Spectral Theory of $\mathbb{Z}^d$ Substitutions

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Abstract

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In this paper, we generalize and develop results of Queffélec allowing us to characterize the spectrum of an aperiodic substitution in $\mathbb{Z}^d$ by describing the Fourier coefficients of mutually singular measures of pure type giving rise to the maximal spectral type of the translation operator on L^2. This is done without any assumptions on primitivity or height, and provides a simple algorithm for determining singularity to Lebesgue spectrum for such substitutions, and we use this to show singularity of the spectrum for Queffélec's noncommutative bijective substitution, as well as the Table tiling, answering an open question of Solomyak. Moreover, we also prove that the spectrum of any aperiodic bijective commutative $\mathbb{Z}^d$ substitution on a finite alphabet is purely singular. Finally, we show that every ergodic matrix of measures on a compact metric space can be diagonalized, which we use in the proof of the main result.

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Chapter 1

INTRODUCTION

Substitutions are of interest to us as models of aperiodic phenomena - for example, mathematical quasicrystals, see [3]. Here, we consider higher dimensional analogues of substitutions of constant length, or $\mathbb{Z}^d$-substitutions which replace letters in a finite alphabet $\mathcal{A}$ with a rectangular block (indexed over the semirectangle $[\mathbf{0}, \mathbf{q})$, see notation at the end of this section) of letters, and which we call $\mathbf{q}$-substitutions. As usual, we study the translation operator T on the hull $X_{\mathcal{S}}$ of a substitution $\mathcal{S}$, or the collection of all $\mathbb{Z}^d$-indexed sequences whose local patterns can be produced by the substitution. Our primary interest is the spectrum (or the maximal spectral type, $\sigma_{\max}$) of the translation action, and our analysis is based on the work of Queffélec, which describes the spectrum of one-sided primitive and aperiodic constant length substitutions of trivial height in one dimension ($\mathbb{N}$-indexed sequences). Our formulation emphasizes the arithmetic properties of such substitutions and the recognizability properties afforded by *aperiodic substitutions*: those without any periodic sequences in their hull. Using only these two assumptions, we are able to describe the spectrum of such a substitution by identifying its discrete, singular continuous, and absolutely continuous spectral components as a sum of mutually singular measures of pure type with readily computable Fourier coefficients. This is done by using a representation consistent with the abelianization (which gives rise to the substitution matrix) and separates the analysis into two parts: the *correlation measures*, which depend on the position of letters in substituted blocks (which we call the *configuration*), and the *spectral hull*, a convex set depending on a *coincidence matrix* $C_{\mathcal{S}}$ which is configuration-independent.

An important notion in tiling theory is that of *local rules* which force global (aperiodic) order. A result of Mozés [19] asserts that self-similar tilings, corresponding to $\mathbb{Z}^d$ substitutions for $d > 1$, can be obtained by local pattern matching rules, albeit, in general, at the cost of increasing the size of the alphabet, whereas this assertion is false for $d = 1$. (We refer the reader to [3] for details and more recent work in this direction.) Moreover, it is known that the mathematical diffraction spectrum is included in the dynamical spectrum (that of the translation operator on L^2) as a consequence of Dworkin's argument [11] (see also [16], [4] and references therein), and so our extension of Queffélec's results to $\mathbb{Z}^d$ substitutions are of interest as they allow us to describe diffraction (and dynamical) properties of higher dimensional mathematical quasicrystals.

We summarize the paper while highlighting the main ideas. First, we begin with a general introduction to topological dynamics, their invariant measures and spectral theory. Then, in §1.2, we discuss preliminaries for the theory of $\mathbb{Z}^d$-indexed substitution dynamical systems, including their invariant measures in §1.2.1 and spectral theory in §1.2.2. As usual, Michel's theorem 1.2.1 is used to identify the uniquely ergodic measure of a primitive system in terms of the Perron vector of $M_{\mathcal{S}}$, its *substitution matrix.* A *primitive reduced form* (proposition 1.2.2) is used to represent an arbitrary substitution in terms of its *primitive components*, which combine in the nonprimitive case with theorem 1.2.4 of Cortez and Solomyak to describe the invariant measures as the convex hull of the invariant measures of its primitive components. Note that $\mathbf{q}$-substitution dynamical systems are never weakly-mixing as Dekking's theorem 4.4.1 shows they always have nontrivial discrete spectrum; more generally, Dekking and Keane proved in [10] that primitive substitution systems cannot be strongly mixing. This should not be confused with the statement of our main result, theorem 3.3.1, which shows that the spectrum of $\mathcal{S}$ can be decomposed into measures strongly-mixing for the $\mathbf{q}$-shift on $\mathbb{T}^d$, as this is a statement about the *spectrum* of $\mathcal{S}$ (not its shift-invariant measures) relative to an entirely different dynamical system. Following the discussion on invariant measures, we discuss the essentials for the spectral theory of substitution dynamical systems where we give a very brief account of some existing results in the spectral theory of $\mathbf{q}$-substitutions.

In §2 we develop arithmetic and topological properties of aperiodic $\mathbf{q}$-substitutions which combine with a representation of the alphabet $\mathcal{A}$ in the vector space $\mathbb{C}^{\mathcal{A}}$, and of substitutions in the matrix algebra $\mathbf{M}_{\mathcal{A}}(\mathbb{C})$, to give us several useful tools (propositions 2.3.5, 2.3.7 and 2.3.8) relating the configuration of an aperiodic $\mathbf{q}$-substitution to both its topological and measure theoretic structure. In §2.1 we develop some basic notation for $\mathbf{q}$-adic arithmetic in $\mathbb{Z}^d$, as well as a key lemma (2.1.1) which essentially lets us ignore the overlapping of superblocks $T^{\mathbf{k}}\mathcal{S}^n(\alpha)$ arising in a number of contexts; this removes an otherwise significant impediment to a detailed description of the spectrum, as tracking the discrepancies caused by overlaps destroys any hope of a detailed analysis, see the discussion at the end of §3.1.

In §2.2, we give an arithmetic description (proposition 2.2.1) of $\mathbf{q}$-substitutions in terms of *configurations*, which represents a $\mathbf{q}$-substitution in terms of *instructions* $\mathcal{A} \to \mathcal{A}$ *arranged by location*, and an equivalence of substitutions is described, relating two substitutions when there is a bijective rearrangement of their configurations which exchanges them. We also describe two constructions on substitutions: definitions 2.2.3 for *cycled substitutions* and 2.2.6 for the *substitution product*. A cycled $\mathbf{q}$-substitution is obtained by permuting the associated configuration, and is related to the map $T^{\mathbf{k}}\mathcal{S}^n$. The substitution product takes two $\mathbf{q}$-substitutions, $\mathcal{S}$ on $\mathcal{A}$ and $\tilde{\mathcal{S}}$ on $\tilde{\mathcal{A}}$, and produces a third $\mathbf{q}$-substitution, $\mathcal{S}\otimes\tilde{\mathcal{S}}$ on $\mathcal{A}\tilde{\mathcal{A}}$, and is related to the *bisubstitution* and the coincidence matrix $C_{\mathcal{S}} = M_{\mathcal{S}\otimes\mathcal{S}}$ of Queffélec.

In §2.3 we discuss consequences of aperiodicity beginning with a convenient criteria of Pansiot (lemma 2.3.1) allowing for a readily verifiable description of aperiodicity in the $\mathbb{Z}$ case; a generalization to $\mathbb{Z}^d$ would be of significant interest, but no serious attempts by this author have been made. Next, we use a result of Mossé and Solomyak (theorem 2.3.2) to identify aperiodicity with *recognizability*, a property of substitutions allowing for *unique local desubstitution*, and which gives rise to many useful topological properties summarized in proposition 2.3.4. Moreover, this allows us to describe the subshift as an iterated function system on the maps $T^{\mathbf{k}}\mathcal{S}$, for $\mathbf{k}$ in a fixed finite set. Using this perspective, we give a description of the Borel structure of $X_{\mathcal{S}}$ through a collection of *iterated cylinders* (definition 2.3.3) which represent the cylinders of $\mathcal{A}^{\mathbb{Z}^d}$ iterated through the above function system. This description is expressed in three ways: the *partition formula* of proposition 2.3.5 which describes how the lattice of iterated cylinders partition the subshift by degree; the *measure formula* of proposition 2.3.7 which gives recursive identities allowing one to *explicitly* compute the measure of an arbitrary cylinder in $X_{\mathcal{S}}$, for an arbitrary invariant measure supported by the subshift; and finally, the *density of iterated indicators* of proposition 2.3.8, permitting the study of the translation action on $L^2(\mu)$ in terms of the iterated indicator functions $\mathbb{1}_{T^{\mathbf{k}}\mathcal{S}^n[\alpha]}$, and ultimately the indicators $\mathbb{1}_{[\alpha]}$ via theorem 3.1.4 discussed below.

In §3, we state our main result (theorem 3.3.1) which separates the spectral characterization problem of an aperiodic $\mathbf{q}$-substitution into a study of its correlation measures and spectral hull; see theorems 3.1.4 and 3.3.1. We begin our discussion with the $\mathbf{q}$-shift, a topological dynamical system

$$(\mathbb{T}^d, \mathbf{S_q}) \qquad \text{with} \qquad \mathbf{S_q} : \mathbb{T}^d \longrightarrow \mathbb{T}^d \qquad \text{taking} \qquad \mathbf{S_q} : \mathbf{z} \longmapsto \mathbf{z}^{\mathbf{q}}$$

which is conjugate to the $\times\mathbf{q}$ mod $\mathbf{1}$ map on $\mathbb{R}^d/\mathbb{Z}^d$ via the map $\boldsymbol{\theta} \mapsto e^{2\pi i \boldsymbol{\theta}}$, and construct a measure $\boldsymbol{\omega}_{\mathbf{q}}$ in (3.2) which is a pure discrete measure supporting the $\mathbf{q}$-adic rationals in $\mathbb{R}^d/\mathbb{Z}^d \simeq \mathbb{T}^d$. Although $\boldsymbol{\omega}_{\mathbf{q}}$ is not $\mathbf{q}$-shift invariant it is a $\mathbf{q}$-*shift invariant type*, see definition 3.3.2. Due to the many $\mathbf{q}$-adic properties of $\mathbf{q}$-substitutions, the $\mathbf{q}$-shift will play a major role in their spectral theory, as the $\mathbf{q}$-shift represents $\mathbf{q}$-adic arithmetic on the d-torus.

In §3.1, we discuss the *correlation vector* Σ of a $\mathbf{q}$-substitution, whose components are the *correlation measures* $\sigma_{\alpha\beta}$ which are spectral measures for the pairs $\mathbb{1}_{[\alpha]}, \mathbb{1}_{[\beta]}$ of indicators on cylinder sets in $X_{\mathcal{S}}$, and whose Fourier coefficients (3.3) give the frequency with which two letter patterns appear in sequences within the subshift. We then prove the Fourier recursion theorem 3.1.2 (a simple corollary of the measure formula) which expresses the Fourier coefficients of the correlation measures in terms of $\mathbf{q}$-adically smaller coefficients and allows for the explicit computation of the Fourier coefficients of the correlation measures of an arbitrary aperiodic $\mathbf{q}$-substitution. These recursions indicate a $\mathbf{q}$-shift invariance of the autocorrelations $\sigma_{\alpha\alpha}$ for $\alpha \in \mathcal{A}$, which combine in theorem 3.1.4 with the density of the iterated indicators to show that $\sigma_{\max} \sim \boldsymbol{\omega}_{\mathbf{q}} * \sum_{\alpha} \sigma_{\alpha\alpha}$ and the spectrum of an aperiodic $\mathbf{q}$-substitution is a positive linear combination of the autocorrelations. Although these measures are positive by the spectral theorem (in general, the correlations $\sigma_{\alpha\beta}$ are complex) and $\mathbf{q}$-shift invariant, they need not be pure types, nor are they in general mutually singular. Thus, it is difficult to characterize the spectrum of $\mathcal{S}$ from the Fourier coefficients of the autocorrelations $\sigma_{\alpha\alpha}$ using this equivalence alone. In §3.2 we consider more general linear combinations of the correlation measures $\sigma_{\alpha\beta}$ and in definition 3.2.4 describe the *spectral hull* of a $\mathbf{q}$-substitution: a convex cone $\mathcal{K}(\mathcal{S}) \subset \mathbb{C}^{\mathcal{A}^2}$ consisting of those left Q-eigenvectors for the coincidence matrix $C_{\mathcal{S}}$ satisfying a positivity condition (definition 3.2.2) of Queffélec.

This hull is shown in proposition 3.2.5 to give rise (via linear combinations of correlation measures) to $\mathbf{q}$-shift invariant (positive) measures on $\mathbb{T}^d$.

Then, in §3.3, we state our main result, theorem 3.3.1, which shows that if $\mathbf{v} \in \mathcal{K}(\mathcal{S})$ is a strictly positive linear combination of the extremal rays $\mathcal{K}^*$ then $\sigma_{\max} \sim \boldsymbol{\omega}_{\mathbf{q}} * \mathbf{v}^t\Sigma$, and the extremal rays of the spectral hull give rise to linear combinations of correlation measures which are strongly mixing probability measures for the $\mathbf{q}$-shift. This separates the spectrum of an aperiodic $\mathbf{q}$-substitution into three components: the $\mathbf{q}$-adic spectrum $\boldsymbol{\omega}_{\mathbf{q}}$, determined entirely by the expansion $\mathbf{q}$ of the substitution; the spectral hull $\mathcal{K}(\mathcal{S})$, which depends only on the coincidence matrix and strong semipositivity, the latter of which is totally independent of $\mathcal{S}$; the correlation vector Σ, which is determined by the configuration $\mathcal{R}$ and the initial weights $\mu([\alpha])$ given by the invariant measure μ. As the spectral hull has already been shown to give rise to $\mathbf{q}$-shift invariant measures absolutely continuous with respect to the maximal spectral type, the proof relies on showing that these measure give rise to the entire spectrum and that the extremal rays have the necessary ergodic properties; the details are left for the appendix, however, as they are not necessary to state the main result or its consequences. We then briefly discuss purity of ergodic measures and show (corollary 3.3.3) that these extremal rays give rise to linear combinations of the correlation measures which are pure discrete, purely singular continuous, or Lebesgue measure on $\mathbb{T}^d$, and use this to provide a simple test (corollary 3.3.4) for the exclusion of Lebesgue spectral component. Following this discussion, we prove theorem 3.4.1 in §3.4, stating that all *aperiodic bijective and commutative* $\mathbf{q}$-substitutions have singular to Lebesgue spectrum, generalizing a result of Baake and Grimm [2] showing singularity for 2-letter bijective substitutions in $\mathbb{Z}^d$.

With our main results established, we move on to the secondary purpose of the paper: the description of an algorithm allowing one to determine the spectrum of an arbitrary aperiodic $\mathbf{q}$-substitution on a finite alphabet by explicitly computing the Fourier coefficients of the strongly-mixing measures $\mathbf{w}^t\Sigma$ arising from the extremal rays of the spectral hull. We summarize the algorithm here, noting that a more in depth description appears in that section. One begins with an aperiodic $\mathbf{q}$-substitution $\mathcal{S}$ on the alphabet $\mathcal{A}$, and then:

1. Replace $\mathcal{S}$ with an iterate $\mathcal{S}^h$ (its *index of imprimitivity*) to expose reducibility and

compute the primitive reduced forms of $\mathcal{S}$ and its bisubstitution $\mathcal{S}\otimes\mathcal{S}$. This is necessary to find the correct $\mathbf{q}$, noting that this is only relevant in the *nonprimitive* case.

2. Choose a positive convex combination $\mathbf{u}$ of Perron vectors corresponding to the primitive components of $M_{\mathcal{S}}$; this fixes an invariant measure μ supported by every ergodic invariant measure for $X_{\mathcal{S}}$ and $u_\alpha = \mu([\alpha])$ for $\alpha \in \mathcal{A}$ by Michel's theorem 1.2.1.

3. Compute the Fourier coefficients of Σ: as $\widehat{\Sigma}(\mathbf{0}) = \sum_\alpha u_\alpha \mathbf{e}_{\alpha\alpha} \in \mathbb{C}^{\mathcal{A}^2}$, we can find $\widehat{\Sigma}(\mathbf{c})$ for $\mathbf{c} \in [-\mathbf{1}, \mathbf{1}]$ by solving algebraically using the Fourier recursion theorem 3.1.2, and compute $\widehat{\Sigma}(\mathbf{k})$ for all other $\mathbf{k}$ using the Fourier recursion theorem 3.1.2 to reduce.

4. Compute $C_{\mathcal{S}} = M_{\mathcal{S}\otimes\mathcal{S}}$ and its primitive reduced form; determine the spectral hull using lemma 4.0.3 and strong semipositivity. Identify the extremal rays $\mathcal{K}^*$.

5. Characterize $\mathbf{w}^t\Sigma$ for $\mathbf{w} \in \mathcal{K}^*$ using corollaries 3.3.3 and 3.3.4: $\sigma_{\max} \sim \boldsymbol{\omega}_{\mathbf{q}} * \sum_{\mathbf{w}\in\mathcal{K}^*} \mathbf{w}^t\Sigma$.

Following a description of the algorithm, we consider several examples, some classical and well known which serve to familiarize the reader with the techniques, as well as several for which the spectrum is unknown or incorrectly classified. In each case, we determine the discrete, singular continuous, and absolutely continuous components of the spectrum.

Finally, in §5 we prove theorem 3.3.1 in two steps: a diagonalization result (theorem 5.1.8) for operator valued measures ergodic for a continuous transformation of a compact metric space, and then using this to connect the spectrum of $\mathcal{S}$ to the spectral hull. The diagonalization result relies on a localization of the space of linear functionals on complex Borel measures, based on the *generalized functionals* of Šreider, see [29], and is extended significantly from the corresponding result in [22] using less machinery. Then in §5.2, we show how the Fourier recursion theorem 3.1.2 describes a matrix of measures $\mathcal{Z}$ called the *bicorrelation matrix*, after Queffélec, and satisfying $\Sigma = \mathcal{Z}\widehat{\Sigma}(\mathbf{0})$. We then show that there is a projection $\mathcal{P}$ onto the Q-eigenspaces of $C_{\mathcal{S}}$ for which $\mathcal{P}\mathcal{Z}$ is ergodic (as a matrix of measures, see §5) with respect to the $\mathbf{q}$-shift, and use theorem 5.1.8 to diagonalize $\mathcal{P}\mathcal{Z}$ and characterize its eigenmeasures. We then show that $\mathcal{P}$ and $\mathcal{Z}$ both preserve the strong semipositivity condition determining the spectral hull, and use this to show that the maximal spectral type of $\mathcal{S}$ is generated by the eigenmeasures of $\mathcal{PZ}$, and the main result result follows by showing the spectral hull maps onto the positive linear span of the eigenmeasures of $\mathcal{PZ}$.

Before beginning, we make a remark on a particularly important notational convention. We will be working extensively with the ring $\mathbb{Z}^d$, and wish to do so by interpreting all operations and relations on $\mathbb{Z}^d$ coordinatewise. We represent $\mathbb{Z}^d$ integers in boldface $\mathbf{i}, \mathbf{j}, \mathbf{k}$, etc, and denote the components of $\mathbf{k}$ with k_i for $1 \leqslant i \leqslant d$. Note that notation such as $\mathbf{k}_n$ refers to a $\mathbb{Z}^d$ integer; should we need to refer to its coordinates, we will use $(\mathbf{k}_n)_j$ for $1 \leqslant j \leqslant d$. The symbols $\mathbf{0}, \mathbf{1}$ represent the $\mathbb{Z}^d$ integers all of whose coordinates are $0, 1$ respectively. For $1 \leqslant i \leqslant d$, let $\mathbf{1}_i$ be the integer 0 in all coordinates but the i-th, where it is 1, so that $\mathbf{1} = \sum_1^d \mathbf{1}_i$. For $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^d$, the inequalities $\mathbf{a} < \mathbf{b}$ or $\mathbf{a} \leqslant \mathbf{b}$ should be interpreted as holding in each coordinate simultaneously, i.e. $a_i < b_i$, for $1 \leqslant i \leqslant d$, and defines a partial order on $\mathbb{Z}^d$. Additionally, whenever $\mathbf{a} \leqslant \mathbf{b}$, the interval notation $[\mathbf{a}, \mathbf{b}]$ or $[\mathbf{a}, \mathbf{b})$ should be interpreted componentwise in the usual way, giving rise to (semi-)rectangles in $\mathbb{Z}^d$. For $t \in \mathbb{Z}$, we have $t\mathbf{a} = (ta_1, \ldots, ta_d)$, with $\mathbf{a}+\mathbf{b}$ and $\mathbf{ab}$ representing the usual sum and componentwise product, and we define $\frac{\mathbf{a}}{\mathbf{h}} \in \mathbb{Q}^d$ as the componentwise quotient for $\mathbf{h} \geqslant \mathbf{1}$. Finally, for $\mathbf{z} \in \mathbb{T}^d$ write $\mathbf{z}^n = (z_1^n, \ldots, z_d^n)$, so that $\mathbf{z}^n \in \mathbb{T}^d$.

1.1 Dynamical Systems and $\mathbb{Z}^d$ Actions

Let X be a compact metric space, and $\mathrm{Aut}(X)$ the homeomorphisms of X. A *$\mathbb{Z}^d$-action T on X* is a map $T : \mathbb{Z}^d \to \mathrm{Aut}(X)$ sending $\mathbf{k} \mapsto T^{\mathbf{k}}$ and satisfying $T^{\mathbf{j}}T^{\mathbf{k}} = T^{\mathbf{j}+\mathbf{k}}$ and with $T^{\mathbf{0}}$ the identity, and we write (X, T) for the *topological dynamical system* generated by this action. Equipping X with its Borel σ-algebra $\mathcal{B}$, one can consider the space of *T-invariant measures* $\mathcal{M}(X, T)$ consisting of those Borel probability measures with $\mu \circ T^{\mathbf{k}} = \mu$ for all $\mathbf{k} \in \mathbb{Z}^d$. For any $\mu \in \mathcal{M}(X, T)$, the triple (X, T, μ) is a measure preserving dynamical system.

A measurable subset E of X is *T-invariant* provided $T^{\mathbf{k}}E = E$ for all $\mathbf{k} \in \mathbb{Z}^d$, and an invariant probability measure μ is *ergodic* if $\mu(E)$ is either 0 or 1 whenever E is an invariant set: in this case, we refer to (X, T, μ) as an *ergodic system.* Ergodic measures are indecomposable: they cannot be expressed as the sum of (nontrivial) mutually singular invariant measures, and for this reason, one can see that ergodic systems are necessarily aperiodic: if $T^{\mathbf{k}}x = x$ for some $x \in X$, then $\mathbf{k} = \mathbf{0}$, so there are no periodic subsystems. Viewed as a subspace $\mathcal{M}(X,T) \subset \mathcal{C}(X)^*$ of the bounded linear functionals in the weak-star topology, the invariant measures $\mathcal{M}(X,T)$ form a nonempty compact convex set, the extreme points of which are ergodic [30, Thm 6.10]. Thus, every invariant probability measure can be expressed as a convex combination, or integral, of ergodic measures by Choquet's theorem [21]. More can be said: any two distinct ergodic probability measures are mutually singular, so that this *ergodic decomposition* is unique. Whenever $\mathcal{M}(X,T)$ consists of a single measure, we say (X,T) is *uniquely ergodic.* The following is one of the most important properties of ergodic systems

Theorem 1.1.1 (Birkhoff Ergodic Theorem; Ergodic Case)**.**

Let (X,T,μ) be an ergodic system. Then for every $f \in L^1(X,\mu)$,

$$\lim_{n\to\infty} \frac{1}{n^d} \sum\nolimits_{\mathbf{j}\in[\mathbf{0},n\mathbf{1})} f(T^{\mathbf{j}}x) = \int_X f \, d\mu,$$

for μ almost every $x \in X$. We call the terms in the above limit the ergodic averages of f.

We have the following alternate characterization of ergodicity: a measure $\mu \in \mathcal{M}(X,T)$ is ergodic if and only if for every measurable $A, B \subset X$,

$$\lim_{n\to\infty} \frac{1}{n^d} \sum\nolimits_{\mathbf{j}\in[\mathbf{0},n\mathbf{1})} \mu(T^{-\mathbf{j}}A \cap B) = \mu(A)\mu(B),$$

so that B is independent from the orbit of A *on average* as $n \to \infty$. There are two variations on this *asymptotic independence of sets*: a measure $\mu \in \mathcal{M}(X,T)$ is *strongly-mixing* provided $T^{-\mathbf{k}}A$ and B are asymptotically independent as $|\mathbf{k}| \to \infty$ for all $A, B \in \mathcal{B}$, or

$$\lim_{|\mathbf{k}|\to\infty} \mu(T^{-\mathbf{k}}A \cap B) = \mu(A)\mu(B),$$

and *weakly-mixing* provided the above limit converges when avoiding a set of *zero density*: a subset $J \subset \mathbb{Z}^d$ has zero density whenever $\frac{1}{(2n+1)^d}\mathrm{Card}\big(J \cap (-n\mathbf{1}, n\mathbf{1})\big) \to 0$ as $n \to \infty$. Note that strongly-mixing $\implies$ weakly-mixing $\implies$ ergodic (converses are all false in general), see [30, Thm 1.20], all of which are isomorphism invariants [30, Thm 2.13].

1.1.1 Spectral Theory of $\mathbb{Z}^d$-actions

We now discuss the spectral theory of $\mathbb{Z}^d$ actions. Let $\mathcal{M}(\mathbb{T}^d)$ denote the space of complex Borel measures on the d-torus $\mathbb{T}^d$ and equipped with the total variation norm; this agrees with its subspace topology in $\mathcal{C}(X)^*$, the space of continuous linear functionals on continuous complex valued functions on X. For $\lambda, \nu \in \mathcal{M}(\mathbb{T}^d)$, let $|\nu|$ denote the variation of ν, and write $\lambda \ll \nu$ whenever their variations $|\lambda| \ll |\nu|$ are absolutely continuous, $\lambda \perp \nu$ when the measures are mutually singular. The two measures are *equivalent* $\lambda \sim \nu$ whenever they are mutually absolutely continuous: the $\mathcal{L}$-space $\mathcal{L}(\nu)$ and the type $\langle \nu \rangle$ of ν are the collections

$$\mathcal{L}(\nu) := \{\lambda \in \mathcal{M}(\mathbb{T}^d) \ : \ \lambda \ll \nu\} \qquad \text{and} \qquad \langle \nu \rangle := \{\lambda \in \mathcal{M}(\mathbb{T}^d) : \lambda \sim \nu\}$$

and one checks that $\mathcal{L}(\nu)$ is a closed subspace of $\mathcal{M}(X)$, consisting of the complex Borel measures on $\mathbb{T}^d$ absolutely continuous with respect to ν, and depending only on its type. Given a measure $\nu \in \mathcal{M}(\mathbb{T}^d)$, its *Fourier coefficients* are, for $\mathbf{k} \in \mathbb{Z}^d$, given by

$$\widehat{\nu}(\mathbf{k}) := \int_{\mathbb{T}^d} \mathbf{z}^{-\mathbf{k}} d\nu \qquad \text{where} \qquad \mathbf{z}^{-\mathbf{k}} := z_1^{-k_1} \cdots z_d^{-k_d}$$

Fix a measure preserving $\mathbb{Z}^d$-action (X, T, μ) on a compact metric space: note this implies that $L^2(\mu)$, the space of square μ-integrable complex valued functions on X, is separable. Any $\mathbb{Z}^d$ action T on X induces a $\mathbb{Z}^d$ action on $L^2(\mu)$ known as the *Koopman representation* U_T of (X, T, μ) and is the $\mathbb{Z}^d$-action $U_T^{\mathbf{k}} : L^2(\mu) \to L^2(\mu)$ sending $f \mapsto f \circ T^{\mathbf{k}}$. As μ is T-invariant and T acts by homeomorphisms, $U := U_T$ is unitary. For each pair $f, g \in L^2(\mu)$, Bochner's theorem gives us a complex Borel measure $\sigma_{f,g}$ on the d-Torus $\mathbb{T}^d$ called *the spectral measure for* f, g, with Fourier coefficients satisfying

$$\widehat{\sigma_{f,g}}(\mathbf{k}) := \int_{\mathbb{T}^d} z_1^{-k_1} \cdots z_d^{-k_d} \, d\sigma_{f,g} = \int_X f \circ T^{-\mathbf{k}} \, \overline{g} \, d\mu \tag{1.1}$$

which determines a continuous sesquilinear form

$$\sigma : L^2(\mu) \times L^2(\mu) \longrightarrow \mathcal{M}(\mathbb{T}^d) \qquad \text{with} \qquad f, g \longmapsto \sigma_{f,g} \ \text{ satisfying } \ (1.1)$$

which we call the *spectral map*, and write $\sigma_f := \sigma_{f,f}$ which is a nonnegative measure. As U_T is a unitary $\mathbb{Z}^d$ action, the spectral theorem for unitary operators guarantees the existence of a *maximal function* $F \in L^2(\mu)$ such that for any $g \in L^2(\mu)$, the spectral measure for g is absolutely continuous with respect to that of F, or $\sigma_g \ll \sigma_F$, and every measure $\lambda \ll \sigma_F$ is the spectral measure of some $g \in L^2(\mu)$. Although the maximal function is not in general unique, all maximal functions have equivalent spectral measures, and thus have the same type, which we denote by $\sigma_{\max}(\mu)$. The following summarizes the above, see [24] for details on the spectral theorem.

Theorem 1.1.2 (Spectral Theorem for $\mathbb{Z}^d$-actions)**.**

Let (X, T, μ) be a measure preserving $\mathbb{Z}^d$-action. There is a continuous sesquilinear form $\sigma : L^2(\mu) \times L^2(\mu) \to \mathcal{M}(\mathbb{T}^d)$ and a maximal spectral type σ_{max} such that

$$f, g \longmapsto \sigma_{f,g} \qquad \textit{satisfying} \qquad \widehat{\sigma_{f,g}}(\mathbf{k}) := \int_{\mathbb{T}^d} \mathbf{z}^{-\mathbf{k}} d\sigma_{f,g} = \int_X f \circ T^{-\mathbf{k}} \cdot \overline{g}\, d\mu$$

and $f \mapsto \sigma_f$ is surjective onto the positive measures absolutely continuous with respect to σ_{max}. Moreover, there is a sequence of nonzero functions $\{f_n\}$ (finite or infinite) with

- *The maximal function f_0 has spectral measure equivalent to σ_{max}*
- *$L^2(\mu)$ is the orthogonal direct sum of the cyclic subspaces $\overline{Span\{U^{\mathbf{k}} f_n : \mathbf{k} \in \mathbb{Z}^d\}}$*
- *the spectral measures of the f_i satisfy $\sigma_{f_0} \gg \sigma_{f_1} \gg \sigma_{f_2} \gg \dots$*

and any other such sequence of $g_n \in L^2(\mu)$ satisfies $\sigma_{f_n} \sim \sigma_{g_n}$ for all n.

We are primarily interested in the maximal spectral type $\sigma_{\max}$ associated to a subshift. As $\sigma_{\max}$ is only specified up to measure equivalence, characterization of the spectrum largely amounts to finding mutually singular decompositions of $\sigma_{\max}$, or writing $\sigma_{\max} = \lambda + \nu$ with $\lambda \perp \nu$ both nonzero. The most popular such decomposition is separation into *discrete, singular continuous, and absolutely continuous* (with respect to Lebesgue) components via the Lebesgue-Radon-Nikodym theorem, writing $\sigma_{\max} = \sigma_d + \sigma_{sc} + \sigma_{ac}$. Note that the discrete spectrum σ_d is a measure whose support is precisely the eigenvalues of U_T.

Note that for every measure-preserving system, U_T always has 1 as an eigenvalue as constant functions are always T-invariant and in fact, μ is ergodic if and only if the eigenvalue 1 is simple. If U_T has no other eigenvalues, or equivalently 1 is the only atom of $\sigma_{\max}$, we say the invertible ergodic system (X, T, μ) has *continuous spectrum.* This is equivalent to $\sigma_{\max} \sim \delta + \sigma_{sc} + \sigma_{ac}$, where δ is the mass at $1 \in \mathbb{T}$. Note that a measure is purely continuous if and only if l^2-Césaro means of its Fourier coefficients are 0, by Wiener's criterion. Using this, one can show that a measure-preserving dynamical system is weakly-mixing if and only if it has continuous spectrum.

A system (X, T, μ) is said to have *discrete spectrum* provided the maximal spectral type of the Koopman representation is pure discrete, or $\sigma_{\max} = \sigma_d$. Note that a measure σ on $\mathbb{T}$ is pure discrete if and only if its Fourier coefficients are an *almost periodic* sequence. A famous theorem of Halmos and von Neumann (1942) showed that spectral isomorphism is equivalent to isomorphism for ergodic measure preserving transformations with discrete spectrum. As two unitary operators with discrete spectrum are unitarily equivalent if and only if they have the same eigenvalues, the Halmos-von Neumann theorem can be used to show that all ergodic measure-preserving systems with discrete spectrum are isomorphic to an ergodic rotation on a compact abelian group, see [30, Thm 3.4, 3.6]. Thus, rotations on a compact group are never weakly-mixing.

A system (X, T, μ) has *Lebesgue spectrum* whenever $\sigma_{\max} = \delta + \sigma_{ac}$. Criteria for a measure to be absolutely continuous is less clear than the above three cases, with Wiener's criterion or equivalently Riemann-Lebesgue giving us a necessary condition (the coefficients must converge to 0.) Sufficient criteria in general are not available, making identification of Lebesgue components in measures a difficult problem. We say a measure has *countable Lebesgue spectrum* provided any normalized spectral decomposition of U_T by functions $\{f_n\}$ (with $f_0 \equiv 1$) has $\{f_0\} \cup \{U^k f_n : k \in \mathbb{Z}, n > 0\}$ as an orthonormal basis for $L^2(\mu)$. The name comes from the fact that $\sigma_{f_n} \sim m$ for $n > 0$. Any two systems with countable Lebesgue spectrum are spectrally isomorphic, which follows by corresponding their normalized spectral decompositions to each other. Every Bernoulli shift has countable Lebesgue spectrum, which follows from a theorem of Rohlin on Kolmogorov automorphisms. One can show that any measure-preserving system with countable Lebesgue spectrum is necessarily strongly-mixing, see [30, Thm 2.12].

1.2 Substitution Dynamical Systems

Fix a dimension $d \geqslant 1$. An *alphabet* is a finite set $\mathcal{A}$ consisting of at least 2 *letters* which we will frequently denote with symbols $\alpha, \beta, \gamma, \delta$. Often, we will consider alphabets of the form $\{0, 1, \ldots, s-1\}$, and s will always refer to the size of the alphabet. Consider the collection of all functions from $\mathbb{Z}^d \to \mathcal{A}$, which can be identified with the product space $\mathcal{A}^{\mathbb{Z}^d}$ of all $\mathbb{Z}^d$-indexed sequences with values in $\mathcal{A}$. The elements of $\mathcal{A}^{\mathbb{Z}^d}$ are *sequences*, denoting them with letters $\mathbf{A}, \mathbf{B}, \mathbf{C}$, etc, and we will express their values $\mathbf{A}(\mathbf{k}) = \mathbf{A}_{\mathbf{k}}$ interchangeably for convenience of notation. Endowing $\mathcal{A}$ with the discrete topology, consider the topology of pointwise convergence on $\mathcal{A}^{\mathbb{Z}^d}$, or equivalently the product topology when viewed as a sequence space. Addition on $\mathbb{Z}^d$ gives rise to a $\mathbb{Z}^d$-action of commuting automorphisms, which act by translation on $\mathcal{A}^{\mathbb{Z}^d}$ sending $\mathbf{k} \mapsto T^{\mathbf{k}}$ where $T^{\mathbf{k}} : \mathcal{A}^{\mathbb{Z}^d} \to \mathcal{A}^{\mathbb{Z}^d}$ sends $\mathbf{A}$ to the function $T^{\mathbf{k}}\mathbf{A}$ defined by $T^{\mathbf{k}}\mathbf{A}(\mathbf{j}) := \mathbf{A}(\mathbf{j}+\mathbf{k})$. We call this action *the shift* and denote it by T. The pair $(\mathcal{A}^{\mathbb{Z}^d}, T)$ is an invertible topological dynamical system, *the full shift,* and we let $\mathcal{B}$ denote the σ-algebra of Borel measurable sets.

By a *block* (or *word* in the $d = 1$ case) we mean a map ω from a finite subset of $\mathbb{Z}^d$ into $\mathcal{A}$, denoting the domain of a block by $\operatorname{supp}(\omega)$, and letting $\mathcal{A}^+$ denote the collection of all blocks in $\mathcal{A}^{\mathbb{Z}^d}$ and are typically denoted with the greek letters $\eta, \omega \in \mathcal{A}^+$. Here, blocks differ from convention in two significant ways: they need not be contiguous, and they need not start at $\mathbf{0}$ index. We will routinely identify blocks with their *graphs,* or the image of the map $\mathbf{j} \mapsto (\mathbf{j}, \omega(\mathbf{j})) \in \mathbb{Z}^d \times \mathcal{A}$, as this is often convenient and coincides with the tiling perspective of substitutions, as treated by Radin [23]. For $\omega, \eta \in \mathcal{A}^+$ and $\mathbf{A} \in \mathcal{A}^+ \cup \mathcal{A}^{\mathbb{Z}^d}$, we say ω is *extended by* η or $\mathbf{A}$ whenever η or $\mathbf{A}$ extends ω as a function into $\mathcal{A}$ and write $\omega \leqslant \eta$ or $\omega \leqslant \mathbf{A}$ in this case. We say ω is a *subblock* (or *subword* in the $d = 1$ case) of $\mathbf{A}$ if $T^{\mathbf{k}}\omega$ is extended by $\mathbf{A}$, for some $\mathbf{k} \in \mathbb{Z}^d$. If $\omega \in \mathcal{A}^+$ is a block, the *cylinder over* ω is the collection of all sequences extending ω, or

$$[\omega] := \{\mathbf{C} \in \mathcal{A}^{\mathbb{Z}^d} : \mathbf{C}(\mathbf{j}) = \omega(\mathbf{j}) \text{ for } \mathbf{j} \in \operatorname{supp}(\omega)\} \qquad \text{so that} \qquad \omega \leqslant \eta \iff [\omega] \supseteq [\eta]$$

so that cylinders over blocks $\mathcal{A}^+$ correspond to the standard basis for the topology of $\mathcal{A}^{\mathbb{Z}^d}$. The alphabet $\mathcal{A}$ is a discrete compact set, and so $\mathcal{A}^{\mathbb{Z}^d}$ is totally disconnected as every cylinder is both open and closed, and compact by the Tychonoff theorem. As every $\mathbf{A} \in \mathcal{A}^{\mathbb{Z}^d}$ is in the intersection of the cylinders $[\mathbf{A}_n]$ where $\mathbf{A}_n$ is the subblock of $\mathbf{A}$ supported in $[-n\mathbf{1}, n\mathbf{1}]$, this implies $\mathcal{A}^{\mathbb{Z}^d}$ is a perfect set. Thus, the full shift $(\mathcal{A}^{\mathbb{Z}^d}, T)$ is an invertible topological dynamical system, consisting of a $\mathbb{Z}^d$-action of commuting automorphisms acting on a Cantor set of functions $\mathbb{Z}^d \to \mathcal{A}$.

A *substitution* is a map $\mathcal{S} : \mathcal{A} \to \mathcal{A}^+$, replacing each letter by a block. In the $d = 1$ case, substitutions of many types are considered, but we wish to consider the class of substitutions which generalize substitutions of constant length. Fix $\mathbf{q} > \mathbf{1}$, and write $Q := q_1 \cdots q_d = \text{Card}[\mathbf{0}, \mathbf{q})$. A substitution $\mathcal{S}$ is a $\mathbf{q}$-*substitution* if all the blocks $\mathcal{S}(\alpha)$ have $[\mathbf{0}, \mathbf{q})$ as their common support. Briefly, consider $\mathbb{Z}^d \subset \mathbb{R}^d$ and inflate $\mathbb{R}^d$ by $\mathbf{q}$, scaling each cube $[\mathbf{a}, \mathbf{a} + \mathbf{1}]$ to the rectangle $\mathbf{aq} + [\mathbf{0}, \mathbf{q}]$, which can then be subdivided into unit cubes located at points $\mathbf{aq} + \mathbf{b}$, for $\mathbf{b} \in [\mathbf{0}, \mathbf{q})$. If α appears at the $\mathbf{a}^{\text{th}}$ position of a block or sequence, and $\mathcal{S}$ maps α to the block $\mathcal{S}(\alpha)$ defined on $[\mathbf{0}, \mathbf{q})$, we can place the letter in the $\mathbf{b}$-th position of $\mathcal{S}(\alpha)$, denoted $\mathcal{S}(\alpha)_{\mathbf{b}}$, at the $\mathbf{aq} + \mathbf{b}$-th position, extending $\mathcal{S}$ to $\mathcal{A}^+$ as well as $\mathcal{A}^{\mathbb{Z}^d}$. Using this inflate and subdivide process identifies $\mathbb{Z}^d$ with $\mathbf{q}^n\mathbb{Z}^d \times [\mathbf{0}, \mathbf{q}^n)$ for every $n \geqslant 0$, and gives rise to many arithmetic properties of substitutions which we discuss in §2. We now associate a subshift to a given substitution, after which we give some essential details and preliminaries for the spectral theory of substitution subshifts.

The *language* $\mathcal{L}_{\mathcal{S}} \subset \mathcal{A}^+$ of a substitution is the collection of all blocks $\omega \in \mathcal{A}^+$ which appear as subwords of $\mathcal{S}^n(\gamma)$ for some $n \in \mathbb{N}$ and $\gamma \in \mathcal{A}$. Associating blocks with their graphs, this can be thought of as the collection of all finite patterns attainable by the substitution; by definition, it is both shift-invariant and closed under the action of $\mathcal{S}$. The *substitution subshift of* $\mathcal{S}$ is the collection $X_{\mathcal{S}}$ consisting of those sequences in $\mathcal{A}^{\mathbb{Z}^d}$, all of whose subblocks appear in the language of $\mathcal{S}$. By the *reduced language* of $\mathcal{S}$, we mean the collection of all blocks in $\mathcal{A}^+$ that appear in some word $\mathbf{A} \in X_{\mathcal{S}}$, and can be strictly smaller than the language; as this has no effect on the measure theoretic or topological structure of $X_{\mathcal{S}}$, we will often assume our language is reduced.

Let $\mathcal{S}$ be a $\mathbf{q}$-substitution on $\mathcal{A}$ with configuration $\mathcal{R}$. As the alphabet is finite, we can always find some $h > 0$ and $\eta : [-\mathbf{1}, \mathbf{0}] \to \mathcal{A}$ so that $\mathcal{S}^h(\eta)_{\mathbf{c}} = \eta(\mathbf{c})$ for $\mathbf{c} \in [-\mathbf{1}, \mathbf{0}]$. Then η acts as a *seed* for the subshift: by construction, the sequence of blocks $\mathcal{S}^{nh}\eta$ defined on $[-\mathbf{q}^{nh}, \mathbf{q}^{nh})$ are nested, as $\mathcal{S}^{nh}\eta$ extends $\mathcal{S}^{mh}\eta$ whenever $n \geqslant m$. As $\mathcal{A}^{\mathbb{Z}^d}$ is endowed with the topology of point-wise convergence and as $\mathbf{q} > \mathbf{1}$, the sequence $\mathcal{S}^{nh}\eta$ converges to a unique limit point $\mathbf{D}_\eta \in \mathcal{A}^{\mathbb{Z}^d}$ which is also in $X_{\mathcal{S}}$ by construction, and so $X_{\mathcal{S}}$ is nonempty for any $\mathbf{q}$-substitution on $\mathcal{A}$.

As distances in $\mathcal{A}^{\mathbb{Z}^d}$ are determined by agreement on finite subsets, and the condition for $X_{\mathcal{S}}$ prescribes extendability for every finite subset, one can check that $X_{\mathcal{S}}$ defines a nonempty, closed, and shift-invariant subset of $\mathcal{A}^{\mathbb{Z}^d}$, and $(X_{\mathcal{S}}, T)$ forms a topological subshift of the full shift $\mathcal{A}^{\mathbb{Z}^d}$ called a *substitution dynamical system.* A useful property of substitution subshifts is their independence of the iterate used: for $n > 0$, $X_{\mathcal{S}^n} = X_{\mathcal{S}}$, sometimes referred to as *telescope invariance,* see [8, Lemma 2.9]. Note that $\mathcal{S}$ restricts from $\mathcal{A}^{\mathbb{Z}^d}$ to a map on $X_{\mathcal{S}}$, and the Borel σ-algebra for $X_{\mathcal{S}}$ is generated by the cylinders over blocks in the reduced language $\mathcal{L}_{\mathcal{S}}$. As our goal is the spectral theory of substitution dynamical systems, we adopt the convention that all cylinders are intersected with $X_{\mathcal{S}}$. Finally, let $\mathcal{M}(X_{\mathcal{S}}, T)$ denote the space of T-invariant ($\mu = \mu \circ T^{\mathbf{k}}$ for all $\mathbf{k} \in \mathbb{Z}^d$) Borel probability measures on $X_{\mathcal{S}}$; note that $\mathcal{M}(X_{\mathcal{S}}, T)$ is a compact convex set, the extreme points of which are ergodic, see [30].

1.2.1 Invariant Measures

Consider the vector space $\mathbb{C}^{\mathcal{A}}$ of formal linear combinations in the letters of $\mathcal{A}$, with standard basis $\mathbf{e}_\alpha$, for $\alpha \in \mathcal{A}$. Given a substitution $\mathcal{S}$ on $\mathcal{A}$, its *substitution matrix* $M_{\mathcal{S}} \in \mathbf{M}_{\mathcal{A}}(\mathbb{C})$ is the nonnegative $\mathcal{A}$-indexed square matrix whose α, γ entry is the number of times α appears in the word $\mathcal{S}(\gamma)$. This representation is often called the *abelianization of* $\mathcal{S}$ as it represents the symbols produced by the substitution without tracking their position. The *expansion* of a substitution is the spectral radius of its substitution matrix; in the case of $\mathbf{q}$-substitutions, the expansion is $Q = \operatorname{Card}[\mathbf{0}, \mathbf{q})$ as the substitution matrix is Q-column stochastic, see §2.2. A substitution is *primitive* if for some $n > 0$, α appears in the block $\mathcal{S}^n(\gamma)$, for every $\alpha, \gamma \in \mathcal{A}$. Thus, the substitution matrix of a primitive substitution is a primitive matrix: there exists some $n > 0$ so that $M_{\mathcal{S}^n} = M_{\mathcal{S}}^n$ has strictly positive entries. By the Perron-

Frobenius theorem, the spectral radius of a primitive matrix is a simple eigenvalue with a strictly positive eigenvector $\mathbf{u}$, normalized to a probability vector and called the *Perron vector* of $\mathcal{S}$, see [14, Theorem 8.1,2].

The following result of Michel [17] relates the invariant measures of a primitive substitution subshift to the eigenspace of its substitution matrix; see [22, Prop 5.22] for the result concerning the measure of cylinders, and [23, Lemma 1.5] for the extension to $\mathbb{Z}^d$.

Theorem 1.2.1 (Michel)**.** *If $\mathcal{S}$ is a primitive $\mathbf{q}$-substitution, then $(X_{\mathcal{S}}, T)$ has a unique ergodic measure μ, with $\mu \circ \mathcal{S} = \frac{1}{Q}\mu$. The Perron vector $\mathbf{u}$ of $M_{\mathcal{S}}$ satisfies $\mu[\alpha] = u_\alpha$, $\alpha \in \mathcal{A}$.*

As the collection of cylinders over the language of a substitution generate the Borel σ-algebra of its subshift, the measure μ is uniquely determined by the above formula. In the case of non-primitive substitutions, there will be a nonempty proper subset $\mathcal{A}_0 \subsetneq \mathcal{A}$ such that $\mathcal{S}^h$ restricts to a substitution on the subalphabet $\mathcal{A}_0$, for some $h > 0$. Using this, we can express any substitution in a *primitive reduced form,* see [22, §10.1.1]. In the statement below, $\sqcup$ denotes a disjoint union.

Proposition 1.2.2. *Let $\mathcal{S}$ be a substitution on $\mathcal{A}$. Then there is an iterate $h > 0$ and a partition of the alphabet $\mathcal{A} = \mathcal{E}_1 \sqcup \cdots \sqcup \mathcal{E}_K \sqcup \mathcal{T}$ so that*

- *$\mathcal{S}^h : \mathcal{E}_j \to \mathcal{E}_j^+$ is primitive for each $1 \leqslant j \leqslant K$,*
- *$\gamma \in \mathcal{T}$ implies $\mathcal{S}^h(\gamma) \notin \mathcal{T}^+$*

Clearly, $K = 1$ and $\mathcal{T} = \varnothing$ if and only if $\mathcal{S}$ is primitive. We call the partition $\{\mathcal{E}_1, \ldots, \mathcal{E}_K, \mathcal{T}\}$ the *ergodic decomposition of* $\mathcal{S}$, its members $\mathcal{E}_j$ the *ergodic classes of* $\mathcal{S}$, and $\mathcal{T}$ its *transient part*, compare [22, §10.1]. As the transient part $\mathcal{T}$ is equal to $\mathcal{A} \setminus \bigcup \mathcal{E}_j$, it suffices to specify the ergodic classes $\mathcal{E} = \{\mathcal{E}_1, \ldots, \mathcal{E}_K\}$, and we will do this frequently. Restricting $\mathcal{S}$ to each ergodic class gives the *primitive components of* $\mathcal{S}$, and we refer to their collective Perron vectors as the *Perron vectors of* $\mathcal{S}$. The minimal such $h \geqslant 1$ satisfying the above proposition is the *index of imprimitivity*; as telescoping has no effect on the subshift, we will assume that all of our substitutions have index of imprimitivity 1 as this directly impacts the existence of a several limits in our analysis.

We comment briefly on a few aspects of the substitution matrix and Perron-Frobenius theory, as these matrices are nonnegative, see [14] but note that our matrices are represented transpose to that in [14], due to conventions for the substitution matrix. The spectral radius ρ of $M_{\mathcal{S}}$ is the expansion of the substitution (average rate of growth for words or blocks), and Frobenius theory [14, III §2 Theorem 2] tells us that all eigenvalues of modulus ρ correspond to roots of unity (times ρ). The index of imprimitivity is then the smallest $h > 0$ such that $\lambda^h = Q^h$ for every eigenvalue λ of modulus Q for $M_{\mathcal{S}}$, allowing one to find the index of imprimitivity from the eigenvalues of the substitution matrix, see also [14, III §5]. A normal form for reducible matrices is described in [14, III §4 (69)], allowing us to write $M_{\mathcal{S}}$ via a permutation of the basis in block form

$$M_{\mathcal{S}} = \begin{pmatrix} M_1 & & & \tilde{M}_1 \\ & \ddots & & \vdots \\ & & M_K & \tilde{M}_K \\ & & & M_{\mathcal{T}} \end{pmatrix} \tag{1.2}$$

where all unrepresented blocks are $\mathbf{0}$. Moreover, the diagonal blocks $M_1, \ldots, M_K$ are all primitive when exponentiated by the index of imprimitivity, and so this serves as a proof of the above proposition. As $M_{\mathcal{S}}$ is Q-stochastic by column, an application of [14, III §4 Theorem 6] shows the spectral radius of the $M_1, \ldots, M_K$ are all Q, whereas that of $M_{\mathcal{T}}$ is strictly less than Q. As the matrices M_i^h are primitive, their Q^h-eigenvectors are determined by the Perron vectors of $\mathcal{S}$ which by the above representation of $M_{\mathcal{S}}$ will be mutually orthogonal.

Lemma 1.2.3. *The Perron vectors of* $\mathbf{q}$*-substitution* $\mathcal{S}$ *span the* Q*-eigenspace of* $M_{\mathcal{S}}$*.*

A substitution $\mathcal{S}$ is *aperiodic* if $X_{\mathcal{S}}$ contains no shift-periodic elements, or if $T^{\mathbf{k}}\mathbf{A} = \mathbf{A}$ implies $\mathbf{k} = \mathbf{0}$ for all $\mathbf{A} \in X_{\mathcal{S}}$. In [8], Cortez and Solomyak use results of Bezuglyi and others in [6] characterizing the invariant measures on aperiodic and stationary Bratteli diagrams, to extend Michel's theorem 1.2.1 to nonprimitive aperiodic $\mathbf{q}$-substitutions. The result as stated here is a corollary of [8, Theorem 3.8] and is adjusted for our purposes:

Theorem 1.2.4 (Cortez and Solomyak)**.** *Let* $\mathcal{S}$ *be an aperiodic* $\mathbf{q}$*-substitution on* $\mathcal{A}$*. Then the ergodic measures of* $X_{\mathcal{S}}$ *are the uniquely ergodic measures of its primitive components.*

Proof. As $M_{\mathcal{S}}$ is Q-column stochastic, we apply the above lemma which tells us that the eigenvalues corresponding to the transient diagonal blocks are strictly dominated by the expansion of the substitution. Thus, the *distinguished eigenvalues,* see [6], are those corresponding to $M_{\mathcal{S}}$ restricted to its ergodic classes, which is the substitution matrix of the primitive components of $\mathcal{S}$, and the result follows from [6, Corollary 5.6] □

In particular, there are at most Card($\mathcal{A}$) ergodic measures for any **q**-substitution subshift and the study of invariant measures for **q**-substitutions reduces to the primitive case. Note that one can express an invariant measure as a convex sum of the uniquely ergodic measures for the primitive components and thus extend the identity of theorem 1.2.1 to nonprimitive subshifts. Moreover, the vector $\mathbf{u} = (\mu([\alpha]))_{\alpha\in\mathcal{A}}$ will be the same convex combination of the mutually orthogonal Perron vectors $\mathcal{S}$.

1.2.2 Spectral Theory

We now discuss some basics of spectral theory for substitution subshifts, and show that a similar relationship to primitivity holds in that context as well. Given a measure $\mu \in \mathcal{M}(X_{\mathcal{S}}, T)$, recall that the Koopman representation of $(X_{\mathcal{S}}, T, \mu)$ is the (unitary) $\mathbb{Z}^d$ action $U_T^{\mathbf{k}} : f \mapsto f \circ T^{\mathbf{k}}$ on $L^2(\mu)$, the space of complex valued (μ) square integrable functions on $X_{\mathcal{S}}$, the substitution subshift. Let $\sigma_{\max}(\mu)$ be the maximal spectral type for the Koopman representation of U_T on $L^2(\mu)$, and recall that $\mathcal{E}_{\mathcal{S}}$ denotes the extreme points of $\mathcal{M}(X_{\mathcal{S}}, T)$. In light of theorem 1.2.4, we define the *spectrum of* $\mathcal{S}$ as the sum of the maximal spectral types of its primitive components

$$\sigma_{\max} := \textstyle\sum_{\mu\in\mathcal{E}_{\mathcal{S}}} \sigma_{\max}(\mu) \tag{1.3}$$

noting that this notion is well defined up to measure equivalence. Using identity (1.1) and the Krein-Milman theorem, one can see that $\sigma_{f,g}(\mu) \ll \sigma_{\max}$ for every $\mu \in \mathcal{M}(X_{\mathcal{S}}, T)$ and $f, g \in L^2(\mu)$, justifying the definition. By Lebesgue decomposition, $\sigma_{\max}$ can be separated into its *pure types*, or its *discrete, singular and absolutely continuous* components on $\mathbb{T}^d$.

In the case of **q**-substitutions and their substitution subshifts, the discrete component is well understood: it is a multiplicative subgroup of $\mathbb{T}^d$ corresponding to $\mathbf{q}$ and the *height* of a substitution, as described by Dekking in [9] (and extended to $\mathbb{Z}^d$ by Frank in [12]) where he also provided a complete classification of the pure discrete case: see section 4.4 for more details on Dekking's criteria. The study of the continuous spectrum is largely based on the work of Queffélec, in which she relates the maximal spectral type of a substitution to *correlation measures:* for $\alpha, \beta \in \mathcal{A}$, the correlation measure $\sigma_{\alpha\beta}$ is the spectral measure for the pair of indicator functions of cylinders $[\alpha]$ and $[\beta]$, relative to the translation action.

Due to Dekking's work, our interest is primarily in the continuous spectrum of $\mathcal{S}$, and distinguishing the purely singular case from those with Lebesgue components in their spectrum. Note that this is a nontrivial problem: the Thue-Morse substitution is an example of a substitution of constant length 2 on 2 letters with purely singular spectrum, possessing both discrete and continuous components, whereas the Rudin-Shapiro substitution has a discrete and absolutely continuous component, with no singular continuous spectrum. Moreover, one can form a *substitution product* of Thue-Morse and Rudin-Shapiro to obtain a substitution with discrete, singular continuous, and absolutely continuous components in the spectrum, which we consider in example 4.3.2; see also [1, §2]. For higher dimensional examples, we have the work of Baake and Grimm which shows a large class of substitutions on 2 symbols to be purely singular to Lebesgue spectrum, see [2], as well as Frank's paper [13] describing a collection of substitutions with Lebesgue spectral component.

Chapter 2

APERIODIC q-SUBSTITUTIONS

In this section, we discuss arithmetic properties of $\mathbf{q}$-substitutions which simplify their spectral analysis significantly. Then, we describe topological and measure theoretic consequences of aperiodicity allowing us to relate the maximal spectral type to the correlation measures, which we examine in §3.

2.1 Arithmetic Base $\mathbf{q}$ *in* $\mathbb{Z}^d$

We take a moment to establish some basic arithmetic notions, most of which are consequences of the classical division algorithm on $\mathbb{Z}$. Essentially, we are just formalizing base q arithmetic in each coordinate separately. Recall that, for $\mathbf{a} \geqslant \mathbf{0}$,

$$[\mathbf{0}, \mathbf{a}) = \left\{\mathbf{j} \in \mathbb{Z}^d : 0 \leqslant j_i < a_i \text{ for } 1 \leqslant i \leqslant d\right\}$$

Fix $\mathbf{q} > \mathbf{1}$, then for every $n \in \mathbb{N}$ and each $\mathbf{k} \in \mathbb{Z}^d$, the division algorithm applied componentwise to $\mathbb{Z}^d$ provides a unique pair $\lceil \mathbf{k} \rceil_n \in [\mathbf{0}, \mathbf{q}^n)$ and $\lfloor \mathbf{k} \rfloor_n \in \mathbb{Z}^d$ satisfying

$$\mathbf{k} = \lceil \mathbf{k} \rceil_n + \lfloor \mathbf{k} \rfloor_n \mathbf{q}^n$$

The map $\lceil \cdot \rceil_n : \mathbb{Z}^d \to [\mathbf{0}, \mathbf{q}^n)$ denotes the *remainder,* and $\lfloor \cdot \rfloor_n : \mathbb{Z}^d \to \mathbb{Z}^d$ the *quotient,* under division modulo $\mathbf{q}^n$. For $\mathbf{k} \in \mathbb{Z}^d$ and $n \geqslant 0$, letting

$$\mathbf{k}_n := \lceil \lfloor \mathbf{k} \rfloor_n \rceil_1 = \lfloor \lceil \mathbf{k} \rceil_{n+1} \rfloor_n$$

gives a unique *digit sequence* $(\mathbf{k}_j)_{j \in \mathbb{N}} \in [\mathbf{0}, \mathbf{q})^{\mathbb{N}}$ such that for $n \geqslant 1$,

$$\mathbf{k} = \mathbf{k}_0 + \mathbf{k}_1 \mathbf{q} + \cdots + \mathbf{k}_{n-1} \mathbf{q}^{n-1} + \lfloor \mathbf{k} \rfloor_n \mathbf{q}^n$$

referred to as the *n-th* $\mathbf{q}$*-adic expansion* of $\mathbf{k}$, and we call $\mathbf{k}_n$ the *n-th digit* of $\mathbf{k}$. Thus, for $\mathbf{k} \in \mathbb{Z}^d$, $\lceil \mathbf{k} \rceil_n$ can be represented by the digits of $\mathbf{k}$ below the n-th place, and $\lfloor \mathbf{k} \rfloor_n$ by the digits at n-th place and above. The *power* $\mathfrak{p} := \mathfrak{p}(\mathbf{k})$ of $\mathbf{k}$ is the minimal $p \geqslant 0$ with $\mathbf{k} \in (-\mathbf{q}^p, \mathbf{q}^p)$ and is such that $\lfloor \mathbf{k} \rfloor_n = \lfloor \mathbf{k} \rfloor_{\mathfrak{p}(\mathbf{k})}$ and $\mathbf{k}_n = \mathbf{k}_{\mathfrak{p}(\mathbf{k})}$ for $n \geqslant \mathfrak{p}(\mathbf{k})$. One checks that for $p \geqslant \mathfrak{p}(\mathbf{k})$ and $1 \leqslant i \leqslant d$

$$\left(\lfloor \mathbf{k} \rfloor_p \right)_i = \begin{cases} 0 & \text{if } k_i \geqslant 0 \\ -1 & \text{if } k_i < 0 \end{cases} \qquad \text{and} \qquad \left(\mathbf{k}_p \right)_i = \begin{cases} 0 & \text{if } k_i \geqslant 0 \\ q_i - 1 & \text{if } k_i < 0 \end{cases} \tag{2.1}$$

and we call $\lfloor \mathbf{k} \rfloor_{\mathfrak{p}(\mathbf{k})}$ *the sign of* $\mathbf{k}$ and $\mathbf{k}_{\mathfrak{p}(\mathbf{k})}$ its *terminal digit*, and correspond to $\mathbf{k}$'s quadrant.

Consider the action of translation by $\mathbb{Z}^d$ on itself and its effect on digit sequences. An important aspect of this action are *carry operations*, i.e. when addition at the p-th digit gives rise to a number larger than q_i in some coordinate. Define the *p-carry function*

$$\mathfrak{c}_p : \mathbb{Z}^d \times \mathbb{Z}^d \to [\mathbf{0}, \mathbf{1}] \qquad \text{with} \qquad \mathfrak{c}_p(\mathbf{j}, \mathbf{k}) := \lfloor \lceil \mathbf{j} \rceil_p + \lceil \mathbf{k} \rceil_p \rfloor_p$$

and one can check the following *rules of* $\mathbf{q}$*-adic arithmetic*

$$\lceil \mathbf{j} + \mathbf{k} \rceil_p = \lceil \mathbf{j} \rceil_p + \lceil \mathbf{k} \rceil_p - \mathbf{q}^p \mathfrak{c}_p(\mathbf{j}, \mathbf{k}) \qquad \text{and} \qquad \lfloor \mathbf{j} + \mathbf{k} \rfloor_p = \mathfrak{c}_p(\mathbf{j}, \mathbf{k}) + \lfloor \mathbf{j} \rfloor_p + \lfloor \mathbf{k} \rfloor_p \tag{2.2}$$

by uniqueness of the division algorithm. Computing the digits of $\mathbf{j} + \mathbf{k}$ with (2.2) is precisely $\mathbf{q}$*-adic addition,* so that $\mathfrak{c}_p$ is the *amount carried at the p-th place* in each coordinate when adding $\mathbf{j}$ and $\mathbf{k}$. For $p \geqslant 0$, and $\mathbf{k} \in \mathbb{Z}^d$, let the *p-carry set for* $\mathbf{k}$ be the collection

$$\Delta_p(\mathbf{k}) := \left\{ \mathbf{j} \in [\mathbf{0}, \mathbf{q}^p) : \mathfrak{c}_p(\mathbf{j}, \mathbf{k}) \neq \mathbf{0} \right\} \tag{2.3}$$

Thus, the collection $\Delta_p(\mathbf{k}) + \mathbf{q}^p \mathbb{Z}^d$ consists of all $\mathbf{j}$ for which $\mathbf{q}$-adic addition with $\mathbf{k}$ requires a carry operation at the p-th place, and enjoys a useful statistical property:

Lemma 2.1.1. *The frequency of carries at the n-th place of* $\mathbf{q}$*-adic addition with* $\mathbf{k}$ *decays*

$$\lim_{n \to \infty} \frac{1}{Q^n} \, Card \, \Delta_n(\mathbf{k}) = 0$$

for every $\mathbf{k} \in \mathbb{Z}^d$*, and this convergence is exponentially fast.*

Proof. As the cardinality of $\Delta_n(\mathbf{k})$ does not depend on the signs of the components of $\mathbf{k}$, we can assume without loss of generality that $\mathbf{k} \in \mathbb{N}^d$. Then for $n \geqslant \mathfrak{p} := \mathfrak{p}(\mathbf{k})$, we have

$$\Delta_n(\mathbf{k}) \subset \Delta_n(\mathbf{q}^{\mathfrak{p}}) = [\mathbf{0}, \mathbf{q}^n) \setminus [\mathbf{0}, \mathbf{q}^n - \mathbf{q}^{\mathfrak{p}})$$

so that taking cardinalities, as $Q = q_1 \cdots q_d$ we obtain

$$\operatorname{Card} \Delta_n(\mathbf{k}) \leqslant Q^n \left(1 - \prod_{i=1}^d \left(1 - \tfrac{q_i^p}{q_i^n}\right)\right) = Q^n \sum_{j=1}^d q_j^{\mathfrak{p}-n} \prod_{i \neq j} (1 - q_i^{\mathfrak{p}-n}) \leqslant Q^n \sum_{j=1}^d q_j^{\mathfrak{p}-n}$$

Dividing by Q^n and letting $n \to \infty$ gives the desired result. □

We briefly describe the above in the context of $\mathbf{q}$-substitutions. Fix $\mathbf{k} \in \mathbb{Z}^d$, and for every $n \geqslant 0$, imagine $\mathbb{Z}^d$ tiled by the superblocks $[\mathbf{0}, \mathbf{q}^n)$ placed at each location of the $\mathbf{q}^n\mathbb{Z}^d$ lattice. The n-th quotient $\lfloor \mathbf{k} \rfloor_n$ indicates the superblock in the $\mathbf{q}^n\mathbb{Z}^d$ lattice containing $\mathbf{k}$, and the n-th remainder $\lceil \mathbf{k} \rceil_n$ tells us where $\mathbf{k}$ sits inside that superblock. The power $\mathfrak{p}(\mathbf{k})$ represents the smallest p such that $\mathbf{k}$ falls into a superblock of size $\mathbf{q}^p$ attached to the origin at one of its corners. As $\lfloor \mathbf{j} + \mathbf{k} \rfloor_n = \lfloor \mathbf{k} \rfloor_n + \mathfrak{c}_n(\mathbf{j}, \mathbf{k})$ for $\mathbf{j} \in [\mathbf{0}, \mathbf{q}^n)$ by the arithmetic rules (2.2), if we place the superblock $[\mathbf{0}, \mathbf{q}^n)$ at $\mathbf{k}$, then for each $\mathbf{j} \in [\mathbf{0}, \mathbf{q}^n)$ the carry function $\mathfrak{c}_n(\mathbf{j}, \mathbf{k})$ represents which superblock $\mathbf{j} + \mathbf{k}$ is in *relative* to the superblock containing $\mathbf{k}$. Thus, the carry set $\Delta_p(\mathbf{k})$ represents the set of locations in which the superblock $[\mathbf{k}, \mathbf{k} + \mathbf{q}^p)$ overlaps superblocks *other than* the one in the $\mathbf{q}^p\mathbb{Z}^d$ lattice containing $\mathbf{k}$. Moreover, if $n \geqslant \mathfrak{p}(\mathbf{k})$ then $\lfloor \mathbf{j} + \mathbf{k} \rfloor_n = \mathbf{k}_{\mathfrak{p}} + \mathfrak{c}_n(\mathbf{j}, \mathbf{k})$, where $\mathbf{k}_{\mathfrak{p}}$ is the terminal digit of $\mathbf{k}$. Thus, the above lemma tells us that the proportion of subtiles of the superblock $[\mathbf{k}, \mathbf{k} + \mathbf{q}^n)$ which overlap $\mathbf{q}^n\mathbb{Z}^d$ superblocks *other than* $\mathbf{k}$*'s* is negligible as $n \to \infty$.

We now proceed to the next section, where we give a description of $\mathbf{q}$-substitutions emphasizing their arithmetic properties and complimenting the abelianization quite well.

2.2 Instructions and Configurations

Let $\mathcal{S}$ be a $\mathbf{q}$-substitution on $\mathcal{A}$. For each $\mathbf{j} \in [\mathbf{0}, \mathbf{q})$, the map sending γ to $\mathcal{S}\gamma(\mathbf{j})$, the $\mathbf{j}$-th letter of the word $\mathcal{S}(\gamma)$, is a map $\mathcal{A} \to \mathcal{A}$ called the $\mathbf{j}$*-th instruction of* $\mathcal{S}$, and is denoted by $\mathcal{R}_{\mathbf{j}}$ for $\mathbf{j} \in [\mathbf{0}, \mathbf{q})$. The following proposition permits an alternate characterization of $\mathbf{q}$-substitutions and exposes their arithmetic properties, see also [22, §5.1].

Proposition 2.2.1 (Adic Description of $\mathbf{q}$-Substitution)**.** *Let $\mathcal{S}$ be a $\mathbf{q}$-substitution on $\mathcal{A}$. For $\mathbf{A} \in \mathcal{A}^{\mathbb{Z}^d}$, $n > 0$ and $\mathbf{j} = \mathbf{j}_0 + \mathbf{j}_1\mathbf{q} + \ldots + \mathbf{j}_{n-1}\mathbf{q}^{n-1} + \lfloor \mathbf{j} \rfloor_n \mathbf{q}^n \in \mathbb{Z}^d$*

$$(\mathcal{S}^n\mathbf{A})(\mathbf{j}) = \mathcal{R}_{\mathbf{j}_0}\mathcal{R}_{\mathbf{j}_1} \cdots \mathcal{R}_{\mathbf{j}_{n-1}}(\mathbf{A}(\lfloor \mathbf{j} \rfloor_n))$$

where $\mathcal{R}_{\mathbf{j}}$ are the instructions of $\mathcal{S}$, and $\mathbf{j}_i \in [\mathbf{0}, \mathbf{q})$ the digits of $\mathbf{j}$, as is our convention.

Proof. The proof follows by a simple inductive argument on the $n = 1$ case. Fix a $\mathbf{q}$-substitution $\mathcal{S}$ and $\mathbf{A} \in \mathcal{A}^{\mathbb{Z}^d}$. The sequence $\mathcal{S}\mathbf{A}$ is obtained by concatenating the blocks $\mathcal{S}(\mathbf{A}(\mathbf{a}))$ at the coordinates $\mathbf{aq}$ for $\mathbf{a} \in \mathbb{Z}^d$. As $\mathcal{R}_{\mathbf{b}}(\alpha) = \mathcal{S}(\alpha)_{\mathbf{b}}$ for $\mathbf{b} \in [\mathbf{0}, \mathbf{q})$, it follows that the letter in the $\mathbf{b} + \mathbf{aq}$-th position of $\mathcal{S}\mathbf{A}$ comes from the $\mathbf{b}$-th letter of $\mathcal{S}(\mathbf{A}(\mathbf{a}))$, so that $\mathcal{S}\mathbf{A}(\mathbf{b} + \mathbf{aq}) = \mathcal{R}_{\mathbf{b}}(\mathbf{A}(\mathbf{a}))$. This proves the $n = 1$ case, as $\mathbf{b} = \lceil \mathbf{b} + \mathbf{aq} \rceil_1$ and $\mathbf{a} = \lfloor \mathbf{b} + \mathbf{aq} \rfloor_1$. Writing $\mathcal{S}^n\mathbf{A} = \mathcal{S}(\mathcal{S}^{n-1}\mathbf{A})$ gives the inductive step necessary to prove the result. □

Denote the instructions of $\mathcal{S}^n$ by $\mathcal{R}^{(n)}$, which we call the *generalized instructions* of $\mathcal{S}$. Then the above proposition allows us to write for $\mathbf{j} \in \mathbb{Z}^d$

$$\mathcal{R}_{\mathbf{j}}^{(n)} = \mathcal{R}_{\mathbf{j}_0} \cdots \mathcal{R}_{\mathbf{j}_{n-1}} \qquad \text{equivalently} \qquad T^{\mathbf{j}}\mathcal{S}^n = T^{\mathbf{j}_0}\mathcal{S} \cdots T^{\mathbf{j}_{n-1}}\mathcal{S} \circ T^{\lfloor \mathbf{j} \rfloor_n} = T^{\lceil \mathbf{j} \rceil_n}\mathcal{S}^n T^{\lfloor \mathbf{j} \rfloor_n} \quad (2.4)$$

where we have extended the definition of $\mathcal{R}_{\mathbf{j}}^{(n)}$ to all $\mathbf{j} \in \mathbb{Z}^d$ by reducing $\mathbf{j}$ modulo $\mathbf{q}^n$. This has no effect on the substitution, however, as the instructions for $\mathcal{S}^n$ depend only on the first n digits of $\mathbf{j}$. As we are only concerned with $\mathbf{q}$-substitutions, we use proposition 2.2.1 over the definition in §1.2.

If the instructions are all bijections, we say $\mathcal{S}$ is a *bijective substitution;* if they all commute with each other, we say $\mathcal{S}$ is a *commutative substitution.* Our first example is classical and is an example of an aperiodic bijective and commutative $\mathbf{q}$-substitution; we show in theorem 3.4.1 that all such substitutions have totally singular (to Lebesgue) spectrum.

Example 2.2.2. The Thue-Morse substitution is a 2-substitution on the alphabet $\{0,1\}$

$$\tau : \begin{cases} 0 \longmapsto 01 \\ 1 \longmapsto 10 \end{cases} \qquad \text{with instructions} \qquad \mathcal{R}_0 = \begin{cases} 0 \longmapsto 0 \\ 1 \longmapsto 1 \end{cases} \qquad \text{and} \qquad \mathcal{R}_1 = \begin{cases} 0 \longmapsto 1 \\ 1 \longmapsto 0 \end{cases}$$

so that, as $12 = 0 \cdot 2^0 + 0 \cdot 2^1 + 1 \cdot 2^2 + 1 \cdot 2^3$, we have $\mathcal{R}_{12}^{(4)} = \mathcal{R}_0\mathcal{R}_0\mathcal{R}_1\mathcal{R}_1 = \mathcal{R}_0$ as $\mathcal{R}_1^2 = \mathcal{R}_0$ is the identity, and so $(\tau^4\alpha)_{12} = \alpha$ for $\alpha = 0, 1$. In this way, one checks that $\mathcal{R}_{\mathbf{j}}^{(n)}$ is the identity, or transposition, according to the parity of 1's in $\mathbf{j}$'s binary expansion.

A *configuration of instructions* in $\mathcal{A}^\mathcal{A}$ is a map $\mathcal{R} : C \to \mathcal{A}^\mathcal{A}$ from a finite subset $C \subset \mathbb{Z}^d$ assigning to every coordinate in C an instruction in $\mathcal{A}^\mathcal{A}$. A $\mathbf{q}$-configuration is a map $\mathcal{R} : [\mathbf{0}, \mathbf{q}) \to \mathcal{A}^\mathcal{A}$, and $\mathbf{q}$ is its *shape*. By proposition 2.2.1, every $\mathbf{q}$-configuration determines a $\mathbf{q}$-substitution, and conversely by writing $\mathcal{R}_{\mathbf{j}}(\gamma) := \mathcal{S}\gamma(\mathbf{j})$. We will always represent the instructions of a substitution with the symbol of its configuration: $\mathbf{j} \mapsto \mathcal{R}_{\mathbf{j}}$, the $\mathbf{j}$-th instruction. Using the generalized instructions (2.4), we can extend any $\mathbf{q}$-configuration $\mathcal{R}$ representing $\mathcal{S}$ to a $\mathbf{q}^n$-configuration $\mathcal{R}^{(n)}$ representing $\mathcal{S}^n$. Two substitutions $\mathcal{S}$ and $\tilde{\mathcal{S}}$, with configurations $\mathcal{R}$ and $\tilde{\mathcal{R}}$, are *configuration equivalent* if there is a bijection ι of their domains for which $\tilde{\mathcal{R}} \circ \iota = \mathcal{R}$. Properties of a substitution which do not depend on the configuration are called *configuration invariants*, and they depend only on the collection of instructions counted with multiplicity.

An interesting example of configuration equivalent substitutions can be obtained by *cycling* configurations: given a $\mathbf{q}$-configuration $\mathcal{R}$, each $\mathbf{k} \in \mathbb{Z}^d$ and $n \geqslant 0$ determines a $\mathbf{q}^n$-configuration $\mathbf{j} \mapsto \mathcal{R}_{\mathbf{j}+\mathbf{k}}^{(n)}$, inducing a $\mathbf{q}^n$-substitution.

Definition 2.2.3. Let $\mathcal{S}$ be a $\mathbf{q}$-substitution with configuration $\mathcal{R}$. For $\mathbf{k} \in \mathbb{Z}^d$ and $n > 0$ the *cycled substitution* $\mathcal{S}_{\mathbf{k}}^n$ is the $\mathbf{q}^n$-substitution with configuration $\mathbf{j} \mapsto \mathcal{R}_{\mathbf{j}+\mathbf{k}}^{(n)}$ on $[\mathbf{0}, \mathbf{q}^n)$.

Note that the substitutions $(\mathcal{S}_{\mathbf{k}}^n)^p \neq \mathcal{S}_{\mathbf{k}}^{np}$ and $\mathcal{S}_{\mathbf{k}}^n \neq \mathcal{S}_{\mathbf{k}_0}^1 \cdots \mathcal{S}_{\mathbf{k}_{n-1}}^1$ are generally distinct, and there is no significant *a priori* relationship between them. Moreover, cycled substitutions determine distinct subshifts in general. Consider the Thue-Morse substitution:

Example 2.2.4. As τ is a 2-substitution, there are only two cycled substitutions for $n = 1$:

$$\tau_0 = \tau : \begin{cases} 0 \mapsto 01 \\ 1 \mapsto 10 \end{cases} \qquad \text{and} \qquad \tau_1 : \begin{cases} 0 \mapsto 10 \\ 1 \mapsto 01 \end{cases}$$

as the instructions of τ_1 are $\mathcal{R}^{(1)}_{0+1} = \mathcal{R}_1$ and $\mathcal{R}^{(1)}_{1+1} = \mathcal{R}_0$ which is evidently configuration equivalent to Thue-Morse. One checks that both substitutions are bijective and commutative, and aperiodicity can be verified using lemma 2.3.1 below. For $n = 3$ and $\mathbf{k} = 1$, the cycled substitution τ_1^3 is obtained by cyclically permuting words of τ^3 forward one position

$$\tau^3 : \begin{cases} 0 \mapsto 01101001 \\ 1 \mapsto 10010110 \end{cases} \quad \text{so that} \quad \tau_1^3 : \begin{cases} 0 \mapsto 11010010 \\ 1 \mapsto 00101101 \end{cases}$$

and one can check that τ_1^3 is neither $(\tau_1)^3$, nor $\tau_1\tau_0\tau_0$ which is compared as $3 = 1{\cdot}1{+}0{\cdot}3{+}0{\cdot}9$. Finally, one checks that anytime one substitutes with τ_1^3 on the words 01 or 10 either of the patterns 000 or 111 will appear, which cannot occur in Thue-Morse as can be seen from the above description of τ^3, and the two cycled substitutions have distinct subshifts.

These substitutions represent cyclic permutations of the instructions of $\mathcal{S}^n$ for $n > 0$, and are related to the substitutions $T^{\mathbf{k}}\mathcal{S}^n$ taking α to the pattern $\mathcal{S}^n(\alpha)$ on $[\mathbf{k}, \mathbf{k} + \mathbf{q}^n)$. As $T^{\mathbf{k}}\mathcal{S}^n(\gamma)_{\mathbf{j}} = \mathcal{S}^n(\gamma)_{\mathbf{j}+\mathbf{k}} = \mathcal{R}^{(n)}_{\mathbf{j}+\mathbf{k}}(\gamma)$, the substitution $T^{\mathbf{k}}\mathcal{S}^n$ has the same instructions as the cycled substitution $\mathcal{S}^n_{\mathbf{k}}$, corresponding via the map $\iota : \mathbf{j} \mapsto [\mathbf{j} + \mathbf{k}]_n$ from $[\mathbf{k}, \mathbf{k}{+}\mathbf{q}^n) \to [\mathbf{0}, \mathbf{q}^n)$. These *shifted* substitutions $T^{\mathbf{k}}\mathcal{S}^n$ will arise when we discuss aperiodicity in §2.3 following theorem 2.3.2, where we describe how they relate to the topological and measure theoretic structure of the subshift.

For a matrix $\mathbf{R} \in M_{\mathcal{A}}(\mathbb{C})$, the space of $\mathcal{A} \times \mathcal{A}$-matrices with complex coefficients, observe that, if we write $\mathbf{R} = (R_{\alpha,\gamma})_{\alpha,\gamma\in\mathcal{A}}$ then $R_{\alpha,\gamma} = \mathbf{e}_\alpha^* \mathbf{R} \mathbf{e}_\gamma$ where $\mathbf{e}_\alpha$ is the α^{th}-coordinate vector. As we are representing $\mathcal{A}$ in $\mathbb{C}^{\mathcal{A}}$ by sending $\alpha \mapsto \mathbf{e}_\alpha$, this induces a representation of an instruction $\mathcal{R} : \mathcal{A} \to \mathcal{A}$ by its *instruction matrix* $\mathcal{R} \in \mathbf{M}_{\mathcal{A}}(\mathbb{C})$. Formally, we can view an instruction as a $\mathbf{1}$-substitution on $\mathcal{A}$ and compute its substitution matrix $\mathcal{R} \in \mathbf{M}_{\mathcal{A}}(\mathbb{C})$, this is equivalent. One can compute the instruction matrix from its instruction via the relation

$$\alpha = \mathcal{R}(\gamma) \iff \mathbf{e}_\alpha^* \mathcal{R} \mathbf{e}_\gamma = \mathcal{R}_{\alpha,\gamma} = 1 \qquad \text{equivalently} \qquad \alpha \neq \mathcal{R}(\gamma) \iff \mathbf{e}_\alpha^* \mathcal{R} \mathbf{e}_\gamma = \mathcal{R}_{\alpha,\gamma} = 0$$

as instruction matrices have coefficients 0 or 1. As they represent functions, the instruction matrices are naturally column-stochastic: their column sums are all 1. Note that the substitution matrix is the sum of its instruction matrices: $M_{\mathcal{S}} = \sum_{\mathbf{j}\in[\mathbf{0},\mathbf{q})} \mathcal{R}_{\mathbf{j}}$, so that substitution matrices of $\mathbf{q}$-substitutions are Q-column stochastic. Note that the substitution matrix is a configuration invariant, as matrix addition is commutative. As an example:

Example 2.2.5. For the Thue-Morse substitution, the instruction matrices are

$$\mathcal{R}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \mathcal{R}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{so that} \qquad M_\tau = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and note that the substitution matrix M_τ is 2-stochastic. As a (componentwise) positive matrix, it has a unique stochastic eigenvector $\mathbf{u} = (1/2, 1/2)$ its Perron vector, see [14]. The instruction matrices for τ_1 are the same as τ but indexed in the opposite order so $M_{\tau_1} = M_\tau$.

We will always conflate the notation for an instruction with its instruction matrix, extending this to the generalized instructions as well, and so $\mathcal{R}_{\mathbf{j}}^{(n)}$ can be interpreted belonging to $\mathcal{A}^{\mathcal{A}}$ or $\mathbf{M}_{\mathcal{A}}(\mathbb{C})$ interchangeably based on context. We now describe a *product* on $\mathbf{q}$-substitutions: given two alphabets $\mathcal{A}$ and $\tilde{\mathcal{A}}$, let $\mathcal{A}\tilde{\mathcal{A}}$ denote their *product alphabet*, or letter pairs $\alpha\tilde{\gamma}$ with $\alpha \in \mathcal{A}, \tilde{\gamma} \in \tilde{\mathcal{A}}$.

Definition 2.2.6. Let $\mathcal{S}, \tilde{\mathcal{S}}$ be $\mathbf{q}$-substitutions on the alphabets $\mathcal{A}, \tilde{\mathcal{A}}$ respectively. Their *substitution product* $\mathcal{S} \otimes \tilde{\mathcal{S}}$ is the $\mathbf{q}$-substitution on $\mathcal{A}\tilde{\mathcal{A}}$ with configuration $\mathcal{R} \otimes \tilde{\mathcal{R}}$

$$(\mathcal{R} \otimes \tilde{\mathcal{R}})_{\mathbf{j}} : \mathcal{A}\tilde{\mathcal{A}} \to \mathcal{A}\tilde{\mathcal{A}} \qquad \text{with} \qquad (\mathcal{R} \otimes \tilde{\mathcal{R}})_{\mathbf{j}} : \alpha\tilde{\gamma} \longmapsto \mathcal{R}_{\mathbf{j}}(\alpha)\tilde{\mathcal{R}}_{\mathbf{j}}(\tilde{\gamma}),$$

for $\mathbf{j} \in [\mathbf{0}, \mathbf{q})$ and where $\mathcal{R}, \tilde{\mathcal{R}}$ are the configurations of $\mathcal{S}, \tilde{\mathcal{S}}$.

If $\tilde{\mathcal{A}} = \mathcal{A}$ we write $\mathcal{A}\mathcal{A} = \mathcal{A}^2$ and if $\tilde{\mathcal{S}} = \mathcal{S}$ the substitution $\mathcal{S} \otimes \mathcal{S}$ is called the *bisubstitution* of $\mathcal{S}$ and its substitution matrix $C_{\mathcal{S}} := M_{\mathcal{S}\otimes\mathcal{S}}$ the *coincidence matrix* of $\mathcal{S}$, after Queffélec [22, §10]. The substitution product is always aperiodic stable as periodic sequences in the hull of a substitution product would necessarily be periodic in both factors. On the other hand, it is not always primitive stable: the bisubstitution of a primitive substitution is in general itself not primitive. Using the primitive reduced form of proposition 1.2.2, however, it can always be made primitive on its ergodic classes by telescoping appropriately.

We offer some justification for the notation: given two matrices $A, B \in M_{\mathcal{A}}(\mathbb{C})$, their *Kronecker product* is the matrix $A \otimes B \in M_{\mathcal{A}^2}(\mathbb{C})$ whose $(\alpha\beta, \gamma\delta) \in \mathcal{A}^2 \times \mathcal{A}^2$ entry is $A_{\alpha\gamma}B_{\beta\delta}$.

Moreover, if $\omega = \gamma_1\gamma_2\cdots\gamma_N$ is any *word* then we denote $\mathbf{e}_\omega := \mathbf{e}_{\gamma_1} \otimes \mathbf{e}_{\gamma_2} \otimes \cdots \otimes \mathbf{e}_{\gamma_N}$, and so for example $\mathbf{e}_{\alpha\beta} = \mathbf{e}_\alpha \otimes \mathbf{e}_\beta$ is the standard unit vector in $\mathbb{C}^{\mathcal{A}^2}$ corresponding to the word $\alpha\beta$. With this notation, observe that the coefficients of the Kronecker product become

$$(A \otimes B)_{\alpha\beta,\gamma\delta} = \mathbf{e}^*_{\alpha\beta} A \otimes B \mathbf{e}_{\gamma\delta} = (\mathbf{e}^*_\alpha A \mathbf{e}_\gamma)(\mathbf{e}^*_\beta B \mathbf{e}_\delta) = A_{\alpha\gamma} B_{\beta\delta} \tag{2.5}$$

where * is the conjugate transpose operator. This implies the *mixed product property*

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD) \tag{2.6}$$

Using this, one checks that the (generalized) instruction matrices of a substitution product are Kronecker products of the corresponding (generalized) instruction matrices of its factors. For every $n \geqslant 0$, and any finite sequence of m triples $(\mathbf{j}_i, \alpha_i, \gamma_i) \in \mathbb{Z}^d \times \mathcal{A} \times \mathcal{A}$, (2.6) gives

$$\alpha_i = \mathcal{R}^{(n)}_{\mathbf{j}_i}(\gamma_i) \quad \text{for} \quad 1 \leqslant i \leqslant m \qquad \Longleftrightarrow \qquad \mathbf{e}^*_{\alpha_1\cdots\alpha_m} \mathcal{R}^{(n)}_{\mathbf{j}_1} \otimes \cdots \otimes \mathcal{R}^{(n)}_{\mathbf{j}_m} \mathbf{e}_{\gamma_1\cdots\gamma_m} = 1 \tag{2.7}$$

so that for us the Kronecker product is just formalizing the conjunction *and* within a linear algebraic context, and will be used as above to represent simultaneous conditions on substitutions. For more discussion of the Kronecker product, see [15, §4]. We work out some of the above details in the case of the Thue-Morse example, for clarification.

Example 2.2.7. We give instruction matrices for substitution products of τ and τ_1, where empty entries are 0, and the basis for $\mathcal{A}^2$ is ordered $00, 01, 10, 11$.

- For $\tau \otimes \tau$ the instruction matrices are $\mathcal{R}_0 \otimes \mathcal{R}_0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ and $\mathcal{R}_1 \otimes \mathcal{R}_1 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$

- For $\tau_1 \otimes \tau$ the instruction matrices are $\mathcal{R}_1 \otimes \mathcal{R}_0 = \begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{pmatrix}$ and $\mathcal{R}_0 \otimes \mathcal{R}_1 = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}$

- For $\tau_1 \otimes \tau_1$ the instruction matrices are $\mathcal{R}_1 \otimes \mathcal{R}_1 = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$ and $\mathcal{R}_0 \otimes \mathcal{R}_0 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

and one can see from the above that $\mathcal{R} \otimes \tilde{\mathcal{R}}$ for instruction matrices can be computed by *placing a copy* of $\tilde{\mathcal{R}}$ at every 1 in $\mathcal{R}$, and $\mathbf{0} \in \mathbf{M}_{\mathcal{A}}(\mathbb{C})$ at every 0 in $\mathcal{R}$. Note that

$$M_{\tau\otimes\tau} = \begin{pmatrix} 1 & & & 1 \\ & 1 & 1 & \\ & 1 & 1 & \\ 1 & & & 1 \end{pmatrix} \qquad \text{can be made similar to} \qquad \begin{pmatrix} 1 & 1 & & \\ 1 & 1 & & \\ & & 1 & 1 \\ & & 1 & 1 \end{pmatrix}$$

by reordering the basis to $00, 11, 01, 10$ as could be seen by computing letter orbits using the instructions of $\tau \otimes \tau$ above. Thus, the primitive reduced form of $\tau \otimes \tau$ is two copies of τ, one each on the alphabets $\mathcal{E}_0 = \{00, 11\}$ and $\mathcal{E}_1 = \{01, 10\}$ which together form the ergodic classes of the bisubstitution; there is no transient part, as is always the case for bijective substitutions - their instructions are invertible, so every letter is in a closed orbit.

Substitutions of the form $\mathcal{S}^n \otimes \mathcal{S}^n_{\mathbf{k}}$ for $\mathbf{k} \in \mathbb{Z}^d$ are related to the correlation vector Σ in §3.1, as its Fourier coefficients are related to limits of their normalized substitution matrices $1/Q^n M_{\mathcal{S}^n \otimes \mathcal{S}^n_{\mathbf{k}}}$ and we refer to such substitutions as *cycled substitution products*. When $\mathbf{k} = \mathbf{0}$, the cycled substitution products $\mathcal{S}^n \otimes \mathcal{S}^n_{\mathbf{0}} = (\mathcal{S} \otimes \mathcal{S})^n$ are just iterates of the bisubstitution, and are used in §3.2 to characterize the spectral hull of the substitution. In §4.3 we discuss an example of a primitive substitution product of Thue-Morse and Rudin-Shapiro, illustrating an interesting relationship that sometimes holds between the spectrum of a substitution product and that of its factors. We now describe how aperiodicity of the subshift gives rise to topological properties allowing us to exploit the arithmetic properties of **q**-substitutions to study their invariant measures and, more importantly, their spectral theory.

2.3 Aperiodicity and the Subshift

Recall that a **q**-substitution is aperiodic if $X_{\mathcal{S}}$ contains no periodic points. We now describe a condition for the $d = 1$ case which is useful for checking aperiodicity in primitive q-substitutions, based on a result of Pansiot [20, Lemma 1]. The specific advantage in the $\mathbb{Z}$ setting is provided by the following equivalence: a primitive substitution ($d = 1$) is periodic if and only if its subshift is finite, and hence has a word of finite length which generates the entire language. Here, *word* means a block in 1 dimension and its *length* is the cardinality of its domain. By a $\mathcal{S}$-*neighborhood of* α, we mean a word $\gamma\alpha\delta$ which appears in the reduced language of $\mathcal{S}$, or the set of all subwords appearing in some substitution sequence $\mathbf{A} \in X_{\mathcal{S}}$.

Lemma 2.3.1 (Pansiot's Lemma)**.** *A primitive q-substitution (the $\mathbb{Z}$ case) which is one-to-one on $\mathcal{A}$ is aperiodic if and only if $\mathcal{A}$ has a letter with at least two distinct $\mathcal{S}$-neighborhoods.*

As $\mathcal{S} : \mathcal{A} \to \mathcal{A}^+$ is a map on a finite set, injectivity is trivial to verify. One then examines the iterates $\mathcal{S}^{nh}(\gamma)$ for neighbor pairs, where h is the index of imprimitivity; as soon as a letter with distinct neighborhoods appears, aperiodicity is verified for the primitive component of $\mathcal{S}$ operating on γ. At the moment, we have no clear criteria for extending Pansiot's result to the case $d > 1$, as it relies on the fact that, for $d = 1$, periodicity is equivalent to finiteness of the hull and in higher dimensions, where periodicity can occur along a strictly lower dimensional subspace, this is not the case.

Note how theorem 1.2.4 allows us to extend theorem 1.2.1: if $\mu \in \mathcal{M}(X_{\mathcal{S}}, T)$ is an invariant probability measure then $\mu \circ \mathcal{S}^n = Q^{-n}\mu$. Thus, as $\mathcal{B}_{\mathcal{S}}$ is generated by the sets $T^{\mathbf{k}}\mathcal{S}^n[\gamma]$, every invariant measure is determined by its values on these initial cylinders $[\alpha]$ which correspond to a convex combination of the Perron vectors of $M_{\mathcal{S}}$. This does not, however, tell us how to measure arbitrary cylinders $[\omega] \subset X_{\mathcal{S}}$, as although the sets $T^{\mathbf{k}}\mathcal{S}^n[\gamma]$ generate $\mathcal{B}_{\mathcal{S}}$ and therefore can represent $[\omega]$, we do not know *how* they are related. We need more information on how these sets fit together to form the subshift.

We need a property of substitutions allowing for unique local desubstitution - a notion made precise by the following result, proven by Mossé [18] in the one-dimensional case, and extended by Solomyak [28] to self-affine tilings in $\mathbb{R}^d$; we state it here for $\mathbf{q}$-substitutions.

Theorem 2.3.2 (Mossé (96), Solomyak (98))**.** *A $\mathbf{q}$-substitution $\mathcal{S}$ is aperiodic if and only if for every $\mathbf{A} \in X_{\mathcal{S}}$ there exists a unique $\mathbf{k} \in [\mathbf{0}, \mathbf{q})$ and $\mathbf{B} \in X_{\mathcal{S}}$ with $T^{\mathbf{k}}\mathcal{S}(\mathbf{B}) = \mathbf{A}$.*

The above property is known as *recognizability.* For every $n \geqslant 0$, this gives a unique $\mathbf{j} \in [\mathbf{0}, \mathbf{q}^n)$ and $\mathbf{A}^{(n)} \in X_{\mathcal{S}}$ for which $T^{\mathbf{j}}\mathcal{S}^n(\mathbf{A}^{(n)}) = \mathbf{A}$, so that we obtain

$$T^{\mathbf{j}}\mathcal{S}^n(\mathbf{A}) = T^{\mathbf{k}}\mathcal{S}^n(\mathbf{B}) \quad\iff\quad \lceil \mathbf{j} \rceil_n = \lceil \mathbf{k} \rceil_n \quad \text{and} \quad T^{\lfloor \mathbf{j} \rfloor_n}\mathbf{A} = T^{\lfloor \mathbf{k} \rfloor_n}\mathbf{B} \tag{2.8}$$

by the factorizations in (2.4) and so theorem 2.3.2 states that the subshift can be expressed

$$X_{\mathcal{S}} = \bigsqcup_{\mathbf{j} \in [\mathbf{0}, \mathbf{q}^n)} T^{\mathbf{j}}\mathcal{S}^n(X_{\mathcal{S}}) = \bigsqcup_{\mathbf{j} \in [\mathbf{0}, \mathbf{q}^n)} T^{\mathbf{j}_0}\mathcal{S} \circ T^{\mathbf{j}_1}\mathcal{S} \circ \cdots \circ T^{\mathbf{j}_{n-1}}\mathcal{S}(X_{\mathcal{S}}) \tag{2.9}$$

where $\sqcup$ denotes a disjoint union. Consider the collection of substitutions $\mathcal{S}_* := \{T^{\mathbf{k}}\mathcal{S} \,:\, \mathbf{k} \in [\mathbf{0},\mathbf{q})\}$, each mapping $\mathcal{A} \to \mathcal{A}^{\mathbf{k}+[\mathbf{0},\mathbf{q}^n)}$ for some $\mathbf{k} \in [\mathbf{0},\mathbf{q}^n)$, and are $\mathbf{q}$-substitutions in every sense *except* that their substituted blocks lie at $\mathbf{k}$ as opposed to the origin. As $\mathcal{S}$ is $\mathbf{q}$-expansive, these substitutions are all nonstrict contractions (consider distinct fixed points) on the full shift and so the above presents $X_{\mathcal{S}}$ as an attractor for the iterated function system of substitutions $\mathcal{S}_*$. We now consider the orbit of the cylinder sets $[\omega]$ for $\omega \in \mathcal{A}^+$ under this iterated function system.

Definition 2.3.3. For a $\mathbf{q}$-substitution $\mathcal{S}$ on $\mathcal{A}$ and every $n \geqslant 0$ we have the collections

- $\mathcal{P}^n := \{T^{\mathbf{j}}\mathcal{S}^n[\gamma] : \mathbf{j} \in [\mathbf{0},\mathbf{q}^n) \text{ and } \gamma \in \mathcal{A}\}$ of *n-th iterated initial cylinders* of $\mathcal{S}$, and
- $\mathcal{P}_{\mathcal{S}} := \{T^{\mathbf{j}}\mathcal{S}^n[\eta] : n \geqslant 0, \mathbf{j} \in [\mathbf{0},\mathbf{q}^n), \text{ and } \eta \in \mathcal{A}^+\}$ of *iterated cylinders* of $\mathcal{S}$.

As we show in proposition 2.3.4.1, the collections $\mathcal{P}^n$ form a Kakutani-Rokhlin partition on $X_{\mathcal{S}}$, and 2.3.4.3 shows how they are nested by degree. First, write for $n \geqslant 0$ and $\alpha \in \mathcal{A}$

$$\mathcal{S}^{-n}(\alpha) = \{(\mathbf{j},\gamma) \in [\mathbf{0},\mathbf{q}^n) \times \mathcal{A} \,:\, \alpha = \mathcal{R}^{(n)}_{\mathbf{j}}(\gamma) = T^{\mathbf{j}}\mathcal{S}^n(\gamma)_{\mathbf{0}}\}$$

As $\eta \leqslant \eta' \leqslant \mathbf{A}$ for $\eta,\eta' \in \mathcal{A}^+$ and $\mathbf{A} \in \mathcal{A}^{\mathbb{Z}^d}$ implies $\mathbf{A} \in [\eta'] \subset [\eta]$, we have for $\mathbf{j} \in [\mathbf{0},\mathbf{q}^n)$

$$(\mathbf{j},\gamma) \in \mathcal{S}^{-n}(\alpha) \iff \alpha = \mathcal{R}^{(n)}_{\mathbf{j}}(\gamma) \iff \alpha \leqslant T^{\mathbf{j}}\mathcal{S}^n(\gamma) \iff T^{\mathbf{j}}\mathcal{S}^n[\gamma] \subset [\alpha]$$

so that $\mathcal{S}^{-n}(\alpha)$ corresponds to superblocks of size n which are α in the $\mathbf{j}$-th position. By the following proposition, these sets allow us to describe how sets in $\mathcal{P}^n$ are distributed inside $\mathcal{P}^m$ for $n \geqslant m$, describing how the iterated function system interacts with cylinders of $\mathcal{B}_{\mathcal{S}}$.

Proposition 2.3.4. *Let $\mathcal{S}$ be an aperiodic $\mathbf{q}$-substitution on $\mathcal{A}$. For every $n \geqslant 0$*

1. *the n-th iterated initial cylinders $\mathcal{P}^n$ partition $X_{\mathcal{S}}$,*
2. *for every $\mathbf{j} \in [\mathbf{0},\mathbf{q}^n)$, the map $T^{\mathbf{j}}\mathcal{S}^n : X_{\mathcal{S}} \longrightarrow X_{\mathcal{S}}$ is an embedding,*
3. *for every $\alpha \in \mathcal{A}$ we have $[\alpha] = \bigsqcup_{(\mathbf{j},\gamma)\in\mathcal{S}^{-n}(\alpha)} T^{\mathbf{j}}\mathcal{S}^n[\gamma]$.*

Proof. That $\mathcal{P}^n$ covers follows from the existence statement in theorem 2.3.2, and disjointness is immediate from (2.8) as here we are limited to $\mathbf{j} \in [\mathbf{0}, \mathbf{q}^n)$ giving the first statement.

As $\mathcal{S} : X_{\mathcal{S}} \to \mathcal{S}(X_{\mathcal{S}})$ is a continuous surjection from a compact space to a Hausdorff space, the second claim will follow by showing $T^{\mathbf{j}}\mathcal{S}^n$ is injective. This is immediate as each $\mathbf{B} \in X_{\mathcal{S}}$ has at most one $\mathbf{A} \in X_{\mathcal{S}}$ with $\mathcal{S}(\mathbf{A}) = \mathbf{B}$, by theorem 2.3.2. Thus $\mathcal{S} : X_{\mathcal{S}} \xrightarrow{\sim} \mathcal{S}(X_{\mathcal{S}})$ is a homeomorphism, and so $T^{\mathbf{j}}\mathcal{S}^n$ is an embedding on $X_{\mathcal{S}}$ as T is a homeomorphism of $X_{\mathcal{S}}$.

Finally, for $\alpha \in \mathcal{A}^+$ and $n \geqslant 0$, as $\mathcal{P}^n$ partitions $X_{\mathcal{S}}$ we have

$$[\alpha] = \bigsqcup_{\mathbf{j} \in [\mathbf{0}, \mathbf{q}^n)} \bigsqcup_{\gamma \in \mathcal{A}} [\alpha] \cap T^{\mathbf{j}}\mathcal{S}^n[\gamma]$$

and the result follows by dropping pairs with $\alpha \neq \mathcal{R}^{(n)}_{\mathbf{j}}(\gamma)$ and the definition of $\mathcal{S}^{-n}(\alpha)$. □

We now discuss the difference between $\cup_n \mathcal{P}^n$ and $\mathcal{P}_{\mathcal{S}}$. As $\eta \leqslant \omega \iff [\eta] \supseteq [\omega]$ and the Borel sets $\mathcal{B}_{\mathcal{S}}$ are generated by the cylinders $[\omega] \subset X_{\mathcal{S}}$ for $\omega \in \mathcal{L}_{\mathcal{S}}$, the language determines the Borel structure of the subshift. As a block $\omega \in \mathcal{L}_{\mathcal{S}}$ if and only if there exists *some* $n \in \mathbb{N}$, $\mathbf{j} \in \mathbb{Z}^d$, and $\gamma \in \mathcal{A}$ for which $\omega \leqslant T^{\mathbf{j}}\mathcal{S}^n(\gamma)$, and so the collection of $\cup_n \mathcal{P}^n$ for $n \in \mathbb{N}$ determines the topological structure of $X_{\mathcal{S}}$. This is not sufficient for our purposes, however, as we require more control over the parameters $n, \mathbf{k}$, and γ in order to produce recursive identities allowing us to compute the Fourier coefficients of our spectral measures explicitly. Fortunately, as proposition 2.3.5 below shows, the family $\mathcal{P}_{\mathcal{S}}$ can be used to express any cylinder of $X_{\mathcal{S}}$ in terms of iterated cylinders in $\mathcal{P}_{\mathcal{S}}$ while maintaining control over n.

We now extend proposition 2.3.4.3 to $\mathcal{P}_{\mathcal{S}}$ by extending $\mathcal{S}^{-n}$ to $\omega \in \mathcal{A}^+$. If we have $\omega \leqslant T^{\mathbf{j}}\mathcal{S}^n(\eta)$ for some $\mathbf{j} \in [\mathbf{0}, \mathbf{q}^n)$ and $\eta \in \mathcal{A}^+$, then for $\mathbf{k} \in \operatorname{supp}(\omega)$ we must also have

$$\omega_{\mathbf{k}} = T^{\mathbf{j}}\mathcal{S}^n(\eta)_{\mathbf{k}} = \mathcal{S}^n(\eta)_{\mathbf{j}+\mathbf{k}} = \mathcal{R}^{(n)}_{\mathbf{j}+\mathbf{k}}\big(\eta_{\lfloor \mathbf{j}+\mathbf{k} \rfloor_n}\big)$$

by proposition 2.2.1 so that $\omega \leqslant T^{\mathbf{j}}\mathcal{S}^n(\eta)$ implies $\lfloor \mathbf{j} + \operatorname{supp}(\omega) \rfloor_n \subset \operatorname{supp}(\eta)$. For each $\omega \in \mathcal{A}^+$ and $n \geqslant 0$, consider the collection of pairs in $[\mathbf{0}, \mathbf{q}^n) \times \mathcal{A}^+$ given by

$$\mathcal{S}^{-n}(\omega) := \Big\{ (\mathbf{j}, \eta) \in [\mathbf{0}, \mathbf{q}^n) \times \mathcal{A}^+ \,:\, \operatorname{supp}(\eta) = \lfloor \mathbf{j} + \operatorname{supp}(\omega) \rfloor_n \text{ and } \omega \leqslant T^{\mathbf{j}}\mathcal{S}^n(\eta) \Big\} \quad (2.10)$$

If we let $\alpha \in \mathcal{A}$ denote the block $\mathbf{0} \to \alpha$, one checks that $\mathcal{S}^{-n}(\alpha)$ remains well-defined as $\lfloor \mathbf{j} + \operatorname{supp}(\alpha) \rfloor_n = \lfloor \mathbf{j} + \mathbf{0} \rfloor_n = \mathbf{0}$ as $\mathbf{j} < \mathbf{q}^n$ and $\operatorname{supp}(\alpha)$ is $\mathbf{0}$, so the support of $\eta \in \mathcal{S}^{-n}(\alpha)$ is $\mathbf{0}$ also. Moreover, if $\eta \leqslant \eta'$ and $\mathbf{j}, \eta \in \mathcal{S}^{-n}(\omega)$, then $\omega \leqslant T^{\mathbf{j}}\mathcal{S}^n(\eta) \leqslant T^{\mathbf{j}}\mathcal{S}^n(\eta')$, and so the condition on the support merely enforces minimality. As $\eta \leqslant \eta'$ implies $[\eta'] \subset [\eta]$, the extension condition gives $T^{\mathbf{j}}\mathcal{S}^n[\eta'] \subset T^{\mathbf{j}}\mathcal{S}^n[\eta] \subset [\omega]$ for every $\mathbf{j}, \eta \in \mathcal{S}^{-n}(\omega)$, and so $\mathcal{S}^{-n}(\omega)$ describes the *largest* sets in $\mathcal{P}_{\mathcal{S}}$ of degree n contained in $[\omega]$, partitioning it as well:

Proposition 2.3.5. *Let $\mathcal{S}$ be an aperiodic $\mathbf{q}$-substitution on $\mathcal{A}$. For $\omega \in \mathcal{A}^+$ and $n \geqslant 0$*

$$[\omega] = \bigsqcup\nolimits_{(\mathbf{j},\eta) \in \mathcal{S}^{-n}(\omega)} T^{\mathbf{j}} \mathcal{S}^n [\eta]$$

Proof. Fix $\omega \in \mathcal{A}^+$, and $n \geqslant 0$. Writing $\Omega = \operatorname{supp}(\omega)$, let

$$\mathcal{D} := \bigtimes\nolimits_{\mathbf{k} \in \Omega} \mathcal{S}^{-n}(\omega_{\mathbf{k}}) = \left\{ \left((\mathbf{j_k}, \gamma_{\mathbf{k}}) \right)_{\mathbf{k} \in \Omega} \;:\; \omega_{\mathbf{k}} = \mathcal{R}^{(n)}_{\mathbf{j_k}}(\gamma_{\mathbf{k}}) \text{ for all } \mathbf{k} \in \Omega \right\}$$

represent all ways symbols in ω appear in superblocks of size $\mathbf{q}^n$. By proposition 2.3.4.3

$$\begin{aligned}
[\omega] &= \bigcap\nolimits_{\mathbf{k} \in \Omega} T^{-\mathbf{k}}[\omega_{\mathbf{k}}] = \bigcap\nolimits_{\mathbf{k} \in \Omega} \bigsqcup\nolimits_{(\mathbf{j_k}, \gamma_{\mathbf{k}}) \in \mathcal{S}^{-n}(\omega_{\mathbf{k}})} T^{\mathbf{j_k} - \mathbf{k}} \mathcal{S}^n([\omega_{\mathbf{k}}]) \\
&= \bigsqcup\nolimits_{((\mathbf{j_k}, \gamma_{\mathbf{k}}))_\Omega \in \mathcal{D}} \left(\bigcap\nolimits_{\mathbf{k} \in \Omega} T^{\lceil \mathbf{j_k} - \mathbf{k} \rceil_n} \mathcal{S}^n (T^{\lfloor \mathbf{j_k} - \mathbf{k} \rfloor_n} [\omega_{\mathbf{k}}]) \right)
\end{aligned}$$

Now, for each fixed $\left((\mathbf{j_k}, \gamma_{\mathbf{k}}) \right)_{\mathbf{k} \in \Omega} \in \mathcal{D}$ the aperiodicity condition (2.8) implies the above intersection is empty unless $\lceil \mathbf{j_k} - \mathbf{k} \rceil_n = \mathbf{j}$ for some $\mathbf{j} \in [\mathbf{0}, \mathbf{q}^n)$ or, equivalently, $\mathbf{j_k} = \lceil \mathbf{j} + \mathbf{k} \rceil_n$ by modular arithmetic. For such sequences $\{\mathbf{j_k}\}_{\mathbf{k} \in \Omega}$, we can use arithmetic (2.2) to write

$$\lfloor \mathbf{j_k} - \mathbf{k} \rfloor_n = \lfloor \lceil \mathbf{j} + \mathbf{k} \rceil_n - \mathbf{k} \rfloor_n = \lfloor \mathbf{j} + \mathbf{k} - \lfloor \mathbf{j} + \mathbf{k} \rfloor_n \mathbf{q}^n - \mathbf{k} \rfloor_n = \lfloor \mathbf{j} - \lfloor \mathbf{j} + \mathbf{k} \rfloor_n \mathbf{q}^n \rfloor_n = - \lfloor \mathbf{j} + \mathbf{k} \rfloor_n$$

as $\mathfrak{c}_n(\mathbf{j}, \mathbf{a}\mathbf{q}^n) = \mathbf{0}$ for $\mathbf{j} \in [\mathbf{0}, \mathbf{q}^n)$. As $T^{\mathbf{j}}\mathcal{S}^n$ preserves intersections (2.3.4.2) this gives:

Lemma 2.3.6. *In the context of proposition 2.3.5: for $\mathbf{j_k} \in [\mathbf{0}, \mathbf{q}^n)$, $\gamma_{\mathbf{k}} \in \mathcal{A}$, and $\mathbf{k} \in \Omega$*

$$\bigcap\nolimits_{\mathbf{k} \in \Omega} T^{\lceil \mathbf{j_k} - \mathbf{k} \rceil_n} \mathcal{S}^n (T^{\lfloor \mathbf{j_k} - \mathbf{k} \rfloor_n} [\omega_{\mathbf{k}}]) = T^{\mathbf{j}} \mathcal{S}^n \left(\bigcap\nolimits_{\mathbf{k} \in \Omega} T^{-\lfloor \mathbf{j} + \mathbf{k} \rfloor_n} [\omega_{\mathbf{k}}] \right)$$

if there exists a $\mathbf{j} \in [\mathbf{0}, \mathbf{q}^n)$ with $\mathbf{j_k} = \lceil \mathbf{j} + \mathbf{k} \rceil_n$ and $\omega_{\mathbf{k}} = \mathcal{R}^{(n)}_{\mathbf{j}+\mathbf{k}}(\gamma_{\mathbf{k}})$ for all $\mathbf{k} \in \Omega$. In all other cases, the above intersection is $\varnothing$.

Note that when viewed as cylinders in $\mathcal{A}^{\mathbb{Z}^d}$ (as the intersection in $X_{\mathcal{S}}$ may vanish for $\omega \notin \mathcal{L}_{\mathcal{S}}$)

$$\bigcap_{\mathbf{k}\in\Omega} T^{-\lfloor \mathbf{j}+\mathbf{k}\rfloor_n}[\omega_{\mathbf{k}}] \neq \varnothing \qquad \Longleftrightarrow \qquad \eta : \lfloor \mathbf{j}+\mathbf{k}\rfloor_n \longmapsto \gamma_{\mathbf{k}} \text{ defines block in } \mathcal{A}^+$$

For such pairs $\mathbf{j}, \eta$, we have $[\eta] = \bigcap_{\mathbf{k}\in\Omega} T^{-\lfloor \mathbf{j}+\mathbf{k}\rfloor_n}[\gamma_{\mathbf{k}}]$ and so as

$$T^{\mathbf{j}}\mathcal{S}^n(\eta)_{\mathbf{k}} = \mathcal{S}^n(\eta)_{\mathbf{j}+\mathbf{k}} = \mathcal{R}^{(n)}_{\mathbf{j}+\mathbf{k}}(\eta_{\lfloor \mathbf{j}+\mathbf{k}\rfloor_n}) = \mathcal{R}^{(n)}_{\mathbf{j}_{\mathbf{k}}}(\gamma_{\mathbf{k}}) = \omega_{\mathbf{k}}$$

we have $(\mathbf{j}, \eta) \in \mathcal{S}^{-n}(\omega)$. Conversely, if $(\mathbf{j}, \eta) \in \mathcal{S}^{-n}(\omega)$, one checks that writing $\mathbf{j}_{\mathbf{k}} := \lceil \mathbf{j}+\mathbf{k}\rceil_n$ and $\gamma_{\mathbf{k}} := \eta(\lfloor \mathbf{j}+\mathbf{k}\rfloor_n)$ for $\mathbf{k} \in \Omega$ defines an element of $\mathcal{D}$ giving rise to a nonempty intersection, and so

$$[\omega] = \bigsqcup_{\left((\mathbf{j}_{\mathbf{k}},\gamma_{\mathbf{k}})\right)_{\mathbf{k}\in\Omega}\in\mathcal{D}} \bigcap_{\mathbf{k}\in\Omega} T^{\lceil \mathbf{j}_{\mathbf{k}}-\mathbf{k}\rceil_n}\mathcal{S}^n(T^{\lfloor \mathbf{j}_{\mathbf{k}}-\mathbf{k}\rfloor_n}[\omega_{\mathbf{k}}]) = \bigsqcup_{(\mathbf{j},\eta)\in\mathcal{S}^{-n}(\omega)} T^{\mathbf{j}}\mathcal{S}^n[\eta]$$

as the two collections are in bijection, completing the proof. □

As we are interested in shift-invariant properties of $X_{\mathcal{S}}$, we can restrict consideration momentarily to cylinders on blocks supported in $\mathbb{N}^d$ without loss of generality. Fix $\omega \in \mathcal{A}^+$ supported on $\Omega \subset \mathbb{N}^d$ and let $\mathfrak{p}(\Omega) := \max_{\mathbf{k}\in\Omega} \mathfrak{p}(\mathbf{k})$. For $n \geqslant 0$, if $(\mathbf{j}, \eta) \in \mathcal{S}^{-n}(\omega)$, then $\operatorname{supp}\eta = \lfloor \mathbf{j} + \operatorname{supp}(\omega)\rfloor_n$ and the $\mathbf{q}$-adic identitites (2.2) shows that, for $p \geqslant \mathfrak{p}(\Omega)$ and $\mathbf{k} \in \Omega$

$$\lfloor \mathbf{j}+\mathbf{k}\rfloor_n = \lfloor \mathbf{j}\rfloor_n + \lfloor \mathbf{k}\rfloor_n + \mathfrak{c}_n(\mathbf{j},\mathbf{k}) = \lfloor \mathbf{k}\rfloor_n + \mathfrak{c}_n(\mathbf{j},\mathbf{k}) \quad \text{and} \quad \lfloor \mathbf{j}+\Omega\rfloor_p = \mathfrak{c}_p(\mathbf{j},\Omega) \subset [\mathbf{0},\mathbf{1}] \tag{2.11}$$

for $\mathbf{j} \in [\mathbf{0}, \mathbf{q}^n)$ and $[\mathbf{0}, \mathbf{q}^p)$ respectively, and so the carry functions inform us about the support of $\eta \in \mathcal{S}^{-n}(\omega)$. Moreover, if $\mathfrak{c}_p(\mathbf{j}, \mathbf{k}) = \mathfrak{c}_p(\mathbf{j}, \mathbf{k}')$ for $\mathbf{k}, \mathbf{k}' \in \Omega$ then $\omega_{\mathbf{k}}$ and $\omega_{\mathbf{k}'}$ are covered by the same superblock in $T^{\mathbf{j}}\mathcal{S}^p(\eta)$, so they tell us how the recursion formula of proposition 2.3.5 reduces the support of blocks in $\mathcal{S}^{-n}(\omega)$ as n increases. As (2.11) shows by taking n large enough, the partition formula allows one to express any cylinder in the language as a disjoint union of iterated cylinders corresponding to blocks defined on $[\mathbf{0}, \mathbf{1}]$, the cardinality of which depends only on d and $\mathcal{A}$.

An important corollary of the above is the *measure formula* of proposition 2.3.7 below, which shows how aperiodicity and the $\mathbf{q}$-adic structure of $\mathcal{S}$ combine to produce recursions for the measures of cylinders in $X_{\mathcal{S}}$, allowing us to compute the measure of an arbitrary cylinder in $X_{\mathcal{S}}$. Its main application, for us, is theorem 3.1.2, where is it used to compute the Fourier coefficients of spectral measures for translation action on $L^2(\mu)$, for $\mu \in \mathcal{M}(X_{\mathcal{S}}, T)$ and so the following is also stated in terms appropriate for that result.

Proposition 2.3.7. *Let $\mathcal{S}$ be an aperiodic $\mathbf{q}$-substitution on $\mathcal{A}$ and μ a T-invariant measure supported by $X_{\mathcal{S}}$. For every $n \in \mathbb{N}$, $\mathbf{k} \in \mathbb{Z}^d$, and $\alpha, \beta \in \mathcal{A}$*

$$\mu([\alpha] \cap T^{-\mathbf{k}}[\beta]) = \frac{1}{Q^n}\, \mathbf{e}^*_{\alpha\beta} \textstyle\sum_{\mathbf{j}\in[\mathbf{0},\mathbf{q}^n)} \Big(\mathcal{R}^{(n)}_{\mathbf{j}} \otimes \mathcal{R}^{(n)}_{\mathbf{j}+\mathbf{k}} \sum_{\gamma,\delta\in\mathcal{A}} \mu\big([\gamma] \cap T^{-\lfloor \mathbf{k} \rfloor_n - \mathfrak{c}_n(\mathbf{j},\mathbf{k})}[\delta]\big)\, \mathbf{e}_{\gamma\delta}\Big)$$

More generally, for any sequence $\mathbf{k}_1, \ldots, \mathbf{k}_m \in \mathbb{Z}^d$ and word $\alpha_1 \cdots \alpha_m \in \mathcal{A}^m$

$$\mu\Big(\bigcap_{i=1}^{m} T^{-\mathbf{k}_i}[\alpha_i]\Big) = \frac{1}{Q^n} \sum_{\mathbf{j}\in[\mathbf{0},\mathbf{q}^n)} \sum_{\gamma_1\cdots\gamma_m\in\mathcal{A}^m} \Big(\mathbf{e}^*_{\alpha_1\cdots\alpha_m} \bigotimes_{i=1}^{m} \mathcal{R}^{(n)}_{\mathbf{j}+\mathbf{k}_i} \mathbf{e}_{\gamma_1\cdots\gamma_m}\Big)\, \mu\Big(\bigcap_{i=1}^{m} T^{-\lfloor \mathbf{k}_i \rfloor_n - \mathfrak{c}_n(\mathbf{j},\mathbf{k}_i)}[\gamma_i]\Big)$$

Proof. The first identity is the $m = 2$ case of the second: $\mathbf{k}_1 = \mathbf{0}$ and $\mathbf{k}_2 = \mathbf{k}$, $\alpha_1 = \alpha$ and $\alpha_2 = \beta$, and the terms and sums have been rearranged according to index dependencies. Thus, we proceed to the proof of the more general identity, which begins by identifying the intersection $\cap T^{-\mathbf{k}_i}[\alpha_i]$ as a cylinder over a block.

First we assume that $\omega : \mathbf{k}_i \mapsto \alpha_i$ is well defined, so that $\mu([\omega]) = \mu(\bigcap_{i=1}^m T^{-\mathbf{k}_i}[\alpha_i])$ and the proof is immediate from lemma 2.3.6 and proposition 2.3.5 as the relation (2.7) shows that the tensor term $\mathbf{e}^*_{\alpha_1,\ldots,\alpha_m} \bigotimes_{i=1}^m \mathcal{R}^{(n)}_{\mathbf{j}+\mathbf{k}_i} \mathbf{e}_{\gamma_1\cdots\gamma_m}$ selects for $\times_{i=1}^m \mathcal{S}^{-n}(\alpha_i)$, as $\mu(\mathcal{S}^n[\eta]) = {}^1\!/_{Q^n}\mu([\eta])$ for $\eta \in \mathcal{A}^+$, and as $\lfloor \mathbf{j} + \mathbf{k} \rfloor_n = \lfloor \mathbf{k} \rfloor + \mathfrak{c}_n(\mathbf{j}, \mathbf{k})$ for $\mathbf{j} \in [\mathbf{0}, \mathbf{q}^n)$.

Now, if $\omega : \mathbf{k}_i \mapsto \alpha_i$ is *not* well defined, then there are $1 \leqslant a, b \leqslant m$ such that $\mathbf{k}_a = \mathbf{k}_b$ but $\alpha_b \neq \alpha_b$, and the left hand side is 0 as the intersection is empty. Note that if $\lfloor \mathbf{j} + \mathbf{k}_a \rfloor_n = \lfloor \mathbf{j} + \mathbf{k}_b \rfloor_n$, then we cannot have *both* $\gamma_a = \gamma_b$ and $\alpha_i = \mathcal{R}^{(n)}_{\mathbf{j}+\mathbf{k}_i}(\gamma_i)$ for $i = a, b$ as this implies

$$\alpha_a = \mathcal{R}^{(n)}_{\mathbf{j}+\mathbf{k}_a}(\gamma_a) = \mathcal{R}^{(n)}_{\mathbf{j}+\mathbf{k}_b}(\gamma_b) = \alpha_b$$

contradicting the assumption. Thus, either the Kronecker term vanishes, or the intersection on the left is empty, and so both sides agree, completing the proof. □

Before we proceed to the spectral theory of aperiodic $\mathbf{q}$-substitutions, we give a third application of aperiodicity in $L^2(\mu)$ allowing us to use our recursions to study the spectral map. The *iterated initial indicators* for a $\mathbf{q}$-substitution $\mathcal{S}$ on $\mathcal{A}$ are the indicator functions $\mathbb{1}_{T^{\mathbf{k}}\mathcal{S}^n[\gamma]} \in L^2(X_{\mathcal{S}})$ of the iterated cylinders $T^{\mathbf{k}}\mathcal{S}^n[\gamma]$ for $n \geqslant 0$, $\mathbf{k} \in \mathbb{Z}^d$, and $\gamma \in \mathcal{A}$.

Proposition 2.3.8. *If $\mathcal{S}$ is an aperiodic $\mathbf{q}$-substitution on $\mathcal{A}$, the iterated initial indicators have dense span in $L^2(\mu)$, for all $\mu \in \mathcal{M}(X_{\mathcal{S}}, T)$.*

Proof. As the functions $\mathbb{1}_{[\omega]}$ for $\omega \in \mathcal{A}^+$ have dense span in $L^2(\mu)$, we can accomplish this by showing that every $\mathbb{1}_{[\omega]}$ can be approximated by the linear combinations of iterated initial indicators. Note that the partition formula 2.3.5 can be interpreted for indicator functions:

$$\mathbb{1}_{[\omega]} = \sum\nolimits_{(\mathbf{j},\eta)\in\mathcal{S}^{-p}(\omega)} \mathbb{1}_{T^{\mathbf{j}}\mathcal{S}^p([\eta])}$$

By translating ω, we can assume that $\Omega \subset \mathbb{N}^d$ and translate back later without loss of generality, and do so. For such ω, write $\Omega := \operatorname{supp}(\omega) \subset \mathbb{N}^d$ and note that for $p \geqslant \mathfrak{p}(\Omega)$ and $\mathbf{j} \in [\mathbf{0}, \mathbf{q}^p)$, the quotient $\lfloor \mathbf{j} + \Omega \rfloor_p = \mathfrak{c}_p(\mathbf{j}, \Omega) \subset [\mathbf{0}, \mathbf{1}]$. Thus, we can separate the above sum into two parts: those for which $\mathbf{j} \notin \Delta_p(\Omega)$ and those for which $\mathbf{j} \in \Delta_p(\Omega)$, corresponding to those blocks which are supported at $\{\mathbf{0}\}$ vs those blocks with larger support and write

$$\mathbb{1}_{[\omega]} - \sum\nolimits_{(\mathbf{j},\eta)\in\mathcal{S}^{-p}(\omega) \text{ and } \mathbf{j}\notin\Delta_p(\Omega)} \mathbb{1}_{T^{\mathbf{j}}\mathcal{S}^p([\eta])} = \sum\nolimits_{(\mathbf{j},\eta)\in\mathcal{S}^{-p}(\omega) \text{ and } \mathbf{j}\in\Delta_p(\Omega)} \mathbb{1}_{T^{\mathbf{j}}\mathcal{S}^p([\eta])}$$

so that taking absolute values and integrating against μ, we can apply lemma 2.1.1 and so the right hand side goes to 0. This in turn shows that $\mathbb{1}_{[\omega]}$ can be approximated by linear combinations of indicator functions for $T^{\mathbf{j}}\mathcal{S}^n[\gamma]$, as those terms over $\mathbf{j} \notin \Delta_p(\Omega)$ are supported entirely at the origin. Thus, linear combinations of the iterated initial indicators approximate the indicators of the standard cylinders, and thus have dense span in $L^2(\mu)$. □

As we are ultimately interested in the maximal spectral type of the translation action on $L^2(\mu)$ and the spectral map $\sigma : L^2(\mu) \times L^2(\mu) \to \mathcal{M}(\mathbb{T}^d)$ is sesquilinear and continuous, the above proposition allows us to restrict our attention to the iterated initial indicators. In the next section, we further reduce this list to the indicator functions $\mathbb{1}_{[\alpha]}$ for $\alpha \in \mathcal{A}$.

Chapter 3

SPECTRAL THEORY

The goal of this section is to state the central result of this paper, theorem 3.3.1, which allows us to identify the spectrum $\sigma_{\max}$ of an aperiodic $\mathbf{q}$-substitution $\mathcal{S}$ with finitely many measures strongly mixing for the $\mathbf{q}$-*shift*. This is significant as these mutually singular measures of pure type can be parametrized by a *correlation vector* Σ consisting of complex Borel measures on $\mathbb{T}^d$ and a *spectral hull* $\mathcal{K}$, which is a convex cone in $\mathbb{C}^{\mathcal{A}^2}$ depending only on the abelianization of $\mathcal{S}$.

The $\mathbf{q}$-*shift* is the topological dynamical system on $\mathbb{T}^d$ given by the map $\mathbf{S_q} : (z_1, \ldots, z_d) \mapsto (z_1^{q_1}, \ldots, z_d^{q_d})$ which is topologically conjugate to the times $\mathbf{q}$ map $\mathbf{x} \mapsto \mathbf{qx} \pmod{\mathbf{1}}$ on $\mathbb{R}^d/\mathbb{Z}^d$, and $\mathcal{M}(\mathbb{T}^d, \mathbf{S_q})$ is the space of $\mathbf{q}$-shift invariant probability measures. Normalized Lebesgue measure m on $\mathbb{T}^d$ is strong-mixing for the $\mathbf{q}$-shift, which can be verified by the following proposition - the proof is standard, see [22 §3.1.1] for the $d = 1$ case. For a measure $\nu \in \mathcal{M}(\mathbb{T}^d)$ (complex Borel measures on $\mathbb{T}^d$, see §5.1) its Fourier coefficients (1.1) are given by the identities

$$\widehat{\nu}(\mathbf{k}) = \int_{\mathbb{T}^d} \mathbf{z}^{-\mathbf{k}} d\nu \qquad \text{where} \qquad \mathbf{z}^{-\mathbf{k}} := z_1^{-k_1} z_2^{-k_2} \cdots z_d^{-k_d}$$

Proposition 3.0.9. *A measure $\nu \in \mathcal{M}(\mathbb{T}^d)$ is invariant for the $\mathbf{q}$-shift provided $\widehat{\nu}(\mathbf{aq}) = \widehat{\nu}(\mathbf{a})$ for every $\mathbf{a} \in \mathbb{Z}^d$. It is strong-mixing for the $\mathbf{q}$-shift provided it is invariant and for every $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^d$*

$$\lim_{p \to \infty} \widehat{\nu}(\mathbf{b} + \mathbf{aq}^p) = \widehat{\nu}(\mathbf{b})\widehat{\nu}(\mathbf{a})$$

By *the* $\mathbf{h}$-*th roots of unity* we mean the collection of $\mathbf{z} \in \mathbb{T}^d$ satisfying $z_i^{h_i} = 1$ for $1 \leqslant i \leqslant d$. It forms a subgroup of the torus and can be identified with the group of integers $\mathbb{Z}^d$ modulo $\mathbf{h}$. Let $\nu_{\mathbf{h}}$ be the Haar measure for the $\mathbf{h}$-th roots of unity, so that $\nu_{\mathbf{h}}$ distributes mass uniformly. For $\mathbf{k} \in \mathbb{Z}^d$, one checks that

$$\widehat{\nu_{\mathbf{h}}}(\mathbf{k}) = \begin{cases} 1 & \text{if } \mathbf{k} \equiv \mathbf{0} \,(\mathrm{mod}\ \mathbf{h}) \\ 0 & \text{if } \mathbf{k} \not\equiv \mathbf{0} \,(\mathrm{mod}\ \mathbf{h}) \end{cases} \tag{3.1}$$

using the above formulas and usual properties of roots of unity. For $\mathbf{q} \geqslant 1$, define

$$\boldsymbol{\omega}_{\mathbf{q}} := \sum_{n \geqslant 1} 2^{-n} \nu_{\mathbf{q}^n} \tag{3.2}$$

so that $\boldsymbol{\omega}_{\mathbf{q}}$ is a probability measure supported by the $\mathbf{q}$-adic roots of unity, which correspond to the $\mathbf{q}$-adic rationals in $\mathbb{R}^d/\mathbb{Z}^d$, or those rationals which can be expressed with denominator $\mathbf{q}^n$ for some n. Due to the extensive $\mathbf{q}$-adic properties of $\mathbf{q}$-substitutions, $\boldsymbol{\omega}_{\mathbf{q}}$ will play an important role in the spectral theory of $\mathbf{q}$-substitutions.

3.1 *The Correlation Vector -* Σ

Recall that for $\alpha, \beta \in \mathcal{A}$, the *correlation measure* $\sigma_{\alpha\beta} \in \mathcal{M}(\mathbb{T}^d)$ is the spectral measure for the pair $\mathbb{1}_{[\alpha]}$ and $\mathbb{1}_{[\beta]}$. Using the spectral theorem (1.1), we compute the Fourier coefficients of the correlation measure $\sigma_{\alpha\beta}$ using translation invariance and obtain for $\mathbf{k} \in \mathbb{Z}^d$

$$\widehat{\sigma_{\alpha\beta}}(\mathbf{k}) = \int_{X_{\mathcal{S}}} \mathbb{1}_{[\alpha]} \circ T^{-\mathbf{k}} \cdot \mathbb{1}_{[\beta]} d\mu = \mu(T^{\mathbf{k}}[\alpha] \cap [\beta]) = \mu([\alpha] \cap T^{-\mathbf{k}}[\beta]) \tag{3.3}$$

and so $\widehat{\sigma_{\alpha\beta}}(\mathbf{k}) = \overline{\widehat{\sigma_{\beta\alpha}}(-\mathbf{k})} = \widehat{\sigma_{\beta\alpha}}(-\mathbf{k})$ as $\mu \in \mathcal{M}(X_{\mathcal{S}}, T)$ is real valued.

Definition 3.1.1. The *correlation vector* of $\mathcal{S}$ is the vector valued measure $\Sigma = (\sigma_{\alpha\beta})_{\alpha\beta \in \mathcal{A}^2}$.

Queffélec defines Σ as an $\mathcal{A} \times \mathcal{A}$ matrix valued measure, see [22, §7.1.3], although it is more natural for us as a (column) vector. The following *Fourier recursion theorem* provides an infinite family of relations amongst the Fourier coefficients of the correlation measures, and allows for the explicit computation of the correlation measures. It both generalizes and extends [22 §8.3 (8.7)].

Theorem 3.1.2. *Let $\mathcal{S}$ be an aperiodic $\mathbf{q}$-substitution on $\mathcal{A}$. Then for $p \in \mathbb{N}$ and $\mathbf{k} \in \mathbb{Z}^d$*

$$\widehat{\Sigma}(\mathbf{k}) = \frac{1}{Q^p} \sum_{\mathbf{j} \in [\mathbf{0}, \mathbf{q}^p)} \mathcal{R}_{\mathbf{j}}^{(p)} \otimes \mathcal{R}_{\mathbf{j}+\mathbf{k}}^{(p)} \, \widehat{\Sigma}(\lfloor \mathbf{k} \rfloor_p + \mathfrak{c}_p(\mathbf{j}, \mathbf{k})) = \lim_{n \to \infty} \frac{1}{Q^n} \sum_{\mathbf{j} \in [\mathbf{0}, \mathbf{q}^n)} \mathcal{R}_{\mathbf{j}}^{(n)} \otimes \mathcal{R}_{\mathbf{j}+\mathbf{k}}^{(n)} \, \widehat{\Sigma}(\mathbf{0})$$

Proof. Writing $\widehat{\Sigma}(\mathbf{k}) = \sum_{\alpha\beta\in\mathcal{A}^2} \mu([\alpha] \cap T^{-\mathbf{k}}[\beta])\mathbf{e}_{\alpha\beta}$ the first identity follows from the measure formula 2.3.7. Note that $\widehat{\Sigma}(\mathbf{0}) = \sum_{\gamma\in\mathcal{A}} \mu([\gamma])\mathbf{e}_{\gamma\gamma}$, by (3.3).

As $\mathbf{e}^*_{\alpha\beta}\widehat{\Sigma}(\mathbf{k}) = \mu([\alpha] \cap T^{-\mathbf{k}}[\beta])$, we apply the measure formula 2.3.7 again for $p \geqslant \mathfrak{p}(\mathbf{k})$ and separate the sum into those blocks in $\mathcal{S}^{-n}\binom{\mathbf{0}\mapsto\alpha}{\mathbf{k}\mapsto\beta}$ which are supported at $\mathbf{0}$ from those which require larger support due to overlapping of superblocks, and we obtain

$$\begin{aligned}\widehat{\Sigma}(\mathbf{k}) &= \frac{1}{Q^p}\sum_{\mathbf{j}\in[\mathbf{0},\mathbf{q}^p)\smallsetminus\Delta_p(\mathbf{k})} \mathcal{R}^{(p)}_{\mathbf{j}} \otimes \mathcal{R}^{(p)}_{\mathbf{j}+\mathbf{k}}\widehat{\Sigma}(\mathbf{0}) + \frac{1}{Q^p}\sum_{\mathbf{j}\in\Delta_p(\mathbf{k})} \mathcal{R}^{(p)}_{\mathbf{j}} \otimes \mathcal{R}^{(p)}_{\mathbf{j}+\mathbf{k}}\widehat{\Sigma}(\lfloor\mathbf{j}+\mathbf{k}\rfloor_p),\\ &= \frac{1}{Q^p}\sum_{\mathbf{j}\in[\mathbf{0},\mathbf{q}^p)} \mathcal{R}^{(p)}_{\mathbf{j}} \otimes \mathcal{R}^{(p)}_{\mathbf{j}+\mathbf{k}}\widehat{\Sigma}(\mathbf{0}) + \frac{1}{Q^p}\sum_{\mathbf{j}\in\Delta_p(\mathbf{k})} \mathcal{R}^{(p)}_{\mathbf{j}} \otimes \mathcal{R}^{(p)}_{\mathbf{j}+\mathbf{k}}\left(\widehat{\Sigma}(\lfloor\mathbf{j}+\mathbf{k}\rfloor_p) - \widehat{\Sigma}(\mathbf{0})\right)\end{aligned}$$

so that, letting $p \to \infty$ and using the carry estimates of lemma 2.1.1 gives the desired result, as $|\widehat{\Sigma}(\mathbf{a})| \leqslant \mathrm{Card}(\mathcal{A})$ which follows from the fact that the $\mathcal{R}^{(n)}_{\mathbf{j}} \otimes \mathcal{R}^{(n)}_{\mathbf{j}+\mathbf{k}}$ are column stochastic, and (3.3) which shows the coefficients of $\widehat{\Sigma}$ are bounded. □

An important consequence of the above is an invariance property of the maximal spectral type which we discuss at length in the next section, but there is a particular consequence which motivates the study of the correlation measures and so we discuss it here.

Lemma 3.1.3. *Let $\mathcal{S}$ be an aperiodic $\mathbf{q}$-substitution on $\mathcal{A}$. For $p \geqslant 0$, $\mathbf{j}, \mathbf{k} \in \mathbb{Z}^d$, and $\gamma, \delta \in \mathcal{A}$*

$$\sigma_{\mathbb{1}_{T^{\mathbf{j}}\mathcal{S}^p[\gamma]},\mathbb{1}_{T^{\mathbf{k}}\mathcal{S}^p[\delta]}} \ll \nu_{\mathbf{q}^p} * \sigma_{\gamma\gamma} \qquad \textit{and} \qquad \sigma_{\mathbb{1}_{\mathcal{S}^p[\gamma]}} \sim \nu_{\mathbf{q}^p} * \sigma_{\gamma\gamma}$$

Proof. By standard properties of the spectral map [22, prop 2.4] and shift invariance of μ

$$\sigma_{\mathbb{1}_{\mathcal{S}^p[\gamma]}\circ T^{\mathbf{k}},\mathbb{1}_{\mathcal{S}^p[\delta]}\circ T^{\mathbf{j}}} \ll \sigma_{\mathbb{1}_{\mathcal{S}^p[\gamma]}} \text{ and } \ll \sigma_{\mathbb{1}_{\mathcal{S}^p[\delta]}}$$

and so the first relation follows from the second, and we proceed to this case.

Note that, by definition of the spectral map and the relations (2.4), we have for $\mathbf{k} \in \mathbb{Z}^d$

$$\widehat{\sigma_{\mathbb{1}_{\mathcal{S}^p[\gamma]}}}(\mathbf{k}) = \mu\big(T^{\lceil\mathbf{k}\rceil_p}\mathcal{S}^p(T^{\lfloor\mathbf{k}\rfloor_p}[\gamma]) \cap \mathcal{S}^p([\gamma])\big) = \begin{cases} \frac{1}{Q^p}\mu(T^{\lfloor\mathbf{k}\rfloor_p}[\gamma] \cap [\gamma]) & \text{if } \lceil\mathbf{k}\rceil_p = \mathbf{0} \\ 0 & \text{if } \lceil\mathbf{k}\rceil_p \neq \mathbf{0}\end{cases}$$

by aperiodicity (2.8), proposition 2.3.4.2 on embeddings, and the scaling property of theorem

1.2.1. For $p \geqslant 0$ and $\mathbf{a} \in \mathbb{Z}^d$, we have $\mathcal{R}^{(p)}_{\mathbf{j}+\mathbf{a}\mathbf{q}^p} = \mathcal{R}^{(p)}_{\mathbf{j}}$ and $\mathfrak{c}_p(\mathbf{j}, \mathbf{a}\mathbf{q}^p) = \mathbf{0}$ so by theorem 3.1.2

$$\begin{aligned}\widehat{\Sigma}(\mathbf{a}\mathbf{q}^p) &= \frac{1}{Q^p}\sum\nolimits_{\mathbf{j}\in[\mathbf{0},\mathbf{q}^p)} \mathcal{R}^{(p)}_{\mathbf{j}} \otimes \mathcal{R}^{(p)}_{\mathbf{j}+\mathbf{a}\mathbf{q}^p} \widehat{\Sigma}\big(\lfloor \mathbf{a}\mathbf{q}^p \rfloor_p + \mathfrak{c}_p(\mathbf{j}, \mathbf{a}\mathbf{q}^p)\big) \\ &= \frac{1}{Q^p}\sum\nolimits_{\mathbf{j}\in[\mathbf{0},\mathbf{q}^p)} \mathcal{R}^{(p)}_{\mathbf{j}} \otimes \mathcal{R}^{(p)}_{\mathbf{j}} \widehat{\Sigma}(\mathbf{a}) = \frac{1}{Q^p} M^p_{\mathcal{S}\otimes\mathcal{S}}\, \widehat{\Sigma}(\mathbf{a}) = \widehat{\Sigma}(\mathbf{a})\end{aligned}$$

by the stochasticity of $\mathbf{q}$-substitution matrices. Thus, $\widehat{\sigma_{\gamma\gamma}}(\mathbf{a}\mathbf{q}^p) = \widehat{\sigma_{\gamma\gamma}}(\mathbf{a})$, independent of $p, \mathbf{a}, \gamma$ so that by proposition 3.0.9, the autocorrelations $\sigma_{\gamma\gamma}$ are $\mathbf{q}$-shift invariant. If $\lceil \mathbf{k} \rfloor_p = \mathbf{0}$ then $\mathbf{k} = \lfloor \mathbf{k} \rfloor_p \mathbf{q}^p$ and so $\widehat{\sigma_{\gamma\gamma}}(\lfloor \mathbf{k} \rfloor_n) = \widehat{\sigma_{\gamma\gamma}}(\lfloor \mathbf{k} \rfloor_n \mathbf{q}^n) = \widehat{\sigma_{\gamma\gamma}}(\mathbf{k})$ by the above, giving

$$\widehat{\sigma_{\mathbb{1}_{\mathcal{S}^n[\gamma]}}}(\mathbf{k}) = \begin{cases} \frac{1}{Q^p}\widehat{\sigma_{\gamma\gamma}}(\mathbf{k}) & \text{if } \mathbf{k} \equiv \mathbf{0} (\text{mod } \mathbf{q}^p) \\ 0 & \text{otherwise} \end{cases} = \frac{1}{Q^p}\widehat{\sigma_{\gamma\gamma}}(\mathbf{k}) \cdot \widehat{\nu_{\mathbf{q}^p}}(\mathbf{k}) = \frac{1}{Q^p}\widehat{\sigma_{\gamma\gamma} * \nu_{\mathbf{q}^p}}(\mathbf{k})$$

using identity (3.1), giving the final equivalence as we are identifying up to type. □

By proposition 2.3.8 the functions $\mathbb{1}_{T^{\mathbf{k}}\mathcal{S}^n[\gamma]}$ have dense span in $L^2(\mu)$ so that the above lets us represent the spectral map σ on $L^2(\mu)$ entirely in terms of the correlation measures for purposes of studying the maximal spectral type. As the autocorrelations are $\mathbf{q}$-shift invariant, and the spectral map is determined by them, it seems reasonable that $\sigma_{\max}$ should share these invariant properties as well. The following result makes this precise, and is the most important property of the correlation measures; see [22, §7.1.2]. Note that $\sim$ denotes equivalence, and $*$ convolution, of measures.

Theorem 3.1.4. *If $\mathcal{S}$ is an aperiodic $\mathbf{q}$-substitution, then σ_{max} is $\mathbf{q}$-shift invariant, and*

$$\sigma_{max} \;\sim\; \boldsymbol{\omega}_{\mathbf{q}} * \sum\nolimits_{\alpha\in\mathcal{A}} \sigma_{\alpha\alpha} \;\sim\; \boldsymbol{\omega}_{\mathbf{q}} * (\widehat{\Sigma}(\mathbf{0})^t\Sigma)$$

Proof. For $n \geqslant 1$, consider the cyclic subspace $H_n := \overline{\mathrm{Span}\{\mathbb{1}_{\mathcal{S}^n[\gamma]} \circ T^{\mathbf{k}} : \gamma \in \mathcal{A},\ \ \mathbf{k} \in \mathbb{Z}^d\}}$ so that $H_n \subset H_{n+1}$ by corollary 2.3.4.3 and $L^2(\mu) = \overline{\cup H_n}$ as the iterated indicators have dense span by proposition 2.3.8. Note that if $G \in H_n$, then as the spectral map $\sigma : L^2(\mu) \to \mathcal{M}(\mathbb{T}^d)$ taking $f \mapsto \sigma_f$ is continuous and sesquilinear, lemma 3.1.3 gives $\sigma_G \sim \sum_{m=0}^{n}\sum_{\gamma\in\mathcal{A}} 2^{-m}\nu_{\mathbf{q}^m} * \sigma_{\gamma\gamma}$. Finally, let $F \in L^2(\mu)$ be a maximal function, so that $\sigma_F \in \sigma_{\max}$. By proposition 2.3.8, we can approximate F by a sequence of simple functions $F_n \in H_n$ so that $\sigma_{F_n} \to \sigma_F$, and so

$$\lim_{n\to\infty} \sum\nolimits_{m=0}^{n} \sum\nolimits_{\gamma\in\mathcal{A}} 2^{-m} \nu_{\mathbf{q}^m} * \sigma_{\gamma\gamma} = \boldsymbol{\omega}_{\mathbf{q}} * \sum\nolimits_{\gamma\in\mathcal{A}} \sigma_{\gamma\gamma} \;\sim\; \sigma_{\max}$$

as we can rescale our measures without affecting the type. The invariance claim follows from $\mathbf{q}$-shift invariance of the autocorrelations and the support of $\boldsymbol{\omega}_{\mathbf{q}}$. The final equivalence follows as $\mu([\alpha])$ is nonzero if and only if $\sigma_{\alpha\alpha}$ is a nonzero measure. □

Combined with theorem 3.1.2, this allows us to compute Fourier coefficients for $\sigma_{\max}$. Although Fourier coefficients are not a type invariant, one has several tools available for detecting the various pure types (discrete, continuous, singular, or absolutely continuous to Lebesgue) via their Fourier coefficients (Wiener's criterion, Riemann-Lebesgue lemma) and all of these are *a priori* type invariants. Many substitutions, however, exhibit mixed spectra and the measures $\sigma_{\alpha\alpha}$ themselves are not in general pure types, so more is needed.

In the next section, we describe a larger collection of measures in the span of the $\sigma_{\alpha\beta}$ giving rise to the maximal spectral type (via convolution with $\boldsymbol{\omega}_{\mathbf{q}}$) and it is not unrealistic to expect some of them to share properties with $\sum \sigma_{\alpha\alpha}$, a positive probability measure, invariant for the $\mathbf{q}$-shift on $\mathbb{T}^d$.

At this point, we clarify a subtle distinction between our approach and Queffélec's. Consider the remark [22 §7.2, identity (7.5)] which states (this is the $\mathcal{A}^{\mathbb{N}}$ case)

$$\widehat{\sigma}_{\alpha\beta}(k) = \lim_{n\to\infty} \frac{1}{q^n} \mathrm{Card}\big\{0 < j < q^n \;:\; 0 < j+k < q^n, \quad \mathcal{S}^n(\gamma)_{j+k} = \alpha, \text{ and } \mathcal{S}^n(\gamma)_j = \beta\big\}$$

versus our equivalent statement using theorem 3.1.2

$$\widehat{\sigma}_{\alpha\beta}(k) = \mathbf{e}^*_{\alpha\beta}\widehat{\Sigma}(\mathbf{k}) = \lim_{n\to\infty} \frac{1}{Q^n} \sum\nolimits_{\mathbf{j}\in[\mathbf{0},\mathbf{q}^n)} \mathbf{e}^*_{\alpha\beta} \mathcal{R}^{(n)}_{\mathbf{j}} \otimes \mathcal{R}^{(n)}_{\mathbf{j}+\mathbf{k}} \widehat{\Sigma}(\mathbf{0}) = \lim_{n\to\infty} \big({}^{1}\!/\!{}_{Q^n} M_{\mathcal{S}^n \otimes \mathcal{S}^n_{\mathbf{k}}}\big) \widehat{\Sigma}(\mathbf{0})$$

and so one can see that, in an implicit sense, Queffélec's analysis is done by solving simultaneous equations in the substitutions $\mathcal{S}^n$ and $T^{\mathbf{k}}\mathcal{S}^n$, whereas ours is carried out through cycled substitution products $\mathcal{S}^n \otimes \mathcal{S}^n_{\mathbf{k}}$. To be clear: in Queffélec's identity, one requires that both j *and* $j+k$ be in the domain of the word $\mathcal{S}^n(\gamma)$, whereas ours only requires that j sum over a mod $\mathbf{q}^n$ equivalence class in $\mathbb{Z}^d$. Moreover, we handle the overlapping of superblocks by simply ignoring them and letting the cycled substitutions and carry estimates of lemma 2.1.1 handle the discrepancies automatically and simplifying the combinatorics of the above Fourier identities significantly.

3.2 The Spectral Hull - $\mathcal{K}$

Let $\mathcal{S}$ be an aperiodic $\mathbf{q}$-substitution on $\mathcal{A}$, telescoped so that its bisubstitution $\mathcal{S} \otimes \mathcal{S}$ has index of imprimitivity 1; let Σ be its correlation vector, and $C_{\mathcal{S}} = M_{\mathcal{S}\otimes\mathcal{S}}$ its coincidence matrix. Recall that $\mathcal{M}(\mathbb{T}^d)$ is the space of complex Borel measures on $\mathbb{T}^d$, and consider

$$\lambda_{\circ} : \mathbb{C}^{\mathcal{A}^2} \longmapsto \mathcal{M}(\mathbb{T}^d) \qquad \text{with} \qquad \mathbf{v} \longmapsto \lambda_{\mathbf{v}} := \mathbf{v}^t \Sigma = \textstyle\sum_{\alpha\beta \in \mathcal{A}^2} v_{\alpha\beta}\sigma_{\alpha\beta} \tag{3.4}$$

which defines a linear map taking a $\mathbb{C}^{\mathcal{A}^2}$ vector to a linear combination of the correlation vectors. Note that theorem 3.1.4 expresses $\sigma_{\max}$ as the $\widehat{\Sigma}(\mathbf{0})$ span of $\sigma_{\alpha\beta}$ so that

$$\sigma_{\max} \sim \boldsymbol{\omega}_{\mathbf{q}} * \lambda_{\widehat{\Sigma}(\mathbf{0})} \qquad \text{as} \qquad \widehat{\Sigma}(\mathbf{0}) = \textstyle\sum_{\alpha \in \mathcal{A}} \mu([\alpha])\mathbf{e}_{\alpha\alpha}$$

As the autocorrelation measures are $\mathbf{q}$-shift invariant, we consider the effect that $\mathbf{S_q}$ has on the other correlation measures, and one can check with (3.3) that

$$\widehat{\mathbf{S_q} \circ \Sigma}(\mathbf{a}) = \widehat{\Sigma}(\mathbf{aq}) = \tfrac{1}{Q} \textstyle\sum_{\mathbf{j} \in [\mathbf{0},\mathbf{q})} \mathcal{R}^{(1)}_{\mathbf{j}} \otimes \mathcal{R}^{(1)}_{\mathbf{j}+\mathbf{aq}} \widehat{\Sigma}(\lfloor \mathbf{j} + \mathbf{aq} \rfloor_1) = \tfrac{1}{Q} C_{\mathcal{S}} \widehat{\Sigma}(\mathbf{a}) \quad \text{for} \quad \mathbf{a} \in \mathbb{Z}^d \tag{3.5}$$

using theorem 3.1.2 as $\lfloor \mathbf{j} + \mathbf{aq} \rfloor_1 = \mathbf{a}$ and $\lceil \mathbf{j} + \mathbf{aq} \rceil_1 = \mathbf{j}$ for $\mathbf{j} \in [\mathbf{0}, \mathbf{q})$, and as $C_{\mathcal{S}} = \sum \mathcal{R}_{\mathbf{j}} \otimes \mathcal{R}_{\mathbf{j}}$.

Lemma 3.2.1. *If* $\mathbf{v}$ *is a left* Q*-eigenvector of* $C_{\mathcal{S}}$*, then* $\lambda_{\mathbf{v}} = \mathbf{v}^t\Sigma$ *is invariant for the* $\mathbf{q}$*-shift.*

Proof. Let $\mathbf{v}$ be a left Q-eigenvector of $C_{\mathcal{S}}$ so that $C^t_{\mathcal{S}}\mathbf{v} = Q\mathbf{v}$. Then by (3.5)

$$\widehat{\mathcal{S}_{\mathbf{q}} \circ \lambda_{\mathbf{v}}}(\mathbf{a}) = \widehat{\lambda_{\mathbf{v}}}(\mathbf{aq}) = \mathbf{v}^t \widehat{\Sigma}(\mathbf{aq}) = \tfrac{1}{Q}\mathbf{v}^t C_{\mathcal{S}} \widehat{\Sigma}(\mathbf{a}) = \mathbf{v}^t \widehat{\Sigma}(\mathbf{a}) = \widehat{\lambda_{\mathbf{v}}}(\mathbf{a})$$

and so $\lambda_{\mathbf{v}}$ is invariant for the $\mathbf{q}$-shift by proposition 3.0.9. □

As the autocorrelation measures are positive, one can also reinterpret theorem 3.1.4 by saying that any positive linear combination λ of the autocorrelation measures $\sigma_{\alpha\alpha}$ for $\alpha \in \mathcal{A}$

generates the spectrum by convolution with $\boldsymbol{\omega}_{\mathbf{q}}$. As the correlation measures $\sigma_{\alpha\beta}$ for $\alpha \neq \beta$ are not in general positive measures, we cannot say the same in this case: a different kind of positivity is required. We are missing an important ingredient: *strong semipositivity*, based on a condition used by Queffélec [22, Prop 10.3].

Definition 3.2.2. For $\mathbf{v} = (v_{\alpha\beta})_{\alpha\beta\in\mathcal{A}^2} \in \mathbb{C}^{\mathcal{A}^2}$, let $\mathring{\mathbf{v}} = (v_{\alpha\beta})_{\alpha,\beta\in\mathcal{A}} \in \mathbf{M}_{\mathcal{A}}(\mathbb{C})$ be its *associated matrix*. If $\mathring{\mathbf{v}}$ is positive semidefinite, we say $\mathbf{v}$ is *strongly semipositive* and *write* $\mathbf{v} \gg 0$.

One reads the entries of a $\mathbb{C}^{\mathcal{A}^2}$ vector into the entries of its $\mathcal{A} \times \mathcal{A}$ associated matrix along each row sequentially in order. In the $s = 2$ case, the forward and inverse maps are

$$\begin{pmatrix} a & b & c & d \end{pmatrix}^{\mathrm{t}} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad\qquad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \longmapsto \begin{pmatrix} a_{11} & a_{12} & a_{21} & a_{22} \end{pmatrix}^{\mathrm{t}}$$

It also relates the Kronecker and matrix products: for $\mathbf{A}, \mathbf{B} \in \mathbf{M}_{\mathcal{A}}(\mathbb{C})$ and $\mathbf{v} \in \mathbb{C}^{\mathcal{A}^2}$

$$(\mathbf{A} \otimes \mathbf{B})^{\circ}\mathbf{v} = \mathbf{A}\mathring{\mathbf{v}}\mathbf{B}^t \tag{3.6}$$

and this identity can be used to express several of our results in terms of matrix Riesz products and is used in the proof of theorem 3.4.1 for this exact purpose.

Lemma 3.2.3. *If* $\mathbf{v} \in \mathbb{C}^{\mathcal{A}^2}$ *is strongly semipositive, then* $\lambda_{\mathbf{v}} = \mathbf{v}^t\Sigma$ *is a positive measure.*

Proof. By the Schur product theorem, the Hadamard product $(A \circ B)$ of two positive semidefinite matrices is positive semidefinite, see [15, §5]. As $\mathbf{v}^t\mathbf{w} = \sum v_i w_i = \mathbf{1}^t(\mathring{\mathbf{v}} \circ \mathring{\mathbf{w}})\mathbf{1}$, we have $\mathbf{v}^t\mathbf{w} \geqslant 0$ whenever both $\mathbf{v}$ and $\mathbf{w}$ are strongly semipositive. Using sesquilinearity of the spectral map $f, g \mapsto \sigma_{f,g}$ and the spectral theorem, one checks that $\mathring{\Sigma}$ is Hermitian positive definite, so that $\lambda_{\mathbf{v}}$ is a positive measure whenever $\mathbf{v} \gg 0$ as these conditions are determined pointwise for measures. □

An important example is the vector $\mathbf{v} = \mathbf{1}_{\mathcal{A}^2} \in \mathbb{C}^{\mathcal{A}^2}$, which is a left Q-eigenvector of the coincidence matrix $C_{\mathcal{S}}$, being a Q-column stochastic matrix. The matrix $\mathring{\mathbf{1}}_{\mathcal{A}^2}$ is positive semidefinite, and so $\mathbf{1}_{\mathcal{A}^2}$ is strongly semipositive, and

$$\lambda_{\mathbf{1}_{\mathcal{A}^2}} = \sum\nolimits_{\alpha\beta\in\mathcal{A}^2} \sigma_{\alpha\beta} = \sum\nolimits_{\alpha\in\mathcal{A}} \sum\nolimits_{\beta\in\mathcal{A}} \sigma_{\mathbb{1}_{[\alpha]},\mathbb{1}_{[\beta]}} = \sigma_{\sum_\alpha \mathbb{1}_{[\alpha]}} = \boldsymbol{\delta}_{\mathbf{1}}$$

the unit Dirac mass at $\mathbf{1} \in \mathbb{T}^d$, as $\sum_{\alpha\in\mathcal{A}} \mathbb{1}_{[\alpha]}$ is the constant function, and is therefore positive. Note that $\boldsymbol{\delta}_{\mathbf{1}}$ is strong mixing for the $\mathbf{q}$-shift on $\mathbb{T}^d$, and is part of the spectrum of *every* $\mathbb{Z}^d$-action on a compact metric space.

Finally, recall from the Fourier recursion theorem 3.1.2 that $\widehat{\Sigma}(\mathbf{0}) = \sum_{\alpha\in\mathcal{A}} \mu([\alpha])\mathbf{e}_{\alpha\alpha}$ so

$$\lambda_{\mathbf{v}}(\mathbb{T}^d) = \widehat{\lambda_{\mathbf{v}}}(\mathbf{0}) = \mathbf{v}^t\widehat{\Sigma}(\mathbf{0}) = \textstyle\sum_{\alpha\in\mathcal{A}} v_{\alpha\alpha}\mu([\alpha])$$

so that $\mathbf{v} \geqq 0$ gives rise to a probability measure if and only if $\mathbf{v}^t\widehat{\Sigma}(\mathbf{0}) = \sum v_{\alpha\alpha}\mu([\alpha]) = 1$. In the primitive case, this is equivalent to $v_{\alpha\alpha} = 1$ for $\alpha \in \mathcal{A}$, so that the probability measures arising from strong semipositivity via λ is a bounded convex set with finitely many extreme points. The above discussion prompts the following definition, see also [22, Def 11.1].

Definition 3.2.4. The *spectral hull* $\mathcal{K}(\mathcal{S})$ of a $\mathbf{q}$-substitution $\mathcal{S}$ is the collection

$$\mathcal{K}(\mathcal{S}) := \{\mathbf{v} \in \mathbb{C}^{\mathcal{A}^2} : C_{\mathcal{S}}^t\mathbf{v} = Q\mathbf{v} \text{ and } \mathbf{v} \geqq 0\}$$

and let $\mathcal{K}^*$ denote the extreme points of the collection of $\mathbf{v} \in \mathcal{K}(\mathcal{S})$ for which $\mathbf{v}^t\widehat{\Sigma}(\mathbf{0}) = 1$.

Note that by Q-column stochasticity of $\mathbf{q}$-substitutions we always have $C_{\mathcal{S}}^t\mathbf{1}_{\mathcal{A}^2} = Q\mathbf{1}_{\mathcal{A}^2}$, as well as $\mathbf{1}_{\mathcal{A}^2} \geqq 0$, and $\mathbf{1}_{\mathcal{A}^2}^t\widehat{\Sigma} = 1$. Note that $\mathbf{1}_{\mathcal{A}^2}$ will *always* be in $\mathcal{K}^*$, which follows from theorem 3.3.1 in the next section as this is gives rise to the $\mathbf{q}$-adic spectrum and thus must be an extreme point; moreover, this implies $\mathcal{K}$ and $\mathcal{K}^*$ are nonempty. Thus, $\mathcal{K}(\mathcal{S})$ is a nonempty closed convex cone in $\mathbb{C}^{\mathcal{A}^2}$ and, although formally the $\mathbf{0}$ measure is contained in $\mathcal{K}$, we do not consider it to be for the purposes of any statements. This proves the following:

Proposition 3.2.5. *For* $\mathbf{v} \in \mathcal{K}(\mathcal{S})$, $\lambda_{\mathbf{v}}$ *is positive and* $\mathbf{q}$*-shift invariant:* $\lambda_{\mathcal{K}(\mathcal{S})} \subset \mathcal{M}(\mathbb{T}^d, S_{\mathbf{q}})$

Thus, the spectral hull is a nonempty closed convex cone (closed under positive scalar multiples and sums) in $\mathbb{C}^{\mathcal{A}^2}$ whose elements give rise (via $\lambda_\circ$) to $\mathbf{q}$-shift invariant positive measures in the span of the correlation measures. The elements of $\mathcal{K}^*$ are the *extremal rays* of the spectral hull, and give rise to $\mathbf{q}$-shift invariant probability measures in the span of the correlation measures. We note here that our definition differs from Queffélec's slightly: whereas we use strong semipositivity, Queffélec considers vectors whose associated matrices are strictly *positive definite*, and so in Queffélec's case the extremal rays do not lie in the spectral hull, but on its boundary. This distinction is only significant in the case when $\mathcal{S} \otimes \mathcal{S}$ has only 1 ergodic class, in which case there will be no strongly positive eigenvectors. This case, however, corresponds to a pure discrete spectrum, as this implies a coincidence condition among the instructions, see §4.4.

3.3 Queffélec's Theorem

We are now prepared to state the main result of our paper. Recall that theorem 3.1.4 relates the correlation measures to the maximal spectral type as well as establishing its $\mathbf{q}$-shift invariance, and proposition 3.2.5 identifies the spectral hull as a convex cone of coefficients for the linear combinations of the correlation vectors which give rise to $\mathbf{q}$-shift invariant probability measures. The following theorem shows that the extremal rays $\mathcal{K}^*$ correspond via $\lambda_\circ$ to extreme points of $\mathcal{M}(\mathbb{T}^d, \mathbf{S_q})$, and that every strictly positive linear combination these ergodic measures give rise to the spectrum of the aperiodic $\mathbf{q}$-substitution.

Theorem 3.3.1. *If $\mathcal{S}$ is an aperiodic $\mathbf{q}$-substitution on $\mathcal{A}$, then for $\lambda_{\mathbf{v}} = \mathbf{v}^t\Sigma$,*

$$\sigma_{max} \sim \boldsymbol{\omega}_{\mathbf{q}} * \textstyle\sum_{\mathbf{w} \in \mathcal{K}^*} \lambda_{\mathbf{w}}$$

Moreover, the measures $\lambda_{\mathbf{w}}$ for $\mathbf{w} \in \mathcal{K}^*$ are strong-mixing (of all orders) for the $\mathbf{q}$-shift.

The proof appears in §5.2, as it relies on a number of details that are not directly related to the main results, and are totally unnecessary for the computation and analysis of the spectrum. Queffélec proved the above in the case of aperiodic primitive substitutions of constant length on $\mathbb{N}$ of trivial height, see [22, Thms 10.1, 10.2, 11.1]. The proof relies heavily on Queffélec's *bicorrelation matrix* $\mathcal{Z}$, a *matrix valued measure* on $\mathbb{T}^d$ (see §5.1) satisyfing $\Sigma = \mathcal{Z}\widehat{\Sigma}(\mathbf{0})$, and so Σ inherits many properties from $\mathcal{Z}$. If we let $\mathcal{P} := \lim_{n\to\infty} \frac{1}{Q^n} C_{\mathcal{S}}^n$ denote the projection onto the Q-eigenspaces of the coincidence matrix $C_{\mathcal{S}}$, then $\mathcal{P}\mathcal{Z}$ turns out to be both invariant and strong-mixing for the $\mathbf{q}$-shift as a matrix of measures. We then apply theorem 5.1.8 (proven in §5.1) allowing us to diagonalize $\mathcal{P}\mathcal{Z}$ over $\mathbb{C}$ and show it has *ergodic eigenmeasures* which coincide with the linear combinations of correlation measures arising from the extremal rays of the spectral hull. This identification is accomplished by showing that $\mathcal{P}$ and $\mathcal{Z}$ preserve the strong semipositivity condition, and so $\mathcal{P}$ preserves the spectral hull. The relationship between $C_{\mathcal{S}}$ and $\mathcal{Z}$ can be seen using identities similar to the Fourier recursion theorem 3.1.2: for every fixed $\mathbf{k}$, the number of terms for which the m-th digit of $\mathbf{j}$ and $\mathbf{j}+\mathbf{k}$ are different occurs with exponentially decreasing frequency as $m \to \infty$, and so the cycled subsitution products $M_{\mathcal{S}^n \otimes \mathcal{S}^n_{\mathbf{k}}}$ have increasingly more instructions in common with the bisubstitution as $n \to \infty$.

We now return to the discussion of Queffélec's result and its applications. For $\nu \in \mathcal{M}(\mathbb{T}^d)$ the *type of* ν, denoted $\langle \nu \rangle$ is the collection of all measures equivalent to ν.

Definition 3.3.2. A *type* $\langle \nu \rangle$ is an equivalence class of measures in $\mathcal{M}(\mathbb{T}^d)$ under mutual absolute continuity, or measure equivalence. We say a type $\langle \nu \rangle$ is $\mathbf{q}$-shift *invariant* if it intersects $\mathcal{M}(\mathbb{T}^d, \mathbf{S_q})$ and *ergodic* if it contains an ergodic $\mathbf{q}$-shift invariant measure. Finally, we say an invariant type $\langle \nu \rangle$ *is supported by* an ergodic type $\langle \lambda \rangle$ provided $\lambda \ll \nu$.

As an invariant measure can be expressed as a convex sum (more generally, a Choquet integral [21]) of its supporting ergodic measures, every invariant type can be decomposed into mutually singular ergodic types. Thus, theorem 3.3.1 decomposes $\sigma_{\max}$ into finitely many ergodic types, and shows how the $\mathbf{q}$-shift acts as a filter for the maximal spectral type: separating $\sigma_{\max}$ into mutually singular, irreducible $\mathbf{q}$-invariant types in the span of the correlation measures, parametrized by the extremal rays of the spectral hull.

Corollary 3.3.3. *Each* $\mathbf{w} \in \mathcal{K}^*$, $\lambda_{\mathbf{w}}$ *is either purely discrete, purely singular continuous, or Lebesgue measure* m *on* $\mathbb{T}^d$, *describing mutually singular subtypes of the spectrum of* $\mathcal{S}$.

Proof. First, we know that each $\lambda_{\mathbf{w}}$ is an ergodic probability measure for the $\mathbf{q}$-shift on $\mathbb{T}^d$. Therefore, as Lebesgue measure is strong-mixing for the $\mathbf{q}$-shift (proposition 3.0.9), $\lambda_{\mathbf{w}}$ is either m, or mutually singular to m. If $\lambda_{\mathbf{w}}$ has any discrete component, then the discrete component of $\lambda_{\mathbf{w}}$ must be absolutely continuous with respect to the discrete measure $\boldsymbol{\omega}_{\mathbf{q}} * \boldsymbol{\nu}_{\mathbf{h}}$ by Dekking's theorem 4.4.1, which is $\mathbf{q}$-shift invariant as the height lattice is arithmetically independent of $\mathbf{q}$ by definition; it follows that $\lambda_{\mathbf{w}}$ is pure discrete by ergodicity. Finally, the singular continuous case follows as it is a mutually singular type to Lebesgue and discrete types, by definition, and all these measures are pure, ergodic types. As convolution with $\boldsymbol{\omega}_{\mathbf{q}}$ does not change the purity of a measure, λ is pure discrete, purely singular continuous, or Lebesgue measure m, respectively, if and only if $\lambda * \boldsymbol{\omega}_{\mathbf{q}}$ is as well. Thus, the statement about the spectrum follows as the maximal spectral type is a *positive* convex combination of the $\lambda_{\mathbf{w}}$ for $\mathbf{w} \in \mathcal{K}^*$, being linear combinations of the correlation measures themselves. $\square$

Note that one can generalize the strong mixing condition (5.7) on $\mathcal{PZ}$ in the proof of proposition 5.2.3 to show that the eigenmeasures are in fact *strong mixing of all finite orders*, using that the carry frequencies of lemma 2.1.1 decay exponentially. Moreover, in the $\mathbb{Z}$ case, one can use the above with [7, Thm 6] to show that the continuous measures of $\lambda_{\mathcal{K}^*}$ are supported on the entire circle: they have no gaps; although this is self-evident for Lebesgue components, it is not *a priori* the case of singular continuous measures. An additional corollary of the above allows us to readily determine absence of Lebesgue spectral component, its proof is an immediate corollary of the above.

Corollary 3.3.4. *If $\mathcal{S}$ be an aperiodic $\mathbf{q}$-substitution on $\mathcal{A}$, then $m \ll \sigma_{max}$ if and only if there is a $\mathbf{w} \in \mathcal{K}^*$ for which $\widehat{\lambda_{\mathbf{w}}}(\mathbf{0}) = 1$ and $\widehat{\lambda_{\mathbf{w}}}(\mathbf{k}) = 0$ for all $\mathbf{k} \in \mathbb{Z}^d \setminus \mathbf{0}$.*

Recall that two $\mathbf{q}$-substitutions are configuration equivalent if they have the same collection of instructions, counted with multiplicity. A property of a substitution is a *configuration invariant* if all configuration equivalent substitutions share that property or, equivalently, if it does not depend on the particular arrangement of the instructions in its configuration. The following proposition is immediate, as it follows entirely from properties of the abelianization of $\mathcal{S}$ which is *a priori* configuration independent:

Proposition 3.3.5. *If $\mathcal{S}$ is a $\mathbf{q}$-substitution, then $M_{\mathcal{S}}$ and $C_{\mathcal{S}}$, the Perron vectors and ergodic decompositions of $\mathcal{S}$ and $\mathcal{S}\otimes\mathcal{S}$, as well as $\widehat{\Sigma}(\mathbf{0})$ and $\mathcal{K}(\mathcal{S})$, are configuration invariants.*

By Queffélec's theorem 3.3.1, the spectrum of $\mathcal{S}$ is determined by the measures $\lambda_{\mathbf{v}} = \mathbf{v}^t\Sigma$ for $\mathbf{v} \in \mathcal{K}(\mathcal{S})$, and so the spectrum of $\mathcal{S}$ can be separated into the study of its correlation vector and extremal properties of the spectral hull. As $\mathcal{K}^*$ is a configuration invariant, however, this shows us that any property of the spectrum which depends on the configuration of $\mathcal{S}$ is determined by the correlation vector Σ. It is immediate that the spectrum of a substitution is *not* invariant with respect to configuration equivalence: not only can the spectrum exist on different dimensional tori, but the height (see §4.4) of a substitution depends heavily on its configuration. One can, however, use theorem 3.1.2 to study the effect changes in configuration have on a given substitution, and it is evident from identities such as (4.1) that the structure of the configuration relative to the carry sets $\Delta_p(\mathbf{k})$ accounts for much of this difference.

We now give a singularity result for the spectrum of a large class of $\mathbf{q}$-substitutions before moving on to describe an algorithm for computing the spectrum of an aperiodic $\mathbf{q}$-substitution via the Fourier coefficients of $\lambda_{\mathbf{w}}$ for $\mathbf{w} \in \mathcal{K}^*$, including several examples.

3.4 Aperiodic Bijective Commutative q*-Substitutions*

In this section, we state and prove a result classifying all aperiodic bijective and commutative $\mathbf{q}$-substitutions as having purely singular spectrum. This generalizes a result of Baake and Grimm in [2] for $\mathbf{q}$-substitutions on two letters, noting that all bijective substitutions on two letters are necessarily commutative (there are only two bijective instructions). In the $\mathbb{Z}$ case, one can combine [22, Prop 3.19 and Thm 8.2] to show that all commutative bijective substitutions have pure singular spectrum (which was not explicitly stated by Queffélec) and this generalizes to $\mathbb{Z}^d$ substitutions as well.

Theorem 3.4.1. *Aperiodic bijective commutative* $\mathbf{q}$*-substitutions have singular spectrum.*

Proof. Let $\mathcal{S}$ be an aperiodic bijective commutative $\mathbf{q}$-substitution. The argument is essentially the following: in the aperiodic bijective commutative case the sum of the autocorrelations $\sigma_{\alpha\alpha}$ can be expressed as Riesz products, which are singular to Lebesgue measure. The result then follows from theorem 3.1.4 which shows that $\sigma_{\max} \sim \sum_{\alpha\in\mathcal{A}} \boldsymbol{\omega}_{\mathbf{q}} * \sigma_{\alpha\alpha}$, the sum of singular measures.

To realize the autocorrelations as Riesz products, it is convenient to express Σ by its associated matrix $\mathbf{S} = \mathring{\Sigma}$ and use the relation (3.6) to write theorem 3.1.2 as

$$\widehat{\mathbf{S}}(\mathbf{k}) = \lim_{n\to\infty} \frac{1}{Q^n} \sum_{\mathbf{j}<\mathbf{q}^n} \mathcal{R}_{\mathbf{j}}^{(n)} \widehat{\mathbf{S}}(\mathbf{0}) (\mathcal{R}_{\mathbf{j}+\mathbf{k}}^{(n)})^*$$

Using lemma 2.1.1 as in the proof of theorem 3.1.2, one checks that if we write

$$R(\mathbf{x}) = \sum_{\mathbf{j}<\mathbf{q}} \mathcal{R}_{\mathbf{j}} e^{2\pi i \mathbf{j}\cdot\mathbf{x}} \text{ and } \Pi_n(\mathbf{x}) = R(\mathbf{q}^{n-1}\mathbf{x})\cdots R(\mathbf{x}) \implies \mathbf{S} = \text{w*-}\lim_{n\to\infty} \tfrac{1}{Q^n}\Pi_n^*\Pi_n d\mathbf{x}$$

so that $\mathbf{S}$ is a *matrix* Riesz product (see also [22, §8.1]). As the instructions are bijective and commute, the matrices $\mathcal{R}_{\mathbf{j}}$ are (commuting) permutation matrices and thus simultaneously unitarily diagonalizable - let $\mathbf{P} \in \mathbf{M}_{\mathcal{A}}(\mathbb{C})$ be a unitary matrix diagonalizing the instructions of $\mathcal{S}$. From the matrix Riesz product description of $\mathbf{S}$, it follows that

$$\mathbf{PSP}^* = \text{w*-}\lim_{n\to\infty} \tfrac{1}{Q^n}\Lambda(\mathbf{x})^* \cdots \Lambda(\mathbf{q}^{n-1}\mathbf{x})^*\Lambda(\mathbf{q}^{n-1}\mathbf{x})\cdots\Lambda(\mathbf{x})$$

where $\Lambda(\mathbf{x})$ is the diagonal matrix polynomial $\mathbf{P}R(\mathbf{x})\mathbf{P}^*$. Thus, if $\pi(\mathbf{x})$ is a diagonal matrix polynomial of $\mathbf{P}R(\mathbf{x})\mathbf{P}^*$, then the diagonal entries of $\mathbf{PSP}^*$ are of the form

$$\text{w*-}\lim_{n\to\infty} \tfrac{1}{Q^n}\prod_{j=0}^{n-1} |\pi(\mathbf{q}^j\mathbf{x})|^2$$

so that the measures on the diagonal are generalized Riesz products (see [22, §1.3]). Writing

$$P_n(\mathbf{x}) := \prod_{0\leqslant j<n} \pi(\mathbf{q}^j\mathbf{x})$$

if $m = d\mathbf{x}$ denotes Lebesgue measure on $\mathbb{R}^d/\mathbb{Z}^d$, then for $n > 0$, $P_n d\mathbf{x}$ determines a sequence of measures on $\mathbb{R}^d/\mathbb{Z}^d$ which converges in the weak-star topology to a diagonal measure of $\mathbf{PSP}^*$. We conclude the proof with the following lemma.

Lemma 3.4.2. *Weak-star limits of $\{P_n d\mathbf{x}\}$ are singular to Lebesgue measure on $\mathbb{R}^d/\mathbb{Z}^d$.*

Let ρ be a weak-star limit of the measures $P_n d\mathbf{x}$. Let $\mathbf{x}/\mathbf{q} := (x_1/q_1, \ldots, x_d/q_d)$, and $\rho(\mathbf{x}/\mathbf{q})$ denote the push forward of ρ under the map $\mathbf{x} \mapsto \mathbf{x}/\mathbf{q}$. As ρ is the weak-star limit, we have

$$\begin{aligned}\rho(\mathbf{x}/\mathbf{q}) &= \text{weak}^* \lim_{n\to\infty} \prod_{j<n} \pi(\mathbf{q}^j\mathbf{x}/\mathbf{q})d(\mathbf{x}/\mathbf{q}) \\ &= (1/\mathbf{q})\pi(\mathbf{x}/\mathbf{q})\left(\text{w}^* \lim_{n\to\infty} \prod_{j<n-1} \pi(\mathbf{q}^j\mathbf{x})d\mathbf{x}\right) = (1/\mathbf{q})\pi(\mathbf{x}/\mathbf{q})\rho(\mathbf{x})\end{aligned}$$

As $\pi(\mathbf{x})$ vanishes on a set of 0 Lebesgue measure, and the push forward map preserves absolutely continuous components, equality passes to the absolutely continuous part and

$$\rho_{\mathrm{ac}}(\mathbf{x}/\mathbf{q}) = (\mathbf{1}/\mathbf{q})\pi(\mathbf{x}/\mathbf{q})\,\rho_{\mathrm{ac}}(\mathbf{x}) \tag{3.7}$$

If $S_{\mathbf{q}} : [0,1]^d \to [0,1]^d$ is the map $\mathbf{x} \mapsto \mathbf{q}\mathbf{x} (\mathrm{mod}\ \mathbf{1})$, then

$$\rho_{\mathrm{ac}} \circ S_{\mathbf{q}}^{-1} = \sum_{\mathbf{j}\in[\mathbf{0},\mathbf{q})} \rho_{\mathrm{ac}}\left(\frac{\mathbf{x}+\mathbf{j}}{\mathbf{q}}\right) = \frac{\mathbf{1}}{\mathbf{q}} \sum_{\mathbf{j}\in[\mathbf{0},\mathbf{q})} \pi\left(\frac{\mathbf{x}+\mathbf{j}}{\mathbf{q}}\right) \rho_{\mathrm{ac}}(\mathbf{x}+\mathbf{j}) = \frac{\mathbf{1}}{\mathbf{q}} \sum_{\mathbf{j}\in[\mathbf{0},\mathbf{q})} \pi\left(\frac{\mathbf{x}+\mathbf{j}}{\mathbf{q}}\right) \rho_{\mathrm{ac}}(\mathbf{x}) = \rho_{\mathrm{ac}}(\mathbf{x})$$

as ρ is $\mathbf{1}$ periodic (all the $P_n(\mathbf{x})$ are, and Lebesgue measure is translation invariant) and as

$$\textstyle\sum_{\mathbf{j}\in[\mathbf{0},\mathbf{q})} e^{2\pi i \mathbf{k}\cdot\left(\frac{\mathbf{j}\mathbf{x}}{\mathbf{q}}\right)} = 0 \quad \text{for all} \quad \mathbf{k} \in \mathbb{Z}^d$$

so that ρ_{ac} is $S_{\mathbf{q}}$ invariant. As m is ergodic for the $\mathbf{q}$-shift, it follows that $\rho_{\mathrm{ac}} = m$ or $\rho_{\mathrm{ac}} = 0$. However, as m only satisfies (3.7) for constant $\pi(\mathbf{x})$, ρ_{ac} is $\mathbf{0}$ unless $\pi(\mathbf{x})$ is constant.

We conclude the proof by showing that $\pi(\mathbf{x})$ cannot be constant for $\mathbf{q} > \mathbf{0}$. As the functions $e^{2\pi i \mathbf{j}\cdot\mathbf{x}}$ are linearly independent and as $\pi(\mathbf{x})$ is a diagonal measure of $\mathcal{R}(\mathbf{x}) = \sum \mathcal{R}_{\mathbf{j}} e^{2\pi i \mathbf{j}\cdot\mathbf{x}}$, it follows that the eigenfunction π can be constant if and only if there is some vector in the kernel of $\mathcal{R}_{\mathbf{j}}$ for $\mathbf{j} > \mathbf{0}$. As the instruction matrices $\mathcal{R}_{\mathbf{j}}$ are unitary and therefore invertible, this is impossible, so that as $\mathbf{q} \neq \mathbf{0}$ it follows that $\pi(\mathbf{x})$ is not a constant. Thus π is singular to Lebesgue measure, as desired. □

Chapter 4

COMPUTING THE SPECTRUM

In this section, we summarize the results of the paper by illustrating an algorithm allowing one to compute the Fourier coefficients of the measures arising from the extremal rays of the spectral hull, and thus determine the spectrum of $\mathcal{S}$. The intent is to produce the entire spectrum of a given aperiodic $\mathbf{q}$-configuration through a single computation, without making any assumptions on the height or primitivity of the resulting $\mathbf{q}$-substitution. The following algorithm is summarized in the introduction; we begin with a $\mathbf{q}$-configuration $\mathcal{R}$ on $\mathcal{A}$, and compute the spectrum of its substitution.

1. **Compute the index of imprimitivity, then telescope and check aperiodicity**: if the substitution is not primitive, we need to telescope to ensure that the resulting substitution and bisubstitution have index of imprimitivity 1: in order to capture all the primitive components of the substitution, as well as compute the spectral hull, we will need to compute the primitive reduced forms of the (bi)substitutions and this part is simplified considerably by first finding h. Moreover, as the spectrum is determined by $\mathbf{q}$, we need to make sure we have the right scale, as $\boldsymbol{\omega}_{\mathbf{q}^h} \neq \boldsymbol{\omega}_{\mathbf{q}}$. Compute the substitution and coincidence matrices from the instruction matrices, $M_{\mathcal{S}} = \sum_{\mathbf{j}\in[\mathbf{0},\mathbf{q})} \mathcal{R}_{\mathbf{j}}$ and $C_{\mathcal{S}} = \sum_{\mathbf{j}\in[\mathbf{0},\mathbf{q})} \mathcal{R}_{\mathbf{j}} \otimes \mathcal{R}_{\mathbf{j}}$ and then determine the eigenvalues of $M_{\mathcal{S}}$ and $C_{\mathcal{S}}$ of modulus Q. One finds the minimal $h > 0$ so that $\lambda^h = Q^h$ for all eigenvalues λ of modulus Q for *both* $M_{\mathcal{S}}$ and $C_{\mathcal{S}}$. Telescoping the configuration $\mathcal{R}$ to $\mathcal{R}^{(h)}$, a $\mathbf{q}^h$-configuration, the $\mathbf{q}^h$-substitutions $\mathcal{S}^h$ and $\mathcal{S}^h \otimes \mathcal{S}^h$ will have index of imprimitivity 1. We now replace $\mathbf{q}$ with $\mathbf{q}^h$, or equivalently, assume $h = 1$. Now that it has been properly telescoped, one can also check for aperiodicity using Pansiot's lemma 2.3.1 in the $d = 1$ case, or using recognizability for $d > 1$.

2. **Compute Perron vectors, determine invariant measure**: recompute $M_{\mathcal{S}}$ and its primitive reduced form (1.2) by looking at letter orbits under the instructions. Once the primitive components of $M_{\mathcal{S}}$ are found (suppose there are $K \geqslant 1$), compute the Perron

vectors $\{\mathbf{u}_j\}_{j=1}^K$ of $\mathcal{S}$ and let $\mathbf{u} := \frac{1}{K}\sum_j \mathbf{u}_j$ be their normalized sum; note that $\mathbf{u} = (u_\alpha)_{\alpha\in\mathcal{A}}$ determines a $\mu \in \mathcal{M}(X_{\mathcal{S}}, T)$ containing a positive portion of all the ergodic measures for the subshift. By the definition of $\sigma_{\max}$ given in (1.3), the maximal spectral type of μ will contain the spectral components of all the primitive components of $\mathcal{S}$.

3. **Compute $\widehat{\Sigma}(\mathbf{k})$ for $\mathbf{k} \in \mathbb{Z}^d$**: we have $\widehat{\Sigma}(\mathbf{0}) = \sum_{\alpha\in\mathcal{A}} u_\alpha \mathbf{e}_{\alpha\alpha}$ where $u_\alpha = \mu([\alpha])$ as above. The first equality in the Fourier recursion theorem 3.1.2 for $p \in \mathbb{N}$ and $\mathbf{k} \in \mathbb{Z}^d$ gives

$$\widehat{\Sigma}(\mathbf{k}) = \frac{1}{Q^p}\sum_{\mathbf{j}\in[\mathbf{0},\mathbf{q}^p)} \mathcal{R}_{\mathbf{j}}^{(p)} \otimes \mathcal{R}_{\mathbf{j}+\mathbf{k}}^{(p)}\, \widehat{\Sigma}(\lfloor\mathbf{k}\rfloor_p + \mathfrak{c}_p(\mathbf{j},\mathbf{k}))$$

and allows one to solve for $\widehat{\Sigma}(\mathbf{c})$ algebraically for $\mathbf{c} \in [-\mathbf{1},\mathbf{1}] \smallsetminus \mathbf{0}$, a process we now describe. First, note that for $m > 0$ and $\mathbf{j} \in [\mathbf{0}, \mathbf{q}^m)$ the $\mathbb{Z}^d$ integer $\lfloor\mathbf{c}\rfloor_m + \mathfrak{c}_m(\mathbf{j},\mathbf{c})$ will lie in the smallest rectangle containing $\mathbf{0}$ and $\mathbf{c}$, as $\mathbf{c}$ cannot force $\mathbf{j}$ to carry in any direction where $(\mathbf{c})_i = 0$. Recall that $\mathbf{1}_i$ is the i-th coordinate vector in $\mathbb{Z}^d$ and so if $\mathbf{j} \in [\mathbf{0}, \mathbf{q}^p)$ then $\lfloor\mathbf{1}_i\rfloor_p + \mathfrak{c}_p(\mathbf{j}, \mathbf{1}_i) = \mathbf{0}$ or $\mathbf{1}_i$ and we obtain

$$\widehat{\Sigma}(\mathbf{1}_i) = \frac{1}{Q^p}\sum_{\mathbf{j}\in[\mathbf{0},\mathbf{q}^p)\smallsetminus\Delta_p(\mathbf{1}_i)} \mathcal{R}_{\mathbf{j}}^{(p)} \otimes \mathcal{R}_{\mathbf{j}+\mathbf{1}_i}^{(p)}\, \widehat{\Sigma}(\mathbf{0}) + \frac{1}{Q^p}\sum_{\mathbf{j}\in\Delta_p(\mathbf{1}_i)} \mathcal{R}_{\mathbf{j}}^{(p)} \otimes \mathcal{R}_{\mathbf{j}+\mathbf{1}_i}^{(p)}\, \widehat{\Sigma}(\mathbf{1}_i)$$

from which one can solve for $\widehat{\Sigma}(\mathbf{1}_i)$ and obtain

$$\widehat{\Sigma}(\mathbf{1}_i) = \Big(Q^p\mathbf{I} - \sum_{\mathbf{j}\in\Delta_p(\mathbf{1}_i)} \mathcal{R}_{\mathbf{j}}^{(p)} \otimes \mathcal{R}_{\mathbf{j}+\mathbf{1}_i}^{(p)}\Big)^{-1} \sum_{\mathbf{j}\in[\mathbf{0},\mathbf{q}^p)\smallsetminus\Delta_p(\mathbf{1}_i)} \mathcal{R}_{\mathbf{j}}^{(p)} \otimes \mathcal{R}_{\mathbf{j}+\mathbf{1}_i}^{(p)} \widehat{\Sigma}(\mathbf{0}) \tag{4.1}$$

where the inverse matrix above exists for *some* $p > 0$ as the frequency of carries goes to 0 and thus the spectral radius of the sum over $\Delta_p(\mathbf{1}_i)$ as well. In this way, we compute $\widehat{\Sigma}(\mathbf{c})$ for $\mathbf{c} = \mathbf{0}, \mathbf{1}_i$ for $1 \leqslant i \leqslant d$. For $-\mathbf{1}_i$, one can use the identities (2.1) for negative arithmetic and solve for $\widehat{\Sigma}(-\mathbf{1}_j)$ similarly to the above. Now, by considering those $\mathbf{c}$ of the form $\pm\mathbf{1}_i \pm \mathbf{1}_j$, the above recursion expresses $\widehat{\Sigma}(\pm\mathbf{1}_i \pm \mathbf{1}_j)$ in terms of $\widehat{\Sigma}(\mathbf{0}), \widehat{\Sigma}(\pm\mathbf{1}_i), \widehat{\Sigma}(\pm\mathbf{1}_j)$, and $\widehat{\Sigma}(\pm\mathbf{1}_i \pm \mathbf{1}_j)$ and again we can solve for $\widehat{\Sigma}(\pm\mathbf{1}_i \pm \mathbf{1}_j)$ for all i and j distinct. Continuing to $\pm\mathbf{1}_i \pm \mathbf{1}_j \pm \mathbf{1}_k$ for distinct i, j, k, then combinations of 4 distinct ones, etc, until all the values of $\widehat{\Sigma}$ on $[-\mathbf{1}, \mathbf{1}]$ are known. Finally, the Fourier coefficients for all other $\mathbf{k} \in \mathbb{Z}^d \smallsetminus [-\mathbf{1}, \mathbf{1}]$ can then be computed explicitly using the Fourier recursion by taking $p \geqslant \mathfrak{p}(\mathbf{k})$. Note that $\widehat{\Sigma}(-\mathbf{k}) = \widehat{\Sigma}(\mathbf{k})$ for all $\mathbf{k} \in \mathbb{Z}^d$ by equation (3.3).

4. **Determine extremal rays of spectral hull**: using the instruction products $\mathcal{R}_j \otimes \mathcal{R}_j$ recompute the coincidence matrix $C_{\mathcal{S}} = M_{\mathcal{S}\otimes\mathcal{S}}$ and, as in step (2) above, compute the primitive reduced form of the bisubstitution. Recall from definition 3.2.4 that the spectral hull is given by

$$\mathcal{K}(\mathcal{S}) := \{\mathbf{v} \in \mathbb{C}^{\mathcal{A}^2} : C_{\mathcal{S}}^t \mathbf{v} = Q\mathbf{v} \text{ and } \mathbf{v} \geqq 0\}$$

We begin by identifying the left Q-eigenspace of $C_{\mathcal{S}}$, recalling that $\mathcal{S}$ has been telescoped so that $\mathcal{S} \otimes \mathcal{S}$ has index of imprimitivity 1. For the ergodic decomposition $\mathcal{F}$ of $\mathcal{S} \otimes \mathcal{S}$ (proposition 1.2.2) consider the projections $\mathcal{P}_{\mathcal{F}}$ and $\mathcal{P}_{\mathcal{T}}$ onto the spans of the ergodic pairs $\alpha\beta$ in some $\mathcal{F}_j$, and transient pairs $\gamma\delta \in \mathcal{T}$, respectively, and so they are diagonal $0, 1$ matrices satisfying $\mathcal{P}_{\mathcal{F}} + \mathcal{P}_{\mathcal{T}} = \mathbf{I}$, the identity; write $C_{\mathcal{T}} = \mathcal{P}_{\mathcal{T}} C_{\mathcal{S}}$. The reduced normal form (1.2) for $C_{\mathcal{S}}$ gives ($\approx$ indicates permutation equivalent)

$$C_{\mathcal{S}}^t \approx \begin{bmatrix} C_{1,1} & & & \\ & \ddots & & \\ & & C_{J,J} & \\ C_{\mathcal{T},1} & \cdots & C_{\mathcal{T},J} & C_{\mathcal{T},\mathcal{T}} \end{bmatrix} \qquad \text{so that} \qquad C_{\mathcal{T}}^t = C_{\mathcal{S}}^t \mathcal{P}_{\mathcal{T}} = \begin{bmatrix} \mathbf{0} & & & \\ & \ddots & & \\ & & \mathbf{0} & \\ \mathbf{0} & \cdots & \mathbf{0} & C_{\mathcal{T},\mathcal{T}} \end{bmatrix}$$

for the transpose coincidence matrix, where everything not appearing on the block diagonal or bottom block row are zero. The square blocks along the diagonal are primitive, barring the last as it represents the transient pairs and so need not be. Unlike step (2), we are now dealing with a *row* stochastic matrix $C_{\mathcal{S}}^t = M_{\mathcal{S}\otimes\mathcal{S}}^t$, which remains row Q-stochastic under permutations of the basis, as in the above. Thus, when restricted to an ergodic class, any Q-eigenvector of $C_{\mathcal{S}}^t$ must be constant as those diagonal blocks are primitive and stochastic. Write $\vec{E} := \sum_{\gamma\delta\in E} \mathbf{e}_{\gamma\delta} \in \mathbb{C}^{\mathcal{A}^2}$ for the representation of $E \subset \mathcal{A}^2$ in $\mathbb{C}^{\mathcal{A}^2}$. For $w_1, \ldots, w_J \in \mathbb{C}$, we write (for $\mathcal{F}_1, \ldots, \mathcal{F}_J$ are the ergodic classes of $\mathcal{S} \otimes \mathcal{S}$)

$$\mathcal{V}_{\mathcal{F}} := \mathcal{V}_{\mathcal{F}}(w_1, \ldots, w_J) := \textstyle\sum_{j=1}^J w_j \vec{\mathcal{F}}_j \in \mathbb{C}^{\mathcal{A}^2} \tag{4.2}$$

then $\mathcal{V}_{\mathcal{F}}$ represents an arbitrary Q-eigenvector of $C_{\mathcal{S}}^t$ *restricted to its ergodic classes*, or $\mathcal{P}_{\mathcal{F}} C_{\mathcal{S}}^t \mathcal{P}_{\mathcal{F}}$. If there is no transient part (as is the case for bijective substitutions), then $\mathcal{P}_{\mathcal{F}} = \mathbf{I}$ and the above determines the left Q-eigenspace of $C_{\mathcal{S}}$; otherwise the transient class is *nonempty* and $\mathcal{P}_{\mathcal{T}} \neq \mathbf{0}$. The following lemma allows us to characterize the left Q-eigenspace of the coincidence matrix; compare [22, Proposition 10.2].

Lemma 4.0.3. *For a* $\mathbf{q}$*-substitution* $\mathcal{S}$ *on* $\mathcal{A}$ *with ergodic decomposition* $\mathcal{F}$*, then*

$$C_{\mathcal{S}}^t \mathbf{v} = Q\mathbf{v} \qquad \Longleftrightarrow \qquad \mathbf{v} = \mathcal{V}_{\mathcal{F}} - (Q\mathbf{I} - C_{\mathcal{T}}^t)^{-1}(Q\mathbf{I} - C_{\mathcal{S}}^t)\mathcal{V}_{\mathcal{F}}$$

Remark: lemma 1.2.3 shows that the spectral radius of $C_{\mathcal{T}}$ is less than Q so the matrix inverse exists, so the above is well defined.

Proof. Let $\mathbf{v}_{\mathcal{T}} := \mathcal{P}_{\mathcal{T}}\mathbf{v}$. By the above discussion, $C_{\mathcal{S}}^t \mathbf{v} = Q\mathbf{v}$ if and only if $\mathbf{v} = \mathcal{V}_{\mathcal{F}} + \mathbf{v}_{\mathcal{T}}$. Then

$$C_{\mathcal{S}}^t \mathbf{v} = Q\mathbf{v} \qquad \Longleftrightarrow \qquad (C_{\mathcal{S}}^t - Q\mathbf{I})\mathcal{V}_{\mathcal{F}} = (Q\mathbf{I} - C_{\mathcal{S}}^t)\mathbf{v}_{\mathcal{T}} = (Q\mathbf{I} - C_{\mathcal{T}}^t)\mathbf{v}_{\mathcal{T}}$$

as $C_{\mathcal{S}}^t \mathcal{P}_{\mathcal{T}} = C_{\mathcal{T}}^t$ and $\mathbf{v}_{\mathcal{T}} = \mathcal{P}_{\mathcal{T}}\mathbf{v}$, and the result follows from invertibility of $Q\mathbf{I} - C_{\mathcal{T}}^t$. □

Now that we have characterized the left Q-eigenspace of the coincidence matrix, we discuss how to impose the strong semipositivity condition, and this is in general the most difficult step, as it requires finding conditions on the coefficients of a matrix function of several variables guaranteeing positive semidefiniteness. In most cases, the strategy involves diagonalizing and then imposing nonnegativity of eigenvalues. In the examples, several strategies are used and we summarize them here. Given $\mathbf{v}$ satisyfing the identity of lemma 4.0.3:

- Perform simultaneous row and column operations to diagonalize, then enforce $\lambda \geqslant 0$; used in Queffélec's example 4.1.2

- Enforce nonnegativity of principal minors; used in the Table example 4.1.3

- The matrices $\mathring{\vec{\mathcal{F}}}_{\mathbf{j}}$ representing the ergodic classes and $(Q\mathbf{I} - C_{\mathcal{T}}^t)^{-1}(Q\mathbf{I} - C_{\mathcal{S}}^t)$ commute, and can be simultaneously diagonalized, then enforce $\lambda \geqslant 0$; used in both the Thue-Morse and Rudin-Shapiro examples 4.1.1 and 4.2.1, respectively

- Using proposition 4.3.1 and properties of Kronecker products to reduce the problem; this is illustrated in §4.3 and Baake-Gahler-Grimm's example 4.3.2

- Using Python's *sympy* toolbox to diagonalize symbolic matrices, then enforce positivity; used in the nontrivial height example 4.4.2

Note that the extremal rays lie on the intersection of planes forming the faces of $\mathcal{K}$, and so the spectral hull can be obtained by solving systems of linear inequalities in the coefficients w_j of (4.2). Thus, one can find the elements of $\mathcal{K}^*$ by intersecting the hyperplane $\mathbf{v}^t\widehat{\Sigma}(\mathbf{0})$ with different combinations of the planes determined by the 0-eigenvalue conditions on $\mathring{\mathbf{v}}$.

5. **Compute $\lambda_{\mathcal{K}^*}$ and $\sigma_{\mathbf{max}}$**: with the correlation measures computed in step 3, and the spectral hull and its extremal rays identified in step 4, we are ready to identify the spectral hull. For each $\mathbf{w} \in \mathcal{K}^*$ compute

$$\widehat{\lambda_{\mathbf{w}}}(\mathbf{k}) = \mathbf{w}^t\widehat{\Sigma}(\mathbf{k}) := \textstyle\sum_{\alpha\beta\in\mathcal{A}^2} w_{\alpha\beta}\widehat{\sigma_{\alpha\beta}}(\mathbf{k})$$

for several $\mathbf{k} \in [\mathbf{0}, \mathbf{q})$. As these measures are pure, $\mathbf{q}$-shift ergodic types, $\lambda_{\mathbf{w}}$ falls into one of three categories depending on its Fourier coefficients:

- If $\widehat{\lambda_{\mathbf{w}}}(\mathbf{k})$ is periodic in $\mathbf{k}$, then $\lambda_{\mathbf{w}}$ is pure discrete; typically discrete measures have *almost periodic* Fourier coefficients, but here the discrete spectrum is supported on rational coordinates in $\mathbb{R}^d/\mathbb{Z}^d \simeq \mathbb{T}^d$ and hence the coefficients are purely periodic. Our nontrivial height example 4.4.2 illustrates this case. In the trivial height case, one can detect pure discreteness via the spectral hull, as there will be only one ergodic class, and so $\mathcal{K}^* = \{\mathbf{1}_{\mathcal{A}^2}\}$.

- If $\widehat{\lambda_{\mathbf{w}}}(\mathbf{k}) = \mathbf{0}$ for all $\mathbf{k} \neq \mathbf{0}$, then $\lambda_{\mathbf{w}}$ is Lebesgue measure on $\mathbb{T}^d$. As this requires us to compute all nonzero $\widehat{\Sigma}(\mathbf{k})$ to verify, one might hope for an upper bound on the number of coefficients that need to be checked in order to guarantee Lebesgue measure. We know of no q-substitution (the $\mathbb{Z}$ case) singular to Lebesgue component for which all of the first $q + 1$ positive Fourier coefficients vanish, although we have not examined this in detail.

- Otherwise, $\lambda_{\mathbf{w}}$ is purely singular continuous, by the above characterization of the discrete component, and corollary 3.3.3. As mentioned in the discussion following corollary 3.3.3, the measures $\lambda_{\mathbf{w}}$ are always strongly mixing of all orders for the $\mathbf{q}$-shift and, in the $\mathbb{Z}$ case, the singular continuous measures have no gaps in their support.

Finally, repeating the above for every $\mathbf{w} \in \mathcal{K}^*$, one obtains the entire maximal spectral type by taking any positive linear combination of the measures $\boldsymbol{\omega}_{\mathbf{q}} * \lambda_{\mathbf{w}}$ for $\mathbf{w} \in \mathcal{K}^*$. Note that Queffélec considers the question of spectral multiplicity in [22, §11.2.1] but we have not analyzed these results in our context. We now work out several examples, computing the spectrum and showing singularity to Lebesgue when possible.

4.1 Substititutions with Purely Singular Spectrum

Our first example is classical, and comes from the Thue-Morse sequence. As its spectrum is well established, it serves as a test case for the algorithm.

Example 4.1.1 (Thue-Morse)**.** Let τ denote the 2-substitution on $\mathcal{A} = \{0, 1\}$ given by

$$\tau : \begin{cases} 0 \longmapsto 01 \\ 1 \longmapsto 10 \end{cases} \qquad \text{with} \qquad M_\tau = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

with instructions $\mathcal{R}_0, \mathcal{R}_1$ appearing consecutively in the above sum. Clearly, τ is primitive, and as $\tau^2(0) = 0110$ with $\mathcal{R}_0$ the identity, the symbol 1 can be proceeded by both symbols $0, 1$ so that it is aperiodic by Pansiot's lemma. The Perron vector of τ is $(\frac{1}{2}, \frac{1}{2})^t$ so that $\widehat{\Sigma}(0) = \frac{1}{2} \sum_{\gamma \in \mathcal{A}} \mathbf{e}_{\gamma\gamma}$. As $q = 2$, we have $\Delta_1(1) = \{1\}$ and so equation (4.1) gives

$$\widehat{\Sigma}(1) = (2\mathbf{I} - \mathcal{R}_1 \otimes \mathcal{R}_0)^{-1} \mathcal{R}_0 \otimes \mathcal{R}_1 \widehat{\Sigma}(0) = \frac{1}{6}(1, 2, 2, 1)^t$$

where the basis is given the lexicographic order $00, 01, 10, 11$.

Now we compute the ergodic classes of the bisubstitution: as τ is primitive, $\mathcal{F}_1 = \{00, 11\}$, and one checks that $\mathcal{F}_2 = \{01, 10\}$ is also a minimal orbit of the instructions $\mathcal{R}_j \otimes \mathcal{R}_j$. As $\mathcal{A} = \mathcal{F}_1 \sqcup \mathcal{F}_2$, these form the ergodic classes of $\tau \otimes \tau$ with empty transient part. By lemma 4.0.3, $\mathbf{v} \in \mathcal{K}$ is given by $\mathbf{v} = (w_1, w_2, w_2, w_1)^{\mathrm{t}}$ and as the matrices $\mathring{\mathcal{F}}_1$ and

$\mathring{\mathcal{F}}_2$ commute, they can be simultaneously diagonalized, and we obtain

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \mathring{\mathbf{v}} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} w_1 - w_2 & \\ & w_1 + w_2 \end{pmatrix}$$

so that $\mathring{\mathbf{v}} \geqslant 0$ implies $-w_1 \leqslant w_2 \leqslant w_1$. Letting $w_1 = 1$ gives $-1 \leqslant w_2 \leqslant 1$ and $\mathcal{K}^*$ is the pair

$$\mathbf{v}_1 = \mathcal{F}_1 + \mathcal{F}_2 = (1,1,1,1)^t \quad \text{and} \quad \mathbf{v}_2 = \mathcal{F}_1 - \mathcal{F}_2 = (1,-1,-1,1)^t$$

Thus, $\lambda_{\mathbf{v}_1} = \sum_{\alpha\beta\in\mathcal{A}^2} \sigma_{\alpha\beta} = \boldsymbol{\delta}_1$, the Dirac-Delta mass at 1 (as $\widehat{\boldsymbol{\delta}}_1(\mathbf{k}) = 1$ for all $\mathbf{k}$), and

$$\widehat{\lambda_{\mathbf{v}_2}}(1) = \mathbf{v}_2^t \widehat{\Sigma}(1) = -\frac{1}{3} \neq 0$$

so that $\lambda_{\mathbf{v}_2}$ is not Lebesgue measure, and Thue-Morse has purely singular spectrum. Note $\lambda_{\mathbf{v}_1} = \boldsymbol{\delta}_1$ gives rise to the discrete component, $\lambda_{\mathbf{v}_2}$ the singular continuous component, and

$$\sigma_{\max} \sim \boldsymbol{\omega}_2 + \boldsymbol{\omega}_2 * \lambda_{\mathbf{v}_2}$$

and so the spectrum of purely singular as expected.

An interesting question is the importance of commutativity. Our next example, due to Queffélec, is bijective but not commutative. In [22, Examples 9.3, 10.2.2.3, and 11.1.2.3] it was shown to have Lebesgue spectrum, however there were errors in the analysis.

Example 4.1.2 (Queffélec's ζ)**.** Let ζ be the 3-substitution on $\mathcal{A} = \{0,1,2\}$ given by

$$\zeta : \begin{cases} 0 \longmapsto 001 \\ 1 \longmapsto 122 \\ 2 \longmapsto 210 \end{cases} \qquad \text{with} \qquad M_\zeta = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} + \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix} + \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

with the instruction matrices $\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2$ appearing sequentially above. As M_ζ^2 is positive, ζ is primitive. As $\mathcal{R}_0$ is the identity and as 1 can be followed by $0, 1$ and 2 in $\zeta^2(0)$, Pansiot's lemma applies and shows that ζ is aperiodic. One checks that the Perron vector of M_ζ is

$(\frac{1}{3},\frac{1}{3},\frac{1}{3})$, giving $\widehat{\Sigma}(0)=\frac{1}{3}\sum_{\gamma\in\mathcal{A}}\mathbf{e}_{\gamma\gamma}$. Using (4.1), and noting that $\Delta_1(1)=\{2\}$ for $q=3$,

$$\begin{aligned}\widehat{\Sigma}(1)&=\Big(3\mathbf{I}-\textstyle\sum_{j\in\Delta_1(1)}\mathcal{R}_j^{(1)}\otimes\mathcal{R}_{j+1}^{(1)}\Big)^{-1}\textstyle\sum_{j\notin\Delta_1(1)}\mathcal{R}_j^{(1)}\otimes\mathcal{R}_{j+1}^{(1)}\widehat{\Sigma}(0)\\&=\big(3\mathbf{I}-\mathcal{R}_2\otimes\mathcal{R}_0\big)^{-1}\big(\mathcal{R}_0\otimes\mathcal{R}_1+\mathcal{R}_1\otimes\mathcal{R}_2\big)\widehat{\Sigma}(0)\\&=\frac{1}{39}\big(5,6,2,6,2,5,2,5,6\big)^t\end{aligned}$$

Here, we order the basis for $\mathcal{A}^2$ lexicographically $\big(00,01,02,10,11,12,20,21,22\big)$ which will be standard for the bialphabet. Computing $\widehat{\Sigma}(2)$ using $p=1$ in theorem 3.1.2 gives

$$\begin{aligned}\widehat{\Sigma}(2)&=\frac{1}{3}\sum_{j=0}^{2}\mathcal{R}_j\otimes\mathcal{R}_{j+2}\widehat{\Sigma}(\lceil j+2\rceil_1)\\&=\frac{1}{3}\mathcal{R}_0\otimes\mathcal{R}_0\widehat{\Sigma}(0)+\frac{1}{3}(\mathcal{R}_1\otimes\mathcal{R}_0+\mathcal{R}_2\otimes\mathcal{R}_1)\widehat{\Sigma}(1)\\&=\frac{1}{117}\big(7,7,25,25,7,7,7,25,7\big)^t\end{aligned}$$

Now, we compute the ergodic classes of the bisubstitution $\zeta\otimes\zeta$, and obtain the orbits $\mathcal{F}_1=\{00,11,22\}$ and $\mathcal{F}_2=\{01,10,12,21,02,20\}$ with empty transient class (as ζ is bijective), which form the ergodic classes of $\zeta\otimes\zeta$. Thus, by lemma 4.0.3

$$\mathbf{v}=w_1\vec{\mathcal{F}}_1+w_2\vec{\mathcal{F}}_2=\big(w_1,w_2,w_2,w_2,w_1,w_2,w_2,w_2,w_1\big)^t\quad\text{or equivalently}\quad\mathring{\mathbf{v}}=\begin{pmatrix}w_1&w_2&w_2\\w_2&w_1&w_2\\w_2&w_2&w_1\end{pmatrix}$$

Performing simultaneous row and column operations, we arrive at

$$\begin{pmatrix}w_1&0&0\\0&\frac{w_1^2-w_2^2}{w_1}&0\\0&0&\frac{w_1^2-w_2w_1-2w_2^2}{w_1+w_2}\end{pmatrix}\quad\text{and }\mathring{\mathbf{v}}\gg 0\text{ implies}\quad\begin{cases}w_1\geqslant 0,\\(w_1+w_2)(w_1-w_2)\geqslant 0,\\(w_1-w_2)(w_1+2w_2)\geqslant 0\end{cases}$$

so that $w_1\geqslant 0$ and $-\frac{1}{2}w_1\leqslant w_2\leqslant w_1$, giving $\mathcal{K}$ two extremal rays determined by the vectors

$$\mathbf{v}_1=\vec{\mathcal{F}}_1+\vec{\mathcal{F}}_2=\mathbf{1},\qquad\text{and}\qquad\mathbf{v}_2=\vec{\mathcal{F}}_1-\frac{1}{2}\vec{\mathcal{F}}_2=\big(1,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},1,-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},1\big)^t$$

One checks that $\lambda_{\mathbf{v}_1}=\boldsymbol{\delta}_1$, and that $\widehat{\lambda_{\mathbf{v}_2}}(1)=0$, and thus $\widehat{\lambda_{\mathbf{v}_2}}(3a)=0$ for all a by 3-shift invariance. Using the value of $\widehat{\Sigma}(2)$ above, however, we obtain

$$\widehat{\lambda_{\mathbf{v}_2}}(2) = \frac{1}{117}\left(1\big(7\cdot 3\big) - \frac{1}{2}\big(7\cdot 3 + 25\cdot 3\big)\right) \neq 0,$$

so that $\lambda_{\mathbf{v}_2}$ is *not* Lebesgue measure. As $\sigma_{\max}(\zeta) = \boldsymbol{\omega}_3 * (\boldsymbol{\delta}_1 + \lambda_{\mathbf{v}_2})$, the spectrum of ζ is purely singular to Lebesgue spectrum on the circle $\mathbb{T}$, as both $\boldsymbol{\omega}_3$ and $\boldsymbol{\omega}_3 * \lambda_{\mathbf{v}_1} \perp \mathbf{m}$.

Note that ζ is an example of a bijective substitution, as all of its instructions are bijections of $\mathcal{A}$. Correcting this mistake of Queffélec's is significant, as it represented the only known example of a bijective substitution with Lebesgue component. Using software to automate the above algorithm, we have excluded Lebesgue component from the spectrum of all bijective substitutions of constant length (the $\mathbb{Z}$ case) $2, 3, 4$, and 5 on alphabets of $2, 3, 4$, and 5 letters. Together with theorem 3.4.1, this suggests the possibility that all aperiodic bijective $\mathbf{q}$-substitution may have spectrum singular to Lebesgue measure.

Our first example with $d > 1$ comes from a substitution tiling system in the plane known as the Table. In [25], Robinson described it as a substitution on 4 symbols in $\mathbb{Z}^2$.

Example 4.1.3 (The Table)**.** Let $\mathcal{T}$ be the $(2,2)$-substitution on $\{0,1,2,3\}$ given by

$$\mathcal{T}: \qquad 0 \mapsto \begin{matrix} 3\,0 \\ 1\,0 \end{matrix} \qquad 1 \mapsto \begin{matrix} 1\,1 \\ 0\,2 \end{matrix} \qquad 2 \mapsto \begin{matrix} 2\,3 \\ 2\,1 \end{matrix} \qquad 3 \mapsto \begin{matrix} 0\,2 \\ 3\,3 \end{matrix}$$

with substitution matrix $M_{\mathcal{T}} = \mathcal{R}_{(0,0)} + \mathcal{R}_{(1,0)} + \mathcal{R}_{(0,1)} + \mathcal{R}_{(1,1)}$ summed consecutively below:

$$M_{\mathcal{T}} = \begin{pmatrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix} + \begin{pmatrix} & & & 1 \\ & 1 & & \\ & & 1 & \\ 1 & & & \end{pmatrix} + \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{pmatrix} = \begin{pmatrix} 2 & 1 & & 1 \\ 1 & 2 & 1 & \\ & 1 & 2 & 1 \\ 1 & & 1 & 2 \end{pmatrix}$$

As $M_{\mathcal{T}}^2 > 0$, $\mathcal{T}$ is primitive, and aperiodicity follows from recognizability or theorem 2.3.2, see [25]. Note that $R_{(0,0)}$ and $R_{(0,1)}$ do not commute, so that $\mathcal{T}$ is not a commutative substitution, and theorem 3.4.1 does not apply. One checks that the Perron vector of $M_{\mathcal{T}}$ is $\mathbf{u} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})^t$ and so $\widehat{\Sigma}(0,0) = \frac{1}{4}\sum_{\gamma\in\mathcal{A}} \mathbf{e}_{\gamma\gamma}$. Using (4.1), we obtain (with $\mathcal{A}^2$ in the lexicographic order)

$$\widehat{\Sigma}(1,0) = \frac{1}{20}\big(0,2,1,2,0,2,2,1,5,0,0,0,0,1,2,2\big)^t$$

We now compute its spectral hull: one checks (by looking at letter orbits) that the ergodic classes of the bisubstitution are:

$$\mathcal{F}_1 = \{00, 11, 22, 33\} \quad \text{and} \quad \mathcal{F}_2 = \{01, 02, 03, 10, 12, 13, 20, 21, 23, 30, 31, 32\}$$

which partition $\mathcal{A}^2$ completely, leaving no transient part as is always the case for bijective substitutions. Using lemma 4.0.3, we have $\mathbf{v} \in \mathcal{K}$ if and only if

$$\mathring{\mathbf{v}} = \begin{pmatrix} w_1 & w_2 & w_2 & w_2 \\ w_2 & w_1 & w_2 & w_2 \\ w_2 & w_2 & w_1 & w_2 \\ w_2 & w_2 & w_2 & w_1 \end{pmatrix} \gg 0 \quad \Longrightarrow \quad \begin{cases} w_1 \geqslant 0 \\ (w_1 + w_2)(w_1 - w_2) \geqslant 0 \\ w_1(w_1 + w_2)(w_1 - w_2) \geqslant 0 \\ (w_1 - w_2)^3(w_1 + 3w_2) \geqslant 0 \end{cases}$$

as $\mathring{\mathbf{v}} \gg 0$ if and only if its principal minors are positive definite. These inequalities show us that $\mathcal{K}$ has two extremal rays determined by

$$\mathbf{v}_1 = \mathcal{F}_1 + \mathcal{F}_2 \quad \text{and} \quad \mathbf{v}_2 = \mathcal{F}_1 - \frac{1}{3}\mathcal{F}_2$$

and so, along with the above, we have

$$\widehat{\lambda_{\mathbf{v}_1}}((1,0)) = 1 \quad \text{and} \quad \widehat{\lambda_{\mathbf{v}_2}}((1,0)) = \mathbf{v}_2^t \widehat{\Sigma}((1,0)) = -\frac{1}{15},$$

so that none of the measures coming from the spectral hull are Lebesgue, and thus the spectrum of the Table is singular to Lebesgue measure on $\mathbb{T}^2$, the two-torus.

We now discuss a collection of (non-bijective) $\mathbf{q}$-substitutions, due to Frank [13] and based on the Rudin-Shapiro substitution, with Lebesgue component in their spectrum.

4.2 Substitutions with Lebesgue Spectral Component

Although the spectrum of the Rudin-Shapiro substitution is well known, it is interesting to see how the details work out allowing for all the terms $\mathbf{v}^*\widehat{\Sigma}(\mathbf{k})$ for $\mathbf{k} \neq \mathbf{0}$ to vanish.

Example 4.2.1 (Rudin-Shapiro)**.** Let ρ be the 2-substitution on $\{0, 1, 2, 3\}$

$$\rho : \begin{cases} 0 \longmapsto 02 \\ 1 \longmapsto 32 \\ 2 \longmapsto 01 \\ 3 \longmapsto 31 \end{cases} \quad \text{with} \quad M_\rho = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

As $M_\rho^3 > 0$, the Rudin-Shapiro substitution ρ is primitive, and as $\rho^2(0) = 0201$ with $\rho(0)_0 = 0$ the symbol 0 can be proceeded by both the symbols 1 and 2, it follows from Pansiot's lemma that ρ is aperiodic. As $\frac{1}{2}M_\rho$ is row and column stochastic, its Perron vector is $\frac{1}{2}(1,1,1,1)^t$ and so $\widehat{\Sigma}(0) = \sum_{\gamma\in\mathcal{A}} \frac{1}{2}\mathbf{e}_{\gamma\gamma}$. As $q = 2$, $\Delta_1(1) = \{1\}$ and (4.1) gives

$$\widehat{\Sigma}(1) = (2\mathbf{I} - \mathcal{R}_1 \otimes \mathcal{R}_0)^{-1}\mathcal{R}_0 \otimes \mathcal{R}_1\widehat{\Sigma}(0) = \frac{1}{8}(0,1,1,0,1,0,0,1,1,0,0,1,0,1,1,0)^t$$

with the basis ordered lexicographically: $00, 01, 02, 03, 10, 11, 12, 13, 20, 21, 22, 23, 30, 31, 32, 33$. Using theorem 3.1.2 for $k = 2$ and $p = 1$ we obtain

$$\widehat{\Sigma}(2) = \frac{1}{2}(\mathcal{R}_0 \otimes \mathcal{R}_0 + \mathcal{R}_1 \otimes \mathcal{R}_1)\widehat{\Sigma}(0) = \frac{1}{8}(1,0,0,1,0,1,1,0,0,1,1,0,1,0,0,1)^t$$

and one checks $\widehat{\Sigma}(2n) = \widehat{\Sigma}(2)$ for $n \neq 0$ and $\widehat{\Sigma}(2n+1) = \widehat{\Sigma}(1)$ for $n \in \mathbb{Z}$. Note $\widehat{\Sigma}(1) \perp \widehat{\Sigma}(2)$.

We now compute the ergodic decomposition of $\rho \otimes \rho$, the bisubstitution. As usual for primitive substitutions, $\mathcal{F}_1 = \{00, 11, 22, 33\}$, and in this case the only other ergodic class is $\mathcal{F}_2 = \{03, 12, 21, 30\}$, so that $\mathcal{T} = \{01, 02, 10, 13, 20, 23, 31, 32\}$ is the transient part. Using lemma 4.0.3 we have $\mathbf{v} \in \mathcal{K}$ if and only if

$$\mathring{\mathbf{v}} = w_1 \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} + w_2 \begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix} + \frac{1}{2}(w_1 + w_2) \begin{pmatrix} & 1 & 1 & \\ 1 & & & 1 \\ 1 & & & 1 \\ & 1 & 1 & \end{pmatrix} \gg 0$$

as the above matrices commute and are diagonalizable, one checks that for

$$S = \begin{pmatrix} 1 & 0 & -1 & 1 \\ -1 & -1 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \quad \text{we have} \quad S^{-1}\mathring{\mathbf{v}}S = \begin{pmatrix} 0 & & & \\ & w_1 - w_2 & & \\ & & w_1 - w_2 & \\ & & & 2w_1 + 2w_2 \end{pmatrix}$$

so that $\mathbf{v}$ is strongly semipositive if and only if $-w_1 \leqslant w_2 \leqslant w_1$, the extreme points of which are given by the vectors $(w_1, w_2) = (1,1)$ or $(1,-1)$. Thus, the extremal rays of $\mathcal{K}$ are

$$\mathbf{v}_1 = \mathbf{1} \quad \text{and} \quad \mathbf{v}_2 = (1,0,0,-1,0,1,-1,0,0,-1,1,0,-1,0,0,1)^t$$

As usual, $\lambda_{\mathbf{v}_1} = \boldsymbol{\delta}_1$, and using the computed values of $\widehat{\Sigma}(k)$, one checks that $\widehat{\lambda_{\mathbf{v}_2}}(k) = 0$ for $k \neq 0$, and so $\lambda_{\mathbf{v}_2}$ is Lebesgue measure. Thus, $\sigma_{\max} \sim \boldsymbol{\omega}_2 + m$ as m is q-shift invariant.

We now describe Frank's generalizations of Rudin-Shapiro to the $\mathbb{Z}^d$ setting. Consider the alphabet $\mathcal{A} = \{1, \ldots, Q, -1, \ldots, -Q\}$ on $2Q$ letters and the collection of instructions

$$\mathcal{F}_Q := \big\{\mathcal{R} : \mathcal{A} \to \mathcal{A} \text{ so that } \exists\ \gamma \text{ with } \mathcal{R} : \mathcal{A} \to \{\gamma, -\gamma\} \text{ and } \mathcal{R}(-\alpha) = -\mathcal{R}(\alpha)\ \forall\ \alpha \in \mathcal{A}\big\}$$

so that $\mathcal{F}_Q$ consists of instructions which are morphisms for negation and take on exactly one value in $1, \ldots, Q$. If $\mathcal{R} \in \mathcal{F}_Q$ takes on the values γ and $-\gamma$, its *sign vector* is a ± 1 vector in $\mathbb{C}^Q$ whose α component is 1 if $\mathcal{R}(\alpha) = \gamma$ and -1 if $\mathcal{R}(\alpha) = -\gamma$. As instructions in $\mathcal{F}_Q$ preserve negation, every $\mathcal{R} \in \mathcal{F}_Q$ is determined by a letter in $1, \ldots, Q$ and a sign vector. For example, if $Q = 4$, the letter 3 and sign vector $(+1, -1, -1, +1)$ determine the instruction

$$\mathcal{R} : \begin{cases} \ \ 1 \mapsto \ \ 3, & \ \ 2 \mapsto -3, \quad \ \ 3 \mapsto -3, \quad \ \ 4 \mapsto \ \ 3 \\ -1 \mapsto -3, & -2 \mapsto \ \ 3, \quad -3 \mapsto \ \ 3, \quad -4 \mapsto -3 \end{cases}$$

A *Hadamard matrix* is a square ± 1-matrix whose rows (and columns) are orthogonal, and are necessarily even dimensional (more generally, entries are n-th roots of unity, and dimension is divisible by n). Every $Q \times Q$ Hadamard matrix determines Q instructions in $\mathcal{F}_Q$ in the following way: as the rows of a Hadamard matrix are sign vectors, the i-th column paired with the letter i in $\{1, \ldots, Q\}$ gives an instruction via the above association. For example $Q = 2$ corresponds to the Rudin-Shapiro substitution whose Hadamard matrix and corresponding instructions are represented by

$$\mathbf{H} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{with} \quad \mathcal{R}_0 : \begin{cases} \ \ 1 \mapsto \ \ 1, & \ \ 2 \mapsto -1 \\ -1 \mapsto -1, & -2 \mapsto \ \ 1 \end{cases} \quad \text{and} \quad \mathcal{R}_1 : \begin{cases} \ \ 1 \mapsto -2, & \ \ 2 \mapsto \ \ 2 \\ -1 \mapsto \ \ 2, & -2 \mapsto -2 \end{cases}$$

For a Hadamard matrix $\mathbf{H}$, let $\mathcal{I}(\mathbf{H})$ be the corresponding instructions induced by $\mathbf{H}$. If $\mathbf{q} > \mathbf{1}$ has expansion Q, any configuration $\mathcal{R} : [\mathbf{0}, \mathbf{q}) \to \mathcal{I}(\mathbf{H})$ gives rise to a $\mathbf{q}$-substitution.

Theorem 4.2.2 (Frank)**.** *Let* $\mathbf{H}$ *be a* $Q \times Q$ *Hadamard matrix and* $\mathbf{q} > \mathbf{1}$ *in* $\mathbb{Z}^d$ *such that* $Q = q_1 \cdots q_d$. *Let* $\mathcal{R}$ *be any configuration on the instructions induced by* $\mathbf{H}$. *The* $\mathbf{q}$*-substitution determined by the configuration* $\mathcal{R}$ *has Lebesgue spectral components with multiplicity* Q.

As suggested in [13, §5.1], any configuration of the instructions $\mathcal{I}(\mathbf{H})$ give rise to substitutions in $\mathbb{Z}^d$ with Lebesgue spectrum. As convolution with $\boldsymbol{\omega}_\mathbf{q}$ has no effect on absolutely continuous spectrum, theorem 3.3.1 tells us that presence of Lebesgue component is determined by the spectral hull and the correlation vector, the first of which is already configuration invariant. This raises an interesting question: is presence of absolutely continuous component a configuration invariant in general? Note that a positive answer to this question allows us to classify the presence of Lebesgue component as a function of the collection of instructions (counted with multiplicity), or even just as substitutions in 1 dimension. If the configuration $\mathcal{R}$ gives a $\mathbf{q}$-substitution with Lebesgue component in its spectrum, then there is a $\mathbf{v}$ in the spectral hull for which $\widehat{\Sigma}(\mathbf{k})$ is orthogonal to $\mathbf{v}$ for all $\mathbf{k} \neq \mathbf{0}$. As a change in configuration is a relabeling of the indices of the instructions, one can use theorem 3.1.2 to compare correlation vectors for substitutions with equivalent configurations.

4.3 Substitutions with Every Spectral Component

Given the role the Kronecker product plays in our analysis, we take a moment to consider some of its properties. For example, it is an associative product, and while not commutative we still have $A \otimes B \approx B \otimes A$, where $\approx$ means *permutation equivalent:* there exists a $0, 1$ unitary matrix P with $P(A \otimes B)P^* = (B \otimes A)$, and this permutation depends only on the dimensions of A and B. Moreover, the spectrum of Kronecker products are closely related to their factors: if λ_i and μ_j are the eigenvalues of A and B, respectively, with corresponding eigenvectors $\mathbf{x}_i$ and $\mathbf{y}_j$, then the eigenvalues of $A \otimes B$ are the pairs $\lambda_i \mu_j$ corresponding to the eigenvector $\mathbf{x}_i \otimes \mathbf{y}_j$. This follows from the Kronecker products mixed product property:

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

As the instructions for a substitution product are the Kronecker products of the instructions of its factors, and ergodic classes depend only on the orbits of the generalized instructions, it follows from the mixed product property that the ergodic classes for the bisubstitution of a substitution product are the pairwise ergodic classes of the bisubstitutions of its factors, giving the following:

Proposition 4.3.1. *Let $\mathcal{S}$ and $\tilde{\mathcal{S}}$ be $\mathbf{q}$-substitutions on the alphabets $\mathcal{A}$ and $\tilde{\mathcal{A}}$. Let $\mathcal{E}$ and $\tilde{\mathcal{E}}$ denote the ergodic classes of their respective bisubstitutions, with transient classes $\mathcal{T}$ and*

$\tilde{\mathcal{T}}$. *Then the ergodic classes of the bisubstitution of* $\mathcal{S} \otimes \tilde{\mathcal{S}}$ *partition the alphabets* $\mathcal{E}_i \tilde{\mathcal{E}}_j$ *as* i, j *range over the indices for the respective bisubstitutions, and the transient part is* $\mathcal{A}\tilde{\mathcal{T}} \cup \mathcal{T}\tilde{\mathcal{A}}$.

Our last example is a substitution of constant length 2 on 8 symbols possessing all three pure types in its spectrum and allows us to illustrate an interesting property of the substitution product, see also [1, §2].

Example 4.3.2. Consider the Thue-Morse and Rudin-Shapiro substitutions of constant length 2 represented on the alphabets $\mathcal{A}_\tau = \{\overline{\ }, \underline{\ }\}$ and $\mathcal{A}_\rho = \{a, b, c, d\}$ respectively, by

$$\tau : \begin{cases} \overline{\ } \longmapsto \overline{\ }\,\underline{\ } \\ \underline{\ } \longmapsto \underline{\ }\,\overline{\ } \end{cases} \qquad \text{and} \qquad \rho : \begin{cases} a \longmapsto ac, \quad b \longmapsto dc \\ c \longmapsto ab, \quad d \longmapsto db \end{cases}$$

and consider the substitution $\mathcal{S}$ of constant length 2 on $\mathcal{A} = \{\bar{a}, \bar{b}, \bar{c}, \bar{d}, \underline{a}, \underline{b}, \underline{c}, \underline{d}\}$, given by

$$\mathcal{S} : \begin{cases} \bar{a} \longmapsto \bar{a}\underline{c}, \quad \bar{b} \longmapsto \bar{d}\underline{c}, \quad \bar{c} \longmapsto \bar{a}\underline{b}, \quad \bar{d} \longmapsto \bar{d}\underline{b} \\ \underline{a} \longmapsto \underline{a}\bar{c}, \quad \underline{b} \longmapsto \underline{d}\bar{c}, \quad \underline{c} \longmapsto \underline{a}\bar{b}, \quad \underline{d} \longmapsto \underline{d}\bar{b} \end{cases}$$

which is equivalent to the substitution product $\tau \otimes \rho$ via the obvious map $\mathcal{A}_\tau \mathcal{A}_\rho \to \mathcal{A}$. Note that it is aperiodic being the substitution product of aperiodic substitutions, and primitivity follows as $M_{\mathcal{S}}^5$ is positive. The Perron vector of $M_{\mathcal{S}}$ is the vector all of whose entries are 1/8, so that the associated matrix $\widehat{\Sigma}^\circ(0) = \frac{1}{8}\mathbf{I}$.

The ergodic classes for the bisubstitutions of τ and ρ are given by

$$\tau : \begin{cases} \mathcal{E}_1^\tau = \{\overline{\ }\,\overline{\ }, \underline{\ }\,\underline{\ }\} \\ \mathcal{E}_2^\tau = \{\overline{\ }\,\underline{\ }, \underline{\ }\,\overline{\ }\} \end{cases} \qquad \text{and} \qquad \rho : \begin{cases} \mathcal{E}_1^\rho = \{aa, bb, cc, dd\} \\ \mathcal{E}_2^\rho = \{ad, bc, cb, da\} \end{cases} \qquad \text{with} \qquad \mathcal{T}^\rho = \begin{cases} ab, ac, ba, bd \\ ca, cd, db, dc \end{cases}$$

so that one can see the relationship between the ergodic decompositions of the bisubstitutions of τ, ρ and $\tau \otimes \rho$ indicated by the above proposition:

$$\begin{cases} \mathcal{E}_1 = \mathcal{E}_1^\tau \mathcal{E}_1^\rho \quad \mathcal{E}_3 = \mathcal{E}_2^\tau \mathcal{E}_1^\rho \\ \mathcal{E}_2 = \mathcal{E}_1^\tau \mathcal{E}_2^\rho \quad \mathcal{E}_4 = \mathcal{E}_2^\tau \mathcal{E}_2^\rho \end{cases} \qquad \text{and} \qquad \mathcal{T} = \mathcal{E}_1^\tau \mathcal{T}^\rho \cup \mathcal{E}_2^\tau \mathcal{T}^\rho$$

Using lemma 4.0.3, we find that $\mathbf{v} \in \mathcal{K}$ if $\mathring{\mathbf{v}} \gg 0$ where

$$\mathring{\mathbf{v}} = \begin{pmatrix} w_1 & \frac{w_1+w_2}{2} & \frac{w_1+w_2}{2} & w_2 & w_3 & \frac{w_3+w_4}{2} & \frac{w_3+w_4}{2} & w_4 \\ \frac{w_1+w_2}{2} & w_1 & w_2 & \frac{w_1+w_2}{2} & \frac{w_3+w_4}{2} & w_3 & w_4 & \frac{w_3+w_4}{2} \\ \frac{w_1+w_2}{2} & w_2 & w_1 & \frac{w_1+w_2}{2} & \frac{w_3+w_4}{2} & w_4 & w_3 & \frac{w_3+w_4}{2} \\ w_2 & \frac{w_1+w_2}{2} & \frac{w_1+w_2}{2} & w_1 & w_4 & \frac{w_3+w_4}{2} & \frac{w_3+w_4}{2} & w_3 \\ w_3 & \frac{w_3+w_4}{2} & \frac{w_3+w_4}{2} & w_4 & w_1 & \frac{w_1+w_2}{2} & \frac{w_1+w_2}{2} & w_2 \\ \frac{w_3+w_4}{2} & w_3 & w_4 & \frac{w_3+w_4}{2} & \frac{w_1+w_2}{2} & w_1 & w_2 & \frac{w_1+w_2}{2} \\ \frac{w_3+w_4}{2} & w_4 & w_3 & \frac{w_3+w_4}{2} & \frac{w_1+w_2}{2} & w_2 & w_1 & \frac{w_1+w_2}{2} \\ w_4 & \frac{w_3+w_4}{2} & \frac{w_3+w_4}{2} & w_3 & w_2 & \frac{w_1+w_2}{2} & \frac{w_1+w_2}{2} & w_1 \end{pmatrix}$$

and with basis ordered: $(\bar{a}, \bar{b}, \bar{c}, \bar{d}, \underline{a}, \underline{b}, \underline{c}, \underline{d})$. Note that $\mathring{\mathbf{v}}$ can be expressed as the Kronecker product of the $\mathbf{v}$ for Thue-Morse (example 4.1.1) and Rudin-Shapiro (example 4.2.1):

$$\mathring{\mathbf{v}} = \begin{pmatrix} w_1^\tau & w_2^\tau \\ w_2^\tau & w_1^\tau \end{pmatrix} \otimes \begin{pmatrix} w_1^\rho & \frac{w_1^\rho+w_2^\rho}{2} & \frac{w_1^\rho+w_2^\rho}{2} & w_2^\rho \\ \frac{w_1^\rho+w_2^\rho}{2} & w_1^\rho & w_2^\rho & \frac{w_1^\rho+w_2^\rho}{2} \\ \frac{w_1^\rho+w_2^\rho}{2} & w_2^\rho & w_1^\rho & \frac{w_1^\rho+w_2^\rho}{2} \\ w_2^\rho & \frac{w_1^\rho+w_2^\rho}{2} & \frac{w_1^\rho+w_2^\rho}{2} & w_1^\rho \end{pmatrix} \quad \text{with} \quad \begin{cases} w_1 = w_1^\tau w_1^\rho \\ w_2 = w_1^\tau w_2^\rho \\ w_3 = w_2^\tau w_1^\rho \\ w_4 = w_2^\tau w_2^\rho \end{cases}$$

As the eigenvalues of a Kronecker product are the product of the eigenvalues of its factors, it follows that the spectrum of $\mathring{\mathbf{v}}$ is positive if and only if

$$\begin{cases} (w_1^\tau - w_2^\tau)(w_1^\rho - w_2^\rho) > 0 \\ (w_1^\tau - w_2^\tau)(w_1^\rho + w_2^\rho) > 0 \\ (w_1^\tau + w_2^\tau)(w_1^\rho - w_2^\rho) > 0 \\ (w_1^\tau + w_2^\tau)(w_1^\rho + w_2^\rho) > 0 \end{cases} \implies \begin{cases} w_1 - w_2 - w_3 + w_4 > 0 \\ w_1 + w_2 - w_3 - w_4 > 0 \\ w_1 - w_2 + w_3 - w_4 > 0 \\ w_1 + w_2 + w_3 + w_4 > 0 \end{cases} \implies \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}^t = \begin{cases} (1,1,1,1) \\ (1,1,-1,-1) \\ (1,-1,1,-1) \\ (1,-1,-1,1) \end{cases}$$

are the extremal rays of this cone, which are the Kronecker products of the extremal rays of this cone for Thue-Morse and Rudin-Shapiro, see the relevant examples. It follows that the extremal rays of $\mathcal{K}$ are the Kronecker products of the extremal rays of the cones $\mathcal{K}_\tau$ and $\mathcal{K}_\rho$. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ be the respective extremal rays of $\mathcal{K}$ corresponding to the four vectors $\mathbf{w} = (w_1, w_2, w_3, w_4)$ above.

Let $\boldsymbol{\delta}_1$ be the usual Dirac mass at 1, m Lebesgue measure on the circle, and λ_τ the singular continuous measure in the spectrum of τ identified in example 4.1.1. Using software with (4.1) and theorem 3.1.2 to compute $\widehat{\Sigma}(\mathbf{k})$ and using theorem 3.3.1, we can compute the Fourier coefficients of the measures $\lambda_j = \mathbf{v}_j^t \Sigma$; they are vectors in $\mathbb{C}^{64}$ so we do not include the computations here. As usual, $\lambda_1 = \boldsymbol{\delta}_1$, and comparing the first 100 Fourier coefficients suggests that $\lambda_2 = \lambda_4 = m$ and $\lambda_3 = \lambda_\tau$. These computations are confirmed by work of Baake, Gahler, and Grimm in [1], where they consider an identical substitution (although there it is not formulated as a substitution product). Thus, in this case, $\sigma_{\tau\otimes\rho} \sim \sigma_\tau + \sigma_\rho$, so that $\tau \otimes \rho$ has discrete, singular continuous, and Lebesgue components in its spectrum, corresponding to the spectral components of its factors. In general, it would be interesting to know the relationship between the spectrum of a substitution product and its factors and raises an additional question: are there any substitutions with Lebesgue spectrum which do not arise from substitution product with substitutions of Frank type?

We finish with a discussion of height, affecting the discrete component of a substitutions spectrum, and describe a collection of examples with interesting properties.

4.4 Height and Dekking's Criterion

Most of our understanding of the discrete spectrum of constant length substitutions is due to the work of Dekking and the notion of *height* of a substitution, see [9]. The following description of height in the $\mathbb{Z}^d$ case is based on [12, §3.1] and [22, §6.1.1]. Given a primitive and aperiodic $\mathbf{q}$-substitution $\mathcal{S}$ on $\mathcal{A}$, fix a substitution sequence $\mathbf{A}$ in the hull $X_{\mathcal{S}}$. Let $\mathcal{L}$ be a sublattice of $\mathbb{Z}^d$, and let $\mathcal{L}_0$ be a set of class representatives for this lattice, so that $\mathbb{Z}^d$ is the disjoint union of $\mathbf{j} + \mathcal{L}$ for $\mathbf{j} \in \mathcal{L}_0$; let $\mathbf{A}(\mathbf{j} + \mathcal{L})$ denote the letters appearing in positions $\mathbf{j} + \mathbf{l}$ for $\mathbf{l} \in \mathcal{L}$. By primitivity we know $\mathcal{A} = \bigcup_{\mathbf{j}\in\mathcal{L}_0} \mathbf{A}(\mathbf{j} + \mathcal{L})$, though in some cases the $\mathbf{A}(\mathbf{j}+\mathcal{L})$ may form a partition of $\mathcal{A}$. When that happens, one can identify all the letters in each $\mathbf{A}(\mathbf{j}+\mathcal{L})$ with a single representative, and this gives a map $\mathcal{A} \to \mathcal{A}$. When extended pointwise to $\mathcal{A}^{\mathbb{Z}^d}$, this map takes $\mathbf{A}$ to a periodic sequence which, by primitivity, must therefore take $X_{\mathcal{S}}$ onto a finite set. If we let $\mathcal{L} = \mathbb{Z}^d$, this reduces $X_{\mathcal{S}}$ to a singleton. The *height lattice* of $\mathcal{S}$ is the largest lattice $\mathcal{L}$ (in the subset order) for which the above partition property holds. In this case, the factor map which sends each $\mathbf{A}(\mathbf{j} + \mathcal{L})$ to a representative gives rise to the maximal equicontinuous factor of $(X_{\mathcal{S}}, T, \mu)$, see [12].

The following theorem of Dekking completely characterizes the discrete spectrum of aperiodic $\mathbf{q}$-substitutions, see [22, Thm 6.1, 6.2] for the $d = 1$ case, and [12, §3.1] for the general case. For a sublattice $\mathcal{L} \subset \mathbb{Z}^d$, let $\nu_{\mathcal{L}}$ be the uniform probability measure supporting the quotient $\mathbb{Z}^d/\mathcal{L} \subset \mathbb{T}^d$. In the case $\mathcal{L} = \mathfrak{h}\mathbb{Z}^d$ for $\mathfrak{h} \geqslant \mathbf{1}$, the measure $\nu_{\mathcal{L}} = \nu_{\mathfrak{h}}$.

Theorem 4.4.1 (Dekking)**.** *If $\mathcal{S}$ is a primitive and aperiodic $\mathbf{q}$-substitution with height lattice $\mathcal{L}$, then the discrete component of σ_{max} is equivalent to $\boldsymbol{\omega}_{\mathbf{q}} * \nu_{\mathcal{L}}$.*

If $\mathcal{L} = \mathbb{Z}^d$, σ_{max} is pure discrete if and only if a generalized instruction is constant.

The criteria for the pure discrete case are referred to as *trivial height* ($\mathcal{L} = \mathbb{Z}^d$) and the *coincidence condition* (a constant generalized instruction), respectively. In the primitive case the coincidence condition is equivalent to the bisubstitution possessing only one ergodic class, as was observed by Queffélec. More generally, the discrete spectrum of an aperiodic $\mathbf{q}$-substitution is given by the sum of $\boldsymbol{\omega}_{\mathbf{q}} * \nu_{\mathcal{L}}$ as $\mathcal{L}$ ranges over the height lattices of its primitive components. We comment here that although methods exist to compute the height of a given substitution, this is not necessary in order to compute the spectrum, as our algorithm produces the discrete component of the measure and so therefore determines the height.

We now describe a class of aperiodic bijective commutative $\mathbf{q}$-substitutions which attain any height lattice of the form $\mathbf{h}\mathbb{Z}^d$. Fix $\mathbf{h} \geqslant \mathbf{1}$ in $\mathbb{Z}^d$ and let $\mathcal{A} = \mathcal{A}_{\mathbf{h}} := \mathbb{Z}^d/2\mathbf{h}\mathbb{Z}^d$ be the quotient ring of $\mathbb{Z}^d$ integers modulo $2\mathbf{h}$, using the residue class $[\mathbf{0}, 2\mathbf{h})$ to represent the letters. For each $\mathbf{k} \in \mathbb{Z}^d$ let $\pi_{\mathbf{k}} : \mathcal{A} \to \mathcal{A}$ be the map

$$\pi_{\mathbf{k}} : \alpha \longmapsto \alpha + \mathbf{k} \,(\text{mod } 2\mathbf{h}).$$

Note that each $\pi_{\mathbf{k}}$ is bijective as a map on $\mathbb{Z}^d/2\mathbf{h}\mathbb{Z}^d$, and they form the commutative group generated by $\pi_i := \pi_{\mathbf{1}_i}$ for $1 \leqslant i \leqslant d$, which is a subgroup of permutations of $\mathcal{A}$, with $\pi_{\mathbf{0}}$ giving the identity map on $\mathcal{A}$.

Let $\mathbf{q} := \mathbf{h} + \mathbf{1}$, so that $[\mathbf{0}, \mathbf{q}) = [\mathbf{0}, \mathbf{h}]$. Consider the $\mathbf{q}$-substitution $\mathcal{H} := \mathcal{H}_{\mathbf{h}}$ on $\mathcal{A}_{\mathbf{h}}$ determined by the instructions $\mathcal{R}_{\mathbf{k}} = \pi_{\mathbf{k}}$ for $\mathbf{k} \in [\mathbf{0}, \mathbf{h}]$. As no block $\mathcal{H}(\gamma)$ can contain repeated symbols, this can be used to identify superblocks in a substitution sequence and desubstitute, and so $\mathcal{H}$ is recognizable. As the instructions of $\mathcal{H}$ are a subset of a commutative group of permutations, $\mathcal{H}$ is an aperiodic bijective commutative $\mathbf{q}$-substitution. As a map $\mathcal{H} : \mathcal{A} \to \mathcal{A}^+$, the $\mathbf{q}$-substitution $\mathcal{H}$ takes the letter α to the block on $[\mathbf{0}, \mathbf{h}]$ whose value at $\mathbf{j}$ is $\pi_{\mathbf{j}}(\alpha) = \alpha + \mathbf{j} \,(\text{mod } 2\mathbf{h})$. Therefore, as $\{\alpha + \mathbf{j} + \mathbf{k} : \mathbf{j}, \mathbf{k} \in [\mathbf{0}, \mathbf{h}]\}$ covers an entire equivalence class modulo $2\mathbf{h}$, it follows that $\mathcal{H}^2(\alpha)$ covers $\mathcal{A}$ for every α, and $\mathcal{H}$ is primitive.

Using proposition 2.2.1, we have for $n > 0$ and $\mathbf{j} = \mathbf{j}_0 + \mathbf{j}_1\mathbf{q} + \ldots + \mathbf{j}_{n-1}\mathbf{q}^{n-1} \in \mathbb{Z}^d$

$$\mathcal{R}_{\mathbf{j}}^{(n)} = \mathcal{R}_{\mathbf{j}_0} \cdots \mathcal{R}_{\mathbf{j}_{n-1}} \qquad \Longrightarrow \qquad \mathcal{R}_{\mathbf{j}}^{(n)}(\alpha) \equiv \alpha + \mathbf{j}_0 + \ldots + \mathbf{j}_{n-1} \,(\mathrm{mod}\ 2\mathbf{h})$$

so that

$$\mathcal{H}^n(\alpha)_{\mathbf{j}} = \beta \qquad \Longleftrightarrow \qquad \mathbf{j}_0 + \ldots + \mathbf{j}_{n-1} \equiv \beta - \alpha \,(\mathrm{mod}\ 2\mathbf{h}) \tag{4.3}$$

Using this, one checks that both the return times and the correlation vector are *independently invariant under permutations of coordinates in* $\mathbb{Z}^d$ *and order of* $\mathbf{q}$*-adic digits.*

We now compute the height $\mathfrak{h}$ of $\mathcal{H}_{\mathbf{h}}$; let $\mathbf{D}_\eta \in X_{\mathcal{H}}$ be the sequence giving rise to the reduced language, generated by telescoping $\mathcal{H}$ and iterating on a seed patch about the origin: in other words, $\mathbf{D}_\eta$ is a fixed point of some iterate of $\mathcal{H}$. Writing for $\mathbf{a} > \mathbf{1}$

$$C_{\mathbf{j}}^{\mathbf{a}} := \mathbf{D}_\eta\left(\mathbf{j} + \mathbf{a}\mathbb{Z}^d\right) \qquad \text{and} \qquad C^{\mathbf{a}} := \{C_{\mathbf{j}}^{\mathbf{a}} : \mathbf{j} \in [\mathbf{0}, \mathbf{a})\}$$

so that the height $\mathfrak{h}$ is maximal amongst $\mathbf{a}$ with $\mathbf{a}\mathbb{Z}^d + \mathbf{q}\mathbb{Z}^d = \mathbb{Z}^d$ for which $C^{\mathbf{a}}$ partitions $\mathcal{A}$. Using (4.3), one can show that $C^{\mathbf{h}}$ partitions $\mathcal{A}$ as well, and thus $\mathfrak{h} \geqslant \mathbf{h}$. Alternatively, if $2 \leqslant k < n$ and $\mathbf{i} \in [\mathbf{0}, \mathbf{h}]$ we have

$$\mathcal{H}^n(\alpha)_{\mathbf{0}} = \mathcal{H}^n(\alpha)_{\mathbf{h}+(\mathbf{h}-\mathbf{i})\mathbf{q}+\mathbf{i}\mathbf{q}^k} = \mathcal{H}^n(\alpha)_{\mathbf{h}-\mathbf{i}+\mathbf{h}\mathbf{q}+\mathbf{i}\mathbf{q}^k} = \alpha$$

as the $\mathbf{q}$-adic digits of the indices add up to $\mathbf{0}$ modulo $2\mathbf{h}$. The difference between the second pair of indices is $\mathbf{i}\mathbf{h}$, so $\mathfrak{h}$ divides $\mathbf{h}$ as $C^{\mathfrak{h}}$ partitions $\mathcal{A}$ by definition, and $\mathfrak{h} = \mathbf{h}$, as desired.

We now briefly examine the bisubstitution $\mathcal{H} \otimes \mathcal{H}$. As $\mathcal{H}$ is bijective, it has no transient part, and the subalphabets

$$\mathcal{A}_{\mathbf{j}}^2 := \{\alpha\beta \in \mathcal{A}^2 : \alpha - \beta \equiv \mathbf{j} \,(\mathrm{mod}\ 2\mathbf{h})\}$$

for $\mathbf{j} \in [\mathbf{0}, 2\mathbf{h})$ form a partition of the bialphabet $\mathcal{A}^2$ into $2\mathbf{h}$ sets of size $\mathrm{Card}[\mathbf{0}, 2\mathbf{h})$, and with respect to which $\mathcal{H} \otimes \mathcal{H}$ is primitive. It follows that the $\mathcal{A}_{\mathbf{j}}^2$ partition $\mathcal{A}^2$ and thus form the ergodic classes of the bisubstitution. Thus, if $\mathbf{v}$ is a left Q-eigenvector of $C_{\mathcal{S}}$, and its associated matrix is

$$\mathring{\mathbf{v}} \approx \textstyle\sum_{\mathbf{j}\in[\mathbf{0},\mathbf{2h})} w_{\mathbf{j}}\mathcal{R}_{\mathbf{j}},$$

as $\mathcal{R}_{\mathbf{j}}$ corresponds to the ergodic class $\mathcal{A}^2_{\mathbf{j}}$; note also that $\mathcal{R}_{\mathbf{0}} = \mathbf{I}$.

Note that in the one-dimensional setting, the $\mathbf{h} = 1$ case gives the Thue-Morse substitution. As all of these are examples of aperiodic bijective commutative $\mathbf{q}$-substitutions, theorem 3.4.1 shows they are purely singular. What is not clear, however, is the role played by height. In the next example, we use the algorithm to compute the spectrum for the case $\mathbf{h} = 3$, and show how the height comes out in this computation. Moreover, cases of nontrivial height are the only ones where we have encountered complex valued vectors in the spectral hull, of which the following is an example.

Example 4.4.2. Let $\mathcal{H}$ be the 3-substitution $\mathcal{H}_3$ described above, whose bisubstitution has the ergodic classes (represented in matrix form via (4.2) in §4)

$$\mathcal{H}: \begin{cases} 0 & \\ 1 & \\ 2 & \longmapsto \\ 3 & \\ 4 & \\ 5 & \end{cases} \begin{matrix} 0123 \\ 1234 \\ 2345 \\ 3450 \\ 4501 \\ 5012 \end{matrix} \qquad \text{and} \qquad \mathring{\mathbf{v}} = \sum_{j=0}^{5} w_j \mathcal{R}_j = \begin{bmatrix} w_0 & w_1 & w_2 & w_3 & w_4 & w_5 \\ w_5 & w_0 & w_1 & w_2 & w_3 & w_4 \\ w_4 & w_5 & w_0 & w_1 & w_2 & w_3 \\ w_3 & w_4 & w_5 & w_0 & w_1 & w_2 \\ w_2 & w_3 & w_4 & w_5 & w_0 & w_1 \\ w_1 & w_2 & w_3 & w_4 & w_5 & w_0 \end{bmatrix}$$

the eigenvalues of which are (using Python's *sympy* toolbox to compute eigenvalues)

$$\begin{cases} w_0 + w_1 + w_2 + w_3 + w_4 + w_5 \\ w_0 - w_1 + w_2 - w_3 + w_4 - w_5 \\ w_0 + \frac{1}{2}w_1 - \frac{1}{2}w_2 - w_3 - \frac{1}{2}w_4 + \frac{1}{2}w_5 - \frac{\sqrt{3}}{2}\sqrt{-(w_1 + w_2 - w_4 - w_5)^2} \\ w_0 + \frac{1}{2}w_1 - \frac{1}{2}w_2 - w_3 - \frac{1}{2}w_4 + \frac{1}{2}w_5 + \frac{\sqrt{3}}{2}\sqrt{-(w_1 + w_2 - w_4 - w_5)^2} \\ w_0 - \frac{1}{2}w_1 - \frac{1}{2}w_2 + w_3 - \frac{1}{2}w_4 - \frac{1}{2}w_5 - \frac{\sqrt{3}}{2}\sqrt{-(w_1 - w_2 + w_4 - w_5)^2} \\ w_0 - \frac{1}{2}w_1 - \frac{1}{2}w_2 + w_3 - \frac{1}{2}w_4 - \frac{1}{2}w_5 + \frac{\sqrt{3}}{2}\sqrt{-(w_1 - w_2 + w_4 - w_5)^2} \end{cases}$$

so that enforcing positive definiteness forces $w_4 = \overline{w_2}$ and $w_5 = \overline{w_1}$ as it must be Hermitian, and we write $w_1 = \alpha + \beta i$ and $w_2 = a + bi$. Positivity of eigenvalues gives the equations:

$$\begin{cases} w_0 + w_3 + 2\alpha + 2a > 0 \\ w_0 - w_3 - 2\alpha + 2a > 0 \\ w_0 - w_3 + \alpha - a - \sqrt{3}(\beta + b) > 0 \\ w_0 - w_3 + \alpha - a + \sqrt{3}(\beta + b) > 0 \\ w_0 + w_3 - \alpha - a - \sqrt{3}(\beta - b) > 0 \\ w_0 + w_3 - \alpha - a + \sqrt{3}(\beta - b) > 0 \end{cases} \quad \text{with extremal rays} \quad \begin{cases} (1, 1, 1, 1, 0, 0) \\ (1, -1, -1, 1, 0, 0) \\ (1, -1, \frac{1}{2}, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}) \\ (1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}) \\ (1, 1, -\frac{1}{2}, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}) \\ (1, 1, -\frac{1}{2}, -\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}) \end{cases}$$

where the vectors are $(w_0, w_3, \alpha, a, \beta, b)$, found by intersecting any 5 of them and letting $w_0 = 1$. Thus, by lemma 4.0.3, the extremal rays of $\mathcal{K}$ are obtained from the above via the identification $w_1 = \alpha + \beta i$ and $w_2 = a + bi$ and we obtain $\mathbf{v} = \sum_{j=0}^{5} w_j \mathcal{R}_j$ for $(w_0, w_1, w_2, w_3, w_4, w_5)$

$$\begin{cases} \mathbf{v}_1 \approx (1,1,1,1,1,1) \\ \mathbf{v}_2 \approx (1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, 1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i) \\ \mathbf{v}_3 \approx (1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, 1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i) \\ \mathbf{v}_4 \approx (1,-1,1,-1,1,-1) \\ \mathbf{v}_5 \approx (1, \frac{1}{2} - \frac{\sqrt{3}}{2}i, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, -1, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} + \frac{\sqrt{3}}{2}i) \\ \mathbf{v}_6 \approx (1, \frac{1}{2} + \frac{\sqrt{3}}{2}i, -\frac{1}{2} + \frac{\sqrt{3}}{2}i, -1, -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \frac{1}{2} - \frac{\sqrt{3}}{2}i) \end{cases}$$

Now, as $\mathcal{H}$ is bijective, $\frac{1}{6}M_{\mathcal{H}}$ is both row and column stochastic, so its Perron vector is $\frac{1}{6}\mathbf{1}$, and $\widehat{\Sigma}(0) = \sum_{\gamma \in \mathcal{A}} \frac{1}{6}\mathbf{e}_{\gamma\gamma}$. Using (4.1) and theorem 3.1.2 we compute $\widehat{\Sigma}(\mathbf{k})$ (in matrix form)

$$\widehat{\Sigma}^{\circ}(0) = \frac{1}{6}\mathcal{R}_0, \quad \widehat{\Sigma}^{\circ}(1) = \frac{1}{30}\mathcal{R}_2 + \frac{4}{30}\mathcal{R}_5, \quad \widehat{\Sigma}^{\circ}(2) = \frac{2}{30}\mathcal{R}_1 + \frac{3}{30}\mathcal{R}_4, \quad \widehat{\Sigma}^{\circ}(3) = \frac{3}{30}\mathcal{R}_0 + \frac{2}{30}\mathcal{R}_3$$

Letting $\lambda_j := \lambda_{\mathbf{v}_j}$, this gives $\lambda_1 = \boldsymbol{\delta}_1$ as usual, and

$$\widehat{\lambda_1}(1) = 1, \qquad \widehat{\lambda_1}(2) = 1, \qquad \widehat{\lambda_1}(3) = 1$$

$$\widehat{\lambda_2}(1) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \qquad \widehat{\lambda_2}(2) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \qquad \widehat{\lambda_2}(3) = 1$$

$$\widehat{\lambda_3}(1) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \qquad \widehat{\lambda_3}(2) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \qquad \widehat{\lambda_3}(3) = 1$$

and one checks from Fourier unicity and equation (3.1) that $\lambda_1 + 2\lambda_2 + 2\lambda_3 = 5\nu_3$, which by Dekking's theorem is the discrete spectrum of $\mathcal{H}$. Additionally, we have

$$\widehat{\lambda_4}(1) = -\frac{3}{5}, \qquad \widehat{\lambda_4}(2) = \frac{1}{5}, \qquad \widehat{\lambda_4}(3) = \frac{1}{5}$$

$$\widehat{\lambda_5}(1) = \frac{3}{10} + \frac{3\sqrt{3}}{10}i, \qquad \widehat{\lambda_5}(2) = -\frac{1}{10} + \frac{\sqrt{3}}{10}i, \qquad \widehat{\lambda_5}(3) = \frac{1}{5}$$

$$\widehat{\lambda_6}(1) = \frac{3}{10} - \frac{3\sqrt{3}}{10}i, \qquad \widehat{\lambda_6}(2) = -\frac{1}{10} - \frac{\sqrt{3}}{10}i, \qquad \widehat{\lambda_6}(3) = \frac{1}{5}$$

from which one can see that $\lambda_4 + \lambda_5 + \lambda_6$ is equal to $\nu_3 * \lambda$ for some (singular continuous) measure λ on $\mathbb{T}$, as the Fourier coefficients which are not multiples of 3 all vanish and λ is invariant for the times 3 map on $\mathbb{T}$. We know it is continuous as it is singular to ν_3, which is the entire discrete spectrum of $\mathcal{H}$ by Dekking's theorem, and it is singular as it has nonvanishing Fourier coefficients, and so $\sigma_{\max} \sim \boldsymbol{\omega}_4 * \boldsymbol{\nu}_3 + \boldsymbol{\omega}_4 * \boldsymbol{\nu}_3 * \lambda$ is purely singular.

Chapter 5

APPENDIX

The appendix is in two sections, the purpose of which is to prove theorem 3.3.1, extended from Queffélec's results for substitutions of constant length on $\mathcal{A}^{\mathbb{N}}$. First, however, we need to extend [22, Corollary 7.1] of Queffélec which allows for the diagonalization of a strongly mixing matrix of measures on the circle group, which we treat in some detail as Queffélec's treatment is scattered throughout her text and omits enough details that an *a priori* extension to our setting is not possible. Moreover, we extend the result significantly, allowing for the diagonalization of a matrix of measures ergodic for a continuous transformation on a compact metric space.

5.1 Generalized Functionals on $\mathcal{M}(X)$

Let X be a metrizable compact space. By a *measure (on X)* we mean a complex Borel measure of finite total variation $|\mu|$, and denote by $\mathcal{M} := \mathcal{M}(X)$ the Banach space of measures on X under the total variation norm $\|\mu\| := |\mu|(X)$. Let $\mathcal{M}^*$ denote the Banach space dual of $\mathcal{M}$ consisting of continuous linear functionals $\mathcal{M} \to \mathbb{C}$. We extend the notions of absolute continuity, mutual singularity, equivalence of measures, almost-everywhere, and null-sets to complex measures using their total variation measures.

For $\mu \in \mathcal{M}$, let $\mathcal{L}(\mu) := \{\nu \in \mathcal{M} : \nu \ll \mu\}$ be the *$\mathcal{L}$-space for μ*, consisting of all measures absolutely continuous with respect to μ. The Lebesgue decomposition theorem implies that $\mathcal{L}(\mu)$ is a closed subspace, and that $\mathcal{M} = \mathcal{L}(\mu) \oplus \mathcal{L}(\mu)^{\perp}$, where $\mathcal{L}(\mu)^{\perp}$ is the set of all measures mutually singular to μ. Let D_μ denote the projection of $\mathcal{M}$ onto $\mathcal{L}(\mu)$.

Theorem 5.1.1. *For $\mu \in \mathcal{M}$, there exist isometric isomorphisms $\partial_\mu, \partial_\mu^*$, such that for $\nu \ll \mu$,*

$$\partial_\mu : \mathcal{L}(\mu) \xrightarrow{\sim} L^1(\mu) \qquad \textit{with} \qquad d\nu = \partial_\mu \nu \, d\mu \qquad \textit{and} \qquad \partial_\mu|_{\mathcal{L}(\nu)} = \partial_\mu \nu \cdot \partial_\nu$$

$$\partial_\mu^* : \mathcal{L}(\mu)^* \xrightarrow{\sim} L^\infty(\mu) \qquad \textit{with} \qquad F(\nu) = \int_X \partial_\mu^* F d\nu \qquad \textit{and} \qquad \partial_\nu^*|_{\mathcal{L}(\mu)^*} = \partial_\mu^*$$

The statements about ∂_μ follow immediately from the Radon-Nikodym theorem for complex measures and those about ∂_μ^* follow as $L^1(\mu)^*$ is isometrically isomorphic to $L^\infty(\mu)$ when μ has finite total variation. Here, the identity $d\nu = \partial_\mu \nu \, d\mu$ holds in the sense of the Riesz Representation theorem - as continuous linear functionals on $C(X)$. Using density arguments, this extends to integration of $L^1(\nu)$ functions, so that $\nu(A) = \int_A \partial_\mu \nu d\mu$ for A Borel, and they agree pointwise as measures. In this sense, we can think of $\mathcal{L}(\mu)$ as $L^1(\mu)\, d\mu$, and multiplication by $\partial_\mu \nu$ is a map from $\mathcal{L}(\nu) \to \mathcal{L}(\mu)$. We now describe a similar localization of $\mathcal{M}^*$ using the maps ∂_μ^*, and giving rise to an action on $\mathcal{M}$ used by Queffélec.

A *generalized functional* φ on $\mathcal{M}$ is an association $\mu \mapsto \varphi_\mu \in L^\infty(\mu)$ from $\mathcal{M}$ to essentially bounded functions on X such that for all $\mu, \nu \in \mathcal{M}$

$$\nu \ll \mu \quad \Longrightarrow \quad \varphi_\mu = \varphi_\nu \;\; \nu\text{-ae} \qquad \text{and} \qquad \|\varphi\| := \sup_{\|\mu\|=1} \|\varphi_\mu\|_{L^\infty(\mu)} < \infty$$

By the above theorem, it is clear that $\mu \mapsto \partial_\mu^* F$ takes every $F \in \mathcal{M}^*$ to a generalized functional. Moreover, each generalized functional φ determines a functional on $\mathcal{M}$ given by $F(\nu) := \int_X \varphi_\nu d\nu$ with $\|\varphi\| = \|F\|$, which follows from a result of Šreider, see [29, Thm 1].

Theorem 5.1.2 (Šreider)**.** *The dual $\mathcal{M}^*$ coincides with generalized functionals on $\mathcal{M}$.*

As our measures have finite total variation, $L^\infty(\mu) \subset L^1(\mu)$ for all $\mu \in \mathcal{M}$ and we can compose the maps $\partial_\mu^{-1}\partial_\mu^* : \mathcal{L}(\mu)^* \hookrightarrow \mathcal{L}(\mu)$, sending $F \mapsto \partial_\mu^* F d\mu$, and so $\mathcal{M}^*$ acts on $\mathcal{M}$ by

$$\mathcal{M}^* \times \mathcal{M} \to \mathcal{M} \qquad \text{sending} \qquad F, \mu \longmapsto F \cdot \mu \qquad \text{with} \qquad d(F \cdot \mu) = \partial_\mu^* F d\mu$$

A map $\psi : \mathcal{M} \to \mathcal{M}$ is *absolutely continuous* if $\psi : \mathcal{L}(\mu) \to \mathcal{L}(\mu)$ for all $\mu \in \mathcal{M}$.

Proposition 5.1.3. *$\mathcal{M}^*$ acts on $\mathcal{M}$ is by absolutely continuous commuting operators.*

Proof. The action is absolutely continuous as $\partial_\mu^{-1} \circ \partial_\mu^*$ is a map into $\mathcal{L}(\mu)$ and so $\nu \ll \mu$ implies $F \cdot \nu \ll \nu \ll \mu$. For $F, G \in \mathcal{M}^*$, theorem 5.1.1 tells us that $\partial_{G \cdot \mu}^* F = \partial_\mu^* F \in L^1(G \cdot \mu)$ so

$$d(F \cdot G \cdot \mu) = \partial_{G\cdot\mu}^* F d(G \cdot \mu) = \partial_\mu^* F d(G \cdot \mu) = \partial_\mu^* F \partial_\mu^* G d\mu = \partial_{F\cdot\mu}^* G d(F \cdot \mu) = d(G \cdot F \cdot \mu)$$

which shows that the action is commutative. $\square$

We say $F \in \mathcal{M}^*$ *acts invariantly* on $\mu \in \mathcal{M}$ if there exists $F_\mu \in \mathbb{C}$ with $F{\cdot}\mu = F_\mu \mu$, a constant multiple of μ. We refer to F_μ as the *eigenvalue for F on μ*. If $\nu \in \mathcal{L}(\mu)$ then $F_\mu = F_\nu$ ν-ae by theorem 5.1.1 so that if F acts invariantly on μ, it acts invariantly on $\mathcal{L}(\mu)$ with the same eigenvalue.

Proposition 5.1.4. *If $F \in \mathcal{M}^*$ acts invariantly on $\mu, \nu \in \mathcal{M}^*$ and $F_\mu \neq F_\nu$, then $\mu \perp \nu$.*

Proof. If $\rho \in \mathcal{L}(\mu) \cap \mathcal{L}(\nu)$ is not $\mathbf{0}$, then F will also act invariantly on ρ and the eigenvalues will satisfy $F_\mu = F_\rho = F_\nu$, a contradiction, and so the measures are mutually singular. $\square$

We now discuss some preliminaries allowing us to extend the action of $\mathcal{M}^*$ to matrices with entries in $\mathcal{M}$. For $n \geqslant 1$, let $\mathbf{M}_n(\mathbb{C})$ denote the space of $n \times n$ complex matrices and $\mathcal{M}_n := \mathcal{M}_n(X)$ the collection of $\mathbf{M}_n(\mathbb{C})$-valued Borel measures of finite total variation. For $\mathcal{W} = (\omega_{ij})_{1 \leqslant i,j \leqslant n} \in \mathcal{M}_n(X)$ we write $|\mathcal{W}| := \sum_{i,j} |\omega_{ij}|$ for the *total variation measure of $\mathcal{W}$*, so that A is $|\mathcal{W}|$-null if and only if $\mathcal{W}(B) = \mathbf{0}$ for every $B \subset A$, and by the following is a similarity invariant (over $\mathbb{C}$.) For readability, we will often denote $\boldsymbol{\omega} := |\mathcal{W}|$.

Proposition 5.1.5. *If $S \in \mathbf{M}_n(\mathbb{C})$ is invertible and $\mathcal{W} \in \mathcal{M}_n(X)$, then $|\mathcal{W}| \sim |S\mathcal{W}S^{-1}|$.*

Proof. If A is $|\mathcal{W}|$-null then A is ω_{ij}-null for $1 \leqslant i, j \leqslant n$, and $\mathcal{W}(B) = \mathbf{0}$, and $S\mathcal{W}(B)S^{-1} = \mathbf{0}$ for all $B \subset A$. Thus $|S\mathcal{W}S^{-1}| \ll |\mathcal{W}| \ll |S\mathcal{W}S^{-1}|$ as $\mathcal{W} = S^{-1}(S\mathcal{W}S^{-1})S$. $\square$

Integration with respect to $\mathcal{W}$ is done as follows: for $f \in L^1(\boldsymbol{\omega})$ the integral $\int_X f d\mathcal{W} \in \mathbf{M}_n(\mathbb{C})$ is the matrix with the integrals $\int_X f d\omega_{ij}$ as its components. All of the above notions can be extended to rectangular matrices as well as vectors, although the square case is of primary interest to us and we will not need the others beyond formalities. We abuse notation by using the symbol $\mathbf{0}$ to denote both the zero matrix and zero matrix of measures, for any dimension. A matrix of measures is *positive definite* if it is *pointwise positive semidefinite* when evaluated on sets.

Extend the action of $\mathcal{M}^*$ to $\mathcal{M}_n$ componentwise (as $\partial^*_{\boldsymbol{\omega}} F$ is a scalar function for $F \in \mathcal{M}^*$)

$$\mathcal{M}^* \times \mathcal{M}_n \longrightarrow \mathcal{M}_n \quad \text{sending} \quad F, \mathcal{W} \longmapsto F{\cdot}\mathcal{W} \quad \text{with} \quad d(F{\cdot}\mathcal{W}) = \partial^*_{\boldsymbol{\omega}} F d\mathcal{W}$$

We say that F *acts invariantly* on $\mathcal{W} \in \mathcal{M}_n$ if there exists a $F_{\mathcal{W}} \in \mathbf{M}_n(\mathbb{C})$ such that $F \cdot \mathcal{W} = \mathcal{W} F_{\mathcal{W}}$ (or $= F_{\mathcal{W}} \mathcal{W}$) and refer to $F_{\mathcal{W}}$ as a right (or left) *eigenmatrix* for F on $\mathcal{W}$. As $(F \cdot \mathcal{W})^* = \overline{F} \cdot \mathcal{W}^*$

$$F \cdot \mathcal{W} = F_{\mathcal{W}} \mathcal{W} \qquad \Longleftrightarrow \qquad \overline{F} \cdot \mathcal{W}^* = \mathcal{W}^* F_{\mathcal{W}}^*$$

so that we may restrict our attention to pairs $F, \mathcal{W}$ for which the eigenmatrices act from the right without loss of generality. Although in general eigenmatrices for a given functional F need not be unique, they must be whenever $\mathcal{W}(A)$ is invertible for some $A \subset X$, as

$$F \cdot \mathcal{W} = \mathcal{W} F_{\mathcal{W}} \qquad \Longrightarrow \qquad F_{\mathcal{W}} = \mathcal{W}(A)^{-1} \int_A \partial_{\boldsymbol{\omega}}^* F d\mathcal{W}$$

Let $\mathcal{M}_n^\circ$ denote the collection of such matrix measures, or

$$\mathcal{M}_n^\circ := \{\mathcal{W} \in \mathcal{M}_n : \text{ there exists } A \subset X \text{ Borel with } \mathcal{W}(A) \text{ invertible}\}$$

Thus when F acts invariantly on $\mathcal{W} \in \mathcal{M}_n^\circ$ it makes sense to speak of *the eigenmatrix* $F_{\mathcal{W}}$ for the action of F on $\mathcal{W}$. For the action of $\mathcal{M}^*$ on a matrix measure $\mathcal{W}$, we have the collection of (right) eigenmatrices of $\mathcal{W}$, or

$$\mathrm{Eig}_{\mathrm{R}}(\mathcal{W}) := \{\mathbf{B} \in \mathbf{M}_n(\mathbb{C}) : F \cdot \mathcal{W} = \mathcal{W}\mathbf{B} \text{ for some } F \in \mathcal{M}^*\}$$

Proposition 5.1.6. *For $\mathcal{V} \in \mathcal{M}_n^\circ$, matrices in $Eig_R(\mathcal{V})$ are simultaneously diagonalizable.*

Proof. If $F, G \in \mathcal{M}^*$ act invariantly on $\mathcal{V} \in \mathcal{M}_n^\circ$, then as the action of $\mathcal{M}^*$ on $\mathcal{M}_n$ is componentwise, we have $F \cdot G \cdot \mathcal{V} = G \cdot F \cdot \mathcal{V}$ by proposition 5.1.3 and so

$$\mathcal{V} F_{\mathcal{V}} G_{\mathcal{V}} = G \cdot F \cdot \mathcal{V} = F \cdot G \cdot \mathcal{V} = \mathcal{V} G_{\mathcal{V}} F_{\mathcal{V}}$$

so that, as $\mathcal{V}(A)$ is invertible for some A Borel, we can cancel and the eigenmatrices commute. Now, suppose $F \in \mathcal{M}^*$ acts invariantly on $\mathcal{V} \in \mathcal{M}_n^\circ$; we show that $F_{\mathcal{V}}$ is diagonalizable. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $F_{\mathcal{V}}$ and $\mathbf{v} \in \mathbb{C}^n$ a generalized eigenvector of degree $k > 0$, so that

$(F_{\mathcal{V}} - \lambda I)^k \mathbf{v} = 0$. The claim will follow by showing that $k = 1$, as this implies the eigenvalues of $F_{\mathcal{V}}$ are simple, and thus diagonalized by putting into Jordan form. As $F{\cdot}\mathcal{V} = \mathcal{V}F_{\mathcal{V}}$

$$\mathcal{V}(F_{\mathcal{V}} - \lambda I)^k \mathbf{v} = (F - \lambda)^k {\cdot} \mathcal{V}\mathbf{v} = \mathbf{0}$$

where $(F - \lambda)^k$ is the action of the functional $\mu \mapsto F(\mu) - \lambda\mu(X)$ on $\mathcal{V}$, k consecutive times; as $\partial^*_{\boldsymbol{\nu}} F - \lambda \in L^\infty(\boldsymbol{\nu})$, this defines a generalized functional on $\mathcal{L}(\boldsymbol{\nu})$. The above then gives us

$$(\partial^*_{\boldsymbol{\nu}} F - \lambda)^k = 0 \quad \mathcal{V}\mathbf{v}\text{-ae} \qquad \Longrightarrow \qquad \partial^*_{\boldsymbol{\nu}} F = \lambda \quad \mathcal{V}\mathbf{v}\text{-ae}$$

where $\boldsymbol{\nu} := |\mathcal{V}|$. In particular, this shows $\mathcal{V}F_{\mathcal{V}}\mathbf{v} = F{\cdot}\mathcal{V}\mathbf{v} = \lambda\mathcal{V}\mathbf{v}$ and we have $F_{\mathcal{V}}\mathbf{v} = \lambda\mathbf{v}$ as $\mathcal{V} \in \mathcal{M}^\circ_n$. Thus, $\mathbf{v}$ is an eigenvector for $F_{\mathcal{V}}$ and the eigenspaces of $F_{\mathcal{V}}$ are one-dimensional, so that $F_{\mathcal{V}}$ is diagonalizable.

Thus $\mathrm{Eig}_{\mathrm{R}}(\mathcal{V})$ is a commuting family of diagonalizable matrices, and therefore can be simultaneously diagonalized, as commuting matrices share invariant subspaces and diagonalizable matrices have one-dimensional invariant subspaces spanning their domain. □

5.1.1 Diagonalization of Ergodic Matrices of Measures

For a continuous $\mathbb{Z}^d$ action S on X, let $\mathcal{M}(X, S)$ denote the collection of S-invariant complex Borel measures on X; note that distinction from the prequel, where it represents only the positive probability measures. Restricting to S-invariant probability measures gives a compact convex set, the extreme points of which are ergodic and mutually singular. By ergodic decomposition, $\mathcal{M}(X, S)$ is the $\mathbb{C}$-span of $\mathcal{E} = \mathcal{E}(X, S)$, the ergodic S-invariant probability measures, see [30].

For each $\mu \in \mathcal{M}(X, S)$ we have the σ-algebra (where Δ denotes the symmetric difference)

$$\mathcal{B}_\mu = \{A \subset X \text{ Borel such that } A \Delta S^{\mathbf{k}} A \text{ is } \mu\text{-null } \forall\, \mathbf{k} \in \mathbb{Z}^d\}$$

consisting of the *μ-ae S-invariant Borel subsets* of X. The simple functions over this σ-algebra generate a closed subspace of $L^2(\mu)$ which we denote $L^2(\mu, S)$ and the orthogonal projection $\mathbb{E}_\mu : L^2(\mu) \to L^2(\mu, S)$ sends f to $\mathbb{E}_\mu(f) = \mathbb{E}(f|\mathcal{B}_\mu)$, the conditional expectation of f given the σ-algebra of μ-ae S-invariant Borel sets. Then the ergodic averages

$$A_m(f) := \tfrac{1}{m^d} \textstyle\sum_{\mathbf{k} \in [\mathbf{0}, m\mathbf{1}]} f \circ S^{-\mathbf{k}}$$

converge to $\mathbb{E}_\mu(f)$ in $L^2(\mu)$ as $m \to \infty$, by the mean ergodic theorem.

Using this, we construct a collection of functionals on $\mathcal{M}$ determined by a bounded Borel function on X. For such a function, f, the ergodic averages $\mathcal{A}_m(f)$ are all bounded by the supremum norm of f, so that $\|\mathbb{E}_\mu(f)\|_\infty \leqslant \|f\|_\infty$ and $\mathbb{E}_\mu(f) \in L^\infty(\mu)$. As all our measures have finite total variation, L^2 convergence implies convergence in L^1, and so subsequences of ergodic averages converge pointwise almost everywhere with respect to an invariant measure. Thus, if $\nu \ll \mu$ are both invariant, we can pass to the almost everywhere pointwise subsequential limits of $\mathcal{A}_m(f)$ and show that $\mathbb{E}_\mu(f) = \mathbb{E}_\nu(f)$ ν-ae. Thus, for each bounded f, the map $\mu \mapsto \mathbb{E}_\mu(f)$ gives rise to a functional on $\mathcal{M}(X, S)$, a closed subspace of $\mathcal{M}$, which can be extended to $\mathcal{M}$ via the Hahn-Banach theorem; let $[f]$ denote the collection of all such extensions. Thus, we obtain:

Proposition 5.1.7. *For each bounded function f on X, there exists a nonempty collection of functionals $[f] \subset \mathcal{M}^*$ such that $F \in [f]$ and $\mu \in \mathcal{M}(X, S)$ give $\partial_\mu^* F = \mathbb{E}_\mu(f)$ μ-ae*

Note that for every $F \in [f]$ and $\mu \in \mathcal{M}(X, S)$ we have $F(\mu) = \int_X \mathbb{E}_\mu(f) d\mu = \int_X f d\mu$ as X is always an invariant set, and $\mathbb{E}_\mu(f)$ is the conditional expectation. We use the above to prove a diagonalization result for ergodic matrices of measures relative to $S : X \to X$ continuous. Let $\mathcal{M}_n(X, S)$ denote those measures in $\mathcal{M}_n$ all of whose components are in $\mathcal{M}(X, S)$. A matrix of measures $\mathcal{W} \in \mathcal{M}_n(X, S)$ with total variation $\boldsymbol{\omega}$ is ergodic provided for all $f, g \in L^2(\boldsymbol{\omega})$

$$\tfrac{1}{m^d} \textstyle\sum_{\mathbf{k} \in [\mathbf{0}, m\mathbf{1}]} \int_X f \circ S^{-\mathbf{k}} \cdot g \, d\mathcal{W} \longrightarrow \int_X f \, d\mathcal{W} \int_X g \, d\mathcal{W} \tag{5.1}$$

as $m \to \infty$. Characterizations for mixing are similar, and just as in the $n = 1$ case, mixing implies ergodic by looking at component measures, and so the familiar Fourier characterizations for mixing hold in this setting as well. Note that in the above, we do not include the case where the limits converge to $\left(\int g d\mathcal{W}\right) \left(\int f d\mathcal{W}\right)$ as possibilities, as the following theorem shows that these matrices will always commute as long as the matrix of measures is ergodic.

We say a matrix of measures $\mathcal{W}$ is *diagonalizable* if there exists a $\mathbf{Q} \in \mathbf{M}_n(\mathbb{C})$ invertible with $\mathbf{Q}\mathcal{W}\mathbf{Q}^{-1}$ a diagonal matrix of measures; the measures appearing on the diagonal are called *eigenmeasures* of $\mathcal{W}$. Recall that $\mathcal{M}_n^\circ$ is the collection of $n \times n$ matrices of measures which are invertible on some Borel subset of X.

Theorem 5.1.8. *If $\mathcal{W} \in \mathcal{M}_n(X,S) \cap \mathcal{M}_n^\circ$ is ergodic, it is diagonalizable with ergodic eigenmeasures. If $\mathcal{W}$ is mixing, so are its eigenmeasures.*

Proof. The result is almost an immediate consequence of propositions 5.1.6 and 5.1.7. For every $A \subset X$ Borel and $F \in [\mathbb{1}_A]$, consider the measure $F \cdot \mathcal{W} \in \mathcal{M}_n(X)$. As $\mathcal{W}$ is ergodic, for $g \in C(X)$

$$\int_X g\, d(F \cdot \mathcal{W}) = \int_X \partial_{\boldsymbol{\omega}}^* F\, g\, d\mathcal{W} = \int_X \mathbb{E}_{\boldsymbol{\omega}}(\mathbb{1}_A)\, g\, d\mathcal{W} = \lim_{m \to \infty} \int_X \mathcal{A}_m(\mathbb{1}_A)\, g\, d\mathcal{W} = \mathcal{W}(A) \int_X g\, d\mathcal{W}$$

by the Lebesgue dominated convergence theorem and the mean ergodic theorem. As measures in $\mathcal{M}(X)$ are characterized by integration against continuous functions, this implies that $F \cdot \mathcal{W} = \mathcal{W}(A)\mathcal{W}$, and so $\mathcal{W}(A) \in \mathrm{Eig}_{\mathrm{R}}(\mathcal{W})$ for all $A \subset X$ Borel. By proposition 5.1.6, all the matrices $\mathcal{W}(A)$ for A Borel are simultaneously diagonalizable over $\mathbb{C}$, and there exists a matrix $\mathbf{Q} \in \mathbf{M}_n(\mathbb{C})$ such that $\mathbf{Q}\mathcal{W}(A)\mathbf{Q}^{-1}$ is diagonal for all $A \subset X$ Borel. This implies that $\mathbf{Q}\mathcal{W}\mathbf{Q}^{-1}$ is a diagonal matrix of measures, and so $\mathcal{W}$ is diagonalizable over $\mathbb{C}$. The claim of ergodicity and mixing follows as

$$\mathbf{Q} \left(\textstyle\int f d\mathcal{W} \right) \mathbf{Q}^{-1} = \textstyle\int f\, d(\mathbf{Q}\mathcal{W}\mathbf{Q}^{-1})$$

and using the usual descriptions of ergodicity and mixing. □

5.2 Proof of Queffélec's Theorem

In this section, we prove our extension of Queffélec's theorem 3.3.1, which states:

Theorem. *If $\mathcal{S}$ is an aperiodic $\mathbf{q}$-substitution on $\mathcal{A}$, then for $\lambda_{\mathbf{v}} = \mathbf{v}^t \Sigma$,*

$$\sigma_{max} \sim \boldsymbol{\omega}_{\mathbf{q}} * \textstyle\sum_{\mathbf{w} \in \mathcal{K}^*} \lambda_{\mathbf{w}}$$

Moreover, the measures $\lambda_{\mathbf{w}}$ *for* $\mathbf{w} \in \mathcal{K}^*$ *are strong-mixing (of all orders) for the* $\mathbf{q}$*-shift.*

Recall that all our $\mathbf{q}$-substitutions $\mathcal{S}$ on $\mathcal{A}$ have been telescoped so that $\mathcal{S}$ and $\mathcal{S} \otimes \mathcal{S}$ have index of imprimitivity 1, see proposition 1.2.2. For such substitutions, note that both

$$\mathbf{P} := \lim_{n\to\infty} \tfrac{1}{Q^n} M_{\mathcal{S}}^n \quad \text{and} \quad \mathcal{P} := \lim_{n\to\infty} \tfrac{1}{Q^n} M_{\mathcal{S}\otimes\mathcal{S}}^n = \lim_{n\to\infty} \tfrac{1}{Q^n} \textstyle\sum_{\mathbf{j}<\mathbf{q}^n} \mathcal{R}_{\mathbf{j}}^{(n)} \otimes \mathcal{R}_{\mathbf{j}}^{(n)} \tag{5.2}$$

exist using the primitive reduced form of proposition 1.2.2 and results in [14, Chap III §7], and they converge to nonorthogonal projections onto the Q-eigenspaces of the substitution and bisubstitution matrices, respectively.

Proposition 5.2.1. *Let* $\mathcal{S}$ *be a be a* $\mathbf{q}$*-substitution on* $\mathcal{A}$*. If, for* $\mathbf{k} \in \mathbb{Z}^d$ *and* $n \geqslant 0$*, we write*

$$\mathbf{C}_{\mathbf{k}}^{(n)} := \tfrac{1}{Q^n} \textstyle\sum_{\mathbf{j}<\mathbf{q}^n} \mathcal{R}_{\mathbf{j}}^{(n)} \otimes \mathcal{R}_{\mathbf{j}+\mathbf{k}}^{(n)} \in \mathbf{M}_{\mathcal{A}^2}(\mathbb{C}) \qquad \textit{then} \qquad \mathbf{C}_{\mathbf{k}} := \lim_{n\to\infty} \mathbf{C}_{\mathbf{k}}^{(n)} \in \mathbf{M}_{\mathcal{A}^2}(\mathbb{C})$$

exists for every $\mathbf{k} \in \mathbb{Z}^d$*, and are Fourier coefficients for a matrix of measures* $\mathcal{Z} \in \mathcal{M}_{\mathcal{A}^2}(\mathbb{T}^d)$*.*

Proof. For $\omega \in \mathcal{A}^+$ and $n \geqslant 0$, let $\mathbf{h}_\omega^{(n)}$ be the vector in $\mathbb{C}^{\mathcal{A}}$ recording the frequencies with which ω appears in superblocks $\mathcal{S}^n(\gamma)$, so that for each $\gamma \in \mathcal{A}$,

$$\mathbf{e}_\gamma^* \mathbf{h}_\omega^{(n)} := {}^1\!/Q^n \,\mathrm{Card}\{\mathbf{j} \in \mathbb{Z}^d \;:\; \omega \leqslant T^{\mathbf{j}} \mathcal{S}^n(\gamma)\}$$

Although such frequencies are known to exist in general, we show it here for completeness.

Lemma 5.2.2. *If* $\mathcal{S}$ *is a* $\mathbf{q}$*-substitution on* $\mathcal{A}$*, then* $\lim_{n\to\infty} \mathbf{h}_\omega^{(n)}$ *exists for every* $\omega \in \mathcal{A}^+$

Proof. See also [22, Proof of Prop 10.4]. If ω is a subblock of $\mathcal{S}^{n+p}(\gamma)$ for $n, p > 0$, then either: ω appears as a subblock of $\mathcal{S}^n(\alpha)$ for some α appearing in $\mathcal{S}^p(\gamma)$, or ω overlaps two or more such blocks. First, the number of ways α can appear in the $\mathcal{S}^p(\gamma)$ is given by $\mathbf{e}_\alpha^* M_{\mathcal{S}}^p \mathbf{e}_\gamma$. Second, suppose $\mathrm{supp}(\omega)$ is contained in $[\mathbf{j}, \mathbf{j}+\mathbf{k})$ for some $\mathbf{j}, \mathbf{k} \in \mathbb{Z}^d$, then the number of ways ω can overlap multiple such blocks is bounded by $Q^p \mathrm{Card}(\Delta_p(\mathbf{k}))$, as $\mathbf{j}$ will be translated into one of the Q^p subblocks of $[\mathbf{0}, \mathbf{q}^{n+p})$ in the $\mathbf{q}^n\mathbb{Z}^d$ lattice. This gives

$$\sum_{\alpha\in\mathcal{A}} L_B(\mathcal{S}^n(\alpha)) \mathbf{e}_\alpha^* M_{\mathcal{S}}^p \mathbf{e}_\gamma \leqslant L_B(\mathcal{S}^{n+p}(\gamma)) \leqslant Q^p \mathrm{Card}(\Delta_n(\mathbf{k})) + \sum_{\alpha\in\mathcal{A}} L_B(\mathcal{S}^n(\alpha)) \mathbf{e}_\alpha^* M_{\mathcal{S}}^p \mathbf{e}_\gamma$$

Dividing through by Q^{n+p} and writing in vector form gives the componentwise inequalities

$$\frac{1}{Q^p}\mathbf{h}_n^* M_{\mathcal{S}}^p \leqslant \mathbf{h}_{n+p}^* \leqslant \frac{\mathrm{Card}(\Delta_n(\mathbf{k}))}{Q^n}\mathbf{1} + \frac{1}{Q^p}\mathbf{h}_n^* M_{\mathcal{S}}^p \tag{5.3}$$

where $\mathbf{1}$ is again the vector of 1's. Letting $p \to \infty$, we have

$$\mathbf{h}_n^*\mathbf{P} \leqslant \liminf_{p\to\infty} \mathbf{h}_{n+p}^* \leqslant \limsup_{p\to\infty} \mathbf{h}_{n+p}^* \leqslant \frac{\mathrm{Card}(\Delta_n(\mathbf{k}))}{Q^n} + \mathbf{h}_n^*\mathbf{P}$$

Moreover, the first inequality in (5.3) together with the identity $M_{\mathcal{S}}\mathbf{P} = Q\mathbf{P}$ gives $\mathbf{h}_n^*\mathbf{P} \leqslant \mathbf{h}_{n+p}^*\mathbf{P}$ so that, as $\mathbf{0} \leqslant \mathbf{h}_n^* \leqslant \mathbf{1}$ and $\mathbf{P}$ is nonnegative, the limit $\mathbf{h}_\infty^* := \lim_{n\to\infty} \mathbf{h}_n^*\mathbf{P}$ exists. Letting $n \to \infty$ in the above inequalities and using lemma 2.1.1 shows that the limit of $\mathbf{h}_n^*$ exists, as $\mathbf{h}_\infty^* \leqslant \liminf_{p\to\infty} \mathbf{h}_p^* \leqslant \limsup_{p\to\infty} \mathbf{h}_p^* \leqslant \mathbf{h}_\infty^*$, completing the proof. □

We now proceed to prove the corollary, see also [22, Prop 10.4]. By the partition formula 2.3.5, $[\omega] = \sqcup_{(\mathbf{j},\eta)\in\mathcal{S}^{-n}(\omega)} T^{\mathbf{j}}\mathcal{S}^n[\eta]$ where

$$\mathcal{S}^{-n}(\omega) = \{(\mathbf{j},\eta) \in [\mathbf{0},\mathbf{q}^n) \times \mathcal{A}^+ : \mathrm{supp}(\eta) = \lfloor \mathbf{j} + \mathrm{supp}(\omega)\rfloor_n \text{ and } \omega \leqslant T^{\mathbf{j}}\mathcal{S}^n(\eta)\}$$

so that $\mathbf{h}_\omega^{(n)}$ is counting those pairs $(\mathbf{j},\eta)$ for which $\mathrm{supp}(\eta) = \mathbf{0}$, i.e. whenever $\mathfrak{c}_n(\mathbf{j},\mathrm{supp}(\omega)) = \mathbf{0}$. In the case where ω is the block sending $\mathbf{0} \mapsto \alpha$ and $\mathbf{k} \mapsto \beta$, this means $\mathbf{h}_\omega^{(n)}$ is *under-counting*, while $\sum_{\mathbf{j}\in[\mathbf{0},\mathbf{q}^n)} \mathcal{R}_{\mathbf{j}}^{(n)} \otimes \mathcal{R}_{\mathbf{j}+\mathbf{k}}^{(n)}$ is *overcounting* (as in the proof of proposition 2.3.7), $\mathcal{S}^{-n}(\omega)$ by a bounded factor of $\mathrm{Card}(\Delta_n(\mathbf{k}))$. Thus, the above limits exist for every $\mathbf{k} \in \mathbb{Z}^d$.

Now, for $\mathbf{w} \in \mathbb{T}^d$ and $\mathbf{k} \in \mathbb{Z}^d$, write

$$\mathcal{Z}_n(\mathbf{w}) = \tfrac{1}{Q^n}\textstyle\sum_{\mathbf{i},\mathbf{j}<\mathbf{q}^n} \mathcal{R}_{\mathbf{j}}^{(n)} \otimes \mathcal{R}_{\mathbf{i}}^{(n)} \mathbf{w}^{\mathbf{i}-\mathbf{j}} \qquad \text{where} \qquad \mathbf{w}^{\mathbf{k}} = (w_1^{k_1}, \ldots, w_d^{k_d})$$

Viewing $\mathcal{Z}_n(\mathbf{w})d\mathbf{w} \in \mathcal{M}_{\mathcal{A}^2}(\mathbb{T}^d)$, for $\alpha\beta, \gamma\delta \in \mathcal{A}^2$ the total variation norm of $\mathbf{e}_{\alpha\beta}^*\mathcal{Z}_n\mathbf{e}_{\gamma\delta}$ is

$$\left\|\mathbf{e}_{\alpha\beta}^*\mathcal{Z}_n\mathbf{e}_{\gamma\delta}\right\| = \int_{\mathbb{T}^d} \left|\mathbf{e}_{\alpha\beta}^*\mathcal{Z}_n(\mathbf{w})\mathbf{e}_{\gamma\delta}\right| d\mathbf{w} = \frac{1}{Q^n}\int_{\mathbb{T}^d}\Big|\sum_{\mathbf{i},\mathbf{j}<\mathbf{q}^n} (\mathbf{e}_\alpha^*\mathcal{R}_{\mathbf{j}}^{(n)}\mathbf{e}_\gamma\mathbf{w}^{-\mathbf{j}})(\mathbf{e}_\beta^*\mathcal{R}_{\mathbf{i}}^{(n)}\mathbf{e}_\delta\mathbf{w}^{\mathbf{i}})\Big| d\mathbf{w}$$

Using the Cauchy-Schwartz inequality, this gives (where $s = \mathrm{Card}\mathcal{A}$)

$$\begin{aligned}\|\mathbf{e}^*_{\alpha\beta}\mathcal{Z}_n\mathbf{e}_{\gamma\delta}\| &\leqslant \frac{1}{Q^n}\Big(\int|\textstyle\sum_{\mathbf{j}<\mathbf{q}^n}\mathbf{e}^*_\alpha\mathcal{R}^{(n)}_\mathbf{j}\mathbf{e}_\gamma\mathbf{w}^{-\mathbf{j}}|^2 d\mathbf{w}\Big)^{1/2}\Big(\int|\textstyle\sum_{\mathbf{i}<\mathbf{q}^n}\mathbf{e}^*_\beta\mathcal{R}^{(n)}_\mathbf{i}\mathbf{e}_\delta\mathbf{w}^{\mathbf{i}}|^2\Big)^{1/2}\\ &= \frac{1}{Q^n}\Big(\textstyle\sum_{\mathbf{j}<\mathbf{q}^n}(\mathbf{e}^*_\alpha\mathcal{R}^{(n)}_\mathbf{j}\mathbf{e}_\gamma)^2\Big)^{1/2}\Big(\textstyle\sum_{\mathbf{i}<\mathbf{q}^n}(\mathbf{e}^*_\beta\mathcal{R}^{(n)}_\mathbf{i}\mathbf{e}_\delta)^2\Big)^{1/2} = \frac{1}{Q^n}(\mathbf{e}^*_\alpha\mathcal{M}^n_\mathcal{S}\mathbf{e}_\gamma)^{\frac12}(\mathbf{e}^*_\beta\mathcal{M}^n_\mathcal{S}\mathbf{e}_\delta)^{\frac12} \leqslant 1\end{aligned}$$

as $\mathbf{e}^*_\iota\mathcal{R}^{(n)}_\mathbf{k}\mathbf{e}_\kappa = 0$ or 1, and $M^n_\mathcal{S} = \sum_{\mathbf{j}<\mathbf{q}^n}\mathcal{R}^{(n)}_\mathbf{j}$. As $\|\mathcal{Z}_n\| \leqslant s^4$, the measures $\mathcal{Z}_n$ are uniformly bounded and a subsequence converges in the weak-star topology to some measure $\mathcal{Z}$. However, as

$$\widehat{\mathcal{Z}_n}(\mathbf{k}) = \tfrac{1}{Q^n}\textstyle\sum_{\mathbf{j}<\mathbf{q}^n}\mathcal{R}^{(n)}_\mathbf{j}\otimes\mathcal{R}^{(n)}_{\mathbf{j}+\mathbf{k}}$$

the Fourier coefficients have unique limits, so $\mathcal{Z}$ is the unique limit point of this sequence and

$$\widehat{\mathcal{Z}}(\mathbf{k}) = \lim_{n\to\infty}\widehat{\mathcal{Z}_n}(\mathbf{k}) = \lim_{n\to\infty}\tfrac{1}{Q^n}\textstyle\sum_{\mathbf{j}<\mathbf{q}^n}\mathcal{R}^{(n)}_\mathbf{j}\otimes\mathcal{R}^{(n)}_{\mathbf{j}+\mathbf{k}}$$

completing the proof. □

The matrix $\mathcal{Z} = (\sigma^{\gamma\delta}_{\alpha\beta})_{\alpha\beta,\gamma\delta\in\mathcal{A}^2}$ is the *bicorrelation matrix,* after Queffélec. Note that in the definition of $\mathcal{Z}$, the limit above can be split at p $\mathbf{q}$-adicly writing

$$\widehat{\mathcal{Z}}(\mathbf{k}) = \lim_{n\to\infty}\frac{1}{Q^p}\sum_{\lfloor\mathbf{j}\rfloor_p\in[\mathbf{0},\mathbf{q}^p)}\mathcal{R}^{(p)}_\mathbf{j}\otimes\mathcal{R}^{(p)}_{\mathbf{j}+\mathbf{k}}\frac{1}{Q^{n-p}}\sum_{\lfloor\mathbf{j}\rfloor_p\in[\mathbf{0},\mathbf{q}^{n-p})}\mathcal{R}^{(n-p)}_{\lfloor\mathbf{j}\rfloor_p}\otimes\mathcal{R}^{(n-p)}_{\lfloor\mathbf{j}\rfloor_p+\lfloor\mathbf{k}\rfloor_p+\mathfrak{c}_p(\mathbf{j},\mathbf{k})}$$

so that one has an identity for $\mathcal{Z}$ similar to that for Σ in theorem 3.1.2:

$$\widehat{\mathcal{Z}}(\mathbf{k}) = \lim_{n\to\infty}\frac{1}{Q^n}\sum_{j\in[\mathbf{0},\mathbf{q}^n)}\mathcal{R}^{(n)}_\mathbf{j}\otimes\mathcal{R}^{(n)}_{\mathbf{j}+\mathbf{k}} = \frac{1}{Q^p}\sum_{\mathbf{j}\in[\mathbf{0},\mathbf{q}^p)}\mathcal{R}^{(p)}_\mathbf{j}\otimes\mathcal{R}^{(p)}_{\mathbf{j}+\mathbf{k}}\widehat{\mathcal{Z}}\big(\lfloor\mathbf{k}\rfloor_p + \mathfrak{c}_p(\mathbf{j},\mathbf{k})\big) \quad (5.4)$$

For $\mathbf{k} = \mathbf{0}$, the first equality gives $\widehat{\mathcal{Z}}(\mathbf{0}) = \mathcal{P}$ and, by lemma 2.1.1, as $p \to \infty$ in the second,

$$\mathcal{Z} = \mathcal{Z}\widehat{\mathcal{Z}}(\mathbf{0}) = \mathcal{Z}\mathcal{P} \quad (5.5)$$

Proposition 5.2.3. *The matrix of measures $\mathcal{P}\mathcal{Z} \in \mathcal{M}_{\mathcal{A}^2}(\mathbb{T}^d)$ is diagonalizable over $\mathbf{M}_{\mathcal{A}^2}(\mathbb{C})$ with eigenmeasures strongly-mixing of all orders for the $\mathbf{q}$-shift $(\mathbb{T}^d, \mathbf{S}_\mathbf{q})$.*

Proof. First, we show that $\mathcal{P}\mathcal{Z}$ is an invariant matrix of measures for the $\mathbf{q}$-shift: for $\mathbf{a} \in \mathbb{Z}^d$,

$$\widehat{\mathcal{P}\mathcal{Z}}(\mathbf{aq}) = \widehat{\mathcal{P}\mathcal{Z}}(\mathbf{a}) \tag{5.6}$$

by the same operations used in §3 to show (3.5). Now, fix $\mathbf{a}, \mathbf{b} \in \mathbb{Z}^d$. Then, expressing $\mathbf{k} \in [\mathbf{0}, \mathbf{q}^{n+p})$ as $\mathbf{i} + \mathbf{j}\mathbf{q}^p$ for $\mathbf{i} < \mathbf{q}^p$ and $\mathbf{j} < \mathbf{q}^n$,

$$\begin{aligned} \widehat{\mathcal{Z}}(\mathbf{b} + \mathbf{a}\mathbf{q}^p) &= \lim_{n\to\infty} \frac{1}{Q^{n+p}} \sum_{\mathbf{i} < \mathbf{q}^p,\, \mathbf{j} < \mathbf{q}^n} \mathcal{R}^{(n+p)}_{\mathbf{i}+\mathbf{j}\mathbf{q}^p} \otimes \mathcal{R}^{(n+p)}_{\mathbf{i}+\mathbf{j}\mathbf{q}^p+\mathbf{b}+\mathbf{a}\mathbf{q}^p} \\ &= \lim_{n\to\infty} \frac{1}{Q^{n+p}} \sum_{\mathbf{i} < \mathbf{q}^p,\, \mathbf{j} < \mathbf{q}^n} \mathcal{R}^{(p)}_{\mathbf{i}} \mathcal{R}^{(n)}_{\mathbf{j}} \otimes \mathcal{R}^{(p)}_{\mathbf{i}+\mathbf{b}} \mathcal{R}^{(n)}_{\mathbf{j}+\mathbf{a}+\mathfrak{c}_p(\mathbf{i},\mathbf{b})} \end{aligned}$$

using the identities in (2.4). Using the mixed product property of the Kronecker product (see §4.3) the above computation for $\widehat{\mathcal{Z}}(\mathbf{b} + \mathbf{a}\mathbf{q}^p)$ continues as

$$\lim_{n\to\infty} \frac{1}{Q^{n+p}} \sum_{\mathbf{i}<\mathbf{q}^p} \big(\mathcal{R}^{(p)}_{\mathbf{i}} \otimes \mathcal{R}^{(p)}_{\mathbf{i}+\mathbf{b}} \sum_{\mathbf{j}<\mathbf{q}^n} \mathcal{R}^{(n)}_{\mathbf{j}} \otimes \mathcal{R}^{(n)}_{\mathbf{j}+\mathbf{a}+\mathfrak{c}_p(\mathbf{i},\mathbf{b})}\big) = \frac{1}{Q^p} \sum_{\mathbf{i}<\mathbf{q}^p} \mathcal{R}^{(p)}_{\mathbf{i}} \otimes \mathcal{R}^{(p)}_{\mathbf{i}+\mathbf{b}} \widehat{\mathcal{Z}}(\mathbf{a} + \mathfrak{c}_p(\mathbf{i}, \mathbf{b}))$$

As $\mathcal{Z} = \mathcal{Z}\mathcal{P}$, we let $p \to \infty$ and use lemma 2.1.1 again as in the proof of proposition 2.3.7

$$\lim_{p\to\infty} \widehat{\mathcal{P}\mathcal{Z}}(\mathbf{b} + \mathbf{a}\mathbf{q}^p) = \widehat{\mathcal{P}\mathcal{Z}}(\mathbf{b})\widehat{\mathcal{P}\mathcal{Z}}(\mathbf{a}) \tag{5.7}$$

so that $\mathcal{P}\mathcal{Z}$ defines a strong-mixing matrix of measures with respect to the $\mathbf{q}$-shift on $\mathbb{T}^d$. That it $\mathcal{P}\mathcal{Z}$ is strong-mixing of all orders follows just as in the above, using the exponential rate of convergence of lemma 2.1.1.

To finish, we need to extend theorem 5.1.8 to $\mathcal{P}\mathcal{Z}$, which says that an ergodic matrix of measures which is invertible on some Borel set is diagonalizable with ergodic eigenmeasures, although $\mathcal{P}\mathcal{Z}$ may not ever be invertible, as $\mathcal{P}$ is a nonsymmetric projection operator. As identity (5.5) gives $\mathcal{P}\mathcal{Z} = \mathcal{P}\mathcal{Z}\mathcal{P}$, the matrix of measures $\mathcal{P}\mathcal{Z}$ is zero on the kernel of $\mathcal{P}$, and will be diagonal on that subspace with respect to any basis for the kernel of $\mathcal{P}$. Thus, we can restrict to the image of $\mathcal{P}$, where $\mathcal{P}$ is the identity. This implies that $\mathcal{P}\mathcal{Z}(\mathbb{T}^d) = \mathcal{P}\widehat{\mathcal{Z}}(\mathbf{0}) = \mathcal{P}^2 = \mathcal{P}$ is the identity on the image of $\mathcal{P}$, so that we may consider $\mathcal{P}\mathcal{Z} \in \mathcal{M}^\circ_n$ as an *operator valued measure* on the image of $\mathcal{P}$. As $\mathcal{P}\mathcal{Z}$ is strong-mixing for the $\mathbf{q}$-shift, theorem 5.1.8 tells us that $\mathcal{P}\mathcal{Z}$ is diagonalizable with respect to the image of $\mathcal{P}$. As both the image and the kernel are invariant subspaces for $\mathcal{P}\mathcal{Z}$ (in particular as

$\widehat{\mathcal{PZ}}(\mathbf{0})$ is invertible on image of $\mathcal{P}$), it follows that $\mathcal{PZ}$ is diagonalizable. Ergodicity of the eigenmeasures follows from the strong-mixing property of $\mathcal{PZ}$. □

Let $\tilde{\mathcal{Z}}$ be the matrix of measures $(\sigma_{\alpha\beta}^{\gamma\delta})_{\alpha\gamma,\beta\delta\in\mathcal{A}^2}$, noting the change in indexing.

Lemma 5.2.4. *The matrix $\tilde{\mathcal{Z}}$ is a positive definite matrix of measures.*

Proof. Let $\{t_{\alpha\gamma}\}_{\alpha\gamma\in\mathcal{A}^2} \subset \mathbb{C}$. Denote by μ the measure $\mu := \sum_{\alpha\gamma,\beta\delta} t_{\alpha\gamma}\overline{t_{\beta\delta}}\sigma_{\alpha\beta}^{\gamma\delta}$. We will show that μ is a positive measure by showing that its Fourier coefficients form a positive definite sequence, and appealing to Bochner's theorem. Thus, we fix $n > 0$ and let $\{a_{\mathbf{j}}\}_{\mathbf{j}\in[\mathbf{0},\mathbf{q}^n)}$ be a sequence of complex numbers: we are interested in the nonnegativity of

$$\sum_{\mathbf{j},\mathbf{k}<\mathbf{q}^n} a_{\mathbf{k}}\overline{a_{\mathbf{j}}}\,\widehat{\mu}(\mathbf{j}-\mathbf{k}) = \sum_{\alpha\gamma,\beta\delta}\sum_{\mathbf{j},\mathbf{k}<\mathbf{q}^n} a_{\mathbf{k}}\overline{a_{\mathbf{j}}}t_{\alpha\gamma}\overline{t_{\beta\delta}}\,\widehat{\sigma_{\alpha\beta}^{\gamma\delta}}(\mathbf{j}-\mathbf{k})$$

For $\mathbf{j},\mathbf{k} \in \mathbb{Z}^d$, note that

$$\widehat{\sigma_{\alpha\beta}^{\gamma\delta}}(\mathbf{j}-\mathbf{k}) = \lim_{n\to\infty}\frac{1}{Q^n}\sum_{\mathbf{i}<\mathbf{q}^n}\mathbf{e}_{\alpha\beta}\mathcal{R}_{\mathbf{i}}^{(n)}\otimes\mathcal{R}_{\mathbf{i}+\mathbf{j}-\mathbf{k}}^{(n)}\mathbf{e}_{\gamma\delta} = \lim_{n\to\infty}\frac{1}{Q^n}\sum_{\mathbf{i}<\mathbf{q}^n}(\mathbf{e}_\alpha^*\mathcal{R}_{\mathbf{i}+\mathbf{k}}^{(n)}\mathbf{e}_\gamma)(\mathbf{e}_\beta^*\mathcal{R}_{\mathbf{i}+\mathbf{j}}^{(n)}\mathbf{e}_\delta)$$

as the definition of $\mathcal{R}^{(n)}$ gives invariance of the sum under translation in its index ($\mathbf{i} \mapsto \mathbf{i}+\mathbf{k}$) and the defining property of the Kronecker product. Thus, for any $n > 0$, and $\{a_{\mathbf{k}}\}_{\mathbf{k}<\mathbf{q}^n} \subset \mathbb{C}$,

$$\begin{aligned}\sum_{\mathbf{j},\mathbf{k}<\mathbf{q}^n} a_{\mathbf{j}}\overline{a_{\mathbf{k}}}\,\widehat{\mu}(\mathbf{j}-\mathbf{k}) &= \lim_{n\to\infty}\frac{1}{Q^n}\sum_{\alpha\gamma,\beta\delta}\sum_{\mathbf{j},\mathbf{k}<\mathbf{q}^n}\sum_{\mathbf{i}<\mathbf{q}^n}\Big((a_{\mathbf{k}}t_{\alpha\gamma}\mathbf{e}_\alpha^*\mathcal{R}_{\mathbf{i}+\mathbf{k}}^{(n)}\mathbf{e}_\gamma)(\overline{a_{\mathbf{j}}t_{\beta\delta}}\mathbf{e}_\beta\mathcal{R}_{\mathbf{i}+\mathbf{j}}^{(n)}\mathbf{e}_\delta)\Big)\\ &= \lim_{n\to\infty}\frac{1}{Q^n}\Big\|\sum_{\mathbf{i},\mathbf{k}<\mathbf{q}^n}\sum_{\gamma\alpha\in\mathcal{A}^2} a_{\mathbf{i}}t_{\gamma\alpha}\mathbf{e}_\alpha^*\mathcal{R}_{\mathbf{i}+\mathbf{k}}^{(n)}\mathbf{e}_\gamma\Big\|^2 \geqslant 0\end{aligned}$$

and so $\{\widehat{\mu}(\mathbf{k})\}_{\mathbf{k}\in\mathbb{Z}^d}$ forms a positive definite $\mathbb{Z}^d$-sequence. By Bochner's theorem, μ is a positive measure on $\mathbb{T}^d$, and it follows that $\tilde{\mathcal{Z}}$ is a positive definite matrix of measures. □

Corollary 5.2.5. *Both $\mathcal{Z}$ and $\mathcal{P}$ preserve strong semipositivity on $\mathbb{C}^{\mathcal{A}^2}$.*

Proof. As $\mathcal{P} = \widehat{\mathcal{Z}}(\mathbf{0}) = \mathcal{Z}(\mathbb{T}^d)$ and strong semipositivity of a vector-valued measure is determined pointwise, the statement for $\mathcal{P}$ will follow from the statement for $\mathcal{Z}$. Let $\mathbf{v}$ be strongly semipositive. We must show that $\mathcal{Z}\mathbf{v} = \sum_{\gamma\delta\in\mathcal{A}^2} v_{\gamma\delta}\sigma_{\alpha\beta}^{\gamma\delta}$ gives rise to a positive definite matrix of measures, or

$$\mathbf{z}^*(\mathcal{Z}\mathring{\mathbf{v}})\mathbf{z} = \textstyle\sum_{\alpha,\beta\in\mathcal{A}}\sum_{\gamma\delta\in\mathcal{A}^2} \overline{z_\alpha} z_\beta v_{\gamma\delta}\sigma_{\alpha\beta}^{\gamma\delta}$$

is a positive measure for every $\mathbf{z} \in \mathbb{C}^{\mathcal{A}}$. As $\mathring{\mathbf{v}} \geqq 0$, there exists a orthonormal basis $\{\mathbf{w}_\kappa\}_{\kappa\in\mathcal{A}}$ of eigenvectors with eigenvalues $\lambda_\kappa \geqslant 0$ (for $\kappa \in \mathcal{A}$) such that $\mathring{\mathbf{v}} = \sum \lambda_\kappa \mathbf{w}_\kappa \mathbf{w}_\kappa^*$. By the above and positivity of the λ_κ, it suffices to show for $\mathbf{w} \in \mathbb{C}^{\mathcal{A}}$ the expression $\sum_{\alpha,\beta} \overline{z_\alpha} z_\beta \overline{w_\gamma} w_\delta \sigma_{\alpha\beta}^{\gamma\delta}$ gives a positive measure. This follows immediately from lemma 5.2.4 by setting $t_{\alpha\gamma} = z_\alpha w_\gamma$. □

Corollary 5.2.6. *$\mathcal{PZ}$ is diagonalizable with respect to strongly semipositive eigenvectors.*

Proof. We know by proposition 5.2.3 that $\mathcal{PZ}$ can be diagonalized, all that remains is the choice of eigenvectors. Note that *a priori* the eigenmeasures are complex valued. As they are strong-mixing, however, they are ergodic and thus constant multiples of positive measures. As $\widehat{\mathcal{PZ}}(\mathbf{0}) = \mathcal{P}$ is the identity on the image of $\mathcal{P}$, it follows that $\widehat{\lambda}(\mathbf{0}) = 1$ for every eigenmeasure λ of $\mathcal{PZ}$. Thus the eigenmeasures are in fact probability measures, and therefore are either equal or mutually singular by ergodic decomposition. Fix an eigenmeasure λ of $\mathcal{PZ}$, and let D_λ be projection onto $\mathcal{L}(\lambda)$. As $\mathcal{P}$ and $\mathcal{Z}$ preserve strong semipositivity, we have $D_\lambda(\mathcal{PZ})$ preserves strong semipositivity as well, which follows as this property is determined pointwise on Borel sets. Thus, the matrix of measures $D_\lambda(\mathcal{PZ})$ is similar (via the same similarity diagonalizing $\mathcal{PZ}$) to a diagonal matrix of measures, with λ or $\mathbf{0}$ on the diagonal, and so $D_\lambda(\mathcal{PZ}) = \lambda P_\lambda$ for some projection operator $P_\lambda \in \mathbf{M}_{\mathcal{A}^2}(\mathbb{C})$. This, however, implies that P_λ preserves strong semipositivity, and as strongly semipositive vectors span $\mathbb{C}^{\mathcal{A}^2}$ (as $\mathbf{M}_n(\mathbb{C})$ is the $\mathbb{C}$-span of the (semi)positive definite matrices) it follows that the image of P_λ is spanned by strongly semipositive vectors. Thus the eigenspace corresponding to each eigenmeasure λ is spanned by strongly semipositive vectors, and we can therefore choose a basis of strongly semipositive eigenvectors for $\mathcal{PZ}$, as desired. □

Proposition 5.2.7. *There is a strongly semipositive basis with respect to which $\mathcal{Z}$ is similar to $\begin{pmatrix} \Lambda & \mathbf{0} \\ \mathcal{W} & \mathbf{0} \end{pmatrix}$ where Λ is a diagonal matrix of measures, and $|\mathcal{W}| \ll \boldsymbol{\omega}_{\mathbf{q}} * |\Lambda|$*

Proof. The specified basis is provided by proposition 5.2.3: let $\mathbf{w}_1, \ldots, \mathbf{w}_n$ be the strongly semipositive eigenvectors for nonzero eigenmeasures of $\mathcal{PZ}$, and $\tilde{\mathbf{w}}_{n+1}, \ldots, \tilde{\mathbf{w}}_{\mathcal{A}^2}$ those corresponding to the zero eigenmeasures. Then Λ corresponds to $\mathcal{Z}$ on the span of the $\mathbf{w}_j$ and $\mathcal{W}$ corresponds to $\mathcal{Z}$ on the span of the $\tilde{\mathbf{w}}_j$. That the last block column of $\mathcal{Z}$ is zero follows as $\mathcal{Z} = \mathcal{Z}\mathcal{P}$. That Λ is diagonal follows as $\mathcal{P}$ is the identity on the span of the $\mathbf{w}_j$, which diagonalize $\mathcal{P}\mathcal{Z}$. It remains to show that $|\mathcal{W}| \ll \boldsymbol{\omega}_{\mathbf{q}} * |\Lambda|$.

Let $\mathbf{L} = \mathcal{L}(\boldsymbol{\omega}_{\mathbf{q}} * |\Lambda|)$ be the set of measures absolutely continuous with respect to $\sum \boldsymbol{\omega}_{\mathbf{q}} * \lambda$ as λ ranges over eigenmeasures of $\mathcal{P}\mathcal{Z}$, and let $S_{\mathbf{q}}$ be the $\mathbf{q}$-shift on $\mathbb{T}^d$ sending $\mathbf{z} \mapsto \mathbf{z}^{\mathbf{q}}$ or $\mathbf{t} \mapsto \mathbf{qt}(\bmod \mathbf{1})$, writing $S_{\mathbf{q}}\mu := \mu \circ S_{\mathbf{q}}^{-1}$ for measures $\mu \in \mathcal{M}(\mathbb{T}^d)$. Note that the support of $|\Lambda| * \boldsymbol{\omega}_{\mathbf{q}}$ is $S_{\mathbf{q}}$-invariant, as the $\mathbf{q}$-adic rationals and support of Λ is $S_{\mathbf{q}}$-invariant. Similarly, the null sets of $|\Lambda| * \boldsymbol{\omega}_{\mathbf{q}}$ are also $S_{\mathbf{q}}$-invariant, and so it follows that $\mathbf{L}$ and $\mathbf{L}^{\perp}$ are $S_{\mathbf{q}}$-invariant $\mathcal{L}$-spaces of measures; see also [22, Lemma 10.4]. Let D and $D^{\perp}$ represent projections onto $\mathcal{L}$-spaces $\mathbf{L}$ and $\mathbf{L}^{\perp}$, respectively. Writing $\mathbf{w} := \sum \mathbf{w}_i$, then $\mathcal{P}\mathbf{w} = \mathbf{w}$ implies $C_{\mathcal{S}}\mathbf{w} = Q\mathbf{w}$

$$\mathcal{Z}\mathbf{w} = \sum \lambda_i \mathbf{w}_i + \mathcal{W}\mathbf{w} \quad \Longrightarrow \quad C_{\mathcal{S}}\mathcal{Z}\mathbf{w} = Q\sum \lambda_i \mathbf{w}_i + C_{\mathcal{S}}\mathcal{W}\mathbf{w} \quad \text{and} \quad \mathcal{P}\mathcal{W}\mathbf{w} = \mathbf{0}$$

Using identity (5.4) and $\widehat{S_{\mathbf{q}}\mu}(\mathbf{a}) = \hat{\mu}(\mathbf{aq})$, one checks that $S_{\mathbf{q}}\mathcal{Z} = \frac{1}{Q}C_{\mathcal{S}}\mathcal{Z}$, and so

$$S_{\mathbf{q}}\mathcal{Z}\mathbf{w} = \sum \lambda_i \mathbf{w}_i + \frac{1}{Q}S_{\mathbf{q}}\mathcal{W}\mathbf{w} \implies \frac{1}{Q}C_{\mathcal{S}}\mathcal{W}\mathbf{w} = S_{\mathbf{q}}\mathcal{W}\mathbf{w} \implies \frac{1}{Q^n}C_{\mathcal{S}}^n D^{\perp}\mathcal{W}\mathbf{w} = S_{\mathbf{q}}^n D^{\perp}\mathcal{W}\mathbf{w}$$

as $\mathbf{L}$ and $\mathbf{L}^{\perp}$ are $S_{\mathbf{q}}$-invariant. As $\mathcal{P}\mathcal{W}\mathbf{w} = \mathbf{0}$, this implies $S_{\mathbf{q}}^n(D^{\perp}\mathcal{W}\mathbf{w}) \to \mathbf{0}$ in norm and

$$S_{\mathbf{q}}^n\widehat{D^{\perp}\mathcal{W}}\mathbf{w}(\mathbf{0}) = \widehat{D^{\perp}\mathcal{W}}\mathbf{w}(\mathbf{0}) = \mathbf{0}$$

As strong semipositivity is determined pointwise and the $\lambda_i \in \mathbf{L}$, $D^{\perp}(\mathcal{Z}\mathbf{w}) = D^{\perp}\mathcal{W}\mathbf{w}$ is strongly semipositive, and thus $\mathbf{0}$. It follows that $\mathcal{W}\mathbf{w} = D(\mathcal{W}\mathbf{w})$. As the eigenvectors of $\mathcal{P}\mathcal{Z}$ for nonzero eigenmeasures span the image of $\mathcal{P}$, it follows that $|\mathcal{W}| \ll |\Lambda * \boldsymbol{\omega}_{\mathbf{q}}|$. □

The following lets us compute the eigenmeasures of $\mathcal{P}\mathcal{Z}$, see §3.2 for a description of $\mathcal{K}^*$.

Proposition 5.2.8. *The map $\mathbf{v} \mapsto \lambda_{\mathbf{v}} = \mathbf{v}^t\Sigma$ takes $\mathcal{K}^*$ onto the eigenmeasures of $\mathcal{P}\mathcal{Z}$.*

Proof. Let λ be an eigenmeasure and $\mathbf{v}$ strongly semipositive with $\mathbf{v}^t\mathcal{P}\mathcal{Z} = \lambda\mathbf{v}^t$. Then

$$\lambda\mathbf{v}^t = \mathbf{v}^t\mathcal{P}\mathcal{Z} = \mathbf{v}^t\mathcal{P}\mathcal{Z}\mathcal{P} = \lambda\mathbf{v}^t\mathcal{P} \quad \Longrightarrow \quad \mathbf{v}^t\mathcal{P} = \mathbf{v}^t$$

as $\mathcal{Z} = \mathcal{Z}\mathcal{P}$, so that $\mathbf{v}$ is a left Q-eigenvector of $C_{\mathcal{S}}$ and so $v_{\alpha\alpha}$ is constant for $\alpha \in \mathcal{A}$ by primitivity and stochasticity of $C_{\mathcal{S}}$. As $\mathbf{v}$ is strongly semipositive, this implies that $\mathbf{v}_{\alpha\alpha} \neq 0$ and moreover, that $\mathbf{v} \in \mathcal{K}$. Now, if $\mathbf{u}$ is the Perron vector of $M_{\mathcal{S}}$, (or if $u_\gamma = \mu([\gamma])$ for the invariant measure μ on $X_{\mathcal{S}}$) then

$$\lambda_{\mathbf{v}} = \mathbf{v}^t\Sigma = \mathbf{v}^t\mathcal{P}\Sigma = \mathbf{v}^t\mathcal{P}\mathcal{Z}\widehat{\Sigma}(\mathbf{0}) = \lambda\mathbf{v}^t\widehat{\Sigma}(\mathbf{0}) = \lambda\sum_{\gamma\in\mathcal{A}} v_{\gamma\gamma}u_\gamma = \lambda v_{\alpha\alpha}\sum u_\gamma = v_{\alpha\alpha}\lambda,$$

so $\lambda = \lambda_{\mathbf{w}}$ for some $\mathbf{w} \in \mathcal{K}$. As $\mathcal{P}\mathcal{Z}$ is diagonalized by strong semipositive vectors, $\mathcal{P}\mathcal{Z}$ can be written $\sum \lambda_j\mathcal{P}_j$, where $\mathcal{P}_j$ preserves strong semipositivity as in corollary 5.2.6. If $\mathbf{v} \in \mathcal{K}$

$$\mathbf{v}^t\Sigma = \mathbf{v}^t\mathcal{P}\mathcal{Z}\widehat{\Sigma}(\mathbf{0}) = \sum \lambda_j\mathbf{v}^t\mathcal{P}_j\widehat{\Sigma}(\mathbf{0}) = \sum c_j\lambda_j$$

with $c_j \geqslant 0$ by lemma 3.2.3. Thus $\mathbf{v} \mapsto \lambda_{\mathbf{v}}$ takes $\mathcal{K}$ onto the positive span of eigenmeasures of $\mathcal{P}\mathcal{Z}$. As linearity preserves convexity, $\mathcal{K}^*$ maps onto the eigenmeasures of $\mathcal{P}\mathcal{Z}$. □

Recall that theorem 3.1.4 relates Σ and $\sigma_{\max}$, and note that theorem 3.1.2 gives $\Sigma = \mathcal{Z}\widehat{\Sigma}(\mathbf{0})$.

Corollary 5.2.9. *The maximal spectral type of* $(X_{\mathcal{S}}, \mu)$ *is equivalent to* $\boldsymbol{\omega}_{\mathbf{q}} * |\mathcal{Z}|$.

Proof. Note that for $\alpha, \beta, \gamma \in \mathcal{A}$, $\sigma^{\gamma\gamma}_{\alpha\beta} = \sigma_{\alpha\beta}$, which can be seen immediately by comparing the respective Fourier coefficients and using the identities $\mathcal{Z} = \mathcal{Z}\mathcal{P}$ and $\mathcal{P}\mathbf{e}_{\gamma\gamma} = \widehat{\Sigma}(\mathbf{0})$ for $\gamma \in \mathcal{A}$. Let $\mathbf{v} = \mathbf{v}^{\gamma\delta}_{\alpha\beta} \in \mathbb{C}^{s^2}$ be the vector $(v_{\alpha'\beta'})$ where, for some $x, y \in \mathbb{C}$ not both 0,

$$v_{\alpha'\beta'} = \begin{cases} x & \text{if } \alpha'\beta' = \gamma\alpha \\ y & \text{if } \alpha'\beta' = \delta\beta \\ 0 & \textit{otherwise.} \end{cases}$$

Then, as $\widetilde{\mathcal{Z}}$ is positive definite, we have for all measurable $A \subset \mathbb{T}^d$,

$$\begin{aligned} \mathbf{v}^*\widetilde{\mathcal{Z}}(A)\mathbf{v} &= |x|^2\sigma^{\gamma\gamma}_{\alpha\alpha}(A) + |y|^2\sigma^{\delta\delta}_{\beta\beta}(A) + x\overline{y}\sigma^{\delta\gamma}_{\beta\alpha}(A) + y\overline{x}\sigma^{\gamma\delta}_{\alpha\beta}(A) \geqslant 0 \\ &= |x|^2\sigma_{\alpha\alpha}(A) + |y|^2\sigma_{\beta\beta}(A) + x\overline{y}\sigma^{\delta\gamma}_{\beta\alpha}(A) + y\overline{x}\sigma^{\gamma\delta}_{\alpha\beta}(A) \geqslant 0 \end{aligned}$$

Let A be such that $\sigma_{\alpha\alpha}(A) = 0$, and let $y = -1$ in the above, so we obtain:

$$\sigma_{\beta\beta}(A) - x\sigma^{\delta\gamma}_{\beta\alpha}(A) - \overline{x}\sigma^{\gamma\delta}_{\alpha\beta}(A) \geqslant 0$$

and letting $x \to \infty$ along the real and imaginary axes we obtain $\sigma^{\delta\gamma}_{\beta\alpha}(A) = \sigma^{\gamma\delta}_{\alpha\beta}(A) = 0$, and so $\sigma^{\gamma\delta}_{\alpha\beta} \ll \sigma_{\alpha\alpha}$. Letting $x = -1$, and $y \to \infty$ gives the corresponding result for $\sigma_{\beta\beta}$. Thus,

$$|\mathcal{Z}| = \sum |\sigma^{\gamma\delta}_{\alpha\beta}| \sim \sum_{\alpha=\beta} |\sigma^{\gamma\delta}_{\alpha\beta}| + \sum_{\alpha\neq\beta} |\sigma^{\gamma\delta}_{\alpha\beta}| \sim \sum |\sigma_{\alpha\alpha}| = \sum \sigma_{\alpha\alpha}$$

as the $\sigma_{\alpha\alpha}$ are positive measures, and the result follows as a corollary of theorem 3.1.4. □

With the above results established, we can prove Queffélec's theorem 3.3.1.

Theorem. *If $\mathcal{S}$ is an aperiodic $\mathbf{q}$-substitution on $\mathcal{A}$, then for $\lambda_{\mathbf{v}} = \mathbf{v}^t\Sigma$,*

$$\sigma_{max} \sim \boldsymbol{\omega}_{\mathbf{q}} * \sum_{\mathbf{w}\in\mathcal{K}^*} \lambda_{\mathbf{w}}$$

Moreover, the measures $\lambda_{\mathbf{w}}$ for $\mathbf{w} \in \mathcal{K}^*$ are strong-mixing (of all orders) for the $\mathbf{q}$-shift.

Proof. Let $S \in \mathbf{M}_{\mathcal{A}^2}(\mathbb{C})$ be the similarity matrix of proposition 5.2.7, and λ_i denote the eigenmeasures of $\mathcal{P}\mathcal{Z}$. Combining propositions 5.1.5 and 5.2.7 with corollary 5.2.9 gives

$$\sigma_{\max} \sim \boldsymbol{\omega}_{\mathbf{q}} * |\mathcal{Z}| \sim \boldsymbol{\omega}_{\mathbf{q}} * |S\mathcal{Z}S^{-1}| = \sum \boldsymbol{\omega}_{\mathbf{q}} * \lambda_i + \boldsymbol{\omega}_{\mathbf{q}} * |\mathcal{W}| \sim \sum \boldsymbol{\omega}_{\mathbf{q}} * \lambda_i$$

Using proposition 5.2.8 and positivity of measures in $\lambda(\mathcal{K}^*)$ gives $\boldsymbol{\omega}_{\mathbf{q}} * \sum_i \lambda_i \sim \boldsymbol{\omega}_{\mathbf{q}} * \sum_{\mathbf{w}\in\mathcal{K}^*} \lambda_{\mathbf{w}}$. The strong-mixing property of the measures $\lambda_{\mathbf{w}}$ follow from proposition 5.7 □

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