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Elliptic Inverse Problems

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A dissertation
submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy

University of Washington

2014

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Abstract

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Inverse problems arise in various areas of science and engineering including medical imaging, computer vision, geophysics, solid mechanics, astronomy, and so forth. A wide range of these problems involve elliptic operators. We call them elliptic inverse problems. In this thesis we discuss three elliptic inverse problems.

The first one is photo-acoustic tomography problem. Photo-acoustic tomography is a hybrid medical imaging modality, it combines a high-resolution modality and a high-contrast modality. The underlying mathematical problem is a typical coupled physics inverse problem which involves two types of waves. We introduce the physical mechanism of photo-acoustic tomography, and focus on the recovery of diffusion coefficient and absorption coefficient from internal measurements in the partial data case.

The second one is electro-seismic conversion problem. Electro-seismic conversion is a phenomenon where electromagnetic waves and seismic waves are coupled. It has been successfully applied in the modern oil prospection. The mathematical model is another coupled physics inverse problem. We concentrate on the reconstruction of the coupling coefficient and the conductivity in the presence of internal measurements.

The third one contains a few identification results for the first order perturbation of the bi-harmonic operator in an infinite slab. We demonstrate the unique determination of a magnetic potential and an electrical potential from knowing the partial Cauchy data set.

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GLOSSARY

Ω : : a bounded open subset of \mathbb{R}^n , $n \geq 3$;

$\partial\Omega$: : the boundary of Ω ;

Σ : : an infinite slab in \mathbb{R}^n , $n \geq 3$;

$\bar{\Omega}$: : the closure of Ω ;

CGO: : complex geometric optics;

PAT: : photo-acoustic tomography;

QPAT: : quantitative photo-acoustic tomography;

TAT: : thermo-acoustic tomography;

QTAT: : quantitative thermo-acoustic tomography;

$H^s(\Omega)$: : Sobolev space of order s over Ω ;

$C^d(\bar{\Omega})$: : the space of functions with continuous derivatives up to order d over $\bar{\Omega}$.

ACKNOWLEDGMENTS

First and foremost, I would like to express my deepest gratitude to my advisor Professor Gunther Uhlmann for his outstanding mentorship throughout my PhD career. I have been amazingly fortunate to be able to work with such an intelligent, supportive and considerate gentleman. Our relationship is always a tremendous benefit both professionally and personally, and I cannot think of a better way to have spent these five years.

I am grateful to Profs Kenneth Bube, Hart Smith and Ulrich Hetmaniuk for serving as my defense committee members.

My special thanks to the professors who have provided professional guide at different stages: Gunther Uhlmann, Thomas Duchamp, John M. Lee, Yu Yuan, James Zhang, Matti Lassas, Mikko Salo, Gabriel Paternain, Katya Krupchyk, Guillaume Bal, Plamen Stefanov, Yaroslav Kurylev. In particular I want to thank Prof. Thomas Duchamp for giving me the chance to study in Seattle, and Prof. Matti Lassas for arranging my visits to Finland and many helpful discussions.

I would like to personally thank my colleagues and friends Jie Chen, Yernat Assylbekov, Lauri Oksanen, Ting Zhou, Francois Monard, Shitao Liu, Lingyun Qiu, Kui Ren, Hanming Zhou, Ru-Yu Lai, Ilker Kocyigit, Kaloyan Marinov, and Eemeli Blåsten. A large fraction of my mathematical work has benefited from the fruitful discussions with them. I would also like to thank my friends Alan Bartlett, Chris Negron Jiashan Wang, ... for making my life colorful and wonderful.

I am grateful to the administrative staff at the Department of Mathematics at University of Washington for their professionalism and friendliness: Brooke Miller, Michael Munz, Mary Sheetz, Chris Bonneu.

Finally, I would like to thank my families for being an endless source of love, inspiration and support.

DEDICATION

to my parents

Jimin Yang and Naining Wang

Chapter 1

INTRODUCTION

1.1 Inverse Problems: An Overview

Inverse problems is an interdisciplinary research area where observed measurements are converted to information about physical parameters of an object. This information is often helpful in revealing the interior structure of the object which is otherwise inaccessible. Some typical fields of applications are:

Medicine: in medical imaging destructive techniques cannot be applied to a patient's body to obtain information inside. Instead in modern imaging methods one attempts to reveal the internal physical properties (e.g. tissue's optical properties, conductive properties, etc.) from boundary probe (e.g. waves, electric currents, etc.). Based on these physical properties, the visual image of tissues and organs is reconstructed.

Geophysics: in geophysics one makes measurements of seismic waves at the Earth's surface and reconstruct the underground density. This technique has been successfully used in geophysics and up to now most knowledge about the inner structure of the Earth is obtained in this manner.

Solid Mechanics: in solid mechanics, inverse problems arise naturally in many areas such as the reconstruction of buried objects of a geometrical nature, the non-destructive testing of solids or structures using mechanical waves, the identification of distributed parameters (e.g. elastic moduli, mass density, wave velocity), and so forth.

There are many other fields where inverse problems have found their applications including, but not limited to, computer vision, remote sensing, astronomy, etc.

1.2 *Elliptic Inverse Problems*

Elliptic operators arise in many mathematical models of inverse problems. In this thesis we mainly focus on such elliptic inverse problems.

Among all the elliptic inverse problems, the one that is most well-known is the so-called Calderón's problem. In 1980 Calderón published the paper [9], where he considered the problem of determining the electrical conductivity of a medium by making voltage and current measurements on the boundary. More precisely, suppose we are given a medium whose conductivity varies from point to point, and due to some reasons the interior is not accessible so that all the operations we can apply are restricted on the boundary; when we impose an electrical potential on the boundary, electrical current occurs inside the medium, as a result one can measure the intensity of the current flowing out of the boundary; Calderón's problem, also known as the inverse conductivity problem, asks whether this data is sufficient to determine the electrical conductivity at each interior point of the medium. Note that throughout this process all we know is how the current flowing out of the boundary is related to the potential imposed on the boundary.

This measuring method is also known as electrical impedance tomography (EIT), which arises in medical imaging. In EIT, a potential is imposed on the skin of a human body and one seeks to determine the electrical conductivity of the interior tissues by measuring the current flowing out of the body. EIT has found its application in the early diagnosis of breast cancer as the conductivity of a malignant tumor is noticeably higher than that of a normal one [30]. EIT has also been utilized to prospect oil and monitor pulmonary functions.

We now formulate this problem mathematically. Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset with smooth boundary, which in EIT represents the object being imaged. Let $\gamma(x)$ be the electrical conductivity at $x \in \Omega$. From now on we assume $\gamma(x)$ is a positive function bounded away from 0 and ∞ . In the absence of source and sink, an imposed voltage potential f on the boundary induces a voltage potential u in Ω . By Ohm's law, u solves the Dirichlet

problem for the conductivity equation

$$\begin{cases} \nabla \cdot \gamma \nabla u = 0 & \text{in } \Omega \\ u = f & \text{on } \partial\Omega. \end{cases} \quad (1.2.1) \quad \boxed{\text{eqn:Dirichle}}$$

With certain regularity condition on γ , for each boundary value $f \in H^{\frac{1}{2}}(\partial\Omega)$, the Dirichlet problem for this second-order elliptic partial differential equation admits a unique weak solution $u \in H^1(\Omega)$. This allows us to define the Dirichlet-to-Neumann map (DN map) by

$$\Lambda_\gamma f = \gamma \frac{\partial u}{\partial \nu} \Big|_{\partial\Omega}.$$

This map turns out to be a bounded linear operator from $H^{\frac{1}{2}}(\partial\Omega)$ to $H^{-\frac{1}{2}}(\partial\Omega)$, and the physical interpretation of this map is as follows: when the voltage potential f is applied on the boundary, the resulting out-flowing current is $\gamma \frac{\partial u}{\partial \nu} \Big|_{\partial\Omega}$; that is, this map encodes how the resulting current depends on the applied voltage. The inverse problem here is to determine γ from the knowledge of the DN map Λ_γ .

Calderón considered the case $\gamma = \text{constant}$, then the conductivity equation becomes the harmonic equation. On the other hand, using integration by parts we obtain

$$(\Lambda_\gamma f, g)_{\partial\Omega} = \int_{\Omega} \gamma \nabla u \cdot \nabla v \, dx$$

where $(\cdot, \cdot)_{\partial\Omega}$ denotes the inner product in $L^2(\partial\Omega)$, u and v solves the conductivity equation (in fact the harmonic equation in this case) in Ω with boundary value f and g respectively. Based on this equality, Calderon's idea is to show that the inner product of gradients of harmonic functions is dense in some appropriate function spaces. To this end he considered complex exponential harmonic functions of the form $u(x) = e^{\rho \cdot x}$ with $\rho \in \mathbb{C}^n$ and $\rho \cdot \rho = 0$, here ρ is a parameter. With these harmonic functions he was able to show the uniqueness of the linearized problem at $\gamma = \text{constant}$.

After Calderón's work, the inverse conductivity problem has been intensively studied and many partial results were obtained. It was not until 1987 that Sylvester and Uhlmann gave the first confirmative answer for C^2 conductivities in dimension $n \geq 3$ in [50].

Theorem 1.2.1 ([50]). *Let $\gamma_i \in C^2(\overline{\Omega})$ be strictly positive, $i = 1, 2$. For $n \geq 3$, $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ implies $\gamma_1 = \gamma_2$ in $\overline{\Omega}$.*

In their proof, Sylvester and Uhlmann reduced the problem to a uniqueness problem of the Shrödinger equation and proved a more general identifiability result. For this purpose they constructed a class of solutions named *complex geometric optics* (CGO) solutions which have the form $u(x) = e^{i\rho \cdot x}(1 + r(x))$, here $i = \sqrt{-1}$, $\rho \in \mathbb{C}^n$ is a parameter with $\rho \cdot \rho = 0$ and $|\rho|$ sufficiently large, $r \in H^1(\Omega)$ satisfying $\|r\|_{L^2(\Omega)} = O(\frac{1}{|\rho|})$ as $|\rho| \rightarrow \infty$. These special solutions of the Shrödinger equation behave asymptotically like Calderón's complex exponential harmonic functions, so they also possess the density property in some sense. From then on, the construction of various CGO solutions has been the key ingredient in solving elliptic inverse problems. In the three topics we are going to discuss, each of them relies heavily on the property of appropriate CGO solutions. We will go through this construction process in detail in the later chapters.

This thesis is organized as follows. In Chapter 2 and Chapter 3 we consider two coupled physics inverse problems, one is the photo-acoustic tomography and the other the electro-seismic conversion. The former is a hybrid medical imaging modality while the latter occurs in modern oil prospection. Our new results are mainly contained in Section 2.3 and Section 3.3. Chapter 4 is devoted to some inverse boundary value problems related to the bi-harmonic operator in an infinite slab. The new results are stated in Section 4.1 and proved in the rest part of the chapter.

Chapter 2

PHOTO-ACOUSTIC TOMOGRAPHY AND THERMO-ACOUSTIC TOMOGRAPHY

2.1 Introduction

In this chapter we discuss some results on photo-acoustic tomography (PAT) and thermo-acoustic tomography (TAT). PAT and TAT are two of the coupled physics inverse problems. Coupled physics inverse problems, also known as hybrid inverse problems or multi-wave inverse problems, arise in various hybrid imaging modalities where one combines two or more types of waves in order to obtain better imaging effects. Before the invention of hybrid imaging modalities, each imaging modality normally involves only one type of wave, for instance optical waves in Optical Tomography (OT), ultrasound waves in Ultrasound Tomography (UT), X-rays in Computerized X-ray Tomography (CT), etc. However, restricted by the physical property of the wave in use, each stand-alone imaging modality has its drawback. For the listed examples above, OT exhibits high contrast but low resolution, while UT and CT have low contrast but high resolution. This phenomenon motivates scientists to develop physical mechanism that couples two modalities with complementary properties. These coupled modalities are known as the hybrid imaging modalities, and the mathematical model yields coupled physics inverse problems. Typical hybrid modalities as well as the involved waves are listed in the table below.

High Contrast	High Resolution	Hybrid Inverse Problems
Optical - OT (non-linear)	Ultrasound (non-linear)	Photo-acoustic tomography - PAT
		Thermo-acoustic tomography - TAT
	Ultrasound modulates optical tomography - UMOT or Acoustic optic tomography - AOT	
	MRI (linear)	
Electrical - EIT (non-linear)	Ultrasound (non-linear)	Ultrasound modulated electrical impedance tomography - UMEIT or Electro acoustic tomography - EAT
	MRI (linear)	Magnetic resonance electrical impedance tomography - MREIT and Current density impedance imaging - CDII
Elastic	Ultrasound (non-linear)	Transient elastography - TE
	MRI (linear)	Magnetic resonance elastography - MRE

We refer the interested readers to the books [1, 46, 52] and their references for the general information about practical and theoretical aspects of medical imaging.

As two of typical hybrid imaging modalities, PAT and TAT are based on the photoacoustic effect. The photo-acoustic effect can be described as follows. When an object (usually animal tissues) is exposed to a short pulse of radiation, a fraction of the radiation is absorbed by the medium, resulting in a thermal expansion. The expansion then emits acoustic waves, which propagate to the boundary of the domain. This physical coupling between the absorbed radiation and the emitted acoustic waves is called the photoacoustic effect. What distinguishes PAT and TAT is the frequency of the radiation: in PAT, high frequency radiation such as near-infra-red with sub-m wavelength is used; while in TAT, low frequency microwave with wavelengths comparable to 1m is used.

The reconstruction in PAT and TAT involve two steps. The first step in both PAT and TAT is the reconstruction of the absorbed radiation, or deposited energy, from time-dependent boundary measurements of acoustic signals. The deposited energy obtained

from this step then serves as the internal data for the second step. The second step of PAT and TAT is modeled differently due to the difference of the radiation used. In PAT the high frequency radiation is modeled by the diffusion equation, while in TAT the low frequency microwave is modeled by Maxwell's equations. The second step in PAT and TAT is sometimes referred to as Quantitative Photo-acoustic Tomography (QPAT) and Quantitative Thermo-acoustic Tomography (QTAT), respectively.

This chapter is structured as follows: in Section 2.2 we consider the mathematical model for the first step of PAT and TAT. We discuss the inverse problem for QPAT in Section 2.3, and the inverse problem for QTAT in Section 2.4.

2.2 The First Step

In both PAT and TAT, the first step of the recovery procedure is to reconstruct the absorbed radiation from the boundary measurements of acoustic waves. To describe the mathematical model for this step, let $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) be a bounded open subset representing the tissue, $u(t, x)$ the pressure at time coordinate t and spatial coordinate x , $c(x)$ the wave speed at x , $f(x)$ the absorbed radiation energy (this is renamed as $d(x)$ in next section when we discuss the second step), Δ the Euclidean Laplacian, $T > 0$ a fixed number, then the propagation of the acoustic waves is modeled by the following wave equation

$$\left\{ \begin{array}{l} \frac{1}{c^2(x)} \frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \quad \text{in } (0, T) \times \mathbb{R}^n \\ u(0, x) = f(x) \\ \frac{\partial u}{\partial t}(0, x) = 0. \end{array} \right. \quad (2.2.1) \quad \text{eqn:wave_ibv}$$

As the acoustic waves propagate in fairly homogeneous domains, we may assume the sound speed $c(x)$ is known everywhere. The measurement in this step is the boundary data

$$\Lambda f := u|_{[0, T] \times \partial \Omega},$$

and the inversion procedure aims to recover the initial condition $f(x)$ by inverting the operator Λ . This step has been studied extensively in the mathematical literature, see e.g. [2, 13, 14, 15, 18, 19, 20, 24, 25, 27, 37, 43, 47, 54, 55]. In this section we briefly review

the uniqueness and stability results in [47]. We assume the wave speed $c(x)$ is smooth and known, and consider both the full data and the partial data cases.

2.2.1 Smooth Wave Speed: Full Data

Suppose that f is supported in the closure $\bar{\Omega}$. In the case when $T = \infty$, we can invert Λ by solving the problem with Cauchy data 0 at ∞ and boundary data Λf . The zero Cauchy data is justified by the energy decay.

However, in the case when $T < \infty$, this will not work out. One method to get an approximation solution of f is the following time reversal method. Given Λf , let v_0 solve

$$\left\{ \begin{array}{l} \frac{1}{c^2} \frac{\partial^2 v_0}{\partial t^2} - \Delta v_0 = 0 \quad \text{in } (0, T) \times \mathbb{R}^n \\ v_0|_{[0, T] \times \partial\Omega} = \Lambda f \\ \partial_t v_0|_{t=T} = 0 \\ v_0|_{t=T} = 0. \end{array} \right.$$

Then we define the following operator which, roughly speaking, serves as an approximate inverse of Λ :

$$A_0(\Lambda f) := v_0(0, \cdot).$$

We would expect $A_0\Lambda f$ to be an approximate of f in a certain sense, since this is true asymptotically as $T \rightarrow \infty$. However, as the function Λf may not vanish on $\{T\} \times \partial\Omega$, the above mixed problem has boundary data with a possible jump type singularity at $\{T\} \times \partial\Omega$. This singularity will propagate back to $t = 0$ and affect v_0 . For this reason, it is necessary to introduce a smooth cut-off function near $t = T > T_0$, i.e. Λf is replaced by $\chi(t)\Lambda f$ where $\chi \in C^\infty(\mathbb{R})$ with $\chi = 0$ at $t = T$ and $\chi = 1$ in a neighborhood of $(-\infty, T_0]$. This time reversal method produces only a rough approximation of f , and thus does not perform satisfactorily in numerics.

In order to overcome this difficulty, the authors in [47] come up with a new approximation method, which we now describe in details. Let Λf be the measurement and u be the solution of (2.2.1) as above. Define the energy of u in Ω by

$$E_\Omega(t, x) := \int_\Omega (|\nabla u|^2 + c^{-2}|u_t|^2) dx.$$

Denote by $H_D(\Omega)$ the completion of $C_c^\infty(\Omega)$ under the Dirichlet norm

$$\|f\|_{H_D(\Omega)}^2 := \int_{\Omega} |\nabla u|^2 dx.$$

Assume the sound speed $c > 0$ is smooth so that $c^{-2}dx^2$ defines a Riemannian metric on Ω . As usual, the distance function $\text{dist}(x, y)$ with respect to this metric is defined as the infimum of the lengths of all the piecewise smooth curves connecting x and y . Let $T(\Omega)$ be the supremum of the lengths of all geodesics with respect to this metric. The Riemannian manifold $(\Omega, c^{-2}dx^2)$ is called non-trapping if $T(\Omega) < \infty$.

The strategy in [47] is to consider the solution v_1 of the following problem

$$\left\{ \begin{array}{l} \frac{1}{c^2} \frac{\partial^2 v_1}{\partial t^2} - \Delta v_1 = 0 \quad \text{in } (0, T) \times \mathbb{R}^n \\ v_1|_{[0, T] \times \partial\Omega} = \Lambda f \\ \partial_t v_1|_{t=T} = 0 \\ v_1|_{t=T} = \phi \end{array} \right.$$

where ϕ solves the elliptic boundary value problem

$$\Delta\phi = 0, \quad \phi|_{\partial\Omega} = \Lambda f(T, \cdot). \quad (2.2.2) \quad \boxed{\text{eqn:harmonic}}$$

Since Δ is a positive operator, 0 is not a Dirichlet eigenvalue of Δ in Ω , and therefore (2.2.2) is uniquely solvable. Notice that the initial data at $t = T$ satisfy compatibility condition of first order, i.e. there is no jump at $\{T\} \times \partial\Omega$. Then we define the following pseudo-inverse

$$A_1(\Lambda f) := v_1(0, \cdot) \quad \text{in } \bar{\Omega}.$$

The operator A_1 maps continuously the closed subspace of $H^1([0, T] \times \partial\Omega)$ consisting of functions that vanish at $t = T$ (compatibility condition) to $H^1(\Omega)$, see [38]. It also sends the range of Λ to $H_0^1(\Omega) \cong H_D(\Omega)$.

Theorem 2.2.1 ([47]). *Let $(\Omega, c^{-2}dx^2)$ be non-trapping, and let $T > T(\Omega)$. Then $A_1\Lambda = Id - K$ where K is compact on $H_D(\Omega)$ and $\|K\|_{H_D(\Omega) \rightarrow H_D(\Omega)} < 1$. In particular, $Id - K$ is invertible on $H_D(\Omega)$, and the inverse thermo-acoustic problem has an explicit solution of the form*

$$f = \sum_{m=0}^{\infty} K^m A_1(\Lambda f).$$

In addition, the proof of the theorem also provides the following estimate on the error in the construction procedure.

Corollary 2.2.2 ([47]).

$$\|f - A_1(\Lambda f)\|_{H_D(\Omega)} \leq \left(\frac{E_\Omega(u, T)}{E_\Omega(u, 0)} \right)^{\frac{1}{2}} \|f\|_{H_D(\Omega)}, \quad \forall f \in H_D(\Omega), f \neq 0.$$

where u is the solution of (2.2.1).

2.2.2 Smooth Wave Speed: Partial Data

The authors in [47] also considered uniqueness and stability of thermo-acoustic tomography when only partial boundary data is accessible. Let $\Gamma \subset \partial\Omega$ be an open subset of $\partial\Omega$. Set

$$\mathcal{G} := [0, T] \times \Gamma.$$

In the partial data setting we assume only the restriction $\Lambda f|_{\mathcal{G}}$ is known, and we would like to know when f could be uniquely and stably recovered.

Let \mathcal{K} be a compact subset of Ω . Intuitively, to be able to uniquely recover all f supported in \mathcal{K} , we hope for any $x \in \mathcal{K}$, at least one signal from x to reach \mathcal{G} , that is, we want to have a signal that reaches some $z \in \Gamma$ for $t < T$. Therefore, we should at least require that

$$\forall x \in \mathcal{K}, \exists z \in \Gamma \text{ so that } \text{dist}(x, z) < T. \quad (2.2.3) \quad \boxed{\text{cdn:uniqueness}}$$

Theorem 2.2.3 ([47]). *Let $\partial\Omega$ be strictly convex. Then under the assumption (2.2.3), if $\Lambda f = 0$ on \mathcal{G} for all $f \in H_D(\Omega)$ with $\text{supp } f \subset \mathcal{K}$, then $f = 0$.*

If we would like the recovery to be stable, we need further be able to recover all singularities of f in a stable way. By the zero initial velocity condition, each singularity (x, ξ) splits into two parts: one propagates in the direction of ξ and the other in the direction of $-\xi$. Moreover, neither one of those singularities vanishes at $t = 0$. Thus to obtain a stable recovery, we need at least one of them to reach \mathcal{G} . Define

$$\tau_{\pm}(x, \xi) := \max\{\tau \geq 0 : \gamma_{x, \xi}(\pm\tau) \in \overline{\Omega}\}.$$

The above argument motivates the next hypothesis:

$$\forall(x, \xi) \in S^*\mathcal{K}, (\tau_\sigma(x, \xi), \gamma_{x,\xi}(\tau_\sigma(x, \xi))) \in \mathcal{G} \text{ for either } \sigma = + \text{ or } \sigma = -. \quad (2.2.4) \quad \boxed{\text{cdn:stabilit}}$$

Compared to condition (2.2.3), this means that for each $x \in \mathcal{K}$ and each unit direction ξ , at least one of the signals from (x, ξ) and $(x, -\xi)$ reaches \mathcal{G} .

Theorem 2.2.4 ([47]). $A_1\chi\Lambda$ is a zero order classical Ψ DO in some neighborhood of \mathcal{K} with principle symbol

$$\frac{1}{2}(\chi(\gamma_{x,\xi}(\tau_+(x, \xi))) + \chi(\gamma_{x,\xi}(\tau_-(x, \xi)))).$$

If \mathcal{G} satisfies (2.2.4), then

1. $A_1\chi\Lambda$ is elliptic,
2. $A_1\chi\Lambda$ is a Fredholm operator on $H_D(\mathcal{K})$,
3. there exists $C > 0$ such that

$$\|f\|_{H_D(\mathcal{K})} \leq C\|\Lambda f\|_{H^1(\mathcal{G})}.$$

2.3 The Second Step of Photo-Acoustic Tomography: Partial Data

In the QPAT model, radiation propagation is modeled by the following diffusion equation

$$\begin{cases} -\nabla \cdot D(x)\nabla u + \sigma_a(x)u = 0 & \text{in } \Omega \\ u|_{\partial\Omega} = g. \end{cases} \quad (2.3.1) \quad \boxed{\text{diff_eq}}$$

Here Ω is an open and bounded subset of \mathbb{R}^n ($n \geq 3$) with C^2 boundary $\partial\Omega$, $D(x)$ is the diffusion coefficient, $\sigma_a(x)$ is the absorption coefficient, g is the prescribed illumination on $\partial\Omega$. The measurement here is the absorbed radiation $d(x)$ which we assume to be known from the first step. The relation between this absorbed radiation and the coefficients in QPAT takes the following form:

$$d(x) = G(x)\sigma_a(x)u(x),$$

where $G(x)$ is the Grüneisen coefficient and it models the strength of the photoacoustic effect. In this setting whenever a boundary illumination g is imposed, a measurement of $d(x)$ is obtained. The objective of QPAT is to reconstruct (D, σ_a, G) from knowledge of $d(x)$ obtained for a given number of illuminations g . For references on QPAT, see e.g., [4, 6, 14, 15, 44, 56].

However, it is shown in [3] that simultaneous determination of (D, σ_a, G) is impossible due to the existence of a gauge transformation. On the other hand, the authors of [3] show that if any one in (D, σ_a, G) is known, then the other two can be uniquely determined from $d(x)$. From now on we assume $G \equiv 1$ is constant, and aim to recover $D(x)$ and $\sigma_a(x)$. Without loss of generality, we may further assume these two functions are bounded from above and below by positive constants.

2.3.1 Main Results

In this section, we will consider the partial data problem, i.e., $g \in C^{k,\alpha}(\Omega)$ is supported only on a subset of the boundary. Precisely, let $x_0 \in \mathbb{R}^n \setminus \overline{\text{ch}(\Omega)}$, where $\text{ch}(\Omega)$ denotes the convex hull of Ω . We define the front and back sides of $\partial\Omega$ with respect to x_0 by

$$\partial\Omega_{\pm} = \{x \in \partial\Omega : \pm(x_0 - x) \cdot n(x) > 0\}, \quad (2.3.2) \quad \boxed{\text{frontbackbd}}$$

where $n(x)$ is the unit exterior norm at x . Let Γ be an open subset of $\partial\Omega$, such that $\partial\Omega_+ \subset \Gamma$. We assume that $\text{supp}(g) \subseteq \Gamma$.

The main purpose of this section is to prove the uniqueness and the stability of the coefficient reconstruction. We define the set of coefficients $(D(x), \sigma_a(x)) \in \mathcal{M}$ as

$$\mathcal{M} = \{(D(x), \sigma_a(x)) : (\sqrt{D}, \sigma_a) \in Y \times C^{k+1}(\bar{\Omega}), \|\sqrt{D}\|_Y + \|\sigma_a\|_{C^{k+1}(\bar{\Omega})} \leq M\},$$

Where $Y = H^{n/2+k+2+\varepsilon}(\bar{\Omega}) \subseteq C^{k+2}(\bar{\Omega})$ and $M > 0$ is fixed.

We define a subset $\tilde{\Omega} \subseteq \Omega$ to be the complement of a neighborhood of $\partial\Omega_-$ in Ω , i.e., $\tilde{\Omega} = \Omega \setminus \mathcal{N}(\partial\Omega_-)$, where $\mathcal{N}(\partial\Omega_-)$ is a neighborhood of $\partial\Omega_-$ in Ω .

The main results for the inverse diffusion problem with internal data and partial boundary data are as follows, where the measurements g and d are real-valued.

main1 **Theorem 2.3.1.** *Let Ω be an open, bounded, connected domain with C^2 boundary $\partial\Omega$. Let Γ and $\tilde{\Omega}$ be defined as above. Assume that $(D(x), \sigma_a(x))$ and $(\tilde{D}(x), \tilde{\sigma}_a(x))$ are in \mathcal{M} with $D|_{\Gamma} = \tilde{D}|_{\Gamma}$. Let $d = (d_j)$ and $\tilde{d} = (\tilde{d}_j)$, $j = 1, \dots, 4$, be the internal data for coefficients $(D(x), \sigma_a(x))$ and $(\tilde{D}(x), \tilde{\sigma}_a(x))$, respectively and with boundary conditions $g = (g_j)$, $j = 1, \dots, 4$. Then there is a set of illuminations $g \in (C^{k,\alpha}(\partial\Omega))^4$, $\text{supp}(g) \subseteq \Gamma$, for some $\alpha > 1/2$, such that if $d = \tilde{d}$, then $(D(x), \sigma_a(x)) = (\tilde{D}(x), \tilde{\sigma}_a(x))$ in $\tilde{\Omega}$.*

To consider the stability of the reconstruction, additional geometric information on $\partial\Omega$ is needed. We impose the following hypothesis.

ConvHypo **Hypothesis 2.3.2.** *Let Ω and $\partial\Omega_+$ be as above. There exists $R < \infty$ such that for each $y \in \partial\Omega_+ \subseteq \partial\Omega$, we have $\Omega \subset B_y(R)$, where $B_y(R)$ is a ball of radius R that is tangent to $\partial\Omega_+$ at y . Also, the line segment from $\forall x \in \Omega$ to $\forall y \in \partial\Omega_+$ lies in Ω .*

The stability result follows.

main2 **Theorem 2.3.3.** *Let $k \geq 3$. Let Ω satisfy Hypothesis 2.3.2 with $\partial\Omega$ of class C^{k+1} . Let Γ and $\tilde{\Omega}$ be defined as above. Assume that $(D(x), \sigma_a(x))$ and $(\tilde{D}(x), \tilde{\sigma}_a(x))$ are in \mathcal{M} with $D|_{\Gamma} = \tilde{D}|_{\Gamma}$. Let $d = (d_j)$ and $\tilde{d} = (\tilde{d}_j)$, $j = 1, \dots, 4$, be the internal data for coefficients $(D(x), \sigma_a(x))$ and $(\tilde{D}(x), \tilde{\sigma}_a(x))$, respectively and with boundary conditions $g = (g_j)$, $j = 1, \dots, 4$. Then there is a set of illuminations $g \in (C^{k,\alpha}(\partial\Omega))^4$, $\text{supp}(g) \subseteq \Gamma$, for some $\alpha > 1/2$ and a constant C such that*

$$\|D - \tilde{D}\|_{C^{k-1}(\tilde{\Omega})} + \|\sigma_a - \tilde{\sigma}_a\|_{C^{k-1}(\tilde{\Omega})} \leq C \|d - \tilde{d}\|_{(C^k(\tilde{\Omega}))^4}. \quad (2.3.3)$$

Instead of imposing geometric hypothesis, the stability of the reconstruction could also follow from $4n$ well-chosen measurements. In this case, we choose x_j , $j = 1, \dots, n$ such that, for $\forall x \in \Omega$, $\{x - x_j : j = 1, \dots, n\}$ are linearly independent. We assume the illumination sources are located at x_j , $j = 1, \dots, n$ and define $\partial\Omega_+^j$ by (2.3.2) and $\partial\Omega_+ = \cup \partial\Omega_+^j$. Furthermore, we can define Γ and $\tilde{\Omega}$ according to $\partial\Omega_+$ as above.

main3 **Theorem 2.3.4.** *Let $k \geq 2$. Let Ω be an arbitrary bounded domain with $\partial\Omega$ of class C^{k+1} . Assume that $(D(x), \sigma_a(x))$ and $(\tilde{D}(x), \tilde{\sigma}_a(x))$ are in \mathcal{M} with $D|_{\Gamma} = \tilde{D}|_{\Gamma}$. Let $d = (d_j)$ and*

$\tilde{d} = (\tilde{d}_j)$, $j = 1, \dots, 4n$, be the internal data for coefficients $(D(x), \sigma_a(x))$ and $(\tilde{D}(x), \tilde{\sigma}_a(x))$, respectively and with boundary condition $g = (g_j)$, $j = 1, \dots, 4n$. Then there is a set of illuminations $g \in (C^{k,\alpha}(\partial\Omega))^{4n}$, $\text{supp}(g) \subseteq \Gamma$, for some $\alpha > 1/2$ and a constant C such that

$$\|D - \tilde{D}\|_{C^k(\tilde{\Omega})} + \|\sigma_a - \tilde{\sigma}_a\|_{C^k(\tilde{\Omega})} \leq C\|d - \tilde{d}\|_{(C^{k+1}(\tilde{\Omega}))^{4n}}.$$

The proof of these results are mainly based on the corresponding results for the inverse Schrödinger equation. In the following section, we will focus on Schrödinger equation and prove similar results.

2.3.2 Reduction to Schrödinger equation

ISID

Let Ω be an open, bounded, connected domain in \mathbb{R}^n with smooth boundary $\partial\Omega$. Let $\partial\Omega_+$ be defined in (2.3.2) and let $\Gamma \subset \partial\Omega$ be an open subset such that $\partial\Omega_+ \subset \Gamma$. Consider the diffusion equation with unknown diffusion coefficient D and unknown attenuation coefficient σ_a :

$$-\nabla \cdot D\nabla u + \sigma_a u = 0, \quad \text{in } \Omega, \quad u = g, \quad \text{on } \partial\Omega,$$

where $\text{supp}(g) \subseteq \Gamma$. Using the standard Liouville change of variables, $v = \sqrt{D}u$ solves

$$\Delta v + qv = 0, \tag{2.3.4} \quad \text{Liou1}$$

with

$$q = -\frac{\Delta\sqrt{D}}{\sqrt{D}} - \frac{\sigma_a}{D}. \tag{2.3.5} \quad \text{Liou2}$$

The internal data in photo-acoustics is given by

$$d = \sigma_a u = \frac{\sigma_a}{\sqrt{D}}v = \mu v, \quad \mu := \frac{\sigma_a}{\sqrt{D}}, \tag{2.3.6} \quad \text{Liou3}$$

while the new boundary condition is given by $\sqrt{D}g$ on $\partial\Omega$, assuming D is known on $\partial\Omega$.

After relabeling, we have a Schrödinger equation of the form

$$\begin{aligned} \Delta u_j + q u_j &= 0 & \text{in } \Omega \\ u_j &= g_j & \text{on } \partial\Omega, \end{aligned} \tag{2.3.7} \quad \text{SchrEq}$$

where $1 \leq j \leq J$, $J \in \mathbb{N}^*$ is the number of illuminations, q is an unknown potential, and $\text{supp}(g_j) \subseteq \Gamma$. We assume that the homogeneous problem with $g_j = 0$ admits the unique

solution $u_j \equiv 0$ so that $\lambda = 0$ is not in the spectrum of $\Delta + q$. We assume that q on Ω is the restriction to Ω of a function \tilde{q} compactly supported on \mathbb{R}^n and such that $\tilde{q} \in H^{n/2+k+\varepsilon}(\mathbb{R}^n)$ with $\varepsilon > 0$ for $k \geq 1$. Moreover, we assume that the extension is chosen so that

$$\|q\|_{H^{n/2+k+\varepsilon}(\Omega)} \leq C\|\tilde{q}\|_{H^{n/2+k+\varepsilon}(\mathbb{R}^n)},$$

for some constant $C = C(\Omega, k, n)$ independent of q . That such a constant exists may be found e.g. in [48], Chapter VI, Theorem 5.

We assume that $g_j \in C^{k,\alpha}(\partial\Omega)$ with $\alpha > \frac{1}{2}$ and $\partial\Omega$ is of class C^{k+1} so that (2.3.7) admits a unique solution $u_j \in C^{k+1}(\Omega)$, see [22], Theorem 6.19. The internal data are of the form

$$d_j(x) = \mu(x)u_j(x), \quad \text{in } \Omega, 1 \leq j \leq J.$$

Here $\mu \in C^{k+1}(\bar{\Omega})$ verifies $0 < \mu_0 \leq \mu(x) \leq \mu_0^{-1}$.

The *partial data inverse Schrödinger problem with internal data* (PISID) consists of reconstructing (q, μ) in Ω from knowledge of $d = (d_j)_{1 \leq j \leq J} \in (C^{k+1}(\Omega))^J$ and illuminations $g = (g_j)_{1 \leq j \leq J} \in (C^{k,\alpha}(\partial\Omega))^J$, while $\text{supp}(g_j) \subseteq \Gamma$. We will mostly be concerned with the case $J = 4$ and $J = 4n$ with g_j and d_j real-valued measurements.

2.3.3 Complex Geometric Optics Solutions

The proof mainly depends on the *Complex Geometrical Optics*(CGO) solutions. Let $P_0 = -h^2\Delta = \sum (hD_{x_k})^2$ be a differential operator, with h a small parameter and $D_{x_k} = -i\partial_{x_k}$. Let $\varphi \in C^\infty(\Omega; \mathbb{R})$, with $\nabla\varphi \neq 0$ everywhere. Consider the conjugated operator

$$e^{\varphi/h} \circ P_0 \circ e^{-\varphi/h} = \sum_{k=1}^n (hD_{x_k} + i\partial_{x_k}\varphi)^2 = A + iB,$$

where A, B are the formally selfadjoint operators,

$$\begin{aligned} A &= (hD)^2 - (\nabla\varphi)^2, \\ B &= \nabla\varphi \cdot hD + hD \cdot \nabla\varphi, \end{aligned}$$

having the principal symbols

$$a = \xi^2 - (\nabla\varphi)^2, \quad b = 2\nabla\varphi \cdot \xi. \quad (2.3.8) \quad \boxed{\text{PricSym}}$$

We want the conjugated operator $e^{\varphi/h} \circ P_0 \circ e^{-\varphi/h}$ to be locally solvable in a semi-classical sense, which means its principal symbol satisfies Hörmander's condition

$$\{a, b\} = 0, \quad \text{when } a = b = 0, \quad (2.3.9) \quad \boxed{\text{Hörmander}}$$

where $\{a, b\} = \sum_k (\partial_{\xi_k} a \partial_{x_k} b - \partial_{x_k} a \partial_{\xi_k} b)$ is the Poisson bracket.

Definition 2.3.5. *A real smooth function φ on an open set Ω is said to be a limiting Carleman weight if it has non-vanishing gradient on Ω and if the symbols (2.3.8) satisfy the condition (2.3.9). This is equivalent to saying that*

$$\langle \varphi'' \nabla \varphi, \nabla \varphi \rangle + \langle \varphi'' \xi, \xi \rangle = 0 \quad \text{when } \xi^2 = (\nabla \varphi)^2 \text{ and } \nabla \varphi \cdot \xi = 0,$$

where φ'' is the Hessian matrix of φ .

Let $H^s(\Omega)$ be the Sobolev space. We denote by $H_{scl}^1(\Omega)$ the semi-classical Sobolev space of order 1 on Ω with associated norm

$$\|u\|_{H_{scl}^1(\Omega)}^2 = \|h \nabla u\|^2 + \|u\|^2$$

and by $H_{scl}^s(\mathbb{R}^n)$ the semi-classical Sobolev space of order s on \mathbb{R}^n with associated norm

$$\|u\|_{H_{scl}^s(\mathbb{R}^n)}^2 = \|\langle hD \rangle^s u\|_{L^2(\mathbb{R}^n)}^2 = \int (1 + h^2 \xi^2)^s |\hat{u}(\xi)|^2 d\xi.$$

Let $P = P_0 + h^2 q$. Carleman estimates give the following solvability result [16, 32]:

Hahn-Banach

Proposition 2.3.6. *Let φ be a limiting Carleman weight and $q \in L^\infty(\Omega)$. Let $0 \leq s \leq 1$. There exists $h_0 > 0$ such that for $0 < h \leq h_0$ and for every $v \in H_{scl}^{s-1}(\Omega)$, there exists $u \in H_{scl}^s(\Omega)$ such that*

$$e^{\frac{\varphi}{h}} P e^{-\frac{\varphi}{h}} u = v, \quad h \|u\|_{H_{scl}^s(\Omega)} \leq C \|v\|_{H_{scl}^{s-1}(\Omega)}.$$

Now we start to construct CGO solutions by choosing $\varphi \in C^\infty(\Omega)$ to be a limiting Carleman weight and $\psi \in C^\infty(\Omega)$ such that

$$(\nabla \psi)^2 = (\nabla \varphi)^2, \quad \nabla \psi \cdot \nabla \varphi = 0.$$

Then ψ is a local solution to the Hamilton-Jacobi problem

$$a(x, \nabla\psi) = b(x, \nabla\psi) = 0.$$

Therefore,

$$\begin{aligned} e^{-\frac{1}{h}(-\varphi+i\psi)} P_0 e^{\frac{1}{h}(-\varphi+i\psi)} a &= [((hD + \nabla\psi)^2 - \nabla\varphi^2) + i(\nabla\varphi(hD + \nabla\psi) \\ &\quad + (hD + \nabla\psi)\nabla\varphi)] a \\ &= (hL - h^2\Delta)a, \end{aligned} \tag{2.3.10}$$

where L is the transport operator given by

$$L = \nabla\psi D + D\nabla\psi + i(\nabla\varphi D + D\nabla\varphi).$$

There exists a non-vanishing smooth function $a \in C^\infty(\Omega)$, see [17, 32], such that

$$La = 0. \tag{2.3.11} \quad \boxed{\text{a:constrain}}$$

Assume that $q \in L^\infty(\Omega)$ and recall $P = h^2(-\Delta + q) = P_0 + h^2q$. Then (2.3.10) implies that

$$P e^{\frac{1}{h}(-\varphi+i\psi)} a = e^{-\varphi/h} h^2 \kappa,$$

with $\kappa = \mathcal{O}(1)$ in L^∞ and hence in L^2 . Now Proposition 2.3.6 implies there exists $r(x, h) \in H_{scl}^1(\Omega)$ such that

$$\begin{aligned} e^{\varphi/h} P e^{\frac{1}{h}(-\varphi+i\psi)} r &= -h^2 \kappa, \quad \text{and} \\ \|r\|_{H_{scl}^1(\Omega)} &= \|h\nabla r\| + \|r\| \leq Ch, \quad \text{for some } C. \end{aligned} \tag{2.3.12} \quad \boxed{\text{r:constrain}}$$

Hence,

$$P(e^{\frac{1}{h}(-\varphi+i\psi)}(a + r)) = 0,$$

i.e., we constructed a solution of the Schrödinger equation of the form

$$e^{\frac{1}{h}(-\varphi+i\psi)}(a + r). \tag{2.3.13} \quad \boxed{\text{CGO_sol}}$$

In our partial data model, we illuminate on part of the boundary, i.e., $\text{supp}(g) \subseteq \Gamma$. But the CGO solution in (2.3.13) is non-vanishing everywhere. Fortunately, using [32], we can modify the CGO solution to match our model.

Define $\Gamma_- = (\partial\Omega \setminus \Gamma)^{int}$. Then, Γ_- is a strictly open subset of $\partial\Omega_-$.

CG0prop **Proposition 2.3.7.** *We can construct a solution of*

$$Pu = 0, \quad u|_{\Gamma_-} = 0$$

of the form

$$u = e^{\frac{1}{h}(\varphi+i\psi)}(a+r) + z \tag{2.3.14} \span style="float: right; border: 1px solid black; padding: 2px;">CG0solution$$

where φ, ψ and a are chosen as above and $z = e^{i\frac{1}{h}}b(x; h)$ with b a symbol of order zero in h and

$$\text{Im } l(x) = -\varphi(x) + k(x)$$

where $k(x) \sim \text{dist}(x, \partial\Omega_-)$ in a neighborhood of $\partial\Omega_-$ and b has its support in that neighborhood. Moreover, $\|r\|_{H^0} = \mathcal{O}(h)$, $r|_{\partial\Omega_-} = 0$, $\|(\nabla\varphi \cdot \nu)^{\frac{1}{2}}r\|_{\partial\Omega_+} = \mathcal{O}(h^{\frac{1}{2}})$.

We remark that according to the proof in [32], $\text{supp}(z)$ can be arbitrarily close to $\partial\Omega_-$ in Ω .

Notice that φ is a limiting Carleman weight, so is $-\varphi$. We construct the CGO solutions of the form

$$\begin{aligned} \check{u}_1 &= e^{\frac{1}{h}(\varphi+i\psi)}(a_1+r_1) + z_1 \\ \check{u}_2 &= e^{\frac{1}{h}(-\varphi+i\psi)}(a_2+r_2) + z_2. \end{aligned} \tag{2.3.15} \span style="float: right; border: 1px solid black; padding: 2px;">CG0u12$$

In particular, we choose

$$\varphi(x) = \log|x-x_0| \quad \text{and} \quad \psi(x) = d_{S^{n-1}}\left(\frac{x-x_0}{|x-x_0|}, \omega\right), \tag{2.3.16} \span style="float: right; border: 1px solid black; padding: 2px;">phipsi_def$$

where $x_0 \in \mathbb{R}^n \setminus \overline{\text{ch}(\Omega)}$ and $\omega \in S^{n-1}$.

Note that z_j , $j = 1, 2$, are supported only in an arbitrary neighborhood of $\partial\Omega_-$. For the rest of this section, we will mainly consider the uniqueness and the stability of the solutions on the subregion defined by

$$\tilde{\Omega} = \Omega \setminus (\text{supp}(z_1) \cup \text{supp}(z_2)). \tag{2.3.17} \span style="float: right; border: 1px solid black; padding: 2px;">tildeomega$$

Remark that a neighborhood of the point $y \in \partial\Omega_+ \cap \partial\Omega_- = \{y \in \partial\Omega \mid -(y-x_0) \cdot n(y) = 0\}$ is excluded from $\tilde{\Omega}$. Thus, there exists a fixed constant $\eta > 0$, such that if $x \in \tilde{\Omega}$ and $y \in \partial\Omega_+ \cap \partial\tilde{\Omega}$, then $-(x-x_0) \cdot n(y) \geq \eta$. In the following proof, we mainly focus on $\tilde{\Omega}$, where z_1 and z_2 vanish.

2.3.4 Construct of Vector Fields and Uniqueness

We assume that we can impose the complex-valued illuminations g_j on $\partial\Omega$ with $\text{supp}(g_j) \subseteq \Gamma$ and observe the complex-valued internal data d_j , $j = 1, 2$. Note that, to make up complex-valued g_j and d_j , $j = 1, 2$, we need to make four real observations. d_j are of the form $d_j = \mu u_j$ in Ω , where u_j is the solutions of

$$\Delta u_j + q u_j = 0, \quad \text{in } \Omega \quad u_j = g_j, \quad \text{on } \partial\Omega, \quad j = 1, 2. \quad (2.3.18) \quad \boxed{\text{scheqn2}}$$

Direct calculation gives us that

$$u_1 \Delta u_2 - u_2 \Delta u_1 = 0.$$

We assume that $\mu \in C^{k+1}(\bar{\Omega})$ is bounded above and below by positive constants. By substituting $u_j = d_j/\mu$, we obtain that

$$2(d_1 \nabla d_2 - d_2 \nabla d_1) \cdot \nabla \mu - (d_1 \Delta d_2 - d_2 \Delta d_1) \mu = 0,$$

or equivalently,

$$\beta_d \cdot \nabla \mu + \gamma_d \mu = 0, \quad (2.3.19) \quad \boxed{\text{vectorode}}$$

where

$$\begin{aligned} \beta_d &:= \chi(x)(d_1 \nabla d_2 - d_2 \nabla d_1) \\ \gamma_d &:= -\frac{1}{2} \chi(x)(d_1 \Delta d_2 - d_2 \Delta d_1) = \frac{-\beta_d \cdot \nabla \mu}{\mu}. \end{aligned} \quad (2.3.20) \quad \boxed{\text{betagamma}}$$

Here, $\chi(x)$ is any smooth known complex-valued function with $|\chi(x)|$ uniformly bounded below by a positive constant on $\bar{\Omega}$. Note that by assumption on μ , we have that $\beta_d \in (C^k(\bar{\Omega}; \mathbb{C}))^n$ and $\gamma_d \in C^k(\bar{\Omega}; \mathbb{C})$.

The methodology for the reconstruction of (μ, q) will be the same as in [6]: we first reconstruct μ using the real part or the imaginary part of (2.3.19) and the boundary condition $\mu = d/g$ on $\partial\Omega_+$. Then we may recover $u_j = d_j/\mu$, $j = 1, 2$, explicitly and therefore q from the Schrödinger equation (2.3.18).

The reconstruction of μ is unique as long as the integral curves of (the real part or the imaginary part of) β_d join any interior point $x \in \tilde{\Omega}$ to a point on $\partial\Omega_+$, denoted as $x_+(x)$. This property of $\beta - d$ is not guaranteed in general, unless the boundary conditions g_j ,

$j = 1, 2$, are properly chosen. CGO solutions are then employed to construct a family of boundary conditions g_j , $j = 1, 2$, which could produce well-performed vector fields β_d .

To make a distinction from the observed data, we denote the CGO solutions by \check{u}_1 and \check{u}_2 , and denote the internal data and the vector field constructed from CGO solutions by \check{d} and $\check{\beta}_d$, respectively.

uniquemu **Proposition 2.3.8.** *Let β be the the normalized vector field, defined by*

$$\beta = \frac{h}{2} \beta_d. \quad (2.3.21) \quad \text{normbeta}$$

For any small ϵ , there is an open set of g in $C^{k,\alpha}(\partial\Omega)$, with $\text{supp}(g) \subseteq \Gamma$, such that,

$$\left\| \beta(x) - \mu^2 \varpi \frac{x_0 - x}{|x_0 - x|^2} \right\|_{C^k(\tilde{\Omega}, \mathbb{C})} \leq C h(1 + \epsilon) \quad \text{on } \tilde{\Omega}, \quad (2.3.22) \quad \text{betaapprox}$$

where $x_0 \in \mathbb{R}^n \setminus \overline{\text{ch}(\Omega)}$ and ϖ is a non-vanishing function of class $C^k(\Omega)$. Therefore, (2.3.19) admits a unique solution μ on $\tilde{\Omega}$.

Proof. Let \check{u}_1, \check{u}_2 be CGO solutions in (2.3.15). By substituting $\check{d}_j = \mu \check{u}_j$ and $\chi(x) = e^{-\frac{2i}{h}\psi}$ in (2.3.20), we find $\check{\beta}_d$, restricted to $\tilde{\Omega}$, is given by

$$\begin{aligned} \check{\beta}_d = \mu^2 \left(-\frac{2\nabla\varphi}{h} (a_1 + r_1)(a_2 + r_2) + (a_1 + r_1)(\nabla a_2 + \nabla r_2) \right. \\ \left. - (a_2 + r_2)(\nabla a_1 + \nabla r_1) \right). \end{aligned}$$

We may then define, on $\tilde{\Omega}$,

$$\check{\beta} = \frac{h}{2} \check{\beta}_d = -\mu^2 \varphi \nabla \varphi + \mu^2 \varsigma, \quad (2.3.23) \quad \text{vectorest}$$

where $\varpi = (a_1 + r_1)(a_2 + r_2)$ and $\varsigma = \frac{h}{2}((a_1 + r_1)(\nabla a_2 + \nabla r_2) - (a_2 + r_2)(\nabla a_1 + \nabla r_1))$. Notice that ϖ is non-vanishing by (2.3.11) and (2.3.12), while $\|\varsigma\| \leq C_0 h$ for some constant C_0 . Also note that, from (2.3.16), $\nabla\varphi = -\frac{x_0 - x}{|x_0 - x|^2}$.

We will next choose appropriate boundary conditions g_j , $j = 1, 2$, on $\partial\Omega$, which could lead to small $\|\beta_d - \check{\beta}_d\|$ in Ω . In particular, recall that $\check{u}_j = 0$ on $\partial\Omega_-$, for some $\epsilon > 0$, we choose $g_j \in C^{k,\alpha}(\partial\Omega)$ and $g_j = 0$ on $\partial\Omega_-$, such that,

$$\|g_j - \check{u}_j\|_{C^{k,\alpha}(\partial\Omega)} \leq \epsilon \quad \text{on } \partial\Omega, \quad j = 1, 2. \quad (2.3.24) \quad \text{gconstrain}$$

Let u_j be the solutions of (2.3.18) with boundary conditions g_j from (2.3.24). By elliptic regularity, we have that

$$\|u_j - \check{u}_j\|_{C^{k+1}(\bar{\Omega}, \mathbb{C})} \leq C\epsilon \quad \text{on } \bar{\Omega}, \quad j = 1, 2, \quad (2.3.25) \quad \boxed{\text{uconstrain}}$$

for some positive constant C . Notice that $d_j = \mu u_j$ and $\mu \in C^{k+1}(\bar{\Omega})$, we deduce that

$$\|d_j - \check{d}_j\|_{C^{k+1}(\bar{\Omega}, \mathbb{C})} \leq C_0\epsilon \quad \text{on } \bar{\Omega}, \quad j = 1, 2. \quad (2.3.26) \quad \boxed{\text{dapprox}}$$

Thus, restricting to $\tilde{\Omega}$, (2.3.23) and (2.3.26) induce (2.3.22), which indicates, when h, ϵ are sufficiently small, β is close to a non-vanishing vector $-\mu^2 \varpi \nabla \varphi = \mu^2 \varpi \frac{x_0 - x}{|x_0 - x|^2}$. Approximately, the integral curves of β are rays from any $x \in \tilde{\Omega}$ to $x_0 \in \mathbb{R}^n \setminus \overline{\text{ch}(\tilde{\Omega})}$, intersecting $\partial\Omega_+$ at a point $x_+(x)$. Therefore, with $\mu = d_1/g_1 = d_2/g_2$ known on $\partial\Omega_+$,

$$\beta \cdot \nabla \mu + \gamma \mu = 0, \quad \gamma = \frac{h}{2} \gamma_d \quad (2.3.27) \quad \boxed{\text{normvecode}}$$

provides a unique reconstruction for μ , so does (2.3.19). More precisely, consider the flow $\theta_x(t)$ associated to β , i.e., the solution to

$$\dot{\theta}_x(t) = \beta(\theta_x(t)), \quad \theta_x(0) = x \in \tilde{\Omega}.$$

By the Picard-Lindelöf theorem, the above equations admit unique solution θ_x while β is of class C^1 . Since β is non-vanishing, θ_x reaches $x_+(x) \in \partial\Omega_+$ in a finite time, denoted as $t_+(x)$, i.e.,

$$\theta_x(t_+(x)) = x_+(x).$$

Then by the method of characteristics, $\mu(x)$ is uniquely determined by

$$\mu(x) = \mu_0(x_+(x)) e^{-\int_0^{t_+(x)} \gamma(\theta_x(s)) ds}, \quad (2.3.28) \quad \boxed{\text{muchar}}$$

where $\mu_0 = d/g$ on $\partial\Omega_+$. This finishes the proof. \square

Let us define the set of parameters

$$\mathcal{P} = \left\{ (\mu, q) \in C^{k+1}(\Omega) \times H^{\frac{n}{2}+k+\epsilon}(\Omega); 0 \text{ is not an eigenvalue of } \Delta + q, \right. \\ \left. \|\mu\|_{C^{k+1}(\Omega)} + \|q\|_{H^{\frac{n}{2}+k+\epsilon}(\Omega)} \leq P < \infty \right\}.$$

The above construction of the vector field allows us to obtain the following uniqueness result.

uniquenesssthm

Theorem 2.3.9. *Let Ω be a bounded, open subset of \mathbb{R}^n with boundary of class C^{k+1} . Let (μ, q) and $(\tilde{\mu}, \tilde{q})$ be two elements in \mathcal{P} and $\tilde{\Omega}$ be defined in (2.3.17). When h and ϵ are sufficiently small, for $j = 1, 2$, let g_j be constructed according to (2.3.24) with the CGO solutions \check{u}_j . d_j and \tilde{d}_j are two sets of observations of the internal data on Ω .*

Restricting to $\tilde{\Omega}$, $d_j = \tilde{d}_j$ implies that $(\mu, q) = (\tilde{\mu}, \tilde{q})$.

Proof. We have proved that, for $j = 1, 2$, when g_j is properly chosen, μ is uniquely reconstructed on $\tilde{\Omega}$, i.e., $\mu = \tilde{\mu}$. Directly, $d_j = \tilde{d}_j$ also implies $u := u_j = \tilde{u}_j$. By unique continuation, u cannot vanish on an open set in $\tilde{\Omega}$ different from the empty set. Otherwise u vanishes everywhere and this is impossible to satisfy the boundary conditions. Therefore, the set $F = \{|u| > 0\} \cap \tilde{\Omega}$ is open and $\bar{F} = \tilde{\Omega}$ since the complement of \bar{F} has to be empty. By continuity, this shows that $q = \tilde{q}$ on $\tilde{\Omega}$. \square

Since the coefficient q in the Schrödinger equation is unknown, the CGO solutions \check{u}_j , $j = 1, 2$, cannot be explicitly determined. Therefore, although we know that g_j can be chosen from a open set close to \check{u}_j , a more explicit characterization of the open set is lacking.

Also notice that the parameters h and ϵ need to be small to make β flat enough, while cannot be too small, otherwise g_1 will be so large that the imposed illuminations become physically infeasible and g_2 will be so small that the imposed illuminations become physically undetectable.

2.3.5 Stability with 2 Complex Internal Measurements

In this section, we consider the stability of the proposed reconstruction method. When two complex-valued internal measurements are available, a star-shaped condition is assumed on the domain of interest in order to obtain the stability. When sufficiently many internal measurements are available, this condition can be removed, this will be discussed later.

With 2 complex internal measurements, the stability of the reconstruction may fail if the vector field is approximately parallel to the boundary $\partial\Omega$, i.e., $\beta(x_+(x)) \cdot n(x_+(x))$ is small for a long time. To avoid this situation, we impose the convexity condition as established in Hypothesis (2.3.2).

Recall that $\theta_x(t)$ is the flow associated to β . Assume $\theta_x(t)$ reaches the boundary at x_+ and at time t_+ , i.e., $\theta_x(t_+) = x_+ \in \partial\Omega$. Similar notations are use for $\tilde{\beta}$. We have the equality

$$\theta_x(t) - \tilde{\theta}_x(t) = \int_0^t [\beta(\theta_x(s)) - \tilde{\beta}(\tilde{\theta}_x(s))] ds.$$

Using the Lipschitz continuity of β and Gronwall's lemma, we thus deduce the existence of a constant C such that

$$|\theta_x(t) - \tilde{\theta}_x(t)| \leq Ct \|\beta - \tilde{\beta}\|_{C^0(\bar{\Omega})}$$

uniformly in t knowing that all characteristics exit $\bar{\Omega}$ in finite time and provided that $\theta_x(t)$ and $\tilde{\theta}_x(t)$ are in $\bar{\Omega}$.

Such estimates are stable with respect to perturbations in the initial conditions. Let us define $W(t) = D_x \theta_x(t)$. Then W solves the equation $\dot{W} = D_x \beta(\theta_x) W$ with $W(0) = I$ and by using Gronwall's lemma once more, we deduce that

$$|W(t) - \tilde{W}(t)| \leq Ct \|D_x \beta - D_x \tilde{\beta}\|_{C^0(\bar{\Omega})},$$

for all times provided that $\theta_x(t)$ and $\tilde{\theta}_x(t)$ are in $\bar{\Omega}$. As a consequence, since β and $\tilde{\beta}$ are in $C^k(\bar{\Omega})$, then we obtain similarly that:

$$|D_x^{k-1} \theta_x(t) - D_x^{k-1} \tilde{\theta}_x(t)| \leq Ct \|\beta - \tilde{\beta}\|_{C^{k-1}(\bar{\Omega})}, \quad (2.3.29) \quad \boxed{\text{thetastable}}$$

and this again for all times $\theta_x(t)$ and $\tilde{\theta}_x(t)$ are in $\bar{\Omega}$. To simplify the notation, we define $\delta = \|\beta - \tilde{\beta}\|_{C^{k-1}(\bar{\Omega})}$.

Recall that by the definition of $\tilde{\Omega}$ in (2.3.17). Let $x \in \tilde{\Omega}$ and $y \in \partial\Omega_+ \cap \partial\tilde{\Omega}$, then $\beta(x) \cdot n(y) > \eta$, for some constant $\eta > 0$, where β satisfies (2.3.22) and $n(y)$ is the unit outer normal.

xtlemma **Lemma 2.3.10.** *Let $k \geq 1$ and assume that β and $\tilde{\beta}$ are $C^k(\bar{\Omega})$ vector fields that are sufficiently flat, i.e., h is sufficiently small. Let us assume that $\partial\Omega_+$ is sufficiently convex so that Hypothesis 2.3.2 holds for some $R < \infty$. Then, restricting to $\tilde{\Omega}$, we have that*

$$\|x_+ - \tilde{x}_+\|_{C^{k-1}(\bar{\Omega})} + \|t_+ - \tilde{t}_+\|_{C^{k-1}(\bar{\Omega})} \leq C \|\beta - \tilde{\beta}\|_{C^{k-1}(\bar{\Omega})},$$

where C is a constant depending on h and R .

Proof. Let $x \in \tilde{\Omega}$. Without loss of generality, we assume $t_+(x) < \tilde{t}_+(x)$. Draw a circle with radius R and tangent to Ω at x_+ . We impose a coordinate system by choosing the circle center to be the origin. Equation (2.3.22) and (2.3.29) gives that

$$|\tilde{x}_+ - [\tilde{\theta}_x(t_+) + \tilde{\beta}(\tilde{\theta}_x(t_+))(\tilde{t}_+ - t_+)]| \leq C(\tilde{t}_+ - t_+)\delta.$$

Thus,

$$|\tilde{\theta}_x(t_+) + \tilde{\beta}(\tilde{\theta}_x(t_+))(\tilde{t}_+ - t_+)| \leq R + C(\tilde{t}_+ - t_+)\delta.$$

Directly, we have

$$\begin{aligned} & |x_+ + \tilde{\beta}(\tilde{\theta}_x(t_+))(\tilde{t}_+ - t_+)|^2 - R^2 \\ &= |\tilde{\theta}_x(t_+) + \tilde{\beta}(\tilde{\theta}_x(t_+))(\tilde{t}_+ - t_+) - (\tilde{\theta}_x(t_+) - x_+)|^2 - R^2 \\ &= K[(R + C(\tilde{t}_+ - t_+)\delta + |\tilde{\theta}_x(t_+) - x_+|)^2 - R^2]. \end{aligned} \quad (2.3.30) \quad \boxed{\text{lemmaeq1}}$$

where $0 < K \leq 1$ is defined by ratio and $K \in C^k$ when $|\tilde{\theta}_x(t_+) - x_+| > 0$. On the other hand,

$$\begin{aligned} & |x_+ + \tilde{\beta}(\tilde{\theta}_x(t_+))(\tilde{t}_+ - t_+)|^2 - R^2 \\ &= 2R\tilde{\beta}(\tilde{\theta}_x(t_+)) \cdot n(x_+)(\tilde{t}_+ - t_+) + |\tilde{\beta}(\tilde{\theta}_x(t_+))|^2(\tilde{t}_+ - t_+)^2. \end{aligned} \quad (2.3.31) \quad \boxed{\text{lemmaeq2}}$$

Substituting (2.3.30) into (2.3.31), we have

$$A(\tilde{t}_+ - t_+)^2 + B(\tilde{t}_+ - t_+) + C = 0,$$

where

$$\begin{aligned} A &= |\tilde{\beta}(\tilde{\theta}_x(t_+))|^2 - KC^2\delta^2, \\ B &= 2R\tilde{\beta}(\tilde{\theta}_x(t_+)) \cdot n(x_+) - 2KC\delta(R + |\tilde{\theta}_x(t_+) - x_+|), \text{ and} \\ C &= -2KR|\tilde{\theta}_x(t_+) - x_+| - K|\tilde{\theta}_x(t_+) - x_+|^2. \end{aligned}$$

By the quadratic formula,

$$\tilde{t}_+ - t_+ = \frac{-B + \sqrt{B^2 - 4AC}}{2A} = \frac{-2C}{B + \sqrt{B^2 - 4AC}}. \quad (2.3.32) \quad \boxed{\text{t_est}}$$

Notice that when $x \in \tilde{\Omega}$ and $x_+ \in \partial\Omega_+ \cap \partial\tilde{\Omega}$, for h sufficiently small, $\tilde{\beta}(\tilde{\theta}_x(t_+)) \cdot n(x_+) > \eta$ for some fixed $\eta > 0$. When δ is small, we see that $A > 0$ and $B > \eta' > 0$ for some η' . Thus, (2.3.29) and (2.3.32) implies

$$\tilde{t}_+ - t_+ \leq C' |\tilde{\theta}_x(t_+) - x_+| \leq C'' \|\beta - \tilde{\beta}\|_{C^0(\tilde{\Omega})}. \quad (2.3.33) \quad \boxed{\text{t_est2}}$$

By taking derivatives of (2.3.32),

$$\partial_x^k (\tilde{t}_+ - t_+) = \sum_{i=0}^k \alpha_i \partial_x^i |\tilde{\theta}_x(t_+) - x_+|,$$

where $|\alpha_i|$ is bounded when δ is small.

We next need to estimate $\partial_x^i |\tilde{\theta}_x(t_+) - x_+|$. When $i = 0$, it is directly from (2.3.29); When $i = 1$, we have

$$\begin{aligned} \partial_x |\tilde{\theta}_x(t_+) - x_+| &\leq |\partial_x(\tilde{\theta}_x(t_+) - x_+)| \\ &= |(\tilde{W}(t_+) - W(t_+)) + (\tilde{\beta}(\tilde{\theta}_x(t_+)) - \beta(\theta_x(t_+))) \partial_x t_+| \\ &\leq C \|\beta - \tilde{\beta}\|_{C^1(\tilde{\Omega})}, \end{aligned}$$

where the inequality follows from (2.3.29) and the continuity of β . Similar estimate works for high order derivatives. Thus,

$$\|\tilde{t}_+ - t_+\|_{C^1(\tilde{\Omega})} \leq \|\beta - \tilde{\beta}\|_{C^1(\tilde{\Omega})}. \quad (2.3.34) \quad \boxed{\text{t_est3}}$$

Furthermore, the estimate of $|x_+ - \tilde{x}_+|$ follows easily.

$$\begin{aligned} |\partial_x(x_+ - \tilde{x}_+)| &= |\partial_x(\tilde{\theta}_x(\tilde{t}_+) - \theta_x(t_+))| \\ &= |(\tilde{W}(\tilde{t}_+) - W(t_+)) + (\tilde{\beta}(\tilde{t}_+) \partial_x \tilde{t}_+ - \beta(t_+) \partial_x t_+)| \\ &\leq C \|\beta - \tilde{\beta}\|_{C^1(\tilde{\Omega})}, \end{aligned}$$

where the inequality follows from (2.3.29), (2.3.34) and the continuity of β and W . High order derivatives follow similarly. This finishes the proof. \square

$\boxed{\text{stableprop}}$

Proposition 2.3.11. *Let $k \geq 1$. Let μ and $\tilde{\mu}$ be solutions to (2.3.27) corresponding to coefficients (β, γ) and $(\tilde{\beta}, \tilde{\gamma})$, respectively, where (2.3.22) holds for both β and $\tilde{\beta}$.*

Let us define $\mu_0 = \mu|_{\partial\Omega}$ and $\tilde{\mu}_0 = \tilde{\mu}|_{\partial\Omega}$, thus $\mu_0, \tilde{\mu}_0 \in C^k(\partial\Omega)$. We also assume h is sufficiently small and Ω is sufficiently convex that Hypothesis 2.3.2 holds for some $R < \infty$. Then there is a constant C such that

$$\begin{aligned} & \|\mu - \tilde{\mu}\|_{C^{k-1}(\tilde{\Omega})} \\ & \leq C\|\mu_0\|_{C^k(\partial\Omega_+)}(\|\beta - \tilde{\beta}\|_{C^{k-1}(\tilde{\Omega})} + \|\gamma - \tilde{\gamma}\|_{C^{k-1}(\tilde{\Omega})}) + C\|\mu_0 - \tilde{\mu}_0\|_{C^k(\partial\Omega_+)}. \end{aligned} \quad (2.3.35)$$

Proof. By the method of characteristics, $\mu(x)$ is determined explicitly in (2.3.28), while $\tilde{\mu}(x)$ has a similar expression.

$$\begin{aligned} |\mu(x) - \tilde{\mu}(x)| & \leq \left| (\mu_0(x_+(x)) - \tilde{\mu}_0(\tilde{x}_+(x)))e^{-\int_0^{t_+(x)} \gamma(\theta_x(s))ds} \right| + \\ & \left| \tilde{\mu}_0(\tilde{x}_+(x)) \left(e^{-\int_0^{t_+(x)} \gamma(\theta_x(s))ds} - e^{-\int_0^{\tilde{t}_+(x)} \tilde{\gamma}(\tilde{\theta}_x(s))ds} \right) \right| \end{aligned}$$

Applying Lemma 2.3.10, we deduce that

$$|D_x^{k-1}[\mu_0(x_+(x)) - \tilde{\mu}_0(\tilde{x}_+(x))]| \leq \|\mu_0 - \tilde{\mu}_0\|_{C^{k-1}(\partial\Omega_+)} + C\|\mu_0\|_{C^{k-1}(\partial\Omega_+)}\|\beta - \tilde{\beta}\|_{C^{k-1}(\tilde{\Omega})}.$$

This proves the $\mu_0(x_+(x))$ is stable. To consider the second term, by the Leibniz rule it is sufficient to prove the stability result for $\int_0^{t_+(x)} \gamma(\theta_x(s))ds$.

Assume without loss of generality that $t_+(x) < \tilde{t}_+(x)$. Then we have, applying (2.3.29),

$$\begin{aligned} \int_0^{t_+(x)} [\gamma(\theta_x(s)) - \tilde{\gamma}(\tilde{\theta}_x(s))]ds & = \int_0^{t_+(x)} [(\gamma(\theta_x(s)) - \gamma(\tilde{\theta}_x(s))) + (\gamma - \tilde{\gamma})(\tilde{\theta}_x(s))]ds \\ & \leq C\|\gamma\|_{C^0(\tilde{\Omega})}\|\beta - \tilde{\beta}\|_{C^0(\tilde{\Omega})} + C\|\gamma - \tilde{\gamma}\|_{C^0(\tilde{\Omega})}. \end{aligned}$$

Derivatives of order $k-1$ of the above expression are uniformly bounded since $t_+(x) \in C^{k-1}(\tilde{\Omega})$, γ has C^k derivatives bounded on $\tilde{\Omega}$ and $\theta_x(t)$ is stable as in (2.3.29).

It remains to handle the term

$$v(x) = \int_{t_+(x)}^{\tilde{t}_+(x)} \tilde{\gamma}(\tilde{\theta}_x(s))ds.$$

$\tilde{\beta}$ and $\tilde{\gamma}$ are of class $C^k(\Omega)$, then so is the function $x \rightarrow \tilde{\gamma}(\tilde{\theta}(s))$. Derivatives of order $k-1$ of $v(x)$ involve terms of size $\tilde{t}_+(x) - t_+(x)$ and terms of form

$$D_x^m(\tilde{t}_+ D_x^{k-1-m} \tilde{\gamma}(\tilde{\theta}_x(\tilde{t}_+)) - t_+ D_x^{k-1-m} \tilde{\gamma}(\tilde{\theta}_x(t_+))), \quad 0 \leq m \leq k-1.$$

Since the function has $k - 1$ derivatives that are Lipschitz continuous, we thus have

$$|D_x^{k-1}v(x)| \leq C\|\tilde{t}_+ - t_+\|_{C^{k-1}(\Omega)}.$$

The rest of the proof follows Lemma 2.3.10. \square

With all prepared, we are ready to prove the following stability result.

stabthm

Theorem 2.3.12. *Let $k \geq 3$. Assume that (μ, q) and $(\tilde{\mu}, \tilde{q})$ are in \mathcal{P} and that $|g_j - \check{u}_j|_{\partial\Omega}$, $j = 1, 2$, are sufficiently small. Let $\mu, \tilde{\mu}$ be solutions of (2.3.27) with $(\beta, \gamma), (\tilde{\beta}, \tilde{\gamma})$ and h sufficiently small. Assume on the boundary, $\mu_0 = \tilde{\mu}_0$. Then we have that*

$$\|\mu - \tilde{\mu}\|_{C^{k-1}(\tilde{\Omega})} + \|q - \tilde{q}\|_{C^{k-3}(\tilde{\Omega})} \leq C\|d - \tilde{d}\|_{(C^k(\tilde{\Omega}))^2}.$$

Proof. The first part follows directly from (2.3.20) and Proposition 2.3.11. This provides a stability result for $\nu = 1/\mu$ and thus for $u_j = \nu d_j$. Notice \check{u}_j is non-vanishing on $\tilde{\Omega}$. So when choosing $|g_j - \check{u}_j|_{\partial\Omega}$ sufficiently small, the arguments in (2.3.24) and (2.3.25) show that u_j is non-vanishing in $\tilde{\Omega}$. Thus $\Delta u_j + u_j q = 0$ gives the stability control of q . \square

2.3.6 Stability with 2 Complex Internal Measurements

We now consider the case that $4n$ real-valued observations are taken, with which we can construct $2n$ sets of complex-valued boundary and internal data, denoted as $g_{1,2}^j = \{g_1^j, g_2^j\}$ and $d_{1,2}^j = \{d_1^j, d_2^j\}$. For the rest of this section, we choose $j = 1, \dots, n$.

Let H be a hyperplane in $\mathbb{R}^n \setminus \overline{\text{ch}(\Omega)}$. Choose $x^j \in H$, such that $\{x^j - x^1\}$ form a basis of H , thus $\text{span}\{x^j - x^1\}$ has dimension $n - 1$. Then for $\forall x \in \Omega$, $\{x^j - x\}$ form a basis of \mathbb{R}^n . In fact, since $x \notin H$,

$$\text{span}\{x^j - x\} = \text{span}\{x^j - x^1, x^1 - x\}$$

has dimension n . Based on x^j , we define the front and back sides of the boundary, $\partial\Omega_+^j$ and $\partial\Omega_-^j$, by (2.3.2), and $\partial\Omega_+ = \cup\partial\Omega_+^j$. Accordingly, we furthermore define the open boundary set Γ and the subregion $\tilde{\Omega}$.

Choose

$$\varphi^j = \log|x - x^j|, \quad \psi = \text{dist}\left(\frac{x - x^j}{|x - x^j|}, \omega\right), \quad j = 1, \dots, n.$$

Then the matrix $B_\varphi := (\nabla\varphi^j)$ is invertible. Let $\check{u}_{1,2}^j$ be the CGO solutions by (2.3.15).

Let us now choose boundary conditions $g_{1,2}^j$ close to $\check{u}_{1,2}^j$ on $\partial\Omega$, precisely, by (2.3.24). With internal data $d_{1,2}^j$, we can define β^j by (2.3.20) and (2.3.21).

Proposition 2.3.8 shows that the matrix $B = (\beta^j)$ is close to an invertible matrix B_φ on a subregion $\tilde{\Omega}$. Therefore, we can make B invertible on $\tilde{\Omega}$ with inverse of class $C^k(\tilde{\Omega})$ by choosing h sufficiently small. Consequently, (2.3.27) can be rewrite

$$\nabla\mu + \Lambda\mu = 0, \quad (2.3.36) \quad \boxed{4nstab1}$$

where Λ is a vector-valued function in $(C^k(\tilde{\Omega}))^n$. Finally, the construction of Λ is stable under small perturbations in the data d^j . Indeed, let β and $\tilde{\beta}$ be two vector fields constructed from the internal measurements $d_{1,2}^j$ and $\tilde{d}_{1,2}^j$, respectively. Then when h is sufficiently small,

$$\|\beta - \tilde{\beta}\|_{(C^k(\tilde{\Omega}))^n} \leq \|d - \tilde{d}\|_{(C^{k+1}(\tilde{\Omega}))^{2n}}. \quad (2.3.37) \quad \boxed{4nGammastab}$$

Now we consider equation (2.3.36) with boundary condition $\mu = \mu_0$. Assume Ω is bounded and connected and $\partial\Omega$ is smooth. Let $x \in \tilde{\Omega}$. Find a smooth curve from x to a point on the boundary. Restricted to this curve, (2.3.36) is a stable ordinary differential equation. Keep the curve fixed. Let $\mu, \tilde{\mu}$ be solutions to (2.3.36) with respect to $\beta, \tilde{\beta}$, respectively. Assume $\mu_0 = \tilde{\mu}_0$ on $\partial\Omega$. By solving the equation explicitly and applying (2.3.37), we find that

$$\|\mu - \tilde{\mu}\|_{C^k(\tilde{\Omega})} \leq \|d - \tilde{d}\|_{(C^{k+1}(\tilde{\Omega}))^{2n}}. \quad (2.3.38) \quad \boxed{4nmustab}$$

We can now state the main theorem of this section.

Theorem 2.3.13. *Let $k \geq 2$. Assume that (μ, q) and $(\tilde{\mu}, \tilde{q})$ are in \mathcal{P} and that we have $2n$ well-chosen complex valued measurement, such that $|g_{1,2}^j - \check{u}_{1,2}^j|_{\partial\Omega}$, $j = 1, \dots, n$, are sufficiently small. Let $\mu, \tilde{\mu}$ be solutions of (2.3.36) with $\beta, \tilde{\beta}$ and h sufficiently small. Assume on the boundary, $\mu_0 = \tilde{\mu}_0$. Then we have that*

$$\|\mu - \tilde{\mu}\|_{C^k(\tilde{\Omega})} + \|q - \tilde{q}\|_{C^{k-2}(\tilde{\Omega})} \leq C\|d - \tilde{d}\|_{(C^{k+1}(\tilde{\Omega}))^{2n}}.$$

Proof. The first result is directly from (2.3.38). The proof of the stability of q is exactly that same as in the proof of Theorem 2.3.12. \square

2.3.7 End of the Proof

In this section, we return to the inverse diffusion problem with internal data (2.3.1) and prove the main theorems. We will invert the standard Liouville change of variables mentioned at the beginning of section 2.3.2. Recall that by defining $v = \sqrt{D}u$, we had (2.3.4, 2.3.5, 2.3.6), which lead to

$$-\Delta\sqrt{D} - q\sqrt{D} = \mu. \quad (2.3.39) \quad \boxed{\text{Liou4}}$$

Section 2.3.2 allows us to reconstruct μ and q , while D is given on $\partial\Omega$. Thus we can solve for \sqrt{D} from (2.3.39) and then $\sigma_a = \mu\sqrt{D}$.

The uniqueness of thus a solution \sqrt{D} of (2.3.39) is based on that 0 is not an eigenvalue of $\Delta + q$. It is enough to prove that $(D, \sigma_a) \in \mathcal{M}$ implies $(q, \mu) \in \mathcal{P}$. Indeed, the inverse of $\Delta + q$ is compact and $\sqrt{D} \in Y = H^{\frac{n}{2}+k+2+\epsilon}(\Omega) \subset C^{k+2}(\bar{\Omega})$. These imply $\Delta + q$ is surjective and thus does not have 0 as an eigenvalue by the Fredholm alternative.

So far we proved the unique reconstruction of D, σ_a form internal data for well-chosen partial boundary illuminations as stated in Theorem 2.3.1.

Proof of Theorem 2.3.3. Since $\sigma = \mu\sqrt{D}$, we only need to prove the stability of \sqrt{D} . When $k \geq 3$, we have the stability of the reconstructions of $q \in C^k$ and $\mu \in C^{k+1}$ in Theorem 2.3.12, while the boundary illuminations are chosen from a open set in $C^{k,\alpha}$ with $\alpha > 1/2$. According to (2.3.39), we have

$$-(\Delta + q)(\sqrt{D} - \sqrt{\tilde{D}}) = \mu - \tilde{\mu} + (q - \tilde{q})\sqrt{\tilde{D}}.$$

By elliptic regularity, we deduce that $(\sqrt{D} - \sqrt{\tilde{D}})$ is bounded in $C^k(\bar{\Omega})$, and hence the theorem. \square

The proof of Theorem 2.3.4 is the same as above.

2.4 The Second Step of Thermo-Acoustic Tomography

In this section we shortly introduce the second step of thermo-acoustic tomography to illustrate the difference with the second step of photo-acoustic tomography. The result in this section is due to [5].

As we discussed at the beginning of this chapter, quantitative thermo-acoustic tomography is distinguished from quantitative photo-acoustic tomography by the wave length of the radiation in use. The high-frequency radiation sent in photo-acoustic tomography is normally laser and hence is modelled by the diffusion equation in (2.3.1), while the low-frequency radiation sent in thermo-acoustic tomography is often electromagnetic waves and thus is modelled by

$$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} E + \sigma \mu \frac{\partial}{\partial t} E + \nabla \times \nabla \times E = S(t, x)$$

for $t \in \mathbb{R}$ and $x \in \mathbb{R}^3$. Here $c^2 = (\varepsilon\mu)^{-1}$ is the speed of light, ε the permittivity, μ the permeability, $\sigma = \sigma(x)$ the conductivity to be recovered.

Let us consider the following scalar approximation to the above problem:

$$\frac{\partial^2}{\partial t^2} u + \sigma \mu \frac{\partial}{\partial t} u - \Delta u = S(t, x) \quad (2.4.1) \quad \boxed{\text{eqn: approximate}}$$

for $t \in \mathbb{R}$ and $x \in \mathbb{R}^3$. We assume that $S(t, x)$ is a narrowband pulse with central frequency ω_c of the form $S(t, x) = -e^{-i\omega_c t} \phi(t) S(x)$, where $\phi(t)$ is the envelope of the pulse and $S(x)$ is a superposition of plane waves with wave number k such that $\omega_c = ck$. By taking the Fourier transform of (2.4.1) we have

$$u(t, x) \sim \phi(t) u(x),$$

where $u(x)$ satisfies

$$\Delta u + (k^2 + ikc\mu\sigma)u = S(x).$$

This equation can be recast as a scattering problem with incoming radiation $u^i(x)$ and scattered radiation $u^s(x)$ satisfying the proper Sommerfeld radiation conditions at infinity. The model we finally obtain for the electromagnetic propagation looks like

$$\Delta u + k^2 u + ikc\mu\sigma(x)u = 0, \quad u = u^i + u^s.$$

The incoming radiation u^i is a superposition of plane waves and is assumed to be controllable.

Taking into consideration that the amount of absorbed radiation is given by

$$d(t, x) = \sigma |u(t, x)|^2 \sim \phi^2(t) \sigma(x) |u(x)|^2,$$

we ultimately arrive at the QTAT model:

$$\begin{cases} \Delta u + q(x)u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \quad (2.4.2) \quad \boxed{\text{eqn:QTAT}}$$

where g is the boundary illumination, $q(x) = k^2 + ikc\mu\sigma$. The internal measurements are of the form

$$d(x) = \sigma|u(x)|^2. \quad (2.4.3) \quad \boxed{\text{def:internal}}$$

The inverse problem in QTAT is then to recover the function $\sigma = \sigma(x)$ from the measurement $d(x)$.

To state the uniqueness and stability result we introduce a few function spaces. For $\epsilon > 0$ let $Y := H^{\frac{n}{2}+\epsilon}(\Omega)$ and $Z := H^{\frac{n}{2}+\epsilon-\frac{1}{2}}(\partial\Omega)$ be the usual Sobolev spaces. Let $M > 0$ be a fixed number, set

$$\mathcal{M} := \{f \in Y : \|f\|_Y \leq M\}.$$

Theorem 2.4.1 ([5]). *Let $\rho \in \mathbb{C}^n$ be such that $\rho \cdot \rho = 0$ and $|\rho|$ is sufficiently large. Let σ and $\tilde{\sigma}$ be functions in \mathcal{M} .*

Let $g \in Z$ be a given illumination and $d(x)$ be the measurement given in (2.4.3) for u a solution of (2.4.2). Let $\tilde{d}(x)$ be the measurement constructed by replacing σ by $\tilde{\sigma}$ in (2.4.2) and (2.4.3).

Then there is an open set of illuminations in Z such that $d(x) = \tilde{d}(x)$ in Y implies that $\sigma(x) = \tilde{\sigma}(x)$ in Y . Moreover, there exists a constant C independent of σ and $\tilde{\sigma}$ such that

$$\|\sigma - \tilde{\sigma}\|_Y \leq C\|d - \tilde{d}\|_Y.$$

More precisely, we can write the reconstruction of σ as finding the unique fixed point to the equation

$$\sigma(x) = e^{-(\rho+\bar{\rho}) \cdot x} d(x) - \mathcal{H}_g[\sigma](x), \quad \text{in } Y.$$

Here the functional $\mathcal{H}_g[\sigma]$ is defined as

$$\mathcal{H}_g[\sigma](x) = \sigma(x)(\psi_g(x) + \overline{\psi_g(x)} + \psi_g(x)\overline{\psi_g(x)})$$

is a contraction map for g in the open set described above, where ψ_g is defined as the solution to

$$(\Delta + 2\rho \cdot \nabla)\psi_g = -q(q + \psi_g) \quad \text{in } X, \quad \psi_g = e^{-\rho \cdot x}g - 1 \quad \text{on } \partial X.$$

We thus deduce the reconstruction algorithm

$$\sigma = \lim_{m \rightarrow \infty} \sigma_m, \quad \sigma_0 = 0, \quad \sigma_m = e^{-(\rho + \bar{\rho}) \cdot x}d(x) - \mathcal{H}_g[\sigma_{m-1}](x), \quad m \geq 1.$$

Chapter 3

ELECTRO-SEISMIC CONVERSION**3.1 Introduction**

Hybrid methods arise not only in modern medical imaging modalities, but also in many other fields. In this section we discuss a coupled-physics inverse problem in seismic imaging and oil exploration.

When a porous rock is saturated with an electrolyte, an electric double layer is formed at the interface of the solid and the fluid. One side of the interface is negatively charged and the other side is positively charged. Such electric double layer(EDL) system is also called Debye layer. Due to the EDL system, electromagnetic(EM) fields and mechanical waves are coupled through the phenomenon of electro-kinetics. Precisely, electrical fields or EM waves acting on the EDL will move the charges, creating relative movement of fluid and solid. This is called electro-seismic conversion. Conversely, mechanical waves moving fluid and solid will generate EM fields. This is called seismo-electric conversion. Thompson and Gist [51] have made field measurement clearly demonstrating seismo-electric conversion in saturated sediments. Zhu et al. [57, 58, 59] made laboratory experiments and observed the seismo-electric conversion in model wells, and their experimental results confirm that seismo-electric logging could be a new bore-hole logging technique.

The investigation of wave propagation in fluid-saturated porous media was early developed by Biot [7, 8]. The governing equations of the electro-seismic conversion was derived

by Pride [41] as following.

$$\nabla \times E = i\omega\mu H, \quad (3.1.1)$$

$$\nabla \times H = (\sigma - i\epsilon\omega)E + L(-\nabla p + \omega^2\rho_f u) + J_s, \quad (3.1.2)$$

$$-\omega^2(\rho u + \rho_f w) = \nabla \cdot \tau, \quad (3.1.3)$$

$$-i\omega w = LE + \frac{\kappa}{\eta}(-\nabla p + \omega^2\rho_f u), \quad (3.1.4)$$

$$\tau = (\lambda\nabla \cdot u + c\nabla \cdot w)I + G(\nabla u + \nabla u^T), \quad (3.1.5)$$

$$-p = c\nabla \cdot u + M\nabla \cdot w, \quad (3.1.6)$$

where the first two are Maxwell's equations, the remaining are Biot's equations. The notation is as follows:

E electric field,

H magnetizing field or magnetic field intensity,

ω seismic wave frequency,

σ conductivity,

ϵ dielectric constant or relative permittivity,

μ magnetic permeability,

J_s source current,

p pore pressure,

ρ_f density of pore fluid,

L electro-kinetic mobility parameter,

κ fluid flow permeability,

u solid displacement,

w fluid displacement,

τ bulk stress tensor,

η viscosity of pore fluid,

λ, G Lamé parameters of elasticity,

C, M Biot moduli parameters.

Pride and Haartsen [42] also analyzed the basic properties of seismo-electric waves.

Notice that the coupling is non-linear, namely electro-seismic and seismo-electric conversions happen simultaneously. Under the assumptions that the coupling is so weak that multiple coupling is neglectable, we can linearize the forward system in two steps. Particularly, we focus on the electro-seismic conversion and ignore the seismo-electric conversion. The first step in the forward system is modeled by Maxwell equations without the effect of the seismic waves, i.e., $L = 0$ in (3.1.2). While the electro-seismic conversion happens, the seismic waves are generated and modeled by Biot's equations with potential LE in (3.1.4).

In the present chapter, we mainly focus on the inverse problem of the linearized electro-seismic conversion, which is a hybrid problem and consists of two steps. The first step of the inverse problem is to invert Biot's equations, i.e., to recover the potential LE in (3.1.4) from any measurements observed on the domain boundary. Williams [53] presented an approximation to Biot's equations, which could be a useful tool to study the inverse problem.

Assuming the first step is implemented successfully, the second step of the inverse problem is to invert Maxwell's equations, which consists of reconstructing the conductivity σ and the electro-kinetic mobility parameter or the coupling coefficient L from boundary measurements of the electrical fields and the internal data LE obtained in the first step.

The problem of interest in this chapter is the second step of the inverse problem. We study the reconstruction of the conductivity σ and the coupling coefficient L and prove

uniqueness and stability results of the reconstructions. Particularly, we show that σ, L are uniquely determined by 2 well-chosen electrical fields at the domain boundary. The explicit reconstruction procedure is presented. The stability of the reconstruction is established from either 2 measurements under geometrical conditions or from 6 well-chosen boundary conditions.

Mathematically, our proof relies on explicit solutions to Maxwell's equations, namely Complex Geometrical Optics (CGO) solutions, constructed by Colton and Päiväranta [12]. In our reconstruction procedure, the coupling coefficient L satisfies a transport equation with vector field β . With CGO solutions, we can prove the integral curves of the vector field β are close to straight lines and exit the domain in finite time. Therefore, L can be uniquely and explicitly solved by the characteristic method. Stability follows the analysis of the method of characteristic.

The rest of the chapter is structured as follows. Section 3.2 is devoted to the first step, that is, the inversion of Biot's system. We present a model suggested by Williams [53] to simplify the formulation. In Section 3.3 we illustrate in detail our uniqueness and stability results on the inversion of Maxwell's equations with internal measurements.

3.2 Inversion of Biot's system

For this step we perform a reduction process and reduce the inversion of the complicated Biot's system to the inversion of an inhomogeneous Helmholtz equation.

The wave propagation in a porous medium saturated with fluid is modeled by Biot's system. The electro-seismic effect is modeled by the following system, where $D = LE$ is an internal potential.

$$\begin{aligned}
 -\omega^2(\rho u + \rho_f w) &= \nabla \cdot \tau, \\
 -i\omega w &= D + \frac{\kappa}{\eta}(-\nabla p + \omega^2 \rho_f u), \\
 \tau &= (\lambda \nabla \cdot u + C \nabla \cdot w)I + G(\nabla u + \nabla u^T), \\
 -p &= C \nabla \cdot u + M \nabla \cdot \omega.
 \end{aligned} \tag{3.2.1}$$

eqn:Biot

The notations are identical to those in the introduction of this chapter. The inverse problem here is to recover the internal potential from boundary measurements of the acoustic waves.

The direct inversion of Biot's system is challenging in two aspects: first, the seismic wave

generated by electro-seismic conversion is normally very weak; second, the Biot slow wave is a diffusive wave which decays rapidly to zero with propagation distance and is therefore difficult to observe.

Williams [53] presented the effective density fluid model (EDFM), which is an accurate approximation to Biot's equations and could dramatically simplify the formulation. We now demonstrate this simplification.

Let $u = u_s$ be the displacement of the solid frame and u_f be the displacement of the saturated fluid. Define $w = \beta(u_s - u_f)$ with β the porosity of the medium. Stoll and Kan [49] introduced potentials $(\Phi_s, \Phi_f, \Psi_s, \Psi_f)$ defined by

$$\begin{aligned} u &= \nabla\Phi_s + \nabla \times \Psi_s \\ u &= \nabla\Phi_f + \nabla \times \Psi_f. \end{aligned} \tag{3.2.2} \quad \text{eqn:Stoll}$$

Assume $D = 0$ for the moment. Substituting (3.2.2) into (3.2.1) we have

$$\begin{aligned} H\Delta^2\Phi_s + C\Delta^2\Phi_f &= -\omega\rho\Delta\Phi_s - \omega^2\rho_f\Delta\Phi_f, \\ C\Delta^2\Phi_s + M\Delta^2\Phi_f &= -\omega^2\rho_f\Delta\Phi_s - \frac{i\omega\eta}{\kappa}\Delta\Phi_f. \end{aligned} \tag{3.2.3} \quad \text{eqn:transiti}$$

Here the notations are as follows.

$$\begin{aligned} \rho &= \beta\rho_f + (1 - \beta)\rho_s, \\ H &= \lambda + 2G = \frac{(K_r - K_b)^2}{\Xi - K_b} + K_b + \frac{4\mu}{3}, \\ C &= \frac{K_r(K_r - K_b)}{\Xi - K_b}, \\ M &= \frac{K_r^2}{\Xi - K_b}, \\ \Xi &= K_r \left[1 + \beta \left(\frac{K_r}{K_f} - 1 \right) \right], \end{aligned}$$

where ρ is the total mass density, K_r is the bulk modulus of individual sediment grains, and K_f is the bulk modulus of the pore fluid.

Due to the fact that for sand sediments the frame and shear moduli are much lower than other moduli, we may take $K_b = \mu = 0$, which also implies that the slow wave and shear wave are neglected in this model. We then have

$$H = C = M = \left(\frac{1 - \beta}{K_r} + \frac{\beta}{K_f} \right)^{-1}.$$

Let $U = \Delta\Phi_s$ and $W = \Delta\Phi_f$. Taking the Fourier transform of (3.2.3) gives

$$\begin{aligned} Hk^2\hat{U} + Hk^2\hat{W} &= \omega\rho\hat{U} + \omega^2\rho_f\hat{W}, \\ Hk^2\hat{U} + Hk^2\hat{W} &= \omega^2\rho_f\hat{U} + \frac{i\omega\eta}{\kappa}\hat{W}. \end{aligned} \tag{3.2.4} \quad \text{eqn:transition2}$$

Williams [[53]] introduced the effective mass density ρ_{eff} by

$$k^2 = \frac{\omega^2\rho_{\text{eff}}(\omega)}{H},$$

or equivalently

$$\rho_{\text{eff}}(\omega) = \rho_f \left(\frac{\omega^2\rho_f^2 + \rho\frac{i\eta}{\kappa}}{2\omega^2\rho_f - \omega\rho + \frac{i\omega\eta}{\kappa}} \right). \tag{3.2.5} \quad \text{eqn:rhoeff}$$

Using (3.2.4) we can write \hat{W} in terms of \hat{U} as

$$\hat{W} = \left(\frac{\rho_f - \rho_{\text{eff}}(\omega)}{\rho_{\text{eff}}(\omega) - \frac{i\eta}{\omega\kappa}} \right) \hat{U}. \tag{3.2.6} \quad \text{eqn:What}$$

Then using (3.2.4), (3.2.5) and (3.2.6) we derive that

$$k^2H(\hat{U} + \hat{W}) = \omega^2\rho_{\text{eff}}(\omega)(\hat{U} + \hat{W}).$$

The inverse Fourier transform then gives that

$$H\Delta\Phi + \omega^2\rho_{\text{eff}}(\omega)\Phi = 0$$

where $\Phi = U + W$.

In the case when $D \neq 0$, we define $\rho_{\text{eff}}(\omega)$ by (3.2.5) and come to the expression

$$H\Delta\Phi + \omega^2\rho_{\text{eff}}(\omega)\Phi = \nabla \cdot D.$$

Then the inverse problem becomes the inversion of this inhomogeneous Helmholtz equation to recover $\nabla \cdot D$, from which we can calculate D .

3.3 Inversion of Maxwell's equations

In this section we study the second step of the linearized electro-seismic conversion. We aim to invert the Maxwell's equations and recover the coupling coefficient L and the conductivity σ from boundary measurements of the electrical field E and the internal potential $D = LE$ which is obtained from the first step. We will demonstrate a procedure which allows the reconstruction to be unique and stable, provided the boundary electrical field E is well-chosen.

3.3.1 Main Results

Let Ω be an open, bounded and connected domain in \mathbb{R}^3 with C^2 boundary $\partial\Omega$. In the second step of the electro-seismic conversion, the propagation of the electrical fields is modeled by Maxwell's equations in Ω ,

$$\begin{cases} \nabla \times E &= i\omega\mu H, \\ \nabla \times H &= (\sigma - i\epsilon\omega)E + J_s. \end{cases} \quad (3.3.1) \quad \text{Maxwell}$$

The measurements available for the inverse problem include the internal data from the first step

$$D := LE, \text{ in } \Omega$$

and the boundary illumination, i.e., the tangential boundary measurement of the electrical field

$$G := tE, \text{ on } \partial\Omega.$$

Define the operator

$$\Lambda_M(L, \sigma) := (J_s, D, G). \quad (3.3.2) \quad \text{Max_Meas_map}$$

The problem now is to invert the operator Λ_M , or namely, to reconstruct (L, σ) from some measurements $(J_{s,j}, D_j, G_j)$ indexed by j , assuming μ and ϵ are given.

The main purpose of this section is to prove the uniqueness and stability of the coefficient reconstructions. For small $\iota > 0$, define the set of coefficients $(L, \sigma) \in \mathcal{M}$ as

$$\mathcal{M} = \{(L, \sigma) \in C^{d+1}(\overline{\Omega}) \times H^{\frac{3}{2}+3+d+\iota}(\Omega) : \quad (3.3.3) \quad \text{eq:para_space}$$

and 0 is not an eigenvalue of $\nabla \times \nabla \times \cdot - k^2 n\}$,

where the wave number $k > 0$ and the refractive index n are given by

$$k = \omega\sqrt{\epsilon_0\mu_0}, \quad n = \frac{1}{\epsilon_0} \left(\epsilon + i\frac{\sigma}{\omega} \right). \quad (3.3.4) \quad \text{Maxwellkn}$$

The main results are as follows, where the measurements G and D are complex-valued.

thm:unique

Theorem 3.3.1. *Let Ω be an open, bounded subset of \mathbb{R}^3 with boundary $\partial\Omega$ of class C^d . Let (L, σ) and $(\tilde{L}, \tilde{\sigma})$ be two elements in \mathcal{M} with $L|_{\partial\Omega} = \tilde{L}|_{\partial\Omega}$. Let $D := (D_1, D_2)$ and $\tilde{D} :=$*

$(\tilde{D}_1, \tilde{D}_2)$, be two sets of internal data on Ω for the coefficients (L, σ) , $(\tilde{L}, \tilde{\sigma})$, respectively and with boundary illuminations $G := (G_1, G_2)$.

Then there is a non-empty open set of $G \in (C^{d+4}(\partial\Omega))^2$ such that if $D_j = \tilde{D}_j$, $j = 1, 2$, we have $(L, \sigma) = (\tilde{L}, \tilde{\sigma})$.

Here and in the following, we shall abuse the notation and use $C^d(\bar{\Omega})$ to denote either set of complex-valued functions or set of vector-valued functions whose elements have up to d order continuous derivatives. It should be clear from the context which one it is. The function space $(C^{d+4}(\partial\Omega))^2$ is an abbreviation of the product space $C^{d+4}(\partial\Omega) \times C^{d+4}(\partial\Omega)$.

To consider the stability of the reconstruction, we need to restrict to a subset of Ω . Let ζ_0 be a constant unit vector. Let $x_0 \in \partial\Omega$ be the tangent point of $\partial\Omega$ with respect to ζ_0 , i.e., the tangent line of $\partial\Omega$ at x_0 is parallel to ζ_0 . Define Ω_1 to be the subset of Ω by removing a neighborhood of each tangent point $x_0 \in \partial\Omega$.

thm:stab

Theorem 3.3.2. *let $d \geq 2$. Let Ω be convex with C^d boundary $\partial\Omega$ and Ω_1 is defined as above. Assume that (L, σ) and $(\tilde{L}, \tilde{\sigma})$ are two elements in \mathcal{M} with $L|_{\partial\Omega} = \tilde{L}|_{\partial\Omega}$. Let $D = (D_j)$ and $\tilde{D} = (\tilde{D}_j)$, $j = 1, 2$, be the internal data for coefficients (L, σ) and $(\tilde{L}, \tilde{\sigma})$, respectively, with boundary conditions $G = (G_j)$, $j = 1, 2$.*

Then there exists a non-empty open set of illuminations $G \in (C^{d+4}(\partial\Omega))^2$ such that restricting to Ω_1 , we have

$$\|L - \tilde{L}\|_{C^{d-1}(\bar{\Omega}_1)} + \|\sigma - \tilde{\sigma}\|_{C^{d-3}(\bar{\Omega}_1)} \leq C\|D - \tilde{D}\|_{(C^{d+1}(\bar{\Omega}_1))^2}. \quad (3.3.5) \quad \text{eq:stable}$$

The geometric conditions can be removed when more measurements are available. In particular, when 6 complex measurements are provided, we have the following stability result.

thm:stab2n

Theorem 3.3.3. *let $d \geq 3$. Let Ω be convex and Ω_1 is defined as above. Assume that (L, σ) and $(\tilde{L}, \tilde{\sigma})$ are two elements in \mathcal{M} with $L|_{\partial\Omega} = \tilde{L}|_{\partial\Omega}$. Let $D = (D_1^j, D_2^j)$ and $\tilde{D} = (\tilde{D}_1^j, \tilde{D}_2^j)$, $j = 1, 2, 3$, be the internal data for coefficients (L, σ) and $(\tilde{L}, \tilde{\sigma})$, respectively, with boundary conditions $G = (G_1^j, G_2^j)$, $j = 1, 2, 3$.*

Then there exists a non-empty open set of illuminations $G \in (C^{d+4}(\partial\Omega))^6$, such that

$$\|L - \tilde{L}\|_{C^d(\bar{\Omega})} + \|\sigma - \tilde{\sigma}\|_{C^{d-2}(\bar{\Omega})} \leq C\|D - \tilde{D}\|_{(C^{d+1}(\bar{\Omega}))^6}. \quad (3.3.6) \quad \text{eq:stable9}$$

Note that the above measurements are all complex-valued. We will need two real measurements to make up one complex data.

3.3.2 Complex Geometric Optics Solutions

First we construct the complex geometric optics solutions which will be a key ingredient in the proof of our main results. Let Ω be an open, bounded and connected domain in \mathbb{R}^3 with C^2 boundary $\partial\Omega$. In the case when $\mu \equiv \mu_0$ is constant and $J_s = 0$ in Ω , we can rewrite the system in (3.3.1) as

$$\nabla \times \nabla \times E - k^2 n E = 0, \quad (3.3.7) \quad \boxed{\text{Maxwell_E1}}$$

and

$$\nabla \cdot n E = 0 \quad (3.3.8) \quad \boxed{\text{Maxwell_E2}}$$

where the wave number $k > 0$ and the refractive index n are given by (3.3.4). We would like to consider the equations (3.3.7) and (3.3.8) in the whole \mathbb{R}^3 . For this purpose, we extend $n \in H^{\frac{3}{2}+3+d+\iota}(\Omega)$ to be a function defined in the whole \mathbb{R}^3 in such a way that $1 - n \in H^{\frac{3}{2}+3+d+\iota}(\mathbb{R}^n)$ is compactly supported. We denote this extension still by n .

Colton and Päiväranta [12] constructed CGO solutions to the Maxwell's equation (3.3.7) and (3.3.8). We follow the construction of CGO solutions in [12] and extend their properties from L^2 space to higher order Sobolev spaces. The CGO solutions we need are of the form

$$E(x) = e^{i\zeta \cdot x} (\eta + R_\zeta(x)), \quad (3.3.9) \quad \boxed{\text{CGOs}}$$

where $\zeta \in \mathbb{C}^3 \setminus \mathbb{R}^3$, $\eta \in \mathbb{C}^3$, are constant vectors satisfying

$$\zeta \cdot \zeta = k^2, \quad \zeta \cdot \eta = 0. \quad (3.3.10) \quad \boxed{\text{CGO_para_con}}$$

Substituting (3.3.9) into (3.3.7) and (3.3.8) gives

$$\tilde{\nabla} \times \tilde{\nabla} \times R_\zeta = k^2(n-1)\eta + k^2 n R_\zeta, \quad (3.3.11)$$

$$\tilde{\nabla} \cdot R_\zeta = -\alpha \cdot (\eta + R_\zeta) \quad (3.3.12)$$

where $\tilde{\nabla} := \nabla + i\zeta$ and $\alpha := \nabla n(x)/n(x)$. We further define $\tilde{\Delta} := \Delta + 2i\zeta \cdot \nabla - k^2$. By substituting the formula $\tilde{\nabla} \times \tilde{\nabla} \times R_\zeta = -\tilde{\Delta} R_\zeta + \tilde{\nabla} \tilde{\nabla} \cdot R_\zeta$ into (3.3.11) and (3.3.12), we see

that R_ζ is a solution to

$$(\Delta + 2i\zeta \cdot \nabla)R_\zeta = -\tilde{\nabla}(\alpha \cdot (\eta + R_\zeta)) + k^2(1-n)(\eta + R_\zeta). \quad (3.3.13) \quad \boxed{\text{Maxwell15}}$$

It was proved in [12] the existence of R_ζ to (3.3.13) as a $C^2(\mathbb{R}^3)$ functions. For our analysis, we need to extend the results of CGO solutions in [12] to smoother function spaces.

Let the space L_δ^2 for $\delta \in \mathbb{R}$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|\cdot\|_{L_\delta^2}$ defined as

$$\|u\|_{L_\delta^2} = \left(\int_{\mathbb{R}^3} \langle x \rangle^{2\delta} |u|^2 dx \right)^{\frac{1}{2}}, \quad \langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}.$$

To get smoother CGO solutions than that constructed in [12], we introduce the space H_δ^s for $s > 0$ as the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|\cdot\|_{H_\delta^s}$ defined as

$$\|u\|_{H_\delta^s} = \left(\int_{\mathbb{R}^3} \langle x \rangle^{2\delta} |(I - \Delta)^{\frac{s}{2}} u|^2 dx \right)^{\frac{1}{2}}$$

Here $(I - \Delta)^{\frac{s}{2}} u$ is defined as the inverse Fourier transform of $\langle \xi \rangle^s \hat{u}(\xi)$, where $\hat{u}(\xi)$ is the Fourier transform of $u(x)$. When $\delta = 0$, this is the standard Sobolev space of order s , which we denote by $H^s(\mathbb{R}^3)$.

We recall [50] for $|\zeta| \geq c > 0$ and $v \in L_{\delta+1}^2$ with $-1 < \delta < 0$, the equation

$$(\Delta + 2i\zeta \cdot \nabla)u = v$$

admits a unique weak solution $u \in L_\delta^2$ with

$$\|u\|_{L_\delta^2} \leq C(\delta, c) \frac{\|v\|_{L_{\delta+1}^2}}{|\zeta|}.$$

Since $(\Delta + 2i\zeta \cdot \nabla)$ and $(I - \Delta)^s$ are constant coefficient operators and hence commute, we deduce that when $v \in H_{\delta+1}^s$, for $s > 0$, then

$$\|u\|_{H_\delta^s} \leq C(\delta, c) \frac{\|v\|_{H_{\delta+1}^s}}{|\zeta|}.$$

We define the integral operator $G_\zeta : H_{\delta+1}^s(\mathbb{R}^3) \rightarrow H_\delta^s(\mathbb{R}^3)$ by

$$G_\zeta(v) := F^{-1} \left(\frac{\hat{v}}{|\xi|^2 + 2\zeta \cdot \xi} \right),$$

where F^{-1} is the inverse Fourier transform. We see that G_ζ is bounded and there exists a positive constant $C(\delta)$ such that

$$\|G_\zeta\| \leq \frac{C}{|\zeta|}. \quad (3.3.14) \quad \boxed{\text{G_bd}}$$

Before we can prove the existence of a unique solution to (3.3.13), we first prove the following lemma.

thm:lemma1 **Lemma 3.3.4.** *For any $v \in H_{\delta+1}^{\frac{5}{2}+d+\iota}(\mathbb{R}^3)$ and $|\zeta|$ sufficiently large, the equation*

$$(\Delta + 2i\zeta \cdot \nabla + \alpha \cdot \tilde{\nabla})u = v \quad (3.3.15) \quad \boxed{\text{Maxwell16}}$$

has a unique solution $u \in H_\delta^{\frac{5}{2}+d+\iota}(\mathbb{R}^3)$ satisfying

$$\|u + n^{-1/2}G_\zeta(n^{1/2}v)\|_{H_\delta^{\frac{5}{2}+d+\iota}} \leq \frac{C}{|\zeta|^2},$$

for some positive constant C independent of ζ .

Here the complex-valued function $n^{1/2}$ is defined as $n^{1/2} := e^{\frac{1}{2}\text{Log}(n)}$ where Log is the principal branch of the complex Logarithmic function, i.e. a branch cut along the negative x -axis with $\text{Log}(1) = 0$. This is possible since $\text{Re } n = \frac{\epsilon}{\epsilon_0}$ is always positive. Lemma 3.1 in [12] proves the case when $s = 0$. We study $s = \frac{3}{2} + d + \iota$ here.

Proof. From the identity

$$n^{-1/2}(\Delta + 2i\zeta \cdot \nabla)(n^{1/2}u) = (\Delta + 2i\zeta \cdot \nabla + \alpha \cdot \tilde{\nabla})u + qu, \quad (3.3.16) \quad \boxed{\text{lemma_transf}}$$

where $q := \Delta n^{1/2}/n^{1/2} \in H^{\frac{5}{2}+d+\iota}(\mathbb{R}^3)$, we can rewrite (3.3.15) as

$$(\Delta + 2i\zeta \cdot \nabla - q)f = g,$$

where $f := n^{1/2}u$ and $g := n^{1/2}v$. The assumption on n ensures that $1 - n^{1/2} \in H^{\frac{5}{2}+d+\iota}(\mathbb{R}^3)$ and is compactly supported, so $1 - n^{1/2} \in H_{\delta+1}^{\frac{5}{2}+d+\iota}(\mathbb{R}^3)$. Therefore,

$$g = n^{1/2}v = v - (1 - n^{1/2})v \in H_{\delta+1}^{\frac{5}{2}+d+\iota}(\mathbb{R}^3).$$

Applying the integral operator $-G_\zeta$ gives

$$f + G_\zeta(qf) = -G_\zeta(g). \quad (3.3.17) \quad \boxed{\text{lemma_f}}$$

Since $q \in H^{\frac{5}{2}+d+\iota}(\mathbb{R}^3)$ is compactly supported, multiplication by q is a bounded operator mapping $H^{\frac{5}{2}+d+\iota}(\mathbb{R}^3)$ into $H^{\frac{5}{2}+d+\iota}(\mathbb{R}^3)$, so $I + G_\zeta(q \cdot)$ is invertible on $H^{\frac{5}{2}+d+\iota}(\mathbb{R}^3)$ for $|\zeta|$ sufficiently large. This shows that (3.3.17) has a unique solution f in $H^{\frac{5}{2}+d+\iota}(\mathbb{R}^3)$, correspondingly $u = n^{-1/2}f \in H^{\frac{5}{2}+d+\iota}(\mathbb{R}^3)$ is the unique solution of (3.3.15). Eq. (3.3.14) also gives

$$\|f + G_\zeta(g)\|_{H^{\frac{5}{2}+d+\iota}} = \|G_\zeta(q(G_\zeta(qf) + G_\zeta(g)))\|_{H^{\frac{5}{2}+d+\iota}} \leq \frac{C}{|\zeta|^2},$$

for some positive constant C independent of ζ . This proves the lemma. \square

thm:CGO **Proposition 3.3.5.** *For $|\zeta|$ sufficiently large, there is a unique solution $R_\zeta \in H^{\frac{5}{2}+d+\iota}(\mathbb{R}^3)$ to (3.3.13). Thus, the CGO solution E defined by (3.3.9) satisfies (3.3.7) and (3.3.8). Moreover, R_ζ satisfies*

$$\|R - in^{-1/2}G_\zeta(n^{1/2}\alpha \cdot \eta)\zeta\|_{H^{\frac{5}{2}+d+\iota}} = O\left(\frac{1}{|\zeta|}\right). \quad (3.3.18) \quad \text{Maxwell19}$$

Proof. By applying the vector identity

$$\nabla(A \cdot B) = A \times (\nabla \times B) + B \times (\nabla \times A) + (A \cdot \nabla)B + (B \cdot \nabla)A,$$

we see that

$$\tilde{\nabla}(\alpha \cdot (\eta + R_\zeta)) = \alpha \times (\tilde{\nabla} \times R_\zeta) + (\alpha \cdot \tilde{\nabla})R_\zeta + (R_\zeta \cdot \nabla)\alpha + \tilde{\nabla}(\alpha \cdot \eta). \quad (3.3.19) \quad \text{vectoridentity}$$

The terms which are potentially troublesome are $\alpha \times (\tilde{\nabla} \times R_\zeta)$ and $(\alpha \cdot \tilde{\nabla})R_\zeta$. The latter can be dealt with using Lemma 3.3.4, so we need only consider the term $\alpha \times (\tilde{\nabla} \times R_\zeta)$. Denote $Q := \tilde{\nabla} \times R_\zeta$, by (3.3.11) we have that

$$\tilde{\nabla} \times Q = k^2(n-1)\eta + k^2nR_\zeta$$

and hence

$$\tilde{\nabla} \times \tilde{\nabla} \times Q = k^2\nabla n \times (\eta + R_\zeta) + k^2(n-1)\zeta \times \eta + k^2nQ.$$

Since $\tilde{\nabla} \cdot Q = 0$, we now have

$$\Delta Q + 2i\zeta \cdot \nabla Q = k^2\nabla n \times (\eta + R_\zeta) + k^2(1-n)(i\zeta \times \eta + Q). \quad (3.3.20) \quad \text{prop:Q}$$

Rearrange the terms to get

$$(\Delta + 2i\zeta \cdot \nabla - k^2(1-n))Q = k^2\nabla(1-n) \times (\eta + R_\zeta) + k^2(1-n)(i\zeta \times \eta).$$

Applying $-G_\zeta$ to this identity yields

$$Q = -(I + k^2G_\zeta(1-n))^{-1}G_\zeta(k^2\nabla(1-n) \times (\eta + R_\zeta) + k^2(1-n)(i\zeta \times \eta)) \quad (3.3.21) \quad \boxed{\text{Q}}$$

for large $|\zeta|$, since the operator $I + k^2G_\zeta(1-n)$ is invertible on $H_\delta^{\frac{5}{2}+d+\iota}(\mathbb{R}^3)$ for large $|\zeta|$.

On the other hand, applying $-G_\zeta$ to (3.3.13) and using (3.3.19) gives

$$R_\zeta = G_\zeta[\alpha \times Q] + G_\zeta[(\alpha \cdot \tilde{\nabla})R_\zeta] + G_\zeta[(R_\zeta \cdot \nabla)\alpha] + G_\zeta[\tilde{\nabla}\alpha \cdot \eta] - k^2G_\zeta[(1-n)(\eta + R_\zeta)] \quad (3.3.22) \quad \boxed{\text{R}}$$

where Q is given by (3.3.21). The integral equation (3.3.21), (3.3.22) has a unique solution in $H_\delta^{\frac{5}{2}+d+\iota}(\mathbb{R}^3)$ due to (3.3.14) and Lemma 3.3.4.

Finally, we use the unique solvability of (3.3.21) and (3.3.22) to deduce the unique solvability of (3.3.13). To do this, let $B \subset \mathbb{R}^3$ be a ball containing Ω . Applying G_ζ to (3.3.13) yields

$$R_\zeta = G_\zeta[\tilde{\nabla}(\alpha \cdot (\eta + R_\zeta))] - k^2G_\zeta[(1-n)(\eta + R_\zeta)]. \quad (3.3.23) \quad \boxed{\text{integral}}$$

This integral equation is of Fredholm type in $H_\delta^{\frac{5}{2}+d+\iota}(B)$ as both $G_\zeta(1-n)$ and $R_\zeta \mapsto G_\zeta[\tilde{\nabla}(\alpha \cdot R_\zeta)]$ are smoothing operators. Now, suppose R_ζ^h is a solution of the following homogeneous equation in $H_\delta^{\frac{5}{2}+d+\iota}(B)$,

$$R_\zeta^h = G_\zeta[\tilde{\nabla}(\alpha \cdot R_\zeta^h)] - k^2G_\zeta[(1-n)R_\zeta^h].$$

Then R_ζ^h also satisfies the homogeneous equation corresponding to (3.3.21), (3.3.22). Since these equations are uniquely solvable, we conclude that $R_\zeta^h = 0$. Therefore, by Fredholm alternative, we conclude that (3.3.23) admits a unique solution in $H_\delta^{\frac{5}{2}+d+\iota}(B)$. Defining $R_\zeta(x)$ for $x \in \mathbb{R}^3$ by the right-hand side of (3.3.23) and recalling $\Omega \subset B$ yields a solution of (3.3.23), which is defined in \mathbb{R}^3 . This shows that (3.3.13) has a unique solution in $H_\delta^{\frac{5}{2}+d+\iota}(\mathbb{R}^3)$.

Furthermore, by Lemma 3.3.4, we see that

$$\begin{aligned} \|R_\zeta + n^{-1/2}G_\zeta(n^{1/2}[-\alpha \times (\tilde{\nabla} \times R_\zeta) - (R_\zeta \cdot \nabla)\alpha \\ - \tilde{\nabla}(\alpha \cdot \eta) + k^2(1-n)(\eta + R_\zeta)])\|_{H_\delta^{\frac{5}{2}+d+\iota}} = O\left(\frac{1}{|\zeta|^2}\right). \end{aligned} \quad (3.3.24) \quad \boxed{\text{Maxwell18}}$$

Substituting $\tilde{\nabla}(\alpha \cdot \eta) = (\nabla + i\zeta)(\alpha \cdot \eta)$, (3.3.24) implies that

$$\|R_\zeta + in^{-1/2}G_\zeta(n^{1/2}\alpha \cdot \eta)\zeta\|_{H_\delta^{\frac{5}{2}+d+\iota}} \leq \|n^{-1/2}G_\zeta(n^{1/2}[-\alpha \times (\tilde{\nabla} \times R_\zeta) - (R_\zeta \cdot \nabla)\alpha - \nabla\alpha \cdot \eta] + k^2(1-n)(\eta + R_\zeta))\|_{H_\delta^{\frac{5}{2}+d+\iota}} + O\left(\frac{1}{|\zeta|^2}\right). \quad (3.3.25)$$

$$\begin{aligned} &= O\left(\frac{1}{|\zeta|}\right) \end{aligned} \quad (3.3.26)$$

This complete the proof. □

By Sobolev embedding theorem, we have the estimate in $C^{d+1}(\bar{\Omega})$.

Corollary 3.3.6. *Let Ω be an open, bounded domain in \mathbb{R}^3 . With the same hypotheses as the previous proposition, we then have*

$$\|R_\zeta - in^{-1/2}G_\zeta(n^{1/2}\alpha \cdot \eta)\zeta\|_{C^{d+1}(\bar{\Omega})} \leq \frac{C}{|\zeta|}, \quad (3.3.27) \quad \boxed{\text{Maxwell119}}$$

for some positive constant C .

thm:CG02 **Proposition 3.3.7.** *Suppose $\zeta \in \mathbb{C}^3 \setminus \mathbb{R}^3$, $\eta \in \mathbb{C}^3$, satisfy $\zeta \cdot \zeta = k^2$ and $\zeta \cdot \eta = 0$ such that as $|\zeta| \rightarrow \infty$ the limits $\zeta/|\zeta|$ and η exist and,*

$$|\zeta/|\zeta| - \zeta_0| = O\left(\frac{1}{|\zeta|}\right), \quad |\eta - \eta_0| = O\left(\frac{1}{|\zeta|}\right). \quad (3.3.28) \quad \boxed{\text{prop_limit}}$$

$R_\zeta \in H_\delta^{\frac{5}{2}+d+\iota}(\mathbb{R}^3)$ is the unique solution to (3.3.13) in Proposition 3.3.5. For $|\zeta|$ large,

$$\|R_\zeta - i|\zeta|n^{-1/2}G_\zeta(n^{1/2}\alpha \cdot \eta_0)\zeta_0\|_{C^{d+1}(\bar{\Omega})} = O\left(\frac{1}{|\zeta|}\right).$$

Proof. The proof follows directly by substituting (3.3.28) into (3.3.27). □

We choose the specific sets of ζ, η as in [12]. Precisely, Let h be a small real parameter and choose arbitrary $a \in \mathbb{R}$. We define ζ_1, ζ_2 and η_1, η_2 by

$$\begin{aligned} \zeta_1 &= (a/2, i\sqrt{1/h^2 + a^2/4 - k^2}, 1/h), \\ \zeta_2 &= (a/2, -i\sqrt{1/h^2 + a^2/4 - k^2}, -1/h), \\ \eta_1 &= \frac{1}{\sqrt{1/h^2 + a^2}}(1/h, 0, -a/2), \\ \eta_2 &= \frac{1}{\sqrt{1/h^2 + a^2}}(1/h, 0, a/2), \end{aligned} \quad (3.3.29) \quad \boxed{\text{eq:zetaeta}}$$

and note that

$$\begin{aligned}\lim_{h \rightarrow 0} \eta_j &= \eta_0 := (1, 0, 0), \quad j = 1, 2, \\ \lim_{h \rightarrow 0} \zeta_1 / |\zeta_1| &= \zeta_0 := \frac{1}{\sqrt{2}}(0, i, 1), \\ \lim_{h \rightarrow 0} \zeta_2 / |\zeta_2| &= -\zeta_0,\end{aligned}$$

and

$$\zeta_1 + \zeta_2 = (a, 0, 0), \quad \zeta_0 \cdot \zeta_0 = 0, \quad \eta_0 \cdot \zeta_0 = 0.$$

Proposition 3.3.7 implies that

$$(\eta_1 + R_{\zeta_1}) \cdot (\eta_2 + R_{\zeta_2}) = 1 + o(1)$$

in $C^{d+1}(\bar{\Omega})$.

3.3.3 Construction of Vector Fields and Uniqueness

Let us now consider the reconstruction of (L, σ) . Assume E_j , $j = 1, 2$, be two complex solutions to

$$\nabla \times \nabla \times E_j - k^2 n E_j = 0 \text{ in } \Omega, \quad (3.3.30) \quad \boxed{\text{Maxwell_E_j}}$$

with the tangential boundary conditions

$$tE_j = G_j \text{ on } \partial\Omega, \quad (3.3.31) \quad \boxed{\text{Maxwell_E_bd}}$$

with G_j well-chosen boundary values and $j = 1, 2$. We will see that

$$\nabla \times \nabla \times E_1 \cdot E_2 - \nabla \times \nabla \times E_2 \cdot E_1 = 0. \quad (3.3.32) \quad \boxed{\text{eq:E1_E2}}$$

Let $D_j = LE_j$, $j = 1, 2$, be the internal complex-valued measurements. Assume $L \in C^{d+1}(\bar{\Omega})$ is non-vanishing, then $D_j = LE_j \in C^{d+1}(\bar{\Omega})$ since $E_j \in C^{d+1}(\bar{\Omega})$. Substituting $E_j = D_j/L$ in (3.3.32), we have, after some algebraic calculation,

$$\beta \cdot \nabla L + \gamma L = 0, \quad (3.3.33) \quad \boxed{\text{eq:vector_fi}}$$

where

$$\begin{aligned}\beta &= \chi(x) \{ [(\nabla D_1) D_2 - (\nabla D_2) D_1] + [(\nabla \cdot D_1) D_2 - (\nabla \cdot D_2) D_1] \\ &\quad - 2[(\nabla D_1)^T D_2 - (\nabla D_2) D_1] \},\end{aligned} \quad (3.3.34)$$

$$\gamma = \chi(x) \{ [\nabla(\nabla \cdot D_1) \cdot D_2 - \nabla(\nabla \cdot D_2) \cdot D_1] - [\nabla^2 D_1 \cdot D_2 - \nabla^2 D_2 \cdot D_1] \}. \quad (3.3.35)$$

Here, $\chi(x)$ is a smooth known complex-valued function with $|\chi(x)|$ uniformly bounded from below by a positive constant on $\bar{\Omega}$.

To show the transport equation (3.3.33) has a unique solution, it suffices to prove that the direction of the vector field β is close to a vector field which has fixed direction and thus the integral curves of β connects every internal point to two boundary points.

Let \check{E}_1, \check{E}_2 be two CGO solutions with parameters ζ_1, ζ_2 and η_1, η_2 defined in (3.3.29), i.e.,

$$\check{E}_j = e^{i\zeta_j \cdot x}(\eta_j + R_{\zeta_j}), \quad j = 1, 2.$$

Let $\check{D}_j = L\check{E}_j$, $j = 1, 2$, be the corresponding internal data. By choosing $\chi(x) = -e^{-i(\zeta_1 + \zeta_2) \cdot x} \frac{h}{4\sqrt{2}}$ and substituting \check{D}_j into (3.3.34), we can analyze the asymptotic behavior of the vector field $\check{\beta}$ as $|\zeta_j| \rightarrow \infty$, or equivalently, $h \rightarrow 0$. Indeed, we have

$$\nabla \check{D}_j = e^{i\zeta_j \cdot x} [(\eta_j + R_{\zeta_j})(\nabla L)^T + i\mu(\eta_j + R_{\zeta_j})\zeta_j^T + \mu\nabla(\eta_j + R_{\zeta_j})], \quad j = 1, 2,$$

and

$$\begin{aligned} & \chi(x)(\nabla \check{D}_1)\check{D}_2 \\ &= -\frac{Lh}{4\sqrt{2}} [(\eta_1 + R_{\zeta_1})(\nabla L)^T + iL(\eta_1 + R_{\zeta_1})\zeta_1^T + \mu\nabla(\eta_1 + R_{\zeta_1})](\eta_2 + R_{\zeta_2}) \\ &= -\frac{Lh}{4\sqrt{2}} [(\eta_1 + R_{\zeta_1})[\nabla L^T(\eta_2 + R_{\zeta_2}) + iL\zeta_1^T(\eta_2 + R_{\zeta_2})] \\ & \quad + \mu\nabla(\eta_1 + R_{\zeta_1})(\eta_2 + R_{\zeta_2})] \\ &= -i\frac{L^2h}{4\sqrt{2}}(\eta_1 + R_{\zeta_1})[\zeta_1 \cdot (\eta_2 + R_{\zeta_2})] + O(h). \end{aligned} \tag{3.3.36}$$

Therefore, by Proposition 3.3.7, $\chi(x)(\nabla \check{D}_1)\check{D}_2 \rightarrow 0$ in $C^d(\bar{\Omega})$ norm as $h \rightarrow 0$. Similarly,

$$\begin{aligned} & \chi(x)(\nabla \check{D}_1)^T \check{D}_2 \\ &= -\frac{Lh}{4\sqrt{2}} [\nabla L(\eta_1 + R_{\zeta_1})^T(\eta_2 + R_{\zeta_2}) + iL\zeta_1(\eta_1 + R_{\zeta_1})^T(\eta_2 + R_{\zeta_2}) \\ & \quad + L\nabla(\eta_1 + R_{\zeta_1})^T(\eta_2 + R_{\zeta_2})] \\ &= -iL^2\frac{\zeta_1 h}{4\sqrt{2}}(\eta_1 + R_{\zeta_1}) \cdot (\eta_2 + R_{\zeta_2}) + O(h) \\ &\rightarrow -\frac{iL^2\zeta_0}{4} \quad \text{in } C^d(\bar{\Omega}) \quad \text{as } h \rightarrow 0. \end{aligned} \tag{3.3.37}$$

More calculation gives that

$$\begin{aligned}
& \chi(x)(\nabla \cdot \check{D}_1)\check{D}_2 \\
= & -\frac{L}{4\sqrt{2}c}[(\nabla\mu \cdot (\eta_1 + R_{\zeta_1}))(\eta_2 + R_{\zeta_2}) + iL(\zeta_1 \cdot (\eta_1 + R_{\zeta_1}))(\eta_2 + R_{\zeta_2}) \\
& + L\nabla \cdot (\eta_1 + R_{\zeta_1})(\eta_2 + R_{\zeta_2})] \quad (3.3.38) \\
\rightarrow & 0 \quad \text{in } C^d(\bar{\Omega}) \quad \text{as } h \rightarrow 0.
\end{aligned}$$

By substituting (3.3.36), (3.3.37) and (3.3.38) into (3.3.34), we have

$$\lim_{h \rightarrow 0} \|\check{\beta} - iL^2\zeta_0\|_{C^d(\bar{\Omega})} = 0, \quad (3.3.39) \quad \boxed{\text{eq:beta4}}$$

i.e., the vector fields have approximately constant directions for small h and their integral curves connect every internal point to two boundary points. Thus, the transport equation (3.3.33) admits a unique solution.

To see the dependence of vector fields on the boundary conditions, we need to introduce a regularity theorem of Maxwell's equations. Let tE be the tangential boundary condition of E . The following function spaces were introduced in [31].

Define the Div-spaces as

$$\begin{aligned}
H_{\text{Div}}^s(\Omega) &= \{u \in H^s\Omega^1(\Omega) : \text{Div}(tu) \in H^{s-1/2}(\partial\Omega)\}, \\
TH_{\text{Div}}^s(\partial\Omega) &= \{g \in H^s\Omega^1(\partial\Omega) : \text{Div}(g) \in H^s(\partial\Omega)\},
\end{aligned}$$

where $H^s\Omega^1(\Omega)$ is a space of vector functions of which each component is in $H^s(\Omega)$. These are Hilbert spaces with norms

$$\begin{aligned}
\|u\|_{H_{\text{Div}}^s(\Omega)} &= \|u\|_{H^s(\Omega)} + \|\text{Div}(tu)\|_{H^{s-1/2}(\partial\Omega)}, \\
\|g\|_{TH_{\text{Div}}^s(\partial\Omega)} &= \|g\|_{H^s(\partial\Omega)} + \|\text{Div}(g)\|_{H^s(\partial\Omega)}
\end{aligned}$$

It is clear that $t(H_{\text{Div}}^s(\Omega)) = TH_{\text{Div}}^{s-1/2}(\partial\Omega)$.

thm:regularity

Proposition 3.3.8 ([31]). *Let $\epsilon, \mu \in C^s$, $s > 2$, be positive functions. There is a discrete subset $\Sigma \subset \mathbb{C}$ such that if ω is outside this set, then one has a unique solution $E \in H_{\text{Div}}^d$ to (3.3.7) given any tangential boundary condition $G \in TH_{\text{Div}}^{s-1/2}(\partial\Omega)$. The solution satisfies*

$$\|E\|_{H_{\text{Div}}^s(\Omega)} \leq C\|G\|_{TH_{\text{Div}}^{s-1/2}(\partial\Omega)}$$

with C independent of G .

Note that when the tangential boundary condition is prescribed by CGO solutions, i.e., $\check{G}_j = t\check{E}_j$, $j = 1, 2$. By Proposition 3.3.8, E_j is the unique solution to (3.3.30) and (3.3.31). Then the corresponding vector field $\check{\beta}$ defined in (3.3.34) satisfies (3.3.39), which implies that the direction of $\check{\beta}$ is close to constant direction and thus its integral curves connect every internal point to two boundary points. Therefore, (3.3.33) admits a unique solution.

Furthermore, Proposition 3.3.8 also allows one to relax the boundary condition $\check{G}_j = t\check{E}_j$ and still to get the uniqueness of the solution to (3.3.33).

Proposition 3.3.9. *Under the assumption of Proposition 3.3.8, when G_j is in a neighborhood of $\check{G}_j = t\check{E}_j$ in $C^{d+4}(\partial\Omega)$, $j = 1, 2$, the corresponding vector field β defined in (3.3.34) satisfies*

$$\|\beta - iL^2\zeta_0\|_{C^d(\Omega)} = O(h), \quad (3.3.40) \quad \text{eq:beta5}$$

for small h .

Proof. By definition $\|E\|_{H^s(\Omega)} \leq \|E\|_{H_{Div}^s(\Omega)}$ and $\|G\|_{TH_{Div}^s(\partial\Omega)} \leq \|G\|_{H^{s+1}(\partial\Omega)}$. In particular, when $s = \frac{5}{2} + d + \iota$, from Sobolev embedding theorem and Proposition 3.3.8 we have that

$$\begin{aligned} \|E\|_{C^{d+1}(\bar{\Omega})} &\leq C\|E\|_{H^{\frac{5}{2}+d+\iota}(\Omega)} \leq C\|E\|_{H_{Div}^{\frac{5}{2}+d+\iota}(\Omega)} \\ &\leq C\|G\|_{TH_{Div}^{d+2+\iota}(\partial\Omega)} \leq C\|G\|_{H^{d+3+\iota}(\partial\Omega)} \leq C\|G\|_{C^{d+4}(\partial\Omega)}, \end{aligned} \quad (3.3.41)$$

where various constants are all named ‘‘C’’. Hence

$$\|E\|_{C^{d+1}(\bar{\Omega})} \leq C\|G\|_{C^{d+4}(\partial\Omega)}. \quad (3.3.42) \quad \text{eq:regularity1}$$

Let us now define boundary conditions $G_j \in C^{d+4}(\partial\Omega)$, $j = 1, 2$, such that

$$\|G_j - t\check{E}_j\|_{C^{d+4}(\partial\Omega)} \leq \varepsilon, \quad (3.3.43) \quad \text{eq:regularity2}$$

for some $\varepsilon > 0$ sufficiently small. let E_j be the solution to (3.3.7) and (3.3.8) with $tE_j = G_j$.

By (3.3.42), we thus have

$$\|E_j - \check{E}_j\|_{C^{d+1}(\bar{\Omega})} \leq C\varepsilon, \quad (3.3.44) \quad \text{eq:regularity3}$$

for some positive constant C . Define the complex valued internal data $D_j = LE_j$. We deduce that

$$\|D_j - \check{D}_j\|_{C^{d+1}(\bar{\Omega})} \leq C\varepsilon. \quad (3.3.45) \quad \text{eq:regularity4}$$

Define β by (3.3.34). We can easily deduce (3.3.40) from (3.3.39) and (3.3.45). This finishes the proof. \square

Recall \mathcal{M} is the parameter space of (L, σ) defined in (3.3.3) and h is the parameter in (3.3.29). We are in the place to prove Theorem 3.3.1.

Proof of Theorem 3.3.1. By Proposition 3.3.9, we choose the set of illuminations as a neighborhood of $(\check{G}_j) = (t\check{E}_j)$ in $(C^{d+4}(\partial\Omega))^2$. Since the measurements $D = \check{D}$, we have that L and \check{L} solve the same transport equation (3.3.33) while $L = \check{L} = D/G$ on $\partial\Omega$. As β satisfies (3.3.40), we deduce that $L = \check{L}$ since the integral curves of β map any $x \in \Omega$ to the boundary $\partial\Omega$. More precisely, consider the flow $\theta_x(t)$ associated to β , i.e., the solution to

$$\dot{\theta}_x(t) = \beta(\theta_x(t)), \quad \theta(0) = x \in \bar{\Omega}. \quad (3.3.46) \quad \text{eq:beta6}$$

By the Picard-Lindelöf theorem, (3.3.46) admits a unique solution since β is of class $C^1(\Omega)$. For $x \in \Omega$, let $x_{\pm}(x) \in \partial\Omega$ and $t_{\pm}(x) > 0$ such that

$$\theta_x(t_{\pm}(x)) = x_{\pm}(x) \in \partial\Omega. \quad (3.3.47) \quad \text{eq:beta7}$$

By the method of characteristics, the solution L to the transport equation (3.3.33) is given by

$$L(x) = L_0(x_{\pm}(x))e^{-\int_0^{t_{\pm}(x)} \gamma(\theta_x(s))ds} \quad (3.3.48) \quad \text{eq:beta8}$$

where $L_0 := L|_{\partial\Omega}$ is the restriction of L on the boundary. The solution \check{L} is given by the same formula since $\theta_x(t) = \check{\theta}_x(t)$. This implies $E_j = \check{E}_j = D_j/L$, $j = 1, 2$. By the choice of illuminations, we have $|E_j| \neq 0$ due to (3.3.44) and $|\check{E}_j| \neq 0$. Under the assumption that $D_j = \check{D}_j$, we have $E_j = \check{E}_j$, $j = 1, 2$. Therefore, $k^2n = k^2\check{n}$ by (3.3.30) and thus $\sigma = \check{\sigma}$. \square

3.3.4 Stability with 2 Complex Internal Measurements

Recall that $\theta_x(t)$ is the flow associated with β . From the equality

$$\theta_x(t) - \check{\theta}_x(t) = \int_0^t [\beta(\theta_x(s)) - \check{\beta}(\check{\theta}_x(s))]ds, \quad (3.3.49) \quad \text{eq:beta_9}$$

and using the Lipschitz continuity of β and Gronwall's lemma, we deduce the existence of a constant C such that

$$|\theta_x(t) - \check{\theta}_x(t)| \leq Ct\|\beta - \check{\beta}\|_{C^0(\bar{\Omega})}, \quad (3.3.50) \quad \text{eq:beta10}$$

when $\theta_x(t)$ and $\tilde{\theta}_x(t)$ are in $\bar{\Omega}$. The inequality (3.3.50) is uniform in t as all characteristics exit $\bar{\Omega}$ in finite time.

To see higher order estimates, we define $W := D_x\theta_x(t)$, which solves the equation, $\dot{W} = D_x\beta(\theta_x)W$, with $W(0) = I$. Define \tilde{W} similarly. By using Gronwall's lemma again, we deduce that

$$|W - \tilde{W}| \leq Ct \|D_x\beta - D_x\tilde{\beta}\|_{C^0(\bar{\Omega})} \quad (3.3.51) \quad \boxed{\text{eq:beta11}}$$

when $\theta_x(t)$ and $\tilde{\theta}_x(t)$ are in $\bar{\Omega}$. Since β and $\tilde{\beta}$ are of class $C^d(\bar{\Omega})$, then we obtain iteratively that

$$|D_x^{d-1}\theta_x(t) - D_x^{d-1}\tilde{\theta}_x(t)| \leq Ct \|D_x\beta - D_x\tilde{\beta}\|_{C^{d-1}(\bar{\Omega})}, \quad (3.3.52) \quad \boxed{\text{eq:beta12}}$$

when $\theta_x(t)$ and $\tilde{\theta}_x(t)$ are in $\bar{\Omega}$.

Recall that Ω_1 is defined to be the subset of Ω by removing a neighborhood of each tangent point of $\partial\Omega$ with respect to ζ_0 .

thm:stab1 **Lemma 3.3.10.** *Let Ω be an open bounded and convex subset in \mathbb{R}^3 with C^d boundary. Let $d \geq 2$ and assume β and $\tilde{\beta}$ are $C^d(\bar{\Omega})$ vector fields which satisfy (3.3.40). Restricting to Ω_1 , we have that*

$$\|x_+ - \tilde{x}_+\|_{C^{d-1}(\bar{\Omega}_1)} + \|t_+ - \tilde{t}_+\|_{C^{d-1}(\bar{\Omega}_1)} \leq C \|\beta - \tilde{\beta}\|_{C^{d-1}(\bar{\Omega}_1)}, \quad (3.3.53) \quad \boxed{\text{eq:beta13}}$$

where C is a constant depending on Ω .

This lemma is similar to the lemma 3.8 in [6] and the lemma 4.1 in [10], but uses a different proof.

Proof. For $x \in \Omega_1$, let $\theta_x(t)$ and $\tilde{\theta}_x(t)$ be two flows associated to vector fields β and $\tilde{\beta}$, respectively. Denote $A := \theta_x(t_+(x)) \in \partial\Omega$ and $B := \tilde{\theta}_x(\tilde{t}_+(x)) \in \partial\Omega$. Without loss of generality, we assume $t_+(x) \leq \tilde{t}_+(x)$. We also denote $C := \tilde{\theta}_x(t_+(x)) \in \Omega$. As in Fig. 3.1, we connect points by line segments, which approximately indicate the integral curves of β and $\tilde{\beta}$. Also notice that point C does not necessarily lie on the line xB or in the plane AxB . To simplify the writing, let $\delta := \|\beta - \tilde{\beta}\|_{C^{d-1}(\bar{\Omega}_1)}$.

We first want to show that the angle $\angle AxB$ is controlled by

$$\angle AxB \leq C_1\delta + C_2h, \quad (3.3.54) \quad \boxed{\text{eq:beta14}}$$

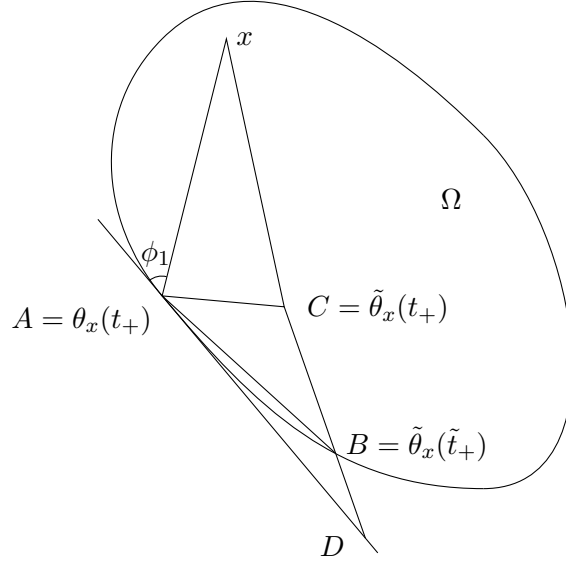
Figure 3.1: Vector fields β and $\tilde{\beta}$

fig:vector_f

for some C_1, C_2 . Indeed, by applying (3.3.50) and sine theorem, we can see that $\angle AxC$ is bounded by $C_1\delta$. Also notice that $\tilde{\beta}$ satisfies (3.3.40). Therefore, similar argument shows that, for any t_1, t_2 , the angle between the vector from x to $\tilde{\theta}_x(t_1)$ and the vector from x to $\tilde{\theta}_x(t_2)$ is bounded by C_2h . Thus $\angle CxB \leq C_2h$. This proves (3.3.54).

By the definition of Ω_1 , a neighborhood of the boundary point at which the tangent plane of $\partial\Omega$ is parallel to ζ_0 is removed. therefore, there exists a constant value $\phi_0 > 0$ depending only on Ω_1 such that, for any $x \in \Omega_1$, $\phi_1 \geq \phi_0$, where ϕ_1 is the angle between the vector \overline{xA} and the tangent plane of $\partial\Omega$ at A , as in Fig. 3.1. Then by (3.3.54), when δ and h are so small that $\phi'_0 := \phi_0 - C_1\delta - C_2h > 0$, the extension of \overline{CB} will intersect the tangent plane of $\partial\Omega$ at A , with intersection point D . Then it is easy to check that

$$\angle ABC > \angle ADx = \phi_1 - \angle AxB > \phi'_0 > 0. \quad (3.3.55) \quad \text{eq:beta15}$$

The sine theorem gives that

$$|AB| = \frac{|AC|}{\sin(\angle ABC)} \sin(\angle ACB). \quad (3.3.56) \quad \text{eq:beta16}$$

(3.3.50) directly implies $|AB| = |x_+ - \tilde{x}_+| \leq C'\delta$.

Since $\beta, \tilde{\beta} \in C^d(\Omega)$ and $\partial\Omega$ is of class C^d , it is clear that $\angle ABC$ and $\angle ACB$ are C^d functions with respect to $x \in \Omega$. By differentiating (3.3.56) and applying (3.3.52), we get higher order estimates

$$\|x_+ - \tilde{x}_+\|_{C^{d-1}(\bar{\Omega}_1)} \leq C''\delta. \quad (3.3.57) \quad \text{eq:beta17}$$

To see the second part in (3.3.53), we have that

$$|CB| = \int_{t_+(x)}^{\tilde{t}_+(x)} \tilde{\beta}(\tilde{\theta}_x(d)) ds = |\tilde{\beta}(\tilde{\theta}_x(\tau))|(\tilde{t}_+(x) - t_+(x)), \quad (3.3.58) \quad \text{eq:beta18}$$

for $t_+(x) \leq \tau \leq \tilde{t}_+(x)$. Similar argument shows the estimate of $t_+ - \tilde{t}_+$ in (3.3.53). \square

thm:stab2 **Proposition 3.3.11.** *Let $d \geq 1$. Let L and \tilde{L} be solutions to (3.3.33) corresponding to coefficients (β, γ) and $(\tilde{\beta}, \tilde{\gamma})$, respectively, where (3.3.40) holds for both β and $\tilde{\beta}$.*

Let $L_0 = L|_{\partial\Omega}$ and $\tilde{L}_0 = \tilde{L}|_{\partial\Omega}$, thus $L_0, \tilde{L}_0 \in C^d(\partial\Omega)$. We also assume h is sufficiently small and Ω is convex. Then there is a constant C such that restricting to Ω_1

$$\begin{aligned} \|L - \tilde{L}\|_{C^{d-1}(\bar{\Omega}_1)} \leq & C\|L_0\|_{C^{d-1}(\partial\Omega_1)}[\|\beta - \tilde{\beta}\|_{C^{d-1}(\bar{\Omega}_1)} \\ & + \|\gamma - \tilde{\gamma}\|_{C^{d-1}(\bar{\Omega}_1)}] + C\|L_0 - \tilde{L}_0\|_{C^{d-1}(\partial\Omega_1)} \end{aligned} \quad (3.3.59) \quad \text{eq:stable1}$$

The proof is omitted as it follows exactly the proof of Proposition 4.2 in [10]. Now we can prove the main stability theorem.

Proof of Theorem 3.3.2. From (3.3.34) (3.3.35) it is easy to check that

$$\|\beta - \tilde{\beta}\|_{C^{d-1}(\bar{\Omega}_1)} \leq C\|D - \tilde{D}\|_{C^d(\bar{\Omega}_1)} \quad \|\gamma - \tilde{\gamma}\|_{C^{d-1}(\bar{\Omega}_1)} \leq C\|D - \tilde{D}\|_{C^{d+1}(\bar{\Omega}_1)}$$

where $C > 0$ is a positive constant. The first part then follows directly from Proposition 3.3.11. To estimate the difference between σ and $\tilde{\sigma}$, notice that

$$E - \tilde{E} = \frac{D}{L} - \frac{\tilde{D}}{\tilde{L}} = \frac{L(D - \tilde{D}) - D(L - \tilde{L})}{L\tilde{L}}.$$

Since L and \tilde{L} are non-vanishing, by the stability result for L we obtain

$$\|E - \tilde{E}\|_{C^{d-1}(\bar{\Omega}_1)} \leq C\|D - \tilde{D}\|_{C^{d+1}(\bar{\Omega}_1)}. \quad (3.3.60) \quad \text{Estability}$$

By choosing the boundary illuminations close to the boundary conditions of CGO solutions, (3.3.43) and (3.3.44) imply that E_j is non-vanishing since the CGO solutions are non-vanishing. From (3.3.7) we have

$$n = \frac{1}{k^2} \frac{(-\Delta E + \nabla \nabla \cdot E) \cdot \bar{E}}{|E|^2},$$

$$\tilde{n} = \frac{1}{k^2} \frac{(-\Delta \tilde{E} + \nabla \nabla \cdot \tilde{E}) \cdot \bar{\tilde{E}}}{|\tilde{E}|^2}.$$

By taking the difference and using (3.3.60) we derive

$$\|n - \tilde{n}\|_{C^{d-3}(\bar{\Omega}_1)} \leq \|D - \tilde{D}\|_{C^{d+1}(\bar{\Omega}_1)}.$$

Similar stability holds for σ since $\sigma = \epsilon_0 w \operatorname{Im} n$. □

3.3.5 Stability with 6 Complex Internal Measurements

Rather than applying the characteristics method to (3.3.33), we can rewrite (3.3.33) into matrix form by introducing more internal measurements. We first construct proper CGO solutions. Let $j = 1, 2, 3$ in this section. We can choose unit vectors ζ_0^j and η_0^j , such that $\zeta_0^j \cdot \zeta_0^j = 0$, $\zeta_0^j \cdot \eta_0^j = 0$ and $\{\zeta_0^j\}$ are linearly independent. Also, choose (ζ_1^j, ζ_2^j) and (η_1^j, η_2^j) such that $|\zeta| := |\zeta_1^j| = |\zeta_2^j|$,

$$\lim_{|\zeta| \rightarrow \infty} \frac{\zeta_1^j}{|\zeta|} = \lim_{|\zeta| \rightarrow \infty} \frac{\zeta_2^j}{|\zeta|} = \zeta_0^j, \quad \text{and} \quad \lim_{|\zeta| \rightarrow \infty} \eta^j = \eta_0^j. \quad (3.3.61) \quad \boxed{\text{eq:stable4}}$$

We construct CGO solutions $\check{E}_1^j, \check{E}_2^j$ corresponding to (ζ_1^j, η_1^j) and (ζ_2^j, η_2^j) . Let the boundary illuminations G_1^j, G_2^j be chosen according to (3.3.43) for ε small enough. The measured internal data are then given by D_1^j, D_2^j . Proposition 3.3.8 shows that the vector field defined by (3.3.34) satisfies that

$$\|\beta^j - L^2 \zeta_0^j\|_{C^d(\bar{\Omega})} \leq \frac{C}{|\zeta|}. \quad (3.3.62) \quad \boxed{\text{eq:stable5}}$$

While $|\zeta|$ is sufficiently large and $L \neq 0$ on $\bar{\Omega}$, we obtain that the vector $\{\beta^j(x)\}$ are linear independent at every $x \in \Omega$. Thus matrix $(\beta^j(x))$ is invertible with inverse of class $C^d(\bar{\Omega})$.

By constructing vector-valued function $\Gamma(x) \in (C^d(\bar{\Omega}))^3$, the transport equation (3.3.33) now becomes the matrix equation

$$\nabla L + \Gamma(x)L = 0. \quad (3.3.63) \quad \boxed{\text{eq:stable6}}$$

Notice that $\Gamma(x)$ is stable under small perturbations in the data $D := (D_1^j, D_2^j) \in (C^{d+1}(\bar{\Omega}))^6$, i.e.,

$$\|\Gamma - \tilde{\Gamma}\|_{(C^d(\bar{\Omega}))^3} \leq C\|D - \tilde{D}\|_{(C^{d+1}(\bar{\Omega}))^6}. \quad (3.3.64) \quad \boxed{\text{eq:stable7}}$$

Assume Ω is connected and $L_0 = L|_{\partial\Omega}$ is known. Choose a smooth curve from $x \in \Omega$ to a point on the boundary. Restricting to the curve, (3.3.63) is a stable ordinary differential equation. Keep the curve fixed. Let L and \tilde{L} be solutions to (3.3.63) with respect to Γ and $\tilde{\Gamma}$, respectively. By solving the equation explicitly and (3.3.64), we find that

$$\|L - \tilde{L}\|_{C^d(\bar{\Omega})} \leq C\|D - \tilde{D}\|_{(C^{d+1}(\bar{\Omega}))^6}. \quad (3.3.65) \quad \boxed{\text{eq:stable8}}$$

Proof of Theorem 3.3.3. The first result in (3.3.6) is directly from (3.3.65). The proof of the stability of σ is exactly the same as in the proof of theorem 3.3.2. \square

Chapter 4

IBVP FOR THE BI-HARMONIC OPERATOR

In this chapter we consider several inverse boundary value problems for the elliptic operator Δ^2 , which is usually referred to as the bi-harmonic operator.

4.1 Introduction

The bi-harmonic operator arises in physics when considering the equilibrium configuration of an elastic plate hinged along the boundary. It is also widely used in other physical models, see [21]. We will concentrate on the identifiability of the first order perturbation of a bi-harmonic operator from partial boundary measurements. A bi-harmonic operator with first order perturbation is a differential operator of the form

$$\mathcal{L}_{A,q}(x, D) := \Delta^2 + A(x) \cdot D + q(x)$$

with $D = \frac{1}{i}\nabla$. Here A is a complex-valued vector field called the magnetic potential, q is a complex-valued function called the electric potential. We will study inverse boundary value problems on two types of open subsets of \mathbb{R}^n , the first type is an infinite slab, and the second type is a bounded domain with C^∞ boundary.

First we consider an infinite slab Σ . The geometry of an infinite slab arises in many applications, for instance, in the study of wave propagation in marine acoustics. It is also a simple geometric setting in medical imaging. By choosing appropriate coordinates, we may assume that

$$\Sigma := \{x = (x', x_n) \in \mathbb{R}^n : x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}, 0 < x_n < L\}, \quad L > 0.$$

Its boundary consists of two parallel hyperplanes

$$\Gamma_1 := \{x \in \mathbb{R}^n : x_n = L\} \quad \Gamma_2 := \{x \in \mathbb{R}^n : x_n = 0\}.$$

Given $(f_1, f_2) \in H^{\frac{7}{2}}(\Gamma_1) \times H^{\frac{3}{2}}(\Gamma_1)$ with f_1, f_2 compactly supported on Γ_1 , we are interested in the following Dirichlet problem

$$\left\{ \begin{array}{lll} \mathcal{L}_{A,q}u = 0 & \text{in } \Sigma \\ u = f_1 \quad \Delta u = f_2 & \text{on } \Gamma_1 \\ u = 0 \quad \Delta u = 0 & \text{on } \Gamma_2. \end{array} \right. \quad (4.1.1) \quad \boxed{\text{Dirichlet1}}$$

In Appendix A we show that problem (4.1.1) has a unique solution in $H^4(\Sigma)$, where $H^4(\Sigma)$ is the standard Sobolev space on Σ . Let $\gamma_1 \subset \Gamma_1$, $\gamma_2 \subset \partial\Sigma$ be non-empty open subsets, we define the partial Cauchy data set for the boundary value problem (4.1.1) as

$$\begin{aligned} C_{A,q}^{\gamma_1, \gamma_2}(\Sigma) := & \{(u|_{\gamma_1}, \Delta u|_{\gamma_1}, \partial_\nu u|_{\gamma_2}, \partial_\nu(\Delta u)|_{\gamma_2}) : \mathcal{L}_{A,q}u = 0 \text{ in } \Sigma, u \in H^4(\Sigma), \\ & \text{supp}(u|_{\Gamma_1}) \text{ and } \text{supp}(\Delta u|_{\Gamma_1}) \text{ are compact and contained in } \gamma_1, \\ & u|_{\Gamma_2} = \Delta u|_{\Gamma_2} = 0\}. \end{aligned}$$

where ν is the unit outer normal vector field to $\partial\Sigma = \Gamma_1 \cup \Gamma_2$. Here we think of $(u|_{\gamma_1}, \Delta u|_{\gamma_1})$ as the Dirichlet data prescribed only on γ_1 , and $\partial_\nu u|_{\gamma_2}, \partial_\nu(\Delta u)|_{\gamma_2}$ as the Neumann data measured on γ_2 . The inverse problem we will study is as follows: for fixed γ_1 and γ_2 , assuming that

$$C_{A^{(1)}, q^{(1)}}^{\gamma_1, \gamma_2}(\Sigma) = C_{A^{(2)}, q^{(2)}}^{\gamma_1, \gamma_2}(\Sigma),$$

can we conclude $A^{(1)} = A^{(2)}$ and $q^{(1)} = q^{(2)}$ in Σ ?

We will show this is valid for some open subsets γ_1, γ_2 , if $A^{(j)}, q^{(j)}, j = 1, 2$ are compactly supported in the closure $\bar{\Sigma}$. Our first result considers the case when the data and the measurements are on different boundary hyperplanes.

Theorem 4.1.1. *Let $\Sigma \subset \mathbb{R}^n (n \geq 3)$ be an infinite slab with boundary hyperplanes Γ_1 and Γ_2 . Let $A^{(j)} \in W^{1,\infty}(\Sigma; \mathbb{C}^n) \cap \mathcal{E}'(\bar{\Sigma}; \mathbb{C}^n)$, $q^{(j)} \in L^\infty(\Sigma; \mathbb{C}) \cap \mathcal{E}'(\bar{\Sigma}; \mathbb{C}), j = 1, 2$. Denote by $B \subset \mathbb{R}^n$ an open ball containing the supports of $A^{(j)}, q^{(j)}, j = 1, 2$. Let $\gamma_j \subset \Gamma_j$ be open sets such that $\Gamma_j \cap \bar{B} \subset \gamma_j, j = 1, 2$. If*

$$C_{A^{(1)}, q^{(1)}}^{\gamma_1, \gamma_2}(\Sigma) = C_{A^{(2)}, q^{(2)}}^{\gamma_1, \gamma_2}(\Sigma),$$

then $A^{(1)} = A^{(2)}$ and $q^{(1)} = q^{(2)}$ in Σ .

We would like to remark that when the supports of $A^{(j)}, q^{(j)}$ are strictly contained in the interior of the slab, then γ_1 and γ_2 in the above theorem can be chosen to be arbitrarily small.

Our next result considers the case when the Dirichlet data and the Neumann data are on the same boundary hyperplane.

Theorem 4.1.2. *Let $\Sigma \subset \mathbb{R}^n (n \geq 3)$ be an infinite slab between two parallel hyperplanes Γ_1 and Γ_2 . Let $A^{(j)} \in W^{1,\infty}(\Sigma; \mathbb{C}^n) \cap \mathcal{E}'(\bar{\Sigma}; \mathbb{C}^n)$, $q^{(j)} \in L^\infty(\Sigma; \mathbb{C}) \cap \mathcal{E}'(\bar{\Sigma}; \mathbb{C})$, $j = 1, 2$. Denote by $B \subset \mathbb{R}^n$ an open ball containing the supports of $A^{(j)}, q^{(j)}$, $j = 1, 2$. Let $\gamma_1, \gamma'_1 \subset \Gamma_1$ be open sets such that $\Gamma_1 \cap \bar{B} \subset \gamma_1$ and $\Gamma_1 \cap \bar{B} \subset \gamma'_1$. If*

$$C_{A^{(1)}, q^{(1)}}^{\gamma_1, \gamma'_1}(\Sigma) = C_{A^{(2)}, q^{(2)}}^{\gamma_1, \gamma'_1}(\Sigma),$$

then $A^{(1)} = A^{(2)}$ and $q^{(1)} = q^{(2)}$ in Σ .

In Theorem 1.1 and Theorem 1.2, the assumptions on the partial Cauchy data sets can also be stated using the Dirichlet-to-Neumann map. The Dirichlet-to-Neumann map for the boundary value problem (4.1.1) is defined as

$$\begin{aligned} \Lambda_{A,q} : \quad (H^{\frac{7}{2}}(\Gamma_1) \cap \mathcal{E}'(\Gamma_1)) \times (H^{\frac{3}{2}}(\Gamma_1) \cap \mathcal{E}'(\Gamma_1)) &\rightarrow H_{loc}^{\frac{5}{2}}(\partial\Sigma) \times H_{loc}^{\frac{1}{2}}(\partial\Sigma) \\ (f_1, f_2) &\mapsto (\partial_\nu u|_{\partial\Sigma}, \partial_\nu(\Delta u)|_{\partial\Sigma}), \end{aligned}$$

where u is the solution of (4.1.1), $\mathcal{E}'(\Gamma_1)$ is the set of compactly supported distributions on Γ_1 . In fact if 0 is not an eigenvalue of $\Lambda_{A,q}$, then the partial Cauchy data set $C_{A,q}^{\gamma_1, \gamma_2}(\Sigma)$ is the restriction of the graph of $\Lambda_{A,q}$. In this case, the condition in Theorem 1.1 that

$$C_{A^{(1)}, q^{(1)}}^{\gamma_1, \gamma_2}(\Sigma) = C_{A^{(2)}, q^{(2)}}^{\gamma_1, \gamma_2}(\Sigma)$$

is equivalent to requiring

$$\Lambda_{A^{(1)}, q^{(1)}}(f_1, f_2)|_{\gamma_2 \times \gamma_2} = \Lambda_{A^{(2)}, q^{(2)}}(f_1, f_2)|_{\gamma_2 \times \gamma_2}$$

for all $(f_1, f_2) \in (H^{\frac{7}{2}}(\Gamma_1) \cap \mathcal{E}'(\Gamma_1)) \times (H^{\frac{3}{2}}(\Gamma_1) \cap \mathcal{E}'(\Gamma_1))$ with $\text{supp}(f_1) \subset \gamma_1$, $\text{supp}(f_2) \subset \gamma_1$. Similarly the hypothesis in Theorem 1.2 can be reformulated using the Dirichlet-to-Neumann map.

Proofs of Theorem 4.1.1 and Theorem 4.1.2 are based on the construction of a special class of complex geometric optics (CGO) solutions which vanish on appropriate boundary hyperplanes, using a reflection argument. The idea of constructing such solutions for the Schrödinger operator goes back to [50]. Constructing complex geometric optics solutions using a reflection argument was initiated in [29].

Inverse problems of identifying an embedded object in a slab have been studied by many authors in [28, 34, 39, 45]. In [39] the authors considered the Schrödinger operator $\Delta + q$ in a slab and showed that the electric potential q can be uniquely determined from partial boundary measurements. In [34] the authors considered the magnetic Schrödinger operator $\Delta + A(x) \cdot D + q$ and showed that the magnetic field dA and the electric potential q can be uniquely determined from partial boundary measurements. Here dA is the exterior differentiation of the magnetic potential vector field A , and notice that determining dA is equivalent to determining the equivalence class $\{\tilde{A} : \tilde{A} = A + \nabla\Phi \text{ for some } \Phi \in C^{1,1}(\bar{\Sigma})\}$. It was also pointed out in [34] that, by only looking at the Dirichlet-to-Neumann map, such a gauge transformation obstruction always exists, so the best one can hope for the magnetic Schrödinger operator is to determine dA . However, for the perturbed bi-harmonic operator, our results indicate that this type of obstruction can be overcome and one therefore determines not only dA , but also A itself. This is due to the fact that in our proof we are able to construct more CGO solutions than for the magnetic Schrödinger operator thanks to the higher order of the bi-harmonic operator.

In the remaining part of this section we shall discuss an inverse boundary value problem for the perturbed bi-harmonic operator on a bounded domain. Let $\Omega \subset \mathbb{R}^n (n \geq 3)$ be a bounded open subset with C^∞ boundary. Consider the Dirichlet problem

$$\begin{cases} \mathcal{L}_{A,q}u = 0 & \text{in } \Omega \\ u = f_1 & \text{on } \partial\Omega \\ \Delta u = f_2 & \text{on } \partial\Omega \end{cases} \quad (4.1.2) \quad \boxed{\text{Dirichlet2}}$$

with $A \in W^{1,\infty}(\Omega; \mathbb{C}^n)$, $q \in L^\infty(\Omega; \mathbb{C})$ and $(f_1, f_2) \in H^{\frac{7}{2}}(\partial\Omega) \times H^{\frac{3}{2}}(\partial\Omega)$. The operator $\mathcal{L}_{A,q}$,

equipped with the domain

$$\mathcal{D}(\mathcal{L}_{A,q}) := \{u \in H^4(\Omega) : u|_{\partial\Omega} = (\Delta u)|_{\partial\Omega} = 0\}$$

is an unbounded closed operator on $L^2(\Omega)$ with purely discrete spectrum, see [23].

Let γ_1, γ_2 be non-empty open subsets of the boundary $\partial\Omega$, we define the partial Cauchy data set to the boundary value problem (4.1.2) as

$$\begin{aligned} C_{A,q}^{\gamma_1, \gamma_2}(\Omega) := & \{(u|_{\gamma_1}, \Delta u|_{\gamma_1}, \partial_\nu u|_{\gamma_2}, \partial_\nu(\Delta u)|_{\gamma_2}) : \mathcal{L}_{A,q}u = 0 \text{ in } \Omega, u \in H^4(\Omega), \\ & \text{supp}(u|_{\partial\Omega}) \subset \gamma_1, \text{supp}(\Delta u|_{\partial\Omega}) \subset \gamma_1\} \end{aligned}$$

In this chapter we are interested in the inverse boundary value problem for the operator $\mathcal{L}_{A,q}$ with partial boundary measurements: assuming that

$$C_{A^{(1)}, q^{(1)}}^{\gamma_1, \gamma_2}(\Omega) = C_{A^{(2)}, q^{(2)}}^{\gamma_1, \gamma_2}(\Omega),$$

can we conclude that $A^{(1)} = A^{(2)}$ and $q^{(1)} = q^{(2)}$ in Ω ?

For the bi-harmonic operator, determination of the first order perturbation on a bounded domain Ω was considered in [36] with partial boundary measurements. The authors showed that, from the Dirichlet-to-Neumann map, one can uniquely determine not only the electric potential q , but also the magnetic potential A . Again this is different from the situation for the magnetic Schrödinger operator where the gauge transformation exists as an obstruction for the recovery of A . In this chapter, we will improve the uniqueness result in [36] under two different assumptions: in Theorem 4.1.3, we assume $A^{(1)} = A^{(2)}$ and $q^{(1)} = q^{(2)}$ in a neighborhood of $\partial\Omega$, and show that we still have the uniqueness even when both γ_1 and γ_2 are arbitrarily small; in Theorem 4.1.4, we assume the inaccessible part of the boundary is contained in a plane, and prove the uniqueness with local data.

Theorem 4.1.3. *Let $\Omega \subset \mathbb{R}^n (n \geq 3)$ be a bounded domain with C^∞ connected boundary. Let $A^{(j)} \in W^{1,\infty}(\Omega; \mathbb{C}^n)$, $q^{(j)} \in L^\infty(\Omega; \mathbb{C})$, $j = 1, 2$. Assume that $A^{(1)} = A^{(2)}$ and $q^{(1)} = q^{(2)}$ in a neighborhood of the boundary $\partial\Omega$. Let $\gamma_1, \gamma_2 \subset \partial\Omega$ be non-empty open subsets of $\partial\Omega$. If*

$$C_{A^{(1)}, q^{(1)}}^{\gamma_1, \gamma_2}(\Omega) = C_{A^{(2)}, q^{(2)}}^{\gamma_1, \gamma_2}(\Omega),$$

then $A^{(1)} = A^{(2)}$ and $q^{(1)} = q^{(2)}$ in Ω .

In the following theorem, notice that we need the magnetic potential A and electric potential q to be smooth. This is due to the fact that our proof relies on determination of the boundary value of A from the Cauchy data set $\mathcal{C}_{A,q}$. For the bi-harmonic operator, or more generally for poly-harmonic operators, this result was proved only for smooth A and q in [35].

Theorem 4.1.4. *Let $\Omega \subset \{x \in \mathbb{R}^n : x_n > 0\}$ ($n \geq 3$) be a bounded domain with C^∞ connected boundary, and let $\partial\Omega \cap \{x \in \mathbb{R}^n : x_n = 0\} \neq \emptyset$ and $\gamma := \partial\Omega \setminus \{x \in \mathbb{R}^n : x_n = 0\}$. Let $A^{(j)} \in C^\infty(\bar{\Omega}; \mathbb{C}^n)$, $q^{(j)} \in C^\infty(\bar{\Omega}; \mathbb{C})$, $j = 1, 2$. If*

$$C_{A^{(1)}, q^{(1)}}^{\gamma, \gamma}(\Omega) = C_{A^{(2)}, q^{(2)}}^{\gamma, \gamma}(\Omega),$$

then $A^{(1)} = A^{(2)}$ and $q^{(1)} = q^{(2)}$ in Ω .

This chapter is structured as follows: in Section 4.2 we establish a Carleman type estimate for the bi-harmonic operator and then construct a class of CGO solutions on a bounded domain; in Section 4.3 we show an integral identity and a Runge type approximation theorem; in Section 4.4 we construct the CGO solutions we desire in the slab by reflecting the CGO solutions constructed in Section 4.2; Section 4.5 is devoted to the proofs of the main theorems in this chapter. In Appendix A we prove the solvability of the boundary value problem (4.1.1) and some identities used in the proofs of the main theorems.

4.2 Carleman Estimate

In this section we construct some CGO solutions on a bounded domain to the equation $\mathcal{L}_{A,q}u = 0$. CGO solutions have been intensively utilized in establishing uniqueness result in elliptic inverse boundary value problems. For the construction of various CGO solutions and their application, see [5, 10, 11, 16, 32, 50].

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded domain with C^∞ boundary. Consider the equation $\mathcal{L}_{A,q}u = 0$ in Ω with $A \in W^{1,\infty}(\Omega; \mathbb{C}^n)$ and $q \in L^\infty(\Omega; \mathbb{C})$. We will construct CGO solutions of the form

$$u(x, \zeta, h) = e^{x \cdot \zeta / h} (a(x, \zeta) + r(x, \zeta, h)). \quad (4.2.1) \quad \boxed{\text{form}}$$

based on a Carleman estimate. Here $\zeta \in \mathbb{C}^n$ is a complex vector satisfying $\zeta \cdot \zeta = 0$, a is a smooth amplitude, r is a correction term, $h > 0$ is a small semiclassical parameter. To deal with the perturbation, we extend $A \in W^{1,\infty}(\Omega; \mathbb{C}^n)$ to a Lipschitz vector field compactly supported in \mathbb{R}^n , extend $q \in L^\infty(\Omega; \mathbb{C})$ as zero to \mathbb{R}^n . We shall work with ζ depending slightly on h , i.e. $\zeta = \zeta^{(0)} + \zeta^{(1)}$ with $\zeta^{(0)}$ independent of h , $\zeta^{(1)} = \mathcal{O}(h)$, and $|\operatorname{Re} \zeta^{(0)}| = |\operatorname{Im} \zeta^{(0)}| = 1$.

Consider the conjugated operator

$$h^4 e^{-x \cdot \zeta / h} \mathcal{L}_{A,q} e^{x \cdot \zeta / h} = (h^2 \Delta + 2ih\zeta \cdot \nabla)^2 + h^3 A \cdot hD - ih^3 A \cdot \zeta + h^4 q$$

In order to eliminate the lowest order term involving h in this expression, we require

$$(\zeta^{(0)} \cdot \nabla)^2 a = 0 \quad \text{in } \Omega. \quad (4.2.2) \quad \boxed{\text{eq_a}}$$

As $|\operatorname{Re} \zeta^{(0)}| = |\operatorname{Im} \zeta^{(0)}| = 1$, $\zeta^{(0)} \cdot \nabla$ is a $\bar{\partial}$ -operator in appropriate coordinates, so the above equation admits a solution $a = a(x, \zeta^{(0)}) \in C^\infty(\bar{\Omega})$. To find an appropriate correction term, we need a Carleman type estimate. We will use the semiclassical Sobolev spaces $H_{\text{scl}}^s(\mathbb{R}^n)$ ($s \in \mathbb{R}$) with the norm $\|f\|_{H_{\text{scl}}^s(\mathbb{R}^n)} = \|\langle hD \rangle^s f\|_{L^2(\mathbb{R}^n)}$ where $\langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$.

Carleman **Proposition 4.2.1.** *Suppose $A \in W^{1,\infty}(\Omega; \mathbb{C}^n)$, $q \in L^\infty(\Omega; \mathbb{C})$. Then for $h > 0$ sufficiently small, there exists a constant $C > 0$ independent of h such that*

$$\|u\|_{L^2(\mathbb{R}^n)} \leq Ch^2 \|e^{x \cdot \zeta / h} \mathcal{L}_{A,q} e^{-x \cdot \zeta / h} u\|_{H_{\text{scl}}^{-1}(\mathbb{R}^n)} \quad u \in C_c^\infty(\Omega).$$

Proof. From [33, Proposition 4.2], we can find a constant $C_1 > 0$ independent of h such that for all $u \in C_c^\infty(\Omega)$

$$\|u\|_{L^2(\mathbb{R}^n)} \leq C_1 h \|e^{x \cdot \zeta / h} \Delta e^{-x \cdot \zeta / h} u\|_{H_{\text{scl}}^{-1}(\mathbb{R}^n)}$$

Iterate to get

$$\|u\|_{L^2(\mathbb{R}^n)} \leq C_1^2 h^2 \|e^{x \cdot \zeta / h} \Delta^2 e^{-x \cdot \zeta / h} u\|_{H_{\text{scl}}^{-1}(\mathbb{R}^n)} \quad (4.2.3) \quad \boxed{\text{CE}}$$

We can add the zeroth order term $h^2 q$ to (4.2.3) since

$$h^2 \|qu\|_{H_{\text{scl}}^{-1}(\mathbb{R}^n)} \leq h^2 \|qu\|_{L^2(\mathbb{R}^n)} \leq h^2 \|q\|_{L^\infty(\mathbb{R}^n)} \|u\|_{L^2(\mathbb{R}^n)}.$$

We can add the first order term $h^2 e^{x \cdot \zeta / h} A \cdot D e^{-x \cdot \zeta / h}$ as

$$h^2 e^{x \cdot \zeta / h} A \cdot D e^{-x \cdot \zeta / h} = h(A \cdot hD + iA \cdot \zeta)$$

and

$$\begin{aligned} h \|A \cdot \zeta u\|_{H_{scl}^{-1}(\mathbb{R}^n)} &\leq h \|A \cdot \zeta u\|_{L^2(\mathbb{R}^2)} \leq h \|A \cdot \zeta\|_{L^\infty(\mathbb{R}^n)} \|u\|_{L^2(\mathbb{R}^n)}. \\ h \|A \cdot hD u\|_{H_{scl}^{-1}(\mathbb{R}^n)} &\leq h \sum_{j=1}^n \|hD_j(A_j u)\|_{H_{scl}^{-1}(\mathbb{R}^n)} + \mathcal{O}(h^2) \|(\operatorname{div} A)u\|_{H_{scl}^{-1}(\mathbb{R}^n)} \\ &\leq \mathcal{O}(h) \sum_{j=1}^n \|A_j u\|_{L^2(\mathbb{R}^n)} + \mathcal{O}(h^2) \|u\|_{L^2(\mathbb{R}^2)} \\ &\leq \mathcal{O}(h) \|u\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

After adding these perturbation terms, we get the desired result. \square

Denote $\|f\|_{H_{scl}^1(\Omega)}^2 := \|f\|_{L^2(\Omega)}^2 + h^2 \|\nabla f\|_{L^2(\Omega)}^2$. The following solvability result is an immediate consequence of the above Carleman estimate and the Hahn-Banach Theorem.

solvability

Proposition 4.2.2. *Suppose $A \in W^{1,\infty}(\Omega; \mathbb{C}^n)$, $q \in L^\infty(\Omega; \mathbb{C})$. Then for any $f \in L^2(\Omega)$, the equation*

$$e^{-x \cdot \zeta / h} \mathcal{L}_{A,q} e^{x \cdot \zeta / h} r = f \quad \text{in } \Omega$$

has a solution $r \in H^1(\Omega)$ with $\|r\|_{H_{scl}^1(\Omega)} \leq \mathcal{O}(h^2) \|f\|_{L^2(\Omega)}$.

Proof. We extend A to a compactly supported Lipschitz vector field in \mathbb{R}^n , extend q and f as zero, and solve the equation in \mathbb{R}^n . Denote $\mathcal{L}_\zeta := e^{-x \cdot \zeta / h} \mathcal{L}_{A,q} e^{x \cdot \zeta / h}$, the L^2 -adjoint of \mathcal{L}_ζ is given by

$$\mathcal{L}_\zeta^* := e^{x \cdot \zeta / h} \mathcal{L}_{A,q}^* e^{-x \cdot \zeta / h} = e^{x \cdot \zeta / h} \mathcal{L}_{\bar{A}, i^{-1} \nabla \cdot \bar{A} + \bar{q}} e^{-x \cdot \zeta / h}.$$

Consider the complex linear functional

$$L : \mathcal{L}_\zeta^* C_c^\infty(\Omega) \rightarrow \mathbb{C} \quad \mathcal{L}_\zeta^* u \mapsto (u, f)_{L^2(\mathbb{R}^n)}.$$

Applying Proposition 4.2.1 with $\mathcal{L}_{A,q}$ replaced by $\mathcal{L}_{A,q}^*$, we see the map L is well-defined and for any $u \in C_c^\infty(\Omega)$, we have

$$|L(\mathcal{L}_\zeta^* u)| = |(u, f)_{L^2(\mathbb{R}^n)}| \leq \|u\|_{L^2(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)} \leq Ch^2 \|\mathcal{L}_\zeta^* u\|_{H_{scl}^{-1}(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)}.$$

This shows that L is bounded in the $H^{-1}(\mathbb{R}^n)$ -norm. By the Hahn-Banach theorem we can extend L to a bounded linear functional \tilde{L} on $H^{-1}(\mathbb{R}^n)$ without increasing the norm. Thus, by Riesz representation theorem, there exists $r \in H^1(\mathbb{R}^n)$ such that for all $u \in H^{-1}(\mathbb{R}^n)$ we have

$$\tilde{L}(u) = (u, r)_{H^{-1}(\mathbb{R}^n), H^1(\mathbb{R}^n)} \quad \text{and} \quad \|r\|_{H^1_{\text{scl}}(\mathbb{R}^n)} \leq Ch^2 \|f\|_{L^2(\mathbb{R}^n)}.$$

Here $(u, r)_{H^{-1}(\mathbb{R}^n), H^1(\mathbb{R}^n)}$ stands for the L^2 -duality. It follows that $\mathcal{L}_\zeta r = f$ in \mathbb{R}^n , hence also in Ω , and $\|r\|_{H^1_{\text{scl}}(\Omega)} \leq \|r\|_{H^1_{\text{scl}}(\mathbb{R}^n)} \leq Ch^2 \|f\|_{L^2(\mathbb{R}^n)} = Ch^2 \|f\|_{L^2(\Omega)}$. \square

Now we can complete the construction of the CGO solution in (4.2.1). Equation (4.2.2) gives $e^{-x \cdot \zeta / h} \mathcal{L}_{A,q} e^{x \cdot \zeta / h} a = \mathcal{O}(h^{-1})$. From Proposition 4.2.2, we can find $r \in H^1(\Omega)$ with $\|r\|_{H^1_{\text{scl}}(\Omega)} = \mathcal{O}(h)$ such that

$$e^{-x \cdot \zeta / h} \mathcal{L}_{A,q} e^{x \cdot \zeta / h} r = -e^{-x \cdot \zeta / h} \mathcal{L}_{A,q} e^{x \cdot \zeta / h} a.$$

Summing up, we have proved

existence **Proposition 4.2.3.** *Let $A \in W^{1,\infty}(\Omega; \mathbb{C}^n)$, $q \in L^\infty(\Omega; \mathbb{C})$, and $\zeta \in \mathbb{C}^n$ be such that $\zeta \cdot \zeta = 0$. Then for all $h > 0$ small enough, there exist solutions $u \in H^1(\Omega)$ to the equation $\mathcal{L}_{A,q} u = 0$ in Ω of the form*

$$u(x, \zeta, h) = e^{x \cdot \zeta / h} (a(x, \zeta^{(0)}) + r(x, \zeta, h)),$$

where $a \in C^\infty(\overline{\Omega})$ satisfies (4.2.2) and $\|r\|_{H^1_{\text{scl}}(\Omega)} = \mathcal{O}(h)$.

Remark: Sometimes we may need complex geometric optics solutions belonging to $H^4(\Omega)$, we can obtain such solutions as follows. Let $\Omega' \supset \supset \Omega$ be a bounded domain with smooth boundary. Extend $A \in W^{1,\infty}(\Omega; \mathbb{C}^n)$ and $q \in L^\infty(\Omega; \mathbb{C})$ to functions in $W^{1,\infty}(\Omega'; \mathbb{C}^n)$ and $L^\infty(\Omega'; \mathbb{C})$, respectively. By elliptic regularity, the complex geometric optics solutions constructed as above in Ω' will belong to $H^4(\Omega)$.

4.3 Integral Identity and Runge's Approximation

For the bi-harmonic operator, Green's formula gives

$$\begin{aligned} \int_{\Omega} (\mathcal{L}_{A,q} u) \bar{v} \, dx - \int_{\Omega} u \overline{\mathcal{L}_{A,q}^* v} \, dx &= -i \int_{\partial\Omega} \nu(x) \cdot Au \bar{v} \, dS - \int_{\partial\Omega} \partial_\nu(-\Delta u) \bar{v} \, dS \\ &+ \int_{\partial\Omega} (-\Delta u) \overline{\partial_\nu v} \, dS - \int_{\partial\Omega} \partial_\nu u \overline{(-\Delta v)} \, dS + \int_{\partial\Omega} u \overline{(\partial_\nu(-\Delta v))} \, dS \end{aligned} \quad (4.3.1) \quad \text{green}$$

for all $u, v \in H^4(\Omega)$. Here $\mathcal{L}_{A,q}^* := \mathcal{L}_{\bar{A}, i^{-1}\nabla \cdot \bar{A} + \bar{q}}$ is the adjoint of $\mathcal{L}_{A,q}$, ν is the unit outer normal vector to the boundary $\partial\Omega$, and dS is the surface measure on $\partial\Omega$.

For $(f_1, f_2) \in (H^{\frac{7}{2}}(\Gamma_1) \cap \mathcal{E}'(\Gamma_1)) \times (H^{\frac{3}{2}}(\Gamma_1) \cap \mathcal{E}'(\Gamma_1))$ with $\text{supp}(f_1) \subset \gamma_1$ and $\text{supp}(f_2) \subset \gamma_1$, let $u_1 \in H^4(\Sigma)$ solve

$$\begin{cases} \mathcal{L}_{A^{(1)}, q^{(1)}} u_1 = 0 & \text{in } \Sigma \\ u_1 = f_1 \quad \Delta u_1 = f_2 & \text{on } \Gamma_1 \\ u_1 = 0 \quad \Delta u_1 = 0 & \text{on } \Gamma_2 \end{cases}$$

Since $C_{A^{(1)}, q^{(1)}}^{\gamma_1, \gamma_2}(\Sigma) = C_{A^{(2)}, q^{(2)}}^{\gamma_1, \gamma_2}(\Sigma)$, there is $u_2 \in H^4(\Sigma)$ which satisfies

$$\begin{aligned} \mathcal{L}_{A^{(2)}, q^{(2)}} u_2 &= 0 \quad \text{in } \Sigma, \\ u_2 &= u_1 \quad \Delta u_2 = \Delta u_1 \quad \text{on } \Gamma_1 \cup \Gamma_2, \\ \partial_\nu u_1|_{\gamma_2} &= \partial_\nu u_2|_{\gamma_2}, \quad \partial_\nu(\Delta u_1)|_{\gamma_2} = \partial_\nu(\Delta u_2)|_{\gamma_2}. \end{aligned} \tag{4.3.2} \quad \boxed{\text{normal}}$$

Let $w := u_2 - u_1$, then

$$\mathcal{L}_{A^{(2)}, q^{(2)}} w = (A^{(1)} - A^{(2)}) \cdot Du_1 + (q^{(1)} - q^{(2)})u_1. \tag{4.3.3} \quad \boxed{\text{difference}}$$

From (4.3.2) it holds that $\partial_\nu w = \partial_\nu(\Delta w) = 0$ on γ_2 . We denote

$$l_1 := \Gamma_1 \cap \bar{B} \subset \gamma_1, \quad l_2 := \Gamma_2 \cap \bar{B} \subset \gamma_2, \quad l_3 := \Sigma \cap \partial B.$$

Apparently $\partial(\Sigma \cap B) = l_1 \cup l_2 \cup l_3$. It follows from (4.3.3) that $w \in H^4(\Sigma)$ is a solution to

$$\Delta^2 w = 0 \quad \text{in } \Sigma \setminus \bar{B}.$$

As $w = \Delta w = 0$ on $\gamma_2 \setminus \bar{l}_2$, by unique continuation, $w = 0$ in $\Sigma \setminus \bar{B}$. Therefore $w = \Delta w = \partial_\nu w = \partial_\nu(\Delta w) = 0$ on l_3 . We record these results here:

$$\begin{aligned} w &= 0 & \text{on } l_1 \\ w = \partial_\nu w = \partial_\nu(\Delta w) &= 0 & \text{on } l_2 \\ w = \partial_\nu w = \Delta w = \partial_\nu(\Delta w) &= 0 & \text{on } l_3 \end{aligned} \tag{4.3.4} \quad \boxed{\text{results}}$$

Let v be a solution to the equation

$$\mathcal{L}_{A^{(2)}, q^{(2)}}^* v = 0 \quad \text{in } \Sigma \cap B \tag{4.3.5} \quad \boxed{\text{eqn}_v}$$

such that

$$v = \Delta v = 0 \quad \text{on } l_1. \quad (4.3.6) \quad \boxed{\text{condition_v}}$$

Taking into consideration of (4.3.3) and (4.3.5), we apply (4.3.1) to w and v over $\Sigma \cap B$ to get

$$\begin{aligned} & \int_{\Sigma \cap B} ((A^{(1)} - A^{(2)}) \cdot Du_1) \bar{v} \, dx + \int_{\Sigma \cap B} (q^{(1)} - q^{(2)}) u_1 \bar{v} \, dx \\ = & -i \int_{\partial(\Sigma \cap B)} \nu(x) \cdot A^{(2)} w \bar{v} \, dS - \int_{\partial(\Sigma \cap B)} \partial_\nu(-\Delta w) \bar{v} \, dS \\ & + \int_{\partial(\Sigma \cap B)} (-\Delta w) \overline{\partial_\nu v} \, dS - \int_{\partial(\Sigma \cap B)} \partial_\nu w \overline{(-\Delta v)} \, dS \\ & + \int_{\partial(\Sigma \cap B)} w \overline{(\partial_\nu(-\Delta v))} \, dS \\ := & I_1 + I_2 + I_3 + I_4 + I_5 \end{aligned} \quad (4.3.7) \quad \boxed{\text{identity1}}$$

We analyze each term on the right-hand side and show $I_j = 0, j = 1, \dots, 5$.

$$I_1 := -i \int_{\partial(\Sigma \cap B)} \nu(x) \cdot A^{(2)} w \bar{v} \, dS.$$

By (4.3.4), $w = 0$ on $\partial(\Sigma \cap B)$; hence $I_1 = 0$.

$$I_2 := - \int_{\partial(\Sigma \cap B)} \partial_\nu(-\Delta w) \bar{v} \, dS.$$

By (4.3.6), $v = 0$ on l_1 ; by (4.3.4), $\partial_\nu(\Delta w) = 0$ on $l_2 \cup l_3$; hence $I_2 = 0$.

$$I_3 := \int_{\partial(\Sigma \cap B)} (-\Delta w) \overline{\partial_\nu v} \, dS.$$

By definition, $\Delta w = 0$ on $l_1 \cup l_2$; by (4.3.4), $\Delta w = 0$ on l_3 ; hence $I_3 = 0$.

$$I_4 := - \int_{\partial(\Sigma \cap B)} \partial_\nu w \overline{(-\Delta v)} \, dS.$$

By (4.3.6), $\Delta v = 0$ on l_1 ; by (4.3.4), $\partial_\nu w = 0$ on $l_2 \cup l_3$; hence $I_4 = 0$.

$$I_5 := \int_{\partial(\Sigma \cap B)} w \overline{(\partial_\nu(-\Delta v))} \, dS.$$

By definition, $w = 0$ on $l_1 \cup l_2$; by (4.3.4), $w = 0$ on l_3 ; hence $I_5 = 0$.

Putting these together, from (4.3.7) we obtain

$$\int_{\Sigma \cap B} ((A^{(1)} - A^{(2)}) \cdot Du_1) \bar{v} \, dx + \int_{\Sigma \cap B} (q^{(1)} - q^{(2)}) u_1 \bar{v} \, dx = 0. \quad (4.3.8) \quad \boxed{\text{identity2}}$$

for all $u_1 \in \mathcal{W}(\Sigma)$ and $v \in \mathcal{V}_{l_1}(\Sigma \cap B)$. Here for $j = 1, 2$, we define some function spaces for later use:

$$\begin{aligned} \mathcal{W}(\Sigma) := & \{u \in H^4(\Sigma) : \mathcal{L}_{A^{(1)}, q^{(1)}} u = 0 \text{ in } \Sigma, u|_{\Gamma_2} = \Delta u|_{\Gamma_2} = 0, \\ & \text{supp}(u|_{\Gamma_1}) \subset \gamma_1, \text{supp}(\Delta u|_{\Gamma_1}) \subset \gamma_1\}. \end{aligned}$$

$$\mathcal{V}_{l_j}(\Sigma \cap B) := \{v \in H^4(\Sigma \cap B) : \mathcal{L}_{A^{(2)}, q^{(2)}}^* v = 0 \text{ in } \Sigma \cap B, v|_{l_j} = \Delta v|_{l_j} = 0\}.$$

$$\mathcal{W}_{l_j}(\Sigma \cap B) := \{u \in H^4(\Sigma \cap B) : \mathcal{L}_{A^{(1)}, q^{(1)}} u = 0 \text{ in } \Sigma \cap B, u|_{l_j} = \Delta u|_{l_j} = 0\}.$$

We would like to replace $u_1 \in \mathcal{W}(\Sigma)$ in (4.3.8) by elements of the space $\mathcal{W}_{l_2}(\Sigma \cap B)$. This can be achieved by the following Runge type approximation result.

runge **Proposition 4.3.1.** *$\mathcal{W}(\Sigma)$ is a dense subspace of $\mathcal{W}_{l_2}(\Sigma \cap B)$ in $L^2(\Sigma \cap B)$ topology.*

Proof. It suffices to establish the following fact: for any $g \in L^2(\Sigma \cap B)$ such that

$$\int_{\Sigma \cap B} u \bar{g} \, dx = 0 \quad \forall u \in \mathcal{W}(\Sigma),$$

we have

$$\int_{\Sigma \cap B} v \bar{g} \, dx = 0 \quad \forall v \in \mathcal{W}_{l_2}(\Sigma \cap B).$$

To prove this fact, we extend g by zero to $\Sigma \setminus \Sigma \cap B$. Let $U \in H^4(\Sigma)$ be the solution of the problem

$$\begin{aligned} \mathcal{L}_{A^{(1)}, q^{(1)}}^* U &= g & \text{in } \Sigma \\ U = \Delta U &= 0 & \text{on } \Gamma_1 \cup \Gamma_2. \end{aligned}$$

For any $u \in \mathcal{W}(\Sigma)$, Green's formula in the infinite slab Σ (see appendix B) gives

$$0 = \int_{\Sigma} u \bar{g} \, dx = \int_{\Sigma} u \overline{(\mathcal{L}_{A^{(1)}, q^{(1)}}^* U)} \, dx = \int_{\Gamma_1} \overline{\partial_{\nu} U} \Delta u \, dS + \int_{\Gamma_1} \overline{\partial_{\nu} \Delta U} u \, dS.$$

Since $u|_{\Gamma_1}$ and $\Delta u|_{\Gamma_1}$ can be arbitrary smooth functions supported in γ_1 , we conclude that $\partial_{\nu} U|_{\gamma_1} = \partial_{\nu} \Delta U|_{\gamma_1} = 0$. Hence U satisfies $\Delta^2 U = 0$ in $\Sigma \setminus B$, and moreover, $U = \Delta U = 0$

on $\gamma_1 \setminus l_1$. Thus, by unique continuation, $U = 0$ in $\Sigma \setminus B$, and we have $U = \partial_\nu U = \Delta U = \partial_\nu \Delta U = 0$ on l_3 .

For any $v \in \mathcal{W}_{l_2}(\Sigma \cap B)$, using Green's formula on the bounded domain $\Sigma \cap B$ we get

$$\begin{aligned}
\int_{\Sigma \cap B} v \bar{g} \, dx &= \int_{\Sigma \cap B} v \overline{(\mathcal{L}_{A^{(1)}, q^{(1)}}^* U)} \, dx \\
&= \int_{\Sigma \cap B} \left(\mathcal{L}_{A^{(1)}, q^{(1)}} v \right) \bar{U} \, dx + i \int_{\partial(\Sigma \cap B)} \nu(x) \cdot Av \bar{U} \, dS \\
&\quad + \int_{\partial(\Sigma \cap B)} \partial_\nu(-\Delta v) \bar{U} \, dS - \int_{\partial(\Sigma \cap B)} (-\Delta v) \overline{\partial_\nu U} \, dS \\
&\quad + \int_{\partial(\Sigma \cap B)} \partial_\nu v \overline{(-\Delta U)} \, dS - \int_{\partial(\Sigma \cap B)} v \overline{(\partial_\nu(-\Delta U))} \, dS \\
&= 0.
\end{aligned}$$

□

Combining (4.3.8) with Proposition 4.3.1 we conclude

Proposition 4.3.2.

$$\int_{\Sigma \cap B} ((A^{(1)} - A^{(2)}) \cdot Du_1) \bar{v} \, dx + \int_{\Sigma \cap B} (q^{(1)} - q^{(2)}) u_1 \bar{v} \, dx = 0. \quad (4.3.9) \quad \boxed{\text{identity3}}$$

for all $u_1 \in \mathcal{W}_{l_2}(\Sigma \cap B)$ and $v \in \mathcal{V}_{l_1}(\Sigma \cap B)$.

4.4 Complex Geometric Optics Solutions in an Infinite Slab

In this section we construct CGO solutions $u_1 \in \mathcal{W}_{l_2}(\Sigma \cap B)$ and $v \in \mathcal{V}_{l_1}(\Sigma \cap B)$. Let $\xi, \mu^{(1)}, \mu^{(2)} \in \mathbb{R}^n$ be such that $|\mu^{(1)}| = |\mu^{(2)}| = 1$ and $\mu^{(1)} \cdot \mu^{(2)} = \mu^{(1)} \cdot \xi = \mu^{(2)} \cdot \xi = 0$. We set

$$\zeta_1 := \frac{ih\xi}{2} + i\sqrt{1 - h^2 \frac{|\xi|^2}{4}} \mu^{(1)} + \mu^{(2)}, \quad \zeta_2 := -\frac{ih\xi}{2} + i\sqrt{1 - h^2 \frac{|\xi|^2}{4}} \mu^{(1)} - \mu^{(2)}. \quad (4.4.1)$$

Note that $\zeta_1 \cdot \zeta_1 = \zeta_2 \cdot \zeta_2 = 0$, and $(\zeta_1 + \bar{\zeta}_2)/h = i\xi$. Here $h > 0$ is a small semiclassical parameter. Note also that

$$\zeta_1 = i\mu^{(1)} + \mu^{(2)} + \mathcal{O}(h) \text{ and } \zeta_2 = i\mu^{(1)} - \mu^{(2)} + \mathcal{O}(h) \text{ as } h \rightarrow 0, \quad (4.4.2) \quad \boxed{\text{zeta}}$$

so $\zeta_1^{(0)} = i\mu^{(1)} + \mu^{(2)}$, $\zeta_2^{(0)} = i\mu^{(1)} - \mu^{(2)}$.

We first construct $u_1 \in \mathcal{W}_{l_2}(\Sigma \cap B)$. To satisfy the condition $u_1|_{l_2} = \Delta u_1|_{l_2} = 0$, we reflect $\Sigma \cap B$ with respect to the plane $x_n = 0$ and denote this reflection by $(\Sigma \cap B)_0^* := \{(x', -x_n) : x = (x', x_n) \in \Sigma \cap B\}$ where $x' = (x_1, \dots, x_{n-1})$. We extend the coefficients $A^{(1)}$ and $q^{(1)}$ to $(\Sigma \cap B)_0^*$ as follows: for the components $A_j^{(1)}, j = 1, \dots, n-1$ and $q^{(1)}$, we extend them as even functions with respect to $x_n = 0$, for $A_n^{(1)}$ we extend it as an odd function with respect to $x_n = 0$, i.e. we set

$$\begin{aligned} \tilde{A}_j^{(1)}(x) &= \begin{cases} A_j^{(1)}(x', x_n) & 0 < x_n < L \\ A_j^{(1)}(x', -x_n) & -L < x_n < 0 \end{cases}, \quad j = 1, \dots, n-1 \\ \tilde{A}_n^{(1)}(x) &= \begin{cases} A_n^{(1)}(x', x_n) & 0 < x_n < L \\ -A_n^{(1)}(x', -x_n) & -L < x_n < 0 \end{cases} \\ \tilde{q}^{(1)}(x) &= \begin{cases} q^{(1)}(x', x_n) & 0 < x_n < L \\ q^{(1)}(x', -x_n) & -L < x_n < 0 \end{cases}. \end{aligned}$$

For the moment, let us assume $A_n^{(1)}|_{x_n=0} = 0$ so that $\tilde{A}^{(1)} \in W^{1,\infty}((\Sigma \cap B) \cup (\Sigma \cap B)_0^*)$ and $\tilde{q}^{(1)} \in L^\infty((\Sigma \cap B) \cup (\Sigma \cap B)_0^*)$. We will come back to the general case after establishing Proposition 4.5.1.

Proposition 4.2.3 implies that there exist CGO solutions of the form

$$\tilde{u}_1(x, \zeta_1, h) = e^{x \cdot \zeta_1 / h} (a_1(x, \zeta_1^{(0)}) + r_1(x, \zeta_1, h)) \in H^2((\Sigma \cap B) \cup (\Sigma \cap B)_0^*)$$

which satisfy the equation $\mathcal{L}_{\tilde{A}^{(1)}, \tilde{q}^{(1)}} \tilde{u}_1 = 0$ in the bounded region $(\Sigma \cap B) \cup (\Sigma \cap B)_0^*$ with

$$((i\mu^{(1)} + \mu^{(2)}) \cdot \nabla)^2 a_1 = 0 \quad \text{in } (\Sigma \cap B) \cup (\Sigma \cap B)_0^*, \quad (4.4.3)$$

$$\|r_1\|_{H_{\text{scl}}^1((\Sigma \cap B) \cup (\Sigma \cap B)_0^*)} = \mathcal{O}(h). \quad (4.4.4)$$

Let

$$u_1(x) := \tilde{u}_1(x', x_n) - \tilde{u}_1(x', -x_n) \quad x \in \Sigma \cap B. \quad (4.4.5) \quad \boxed{\text{CGO}_u}$$

Then it is easy to check that $u_1 \in \mathcal{W}_{l_2}(\Sigma \cap B)$.

To construct $v \in \mathcal{V}_{l_1}(\Sigma \cap B)$, we notice that $\mathcal{L}_{A,q}^* = \mathcal{L}_{\bar{A}, i^{-1} \nabla \cdot \bar{A} + \bar{q}}$, so $\mathcal{L}_{A,q}^* v = 0$ is equivalent to $\mathcal{L}_{A^{(3)}, q^{(3)}} v = 0$ where $A^{(3)} := \overline{A^{(2)}}$ and $q^{(3)} := i^{-1} \nabla \cdot \overline{A^{(2)}} + \overline{q^{(2)}}$. In the

following we will construct v such that $\mathcal{L}_{A^{(3)},q^{(3)}}v = 0$ with $v|_{l_1} = \Delta v|_{l_1} = 0$. To this end, we reflect $\Sigma \cap B$ with respect to the plane $x_n = L$ and denote this reflection by $(\Sigma \cap B)_L^* := \{(x', -x_n + 2L) : x = (x', x_n) \in \Sigma \cap B\}$. We extend the coefficients $A^{(3)}$ and $q^{(3)}$ to $(\Sigma \cap B)_L^*$ as follows: for $A_j^{(3)}, j = 1, \dots, n-1$ and $q^{(3)}$ we extend them as even functions with respect to $x_n = L$, for $A_n^{(3)}$ we extend it as an odd function with respect to $x_n = L$, i.e.

$$\begin{aligned} \tilde{A}_j^{(3)}(x) &= \begin{cases} A_j^{(3)}(x', x_n) & 0 < x_n < L \\ A_j^{(3)}(x', -x_n + 2L) & L < x_n < 2L \end{cases}, \quad j = 1, \dots, n-1 \\ \tilde{A}_n^{(3)}(x) &= \begin{cases} A_n^{(3)}(x', x_n) & 0 < x_n < L \\ -A_n^{(3)}(x', -x_n + 2L) & L < x_n < 2L \end{cases} \\ \tilde{q}^{(3)}(x) &= \begin{cases} q^{(3)}(x', x_n) & 0 < x_n < L \\ q^{(3)}(x', -x_n + 2L) & L < x_n < 2L \end{cases}. \end{aligned}$$

Again, first we assume $A_n^{(2)}|_{x_n=L} = 0$ so that $A_n^{(3)}|_{x_n=L} = 0$, $\tilde{A}^{(3)} \in W^{1,\infty}((\Sigma \cap B) \cup (\Sigma \cap B)_L^*)$ and $\tilde{q}^{(3)} \in L^\infty((\Sigma \cap B) \cup (\Sigma \cap B)_L^*)$. The general case will be dealt with below Proposition 4.5.1.

Proposition 4.2.3 implies that there exist CGO solutions of the form

$$\tilde{v}(x, \zeta_2, h) = e^{x \cdot \zeta_2/h} (a_2(x, \zeta_2^{(0)}) + r_2(x, \zeta_2, h)) \in H^2((\Sigma \cap B) \cup (\Sigma \cap B)_L^*)$$

which satisfy the equation $\mathcal{L}_{\tilde{A}^{(3)}, \tilde{q}^{(3)}} \tilde{v} = 0$ in the bounded region $(\Sigma \cap B) \cup (\Sigma \cap B)_L^*$ with

$$((i\mu^{(1)} - \mu^{(2)}) \cdot \nabla)^2 a_2 = 0 \quad \text{in } (\Sigma \cap B) \cup (\Sigma \cap B)_L^*,$$

$$\|r_2\|_{H_{\text{sc1}}^1((\Sigma \cap B) \cup (\Sigma \cap B)_L^*)} = \mathcal{O}(h).$$

Let

$$v(x) := \tilde{v}(x', x_n) - \tilde{v}(x', -x_n + 2L) \quad x \in \Sigma \cap B. \quad (4.4.6) \quad \boxed{\text{CGO}_v}$$

Then it is easy to check that $v \in \mathcal{V}_{l_1}(\Sigma \cap B)$.

We write down the CGO solutions (4.4.5) and (4.4.6) explicitly for future references:

$$u_1(x) = e^{x \cdot \zeta_1/h} (a_1(x) + r_1(x)) - e^{(x', -x_n) \cdot \zeta_1/h} (a_1(x', -x_n) + r_1(x', -x_n)) \quad (4.4.7) \quad \boxed{\text{CGO1}}$$

$$v(x) = e^{x \cdot \zeta_2/h} (a_2(x) + r_2(x)) - e^{(x', -x_n + 2L) \cdot \bar{\zeta}_2/h} (a_2(x', -x_n + 2L) + r_2(x', -x_n + 2L)) \quad (4.4.8) \quad \boxed{\text{CG02}}$$

where $a_1 \in C^\infty(\overline{(\Sigma \cap B) \cup (\Sigma \cap B)_0^*})$, $a_2 \in C^\infty(\overline{(\Sigma \cap B) \cup (\Sigma \cap B)_L^*})$ and

$$((i\mu^{(1)} + \mu^{(2)}) \cdot \nabla)^2 a_1 = 0 \quad \text{in } (\Sigma \cap B) \cup (\Sigma \cap B)_0^*, \quad (4.4.9) \quad \boxed{\text{condition1}}$$

$$((i\mu^{(1)} - \mu^{(2)}) \cdot \nabla)^2 a_2 = 0 \quad \text{in } (\Sigma \cap B) \cup (\Sigma \cap B)_L^*, \quad (4.4.10) \quad \boxed{\text{condition2}}$$

$$\|r_1\|_{H_{\text{scl}}^1((\Sigma \cap B) \cup (\Sigma \cap B)_0^*)} = \mathcal{O}(h), \quad (4.4.11) \quad \boxed{\text{condition3}}$$

$$\|r_2\|_{H_{\text{scl}}^1((\Sigma \cap B) \cup (\Sigma \cap B)_L^*)} = \mathcal{O}(h). \quad (4.4.12) \quad \boxed{\text{condition4}}$$

4.5 Proof of the Theorems

4.5.1 Proof of Theorem 4.1.1

We are ready to prove our first main theorem. We will substitute the CGO solutions constructed in the last section into (4.3.9). To this end we compute

$$\begin{aligned} e^{x \cdot \zeta_1/h} e^{x \cdot \bar{\zeta}_2/h} &= e^{ix \cdot \xi} \\ e^{(x', -x_n) \cdot \zeta_1/h} e^{x \cdot \bar{\zeta}_2/h} &= e^{-2\mu_n^{(2)} x_n/h + ib_1} \\ e^{x \cdot \zeta_1/h} e^{(x', -x_n + 2L) \cdot \bar{\zeta}_2/h} &= e^{2\mu_n^{(2)} (x_n - L)/h + ib_2} \\ e^{(x', -x_n) \cdot \zeta_1/h} e^{(x', -x_n + 2L) \cdot \bar{\zeta}_2/h} &= e^{-2L\mu_n^{(2)}/h + ib_3} \end{aligned} \quad (4.5.1) \quad \boxed{\text{exp1}}$$

where $b_1, b_2, b_3 \in \mathbb{R}^n$ are defined by

$$\begin{aligned} b_1 &:= x' \cdot \xi' - \frac{2}{h} \sqrt{1 - h^2 \frac{|\xi|^2}{4}} \mu_n^{(1)} x_n, \\ b_2 &:= x' \cdot \xi' + \frac{2}{h} \sqrt{1 - h^2 \frac{|\xi|^2}{4}} \mu_n^{(1)} (x_n - L) + L\xi_n, \\ b_3 &:= x' \cdot \xi' - \frac{2L}{h} \sqrt{1 - h^2 \frac{|\xi|^2}{4}} \mu_n^{(1)} - x_n \xi_n + L\xi_n. \end{aligned}$$

We further assume that $\mu_n^{(2)} > 0$, hence for $0 < x_n < L$ the following pointwise convergence holds as $h \rightarrow 0+$:

$$\begin{aligned} |e^{(x', -x_n) \cdot \zeta_1/h} e^{x \cdot \bar{\zeta}_2/h}| &\rightarrow 0 \text{ as } h \rightarrow 0+, \\ |e^{x \cdot \zeta_1/h} e^{(x', -x_n + 2L) \cdot \bar{\zeta}_2/h}| &\rightarrow 0 \text{ as } h \rightarrow 0+, \\ |e^{(x', -x_n) \cdot \zeta_1/h} e^{(x', -x_n + 2L) \cdot \bar{\zeta}_2/h}| &\rightarrow 0 \text{ as } h \rightarrow 0+. \end{aligned} \quad (4.5.2) \quad \boxed{\text{vanish}}$$

Therefore, with the CGO solutions u_1 and v given by (4.4.7) and (4.4.8), we conclude from (4.4.11) (4.4.12) and (4.5.2) that

$$h \int_{\Sigma \cap B} (q^{(1)} - q^{(2)}) u_1 \bar{v} dx \rightarrow 0 \quad \text{as } h \rightarrow 0+. \quad (4.5.3) \quad \boxed{\text{term2}}$$

On the other hand, denote $\zeta_j^* = (\zeta_j', -(\zeta_j)_n)$ for $\zeta_j = (\zeta_j', (\zeta_j)_n)$, $j = 1, 2$. Using (4.4.7) we compute

$$\begin{aligned} Du_1(x) = & -\frac{i\zeta_1}{h} e^{x \cdot \zeta_1/h} (a_1(x) + r_1(x)) + e^{x \cdot \zeta_1/h} (Da_1(x) + Dr_1(x)) \\ & + \frac{i\zeta_1^*}{h} e^{(x', -x_n) \cdot \zeta_1/h} (a_1(x', -x_n) + r_1(x', -x_n)) \\ & - e^{(x', -x_n) \cdot \zeta_1/h} (Da_1(x', -x_n) + Dr_1(x', -x_n)). \end{aligned} \quad (4.5.4)$$

Therefore, with the CGO solutions u_1 and v given by (4.4.7) and (4.4.8), we have from (4.4.2) (4.4.11) (4.4.12) (4.5.2) and the dominant convergence theorem that

$$\begin{aligned} & h \int_{\Sigma \cap B} (A^{(1)} - A^{(2)}) \cdot Du_1 \bar{v} dx \\ \rightarrow & (\mu^{(1)} - i\mu^{(2)}) \cdot \int_{\Sigma \cap B} (A^{(1)} - A^{(2)}) e^{ix \cdot \xi} a_1 \bar{a}_2 dx \quad \text{as } h \rightarrow 0+. \end{aligned} \quad (4.5.5) \quad \boxed{\text{term1}}$$

Multiplying (4.3.9) by h and letting $h \rightarrow 0+$ for the constructed solutions u_1 and v , we obtain from (4.5.3) and (4.5.5) that

$$(\mu^{(1)} - i\mu^{(2)}) \cdot \int_{\Sigma \cap B} (A^{(1)} - A^{(2)}) e^{ix \cdot \xi} a_1 \bar{a}_2 dx = 0. \quad (4.5.6) \quad \boxed{\text{identity_A1}}$$

This identity holds for all a_1 satisfying (4.4.9), a_2 satisfying (4.4.10), and for all $\mu^{(1)}, \mu^{(2)}, \xi \in \mathbb{R}^n$ such that $\mu^{(1)} \cdot \mu^{(2)} = \mu^{(1)} \cdot \xi = \mu^{(2)} \cdot \xi = 0$ and $\mu_n^{(2)} > 0$. Replace $\mu^{(1)}$ by $-\mu^{(1)}$ and subtract to find

$$\mu^{(1)} \cdot \int_{\Sigma \cap B} (A^{(1)} - A^{(2)}) e^{ix \cdot \xi} a_1 \bar{a}_2 dx = 0. \quad (4.5.7) \quad \boxed{\text{identity_A2}}$$

curlvanish **Proposition 4.5.1.**

$$\partial_j (A_k^{(1)} - A_k^{(2)}) - \partial_k (A_j^{(1)} - A_j^{(2)}) = 0 \quad \text{in } \Sigma \cap B, \quad 1 \leq j, k \leq n. \quad (4.5.8) \quad \boxed{\text{curl1}}$$

Proof. Obviously $a_1 = a_2 = 1$ satisfies (4.4.9) and (4.4.10). Inserting $a_1 = a_2 = 1$ in (4.5.7)

we get

$$\mu^{(1)} \cdot (\widehat{(A^{(1)})\chi_{\Sigma \cap B}}(\xi) - \widehat{(A^{(2)})\chi_{\Sigma \cap B}}(\xi)) = 0 \quad (4.5.9) \quad \boxed{\text{fourier}}$$

where $\chi_{\Sigma \cap B}$ stands for the characteristic function of the set $\Sigma \cap B$ and $\widehat{A^{(j)}}_{\chi_{\Sigma \cap B}}$ denotes the Fourier transform of $A^{(j)}\chi_{\Sigma \cap B}$.

To show the proposition, it suffices to consider the case when $j \neq k$. Let e_1, \dots, e_n be the standard orthonormal basis in \mathbb{R}^n . Let $\xi = (\xi_1, \dots, \xi_n)$ with $\xi_j > 0, j = 1, \dots, n$. Define

$$\mu^{(1)} = -\xi_k e_j + \xi_j e_k \quad 1 \leq j, k \leq n, j \neq k.$$

To define $\mu^{(2)}$ we consider two cases: if j, k are such that $1 \leq j, k < n$, define

$$\mu^{(2)} = -\xi_j \xi_n e_j - \xi_k \xi_n e_k + (\xi_j^2 + \xi_k^2) e_n;$$

if $k = n$ and j is such that $1 \leq j < n$, define

$$\mu^{(2)} = (-\xi_j^2 - \xi_n^2) e_l + \xi_l \xi_j e_j + \xi_l \xi_n e_n$$

with some $l \neq j, n$, which exists since $n \geq 3$. In either case it is easy to check $\mu^{(1)} \cdot \mu^{(2)} = \mu^{(1)} \cdot \xi = \mu^{(2)} \cdot \xi = 0$ and $\mu_n^{(2)} > 0$. For such $\mu^{(1)}$ and ξ we get from (4.5.9) that

$$\xi_j \cdot (\widehat{A_k^{(2)}}_{\chi_{\Sigma \cap B}}(\xi) - \widehat{A_k^{(1)}}_{\chi_{\Sigma \cap B}}(\xi)) - \xi_k \cdot (\widehat{A_j^{(2)}}_{\chi_{\Sigma \cap B}}(\xi) - \widehat{A_j^{(1)}}_{\chi_{\Sigma \cap B}}(\xi)) = 0,$$

$1 \leq j, k \leq n, j \neq k$ for all $\xi \in \mathbb{R}^n, \xi_1 > 0, \dots, \xi_n > 0$, and thus everywhere by analyticity of the Fourier transform. This completes the proof. \square

By Proposition 4.5.1, we conclude $dA^{(1)} = dA^{(2)}$ in Σ . As Σ is simply connected, there exists a compactly supported $\Phi \in C^{1,1}(\overline{\Sigma})$ such that

$$A^{(1)} - A^{(2)} = \nabla \Phi \quad \text{in } \Sigma.$$

In particular, $\Phi = 0$ along $\partial B \cap \Sigma$.

Recall that in the construction of the CGO solutions above, we have assumed that $A_n^{(1)}|_{x_n=0} = 0$ and $A_n^{(2)}|_{x_n=L} = 0$. Now we show why our results are independent of such assumptions. Indeed, for $A^{(1)}$, there exists $\Psi^{(1)} \in C^{1,1}(\overline{\Sigma})$ with compact support such that $\Psi^{(1)}|_{\partial \Sigma} = 0$ and $\partial_\nu \Psi^{(1)} = -A^{(1)} \cdot \nu$ on $\partial \Sigma$, where as usual ν is the unit outer normal vector on $\partial \Sigma$. Then $A^{(1)} + \nabla \Psi^{(1)}$ satisfies $(A_n^{(1)} + \nabla \Psi_n^{(1)})|_{x_n=0} = 0$. See [26, Theorem 1.3.3] for the existence of $\Psi^{(1)}$. Similarly, we can find $\Psi^{(2)} \in C^{1,1}(\overline{\Sigma})$ with compact support such that

$\Psi^{(2)}|_{\partial\Sigma} = 0$ and $\partial_\nu \Psi^{(2)} = -A^{(2)} \cdot \nu$ on $\partial\Sigma$. Then $(A_n^{(2)} + \nabla \Psi_n^{(2)})|_{x_n=L} = 0$. Therefore, we may replace $A^{(j)}$ by $A^{(j)} + \nabla \Psi^{(j)}$, $j = 1, 2$ to fulfill the assumption. After the replacement, Proposition 4.5.1 will give $d(A^{(1)} + \nabla \Psi^{(1)}) = d(A^{(2)} + \nabla \Psi^{(2)})$ in Σ . As above we can find a compactly supported function $\Phi' \in C^{1,1}(\overline{\Sigma})$ such that

$$A^{(1)} + \nabla \Psi^{(1)} - A^{(2)} - \nabla \Psi^{(2)} = \nabla \Phi' \quad \text{in } \Sigma.$$

Define $\Phi := \Phi' - \Psi^{(1)} + \Psi^{(2)}$, then $\Phi \in C^{1,1}(\overline{\Sigma})$ is compactly supported and satisfies $A^{(1)} - A^{(2)} = \nabla \Phi$. In particular $\Phi = 0$ on $\partial B \cap \Sigma$. We are back to the same situation.

Next, we establish a proposition which asserts that $\Phi = 0$ on $\Gamma_1 \cup \Gamma_2$.

Proposition 4.5.2. $\Phi = 0$ along $\partial(\Sigma \cap B)$.

Proof. Notice (4.4.9) implies that in the expression (4.4.7), we may replace a_1 by $g_1 a_1$ if $g_1 \in C^\infty(\overline{(\Sigma \cap B) \cup (\Sigma \cap B)_0^* \cup (\Sigma \cap B)_L^*})$ satisfies

$$(i\mu^{(1)} + \mu^{(2)}) \cdot \nabla g_1 = 0 \quad \text{in } (\Sigma \cap B) \cup (\Sigma \cap B)_0^* \cup (\Sigma \cap B)_L^*.$$

Thus (4.5.6) becomes

$$(\mu^{(1)} - i\mu^{(2)}) \cdot \int_{\Sigma \cap B} \nabla \Phi g_1 e^{ix \cdot \xi} a_1 \bar{a}_2 dx = 0.$$

Set $\xi = 0$, $a_1 = a_2 = 1$ and multiply by i :

$$(i\mu^{(1)} + \mu^{(2)}) \cdot \int_{\Sigma \cap B} \nabla \Phi g_1 dx = 0 \quad (4.5.10) \quad \boxed{\mathbf{g}}$$

As $\mu^{(1)} \cdot \mu^{(2)} = 0$ and $|\mu^{(1)}| = |\mu^{(2)}| = 1$, we can make a change of variable so that $(i\mu^{(1)} + \mu^{(2)}) \cdot \nabla$ becomes a $\bar{\partial}$ -operator as follows. Complete the set $\{\mu^{(2)}, \mu^{(1)}\}$ to an orthonormal basis in \mathbb{R}^n , say $\{\mu^{(2)}, \mu^{(1)}, \mu^{(3)}, \dots, \mu^{(n)}\}$; introduce new coordinates $y = (y_1, \dots, y_n) \in \mathbb{R}^n$ with respect to this orthonormal basis by defining $y_1 = x \cdot \mu^{(2)}$, $y_2 = x \cdot \mu^{(1)}$, $y_j = x \cdot \mu^{(j)}$, $j = 3, \dots, n$; in other words, we made an orthogonal transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $T(x) = y$. Denote $z = y_1 + iy_2$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_{y_1} + i\partial_{y_2})$. Then $(i\mu^{(1)} + \mu^{(2)}) \cdot \nabla = 2\partial_{\bar{z}}$, and in the new coordinates (4.5.10) becomes

$$\int_{T(\Sigma \cap B)} g_1 \partial_{\bar{z}} \Phi dy = 0$$

for all $g_1 \in C^\infty(\overline{(\Sigma \cap B) \cup (\Sigma \cap B)_0^* \cup (\Sigma \cap B)_L^*})$ satisfying $\partial_{\bar{z}} g_1 = 0$. Replacing $\mu^{(1)}$ by $-\mu^{(1)}$, in the same way we can show

$$\int_{T(\Sigma \cap B)} g_2 \partial_z \Phi \, dy = 0$$

for all $g_2 \in C^\infty(\overline{(\Sigma \cap B) \cup (\Sigma \cap B)_L^* \cup (\Sigma \cap B)_0^*})$ satisfying $\partial_z g_2 = 0$. Taking $g_j(y) = g'_j(z) \otimes g''_j(y'')$, $j = 1, 2$, $y'' = (y_3, \dots, y_n)$ and varying g''_j yields

$$\int_{T_{y''}} g'_1(z) \partial_{\bar{z}} \Phi \, dz \wedge d\bar{z} = 0 \quad \int_{T_{y''}} g'_2(\bar{z}) \partial_z \Phi \, dz \wedge d\bar{z} = 0$$

Here $T_{y''}$ is the intersection of $T(\Sigma \cap B)$ with the two dimensional plane $\{(y_1, y_2, y'') \in \mathbb{R}^n : y'' \text{ fixed}\}$, $\partial_{\bar{z}} g'_1 = 0$ and $\partial_z g'_2 = 0$. Notice that $\partial T_{y''}$ is piecewise smooth. Since

$$d(g'_1(z) \Phi \, dz) = g'_1(z) \partial_{\bar{z}} \Phi \, d\bar{z} \wedge dz, \quad d(g'_2(\bar{z}) \Phi \, d\bar{z}) = g'_2(\bar{z}) \partial_z \Phi \, dz \wedge d\bar{z},$$

we obtain from Stokes' formula that

$$\int_{\partial T_{y''}} g'_1(z) \Phi \, dz = 0 \quad \int_{\partial T_{y''}} g'_2(\bar{z}) \Phi \, d\bar{z} = 0.$$

Taking $g'_2 = \overline{g'_1}$ we see that

$$\int_{\partial T_{y''}} g'_1(z) \Phi \, dz = 0 \quad \int_{\partial T_{y''}} g'_1(z) \bar{\Phi} \, dz = 0$$

Hence

$$\int_{\partial T_{y''}} g'_1(z) \operatorname{Re} \Phi \, dz = \int_{\partial T_{y''}} g'_1(z) \operatorname{Im} \Phi \, dz = 0.$$

for all holomorphic functions $g'_1 \in C^\infty(\overline{T_{y''}})$. Arguing as in [16, Lemma 5.1], we can find holomorphic functions $F_1, F_2 \in C(\overline{T_{y''}})$ such that

$$F_1|_{\partial T_{y''}} = \operatorname{Re} \Phi|_{\partial T_{y''}} \quad F_2|_{\partial T_{y''}} = \operatorname{Im} \Phi|_{\partial T_{y''}}.$$

Moreover, $\Delta \operatorname{Im} F_j = 0$ in $T_{y''}$ and $\operatorname{Im} F_j|_{\partial T_{y''}} = 0$. Thus, F_j , $j = 1, 2$, are real-valued and thus constant on $T_{y''}$. Therefore, Φ is constant along $\partial T_{y''}$. In the x -coordinate system, we see that the function $\Phi(x)$ is constant on the boundary of the intersection $T^{-1}(\Pi_{y''}) \cap (\Sigma \cap B)$ for all $y'' \in \mathbb{R}^{n-2}$, where $T^{-1}(\Pi_{y''})$ is defined by

$$T^{-1}(\Pi_{y''}) := \{x = y_1 \mu^{(2)} + y_2 \mu^{(1)} + \sum_{j=3}^n y_j \mu^{(j)} : y_1, y_2 \in \mathbb{R}, y'' = (y_3, \dots, y_n)\}.$$

Setting $\mu^{(1)} = e_j$, $j = 1, \dots, n-1$ and $\mu^{(2)} = e_n$, then varying y'' gives that Φ vanishes on $\partial(\Sigma \cap B)$. This completes the proof. \square

To show that $A^{(1)} = A^{(2)}$ consider (4.5.6) with $a_2 = 1$ and a_1 satisfying

$$((i\mu^{(1)} + \mu^{(2)}) \cdot \nabla) a_1 = 1 \quad \text{in } (\Sigma \cap B) \cup (\Sigma \cap B)_0^*.$$

This choice is possible thanks to (4.4.9). We have from (4.5.6) that

$$(i\mu^{(1)} + \mu^{(2)}) \cdot \int_{\Sigma \cap B} (\nabla \Phi) e^{ix \cdot \xi} a_1 dx = 0$$

Integrating by parts and using the fact that $\Phi = 0$ along $\partial(\Sigma \cap B)$ and $\mu^{(1)} \cdot \xi = \mu^{(2)} \cdot \xi = 0$ we obtain

$$0 = \int_{\Sigma \cap B} \Phi(x) e^{ix \cdot \xi} [(i\mu^{(1)} + \mu^{(2)}) \cdot \nabla a_1] dx = \int_{\Sigma \cap B} \Phi(x) e^{ix \cdot \xi} dx.$$

This indicates that Fourier transform of the function $\Phi \chi_{\Sigma \cap B}$ vanishes. Thus $\Phi = 0$ in $\Sigma \cap B$, and therefore $A^{(1)} = A^{(2)}$.

Inserting $A^{(1)} = A^{(2)}$ in (4.3.9) gives

$$\int_{\Sigma \cap B} (q^{(1)} - q^{(2)}) u_1 \bar{v} dx = 0.$$

Let u_1 and v be the CGO solutions given by (4.4.7) and (4.4.8). Taking the limit $h \rightarrow 0+$, from (4.4.11) (4.4.12) (4.5.2) we get

$$\int_{\Sigma \cap B} (q^{(1)} - q^{(2)}) e^{ix \cdot \xi} a_1 \bar{a}_2 dx = 0$$

where a_1 and a_2 satisfy (4.4.9) and (4.4.10) respectively. In particular, for $a_1 = a_2 = 1$ this identity becomes

$$\int_{\Sigma \cap B} (q^{(1)} - q^{(2)}) e^{ix \cdot \xi} dx = 0 \quad (4.5.11) \quad \boxed{\text{identity}_q}$$

for all ξ such that there exist $\mu^{(1)}, \mu^{(2)} \in \mathbb{R}^n$ such that

$$\mu^{(1)} \cdot \mu^{(2)} = \xi \cdot \mu^{(1)} = \xi \cdot \mu^{(2)} = 0, \quad |\mu^{(1)}| = |\mu^{(2)}| = 1, \quad \mu_n^{(2)} > 0.$$

Write $\xi = (\xi', \xi_{n-1}, \xi_n)$ with $\xi' \in \mathbb{R}^{n-2}$. If $\xi_{n-1} \neq 0$, we can choose

$$\mu^{(2)} = \frac{1}{\sqrt{1 + \frac{\xi_n^2}{\xi_{n-1}^2}}} (0_{\mathbb{R}^{n-2}}, \frac{-\xi_n}{\xi_{n-1}}, 1),$$

which satisfies $\xi \cdot \mu^{(2)} = 0$, $|\mu^{(2)}| = 1$ and $\mu_n^{(2)} > 0$. Since $n \geq 3$, we can find a third unit vector $\mu^{(1)}$ so that $\{\mu^{(1)}, \mu^{(2)}, \xi\}$ are mutually orthogonal. Thus (4.5.11) indicates that $q^{(1)} \widehat{\chi_{\Sigma \cap B}}(\xi) = q^{(2)} \widehat{\chi_{\Sigma \cap B}}(\xi)$ for ξ with $\xi_{n-1} \neq 0$, and therefore for all $\xi \in \mathbb{R}^n$ as both Fourier transforms are continuous functions. This completes the proof of Theorem 4.1.1.

4.5.2 Proof of Theorem 4.1.2

In this section we show Theorem 4.1.2. First, arguing as in the proof of Theorem 4.1.1, we derive identity (4.3.9) for all $u_1 \in \mathcal{W}_{l_2}(\Sigma \cap B)$ and $v \in \mathcal{V}_{l_2}(\Sigma \cap B)$. Next, we construct CGO solutions to be used in the proof of Theorem 1.2. We have constructed $u_1 \in \mathcal{W}_{l_2}(\Sigma \cap B)$ in (4.4.5), now we construct $v \in \mathcal{V}_{l_2}(\Sigma \cap B)$. As in the construction of u_1 , we will reflect the coefficients with respect to the plane $x_n = 0$. Recall that we have introduced $A^{(3)} = \overline{A^{(2)}}$ and $q^{(3)} = i^{-1} \nabla \cdot \overline{A^{(2)}} + \overline{q^{(2)}}$ so that $\mathcal{L}_{A^{(2)}, q^{(2)}}^* = \mathcal{L}_{A^{(3)}, q^{(3)}}$. For $A_j^{(3)}, j = 1, \dots, n-1$ and $q^{(3)}$, we extend them as even functions with respect to $x_n = 0$; for $A_n^{(3)}$, we extend it as an odd function with respect to $x_n = 0$, i.e. we set

$$\begin{aligned} \tilde{A}_j^{(3)}(x) &= \begin{cases} A_j^{(3)}(x', x_n) & 0 < x_n < L \\ A_j^{(3)}(x', -x_n) & -L < x_n < 0 \end{cases}, \quad j = 1, \dots, n-1 \\ \tilde{A}_n^{(3)}(x) &= \begin{cases} A_n^{(3)}(x', x_n) & 0 < x_n < L \\ -A_n^{(3)}(x', -x_n) & -L < x_n < 0 \end{cases} \\ \tilde{q}^{(3)}(x) &= \begin{cases} q^{(3)}(x', x_n) & 0 < x_n < L \\ q^{(3)}(x', -x_n) & -L < x_n < 0 \end{cases}. \end{aligned}$$

Without loss of generality we assume $A_n^{(3)}|_{x_n=0} = 0$, as we did before. Then $\tilde{A}^{(3)} \in W^{1,\infty}((\Sigma \cap B) \cup (\Sigma \cap B)_0^*)$ and $\tilde{q}^{(3)} \in L^\infty((\Sigma \cap B) \cup (\Sigma \cap B)_0^*)$. Proposition 4.2.3 implies that there exist CGO solutions of the form

$$\tilde{v}(x, \zeta_2, h) = e^{x \cdot \zeta_2 / h} (a_2(x, \zeta_2^{(0)}) + r_2(x, \zeta_2, h)) \in H^2((\Sigma \cap B) \cup (\Sigma \cap B)_0^*)$$

which satisfy the equation $\mathcal{L}_{\tilde{A}^{(3)}, \tilde{q}^{(3)}} v = 0$ in the bounded region $(\Sigma \cap B) \cup (\Sigma \cap B)_0^*$ with

$$((i\mu^{(1)} - \mu^{(2)}) \cdot \nabla)^2 a_2 = 0 \quad \text{in } (\Sigma \cap B) \cup (\Sigma \cap B)_0^*, \quad (4.5.12)$$

$$\|r_2\|_{H_{\text{scl}}^1((\Sigma \cap B) \cup (\Sigma \cap B)_0^*)} = \mathcal{O}(h). \quad (4.5.13)$$

Let

$$v(x) := \tilde{v}(x', x_n) - \tilde{v}(x', -x_n) \quad x \in \Sigma \cap B. \quad (4.5.14) \quad \boxed{\text{CGO_v2}}$$

Then it is easy to see that $v \in \mathcal{V}_2(\Sigma \cap B)$.

It will be convenient to write down the CGO solutions (4.5.14) explicitly for future references:

$$v(x) = e^{x \cdot \zeta_2/h} (a_2(x) + r_2(x)) - e^{(x', -x_n) \cdot \zeta_2/h} (a_2(x', -x_n) + r_2(x', -x_n)) \quad (4.5.15) \quad \boxed{\text{CGO3}}$$

where $a_2 \in C^\infty(\overline{(\Sigma \cap B) \cup (\Sigma \cap B)_0^*})$ and

$$((i\mu^{(1)} - \mu^{(2)}) \cdot \nabla)^2 a_2 = 0 \quad \text{in } (\Sigma \cap B) \cup (\Sigma \cap B)_0^*, \quad (4.5.16) \quad \boxed{\text{condition5}}$$

$$\|r_2\|_{H_{\text{scl}}^1((\Sigma \cap B) \cup (\Sigma \cap B)_0^*)} = \mathcal{O}(h). \quad (4.5.17) \quad \boxed{\text{condition6}}$$

We will substitute the solutions (4.4.7) and (4.5.15) into (4.3.9). To this end we compute

$$\begin{aligned} e^{x \cdot \zeta_1/h} e^{x \cdot \bar{\zeta}_2/h} &= e^{ix \cdot \xi} \\ e^{x \cdot \zeta_1/h} e^{(x', -x_n) \cdot \bar{\zeta}_2/h} &= e^{ix \cdot \xi_+ + 2\mu_n^{(2)} x_n/h} \\ e^{(x', -x_n) \cdot \zeta_1/h} e^{x \cdot \bar{\zeta}_2/h} &= e^{ix \cdot \xi_- - 2\mu_n^{(2)} x_n/h} \\ e^{(x', -x_n) \cdot \zeta_1/h} e^{(x', -x_n) \cdot \bar{\zeta}_2/h} &= e^{i(x', -x_n) \cdot \xi} \end{aligned} \quad (4.5.18) \quad \boxed{\text{exp2}}$$

where

$$\xi_\pm = \left(\xi', \pm \frac{2}{h} \sqrt{1 - h^2 \frac{|\xi|^2}{4} \mu_n^{(1)}} \right).$$

Moreover, we assume $\mu_n^{(1)} \neq 0$ and $\mu_n^{(2)} = 0$, so $\xi_\pm \rightarrow \infty$ as $h \rightarrow 0$. Then we have

$$\zeta_1 \cdot \int_{\Sigma \cap B} (A^{(1)} - A^{(2)}) e^{x \cdot \zeta_1/h} e^{(x', -x_n) \cdot \bar{\zeta}_2/h} a_1 a_2 dx \rightarrow 0 \quad (4.5.19) \quad \boxed{\text{vanish2}}$$

as $h \rightarrow 0$ by the Riemann-Lebesgue lemma. Similarly

$$\zeta_1 \cdot \int_{\Sigma \cap B} (A^{(1)} - A^{(2)}) e^{(x', -x_n) \cdot \bar{\zeta}_1/h} e^{x \cdot \bar{\zeta}_2/h} a_1 a_2 dx \rightarrow 0 \quad (4.5.20) \quad \boxed{\text{vanish3}}$$

as $h \rightarrow 0$. Therefore, multiplying (4.3.9) by h and taking the limit $h \rightarrow 0$ gives

$$\begin{aligned} & (\mu^{(1)} - i\mu^{(2)}) \cdot \int_{\Sigma \cap B} (A^{(1)} - A^{(2)}) e^{ix \cdot \xi} a_1 \bar{a}_2 dx \\ & + ((\mu^{(1)})' - i(\mu^{(2)})', -(\mu^{(1)} - i\mu^{(2)})) \cdot \int_{\Sigma \cap B} (A^{(1)} - A^{(2)}) e^{i(x', -x_n) \cdot \xi} \\ & a_1(x', -x_n) \overline{a_2(x', -x_n)} dx \rightarrow 0. \end{aligned}$$

Set $\tilde{A}^{(2)} = \overline{\tilde{A}^{(3)}}$. After a change of variable, this expression becomes

$$(\mu^{(1)} - i\mu^{(2)}) \cdot \int_{(\Sigma \cap B) \cup (\Sigma \cap B)_0^*} (\tilde{A}^{(1)} - \tilde{A}^{(2)}) e^{ix \cdot \xi} a_1 \bar{a}_2 dx = 0 \quad (4.5.21) \quad \boxed{\text{mu1}}$$

for all $\xi, \mu^{(1)}, \mu^{(2)} \in \mathbb{R}^n$ with

$$\mu^{(1)} \cdot \mu^{(2)} = \xi \cdot \mu^{(1)} = \xi \cdot \mu^{(2)} = 0, \quad |\mu^{(1)}| = |\mu^{(2)}| = 1, \quad \mu_n^{(2)} = 0, \quad \mu_n^{(1)} \neq 0. \quad (4.5.22) \quad \boxed{\text{perp}}$$

Replacing $\mu^{(1)}$ by $-\mu^{(1)}$ to get

$$(\mu^{(1)} + i\mu^{(2)}) \cdot \int_{(\Sigma \cap B) \cup (\Sigma \cap B)_0^*} (\tilde{A}^{(1)} - \tilde{A}^{(2)}) e^{ix \cdot \xi} a_1 \bar{a}_2 dx = 0. \quad (4.5.23) \quad \boxed{\text{mu2}}$$

Hence, (4.5.21) and (4.5.23) imply that

$$\mu \cdot \int_{(\Sigma \cap B) \cup (\Sigma \cap B)_0^*} (\tilde{A}^{(1)} - \tilde{A}^{(2)}) e^{ix \cdot \xi} a_1 \bar{a}_2 dx = 0. \quad (4.5.24) \quad \boxed{\text{mu3}}$$

for all $\mu \in \text{span}\{\mu^{(1)}, \mu^{(2)}\}$ and all $\xi \in \mathbb{R}^n$ for which (4.5.22) holds.

Next proposition indicates that $d\tilde{A}^{(1)} = d\tilde{A}^{(2)}$ in $(\Sigma \cap B) \cup (\Sigma \cap B)_0^*$.

curlvanish2 **Proposition 4.5.3.**

$$\partial_j(\tilde{A}_k^{(1)} - \tilde{A}_k^{(2)}) - \partial_k(\tilde{A}_j^{(1)} - \tilde{A}_j^{(2)}) = 0 \text{ in } (\Sigma \cap B) \cup (\Sigma \cap B)_0^*, \quad 1 \leq j, k \leq n. \quad (4.5.25) \quad \boxed{\text{curl2}}$$

Proof. If $n = 3$, for any vector $\xi \in \mathbb{R}^3$ with $\xi_1^2 + \xi_2^2 > 0$, it is easy to see the vectors

$$\begin{aligned} \mu^{(1)} &= \frac{\tilde{\mu}^{(1)}}{|\tilde{\mu}^{(1)}|} & \tilde{\mu}^{(1)} &= (-\xi_1 \xi_3, -\xi_2 \xi_3, \xi_1^2 + \xi_2^2), \\ \mu^{(2)} &= \left(\frac{-\xi_2}{\sqrt{\xi_1^2 + \xi_2^2}}, \frac{\xi_1}{\sqrt{\xi_1^2 + \xi_2^2}}, 0 \right), \end{aligned}$$

satisfy (4.5.22). Thus, after choosing $a_1 = a_2 = 1$, (4.5.24) gives

$$\mu \cdot f(\xi) = 0 \quad \text{where } f(\xi) := \widehat{\tilde{A}^{(1)}\chi_{(\Sigma \cap B) \cup (\Sigma \cap B)_0^*}}(\xi) - \widehat{\tilde{A}^{(2)}\chi_{(\Sigma \cap B) \cup (\Sigma \cap B)_0^*}}(\xi) \quad (4.5.26) \quad \boxed{\text{muperp}}$$

for all $\mu \in \text{span}\{\mu^{(1)}, \mu^{(2)}\}$. Here $\chi_{(\Sigma \cap B) \cup (\Sigma \cap B)_0^*}$ stands for the characteristic function of the set $(\Sigma \cap B) \cup (\Sigma \cap B)_0^*$. We decompose $f(\xi) \in \mathbb{R}^3$ as

$$f(\xi) = \alpha(\xi)\xi + f_{\perp}(\xi)$$

where $\text{Re } \alpha(\xi)$, $\text{Im } \alpha(\xi)$ are real numbers, and $\text{Re } f_{\perp}(\xi)$, $\text{Im } f_{\perp}(\xi)$ are orthogonal to ξ . As $n = 3$, we conclude that $\text{Re } f_{\perp}(\xi)$, $\text{Im } f_{\perp}(\xi) \in \text{span}\{\mu^{(1)}, \mu^{(2)}\}$. It follows from (4.5.26) that $f_{\perp}(\xi) = 0$ for all $\xi \in \mathbb{R}^3$ with $\xi_1^2 + \xi_2^2 > 0$. Hence $f(\xi) = \alpha(\xi)\xi$. Choose $\mu = -\xi_k e_j + \xi_j e_k$, $1 \leq j, k \leq 3$, $j \neq k$, where e_j is the standard orthonormal basis of \mathbb{R}^3 . This choice of μ satisfies $\mu \cdot f(\xi) = 0$. Therefore,

$$\xi_j \cdot (\widehat{\tilde{A}_k^{(1)}\chi(\xi)} - \widehat{\tilde{A}_k^{(2)}\chi(\xi)}) - \xi_k \cdot (\widehat{\tilde{A}_j^{(1)}\chi(\xi)} - \widehat{\tilde{A}_j^{(2)}\chi(\xi)}) = 0$$

for all $\xi \in \mathbb{R}^3$ with $\xi_1^2 + \xi_2^2 > 0$, and hence everywhere by analyticity of the Fourier transform.

If $n \geq 4$, for any vector $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$, $\xi_l \neq 0$, $l = 1, \dots, n$, define vectors

$$\mu^{(1)} = (-\xi_j \xi_n) e_j + (-\xi_k \xi_n) e_k + (\xi_j^2 + \xi_k^2) e_n, \quad \mu^{(2)} = -\xi_k e_j + \xi_j e_k$$

where $1 \leq j, k < n$, $j \neq k$. It is easy to check that $\mu^{(1)} \cdot \mu^{(2)} = \mu^{(1)} \cdot \xi = \mu^{(2)} \cdot \xi = 0$, $\mu_n^{(2)} = 0$ and $\mu_n^{(1)} \neq 0$. Thus, after choosing $a_1 = a_2 = 1$ and $\mu = \mu^{(2)}$, (4.5.24) implies

$$\xi_j \cdot (\widehat{\tilde{A}_k^{(1)}\chi(\xi)} - \widehat{\tilde{A}_k^{(2)}\chi(\xi)}) - \xi_k \cdot (\widehat{\tilde{A}_j^{(1)}\chi(\xi)} - \widehat{\tilde{A}_j^{(2)}\chi(\xi)}) = 0 \quad 1 \leq j, k < n, j \neq k \quad (4.5.27) \quad \boxed{\text{vanish4}}$$

for all $\xi \in \mathbb{R}^n$ with $\xi_l \neq 0$, $l = 1, 2, \dots, n$.

Let $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$ with $\xi_l \neq 0$, $l = 1, 2, \dots, n$, and let $1 \leq j < n$. Choose indices k and l so that the set $\{j, k, l, n\}$ consists of four distinct numbers. Define

$$\mu^{(1)} = -\xi_n e_j + \xi_j e_n, \quad \mu^{(2)} = -\xi_k e_l + \xi_l e_k.$$

Again one can check that $\mu^{(1)} \cdot \mu^{(2)} = \mu^{(1)} \cdot \xi = \mu^{(2)} \cdot \xi = 0$, $\mu_n^{(2)} = 0$ and $\mu_n^{(1)} \neq 0$. After choosing $a_1 = a_2 = 1$ and $\mu = \mu^{(1)}$, (4.5.24) implies

$$\xi_j \cdot (\widehat{\tilde{A}_n^{(1)}\chi(\xi)} - \widehat{\tilde{A}_n^{(2)}\chi(\xi)}) - \xi_n \cdot (\widehat{\tilde{A}_j^{(1)}\chi(\xi)} - \widehat{\tilde{A}_j^{(2)}\chi(\xi)}) = 0 \quad 1 \leq j < n \quad (4.5.28) \quad \boxed{\text{vanish5}}$$

for all $\xi \in \mathbb{R}^n$ with $\xi_l \neq 0, l = 1, 2, \dots, n$. The result in the case $n \geq 4$ then follows from (4.5.27) and (4.5.28). \square

Arguing as in the proof of Theorem 4.1.1, we can find compactly supported function $\Phi \in C^{1,1}(\overline{\Sigma \cup \Sigma_0^*})$ such that

$$\tilde{A}^{(1)} - \tilde{A}^{(2)} = \nabla \Phi \quad \text{in } (\Sigma \cap B) \cup (\Sigma \cap B)_0^*$$

and $\Phi = 0$ on $\partial((\Sigma \cap B) \cup (\Sigma \cap B)_0^*)$. In (4.5.24), pick $a_2 = 1$, a_1 satisfying

$$((i\mu^{(1)} + \mu^{(2)}) \cdot \nabla) a_1 = 1 \quad \text{in } (\Sigma \cap B) \cup (\Sigma \cap B)_0^*.$$

and $\mu = i\mu^{(1)} + \mu^{(2)}$. Integrating by parts we obtain

$$\begin{aligned} 0 &= (i\mu^{(1)} + \mu^{(2)}) \cdot \int_{(\Sigma \cap B) \cup (\Sigma \cap B)_0^*} (\nabla \Phi) e^{ix \cdot \xi} a_1 dx \\ &= \int_{(\Sigma \cap B) \cup (\Sigma \cap B)_0^*} \Phi(x) e^{ix \cdot \xi} [(i\mu^{(1)} + \mu^{(2)}) \cdot \nabla a_1] dx \\ &= \int_{(\Sigma \cap B) \cup (\Sigma \cap B)_0^*} \Phi(x) e^{ix \cdot \xi} dx. \end{aligned}$$

This implies that $\Phi = 0$ in $(\Sigma \cap B) \cup (\Sigma \cap B)_0^*$. Hence $\tilde{A}^{(1)} = \tilde{A}^{(2)}$, and therefore, $A^{(1)} = A^{(2)}$ in $\Sigma \cap B$.

As for electric potentials $q^{(1)}$ and $q^{(2)}$, continuing to argue as in the proof of Theorem 1.1, we arrive at

$$\int_{(\Sigma \cap B) \cup (\Sigma \cap B)_0^*} (q^{(1)} - q^{(2)}) e^{ix \cdot \xi} dx = 0 \quad (4.5.29) \quad \boxed{\text{vanish6}}$$

for all $\mu^{(1)}, \mu^{(2)}, \xi \in \mathbb{R}^n$ satisfying (4.5.22). For any vector $\xi \in \mathbb{R}^n$ with $\xi_{n-2}^2 + \xi_{n-1}^2 > 0$, the vectors

$$\begin{aligned} \mu^{(1)} &= \frac{\tilde{\mu}^{(1)}}{|\tilde{\mu}^{(1)}|} \quad \tilde{\mu}^{(1)} = \left(0_{\mathbb{R}^{n-3}}, \frac{-\xi_n \xi_{n-2}}{\sqrt{\xi_{n-2}^2 + \xi_{n-1}^2}}, \frac{-\xi_n \xi_{n-1}}{\sqrt{\xi_{n-2}^2 + \xi_{n-1}^2}}, \sqrt{\xi_{n-2}^2 + \xi_{n-1}^2} \right), \\ \mu^{(2)} &= \left(0_{\mathbb{R}^{n-3}}, \frac{-\xi_{n-1}}{\sqrt{\xi_{n-2}^2 + \xi_{n-1}^2}}, \frac{\xi_{n-2}}{\sqrt{\xi_{n-2}^2 + \xi_{n-1}^2}}, 0 \right), \end{aligned}$$

satisfy (4.5.24). Thus, (4.5.29) holds for all $\xi \in \mathbb{R}^n$ with $\xi_{n-2}^2 + \xi_{n-1}^2 > 0$. We conclude that (4.5.29) also holds for all $\xi \in \mathbb{R}^n$ by the analyticity of the Fourier transform. This completes the proof of Theorem 4.1.2.

4.5.3 Proof of Theorem 4.1.3

In this section we prove Theorem 4.1.3. Let $\Omega_1 \subset\subset \Omega$ be a bounded sub-domain with C^∞ boundary and be such that $\Omega \setminus \bar{\Omega}_1$ is connected and $\text{supp}(A^{(1)} - A^{(2)})$ and $\text{supp}(q^{(1)} - q^{(2)})$ are contained in Ω_1 .

Let $u_1 \in H^4(\Omega)$ be the solution to the Dirichlet problem

$$\begin{cases} \mathcal{L}_{A^{(1)},q^{(1)}} u_1 = 0 & \text{in } \Omega \\ u_1 = f_1 & \text{on } \partial\Omega \\ \Delta u_1 = f_2 & \text{on } \partial\Omega \end{cases}$$

with $(f_1, f_2) \in H^{\frac{7}{2}}(\partial\Omega) \times H^{\frac{3}{2}}(\partial\Omega)$ and $\text{supp}(f_1) \subset \gamma_1, \text{supp}(f_2) \subset \gamma_1$. Since by assumption $C_{A^{(1)},q^{(1)}}^{\gamma_1,\gamma_2}(\Omega) = C_{A^{(2)},q^{(2)}}^{\gamma_1,\gamma_2}(\Omega)$, there is $u_2 \in H^4(\Omega)$ which satisfies

$$\mathcal{L}_{A^{(2)},q^{(2)}} u_2 = 0 \quad \text{in } \Omega,$$

$$u_2 = u_1 \quad \Delta u_2 = \Delta u_1 \quad \text{on } \partial\Omega,$$

$$\partial_\nu u_1|_{\gamma_2} = \partial_\nu u_2|_{\gamma_2}, \quad \partial_\nu(\Delta u_1)|_{\gamma_2} = \partial_\nu(\Delta u_2)|_{\gamma_2}.$$

Setting $w = u_2 - u_1$, then $w = \Delta w = 0$ on $\partial\Omega$, $\partial_\nu w = \partial_\nu(\Delta w) = 0$ on γ_2 , and

$$\mathcal{L}_{A^{(2)},q^{(2)}} w = (A^{(1)} - A^{(2)}) \cdot Du_1 + (q^{(1)} - q^{(2)})u_1 \quad \text{in } \Omega.$$

Therefore, w is a solution of

$$\mathcal{L}_{A^{(2)},q^{(2)}} w = 0 \quad \text{in } \Omega \setminus \bar{\Omega}_1$$

with $w = \partial_\nu w = 0$ on γ_2 . By unique continuation, we obtain that $w = 0$ in $\Omega \setminus \bar{\Omega}_1$. Thus, $w = \Delta w = \partial_\nu w = \partial_\nu \Delta w = 0$ on $\partial\Omega_1$.

Let $v \in H^4(\Omega_1)$ be a solution of

$$\mathcal{L}_{A^{(2)},q^{(2)}}^* v = 0 \quad \text{in } \Omega_1 \tag{4.5.30} \quad \boxed{\text{equation}}$$

Using Green's formula (4.3.1) over Ω_1 , we have

$$\int_{\Omega_1} ((A^{(1)} - A^{(2)}) \cdot Du_1) \bar{v} \, dx + \int_{\Omega_1} (q^{(1)} - q^{(2)}) u_1 \bar{v} \, dx = 0 \tag{4.5.31} \quad \boxed{\text{difference2}}$$

for all $v \in H^4(\Omega_1)$ satisfying (4.5.30) and for all $u_1 \in \mathcal{W}(\Omega)$, where

$$\mathcal{W}(\Omega) := \{u \in H^4(\Omega) : \mathcal{L}_{A^{(1)}, q^{(1)}} u = 0 \text{ in } \Omega, \text{supp}(u|_{\partial\Omega}) \subset \gamma_1, \text{supp}(\Delta u|_{\partial\Omega}) \subset \gamma_1\}.$$

Let

$$\widetilde{\mathcal{W}}(\Omega_1) := \{u \in H^4(\Omega_1) : \mathcal{L}_{A^{(1)}, q^{(1)}} u = 0 \text{ in } \Omega_1\}.$$

Again we need a density result to pass from $\mathcal{W}(\Omega)$ to $\widetilde{\mathcal{W}}(\Omega_1)$.

Proposition 4.5.4. *$\mathcal{W}(\Omega)$ is a dense subspace in $\widetilde{\mathcal{W}}(\Omega_1)$ in $L^2(\Omega_1)$ -topology.*

Proof. It suffices to establish the following fact: for any $g \in L^2(\Omega_1)$ such that

$$\int_{\Omega_1} u \bar{g} \, dx = 0 \quad \forall u \in \mathcal{W}(\Omega),$$

we have

$$\int_{\Omega_1} v \bar{g} \, dx = 0 \quad \forall v \in \widetilde{\mathcal{W}}(\Omega).$$

To this end, extend g by zero to $\Omega \setminus \Omega_1$. Let $U \in H^4(\Omega)$ be the solution of the Dirichlet problem

$$\begin{aligned} \mathcal{L}_{A^{(1)}, q^{(1)}}^* U &= g && \text{in } \Omega \\ U = \Delta U &= 0 && \text{on } \partial\Omega. \end{aligned}$$

For any $u \in \mathcal{W}(\Omega)$, Green's formula on bounded domain Ω gives

$$\begin{aligned} 0 &= \int_{\Omega} u \bar{g} \, dx = \int_{\Omega} u \overline{(\mathcal{L}_{A^{(1)}, q^{(1)}}^* U)} \, dx \\ &= - \int_{\partial\Omega} (-\Delta u) \bar{\partial}_\nu U \, dS - \int_{\partial\Omega} u \overline{(\partial_\nu(-\Delta U))} \, dS \end{aligned}$$

where we have used $U = \Delta U = 0$ on $\partial\Omega$. Since $u|_{\gamma_1}$ and $\Delta u|_{\gamma_1}$ can be arbitrary smooth functions supported in γ_1 , we conclude that $\partial_\nu U|_{\gamma_1} = \partial_\nu \Delta U|_{\gamma_1} = 0$. Hence U satisfies $\mathcal{L}_{A^{(1)}, q^{(1)}}^* U = 0$ in $\Omega \setminus \Omega_1$, and $U = \Delta U = \partial_\nu U = \partial_\nu(\Delta U) = 0$ on γ_1 . By unique continuation, $U = 0$ in $\Omega \setminus \Omega_1$, and therefore, $U = \Delta U = \partial_\nu U = \partial_\nu(\Delta U) = 0$ on $\partial\Omega_1$.

For any $v \in \widetilde{\mathcal{W}}(\Omega_1)$, using Green's formula over Ω_1 we get

$$\begin{aligned}
\int_{\Omega_1} v \bar{g} \, dx &= \int_{\Omega_1} v \overline{(\mathcal{L}_{A^{(1)}, q^{(1)}}^* U)} \, dx \\
&= \int_{\Omega_1} (\mathcal{L}_{A^{(1)}, q^{(1)}} v) \bar{U} \, dx + i \int_{\partial\Omega_1} \nu(x) \cdot A \bar{U} v \, dS \\
&\quad + \int_{\partial\Omega_1} \partial_\nu(-\Delta v) \bar{U} \, dS - \int_{\partial\Omega_1} (-\Delta v) \overline{\partial_\nu U} \, dS \\
&\quad + \int_{\partial\Omega_1} \partial_\nu v \overline{(-\Delta U)} \, dS - \int_{\partial\Omega_1} v \overline{(\partial_\nu(-\Delta U))} \, dS \\
&= 0.
\end{aligned}$$

□

We conclude from this proposition that (4.5.31) holds for all $u \in \widetilde{\mathcal{W}}(\Omega_1)$ and $v \in H^4(\Omega_1)$ satisfying (4.5.30).

Let $B \subset \mathbb{R}^n$ be an open ball such that $\Omega_1 \subset B$. The fact that $A^{(1)} = A^{(2)}$ and $q^{(1)} = q^{(2)}$ on $\partial\Omega_1$ allows to extend $A^{(j)}$ and $q^{(j)}$ to B in such a way that the extensions, still denoted by $A^{(j)}$ and $q^{(j)}$, coincide on $B \setminus \Omega_1$, have compact supports, and satisfy $A^{(j)} \in W^{1,\infty}(B)$, $q^{(j)} \in L^\infty(B)$. It follows from (4.5.31) that

$$\int_B ((A^{(1)} - A^{(2)}) \cdot Du_1) \bar{v} \, dx + \int_B (q^{(1)} - q^{(2)}) u_1 \bar{v} \, dx = 0$$

for all $u_1, v \in H^4(B)$ which are solutions of

$$\mathcal{L}_{A^{(1)}, q^{(1)}} u_1 = 0 \text{ in } B \quad \mathcal{L}_{A^{(2)}, q^{(2)}}^* v = 0 \text{ in } B.$$

Now we are in the same situation as in [35] for the bi-harmonic operator, and as in [36] with full boundary measurements. We can construct complex geometric optics solutions as in Proposition 4.2.3, and proceed as in [35], [36] and the proof of Theorem 4.1.1 to show that $A^{(1)} = A^{(2)}$ and $q^{(1)} = q^{(2)}$ in Ω .

4.5.4 Proof of Theorem 4.4.4

In this section we prove Theorem 4.1.4. First, as in the proof of Theorem 4.1.1 and Theorem 4.1.2, after applying Green's formula over Ω , we obtain the integral identity

$$\int_{\Omega} ((A^{(1)} - A^{(2)}) \cdot Du_1) \bar{v} \, dx + \int_{\Omega} (q^{(1)} - q^{(2)}) u_1 \bar{v} \, dx = 0 \quad (4.5.32) \quad \boxed{\text{difference3}}$$

for all $u_1, v \in H^4(\Omega)$ such that

$$\mathcal{L}_{A^{(1)}, q^{(1)}} u_1 = 0 \text{ in } \Omega, \quad u_1|_{x_n=0} = (\Delta u_1)|_{x_n=0} = 0;$$

$$\mathcal{L}_{A^{(2)}, q^{(2)}}^* v = 0 \text{ in } \Omega, \quad v|_{x_n=0} = (\Delta v)|_{x_n=0} = 0.$$

Applying the reflection argument as in the proof of Theorem 4.1.2, we can construct CGO solutions u_1 and v , as in (4.4.7) and (4.5.15), to the above equations and with the corresponding boundary conditions. Substituting these solutions u_1 and v into (4.5.32) and proceeding as in the proof of Theorem 4.1.2 we get

$$\left(\mu^{(1)} - i\mu^{(2)}\right) \cdot \int_{\Omega \cup \Omega_0^*} \left(\tilde{A}^{(1)} - \tilde{A}^{(2)}\right) e^{ix \cdot \xi} a_1 \bar{a}_2 dx = 0 \quad (4.5.33) \quad \boxed{\mu 4}$$

for all $\xi, \mu^{(1)}, \mu^{(2)} \in \mathbb{R}^n$ such that

$$\mu^{(1)} \cdot \mu^{(2)} = \xi \cdot \mu^{(1)} = \xi \cdot \mu^{(2)} = 0, \quad |\mu^{(1)}| = |\mu^{(2)}| = 1, \quad \mu_n^{(2)} = 0, \quad \mu_n^{(1)} \neq 0,$$

where we have introduced the notation $\Omega_0^* := \{(x', x_n) \in \mathbb{R}^n : (x', -x_n) \in \Omega\}$. Applying the boundary reconstruction result [35, Proposition 4.1] we conclude that $A^{(1)} = A^{(2)}$ on $\bar{\gamma}$, hence $\tilde{A}^{(1)} = \tilde{A}^{(2)}$ on $\partial(\Omega \cup \Omega_0^*)$. This allows us to extend $\tilde{A}^{(j)}$, $j = 1, 2$, to compactly supported vector fields on a large ball B with $\Omega \cup \Omega_0^* \subset\subset B$ and $\tilde{A}^{(1)} = \tilde{A}^{(2)}$ in $B \setminus \Omega \cup \Omega_0^*$. Then (4.5.33) leads to

$$\left(\mu^{(1)} - i\mu^{(2)}\right) \cdot \int_B \left(\tilde{A}^{(1)} - \tilde{A}^{(2)}\right) e^{ix \cdot \xi} a_1 \bar{a}_2 dx = 0.$$

From Proposition 4.5.3 we have $d\tilde{A}^{(1)} = d\tilde{A}^{(2)}$ in B . Therefore, there exists $\Phi \in C^{1,1}(\bar{B})$ so that

$$\tilde{A}^{(1)} - \tilde{A}^{(2)} = \nabla \Phi \quad \text{in } B.$$

As before we can show that $\Phi = 0$ on $\partial(\Omega \cup \Omega_0^*)$; in particular, $\Phi = 0$ on $\bar{\gamma}$. Now we are facing the same situation as in the proof of Theorem 4.1.2. Arguing as there we conclude that $A^{(1)} = A^{(2)}$ and $q^{(1)} = q^{(2)}$. This completes the proof of Theorem 4.1.4.

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Appendices

Appendix A

SOLVABILITY OF THE FORWARD PROBLEM IN AN INFINITE SLAB

In this appendix we provide the proof of the existence of the forward boundary value problem (4.1.1) for the perturbed bi-harmonic operator in an infinite slab. Recall that the perturbed bi-harmonic operator is of the form

$$\mathcal{L}_{A,q}(x, D) := \Delta^2 + A(x) \cdot D + q(x).$$

The infinite slab is written as ($n \geq 3$)

$$\Sigma = \{x = (x', x_n) \in \mathbb{R}^n : x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}, 0 < x_n < L\}, \quad L > 0.$$

whose boundary hyperplanes are

$$\Gamma_1 = \{x \in \mathbb{R}^n : x_n = L\} \quad \Gamma_2 = \{x \in \mathbb{R}^n : x_n = 0\}.$$

We will rewrite the perturbed bi-harmonic equation as a system of equations. For this purpose, let $u = (u_1, u_2)$ with $u_2 = \Delta u_1$, define

$$\mathcal{S}u := \Delta \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ A \cdot D + q & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

then $\mathcal{L}_{A,q}u_1 = 0$ is equivalent to $\mathcal{S}u = 0$. We will show the existence of a unique solution to this system with boundary value $(u_1, u_2)|_{\partial\Sigma} = (f_1, f_2)$.

Poincaré's inequality in an infinite slab indicates that the quadratic form

$$u \mapsto \int_{\Sigma} |\nabla u|^2 dx = \int_{\Sigma} (|\nabla u_1|^2 + |\nabla u_2|^2) dx$$

is non-negative and densely defined closed on $H_0^1(\Sigma) \times H_0^1(\Sigma)$. Associated with this quadratic form, the Laplace operator $-\Delta$ equipped with the domain

$$\mathcal{D}(-\Delta) := \{u \in H_0^1(\Sigma) \times H_0^1(\Sigma) : \Delta u = (\Delta u_1, \Delta u_2) \in L^2(\Sigma) \times L^2(\Sigma)\}$$

is a non-negative self-adjoint operator on $L^2(\Sigma) \times L^2(\Sigma)$. Its spectrum is obtained in the following proposition.

Proposition A.0.5. $\mathcal{D}(-\Delta) = H_0^1(\Sigma) \cap H^2(\Sigma) \times H_0^1(\Sigma) \cap H^2(\Sigma)$. Moreover, the spectrum of $-\Delta$ is purely absolutely continuous and is equal to $[\pi^2/L^2, +\infty)$.

Proof. Let $F = (F_1, F_2) \in L^2(\Sigma) \times L^2(\Sigma)$, we will consider

$$-\Delta u = F, \quad u \in \mathcal{D}(-\Delta).$$

Taking the Fourier series with respect to the variable $x_n \in [0, L]$ we have

$$\begin{aligned} u(x', x_n) &= \sum_{l=1}^{\infty} u_l(x') \sin \frac{l\pi x_n}{L}, & u_l(x') &= \frac{2}{L} \int_0^L u(x) \sin \frac{l\pi x_n}{L} dx_n; \\ F(x', x_n) &= \sum_{l=1}^{\infty} F_l(x') \sin \frac{l\pi x_n}{L}, & F_l(x') &= \frac{2}{L} \int_0^L F(x) \sin \frac{l\pi x_n}{L} dx_n. \end{aligned} \tag{A.0.1} \quad \boxed{\text{Fourier}}$$

As usual Parseval's identities hold

$$\begin{aligned} \|u\|_{L^2(\Sigma) \times L^2(\Sigma)} &= \frac{L}{2} \sum_{l=1}^{\infty} \|u_l\|_{L^2(\mathbb{R}^{n-1}) \times L^2(\mathbb{R}^{n-1})}^2, \\ \|F\|_{L^2(\Sigma) \times L^2(\Sigma)} &= \frac{L}{2} \sum_{l=1}^{\infty} \|F_l\|_{L^2(\mathbb{R}^{n-1}) \times L^2(\mathbb{R}^{n-1})}^2. \end{aligned}$$

Comparing the Fourier coefficients u_l of u and F_l of F we see that they are related by

$$\left(-\Delta_{x'} + \frac{l^2\pi^2}{L^2}\right) u_l(x') = F_l(x'), \quad x' \in \mathbb{R}^{n-1}, l = 1, 2, \dots \tag{A.0.2} \quad \boxed{\text{Fourieridentity}}$$

The operator $-\Delta_{x'} + \frac{l^2\pi^2}{L^2}$ ($l \geq 1$), when equipped with the domain $H^2(\mathbb{R}^{n-1})$, is self-adjoint on $L^2(\mathbb{R}^{n-1})$ with purely absolutely continuous spectrum $[l^2\pi^2/L^2, +\infty)$. Hence (A.0.2) has the unique solution

$$u_l(x') = \left(-\Delta_{x'} + \frac{l^2\pi^2}{L^2}\right)^{-1} F_l(x') \in H^2(\mathbb{R}^{n-1}),$$

and moreover, it satisfies the norm estimate

$$\begin{aligned} \|u_l\|_{L^2(\mathbb{R}^{n-1}) \times L^2(\mathbb{R}^{n-1})} &\leq \frac{L^2}{l^2\pi^2} \|F_l\|_{L^2(\mathbb{R}^{n-1}) \times L^2(\mathbb{R}^{n-1})}; \\ \|u_l\|_{H^2(\mathbb{R}^{n-1}) \times H^2(\mathbb{R}^{n-1})} &\leq C \|F_l\|_{L^2(\mathbb{R}^{n-1}) \times L^2(\mathbb{R}^{n-1})}. \end{aligned} \tag{A.0.3} \quad \boxed{\text{normestimate1}}$$

Here and in the following we will name all the constants independent of l as C . By interpolation we obtain

$$\|u_l\|_{H^1(\mathbb{R}^{n-1}) \times H^1(\mathbb{R}^{n-1})} \leq \frac{C}{l} \|F_l\|_{L^2(\mathbb{R}^{n-1}) \times L^2(\mathbb{R}^{n-1})}. \quad (\text{A.0.4}) \quad \boxed{\text{normestimate}}$$

Parseval's identities and (A.0.3) then give

$$\begin{aligned} \|u\|_{L^2(\Sigma) \times L^2(\Sigma)}^2 &= \frac{L}{2} \sum_{l=1}^{\infty} \|u_l\|_{L^2(\mathbb{R}^{n-1}) \times L^2(\mathbb{R}^{n-1})}^2 \\ &\leq C \sum_{l=1}^{\infty} \frac{1}{l^4} \|F_l\|_{L^2(\mathbb{R}^{n-1}) \times L^2(\mathbb{R}^{n-1})}^2 \leq C \|F\|_{L^2(\Sigma) \times L^2(\Sigma)}^2. \end{aligned}$$

To take care of the first order derivatives, we differentiate with respect to x_n to get

$$\begin{aligned} \|\partial_{x_n} u\|_{L^2(\Sigma) \times L^2(\Sigma)}^2 &= \left\| \sum_{l=1}^{\infty} \frac{l\pi}{L} u_l(x') \cos \frac{l\pi x_n}{L} \right\|_{L^2(\Sigma) \times L^2(\Sigma)}^2 \\ &= \frac{L}{2} \sum_{l=1}^{\infty} \frac{l^2 \pi^2}{L^2} \|u_l\|_{L^2(\mathbb{R}^{n-1}) \times L^2(\mathbb{R}^{n-1})}^2 \leq C \|F\|_{L^2(\Sigma) \times L^2(\Sigma)}^2. \end{aligned}$$

Using (A.0.4) we obtain that for $j = 1, 2, \dots, n-1$,

$$\begin{aligned} \|\partial_{x_j} u\|_{L^2(\Sigma) \times L^2(\Sigma)}^2 &= \frac{L}{2} \sum_{l=1}^{\infty} \|\partial_{x_j} u_l\|_{L^2(\mathbb{R}^{n-1}) \times L^2(\mathbb{R}^{n-1})}^2 \\ &\leq C \sum_{l=1}^{\infty} \frac{1}{l^2} \|F_l\|_{L^2(\mathbb{R}^{n-1}) \times L^2(\mathbb{R}^{n-1})}^2 \leq C \|F\|_{L^2(\Sigma) \times L^2(\Sigma)}^2. \end{aligned}$$

We proceed to estimate the second order derivatives. For $j, k = 1, 2, \dots, n-1$, it follows from (A.0.3) that

$$\begin{aligned} \|\partial_{x_j x_k}^2 u\|_{L^2(\Sigma) \times L^2(\Sigma)}^2 &= \frac{L}{2} \sum_{l=1}^{\infty} \|\partial_{x_j x_k} u_l\|_{L^2(\mathbb{R}^{n-1}) \times L^2(\mathbb{R}^{n-1})}^2 \leq C \|F\|_{L^2(\Sigma) \times L^2(\Sigma)}^2; \\ \|\partial_{x_j x_n}^2 u\|_{L^2(\Sigma) \times L^2(\Sigma)}^2 &= \frac{L}{2} \sum_{l=1}^{\infty} \frac{l^2 \pi^2}{L^2} \|\partial_{x_j} u_l\|_{L^2(\mathbb{R}^{n-1}) \times L^2(\mathbb{R}^{n-1})}^2 \leq C \|F\|_{L^2(\Sigma) \times L^2(\Sigma)}^2; \\ \|\partial_{x_n}^2 u\|_{L^2(\Sigma) \times L^2(\Sigma)}^2 &= \frac{L}{2} \sum_{l=1}^{\infty} \frac{l^4 \pi^4}{L^4} \|u_l\|_{L^2(\mathbb{R}^{n-1}) \times L^2(\mathbb{R}^{n-1})}^2 \leq C \|F\|_{L^2(\Sigma) \times L^2(\Sigma)}^2. \end{aligned}$$

These estimates show that $u \in H^2(\Sigma) \times H^2(\Sigma)$. The statement concerning the spectrum of $-\Delta$ follows from the fact that

$$-\Delta = \bigoplus_{l=1}^{\infty} \left(-\Delta_{x'} + \frac{l^2 \pi^2}{L^2} \right).$$

This completes the proof of the proposition. \square

Proposition A.0.6. *Let $A \in W^{1,\infty}(\Sigma; \mathbb{C}^n) \cap \mathcal{E}'(\bar{\Sigma}; \mathbb{C}^n)$, $q \in L^\infty(\Sigma; \mathbb{C}) \cap \mathcal{E}'(\bar{\Sigma}; \mathbb{C}^n)$. Then the operator \mathcal{S} , equipped with the domain $H_0^1(\Sigma) \cap H^2(\Sigma) \times H_0^1(\Sigma) \cap H^2(\Sigma)$, is closed and its essential spectrum is equal to $[\pi^2/L^2, +\infty)$.*

Proof. This follows from the fact that

$$\begin{pmatrix} 0 & -1 \\ A \cdot D + q & 0 \end{pmatrix} \Delta^{-1} : L^2(\Sigma) \times L^2(\Sigma) \rightarrow L^2(\Sigma) \times L^2(\Sigma)$$

is a compact operator and that the essential spectrum do not change under relatively compact perturbations. \square

This proposition yields the following solvability result. Suppose $A \in W^{1,\infty}(\Sigma; \mathbb{C}^n) \cap \mathcal{E}'(\bar{\Sigma}; \mathbb{C}^n)$ and $q \in L^\infty(\Sigma; \mathbb{C}) \cap \mathcal{E}'(\bar{\Sigma}; \mathbb{C}^n)$, then for any $F = (F_1, F_2) \in L^2(\Sigma) \times L^2(\Sigma)$, the boundary value problem

$$\begin{cases} \mathcal{S}u = F & \text{in } \Sigma \times \Sigma \\ u|_{\partial\Sigma \times \partial\Sigma} = 0. \end{cases} \quad (\text{A.0.5}) \quad \boxed{\text{Dirichlet3}}$$

admits a unique solution $u \in H^2(\Sigma) \times H^2(\Sigma)$.

Given any $f = (f_1, f_2) \in (H^{\frac{7}{2}}(\Gamma_1) \cap \mathcal{E}'(\Gamma_1)) \times (H^{\frac{3}{2}}(\Gamma_1) \cap \mathcal{E}'(\Gamma_1))$, we can establish the existence and uniqueness of the solution to the boundary value problem (4.1.1) as follows. (4.1.1) is equivalent to the following boundary value problem for the system

$$\begin{cases} \mathcal{S}u = 0 & \text{in } \Sigma \times \Sigma \\ u|_{\Gamma_1 \times \Gamma_1} = f \\ u|_{\Gamma_2 \times \Gamma_2} = 0. \end{cases} \quad (\text{A.0.6}) \quad \boxed{\text{Dirichlet4}}$$

Uniqueness of the solution to (A.0.6) follows from the unique solvability of (A.0.5) when $F = 0$. To show that (A.0.6) has at least one solution, choose $G \in H^4(\Sigma) \cap \mathcal{E}'(\bar{\Sigma}) \times H^2(\Sigma) \cap \mathcal{E}'(\bar{\Sigma})$ so that $G|_{\Gamma_1 \times \Gamma_1} = f$ and $G|_{\Gamma_2 \times \Gamma_2} = 0$; choose u_0 to be the unique solution of (A.0.5) when $F = -SG$, then $G + u_0$ is a solution for (A.0.6). This completes the proof that (4.1.1) admits a unique solution in $H^4(\Sigma)$ for f_1 and f_2 .

Appendix B

GREEN'S FORMULA IN A SLAB

In the proof of Proposition 4.3.1 we used the Green's formula in a slab, in this part we establish this identity. For $R > 0$, define Σ_R by

$$\Sigma_R := \{x \in \Sigma : |x'| < R\}.$$

We may choose $R > 0$ sufficiently large so that $\text{supp}(A^{(1)}) \subset \overline{\Sigma_R}$. Introduce the notations

$$d_j(R) := \partial\Sigma_R \cap \Gamma_j, j = 1, 2; \quad d_3(R) = \partial\Sigma_R \cap \Sigma,$$

then $A^{(1)} = 0$ on $d_3(R)$. Let $u \in W(\Sigma)$ and let $U \in H^4(\Sigma)$ be the solution of the problem

$$\begin{aligned} \mathcal{L}_{A^{(1)}, q^{(1)}}^* U &= g & \text{in } \Sigma \\ U = \Delta U &= 0 & \text{on } \Gamma_1 \cup \Gamma_2. \end{aligned}$$

Apply Green's formula (4.3.1) over the region Σ_R we obtain

$$\begin{aligned} & - \int_{\Sigma_R} u g \, dx \\ &= \int_{\Sigma_R} (\mathcal{L}_{A^{(1)}, q^{(1)}} u) \overline{U} \, dx - \int_{\Sigma_R} u \overline{(\mathcal{L}_{A^{(1)}, q^{(1)}}^* U)} \, dx \\ &= - \int_{d_3(R)} \partial_\nu (-\Delta u) \overline{U} \, dS + \int_{d_1(R) \cup d_3(R)} (-\Delta u) \overline{\partial_\nu U} \, dS \\ & \quad - \int_{d_3(R)} \partial_\nu u \overline{(-\Delta U)} \, dS + \int_{d_1(R) \cup d_3(R)} u \overline{\partial_\nu (-\Delta U)} \, dS. \end{aligned} \tag{B.0.1} \quad \boxed{\text{greenslab}}$$

We will show that the right hand side converges to

$$\int_{\Gamma_1} \overline{\partial_\nu U} \Delta u \, dS + \int_{\Gamma_1} \overline{\partial_\nu \Delta U} u \, dS. \tag{B.0.2} \quad \boxed{\text{righthandsid}}$$

To this end, notice that for $R > 0$ sufficiently large,

$$\Delta^2 u = \Delta^2 U = 0 \quad \text{in } \Sigma \setminus \Sigma_R,$$

$$\Delta u = \Delta U = 0 \quad \text{on } \partial(\Sigma \setminus \Sigma_R).$$

According to [40], we have

$$\Delta u, \partial_\nu(\Delta u), \Delta U, \partial_\nu(\Delta U) \text{ are of order } \mathcal{O}(|x'|^{-n}) \text{ as } |x'| \rightarrow \infty. \quad (\text{B.0.3})$$

We can estimate the first term on the right hand side of (B.0.1) as follows

$$\begin{aligned} & \left| - \int_{d_3(R)} \partial_\nu(-\Delta u) \bar{U} \, dS \right| = \left| \int_{|x'|=R, 0 < x_n < L} \partial_\nu(\Delta u) \bar{U} \, dS \right| \\ & \leq \left(\int_{|x'|=R, 0 < x_n < L} |\partial_\nu(\Delta u)|^2 \, dS \right)^{\frac{1}{2}} \left(\int_{|x'|=R, 0 < x_n < L} |U|^2 \, dS \right)^{\frac{1}{2}} \\ & \leq \mathcal{O}(R^{-\frac{n}{2}-1}) \left(\int_{|x'|=R, 0 < x_n < L} |U|^2 \, dS \right)^{\frac{1}{2}} \\ & \leq \mathcal{O}(R^{-\frac{n}{2}-1}) \left(\int_{\partial \Sigma_R} |U| \, dS \right)^{\frac{1}{2}} \\ & \leq C \mathcal{O}(R^{-\frac{n}{2}-1}) \|U\|_{H^2(\Sigma)} \rightarrow 0 \text{ as } R \rightarrow \infty. \end{aligned}$$

where in the last step the constant C comes from the trace theorem. Similarly, all the other terms involving $d_3(R)$ on the right hand side of (B.0.1) will vanish as $R \rightarrow \infty$. Therefore, after taking the limit $R \rightarrow \infty$ in (B.0.1), the right hand side will become (B.0.2), as we have claimed.