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# Non-local operators, jump diffusions and Feynman-Kac transforms

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**Abstract**

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Non-local operators are analytically defined by integrals over the whole space, hence hard to study certain properties. This thesis studies inverse local times at 0 of one-dimensional reflected diffusions on  $[0, \infty)$ , and establishes a new comparison principle for inverse local times. As an application, we obtain the Green function estimates for a class of non-local operators.

We further study diffusions with jumps, which are associated with the combination of local and non-local operators. We show that the two-sided heat kernel estimates for a class of (not necessarily symmetric) diffusions with jumps are stable under non-local Feynman-Kac perturbations.

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## NOTATION INDEX

$B_r(x)$	a ball centered at $x$ with radius $r$
$B_r$	a ball centered at the origin with radius $r$
$\mathbb{P}_x, \mathbb{E}_x$	probability, expectation with $x$ as the starting point
$\hat{f}$	Fourier transform of $f$
$\mathcal{F}_t$	$\mathbb{P}$ -complete $\sigma$ -field generated by $(X_s; s \leq t)$
$G_D(x, y)$	Green function in an open set $D$
$\mathbb{P}, \mathbb{Q}$	probability measures
$\mathbb{R}_+$	$[0, \infty)$
$S_t$	inverse local time at 0 (unless otherwise stated)
$1_D$	indicator function of a set $D$
$\stackrel{d}{=}$	identically distributed
$:=$	is defined as
$a \vee b$	$\max\{a, b\}$
$a \wedge b$	$\min\{a, b\}$
$f \sim g$	$\lim(f/g) = 1$
$f \asymp g$	there is a constant $c \geq 1$ so that $g/c \leq f \leq cg$
$f \lesssim g$	there is a constant $c > 0$ so that $f \leq cg$
$f \gtrsim g$	there is a constant $c > 0$ so that $f \geq cg$

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## **DEDICATION**

to my parents

Bozhen Wang and Rong Li

## Chapter 1

## INTRODUCTION

The first chapter provides the necessary background for this thesis. We will briefly introduce Lévy processes, local times and inverse local times. Then we will introduce diffusions with jumps, and two-sided heat kernel estimates. Finally, we will give definitions of Girsanov and Feynman-Kac transforms.

### 1.1 Lévy processes, infinitesimal generators

In this section, we give definitions of Lévy processes, the Lévy -Khintchine formula, and the infinitesimal generators.

**Definition 1.1.1.** *We say that  $X$  is a **Lévy process** for  $(\Omega, \mathcal{F}, \mathbb{P})$  if  $\mathbb{P}(X_0 = 0) = 1$ , and for any  $s, t \geq 0$ , the increment  $X_{t+s} - X_t$  is independent of the process  $\{X_r, 0 \leq r \leq t\}$  and has the same law as  $X_s$ . There is  $\Omega_0 \in \mathcal{F}$  with  $\mathbb{P}[\Omega_0] = 1$  such that, for every  $\omega \in \Omega_0$ ,  $X_t(\omega)$  is right-continuous in  $t \geq 0$  and has left limits in  $t > 0$ .*

By the definition,  $X_t$  has a characteristic exponent  $\Psi(\xi)$  given by

$$\mathbb{E}[\exp(i\langle \xi, X_t \rangle)] = \exp(-t\Psi(\xi)), \quad \xi \in \mathbb{R}^d.$$

The characteristic exponent  $\Psi(\xi)$  characterizes the law of the Lévy process in the sense that two Lévy processes with the same characteristic exponent have the same law. The Lévy -Khintchine formula gives the expression of  $\Psi(\xi)$  as follows:

$$\Psi(\xi) = i\langle a, \xi \rangle + \frac{1}{2}Q(\xi) + \int_{\mathbb{R}^d} (1 - e^{i\langle x, \xi \rangle} + i\langle x, \xi \rangle 1_{\{|x|<1\}}) \Pi(dx), \quad (1.1.1)$$

where  $a \in \mathbb{R}^d$ ,  $Q$  is a positive semi-definite quadratic form,  $\Pi$  is a measure defined on  $\mathbb{R}^d \setminus \{0\}$  such that  $\int_{\mathbb{R}^d} (1 \wedge |x|^2) \Pi(dx) < \infty$ , and  $\Pi$  is called the Lévy measure of  $X$ .

The Lévy process  $X$  has an infinitesimal generator  $\mathcal{L}$ : for any  $f \in \mathcal{S}$ , where  $\mathcal{S}$  is the Schwartz space of rapidly decreasing functions,

$$\begin{aligned} \mathcal{L}f(x) := & -\langle a, \nabla f(x) \rangle + \frac{1}{2} \sum_{1 \leq i, j \leq d} Q_{ij} \frac{\partial^2}{\partial x_i \partial x_j} f(x) \\ & + \int_{\mathbb{R}^d} (f(x+y) - f(x) - 1_{\{|y| < 1\}} \langle y, \nabla f(x) \rangle) \Pi(dy). \end{aligned} \quad (1.1.2)$$

**Examples 1.1.2.** *Brownian motion on  $\mathbb{R}^d$  is a Lévy process, with characteristic exponent  $\Psi(\xi) = |\xi|^2/2$ . The infinitesimal generator of Brownian motion is  $\Delta/2$ , in other words, for any  $y \in \mathbb{R}^d$ ,  $(t, x) \mapsto p(t, x, y)$  is a solution of the (Fokker-Planck) equation:*

$$\frac{\partial u(t, x)}{\partial t} = \frac{1}{2} \Delta u(t, x). \quad (1.1.3)$$

*Brownian motion has nice properties, for example, it is a Gaussian martingale and has strong Markov property.*

It is well known that aside from Brownian motion with constant drift, all other Lévy processes have discontinuous sample paths. The following example gives one of them.

**Examples 1.1.3.** *One important sub-family of Lévy processes is the class of rotationally symmetric  $\alpha$ -stable processes for  $\alpha \in (0, 2)$ . The characteristic exponent of rotationally symmetric  $\alpha$ -stable process  $X_t$  is given by*

$$\mathbb{E} e^{i \langle \xi, X_t \rangle} = e^{-t|\xi|^\alpha}, \quad \xi \in \mathbb{R}^d, \quad (1.1.4)$$

*its infinitesimal generator is the fractional Laplacian  $-(-\Delta)^{\alpha/2}$ , i.e., the transition density  $p(t, x, y)$  of  $X$  is the fundamental solution of the equation:*

$$\frac{\partial u(t, x)}{\partial t} = -(-\Delta)^{\alpha/2} u(t, x). \quad (1.1.5)$$

*Unlike  $\Delta$ , the fractional Laplacian operator is non-local. Analytically it can be defined in two ways:*

- For  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $(-\Delta)^{\alpha/2} f(x) = C_{d,\alpha} p.v. \int_{\mathbb{R}^d} \frac{f(x) - f(y)}{|x-y|^{d+\alpha}} dy$ , where  $C_{d,\alpha}$  is some normalization constant;

- For  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $(-\Delta)^{\alpha/2} f(\xi) = |\xi|^\alpha \hat{f}(\xi)$ , where  $\hat{f}$  represents the Fourier transform of  $f$ .

**Examples 1.1.4.** A subordinator is a Lévy process that takes values in  $[0, \infty)$  such that  $t \mapsto X_t$  is nondecreasing. For subordinators, one can work with the Laplace transform, and the infinite divisibility of the law of  $X$  implies that

$$\mathbb{E}[\exp(-\lambda X_t)] = \exp(-t\phi(\lambda)), \quad \lambda \geq 0, \quad (1.1.6)$$

where  $\phi(\lambda) : [0, \infty) \rightarrow [0, \infty)$  is called the Laplace exponent of  $X_t$ , there exists a unique pair  $(a, b)$  of nonnegative real numbers and a unique measure  $\nu$  on  $(0, \infty)$  with  $\int_{(0, \infty)} (1 \wedge x) \nu(dx) < \infty$  such that for every  $\lambda \geq 0$ ,

$$\phi(\lambda) = a + b\lambda + \int_{(0, \infty)} (1 - e^{-\lambda x}) \nu(dx), \quad (1.1.7)$$

and  $\nu$  is called the Lévy measure of  $X_t$ .

In particular, when  $\phi(\lambda) = c_s \lambda^s$  with  $s \in (0, 1)$ ,  $X_t$  is an  $s$ -stable subordinator.

Molchanov-Ostrovski [36] realized rotationally symmetric  $\alpha$ -stable processes as Brownian motions on  $\mathbb{R}^d$  time changed by an independent  $\alpha/2$ -stable subordinator, in other words,  $X_t$  is an  $\alpha/2$ -stable subordinator, and  $B_t$  is a  $d$ -dimensional Brownian motion independent of  $X_t$ , then the subordinate Brownian motion  $B_{X_t}$  is an  $\alpha$ -stable process. This realization can also be interpreted as traces of diffusion processes in one-dimension higher, details will be given in Theorem 1.2.12.

Caffarelli-Silvestre [6] rediscovered this result, by an analytic approach, that fractional Laplacian operators can be constructed from an extension problem to the upper half space for a specific degenerate elliptic PDE. For a function  $f(x) : \mathbb{R}^d \rightarrow \mathbb{R}$ , consider the extension function  $u(x, t) : \mathbb{R}^d \times [0, \infty) \rightarrow \mathbb{R}$  that satisfies

$$u(x, 0) = f(x), \quad \text{for } x \in \mathbb{R}^d; \quad (1.1.8)$$

$$\Delta_x u + \frac{1 - \alpha}{t} u_t + u_{tt} = 0, \quad (1.1.9)$$

where  $\Delta_x$  represents Laplacian operator in  $x \in \mathbb{R}^d$ . It can be shown that, up to a constant factor,  $\lim_{t \rightarrow 0^+} t^{1-\alpha} u_t(x, t) = -(-\Delta)^{\alpha/2} f(x)$  for any smooth and bounded function  $f(x) : \mathbb{R}^d \rightarrow \mathbb{R}$ , then using reflection extensions of  $u$ , they obtained a solution to the corresponding extension problem. By using (local) PDE methods, they gave an alternative proof to Harnack inequality and boundary Harnack inequality for fractional Laplacian; see Theorem 5.1 and 5.3 in [6].

**Theorem 1.1.5.** *Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$  and  $(-\Delta)^{\alpha/2} f = 0$  in  $B_r := \{x \in \mathbb{R}^d : |x| \leq r\}$ . Then there is a constant  $C$  (depending only on  $\alpha$  and  $d$ ) such that*

$$\sup_{x \in B_{r/2}} f(x) \leq C \inf_{x \in B_{r/2}} f(x).$$

**Theorem 1.1.6.** *Let  $\Omega$  be a domain such that  $\partial\Omega \cap B_1$  is a Lipschitz graph with Lipschitz constant less than 1. Let  $f, g : \mathbb{R}^d \rightarrow \mathbb{R}_+$  such that  $(-\Delta)^{\alpha/2} f = (-\Delta)^{\alpha/2} g = 0$  in  $\Omega$  and  $f(x) = g(x) = 0$  for any  $x \in B_1 \setminus \Omega$ . Then for any  $x_0 \in \partial\Omega$ , there is a constant  $C$  depending only on  $d$  such that*

$$\sup_{\Omega \cap B_{1/2}(x_0)} \frac{f(x)}{g(x)} \leq C \inf_{\Omega \cap B_{1/2}(x_0)} \frac{f(x)}{g(x)},$$

where  $B_{1/2}(x_0)$  is a ball centered at  $x_0$  with radius  $1/2$ .

## 1.2 Local times, inverse local times

$X_t$  is a Lévy process in  $\mathbb{R}^d$  on the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . The construction of local times is the generalization of the local time of one-dimensional Brownian motions. We denote by  $\mathbb{P}_x, \mathbb{E}_x$  probability and expectation with  $x$  as the starting point, respectively.

**Definition 1.2.1.** *We say that  $x \in \mathbb{R}^d$  is **regular for itself** with respect to  $X$  if  $\mathbb{P}_x(T_x = 0) = 1$ , where  $T_x := \{t > 0 : X_t = x\}$ .*

The *local time* at  $x$  exists if and only if  $x$  is regular for itself, and we define the *local time* at  $x$  of a Lévy process  $(X_t, \mathbb{P})$  as follows:

**Definition 1.2.2.**  $\{L^X(\omega, t, x) : t \geq 0\}$  is called the **local time** at  $x$  of a Lévy process  $(X_t, \mathbb{P})$  if  $L^X(t, x)$  is  $\mathcal{F}_t$ -measurable for any  $t$  and  $\mathbb{E}_y L^X(t, x) > 0$  for some  $t$  and  $y$  and if there is  $\Omega_0 \in \mathcal{F}$  with  $\mathbb{P}_y(\Omega_0) = 1$  for every  $y$  such that for all  $\omega \in \Omega_0$ , the following are satisfied:

(1)  $t \mapsto L^X(\omega, t, x)$  is continuous and increasing, and  $L^X(\omega, 0, x) = 0$ ;

(2)  $L^X(\omega, s + t, x) = L^X(\omega, s, x) + L^X(\theta_s \omega, t, x)$  for all  $s$  and  $t$ , where the shift  $\theta_s$  of sample function is defined by  $(\theta_s \omega)(t) = \omega(s + t)$ ;

(3)  $\int_0^\infty 1_{\mathbb{R}^d \setminus \{x\}}(X_t(\omega)) dL^X(\omega, t, x) = 0$ , where the integral is Stieltjes in  $t$ .

We write  $L^X(\omega, t, x)$  as  $L_t$  when there is no confusion.

It is known that there is a property that if  $\{L_t\}$  and  $\{\tilde{L}_t\}$  are both local times at  $x$ , then there is a constant  $c > 0$  such that for all  $y$ ,  $\mathbb{P}_y(\tilde{L}_t = cL_t \text{ for all } t) = 1$ , see [38].

Given a strong Markov process  $X$ , if the origin is an accessible point, and it is regular for itself, the local time at the origin was constructed in [35] that satisfies

$$\mathbb{E} \left[ \int_t^\infty e^{-s} dL_s \middle| \mathcal{F}_t \right] = \mathbb{E} [e^{-T_0 \circ \theta_t} | \mathcal{F}_t], \quad (1.2.1)$$

where  $T_0$  is the first hitting time at the origin for  $X$ , and  $\theta_t$  is the time shift operator. It is shown that this construction satisfies the conditions in Definition 1.2.2 with  $x$  as the origin, in other words, the two constructions of local times at the origin are equivalent, see Theorem 2 in [35].

The inverse local time at the origin, denoted by  $S_t$ , is defined as

$$S_t := \inf\{s \geq 0, L_s > t\},$$

where  $L_s$  is the local time at the origin. It is a general fact that the inverse local time of a Markov process at a point having positive capacity is always a subordinator, that is, a non-decreasing real-valued Lévy process. Also, the closure of the range of the inverse local

time coincides with the closure of the zero set of  $X$ , see regenerative embedding theory in [2]. Fristedt-Pruitt showed that there exists an increasing function  $g$  on  $[0, \infty)$  such that

$$g - m(S[0, t]) = t, \quad (1.2.2)$$

where the left side represents the Hausdorff measure of the range of  $S$  on the time interval  $[0, t]$  with respect to the function  $g$ . See [30] for details.

An interesting problem is if it is possible to characterize subordinators as inverse local times at zero of diffusions, equivalently, to realize the family of functions that can arise as Laplace exponents of inverse local times at zero of diffusions. This question was raised in Itô-McKean [32], then Knight [33] and Kotani-Watanabe [34] showed independently in 1981-1982 that if one relaxes  $\mathbb{R}_+$ -valued diffusions to generalized diffusions (a family of Markov processes having possibly discontinuous trajectories), then the answer is affirmative.

**Theorem 1.2.3** (Knight [33]). *The class of Lévy measures of inverse local times of regular generalized diffusion (the definition will be given later) on  $[0, \infty)$ , reflected at 0, consists of all*

$$\nu(dx) = \int_0^\infty e^{-xz} \mu(dz) dx \quad (1.2.3)$$

with measure  $\mu(dz) \geq 0$  on  $(0, \infty)$  such that  $\int_0^\infty \frac{1}{x(1+x)} \mu(dx) < \infty$ .

The following gives one way to explain the above theorem.

**Definition 1.2.4.** *A function  $\phi : (0, \infty) \rightarrow \mathbb{R}$  is said to be **completely monotone** if  $\phi$  is  $C^\infty$  and*

$$(-1)^n \phi^{(n)}(\lambda) \geq 0 \text{ for all } n \in \mathbb{N} \cup \{0\} \text{ and } \lambda > 0. \quad (1.2.4)$$

*The family of all completely monotone functions will be denoted by  $\mathcal{CM}$ .*

The condition (1.2.4) is often referred as Bernstein-Hausdorff-Widder condition, and the next theorem is known as Bernstein's theorem.

**Theorem 1.2.5.** *Let  $\phi : (0, \infty) \rightarrow \mathbb{R}$  be a completely monotone function. Then it is the Laplace transform of a unique measure  $\mu$  on  $[0, \infty)$ , i.e., for all  $\lambda > 0$ ,*

$$\phi(\lambda) = \int_{[0, \infty)} e^{-\lambda t} \mu(dt). \quad (1.2.5)$$

*Conversely, if  $\phi(\lambda)$  is given by (1.2.5) for some measure  $\mu$  on  $[0, \infty)$ ,  $\lambda \mapsto \phi(\lambda)$  is a completely monotone function.*

The set  $\mathcal{CM}$  of completely monotone functions is a convex cone and is closed under multiplication. The class of Bernstein functions is closely related to the class of completely monotone functions, and it's defined as follows:

**Definition 1.2.6.** *A function  $\phi : (0, \infty) \rightarrow \mathbb{R}$  is a **Bernstein function** if  $\phi$  is of class  $C^\infty$ ,  $\phi(\lambda) > 0$  and*

$$(-1)^{n-1} \phi^{(n)}(\lambda) \geq 0 \text{ for all } n \in \mathbb{N} \text{ and } \lambda > 0. \quad (1.2.6)$$

*The set of all Bernstein functions will be denoted by  $\mathcal{BF}$ .*

The next theorem is an observation that a non-negative  $C^\infty$ -function  $\phi$  is a Bernstein function if and only if  $\phi'$  is a completely monotone function.

**Theorem 1.2.7.** *A function  $\phi : (0, \infty) \rightarrow \mathbb{R}$  is a Bernstein function if, and only if, it admits the representation (1.1.7), where  $a, b \geq 0$  and  $\nu$  is a measure on  $(0, \infty)$  satisfying  $\int_{(0, \infty)} (1 \wedge x) \nu(dx) < \infty$ .*

**Definition 1.2.8.** *A Bernstein function  $\phi$  is said to be a **complete Bernstein function** if its Lévy measure  $\nu$  in (1.1.7) has a completely monotone density  $\nu(x)$  with respect to Lebesgue measure:*

$$\phi(\lambda) = a + b\lambda + \int_{(0, \infty)} (1 - e^{-\lambda x}) \nu(x) dx. \quad (1.2.7)$$

*We will use  $\mathcal{CBF}$  to denote the collection of all complete Bernstein functions.*

The final crucial tool is Krein's theory of strings, which is introduced as follows.

**Definition 1.2.9.** A *string* is a non-decreasing, right-continuous function  $m : \mathbb{R} \rightarrow [0, \infty)$  satisfying

1.  $m(x) = m(0-) = 0$  for all  $x < 0$ ,
2.  $m(x_0) < \infty$  for some  $x_0 \geq 0$ ,
3.  $m(x) > 0$  for all  $x > 0$ .

The family of all strings will be denoted by  $\mathfrak{M}_+$ .

For any  $m \in \mathfrak{M}_+$ , let  $r := \sup\{x : m(x) < \infty\}$  and  $E_m := \text{supp } m \cap [0, r)$ . Let  $B^+ = (B_t^+, \mathbb{P}_x)_{t \geq 0, 0 \leq x < r}$  be a reflected Brownian motion on  $[0, \infty)$  which is killed upon hitting the point  $r$ , after being killed, the process goes to the cemetery  $\partial$  which we adjoin to the state space  $[0, r)$  as an isolated point. Let  $L = (L(t, x))_{t \geq 0, 0 \leq x < r}$  be the jointly continuous local time of  $B^+$  normalized such that for all Borel functions  $g : [0, \infty) \rightarrow [0, \infty)$ , the following occupation time formula holds:

$$\int_0^t g(B_s^+) ds = \int_{[0, r)} g(x) L(t, x) dx, \quad t \geq 0.$$

Now define a positive continuous additive functional  $C = (C_t)_{t \geq 0}$  of the process  $B^+$  by

$$C_t := \int_{[0, r)} L(t, x) m(dx),$$

with the Revuz measure  $m$ . Let  $\tau_t$  be the right-continuous inverse  $\tau_t := \inf\{s > 0, C_s > t\}$ , then  $X_t := B_{\tau_t}^+$  is an  $m$ -symmetric Hunt process on  $E_m$ , and it is called a **generalized diffusion** on  $[0, r)$ .

Schilling-Song-Vondraček restated Theorem 1.2.3 in terms of strings theory.

**Theorem 1.2.10.** Let  $X$  be a generalized diffusion corresponding to  $m \in \mathfrak{M}_+$  and let  $S^X$  be its inverse local time at zero. Then the Laplace exponent  $f$  of  $S^X$  belongs to  $\mathcal{CBF}$ . Conversely, given any function  $f \in \mathcal{CBF}$ , there exists a generalized diffusion such that  $f$  is the Laplace exponent of its inverse local time at zero.

Now define for  $\lambda > 0$  the  $\lambda$ -potential operator of  $X$  as follows:

$$G_\lambda g(x) := \mathbb{E}_x \int_0^\infty e^{-\lambda t} g(X_t) dt, \quad (1.2.8)$$

where  $x \in E_m$  and  $g$  is a non-negative Borel function. The generalized diffusion  $X$  admits the local time process at  $x$ ,  $L^X = (L^X(t, x))_{t \geq 0, x \in E_m}$ , which can be realized as a time-change of the local time of a reflected Brownian motion. For any bounded measurable function  $g$ , one can have

$$\int_0^s g(X_t) dt = \int_{[0, r)} g(x) L^X(s, x) m(dx).$$

Thus, by Fubini's theorem, the  $\lambda$ -potential can be rewritten as

$$\begin{aligned} G_\lambda g(x) &= \mathbb{E}_x \int_0^\infty \lambda e^{-\lambda s} \left( \int_0^s g(X_t) dt \right) ds \\ &= \mathbb{E}_x \int_0^\infty \lambda e^{-\lambda s} \left( \int_{[0, r)} g(y) L^X(s, y) m(dy) \right) ds \\ &= \int_{[0, r)} \mathbb{E}_x \left( \int_{[0, \infty)} e^{-\lambda s} L^X(ds, y) \right) g(y) m(dy) \end{aligned}$$

We could define the kernel of the  $\lambda$ -potential operator  $G_\lambda$  as

$$G_\lambda(x, y) := \mathbb{E}_x \left( \int_{[0, \infty)} e^{-\lambda s} L^X(ds, y) \right), \quad x, y \in E_m. \quad (1.2.9)$$

Correspondingly, for any bounded measurable function  $g$ ,

$$G_\lambda g(x) = \int_{[0, r)} G_\lambda(x, y) g(y) m(dy).$$

Thus, we can have the following result,

**Proposition 1.2.11.** *Suppose  $X$  is a one-dimensional generalized diffusion corresponding to  $m \in \mathfrak{M}_+$ , it holds that*

$$G_\lambda(0, 0) = \frac{1}{\phi(\lambda)},$$

where  $\phi(\lambda)$  is the Laplace component of the inverse local time at zero.

**Proof.** From (1.2.9), and denote by  $L^X(t, 0), S^X(t)$  the local time and inverse local time at zero,

$$\begin{aligned} G_\lambda(0, 0) &= \mathbb{E}_0 \int_{[0, \infty)} e^{-\lambda t} L^X(dt, 0) = \mathbb{E}_0 \int_0^\infty e^{-tS^X(t)} dt \\ &= \int_0^\infty \mathbb{E}_0(e^{-\lambda S^X(t)}) dt = \int_0^\infty e^{-t\phi(\lambda)} dt \\ &= \frac{1}{\phi(\lambda)}. \end{aligned}$$

□

Applying the above Propostion, we can explicitly realize stable subordinators as inverse local times of diffusions, this is also Theorem 1 and 2 in [36].

**Theorem 1.2.12.** *Let  $X_t$  be a Bessel process with index  $0 < \alpha < 1$  that is generated by*

$$\mathcal{L}_\alpha = \frac{1}{2} \frac{d^2}{dx^2} + \frac{1 - 2\alpha}{2x} \frac{d}{dx}, \quad X_0 = x \geq 0. \quad (1.2.10)$$

*By Proposition I.7.2 in [1], 0 is a reflection boundary. Denote by  $S_t$  the inverse local time at  $x = 0$ , then  $S_t$  is an  $\alpha$ -stable subordinator.*

*Consider the  $d+1$ -dimensional process  $Z_t = (B_t, X_t)$ , where  $B_t$  is a  $d$ -dimensional Brownian motion independent of  $X_t$ . Then, the process  $Z_t = B_{S_t}$  is a symmetric  $2\alpha$ -stable process.*

**Proof.** Bessel process is a diffusion process, from [36] we have the transition density for  $X_t$  as

$$p(t; x, y) = \frac{1}{2t} (xy)^\alpha \exp\left(-\frac{x^2 + y^2}{2t}\right) I_{-\alpha}\left(\frac{xy}{t}\right), \quad (1.2.11)$$

where  $I_a(x) = \sum_{m=0}^\infty \frac{1}{m! \Gamma(m+a+1)} (x/2)^{2m+a}$ . Since the  $I_a(x)$  converges near 0 for all  $-1 < a < 0$ , we get

$$\begin{aligned} p(t; 0, 0) &= \lim_{x \rightarrow 0, y \rightarrow 0} p(t; x, y) \\ &= \lim_{x \rightarrow 0, y \rightarrow 0} \exp\left(-\frac{x^2 + y^2}{2t}\right) \sum_{m=0}^\infty \frac{1}{m! \Gamma(m - \alpha + 1)} \frac{(xy)^{2m}}{(2t)^{2m - \alpha + 1}} \\ &= \frac{1}{\Gamma(\alpha)} \frac{1}{(2t)^{1 - \alpha}}. \end{aligned}$$

By the definition of  $G_\lambda(x, y)$  and dominated convergence theorem,

$$\begin{aligned}
G_\lambda(0, 0) &= \lim_{x \rightarrow 0, y \rightarrow 0} G_\lambda(x, y) = \lim_{x \rightarrow 0, y \rightarrow 0} \int_0^\infty e^{-\lambda t} p(t; x, y) dt \\
&= \int_0^\infty e^{-\lambda t} p(t; 0, 0) dt = \int_0^\infty e^{-\lambda t} \frac{1}{\Gamma(1-\alpha)} 2^{-(1-\alpha)} t^{\alpha-1} dt \\
&= \frac{2^{-(1-\alpha)}}{\Gamma(1-\alpha)} \int_0^\infty e^{-s} \lambda^{1-\alpha} s^{\alpha-1} \frac{1}{\lambda} ds \\
&= \frac{2^{-(1-\alpha)} \Gamma(\alpha)}{\Gamma(1-\alpha)} \lambda^{-\alpha}.
\end{aligned}$$

By Proposition 1.2.11, we can get the Laplace exponent of inverse local time

$$\phi(\lambda) = \frac{1}{G_\lambda(0, 0)} = \frac{2^{1-\alpha} \Gamma(1-\alpha)}{\Gamma(\alpha)} \lambda^\alpha. \quad (1.2.12)$$

It is exactly the Laplace exponent of  $\alpha$ -stable subordinator  $S_t$ .

Now take a Brownian motion  $B_t$  (let it run at twice the usual speed) independent of the above Bessel process, and consider the new process  $Z_t = B_{S_t}$ , we can get a  $2\alpha$ -stable process by looking through the characteristic function,

$$\begin{aligned}
\mathbb{E} e^{i\langle \xi, Z_t \rangle} &= \int_0^\infty \mathbb{P}\{S_t \in ds\} \mathbb{E} e^{i\langle \xi, B_s \rangle} = \int_0^\infty \mathbb{P}\{S_t \in ds\} e^{-s|\xi|^2} \\
&= \exp \left\{ -t \cdot \frac{2^{1-\alpha} \Gamma(1-\alpha) |\xi|^{2\alpha}}{\Gamma(\alpha)} \right\}.
\end{aligned}$$

□

Generally speaking, The class of subordinate Brownian motions that can be realized as boundary traces of diffusion processes in upper half space of one-dimensional higher is in one-to-one correspondence with the class of subordinators that can be realized as the inverse local time at 0 of some reflected diffusions on the half line  $[0, \infty)$ , the latter is known as Krein representation problem.

In Chapter 2, we will get the comparison theorem for inverse local times at 0 of reflected diffusions, under certain conditions. Then we can study a class of subordinate Brownian motions, whose infinitesimal generators are non-local, and get Green function estimates for them. The main results are shown in Section 2.1.

### 1.3 Jump diffusions and heat kernel estimates

There is an intimate interplay between self-adjoint pseudo-differentiable operators on  $\mathbb{R}^d$  and symmetric Markov processes on  $\mathbb{R}^d$ . By a theorem of Ph. Courrège [23], given a class of self-adjoint pseudo-differential operators  $\mathcal{L}$  on  $\mathbb{R}^d$  that has positive maximum property, that is, for each function  $\varphi$  in the domain of  $\mathcal{L}$  which attains its nonnegative maximum in a point  $x_0 \in \mathbb{R}^d$  we have  $\mathcal{L}\varphi(x_0) \leq 0$ , there exists a jump diffusion  $X$  on  $\mathbb{R}^d$  such that  $\mathcal{L}$  is the infinitesimal generator of  $X$ , and vice versa.

Suppose  $X$  is a Markov process on  $\mathbb{R}^d$  with transition density  $p(t, x, y)$  and infinitesimal generator  $\mathcal{L}$ , an important connection between the generator  $\mathcal{L}$  and  $X$  is as follows: the transition density function of  $X$ , or heat kernel, is the fundamental solution of  $\partial_t p(t, x, y) = \mathcal{L}p(t, \cdot, y)(x)$ . The following two examples are typical classes of continuous and purely discontinuous processes.

**Examples 1.3.1.** Consider the local operator

$$\mathcal{L} = \sum_{i,j=1}^d \partial_{x_i} (a_{ij}(x) \partial_{x_j}),$$

where  $(a_{ij}(x))_{1 \leq i,j \leq d}$  is a measurable  $d \times d$  matrix-valued function on  $\mathbb{R}^d$  that is uniformly elliptic and bounded. There exists a symmetric diffusion  $X$  having  $\mathcal{L}$  as its  $L^2$ -infinitesimal generator, and  $\mathcal{L}$  has a jointly continuous heat kernel  $p(t, x, y)$  with respect to the Lebesgue measure on  $\mathbb{R}^d$  that enjoys the Aronson's estimate: there are constants  $c_k > 0$ ,  $k = 1, \dots, 4$  so that

$$c_1 \Gamma_{c_2}(t; x - y) \leq p(t, x, y) \leq c_3 \Gamma_{c_4}(t; x - y),$$

where  $\Gamma_c(t; r) = t^{-d/2} \exp(-c|r|^2/t)$ .

Aronson's estimate also holds for non-divergence form elliptic operators,  $\mathcal{L} = \sum_{i,j=1}^d a_{ij}(x) \partial_{x_i}^2 \partial_{x_j}^2$ , with Hölder continuous coefficients.

**Examples 1.3.2.** Consider the non-local operator

$$\mathcal{L}f(x) = \lim_{\varepsilon \rightarrow 0} \int_{\{y \in \mathbb{R}^d: |y-x| \geq \varepsilon\}} (f(y) - f(x)) \frac{c(x, y)}{|x - y|^{d+\alpha}} dy,$$

where  $c(x, y)$  is a symmetric function bounded between two positive constants and  $\alpha \in (0, 2)$ ,  $\mathcal{L}$  is a symmetric stable-like operator. Chen-Kumagai showed in [16] that it admits a jointly Hölder continuous heat kernel  $p(t, x, y)$  with respect to the Lebesgue measure, which satisfies

$$C^{-1} \frac{t}{(t^{1/\alpha} + |x - y|)^{d+\alpha}} \leq p(t, x, y) \leq C \frac{t}{(t^{1/\alpha} + |x - y|)^{d+\alpha}},$$

for  $t > 0$  and  $x, y \in \mathbb{R}^d$ .  $C$  depends only on the bounds of  $c(x, y)$  and  $d, \alpha$ .

Similar estimate holds on fixed time intervals for non-symmetric stable-like operator in the form of

$$\mathcal{L}u(x) = \lim_{\varepsilon \downarrow 0} \int_{\{z \in \mathbb{R}^d: |z| \geq \varepsilon\}} (u(x+z) - u(x)) \frac{\kappa(x, z)}{|z|^{d+\alpha}} dz,$$

where  $\alpha \in (0, 2)$ , and  $\kappa(x, z)$  is a measurable function symmetric in  $z$  and bounded between two positive constants, uniformly Hölder continuous in  $x$ . Check [22] for details.

A generic strong Markov process may have both the continuous (diffusive) part and the purely discontinuous (jumping) part. We end this section by giving two classes of diffusions with jumps and their heat kernel estimates.

**Examples 1.3.3** (Chen-Kumagai, [17]). Consider the following type of non-local operator  $\mathcal{L}$  on  $\mathbb{R}^d$ :

$$\mathcal{L}u(x) = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial}{\partial x_j} \right) + \lim_{\varepsilon \downarrow 0} \int_{|x-y| > \varepsilon} (u(y) - u(x)) J(x, y) dy, \quad (1.3.1)$$

where  $A(x) = (a_{ij}(x))_{1 \leq i, j \leq d}$  is a measurable  $d \times d$  matrix-valued function on  $\mathbb{R}^d$  that is uniformly elliptic and bounded, and  $J$  is a symmetric non-negative measurable kernel on  $\mathbb{R}^d \times \mathbb{R}^d$  that satisfies

$$J(x, y) \asymp \frac{1}{|x - y|^d \phi(|x - y|)}, \quad (1.3.2)$$

for two positive functions  $f$  and  $g$ , notation  $f \asymp g$  means that there is a constant  $c \geq 1$  so that  $g/c \leq f \leq cg$ , and  $\phi$  is a strictly increasing continuous function defined on  $[0, \infty)$  with  $\phi(0) = 0$ ,  $\phi(1) = 1$  and there exist constants  $c \geq 1, 0 < \beta_1 \leq \beta_2 < 2$  such that

$$c^{-1} \left( \frac{R}{r} \right)^{\beta_1} \leq \frac{\phi(R)}{\phi(r)} \leq c \left( \frac{R}{r} \right)^{\beta_2} \quad \text{for any } 0 < r < R < \infty. \quad (1.3.3)$$

It is shown in [17] that there is a Feller process  $X$  having strong Feller property associated with  $\mathcal{L}$ , and that  $X$  has a jointly continuous transition density function  $p(t, x, y)$  with the following two-sided estimates hold: there exist positive constants  $c_k$ ,  $k = 1, 2, 3, 4$  such that for every  $t > 0$  and  $x, y \in \mathbb{R}^d$ ,

$$\begin{aligned} & c_1(t^{-d/2} \wedge \phi^{-1}(t)^{-d}) \wedge (\Gamma_{c_2}(t; x - y) + p^j(t, |x - y|)) \\ & \leq p(t, x, y) \leq c_3(t^{-d/2} \wedge \phi^{-1}(t)^{-d}) \wedge (\Gamma_{c_4}(t; x - y) + p^j(t, |x - y|)), \end{aligned} \quad (1.3.4)$$

where

$$p^j(t, r) := \left( \phi^{-1}(t)^{-d} \wedge \frac{t}{r^d \phi(r)} \right),$$

and for  $a, b \in \mathbb{R}$ ,  $a \wedge b := \min\{a, b\}$  and  $a \vee b := \max\{a, b\}$ .

**Examples 1.3.4** (Chen-Hu-Xie-Zhang, [13]). For  $d \geq 2$ , there are existence, uniqueness, and estimates of fundamental solutions of time-dependent version of nonsymmetric diffusion operator with jumps,  $\mathcal{L}$ , in the following form:

$$\begin{aligned} \mathcal{L}_t f(x) &= \frac{1}{2} \sum_{i,j=1}^d a_{ij}(t, x) \partial_{ij}^2 f(x) + \sum_{i=1}^d b_i(t, x) \partial_i f(x) \\ &+ \int_{\mathbb{R}^d} (f(x+z) - f(x) - \mathbf{1}_{\{|z| \leq 1\}} z \cdot \nabla f(x)) \frac{\kappa(t, x, z)}{|z|^{d+\alpha}} dz, \end{aligned} \quad (1.3.5)$$

where  $a(t, x) = (a_{ij}(t, x))_{1 \leq i, j \leq d}$  is a  $d \times d$ -symmetric positive definite matrix-valued measurable function on  $[0, \infty) \times \mathbb{R}^d$ ,  $b(t, x) : [0, \infty) \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $\kappa(t, x, z) : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  are measurable functions, and  $\alpha \in (0, 2)$ .

Chen-Hu-Xie-Zhang showed in [12] that under certain conditions for  $a, \kappa, b$ , for every  $T > 0$ , there are positive constants  $C, \lambda \geq 1$  such that for  $(t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ ,

$$C^{-1}(\Gamma_\lambda + m_\kappa \eta)(s - t; y - x) \leq p(t, x; s, y) \leq C(\Gamma_{\lambda^{-1}} + \|\kappa\|_\infty \eta)(s - t; y - x), \quad (1.3.6)$$

where

$$\eta(t; x) := \frac{t}{(t^{1/2} + |x|)^{d+\alpha}}.$$

**Remark 1.3.5.** When  $\phi(r) = r^\alpha$  in Example 1.3.3, on fixed time intervals, (1.3.4) can be rewritten into similar estimates as in (1.3.6). See details in Lemma 3.1.1.

### 1.4 Girsanov and Feynman-Kac transforms

Girsanov transform describes how Brownian motion, or more generally, local martingales behave under changes of the underlying probability measure. Let  $\{W_t\}$  be a standard Brownian motion under the probability measure  $\mathbb{P}$ , and  $\{\mathcal{F}\}_{t \geq 0}$  is the associated filtration. Let  $\{\xi_s\}$  be a real valued adapted process on  $(\Omega, \mathcal{F}_t, \mathbb{P})$ , we define

$$Z_t = \exp \left( \int_0^t \xi_s dW_s - \frac{1}{2} \int_0^t \xi_s^2 ds \right).$$

Novikov showed that if for each  $t \geq 0$ ,  $\mathbb{E}[\exp(\frac{1}{2} \int_0^t |\xi_s|^2 ds)] < \infty$ , then  $Z_t$  is a martingale under the probability  $\mathbb{P}$ . Define

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Z_t|_{\mathcal{F}_t},$$

this gives a new probability measure on  $(\Omega, \mathcal{F})$ . Girsanov's theorem describes the distribution of the stochastic process  $\{W_t\}$  under the new probability measure as

$$\widetilde{W}_t := W_t - \int_0^t \xi_s ds,$$

and under the new probability measure  $\mathbb{Q}$ ,  $\{\widetilde{W}_t\}$  is a standard Wiener process as well.

Feynman-Kac transform is one of the most important transforms for Markov processes. Suppose that  $E$  is a Lusin space, which is a space that is homeomorphic to a Borel subset of a compact metric space. Denote by  $\mathcal{B}(E)$  the Borel  $\sigma$ -algebra on  $E$ . Let  $m$  be a Borel  $\sigma$ -finite measure on  $E$  with full support, and  $X$  on  $(\Omega, \mathcal{F}, \mathbb{P})$  is a  $m$ -symmetric irreducible Borel standard process on  $E$  with lifetime  $\zeta$ , with  $\mathcal{L}$  as the  $L^2$ -infinitesimal generator. For a function  $q$ , one can define Feynman-Kac semigroup:

$$T_t f(x) := \mathbb{E}_x[e^{\int_0^t q(X_s) ds} f(X_t)],$$

under certain condition on  $q$  (see details in [21]), the  $L^2$ -infinitesimal generator of  $\{T_t; t \geq 0\}$  is  $\mathcal{L} + q(x)$ .

More generally, one can define the Feynman-Kac transform for a continuous additive functional  $A^\mu$  of  $X$  having finite variations,

$$T_t f(x) = \mathbb{E}_x[e^{A_t^\mu} f(X_t)], \quad t \geq 0,$$

where  $A_t^\mu$  is defined in a similar way that is analogous to  $\mu(dx) = q(x)m(dx)$ . It's well-known that, under suitable Kato class condition on  $\mu$ ,  $\{T_t; t \geq 0\}$  forms a strongly continuous symmetric semigroup on  $L^p(E; m)$  for  $1 \leq p < \infty$  and its  $L^2$ -infinitesimal generator is  $\mathcal{L}^\mu := \mathcal{L} + \mu$  in distributional sense.

For discontinuous process  $X$ , there are many discontinuous additive functionals. Let  $F$  be a symmetric function on  $E \times E$  vanishing along the diagonal. We can also extend it to be zero off  $E \times E$ . Then  $\sum_{0 < s \leq t} F(X_{s-}, X_s)$ , whenever it is summable, is an additive functional of  $X$ . Therefore, one can perform a non-local Feynman-Kac transform

$$T_t^{\mu, F} f(x) := \mathbb{E}_x \left[ \exp \left( A_t^\mu + \sum_{0 < s \leq t} F(X_{s-}, X_s) \right) f(X_t) \right], \quad t \geq 0.$$

Let  $(N(x, dy), H_t)$  be a Lévy system of  $X$ , the infinitesimal generator for  $\{T_t^{\mu, F}; t \geq 0\}$  is

$$\mathcal{L}^{\mu, F} := \mathcal{L} + \mu_H \mathbf{F} + \mu,$$

where  $\mu_H$  is the Revuz measure of the positive continuous additive functional  $H$  and

$$\mu_H \mathbf{F} f(dx) := \left( \int_E (e^{F(x, y)} - 1) f(y) N(x, dy) \right) \mu_H(dx).$$

An important question related to Feynman-Kac transforms is the stability of various properties. For instance, for Brownian motion, under Feynman-Kac perturbation, one can get a Schrödinger operator  $\Delta/2 + \mu$  and Schrödinger semigroup  $(\exp((\Delta/2 + \mu)t))_{t > 0}$ . Blanchard and Ma [3] showed in 1988 that when  $d \geq 3$ ,  $\mu$  is a signed Radon measure in generalized Kato class, which satisfies

$$\limsup_{r \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_{B(x, r)} \frac{1}{|x - y|^{d-2}} |\mu|(dy) = 0,$$

the Schrödinger semigroup admits a jointly continuous symmetric integral kernel  $q(t, x, y)$  with the following two-sided estimates:

$$C^{-1} e^{-\beta t} \Gamma_{c_1}(t; x - y) \leq q(t, x, y) \leq C e^{\beta t} \Gamma_{c_2}(t; x - y).$$

In [37], sharp two-sided estimates on the densities of (local) Feynman-Kac semigroups of killed Brownian motions in  $C^{1,1}$  domains were established. Non-local Feynman-Kac semigroups for symmetric stable processes and their associated quadratic forms were studied in [41, 42]. More generally, stability of heat kernel estimates for purely discontinuous Markov processes under non-local Feynman-Kac perturbations have been studied in [15, 44]. We also mention that the stability of Martin boundary under non-local Feynman-Kac perturbation is addressed in [14].

In Chapter 3, we will study a class of jump diffusions having a jointly continuous transition density with two-sided estimates (1.3.6), in particular, it contains both continuous and pure jump components. We obtain the stability of heat kernel estimates under non-local Feynman-Kac perturbations. The main theorem is shown in Section 3.1.

## Chapter 2

### INVERSE LOCAL TIME OF ONE-DIMENSIONAL DIFFUSIONS AND ITS COMPARISON THEOREM

#### 2.1 *Diffusions and subordinate Brownian motions*

It is well known that the trace of Brownian motion in  $\mathbb{R}^{d+1}$  (or reflected Brownian motion in the upper half space  $\mathbb{R}_+^{d+1} := \{x = (x_1, \dots, x_d, x_{d+1}) \in \mathbb{R}^{d+1} : x_{d+1} > 0\}$ ) on the hyperplane  $\{x_{d+1} = 0\}$  is a  $d$ -dimensional Cauchy process, which is also a Brownian motion time changed by an independent Cauchy subordinator. Another type of subordinate Brownian motions is Brownian motion time-changed by an independent  $\alpha$ -stable subordinator, which has the same distribution as a rotationally symmetric stable process.

Moreover, relativistic Cauchy process in  $\mathbb{R}^d$  (also called relativistic Brownian motion in the study of relativistic Hamiltonian system in physics [7]) is a subordinate Brownian motion  $X_t$  characterized by

$$\mathbb{E}e^{i\xi \cdot (X_t - X_0)} = e^{t(\sqrt{m} - \sqrt{m + |\xi|^2})}, \quad \xi \in \mathbb{R}^d,$$

where  $m > 0$  stands for the mass of the particle. The infinitesimal generator of  $X_t$  is  $\sqrt{m} - \sqrt{m - \Delta}$ . It is not hard to see that the inverse local time at 0 of the reflected Brownian motion with downward constant drift  $\sqrt{2m}$  on  $[0, \infty)$  is a subordinator  $S_t$  with  $S_0 = 0$  and

$$\mathbb{E}e^{-\lambda S_t} = e^{t(\sqrt{m} - \sqrt{m + \lambda})}, \quad \lambda \geq 0.$$

Hence the relativistic Cauchy process on  $\mathbb{R}^d$  can be regarded as the boundary trace on  $\{x_{d+1} = 0\}$  in the upper half space of  $\mathbb{R}^{d+1}$  where the vertical motion in the  $x_{d+1}$  direction is a Brownian motion with downward constant drift while the horizontal motion is an independent Brownian motion in  $\mathbb{R}^d$ . However for general  $\alpha \neq 1/2$ , relativistic  $\alpha$ -stable processes can not be simply realized, by analogy with rotationally symmetric stable processes, as a diffusion in

the upper half space of one-dimensional high whose vertical motion is a Bessel process with constant drift.

By using Esscher transform, Martin and Yor[26] showed that relativistic  $\alpha$ -subordinator for  $0 < \alpha < 1$  is in fact the inverse local time at 0 of the reflected diffusion on  $[0, \infty)$  determined by generator

$$\mathcal{L}^{(\alpha, m)} = \frac{1}{2} \frac{d^2}{dx^2} + \left( \frac{1 - 2\alpha}{2x} + \frac{\widehat{K}'_\alpha(\sqrt{2mx})}{\widehat{K}_\alpha(\sqrt{2mx})} \right) \frac{d}{dx}, \quad (2.1.1)$$

where  $\widehat{K}(x) = x^\alpha K_\alpha(x)$  and  $K_\alpha(x)$  is the modified Bessel function of the second kind defined in (2.4.4).

Realizing non-local operators as boundary trace of some differential operators is a powerful way to study non-local operators from analytic point of view as one can employ many well developed techniques and ideas from partial differential equations (PDE). It is a natural and interesting question to investigate the scope of non-local operators that can be realized as the boundary trace of differential operators.

In this chapter, we set out a modest goal to investigate properties of the inverse local time at 0 of reflected diffusions on  $[0, \infty)$  with infinitesimal generator of the form

$$\mathcal{L} = \frac{1}{2} \frac{d^2}{dx^2} + \left( \frac{1 - 2\alpha}{2x} - f(x) \right) \frac{d}{dx}, \quad (2.1.2)$$

where  $f \geq 0$  is a function on  $(0, \infty)$ , and the corresponding subordinate Brownian motions.

For two functions  $f$  and  $g$ , notation  $f \lesssim g$  means there is a constant  $c > 0$  so that  $f \leq cg$ . The following are the main results of this chapter.

**Theorem 2.1.1.** *Let  $Y_t$  be the reflected diffusion process on  $[0, \infty)$  determined by the local generator of the form (2.1.2) with*

$$0 \leq f(x) \lesssim (1 \wedge x)^{2\alpha-1} \quad \text{on } (0, \infty).$$

*Let  $S_t$  be the inverse local time of  $Y_t$  at 0. Then there is a constant  $m > 0$  so that stochastically  $S_t^{(\alpha, m)} \leq S_t \leq S_t^{(\alpha)}$  for all  $t \geq 0$ , where  $S_t^{(\alpha)}$  and  $S_t^{(\alpha, m)}$  are the  $\alpha$ -stable subordinator and relativistic  $\alpha$ -stable subordinator with mass  $m$ , respectively.*

In order to achieve the above result, we have obtained the comparison theorem for inverse local times at 0 of reflected diffusions. We would like to mention that in general one can not conclude that the inverse local time at 0 of one diffusion is dominated by that of another. Indeed, there is no monotonicity between  $\alpha$ -stable and  $\beta$ -stable subordinators, as there is no monotonicity between their Laplace exponents. However, in Section 2.3, we get a key comparison result for the inverse local times at 0 for reflected diffusion processes on  $[0, \infty)$  and their corresponding Lévy measures. Hausdorff measure of zero sets and Girsanov transform are the main tools to get this result. Regenerative embedding theory for subordinators are also used.

As an application, we have the following Green function estimates for the trace processes of diffusion processes in  $\mathbb{R}^{d+1}$  whose vertical  $x_{d+1}$ -coordinate is a reflected diffusion on  $[0, \infty)$  with infinitesimal generator (2.1.2), and the horizontal direction is an independent Brownian motion in  $\mathbb{R}^d$ .

**Theorem 2.1.2.** *Under the setting of Theorem 2.1.1, let  $B_t$  be a  $d$ -dimensional Brownian motion independent of  $Y_t$  with variance  $2t$ , and  $\mu(x)$  be the density of the Lévy measure of trace process  $B_{S_t}$ . Denote the density of the Lévy measure of symmetric  $2\alpha$ -stable process by  $\mu^{(\alpha)}(x)$ . Then  $j(x) := \mu^{(\alpha)}(x) - \mu(x) \geq 0$ , and there exists a constant  $C$  such that for  $|x| \leq 1$ ,*

$$j(x) \leq C|x|^{2-2\alpha-d}.$$

*Let  $D \subset \mathbb{R}^d$  be a bounded connected Lipschitz open set. Denote Green functions of the trace process  $B_{S_t}$  in  $D$  by  $G_D(x, y)$ . Then there exists a constant  $C_1 = C_1(d, \alpha, D, C)$  such that*

$$C_1^{-1}G_D^{(2\alpha)}(x, y) \leq G_D(x, y) \leq C_1G_D^{(2\alpha)}(x, y) \text{ for } x, y \in D,$$

*where  $G_D^{(2\alpha)}$  is the Green function of rotationally symmetric  $(2\alpha)$ -stable process, or equivalently of the fractional Laplacian  $\Delta^\alpha$ , in  $D$ .*

The rest of this chapter is organized as follows. Esscher transform (an Economics terminology) and Girsanov transform for reflected diffusions on  $[0, \infty)$  are discussed in Section 2.2.

This extends and refines the corresponding part of Martin-Yor [26] with a more complete and rigorous proof. In Section 2.3, we present the comparison theorem for inverse local times. With the comparison result obtained in Section 2.3, Theorem 2.1.1 and 2.1.2 are established in Section 2.4.

## 2.2 Esscher transforms

Recall that the Laplace exponent and Lévy measure for  $\alpha$ -stable subordinator, where  $0 < \alpha < 1$ , are  $\phi^{(\alpha)}(\lambda) = c_\alpha \lambda^\alpha$  and  $\nu^{(\alpha)}(dx) := c_\alpha x^{-1-\alpha} dx$ , where  $c_\alpha = \alpha/\Gamma(1-\alpha)$ ; while that for relativistic  $\alpha$ -stable subordinator with mass  $m > 0$  are  $\phi^{(\alpha,m)}(\lambda) = c_\alpha[(m+\lambda)^\alpha - m^\alpha]$  and  $\nu^{(\alpha,m)}(dx) := c_\alpha x^{-1-\alpha} e^{-mx} dx$ .

Fix  $0 < \alpha < 1$ . Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space on which a reflected Bessel process  $Y_t$  on  $[0, \infty)$  with generator (2.1.2) is defined. The filtration generated by  $Y_t$  will be denoted as  $\{\mathcal{F}_t; t \geq 0\}$ . Let  $L_t$  be the local time of  $Y$  at 0, and

$$S_t := \inf\{s > 0 : L_s > t\},$$

the inverse of  $L$ , which is a stopping time with respect to the filtration  $\{\mathcal{F}_t; t \geq 0\}$ . We know that  $S_t$  is an  $\alpha$ -stable subordinator. We can define a new probability measure  $\mathbb{Q}$  on  $(\Omega, \mathcal{F})$  by

$$\frac{\mathbb{Q}(dx)}{\mathbb{P}(dx)} = \frac{e^{-mx}}{\mathbb{E}_0[\exp(-mS_t)]} = \exp(tc_\alpha m^\alpha - mx) \quad \text{on } \mathcal{F}_{S_t}. \quad (2.2.1)$$

This change of measure is called Esscher transform in literature (see Chapter VII, 3c, [28]). Note that under the new probability measure  $\mathbb{Q}$ ,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[\exp(-\lambda S_t)] &= \exp(tc_\alpha m^\alpha) \mathbb{E}^{\mathbb{P}}[\exp(-(\lambda+m)S_t)] \\ &= \exp(tc_\alpha m^\alpha - t\phi^{(\alpha)(\lambda+m)}) \\ &= \exp(-t\phi^{(\alpha,m)}(\lambda)). \end{aligned} \quad (2.2.2)$$

In other words, under  $\mathbb{Q}$ ,  $\{S_t; t \geq 0\}$  is a relativistic  $\alpha$ -stable subordinator with mass  $m$ .

We now extend the above Esscher transform to general one-dimensional diffusions.

**Theorem 2.2.1.** *Suppose that  $X_t$  is a reflected diffusion process on  $[0, \infty)$  defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , determined locally by the generator*

$$\mathcal{L} = a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx}.$$

*Let  $L_t$  be the local time of  $X_t$  at 0, and  $S_t = \inf\{s : L_s > t\}$  its inverse local time at 0, which is a subordinator. Denote by  $\phi(\lambda)$  the Laplace exponent of  $\{S_t; t \geq 0\}$ . Define*

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \frac{\exp(-mS_t)}{\mathbb{E}[\exp(-mS_t)]} \quad \text{on } \mathcal{F}_{S_t}, \quad t \geq 0. \quad (2.2.3)$$

*Then*

(i) (2.2.3) defines a new measure  $\mathbb{Q}$  on  $\mathcal{F}_\infty$  in a consistent way;

(ii) Under  $\mathbb{Q}$ , the original diffusion  $X$ , write as  $X^{(m)}$  for emphasis, is a reflected diffusion on  $[0, \infty)$  having generator

$$\mathcal{L}^{(m)} = \mathcal{L} + 2a(x) \frac{\rho'_m(x)}{\rho_m(x)} \frac{d}{dx}, \quad x > 0,$$

where  $\rho_m(x) := \mathbb{E}_x[\exp(-mT_0)]$  and  $T_0$  is the first hitting time of 0 by the process  $X_t$ .

(iii) Denote by  $S_t^{(m)}$  the inverse local time of  $X_t^{(m)}$  at 0 and  $\phi^{(m)}(\lambda)$  the Laplace exponent for subordinator  $S_t^{(m)}$ . Then we have

$$\phi^{(m)}(\lambda) = \phi(\lambda + m) - \phi(m).$$

**Proof.** The definition (2.2.3) gives

$$M_{s,t} := \frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_{s \wedge S_t}} = \frac{\mathbb{E}[\exp(-mS_t) | \mathcal{F}_{s \wedge S_t}]}{\mathbb{E}[\exp(-mS_t)]}. \quad (2.2.4)$$

To see the consistency, we observe for  $r \leq t$ , since  $S_t$  is a subordinator,

$$\begin{aligned}
\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_{s \wedge S_r}} &= \frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_{s \wedge S_r \wedge S_t}} = \frac{\mathbb{E}[\exp(-mS_t) | \mathcal{F}_{s \wedge S_r \wedge S_t}]}{\mathbb{E}[\exp(-mS_t)]} \\
&= \frac{\mathbb{E}[\exp(-mS_t) | \mathcal{F}_{s \wedge S_r}]}{\mathbb{E}[\exp(-mS_t)]} \\
&= \frac{\mathbb{E}[\exp(-mS_r) \exp(-m(S_t - S_r)) | \mathcal{F}_{s \wedge S_r}]}{\mathbb{E}[\exp(-mS_r) \exp(-m(S_t - S_r))]} \\
&= \frac{\mathbb{E}[\exp(-m(S_t - S_r)) | \mathcal{F}_{s \wedge S_r}]}{\mathbb{E}[\exp(-m(S_t - S_r))]} \frac{\mathbb{E}[\exp(-mS_r) | \mathcal{F}_{s \wedge S_r}]}{\mathbb{E}[\exp(-mS_r)]} \\
&= \frac{\mathbb{E}[\exp(-mS_r) | \mathcal{F}_{s \wedge S_r}]}{\mathbb{E}[\exp(-mS_r)]}
\end{aligned}$$

To see uniform integrability, we first have the decomposition of the inverse local time at 0, since  $S_0 = T_0$ ,

$$\begin{aligned}
S_t \text{ under } \mathbb{P} &= \inf\{s : L_s > t \text{ with } X_0 = x\} \\
&= \inf\{s : s = T_0 + r, L_{T_0} + L_r \circ \theta_{T_0} > t \text{ with } X_0 = x\} \\
&= T_0 + \inf\{r : L_r \circ \theta_{T_0} > t \text{ with } X_0 = x\} \\
&= T_0 + \inf\{r : L_r > t \text{ with } X_0 = 0\} \\
&= T_0 + S_t \text{ under } \mathbb{P}_0
\end{aligned} \tag{2.2.5}$$

With the decomposition,

$$\begin{aligned}
\mathbb{E}[\exp(-mS_t)] &= \mathbb{E}[e^{-mT_0}] \mathbb{E}_0[e^{-mS_t}] \\
&= \rho_m(x) \mathbb{E}_0[e^{-mS_t}] \\
&= \rho_m(x) \exp[-t\phi(m)]
\end{aligned} \tag{2.2.6}$$

Similarly,

$$\begin{aligned}
\mathbf{1}_{\{s \leq S_t\}} \mathbb{E}[e^{-mS_t} | \mathcal{F}_s] &= \mathbf{1}_{\{s \leq S_t\}} \mathbb{E}[e^{-mS_t} | \mathcal{F}_s] \\
&= \mathbf{1}_{\{s \leq S_t\}} e^{-ms} \mathbb{E}_{X_s}[\exp(-mS_{t-r})] \Big|_{r=L_s}
\end{aligned}$$

The last equation holds because on  $\{s \leq S_t\}$ ,

$$\begin{aligned} S_t &= \inf\{r + s : L_{r+s} > t\} = s + \inf\{r : L_s + L_r \circ \theta_s > t\} \\ &= s + \inf\{r : L_r \circ \theta_s > t - L_s\} \\ &= s + S_{t-r} \circ \theta_s|_{r=L_s}. \end{aligned}$$

Now using (2.2.6), we have

$$\begin{aligned} \mathbb{E}_{X_s}[\exp(-mS_{t-r})]|_{r=L_s} &= \rho_m(X_s)\mathbb{E}_0[\exp(-mS_{t-r})]|_{r=L_s} \\ &= \rho_m(X_s)\exp(-(t-r)\phi(m))|_{r=L_s} \\ &= \rho_m(X_s)\exp(-(t-L_s)\phi(m)). \end{aligned}$$

Thus, restricted to  $\mathcal{F}_{s \wedge S_t}$ ,

$$\begin{aligned} M_{s,t} &= \frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_{s \wedge S_t}} = \mathbf{1}_{\{s \leq S_t\}} \frac{\mathbb{E}[\exp(-mS_t)|\mathcal{F}_s]}{\mathbb{E}[\exp(-mS_t)]} + \mathbf{1}_{\{s > S_t\}} \frac{\exp(-mS_t)}{\mathbb{E}[\exp(-mS_t)]} \\ &= \mathbf{1}_{\{s \leq S_t\}} \frac{e^{-ms} \rho_m(X_s) \exp(-(t-L_s)\phi(m))}{\rho_m(x) \exp(-t\phi(m))} + \mathbf{1}_{\{s > S_t\}} \frac{\exp(-mS_t)}{\mathbb{E}[\exp(-mS_t)]} \\ &= \mathbf{1}_{\{s \leq S_t\}} \frac{\rho_m(X_s)}{\rho_m(x)} \exp(-ms + L_s\phi(m)) + \mathbf{1}_{\{s > S_t\}} \frac{\exp(-mS_t)}{\mathbb{E}[\exp(-mS_t)]} \end{aligned}$$

As  $t \rightarrow \infty$ ,  $M_{s,t} \rightarrow M_s$ , a.s. and

$$M_s := \frac{\rho_m(X_s)}{\rho_m(x)} \exp(-ms + L_s\phi(m)).$$

It's obvious that  $M_s \in \mathcal{F}_s$ . Also, from the original definition of  $M_{s,t}$  in (2.2.4), for any  $s, t$ ,  $\mathbb{E}M_{s,t} \leq 1$ , so by Fatou's lemma,

$$\mathbb{E}M_s \leq \liminf \mathbb{E}M_{s,t} \leq 1.$$

Thus,  $M_s \in L^1$ , and on the other hand, with  $\mathcal{F}_{s \wedge S_t} \subset \mathcal{F}_s$

$$\begin{aligned} \mathbb{E}[M_s|\mathcal{F}_{s \wedge S_t}] &= \mathbb{E} \left[ \frac{\rho_m(X_s)}{\rho_m(x)} \exp(-ms + L_s\phi(m)) \Big| \mathcal{F}_{s \wedge S_t} \right] \\ &= \mathbf{1}_{\{s \leq S_t\}} \frac{\rho_m(X_s)}{\rho_m(x)} \exp(-ms + L_s\phi(m)) + \mathbf{1}_{\{s > S_t\}} \frac{\rho_m(0)}{\rho_m(x)} \exp(-mS_t + t\phi(m)) \\ &= \mathbf{1}_{\{s \leq S_t\}} \frac{\rho_m(X_s)}{\rho_m(x)} \exp(-ms + L_s\phi(m)) + \mathbf{1}_{\{s > S_t\}} \frac{\exp(-mS_t)}{\mathbb{E}(\exp(-mS_t))} \\ &= M_{s,t}, \end{aligned}$$

the second to the last equality comes from  $\rho_m(0) = 1$  and (2.2.6). Thus  $\{M_{s,t} = \mathbb{E}[M_s | \mathcal{F}_s \wedge S_t]\}_{t \geq 0}$  is uniformly integrable. Taking  $t \rightarrow \infty$  yields

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{F}_s} = \frac{\rho_m(X_s)}{\rho_m(x)} \exp(-ms + L_s \phi(m)). \quad (2.2.7)$$

The above is a combination of Doob's  $h$ -transform and a Feynman-Kac transform by local time  $L_t$ . It follows that for  $x > 0$ ,

$$\begin{aligned} \mathcal{L}^{(m)} f(x) &= \rho_m^{-1}(x) (\mathcal{L} - m) (\rho_m \cdot f)(x) \\ &= \mathcal{L} f(x) + 2a(x) \frac{\rho'_m(x)}{\rho_m(x)} f'(x) + \rho_m^{-1}(x) (\mathcal{L} - m) (\rho_m)(x) f(x) \end{aligned}$$

Since  $\rho_m(x)$  satisfies  $(\mathcal{L} - m)\rho_m(x) = 0$ , under the new measure  $\mathbb{Q}$ , the diffusion process  $X_t$  is a reflected diffusion on  $[0, \infty)$  with generator

$$\mathcal{L}^{(m)} = \mathcal{L} + 2a(x) \frac{\rho'_m(x)}{\rho_m(x)} \frac{d}{dx}, \text{ for } x > 0.$$

By (2.2.3), for every  $\lambda > 0$ ,

$$\mathbb{E}^{\mathbb{Q}} e^{-\lambda S_t} = \frac{\mathbb{E} e^{-(\lambda+m)S_t}}{\exp(-t\phi(m))} = \exp\{-t(\phi(\lambda+m) - \phi(m))\}.$$

This proves that the Laplace exponent of  $S_t^{(m)}$  is  $\phi_m(\lambda) = \phi(\lambda+m) - \phi(m)$ .  $\square$

**Remark 2.2.2.** By Feynman-Kac transformation,  $\rho_m(x) = \mathbb{E}[\exp(-mT_0)]$  is the unique solution to

$$\begin{cases} (\mathcal{L} - m)\rho_m = 0; \\ \rho_m(0) = 1, \rho_m(\infty) = 0. \end{cases}$$

### 2.3 Comparison theorem for inverse local time

Let  $X_t$  and  $Y_t$  be reflected diffusion processes on  $[0, \infty)$  defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and driven by a common Brownian motion, whose generators are

$$\begin{aligned} \mathcal{L}^X &= a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx}; \\ \mathcal{L}^Y &= a(x) \frac{d^2}{dx^2} + B(x) \frac{d}{dx}. \end{aligned}$$

Denote by  $Z^X$  and  $Z^Y$  the zero sets for  $X$  and  $Y$  respectively; that is,

$$Z^X := \{t \in [0, \infty) : X_t = 0\} \quad \text{and} \quad Z^Y := \{t \in [0, \infty) : Y_t = 0\}.$$

These are random closed subsets of  $[0, \infty)$ , and are regenerative (also called Markov) sets in the sense of Maisonneuve (cf [2]).

**Lemma 2.3.1.** *Let  $S_t^X$  and  $S_t^Y$  be inverse local times at 0 for  $X_t, Y_t$ , whose Laplace exponents are denoted by  $\phi^X(\lambda), \phi^Y(\lambda)$ , respectively. Suppose that  $b(x) \leq B(x)$  for all  $x$ . Then  $\phi^Y/\phi^X$  is a completely monotone function.*

**Proof.** If  $X_0 \leq Y_0$ , then by the comparison theorem for one-dimensional diffusions (see, e.g., [1, Theorem I.6.2]), we have, almost surely,  $X_t \leq Y_t$  for all  $t \geq 0$ . Consequently,

$$Z^Y \subset Z^X, \quad \mathbb{P}\text{-a.s.} \quad (2.3.1)$$

Note that  $S_t^X$  and  $S_t^Y$  are the subordinators associated with the regenerative sets  $Z^X$  and  $Z^Y$ , respectively. It follows from the regenerative embedding theorem due to Bertoin (see [2, Theorem 1]) that  $\phi^Y/\phi^X$  is a completely monotone function.  $\square$

**Examples 2.3.2.** Consider two reflected Bessel processes on  $[0, \infty)$ :  $X_t^{(\alpha)}, X_t^{(\beta)}$ , determined by local generators:

$$\begin{aligned} \mathcal{L}^{(\alpha)} &= \frac{1}{2} \frac{d^2}{dx^2} + \frac{1-2\alpha}{2x} \frac{d}{dx}; \\ \mathcal{L}^{(\beta)} &= \frac{1}{2} \frac{d^2}{dx^2} + \frac{1-2\beta}{2x} \frac{d}{dx}, \end{aligned}$$

where  $0 < \beta < \alpha < 1$  and  $X_0^{(\alpha)} \leq X_0^{(\beta)}$ . Denote by  $S_t^{(\alpha)}, S_t^{(\beta)}$  the inverse local times at 0, and  $\phi^{(\alpha)}, \phi^{(\beta)}$  their Laplace exponents.

Since  $\frac{1-2\alpha}{2x} < \frac{1-2\beta}{2x}$  for  $x > 0$ , apply the classic comparison theorem for one-dimensional SDE and Lemma 2.3.1,  $\phi^{(\beta)}/\phi^{(\alpha)}$  is completely monotone.

On the other hand, as we know from [36],  $S_t^{(\alpha)}, S_t^{(\beta)}$  are  $\alpha$ - and  $\beta$ -stable subordinators, respectively. Since  $\phi^{(\alpha)}(\lambda) = c_\alpha \lambda^\alpha$ ,  $\phi^{(\beta)}(\lambda) = c_\beta \lambda^\beta$ ,  $\phi^{(\beta)}(\lambda)/\phi^{(\alpha)}(\lambda) = (c_\beta/c_\alpha) \lambda^{\beta-\alpha}$  is indeed completely monotone in  $\lambda$ .

In general one can not conclude that the inverse local time at 0 of  $Y$  is dominated by that of  $X$ . Indeed, there is no monotonicity between  $\alpha$ -stable and  $\beta$ -stable subordinators, as there is no monotonicity between their Laplace exponents. However we have the following comparison theorem for inverse local times.

**Theorem 2.3.3.** *Suppose  $X_t$  and  $Y_t$  defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  are reflected diffusions on  $[0, \infty)$ , determined by the local generator*

$$\begin{aligned}\mathcal{L}^X &= \frac{1}{2} \frac{d^2}{dx^2} + b(x) \frac{d}{dx}; \\ \mathcal{L}^Y &= \frac{1}{2} \frac{d^2}{dx^2} + B(x) \frac{d}{dx}.\end{aligned}$$

Let  $S_t^X$  and  $S_t^Y$  be the corresponding inverse local times at 0, respectively. Suppose  $f(x) = B(x) - b(x) \geq 0$  satisfies the condition

$$\sup_{x>0} \mathbb{E}_x \left[ \int_0^T |f(X_t)|^2 dt \right] < \infty, \quad \text{for any fixed time } T > 0. \quad (2.3.2)$$

Then stochastically,  $S_t^X \leq S_t^Y$  for all  $t \geq 0$ .

**Proof.** We first define a Girsanov transform between  $X_t$  and  $Y_t$ ,

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \Big|_{\mathcal{F}_t} = \exp \left[ \int_0^t f(X_s) dB_s - \frac{1}{2} \int_0^t f^2(X_s) ds \right].$$

Note that due to condition (2.3.2), by [11, Theorem 3.2], the right hand side of the above is a uniformly integrable martingale. Thus with  $Z^X, Z^Y$  as zero sets, we have the relations

$$(X_t, \mathbb{Q}) \stackrel{d}{=} (Y_t, \mathbb{P}) \Rightarrow (Z^X, \mathbb{Q}) \stackrel{d}{=} (Z^Y, \mathbb{P}). \quad (2.3.3)$$

In other words, under  $\mathbb{Q}$ ,  $X_t$  can be viewed as  $Y_t$ . This leads to the same properties for zero sets of  $X_t$  and  $Y_t$ .

Now let  $L_t^X$  be a choice of the local time for  $X_t$  at 0 such that  $L_t^X$  satisfies (cf. Theorem X.2. in [35])

$$\mathbb{E}_x \left[ \int_t^\infty e^{-s} dL_s^X \Big| \mathcal{F}_t \right] = \mathbb{E}_x [e^{-T_0 \circ \theta_t} | \mathcal{F}_t], \quad (2.3.4)$$

where  $T_0$  is the first hitting time at 0 for  $X_t$ . We claim that  $M_t \triangleq \{e^{-T_0} \mathbf{1}_{\{T_0 \leq t\}} - \int_0^t e^{-s} dL_s^X; t \geq 0\}$  is a  $\mathbb{P}$ -martingale with respect to the filtration  $\{\mathcal{F}_t; t \geq 0\}$ . This is because for every  $t \geq r$ ,

$$\begin{aligned} \mathbb{E}_x[M_t | \mathcal{F}_r] &= M_r + \mathbb{E}_x \left[ e^{-T} \mathbf{1}_{\{r < T \leq t\}} - \int_r^t e^{-s} dL_s^X \middle| \mathcal{F}_r \right] \\ &= M_r + \mathbb{E}_x \left[ e^{-T} \mathbf{1}_{\{r < T \leq t\}} - \int_r^t e^{-s} dL_s^X \middle| \mathcal{F}_r \right] + \mathbb{E}_x \left[ e^{-T \circ \theta_t} - \int_t^\infty e^{-s} dL_s^X \middle| \mathcal{F}_r \right] \\ &= M_r + \mathbb{E}_x \left[ e^{-T \circ \theta_r} - \int_r^\infty e^{-s} dL_s^X \middle| \mathcal{F}_r \right] \\ &= M_r, \end{aligned}$$

where the last equality is due to (2.3.4). This proves the claim that  $\{M\}_t$  is a martingale with respect to  $\{\mathcal{F}_t\}_{t \geq 0}$ . Clearly, it is purely discontinuous martingale of finite variation.

Applying the same Girsanov transform to  $M_t$ ,  $M_t - [M, \int_0^\cdot f(X_s) dB_s]_t$  is a  $\mathbb{Q}$ -martingale. Since  $M_t$  is a purely discontinuous martingale of finite variation and  $\int_0^t f(X_s) dB_s$  is continuous,  $[M, \int_0^\cdot f(X_s) dB_s]_t = 0$ ,  $M_t$  is a  $\mathbb{Q}$ -martingale as well. Thus,

$$\mathbb{E}_x^{\mathbb{Q}}[e^{-T_0} \mathbf{1}_{\{T_0 \leq t\}}] = \mathbb{E}_x^{\mathbb{Q}} \left[ \int_0^t e^{-s} dL_s^X \right]$$

By letting  $t \rightarrow \infty$ , one can have

$$\mathbb{E}_x^{\mathbb{Q}}[e^{-T_0}] = \mathbb{E}_x^{\mathbb{Q}} \left[ \int_0^\infty e^{-s} dL_s^X \right].$$

Thus, we get the relation that  $(L_t^X, \mathbb{Q}) \stackrel{d}{=} (L_t^Y, \mathbb{P})$ .

Fristedt-Pruitt showed in [30] there exists an increasing function  $g$  such that

$$g\text{-}m(S^X[0, t]) = t,$$

where the left hand side represents the Hausdorff measure of the range of  $S^X$  on the time interval  $[0, t]$  with respect to the function  $g$ . Since the closure of the range for  $S^X$  is  $Z^X$ , it follows that  $g\text{-}m(Z^X \cap [0, t]) = L_t^X$ ,  $\mathbb{P}$ -a.s. and

$$(g\text{-}m(Z^X \cap [0, t]); \mathbb{Q}) = (L_t^X, \mathbb{Q}) \stackrel{d}{=} (L_t^Y, \mathbb{P}).$$

Also, by the classic comparison theorem,  $X_t \leq Y_t$  almost surely for all  $t$ , we have  $Z^X \supset Z^Y$ ,  $\mathbb{P}$ -a.s. Together with (2.3.3),

$$\begin{aligned} (L_t^Y, \mathbb{P}) &= (g\text{-}m(Z^X \cap [0, t]); \mathbb{Q}) = (g\text{-}m(Z^Y \cap [0, t]); \mathbb{P}) \\ &\leq (g\text{-}m(Z^X \cap [0, t]); \mathbb{P}) \\ &= (L_t^X, \mathbb{P}) \end{aligned}$$

The conclusion of the theorem now follows.  $\square$

Denote by  $\mu_X$  and  $\mu_Y$  the Lévy measure for the subordinators  $S_t^X$  and  $S_t^Y$ , respectively. The following is a comparison theorem on Lévy measures.

**Theorem 2.3.4.** *Suppose  $X_t$  and  $Y_t$  are reflected diffusions on  $[0, \infty)$  as in Theorem 2.3.3,  $\phi^X$  and  $\phi^Y$  are the Laplace exponents of inverse local times, respectively. Then  $\phi^Y - \phi^X \geq 0$  is completely monotone and, consequently,  $\mu_X \leq \mu_Y$ .*

**Proof.** Applying Theorem 2.3.3, we have  $S_t^X \leq S_t^Y$ ,  $\mathbb{P}$ -a.s. and so  $0 \leq \phi^X \leq \phi^Y$ .

On the other hand, since  $b(x) \leq B(x)$ , by Lemma 2.3.1,  $\phi^Y/\phi^X$  is completely monotone. Combining the two facts, we see that

$$\frac{\phi^Y}{\phi^X} - 1 \geq 0 \text{ is completely monotone.}$$

Since completely monotone relation is preserved under multiplication (check details in Chapter 1, [39]), we would have

$$\phi^Y - \phi^X = \left( \frac{\phi^Y}{\phi^X} - 1 \right) \phi^X \geq 0 \text{ is completely monotone.}$$

This says that  $\phi^Y - \phi^X$  is the Laplace exponent of some subordinator  $Z$ . Hence  $S^Y$  has the same distribution as the independent sum of two subordinators  $S^X$  and  $Z$ . Denote by  $\nu$  the Lévy measure for  $Z$ . It follows then  $\mu_Y - \mu_X = \nu \geq 0$ .  $\square$

## 2.4 Properties of non-local operators

We use the same notations as in Example 2.3.2. For  $0 < \alpha < 1$ , let  $X_t^{(\alpha)}$  is a reflected Bessel process on  $[0, \infty)$  with the local generator

$$\mathcal{L}^{(\alpha)} = \frac{1}{2} \frac{d^2}{dx^2} + \frac{1 - 2\alpha}{2x} \frac{d}{dx}.$$

As we noted earlier, the inverse local time  $S_t^{(\alpha)}$  is an  $\alpha$ -stable subordinator, with the Laplace exponent

$$\phi^{(\alpha)}(\lambda) = c_\alpha \lambda^\alpha. \quad (2.4.1)$$

We know from Theorem 2.2.1 that under the new probability measure  $\mathbb{Q}$  defined by (2.2.3), the inverse local time of the Girsanov transformed diffusion,  $X_t^{(\alpha, m)}$ , is a relativistic  $\alpha$ -stable subordinator, with the Laplace exponent

$$\phi^{(\alpha, m)}(\lambda) = \phi^{(\alpha)}(\lambda + m) - \phi^{(\alpha)}(m) = c_\alpha((\lambda + m)^\alpha - m^\alpha). \quad (2.4.2)$$

The new reflected diffusion  $X_t^{(\alpha, m)}$  on  $[0, \infty)$  has generator

$$\mathcal{L}^{(\alpha, m)} = \frac{1}{2} \frac{d^2}{dx^2} + \left( \frac{1 - 2\alpha}{2x} + \frac{\rho'_m(x)}{\rho_m(x)} \right) \frac{d}{dx}, \text{ for } x > 0, \quad (2.4.3)$$

where  $\rho_m(x) := \mathbb{E}_x[\exp(-mT_0)]$  with  $T_0$  being the first hitting time of 0 by  $X^{(\alpha)}$ . By Remark 2.2.2,  $\rho_m(x)$  is the unique solution to

$$\begin{cases} (\mathcal{L}^{(\alpha)} - m)\rho(x) = 0, \\ \rho(0) = 1; \rho(\infty) = 0. \end{cases}$$

It is know from ODE,

$$\rho_m(x) = \widehat{c}_\alpha \widehat{K}_\alpha(\sqrt{2mx}),$$

where  $\widehat{c}_\alpha$  is a normalizing constant depending on  $\alpha$  only,  $\widehat{K}_\alpha = x^\alpha K_\alpha$  and

$$K_\alpha(x) = \frac{\pi I_{-\alpha}(x) - I_\alpha(x)}{2 \sin(\alpha\pi)}, \quad (2.4.4)$$

where

$$I_\alpha(x) = \sum_{n=0}^{\infty} \frac{1}{n! \Gamma(n + \alpha + 1)} \left(\frac{x}{2}\right)^{2n+\alpha}.$$

The function  $I_\alpha$  is a solution to the following modified Bessel's equation

$$x^2 u''(x) + x u'(x) - (x^2 + \alpha^2) u = 0.$$

Clearly,  $K_\alpha$  also satisfies the above equation, and is called a modified Bessel function of the second kind.

**Examples 2.4.1.** Suppose  $\alpha = 0.5$ . Note that

$$\begin{aligned} K_{0.5}(x) &= \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n}}{n!} \left[ \frac{\left(\frac{x}{2}\right)^{-0.5}}{\Gamma(0.5 + n)} - \frac{\left(\frac{x}{2}\right)^{0.5}}{\Gamma(1.5 + n)} \right] \\ &= \frac{\pi}{\sqrt{2x}} \sum_{n=0}^{\infty} \left[ \frac{\left(\frac{x}{2}\right)^{2n}}{n! \Gamma(0.5 + n)} - \frac{\left(\frac{x}{2}\right)^{2n+1}}{n! \Gamma(1.5 + n)} \right] \\ &= \frac{\pi}{\sqrt{2x}} \sum_{n=0}^{\infty} \left[ \frac{x^{2n}}{\Gamma(0.5)(2n)!} - \frac{x^{2n+1}}{\Gamma(0.5)(2n+1)!} \right] \\ &= \frac{\pi e^{-x}}{\Gamma(0.5) \sqrt{2x}}. \end{aligned}$$

Thus for  $\alpha = 0.5$ ,

$$\rho_m(x) = \frac{\pi}{\Gamma(0.5) \sqrt{2}} \exp(-\sqrt{2m}x).$$

Consequently, we have the perturbation part as

$$\frac{\rho'_m(x)}{\rho_m(x)} = -\sqrt{2m}.$$

Hence if  $X_t^{(0.5,m)}$  is a reflected process on  $[0, \infty)$  with the local generator

$$\mathcal{L}^{(0.5,m)} = \frac{1}{2} \frac{d^2}{dx^2} - \sqrt{2m} \frac{d}{dx},$$

then its inverse local time at 0 is a relativistic Cauchy subordinator with the Laplace exponent

$$\phi^{(0.5,m)}(x) = c(\sqrt{\lambda + m} - \sqrt{m}).$$

**Examples 2.4.2.** Now if we operate another Girsanov transform on  $\mathcal{L}^{(\alpha,m)}$ , that is

$$\mathcal{L}^{(1)} = \mathcal{L}^{(\alpha,m)} + \frac{q'_n(x)}{q_n(x)} \frac{d}{dx},$$

where  $q_n(x)$  is the unique solution to

$$\begin{cases} (\mathcal{L}^{(\alpha,m)} - n)q_n(x) = 0; \\ q_n(0) = 1, \quad q_n(\infty) = 0. \end{cases}$$

Then the inverse local time at 0,  $S_t^{(1)}$ , of the new reflected diffusion generated by the above generator, has the Laplace exponent

$$\phi^{(1)}(\lambda) = \phi^{(\alpha,m)}(\lambda + n) - \phi^{(\alpha,m)}(n) = c_\alpha((\lambda + m + n)^\alpha - (m + n)^\alpha).$$

It can also be viewed as the Laplace exponent of a relativistic  $\alpha$ -stable subordinator with mass  $m + n$ , which is obtained as the inverse local time at 0 for a reflected diffusion determined locally by

$$\mathcal{L}^{(\alpha,m+n)} = \mathcal{L} + \frac{\rho'_{m+n}(x)}{\rho_{m+n}(x)} \frac{d}{dx}.$$

The two generators should be the same, so we get the relation

$$\frac{\rho'_m(x)}{\rho_m(x)} + \frac{q'_n(x)}{q_n(x)} = \frac{\rho'_{m+n}(x)}{\rho_{m+n}(x)}$$

**Remark 2.4.3.** S. Watanabe [43] has defined a conservative diffusion process  $\tilde{X}_t$  on  $[0, \infty)$ , determined by the local generator in the same form

$$\tilde{\mathcal{L}} = \frac{1}{2} \frac{d^2}{dx^2} + \left( \frac{1 - 2\alpha}{2x} + \frac{\rho'_c(x)}{\rho_c(x)} \right) \frac{d}{dx},$$

but with  $\rho_c(x)$  as the unique solution to

$$\begin{cases} (\mathcal{L}^{(\alpha)} - c)\rho(x) = 0; \\ \rho(0) = 1, \quad \rho'(0) = 0. \end{cases}$$

Thus, the generator can be written as

$$\tilde{\mathcal{L}}u = \frac{1}{\rho_c} (\mathcal{L}^{(\alpha)} - c)(\rho_c \cdot u)(x),$$

and this yields an explicit expression of the transition density for  $X_t^{(\alpha,c)}$

$$\tilde{p}(t, x, y) = \frac{e^{-mt} p^{(\alpha)}(t, x, y)}{\rho_c(x) \rho_c(y)}, \quad x, y \geq 0,$$

where  $p^{(\alpha)}(t, x, y)$  is the transition density of the Bessel process  $X_t^{(\alpha)}$  with respect to the measure  $m^{(\alpha)}(dx) = x^{1-2\alpha} dx$ .

We continue to discuss the reflected diffusions,  $X_t^{(\alpha,m)}$ , which is locally determined by the generator (2.4.3). Consider the drift term  $\frac{\rho'_m(x)}{\rho_m(x)}$ , because  $K'_\alpha(x) = -\frac{\alpha}{x} K_\alpha(x) - K_{\alpha-1}(x)$

$$\begin{aligned} \frac{\rho'_m(x)}{\rho_m(x)} &= \sqrt{2m} \frac{\widehat{K}'_\alpha(\sqrt{2mx})}{\widehat{K}_\alpha(\sqrt{2mx})} \\ &= \frac{\alpha}{x} + \sqrt{2m} \frac{K'_\alpha(\sqrt{2mx})}{K_\alpha(\sqrt{2mx})} \\ &= \frac{\alpha}{x} + \sqrt{2m} \frac{-\frac{\alpha}{\sqrt{2mx}} K_\alpha(\sqrt{2mx}) - K_{\alpha-1}(\sqrt{2mx})}{K_\alpha(\sqrt{2mx})} \\ &= -\sqrt{2m} \frac{K_{\alpha-1}(\sqrt{2mx})}{K_\alpha(\sqrt{2mx})}. \end{aligned} \tag{2.4.5}$$

We will focus on the asymptotic behaviors of  $\frac{\rho'_m(x)}{\rho_m(x)}$  near 0 and  $\infty$  and have the following lemma:

**Lemma 2.4.4.** For  $m \geq 0$ ,  $0 < \alpha < 1$ ,

$$\frac{\rho'_m(x)}{\rho_m(x)} = -\sqrt{2m} \frac{K_{\alpha-1}(\sqrt{2mx})}{K_\alpha(\sqrt{2mx})} \sim \begin{cases} -\frac{m^\alpha \Gamma(1-\alpha)}{2^{\alpha-1} \Gamma(\alpha)} x^{2\alpha-1} & \text{as } x \rightarrow 0+; \\ -\sqrt{2m} & \text{as } x \rightarrow \infty, \end{cases} \tag{2.4.6}$$

where  $\sim$  means the ratio between two sides approaches 1 as  $x$  goes to  $0+$  or  $\infty$ .

**Proof.** When  $x \rightarrow 0+$  and  $\nu \notin \mathbb{Z}$ ,  $K_\nu(x)$  has the following series expansion:

$$\begin{aligned} K_\nu(x) \propto \frac{1}{2} \left( \Gamma(\nu) \left(\frac{x}{2}\right)^{-\nu} \left( 1 + \frac{x^2}{4(1-\nu)} + \frac{x^4}{32(1-\nu)(2-\nu)} + \dots \right) \right. \\ \left. + \Gamma(-\nu) \left(\frac{x}{2}\right)^\nu \left( 1 + \frac{x^2}{4(\nu+1)} + \frac{x^4}{32(\nu+1)(\nu+2)} + \dots \right) \right). \end{aligned}$$

Since  $0 < \alpha < 1$ ,  $\alpha, \alpha - 1 \notin \mathbb{Z}$  in (2.4.5), as  $x \rightarrow 0+$ ,

$$\begin{aligned} \frac{\rho'_m(x)}{\rho_m(x)} &\sim -\sqrt{2m} \frac{\Gamma(1-\alpha) \left(\frac{\sqrt{2mx}}{2}\right)^{\alpha-1}}{\Gamma(\alpha) \left(\frac{\sqrt{2mx}}{2}\right)^{-\alpha}} \\ &\sim -\frac{m^\alpha \Gamma(1-\alpha)}{2^{\alpha-1} \Gamma(\alpha)} x^{2\alpha-1}. \end{aligned} \quad (2.4.7)$$

When  $x \rightarrow \infty$ ,  $K_\nu(x)$  can be described as the following formula:

$$K_\nu(x) \propto \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{\sqrt{x}} \left(1 + O\left(\frac{1}{x}\right)\right).$$

The asymptotic behavior near  $\infty$  is independent of the index  $\nu$ . Thus, as  $x \rightarrow \infty$

$$\frac{\rho'_m(x)}{\rho_m(x)} \sim -\sqrt{2m}. \quad (2.4.8)$$

□

We are now in the position to present the proof for Theorem 2.1.1.

**Proof of Theorem 2.1.1.**  $Y_t$  is a reflected diffusion process, determined by the local generator

$$\mathcal{L} = \frac{1}{2} \frac{d^2}{dx^2} + \left( \frac{1-2\alpha}{2x} - f(x) \right) \frac{d}{dx},$$

and there exists a constant  $c_1$  such that

$$0 \leq f(x) \leq c_1(1 \wedge x)^{2\alpha-1}.$$

Now we check the condition (2.3.2) in Theorem 2.3.3,  $f(x)$  is bounded when  $1/2 \leq \alpha < 1$ , so the condition is naturally satisfied. When  $0 < \alpha < 1/2$ , for a fixed  $T > 0$ , with  $p^{(\alpha)}(t, x, y)$  representing the transition density of a Bessel process of index  $\alpha$  with respect to the measure

$$m^{(\alpha)}(dx) = 2x^{1-2\alpha}dx,$$

$$\begin{aligned} \sup_{x>0} \mathbb{E}_x \left[ \int_0^T |f(X_t^{(\alpha)})|^2 dt \right] &\leq c_1 \sup_{x>0} \int_{(0,\infty)} \left( \int_0^T p^{(\alpha)}(t, x, y) (1 \vee y^{4\alpha-2}) dt \right) m^{(\alpha)}(dy) \\ &\leq c_1 T + c_1 \sup_{x>0} \int_0^1 \left( \int_0^T \frac{x^\alpha y^{3\alpha-1}}{t} \exp\left(-\frac{x^2+y^2}{2t}\right) I_{-\alpha}\left(\frac{xy}{t}\right) dt \right) dy \\ &= c_1 T + c_1 \sup_{x>0} \int_0^1 \int_{xy/T}^\infty \frac{x^\alpha y^{3\alpha-1}}{s} \exp\left(-\frac{(x^2+y^2)s}{2xy}\right) I_{-\alpha}(s) ds dy \\ &\leq c_1 T + c_1 \sup_{x>0} \int_0^1 x^\alpha y^{3\alpha-1} \left( \int_{xy/T}^\infty \frac{e^{-s}}{s} I_{-\alpha}(s) ds \right) dy, \end{aligned}$$

where  $I_{-\alpha}(s)$  is the modified Bessel function of the first kind. Then

$$I_{-\alpha}(s) \propto \begin{cases} \frac{1}{\Gamma(1-\alpha)} \left(\frac{s}{2}\right)^{-\alpha} \left(1 + \frac{s^2}{4(1-\alpha)} + \frac{s^4}{32(1-\alpha)(2-\alpha)} + \dots\right), & s \rightarrow 0; \\ \frac{e^s}{\sqrt{2\pi s}} \left(1 + O\left(\frac{1}{s}\right)\right), & s \rightarrow \infty. \end{cases}$$

Since we have  $0 < y < 1$ , by the above asymptotic behavior

$$\int_{xy/T}^\infty \frac{e^{-s}}{s} I_{-\alpha}(s) ds \text{ is dominated by } C_\alpha T^\alpha (xy)^{-\alpha} \text{ as } x \rightarrow 0+; \text{ by } C_\alpha T^{1/2} (xy)^{-1/2} \text{ as } x \rightarrow \infty.$$

We would then get for  $0 < \alpha < 1/2$

$$\sup_{x>0} \mathbb{E}_x \left[ \int_0^T |f(X_t^{(\alpha)})|^2 dt \right] < \infty.$$

Thus, one can set up a Girsanov transform between  $X_t$  and  $X_t^{(\alpha)}$ , or,  $X_t$  and  $X_t^{(\alpha)}$  are absolutely continuous to each other. Applying Theorem 2.3.3, we have  $S_t \leq S_t^{(\alpha)}$ ,  $\mathbb{P}$ -a.s.

For any  $0 < \alpha < 1$ , from the asymptotic behaviors of  $\frac{\rho'_m(x)}{\rho_m(x)}$  near 0 and  $\infty$  shown in (2.4.6), we can always choose a proper value of  $m$  such that

$$c_1 \leq \sqrt{2m} \wedge \frac{m^\alpha \Gamma(1-\alpha)}{2^{\alpha-1} \Gamma(\alpha)},$$

consequently,

$$0 \leq f(x) \leq -\frac{\rho'_m(x)}{\rho_m(x)}, \text{ } m \text{ depends on } c_1.$$

Thus, by the classic Comparison theorem,

$$X_t^{(\alpha, m)} \leq Y_t \leq X_t^{(\alpha)}, \text{ } \mathbb{P}\text{-a.s.} \quad (2.4.9)$$

By Theorem 2.2.1,  $X_t^{(\alpha)}$  and  $X_t^{(\alpha,m)}$  are absolutely continuous to each other. Thus,  $Y_t$  and  $X_t^{(\alpha,m)}$  are absolutely continuous to each other, i.e., there exists a Girsanov transform between them. Applying Theorem 2.3.3 again,  $S_t^{(\alpha,m)} \leq S_t$ ,  $\mathbb{P}$ -a.s.  $\square$

To prove Theorem 2.1.2, we first recall the following result from Grzywny-Ryznar [31] (with slightly different notation here).

**Theorem 2.4.5.** [31, Theorem 1.1] *Let  $D \subset \mathbb{R}^d$  be a bounded Lipschitz open set. Suppose that  $Y_t$  is a symmetric Lévy process on  $\mathbb{R}^d$  with Lévy measure  $\nu(x)dx$ . Denote by  $\nu^{(\alpha)}(x)$  the Lévy density for the isotropic  $\alpha$ -stable process  $Z$  on  $\mathbb{R}^d$ . Denote by  $G_D$  and  $G_D^{(\alpha)}$  the Green functions of  $Y$  and  $Z$  in  $D$ , respectively. Assume that  $j(x) = \nu^{(\alpha)}(x) - \nu(x) \geq 0$  on  $\mathbb{R}^d$ , and that  $j(x) \leq c|x|^{\rho-d}$  for  $|x| \leq 1$ , where  $c, \rho > 0$ . Then there exists a constant  $C = C(d, \alpha, D, \rho, c)$ , such that*

$$C^{-1}G_D^{(\alpha)}(x, y) \leq G_D(x, y) \leq CG_D^{(\alpha)}(x, y) \quad \text{for all } x, y \in D.$$

**Proof of Theorem 2.1.2.** Denote the Laplace exponents of  $S_t$ ,  $S_t^{(\alpha)}$ ,  $S_t^{(\alpha,m)}$ , by  $\phi(\lambda)$ ,  $\phi^{(\alpha)}(\lambda)$ ,  $\phi^{(\alpha,m)}(\lambda)$ , respectively. Now applying Theorem 2.3.4,

$$\phi(\lambda) - \phi^{(\alpha,m)}(\lambda) \text{ and } \phi^{(\alpha)}(\lambda) - \phi(\lambda) \text{ are completely monotone.}$$

Denote  $\nu^{(\alpha,m)}$ ,  $\nu$ ,  $\nu^{(\alpha)}$  as Lévy measures of inverse local times respectively, then

$$\nu^{(\alpha)} - \nu \geq 0; \quad \nu - \nu^{(\alpha,m)} \geq 0$$

and for any  $0 < \alpha < 1$ ,  $t > 0$ ,

$$0 \leq (\nu^{(\alpha)} - \nu)(t) \leq (\nu^{(\alpha)} - \nu^{(\alpha,m)})(t) \leq c_\alpha \frac{1 - e^{-mt}}{t^{\alpha+1}}. \quad (2.4.10)$$

Thus, for  $|x| \leq 1$ , the difference between Lévy measures of trace processes,  $B_{S_t}$ ,  $B_{S_t^{(\alpha)}}$ , would

be

$$\begin{aligned}
j(x) &= c_\alpha \int_0^\infty (4\pi t)^{-d/2} e^{-\frac{|x|^2}{4t}} (\nu^{(\alpha)} - \nu)(t) dt \\
&= c_\alpha \int_0^\infty (4\pi t)^{-d/2} e^{-\frac{|x|^2}{4t}} \frac{1 - e^{-mt}}{t^{\alpha+1}} dt \\
&\leq c_\alpha \int_0^\infty (4\pi t)^{-d/2} e^{-\frac{|x|^2}{4t}} \frac{mt}{t^{\alpha+1}} dt \\
&= c_\alpha \pi m^{-d/2} 4^{\alpha-1} |x|^{-d+2-2\alpha} \int_0^\infty s^{d/2+\alpha-2} e^{-s} ds \\
&\leq C |x|^{-d+2-2\alpha},
\end{aligned}$$

where  $C = C(\alpha, m, d, c_\alpha)$ . The second to the last equality is obtained by doing change of variables  $s = |x|^2/(4t)$ . From the proof of Theorem 2.1.1,  $m$  is fixed once given a  $f(x)$ .

Applying Theorem 2.4.5 with  $\rho = 2 - 2\alpha > 0$ , we conclude that there exists a constant  $C_1 = C(d, \alpha, D, C)$  such that

$$C_1^{-1} G_D^{(2\alpha)}(x, y) \leq G_D(x, y) \leq C_1 G_D^{(2\alpha)}(x, y)$$

for all  $x, y \in D$ . □

## Chapter 3

## DIFFUSIONS WITH JUMPS AND NON-LOCAL FEYNMAN-KAC PERTURBATIONS

### 3.1 Diffusions with jumps and heat kernel estimates

In this chapter, we start with a Hunt process  $X$  on  $\mathbb{R}^d$  that has a jointly continuous transition density function  $p(t, x, y)$  that enjoys two-sided estimates, there exist positive constants  $c_k$ ,  $1 \leq k \leq 4$ , such that for every  $t > 0$  and  $x, y \in \mathbb{R}^d$ ,

$$\begin{aligned} & c_1 (t^{-d/2} \wedge t^{-d/\alpha}) \wedge \left( t^{-d/2} \exp\left(-\frac{c_2|x-y|^2}{t}\right) + t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right) \\ & \leq p(t, x, y) \\ & \leq c_3 (t^{-d/2} \wedge t^{-d/\alpha}) \wedge \left( t^{-d/2} \exp\left(-\frac{c_4|x-y|^2}{t}\right) + t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right). \end{aligned} \quad (3.1.1)$$

on  $(0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ . Example 1.3.3 and 1.3.4 are two typical classes of jump processes that enjoy these heat kernel estimates.

Under this assumption, the Hunt process  $X$  has a Lévy system  $(N(x, dy), dt)$  with  $N(x, dy) = \frac{c(x, y)}{|x-y|^{d+\alpha}} dy$  for some measurable function  $c(x, y)$  bounded between two positive constants; see (3.3.7) below. That is, for every non-negative function  $\varphi(x, y)$  on  $\mathbb{R}^d \times \mathbb{R}^d$  that vanishes along the diagonal,

$$\mathbb{E}_x \left[ \sum_{0 < s \leq t} \varphi(X_{s-}, X_s) \right] = \mathbb{E}_x \left[ \int_0^t \int_{\mathbb{R}^d} \varphi(X_s, y) N(X_s, dy) ds \right], \quad x \in \mathbb{R}^d, \quad t > 0.$$

Here we use the convention that we extend the definition of functions to cemetery point  $\partial$  by setting 0 value there; for example  $\varphi(x, \partial) = 0$ . For convenience, we take  $T = 1$ . We will study the stability of heat kernel estimates under non-local Feynman-Kac transform:

$$T_t f(x) = \mathbb{E}_x \left[ \exp \left( A_t^\mu + \sum_{s \leq t} F(X_{s-}, X_s) \right) f(X_t) \right],$$

where  $A^\mu$  is a continuous additive functional of  $X$  of finite variations having signed Revuz measure  $\mu$  and  $F(x, y)$  is a bounded measurable function vanishing on the diagonals. We point out that in this chapter we do not require  $F$  to be symmetric. Informally, the semigroup  $(T_t^{\mu, F}; t \geq 0)$  has generator

$$\mathcal{A}f(x) = (\mathcal{L} + \mu)f(x) + \int_{\mathbb{R}^d} (e^{F(x, y)} - 1) f(y) N(x, dy), \quad (3.1.2)$$

where  $\mathcal{L}$  is the infinitesimal generator of  $X$ ; see [18, Remark 1 on p.53] for a calculation.

Throughout this chapter, we assume that  $X$  is a Hunt process on  $\mathbb{R}^d$  having a jointly continuous transition density function  $p(t, x, y)$  with respect to the Lebesgue measure on  $\mathbb{R}^d$  and that the two-sided estimates (3.1.1) holds for  $p(t, x, y)$  on  $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ . Since we are concerned with heat kernel estimates of (3.1.1) on fixed time intervals, it is desirable to rewrite the estimates in the following equivalent but more compact form. This equivalent form (3.1.3) is given in [13]. For reader's convenience, we give a proof here.

**Lemma 3.1.1.** *Two-sided estimates (3.1.1) for  $p(t, x, y)$  on  $(0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$  is equivalent to the following. There exist constants  $C \geq 1$  such that for  $0 < t \leq 1$  and  $x, y \in \mathbb{R}^d$ ,*

$$C^{-1} (\Gamma_{c_2}(t; x - y) + \eta(t; x - y)) \leq p(t, x, y) \leq C (\Gamma_{c_4}(t; x - y) + \eta(t; x - y)), \quad (3.1.3)$$

where

$$\Gamma_\lambda(t; x) := t^{-d/2} e^{-\lambda|x|^2/t} \quad \text{and} \quad \eta(t; x) := \frac{t}{(t^{1/2} + |x|)^{d+\alpha}}. \quad (3.1.4)$$

**Proof.** Note that  $t^{-d/2} \leq t^{-d/\alpha}$  for  $t \in (0, 1]$ , and

$$\frac{1}{2}(a \wedge b + a \wedge c) \leq a \wedge (b + c) \leq a \wedge b + a \wedge c \quad \text{for } a, b, c > 0.$$

Thus, for  $\lambda > 0$  and  $t \in (0, 1]$ ,

$$\begin{aligned}
& (t^{-d/2} \wedge t^{-d/\alpha}) \wedge \left( t^{-d/2} \exp\left(-\frac{\lambda r^2}{t}\right) + t^{-d/\alpha} \wedge \frac{t}{r^{d+\alpha}} \right) \\
\asymp & t^{-d/2} \exp\left(-\frac{\lambda r^2}{t}\right) + t^{-d/2} \wedge t^{-d/\alpha} \wedge \frac{t}{r^{d+\alpha}} \\
\asymp & \begin{cases} t^{-d/2} \exp\left(-\frac{\lambda r^2}{t}\right) + t^{-d/2} \wedge \frac{t}{r^{d+\alpha}} \asymp t^{-d/2} \exp\left(-\frac{\lambda r^2}{t}\right) + \frac{t}{(t^{1/2}+r)^{d+\alpha}} & \text{if } r \geq t^{1/2} (\geq t^{1/\alpha}), \\ t^{-d/2} + t^{-d/2} \wedge \frac{t}{r^{d+\alpha}} \asymp t^{-d/2} & \text{if } t^{1/\alpha} < r \leq t^{1/2}, \\ t^{-d/2} + t^{-d/2} \wedge t^{-d/\alpha} \asymp t^{-d/2} & \text{if } 0 < r < t^{1/\alpha} (\leq t^{1/2}) \end{cases} \\
\asymp & t^{-d/2} \exp\left(-\frac{\lambda r^2}{t}\right) + \frac{t}{(t^{1/2}+r)^{d+\alpha}},
\end{aligned}$$

where the last line is due to the fact that for  $0 < r \leq t^{1/2}$ ,  $\frac{t}{(t^{1/2}+r)^{d+\alpha}} \leq t^{1-(d+\alpha)/2} \leq t^{-d/2}$ .

This establishes the lemma.  $\square$

We now consider two conditions on signed measures and functions used in non-local Feynman-Kac perturbation. For a  $\sigma$ -finite signed measure  $\mu$ , we use  $\mu^+$  and  $\mu^-$  to denote its positive and negative part in its Jordan decomposition, and its total variation measure is given by  $|\mu| := \mu^+ + \mu^-$ . We require that  $\mu$  satisfies the following condition:

$$\begin{cases} \limsup_{r \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_{B(x,r)} |x-y|^{(1+\alpha)/3} |\mu|(dy) = 0 & d = 1; \\ \limsup_{r \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_{B(x,r)} \ln \frac{1}{|x-y|} |\mu|(dy) = 0 & d = 2; \\ \limsup_{r \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_{B(x,r)} |x-y|^{2-d} |\mu|(dy) = 0 & d \geq 3; \end{cases} \quad (3.1.5)$$

where  $B(x, r)$  is a ball centered at  $x$  with radius  $r$ .

For a function  $F(x, y)$  defined on  $\mathbb{R}^d \times \mathbb{R}^d$  that vanishes along the diagonal, we require

$$\begin{cases} \limsup_{r \downarrow 0} \sup_{y \in \mathbb{R}^d} \int_{B(y,r)} |y-z|^{(1+\alpha)/3} \left( \int_{\mathbb{R}^d} \frac{|F(z,w)| + |F(w,z)|}{|z-w|^{d+\alpha}} dw \right) dz = 0 & d = 1; \\ \limsup_{r \downarrow 0} \sup_{y \in \mathbb{R}^d} \int_{B(y,r)} \ln \frac{1}{|y-z|} \left( \int_{\mathbb{R}^d} \frac{|F(z,w)| + |F(w,z)|}{|z-w|^{d+\alpha}} dw \right) dz = 0 & d = 2; \\ \limsup_{r \downarrow 0} \sup_{y \in \mathbb{R}^d} \int_{B(y,r)} |y-z|^{2-d} \left( \int_{\mathbb{R}^d} \frac{|F(z,w)| + |F(w,z)|}{|z-w|^{d+\alpha}} dw \right) dz = 0 & d \geq 3; \end{cases} \quad (3.1.6)$$

With  $\mu, F$  satisfying the above conditions, we can define an additive function of  $X$  by

$$A_t^{\mu, F} = A_t^\mu + \sum_{0 < s \leq t} F(X_{s-}, X_s),$$

where  $A_t^\mu$  is a continuous additive functional of  $X$  having  $\mu$  as its Revuz measure. It is easy to check that  $A_t^{\mu, F}$  is well defined and is of finite variations on compact time intervals. We can then define the following non-local Feynman-Kac semigroup of  $X$  by

$$T_t^{\mu, F} f(x) = \mathbb{E}_x \left[ \exp(A_t^{\mu, F}) f(X_t) \right], \quad t \geq 0, x \in \mathbb{R}^d. \quad (3.1.7)$$

The goal of this chapter is to study the stability of heat kernel estimates under the above non-local Feynman-Kac transform. We show that if  $\mu$  and  $F$  satisfy certain conditions, or equivalently, are in certain Kato classes of  $X$ , the non-local Feynman-Kac semigroup  $\{T_t; t \geq 0\}$  has a heat kernel  $q(t, x, y)$  and  $q(t, x, y)$  has two-sided estimates (3.1.1) on  $(0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$  but with a set of possible different constants  $c_k, 1 \leq k \leq 4$ . Comparing with [15, 44, 5], the novelty of our study is that  $X$  has both the diffusive and jumping components, and that the Gaussian bounds in (3.1.1) have different constants  $c_2$  and  $c_4$  in the exponents for the upper and lower bound estimates. These features made the perturbation estimates more challenging.

**Theorem 3.1.2.** *Suppose  $X$  is a Hunt process on  $\mathbb{R}^d$  that has a jointly continuous transition density function  $p(t, x, y)$  with respect to the Lebesgue measure and that the two-sided heat kernel estimates (3.1.1) holds for  $p(t, x, y)$  on  $(0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$ . Let  $\mu$  satisfies condtion (3.1.5) and  $F(x, y)$  be a measurable function so that  $F_1 = e^F - 1$  satisfies condtion (3.1.6). Then the non-local Feynman-Kac semigroup  $(T_t^{\mu, F}; t \geq 0)$  has a jointly continuous kernel  $q(t, x, y)$  so that  $T_t^{\mu, F} f(x) = \int_{\mathbb{R}^d} q(t, x, y) f(y) dy$  for every bounded Borel measurable function  $f$  on  $\mathbb{R}^d$ . Moreover, there exist positive constants  $\tilde{c}_3, K$  that depend on  $(d, \alpha, c_3, c_4, \|F_1\|_\infty, N_\mu^{\alpha, c_4}, N_{F_1}^{\alpha, c_4})$ , where  $N_\mu^{\alpha, c_4}, N_{F_1}^{\alpha, c_4}$  are defined in (3.2.1) and (3.2.2), so that for any  $t > 0$  and  $x, y \in \mathbb{R}^d$ , using the notations in (3.1.4), we have*

$$q(t, x, y) \leq \tilde{c}_3 e^{Kt} (\Gamma_{2c_4/3}(t; x - y) + \eta(t; x - y)).$$

If in addition,  $F$  satisfies condition (3.1.6), then there exist positive constants  $\tilde{c}_1$ ,  $\lambda_1$  and  $K_1$  that depend on  $(d, \alpha, c_1, c_2, c_3, c_4, \|F\|_\infty, N_\mu^{\alpha, c_4}, N_F^{\alpha, c_4})$  so that for any  $t > 0$  and  $x, y \in \mathbb{R}^d$ ,

$$q(t, x, y) \geq \tilde{c}_1 e^{-K_1 t} (\Gamma_{\lambda_1}(t; x - y) + \eta(t; x - y)).$$

The rest of this chapter is organized as follows. Section 3.2 gives the definition of Kato classes and the properties that are related to conditions (3.1.5), (3.1.6), Green function estimates are also given in this section. In Section 3.3, we establish various 3P type inequalities that are needed to study non-local Feynman-Kac perturbations. Proof of the main results, the two-sided estimates for the heat kernel of the Feynman-Kac semigroup, is given in Section 3.4.

### 3.2 Kato classes

In this section, we first introduce Kato classes for signed measures  $\mu$  and for functions  $F$ , then discuss their relations with conditions (3.1.5) and (3.1.6).

For a signed measure  $\mu$  on  $\mathbb{R}^d$ , using the notations in (3.1.4), we define

$$N_\mu^{\alpha, \lambda} := \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} (\Gamma_\lambda(s; x - y) + \eta(s; x - y)) |\mu|(dy) ds. \quad (3.2.1)$$

For a function  $F(x, y)$  defined on  $\mathbb{R}^d \times \mathbb{R}^d$  that vanishes along the diagonal, we define

$$N_F^{\alpha, \lambda}(t) := \sup_{y \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} (\Gamma_\lambda(s; y - z) + \eta(s; y - z)) \frac{|F(z, w)| + |F(w, z)|}{|z - w|^{d+\alpha}} dw dz ds. \quad (3.2.2)$$

Two Kato classes can be defined accordingly as follows:

**Definition 3.2.1.** (i) A signed measure  $\mu$  on  $\mathbb{R}^d$  is said to be in the **Kato class**  $\mathbf{K}_\alpha$  if  $\lim_{t \downarrow 0} N_\mu^{\alpha, \lambda}(t) = 0$  for some and hence for all  $\lambda > 0$ . A measurable function  $f$  on  $\mathbb{R}^d$  is said to be in **Kato class**  $\mathbf{K}_\alpha$  if  $|f(x)|\mu(dx) \in \mathbf{K}_\alpha$ .

(ii) A bounded measurable function  $F$  on  $\mathbb{R}^d \times \mathbb{R}^d$  vanishing on the diagonal, is said to be **Kato class**  $\mathbf{J}_\alpha$  if  $\lim_{t \downarrow 0} N_F^{\alpha, \lambda}(t) = 0$  for some and hence for all  $\lambda > 0$ .

Clearly, by definition, if  $F, G \in \mathbf{J}_\alpha$  and  $a \in \mathbb{R}$ , then so are  $aF$ ,  $e^F - 1$ ,  $F + G$  and  $FG$ . We first have the following property of  $\mathbf{K}_\alpha$ ,

**Proposition 3.2.2.**  $\mu$  is a  $\sigma$ -finite signed measure on  $\mathbb{R}^d$ , if (3.1.5) holds for  $\mu$ , then  $\mu \in \mathbf{K}_\alpha$ . In particular, when  $d \geq 2$ ,  $\mathbf{K}_\alpha$  is independent of  $\alpha$ .

**Proof.** By the definition (3.2.1),

$$N_\mu^{\alpha, \lambda} \asymp \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \left( s^{-d/2} e^{-|x-y|/t} + s^{-d/2} \wedge \frac{s}{|x-y|^{d+\alpha}} \right) |\mu|(dy) ds. \quad (3.2.3)$$

$\lim_{t \downarrow 0} N_\mu^{\alpha, \lambda} = 0$  if and only if

$$\limsup_{t \downarrow 0} \int_0^t \int_{\mathbb{R}^d} s^{-d/2} e^{-|x-y|/t} |\mu|(dy) ds = 0; \quad (3.2.4)$$

and

$$\limsup_{t \downarrow 0} \int_0^t \int_{\mathbb{R}^d} s^{-d/2} \wedge \frac{s}{|x-y|^{d+\alpha}} |\mu|(dy) ds = 0. \quad (3.2.5)$$

By Theorem 3.6 in [22], (3.2.4) is the condition of Kato class for Brownian motion and it is equivalent to the condition (3.1.5) for  $d \geq 2$ ; when  $d = 1$ , (3.2.4) is equivalent to

$$\limsup_{r \downarrow 0} \int_{x \in \mathbb{R}^d} \int_{B(x, r)} |x-y| |\mu|(dy) = 0. \quad (3.2.6)$$

Now we consider (3.2.5), when  $t$  is small enough, let  $U := \{|x-y|^{d+\alpha} < t^{1+d/2}\}$ , then

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^d} s^{-d/2} \wedge \frac{s}{|x-y|^{d+\alpha}} |\mu|(dy) ds \\ &= \int_0^t \int_U s^{-d/2} \wedge \frac{s}{|x-y|^{d+\alpha}} |\mu|(dy) ds + \int_0^t \int_{U^c} \frac{s}{|x-y|^{d+\alpha}} |\mu|(dy) ds \\ &\asymp \int_U \int_{|x-y|^{2(d+\alpha)/(d+2)}}^t s^{-d/2} ds |\mu|(dy) + \int_U \int_0^{|x-y|^{2(d+\alpha)/(d+2)}} \frac{s}{|x-y|^{d+\alpha}} ds |\mu|(dy) \\ &\quad + \int_{U^c} \frac{t^2}{|x-y|^{d+\alpha}} |\mu|(dy) \\ &\lesssim \begin{cases} \int_U |x-y|^{(1+\alpha)/3} |\mu|(dy) & d = 1; \\ \int_U \ln |x-y|^{-1} |\mu|(dy) & d = 2; \\ \int_U |x-y|^{(2-d)(d+\alpha)/(d+2)} |\mu|(dy) & d \geq 3. \end{cases} \end{aligned}$$

Thus, when  $d \geq 2$ , if (3.1.5) holds for  $\mu$ ,

$$\limsup_{t \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} s^{-d/2} \wedge \frac{s}{|x-y|^{d+\alpha}} |\mu|(dy) ds = 0.$$

When  $d = 1$ , as  $|x-y| \leq |x-y|^{(1+\alpha)/3}$ ,  $\lim_{t \downarrow 0} N_\mu^{\alpha,\lambda} = 0$  holds if  $\mu$  satisfies the corresponding condition in (3.1.5).  $\square$

Observe that

$$N_F^{\alpha,\lambda}(t) = \sup_{y \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} (\Gamma_\lambda(s; y-z) + \eta(s; y-z)) \left( \int_{\mathbb{R}^d} \frac{|F(z,w)| + |F(w,z)|}{|z-w|^{d+\alpha}} dw \right) dz ds,$$

Thus,  $F \in \mathbf{J}_\alpha$  is equivalent to

$$\int_{\mathbb{R}^d} \frac{|F(z,w)| + |F(w,z)|}{|z-w|^{d+\alpha}} dw \in \mathbf{K}_\alpha,$$

we can easily get the following property of  $\mathbf{J}_\alpha$  as the corollary of Proposition 3.2.2,

**Corollary 3.2.3.** *F is a bounded measurable function on  $\mathbb{R}^d \times \mathbb{R}^d$  vanishing on the diagonal, if  $F \in \mathbf{J}_\alpha$ , then (3.1.6) holds for F.*

A different way to view the relation between (3.1.5), (3.1.6) and Kato classes is using Green function estimates for  $X$  in small balls (see Theorem 1 in [46]) when  $d \geq 2$ , we first obtain Green function estimates for  $X_t$ .

**Proposition 3.2.4.** *For  $d \geq 2$ , denote the Green function of  $X_t$  by  $G(x,y)$ , then*

$$G(x,y) \asymp \begin{cases} \left(1 \vee \ln \frac{1}{|x-y|}\right) \wedge |x-y|^{\alpha-2} & d=2; \\ |x-y|^{2-d} \wedge |x-y|^{\alpha-d} & d \geq 3. \end{cases} \quad (3.2.7)$$

**Proof.** We define

$$\varphi(t,r) := (t^{-d/2} \wedge t^{-d/\alpha}) \wedge \left( t^{-d/2} e^{-r^2/t} + t^{-d/\alpha} \wedge \frac{t}{r^{d+\alpha}} \right),$$

and  $g(r) := \int_0^\infty \varphi(t,r) dt$  for  $r > 0$ . When  $t \in (0,1)$ ,

$$\begin{aligned} \varphi(t,r) &= t^{-d/2} \wedge \left( t^{-d/2} e^{-r^2/t} + t^{-d/\alpha} \wedge \frac{t}{r^{d+\alpha}} \right) \\ &\asymp t^{-d/2} e^{-r^2/t} + t^{-d/2} \wedge \frac{t}{r^{d+\alpha}}. \end{aligned}$$

When  $t \in (1, \infty)$ ,

$$\begin{aligned}\varphi(t, r) &= t^{-d/\alpha} \wedge \left( t^{-d/2} e^{-r^2/t} + t^{-d/\alpha} \wedge \frac{t}{r^{d+\alpha}} \right) \\ &\asymp t^{-d/\alpha} \wedge t^{-d/2} e^{-r^2/t} + t^{-d/\alpha} \wedge \frac{t}{r^{d+\alpha}}.\end{aligned}$$

We first consider for small  $r$ , when  $t > 1$ ,  $t^{-d/\alpha} < t/r^{d+\alpha}$  for  $r \leq 1/2$ . Thus, we have

$$\begin{aligned}g(r) &\asymp \int_0^1 t^{-d/2} e^{-r^2/t} dt + \int_0^1 t^{-d/2} \wedge \frac{t}{r^{d+\alpha}} dt + \int_1^\infty t^{-d/\alpha} dt \\ &\asymp \int_0^1 t^{-d/2} e^{-r^2/t} dt + \int_0^{r^{2(\alpha+d)/(2+d)}} \frac{t}{r^{d+\alpha}} dt + \int_{r^{2(\alpha+d)/(2+d)}}^1 t^{-d/2} dt \\ &\asymp \begin{cases} \ln \frac{1}{r}, & \text{if } d = 2; \\ r^{2-d} + r^{(2-d)(\alpha+d)/(2+d)}, & \text{if } d \geq 3. \end{cases} \\ &\asymp \begin{cases} \ln \frac{1}{r}, & \text{if } d = 2; \\ r^{2-d}, & \text{if } d \geq 3. \end{cases}\end{aligned}$$

Now we consider for large  $r$ , when  $t < 1$ ,  $t/r^{d+\alpha} < t^{-d/2}$  for  $r > 1$ . We have

$$\begin{aligned}g(r) &\asymp \int_0^1 t^{-d/2} e^{-r^2/t} dt + \int_0^1 \frac{t}{r^{d+\alpha}} dt \\ &\quad + \int_1^\infty \left( t^{-d/\alpha} \wedge t^{-d/2} e^{-r^2/t} \right) dt + \int_1^{r^\alpha} \frac{t}{r^{d+\alpha}} dt + \int_{r^\alpha}^\infty t^{-d/\alpha} dt \\ &\asymp \int_0^\infty t^{-d/2} e^{-r^2/t} dt + r^{\alpha-d} \asymp r^{\alpha-d}.\end{aligned}$$

Since  $g(r)$  is decreasing in  $r$ , we can get Green function estimates as shown in (3.2.7)  $\square$

**Remark 3.2.5.** *By Theorem 1 in [46], we can re-prove Proposition 3.2.2 for  $d \geq 2$ . Since  $\mu \in \mathbf{K}_\alpha$  is equivalent to*

$$\limsup_{r \downarrow 0} \sup_{x \in \mathbb{R}^d} \mathbb{E}_x \left[ \int_0^\infty 1_{B(x,r)}(X_t) dA_t^{|\mu|} \right] = 0,$$

which can also be written as

$$\limsup_{r \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_{B(x,r)} G(x,y) |\mu|(dy) = 0.$$

Now applying Proposition 3.2.4, when  $r \rightarrow 0$ ,

$$G(x,y) \asymp \begin{cases} \ln \frac{1}{|x-y|} & d = 2; \\ |x-y|^{2-d} & d \geq 3. \end{cases}$$

Thus, (3.1.5) holds for  $\mu$  when  $d \geq 2$ .

### 3.3 3P inequalities

In this section we will establish various 3P type inequalities, which are key ingredients in the proof of Theorem 3.1.2. Lemma 3.3.1, Lemma 3.3.3, Lemma 3.3.4, and Lemma 3.3.5 are dealing with  $\Gamma_c(t; x-y)$  and  $\eta(t; x-y)$  as defined in (3.1.4). Theorem 3.3.2 and Theorem 3.3.6 are the main results of this section.

**Lemma 3.3.1.** For  $0 < s < t$ , and  $x, y, z \in \mathbb{R}^d$ ,

(i) There exists a constant  $C_1 = C_1(d, \alpha)$  such that

$$\eta(t-s; x-z)\eta(s; z-y) \leq C_1 \eta(t; x-y)(\eta(t-s; x-z) + \eta(s; z-y)). \quad (3.3.1)$$

(ii) For  $0 < a < b$ , there exists a constant  $C_2 = C_2(d, a, b)$  such that for any measure  $\mu$  on  $\mathbb{R}^d$ ,

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^d} \Gamma_a(t-s; x-z) \Gamma_b(s; z-y) |\mu|(dz) ds \\ & \leq C_2 \Gamma_a(t; x-y) \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \Gamma_c(s; x-y) |\mu|(dy) ds, \end{aligned} \quad (3.3.2)$$

where  $c = (b-a) \wedge \frac{a}{2}$ .

(iii) There exists a constant  $C_3 = C_3(d, \alpha, a)$  such that for any measure  $\mu$  on  $\mathbb{R}^d$ ,

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^d} \Gamma_a(t-s; x-z) \eta(s; z-y) |\mu|(dz) ds \\ & \leq C_3 \left( \Gamma_a(t; x-y) \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \eta(s; x-y) |\mu|(dy) ds \right. \\ & \quad \left. + \eta(t; x-y) \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \Gamma_a(s; x-y) |\mu|(dy) ds \right). \end{aligned} \quad (3.3.3)$$

**Proof.** (i) For  $0 < s < t$  and  $x, y, z \in \mathbb{R}^d$ , we have

$$\begin{aligned} & \frac{\eta(t-s; x-z) \eta(s; z-y)}{\eta(t; x-y)} \\ & = \frac{(t-s)s}{t} \left( \frac{t^{1/2} + |x-y|}{((t-s)^{1/2} + |x-z|)(s^{1/2} + |z-y|)} \right)^{d+\alpha} \\ & \leq ((t-s) \wedge s) \left( \frac{((t-s) + s)^{1/2} + |x-z| + |z-y|}{((t-s)^{1/2} + |x-z|)(s^{1/2} + |z-y|)} \right)^{d+\alpha} \\ & \leq 2^{d+\alpha} ((t-s) \wedge s) \left( \frac{1}{((t-s)^{1/2} + |x-z|)^{d+\alpha}} + \frac{1}{(s^{1/2} + |z-y|)^{d+\alpha}} \right) \\ & \leq 2^{d+\alpha} (\eta(t-s; x-z) + \eta(s; z-y)). \end{aligned}$$

(ii) The proof for this part is similar to that for Lemma 3.1 in [45]. We first write

$$\begin{aligned} J(t, x, y) & := \int_0^t \int_{\mathbb{R}^d} \Gamma_a(t-s; x-z) \Gamma_b(s; z-y) |\mu|(dz) ds \\ & = \int_0^{\rho t} \int_{\mathbb{R}^d} \Gamma_a(t-s; x-z) \Gamma_b(s; z-y) |\mu|(dz) ds \\ & \quad + \int_{\rho t}^t \int_{\mathbb{R}^d} \Gamma_a(t-s; x-z) \Gamma_b(s; z-y) |\mu|(dz) ds \end{aligned}$$

Applying the elementary inequality

$$\frac{|x-z|^2}{t-s} + \frac{|z-y|^2}{s} \geq \frac{|x-y|^2}{t}, \quad \text{for } 0 < s < t, \quad (3.3.4)$$

one can obtain

$$\begin{aligned}
& \int_0^{\rho t} \int_{\mathbb{R}^d} \Gamma_a(t-s; x-z) \Gamma_b(s; z-y) |\mu|(dz) ds \\
&= \int_0^{\rho t} \int_{\mathbb{R}^d} (t-s)^{-d/2} s^{-d/2} \exp\left(-a \frac{|x-z|^2}{t-s}\right) \exp\left(-b \frac{|z-y|^2}{s}\right) |\mu|(dz) ds \\
&= \int_0^{\rho t} \int_{\mathbb{R}^d} (t-s)^{-d/2} s^{-d/2} \exp\left(-a \left(\frac{|x-z|^2}{t-s} + \frac{|z-y|^2}{s}\right)\right) \exp\left(- (b-a) \frac{|z-y|^2}{s}\right) |\mu|(dz) ds \\
&\leq \int_0^{\rho t} \int_{\mathbb{R}^d} (t-s)^{-d/2} s^{-d/2} \exp\left(-a \frac{|x-y|^2}{t}\right) \exp\left(- (b-a) \frac{|z-y|^2}{s}\right) |\mu|(dz) ds \\
&\leq (1-\rho)^{-d/2} \Gamma_a(t; x-y) \int_0^{\rho t} \int_{\mathbb{R}^d} \Gamma_{b-a}(s; z-y) |\mu|(dz) ds.
\end{aligned}$$

For the other term, by defining  $U := \{|z-y| \geq |x-y|(a/b)^{1/2}\}$ , we have

$$\begin{aligned}
& \int_{\rho t}^t \int_{\mathbb{R}^d} \Gamma_a(t-s; x-z) \Gamma_b(s; z-y) |\mu|(dz) ds \\
&= \int_{\rho t}^t \int_U \Gamma_a(t-s; x-z) \Gamma_b(s; z-y) |\mu|(dz) ds + \int_{\rho t}^t \int_{U^c} \Gamma_a(t-s; x-z) \Gamma_b(s; z-y) |\mu|(dz) ds \\
&\leq (\rho t)^{-d/2} \exp\left(-\frac{a|x-y|^2}{t}\right) \int_{\rho t}^t \int_U (t-s)^{-d/2} \exp\left(-\frac{a|x-z|^2}{t-s}\right) |\mu|(dz) ds \\
&\quad + (\rho t)^{-d/2} \int_{\rho t}^t \int_{U^c} (t-s)^{-d/2} \exp\left(-\frac{a|x-z|^2}{t-s}\right) |\mu|(dz) ds.
\end{aligned}$$

On  $U^c$ , we have the inequality,

$$|x-z| \geq |x-y| - |y-z| \geq |x-y|(1 - (a/b)^{1/2}),$$

thus,

$$\begin{aligned}
& \int_{\rho t}^t \int_{\mathbb{R}^d} \Gamma_a(t-s; x-z) \Gamma_b(s; z-y) |\mu|(dz) ds \\
&\leq \rho^{-d/2} \Gamma_a(t; x-y) \int_{\rho t}^t \Gamma_a(t-s; x-z) |\mu|(dz) ds \\
&\quad + (\rho t)^{-d/2} \int_{\rho t}^t \int_{U^c} (t-s)^{-d/2} \exp\left(-\frac{a|x-z|^2}{2(t-s)}\right) \exp\left(-\frac{a(1-(a/b)^{1/2})^2|x-y|^2}{2(t-s)}\right) |\mu|(dz) ds \\
&\leq \rho^{-d/2} \Gamma_a(t; x-y) \int_{\rho t}^t \Gamma_a(t-s; x-z) |\mu|(dz) ds \\
&\quad + (\rho t)^{-d/2} \exp\left(-\frac{a(1-(a/b)^{1/2})^2|x-y|^2}{2(1-\rho)t}\right) \int_{\rho t}^t \int_{U^c} (t-s)^{-d/2} \exp\left(-\frac{a|x-z|^2}{2(t-s)}\right) |\mu|(dz) ds,
\end{aligned}$$

by selecting  $\rho$  such that  $2(1 - \rho) = (1 - (a/b)^{1/2})^2$ , we achieve the estimate in (3.3.2), with  $c = (b - a) \wedge \frac{a}{2}$ , and  $C_2$  depends on  $d, a, b$ .

(iii) For  $0 < s < t$ , if  $|x - y| \leq t^{1/2}$ , we have

$$\begin{aligned} \Gamma_a(t - s; x - z) &\leq 2^{d/2} t^{-d/2} \leq 2^{d/2} e^a \Gamma_a(t; x - y), & \text{for } s \in (0, t/2]; \\ \eta(s; z - y) &\leq 4^{d+\alpha} \eta(t; x - y), & \text{for } s \in (t/2, t). \end{aligned}$$

If  $|x - y| > t^{1/2}$ , consider on  $V := \{|y - z| \geq |x - y|/2\}$ , one have  $\eta(s; z - y) \leq 2^{d+\alpha} \eta(t; x - y)$  for all  $0 < s < t$ .

On  $V^c$ ,  $|x - z| \geq |x - y| - |y - z| \geq |x - y|/2$ , we have  $\Gamma_a(t - s; x - z) \leq \gamma \Gamma_a(t; x - y)$ , where  $\gamma$  depends on  $a, d$ .

The estimate (3.3.3) directly follows from the above discussion.  $\square$

Recall the definition of  $N_\mu^{\alpha, \lambda}$  from (3.2.1). We next derive an integral 3P type inequality for  $p(t, x, y)$  in small time, by using two-sided heat kernel estimates in Lemma 3.1.1. For notational convenience, let  $\lambda = c_4$ , where  $c_4$  is the positive constant in (3.1.1).

**Theorem 3.3.2.** *For any  $\mu$  that satisfies (3.1.5), and any  $(t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$ , there exists a constant  $M_1$  depending on  $d, \alpha, C, \lambda$  such that*

$$\int_0^t \int_{\mathbb{R}^d} p(t - s, x, z) p_{2\lambda/3}(s, z, y) |\mu|(dz) ds \leq M_1 p_{2\lambda/3}(t, x, y) N_\mu^{\alpha, \lambda/3}(t), \quad (3.3.5)$$

where  $p_{2\lambda/3}(t, x, y) := \Gamma_{2\lambda/3}(t; x - y) + \eta(t; x - y)$ .

**Proof.** By Lemma 3.1.1, we have

$$p(t, x, y) \leq C (\Gamma_\lambda(t; x - y) + \eta(t; x - y)) \quad \text{for } t \in (0, 1].$$

Thus for  $(t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$ ,

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}^d} p(t - s, x, z) p_{2\lambda/3}(s, z, y) |\mu|(dz) ds \\ &\leq C \left( \int_0^t \int_{\mathbb{R}^d} \Gamma_\lambda(t - s; x - z) \Gamma_{2\lambda/3}(s; z - y) |\mu|(dy) ds + \int_0^t \int_{\mathbb{R}^d} \eta(t - s; x - z) \eta(s; z - y) |\mu|(dz) ds \right. \\ &\quad \left. + \int_0^t \int_{\mathbb{R}^d} \Gamma_\lambda(t - s; x - z) \eta(s; z - y) |\mu|(dz) ds + \int_0^t \int_{\mathbb{R}^d} \eta(t - s; x - z) \Gamma_{2\lambda/3}(s; z - y) |\mu|(dz) ds \right) \end{aligned}$$

Applying Lemma 3.3.1, we have

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}^d} p(t-s, x, z) p_{2\lambda/3}(s, z, y) |\mu|(dz) ds \\
& \leq C \left( C_2 \Gamma_{2\lambda/3}(t, x-y) \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \Gamma_{\lambda/3}(s; x-y) |\mu|(dy) ds \right. \\
& \quad + 2C_1 \eta(t; x-y) \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \eta(s; x-y) |\mu|(dy) ds \\
& \quad + 2C_3 \Gamma_{2\lambda/3}(t; x-y) \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \eta(s; x-y) |\mu|(dy) ds \\
& \quad \left. + 2C_3 \eta(t; x-y) \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} \Gamma_{2\lambda/3}(s; x-y) |\mu|(dy) ds \right),
\end{aligned}$$

where  $C_1, C_2, C_3$  depend on  $d, \alpha, C, \lambda$ . Altogether, let  $M_1 = C(2C_1 \vee C_2 \vee 2C_3)$ , we have

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}^d} p(t-s, x, z) p_{2\lambda/3}(s, z, y) |\mu|(dz) ds \\
& \leq M_1 (\Gamma_{2\lambda/3}(t, x-y) + \eta(t; x-y)) \sup_{x \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} (\Gamma_{\lambda/3}(s; x-y) + \eta(s; x-y)) |\mu|(dy) ds \\
& = M_1 p_{2\lambda/3}(t, x, y) N_\mu^{\alpha, \lambda/3}(t).
\end{aligned}$$

□

We will use the following notations: for any  $(x, y) \in \mathbb{R}^d \times \mathbb{R}^d$ ,

$$\begin{aligned}
V_{x,y} & := \{(z, w) \in \mathbb{R}^d \times \mathbb{R}^d : |x-y| \geq 4(|y-w| \wedge |x-z|)\}; \\
U_{x,y} & := V_{x,y}^c.
\end{aligned}$$

First, similar as the discussion in [15] (see Theorem 2.7), we get the generalized integral 3P inequality for  $\eta(t; x-y)$ .

**Lemma 3.3.3.** *There exists a constant  $C_4 = C_4(\alpha, d)$  such that for any non-negative bounded function  $F(x, y)$  on  $\mathbb{R}^d \times \mathbb{R}^d$ , the followings are true for  $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ .*

(i) *If  $|x-y| \leq t^{1/2}$ , then*

$$\begin{aligned}
& \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \eta(t-s; x-z) \eta(s; w-y) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds \\
& \leq C_4 \eta(t; x-y) \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} (\eta(s; x-z) + \eta(s; w-y)) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds.
\end{aligned}$$

(ii) If  $|x - y| > t^{1/2}$ , then

$$\begin{aligned} & \int_0^t \int_{U_{x,y}} \eta(t-s; x-z) \eta(s; w-y) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds \\ & \leq C_4 \eta(t; x-y) \int_0^t \int_{U_{x,y}} (\eta(s; x-z) + \eta(s; w-y)) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds. \end{aligned}$$

(iii) If  $|x - y| > t^{1/2}$ , then

$$\int_0^t \int_{V_{x,y}} \eta(t-s; x-z) \eta(s; w-y) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds \leq C_4 \|F\|_\infty \eta(t; x-y),$$

where  $\|F\|_\infty$  denotes the  $L^\infty$ -norm of  $F$  on  $\mathbb{R}^d \times \mathbb{R}^d$ .

Now we proceed to get the generalized integral 3P inequality for  $\Gamma_c(t; x-y)$ .

**Lemma 3.3.4.** For  $0 < a < b$ , there exists a constant  $C_5 = C_5(a, b, d)$  such that for any non-negative bounded function  $F(x, y)$  on  $\mathbb{R}^d \times \mathbb{R}^d$ , the followings are true for  $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ .

(i) If  $|x - y| \leq t^{1/2}$ , then

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \Gamma_a(t-s; x-z) \Gamma_b(s; w-y) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds \\ & \leq C_5 \Gamma_b(t; x-y) \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} (\Gamma_a(s; x-z) + \Gamma_b(s; w-y)) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds. \end{aligned}$$

(ii) If  $|x - y| > t^{1/2}$ , then

$$\begin{aligned} & \int_0^t \int_{U_{x,y}} \Gamma_a(t-s; x-z) \Gamma_b(s; w-y) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds \\ & \leq C_5 \Gamma_a(t; x-y) \int_0^t \int_{U_{x,y}} (\Gamma_a(s; x-z) + \Gamma_b(s; w-y)) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds. \end{aligned}$$

(iii) If  $|x - y| > t^{1/2}$ , then

$$\int_0^t \int_{V_{x,y}} \Gamma_a(t-s; x-z) \Gamma_b(s; w-y) \frac{F(z, w)}{|z-w|^{d+\alpha}} dz dw ds \leq C_5 \|F\|_\infty \eta(t; x-y).$$

**Proof.** (i) If  $|x - y| \leq t^{1/2}$ , we have  $\Gamma_a(t - s; x - z) \leq 2^{d/2} e^b \Gamma_b(t; x - y)$  when  $s \in (0, t/2]$ ;  $\Gamma_b(s; w - y) \leq 2^{d/2} e^b \Gamma_b(t; x - y)$  when  $s \in (t/2, t)$ . Then (i) follows naturally.

(ii) If  $|x - y| > t^{1/2}$ , we let

$$U_1 := \{(z, w) \in \mathbb{R}^d \times \mathbb{R}^d : |y - w| > 4^{-1}|x - y|, |y - w| \geq |x - z|\};$$

$$U_2 := \{(z, w) \in \mathbb{R}^d \times \mathbb{R}^d : |x - z| > 4^{-1}|x - y|\}.$$

Note that  $\Gamma_b(s; w - y) \leq \gamma_1 \Gamma_b(t; x - y)$  on  $U_1$  for  $s \in (0, t)$ ;  $\Gamma_a(t - s; x - z) \leq \gamma_2 \Gamma_a(t; x - y)$  on  $U_2$  for  $s \in (0, t)$ , where  $\gamma_1 := \gamma_1(b, d)$  and  $\gamma_2 := \gamma_2(a, d)$ . Since  $U_{x,y} = U_1 \cup U_2$ , (ii) follows directly.

(iii) On  $V_{x,y}$ ,  $|z - w| \geq 2^{-1}|x - y|$ . Hence

$$\begin{aligned} & \int_0^t \int_{V_{x,y}} \Gamma_a(t - s; x - z) \Gamma_b(s; w - y) \frac{F(z, w)}{|z - w|^{d+\alpha}} dz dw ds \\ & \leq 2^{d+\alpha} |x - y|^{-(d+\alpha)} \|F\|_\infty \int_0^t \int_{V_{x,y}} \Gamma_a(t - s; x - z) \Gamma_b(s; w - y) dz dw ds \\ & \lesssim \frac{1}{t} \eta(t; x - y) \|F\|_\infty \int_0^t \left( \int_{\mathbb{R}^d} \Gamma_a(t - s; z) dz \right) \left( \int_{\mathbb{R}^d} \Gamma_b(s; w) dw \right) ds \\ & \lesssim \eta(t; x - y) \|F\|_\infty. \end{aligned}$$

This completes the proof of the lemma.  $\square$

We next establish a generalized integral 3P inequality involving both  $\Gamma_c(t; x - y)$  and  $\eta(t; x - y)$ .

**Lemma 3.3.5.** *There exists a constant  $C_6 = C_6(c, \alpha, d)$  such that for any non-negative bounded function  $F(x, y)$  on  $\mathbb{R}^d \times \mathbb{R}^d$ , the followings are true for  $(t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$ .*

(i) *If  $|x - y| \leq t^{1/2}$ , then*

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \Gamma_c(t - s; x - z) \eta(s; w - y) \frac{F(z, w)}{|z - w|^{d+\alpha}} dz dw ds \\ & \leq C_6 \left( \Gamma_c(t; x - y) \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \eta(s; w - y) \frac{F(z, w)}{|z - w|^{d+\alpha}} dz dw ds \right. \\ & \quad \left. + \eta(t; x - y) \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \Gamma_c(s; x - z) \frac{F(z, w)}{|z - w|^{d+\alpha}} dz dw ds \right). \end{aligned}$$

(ii) If  $|x - y| > t^{1/2}$ , then

$$\begin{aligned} & \int_0^t \int_{U_{x,y}} \Gamma_c(t-s; x-z) \eta(s; w-y) \frac{F(z,w)}{|z-w|^{d+\alpha}} dz dw ds \\ & \leq C_6 \left( \Gamma_c(t; x-y) \int_0^t \int_{U_{x,y}} \eta(s; w-y) \frac{F(z,w)}{|z-w|^{d+\alpha}} dz dw ds \right. \\ & \quad \left. + \eta(t; x-y) \int_0^t \int_{U_{x,y}} \Gamma_c(s; x-z) \frac{F(z,w)}{|z-w|^{d+\alpha}} dz dw ds \right). \end{aligned}$$

(iii) If  $|x - y| > t^{1/2}$ , then

$$\int_0^t \int_{V_{x,y}} \Gamma_c(t-s; x-z) \eta(s; w-y) \frac{F(z,w)}{|z-w|^{d+\alpha}} dz dw ds \leq C_6 \|F\|_\infty \eta(t; x-y).$$

**Proof.** (i) If  $|x - y| \leq t^{1/2}$ , we have  $\Gamma_c(t-s; x-z) \leq 2^{d/2} e^c \Gamma_c(t; x-y)$  when  $s \in (0, t/2]$ ;  $\eta(s; w-y) \leq 4^{d+\alpha} \eta(t; x-y)$  when  $s \in (t/2, t]$ . Thus, we have (i) hold naturally.

(ii) If  $|x - y| > t^{1/2}$ , we continue to use the decomposition  $U_{x,y} = U_1 \cup U_2$  in the proof of Lemma 3.3.4, and observe that  $\Gamma_c(t-s; x-z) \leq \gamma_3 \Gamma_c(t; x-y)$  on  $U_2$  for  $s \in (0, t)$ , where  $\gamma_3$  depends on  $d, c$ . Thus, we first have

$$\begin{aligned} & \int_0^t \int_{U_2} \Gamma_c(t-s; x-z) \eta(s; w-y) \frac{F(z,w)}{|z-w|^{d+\alpha}} dz dw ds \\ & \leq \gamma_3 \Gamma_c(t; x-y) \int_0^t \int_{U_2} \eta(s; w-y) \frac{F(z,w)}{|z-w|^{d+\alpha}} dz dw ds. \end{aligned}$$

Also, observe that  $\eta(s; w-y) \leq 4^{d+\alpha} \eta(t; x-y)$  when  $s \in (0, t)$  and  $(z, w) \in U_1$ . Thus,

$$\begin{aligned} & \int_0^t \int_{U_1} \Gamma_c(t-s; x-z) \eta(s; w-y) \frac{F(z,w)}{|z-w|^{d+\alpha}} dz dw ds \\ & \leq 4^{d+\alpha} \eta(t; x-y) \int_0^t \int_{U_1} \Gamma_c(s; x-z) \frac{F(z,w)}{|z-w|^{d+\alpha}} dz dw ds. \end{aligned}$$

Altogether, (ii) holds directly.

(iii) Note that on  $V_{x,y}$ ,  $|z - w| \geq 2^{-1}|x - y|$ . Also, there exists  $\gamma_4$  depending on  $d, c, \alpha$

such that  $\int_{\mathbb{R}^d} \Gamma_c(t; x - y) dy \leq \gamma_4$  and  $\int_{\mathbb{R}^d} \eta(t; x - y) dy \leq \gamma_4 t^{(2-\alpha)/2}$ . Thus,

$$\begin{aligned} & \int_0^t \int_{V_{x,y}} \Gamma_c(t - s; x - z) \eta(s; w - y) \frac{F(z, w)}{|z - w|^{d+\alpha}} dz dw ds \\ & \leq 2^{d+\alpha} |x - y|^{-(d+\alpha)} \|F\|_\infty \int_0^t \gamma_4^2 s^{(2-\alpha)/2} ds \\ & \leq 2^{d+\alpha} \gamma_4^2 \|F\|_\infty |x - y|^{-(d+\alpha)} t^{(4-\alpha)/2}, \end{aligned}$$

for  $t \leq 1$  and  $|x - y| > t^{1/2}$ , there exists  $C_6$  depending on  $d, \alpha, c$  such that

$$\int_0^t \int_{V_{x,y}} \Gamma_c(t - s; x - z) \eta(s; w - y) \frac{F(z, w)}{|z - w|^{d+\alpha}} dz dw ds \leq C_6 \|F\|_\infty \eta(t; x - y).$$

□

Recall that the definition of  $N_F^{\alpha, \lambda}(t)$  from (3.2.2). Note that a Hunt process  $X_t$  admits a Lévy system  $(N(x, dy), H_t)$ , where  $N(x, dy)$  is a kernel and  $H_t$  is a positive continuous additive functional of  $X_t$ ; that is, for any  $x \in \mathbb{R}^d$ , any stopping time  $T$  and any non-negative measurable function  $\varphi$  on  $[0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ , vanishing on the diagonal,

$$\mathbb{E}_x \left[ \sum_{s \leq T} \varphi(s, X_{s-}, X_s) \right] = \mathbb{E}_x \left[ \int_0^T \int_{\mathbb{R}^d} \varphi(s, X_s, y) N(X_s, dy) dH_s \right]. \quad (3.3.6)$$

Since  $X_t$  has transition density function  $p(t, x, y)$  with respect to the Lebesgue measure, it follows that the Revuz measure  $\mu_H$  of  $H$  is absolutely continuous with respect to the Lebesgue measure. So we can take  $\mu_H(dx) = dx$ , in other words, we can take  $H_t \equiv t$ . By two-sided heat kernel estimates (3.1.1) for the Hunt process  $X$  and the fact that  $N(x, dy)$  is the weak limit of  $p(t, x, y) dy / t$  as  $t \rightarrow 0$ , we have

$$H_t = t \quad \text{and} \quad N(x, dy) = \frac{c(x, y)}{|x - y|^{d+\alpha}} dy \quad (3.3.7)$$

for some measurable function  $c(x, y)$  on  $\mathbb{R}^d \times \mathbb{R}^d$  that is bounded between two positive constants.

**Theorem 3.3.6.** *Suppose  $F(x, y)$  is a measurable function so that  $F_1 = e^F - 1$  satisfies (3.1.6). There is a constant  $M_2 > 0$  so that for any  $(t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$ ,*

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} p(t-s, x, z) p_{2\lambda/3}(s, w, y) \frac{|F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw ds \\ & \leq M_2 p_{2\lambda/3}(t, x, y) \left( N_{F_1}^{\alpha, \lambda/3}(t) + \|F_1\|_\infty 1_{\{|x-y| > t^{1/2}\}} \right), \end{aligned} \quad (3.3.8)$$

In particular, on  $U_{x,y} = \{(z, w) \in \mathbb{R}^d \times \mathbb{R}^d : |x-y| \geq 4(|y-w| \wedge |x-z|)\}^c$ ,

$$\int_0^t \int_{U_{x,y}} p(t-s, x, z) p_{2\lambda/3}(s, w, y) \frac{|F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw ds \leq M_2 p_{2\lambda/3}(t, x, y) N_{F_1}^{\alpha, \lambda/3}(t). \quad (3.3.9)$$

**Proof.** By Lemma 3.1.1,

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} p(t-s, x, z) p_{2\lambda/3}(s, w, y) \frac{|F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw ds \\ & \leq C \left( \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \Gamma_\lambda(t-s; x-z) \Gamma_{2\lambda/3}(s; w-y) \frac{|F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw ds \right. \\ & \quad + \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \eta(t-s; x-z) \eta(s; w-y) \frac{|F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw ds \\ & \quad + \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \Gamma_\lambda(t-s; x-z) \eta(s; w-y) \frac{|F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw ds \\ & \quad \left. + \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \eta(t-s; x-z) \Gamma_{2\lambda/3}(s; w-y) \frac{|F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw ds \right). \end{aligned}$$

Applying (i) and (ii) in Lemma 3.3.3, 3.3.4 and 3.3.5, we first have for  $|x-y| \leq t^{1/2}$ , and for  $\{|x-y| > t^{1/2}\} \cap U_{x,y}$ ,

$$\int_0^t \int_{U_{x,y}} p(t-s, x, z) p_{2\lambda/3}(s, w, y) \frac{|F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw ds \lesssim p_{2\lambda/3}(t, x, y) N_{F_1}^{\alpha, \lambda/3}(t).$$

This establishes (3.3.9). For  $|x-y| > t^{1/2}$  and  $(z, w) \in V_{x,y}$ , we apply (iii) in Lemma 3.3.3, 3.3.4 and 3.3.5 to deduce

$$\int_0^t \int_{V_{x,y}} p(t-s, x, z) p_{2\lambda/3}(s, w, y) \frac{|F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw ds \lesssim \eta(t; x-y) \|F_1\|_\infty.$$

Hence inequality (3.3.8) holds.  $\square$

**Lemma 3.3.7.** *For every  $t \in (0, 2)$  and  $x, y \in \mathbb{R}^d$ , the following inequality holds:*

$$\int_{\mathbb{R}^d} p_{2\lambda/3}(t/2, x, z)p_{2\lambda/3}(t/2, z, y)dz \lesssim p_{2\lambda/3}(t, x, y). \quad (3.3.10)$$

**Proof.** It follows from the 3P inequality for  $\eta$  in Lemma 3.3.1, we have

$$\begin{aligned} & \int_{\mathbb{R}^d} p_{2\lambda/3}(t/2, x, z)p_{2\lambda/3}(t/2, z, y)dz \\ \leq & \int_{\mathbb{R}^d} \Gamma_{2\lambda/3}(t/2; x - z)\Gamma_{2\lambda/3}(t/2; z - y)dz + \int_{\mathbb{R}^d} \eta(t/2; x - z)\eta(t/2; z - y)dz \\ & + \int_{\mathbb{R}^d} \Gamma_{2\lambda/3}(t/2; x - z)\eta(t/2; z - y)dz + \int_{\mathbb{R}^d} \Gamma_{2\lambda/3}(t/2; z - y)\eta(t/2; x - z)dz \\ \lesssim & \Gamma_{2\lambda/3}(t; x - y) + \eta(t; x - y) + \int_{\mathbb{R}^d} \Gamma_{2\lambda/3}(t/2; x - z)\eta(t/2; z - y)dz \\ & + \int_{\mathbb{R}^d} \Gamma_{2\lambda/3}(t/2; z - y)\eta(t/2; x - z)dz, \end{aligned}$$

for the second to the last term, when  $|x - z| \geq \sqrt{2}|x - y|/2$ ,  $\Gamma_{2\lambda/3}(t/2; x - z) \leq 2^{d/2}\Gamma_{2\lambda/3}(t; x - y)$ ; when  $|x - z| < \sqrt{2}|x - y|/2$ , then  $|y - z| \geq |x - y| - |x - z| \geq \left(1 - \frac{\sqrt{2}}{2}\right)|x - y|$ , we have  $\eta(t/2; z - y) \leq (2/(2 - \sqrt{2}))^{d+\alpha} \eta(t; x - y)$ . Thus

$$\int_{\mathbb{R}^d} \Gamma_{2\lambda/3}(t/2; x - z)\eta(t/2; z - y)dz \lesssim p_{2\lambda/3}(t, x, y).$$

With similar discussion for the last term, we conclude that (3.3.10) holds.  $\square$

### 3.4 Heat kernel estimates

In the study of non-local Feynman-Kac perturbation, it is convenient to use Stieltjes exponential rather than the standard exponential. Recall that if  $K_t$  is a right continuous function with left limits on  $\mathbb{R}_+$  with  $K_0 = 1$  and  $\Delta K_t := K_t - K_{t-} > -1$  for every  $t > 0$ , and if  $K_t$  is of finite variation on each compact time interval, then the Stieltjes exponential  $\text{Exp}(K)_t$  of  $K_t$  is the unique solution  $Z_t$  of

$$Z_t = 1 + \int_{(0,t]} Z_{s-} dK_s, \quad t > 0.$$

It is known that

$$\text{Exp}(K)_t = e^{K_t^c} \prod_{0 < s \leq t} (1 + \Delta K_s),$$

where  $K_t^c$  denotes the continuous part of  $K_t$ . The above formula gives a one-to-one correspondence between Stieltjes exponential and the natural exponential. The reason of  $\text{Exp}(K)_t$  being called the *Stieltjes* exponential of  $K_t$  is that, by [25, p. 184],  $\text{Exp}(K)_t$  can be expressed as the following infinite sum of Lebesgue-Stieltjes integrals:

$$\text{Exp}(K)_t = 1 + \sum_{n=1}^{\infty} \int_{[0,t]} dK_{s_n} \int_{[0,s_n)} dK_{s_{n-1}} \cdots \int_{[0,s_2)} dK_{s_1}. \quad (3.4.1)$$

The advantage of using the Stieltjes exponential  $\text{Exp}(K)_t$  over the usual exponential  $\text{Exp}(K_t)$  is the identity (3.4.1), which allows one to apply the Markov property of  $X$ .

### 3.4.1 Upper bound estimate

Throughout this subsection,  $\mu$  satisfies (3.1.5) and  $F$  is a measurable function so that  $F_1 := e^F - 1$  satisfies (3.1.6). By Proposition 3.2.2 and Corollary 3.2.3,  $\mu \in \mathbf{K}_\alpha$ ,  $F_1 \in \mathbf{J}_\alpha$ . We will adopt the approach of [15] to construct and derive its upper bound estimate for the heat kernel of the non-local Feynman-Kac semigroup. Define

$$N_{\mu, F_1}^{\alpha, \lambda}(t) := N_\mu^{\alpha, \lambda}(t) + N_{F_1}^{\alpha, \lambda}(t)$$

and let

$$K_t := A_t^\mu + \sum_{s \leq t} F_1(X_{s-}, X_s). \quad (3.4.2)$$

Then  $\exp(A_t^\mu + \sum_{s \leq t} F(X_{s-}, X_s)) = \text{Exp}(K)_t$ . So it follows from (3.4.1) that

$$T_t^{\mu, F} f(x) = P_t f(x) + \mathbb{E}_x \left[ f(X_t) \sum_{n=1}^{\infty} \int_{[0,t]} dK_{s_n} \int_{[0,s_n)} dK_{s_{n-1}} \cdots \int_{[0,s_2)} dK_{s_1} \right]. \quad (3.4.3)$$

In view of Theorem 3.3.2 and Theorem 3.3.6, we can interchange the order of the expectation and the unfinite sum (see the proof of Theorem 3.4.3 for details). Using the Markov property

of  $X$  and setting  $h_1(s) := 1$ ,  $h_{n-1}(s) := \int_{[0,s)} dK_{s_{n-1}} \cdots \int_{[0,s_2)} dK_{s_1}$ , we have

$$\begin{aligned}
T_t^{\mu, F} f(x) &= P_t f(x) + \sum_{n=1}^{\infty} \mathbb{E}_x \left[ f(X_t) \int_{[0,t]} dK_{s_n} \int_{[0,s_n)} dK_{s_{n-1}} \cdots \int_{[0,s_2)} dK_{s_1} \right] \\
&= P_t f(x) + \sum_{n=1}^{\infty} \mathbb{E}_x \left[ \int_{[0,t]} P_{t-s_n} f(X_{s_n}) dK_{s_n} \int_{[0,s_n)} dK_{s_{n-1}} \cdots \int_{[0,s_2)} dK_{s_1} \right] \\
&= P_t f(x) + \sum_{n=1}^{\infty} \mathbb{E}_x \left[ \int_{[0,t]} \left( \int_{[0,s_n)} P_{t-s_n} f(X_{s_n}) h_{n-1}(s_{n-1}) dK_{s_{n-1}} \right) dK_{s_n} \right] \\
&= P_t f(x) + \mathbb{E}_x \left[ \int_{[0,t]} \left( \mathbb{E}_{X_{s_{n-1}}} \left[ \int_{(0,t-s_{n-1}]} P_{t-s_{n-1}-r} f(X_r) dK_r \right] \right. \right. \\
&\quad \left. \left. \times \int_{[0,s_{n-1})} dK_{s_{n-2}} \cdots \int_{[0,s_2)} dK_{s_1} \right) dK_{s_{n-1}} \right]. \tag{3.4.4}
\end{aligned}$$

For any bounded measurable  $g \geq 0$  on  $[0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ , by Lévy system of  $X$  in (3.3.6)-(3.3.7),

$$\begin{aligned}
\mathbb{E}_x \left[ \int_{(0,s]} g(s-r, X_r) dK_r \right] &= \mathbb{E}_x \left[ \int_{(0,s]} g(s-r, X_r) dA_r^\mu + \sum_{r \leq s} g(s-r, X_r) F_1(X_{r-}, X_r) \right] \\
&= \int_0^s \int_{\mathbb{R}^d} p(r, x, y) g(s-r, y) \mu(dy) dr \\
&\quad + \mathbb{E}_x \left[ \int_0^s \left( \int_{\mathbb{R}^d} g(s-r, y) F_1(X_r, y) \frac{c(X_r, y)}{|X_r - y|^{d+\alpha}} dy \right) dr \right] \\
&= \int_0^s \int_{\mathbb{R}^d} p(r, x, y) g(s-r, y) \mu(dy) dr \\
&\quad + \int_0^s \int_{\mathbb{R}^d} p(r, x, y) \left( \int_{\mathbb{R}^d} g(s-r, y) F_1(z, y) \frac{c(z, y)}{|z - y|^{d+\alpha}} dy \right) dz dr. \tag{3.4.5}
\end{aligned}$$

Define  $p^{(0)}(t, x, y) := p(t, x, y)$ , and for  $k \geq 1$ ,

$$\begin{aligned}
p^{(k)}(t, x, y) &:= \int_0^t \left( \int_{\mathbb{R}^d} p(t-s, x, z) p^{(k-1)}(s, z, y) \mu(dz) \right) ds \\
&\quad + \int_0^t \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} p(t-s, x, z) p^{(k-1)}(s, w, y) \frac{c(z, w) F_1(z, w)}{|z - w|^{d+\alpha}} dz dw \right) ds. \tag{3.4.6}
\end{aligned}$$

Let

$$q(t, x, y) := \sum_{n=0}^{\infty} p^{(k)}(t, x, y), \tag{3.4.7}$$

which will be shown in the proof of Theorem 3.4.3 to be absolutely convergent under the assumption that  $\mu$  and  $F_1$  satisfy (3.1.5), (3.1.6), respectively. Then it follows from (3.4.4) and (3.4.5) that

$$T_t^{\mu, F} f(x) = \int_{\mathbb{R}^d} q(t, x, y) f(y) dy. \quad (3.4.8)$$

So  $q(t, x, y)$  is the heat kernel for the Feynman-Kac semigroup  $\{T_t^{\mu, F}; t \geq 0\}$ . We will derive upper bound estimate of  $q(t, x, y)$  by estimating each  $p^{(k)}(t, x, y)$ .

**Lemma 3.4.1.** *There are constants  $C_0 \geq 1$  and  $M \geq 1$  such that for every  $k \geq 0$  and  $(t, x) \in (0, 1] \times \mathbb{R}^d$ ,*

$$\int_{\mathbb{R}^d} |p^{(k)}(t, x, y)| dy \leq C_0 \left( MN_{\mu, F_1}^{\alpha, \lambda}(t) \right)^k. \quad (3.4.9)$$

**Proof.** We prove this lemma by induction. When  $k = 0$ , by Lemma 3.1.1, we have the inequality hold naturally by selecting  $C_0 \geq 1$  such that

$$C_0 \geq \int_{\mathbb{R}^d} C(\Gamma_{c_4}(t; x) + \eta(t; x)) dx.$$

Suppose that (3.4.9) is true for  $k - 1$ . By (3.4.6),

$$\begin{aligned} \int_{\mathbb{R}^d} |p^{(k)}(t, x, y)| dy &\leq \int_0^t \left( \int_{\mathbb{R}^d} p(t-s, x, z) \left( \int_{\mathbb{R}^d} p^{(k-1)}(s, z, y) dy \right) |\mu|(dz) \right) ds \\ &\quad + \int_0^t \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} p(t-s, x, z) \frac{c(z, w) |F_1|(z, w)}{|z-w|^{d+\alpha}} \left( \int_{\mathbb{R}^d} p^{(k-1)}(s, w, y) dy \right) dz dw \right) ds \\ &\leq C_0 \left( MN_{\mu, F_1}^{\alpha, \lambda}(t) \right)^{k-1} \int_0^t \int_{\mathbb{R}^d} p(t-s, x, z) |\mu|(dz) ds \\ &\quad + C_0 \left( MN_{\mu, F_1}^{\alpha, \lambda}(t) \right)^{k-1} \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} p(t-s, x, z) \frac{c(z, w) |F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw ds \\ &\leq C_0 C(1 + \|c\|_\infty) M^{k-1} \left( N_{\mu, F_1}^{\alpha, \lambda}(t) \right)^k \leq C_0 \left( MN_{\mu, F_1}^{\alpha, \lambda}(t) \right)^k, \end{aligned}$$

For the last inequality, we increase the value of  $M$  if necessary so that  $M \geq C(1 + \|c\|_\infty)$ . Here  $C \geq 1$  is the constant in Lemma 3.1.1. The Lemma is proved.  $\square$

**Lemma 3.4.2.** *For any  $k \geq 0$  and  $(t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$ ,*

$$|p^{(k)}(t, x, y)| \leq Cp_{2\lambda/3}(t, x, y) \left( (MN_{\mu, F_1}^{\alpha, \lambda/3}(t))^k + k \|F_1\|_\infty M (MN_{\mu, F_1}^{\alpha, \lambda/3}(t))^{k-1} \right), \quad (3.4.10)$$

where  $C \geq 1$  and  $M \geq 1$  are the constants in Lemma 3.1.1 and Lemma 3.4.1, respectively.

**Proof.** Inequality holds trivially for  $k = 0$ . Suppose it is true for  $k - 1 \geq 0$ , then if  $|x - y| \leq t^{1/2}$ , using the induction hypothesis and applying Theorem 3.3.2 and Theorem 3.3.6,

$$\begin{aligned}
|p^{(k)}(t, x, y)| &\leq \int_0^t \left( \int_{\mathbb{R}^d} p(t-s, x, z) |p^{(k-1)}(s, z, y)| |\mu|(dz) \right) ds \\
&\quad + \int_0^t \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} p(t-s, x, z) \frac{c(z, w) |F_1|(z, w)}{|z-w|^{d+\alpha}} |p^{(k-1)}(s, w, y)| dz dw \right) ds \\
&\leq C \left( (MN_{\mu, F_1}^{\alpha, \lambda/3})^{k-1} + (k-1) \|F_1\|_\infty M (MN_{\mu, F_1}^{\alpha, \lambda/3})^{k-2} \right) \\
&\quad \times \left( \int_0^t \left( \int_{\mathbb{R}^d} p(t-s, x, z) p_{2\lambda/3}(s, z, y) |\mu|(dz) \right) ds \right. \\
&\quad \left. + \int_0^t \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} p(t-s, x, z) p_{2\lambda/3}(s, w, y) \frac{c(z, w) |F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw \right) ds \right) \\
&\leq C p_{2\lambda/3}(t, x, y) \left( (MN_{\mu, F_1}^{\alpha, \lambda/3})^{k-1} + (k-1) \|F_1\|_\infty M (MN_{\mu, F_1}^{\alpha, \lambda/3})^{k-2} \right) MN_{\mu, F_1}^{\alpha, \lambda/3}(t),
\end{aligned}$$

we increase the value of  $M$  if necessary so that  $M \geq M_1 \vee M_2$ , where  $M_1, M_2$  are constants in Theorem 3.3.2 and Theorem 3.3.6.

If  $|x - y| > t^{1/2}$ , we have

$$\begin{aligned}
|p^{(k)}(t, x, y)| &\leq \int_0^t \left( \int_{\mathbb{R}^d} p(t-s, x, z) |p^{(k-1)}(s, z, y)| |\mu|(dz) \right) ds \\
&\quad + \int_0^t \left( \int_{U_{x,y}} p(t-s, x, z) \frac{c(z, w) |F_1|(z, w)}{|z-w|^{d+\alpha}} |p^{(k-1)}(s, w, y)| dz dw \right) ds \\
&\quad + \int_0^t \left( \int_{V_{x,y}} p(t-s, x, z) \frac{c(z, w) |F_1|(z, w)}{|z-w|^{d+\alpha}} |p^{(k-1)}(s, w, y)| dz dw \right) ds \\
&= J_1 + J_2 + J_3.
\end{aligned}$$

Applying Theorem 3.3.2 to  $J_1$  and Theorem 3.3.6 to  $J_2$ ,

$$\begin{aligned}
J_1 + J_2 &\leq C \left( (MN_{\mu, F_1}^{\alpha, \lambda/3})^{k-1} + (k-1) \|F_1\|_\infty M (MN_{\mu, F_1}^{\alpha, \lambda/3})^{k-2} \right) \\
&\quad \times \left( \int_0^t \int_{\mathbb{R}^d} p(t-s, x, z) p_{2\lambda/3}(t, x, y) |\mu|(dz) ds \right. \\
&\quad \left. + \int_0^t \left( \int_{U_{x,y}} p(t-s, x, z) p_{2\lambda/3}(s, w, y) \frac{c(z, w) |F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw \right) ds \right) \\
&\leq C p_{2\lambda/3}(t, x, y) \left( (MN_{\mu, F_1}^{\alpha, \lambda/3})^{k-1} + (k-1) \|F_1\|_\infty M (MN_{\mu, F_1}^{\alpha, \lambda/3})^{k-2} \right) MN_{\mu, F_1}^{\alpha, \lambda/3}.
\end{aligned}$$

For  $J_3$ , use the fact that  $|z-w| \geq 2^{-1}|x-y|$  and Lemma 3.4.1,

$$\begin{aligned}
J_3 &\leq \frac{2^{d+\alpha} \|F_1\|_\infty}{|x-y|^{d+\alpha}} \int_0^t \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} p(t-s, x, z) |p^{(k-1)}(s, w, y)| dz dw \right) ds \\
&\leq 2^{d+\alpha} \|F_1\|_\infty \frac{t}{|x-y|^{d+\alpha}} C_0^2 (MN_{\mu, F_1}^{\alpha, \lambda})^{k-1} \\
&\leq M \|F_1\|_\infty p_{2\lambda/3}(t, x, y) (MN_{\mu, F_1}^{\alpha, \lambda/3})^{k-1}.
\end{aligned}$$

This completes the proof.  $\square$

The following result gives the existence and the desired upper bound estimates of the heat kernel for the non-local Feynman-Kac semigroup  $\{T_t^{\mu, F}; t \geq 0\}$ , as stated in Theorem 3.1.2.

**Theorem 3.4.3.** *The series  $\sum_{k=0}^\infty p^{(k)}(t, x, y)$  converges absolutely to a jointly continuous function  $q(t, x, y)$  on  $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$ . The function  $q(t, x, y)$  is the integral kernel (or, heat kernel) for the Feynman-Kac semigroup  $\{T_t^{\mu, F}; t \geq 0\}$ , and there exist constants  $\tilde{c}_3, K$  depending on  $d, \alpha, \|F_1\|_\infty, N_{\mu, F_1}^{\alpha, c_4}$  and the constants  $C$  and  $\lambda := c_4$  in Lemma 3.1.1 such that*

$$q(t, x, y) \leq \tilde{c}_3 e^{Kt} p_{2\lambda/3}(t, x, y) \quad \text{for every } (t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d. \quad (3.4.11)$$

**Proof.** Let  $\hat{p}^{(k)}(t, x, y)$  be defined as in (3.4.6) but with  $|\mu|$  and  $|F_1|$  in place of  $\mu$  and  $F_1$ ; that is,  $\hat{p}^{(0)}(t, x, y) = p(t, x, y)$ , and for  $k \geq 1$ ,

$$\begin{aligned}
\hat{p}^{(k)}(t, x, y) &:= \int_0^t \left( \int_{\mathbb{R}^d} p(t-s, x, z) \hat{p}^{(k-1)}(s, z, y) |\mu|(dz) \right) ds \\
&\quad + \int_0^t \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} p(t-s, x, z) \hat{p}^{(k-1)}(s, w, y) \frac{c(z, w) |F_1|(z, w)}{|z-w|^{d+\alpha}} dz dw \right) ds. \quad (3.4.12)
\end{aligned}$$

Clearly,  $|p^{(k)}(t, x, y)| \leq \widehat{p}^{(k)}(t, x, y)$ . Since  $F_1$  satisfies (3.1.6), by Corollary 3.2.3,  $F_1 \in \mathbf{J}_\alpha$ , there is  $t_1 \in (0, 1]$  so that  $N_{\mu, F_1}^{\alpha, \lambda/3}(t_1) \leq (2M)^{-1}$ . Now it follows from Lemma 3.4.2 that

$$\begin{aligned}
\widehat{q}(t, x, y) &:= \sum_{k=0}^{\infty} \widehat{p}^{(k)}(t, x, y) \\
&\leq Cp_{2\lambda/3}(t, x, y) + Cp_{2\lambda/3}(t, x, y) \sum_{k=1}^{\infty} \left( (MN_{\mu, F_1}^{\alpha, \lambda/3}(t))^k + k\|F_1\|_\infty M (MN_{\mu, F_1}^{\alpha, \lambda/3}(t))^{k-1} \right) \\
&\leq Cp_{2\lambda/3}(t, x, y) + Cp_{2\lambda/3}(t, x, y) (1 + 4\|F_1\|_\infty M) \\
&\leq C(2 + 4\|F_1\|_\infty M)p_{2\lambda/3}(t, x, y) =: \gamma_1 p_{2\lambda/3}(t, x, y).
\end{aligned} \tag{3.4.13}$$

This in particular implies that  $\widehat{q}(t, x, y)$  is jointly continuous on  $(0, t_1] \times \mathbb{R}^d \times \mathbb{R}^d$ . Repeating the procedure (3.4.3), (3.4.4) and (3.4.5) with  $|\mu|$ ,  $|F_1|$  in place of  $\mu$ ,  $F_1$ . and by Fubini's theorem, we have for any bounded function  $f \geq 0$  on  $\mathbb{R}^d$  and  $t \in (0, t_1]$ ,

$$T_t f(x) := \mathbb{E}_x \left[ f(X_t) \exp \left( A^{|\mu|} + \sum_{s \leq t} |F_1|(X_{s-}, X_s) \right) \right] = \int_{\mathbb{R}^d} \widehat{q}(t, x, y) f(y) dy. \tag{3.4.14}$$

Note that  $T_t \circ T_s = T_{t+s}$  for any  $t, s \geq 0$ . Extend the definition of  $\widehat{q}(t, x, y)$  to  $(0, 2t_1] \times \mathbb{R}^d \times \mathbb{R}^d$  by

$$\widehat{q}(t+s, x, y) = \int_{\mathbb{R}^d} \widehat{q}(t, x, z) \widehat{q}(s, z, y) dz$$

for  $s, t \in (0, t_1]$ . The above is well defined and, in view of (3.4.13) and Lemma 3.3.7,  $\widehat{q}(t, x, y)$  is jointly continuous on  $[0, 2t_1] \times \mathbb{R}^d \times \mathbb{R}^d$  and there is constant  $\gamma_2$  so that  $\widehat{q}(t, x, y) \leq \gamma_1^2 \gamma_2 p_{2\lambda/3}(t, x, y)$  on  $(0, 2t_1] \times \mathbb{R}^d \times \mathbb{R}^d$ . Clearly,

$$T_t f(x) = \int_{\mathbb{R}^d} \widehat{q}(t, x, y) f(y) dy,$$

for every  $f \geq 0$  on  $\mathbb{R}^d$  and  $(t, x) \in (0, 2t_1] \times \mathbb{R}^d$ . Repeat the above procedure, we can extend  $\widehat{q}(t, x, y)$  to be a jointly continuous function on  $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$  so that (3.4.14) holds for every  $f \geq 0$  on  $\mathbb{R}^d$  and  $(t, x) \in (0, \infty) \times \mathbb{R}^d$ , and that there exist constants  $\tilde{c}_3, K > 0$  depending on  $d, \alpha, C, \lambda, \|F_1\|_\infty, N_{\mu, F_1}^{\alpha, \lambda/3}$  so that for any  $t > 0$  and  $x, y \in \mathbb{R}^d$

$$\widehat{q}(t, x, y) \leq \tilde{c}_3 e^{Kt} p_{2\lambda/3}(t, x, y).$$

This proves the theorem as  $q(t, x, y) \leq \widehat{q}(t, x, y)$ .  $\square$

### 3.4.2 Lower bound estimate

In this subsection, we assume in addition  $F$  satisfies (3.1.6), then by Corollary 3.2.3,  $F \in \mathbf{J}_\alpha$ . Clearly,  $F_1 := e^F - 1 \in \mathbf{J}_\alpha$ . Due to the presence of the Gaussian component in (3.1.1), the approach in [15] of obtaining lower bound estimates for  $q(t, x, y)$  is not applicable here. We will employ a probabilistic approach from [20, 21] to get the desired lower bound estimates.

Let  $\tilde{p}^{(1)}(t, x, y)$  be defined as in (3.4.12) but with  $|F|$  in place of  $|F_1|$ . Thus by (3.4.13), there is a constant  $\gamma > 0$  so that

$$\tilde{p}^{(1)}(t, x, y) \leq \gamma p_{2\lambda/3}(t, x, y) \quad \text{for } (t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d.$$

In particular, there is a constant  $K_1 > 0$  so that  $\tilde{p}^{(1)}(t, x, y) \leq K_1 t^{-d/2}$  for  $t \in (0, 1]$  and  $|x - y| \leq \sqrt{t}$ . On the other hand, it follows from (3.1.3) that there exists a constant  $\tilde{C} \geq 1$  so that

$$\tilde{C}^{-1} t^{-d/2} \leq p(t, x, y) \leq \tilde{C} t^{-d/2} \quad \text{for every } t \in (0, 1] \text{ and } |x - y| \leq \sqrt{t}.$$

Let  $k \geq 2$  be an integer so that  $k \geq 2K_1 \tilde{C}$ . Then for every  $t \in (0, 1]$  and  $x, y \in \mathbb{R}^d$  with  $|x - y| \leq \sqrt{t}$ ,

$$p(t, x, y) - \frac{1}{k} \tilde{p}^{(1)}(t, x, y) \geq \frac{1}{2\tilde{C}} t^{-d/2} \geq \frac{1}{2\tilde{C}^2} p(t, x, y). \quad (3.4.15)$$

Note that

$$\mathbb{E}_x \left[ A_t^{|\mu|, |F|} f(X_t) \right] = \int_{\mathbb{R}^d} \tilde{p}^{(1)}(t, x, y) f(y) dy.$$

Using the elementary inequality that

$$1 - A_t^{|\mu|, |F|} / k \leq \exp(-A_t^{|\mu|, |F|} / k) \leq \exp(A_t^{\mu, F} / k),$$

we have for any ball  $B(x, r)$  centered at  $x$  with radius  $r$  and any  $(t, y) \in (0, 1] \times \mathbb{R}^d$ ,

$$\frac{1}{|B(x, r)|} \mathbb{E}_y \left[ (1 - A_t^{|\mu|, |F|} / k) 1_{B(x, r)}(X_t) \right] \leq \frac{1}{|B(x, r)|} \mathbb{E}_y \left[ \exp(A_t^{\mu, F} / k) 1_{B(x, r)}(X_t) \right].$$

Hence by (3.4.15) and Hölder's inequality, we have for  $0 < t \leq 1$  and  $|x - y| \leq \sqrt{t}$ ,

$$\begin{aligned} & \frac{1}{2\tilde{C}^2} \frac{1}{|B(x, r)|} \mathbb{E}_y [1_{B(x, r)}(X_t)] \leq \frac{1}{|B(x, r)|} \mathbb{E}_y [\exp(A_t^{\mu, F} / k) 1_{B(x, r)}(X_t)] \\ & \leq \left( \frac{1}{|B(x, r)|} \mathbb{E}_y [\exp(A_t^{\mu, F}) 1_{B(x, r)}(X_t)] \right)^{1/k} \left( \frac{1}{|B(x, r)|} \mathbb{E}_y [1_{B(x, r)}(X_t)] \right)^{1-1/k}. \end{aligned}$$

Thus

$$\frac{1}{2^k \tilde{C}^{2k}} \frac{1}{B(x, r)} \mathbb{E}_y[1_{B(x, r)}(X_t)] \leq \frac{1}{B(x, r)} \mathbb{E}_y[\exp(A_t^{\mu, F}) 1_{B(x, r)}(X_t)].$$

By taking  $r \downarrow 0$ , we conclude from above as well as Lemma 3.1.1 that

$$q(t, x, y) \geq 2^{-k} \tilde{C}^{-2k} p(t, x, y) \gtrsim t^{-d/2} \quad \text{for every } t \in (0, 1] \text{ and } |x - y| \leq \sqrt{t}. \quad (3.4.16)$$

By a standard chaining argument (see, e.g., [29]), it follows that there exist constants  $K_2, \lambda_1 > 0$  so that

$$q(t, x, y) \geq K_2 \Gamma_{\lambda_1}(t; x - y) \quad \text{for } (t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d. \quad (3.4.17)$$

To get the jumping component in the lower bound estimate for  $q(t, x, y)$ , we consider a sub-Markovian semigroup  $\{Q_t; t \geq 0\}$  defined by

$$Q_t f(x) := \mathbb{E}_x \left[ \exp \left( -A_t^{|\mu|} - \sum_{s \leq t} |F|(X_{s-}, X_s) \right) f(X_t) \right].$$

Since  $|\mu| \in \mathbf{K}_\alpha$  and  $|F| \in \mathbf{J}_\alpha$ , we know that  $\{Q_t; t \geq 0\}$  has a jointly continuous transition kernel  $\bar{p}(t, x, y)$ . Clearly,  $q(t, x, y) \geq \bar{p}(t, x, y)$  for every  $t > 0$  and  $x, y \in \mathbb{R}^d$ . Since  $\{Q_t; t \geq 0\}$  forms a Feller semigroup, there exists a Feller process  $Y = \{Y_t, \mathbb{P}_x, x \in \mathbb{R}^d, \zeta^Y\}$  such that  $Q_t f(x) = \mathbb{E}_x[f(Y_t)]$ . We will derive a lower bound estimate on  $q(t, x, y)$  through the Feller process  $Y$ .

It follows from the definition of  $\mu \in \mathbf{K}_\alpha$  and  $F \in \mathbf{J}_\alpha$  that  $\sup_{x \in \mathbb{R}^d} \mathbb{E}_x \left[ A_1^{|\mu|, |F|} \right] < \infty$ . Thus by Jensen's inequality,

$$\inf_{x \in \mathbb{R}^d} \mathbb{E}_x \left[ \exp(A^{-|\mu|, -|F|})_1 \right] \geq \exp \left( - \sup_{x \in \mathbb{R}^d} \mathbb{E}_x \left[ A_1^{|\mu|, |F|} \right] \right) =: \gamma_0 > 0. \quad (3.4.18)$$

Let  $\eta$  be the random time whose distribution is determined by  $\mathbb{P}_x(\zeta > t) = \mathbb{E}_x \left[ \exp(A^{-|\mu|, -|F|})_t \right]$ .

We can couple the processes  $X$  and  $Y$  in such a way that on  $\{\eta > t\}$ ,  $Y_s = X_s$  for every  $s \leq t$ .

We define the first hitting time and exit time of a Borel set  $D \subset \mathbb{R}^d$  by  $X_t$  and  $Y_t$  as follows;

$$\begin{aligned} \sigma_D^X &:= \inf\{s \geq 0, X_s \in D\}, & \tau_D^X &:= \inf\{s \geq 0, X_s \notin D\}; \\ \sigma_D^Y &:= \inf\{s \geq 0, Y_s \in D\}, & \tau_D^Y &:= \inf\{s \geq 0, Y_s \notin D\}. \end{aligned}$$

**Lemma 3.4.4.** *Let  $\gamma_0 \in (0, 1)$  be the constant in (3.4.18). There exists a constant  $\kappa_0 \in (0, 1)$  depending on  $d, C, \lambda, \alpha, \gamma_0$  such that for any  $0 < r \leq 1$ ,*

$$\sup_{x \in \mathbb{R}^d} \mathbb{P}_x (\tau_{B(x,r)}^X \leq \kappa_0 r^2) \leq \gamma_0/2. \quad (3.4.19)$$

Consequently, for every  $x \in \mathbb{R}^d$  and  $r \in (0, 1]$ ,

$$\mathbb{P}_x (\tau_{B(x,r)}^Y > \kappa_0 r^2) \geq \gamma_0/2. \quad (3.4.20)$$

**Proof.** First note that by (3.1.1), for every  $s \leq t \leq 1$ ,  $x \in \mathbb{R}^d$  and  $r > 0$ ,

$$\int_{B(x,r)^c} p(s, x, y) dy \leq c \left( \int_{B(0,r/2\sqrt{s})^c} e^{-c_4|z|^2} dz + sr^{-\alpha} \right) \leq c \left( e^{-c_4 r^2/8t} + tr^{-\alpha} \right).$$

Let  $\kappa_0 > 0$  be sufficiently small so that  $c(e^{-c_4/8\kappa_0} + \kappa_0) < \gamma_0/4$ . Then by taking  $t = \kappa_0 r^2$ , we have from the above that for every  $x \in \mathbb{R}^d$  and  $r \in (0, 1]$ ,

$$\sup_{s \in (0, \kappa_0 r^2]} \int_{B(x,r/2)^c} p(s, x, y) dy \leq \gamma_0/4. \quad (3.4.21)$$

For simplicity, denote  $\tau_{B(x,r)}^X$  by  $\tau$ . We have by the strong Markov property of  $X_t$  and (3.4.21) that

$$\begin{aligned} \mathbb{P}_x(\tau \leq \kappa_0 r^2) &\leq \mathbb{P}_x(\tau \leq \kappa_0 r^2; X_{\kappa_0 r^2} \in B(x, r/2)) + \mathbb{P}_x(X_{\kappa_0 r^2} \notin B(x, r/2)) \\ &\leq \mathbb{P}_x(\mathbb{P}_{X_\tau}(|X_{\kappa_0 r^2 - \tau} - X_0| \geq r/2); \tau \leq \kappa_0 r^2) + \frac{\gamma_0}{4} \\ &\leq \frac{\gamma_0}{4} + \frac{\gamma_0}{4} = \frac{\gamma_0}{2}. \end{aligned}$$

Hence

$$\begin{aligned} \mathbb{P}_x(\tau_{B(x,r)}^Y > \kappa_0 r^2) &\geq \mathbb{P}_x(\eta > \kappa_0 r^2 \text{ and } \tau_{B(x,r)}^X > \kappa_0 r^2) \\ &\geq \mathbb{P}_x(\eta > \kappa_0 r^2) - \mathbb{P}_x(\tau_{B(x,r)}^X \leq \kappa_0 r^2) \geq \gamma_0/2, \end{aligned}$$

where in the last inequality, we used (3.4.18).  $\square$

**Lemma 3.4.5.** *Let  $0 < \kappa_0 < 1$  be the constant in Lemma 3.4.4. There exists a constant  $\gamma_1 > 0$  so that for any  $r > 0$  and  $x_0, y_0 \in \mathbb{R}^d$  with  $|y_0 - x_0| \geq 3r$ ,*

$$\mathbb{P}_{x_0} \left( \sigma_{B(y_0, r)}^Y \leq \kappa_0 r^2 \right) \geq \gamma_1 \frac{r^{d+2}}{|y_0 - x_0|^{d+\alpha}}. \quad (3.4.22)$$

**Proof.** Define  $f(x, y) = 1_{B(x_0, r)}(x)1_{B(y_0, r)}(y)$ . Then

$$M_t := \sum_{s \leq t \wedge (\kappa_0 r^2)} f(X_{s-}, X_s) - \int_0^{t \wedge (\kappa_0 r^2)} f(X_s, y) N(X_s, dy) ds, \quad t \geq 0,$$

is a martingale additive functional of  $X$  that is uniformly integrable under  $\mathbb{P}_x$  for every  $x \in \mathbb{R}^d$ . Let

$$A_t := A_t^{-|\mu|, |F|} = -A_t^{|\mu|} - \sum_{s \leq t} F(X_{s-}, X_s), \quad t \geq 0,$$

which is a non-increasing additive functional of  $X$ . By stochastic integration by parts formula,

$$e^{A_t} M_t = \int_0^t e^{A_{s-}} dM_s + \int_0^t M_{s-} de^{A_s} + \sum_{s \leq t} (e^{A_s} - e^{A_{s-}})(M_s - M_{s-}).$$

For  $\tau := \tau_{B(x_0, r)}^X \wedge (\kappa_0 r^2)$ ,  $M_t = - \int_0^{t \wedge (\kappa_0 r^2)} f(X_s, y) N(X_s, dy) ds \leq 0$  for  $t \in [0, \tau]$ . It follows that

$$e^{A_\tau} M_\tau \geq \int_0^\tau e^{A_{s-}} dM_s + (e^{A_\tau} - e^{A_{\tau-}}) M_\tau.$$

Thus

$$\mathbb{E}_{x_0} [e^{A_{\tau-}} M_\tau] \geq \mathbb{E}_{x_0} \int_0^\tau e^{A_{s-}} dM_s = 0.$$

This together with (3.3.7) implies that

$$\begin{aligned} \mathbb{E}_{x_0} [e^{A_{\tau-}} 1_{B(y_0, r)}(X_\tau)] &\geq \mathbb{E}_{x_0} \left[ e^{A_{\tau-}} \int_0^{\tau \wedge (\kappa_0 r^2)} \int_{B(y_0, r)} N(X_s, dy) ds \right] \\ &\geq \frac{c \kappa_0 r^2 r^d}{|x_0 - y_0|^{d+\alpha}} \mathbb{E}_{x_0} \left[ e^{A_{\kappa_0 r^2}}; \tau_{B(x_0, r)}^X \geq \kappa_0 r^2 \right] \\ &= \frac{c \kappa_0 r^{d+2}}{|x_0 - y_0|^{d+\alpha}} \mathbb{P}_{x_0} \left( \tau_{B(x_0, r)}^Y \geq \kappa_0 r^2 \right) \\ &\geq \frac{c \gamma_0 \kappa_0 r^{d+2}}{2|x_0 - y_0|^{d+\alpha}}, \end{aligned}$$

where the last inequality is due to (3.4.20). Consequently,

$$\begin{aligned}
\mathbb{P}_{x_0}(\sigma_{B(y_0,r)}^Y \leq \kappa_0 r^2) &\geq \mathbb{P}_{x_0}\left(\tau_{B(x_0,r)}^Y \leq \kappa_0 r^2 \text{ and } Y_{\tau_{B(x_0,r)}^Y} \in B(y_0,r)\right) \\
&= \mathbb{E}_{x_0}\left[e^{A\tau} B(y_0,r)(X_\tau)\right] \\
&\geq e^{-\|F\|_\infty} \mathbb{E}_{x_0}\left[e^{A\tau-1} B(y_0,r)(X_\tau)\right] \\
&\geq \frac{c e^{-\|F\|_\infty} \gamma_0 \kappa_0 r^{d+2}}{2|x_0 - y_0|^{d+\alpha}}.
\end{aligned}$$

The lemma is proved.  $\square$

We now derive lower bound heat kernel estimate for the heat kernel  $q(t, x, y)$  of the Feynman-Kac semigroup  $\{T_t^{\mu,F}; t \geq 0\}$ .

**Theorem 3.4.6.** *Suppose  $\mu$  satisfies (3.1.5) and  $F$  is a measurable function satisfying (3.1.6). Then there exist positive constants  $\tilde{K} \geq 1$  and  $\lambda_1 > 0$  depending on  $d, \alpha, N_{\alpha,F}^{\alpha,\lambda/3}, \|F\|_\infty$  and the constants in (3.1.1) such that*

$$\tilde{K}^{-1} p_{\lambda_1}(t, x, y) \leq q(t, x, y) \leq \tilde{K} p_{2\lambda/3}(t, x, y) \quad (3.4.23)$$

for  $(t, x, y) \in (0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$ .

**Proof.** The upper bound estimates follows from Theorem 3.4.3 so it remains to establish the lower bound estimate for  $q(t, x, y)$ . If  $|x - y| \leq \sqrt{t}$ , the desired lower bound heat kernel estimate follows from (3.4.17). So it suffices to consider the case that  $|x - y| > \sqrt{t}$ . Set  $r = \sqrt{t}/3$ . It follows from Lemma 3.4.4 and Lemma 3.4.5 that

$$\begin{aligned}
\mathbb{P}_x(Y_{2\kappa_0 r^2} \in B(y, 2r)) &\geq \mathbb{P}_x\left(\sigma_{B(y,r)}^Y < \kappa_0 r^2; \sup_{s \in [\sigma, \sigma + \kappa_0 r^2]} |Y_s - Y_\sigma| < r\right) \\
&= \mathbb{E}_x\left[\mathbb{P}_{Y_\sigma}\left(\sup_{s \in [\sigma, \sigma + \kappa_0 r^2]} |Y_s - Y_\sigma| < r\right); \sigma < \kappa_0 r^2\right] \\
&\geq \mathbb{P}_x(\tau_{B(x,r)}^Y > \kappa_0 r^2) \mathbb{P}_x(\sigma_{B(y,r)}^Y < \kappa_0 r^2) \\
&\geq \frac{\gamma_0 \gamma_1}{2} \frac{r^{d+2}}{|x - y|^{d+\alpha}}.
\end{aligned}$$

Thus

$$\int_{B(y,2r)} q(2\kappa_0 r^2, x, z) dz \geq \mathbb{P}_x(Y_{2\kappa_0 r^2} \in B(y, 2r)) \geq \frac{\gamma_0 \gamma_1 r^{d+2}}{2|x - y|^{d+\alpha}}.$$

Since  $|y - z| < 2r < \sqrt{t - 2\kappa_0 r^2}$ , one has by (3.4.16) that

$$\begin{aligned}
q(t, x, y) &\geq \int_{B(y, 2r)} q(2\kappa_0 r^2, x, z) q(t - 2\kappa_0 r^2, z, y) dz \\
&\geq \inf_{z \in B(y, 2r)} q(t - 2\kappa_0 r^2, y, z) \frac{\gamma_0 \gamma_1 r^{d+2}}{2|x - y|^{d+\alpha}} \\
&\geq K_2 e^{-\lambda_1 t} t^{-d/2} \frac{\gamma_0 \gamma_1 r^{d+2}}{2|x - y|^{d+\alpha}} \\
&\geq K_2 e^{-\lambda_1 t} \frac{\gamma_0 \gamma_1}{2 \cdot 3^{d+2}} \eta(t; x - y).
\end{aligned}$$

This together with (3.4.17) establishes the lower bound estimate for  $q(t, x, y)$  in (3.4.23).  $\square$

## BIBLIOGRAPHY

- [1] R. Bass, Diffusions and Elliptic Operators, *Springer* 1998.
- [2] J. Bertoin, Regenerative embedding of Markov sets, *Probab. Theory Relat. Fields* **108** (1997), 559-571.
- [3] Ph. Blanchard and Z.-M. Ma, Semigroup of Schrödinger operators with potentials given by Radon measures, *Stochastic processes, physics and geometry (Ascona and Locarno, 1988)*, World Sci. Publ., Teaneck, NJ (1990), 160-195.
- [4] R.M. Blumenthal, R. K. Gettoor, Markov processes and potential theory, *Academic Press*, New York, 1968.
- [5] K. Bogdan and S. Sydor, On nonlocal perturbations of integral kernels. *Semigroups of operators: theory and applications, Proc. Math. Stat.*, **113**, Springer, Cham (2015), 27-42.
- [6] L. Caffarelli and L. Silvestre, An extension problem related to the fractional Laplacian, *Communications in Partial Differential Equations*, **32:8** (2007), 1245-1260.
- [7] R. Carmona, W.C. Masters, B. Simon, Relativistic Schrödinger operators: asymptotic behavior of the eigenvalues, *J. Funct. Anal.* **91** (1990), 117-142.
- [8] Z.-Q. Chen, Gaugeability and conditional gaugeability. *Trans. Amer. Math. Soc.* **354** (2002), 4639–4679.
- [9] Z.-Q. Chen, Analytic characterization of conditional gaugeability for non-local Feynman-Kac transforms. *J. Funct. Anal.* **202** (2003), 226-246.
- [10] Z.-Q. Chen,  $L^p$ -independence of spectral bounds of generalized non-local Feynman-Kac semigroups. *J. Funct. Anal.* **262** (2012), 4120-4139.
- [11] Z.-Q. Chen, Uniform integrability of exponential martingales and spectral bounds of non-local Feynman-Kac semigroups, *Stochastic Analysis and Applications to Finance, Interdiscip. Math. Sci., vol.13* (2012), 55-75.
- [12] Z.-Q. Chen, P. J. Fitzsimmons, K. Kuwae and T.-S. Zhang, Perturbation of symmetric Markov processes. *Probab. Theory Relat. Fields* **140** (2008), 239-275.

- [13] Z.-Q. Chen, E. Hu, L. Xie and X. Zhang, Heat kernels for non-symmetric diffusion operators with jumps, Preprint, arXiv:1611.05762.
- [14] Z.-Q. Chen and P. Kim, Stability of Martin boundary under non-local Feynman-Kac perturbations. *Probab. Theory Relat. Fields*, **128** (2004), 525-564.
- [15] Z.-Q. Chen, P. Kim and R. Song, Stability of Dirichlet heat kernel estimates for non-local operators under Feynman-Kac perturbation, *Trans. Amer. Math. Soc.* **367** (2015), 5237-5270.
- [16] Z.-Q. Chen and T. Kumagai, Heat kernel estimates for stable-like processes on  $d$ -sets. *Stochastic Process Appl.* **108** (2003), 27-62.
- [17] Z.-Q. Chen and T. Kumagai, A priori Hölder estimate, parabolic Harnack principle and heat kernel estimates for diffusions with jumps, *Rev. Mat. Iberoamericana*, **26** (2010), 551-589.
- [18] Z.-Q. Chen and R. Song, Conditional gauge theorem for non-local Feynman-Kac transforms, *Probab. Theory Related Fields* **125** (2003), no. 1, 45-72.
- [19] Z.-Q. Chen and R. Song, Drift transforms and Green function estimates for discontinuous processes, *J. Funct. Anal.* **201** (2003), 262-281.
- [20] Z.-Q. Chen and J.-M. Wang, Perturbation by non-local operators, to appear in *Ann. Inst. Henri Poincaré Probab. Statist.*
- [21] Z.-Q. Chen and X. Zhang, Heat kernels and analyticity of non-symmetric jump diffusion semigroups. *Probab. Theory Relat. Fields* **165** (2016), 267-312.
- [22] K. L. Chung and Z. Zhao, *From Brownian Motion to Schrödinger's Equation*, Springer-Verlag, Berlin, 1995.
- [23] Ph. Courrège, Sur la forme intégrô-différentielle des opérateurs de  $C_k^\infty$  dans  $C$  satisfaisant au principe du maximum. *Sém. Théorie du Potentiel* (1965/66) Exposé 2.
- [24] C. Dellacherie, P. A. Meyer, Probabilités et Potentiel, **vol. IV** *Théorie du potentiel, processus de Markov*. Hermann, Paris. 1987
- [25] C. Doléans-Dade. Quelques applications de la formule de changement de variables pour les semimartingales. *Z. Wahrsch.* **16** (1970), 181-194.

- [26] C. Donati-Martin, M. Yor, Some explicit Krein representations of certain subordinators, including the Gamma process, *Publ. Res. Ins. Math. Sci.* **42** (2006), 879-895.
- [27] C. Donati-Martin, M. Yor, Further examples of explicit Krein representations of certain subordinators, *Publ. Res. Ins. Math. Sci.* **43** (2007), 315-328.
- [28] F. Esscher, On the probability function in the collective theory of risk, *Scandinavisk Aktuarietidskrift* **15** (1932).
- [29] E. B. Fabes and D. W. Stroock, A new proof of Moser's parabolic Harnack inequality using the old ideas of Nash. *Arch. Rational Mech. Anal.* **96**(4) (1986), 327-338.
- [30] B. E. Fristedt and W. E. Pruitt, Lower functions for increasing random walks and subordinators, *Z. Wahrscheinlichkeitstheorie verw. Geb.* **18** (1971), 167-182.
- [31] T. Grzywny and M. Ryznar, Estimates of Green functions for some perturbations of fractional Laplacian, *Illinois Journal of Mathematics.* **51** (2007), 1409-1438.
- [32] K. Itô and H. P. McKean, *Diffusion Processes and Their Sample Paths*. Springer, Grundlehren Math. Wiss. **125**, 1974.
- [33] F. B. Knight, Characterization of the Lévy measures of inverse local times of gap diffusion, *Seminar on Stochastic processes, 1981 Prog. Prob. Statis.* **1** (1981), 53-78.
- [34] S. Kotani and S. Watanabe, Krein's spectral theory of strings and generalized diffusion processes, *Functional Analysis in Markov processes (Katata/Kyoto 1981)* (1982), 235-259.
- [35] B. Maisonneuve, Systèmes Régénératifs, *Société Mathématique de France, Astérisque* **15** (1974).
- [36] M. A. Molchanov and E. Ostrovskii, Symmetric stable processes as traces of degenerate diffusion processes, *Theor. Prob. App.* **14** (1969), 128-131.
- [37] L. Riahi, Estimates of Green functions and their applications for parabolic operators with singular potentials, *Colloq. Math.* **95** (2003), 267-283.
- [38] K. Sato, Lévy Processes and Infinitely Divisible Distributions, *Cambridge University Press* (1999).
- [39] R. L. Schilling, R. Song and Z. Vondraček, Bernstein Functions, *de Gruyter Studies in Mathematics* **37** (2010).

- [40] B. Simon, Schrödinger semigroups, *Bull. Amer. Math. Sci. (N.S.)* **7** (1982), 447-526.
- [41] R. Song, Probabilistic approach to the Dirichlet problem of perturbed stable processes, *Probab. Theory Related Fields* **95** (1993), 371-389.
- [42] R. Song, Feynman-Kac semigroup with discontinuous additive functionals, *J. Theoret. Probab.* **8** (1995), 727-762.
- [43] S. Watanabe, On time inversion of one-dimensional diffusion processes, *Z. Wahrsch. verw. Geb.* **31** (1975), 115-124.
- [44] C. Wang, On estimates of the density of Feynman-Kac semigroups of  $\alpha$ -stable-like processes, *J. Math. Anal. Appl.* **348** (2008), 938-970.
- [45] Q. S. Zhang, Gaussian bounds for the fundamental solutions of  $\nabla(A\nabla u) + B\nabla u - u_t = 0$ , *Manuscripta Math.* **93** (1997), 381-390.
- [46] Z. Zhao, A probabilistic principle and generalized Schrödinger perturbation. *J. Funct. Anal.* **101** (1991), 162-176.