finite gr-group variety of height $\leq r$ with the SFB-property, then for any $z \in H^{n,m}(G, \mathbf{k})$, z is nilpotent whenever $\psi(z) \in \mathbf{k}[V_r^*(G)]$ is nilpotent.*

Proof. Our proof is based on that in [SFB97a, 5.1]. Let $z \in H^{n,m}(G, \mathbf{k})$ with $\psi(z)$ nilpotent. Since G has the SFB-property, it is enough to prove that, for all field extensions K of \mathbf{k} and every gr-group scheme homomorphism $\nu : \mathbf{G}_{\mathbf{a}(r)}^* \otimes A \rightarrow G \otimes A$, $\nu^*(z_A) \in H^{n,m}(\mathbf{G}_{\mathbf{a}(r)}^* \otimes A, A)$ is nilpotent, where $A = K[X^\pm]$. Without loss of generality, let K be algebraically closed. Let $\overline{\nu^*(z_A)} \in A[x_1, \dots, x_r, y]^{gr}$ be the element in $H^{n,m}(\mathbf{G}_{\mathbf{a}(r)}^* \otimes A, A)_{red}$ (the reduce cohomology) corresponding to $\nu^*(z_A) \in H^{n,m}(\mathbf{G}_{\mathbf{a}(r)}^* \otimes A, A)$. Then $\overline{\nu^*(z_A)}$ equals

$$\sum_{\substack{2(i_1 + \dots + i_r) + j = n \\ |t|(p i_1 + p^2 i_2 + \dots + p^r i_r) + |s|j = m}} a_{(i,j)} x_1^{i_1} x_2^{i_2} \dots x_r^{i_r} y^j,$$

where $(\underline{i}, j) = (i_1, \dots, i_r, j)$. We will show that $\overline{\nu^*(z_A)} = 0$ and since G has the SFB-detection property, it will follow that z is nilpotent.

Let $(\underline{c}, d) = (c_1, \dots, c_r, d)$ be an $r + 1$ -tuple in A , where $|c_i| = |t|(1 - p^{i-1})$ and $|d| = 0$. Note that $|c_i|$ depends only on i and not on the choice of c_i . We can write $c_i = \tilde{c}_i X^{|c_i|}$ and $(\tilde{c}, d) = (\tilde{c}_1, \dots, \tilde{c}_r, d)$ an $r + 1$ -tuple in K .

Let E be the elementary gr-group scheme over A with coordinate ring

$$A[E] = A[t_1, \dots, t_r, s]^{gr} / (t_1^{p^r}, \dots, t_r^{p^r}),$$

where $|t_i| = |t|$ and s is graded as in $\mathbf{G}_{\mathbf{a}(r)}^*$. Let $\gamma_{(\underline{c}, d)} : \mathbf{G}_{\mathbf{a}(r)}^* \otimes A \rightarrow \mathbf{G}_{\mathbf{a}(r)}^* \otimes A$ denote the composition

$$\mathbf{G}_{\mathbf{a}(r)}^* \otimes A \xrightarrow{(\Delta^{ev})^r \times \Delta^{od}} E \otimes A \xrightarrow{F_{(\underline{c}, d)}} E \otimes A \xrightarrow{m} \mathbf{G}_{\mathbf{a}(r)}^* \otimes A,$$

where

- $m^* : A[\mathbf{G}_{\mathbf{a}(r)}^*] \rightarrow A[E]$ is given by $t \mapsto \Delta^{r-1}(t) \otimes 1$, $s \mapsto 1 \otimes \dots \otimes 1 \otimes s$, followed by the map $(t \mapsto t_1) \otimes \dots \otimes (t \mapsto t_r) \otimes (s \mapsto s)$ and followed by multiplication.
- $F_{(\underline{c}, d)}^* : A[E] \rightarrow A[E]$ is given by $t_i \mapsto c_i t_i^{p^{i-1}}$, $s \mapsto ds$, and finally
- $((\Delta^{ev})^r \times \Delta^{od})^* : A[E] \rightarrow A[\mathbf{G}_{\mathbf{a}(r)}^*]$ is given by $t_i \mapsto t$, and $s \mapsto s$.

It follows that on cohomology

$$\gamma_{(\underline{c}, d)}^*(x_i) = c_1^{p^{i-1}} x_i + c_2^{p^{i-1}} x_{i+1} + \dots + c_{r-i+1}^{p^{i-1}} x_r, \text{ and } \gamma_{(\underline{c}, d)}^*(y) = dy.$$

Recall from 3.5.3 that $u : \mathbf{G}_{\mathbf{a}(r)}^* \otimes \mathbf{k}[V_r^*(G)] \rightarrow G \otimes \mathbf{k}[V_r^*(G)]$ is defined to be the identity map $1 \in V_r^*(G)(\mathbf{k}[V_r^*(G)])$. The universality of u gives that for any graded commutative \mathbf{k} -algebra A and any $\omega : \mathbf{G}_{\mathbf{a}(r)}^* \otimes A \rightarrow G \otimes A$ the following diagram commutes.

$$\begin{array}{ccc}
H^{*,*}(G, \mathbf{k}) & \xrightarrow{=} & H^{*,*}(G, \mathbf{k}) \\
\downarrow & & \downarrow \\
H^{*,*}(G, \mathbf{k}) \otimes \mathbf{k}[V_r^*(G)] & & H^{*,*}(G, \mathbf{k}) \otimes A \\
\downarrow = & & \downarrow = \\
H^{*,*}(G \otimes \mathbf{k}[V_r^*(G)]) & & H^{*,*}(G \otimes A, A) \\
\downarrow & & \downarrow \\
H^{*,*}(\mathbf{G}_{\mathbf{a}(r)}^* \otimes \mathbf{k}[V_r^*(G)], \mathbf{k}[V_r^*(G)]) & & H^{*,*}(\mathbf{G}_{\mathbf{a}(r)}^* \otimes A, A) \\
\downarrow = & & \downarrow = \\
H^{*,*}(\mathbf{G}_{\mathbf{a}(r)}^*, \mathbf{k}) \otimes \mathbf{k}[V_r^*(G)] & & H^{*,*}(\mathbf{G}_{\mathbf{a}(r)}^*, \mathbf{k}) \otimes A \\
\downarrow & & \downarrow \\
\mathbf{k}[V_r^*(G)] & \xrightarrow{\quad} & A
\end{array}$$

u^* (left side of the diagram) and ω^* (right side of the diagram) are indicated by curved arrows pointing from the top-left and top-right nodes to the bottom-left and bottom-right nodes respectively.

Now $\psi(z)$ is given by following $z \in H^{*,*}(G, \mathbf{k})$ through the left side of the diagram and then looking at the sum of all the coefficients for $x_r^l y^k$ such that $n = 2l + k$ in $\mathbf{k}[V_r^*(G)]$.

Let $\nu' = \nu \circ \gamma_{(\underline{c}, d)} : \mathbf{G}_{\mathbf{a}(r)}^* \otimes A \rightarrow G \otimes A$, then $\overline{\nu'^*(z_A)} \in H^{n,m}(\mathbf{G}_{\mathbf{a}(r)}^* \otimes A, A)_{red}$ equals

$$\sum_{\substack{2(i_1 + \dots + i_r) + j = n \\ |t|(pi_1 + p^2 i_2 + \dots + p^r i_r) + |s|j = m}} a_{(\underline{i}, j)} (c_1 x_1 + \dots + c_r x_r)^{i_1} \dots (c_1^{p^{r-1}} x_r)^{i_r} (dy)^j.$$

If $\psi(z)$ is nilpotent then it follows from the diagram, in the case of $\omega = \nu'$, that the sum of all the coefficients of $x_r^l y^k$ such that $n = 2l + k$ for $\overline{\nu'^*(z_A)}$ is zero. This sum is equal to

$$\sum_{\substack{2(i_1 + \dots + i_r) + j = n \\ |t|(pi_1 + p^2 i_2 + \dots + p^r i_r) + |s|j = m}} a_{(\underline{i}, j)} (c_r)^{i_1} (c_{r-1}^p)^{i_2} \dots (c_1^{p^{r-1}})^{i_r} d^j.$$

We can rewrite this equation as

$$\sum_{\substack{2(i_1 + \dots + i_r) + j = n \\ |t|(pi_1 + p^2 i_2 + \dots + p^r i_r) + |s|j = m}} \tilde{a}_{(\underline{i}, j), l} (\tilde{c}_r)^{i_1} (\tilde{c}_{r-1}^p)^{i_2} \dots (\tilde{c}_1^{p^{r-1}})^{i_r} d^j X^{q(\underline{i}, l)},$$

where $a_{(\underline{i}, j)} = \sum_l \tilde{a}_{(\underline{i}, j), l} X^l$ and $q(\underline{i}, l) = i_1 |c_r| + p i_2 |c_{r-1}| + \dots + p^{r-1} i_r |c_1| + l$.

Let

$$f_{\underline{i},l}(\underline{c}, d) = \sum_{\substack{2(i_1+\dots+i_r)+j=n \\ |t|(pi_1+p^2i_2+\dots+p^ri_r)+|s|j=m}} \tilde{a}_{(\underline{i},j),l}(\tilde{c}_r)^{i_1}(\tilde{c}_{r-1}^p)^{i_2}\dots(\tilde{c}_1^{p^{r-1}})^{i_r}d^j,$$

then each homogeneous term $f_{\underline{i},l}(\underline{c}, d) = 0$ for any choice of $r + 1$ -tuple $(\underline{c}, d) \in K$. Hence by the Nullstellensatz $f_{\underline{i},j} \equiv 0$ as polynomial with coefficients in K and $\tilde{a}_{(\underline{i},j),l} = 0$. Therefore, $\overline{\nu^*(z_A)} = 0$ and since G has the SFB-detection property, it follows that z is nilpotent as desired. \square

Our main theorem of this chapter is now a direct consequence of 3.6.15. We also get some corollaries from it.

Theorem 3.6.16. *Let G be a finite gr-group variety of height $\leq r$. If G has the SFB-detection property, then $\psi : H^{*,*}(G, \mathbf{k}) \rightarrow \mathbf{k}[V_r^*(G)]$ is an F -monomorphism, that is, its kernel consists of nilpotent elements.*

Corollary 3.6.17. *Let G be an evenly gr-group variety of height $\leq r$, then $\psi : H^{*,*}(G, \mathbf{k}) \rightarrow \mathbf{k}[V_r^*(G)]$ is an F -monomorphism.*

Corollary 3.6.18. *Let G be an elementary gr-group scheme of height $\leq r$, then $\psi : H^{*,*}(G, \mathbf{k}) \rightarrow \mathbf{k}[V_r^*(G)]$ is an F -monomorphism.*

Chapter 4

GRADED P -POINTS

We would like to define and study graded p -points, as a way of understanding the cohomology of gr-group schemes.

Let \mathbf{k} be a perfect field of characteristic p . In [FP05], given a finite group scheme G , they introduce the concept of p -points. We define an equivalent definition for finite gr-group schemes.

We first recall the definition of p -points introduced in [FP05]. Also recall from 1.0.7 that if G is a finite (gr-)group scheme, $\mathbf{k}G$ is the (gr-)dual of $\mathbf{k}[G]$ and is also a (gr-)Hopf algebra.

Definition 4.0.1. A p -point of a finite group scheme G is defined to be a flat map of \mathbf{k} -algebras

$$\alpha : \mathbf{k}\mathbf{Z}/p \rightarrow \mathbf{k}G,$$

such that α is a composition of some flat map $\mathbf{k}\mathbf{Z}/p \rightarrow \mathbf{k}C$ followed by the map $\mathbf{k}C \rightarrow \mathbf{k}G$ given by the embedding $C \subset G$, where C is a unipotent abelian subgroup scheme.

From [SFB97a], a finite group scheme C is unipotent if it admits an embedding as a closed subgroup scheme of some U_n ; the group scheme of strictly upper triangular matrices of \mathbf{GL}_n .

Definition 4.0.2. Two p -points α and β are said to be *equivalent* provided that $\alpha^*(M)$ is free as a \mathbf{Z}/p -module if and only if $\beta^*(M)$ is free as a \mathbf{Z}/p -module for all finite dimensional G -modules M . The set of all p -points of G up to this equivalence relation is denoted $P(G)$.

Based on the definition of p -points, we introduce a similar definition in the case of gr-group schemes. Our definition includes maps that ‘pick up’ the odd degree part of $\mathbf{k}G$.

Definition 4.0.3. The gr-group scheme of *strictly upper triangular matrices* denoted by U_I , is the gr-subgroup scheme of \mathbf{GL}_I , with coordinate ring $\mathbf{k}[U_I] = \mathbf{k}[x_{ij}]_{1 \leq i < j \leq n}^{gr}$.

Definition 4.0.4. A finite gr-group scheme C is *unipotent* if it admits an embedding as a closed gr-subgroup scheme of some U_I .

Definition 4.0.5. Let G be a finite gr-group scheme.

Let $p = 2$, and $\mathbf{k}E^i = \mathbf{k}[t]/t^2$, where t is primitive, and $|t| = i$ is either even or odd. A *graded p -point* of G is a flat map of graded algebras, $\alpha : \mathbf{k}E^i \rightarrow \mathbf{k}G$ such that α is a composition of some flat map $\mathbf{k}E^i \rightarrow \mathbf{k}C$ followed by the map $\mathbf{k}C \rightarrow \mathbf{k}G$ given by the embedding $C \subset G$, where C is a unipotent abelian gr-subgroup scheme.

Let $p > 2$, and $\mathbf{k}E^i = \mathbf{k}[t]^{gr}/t^p$ or $\mathbf{k}E^j = \mathbf{k}[s]^{gr}$, where $|t| = i$ is even and $|s| = j$ is odd. A *graded p -point* of G is a graded flat map as above where α goes from $\mathbf{k}E^i$ or $\mathbf{k}E^j$.

We will use $\mathbf{k}E^*$ when we do not wish to specify if we are in the case of t or s and we also do not wish to specify their degrees.

We can identify $\mathbf{k}[t]^{gr}/t^p$ with $\mathbf{k}\mathbf{Z}/p$ by the map $t \mapsto g - 1$ where g is a generator for \mathbf{Z}/p , similarly $\mathbf{k}[s]^{gr} \cong \mathbf{k}\mathbf{Z}/2$.

Definition 4.0.6. Two graded p -points α and β are said to be *equivalent* if for any finite dimensional graded G -module M , $\alpha^*(M)$ is gr-projective as a $\mathbf{k}E^*$ -module if and only if $\beta^*(M)$ is gr-projective as a $\mathbf{k}E^*$ -module. Let $P^*(G)$ denote the set of graded p -points up to this equivalence relation.

Remark 4.0.7. It may not be the case that for a graded module, free and graded implies graded free, but gr-projective is the same as graded and projective (same with gr-flat), that is why we prefer to use projectivity in our condition for the equivalence relation. By A.1.6 it turns out that since $\mathbf{k}E^*$ is a local ring, gr-flat, gr-free and gr-projective are the same. In fact they are the same as flat, free, projective plus graded.

Example 4.0.8 (Graded p -points of $A(1)$). Let $A(1)$ be the dual Steenrod subalgebra from 1.0.12. To compute the graded p -points of $A(1)$ we need to find all the flat maps $\mathbf{F}_2 E^* \rightarrow A(1)^*$ where $A(1)^*$ is the dual of $A(1)$. The dual is

$$A(1)^* = \mathbf{F}_2\langle x, y \mid x^2 = 0, y^2 = xyx \rangle,$$

where $|x| = 1, |y| = 2$.

It has a basis $1, x, y, xy, yx, y^2 = xyx, yxy, y^3 = yxyx = xyxy$. The comultiplication is given by $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $\Delta(y) = y \otimes 1 + x \otimes x + 1 \otimes y$.

Usually x is denoted $Sq(1)$ and y by $Sq(2)$. The possible graded algebra maps $\mathbf{F}_2E^* \rightarrow A(1)$, are $\alpha_1(t) = x, \alpha_3(t) = xy + yx, \alpha_4(t) = y^2, \alpha_5(t) = yxy$ and $\alpha_6(t) = y^3$. It can be checked that the only flat maps are $\alpha_1(t) = x$ giving $A(1)^*$ a free module structure over \mathbf{F}_2E^* with basis $1, y, yx, yxy$, and $\alpha_3(t) = xy + yx$ with basis $1, x, y, xy$.

4.1 Graded projective spectrum of G .

For the last sections in this chapter we will assume that G is a finite gr-group variety. Recall that by A.4.27, $H^{*,*}(G, \mathbf{k})$ is a graded commutative algebra over \mathbf{k} . We now define the graded even cohomology of G , $H^{gr-ev}(G, \mathbf{k})$.

Definition 4.1.1. Let A be a gr-Hopf algebra. We define

$$H^{gr-ev}(A, \mathbf{k}) = \begin{cases} \bigoplus_{i+j=2n} H^{i,j}(A, \mathbf{k}) & \text{if } p > 2 \\ H^{*,*}(A, \mathbf{k}) & \text{if } p = 2. \end{cases}$$

Definition 4.1.2. Let G be a finite group scheme, $\text{Proj}|G|$ is the set of prime homogeneous ideals of $H^{ev}(G, \mathbf{k})$ which are maximal with respect to not containing the irrelevant ideal.

Since $H^{ev}(G, \mathbf{k})$ is an algebraically connected algebra, an element in $\text{Proj}|G|$ not containing the irrelevant ideal of $H^{ev}(G, \mathbf{k})$ is the same as not being equal to the irrelevant ideal, that is, being properly contained in the irrelevant ideal.

In [FP05], given a finite group scheme G , they establish a homeomorphism between $\text{Proj}|G|$ and $P(G)$. This homeomorphism comes loosely from the fact that if $\alpha : \mathbf{k}\mathbf{Z}/p\mathbf{Z} \rightarrow \mathbf{k}G$ is a p -point then the kernel of its induced map in cohomology $\alpha^* : H^{ev}(G, \mathbf{k}) \rightarrow H^{ev}(\mathbf{Z}/p\mathbf{Z}, \mathbf{k})$ is an element in $\text{Proj}|G|$. We would like to do something similar with $P^*(G)$ where G is a gr-group variety.

Given a graded p -point $\alpha : \mathbf{k}E^* \rightarrow \mathbf{k}G$ of a gr-group scheme G , we have an induced bigraded map $\alpha^* : H^{gr-ev}(G, \mathbf{k}) \rightarrow H^{gr-ev}(E^*, \mathbf{k})$. By 3.4.10,

$$H^{gr-ev}(E^*, \mathbf{k}) = \begin{cases} \mathbf{k}[\lambda] & \text{where } \|\lambda\| = (1, |t|), \text{ if } p = 2, \\ \mathbf{k}[x] \text{ or } \mathbf{k}[y] & \text{where } \|x\| = (2, |t|p) \text{ and } \|y\| = (1, |s|), \text{ if } p > 2. \end{cases}$$

The kernel of α^* is an element in $\text{Proj}|G|$ (when we forget the internal grading). This suggests that there is a correspondence between graded p -points and a subset of $\text{Proj}|G|$ that somehow captures the internal grading of $\mathbf{k}G$. We define a subset of $\text{Proj}|G|$ that we hope will do the trick.

Definition 4.1.3. Let S be a positively bigraded ring, where $S_{0,0} = \mathbf{k}$ (biconnected). Let $\text{Proj}_{\mathbf{k}}^{gr}(S)$ denote the set of kernels of finite bigraded maps from $S \rightarrow \mathbf{k}[z]$, where $\|z\| = (m, n)$ is arbitrary, as long as $m, n \geq 0$ and $(m, n) \neq (0, 0)$. Similarly if S is graded connected we can define $\text{Proj}_{\mathbf{k}}(S)$.

Definition 4.1.4. Let $\text{Proj}_{\mathbf{k}}^{gr}|G|$ denote the set of kernels of finite bigraded maps from $H^{gr-ev}(G, \mathbf{k}) \rightarrow \mathbf{k}[z]$.

Example 4.1.5 (Correspondence $P^*(G) \leftrightarrow \text{Proj}_{\mathbf{k}}^{gr}|G|$ in the case of $A(1)$). Recall from 4.0.8 that $\mathbf{S}_1(R) = \text{Hom}_{\mathfrak{gR}}(A(1), R)$ and that $P^*(\mathbf{S}_1) = \{\alpha_1, \alpha_3\}$ (we did not show that α_1 is not equivalent to α_3 , but we suspect that they are not, by a degree argument) where $\alpha_1(t) = x$ and $\alpha_3(t) = xy + yx$.

From [Liu62, 3] $H^{*,*}(\mathbf{S}_1, \mathbf{F}_2) \cong \mathbf{F}_2[h_0, h_1, u, \omega]/(h_0h_1, h_1^3, h_1u, u^2 + h_0^2\omega)$ where $\|h_0\| = (1, 1)$, $\|h_1\| = (1, 2)$, $\|u\| = (3, 7)$ and $\|\omega\| = (4, 12)$. Let us denote this algebra by $B(1)$.

The maps that these graded p -points induce in cohomology are $\alpha_1^* : B(1) \rightarrow \mathbf{F}_2[z]$ where $\|z\| = (1, 1)$ and $\alpha_3^* : B(1) \rightarrow \mathbf{F}_2[z]$ where $\|z\| = (1, 3)$.

To compute $\text{Proj}_{\mathbf{k}}^{gr}|\mathbf{S}_1|$ we find all the possible bigraded finite maps from $B(1)$ to $\mathbf{F}_2[z]$.

Finite maps from $B(1)$ to $\mathbf{F}_2[z]$ are equivalent to finite bigraded maps from $B(1)/\text{nil}(B(1)) \cong \mathbf{F}_2[h_0, u, \omega]/(u^2 + h_0^2\omega)$ to $\mathbf{F}_2[z]$ where $\text{nil}(B(1)) = (h_1)$. We denote the quotient by $C(1) = B(1)/\text{nil}(B(1))$.

Any bigraded map from $C(1)$ to $\mathbf{F}_2[z]$ where $\|z\| = (i, j)$ must send each generator to a power of z or to zero. Say $h_0 \mapsto z^a$ or 0, $u \mapsto z^b$ or 0 and $\omega \mapsto z^c$ or 0. For each generator h_0, u, ω which is not sent to zero we get the following relations respectively $(ai, aj) = (1, 1)$, $(bi, bj) = (3, 7)$ and $(ci, cj) = (4, 12)$.

1. If $h_0 \mapsto z^a$ then $(ai, aj) = (1, 1)$ which implies that $a = 1$ and $(i, j) = (1, 1)$. In this case $h_0 \mapsto z$ and by degree arguments $u, \omega \mapsto 0$. This is the only map of such degree, hence it must correspond to α_1^* .
2. If $u \mapsto z^b$ then $(bi, bj) = (3, 7)$ which implies that $b = 1$ and $(i, j) = (3, 7)$. Then again by degree arguments $h_0, \omega \mapsto 0$, but since $u^2 = h_0^2 \omega$ it follows that $u \mapsto 0$, hence such map cannot exist.
3. If $\omega \mapsto z^c$ then $(ci, cj) = (4, 12)$ and we have several possibilities.

- Case $c = 1$, $(i, j) = (4, 12)$, then $\omega \mapsto z$ and $h_0, u \mapsto 0$ and $\ker = (h_0, u)$.
- Case $c = 2$, $(i, j) = (2, 6)$, then $\omega \mapsto z^2$ and $h_0, u \mapsto 0$ and $\ker = (h_0, u)$.
- Case $c = 4$, $(i, j) = (1, 3)$, then $\omega \mapsto z^4$ and $h_0, u \mapsto 0$ and $\ker = (h_0, u)$.

These last three cases give the same kernel therefore they correspond to the same element in $\text{Proj}_{\mathbf{k}}^{gr}|\mathbf{S}_1|$ which must be α_3^* .

Hence $\text{Proj}_{\mathbf{k}}^{gr}|\mathbf{S}_1|$ consists of two elements one given by the map sending $h_0 \rightarrow z$ and the map sending $\omega \rightarrow z$; moreover these maps correspond to α_1^* and α_3^* respectively.

Example 4.1.6 (Correspondence $P^*(G) \leftrightarrow \text{Proj}_{\mathbf{k}}^{gr}(V_r^*(G))$). Recall example 3.4.16: the quotient of the dual of the Steenrod algebra given by

$$\mathbf{F}_3[G] = \frac{\mathbf{F}_3[\xi_1, \tau_0, \tau_1]^{gr}}{(\xi_1^p)}.$$

Its dual is the following free algebra

$$\mathbf{F}_3G = \mathbf{F}_3\langle Q_0, Q_1, P_1, P_2 \rangle / I,$$

where Q_0, Q_1, P_1, P_2 are the duals of $\tau_0, \tau_1, \xi_1, \xi_1^2$ respectively, and I is generated by the following relations:

- $Q_0^2 = Q_1^2 = P_1^3 = 0,$
- $Q_0Q_1 = -Q_1Q_0,$
- $P_1^2 = 2P_2, P_1P_2 = P_2P_1 = 0,$
- $P_1Q_0 = Q_0P_1 + Q_1, P_2Q_0 = Q_0P_2 + Q_1P_1,$ and
- $P_iQ_1 = Q_1P_i,$ for $i = 1, 2.$

A basis for \mathbf{F}_3G is

$$\{1, P_1, P_2, Q_0, Q_0P_1, Q_0P_2, Q_1, Q_1P_1, Q_1P_2, Q_0Q_1, Q_0Q_1P_1, Q_0Q_1P_2\}.$$

The only flat maps (up to scalars) are $\alpha_1(s) = Q_0, \alpha_5(s) = Q_1,$ and $\alpha_4(t) = P_1,$ hence $P^*(G) = \{\alpha_1, \alpha_4, \alpha_5\}.$

To compute $\text{Proj}_{\mathbf{k}}^{gr}(V_1^*(G))$ we find all the possible bigraded finite maps from $\mathbf{F}_3[V_1^*(G)]$ to $\mathbf{F}_3[z].$ From 3.4.16, $\mathbf{F}_3[V_1^*(G)]$ is F-isomorphic to

$$\frac{\mathbf{F}_3[X_{12}, Y_{13}, Y_{14}]^{gr}}{X_{12}Y_{13}}.$$

The finite bigraded maps (up to scalars) are

- $X_{12} \mapsto z, z^2$ with $\ker = (Y_{13}, Y_{14}),$
- $Y_{13} \mapsto z, z^3,$ with $\ker = (X_{12}, Y_{14}),$ and
- $Y_{14} \mapsto z,$ with $\ker = (X_{12}, Y_{13}).$

Hence $\text{Proj}_{\mathbf{k}}^{gr}(V_1^*(G))$ consists of three elements as well.

Chapter 5

FUTURE WORK

5.1 Do finite gr-group varieties have the SFB-detection property?

In [Wil81] Wilkerson showed that examples W_1 and W_2 in 3.4.14 and 3.4.15, do not satisfy the W-detection property. However, as discussed in chapter 3 they do satisfy the SFB-detection property. This evidence seems to indicate that the SFB-detection property is the concept needed in order to understand the cohomology of finite gr-group varieties (up to nilpotents). We conjecture that all finite gr-group varieties satisfy the SFB-detection property, just like all infinitesimal group schemes satisfy the ungraded version (see 3.6.9). If this is the case, we get 3.6.16 for all finite gr-group varieties.

Conjecture 5.1.1. *Let G be a finite gr-group variety, then G satisfies the SFB-detection property.*

Conjecture 5.1.2. *Let G be a finite gr-group variety of height $\leq r$, then $\psi : H^{**}(G, \mathbf{k}) \rightarrow \mathbf{k}[V_r^*(G)]$ is an F -monomorphism. That is, its kernel consists of nilpotent elements.*

In order to prove 5.1.1 and consequently 5.1.2 we want to develop as future work: an analogue of Serre's structure theorem characterizing elementary abelian p -groups, a spectral sequence for graded Frobenius kernels, among other things, and proceed as in [SFB97a] in our context of gr-group schemes.

As usual, we cannot just do as in [SFB97a] but keeping track the internal grading, instead we need construct the theory in our context. One main hindrance is the fact that we need some sort of graded faithfully flat descent theory. We would like to develop the theory of general graded schemes, expanding our work on affine graded group schemes to this broader class.

5.2 Is ψ an F -epimorphism?

In [SFB97b] they show that the ungraded version of ψ is an F -epimorphism. To do that they use the theory of strict polynomial functors to construct universal classes in $H^*(Gl_n(r), \mathbf{k}^n)$ and characteristic classes in $H^*(\mathbf{G}_{\mathbf{a}(r)}, \mathbf{k}^n)$. These computations gives that the composition of schemes

$$V_r(G) \xrightarrow{\Psi} \text{Spec}(H^{ev}(G, \mathbf{k})) \xrightarrow{\Phi} gl_n^{(r) \times r},$$

is the r -th Frobenius twist of the embedding $V_r(G) \subset gl_n^{\times r}$, where Ψ is the map given by $\psi : H^{ev}(G, \mathbf{k}) \rightarrow \mathbf{k}[V_r(G)]$. This gives that ψ is an F -epimorphism.

We would like to develop a theory of graded polynomial functors in the hopes of getting results as above for gr-group schemes.

5.3 Finite generation of cohomology for finite gr-group schemes

In 2.4 we related the cohomology of the gr-group scheme G with that of the group scheme G_0 and $\kappa(G)$. We would like to use this result to show the finite generation of $H^{*,*}(G, \mathbf{k})$ for finite gr-group schemes. Moreover we would like to show that $H^{*,*}(G, M)$ is a finite $H^{*,*}(G, \mathbf{k})$ -module for M finite dimensional.

Conjecture 5.3.1. *Let G be a finite gr-group scheme and M a finite dimensional G -module. Then $H^{*,*}(G, \mathbf{k})$ is a finitely generated \mathbf{k} -algebra and $H^{*,*}(G, M)$ is a finite $H^{*,*}(G, \mathbf{k})$ -module.*

5.4 Homeomorphism $\Psi : \text{Proj}_{\mathbf{k}}^{gr} |G| \rightarrow P^*(G)$

We want to show that graded p -points $P^*(G)$ correspond to elements in $\text{Proj}_{\mathbf{k}}^{gr} |G|$, define a topology on $P^*(G)$ and $\text{Proj}_{\mathbf{k}} |G|$ and construct a homeomorphism between them. We would also like to compare $\text{Proj}_{\mathbf{k}}^{gr}(V_r^*(G)), \text{Proj}_{\mathbf{k}}^{gr} |G|$ and $P^*(G)$ in the case of finite gr-group varieties.

Conjecture 5.4.1. *Let G be a finite gr-group scheme. Sending a graded p -point $\alpha : \mathbf{k}E^* \rightarrow \mathbf{k}G$ to $\ker(\alpha^* : H^{*,*}(G, \mathbf{k}) \rightarrow H^{*,*}(E^*, \mathbf{k}))$ gives a homeomorphism*

$$\Psi : P^*(G) \rightarrow \mathrm{Proj}_{\mathbf{k}}^{gr} |G|.$$

We want to verify for a finite gr-group variety G , that $P^*(G)$ corresponds to $V_r^*(G)$.

Conjecture 5.4.2. *(From [FP05, 3.8]) Let G be a finite gr-group variety of height $\leq r$. Then there are bijections*

$$\mathrm{Proj}_{\mathbf{k}}^{gr}(V_r^*(G)) \rightarrow P^*(G) \rightarrow \mathrm{Proj}_{\mathbf{k}}^{gr} |G|.$$

Appendix A

A.1 Graded spectrum and graded local rings

Definition A.1.1. Let R be a positively graded ring, then $R_+ = \sum_{i>0} R_i$ is a homogeneous ideal and it is called the *irrelevant ideal*.

Proposition A.1.2 (Nakayama's lemma, 1 for graded rings). *Let R be a positively graded ring. Let I be an ideal contained in R_+ , and let M be a graded R -module such that $M_n = 0$ for $n \ll 0$. If $IM = M$, then $M = 0$.*

Proof. By contradiction, let $M \neq 0$. Let then n be the first such that $M_n \neq 0$, and let k be the first such that $I_k \neq 0$. Now IM is all contained above degree $n + k > n$ since $I \subseteq R$, and it is impossible to obtain any non-zero element in M_n . \square

For the next lemmas we can assume the weaker hypothesis for M , that $M_n = 0$ for $n \ll 0$, but for most of the time we will assume M is finitely generated.

Corollary A.1.3 (Nakayama's lemma, 2). *Let R be a positively graded ring. Let M be a finitely generated graded R -module and I be an ideal contained in R_+ . Let N be a graded submodule of M such that $M = N + IM$, then $M = N$.*

Proof. Modding out by N , $M/N = N/N + I(M/N) = I(M/N)$. By Nakayama's lemma 1, $M/N = 0$, that is, $M = N$. \square

Corollary A.1.4 (Nakayama's lemma, 3). *Let R be a positively graded ring. Let M be a finitely generated graded R -module and let I be an ideal contained in R_+ . If m_1, \dots, m_n homogeneous elements in M have images in M/IM that generate it as graded R -module, then m_1, \dots, m_n generate M as a graded R -module.*

Proof. Let $N = \sum Rm_i$. Since the m_i 's generate M/IM , $N + IM = \sum Rm_i + IM = M$. By Nakayama's lemma 2, $N = M$. \square

Definition A.1.5. A graded module F is *gr-free* if it has an R -basis consisting of homogeneous elements, that is if, $F \cong \bigoplus_i R(a_i)$.

Proposition A.1.6. *If R is a positively graded ring, with R_0 a field, and M is a finitely generated graded module that is projective, then M is a graded free module (that is, a direct sum of the form $\bigoplus R(a_i)$ for some integers a_i).*

Proof. Since M is graded projective then there is a graded module N and a graded free module F such that $M \oplus N = F$. Consider $F/R_+F = M/R_+M \oplus N/R_+N$; all F/R_+F , M/R_+M , and N/R_+N are vector spaces over $R/R_+ = R_0$ which is field. Let $\{m_i\}$ and $\{n_j\}$ be bases for M/R_+M and N/R_+N respectively corresponding to homogeneous elements in R ; then together they form a basis for F/R_+F . By Nakayama's lemma $F = \sum Rm_i \oplus \sum Rn_i$ and $M = \sum Rm_i$ and $N = \sum Rn_i$, hence M is graded free as desired. \square

Remark A.1.7. It is enough to assume that R_0 is a local ring, instead of a field, since in our argument we can replace R_+ with $R_+ \oplus \mathfrak{m}_0$, where \mathfrak{m}_0 is the unique maximal ideal of R_0 .

Definition A.1.8. A graded ring R is said to be *gr-local* if there exists a unique homogeneous maximal ideal \mathfrak{m} .

Remark A.1.9. Let R be a positively graded ring, for all ideals I of R_0 , $I \oplus R_+$ is a homogeneous ideal for R .

The following proposition is a consequence of the previous remark.

Proposition A.1.10. *Let R be a positively graded ring, then R_0 is a local ring if and only if R is gr-local.*

Proposition A.1.11. *If R is positively graded and local (in the ungraded sense), then R is gr-local.*

Proof. Let \mathfrak{m} be the unique maximal ideal of R . Therefore $R_+ \subset \mathfrak{m}$. Then \mathfrak{m} is necessary homogeneous since $\mathfrak{m} = (R_0 \cap \mathfrak{m}) \oplus R_+$. \square

Remark A.1.12. A graded ring may be gr-local but not graded and local. For example, consider $R = \mathbf{k}[x]$ with $|x| = 1$, then R_+ is the unique maximal homogeneous ideal of R but R is not local. The reason is that any other maximal ideal of R is not homogeneous.

Definition A.1.13. Let R be a graded ring and M a graded R -module. Then the *graded Jacobson radical* of M denoted by $J^{gr}(M)$ is the intersection of all gr-maximal submodules of M .

Remark A.1.14. By [NVO04, 2.9.1.vi] for a graded ring R , $J^{gr}(R)$ is the largest proper ideal I such that any $a \in R$ homogeneous is invertible, if the class of a in R/I is invertible. In the case of R a gr-local ring with gr-maximal ideal \mathfrak{m} then $J^{gr}(R) = \mathfrak{m}$. Hence for R gr-local, R/\mathfrak{m} is a gr-division ring. When R is commutative we get that R/\mathfrak{m} is a gr-field. We called that field *the residue gr-field* of R .

Definition A.1.15. (From [NVO04, 2.11]) Let R be a graded ring a graded ideal P of R is *gr-prime* if $P \neq R$ and for graded ideals I and J of R we have $I \subset P$ or $J \subset P$ only when $IJ \subset P$.

Definition A.1.16. (From [NVO04, 2.11]) The set of all gr-prime ideals of R is denoted by $\text{Spec}^{gr}(R)$ and it is called the graded (prime) spectrum of R .

Proposition A.1.17. (From [CVO88, II.2.11]) *The idempotents of a graded ring R are homogeneous of degree zero.*

Proposition A.1.18. *Let R be a graded ring, then $\text{Spec}^{gr}(R)$ is connected if and only if $\text{Spec}(R_0)$ is connected, where R_0 is the degree zero part of R .*

Proof. Note that $\text{Spec}^{gr}(R)$ is connected if and only if R contains no nontrivial idempotents, then by A.1.17 it is equivalent to R_0 having no nontrivial idempotents which is the case if and only if $\text{Spec}(R_0)$ is connected. \square

Proposition A.1.19. *Let R be a local ring, respectively gr-local, then $\text{Spec}(R)$, respectively $\text{Spec}^{gr}(R)$, is connected.*

Proof. Let P and Q be (gr-)prime ideals such that $P + Q = R$ and $P \cap Q = (0)$ then there exists a (gr-)maximal ideals \mathfrak{m}_1 and \mathfrak{m}_2 such that $P \subset \mathfrak{m}_1$ and $Q \subset \mathfrak{m}_2$. Now if R is (gr-)local then $\mathfrak{m}_1 = \mathfrak{m}_2 = \mathfrak{m}$ is the unique (gr-)maximal ideal, hence $R = P + Q \subset \mathfrak{m}$ which is a contradiction. \square

A.2 Graded Henselian rings

Proposition A.2.1. *Any finitely generated gr-module M over a gr-division ring R has a well defined notion of dimension and $M \cong \sum_{i=1}^n R(a_i)$ where a_1, a_2, \dots, a_n is a minimal set of homogeneous generators or a basis for M . We then denote the dimension by $\dim_R(M)$.*

Proof. We claim that a homogeneous set of elements is linearly independent if and only if they are ‘homogeneously’ linearly independent. That is, $\sum_{i=1}^n r_i a_i = 0$ for $r_i \in R$ implies that $r_i = 0$ for all i if and only if $\sum_{i=1}^n s_i a_i = 0$ for s_i homogeneous in R implies that $s_i = 0$.

One direction is clear. Now assume that a_1, \dots, a_n are homogeneously independent, and let $\sum_{i=1}^n r_i a_i = 0$ where the r_i ’s are not necessarily homogeneous. We can rewrite this sum in terms of homogeneous elements by $\sum_{i=1}^n r_i a_i = \sum_{i=1}^n \sum_{j \in \mathbf{Z}} (s_i)_{j-|a_i|} a_i = 0$. Each $\sum_{j \in \mathbf{Z}} (s_i)_{j-|a_i|} a_i$ is of a different degree, hence for the whole sum over i to be zero we need each $\sum_{j \in \mathbf{Z}} (s_i)_{j-|a_i|} a_i = 0$. Each $\sum_{j \in \mathbf{Z}} (s_i)_{j-|a_i|} a_i$ is a homogeneous linear combination of the a_i ’s, hence each $(s_i)_{j-|a_i|} = 0$, which gives us that each $r_i = 0$ as desired. \square

Remark A.2.2. The above proposition allows us to proceed as usual and work with homogeneous linear combinations where the ‘scalars’ are homogeneous therefore when nonzero they are invertible.

Proposition A.2.3. *Any finite graded algebra over a graded field is gr-artinian.*

Proof. Let A be a finite graded algebra over the gr-field R . Then by A.2.1 we have $\dim_R(A)$ is finite. Let $\dots \subset I_2 \subset I_1 \subset A$ be a descending chain of ideals. Then by A.2.1 we have a

chain of inequalities

$$\cdots \leq \dim_R I_2 \leq \dim_R I_1 \leq \dim_R A.$$

This way we get a decreasing chain of inequalities that cannot continue indefinitely hence the descending sequence of graded ideals stabilizes. \square

Definition A.2.4. A commutative gr-local ring R is *gr-henselian* if every commutative finite graded R -algebra is *gr-decomposed*, that is, if it is the direct sum of gr-local rings.

Remark A.2.5. By [CVO88, II.3.14] a gr-field is gr-henselian. This follows from A.2.3.

Corollary A.2.6 (A.2.5). *Any commutative finite graded algebra over a gr-field is the direct sum of gr-local rings.*

Corollary A.2.7. *If A is a commutative finite graded algebra over a field \mathbf{k} , then $A = \prod_{i=1}^n A_i$, where each A_i is a gr-local ring where \mathbf{k}_i are the residue gr-field for A_i with gr-maximal ideal \mathfrak{m}_i . Moreover the homogeneous elements of the gr-maximal ideals \mathfrak{m}_i are nilpotent. [Wat79, A.3]*

A.3 Graded separable extensions

Definition A.3.1. Let L be a finite graded field extension of the graded field K . Then L is a *graded separable extension* of K if, for each homogeneous element $l \in L$, the minimal homogeneous polynomial of l over K has distinct roots.

Definition A.3.2. A finite dimensional graded algebra A is a *gr-separable algebra* if it is the product of gr-separable field extensions.

Remark A.3.3. Let \mathbf{k} be a field. A graded field extension of \mathbf{k} may be one of the following; $L = l$ where l is a field extension in the usual sense, or $L = l[X, X^{-1}]$ where l is a field extension in the usual sense.

Remark A.3.4. Any gr-field of the form $L[T, T^{-1}]$ cannot be a gr-separable extension for \mathbf{k} since T is transcendental over \mathbf{k} , even if L is a separable extension of \mathbf{k} .

A.4 Cohomology of graded algebras

For reference and completeness we provide some result and definitions regarding the cohomology of gr-group schemes. Let R be a graded ring.

Definition A.4.1. Let M and N be graded R -modules, a map $f : M \rightarrow N$ is a *graded map of degree n* if $f(M_i) \subset N_{i+n}$ for all i .

Definition A.4.2. The *i -suspension* of a graded R -module M is defined to be the graded module $M(i)$ given by M so that $M(i)_j = M_{i+j}$.

Definition A.4.3. Let M and N be graded R -modules. Let $\underline{\text{Hom}}_R(M, N)_n$ denote the subgroup of $\text{Hom}_R(M, N)$ consisting of maps of degree n . We define $\underline{\text{Hom}}_R(M, N) = \bigoplus_i \underline{\text{Hom}}_R(M, N)_i$.

Definition A.4.4. Since $\underline{\text{Hom}}_R(-, -)$ is left exact we can defined the right derived functor denoted by $\underline{\text{Ext}}_R^n(-, -)$.

Proposition A.4.5. ([NVO04, 2.4.4]) *If M, N are graded modules over R and M is finitely generated then*

$$\underline{\text{Hom}}_R(M, N) = \text{Hom}_R(M, N).$$

Definition A.4.6. A graded module M is *left gr-Noetherian* if M satisfies the ascending chain condition for graded left R -submodules.

Proposition A.4.7 (P. Samuel). *A \mathbf{Z} -graded ring $R = \bigoplus_{i \in \mathbf{Z}} R_i$, is gr-Noetherian if and only if R_0 is a Noetherian ring and R is finitely generated as an R_0 algebra.*

Proposition A.4.8. (From [NVO04, 2.4.7]) *If R is left gr-Noetherian and M is finitely generated then, for every $n \geq 0$*

$$\underline{\text{Ext}}_R^n(M, N) = \text{Ext}_R^n(M, N).$$

Definition A.4.9. A graded R -module P is *gr-projective* if it is projective as an object in the category of graded R -modules. That is, if P satisfies the universal lifting property of projective modules, where the maps are graded maps between graded modules. It can be

shown (with the usual proofs following verbatim), that P is gr-projective if and only if P is a direct summand of a graded free module.

Proposition A.4.10. *A graded module P is gr-projective if and only if P is graded and projective.*

Proof. One direction is clear, if P is gr-projective, then P is a direct summand of a gr-free, forgetting the grading on F it is then a direct summand of a free module, hence it is projective and also graded.

On the other hand if P is a graded module which is also a projective module, then let $f : M \rightarrow N$ be a graded surjective map and $g : P \rightarrow N$ a graded map, then since P is projective there exists a map $h : P \rightarrow M$ (not necessarily graded) such that the following diagram commutes.

$$\begin{array}{ccc} & & M \\ & \nearrow \exists h & \downarrow f \\ P & \xrightarrow{g} & N \end{array}$$

We construct a graded map $\tilde{h} : P \rightarrow M$ from h , such that the diagram will commute if we substitute h with \tilde{h} . Consider the group homomorphisms $\pi_i : M \rightarrow M_i$ which are the projections of M onto its degree i part M_i and define the group homomorphisms $\tilde{h}_i : P_i \rightarrow M_i$ to be $\tilde{h}_i = \pi_i \circ h|_{P_i}$. We then define $\tilde{h} = \bigoplus_i \tilde{h}_i : P \rightarrow M$.

The map \tilde{h} is degree preserving. Let $p_i \in P_i$, then $\tilde{h}(p_i) = \tilde{h}_i(p_i) = \pi_i \circ (h(p_i))$. If we write $h(p_i) = \sum_j m_j$, then $\pi_i(h(p_i)) = m_i$.

The map \tilde{h} is in fact an R -module map, since \tilde{h} is defined as composition of group homomorphisms $\tilde{h}(p + q) = \tilde{h}(p) + \tilde{h}(q)$. Since \tilde{h} distributes with the sum, to check if elements of R factor out it is enough to check on homogeneous elements. Let $r \in R_i$ and $p \in P_j$, then $rp \in P_{i+j}$ hence $\tilde{h}(rp) = \pi_{i+j} \circ h(rp) = \pi_{i+j}(rh(p)) = r\pi_j(h(p)) = r\tilde{h}(p)$.

To show $g = f \circ \tilde{h}$ we again assume $p \in P_i$, homogeneous, then if $h(p) = \sum_j m_j$, then $f \circ \tilde{h}(p) = f(\pi_i(h(p))) = f(m_i)$; on the other hand $f \circ h(p) = g(p)$ which gives that

$f(h(p)) = f(\sum_j m_j) = \sum_j f(m_j) = f(m_i)$ since g and f are graded maps. \square

Remark A.4.11. (From [NVO04, 2.2]) Note that the proposition above is not true if we replace gr-projective with gr-free as we may have a module which is free with respect to a non-homogeneous basis and not free with respect to a graded basis. For example let $R = \mathbf{Z} \times \mathbf{Z}$ with trivial grading and M the graded R module where $M_0 = \mathbf{Z} \times 0$, $M_1 = 0 \times \mathbf{Z}$ and $M_i = 0$ for $i \neq 0, 1$, then M cannot have a homogeneous basis. Hence, gr-free is a stronger property than graded and free.

Proposition A.4.12. *If Q in R -gr is injective when considered as an ungraded module, then Q is gr-injective.*

Proof. From [NVO04, 2.3.2]. The proof is very similar to the one above about projective module except that a gr-injective module, may not be injective as an ungraded module, so we do not get an if and only if statement. \square

We get the following corollary.

Corollary A.4.13. *Let P be an R -module, then P is a gr-projective if and only if P is projective as an object in the category where the objects are graded modules and morphisms are $\underline{\text{Hom}}_R(-, -)$.*

We can do similar argument about gr-flat modules.

Definition A.4.14. A graded module M is gr-flat if the functor $-\otimes_{\mathbf{Z}} M$ is exact.

We can show that M is gr-flat if and only if M is graded and flat, c.f. [NVO04, 2.12.11].

Given a graded algebra A , we want to compute its cohomology, which will turn out to be a bigraded ring, that is $H^{*,*}(A, \mathbf{k})$. Since a gr-projective module is the same as a projective and graded module, a gr-projective resolution of \mathbf{k} as a graded A -module will also be a projective resolution of \mathbf{k} as an A -module. We then apply $\text{Hom}_A(-, \mathbf{k})$ to compute $\text{Ext}_A^*(\mathbf{k}, \mathbf{k})$ in order to compute the usual cohomology of A , without keeping track of the grading.

If instead we want to compute the graded cohomology, we would need to apply the functor $\underline{\text{Hom}}_A(-, \mathbf{k})$. Let

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \mathbf{k} \longrightarrow 0$$

be a graded projective resolution of \mathbf{k} as an A -gr-module. We then apply the functor $\underline{\text{Hom}}_A(-, \mathbf{k})$ to the resolution to get complex

$$0 \longrightarrow \underline{\text{Hom}}_A(P_0, \mathbf{k}) \xrightarrow{\delta_0} \underline{\text{Hom}}_A(P_1, \mathbf{k}) \xrightarrow{\delta_1} \underline{\text{Hom}}_A(P_2, \mathbf{k}) \xrightarrow{\delta_2} \cdots$$

where $\delta_i : \underline{\text{Hom}}_A(P_i, \mathbf{k}) \rightarrow \underline{\text{Hom}}_A(P_{i+1}, \mathbf{k})$ is given by $\delta_i(f)(p) = f(d_{i+1}(p))$ for $p \in P_{i+1}$ and $f \in \underline{\text{Hom}}_A(P_i, \mathbf{k})$.

The n^{th} graded cohomology is defined by $H^n(A, \mathbf{k}) = \frac{\ker(\delta_n)}{\text{im}(\delta_{n-1})}$.

If A is gr-Noetherian then by A.4.8 we would get the same cohomology as the ungraded one.

Proposition A.4.15. *If*

$$\cdots \xrightarrow{f_3} M_2 \xrightarrow{f_2} M_1 \xrightarrow{f_1} M_0$$

is a sequence of graded maps where each f_i may be of a nonzero degree, then this sequence is equivalent to one where the maps are all graded of zero degree.

Proof. This follows from the fact that the suspension A.4.2 is a functor and that the suspension of a gr-projective module is still gr-projective. We also use that $\underline{\text{Hom}}_R(M, N)_i = \text{Hom}_{R\text{-gr}}(M, N(i)) = \text{Hom}_{R\text{-gr}}(M(-n), N)$. \square

Remark A.4.16. The above observations tell us that given a graded algebra A , we can assume our category to be the one of graded A modules, where the morphisms are graded maps of various degrees. Then when computing the cohomology, the gr-projective resolution can consist of maps of degree zero and to the resolution we apply the functor $\underline{\text{Hom}}_A(-, \mathbf{k})$.

Proposition A.4.17. *Let A be a graded algebra then $H^{*,*}(A, \mathbf{k}) = \underline{\text{Ext}}_A^{*,*}(\mathbf{k}, \mathbf{k})$ is a bigraded \mathbf{k} -module.*

Proof. Let

$$\cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \mathbf{k} \longrightarrow 0$$

be a graded projective resolution of \mathbf{k} as an A -gr-module. By A.4.15 we can assume that the maps d_i are graded of degree zero. Let $f = \sum_s f_s \in \underline{\text{Hom}}_A(P_n, \mathbf{k})$, where $f_s \in \underline{\text{Hom}}_A(P_n, \mathbf{k})_s$, that is $f_s : P_n \rightarrow \mathbf{k}$ is a map of degree s . We want to show that if $f \in \ker(\delta_n)$ then each f_s is in the kernel as well. Hence $\ker(\delta_n)$ is generated by homogeneous elements of $\underline{\text{Hom}}_A(P_n, \mathbf{k})$.

For $p \in P_{n+1}$, $\delta_n(f)(p) = f(d_{n+1}(p)) = \sum_s f_s(d_{n+1}(p))$, and each $f_s \circ d_{n+1}$ is a map of degree s . Now $f \in \ker(\delta_n)$ implies that $\delta_n(f)(p) = 0$ for all $p \in P_{n+1}$.

On a homogeneous element p of P_{n+1} ,

$$\delta_n(f_s)(p) = \begin{cases} \lambda_s & \text{if } |p| = -s \\ 0 & \text{otherwise} \end{cases}$$

Hence

$$\delta_n(f)(p) = \sum_s \delta_n(f_s)(p) = \begin{cases} \lambda_s & \text{if } |p| = -s \\ 0 & \text{otherwise} \end{cases}$$

This gives that if $f \in \ker(\delta_n)$ then $\lambda_s = 0$, thus $\delta_n(f_s)(p) = 0$ for each homogeneous elements; therefore $f_s \in \ker(\delta_n)$ as well.

We want to show that $\text{im}(\delta_n)$ is also generated by homogeneous elements of $\underline{\text{Hom}}_A(P_{n+1}, \mathbf{k})$. In this case $f = \sum f_s \in \underline{\text{Hom}}_A(P_n, \mathbf{k})$ and $\delta_n(f)(p) = \sum \delta_n(f_s)$ and clearly each $\delta_n(f_s)$ is homogeneous of degree s . Therefore we have $\underline{\text{Ext}}_A^n(\mathbf{k}, \mathbf{k}) = \bigoplus_s \underline{\text{Ext}}_A^{n,s}(\mathbf{k}, \mathbf{k})$. That is, a $\zeta \in H^n(A, \mathbf{k})$ corresponds to a map $\hat{\zeta} : P_n \rightarrow \mathbf{k}$ of degree s . \square

Similarly we can define the graded cohomology of A with coefficients in M by $H^{*,*}(A, M)$ where M is a graded A -module.

We now give other characterizations for $H^{*,*}(A, \mathbf{k})$.

Definition A.4.18. Given two graded modules M, N a *graded n -extension of M to N* is an exact sequence of graded maps of various degrees of the form

$$0 \longrightarrow N \longrightarrow M_{n-1} \longrightarrow \cdots \longrightarrow M_0 \longrightarrow M \longrightarrow 0$$

Definition A.4.19. Two graded n -extensions are *equivalent* if there exist graded maps, of possibly nonzero degree, h_0, \dots, h_{n-1} such that the diagram commutes.

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & N & \xrightarrow{g_n} & M_{n-1} & \xrightarrow{g_{n-1}} & \cdots & \longrightarrow & M_0 & \xrightarrow{g_0} & M & \longrightarrow & 0 \\
& & \downarrow = & & \downarrow h_{n-1} & & \downarrow & & \downarrow h_0 & & \downarrow = & & \\
0 & \longrightarrow & N & \xrightarrow{g'_n} & M'_{n-1} & \xrightarrow{g'_{n-1}} & \cdots & \longrightarrow & M'_0 & \xrightarrow{g'_0} & M & \longrightarrow & 0
\end{array}$$

Two equivalent extension will have as a consequence that $\sum |g_i| = \sum |g'_i|$. Using that it can be shown that two equivalent extensions correspond to the same element in $\underline{\text{Ext}}^{n,s}(M, N)$, where $s = -\sum |g_i|$. We complete this to an equivalence relation by symmetry and transitivity in the usual way.

Proposition A.4.20. *An equivalence class of graded n -extension of M to N of the form*

$$0 \longrightarrow N \xrightarrow{g_n} M_{n-1} \xrightarrow{g_{n-1}} \cdots \xrightarrow{g_1} M_0 \xrightarrow{g_0} M \longrightarrow 0$$

corresponds to an element in $\underline{\text{Ext}}^{n,s}(M, N)$ where $s = -\sum_{i=0}^n |g_i|$.

Proof. We use the fact that the gr-projective resolution is in fact a projective resolution and we can get maps from the resolution to the extension via the usual discussion of extensions and projective resolutions in the ungraded case.

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & P_n & \xrightarrow{d_n} & P_{n-1} & \xrightarrow{d_{n-1}} & \cdots & \longrightarrow & P_0 & \xrightarrow{d_0} & M & \longrightarrow & 0 \\
& & \downarrow f_n & & \downarrow f_{n-1} & & & & \downarrow f_0 & & \downarrow = & & \\
0 & \longrightarrow & N & \xrightarrow{g_n} & M_{n-1} & \xrightarrow{g_{n-1}} & \cdots & \xrightarrow{g_1} & M_0 & \xrightarrow{g_0} & M & \longrightarrow & 0
\end{array}$$

It can be checked that these maps can be assumed to be graded and the last map f_n will then have degree $s = -\sum_{i=0}^n |g_i|$ and will correspond to an element in $\underline{\text{Ext}}^{n,s}(M, N)$. \square

We relate an element in $\zeta \in \underline{\text{Ext}}^{n,s}(M, N)$ with a map $\widehat{\zeta} \in \Omega^n(M) \rightarrow N$. Here we will be working on the stable category.

Proposition A.4.21. *Let $\zeta \in \underline{\text{Ext}}^{n,s}(M, N)$, then there is a map of degree s , $\widehat{\zeta} : \Omega^n(M) \rightarrow N$ corresponding to ζ , if two such maps exist they correspond to equivalent extensions. Where $\Omega^n(M)$ is well-defined up to projective summands.*

Proof. Recall that given a projective resolution of M ,

$$\cdots \longrightarrow P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \longrightarrow P_0 \xrightarrow{d_0} M \longrightarrow 0,$$

$\Omega^n(M) = \ker(d_{n-1}) = \text{im}(d_n) \subset P_{n-1}$. An element $\zeta \in \underline{\text{Ext}}^{n,s}(M, N)$ corresponds to an extension, where the following diagram commutes.

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & P_n & \xrightarrow{d_n} & P_{n-1} & \xrightarrow{d_{n-1}} & \cdots & \longrightarrow & P_0 & \xrightarrow{d_0} & M & \longrightarrow & 0 \\ & & \downarrow \zeta & & \downarrow f_{n-1} & & & & \downarrow f_0 & & \downarrow = & & \\ 0 & \longrightarrow & N & \xrightarrow{g_n} & M_{n-1} & \xrightarrow{g_{n-1}} & \cdots & \xrightarrow{g_1} & M_0 & \xrightarrow{g_0} & M & \longrightarrow & 0 \end{array}$$

Let $\text{id}(-|g_n|) : N(|g_n|) \rightarrow N$ the map of degree $-|g_n|$ that suspends back $N(|g_n|)$ to N , that is $\text{id}(-|g_n|)(N(|g_n|)) = N$. We can then define $\widehat{\zeta} = \text{id}(-|g_n|) \circ f_{n-1}|_{\Omega^n(M)}$. Since $|f_{n-1}| = -\sum_{i=0}^{n-1} |g_i|$, then $|\widehat{\zeta}| = -|g_n| - \sum_{i=0}^{n-1} |g_i| = -\sum_{i=0}^n |g_i| = s$ as desired. The last statement of the proposition can be proven as in the ungraded case. \square

There are several ways of describing the ring structure on $H^{*,*}(A, \mathbf{k})$. One way is doing the *Yoneda splice* of an extension. We use the Yoneda splice to prove A.4.23.

Definition A.4.22. Let A be a graded algebra and let M, M' and M'' be graded A -modules. Let $\zeta \in \underline{\text{Ext}}^{n,s}(M, M')$ and $\eta \in \underline{\text{Ext}}^{m,r}(M', M'')$, then ζ and η correspond to the following extensions:

$$\zeta : 0 \longrightarrow M' \xrightarrow{g_n} M_{n-1} \xrightarrow{g_{n-1}} \cdots \xrightarrow{g_1} M_0 \xrightarrow{g_0} M \longrightarrow 0$$

$$\eta : 0 \longrightarrow M'' \xrightarrow{g'_m} M'_{m-1} \xrightarrow{g'_{m-1}} \cdots \xrightarrow{g'_1} M'_0 \xrightarrow{g_0} M' \longrightarrow 0.$$

Then $s = -\sum |g_i|$ and $r = -\sum |g'_i|$. The *Yoneda splice* is defined as the sequence given by

$$\begin{array}{ccccccccccccccc} 0 & \rightarrow & M'' & \xrightarrow{g'_m} & M'_{m-1} & \xrightarrow{g'_{m-1}} & \cdots & \xrightarrow{g'_1} & M'_0 & \longrightarrow & M_{n-1} & \xrightarrow{g_{n-1}} & \cdots & \xrightarrow{g_1} & M_0 & \xrightarrow{g_0} & M & \rightarrow & 0 \\ & & & & & & & & & & \searrow^{g'_0} & & \nearrow^{g_n} & & & & & & \\ & & & & & & & & & & & M' & & & & & & & & \\ & & & & & & & & & & \nearrow & & \searrow & & & & & & & \\ & & & & & & & & & & 0 & & & & & & & & & 0 \end{array}$$

Then the Yoneda splice gives an element $\eta \circ \zeta \in \underline{\text{Ext}}^{n+m,s+r}(M, M'')$.

Note that the Yoneda splice makes $\underline{\text{Ext}}^{*,*}(M, M)$ into a ring, hence we get the following result.

Theorem A.4.23. *Let A be a graded algebra then $H^{*,*}(A, \mathbf{k}) = \underline{\text{Ext}}_A^{*,*}(\mathbf{k}, \mathbf{k})$ is a bigraded ring with $H^{n,s}(A, \mathbf{k}) = \underline{\text{Ext}}_A^{n,s}(\mathbf{k}, \mathbf{k})$, where n is the cohomological degree and s is the internal degree.*

Let A be a gr-Hopf algebra and C and D be graded A -modules. Then $(C \otimes D)$ is a graded A -module as well, where $(C \otimes D)_n = \bigoplus_{i+j=n} C_i \otimes D_j$. The action of A on $C \otimes D$ is defined the following way: for $a \in A$, write $\Delta(a) = \sum a_1 \otimes a_2$, then $a \cdot (x \otimes y) = \Delta(a)(x \otimes y) = \sum (-1)^{|a_2||x|} a_1 x \otimes a_2 y$.

Lemma A.4.24. *Let A be a cocommutative gr-Hopf algebra and C and D be graded A -modules. Let $T : C \otimes D \rightarrow D \otimes C$ be the map given by $x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$ for x and y homogeneous elements. Then T is a graded A -module morphism.*

Proof. Recall that A is cocommutative if for any homogeneous element a , $\Delta(a) = \sum a_1 \otimes a_2 = \sum (-1)^{|a_1||a_2|} a_2 \otimes a_1$. We just need to write out and check the sign conventions on homogeneous elements.

$$\begin{aligned}
T(a \cdot (x \otimes y)) &= T\left(\sum (-1)^{|a_2||x|} a_1 x \otimes a_2 y\right) \\
&= \sum (-1)^{|a_2||x|+|a_1x||a_2y|} a_2 y \otimes a_1 x \\
&= \left(\sum (-1)^{|a_1||a_2|} a_1 \otimes a_2\right) (-1)^{|x||y|} y \otimes x \\
&= \Delta(a)T(x \otimes y) \\
&= a \cdot T(x \otimes y)
\end{aligned}$$

□

Definition A.4.25. Let C and D be complexes of graded A -modules. The *tensor product of complexes* $C \otimes D$ is defined as, $(C \otimes D)_n = \bigoplus_{i+j=n} C_i \otimes D_j$, where $\delta^{C \otimes D}(x \otimes y) = \delta^C(x) \otimes y + (-1)^{\text{tot}(x)} x \otimes \delta^D(y)$. A homogeneous element $x \otimes y \in (C \otimes D)_n$ corresponds to

$x \in C_l^i, y \in D_k^j$ where $i + j = n$ and C_l^i and D_k^j are the degree l and k part of C^i and D^j respectively. Then $\text{tot}(x) = i + l$ is the total degree of x .

Proposition A.4.26. *Let A be a cocommutative gr-Hopf algebra, then the tensor product of graded A -complexes is gr-commutative, in the sense that given \mathbf{C} and \mathbf{D} graded A -complexes, the map $T : \mathbf{C} \otimes \mathbf{D} \rightarrow \mathbf{D} \otimes \mathbf{C}$, given by $x \otimes y \mapsto (-1)^{\text{tot}(x)\text{tot}(y)} y \otimes x$ is an isomorphism of graded complexes.*

Proof. By A.4.24 T is a map of graded A -modules. We are left to show that it is an isomorphism of graded A -complexes. We follow the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & (C \otimes D)_{n+1} & \xrightarrow{\delta_{n+1}^{C \otimes D}} & (C \otimes D)_n & \xrightarrow{\delta_n^{C \otimes D}} & (C \otimes D)_{n-1} \xrightarrow{\delta_{n-1}^{C \otimes D}} \cdots \\ & & \downarrow T & & \downarrow T & & \downarrow T \\ \cdots & \longrightarrow & (D \otimes C)_{n+1} & \xrightarrow{\delta_{n+1}^{D \otimes C}} & (D \otimes C)_n & \xrightarrow{\delta_n^{D \otimes C}} & (D \otimes C)_{n-1} \xrightarrow{\delta_{n-1}^{D \otimes C}} \cdots \end{array}$$

Note that

$$T \circ \delta^{C \otimes D}(x \otimes y) = (-1)^{(\text{tot}(x)-1)\text{tot}(y)} y \otimes \delta^C(x) + (-1)^{\text{tot}(x)+\text{tot}(x)(\text{tot}(y)-1)} \delta^D(y) \otimes x,$$

and

$$\delta^{D \otimes C} \circ T(x \otimes y) = (-1)^{\text{tot}(y)\text{tot}(x)} (\delta^D(y) \otimes x + (-1)^{\text{tot}(y)} y \otimes \delta^C(x)).$$

are equal since the signs have the same parity. \square

Theorem A.4.27. *Let A be a cocommutative gr-Hopf algebra, then $H^{*,*}(A, \mathbf{k})$ is a gr-commutative ring with respect the total degree.*

Proof. Let

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \mathbf{k} \longrightarrow 0$$

be a graded projective resolution of \mathbf{k} as an A -gr-module and let \mathbf{C} be the complex

$$0 \longrightarrow \underline{\text{Hom}}_A(P_0, \mathbf{k}) \xrightarrow{\delta_0} \underline{\text{Hom}}_A(P_1, \mathbf{k}) \xrightarrow{\delta_1} \underline{\text{Hom}}_A(P_2, \mathbf{k}) \xrightarrow{\delta_2} \cdots$$

Then $\mathbf{C} \otimes \mathbf{C}$ still corresponds to $\underline{\text{Hom}}$ applied to a gr-projective resolution of \mathbf{k} . Moreover, if $x \in C_n^s = \underline{\text{Hom}}(P_n, \mathbf{k})_s$ and $y \in C_m^r = \underline{\text{Hom}}(P_m, \mathbf{k})_r$ then $x \otimes y \in \underline{\text{Hom}}(P_n \otimes P_m, \mathbf{k})_{s+r} \subset$

$(C \otimes C)_{n+m}^{s+r}$, Hence the tensor product of the complex C makes $H^{*,*}(A, \mathbf{k})$ into a ring. In fact this product on $\underline{\text{Ext}}$ is called the *cup product* and in the case of $H^{*,*}(A, \mathbf{k})$ it coincides with the Yoneda splice (for more details refer to [Ben98, 3.2]). We can check that $\delta^{C \otimes C}(x \otimes y) = \delta^C(x) \otimes y + (-1)^{\text{tot}(x)} x \otimes \delta^C(y)$ passes down to a well defined product on $H^{*,*}(A, \mathbf{k})$.

Let $T : C \otimes C \rightarrow C \otimes C$ as in A.4.25, then by A.4.26 T is an isomorphism of graded A -complexes. Hence $H^{*,*}(A, \mathbf{k})$ is a gr-commutative ring with respect the total degree as desired. \square

Remark A.4.28. Graded comodules over a gr-commutative Hopf algebra of finite type B correspond to gr-modules over its dual $B^\#$ which is a gr-cocommutative Hopf algebra.

Hence, when working in the category of graded B -comodules, $\underline{\text{Ext}}_B^{*,*}(\mathbf{k}, \mathbf{k})$ is gr-commutative by A.4.27.

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