Analytical System Dynamics

by

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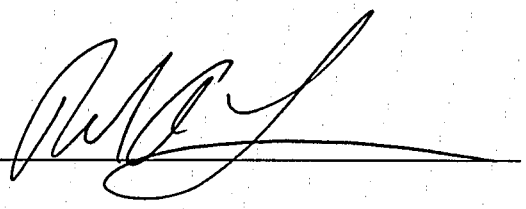
Signature 

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Abstract

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A unified approach for modeling engineering systems is presented for systems comprised of mechanical, electrical, fluid and thermal elements. The method of analysis is based on the energy methods of Lagrange, generalized to encompass constrained multidiscipline systems. The physical systems theory is developed within the framework established by Paynter. The result of analysis is a set of differential–algebraic equations (DAE) systematically formulated for numerical solution, thereby providing the engineer with a means of modeling and computation suitable for the analysis of complex engineering systems.

A new derivation of Lagrange's equation is given based on a differential–variational form of the first law of thermodynamics. Equations of motion are formulated without posing the problem as a calculus of variations problem and without invoking Hamilton's principle. Undetermined multipliers are used, and a Lagrangian DAE of motion is formulated.

By applying a partial Legendre transform to the Lagrangian DAE, a differential–algebraic form of Hamilton's equation is derived. Complementary forms of the Lagrangian DAE and the Hamiltonian DAE are derived. In all cases a nonlinear model is formulated for systematic application to a multidiscipline system. Lagrange's equation and its dual are given as a set of linearly implicit DAEs in descriptor form. Hamilton's equation and its dual are given as a set of semi–explicit DAEs.

The formulation of a model as a DAE avoids some of the drawbacks of conventional

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NOMENCLATURE

β	transformer/transducer modulus	e^d	dissipation effort
δ	variational operator	e^g	given effort
δp	virtual momentum	e^n	nonpotential effort
δq	virtual displacement	e^p	potential effort
δW	virtual work	e^s	source effort
γ	dynamic constraint	e^{t^*}, e^t	kinetic effort
Γ	vector of dynamic constraints	E	energy
κ, λ, μ	Lagrange multipliers	f	generalized flow
ϱ	momentum component	G	co–content
$\dot{\varrho}$	effort component	I	generalized inductance
Υ	right–hand side term of DAE	M	inertial matrix
ϕ	displacement constraint	p	generalized momentum
Φ	vector of displacement constraints	P	power, generalized flow vector
$\tilde{\phi}$	momentum constraint	q	generalized displacement
$\tilde{\Phi}$	vector of momentum constraints	Q	heat, generalized effort vector
ψ	flow constraint	R	generalized resistance
Ψ	vector of flow constraints	T	kinetic energy
A	compliance matrix	T^*	kinetic coenergy
C	generalized capacitance	u	displacement component
$\mathcal{C}\ell$	constituitive law	$\dot{u}$	flow component
D	content	V	potential energy
e	generalized effort	V^*	potential coenergy
e^γ	implicit effort	W	work
e^ϕ	workless constraint effort		

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To Andrea

all ways & always

graphical techniques: a model need not be reduced to a set of independent equations; nonholonomic constraints and system nonlinearities are readily accommodated; and the primary effort of the analyst is directed at obtaining energy functions and equations of constraint rather than drawing a graph to represent the system. A secondary effort is directed at manipulating these functions in a systematic way to obtain the desired set of DAEs. This step is readily automated, and produces a set of equations that are suitable for numerical solution.

The procedures of analysis, function manipulation, numerical solution and automation are described. This research, emphasizing analysis and model formulation, lays the foundation for future work in numerical solution and automated modeling of complex engineering systems.

Chapter 1

INTRODUCTION

Products and systems designed by engineers are comprised often of interacting subsystems, and the design of each subsystem is typically the responsibility of a functional group organized by engineering discipline. Under this organizational scheme, the contribution of the individual engineer to the design of a product is limited usually to the discipline which he or she practices. A drawback of this approach to design is that while each discipline–specific subsystem may be well–designed, the overall system may not be as efficient or robust as a comparable system designed in a unified way. For this reason, a multidiscipline approach to design is gaining favor in both industry and academe. This approach to the synthesis of engineering systems requires methods of analysis and computation suitable for multidiscipline design, control and optimization. This research addresses this need.

The method of analysis developed in this research provides a unified approach to the modeling of engineering systems. The engineering disciplines of interest are mechanical, electrical, fluid and thermal. An illustrative example of such a system is shown in Fig. 1.1. The basic physics of such systems are described in terms of set of unified variables — effort, flow, momentum and displacement — established by Paynter [30]. The purpose of analysis is to obtain a mathematical model of a system and to solve the resulting equations of motion. The method of analysis presented herein is based on the energy methods of Lagrange, generalized to encompass constrained, multidiscipline engineering systems, resulting in a set of differential–algebraic equations suitable for numerical integration.

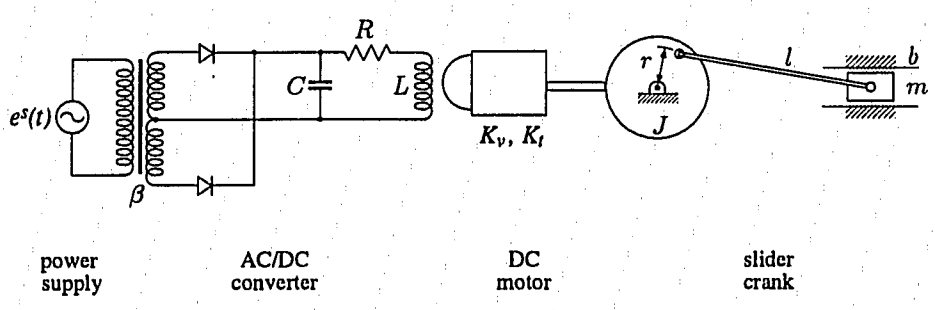


Figure 1.1. An illustrative electromechanical system.

Graphical methods for modeling multidiscipline systems (for example, bond graphs and linear graphs) are in widespread use today. Although graphical methods are systematic, the graphs can become quite cumbersome for complex systems. The analytical method presented herein can reduce that difficulty, primarily through the use of scalar energy functions and the ability to formulate equations of motion with excess coordinates and redundant constraints. However, the engineer is not relieved of the necessity of insight into the physics of the modeled system. Developing this insight is still the single most important aspect of modeling, for all methods.

Solving the dynamics problem entails more than just formulating equations. A useful modeling technique should be suitable also for computer implementation and numerical solution, if one expects to systematically tackle complicated multidiscipline systems without resorting to *ad hoc* measures. The analytical approach provides a useful framework from within which such computational issues can be addressed.

1.1 Objectives

The broad objective of this research is to develop techniques for systematically modeling multidiscipline systems based on the energy methods of Lagrange. This purpose is pursued by applying new, generalized forms of the principles of analytical mechanics to the problem of multidiscipline system dynamics.

The first specific objective of this research is to show how a differential–variational

method of analysis leads to Lagrange's equation for multidiscipline systems. The second objective is to examine how the equations of motion are transformed when variables other than those used by Lagrange (displacement and flow) are chosen to represent the motion of the system. And the third objective is to develop modeling techniques based on this approach to system dynamics such that the equations of motion for multidiscipline systems can be systematically formulated for numerical solution in the time domain.

1.2 Background

The approach to modeling multidiscipline systems presented herein is based on two fields of study: analytical mechanics and system dynamics. Because of its origins in these two fields of study, this unified approach to system dynamics is given the name *analytical system dynamics.*

Analytical mechanics or classical dynamics is a mature discipline having its origins in the 18th and 19th century work of Lagrange, Hamilton and Jacobi. Pars' [29] excellent treatise gives a detailed exposition. Lanczos [22] describes variational principles. R. M. Rosenberg [35] gives a thorough description of d'Alembert's principle, constraints and configuration space. Goldstein [15] reviews classical mechanics. Von Flotow [44] examines Kane's equation and its relationship to Lagrange's equation. Haug [19] examines modern methods of modeling mechanical systems. From this field of study are taken the concept of a variational operator, the classification of constraints, the use of undetermined multipliers and the concept of virtual work.

System dynamics is the study of unified methods of modeling multidiscipline engineering systems. A brief chronology of contributors shows that such methods have been in development for at least 50 years: Olson [28]; White and Woodson [49]; Paynter [30]; Ogar and D'Azzo [27]; Shearer, Murphy and Richardson [39]; Crandall, Karnopp, Kurtz and Pridmore–Brown [9]; MacFarlane [24]; Wellstead [48]; Karnopp, Margolis and Rosenberg [21]; and Rowell and Wormley [37]. The methods propounded by these au-

thors include the three common methods of analysis: linear graphs, bond graphs, and integral–variational methods. From this field of study are taken the concept of reticulated systems, the power postulate, the classification of system elements, and the use of a unified set of variables.

The idea of a unified approach to multidiscipline systems is so well established today that a course called *System Dynamics* is a staple of the undergraduate engineering curriculum. However, analytical dynamics for systems is not widely taught. System dynamics is usually taught using one of the graphical methods. In industry, modeling is largely discipline–specific and often empirically modified to suit particular system configurations. So even though the analytical approach, particularly Lagrange's equation, is known to be applicable to the system dynamics problem, current literature does not include a systematic exposition of the principles of analytical dynamics applied to multidiscipline systems. This dissertation fills that gap.

1.3 Significance of this research

Analytical methods are known to be applicable to the multidiscipline problem. For example, White and Woodson [49] use Lagrange's equation in formulating the equations of motion for electromechanical devices. Crandall *et al.* [9] utilize an integral variational principle to the same end. Gossick [16] examines the application of Hamilton's principle to physical systems. Nevertheless, the literature contains no systematic exposition of the basic principles of analytical dynamics applied to multidiscipline systems subject to holonomic and nonholonomic constraints. The primary significance of this research is the contribution of just such a systematic exposition.

The second significant aspect of the analytical method developed here is that it is based on a differential–variational form of the first law of thermodynamics rather than the integral approach (Hamilton's principle) favored by most authors. One advantage of this approach is that the first law of thermodynamics provides a foundation for a physi-

cal interpretation of the equations of analytical dynamics that is not provided as clearly by Hamilton's principle. Furthermore, the usual physical interpretation of Hamilton's principle — that the equations of motion minimize an action integral — is lost if Hamilton's principle is extended to admit nonholonomic constraints [25,35]. The differential approach used here admits nonholonomic constraints and the resulting equations of motion retain physical significance based on the work–energy relationships inherent in the first law.

Redfield [31] uses the first law in the form of a rate equation as a basis for deriving Lagrange's equation. Redfield's development, while having many features in common with the development given in this dissertation, is restricted to holonomic systems that are described in terms of a coordinate set of minimum dimension (generalized coordinates). The development given here admits nonholonomic constraints as well as excess coordinates.

Another attractive feature of the differential approach compared to the integral approach is that the calculus of variations is not invoked. The requisite mathematics for understanding the theory of the differential approach is appropriate to the senior undergraduate, making this method accessible to the engineer who may have only casual knowledge, if any, of analytical methods. So while it is assumed that the reader has a grasp of engineering fundamentals, the reader is not expected to have a background in system dynamics or advanced engineering mathematics.

Third, a methodical examination of the use of alternate pairs of variables to represent the motion of a system fills a gap in the literature. Many authors note the existence of dual or complementary formulations of the equations of motion. Here, these complementary forms are derived from first principles. The result is a new formulation of Hamilton's equation as well as complementary forms of both Lagrange's equation and Hamilton's equation.

Last, this research offers the engineer a technique of analysis and computation for

synthesis of engineering systems that avoids some of the drawbacks of the more conventional graphical techniques. Conventional methods of modeling multidiscipline systems produce sets of independent ordinary differential equations (ODE's). The equations of motion developed in this work are systematically formulated as sets of differential–algebraic equations (DAEs). This formulation is eminently suitable for nonlinear systems and foregoes the mathematical manipulation of graphical techniques aimed at reducing a model to a set of independent equations.

1.4 Method

The system models developed through the application of the analytical method presented herein are continuous–time, discrete models with deterministic excitations. Nonholonomic equality constraints and time–varying parameters are accommodated. Nonlinear equations of motion are formulated in terms of a unified set of variables. These formulations are manipulated to obtain differential–algebraic equations suitable for systematic application to multidiscipline systems. Within this basic framework, the dissertation proceeds as follows.

Chapter 2 The fundamental concepts of the unified approach to system dynamics are presented. A unified set of system variables is defined. Discrete system elements are classified as energy sources, stores, dissipators and transducers. Stored energy is represented in terms of the energy and coenergy state functions. Dissipative state functions are defined. The concepts of configuration space and state space are introduced to represent the motion of a multidiscipline system in a systematic manner.

Chapter 3 Constraints are classified and the concept of dynamic constraints is introduced. Virtual displacements and virtual work for multidiscipline systems are introduced. Efforts are classified and a multidiscipline form of Lagrange's principle is introduced to eliminate constraint efforts from the analysis. A geometric interpretation of

constraints in configuration space is given, leading to a visualization of the relationships among virtual displacements, virtual work, constraint efforts and Lagrange multipliers.

Chapter 4 The contribution of sources, transformers and transducers to the dynamic behavior of multidiscipline systems is characterized, based on the constraint concepts developed in Chapter 3.

Chapter 5 A new derivation of Lagrange's equation is given based on the first law of thermodynamics in differential–variational form, given by $\delta E = \delta W$, where δE represents the variation in stored energy and δW represents the virtual work of nonpotential efforts. Undetermined multipliers are introduced and a Lagrangian, linearly implicit DAE in descriptor form is obtained.

Chapter 6 A new derivation of Hamilton's equation is given based on the application of the Legendre transform to the energy of kinetic stores in the Lagrangian DAE. A Hamiltonian, semi–explicit DAE is obtained. From this new formulation of Hamilton's equation, the canonical form of Hamilton's equation is derived as a special case.

Chapter 7 Complementary or dual forms of Lagrange's equation and Hamilton's equation are derived. Selection of alternate variable pairs to represent the motion of a system results in the formulation of a complementary Lagrangian DAE and a complementary Hamiltonian DAE. In all cases, the set of differential–algebraic equations is formulated for systematic application to a multidiscipline system.

Chapter 8 A systematic method of modeling multidiscipline systems is presented. The process of analysis and function manipulation that results in a DAE of the desired form is described. The process is illustrated by example. Also presented is an overview of two numerical methods for solving the DAE — Euler's method and backward difference formula (BDF) methods. Numerical examples are given. A logical structure for automating the process of formulating and solving the DAE is outlined.

Chapter 9 The basic contributions of this research to the synthesis of engineering systems are summarized. Future work that is an extension of this research is outlined. Related areas of research are described.

Chapter 2

FUNDAMENTALS OF SYSTEM DYNAMICS

Unified procedures for modeling multidiscipline systems are based on the idea that systems are manipulators of energy. Wellstead [48] expresses the idea succinctly:

> *A physical system can be thought of as operating upon a pair of variables whose product is power. The physical components which make up the system may be thought of as energy manipulators which, depending upon the way they are interconnected, process the energy injected into the system in a characteristic fashion which is observed as the system dynamic response.*

This brief statement contains all the essential concepts of analytical system dynamics. By examining the consequences of these concepts, a unified method of analysis is developed for obtaining the set of differential–algebraic equations that describe the motion of a multidiscipline system.

2.1 A unified set of system variables

At the heart of the unified approach to dynamics is the idea that power and energy are phenomena that have the same meaning in all the engineering disciplines. For example, the fluid power of a river can be converted into the rotational power of turbines, which is converted into the electrical power of high voltage distribution lines and transformed to low voltage power for consumption. This power is then converted into various forms of useful work by common machines and appliances. One such machine could be a copier, in which electrical power becomes the rotational power of motors, the thermal power of heating elements and the translational power applied to individual sheets of paper. Because of inefficiencies, thermal power or heat is dissipated throughout the system. All

these engineering environments, from the flowing river to the moving sheets of paper, can be considered conceptually in a unified fashion if the system elements are thought of as power and energy manipulators.

2.1.1 Effort and flow

The observable manifestations of power take different forms that depend on the engineering discipline involved, yet power itself is an invariant quantity. This invariance is expressed by way of the *power postulate* [5].

POSTULATE 1 (Power postulate) *Every power term P_i in a multidiscipline system is the product of two* power variables, *an* effort $e_i(t)$ *and a* flow $f_i(t)$, *such that the total power of the system is given by* [1]

$$P(t) = \sum_i P_i(t) = \sum_i e_i(t) f_i(t) = \langle e, f \rangle. \tag{2.1}$$

The choice of which variable should be called effort and which should be called flow differs among the various unified methods of modeling multidiscipline systems. In this work the following convention is adopted. In mechanical translation, force F is considered an effort, velocity v is a flow and the product Fv is power. In mechanical rotation, torque τ is effort, angular velocity ω is flow and the product $\tau\omega$ is power. In electrical circuits, voltage v is considered an effort and current i is a flow (corresponding to the traditional force–voltage analogy) and the product vi is power. In fluid systems, pressure P is effort, volumetric flow rate Q is flow and the product PQ is power. In thermodynamic systems, temperature T is effort, entropy flow rate $\dot{S}$ is flow and the product $T\dot{S}$ is power. The assignment of effort and flow used in this research is summarized in Table 2.1.

For fluid systems, both hydraulic (incompressible) and pneumatic (compressible) flow may be modeled. In both cases, only constant mass or Lagrangian systems are

[1] The notation $\langle a, b \rangle$ represents the inner product between two appropriately dimensioned vectors a and b, usually written $\mathbf{a} \bullet \mathbf{b}$ or $\mathbf{a}^T\mathbf{b}$.

Table 2.1. Power variables.

Discipline	Effort e	Flow f	Power ef
translational	force F	velocity v	Fv
rotational	torque τ	angular velocity ω	$\tau\omega$
electrical	voltage v	current i	vi
fluid	pressure P	volumetric flow rate Q	PQ
thermal	temperature T	entropy flow rate $\dot{S}$	$T\dot{S}$

considered. The control volume or Eulerian approach is not considered.[2]

For thermal systems, heat transfer rate q is cited often as the flow quantity instead of entropy flow rate $\dot{S}$ [37,39]. For example, q can be considered a flow quantity in the Fourier conduction law given by

$$q = \frac{kA}{L}\Delta T, \tag{2.2}$$

where ΔT is the temperature difference across a quantity of material, A is the surface area, L is the distance through which heat flows and k is thermal conductivity. Similarly, heat transfer rate q can be considered a flow quantity in the simple convection law given by

$$q = hA\,\Delta T, \tag{2.3}$$

where h is the heat-transfer coefficient. In these equations, temperature T is considered analogous to electrical voltage, heat flow rate $\dot{q}$ is considered analogous to electrical current, and the terms L/kA and $1/hA$ represent thermal resistances R_t. In this context, these heat equations have the same form as the voltage-current relationship for an electrical resistor given by

$$i = \frac{1}{R}v. \tag{2.4}$$

This common analogy is based primarily on the algebraic similarity among these three equations, even though the analogy is inconsistent with the power postulate. In

[2]The extension of the analytical method to the Eulerian point of view is left to the future.

the electrical domain, the product of voltage and current vi is power, thus v and i satisfy the power postulate. In the thermal domain, the product of temperature and heat flow rate $T\dot{q}$ is not power, hence T and $\dot{q}$ do not satisfy the power postulate. To distinguish this variable pair from the true power variables, the variables $(T, \dot{q})$ are called the *pseudo–thermal* variables [21].

The choice of true thermal power variables, indicated in Table 2.1, is justified by considering the relationship between internal energy U, entropy S and volume V given by the Gibbs equation

$$dU = TdS - PdV, \tag{2.5}$$

where P is pressure and T is the temperature of a simple compressible substance. Assuming that changes occur slowly enough such that temperature and pressure are uniform throughout the substance, (2.5) can be written as a rate equation given by

$$\dot{U} = T\dot{S} - P\dot{V}, \tag{2.6}$$

in which all the terms represent time rates of change of energy. $\dot{U}$ is the rate of change of internal energy, $P\dot{V}$ is the rate of change of work energy similar to the fluid power expression in Table 2.1, and $T\dot{S}$ is an expression of the rate of change of heat energy. The rate of change of energy is power, thus the product $T\dot{S}$ is power. Consequently, T is an effort variable and $\dot{S}$ is a flow variable.

2.1.2 Momentum and displacement

The second pair of unified variables, *momentum* $p(t)$ and *displacement* $q(t)$, are defined through consideration of the cumulative effect of the power variables acting over time. Momentum and displacement are called *energy variables* because of the importance of these two quantities in the definition of energy functions.

Consider an effort acting over time. The cumulative effect of a translational force acting over time is a change in linear momentum. The cumulative effect of torque acting

over time is a change in angular momentum. The cumulative effect of voltage acting over time is a change in flux linkage. These observations are generalized to define a unified momentum variable $p(t)$ as the integrated or cumulative effect of effort acting over time, given by[3]

$$p(t) \;:=\; \int e(t)\,dt, \tag{2.7}$$

or in differential form,

$$\dot{p} \;=\; e. \tag{2.8}$$

The physical phenomenon described by this equation is referred to as the inertial property of systems. This property is the tendency of a system to resist change, and the resistance is proportional to the applied effort. The relationship given by (2.8) implies that an applied effort e is resisted by an equivalent effort $\dot{p}$ equal to the time derivative of generalized momentum. This property is called the *dynamic requirement* of system modeling and the variables (e, p) are called the *dynamic variables.*

For example, Newton's second law states that a particle resists a change in motion according to the momentum relation $F = \dot{p}$, where a change in F represents a force imbalance and p represents linear momentum. In rotational motion, a solid body resists a change in applied torque according to the momentum relation $\tau = \dot{H}$, where τ is applied torque and H is angular momentum. In fluid systems there exists a form of dynamic pressure called pressure momentum Γ that represents an inertial opposition to pressure changes, such that $P = \dot{\Gamma}$, where P is pressure. And in the electrical domain, Faraday's induction law states that a magnetic field generates a back-emf to oppose a change in current such that $v = \dot{\lambda}$, where v is voltage and λ is flux linkage.

Consider flows acting over time. The cumulative effect of velocity acting over time is a change in displacement. The cumulative effect of current acting over time is a change in accumulated charge. The cumulative effect of fluid flow acting over time is a change

[3]The symbol ':=' denotes a definition.

in volume. And the cumulative effect of entropy flow over time is a change in entropy. These observations are generalized to define a unified displacement variable $q(t)$ as the integrated effect of flow acting over time, given by

$$q(t) \;:=\; \int f(t)\,dt, \tag{2.9}$$

or in differential form,

$$\dot{q} \;=\; f. \tag{2.10}$$

The physical manifestation of this equation is simply that flow is the time rate of change of displacement, velocity is the time rate of change of position, current is the time rate of change of charge, and so forth. Since these relationships concern the kinematics rather than the dynamics of a system, (2.10) is called the *kinematic requirement* of system modeling and the variables (f, q) are called the *kinematic variables.*

If a model of a dynamic system is to accurately predict system behavior, both the kinematic and dynamic requirements must be satisfied. The remaining requirements that must be satisfied — the constituitive laws and constraints — are the subjects of subsequent sections.

Effort, flow, momentum and displacement comprise the unified set of variables for multidiscipline systems. The physical quantities these variables represent are listed in Table 2.2. This table corresponds to the "Paynterian framework of physical systems theory" [5].

While most of the variables in Table 2.2 are conventional physical quantities, pressure momentum requires additional explanation. The standard variable used in fluid mechanics to represent the inertial property of fluids is dynamic pressure P_d. Pressure momentum is a function of dynamic pressure. For example, for incompressible flow in a constant diameter pipe of length l and cross-sectional area A, it can be shown [37,39] that pressure momentum Γ is given by

$$\Gamma = \frac{\rho l}{A} Q, \tag{2.11}$$

Table 2.2. Unified set of variables for multidiscipline systems.

Discipline	Effort e	Flow f	Displacement q	Momentum p
translational	force F	velocity v	position x	linear momentum p
rotational	torque τ	angular velocity ω	angle θ	angular momentum H
electrical	voltage v	current i	charge q	flux linkage λ
fluid	pressure P	volumetric flow rate Q	volume V	pressure momentum Γ
thermal	temperature T	entropy flow rate $\dot{S}$	entropy S	(none)

where ρ is the fluid density and Q is flow rate. Substituting $Q = Av$ where v is the velocity of the fluid yields

$$\Gamma = \rho l v = 2\frac{l}{v}\frac{1}{2}\rho v^2 = 2\frac{l}{v}P_d, \tag{2.12}$$

where $P_d = \frac{1}{2}\rho v^2$ is dynamic pressure.

2.2 Classification of system elements

System elements found in the engineering disciplines are classified according to the manner in which they manipulate energy.

1. Energy *sources* supply energy to a system and are generally of two types: sources of effort or sources of flow. Sources are assumed tc be known functions of time.

2. Energy *stores* store energy within a system. Stored energy is of two types: kinetic or potential. Both types of stored energy are state–functions, independent of the path taken by the system to attain an instantaneous state.

3. Energy *dissipators* release energy to the surroundings of the system. Energy dissipation is a path–dependent function in the general case, although some dissipation can be modeled using a dissipation state function.

4. Transformers and transducers are system elements that couple two dynamic subsystems. *Transformers* couple subsystems of the same energy domain. *Transducers*

couple subsystems of different energy domains. The coupling property of transformers and transducers imposes constraints on the system variables.

The distinguishing characteristic of each pure element is described by a constituitive law which specifies the behavior of the element in terms of the unified set of variables. These constituitive laws are the subject of this section.

2.2.1 Sources

Energy sources supply energy to a system and are generally of two types: sources of effort or sources of flow. Sources are assumed to be known functions of time. Sources can be thought of as system elements that impose prescribed conditions of effort or flow at a system boundary. This prescribed condition is the constituitive law of the source.

Since either the effort or the flow of a source is prescribed, the other power variable is free to increase as large as the system demands. This implies that the pure source can supply an indefinitely large amount of power, which of course is not true of real devices. Such limits on the performance of real devices are modeled separately from the prescribed effort or flow characteristic of the pure source.

In Chapter 4, it is shown that effort sources are classified among the nonpotential efforts of a system and that flow sources are included among the kinematic constraints on a system. Examples of real devices that approximate pure sources are shown in Table 2.3, adapted from Shearer *et al.* [39].

Table 2.3. Physical systems that approximate pure sources.

System	Source medium	Type of source
battery	electrical voltage	effort
melting ice	liquid temperature	effort
reservoir	pressure	effort
hydraulic pump	fluid flow	flow
constant speed motor	angular velocity	flow

2.2.2 Potential stores

Consider energy storage devices characterized by a relationship between effort and displacement such that effort is a function of displacement only, that is, $e_{\mathcal{C}\ell} = e_{\mathcal{C}\ell}(q)$, where the subscript $\mathcal{C}\ell$ indicates an experimentally determined constituitive law. One such device is a linear spring, characterized by the constituitive law $F_{\mathcal{C}\ell}(x) = kx$, where k is the spring stiffness constant and $F_{\mathcal{C}\ell}$ is the force applied to the spring to obtain displacement x. Another example is a linear capacitor, characterized by the constituitive law $v_{\mathcal{C}\ell}(q) = q/C$, where C is capacitance and $v_{\mathcal{C}\ell}$ is the voltage applied to obtain charge q. Devices characterized by such laws are called potential stores or *generalized capacitors*.

Constituitive efforts $e_{\mathcal{C}\ell}$ are so designated because constituitive laws are usually written in terms of efforts applied to the element, while in the analytical method the efforts of interest are those the element applies to the system. Thus the effort e associated with an element is the negative of its constituitive effort, that is, $e = -e_{\mathcal{C}\ell}$.

The constituitive laws of generalized capacitors need not be linear. For example, spring stiffness may vary with displacement or capacitance may vary with charge. This type of constituitive law is indicated by the nonlinear curve in Fig. 2.1.

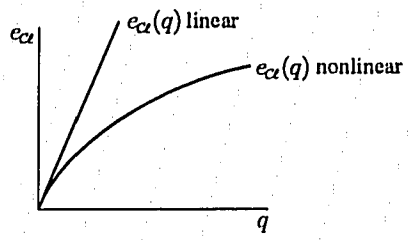


Figure 2.1. Constituitive laws of pure capacitors.

However, many common devices are characterized by linear or *ideal* constituitive laws. Such a constituitive law is indicated by the linear curve in Fig. 2.1. Common potential stores with ideal constituitive laws are listed in Table 2.4.

Table 2.4. Constituitive laws of ideal capacitors.

Pure relation: Ideal relation:	$e_\alpha = e_\alpha(q)$ $e_\alpha = q/C$	Proportionality constant
translational spring	$F_\alpha = kx$	k spring constant
torsional spring	$\tau_\alpha = k_\theta \theta$	k_θ spring constant
electrical capacitor	$v_\alpha = q/C$	C capacitance
fluid capacitor	$P_\alpha = V/C_f$	C_f fluid capacitance

2.2.3 Kinetic stores

Some energy stores are characterized by a relationship between flow and momentum such that flow is a function of momentum only, that is, $f_\alpha = f_\alpha(p)$. One such element is a mass, characterized by the constituitive law $v_\alpha(p) = p/m$, where m is mass and v_α is the velocity applied to the mass to obtain momentum p. Another example is a linear electrical inductor, characterized by the constituitive law $i_\alpha(\lambda) = \lambda/L$, where L is inductance and i_α is the current applied to obtain flux linkage λ. Devices characterized by such laws are called kinetic stores, inertial elements or *generalized inductors.*

The constituitive laws of generalized inductors need not be linear. For example, inductance may vary with flux linkage as in an electromagnetic suspension. Or mass may vary with momentum at relativistic speeds. This type of constituitive law is indicated by the nonlinear curve in Figure 2.2.

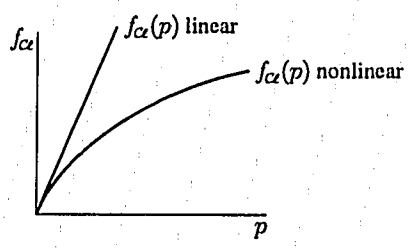


Figure 2.2. Constituitive laws of pure inductors.

Many common devices are characterized by ideal constituitive laws. Such a constituitive law is indicated by the linear curve in Figure 2.2. Common kinetic stores with ideal constituitive laws are listed in Table 2.5.

Table 2.5. Constituitive laws of ideal inductors.

Pure relation:	$f_\alpha = f_\alpha(p)$	Proportionality
Ideal relation:	$f_\alpha = p/I$	constant
translating mass	$v_\alpha = p/m$	m mass
rotating mass	$\omega_\alpha = H/I_o$	I_o moment of inertia
electrical inductor	$i_\alpha = \lambda/L$	L inductance
fluid inductor	$Q_\alpha = \Gamma/I_f$	I_f fluid inertance

2.2.4 Dissipators

The pure element that dissipates power is the *generalized resistor*. The generalized resistor has a constituitive law that relates effort and flow, distinguishing it from the generalized storage elements which have constituitive laws that are functions of displacement or momentum. In the following discussion, it is convenient to select $e_\alpha = e_\alpha(f)$ rather than $f_\alpha = f_\alpha(e)$ as the form for this constituitive law.

One such device is a linear damper, characterized by the constituitive law $F_\alpha(v) = bv$, where b is the damping coefficient and F_α is the force applied to the damper to obtain velocity v. Another example is a linear resistor, characterized by the constituitive law $v_\alpha(i) = Ri$, where R is resistance and v_α is the voltage applied to obtain current i.

The constituitive laws of generalized resistors need not be linear. For example, the damping in a dashpot may vary with velocity. Or the fluid resistance in a section of pipe may vary with flow rate. This type of constituitive law is indicated by the nonlinear curve in Figure 2.3.

Many common devices are characterized by ideal constituitive laws. Such a constituitive law is indicated by the linear curve in Figure 2.2. Common dissipators with ideal

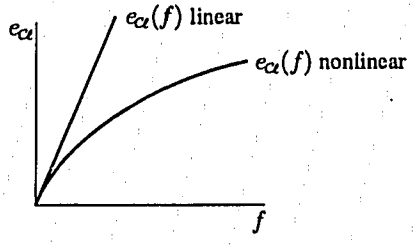


Figure 2.3. Constituitive laws of pure resistors.

constituitive laws are listed in Table 2.6. As with the constituitive laws of generalized capacitors, the effort of interest is the effort e the element applies to the system while the constituitive laws are written in terms of an effort $e_{c\ell}$ applied to the element. The desired effort e and the constituitive effort $e_{c\ell}$ are related by $e(f) = -e_{c\ell}(f)$.

Table 2.6. Constituitive laws of ideal resistors.

Pure relation: Ideal relation:	$e_{c\ell} = e_{c\ell}(f)$ $e_{c\ell} = Rf$	Proportionality constant
translational damper	$F_{c\ell} = bv$	b damping coefficient
rotational damper	$\tau_{c\ell} = b_\theta \omega$	b_θ damping coefficient
electrical resistor	$v_{c\ell} = Ri$	R resistance
fluid resistor	$P_{c\ell} = R_f Q$	R_f fluid resistance

2.2.5 Transformers and transducers

Transformers and transducers are system elements that couple two dynamic subsystems. *Transformers* couple subsystems of the same energy domain. *Transducers* couple subsystems of different energy domains. The coupling characteristic of transformers and transducers is described by a constituitive law that relates the system variables at one port of the device to the system variables at the second port of the device. For ideal power transformation or transduction, power coupling is assumed to be perfectly efficient.

The coupling property of transformers and transducers imposes constraints on the

system variables. A more exhaustive description of these elements and constraints is given in Chapter 4.

2.2.6 Summary of elemental constituitive laws

The constituitive equations for the ideal energy stores and dissipators are summarized in Table 2.7.

Table 2.7. Summary of ideal constituitive laws.

Element:	Kinetic store	Potential store	Dissipator
Pure relation:	$f_\alpha = f_\alpha(p)$	$e_\alpha = e_\alpha(q)$	$e_\alpha = e_\alpha(f)$
Ideal relation:	$f_\alpha = p/I$	$e_\alpha = q/C$	$e_\alpha = Rf$
translational	$v_\alpha = p/m$	$F_\alpha = kx$	$F_\alpha = bv$
rotational	$\omega_\alpha = H/I_o$[a]	$\tau_\alpha = k_\theta\theta$	$\tau_\alpha = b_\theta\omega$
electrical	$i_\alpha = \lambda/L$	$v_\alpha = q/C$	$v_\alpha = Ri$
fluid	$Q_\alpha = \Gamma/I_f$	$P_\alpha = V/C_f$	$P_\alpha = R_fQ$
Legend	m mass	k spring constant	b damping coefficient
	I_o moment of inertia	k_θ spring constant	b_θ damping coefficient
	L inductance	C capacitance	R resistance
	I_f fluid inertance	C_f fluid capacitance	R_f fluid resistance

[a]This relationship is valid for three types of motion: rotation about the center of mass; rotation about a fixed point; or rotation such that all forces act through the center of mass. A method of including noncentroidal angular momentum in the equations of motion is presented in Appendix D.

A convenient mnemonic device for displaying the relationships among the unified set of variables is given in Fig. 2.4. This figure is adapted from Paynter [30] and called herein *Paynter's diagram.* The vertices of the figure are labeled with the generalized variables (e, f, p, q). The line segments connecting the vertices represent the manner in which the generalized variables are interrelated. Thus the constituitive law of a potential store $e_\alpha(q)$ relates e and q, the constituitive law of a dissipator $e_\alpha(f)$ relates e and f, and the constituitive law of a kinetic store $f_\alpha(p)$ relates f and p. Furthermore, the two line segments marked as time integrals represent the time-dependent relationships between

the kinematic variables q and f and between the dynamic variables p and e.

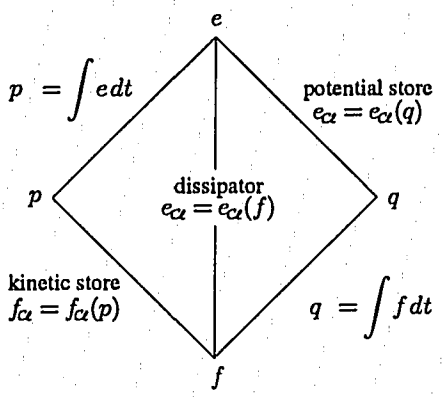


Figure 2.4. Paynter's diagram, illustrating the relationships among the generalized variables.

2.3 Work and energy

The basic postulate of system dynamics is that power is the physical quantity underlying the unified approach to modeling. All other quantities are derived in terms of the single axiomatic quantity, power. Consequently *energy* $E(t)$ is defined as the time integral of power given by

$$E(t) := \int P(t)\, dt, \tag{2.13}$$

or in terms of the power variables,

$$E(t) = \int e(t) f(t)\, dt. \tag{2.14}$$

This equation implies that energy is a manifestation of effort and flow acting over time. A distinguishing characteristic of the energy, effort and flow of this equation is the dependence or independence of these quantities on their time history or path of integration.

2.3.1 Paths of integration

The significance of paths of integration can be appreciated by considering the first law of thermodynamics. The first law for a closed system can be written

$$đQ + đW = dE, \tag{2.15}$$

where $đQ$ is an increment of heat added to a system, $đW$ is an increment of work done on a system and dE is an incremental change in the energy stored in the system. The left–hand terms are inexact differentials (denoted by the symbol $đ$) representing energy in transition across a system boundary as heat and work. The term dE is an exact differential representing energy stored within the system [43].

Integrating this equation yields

$$\int_C đQ + \int_C đW = E_2 - E_1. \tag{2.16}$$

Integration of inexact differentials follows the path C. The resulting integrals are path–dependent functions. By (2.14) the existence of path–dependent energy functions implies the existence of path–dependent efforts and flows. The first law indicates that such quantities are observed as either heat or work. Dissipated energy, work sources and nonpotential efforts are generally of this form.

Integration of exact differentials, on the other hand, depends only on the endpoints of the path, that is, the state of the system at the initial and final points. Such integrals are path–independent functions or *state functions* that imply the existence of effort, flow and energy functions that are independent of path. The first law indicates that such quantities are observed as energy stored within a system. Potential and kinetic energy are of this form.[4]

The sign convention adopted here is that power or energy flowing into a system is positive and that power or energy flowing out of a system is negative. This convention

[4]Additional commentary on path and point functions is given in Appendix E.

is implicit in the manner in which the first law is stated. Note that this convention differs from some of the discipline–specific sign conventions with which the reader may be familiar. For example, the usual statement of the first law of thermodynamics is given by $đQ = đW + dE$, where positive $đW$ represents work done by the system, that is, positive work is energy leaving the system. Likewise, in electrical network theory, when summing voltages around a loop in accordance with Kirchhoff's voltage law, a voltage drop (energy sink) is considered a positive quantity and a voltage source (energy source) is considered a negative quantity. One of the advantages of a unified method is that such discipline–specific sign conventions are foregone in favor of the sufficient and consistent sign convention of "positive power in."

2.3.2 Potential energy

To obtain an expression for the energy stored in a generalized capacitor, the relationship $dq = f\,dt$ from the kinematic requirement (2.10) is substituted into (2.14) to obtain

$$E = \int e\,dq, \tag{2.17}$$

or in differential form,

$$dE = e\,dq, \tag{2.18}$$

where e is the effort the element applies to the system.

The term $e\,dq$ is a differential of work. It can represent a force F acting through a distance dx; a torque τ acting through an angular displacement $d\theta$; a voltage v acting to displace a charge dq; or a pressure P acting through a volume differential dV. Since work is in general a path–dependent function, this type of energy is given in its most general form by $đW = e\,dq$.

The constituitive law of the generalized capacitor, $e_{C_i} = e_{C_i}(q)$, states that the instantaneous effort of this element depends only on its instantaneous displacement. Such efforts are independent of the path or displacement history taken by the system to arrive

at displacement q. It follows that the work done by such efforts is also independent of path. Thus the work expression $đW = e\,dq$ can be written as an exact differential $dW(q) = e(q)\,dq$ and integrated to obtain

$$W(q) = \int e(q)\,dq. \tag{2.19}$$

The negative of this quantity is called the *potential energy* $V(q)$, defined by

$$V(q) := -\int e(q)\,dq. \tag{2.20}$$

Displacement is designated an energy variable because of this relationship.

Introducing limits of integration, such that $q(t_o) = q_o$ and $q(t) = q$, there exists a constant of integration $V_o = V(q_o)$ such that potential energy is given by

$$V(q) = V_o - \int_{q_o}^{q} e(q)\,dq. \tag{2.21}$$

Thus the energy of a potential store is defined as the work the element is capable of doing by virtue of its displacement q if the element was brought from q to the origin of the coordinate reference frame q_o plus a reference value. The origin of the reference frame is often selected to be a convenient datum or equilibrium position such as the unextended position of a spring or the completely discharged state of a capacitor. In these cases, $q_o = 0$ and V_o can be assigned a zero value.[5]

Finally, summing over n potential stores in a system, the potential energy function $V(q)$ for a multidiscipline system is given by

$$V(q) = V_o - \sum_{j=1}^{n} \int_{q_o}^{q} e_j(q)\,dq_j, \tag{2.22}$$

yielding the differential form,

$$-\frac{\partial V}{\partial q_j} = e_j, \tag{2.23}$$

[5]The integral in (2.21) could be written in a less ambiguous fashion using a dummy variable of integration ξ, as in $\int^q e(\xi)\,d\xi$, but dummy variables are not used in this work. Given the number of variables required by the multidiscipline nature of the work, and given the number of variable transformations performed, the use of dummy variables detracts from the clarity of the exposition.

where $e_j = -(e_\alpha)_j$. Efforts satisfying this relationship are called potential efforts e^p. In the linear case, $e^p = -q/C$, where C is generalized capacitance or compliance. Thus, for a system containing all linear potential stores, and assuming zero reference conditions, the potential energy of the system is given by

$$V(q) = \frac{1}{2}\sum_{j=1}^{n} \frac{q_j^2}{C_j}. \tag{2.24}$$

These definitions are consistent with usual definition of a potential, as follows. If an n–dimensional effort $e \in \mathcal{R}^n$ and a scalar function $U(t)$ satisfy the relationship $e = \nabla U$ with U continuously differentiable, then e is defined as a potential effort, U is defined as a potential, and the negative of the potential is defined as potential energy $V = -U$. It follows that $e = -\nabla V$, which is the vector form of (2.23).

EXAMPLE 2.1 A linear spring, extended a distance x from its unextended equilibrium position $x_o = 0$, is shown in Figure 2.5. The effort $e(q)$ the spring exerts is given by $F(x) = -kx$, where F is the spring force and k is the spring constant. The force opposes the positive displacement x.

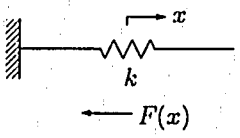


Figure 2.5. Effort and displacement of a linear spring.

The definition of potential energy yields

$$V(x) = -\int_0^x F(x)\,dx = -\int_0^x -kx\,dx = \tfrac{1}{2}kx^2, \tag{2.25}$$

which is the well–known expression for the energy stored in a linear spring. ◇

Potential energy can be given an area interpretation. Assuming zero initial conditions

and substituting the relationship $e = -e_\alpha$ into (2.21) yields

$$V(q) = \int_0^q e_\alpha(q)\, dq. \tag{2.26}$$

Thus, if the constituitive law $e_\alpha = e_\alpha(q)$ is plotted as in Figure 2.6, the area under the curve from 0 to $\hat{q}$ represents the potential energy $V(\hat{q})$ stored in the element. This graphical interpretation illustrates that potential energy is a scalar function that varies with the instantaneous displacement of the potential store.

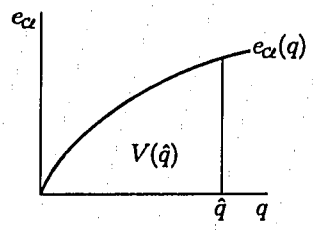


Figure 2.6. Area representation of potential energy.

2.3.3 Potential coenergy

A complementary energy function, the potential coenergy V^*, is obtained through application of the Legendre transform to the potential energy function.

The Legendre transform operates on a function of a single variable $f(y)$ such that the transform of $f(y)$ is a new function $g(\xi)$, where $\xi := \partial f/\partial y$.[6] The transform yields two results of interest:

$$f(y) + g(\xi) = y\xi, \tag{2.27}$$

$$g(\xi) = \int y(\xi)\, d\xi. \tag{2.28}$$

In the case of a potential store, the univariate function to be transformed is the potential energy function $V(q)$, which is given by

$$V(q) = -\int e\, dq. \tag{2.29}$$

[6]The nomenclature and basic principles of the Legendre transform are given in Appendix A.

An expression for ξ is found from

$$\xi := \frac{\partial V}{\partial q} = -e. \tag{2.30}$$

The Legendre transform of $V(q)$ is the new function $V^*(e)$ called the *potential coenergy*, defined by

$$V^*(e) := -qe - V(q), \tag{2.31}$$

yielding the first result of interest, similar to (2.27), given by

$$V(q) + V^*(e) = -qe. \tag{2.32}$$

Summing over n potential stores in a system, the potential energy function and potential coenergy function satisfy

$$V(q) + V^*(e) = -\sum_{j=1}^{n} q_j e_j. \tag{2.33}$$

Furthermore, as a consequence of (2.28), potential coenergy can also be given by

$$V^*(e) = V_o^* - \sum_{j=1}^{n} \int_{e_o}^{e} q_j(e)\, de_j. \tag{2.34}$$

This expression yields the differential form

$$-\frac{\partial V^*}{\partial e_j} = q_j, \tag{2.35}$$

where q and e are related by the constituitive law for the potential store, that is, $e = -e_\alpha(q)$. Displacements satisfying this relationship are called potential displacements q^p. In the linear case, $q^p = -Ce$, where C is generalized capacitance or compliance. Thus, for a system containing all linear potential stores, and assuming zero reference conditions, the potential coenergy of the system is given by

$$V^*(e) = \tfrac{1}{2} \sum_{j=1}^{n} C_j e_j^2. \tag{2.36}$$

EXAMPLE 2.2 The charge q of a linear electrical capacitor is given by $q = -Ce$, where C is capacitance and e is the voltage applied by the capacitor to the network to which it is connected. The potential coenergy of the capacitor is given by

$$V^*(e) = -\int_o^e q(e)\,de = -\int_o^e -Ce\,de = \tfrac{1}{2}Ce^2. \quad \diamond \tag{2.37}$$

This example illustrates a common textbook error. The quantity $\frac{1}{2}Ce^2$ is almost always cited as the energy of a linear capacitor. Given the assignment of effort and flow used in this work, this quantity is defined more rigorously as the *coenergy* of a capacitor. The *energy* of a capacitor is given by

$$V(q) = \frac{q^2}{2C}. \tag{2.38}$$

Of course for linear capacitors, $q = -Ce$, and the two representations of energy are identical quantities, that is,

$$V(q) = \frac{q^2}{2C} = \frac{(-Ce)^2}{2C} = \tfrac{1}{2}Ce^2 = V^*(e). \tag{2.39}$$

The equivalence of energy and coenergy only holds for elements with linear constitutive laws. Expressions for the potential energy and coenergy for the common linear potential stores are listed in Table 2.8.

Table 2.8. Potential energy and coenergy of ideal capacitors.

		translational spring	torsional spring	electrical capacitor	fluid capacitor
Energy	$V(q)$	$\frac{1}{2}kx^2$	$\frac{1}{2}k_\theta\theta^2$	$\frac{q^2}{2C}$	$\frac{V^2}{2C_f}$
Coenergy	$V^*(e)$	$\frac{F^2}{2k}$	$\frac{\tau^2}{2k_\theta}$	$\frac{1}{2}Ce^2$	$\frac{1}{2}C_fP^2$

Energy and coenergy of a pure potential store can be given the graphical interpretation shown in Fig. 2.7. Potential energy can be interpreted as the area under the curve

of the constituitive law $e_\alpha(q)$. Potential coenergy can be interpreted as the difference in areas given by $V^* = qe_\alpha - V$. If $e_\alpha(q)$ is linear, the areas representing V and V^* are equal, which is an area representation of (2.39).

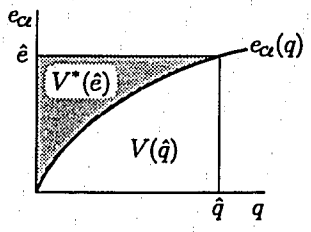


Figure 2.7. Area representation of potential energy and coenergy.

Potential energy and coenergy are scalar quantities representing the same constituitive law, differing only in the choice of independent variable. Energy varies with displacement. Coenergy varies with effort. And effort and displacement are related through the constituitive law of the potential store.

2.3.4 Kinetic energy

To obtain an expression for the energy stored in a generalized inductor, the relationship $dp = e\,dt$ from the dynamic requirement (2.8) is substituted into (2.14) to obtain

$$E = \int f\,dp, \tag{2.40}$$

or in differential form,

$$dE = f\,dp, \tag{2.41}$$

where f is the flow of the element.

Again, the right-hand term is a differential of work, as follows

$$f\,dp = f\frac{dp}{dt}dt = \dot{p}\,f\,dt = e\,dq = đW, \tag{2.42}$$

where from Paynter's diagram $\dot{p} = e$ and $f\,dt = dq$.

The constituitive law of the generalized inductor, $f_\alpha = f_\alpha(p)$, states that the instantaneous flow of this element depends only on its instantaneous momentum. Such flows are independent of the path or momentum history taken by the system to arrive at momentum p. It follows that the work done by such flows is also independent of path. Thus the work expression $đW = f\,dp$ can be written as an exact differential $dW(p) = f(p)\,dp$. The integral of this differential is called the *kinetic energy* $T(p)$, defined by

$$T(p) := \int f(p)\,dp. \tag{2.43}$$

Momentum is designated an energy variable because of this relationship.

Introducing limits of integration, such that $p(t_o) = p_o$ and $p(t) = p$, there exists a constant of integration $T_o = T(p_o)$ such that kinetic energy is given by

$$T(p) = T_o + \int_{p_o}^{p} f(p)\,dp. \tag{2.44}$$

Thus the energy of a kinetic store is defined as the work necessary to impart momentum p to an element with reference to the origin of a inertial reference frame p_o plus a reference value. It is usually convenient to select a reference frame for which $p_o = 0$ and $T_o = 0$, such as the zero-current state of an electrical inductor or the at-rest state of a mass in an inertial reference frame.

Finally, summing over n kinetic stores in a system, the kinetic energy function $T(p)$ for a multidiscipline system is given by

$$T(p) = T_o + \sum_{j=1}^{n} \int_{p_o}^{p} f_j(p)\,dp_j, \tag{2.45}$$

yielding the differential form,

$$\frac{\partial T}{\partial p_j} = f_j, \tag{2.46}$$

where $f_j = (f_\alpha)_j$. Flows satisfying this relationship are called kinetic flows f^t. In the linear case, $f^t = p/I$, where I is generalized inertance. Thus, for a system containing all linear kinetic stores, and assuming zero reference conditions, the kinetic energy of the

system is given by

$$T(p) = \frac{1}{2}\sum_{j=1}^{n}\frac{p_j^2}{I_j}. \tag{2.47}$$

EXAMPLE 2.3 The flow $f(p)$ in a linear inductor can be written $i(\lambda) = \lambda/L$, where i is current, λ is the flux linkage, and L is inductance. The definition of kinetic energy yields

$$T(\lambda) = \int_0^{\lambda} i(\lambda)\, d\lambda = \int_0^{\lambda} \frac{\lambda}{L}\, d\lambda = \frac{\lambda^2}{2L}, \tag{2.48}$$

which is the usual flux–linkage form of the energy stored in a linear inductor. ◇

Kinetic energy $T(p)$ can be interpreted as the area under the constituitive law $f = f(p)$ as shown in Figure 2.8. This graphical interpretation illustrates that kinetic energy is a scalar function that varies with the instantaneous momentum of the kinetic store.

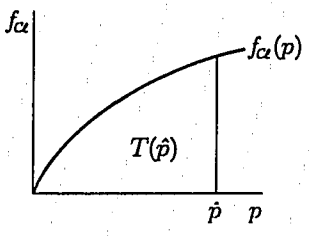


Figure 2.8. Area representation of kinetic energy.

2.3.5 Kinetic coenergy

A complementary energy function, the kinetic coenergy T^*, is obtained through application of the Legendre transform to the kinetic energy function $T(p)$. The details of the transform parallel the transform of potential energy given in a previous section and so are not repeated here.

The Legendre transform of $T(p)$ is the new function $T^*(f)$ called the *kinetic coenergy*, defined by

$$T^*(f) := pf - T(p), \tag{2.49}$$

yielding the first result of interest, similar to (2.27), given by

$$T(p) + T^*(f) = pf. \tag{2.50}$$

Summing over n kinetic stores in a system, the kinetic energy function and kinetic coenergy function satisfy

$$T(p) + T^*(f) = \sum_{j=1}^{n} p_j f_j. \tag{2.51}$$

Furthermore, as a consequence of (2.28), kinetic coenergy can also be given by

$$T^*(f) = T_o^* + \sum_{j=1}^{n} \int_{f_o}^{f} p_j(f)\, df_j. \tag{2.52}$$

This expression yields the differential form

$$\frac{\partial T^*}{\partial f_j} = p_j, \tag{2.53}$$

where $p = p_{\alpha t}(f)$ is the inverse of the constituitive law for the kinetic store. Momenta satisfying this relationship are called kinetic momenta p^t. In the linear case, $p^t = If$, where I is generalized inertance. Thus, for a system containing all linear kinetic stores, and assuming zero reference conditions, the kinetic coenergy of the system is given by

$$T^*(f) = \tfrac{1}{2} \sum_{j=1}^{n} I_j f_j^2. \tag{2.54}$$

EXAMPLE 2.4 A mass m translates with velocity v. The linear momentum of the mass is given by $p = mv$. The kinetic coenergy of the mass is given by

$$T^*(v) = \int_o^v p(v)\, dv = \int_o^v mv\, dv = \tfrac{1}{2} mv^2. \quad \diamond \tag{2.55}$$

This example illustrates another common textbook error. The quantity $\frac{1}{2}mv^2$ is traditionally called the kinetic energy of a mass. As shown above, this quantity is more rigorously defined as the kinetic *coenergy*. Kinetic *energy* is given by

$$T(p) = \frac{p^2}{2m}. \tag{2.56}$$

Of course for nonrelativistic motion, $p = mv$, and the two representations of energy are identical quantities, that is,

$$T(p) = \frac{p^2}{2m} = \frac{(mv)^2}{2m} = \tfrac{1}{2}mv^2 = T^*(f). \tag{2.57}$$

This equivalence of energy and coenergy only holds for elements with linear constituitive laws.

The distinction between energy and coenergy is important. Kinetic energy $T(p)$ is the proper expression for stored energy when applying the first law of thermodynamics, such that the total stored energy is given by $E := T(p) + V(q)$. The correct definition of the Hamiltonian function H used in Hamiltonian dynamics also is given by $T(p) + V(q)$. Furthermore, the common expression for the Lagrangian L used in most texts on Lagrangian dynamics, given by $T - V$, is misleading. The proper definition of the the Lagrangian is $L := T^*(f) - V(q)$.

Expressions for the kinetic energy and coenergy for the common linear kinetic stores are given in Table 2.9.

Table 2.9. Kinetic energy and coenergy of ideal inductors.

		translating mass	rotating mass	electrical inductor	fluid inductor
Energy	$T(p)$	$\frac{p^2}{2m}$	$\frac{H^2}{2I_o}$	$\frac{\lambda^2}{2L}$	$\frac{\Gamma^2}{2I_f}$
Coenergy	$T^*(f)$	$\frac{1}{2}mv^2$	$\frac{1}{2}I_o\omega^2$	$\frac{1}{2}Li^2$	$\frac{1}{2}I_fQ^2$

Energy and coenergy of a pure kinetic store can be given the graphical interpretation shown in Fig. 2.9. Kinetic energy can be interpreted as the area under the curve of the constituitive law $f_\alpha(p)$. Kinetic coenergy can be interpreted as the difference in areas given by $T^* = pf_\alpha - T$. If $f_\alpha(p)$ is linear, the areas representing T and T^* are equal, which is an area interpretation of (2.57).

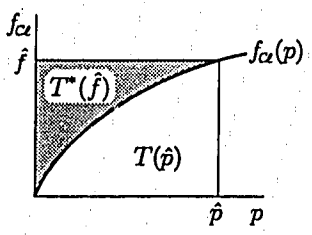


Figure 2.9. Area representation of kinetic energy and coenergy.

Kinetic energy and coenergy are scalar quantities representing the same constituitive law, differing only in the choice of independent variable. Energy varies with momentum. Coenergy varies with flow. And flow and momentum are related through the constituitive law of the kinetic store.

2.3.6 Dissipated energy

Dissipated energy is in general a path–dependent function. The path or time–history of a dissipative element must be accounted for in a model. Dissipated power is given by $P(t) := e(t)f(t)$. The dissipated energy E_D is the time integral of power given by

$$E_D(t) := \int P(t)\,dt. \tag{2.58}$$

Unlike kinetic and potential energy, dissipated energy can not be interpreted as the area under the curve of a constituitive equation. However, dissipated energy can be given an area interpretation in terms of dissipated power. If dissipated power P_α is plotted as a function of time as in Fig. 2.10, dissipated energy at time $\hat{t}$ is represented by the area under the curve.

This graphical interpretation illustrates that dissipated energy is a function that is both path-dependent and nondecreasing, as follows. The abscissa, time, always increases. Dissipated power P_α is nonnegative because energy in a dissipator can be lost but not gained. (Energy gained is modeled with a source.) So the area representing dissipated energy E_D is a function that cannot decrease with time. And since the area depends on

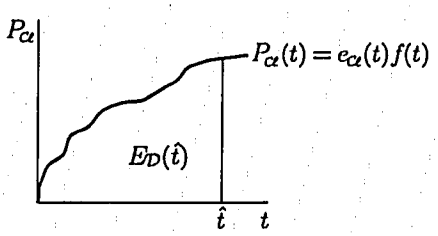


Figure 2.10. Area representation of dissipated energy.

the path or total time history of power on the time interval $[t_o, t]$, dissipated energy is a path-dependent function. From the first law of thermodynamics (2.16) path-dependent energy in a system is expected to occur in the form of either heat or work. Lastly, by the sign convention of "positive power in," dissipated energy is by definition a negative quantity.

EXAMPLE 2.5 The power dissipated in an electrical resistor given by

$$P = ef = vi. \tag{2.59}$$

If the resistor has resistance R and a linear constituitive equation given by Ohm's law $v_\alpha = iR = -v$, then the dissipated energy is given by

$$E_D = \int P\,dt = \int vi\,dt = -\int_{t_o}^{t} i^2 R\,dt. \tag{2.60}$$

Since E_D depends on the path or time history of the current $i(t)$, the dissipated energy is a path-dependent function. In this case, the energy is dissipated as heat. $\diamond$

Not all dissipated energy can be characterized by a constituitive law of the form $e_\alpha = e_\alpha(f)$. Dissipation due to dry friction is an example. In such cases, the basic definition (2.58) must be used to account for energy losses.

EXAMPLE 2.6 Consider a mass sliding on a surface along a general trajectory C with velocity v. The force or effort e acting on the mass due to dry friction is $-\mu N$, where μ

is the friction coefficient, and N is the normal force acting on the mass due to gravity and the geometry of the surface. The power dissipated is given by

$$P = ef = -\mu N v. \tag{2.61}$$

Dissipated energy is the integral of power given by

$$E_D = \int P\,dt = -\int \mu N v\,dt. \tag{2.62}$$

This is a valid expression describing dissipated energy, but it cannot be characterized in terms of the resistive constituitive relation $e_{\alpha} = e_{\alpha}(f)$ since effort $e = -\mu N$ is not a function of flow, nor is flow v a function of effort.

If ds is an infinitesimal element of C, where C is the path followed during the time interval $[t_o, t]$, then instantaneous velocity is given by $v = ds/dt$, and dissipated energy can be written

$$E_D = -\int_{t_0}^{t} \mu N \frac{ds}{dt}\,dt = -\int_C \mu N\,ds. \tag{2.63}$$

Since E_D depends on path C, it is clearly a path-dependent function, and since the right-hand term $\mu N\,ds$ has the form of the work differential $e\,dq$, dissipation in this case represents work done by the element. The minus sign indicates that element does indeed dissipate energy. ◇

As illustrated by these examples, dissipated energy is generally path-dependent, distinguishing it from potential and kinetic energy, which are path-independent. The importance of this distinction is that in modeling an element characterized by a path-dependent energy function, the path or time history of the energy must be accounted for in the model. On the other hand, elements that are characterized by state-determined energy functions do not require a knowledge of past states. This distinction determines the method by which dissipative elements are modeled.

In some cases, it is possible to account for dissipation independent of path. It is shown below that such dissipative terms are included in a state function $D(f)$ constructed for

the purpose. In other cases, dissipation must be accounted for as a path function, expressed in terms of the nonpotential efforts or flows of a system.

2.3.7 Content

The dissipation state function is a multidiscipline form of Rayleigh's dissipation function, and is constructed as follows.

The effort of an ideal resistor is a function of flow $e(f) = -e_\alpha(f)$. The negative integral of this effort with respect to flow is the dissipation function or *content* $D(f)$ defined by

$$D(f) := -\int e(f)\,df. \tag{2.64}$$

Introducing limits of integration, such that $f(t_o) = f_o$ and $f(t) = f$, there exists a constant of integration $D_o = D(f_o)$ such that the content is given by

$$D(f) = D_o - \int_{f_o}^{f} e(f)\,df. \tag{2.65}$$

Summing over n generalized resistors in a system, the dissipation state function or content $D(f)$ for a multidiscipline system is given by

$$D(f) = D_o - \sum_{j=1}^{n} \int_{f_o}^{f} e_j(f)\,df_j, \tag{2.66}$$

yielding the differential form

$$-\frac{\partial D}{\partial f_j} = e_j, \tag{2.67}$$

where $e_j = -(e_\alpha)_j$. Efforts satisfying this relationship are called dissipation efforts e^d. In the linear case, $e^d = -Rf$, where R is generalized resistance. Thus, for a system containing all linear dissipators, and assuming zero reference conditions, the content of the system is given by

$$D(f) = \frac{1}{2}\sum_{j=1}^{n} R_j f_j^2. \tag{2.68}$$

EXAMPLE 2.7 The voltage $e(f)$ in a linear resistor can be written $e(i) = -Ri$, where i is current and R is resistance. The definition of content yields

$$D(i) = -\int_0^i e(i)\,di = -\int_0^i -Ri\,di = \tfrac{1}{2}Ri^2. \tag{2.69}$$

This expression is identical to one-half the dissipated power, which is typical of the relationship between content and power. ◇

Content $D(f)$ can be interpreted as the area under the constituitive law $e_\alpha = e_\alpha(f)$ as shown in Fig. 2.11. This graphical interpretation illustrates that content is a scalar function that varies with the instantaneous flow of the ideal resistor.

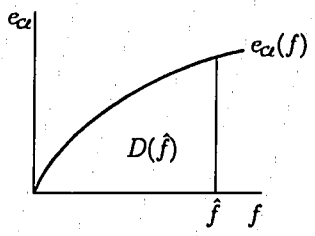


Figure 2.11. Area representation of content.

2.3.8 Co-content

A complementary dissipation function, the co-content G, is obtained through application of the Legendre transform to the content $D(f)$. The details of the transform parallel the transform of potential energy given in a previous section and so are not repeated here.

The Legendre transform of $D(f)$ is the new function $G(e)$ called the *co-content*, defined by

$$G(e) := -fe - D(f), \tag{2.70}$$

yielding the first result of interest, similar to (2.27), given by

$$D(f) + G(e) = -fe. \tag{2.71}$$

Summing over n dissipators in a system, the content and co-content satisfy

$$D(f) + G(e) = -\sum_{j=1}^{n} f_j e_j. \tag{2.72}$$

Furthermore, as a consequence of (2.28), co-content can also be given by

$$G(e) = G_o - \sum_{j=1}^{n} \int_{e_o}^{e} f_j(e)\, de_j. \tag{2.73}$$

This expression yields the differential form

$$-\frac{\partial G}{\partial e_j} = f_j, \tag{2.74}$$

where $f = -f_\alpha(e)$ is the inverse of the constituitive law for the dissipator. Flows satisfying this relationship are called dissipative flows f^d. In the linear case, $f^d = -e/R$, where R is generalized resistance. Thus, for a system containing all linear resistors, and assuming zero reference conditions, the co-content of the system is given by

$$G(e) = \sum_{j=1}^{n} \frac{e^2}{2R}. \tag{2.75}$$

EXAMPLE 2.8 A linear damper with damping coefficient b translates with velocity v. The force the damper exerts on the systems is given by $F = -bv$. The co-content of the damper is given by

$$G(F) = -\int_o^F v(F)\, dF = -\int_o^F -\frac{F}{b}\, dF = \frac{F^2}{2b}. \quad \diamond \tag{2.76}$$

Expressions for the content and co-content for the common linear resistors are given in Table 2.10.

Content and co-content of a pure resistor can be given the graphical interpretation shown in Fig. 2.12. Content can be interpreted as the area under the curve of the constituitive law $e_\alpha(f)$. Co-content can be interpreted as the difference in areas given by $G = fe_\alpha - D$. If $e_\alpha(f)$ is linear, the areas representing D and G are equal.

Content and co-content are scalar quantities representing the same constituitive law, differing only in the choice of independent variable. Content varies with flow. Co-content

Table 2.10. Content and co-content of ideal resistors.

		translational damper	torsional damper	electrical resistor	fluid resistor
Content	$D(f)$	$\frac{1}{2}bv^2$	$\frac{1}{2}b_\theta\omega^2$	$\frac{1}{2}Ri^2$	$\frac{1}{2}R_fQ^2$
Co-content	$G(e)$	$\frac{F^2}{2b}$	$\frac{\tau^2}{2b_\theta}$	$\frac{e^2}{2R}$	$\frac{P^2}{2R_f}$

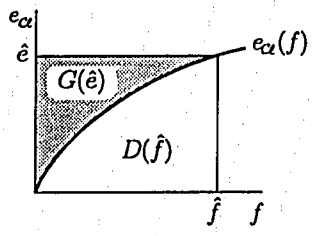


Figure 2.12. Area representation of content and co-content.

varies with effort. And effort and flow are related through the constituitive law of the ideal dissipator.

2.3.9 Paynter's diagram

To summarize the energy relationships developed in this section, Paynter's diagram is labeled with the energy functions as shown in Fig. 2.13. The vertices of the figure are labeled with the generalized variables (e, f, p, q) and the line segments connecting the vertices represent the manner in which the generalized variables are interrelated. Thus the energy functions of a potential store $V(q)$ and $V^*(e)$ relate q and e, the energy functions of a dissipator $D(f)$ and $G(e)$ relate f and e, and the energy functions of a kinetic store $T(p)$ and $T^*(f)$ relate p and f. Again the time integrals represent the time–dependent relationships between the kinematic variables q and f and between the dynamic variables p and e. Choosing the proper form of energy function to represent potential stores, kinetic stores and dissipators in a model depends on the system variables

selected by the analyst to represent the motion of the system.

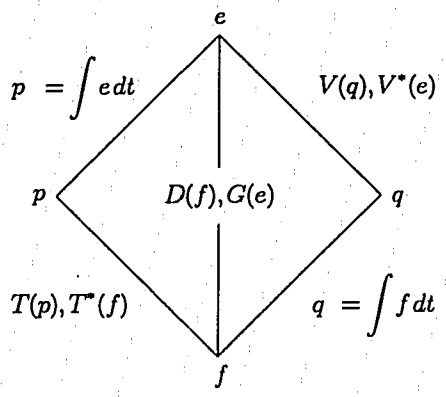


Figure 2.13. Paynter's diagram and the energy state functions.

2.4 Representation of motion

Unified methods of modeling multidiscipline systems are distinguished in part by the choice of variable pairs used to represent the motion of a system. In this section is presented an overview of the implications of the choice of representational variable pairs. Furthermore, in multidiscipline systems both vector and scalar quantities are required to describe the physical quantities represented by the generalized variables effort, flow, momentum and displacement. The concepts of configuration space and state space provide a unified approach to manipulating these different quantities.

2.4.1 Variable pairs

The relationships indicated on Paynter's diagram imply that a pair of variables from the set of generalized variables (e, f, p, q) is sufficient to represent the motion of a system. Using the variable pair (e, f) for example, potential stores are represented by $V^*(e)$, dissipators can be represented by either $G(e)$ or $D(f)$ and kinetic stores are represented by $T^*(f)$. Or using the variable pair (p, q), potential stores are represented by $V(q)$, dissipators are represented by $D(\dot{q})$ and kinetic stores are represented by $T(p)$. The selection

of a particular pair of variables to represent the motion of a system is a distinguishing characteristic of the various unified methods of modeling multidiscipline systems.

For example, the bond graph technique utilizes effort and flow (e, f) as representational variables (an implementation of the traditional force-voltage analogy). Linear graph techniques also utilize effort and flow, but in the form of *through* and *across* variables (the force–current analogy). In Lagrangian dynamics displacement and flow (q, f) are used (see Chapter 5). In Hamiltonian dynamics momentum and displacement (p, q) are used (see Chapter 6). And in the dual to the Lagrangian formulation, the pair (e, p) is used (see Chapter 7).

The remaining variable–pair combinations (e, q) and (f, p) are not commonly used in multidiscipline dynamics. Consider, for example, the selection of the variable pair (e, q) to represent the motion of a system. The energy of kinetic stores cannot be directly expressed in terms of e and q since kinetic energy and coenergy depend on p and f respectively. Thus the use of the variable pair (e, q) is reasonable only for systems containing no kinetic stores. Similarly, selecting the variable pair (f, p) implies that the energy of potential stores cannot be directly expressed since potential energy and coenergy depend on q and e respectively. Thus the use of the variable pair (f, p) is reasonable only for systems containing no potential stores. These variable pairs may be useful in these special cases and merit consideration for the purpose of rounding out the theory of the analytical method. However, these variable pairs are not applicable in the general multidiscipline case and so are not examined in this work.

Following the precedent set by Lagrange, displacement and flow (q, f) are selected as the representational pair of variables in terms of which Lagrange's equation is derived. The energy functions having the desired form are potential energy $V(q)$, the dissipation function $D(f)$, and the kinetic coenergy function $T^*(f)$. Paynter's diagram in Figure 2.14 summarizes the relationships among the generalized variables in Lagrangian form.

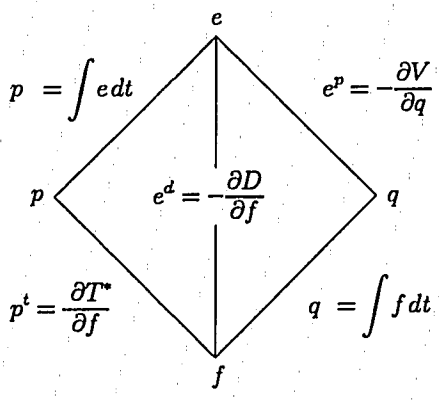


Figure 2.14. Paynter's diagram in Lagrangian form. Relationships are given in terms of displacement and flow.

2.4.2 Configuration space and state space

The position of a particle P in a system can be denoted by an ordered tuple of scalar coefficients (x, y, z) in a Cartesian coordinate frame. For a system of N_p particles, there are N_p of these tuples. These tuples can be assembled into an ordered set S_u of dimension $3N_p$, that is,

$$S_u = (x_1, y_1, z_1, x_2, y_2, z_2, \ldots, x_{N_p}, y_{N_p}, z_{N_p})\,. \tag{2.77}$$

The members of S_u are renamed u_i to obtain a scalar set u of dimension $3N_p$, that is, $u = (u_1, \ldots, u_{3N_p})$.

Electrical elements in a system have displacement variables of charge q. For a system with N_e electrical coordinates there is an ordered set of charges $q = (q_1, \ldots, q_{N_e})$. The elements of this set are designated u_i, where $i = 3N_p + 1, \ldots, 3N_p + 1 + N_e$, and this set is appended to the set u. A similar procedure is followed for the rotational, fluid and thermal coordinates in the system. All components of displacement are appended to the set u.

Let N be the total number of displacement components in a system. Then the set of scalars or coordinates $u = (u_1, \ldots, u_N)$ uniquely defines the *configuration* of the

system. The u_i are called *configuration coordinates.* At an instant, u can be considered a point in an N-dimensional *configuration space.* Over time, u describes a trajectory in configuration space.

Each component of flow in a system is represented by the scalar $\dot{u}_i$. An ordered set of displacement and flow variables is constructed which uniquely defines the *state* of the system, given by

$$(u, \dot{u}) = (u_1, \ldots, u_N, \dot{u}_1, \ldots, \dot{u}_N). \tag{2.78}$$

The elements of this set are the *state variables* of the system. At an instant, $(u, \dot{u})$ can be considered a point in a $2N$-dimensional *state space.* Over time, $(u, \dot{u})$ describes a trajectory in state space.

The results of this change in nomenclature are summarized on Paynter's diagram, Fig. 2.15. All variables are expressed in component form, and the Lagrangian choice of state variables $(u, \dot{u})$ is implicit in the derivatives shown. The component form indicated here is the form in which Lagrange's equation for multidiscipline systems is derived.

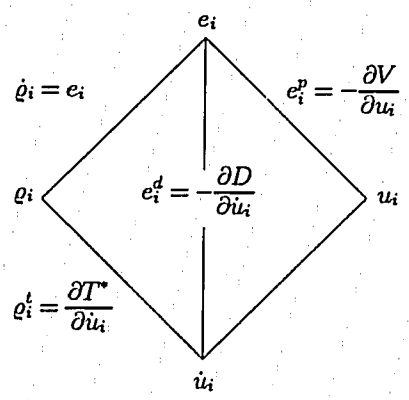


Figure 2.15. Paynter's diagram in component form.

EXAMPLE 2.9 To illustrate the concepts of configuration space and state space, coordinates are assigned to the electromechanical system in Fig. 2.16.

Diode currents are assigned the flow variables $\dot{u}_1$ and $\dot{u}_2$. The capacitor, resistor

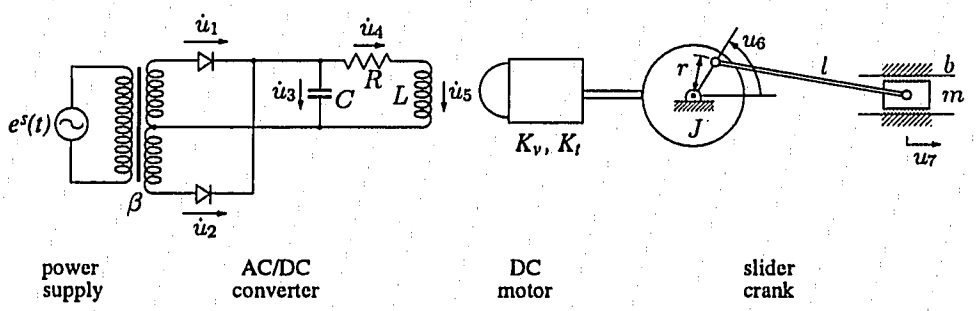


Figure 2.16. Assignment of configuration coordinates in an electromechanical system.

and inductor currents are assigned the flow variables $\dot{u}_3$ through $\dot{u}_5$. The angular displacement of the crank is assigned the displacement variable u_6, and the translational displacement of the mass is assigned the variable u_7. The configuration of the system is given by

$$u = (u_1, \ldots, u_7), \tag{2.79}$$

and the state of the system is given by

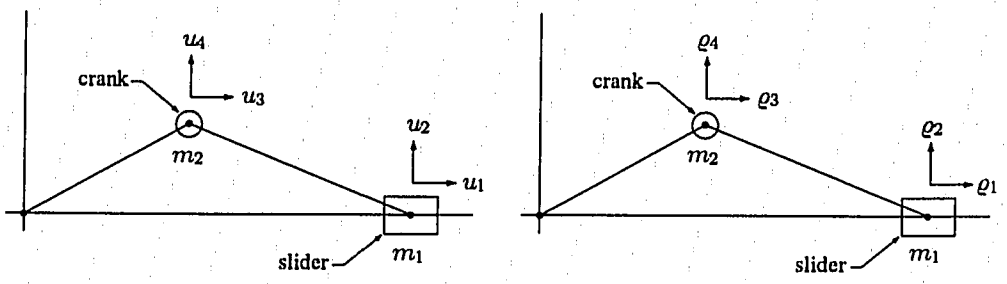
$$(u, \dot{u}) = (u_1, \ldots, u_7, \dot{u}_1, \ldots, \dot{u}_7). \quad \diamond \tag{2.80}$$

This example illustrates that the choice of displacement and flow as the representational pair of system variables does not imply that the components of the set of state variables are mutually independent. In the previous example for instance, the connections among the elements impose on the state variables two conditions described by Kirchhoff's current law, namely, $\dot{u}_4 = \dot{u}_5$ and $\dot{u}_1 + \dot{u}_2 = \dot{u}_3 + \dot{u}_4$. These relationships constrain the flow coordinates, consequently not all the state variables are independent. Thus, while the set of state variables $(u, \dot{u})$ is sufficient to describe the motion of a system, the set may be overdetermined. In such a case the set of configuration coordinates is said to contain *excess* coordinates.

Similar to the component representation of displacement, the components of momentum p can be represented by the set $\varrho = (\varrho_1, \ldots, \varrho_N)$, defining a complement to

the configuration space of a system. If each component of effort is represented by the scalar $\dot{\varrho}_i$, then an ordered set $(\varrho, \dot{\varrho}) = (\varrho_1, \ldots, \varrho_N, \dot{\varrho}_1, \ldots, \dot{\varrho}_N)$ can be constructed which uniquely defines a complement to the state space.

EXAMPLE 2.10 Assignment of components of displacement u_i to a slider–crank mechanism is shown in Fig. 2.17(a). The corresponding components of momenta ϱ_i are shown in Fig. 2.17(b). In both cases, the coordinates are assigned to the lumped masses m_1 and m_2, and planar motion is assumed. $\diamond$



(a) Components of displacement

(b) Components of momenta

Figure 2.17. Displacement and momentum in component form for a slider–crank mechanism.

2.4.3 Reduced–order coordinates

When a system is constrained, the elements of $(u, \dot{u})$ or $(\varrho, \dot{\varrho})$ or both are not independent. In such a case, a coordinate set of reduced dimension n exists $(n < N)$ that is sufficient to define the configuration of the system. Such a set is not necessarily unique, nor is it necessarily of minimum dimension. These coordinates, represented by the displacement vector $q = (q_1, \ldots, q_n)$ are called herein the *reduced–order* configuration coordinates. The term *generalized coordinates* or *Lagrange coordinates* is reserved for the reduced–order coordinate set of minimum dimension. The associated momentum vector is given by $p = (p_1, \ldots, p_n)$.

Reduced–order coordinates are related to the configuration coordinates through N

transformation equations having the form $u_i = u_i(q_1, \ldots, q_n, t)$, where $i = 1, \ldots, N$. The time derivative of the coordinate transformation equations is given by N flow transformation equations as follows

$$\dot{u}_i = \sum_{j=1}^{n} \frac{\partial u_i}{\partial q_j} \dot{q}_j + \frac{\partial u_i}{\partial t} \qquad i = 1, \ldots, N. \tag{2.81}$$

Since the coefficients given by $\partial u_i / \partial q_j$ are functions of q, the flow transformation equations have the general form $\dot{u}_i = \dot{u}_i(\dot{q}_1, \ldots, \dot{q}_n, q_1, \ldots, q_n, t)$.

EXAMPLE 2.11 The pendulum shown in Fig. 2.18 is used to illustrate the concept of reduced-order coordinates. The configuration coordinates of the mass m at the end of a rod of fixed length l is given by $u = (u_1, u_2, u_3)$. Assuming planar motion such that $u_3 = 0$, the angle q is sufficient to describe the motion of the system.

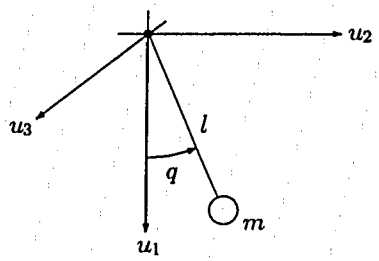


Figure 2.18. Reduced-order coordinates for a simple pendulum.

The configuration space has dimension $N = 3$. The set of three transformation equations is given by

$$\begin{aligned} u_1(q) &= l \cos q \\ u_2(q) &= l \sin q \\ u_3(q) &= 0. \end{aligned} \tag{2.82}$$

The time derivatives of these equations are given by

$$\begin{aligned} \dot{u}_1(\dot{q}, q) &= -l\dot{q} \sin q \\ \dot{u}_2(\dot{q}, q) &= l\dot{q} \cos q \\ \dot{u}_3 &= 0, \end{aligned} \tag{2.83}$$

Since the angle q is the reduced-order coordinate set of minimum dimension, q is a Lagrange or generalized coordinate for the system. ◇

Chapter 3

CONSTRAINTS

The manner in which a system processes energy depends on the characteristics of the elements of the system as well as the way in which the system elements are interconnected. The concept of constraint is a means of systematically modeling the contribution of energy sources, transformers and transducers, as well as the contribution of system interconnections or geometry, to the dynamic behavior of a system.

A *constraint* is defined as an algebraic condition imposed on a system due to system geometry or interconnectivity, or due to the constituitive relations of system elements that are not subsumed under the energy state functions. Constraints are divided into two broad classifications. *Kinematic* constraints are described by relationships involving displacements or flows. *Dynamic* constraints are described by relationships involving efforts or momenta. Additional characteristics that distinguish the two basic types of constraint are presented in due course.

The concept of constraint is central to analytical dynamics. The types of constraints in a system have broad consequences regarding the tractability of a modeling problem. Constraints are used to define the concepts of virtual displacements and virtual work, both of which are important to the development of the equations of motion. Constraints also are a fundamental idea underlying Lagrange's principle, which allows constraint efforts to be neglected in the analysis of multidiscipline systems. And it is the ubiquitous presence of constraints in multidiscipline systems that underlies one of the primary goals of this work — to systematically formulate the equations of motion as a set of differential–algebraic equations.

3.1 Kinematic constraints

Kinematic constraints impose conditions on displacement or flow. These conditions may take several forms. Kinematic constraints in holonomic form place conditions on u or possibly $\dot{u}$, where $u = (u_1, \ldots, u_N)$ represents the components of system displacements in configuration space. Kinematic constraints that are nonholonomic are not as easily characterized, but in general nonholonomic constraints impose conditions on differentials of u.

3.1.1 Holonomic constraints

A *holonomic* constraint is an algebraic condition imposed on a system that can be expressed as, or is reducible to, a function of displacement and possibly time having the general form

$$\phi(u, t) = 0. \tag{3.1}$$

A set of such constraints $\Phi \in \mathcal{R}^m$ is given in vector form by

$$\Phi(u, t) = \begin{bmatrix} \phi_1(u, t) \\ \vdots \\ \phi_m(u, t) \end{bmatrix} = 0, \tag{3.2}$$

where $u = (u_1, \ldots, u_N)$.

The effect of a set of holonomic constraints on the motion of a system is to render certain states impossible to attain. Such states, called *inadmissible states*, can be thought of in geometric terms as regions of the state space that the system may not occupy. The various conditions which give rise to holonomic constraints are best illustrated by example.

EXAMPLE 3.1 In the slider-crank mechanism shown in Fig. 3.1, the translational displacement x of mass m and the angular displacement θ of crank J are constrained by the rod of fixed length l. The displacement constraint ϕ is given by

$$\phi(x, \theta) := (x - r\cos\theta)^2 + (r\sin\theta)^2 - l^2 = 0. \tag{3.3}$$

Configurations (x, θ) that are inconsistent with this constraint are inadmissible. $\diamond$

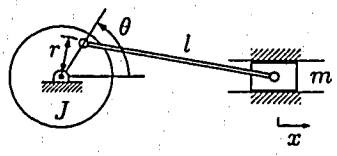


Figure 3.1. A slider-crank mechanism.

EXAMPLE 3.2 A wheel of radius r rolls without slipping on a horizontal surface, as shown in Fig. 3.2. The wheel is attached to a larger cylinder such that the combined rotating element has mass m_2 and mass moment of inertia J. The wheel is wrapped with an inextensible cable connected to mass m_1. The initial displacement of m_1 is x_{1_0} and the initial displacement of m_2 is x_{2_0}.

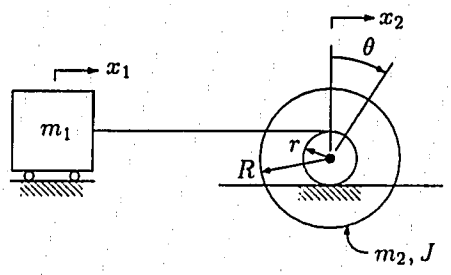


Figure 3.2. Mechanical displacement constraints.

The inextensible cable imposes a constraint ϕ_1 given by

$$\phi_1(x_1, x_2, \theta) := (x_1 - x_{1_0}) - (x_2 - x_{2_0}) - r(\theta - \theta_o) = 0. \tag{3.4}$$

The no-slip condition imposes a constraint ϕ_2 given by

$$\phi_2(x_1, x_2, \theta) := (x_2 - x_{2_0}) - r(\theta - \theta_o) = 0. \tag{3.5}$$

Configurations (x_1, x_2, θ) that are inconsistent with these constraints are inadmissible. $\diamond$

Flow constraints

Constraints on displacement often imply corresponding constraints on flow. For instance, the time derivative of the holonomic constraint in the slider–crank example is given by

$$\psi(\dot{x}, \dot{\theta}, x, \theta) := (x - r\cos\theta)\dot{x} + (rx\sin\theta)\dot{\theta} = 0. \tag{3.6}$$

States $(x, \theta, \dot{x}, \dot{\theta})$ that are inconsistent with either this constraint or the displacement constraint given by (3.3) are inadmissible states.

The constraint in (3.6) is representative of a class of constraints called flow continuity relations or simply *flow constraints* ψ. Flow constraints that can be integrated, such that the holonomic form given by $\phi(u, t) = 0$ can be obtained, are called holonomic or integrable flow constraints.

EXAMPLE 3.3 Referring again to the wheel and mass example shown in Fig. 3.2, the flow constraints of the system are given by

$$\begin{aligned} \psi_1(\dot{x}_1, \dot{x}_2, \dot{\theta}) &:= \dot{x}_1 - \dot{x}_2 - r\dot{\theta} = 0 \\ \psi_2(\dot{x}_1, \dot{x}_2, \dot{\theta}) &:= \dot{x}_2 - r\dot{\theta} = 0. \end{aligned} \tag{3.7}$$

Integrating these functions with respect to time yields a pair of equations identical to the displacement constraints given in Ex. 3.2. The flow constraints in this example are integrable, hence the constraints are holonomic. ◇

Holonomic constraints are not always perceived fundamentally as conditions on displacement, even though they may be cast in that form through integration. For instance, the no-slip condition in the previous example is usually written in terms of flow as in (3.7) rather than in terms of displacement as in (3.5). Kirchhoff's current law and the fluid continuity equation are flow constraints also of this type.

EXAMPLE 3.4 By Kirchhoff's law the currents at node a in the circuit shown in Fig. 3.3 satisfy the flow constraint given by

$$\psi(i) := i_1 - i_2 - i_3 = 0. \tag{3.8}$$

This equation can be integrated with respect to time to obtain

$$\phi(q) := q_1 - q_2 - q_3 + C_o = 0, \tag{3.9}$$

where C_o represents an appropriate sum of initial charges q_{i_0}. $\diamond$

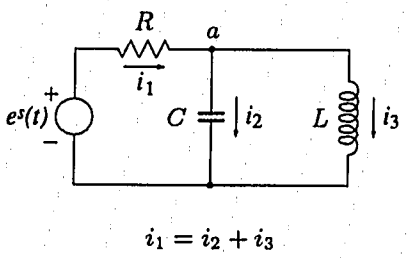


Figure 3.3. A simple electrical flow constraint.

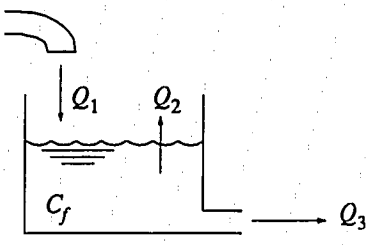
The electrical displacement variable q does not necessarily represent a physical accumulation of charge-carrying particles. For example, q_3 in (3.9) does not represent an accumulation of charge in the inductor L. By definition, the displacement q of an element is the integral of the flow of that element. In the electrical case, displacement is given by $q = \int i\,dt$. Thus an inductor, for example, does not accumulate *charge* q but does undergo a *displacement* q. It is in this sense that a flow constraint equation like (3.8) is integrable.

EXAMPLE 3.5 The inlet fluid flow rate Q_1, the rate of change of storage Q_2 and outlet flow rate Q_3 shown in Figure 3.4 are related by the flow constraint given by

$$\psi(Q) := Q_1 - Q_2 - Q_3 = 0. \tag{3.10}$$

Letting C_o represents the sum of initial volume conditions V_{i_0}, this expression can be integrated to obtain the holonomic form

$$\phi(V) := V_1 - V_2 - V_3 + C_o = 0. \quad \diamond \tag{3.11}$$



$$Q_1 = Q_2 + Q_3$$

Figure 3.4. A simple fluid flow constraint.

These examples illustrate that although a constraint may be conveniently expressed in terms of flow, it is holonomic if and only if it can be integrated to obtain the form given by (3.1). It is preferable in some cases to model holonomic constraints in flow constraint form. Such cases typically arise when equations of motion contain a component of flow $\dot{u}_i$ but do not explicitly contain the displacement variable u_i corresponding to that flow. Such displacement variables are called in traditional nomenclature *ignorable coordinates.* The significance of ignorable coordinates here is that holonomic constraints involving only ignorable coordinates need not be integrated in formulating the differential–algebraic equations of motion.[1] There are numerical advantages to leaving such constraints in flow variable form.

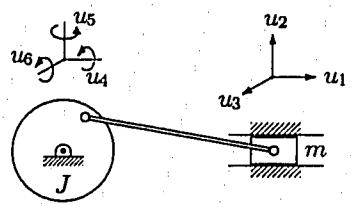
Unstated constraints

Another type of constraint is the *unstated constraint* which, like the flow constraint, is often holonomic. Such constraints are commonly part of the assumptions associated with

[1]The traditional utility of ignorable coordinates is to reduce the dimension of a problem, which is not a primary concern in this work.

a particular problem, and often go unstated.

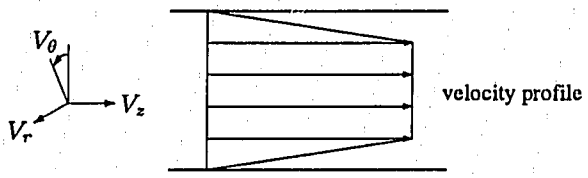
EXAMPLE 3.6 To describe the motion of the slider–crank in Ex. 3.1, only a single direction of translation x and a single direction of rotation θ are indicated. The system must be constrained in an unstated fashion such that no displacement takes place in any other direction. If the figure is redrawn with the three-dimensional components of displacement u_i explicitly shown, as in Figure 3.5, the unstated constraints are readily obtained. With motion admissible only in the translational direction of u_1 and the rotational direction of u_6, it follows that $u_2 = u_3 = u_4 = u_5 = 0$. This expression contains four constraint equations in holonomic form. ◇



$$u_2 = u_3 = u_4 = u_5 = 0$$

Figure 3.5. Unstated constraints in mechanical motion due to the assumption of one–dimensional translation and rotation.

EXAMPLE 3.7 The assumption of one–dimensional fluid flow also illustrates the idea of an unstated constraint. In Figure 3.6, a typical velocity profile is shown for an incompressible fluid flowing in a circular pipe. This idealized velocity profile implies that the flow is non–radial and irrotational. If the motion is described in terms of cylindrical coordinates (r, θ, z) and the components of fluid displacement are the volumetric displacement terms $(u_1, u_2, u_3) = (V_r, V_\theta, V_z)$, then the non-radial constraint is written $u_1 = 0$ and the irrotational constraint is written $u_2 = 0$. ◇



$$V_r = V_\theta = 0$$

Figure 3.6. Unstated constraints in fluid motion due to the assumption of non–radial and irrotational flow.

Holonomic constraints in Pfaffian form

To conclude this section on holonomic constraints, consider a system having m holonomic constraints of the form $\phi_k(u,t) = 0$ where $k = 1,\ldots,m$; $u = (u_1,\ldots,u_N)$; and $m < N$, that is, the number of constraints is less than dimension of the configuration space. The differential of each constraint ϕ_k is given by

$$d\phi_k(u,t) = \sum_{i=1}^{N} \frac{\partial \phi_k}{\partial u_i}(u,t)\, du_i + \frac{\partial \phi_k}{\partial t}(u,t)\, dt = 0 \qquad k = 1,\ldots,m. \tag{3.12}$$

Dividing (3.12) by dt yields the holonomic constraint equation in rate form given by

$$\sum_{i=1}^{N} \frac{\partial \phi_k}{\partial u_i}(u,t)\, \dot{u}_i + \frac{\partial \phi_k}{\partial t}(u,t) = 0 \qquad k = 1,\ldots,m. \tag{3.13}$$

In vector form, these two equations are given by

$$\begin{aligned} \Phi_u\, du + \Phi_t\, dt &= 0 \\ \Phi_u\, \dot{u} + \Phi_t &= 0, \end{aligned} \tag{3.14}$$

where $\Phi_t := \partial\Phi/\partial t \in \mathcal{R}^m$, and Φ_u is the Jacobian matrix $\partial\Phi/\partial u \in \mathcal{R}^{m\times N}$, given by

$$\frac{\partial \Phi}{\partial u} = \begin{bmatrix} \dfrac{\partial \phi_1}{\partial u_1} & \cdots & \dfrac{\partial \phi_1}{\partial u_N} \\ \vdots & & \vdots \\ \dfrac{\partial \phi_m}{\partial u_1} & \cdots & \dfrac{\partial \phi_m}{\partial u_N} \end{bmatrix}. \tag{3.15}$$

These representations of holonomic constraints are in *Pfaffian form.* The general Pfaffian form is given by

$$\sum_{i=1}^{N} a_{ki}(u,t)du_i + a_k(u,t)dt = 0 \qquad k = 1,\ldots,m, \tag{3.16}$$

or in vector form,

$$A\,du + a\,dt \;=\; 0, \tag{3.17}$$

where

$$\begin{aligned} A(u,t) &:= \text{matrix}\{a_{ki}\} \in \mathcal{R}^{m\times N} \\ a(u,t) &:= \text{vector}\{a_k\} \in \mathcal{R}^{m}. \end{aligned} \tag{3.18}$$

Dividing by dt yields the Pfaffian flow constraint equation given by

$$\sum_{i=1}^{N} a_{ki}(u,t)\dot{u}_i + a_k(u,t) = 0 \qquad k = 1,\ldots,m, \tag{3.19}$$

or in vector form,

$$A\,\dot{u} + a = 0. \tag{3.20}$$

For a set of holonomic constraints, the coefficients of the Pfaffian form are given by $a_{ki} := \partial\phi_k/\partial u_i$ and $a_k := \partial\phi_k/\partial t$, or in vector form $A := \Phi_u$ and $a := \Phi_t$. The Pfaffian form expresses holonomic constraints as a set of restrictions on the infinitesimal quantities (du, dt) instead of as a set of restrictions on the finite quantities (u, t). However, holonomic constraints are not inherently conditions on differentials. The Pfaffian form in this case is simply the differential representation of constraints that restrict the finite quantities u_i. Holonomic constraints in Pfaffian form are integrable, though in practice the integration may be intractable. One of the useful properties of the Pfaffian form is that it permits the representation of holonomic constraints without reducing the constraints to the form $\phi(u,t)$.

EXAMPLE 3.8 As shown in Fig. 3.7, the motion of mass m at the end of a rod of fixed length l is constrained such that the distance from the mass to the pivot is constant. In holonomic form, this constraint is given by

$$\phi(u) := u_1^2 + u_2^2 - l^2 = 0. \tag{3.21}$$

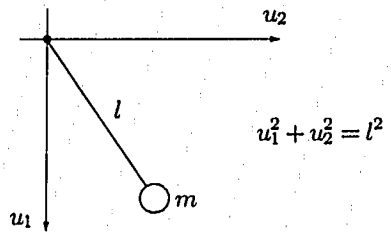


Figure 3.7. A holonomic constraint for a simple pendulum.

The differential of this constraint yields the Pfaffian form, as follows.

$$d\phi = 0 = \frac{\partial \phi}{\partial u_1} du_1 + \frac{\partial \phi}{\partial u_2} du_2 + \frac{\partial \phi}{\partial t} dt \tag{3.22}$$

$$= 2u_1\, du_1 + 2u_2\, du_2 \tag{3.23}$$

$$= u_1\, du_1 + u_2\, du_2. \tag{3.24}$$

Dividing by dt yields the flow-constraint form given by

$$\psi(\dot{u}, u) := u_1 \dot{u}_1 + u_2 \dot{u}_2 = 0, \tag{3.25}$$

which is a special case of the general form

$$\psi(\dot{u}, u, t) := a_{11}(u,t)\dot{u}_1 + a_{21}(u,t)\dot{u}_2 + a_1(u,t) = 0, \tag{3.26}$$

which was to be shown. $\diamond$

The Pfaffian form of a holonomic constraint does not contain as much information as the general form $\phi(u,t) = 0$. For example, if the differential form of the pendulum constraint is integrated, the resulting equation contains an unknown constant of integration

which can be evaluated only if additional information is known about the physical system. In general, such information is found from initial conditions, boundary conditions, or the geometry of the system.

3.1.2 Nonholonomic constraints

Kinematic constraints that are not holonomic are called *nonholonomic.* Nonholonomic constraints are more difficult to characterize than holonomic constraints. As Rosenberg [35] states,

> *One will readily understand that it is not possible to give a general discussion of nonholonomic constraints such as can be done for holonomic ones because the latter is a narrowly circumscribed class while the former is not. (Thus, bananas are readily discussed, while nonbananas are not.) Nevertheless, some classification of frequently encountered nonholonomic constraints is possible.*

The classification of nonholonomic constraints given in this section generally follows Rosenberg's development in analytical mechanics, though the material is expanded to account for the multidiscipline nature of system dynamics.

Impenetrability constraints

The first type of nonholonomic constraint to be considered is the *impenetrability constraint.* Consider a mass moving moving on an impenetrable surface, with spatial coordinates (x, y) tangent and normal to the surface, as shown in Figure 3.8(a). Since the mass may not penetrate the surface, motion normal to the surface is restricted such that $y \geq 0$; this inequality represents a physical condition that constrains the mass such that motion below the surface is inadmissible. In differential form, this constraint is written $dy \geq 0$.[2]

[2]It is common practice to express inequality constraints in the form of a function ϕ such that $\phi \leq 0$. Inequality constraints can be multiplied by -1, if necessary, to obtain the standard form.

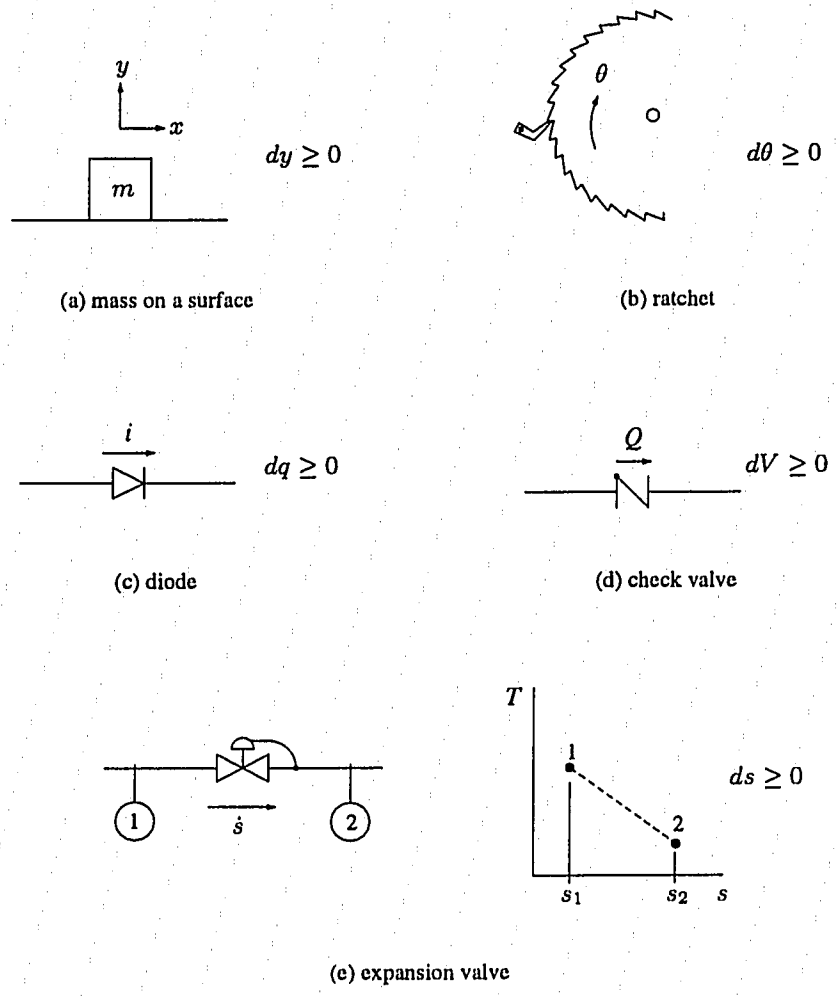


Figure 3.8. Impenetrability constraints having the form $du \geq 0$: (a) translational; (b) rotational; (c) electrical; (d) fluid; and (e) thermal.

The same type of constraint occurs in the motion of a ratchet and pawl, shown in Figure 3.8(b). Rotation is constrained to occur in one direction only, that is, $\theta \geq 0$, or in differential form, $d\theta \geq 0$. Negative angular displacements are inadmissible.

An electrical example of an impenetrability constraint is the effect a diode imposes on current, illustrated in Figure 3.8(c).[3] Current is constrained to flow in one direction only, that is, $i \geq 0$, and since $i = dq/dt$, the constraint can be written $dq \geq 0$. Thus negative displacements of charge are inadmissible. In one-dimensional fluid flow, a check valve imposes the same type of constraint, illustrated in Figure 3.8(d), such that fluid flow Q moves in one direction only, and since $Q = dV/dt$, the constraint can be written $dV \geq 0$. Thus negative displacements of volume are inadmissible.

An expansion valve as used in a refrigeration cycle is an example of this type of constraint in thermal terms, as shown in Figure 3.8(e). The second law of thermodynamics applied to the expansion valve states that $Tds \geq đq$, where q is heat. The ideal expansion valve is assumed to be adiabatic, so $đq = 0$ and $Tds \geq 0$. Thermodynamic temperature or absolute temperature is always positive, implying that $ds \geq 0$, which is a constraint stating that the thermal displacement, entropy, can only increase across an expansion valve. Negative displacements of entropy are inadmissible. On the Ts–diagram, inadmissible states are those to the left of the line $s = s_1$.

The constraints shown in Figure 3.8 are special cases of constraints on either finite displacements u or infinitesimal displacements du given by

$$\begin{aligned} \phi(u,t) &\leq 0 \\ \phi(du,dt) &\leq 0. \end{aligned} \tag{3.27}$$

Such constraints are prevalent in physical systems, yet there exists no systematic method of accounting for such constraints in a model. No such method is offered in this work.

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[3]The impenetrability model of a diode is an elementary view of diode characteristics. It is shown in a subsequent section how an equivalent–circuit model of a diode is used in the analytical method.

Nonholonomic equality constraints

The second type of nonholonomic constraint to be considered is the *equality constraint.* The nonholonomic equality constraints considered in this work are those that are reducible to Pfaffian form. The distinguishing characteristic of the Pfaffian form is that it imposes restrictions on the infinitesimal quantities du_i. Such constraints are incorporated in the equations of motion through the use of Lagrange multipliers, which are introduced in a subsequent section.

The general Pfaffian form (from p. 57) is given by

$$\sum_{i=1}^{N} a_{ki}(u,t)du_i + a_k(u,t)dt = 0 \qquad k = 1,\ldots,m, \tag{3.28}$$

or in vector form,

$$A\,du + a\,dt \;=\; 0, \tag{3.29}$$

where

$$\begin{aligned} A(u,t) &:= \text{matrix}\{a_{ki}\} \in \mathcal{R}^{m\times N} \\ a(u,t) &:= \text{vector}\{a_k\} \in \mathcal{R}^{m}. \end{aligned} \tag{3.30}$$

In this form, the conditions imposed on du are explicitly shown. Dividing by dt yields the flow rate form of the nonholonomic equality constraint,

$$\sum_{i=1}^{N} a_{ki}(u,t)\dot{u}_i + a_k(u,t) = 0 \qquad k = 1,\ldots,m, \tag{3.31}$$

or in vector form,

$$A\,\dot{u} + a = 0. \tag{3.32}$$

In contradistinction to the Pfaffian form of the holonomic constraints, the Pfaffian form of a nonholonomic constraint is not an exact differential, nor is it integrable to obtain the holonomic form $\phi(u,t) = 0$.

The rate form of these equations is linear in flow $\dot{u}$. Such constraints can be holonomic as illustrated by (3.25), or nonholonomic as illustrated by the example below. Thus the

Pfaffian form has the useful properties that 1) it can represent both holonomic and nonholonomic constraints, and 2) that these constraints can be represented as linear in flow.

EXAMPLE 3.9 Consider the motion of a boat in a plane.[4] Assuming that the roll, pitch and heave motions of the boat are negligible, the position and orientation of the boat are given by the coordinates $u = (x, y, \theta)$ as shown in Fig. 3.9.

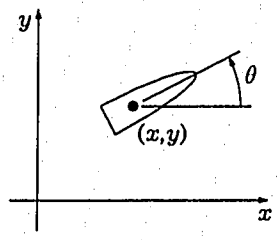


Figure 3.9. Motion of a boat in a plane.

The constraint is that at any instant the center of mass of the boat must move in the direction of its heading. This constraint is expressed by the equation

$$\frac{dy}{dx} = \tan\theta. \tag{3.33}$$

Rearranging yields a constraint on $(dx, dy, d\theta)$ given by

$$(\tan\theta)\ dx - dy = 0, \tag{3.34}$$

which is in Pfaffian form and the coefficient of $d\theta$ is zero. This constraint can be shown to be nonintegrable and therefore nonholonomic.[5] The flow-constraint form is given by

$$(\tan\theta)\,\dot{x} - \dot{y} = 0, \tag{3.35}$$

[4]This is a common example used to illustrate a nonholonomic constraint. This particular case is taken from D'Souza and Garg [11].

[5]See Appendix A.

which is a special case of the general form

$$\psi(\dot{u}, u, t) := \sum_{i=1}^{3} a_{1i}(u,t)\dot{u}_i + a_1(u,t) = 0, \tag{3.36}$$

which was to be shown. ◇

3.2 Dynamic constraints

Dynamic constraints involve at least one of the dynamic variables and possibly a kinematic variable or time. Dynamic constraints arise primarily because of the presence of elements whose contribution to the dynamic behavior of the system cannot be subsumed under any of the energy functions. The following examples illustrate the type of conditions that give rise to dynamic constraints.

EXAMPLE 3.10 In the simplest model of a DC motor, shown in Fig. 3.10, torque τ is proportional to current i, and speed ω is proportional to voltage e. These relationships are modeled as two dynamic constraints γ_1 and γ_2 given by

$$\begin{aligned} \gamma_1(\tau, i) &:= \quad \tau + K_t\, i \quad = 0 \\ \gamma_2(e, \omega) &:= \quad e - K_v\, \omega \quad = 0. \quad \diamond \end{aligned} \tag{3.37}$$

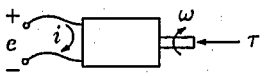


Figure 3.10. A DC motor.

The efforts (τ, e) are not subsumed under any of the energy functions. Such efforts arise due to the presence of dynamic constraints. In general, the dynamic constraints $\gamma = 0$ are implicit equations with respect to these efforts. Hence, these efforts are designated *implicit efforts* e^γ.

EXAMPLE 3.11 The current–voltage relationship of an ideal diode may be written as $i = i_s\,(e^{\alpha v} - 1)$, where i is current, v is voltage and i_s and α are known parameters. If the voltage is large and negative, the current approaches $-i_s$, which is usually referred to as the saturation current of the diode [17].

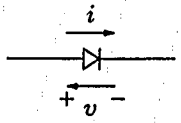


Figure 3.11. Effort and flow of a diode.

The voltage v is an effort that is not subsumed under any of the energy functions T^*, V or D, nor is it a known function of time. Therefore, this v–i relationship imposes a dynamic constraint given by

$$\gamma(v, i) := i - i_s\,(e^{\alpha v} - 1) = 0, \tag{3.38}$$

where voltage v is an implicit effort. ◇

This example illustrates a constraint equation implicit in v. This equation could be solved explicitly for v to obtain

$$v = \frac{1}{\alpha} \ln\left(\frac{i}{i_s} + 1\right), \tag{3.39}$$

which could be included in the equations of motion among the generalized nonpotential efforts. But the implicit form given by (3.38) is more tractable numerically than the explicit form given by (3.39).[6] Hence the implicit form of the dynamic constraint is retained and the implicit effort e^γ is an additional unknown for which a solution is sought in solving the equations of motion.

EXAMPLE 3.12 A mass slides with dry friction on a horizontal surface in the xy–plane. A friction force with magnitude μmg opposes the motion. From Newton's second law,

[6]DAEs containing the diode equation in implicit form have been successfully solved numerically. The solver does not converge, however, for the same DAE containing the diode equation in explicit, logarithmic form. Such numerical issues are not examined in this work.

the equations of motion are given in component form by

$$\begin{aligned} m\ddot{x} &= F_x(t) - e_x^\gamma \\ m\ddot{y} &= F_y(t) - e_y^\gamma, \end{aligned} \tag{3.40}$$

where F_x and F_y are known components of a force source acting on the mass, and e_x^γ and e_y^γ are unknown components of the force due to to friction.

In the absence of the friction force, the two differential equations would be sufficient for solving for the two unknowns x, y. By including the friction force, however, two additional unknowns (e_x^γ, e_y^γ) have been added to the problem. In order to solve for these two additional unknowns, two additional equations are required. These additional equations are the dynamic constraints.

The first dynamic constraint relates the implicit efforts to the total magnitude μmg of the friction force. This constraint is given by

$$\gamma_1 := (e_y^\gamma)^2 + (e_y^\gamma)^2 - (\mu mg)^2 = 0. \tag{3.41}$$

The second dynamic constraint arises from the geometric relationship between the two components of implicit effort. The friction force opposes the instantaneous motion of the mass. Thus the components of the friction force are in the same proportion as the components of velocity, that is, $e_y^\gamma / e_x^\gamma = \dot{y}/\dot{x}$, or

$$\gamma_2 := \dot{y} e_x^\gamma - \dot{x} e_y^\gamma = 0. \tag{3.42}$$

These relationships are illustrated in Fig. 3.12. $\diamond$

This example illustrates that in adding implicit efforts to the solution space of a problem, the number of unknowns is increased by the number of implicit efforts. The additional equations required to determine a solution are the dynamic constraint equations.

In summary, a *dynamic constraint* is an algebraic condition imposed on a system that is expressed as a function of implicit efforts, state variables, and possibly time, having

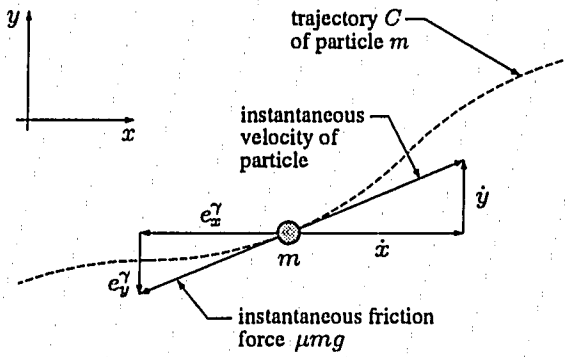


Figure 3.12. Dry friction modeled as a dynamic constraint.

the form

$$\gamma(e^\gamma, \dot{u}, u, t) = 0, \qquad k = 1, \ldots, m. \tag{3.43}$$

A set of such constraints is denoted by the vector Γ such that

$$\Gamma(e^\gamma, \dot{u}, u, t) = \begin{bmatrix} \gamma_1(e^\gamma, \dot{u}, u, t) \\ \vdots \\ \gamma_m(e^\gamma, \dot{u}, u, t) \end{bmatrix} = 0. \tag{3.44}$$

3.3 Classification of displacements

In analytical dynamics, three types of displacement arise: *actual*, *possible* and *virtual* displacements. This nomenclature, as well as the direct quotations below, are adapted for multidiscipline use from Rosenberg [35].

Actual displacements, as the name implies, give the actual motion. The class of finite quantities $u_i(t)$ that satisfy the dynamic and kinematic requirements of the model as well as the constraint equations is called the class of *actual displacements.* Actual displacements satisfy the differential–algebraic equations of motion. Solving for trajectories of actual displacements is often the explicit purpose of modeling.

Displacements that satisfy the constraints but not necessarily the differential equations of motion are called *possible displacements.* Possible displacements are the class of

infinitesimal quantities du_i that satisfy

$$\sum_{i=1}^{N} a_{ki}(u,t)\,du_i + a_k(u,t)\,dt = 0 \qquad k = 1,\ldots,m, \tag{3.45}$$

where m is the total number of kinematic constraints. This class of constraints includes holonomic constraints and nonholonomic equality constraints — all constraints, in other words, that can be expressed in Pfaffian form. Thus the vector of possible displacements $du = (du_1,\ldots,du_N)$ satisfies

$$A(u,t)\,du + a(u,t)\,dt = 0, \tag{3.46}$$

where A and a are the matrix coefficients of the Pfaffian form, previously given by (3.17) and (3.29). In geometric terms, this definition states that "every vector of infinitesimal length which lies in the tangent plane defined by the constraint equations is a possible displacement vector." Possible displacements can be thought of as *true* infinitesimal displacements.

The third class of displacements are the "virtual" displacements which are so important to analytical dynamics. From a given configuration at an instant, each displacement variable u_i may be imagined to vary an infinitesimal amount to a neighboring configuration that the system might have had at that instant. By definition this variation does not violate the constraints and time is not varied. Such variations from the actual configuration are called by Crandall *et al.* [9] *admissible variations*, or in traditional nomenclature *virtual displacements*. Virtual displacements are the class of infinitesimal quantities δu_i satisfying

$$\sum_{i=1}^{N} a_{ki}(u,t)\,\delta u_i = 0 \qquad k = 1,\ldots,m. \tag{3.47}$$

This equation is obtained from the Pfaffian form of the constraint equation by substituting δu for du and with $dt = 0$. The vector of virtual displacements $\delta u = (\delta u_1,\ldots,\delta u_N)$ satisfies

$$A(u,t)\,\delta u = 0. \tag{3.48}$$

In geometric terms this definition states that "every vector of infinitesimal length which lies in the tangent plane defined by (3.47) is a virtual displacement vector."

The variational operator δ introduced here is consistent with the variational operator used in the calculus of variations and has properties similar to those of the differential operator d. The definition of the δ operator from a variational calculus standpoint as well as a summary of its properties are provided in Appendix A. However, the basic variational principle from classical dynamics — that the variation of a certain integral vanishes — is not invoked in this work. Indeed, equations of motion for nonholonomic systems are not derivable from such a principle [22].

To illustrate the different classes of displacements, a bead that is free to slide on a rotating rod as shown in Figure 3.13 is a commonly cited example. In this instance, the example is taken from D'Souza and Garg [11].

EXAMPLE 3.13 A bead is free to slide along a rod which rotates about the origin in the xy–plane with a constant angular velocity ω_o. At time t_o the bead has position (x_o, y_o) and a velocity v given by $(\dot{x}_o, \dot{y}_o)$. In Figure 3.13(a), the state of the system is shown at time t_o.

Finite *actual displacements* $(\Delta x, \Delta y)$ resulting from the actual motion of the bead along trajectory C on the finite time interval $[t_o, t]$ are indicated in (b). Infinitesimal *possible displacements* (dx, dy) associated with the infinitesimal time interval dt are exemplified in (c). Note that possible displacements are consistent with the constraints but do not necessarily follow the trajectory C since possible displacements do not necessarily satisfy the equations of motion. Infinitesimal *virtual displacements* which are imagined to occur without a corresponding variation in time, that is, $\delta t = 0$, are indicated in (d). The displacement of the bead in (d) lies along the line of the rod since virtual displacements may not violate the constraints at the instant the variation is considered. $\diamond$

The figure illustrates that virtual displacements and possible displacements are not

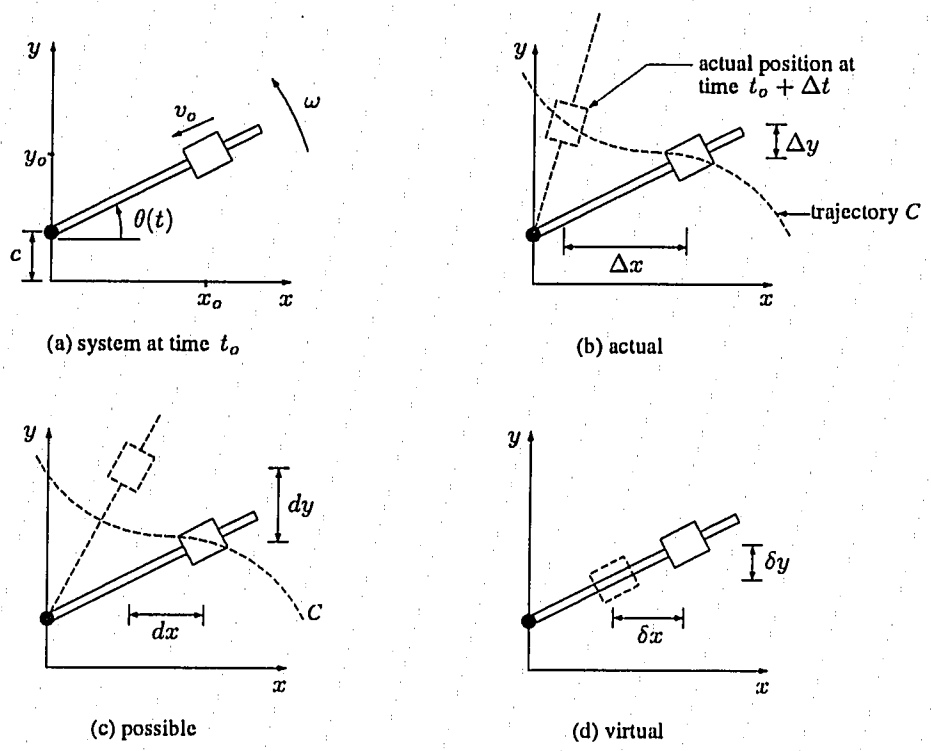


Figure 3.13. Classification of displacements: (a) state of the system at time t_o; (b) actual displacement; (c) possible displacement; and (d) virtual displacement.

identical quantities. This follows from the definitions, since a single set of infinitesimal quantities is unlikely to satisfy both (3.45) and (3.47).

EXAMPLE 3.14 Consider again the bead and rod from the previous example. The system is constrained such that the coordinates (x, y) of the bead and the angle θ of the rod must satisfy

$$\tan\theta = \frac{y-c}{x}. \tag{3.49}$$

Since $\theta = \omega_o t$, the constraint can be written as

$$\phi(x, y, t) := x\tan\omega_o t - y + c = 0, \tag{3.50}$$

which is a holonomic constraint in standard form. The Pfaffian form is given by

$$(\tan\omega_o t)\,dx - dy + (x\omega_o \sec^2\omega_o t)\,dt = 0. \tag{3.51}$$

Displacements (dx, dy) satisfying this equation are possible displacements. Virtual displacements $(\delta x, \delta y)$, on the other hand, satisfy

$$(\tan\omega_o t)\,\delta x - \delta y = 0. \tag{3.52}$$

The only condition for which the two sets of displacements are identical is $\omega_o = 0$. In that case, θ is constant and (dx, dy) satisfies

$$(\tan\theta)\,dx - dy = 0, \tag{3.53}$$

and $(\delta x, \delta y)$ satisfies

$$(\tan\theta)\,\delta x - \delta y = 0. \quad \diamond \tag{3.54}$$

This example illustrates that virtual and possible displacements are identical only for a system having constraints that are not explicit functions of time. Such systems are called *catastatic*. Systems with constraints that are explicit functions of time are called *acatastatic*. For acatastatic systems the set of possible displacements and the set of virtual displacements have no elements in common except the null set.

In summary, a possible displacement du implies a small change in the variable $u(t)$ that depends on a small change in time dt. A virtual displacement δu, on the other hand, is a deliberately introduced variation in the variable u independent of time. Both du and δu satisfy the constraint equations.

3.4 Virtual work

Work is a path–dependent function given in differential form by $đW = e\,dq$, where dq is an infinitesimal increment of the path. In terms of configuration coordinates, the work of a system is given by $đW = \sum e_i\,du_i$. If the infinitesimal displacements are virtual displacements δu_i rather than possible displacements du_i, then work is given by

$$\delta W := \sum_{i=1}^{N} e_i\,\delta u_i, \tag{3.55}$$

which is defined as *virtual work*. In general, virtual work is not the variation of a work function W, that is, the δ symbol in this expression does not represent the variational operator. The term δW simply represents the summation shown in (3.55). The summation over all the elements in a multidiscipline system is possible because work, like energy, is a quantity with a consistent meaning among the engineering disciplines, even though the physical manifestations of work vary.

It is shown in (2.42) that substituting $\dot{p} = e$ and $dq = f\,dt$ from Paynter's diagram into the definition of $đW$ yields an alternate work expression given by $đW = f\,dp$. In component form, this expression of work is given by $đW = \sum f_i\,d\varrho_i$. If the infinitesimal momenta are variational quantities $\delta\varrho_i$ rather than differential quantities $d\varrho_i$, then virtual work is given by

$$\delta W = \sum_{i=1}^{N} f_i\,\delta\varrho_i. \tag{3.56}$$

The infinitesimal quantity $\delta\varrho$ is called *virtual momentum*, defined as a variation of momentum consistent with constraints and independent of time. Virtual momentum is an admissible variation in the same sense as is a virtual displacement. Moreover, the two

quantities are not independent. From the equivalence of the differential work expressions $đW = e\,dq = f\,dp$ is inferred the equivalence of the virtual work expressions

$$\sum_{i=1}^{N} e_i\,\delta u_i = \sum_{i=1}^{N} f_i\,\delta \varrho_i. \tag{3.57}$$

Virtual momentum $\delta\varrho$ is defined as the infinitesimal quantity that satisfies this relationship with virtual displacement δu.[7]

The equivalent forms of virtual work among the engineering disciplines are shown in Table 3.1.

Table 3.1. Virtual work equivalents.

Effort e	Displ. δu	Work $e\,\delta u$	Flow f	Momentum $\delta\varrho$	Work $f\,\delta\varrho$
force F	position δx	$F\,\delta x$	velocity v	lin. mom'n. δp	$v\,\delta p$
torque τ	angle $\delta\theta$	$\tau\,\delta\theta$	ang. velocity ω	ang. mom'n. δH	$\omega\,\delta H$
voltage v	charge δq	$v\,\delta q$	current i	flux linkage $\delta\lambda$	$i\,\delta\lambda$
press. P	volume δV	$P\,\delta V$	vol. rate Q	press. mom'n. $\delta\Gamma$	$Q\,\delta\Gamma$
temp. T	entropy δS	$T\delta S$	entropy rate $\dot{S}$	(none)	—

3.5 Lagrange's principle

Constraints give rise to efforts that alter the motion of a system. In effect, constraint efforts force the configuration of a system into compliance with the constraints. Lagrange used this concept to formulate what is now called Lagrange's principle, as follows.

D'Alembert's principle in mechanics states that the totality of constraint forces in a system of particles does not contribute to the motion of the system [34]. Thus the constraint efforts comprise a set of forces in equilibrium.

Johann Bernoulli's virtual work principle states that static equilibrium may be characterized through requiring that the work done by forces in equilibrium, during a small displacement from equilibrium, should vanish [41]. Lagrange inferred from d'Alembert's

[7] An interpretation of virtual momentum is given in Appendix C.

principle and Bernoulli's principle of virtual work that mechanical constraint forces in their totality, being a system of forces in equilibrium, do no virtual work. This principle is called Lagrange's principle.

Being a condition on work, Lagrange's principle is readily applied to a multidiscipline system. The new principle may be stated:

PRINCIPLE 1 (Lagrange's principle) *At each instant in the motion of a multidiscipline system, the virtual work of the constraint efforts in their totality vanishes.*

Let all efforts acting on the elements of a multidiscipline system be represented in component form by the effort vector $e = (e_1, \ldots, e_N)$ and the vector of virtual displacements be given by $\delta u = (\delta u_1, \ldots, \delta u_N)$. Then the virtual work of the system is given by $\delta W = \langle e, \delta u \rangle$. By Principle 1, all efforts e^ϕ satisfying

$$\delta W^{(\phi)} := \langle e^\phi, \delta u \rangle = 0 \tag{3.58}$$

are called *constraint efforts.* All other efforts are called *given efforts* e^g. The effort vector e can therefore be expressed as the sum of the given efforts and constraint efforts, that is,

$$\text{all efforts } e := e^\phi + e^g. \tag{3.59}$$

EXAMPLE 3.15 Consider the efforts acting on a mass sliding down an inclined surface as shown in Figure 3.14. Mass m moves with velocity $\dot{x}$ down an incline at angle θ subject to the given force $F(t)$, the force due to gravity mg, a normal constraint force N and a friction force μN.

A virtual displacement δx must comply with the constraint, thus δx is collinear with x. The normal force and a virtual displacement are therefore orthogonal and

$$\langle N, \delta x \rangle = 0. \tag{3.60}$$

Thus the virtual work of the constraint force N vanishes. $\diamond$

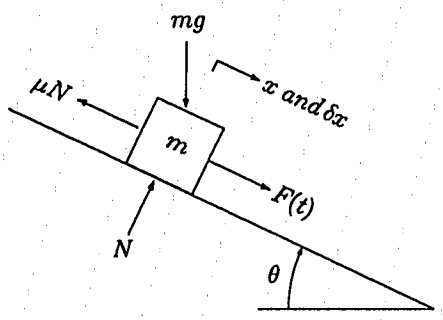


Figure 3.14. A mechanical illustration of Lagrange's principle.

This example has other notable features. From Newton's second law, the equation of motion is given by

$$m\ddot{x} = F + mg\sin\theta - \mu N. \tag{3.61}$$

The constraint force N does not itself contribute to the motion of the mass, that is, there is no component of motion collinear with N. The friction force μN does appear, however, illustrating an important distinction between constraint efforts and given efforts. Constraint efforts satisfy (3.58) but efforts that are functions of constraints, such as the force due to dry friction, are classified among the given efforts.

Lagrange's principle states that the virtual work of the constraint efforts in their totality vanishes. This implies that the virtual work of individual constraint efforts need not vanish. Such a condition is illustrated by the following example.

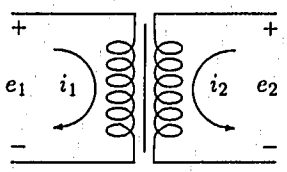


Figure 3.15. An electrical illustration of Lagrange's principle.

EXAMPLE 3.16 The electrical transformer shown in Fig. 3.15 is represented in idealized form as a power–conserving, two–port device. The power variables associated with the

two ports are the two voltage–current pairs (e_1, i_1) and (e_2, i_2). The conservation of power requires that

$$e_1 i_1 + e_2 i_2 = 0. \tag{3.62}$$

The transformer imposes two constraints on the system to which it is connected. The first is a kinematic constraint given by

$$\beta i_1 + i_2 = 0, \tag{3.63}$$

where β is the turn ratio N_1/N_2. This is a holonomic flow constraint that can be integrated to obtain

$$\beta q_1 + q_2 = 0, \tag{3.64}$$

assuming zero initial conditions for q. The second constraint is a dynamic constraint of the form

$$e_1 - \beta e_2 = 0. \tag{3.65}$$

Consider the virtual work of the voltages (e_1, e_2), given by

$$\delta W = e_1\, \delta q_1 + e_2\, \delta q_2. \tag{3.66}$$

Solving (3.65) for e_1 and substituting into the virtual work equation yields

$$\delta W = e_2(\beta\, \delta q_1 + \delta q_2). \tag{3.67}$$

By definition, the virtual displacements satisfy the kinematic constraint, that is,

$$\beta \delta q_1 + \delta q_2 = 0. \tag{3.68}$$

Thus the parenthetical term in (3.67) is zero and the virtual work δW vanishes. It follows that

$$\delta W = e_1\, \delta q_1 + e_2\, \delta q_2 = 0, \tag{3.69}$$

and the voltages (e_1, e_2) are constraint efforts e^ϕ since, in their totality, they do no virtual work. $\diamond$

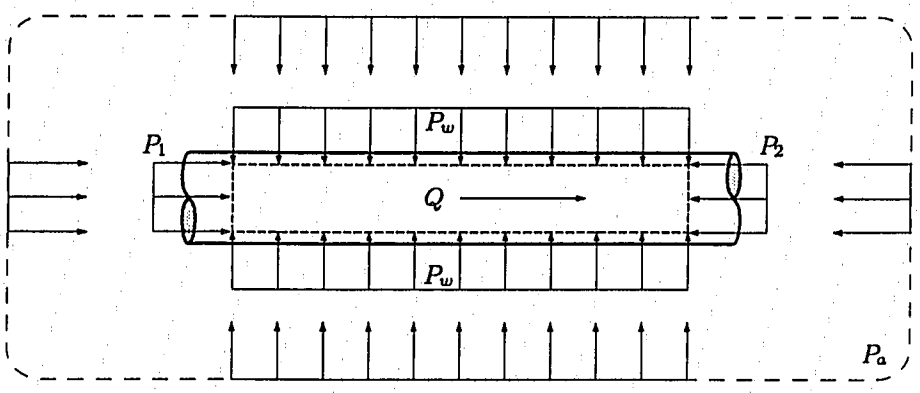


Figure 3.16. A fluid illustration of Lagrange's principle.

EXAMPLE 3.17 A quantity of incompressible fluid flows at rate Q in a pipe, subject to the pressure of the pipe walls P_w, inlet pressure P_1, outlet pressure P_2 and atmospheric pressure P_a as indicated in Figure 3.16. Flow is assumed to be irrotational and nonradial.

The wall pressure P_w constrains the motion of the fluid. The inlet and outlet pressures affect the dynamic characteristics of the fluid. And the atmospheric pressure acts equally at all points. In the event that atmospheric pressure must be accounted for in a problem, only that component of P_a acting in the line of motion need be considered. All other components of P_a are normal to the line of motion. Letting P_N represent the totality of the normal components of P_a plus wall pressure P_w, and letting δV represent an admissible variation of volumetric displacement, the virtual work of P_N is given by

$$\delta W = \langle P_N, \delta V \rangle = 0, \tag{3.70}$$

since P_N and δV are everywhere orthogonal. Hence the wall pressure and normal components of atmospheric pressure are constraint efforts. $\diamond$

The basic classification of efforts in analytical dynamics as either constraint efforts or given efforts has the result that constraint efforts can be neglected in the analysis of a system since the virtual work of the constraint efforts vanishes. While it may seem inappropriate to disregard constraint efforts, which are clearly important to the dynamic

behavior of a system, the reader may find the idea more agreeable after considering an analogous situation in the mechanics of Newton.

A distinguishing characteristic of Newtonian mechanics is the basic classification of forces as either internal forces or external forces. Newton's third law of motion is the postulate that allows the class of internal forces to be neglected in the analysis of dynamic systems.

A similar classification exists in analytical dynamics. A distinguishing characteristic of analytical dynamics is the basic classification of efforts as either constraint efforts or given efforts. The generalized form of Lagrange's principle is the postulate that allows the class of constraint efforts to be neglected in the analysis of multidiscipline systems.

3.6 Classification of efforts

For convenience, the basic classification of efforts as constraint efforts and given efforts is restated. All efforts e^{ϕ} satisfying

$$\delta W^{(\phi)} := \langle e^{\phi}, \delta u \rangle = 0, \tag{3.71}$$

are called constraint efforts. All other efforts are called given efforts e^{g}. Thus the effort vector e can be expressed as the sum given by

$$\text{all efforts } e := e^{\phi} + e^{g}. \tag{3.72}$$

Given efforts are subclassified as either potential efforts e^{p} or nonpotential efforts e^{n} such that $e^{g} = e^{p} + e^{n}$. Potential efforts satisfy the definition of potential energy V, that is,

$$e^{p} = -\frac{\partial V}{\partial u}. \tag{3.73}$$

All given efforts that are not potential efforts are called nonpotential efforts. Thus the effort vector e can be expressed as the sum given by

$$e := e^{\phi} + e^{p} + e^{n}. \tag{3.74}$$

Nonpotential efforts are subclassified as source efforts e^s, dissipation efforts e^d and implicit efforts e^γ such that $e^n = e^s + e^d + e^\gamma$. Source efforts arise from source elements and are assumed to be known functions of time. Dissipation efforts satisfy the definition of the dissipation function D, that is,

$$e^d = -\frac{\partial D}{\partial \dot{u}}. \tag{3.75}$$

All nonpotential efforts that are neither source efforts nor dissipation efforts are called implicit efforts e^γ. Thus the effort vector e can be expressed as the sum given by

$$e := e^\phi + e^p + e^s + e^d + e^\gamma. \tag{3.76}$$

The appearance of implicit efforts in the equations of motion implies the presence of dynamic constraints.

Comment

The classification of efforts presented above, in conjunction with the concept of virtual work, permits a final characterization by which dynamic constraints are distinguished from kinematic constraints, as follows.

The efforts associated with *kinematic* constraints are constraint efforts e^ϕ that in their totality do no virtual work, that is,

$$\delta W^{(\phi)} = \langle e^\phi, \delta u \rangle = 0. \tag{3.77}$$

The efforts associated with *dynamic* constraints are implicit efforts e^γ that do nonzero virtual work, that is,

$$\delta W^{(\gamma)} = \langle e^\gamma, \delta u \rangle \neq 0. \tag{3.78}$$

3.7 The geometry of constraints

In this section is presented a geometric interpretation of the constraint concepts introduced in this chapter.

3.7.1 Geometric interpretation of holonomic constraints

An m–vector of kinematic constraints is given by $\Phi(u,t) = 0$, where $u = (u_1, \ldots, u_N)$. This function describes a surface in N–dimensional configuration space. From calculus, all vectors τ tangent to this surface satisfy $\Phi_u \tau = 0$, where Φ_u is the Jacobian matrix $\partial\Phi/\partial u$, given by (3.15).

By definition, virtual displacements satisfy (3.48), which for a holonomic system is given by

$$\Phi_u(u,t)\,\delta u = 0. \tag{3.79}$$

Thus $\delta u \in \tau$, and the virtual displacement vector δu can be interpreted geometrically as a variation of configuration that is instantaneously tangent to the constraint surface $\Phi(u,t)$ for all $u(t)$.

By Lagrange's principle, constraint efforts satisfy $\langle e^\phi, \delta u\rangle = 0$. Thus e^ϕ and δu are orthogonal, and since δu is tangent to the constraint surface, e^ϕ must be normal to the constraint surface. Hence the constraint effort vector e^ϕ can be interpreted geometrically as an effort that is instantaneously normal to the constraint surface $\Phi(u,t)$ for all $u(t)$.

Lastly, the k^{th} column of Φ_u^T is identical to the gradient of the k^{th} constraint, that is,

$$\nabla\phi_k = \left[\begin{array}{ccc} \dfrac{\partial\phi_k}{\partial u_1} & \cdots & \dfrac{\partial\phi_k}{\partial u_N} \end{array}\right]^T. \tag{3.80}$$

These gradients $\nabla\phi_k$ are normal to the constraint surface, and if the gradients are independent, they form a basis for the normal space. Since constraint efforts lie in the normal space, they can be expressed as a linear combination of these basis vectors. This linear combination can be written as

$$e^\phi = -\left(\lambda_1\nabla\phi_1 + \cdots + \lambda_m\nabla\phi_m\right), \tag{3.81}$$

or in vector form,

$$e^\phi = -\Phi_u^T\lambda, \tag{3.82}$$

where $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a vector of undetermined coefficients, or *Lagrange multipliers.*[8]

The geometric interpretation of virtual displacements and constraint efforts is illustrated in Fig. 3.17. A two-dimensional configuration space is shown subject to a single constraint $\phi(u_1, u_2, t) = 0$. The configuration of the system at time $\hat{t}$ is represented by the point $u(\hat{t})$. The line tangent to ϕ at $u(\hat{t})$ defines the line of action of δu. This is the line of admissible variations. The line normal to ϕ at $u(\hat{t})$ defines the line of action of e^{ϕ}, where e^{ϕ} is a scalar multiple of the gradient $\nabla\phi$. The undetermined scalar multiplier is called a Lagrange multiplier λ.

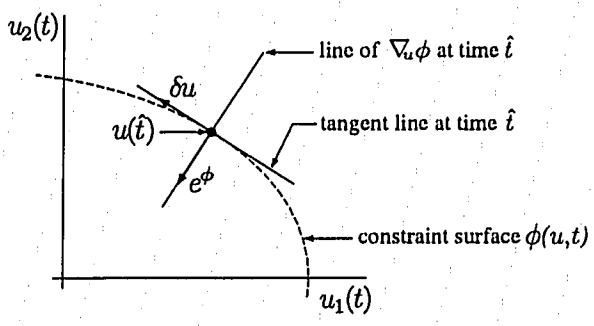


Figure 3.17. The geometry of constraints in a 2-dimensional configuration space.

EXAMPLE 3.18 Consider again a mass sliding down an inclined surface. A free-body diagram is given in Fig. 3.18. The geometry of this example illustrates the geometric interpretation outlined in this section.

First, the virtual displacement δx is tangential to the literal constraint surface defined by the inclined plane. Second, the constraint force N is normal to the constraint surface and satisfies Lagrange's principle, that is, $\langle N, \delta x \rangle = 0$. Third, the constraint force at x is collinear with the gradient of the constraint surface at x. And last, the forces normal to the constraint surface form a set of forces in equilibrium, that is, $N - mg\cos\theta = 0$.

Components of effort that lie in the plane normal to the constraint surface do not

[8]Multipliers are presented in more detail in Chapter 5.

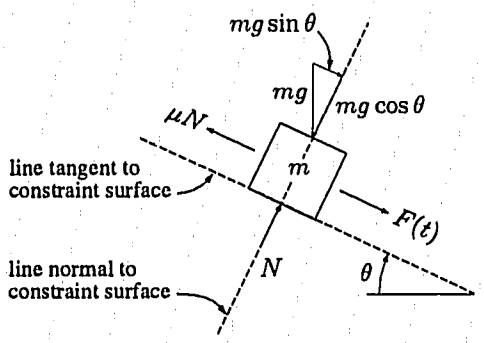


Figure 3.18. Free–body diagram of a mass on a plane.

contribute to the motion of the system. Conversely, only those components of effort that lie in the plane tangential to the constraint surface do contribute to the motion of the system. For example, the component of the force due to gravity ($mg\cos\theta$) that is normal to the constraint surface does not contribute to the motion of the mass, and the component of the force due to gravity ($mg\sin\theta$) that is tangent to the constraint surface does contribute to the motion of the mass. ◇

3.7.2 Geometric interpretation of nonholonomic constraints

It has been shown that holonomic constraints define at an instant a constraint surface $\Phi(u,t) = 0$, and that the virtual displacement vector δu satisfies $\Phi_u(u,t)\delta u = 0$ and therefore lies in a space that is tangent to the constraint surface.

Nonholonomic constraints impose additional requirements on the virtual displacement vector within the tangent space. These additional restrictions on δu are given by $A(u,t)\delta u = 0$. The geometric interpretation of this relationship is that each nonholonomic constraint reduces by one the number of possible directions available to a system at an instant. Thus while holonomic constraints make certain portions of the configuration space inaccessible to the system for all possible trajectories, nonholonomic constraints limit the possible directions of trajectory from any given point. Holonomic constraints define the tangent space in which the virtual displacements lie, and nonholonomic constraints reduce the dimension of that tangent space.

3.7.3 Geometric interpretation of dynamic constraints

In adding implicit efforts to the solution space of a problem, the number of unknowns is increased by the number of implicit efforts. The additional equations required to determine a solution are the dynamic constraint equations. The geometric interpretation of this procedure is that adding implicit efforts to a model increases the dimension of the solution space, and that the dynamic constraints reduce the dimension of the solution space to the same degree by defining constraint surfaces in the enlarged solution space on which the solution must lie.

3.8 Summary

Kinematic constraints

1. Kinematic constraints are associated with constraint efforts e^{ϕ} that do no virtual work, that is,

$$\delta W^{(\phi)} = \langle e^{\phi}, \delta u \rangle = 0. \tag{3.83}$$

2. Kinematic constraints arise from system geometry and from constituitive laws involving displacement and flow that are not subsumed under the energy functions.

3. Kinematic constraints may be subclassified as either holonomic or nonholonomic. Nonholonomic constraints may be further subclassified as either equality or inequality constraints.

4. Holonomic constraints reduce the dimension of the configuration space. Nonholonomic constraints reduce the size of the space tangent to the configuration space.

Dynamic constraints

1. Dynamic constraints are associated with the implicit efforts e^{γ} that do nonzero virtual work, that is,

$$\delta W^{(\gamma)} = \langle e^{\gamma}, \delta u \rangle \neq 0. \tag{3.84}$$

2. Dynamic constraints arise from constituitive laws involving momentum and effort that are not subsumed under the energy functions.

3. Each implicit effort adds a dimension to the solution space, and each dynamic constraint reduces the dimension of the solution space to the same degree.

Differentiability

The requirement that kinematic constraints be expressible in Pfaffian form is tantamount to requiring that constraints be continuously differentiable on the time interval of interest, that is, $\phi, \psi \in C_n^1 [t_o, t_f]$.

Chapter 4

SOURCES, TRANSFORMERS AND TRANSDUCERS

Having established the fundamentals of analytical system dynamics in previous chapters, the contributions of transformers, transducers and sources to the dynamic behavior of a system are readily classified.

4.1 Sources

Power or energy sources can be thought of as system elements that impose certain boundary conditions on a system [30]. In most cases, sources are assumed to be known functions of time, although in some problems these boundary conditions are not known *a priori*. The optimal control problem is typical of this class of problems, where control inputs to a system can be thought of as sources of effort and flow that are to be determined. For the purposes of this research, however, sources are assumed to be known.

In accordance with the power principle, sources are generally of two types, effort sources and flow sources. As shown in Chapter 3, effort sources e^s are classified among the nonpotential efforts of a system. Hence, when formulating the equations of motion, the known effort-source functions are included among the nonpotential effort terms. Common examples of such functions are the sinusoidal voltage at a wall outlet, the force due to gravity, or the pressure output of a constant pressure pump.

The following example illustrates the manner in which effort sources appear in the equations of motion. The method by which these equations are obtained is the subject of Chapter 5.

EXAMPLE 4.1 For the electrical circuit shown in Fig. 4.1 displacement variables q_1 and q_3 can be selected as independent variables. It can be shown that the equations of motion

for this circuit are given by

$$\begin{aligned} L_1(\ddot{q}_1 - \ddot{q}_3) + \frac{q_1}{C_1} &= e^s(t) \\ -L_1(\ddot{q}_1 - \ddot{q}_3) + L_2\ddot{q}_3 + \frac{q_3}{C_2} &= 0. \end{aligned} \tag{4.1}$$

This is a set of two differential equations with unknowns (q_1, q_3). The voltage source $e^s(t)$ is a nonpotential effort acting in the direction of the q_1 displacement. $\diamond$

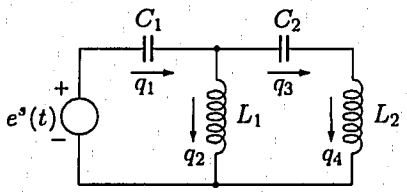


Figure 4.1. An effort source as a nonpotential effort.

Flow sources, on the other hand, impose conditions on flow variables. By definition such conditions impose kinematic constraints on a system. If the source function is integrable, it can be written as a constraint on displacement and is therefore holonomic. If the source function is not integrable, it is written as a nonholonomic constraint. In either case, the flow or displacement source is included among the kinematic constraints of the system model. Common examples of flow sources include an electrical current source, a constant flow pump, and a mechanical shaker with a prescribed velocity profile.

The following example illustrates the manner in which flow sources appear in the equations of motion

EXAMPLE 4.2 A simple fluid system is shown in Fig. 4.2 consisting of a flow source $Q^s(t)$, a storage tank with capacitance C_f and an outlet pipe with resistance R_f. The equations of motion for this system are given by

$$\frac{u_1}{C_f} + \lambda = 0$$

$$
\begin{aligned}
R_f \dot{u}_2 + \lambda &= 0 \\
\dot{u}_1 + \dot{u}_2 - Q^s(t) &= 0,
\end{aligned}
\tag{4.2}
$$

where (u_1, u_2) represents the configuration of the system and λ is a Lagrange multiplier. This is a set of three equations (two differential and one algebraic) with unknowns (u_1, u_2, λ). The flow source is incorporated in the algebraic flow constraint equation. $\diamond$

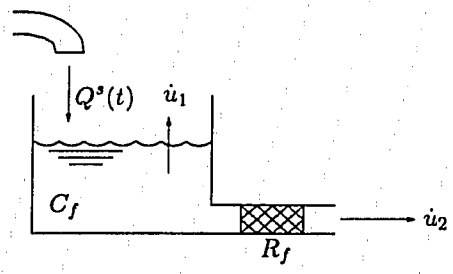


Figure 4.2. A flow source as a constraint.

4.2 Introduction to transformers and transducers

Transformers and transducers are system elements that couple two dynamic subsystems. *Transformers* couple subsystems of the same energy domain. *Transducers* couple subsystems of different energy domains. An overview of the types of transformation and transduction among the energy domains is shown in Fig. 4.3.

The efficiency of power coupling is often important. For example, motors and transmissions are designed to transmit power efficiently. In such devices, sometimes called *power* transformers and transducers, power coupling is assumed to be perfectly efficient. Sensors, on the other hand, are designed to extract as little power as possible from the system they measure. These devices are often called *signal* transducers. In either case, energy storage or dissipation in a real device is modeled externally to the ideal transforming or transducing properties [21].

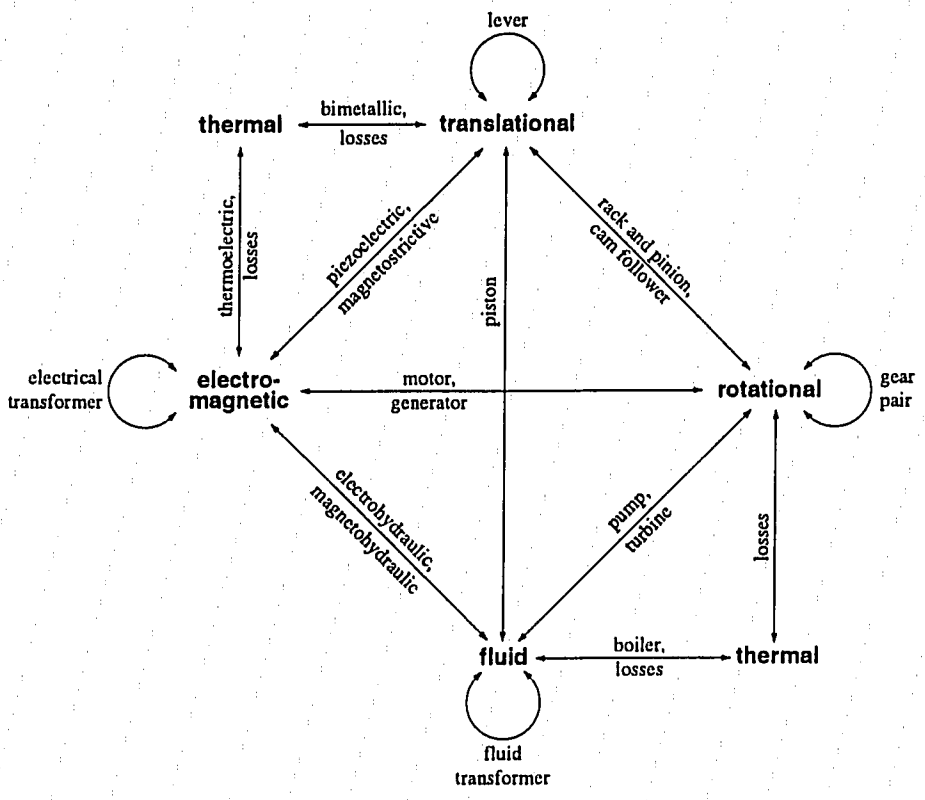


Figure 4.3. Energy transformation and transduction. Energy domains are shown at the vertices of the figure. For clarity, thermal energy is shown twice. Transformers are represented by semi-circles and transducers are represented by straight lines. (Adapted from [30] and [37].)

The coupling characteristic of these devices requires that their constituitive laws relate the system variables (e_1, f_1, p_1, q_1) at one port of the device to the system variables (e_2, f_2, p_2, q_2) at the other port of the device. These constituitive laws are classified according to the manner in which the variables at the two ports are related.

The first type of constituitive law is comprised of a pair of functions. The first function represents a kinematic constraints given by $\phi(q_1, q_2) = 0$ or $\psi(f_1, f_2) = 0$. The second function represents a dynamic constraint given by $\gamma(e_1, e_2) = 0$. The second type of constituitive law is comprised of a function or a pair of functions that represents a dynamic constraint having the form $\gamma(q_1, e_2) = 0$ or $\gamma(f_1, e_2) = 0$.

In all cases, these constituitive relationships are algebraic, not differential. In this sense, pure transformers and transducers are static devices. Furthermore, for those devices for which perfect efficiency is assumed, the net instantaneous power at the two ports sums to zero, that is

$$P_1 + P_2 = e_1 f_1 + e_2 f_2 = 0. \tag{4.3}$$

Presented in the following sections are descriptions of common transformers and transducers. Constituitive laws are presented, and the constraints associated with the devices are described.

4.3 Transformers

The lever, shown in Fig. 4.4, is represented in idealized form as a power transformer. The power variables associated with the two ports of this device are the two force–velocity pairs $(F_1, \dot{x}_1)$ and $(F_2, \dot{x}_2)$. The lever is rigid and massless, and the pivot is frictionless.

For small displacements, this device imposes the relationship $x_2/l_2 = -x_1/l_1$ on the subsystems to which it is connected. This relationship is a constituitive law of the first type, $\phi(q_1, q_2) = 0$, and imposes on a system the following kinematic constraint:

$$\phi(x) := \frac{l_2}{l_1} x_1 + x_2 = 0. \tag{4.4}$$

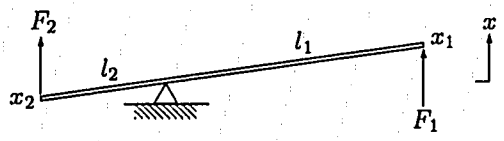


Figure 4.4. A lever represented as a transformer for small displacements x.

The ratio l_2/l_1 is called the transformation ratio or *modulus* β of the transformer. Thus, the constraint is given by

$$\phi(x) := \beta x_1 + x_2 = 0. \tag{4.5}$$

The conservation of power requires that

$$F_1\dot{x}_1 + F_2\dot{x}_2 = 0. \tag{4.6}$$

Assuming a constant modulus β, the time derivative of the kinematic constraint equation (4.4) yields $\dot{x}_2 = -\beta\dot{x}_1$, which is substituted into the power equation to obtain

$$F_1 - \beta F_2 = 0. \tag{4.7}$$

This equation has the form $\gamma(e_1, e_2) = 0$. The time derivative of the kinematic constraint, $\beta\dot{x}_1 + \dot{x}_2 = 0$, has the form $\psi(f_1, f_2) = 0$. Together, this pair of equations is sometimes called the *ideal transformer law*, which is given in generalized form by

$$\begin{aligned} \psi(f) &:= \beta f_1 + f_2 = 0 \\ \gamma(e) &:= e_1 - \beta e_2 = 0. \end{aligned} \tag{4.8}$$

In the context of analytical dynamics, $\gamma(e)$ can be neglected as follows. The virtual work of the forces F_1 and F_2 over small displacements is given by

$$\delta W = F_1\delta x_1 + F_2\delta x_2. \tag{4.9}$$

Substituting for F_1 from (4.7) yields

$$\begin{aligned} \delta W &= \beta F_2\delta x_1 + F_2\delta x_2 \\ &= F_2\left(\beta\delta x_1 + \delta x_2\right). \end{aligned} \tag{4.10}$$

By definition, virtual displacements satisfy the constraint, that is, $\beta\delta x_1 + \delta x_2 = 0$. Thus the parenthetical term in (4.10) vanishes and the virtual work δW is zero. Consequently,

$$\delta W = F_1\,\delta x_1 + F_2\,\delta x_2 = 0. \tag{4.11}$$

By definition, efforts satisfying the relationship $\delta W = 0$ are constraint efforts e^ϕ that can be neglected in the analysis of the system. Thus the constituitive law $\gamma(e)$ is also neglected. Only the kinematic constraint equation $\phi(q)$ or $\psi(f)$ is required in the system model.

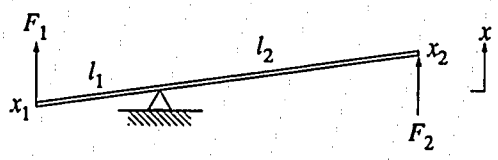
The constituitive law $\gamma(e)$ represents a special class of dynamic constraint. Dynamic constraints in general involve efforts that do virtual work. The constraint $\gamma(e)$ involves efforts that do no virtual work. Hence the dynamic constraint equation $\gamma(e) = 0$ for transformers is a special case of Lagrange's principle.

In this example, the virtual work of the individual constraint efforts does not vanish, that is, $F_1\delta x_1 \neq 0$ and $F_2\delta x_2 \neq 0$, since these forces are collinear with their respective virtual displacements. Nevertheless, the virtual work of the totality of the constraint efforts does vanish, as required by Lagrange's principle.

A gear pair, an electrical transformer and a fluid differential transformer are power transformers also. Ideal models are described by the linear transformer law given in (4.8). Analysis similar to that used in the case of the lever shows that in the case of each transformer, the constituitive function $\psi(f)$ represents a kinematic constraint, and the dynamic constraint $\gamma(e)$ may be neglected because the efforts in $\gamma(e)$ are constraint efforts e^ϕ that do no virtual work. The moduli and constituitive relations for these transformers are summarized in Fig. 4.5.

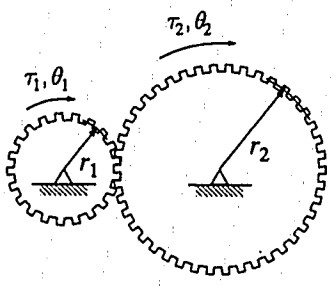
4.4 Transducers

Some power transducers have constituitive laws similar to those of the transformers described above. In these cases, one of the pair of constituitive relations imposes a



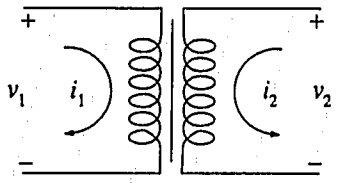
$$\psi(\dot{x}) := \beta\dot{x}_1 + \dot{x}_2 = 0$$
$$\gamma(F) := F_1 - \beta F_2 = 0$$
$$\text{lever ratio } \frac{l_2}{l_1} := \beta$$

(a) lever



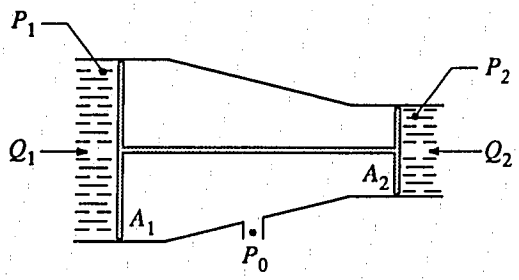
$$\psi(\dot{\theta}) := \beta\dot{\theta}_1 + \dot{\theta}_2 = 0$$
$$\gamma(\tau) := \tau_1 - \beta\tau_2 = 0$$
$$\text{gear ratio } \frac{r_1}{r_2} := \beta$$

(b) gear pair



$$\psi(i) := \beta i_1 + i_2 = 0$$
$$\gamma(v) := v_1 - \beta v_2 = 0$$
$$\text{turn ratio } \frac{N_1}{N_2} := \beta$$

(c) electrical transformer



$$\psi(Q) := \beta Q_1 + Q_2 = 0$$
$$\gamma(P) := (P_1 - P_o) - \beta(P_2 - P_o) = 0$$
$$\text{area ratio } \frac{A_2}{A_1} := \beta$$

(d) fluid differential transformer

Figure 4.5. Ideal power transformers: (a) translational; (b) rotational; (c) electrical; (d) fluid. For each transformer the modulus β and the pair of equations $\psi(f) = 0$ and $\gamma(e) = 0$ which comprise the ideal transformer law are given.

kinematic constraint $\phi(q)$ or $\psi(f)$ on a system and the other relationship is a dynamic constraint $\gamma(e)$ involving workless constraint efforts only. A power transducer of this type is called a *transforming transducer*. A rack and pinion, a pump, and a piston are transducers of this type.

A second type of power transducers impose dynamic constraints $\gamma(e, f)$ that are not workless. A power transducer of this type is called a gyrating transducer or a *gyrator*. The implicit efforts e^γ in these constituitive relations are among the unknown variables for which a solution is sought. Motors and generators are power transducers of this type, as are many signal transducers such as bimetallic sensors and thermocouples. Examples of these types of transducers are presented in this section.

4.4.1 Transforming transducers

A rack and pinion, shown in Fig. 4.6, is representative of the type of transducer having a constituitive law similar to that of a transformer. The power variables associated with the two ports of this device are the two effort–flow pairs $(F, \dot{x})$ and $(\tau, \dot{\theta})$. The device is rigid, massless and frictionless.

The rack and pinion imposes the relationship $x = -r\theta$ on the subsystems to which it is connected. The modulus β of this transducer is the pinion radius r, yielding the kinematic constraint equation:

$$\phi(x, \theta) := \beta\theta + x = 0. \tag{4.12}$$

The conservation of power requires that

$$\tau\dot{\theta} + F\dot{x} = 0. \tag{4.13}$$

For a constant modulus β, the time derivative of the constraint equation yields $\dot{x} = -\beta\dot{\theta}$, which is substituted into the power equation to obtain

$$\tau - \beta F = 0. \tag{4.14}$$

This equation has the form $\gamma(e) = 0$, and with the time derivative of the constraint, $\psi(f) := \beta\dot{\theta} + \dot{x} = 0$, forms a constituitive law similar to the ideal transformer law. The efforts in $\gamma(e)$ are constraint efforts e^{ϕ} which do no virtual work in their totality, that is,

$$\delta W = \tau\,\delta\theta + F\,\delta x = 0. \tag{4.15}$$

Thus only the kinematic constraint equation is required to model the energy transduction of transducers of this type. Energy storage or dissipation in a real device is modeled separately. The moduli and constituitive relations for common ideal power transducers of the transformer type are summarized in Fig. 4.6.

4.4.2 Gyrators

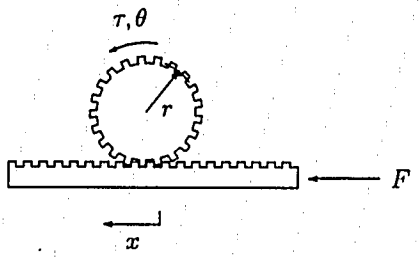
A DC motor, shown in Fig. 4.7, is an example of the type of energy transducer that imposes dynamic constraints on a system that are not workless. In such a case, the efforts associated the device are implicit efforts e^{γ}. The power variables associated with the two ports of this device are the two effort–flow pairs (τ, ω) and (v, i). The constituitive relations of the DC motor are given by

$$\begin{aligned} \tau &= -K_t\, i \\ v &= K_v\, \omega. \end{aligned} \tag{4.16}$$

These equations have the form $\gamma(e_i, f_j)$, that is, they relate effort at one port to flow at the other port. The induced torque τ and the back-emf v are implicit efforts that can be designated e_1^{γ} and e_2^{γ} respectively. The modulus β of this transducer is the motor constant $K_v = K_t$.[1] The dynamic constraints are given in general form by

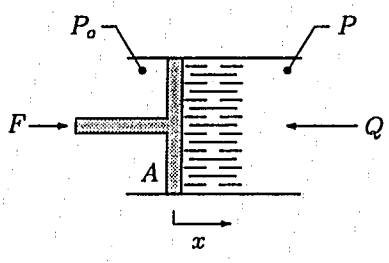
$$\begin{aligned} \gamma_1(e^{\gamma}, f) &:= e_1^{\gamma} + \beta f_2 = 0 \\ \gamma_2(e^{\gamma}, f) &:= e_2^{\gamma} - \beta f_1 = 0. \end{aligned} \tag{4.17}$$

[1]This equality holds for SI units but not for EES (English Engineering System) units.



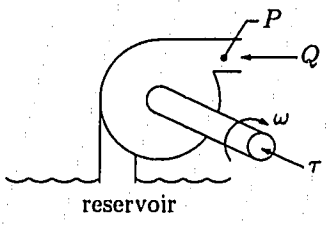
$\psi(f) := \beta\dot{\theta} + \dot{x} = 0$
$\gamma(e) := \tau - \beta F = 0$
pinion radius $r := \beta$

(a) rack and pinion



$\psi(f) := \beta v + Q = 0$
$\gamma(e) := F - \beta(P - P_o) = 0$
piston area $A := \beta$

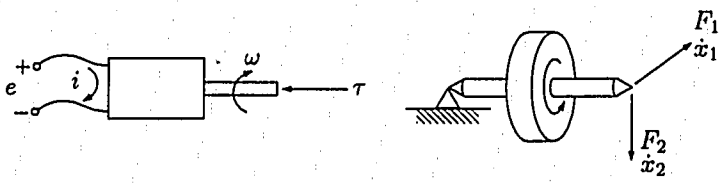
(b) ram–cylinder



$\psi(f) := \beta\omega + Q = 0$
$\gamma(e) := \tau - \beta P = 0$
pump displacement $D := \beta$

(c) positive displacement pump

Figure 4.6. Ideal power transducers of the transforming type. For each transducer the modulus β and the pair of equations $\psi(f) = 0$ and $\gamma(e) = 0$ which comprise the ideal transforming transduction law are given.

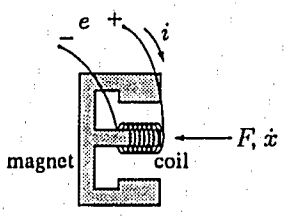


$\gamma_1(e^\gamma, f) := \beta i + \tau = 0$
$\gamma_2(e^\gamma, f) := e - \beta\omega = 0$
motor constant $K_v := \beta$

(a) DC motor

$\gamma_1(e^\gamma, f) := \beta\dot{x}_2 + F_1 = 0$
$\gamma_2(e^\gamma, f) := F_2 - \beta\dot{x}_1 = 0$
gyrator modulus β

(b) mechanical gyrator



$\gamma_1(e^\gamma, f) := \beta\dot{x} + e = 0$
$\gamma_2(e^\gamma, f) := F - \beta i = 0$
current constant $K_a := \beta$

(c) voice coil

Figure 4.7. Ideal gyrators. For each gyrating transducer the modulus β and the pair of dynamic constraints $\gamma_i(e^\gamma, f) = 0$ which comprise the ideal gyrating transduction law are given.

A gyroscope is another common device that obeys a gyrating transduction law. Over a restricted range of speeds and forces a gyroscope is approximately a gyrator and gives the gyrator its name [21]. A gyrating transduction law also describes the simplest model of a voice-coil. The moduli and constituitive relations for these common ideal gyrators are summarized in Fig. 4.7.

4.4.3 Signal transducers

Signal transducers are the final category of transducers presented herein. For signal transducers, power transfer from one energy domain to another is assumed to be inefficient, that is, the transducer draws very little energy from the subsystems to which it

is connected. Unless the dynamic behavior of the transducer itself is of interest, signal transducers are typically represented by a single constituitive relationship.

For example, the behavior of a piezoelectric transducer is usually described by a kinematic relationship between displacement x and charge q, as shown in Fig. 4.8, that is,

$$\phi(q) := q - K_q x = 0. \tag{4.18}$$

This kinematic relationship is sufficient to describe the transduction property of the transducer. However, if charge is related to voltage e by $q = Ce$, where C is capacitance, then the constituitive relation might be given by $Ce - K_q x = 0$ which is in the form of a dynamic constraint $\gamma(e, q) = 0$. Either of these two relationships is sufficient to describe the constraint the element imposes on a system, but the kinematic constraint describes strictly the transduction property while the dynamic constraint describes an energy storage property.

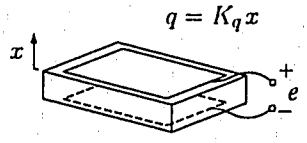


Figure 4.8. A piezoelectric transducer.

In contrast to the constituitive law $\phi(q) = 0$ of the piezoelectric transducer, temperature sensors are signal transducers characterized by relationships among efforts. For example, both a thermocouple and a resistance temperature detector (RTD) generate a potential in response to temperature. Both temperature and voltage are efforts. The relationship between these efforts is a dynamic constraint of the form $\gamma(e_1, e_2) = 0$.

4.5 Transactors

A final category of two–port device is the *transactor*. Such devices include current–controlled voltage sources, current–controlled current sources, and so on. These devices are characterized by constituitive laws similar to those of transformers $\psi : f_1 - \beta f_2 = 0$ or transducers $\gamma : e_1 - \beta f_2 = 0$. However, unlike ideal transformers and transducers, the ideal transactor is not power–conserving. Therefore the transactor effort does virtual work and must be included among the nonpotential effoirts of the system. A detailed development of transactors is given in [38].

Chapter 5

THE LAGRANGIAN DAE OF MOTION

In this chapter the fundamental concepts of analytical system dynamics are used to obtain a multidiscipline form of Lagrange's equation from the first law of thermodynamics. Undetermined multipliers are introduced and a Lagrangian DAE is formulated.

5.1 A variational form of the first law

The first law of thermodynamics is an appropriate starting point from which to derive a multidiscipline form of Lagrange's equation, since both the first law and Lagrange's equation are based on considerations of work and energy.

The first law for a closed system is given in differential form by

$$dE = đW + đQ. \tag{5.1}$$

This law describes an infinitesimal change in a system that takes place over an infinitesimal interval of time. Energy E is a state function while work W and heat Q are both path functions. (See Chapter 2.)

To consider the energy of a system at an instant in time, the first law may be written using the variational operator δ rather than the differential operator d to obtain

$$\delta E = \delta W + \delta Q. \tag{5.2}$$

Applied to the state function E, the δ operator signifies a variation, or a contemporaneous perturbation, in the stored energy of the system. In the case of the path-dependent functions however, δW represents virtual work and δQ represents virtual heat. This distinction between two types of quantities, where δE is a variation and δW and δQ are not variations, is parallel to the distinction made in (5.1) in which dE is an exact differential and $đW$ and $đQ$ are inexact differentials.

Let the class of systems under consideration be restricted to those having thermal elements that can be modeled with temperature T as an effort acting through an entropy displacement δS. Then virtual heat, given by $\delta Q = T\,\delta S$, is a form of virtual work $\delta W := e\,\delta q$ and $\delta Q \in \delta W$. With these considerations, the first law is given by

$$\delta E = \delta W. \tag{5.3}$$

5.2 Derivation of Lagrange's equation

Stored energy is defined herein as the sum of the energies of kinetic and potential stores, that is, $E := T + V$, where both T and V are state functions. The variation of stored energy is given by

$$\delta E = \delta T + \delta V. \tag{5.4}$$

The total virtual work in a system is given in component form by

$$\delta W_{\text{total}} = \sum_{i=1}^{N} e_i\,\delta u_i. \tag{5.5}$$

Efforts are classified as constraint efforts e^ϕ, potential efforts e^p, and nonpotential efforts e^n. It follows that

$$\delta W_{\text{total}} = \sum_{i=1}^{N} (e_i^\phi + e_i^p + e_i^n)\,\delta u_i. \tag{5.6}$$

The work of potential efforts e^p is invested in the potential energy stored in the system, that is, $\sum e^p\,\delta u = -\delta V$. Therefore, the virtual work in the first law is the work only of nonpotential efforts e^n and constraint efforts e^ϕ acting on the system through a virtual displacement, that is,

$$\delta W^{(n\phi)} = \sum_{i=1}^{N} \left(e_i^n + e_i^\phi\right)\,\delta u_i. \tag{5.7}$$

Thus, the first law in differential–variational form is given by

$$\delta T + \delta V = \sum_{i=1}^{N} \left(e_i^n + e_i^\phi\right)\,\delta u_i. \tag{5.8}$$

Applying the δ operator to the energy functions $T(\varrho)$ and $V(u)$ yields

$$\sum_{i=1}^{N} \frac{\partial T}{\partial \varrho_i}\,\delta\varrho_i + \sum_{i=1}^{N} \frac{\partial V}{\partial u_i}\,\delta u_i = \sum_{i=1}^{N} \left(e_i^n + e_i^\phi\right)\,\delta u_i. \tag{5.9}$$

In Lagrange's equation, the variable pair $(u, \dot{u})$ is selected to represent the motion of the system. This choice requires that the term involving $\delta\varrho$ in the first law be replaced with an equivalent expression in terms of δu. To this end, the work of kinetic stores is considered in its two equivalent forms, namely,

$$\sum_{i=1}^{N} f_i\,\delta\varrho_i = \sum_{i=1}^{N} e_i\,\delta u_i. \tag{5.10}$$

Substituting $f_i = \partial T/\partial \varrho_i$ and $e_i = \dot{\varrho}_i$ yields

$$\sum_{i=1}^{N} \frac{\partial T}{\partial \varrho_i}\,\delta\varrho_i = \sum_{i=1}^{N} \dot{\varrho}_i\,\delta u_i. \tag{5.11}$$

Kinetic momenta satisfy $\varrho_i = \partial T^*/\partial \dot{u}_i$. Substituting this expression in (5.11) yields

$$\sum_{i=1}^{N} \frac{\partial T}{\partial \varrho_i}\,\delta\varrho_i = \sum_{i=1}^{N} \frac{d}{dt}\frac{\partial T^*}{\partial \dot{u}_i}\,\delta u_i. \tag{5.12}$$

Substituting this term into the first law (5.9) and collecting terms yields

$$\sum_{i=1}^{N} \left(\frac{d}{dt}\frac{\partial T^*}{\partial \dot{u}_i} + \frac{\partial V}{\partial u_i} - e_i^n\right)\delta u_i = \sum_{i=1}^{N} e_i^\phi\,\delta u_i. \tag{5.13}$$

By Lagrange's principle, the virtual work of the constraint efforts vanishes. Thus the right–hand side of (5.13) is zero, yielding

$$\sum_{i=1}^{N} \left(\frac{d}{dt}\frac{\partial T^*}{\partial \dot{u}_i} + \frac{\partial V}{\partial u_i} - e_i^n\right)\delta u_i = 0. \tag{5.14}$$

If the virtual displacements δu_i are independent, then the parenthetical term in (5.14) must vanish, yielding what Paynter [30] calls the *classical form* of Lagrange's equation, given by

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{u}_i} + \frac{\partial V}{\partial u_i} = e_i^n \qquad i = 1, \ldots, N. \tag{5.15}$$

It is assumed that the nonpotential efforts are known functions of time. This equation is given in vector form by

$$\tfrac{d}{dt}\nabla_{\dot{u}} T^* + \nabla_u V = e^n. \tag{5.16}$$

Comment on virtual displacements

To paraphrase Rosenberg [34], one of the properties of virtual displacements is that they are said to take place in "zero time." The science of dynamics is concerned with motion in time, that is, with the sequence of configurations of system elements relative to a reference frame, *as time progresses.* How is it possible to deduce results, useful in this science, from the introduction of virtual displacements occurring in zero time?

The justification for using virtual displacements in dynamics rests altogether in Lagrange's principle, which is applied to (5.13) to obtain (5.14). The work of the constraint efforts in *possible* displacements does not vanish, that is, $dW^{(\phi)} = \sum e^{\phi}\, du \neq 0$. It is the work of the constraint efforts in *virtual* displacements that vanishes, that is, $\delta W^{(\phi)} = \sum e^{\phi}\, \delta u = 0$. Lagrange's equation is a direct consequence of the application of this principle.

Since Lagrange's equation in virtual work form (5.14) holds for each instant t in which a virtual displacement δu_i is considered, it holds for all time $t \in [t_o, t_f]$. Thus Lagrange's equation (5.15), which is a useful result concerned with motion over an interval in time, is deduced through the use of virtual displacements occurring in zero time.

5.2.1 Kinetic effort

The usual form of Lagrange's equation, given by

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{u}_i} - \frac{\partial T^*}{\partial u_i} + \frac{\partial V}{\partial u_i} = e_i^n \qquad i = 1, \ldots, N, \tag{5.17}$$

differs from (5.15) by the term $\partial T^*/\partial u_i$. For reasons that are presented in this section and in the following section, the term $\partial T^*/\partial u_i$ is classified as a *kinetic effort.* There are two means by which kinetic efforts arise in the equations of motion. First, kinetic efforts have a physical basis in multidiscipline systems in which the energy of a kinetic store T_i^* is also a function of displacement u_j, that is, $T_i^* = T_i^*(\dot{u}_i, u_j)$, where $i \neq j$. An example of this type of kinetic effort is presented in this section.

Second, the kinetic effort term $\partial T^*/\partial q_j$ arises through a coordinate transformation from configuration coordinates u to reduced-order coordinates q. This means by which kinetic efforts arise in the equations of motion is the subject of Section 5.2.2.

EXAMPLE 5.1 The kinetic coenergy T^* of a solenoid is a function of the velocity $\dot{x}$ of the iron core and the current $\dot{q}$ of the coil. Since the coil inductance $L(x)$ depends on the displacement of the core, T^* depends also on x. The coordinates are shown in Fig. 5.1.

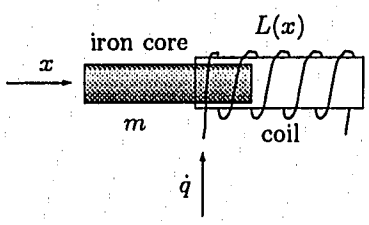


Figure 5.1. A simple model of a solenoid.

It is assumed that the inductance of the solenoid is at a maximum when the iron core is centered and drops off symmetrically as the core is removed from either end [9]. This function is sketched in Fig. 5.2. A possible expression for such a function is the Gaussian distribution function given by

$$L(x) = \frac{A}{\sigma\sqrt{2\pi}} \exp\left[-\tfrac{1}{2}\left(\frac{x - x_c}{\sigma}\right)^2\right], \tag{5.18}$$

where A is an amplitude coefficient, σ is a standard deviation and x_c is the value of x at the magnetic center.

The kinetic coenergy of the solenoid is given by

$$\begin{aligned} T^* &= \int m\dot{x}\, d\dot{x} + \int L(x)\dot{q}\, d\dot{q} \\ &= \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}L(x)\dot{q}^2 \\ &= T^*(\dot{x}, \dot{q}, x), \end{aligned} \tag{5.19}$$

which is a special case of the general form

$$T^* = T^*(\dot{u}, u, t). \tag{5.20}$$

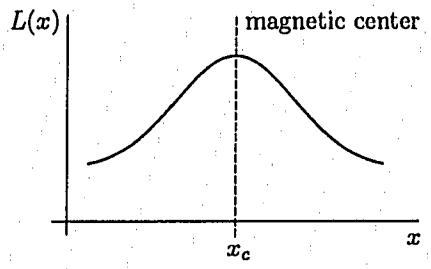


Figure 5.2. Inductance function for a solenoid.

The electromechanical coupling of the solenoid produces a force F that acts on the mass such that the equation of motion for the mass, by Newton's second law, is given by

$$m\ddot{x} = F, \tag{5.21}$$

where F is given by

$$\begin{aligned} F &= \frac{\partial T^*}{\partial x} = \frac{\partial T^*}{\partial L}\frac{\partial L}{\partial x} \\ &= -\frac{A\dot{q}^2\,(x - x_c)}{2\sigma^2\sqrt{2\pi}} \exp\left[-\tfrac{1}{2}\left(\frac{x - x_c}{\sigma}\right)^2\right]. \end{aligned} \tag{5.22}$$

Kinetic coenergy T^* in this example is a function of the displacement coordinate x, and the applied force F is a function of T^* that acts in the x–direction. ◇

This example illustrates two concepts. First, kinetic coenergy T^* can be a function of displacement u. Second, the dependence of T^* on a particular coordinate u_i implies the presence of a nonpotential effort given by $\partial T^*/\partial u_i$ that acts in the direction of u_i.[1] Efforts of this type are called *kinetic efforts* e^{t^*}, and are defined by

$$e_i^{t^*} := \frac{\partial T^*}{\partial u_i}. \tag{5.23}$$

Kinetic efforts are a subset of nonpotential efforts, that is, $e^{t^*} \in e^n$. Thus the classical form of Lagrange's equation can be written

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{u}_i} + \frac{\partial V}{\partial u_i} = e_i^n + e_i^{t^*} \qquad i = 1, \ldots, N, \tag{5.24}$$

[1] In a similar manner, the dependence of the potential energy function V on a particular coordinate u_i implies the presence of a potential effort e_i^p given by $-\partial V/\partial u_i$ that acts in the direction of u_i.

where e^n represents all nonpotential efforts that are not of kinetic origin. Substituting for e^{t^*} and rearranging yields Lagrange's equation in *configuration-space* form, given by

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{u}_i} - \frac{\partial T^*}{\partial u_i} + \frac{\partial V}{\partial u_i} = e_i^n \qquad i = 1, \ldots, N, \tag{5.25}$$

which is suitable for obtaining the equations of motion for systems described in terms of independent configuration coordinates u. This equation is given in vector form by

$$\frac{d}{dt}\nabla_{\dot{u}} T^* - \nabla_u T^* + \nabla_u V = e^n. \tag{5.26}$$

To apply this equation, it is assumed that the nonpotential efforts are known functions of time. The following example from [9] illustrates such a system.

EXAMPLE 5.2 A model of a door chime, shown in Fig. 5.3, consists of a solenoid driven by a voltage source. A striker of mass m is connected to a return spring k.

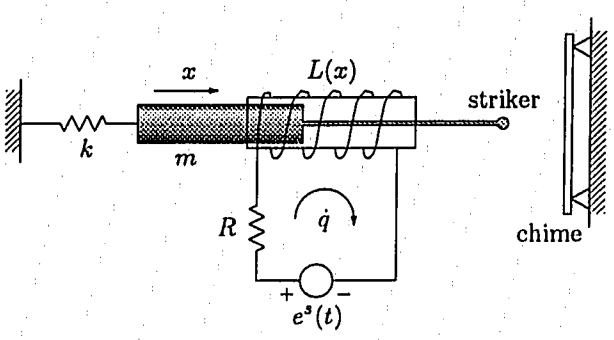


Figure 5.3. A model of a door chime.

The energy functions are given by

$$\begin{aligned} T^* &= \tfrac{1}{2}m\dot{x}^2 + \tfrac{1}{2}L(x)\dot{q}^2 \\ V &= \tfrac{1}{2}kx^2, \end{aligned} \tag{5.27}$$

where $L(x)$ is given by (5.18). Nonpotential efforts e_i^n that are not of kinetic origin act in the electrical domain only, that is,

$$e_x^n = 0 \tag{5.28}$$

$$e_q^n = e^s(t) - R\dot{q}.$$

Lagrange's equation is given by

$$\begin{aligned} \frac{d}{dt}\frac{\partial T^*}{\partial \dot{x}} - \frac{\partial T^*}{\partial x} + \frac{\partial V}{\partial x} &= e_x^n \\ \frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}} - \frac{\partial T^*}{\partial q} + \frac{\partial V}{\partial q} &= e_q^n, \end{aligned} \tag{5.29}$$

yielding the equations of motion

$$\begin{aligned} m\ddot{x} - \tfrac{1}{2}\dot{q}^2 L_x + kx &= 0 \\ L\ddot{q} + L_x\dot{q}\dot{x} &= e^s - R\dot{q}, \end{aligned} \tag{5.30}$$

where

$$L_x := \frac{\partial L}{\partial x} = -\frac{A\,(x - x_c)}{\sigma^2\sqrt{2\pi}} \exp\left[-\tfrac{1}{2}\left(\frac{x - x_c}{\sigma}\right)^2\right]. \quad \diamond \tag{5.31}$$

5.2.2 Reduced–order coordinates

A second means by which the kinetic effort term appears in Lagrange's equation is through the transformation of the set of configuration coordinates u to a set of reduced–order coordinates q. Due to this change in coordinates, the displacement coordinate q and possibly time appear in the energy functions of the kinetic stores, that is, kinetic coenergy is given by $T^*(\dot{q}, q, t)$ and kinetic energy is given by $T(p, q, t)$. As shown in this section, is the use of these forms of kinetic energy and coenergy that give rise to the kinetic effort term in Lagrange's equation.[2]

If the set of configuration coordinates $u = (u_1, \ldots, u_N)$ is transformed via a set of N transformation equations to obtain a reduced–order set of coordinates $q = (q_1, \ldots, q_n)$, then $u_i = u_i(q, t)$ and $\dot{u}_i = \dot{u}_i(\dot{q}, q, t)$. Consequently the kinetic coenergy function has the general form $T^* = T^*(\dot{u}) = T^*(\dot{q}, q, t)$, as illustrated by the following example.

[2]In mechanics, this is the usual means by which kinetic efforts appear in the equations of motion. See, for instance, Meirovitch [25] or Wells [47].

EXAMPLE 5.3 A two-link robot arm is shown in Fig. 5.4. The mass of the robot is lumped in two locations m_1 and m_2 and the links are considered rigid and massless. The robot moves in the xy-plane, and given forces act on both masses as shown. Rotational coordinates (θ_1, θ_2) are measured from the horizontal.

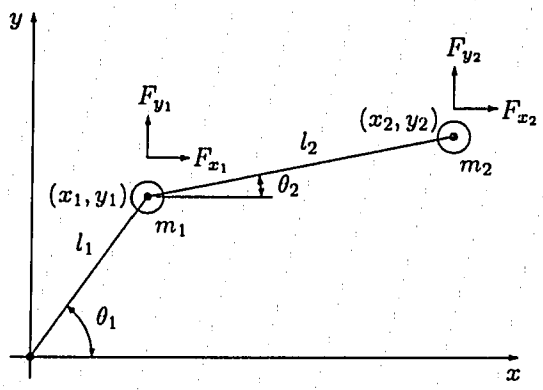


Figure 5.4. A two-link robot arm.

The kinetic coenergy function is given by

$$T^* = \tfrac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2). \tag{5.32}$$

The rotational coordinates (θ_1, θ_2) are a set of independent coordinates. Since these coordinates are of minimum dimension for describing the motion of this system, they are a set of generalized or Lagrange coordinates. Coordinate transformation equations between Cartesian coordinates (x, y) and generalized coordinates (θ_1, θ_2) are given by:

$$\begin{aligned} x_1 &= l_1 \cos\theta_1 \\ y_1 &= l_1 \sin\theta_1 \\ x_2 &= l_1 \cos\theta_1 + l_2 \cos\theta_2 \\ y_2 &= l_1 \sin\theta_1 + l_2 \sin\theta_2. \end{aligned} \tag{5.33}$$

The time derivatives of the transformation equations yield

$$\dot{x}_1 = -l_1\dot{\theta}_1 \sin\theta_1$$

$$\begin{aligned} \dot{y}_1 &= l_1\dot{\theta}_1\cos\theta_1 \\ \dot{x}_2 &= -l_1\dot{\theta}_1\sin\theta_1 - l_2\dot{\theta}_2\sin\theta_2 \\ \dot{y}_2 &= l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2, \end{aligned} \tag{5.34}$$

and kinetic coenergy is given by

$$\begin{aligned} T^* &= \tfrac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \tfrac{1}{2}m_2\left[l_1^2\dot{\theta}_1^2 + l_2^2\dot{\theta}_2^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right] \\ &= T^*(\dot{\theta},\theta), \end{aligned} \tag{5.35}$$

which is a special case of the general form

$$T^* = T^*(\dot{q},q,t), \tag{5.36}$$

which was to be shown. ◇

In this example the kinetic coenergy in configuration–coordinate form (5.32) is a function of flow only, that is, $T^* = T^*(\dot{x},\dot{y})$, hence a kinetic effort $\partial T^*/\partial x$ or $\partial T^*/\partial y$ would not appear in the equations of motion. In terms of reduced–order coordinates however, kinetic coenergy (5.35) is a function of flow and displacement, that is, $T^* = T^*(\dot{\theta},\theta)$. In this case a kinetic effort $\partial T^*/\partial\theta$ does appear in the equations of motion. This is not to say that physical efforts in a system appear or disappear based on the selection of a coordinate system. The actual motion of a physical system is not affected by the choice of variables by which that motion is represented. The representation of that motion, however, does depend on the coordinates selected.

The selection of a particular set of coordinates imposes the restriction on a model that the motion of a system be resolved into components which act in the direction of the selected coordinates. Thus in the two–link robot arm example, selecting the coordinate set $q = (\theta_1,\theta_2)$ to represent the motion of the system requires that all efforts and momenta in the system be resolved into two components: a sum of efforts e_1 that act in the direction of θ_1, and a sum of efforts e_2 that act in the direction of θ_2. Kinetic efforts are an outcome of this resolution of the system motion into these two components. Since

the coordinates in this case are angular displacements, the efforts must be expressions of torque.

EXAMPLE 5.4 The kinetic efforts of the two–link robot arm are given by

$$\begin{aligned} \frac{\partial T^*}{\partial \theta_1} &= -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \\ \frac{\partial T^*}{\partial \theta_2} &= m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) . \end{aligned} \tag{5.37}$$

Dimensional analysis yields

$$\text{units of } \frac{\partial T^*}{\partial \theta_j} = (\text{kg})(\text{m}^2)(\text{sec}^{-2}) = \text{Nm} = \text{torque}, \tag{5.38}$$

which is the effort appropriate to the coordinates selected to represent the motion of this system. ◇

5.2.3 Kinetic energy

The dependence of kinetic coenergy T^* on displacement q implies the dependence of kinetic energy T on displacement q, that is, $T^* = T^*(\dot{q}, q, t)$ implies $T = T(p, q, t)$. This dependence is illustrated by the following example.

EXAMPLE 5.5 For the two–link robot arm, components of momenta $(\varrho_1, \varrho_2, \varrho_3, \varrho_4)$ corresponding to the original configuration coordinates are shown in Fig. 5.5. The reduced–order components of momentum corresponding to the Lagrange coordinates (θ_1, θ_2) are designated (p_1, p_2) and satisfy

$$p_i = \frac{\partial T^*}{\partial \dot{\theta}_i}, \tag{5.39}$$

yielding

$$\begin{aligned} p_1 &= \frac{\partial T^*}{\partial \dot{\theta}_1} = l_1^2 (m_1 + m_2)\dot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_2, \\ p_2 &= \frac{\partial T^*}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1. \end{aligned} \tag{5.40}$$

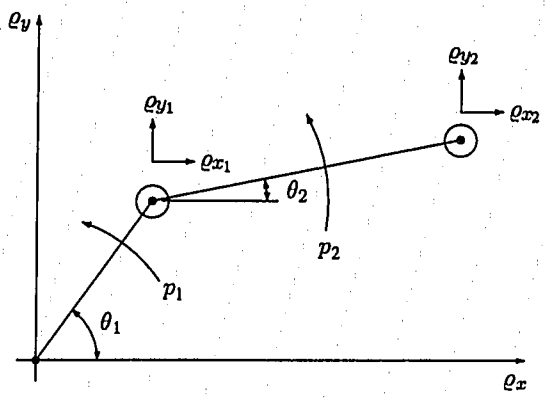


Figure 5.5. Momentum coordinates for a two-link robot arm.

Using Cramer's rule, these two equations can be solved for flows $\dot{\theta}_1$ and $\dot{\theta}_2$, resulting in

$$\begin{aligned} \dot{\theta}_1 &= \frac{m_2 l_2^2}{\Delta} p_1 - \frac{m_2 l_1 l_2 \cos(\theta_1 - \theta_2)}{\Delta} p_2 \\ \dot{\theta}_2 &= -\frac{m_2 l_1 l_2 \cos(\theta_1 - \theta_2)}{\Delta} p_1 + \frac{(m_1 + m_2) l_1^2}{\Delta} p_2 \end{aligned} \tag{5.41}$$

where

$$\Delta := m_2(m_1 + m_2) l_1^2 l_2^2 \sin^2(\theta_1 - \theta_2). \tag{5.42}$$

Both flow expressions have the form

$$\dot{\theta} = \dot{\theta}(p_1, p_2, \theta_1, \theta_2). \tag{5.43}$$

Kinetic energy is given by

$$\begin{aligned} T &= \int \dot{\theta}(p, \theta)\, dp \\ &= T(p, \theta), \end{aligned} \tag{5.44}$$

which is a special case of the general form

$$T = T(p, q, t), \tag{5.45}$$

which was to be shown. ◇

In summary, the transformation of configuration coordinates u to a set of reduced–order coordinates q results in a kinetic coenergy function given by $T^* = T^*(\dot{u}) = T^*(\dot{q}, q, t)$. This representation of kinetic coenergy implies a transformation of momentum coordinates ϱ to a reduced–order set of coordinates p such that the kinetic energy is is given by $T = T(\varrho) = T(p, q, t)$. Given these forms of kinetic energy and coenergy, the derivation of the generalized–coordinate form of Lagrange's equation can be given.

5.2.4 Lagrange's equation in generalized–coordinate form

The total energy stored in a system is given by $E(p, q, t) := T(p, q, t) + V(q, t)$. The variational form of the first law $\delta E = \delta W^{(n)}$ yields

$$\sum_{j=1}^{n} \frac{\partial T}{\partial p_j} \delta p_j + \sum_{j=1}^{n} \frac{\partial T}{\partial q_j} \delta q_j + \sum_{j=1}^{n} \frac{\partial V}{\partial q_j} \delta q_j = \sum_{j=1}^{n} e_j^n \, \delta q_j. \tag{5.47}$$

A term like $\partial E/\partial t$ does not appear since time t is not varied. Constraint efforts e^ϕ do not appear on the right–hand side since they do no virtual work.

As in (5.12), the work equivalence $\sum e \, \delta q = \sum f \, \delta p$ for kinetic stores yields

$$\sum_{j=1}^{n} \frac{\partial T}{\partial p_j} \delta p_j = \sum_{j=1}^{n} \frac{d}{dt} \frac{\partial T^*}{\partial \dot{q}_j} \delta q_j. \tag{5.48}$$

This expression is to be substituted for the first term in (5.47).

Kinetic energy and coenergy satisfy $T(p, q, t) + T^*(f, q, t) = \sum pf$. Applying the variational operator to both sides of this equation yields

$$\sum_{j=1}^{n} \frac{\partial T}{\partial p_j} \delta p_j + \sum_{j=1}^{n} \frac{\partial T}{\partial q_j} \delta q_j + \sum_{j=1}^{n} \frac{\partial T^*}{\partial f_j} \delta f_j + \sum_{j=1}^{n} \frac{\partial T^*}{\partial q_j} \delta q_j = \sum_{j=1}^{n} f_j \, \delta p_j + \sum_{j=1}^{n} p_j \, \delta f_j. \tag{5.49}$$

Collecting terms yields

$$\sum_{j=1}^{n} \left(\frac{\partial T}{\partial p_j} - f_j \right) \delta p_j + \sum_{j=1}^{n} \left(\frac{\partial T^*}{\partial f_j} - p_j \right) \delta f_j + \sum_{j=1}^{n} \left(\frac{\partial T}{\partial q_j} + \frac{\partial T^*}{\partial q_j} \right) \delta q_j = 0. \tag{5.50}$$

By definition, first two parenthetic terms vanish, yielding

$$\sum_{j=1}^{n} \left(\frac{\partial T}{\partial q_j} + \frac{\partial T^*}{\partial q_j} \right) \delta q_j = 0, \tag{5.51}$$

which is an energy equation of the form $\langle e, \delta q \rangle = 0$, implying that the partial derivative terms do indeed represent components of effort acting on a system. These efforts are designated kinetic efforts e^t and e^{t^*} such that

$$\sum_{j=1}^{n} \left(e^t + e^{t^*} \right) \delta q_j = 0. \tag{5.52}$$

It follows that

$$\sum_{j=1}^{n} \frac{\partial T}{\partial q_j} \delta q_j = -\sum_{j=1}^{n} \frac{\partial T^*}{\partial q_j} \delta q_j. \tag{5.53}$$

This expression is to be substituted for the second term in (5.47).

From the definition of nonpotential efforts $e^n = e^s + e^\gamma + e^d$, where dissipative efforts satisfy

$$e_j^d = -\frac{\partial D}{\partial \dot{q}_j}. \tag{5.54}$$

The remaining nonpotential efforts are designated Q such that

$$Q_j := e_j^s + e_j^\gamma. \tag{5.55}$$

In mechanics, the Q_j are usually called generalized forces. Here the Q_j are generalized efforts representing the resolution of all source efforts and implicit efforts into components which act in the direction of the reduced-order coordinates q. A definition of generalized effort is obtained as follows.

The virtual work of the source efforts and implicit efforts in terms of configuration coordinates is given by

$$\delta W^{(s\gamma)} = \sum_{i=1}^{N} (e_i^s + e_i^\gamma) \, \delta u_i. \tag{5.56}$$

The transformation equations relating u to q have the form $u_i = u_i(q_1, \ldots, q_n, t)$, where $i = 1, \ldots, N$. Applying the variational operator to each transformation equation yields the following relationship between virtual displacements δu_i and δq_j

$$\delta u_i = \sum_{j=1}^{n} \frac{\partial u_i}{\partial q_j} \delta q_j. \tag{5.57}$$

Substituting for δu_i in (5.56) yields

$$\begin{aligned}
\delta W^{(s\gamma)} &= \sum_{i=1}^{N} (e_i^s + e_i^\gamma) \sum_{j=1}^{n} \frac{\partial u_i}{\partial q_j}\, \delta q_j \\
&= \sum_{j=1}^{n} \left(\underbrace{\sum_{i=1}^{N} e_i^s \frac{\partial u_i}{\partial q_j}}_{e_j^s} + \underbrace{\sum_{i=1}^{N} e_i^\gamma \frac{\partial u_i}{\partial q_j}}_{e_j^\gamma} \right) \delta q_j \\
&= \sum_{j=1}^{n} \left(e_j^s + e_j^\gamma \right) \delta q_j \\
&= \sum_{j=1}^{n} Q_j\, \delta q_j. \qquad (5.58)
\end{aligned}$$

Thus the virtual work of the nonpotential efforts is given by

$$\delta W^{(n)} = \sum_{j=1}^{n} e_j^n\, \delta q_j = \sum_{j=1}^{n} \left(Q_j - \frac{\partial D}{\partial \dot{q}_j} \right) \delta q_j, \qquad (5.59)$$

which is to be substituted for the right–hand side of (5.47).

Making the aforementioned substitutions into (5.47) and collecting terms yields the reduced–order, virtual–work form of Lagrange's equation, given by

$$\sum_{j=1}^{n} \left(\frac{d}{dt} \frac{\partial T^*}{\partial \dot{q}_j} - \frac{\partial T^*}{\partial q_j} + \frac{\partial V}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} - Q_j \right) \delta q_j = 0. \qquad (5.60)$$

If the virtual displacements δq_j are independent, then the coefficients of δq_j in (5.60) must vanish. In such a case the coordinates $(q_1, \ldots, q_n)$ are generalized coordinates, that is, a coordinate set of minimum dimension. The vanishing of the coefficients of δq yields the *generalized–coordinate form* of Lagrange's equation, given by

$$\frac{d}{dt} \frac{\partial T^*}{\partial \dot{q}_j} - \frac{\partial T^*}{\partial q_j} + \frac{\partial V}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} = Q_j \qquad j = 1, \ldots, n. \qquad (5.61)$$

This form of Lagrange's equation is suitable for obtaining the equations of motion for holonomic systems in terms of a set of generalized or Lagrange coordinates. If the nonpotential effort Q contains m_3 unknown implicit efforts e^γ, then constraint equations

$$\gamma_k(e^\gamma, \dot{q}, q, t) = 0 \qquad k = 1, \ldots, m_3, \qquad (5.62)$$

are required to solve the equations of motion. Lagrange's differential equations plus these algebraic constraint equations form a set of *differential–algebraic equations* (DAE). In vector form the set of constraints is given by

$$\Gamma(e^\gamma, \dot{q}, q, t) := \begin{bmatrix} \gamma_1(e^\gamma, \dot{q}, q, t) \\ \vdots \\ \gamma_{m_3}(e^\gamma, \dot{q}, q, t) \end{bmatrix}, \tag{5.63}$$

and the DAE is given in vector form by

$$\begin{aligned} \tfrac{d}{dt}\nabla_{\dot{q}} T^* - \nabla_q T^* + \nabla_{\dot{q}} D + \nabla_q V &= Q_q \\ \Gamma &= 0. \end{aligned} \tag{5.64}$$

EXAMPLE 5.6 The kinetic coenergy of the two–link robot arm system is given in generalized–coordinate form by (5.35) which is repeated here

$$T^* = \tfrac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \tfrac{1}{2} m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right]. \tag{5.65}$$

The virtual work of the force sources acting on the masses is given by

$$\delta W = F_{x_1}\, \delta x_1 + F_{y_1}\, \delta y_1 + F_{x_2}\, \delta x_2 + F_{y_2}\, \delta y_2. \tag{5.66}$$

This work expression is to be rewritten in terms of the generalized coordinates (θ_1, θ_2) using the coordinate transformation equations. Applying the variational operator to these transformation equations yields

$$\begin{aligned} \delta x_1 &= \delta(l_1 \cos\theta_1) = -l_1 \sin\theta_1\, \delta\theta_1 \\ \delta y_1 &= \delta(l_1 \sin\theta_1) = l_1 \cos\theta_1\, \delta\theta_1 \\ \delta x_2 &= \delta(l_1 \cos\theta_1 + l_2 \cos\theta_2) \\ &= -l_1 \sin\theta_1\, \delta\theta_1 - l_2 \sin\theta_2\, \delta\theta_2 \\ \delta y_2 &= \delta(l_1 \sin\theta_1 + l_2 \sin\theta_2) \\ &= l_1 \cos\theta_1\, \delta\theta_1 + l_2 \cos\theta_2\, \delta\theta_2. \end{aligned} \tag{5.67}$$

Substituting these relationships into the virtual work equation (5.66) and collecting terms yields the virtual work expression in terms of the generalized coordinates. In the resulting equation, given by

$$\begin{aligned}\delta W &= \left[(F_{y_1}+F_{y_2})\,l_1\cos\theta_1-(F_{x_1}+F_{x_2})\,l_1\sin\theta_1\right]\delta\theta_1 \\ &\quad+\left[l_2\,(F_{y_2}\cos\theta_2-F_{x_2}\sin\theta_2)\right]\delta\theta_2,\end{aligned}\tag{5.68}$$

the coefficient of $\delta\theta_1$ is the generalized effort Q_{θ_1} and the coefficient of $\delta\theta_2$ is the generalized effort Q_{θ_2}. Lagrange's equation for this example is given by

$$\tfrac{d}{dt}\nabla_{\dot\theta}T^*-\nabla_\theta T^*=Q_\theta,\tag{5.69}$$

yielding the unconstrained equations of motion:

$$\begin{aligned}l_1^2(m_1+m_2)\ddot\theta_1+m_2l_1l_2\cos(\theta_1-\theta_2)\ddot\theta_2+m_2l_1l_2\dot\theta_2^2\sin(\theta_1-\theta_2) &= Q_{\theta_1}\\ m_2l_1l_2\cos(\theta_1-\theta_2)\ddot\theta_1+m_2l_2^2\ddot\theta_2-m_2l_1l_2\dot\theta_1^2\sin(\theta_1-\theta_2) &= Q_{\theta_2},\end{aligned}\tag{5.70}$$

where the Q_{θ_j} are given by

$$\begin{aligned}Q_{\theta_1} &= (F_{y_1}+F_{y_2})\,l_1\cos\theta_1-(F_{x_1}+F_{x_2})\,l_1\sin\theta_1\\ Q_{\theta_2} &= l_2\,(F_{y_2}\cos\theta_2-F_{x_2}\sin\theta_2)\,.\quad\diamond\end{aligned}\tag{5.71}$$

5.2.5 Lagrange multipliers

The form of Lagrange's equation obtained in the previous section depends on the assumption of independent virtual displacements. To use this equation, therefore, the energy functions and nonpotential efforts must be expressed in terms of a set of independent coordinates. For complex multidiscipline systems, selecting a set of independent coordinates can be a daunting task. In such cases, it is preferable to utilize the *method of undetermined multipliers*, devised by Lagrange, which admits the use of dependent coordinates in formulating the dynamic equations of motion.[3]

[3]This introduction to the application and theory of Lagrange multipliers is adapted from Haug [19] and is sufficient for the purposes of this work. Multipliers have a significant role to play in the calculus

THEOREM 1 (Lagrange multipliers) *Given a matrix $A \in \mathcal{R}^{m\times n}$, let $s \in \mathcal{R}^n$ represent all vectors that satisfy $As = 0$. If, for these vectors s, it is also true that $s^T b = 0$ for a given vector $b \in \mathcal{R}^n$, then there exists a vector $\lambda \in \mathcal{R}^m$ such that*

$$s^T b + s^T A^T \lambda = 0 \tag{5.72}$$

for all $s \in \mathcal{R}^n$. *The vector λ is called the vector of* Lagrange multipliers. *If A has full row rank, then λ is unique.*

For holonomic systems, the multiplier theorem is applied as follows. It follows from (3.47) that for systems subject to a set of holonomic constraints $\Phi(q,t) = 0$, the vector of virtual displacements $\delta q \in \mathcal{R}^n$ satisfies

$$\Phi_q \, \delta q = 0, \tag{5.73}$$

where $\Phi \in \mathcal{R}^{m_1}$ is a vector of holonomic constraint equations and $\Phi_q \in \mathcal{R}^{m_1 \times n}$ is its Jacobian. Let the parenthetical expression in (5.60) be denoted by the effort vector $\mathcal{E}$, that is,

$$\mathcal{E}_j := \frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_j} - \frac{\partial T^*}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} + \frac{\partial V}{\partial q_j} - Q_j. \tag{5.74}$$

Then (5.60) is given by

$$\sum_{j=1}^{n} \mathcal{E}_j \, \delta q_j = 0, \tag{5.75}$$

which can be written as the inner product

$$\begin{bmatrix} \delta q_1 & \cdots & \delta q_n \end{bmatrix} \begin{bmatrix} \mathcal{E}_1 \\ \vdots \\ \mathcal{E}_n \end{bmatrix} = 0, \tag{5.76}$$

or

$$\delta q^T \mathcal{E} = 0. \tag{5.77}$$

---

of variations, optimization theory and game theory, where they known also as Karush–Kuhn–Tucker (KKT) multipliers. Conditions for the rigorous application of multiplier theory, known as the KKT conditions, can be found in texts on optimization. Gill *et al.* [14] give a conventional treatment, while Rockafellar [33] gives a contemporary treatment.

Given these relationships among δq, $\mathcal{E}$ and Φ_q, the multiplier theorem states that a multiplier vector $\kappa \in \mathcal{R}^{m_1}$ exists such that

$$\delta q^T \mathcal{E} + \delta q^T \Phi_q^T \kappa = 0. \tag{5.78}$$

This can be rearranged to obtain

$$\delta q^T \left(\mathcal{E} + \Phi_q^T \kappa \right) = 0. \tag{5.79}$$

By the theorem, this equality holds for all $\delta q \in \mathcal{R}^n$. In other words, the independence or dependence of the virtual displacements is no longer an issue. The virtual work expression (5.79) is satisfied for arbitrary displacements δq. It follows that the term in the parentheses is identically zero, that is,

$$\mathcal{E} + \Phi_q^T \kappa = 0. \tag{5.80}$$

With the matrix elements explicitly shown, this equation is given by

$$\begin{bmatrix} \mathcal{E}_1 \\ \vdots \\ \mathcal{E}_n \end{bmatrix} + \begin{bmatrix} \dfrac{\partial \phi_1}{\partial q_1} & \cdots & \dfrac{\partial \phi_{m_1}}{\partial q_1} \\ \vdots & & \vdots \\ \dfrac{\partial \phi_1}{\partial q_n} & \cdots & \dfrac{\partial \phi_{m_1}}{\partial q_n} \end{bmatrix} \begin{bmatrix} \kappa_1 \\ \vdots \\ \kappa_{m_1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{5.81}$$

from which is obtained the scalar formulation

$$\mathcal{E}_j + \sum_{k=1}^{m_1} \frac{\partial \phi_k}{\partial q_j} \kappa_k = 0 \qquad j = 1, \ldots, n. \tag{5.82}$$

Substituting for $\mathcal{E}_j$ yields

$$\frac{d}{dt} \frac{\partial T^*}{\partial \dot{q}_j} - \frac{\partial T^*}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} + \frac{\partial V}{\partial q_j} + \sum_{k=1}^{m_1} \frac{\partial \phi_k}{\partial q_j} \kappa_k = Q_j, \tag{5.83}$$

for $j = 1, \ldots, n$. This is Lagrange's equation for holonomic systems with excess coordinates. This is a set of n equations in $n+m_1+m_3$ unknowns (q, κ, e^γ). The displacement constraint equations

$$\phi_k(q, t) = 0 \qquad k = 1, \ldots, m_1, \tag{5.84}$$

and the implicit effort constraint equations

$$\gamma_k(e^\gamma, \dot{q}, q, t) = 0 \qquad k = 1, \ldots, m_3, \tag{5.85}$$

provide the additional equations necessary to solve for the unknowns. The vector of displacement constraints is given by

$$\Phi(q,t) := \begin{bmatrix} \phi_1(q,t) \\ \vdots \\ \phi_{m_1}(q,t) \end{bmatrix}, \tag{5.86}$$

and the DAE is given in vector form by

$$\begin{aligned} \tfrac{d}{dt}\nabla_{\dot{q}} T^* - \nabla_q T + \nabla_q V + \nabla_{\dot{q}} D + \Phi_q^T \kappa &= Q \\ \Phi &= 0 \\ \Gamma &= 0, \end{aligned} \tag{5.87}$$

where Γ is the vector of dynamic constraints given by (5.63).

EXAMPLE 5.7 For the two–link robot arm, let the displacement vector be defined as $u = (x_1, y_1, x_2, y_2)$. Then the kinetic coenergy function (5.32) is given by

$$T^* = \tfrac{1}{2} m_1 (\dot{u}_1^2 + \dot{u}_2^2) + \tfrac{1}{2} m_2 (\dot{u}_3^2 + \dot{u}_4^2). \tag{5.88}$$

Two holonomic constraints are given by

$$\begin{aligned} \phi_1 &:= u_1^2 + u_2^2 - l_1^2 = 0 \\ \phi_2 &:= (u_3 - u_1)^2 + (u_4 - u_2)^2 - l_2^2 = 0. \end{aligned} \tag{5.89}$$

The virtual work of the force sources (5.66) is given by

$$\delta W = F_{u_1}\, \delta u_1 + F_{u_2}\, \delta u_2 + F_{u_3}\, \delta u_3 + F_{u_4}\, \delta u_4. \tag{5.90}$$

The coefficients of δu are the generalized efforts Q_u.[4]

Lagrange's equation for this example is given by

$$\begin{aligned} \frac{d}{dt}\nabla_{\dot{u}}T^* + \Phi_u^T\kappa &= Q_u \\ \Phi &= 0. \end{aligned} \tag{5.91}$$

which yields the following set of differential–algebraic equations of motion

$$\begin{bmatrix} m_1\ddot{u}_1 \\ m_1\ddot{u}_2 \\ m_2\ddot{u}_3 \\ m_2\ddot{u}_4 \end{bmatrix} + \begin{bmatrix} 2u_1 & 2(u_1-u_3) \\ 2u_2 & 2(u_2-u_4) \\ 0 & 2(u_3-u_1) \\ 0 & 2(u_4-u_2) \end{bmatrix} \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix} = \begin{bmatrix} F_{u_1} \\ F_{u_2} \\ F_{u_3} \\ F_{u_4} \end{bmatrix}$$

$$\begin{bmatrix} u_1^2+u_2^2-l_1^2 \\ (u_3-u_1)^2+(u_4-u_2)^2-l_2^2 \end{bmatrix} = 0. \tag{5.92}$$

This is a set of six equations with unknowns $(u_1, u_2, u_3, u_4, \kappa_1, \kappa_2)$. This set of equations is substantially simpler in form than the set of equations written in generalized coordinate form, illustrating one of the advantages of the multiplier form compared to the generalized coordinate form. Using the multiplier form, the analyst need not perform arithmetic manipulations aimed at reducing the set of equations to a set of minimum dimension. ◇

5.2.6 Nonholonomic constraints

It is shown in Chapter 3 that a system of constraints in Pfaffian form can include nonholonomic equality constraints as well as holonomic flow constraints which the analyst decides to leave in flow–constraint form. Such constraints are accommodated in Lagrange's equation through the use of Lagrange multipliers in much the same manner as

[4]Since this expression is already written in terms of the coordinates used to represent the motion of the system, no further manipulation of the virtual work expression is required. If the force sources were replaced with torque sources, the virtual work expression would initially be given in terms of angular virtual displacements $\delta\theta_j$. In this event the virtual work expression would have to be manipulated to obtain an expression written in terms of the translational virtual displacements δu_j.

holonomic constraints are accommodated. The only restriction on these constraints is that they must be expressed in, or be reducible to, the general Pfaffian form given by

$$\sum_{i=1}^{N} a_{ki}(u,t)\,du_i + a_k(u,t)\,dt = 0 \qquad k = 1,\ldots,m. \tag{5.93}$$

To express this set of equations in terms of reduced–order coordinates, the differential of u, given by

$$du_i = \sum_{j=1}^{n} \frac{\partial u_i}{\partial q_j}\,dq_j + \frac{\partial u_i}{\partial t}\,dt \qquad i = 1,\ldots,N, \tag{5.94}$$

is substituted into the Pfaffian constraint equation to obtain

$$\sum_{i=1}^{N} a_{ki}\left(\sum_{j=1}^{n} \frac{\partial u_i}{\partial q_j}\,dq_j + \frac{\partial u_i}{\partial t}\,dt\right) + a_k\,dt = 0, \tag{5.95}$$

which can be rearranged to obtain

$$\sum_{j=1}^{n}\left(\underbrace{\sum_{i=1}^{N} a_{ki}\frac{\partial u_i}{\partial q_j}}_{b_{kj}}\right)dq_j + \left(\underbrace{\sum_{i=1}^{N} a_{ki}\frac{\partial u_i}{\partial t} + a_k}_{b_k}\right)dt = 0. \tag{5.96}$$

Denoting the parenthetical terms b_{kj} and b_k respectively, this system of constraints is given by

$$\sum_{j=1}^{n} b_{kj}(q,t)\,dq_j + b_k(q,t)\,dt = 0 \qquad k = 1,\ldots,m_2. \tag{5.97}$$

Thus, a set of constraints in Pfaffian form remain in Pfaffian form after transformation to a reduced–order set of coordinates. Defining

$$\begin{aligned} B(q,t) &:= \text{matrix}\{b_{kj}\} \in \mathcal{R}^{m_2\times n} \\ b(q,t) &:= \text{vector}\{b_k\} \in \mathcal{R}^{m_2}, \end{aligned} \tag{5.98}$$

the Pfaffian constraint is given in vector form by

$$B\,dq + b\,dt = 0. \tag{5.99}$$

These constraints are expressed in rate form by dividing by dt. It is convenient to denote the resulting flow constraints, which are linear in flow $\dot{q}$, by the symbol ψ_k such that

$$\psi_k(\dot{q}, q, t) := \sum_{j=1}^{n} b_{kj}(q,t)\,\dot{q}_j + b_k(q,t) = 0 \qquad k = 1, \ldots, m_2. \tag{5.100}$$

The set of flow constraints is represented by the vector Ψ given by

$$\Psi(\dot{q}, q, t) := \begin{bmatrix} \psi_1(\dot{q}, q, t) \\ \vdots \\ \psi_{m_2}(\dot{q}, q, t) \end{bmatrix} = B\,\dot{q} + b = 0. \tag{5.101}$$

Since Ψ is linear in flow, the Jacobian of Ψ with respect to flow is given by $\Psi_{\dot{q}} = B(q,t)$.

It follows from the reduced-order form of the nonholonomic constraint equation $B\,dq + b\,dt = 0$, that this class of constraints imposes on the virtual displacement vector $\delta q \in \mathcal{R}^n$ the condition

$$B\,\delta q = 0. \tag{5.102}$$

The virtual displacement vector, from (5.79), also satisfies

$$\delta q^T \tilde{\mathcal{E}} = 0, \tag{5.103}$$

where

$$\tilde{\mathcal{E}} = \mathcal{E} + \Phi_q^T \kappa. \tag{5.104}$$

By the multiplier theorem, a multiplier vector $\mu \in \mathcal{R}^{m_2}$ exists such that

$$\delta q^T \left(\tilde{\mathcal{E}} + B^T \mu\right) = 0, \tag{5.105}$$

Substituting for $\tilde{\mathcal{E}}$, and writing the equation in component form yields the virtual work expression given by

$$\sum_{j=1}^{n} \left(\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_j} - \frac{\partial T^*}{\partial q_j} + \frac{\partial V}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} + \sum_{k=1}^{m_1} \frac{\partial \phi_k}{\partial q_j}\kappa_k + \sum_{k=1}^{m_2} b_{kj}\mu_k - Q_j \right) \delta q_j = 0, \tag{5.106}$$

where $j = 1, \ldots, n$. By the multiplier theorem, this equation is rendered true for *all* $\delta q_j \in \mathcal{R}^n$. Thus the coefficient of each δq_j must vanish, yielding

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_j} - \frac{\partial T^*}{\partial q_j} + \frac{\partial V}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} + \sum_{k=1}^{m_1} \frac{\partial \phi_k}{\partial q_j}\kappa_k + \sum_{k=1}^{m_2} b_{kj}\mu_k = Q_j, \tag{5.107}$$

where $j = 1, \ldots, n$. This is the *multiplier form* of Lagrange's equation for systems subject to both holonomic and nonholonomic constraints. This is a set of n equations with $n+m_1+m_2+m_3$ unknowns $(q, \kappa, \mu, e^\gamma)$.

The displacement constraints, the Pfaffian flow constraints and the implicit effort constraints provide the additional equations necessary to determine the motion of a system. The kinematic constraint equations $\phi = 0$ and $\psi = 0$ are added to the equations of motion to solve for the unknown multipliers κ and μ. The dynamic constraint equations $\gamma = 0$ are added to the equations of motion to solve for the implicit efforts $e^\gamma \in Q$. Lagrange's equation, an ordinary differential equation (ODE), plus the algebraic constraint equations comprise an $n+m_1+m_2+m_3$ set of differential–algebraic equations called herein the *Lagrangian DAE*. The Lagrangian DAE is given by

$$\begin{aligned}
\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_j} - \frac{\partial T^*}{\partial q_j} + \frac{\partial V}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} + \sum_{k=1}^{m_1} \frac{\partial \phi_k}{\partial q_j}\kappa_k \;+\; \sum_{k=1}^{m_2} b_{kj}\mu_k &= Q_j, \quad j = 1, \ldots, n \\
\phi_k(q,t) &= 0 \qquad k = 1, \ldots, m_1 \\
\psi_k(\dot{q}, q, t) &= 0 \qquad k = 1, \ldots, m_2 \\
\gamma_k(e^\gamma, \dot{q}, q, t) &= 0 \qquad k = 1, \ldots, m_3.
\end{aligned} \tag{5.108}$$

Let the multipliers κ and μ be adjoined into a single vector of multipliers λ such that

$$\lambda(t) := \begin{bmatrix} \kappa(t) \\ \mu(t) \end{bmatrix} \in \mathcal{R}^{m_o}, \tag{5.109}$$

and let the kinematic constraints Φ and Ψ be adjoined into a single vector C such that

$$C(\dot{q}, q, t) := \begin{bmatrix} \Phi(q,t) \\ \Psi(\dot{q}, q, t) \end{bmatrix} \in \mathcal{R}^{m_o}, \tag{5.110}$$

where the index $m_o = m_1 + m_2$ is the total number of kinematic constraints. Then the constraint Jacobian C_n is defined as

$$C_n(q,t) := \begin{bmatrix} \Phi_q(q,t) \\ B(q,t) \end{bmatrix} \in \mathcal{R}^{m_o \times n}, \tag{5.111}$$

and the Lagrangian DAE is given by

$$\begin{aligned} \tfrac{d}{dt}\nabla_{\dot{q}} T^* - \nabla_q T^* + \nabla_q V + \nabla_{\dot{q}} D + C_n^T \lambda &= Q \\ C &= 0 \\ \Gamma &= 0. \end{aligned} \tag{5.112}$$

5.3 Descriptor form

Carrying out the time derivative in Lagrange's equation and rearranging terms yields

$$\underbrace{\nabla_{\dot{q}}^2 T^*}_{M} \ddot{q} + C_n^T \lambda = \underbrace{Q - (\nabla_{\dot{q}} T^*)_q \dot{q} + (\nabla_{\dot{q}} T^*)_t + \nabla_q T^* - \nabla_q V - \nabla_{\dot{q}} D}_{\Upsilon_0}, \tag{5.113}$$

Introducing $M(\dot{q}, q, t) \in \mathcal{R}^{n \times n}$ to represent the Hessian matrix $\nabla_{\dot{q}}^2 T^*$ and $\Upsilon_0(e^\gamma, \dot{q}, q, t) \in \mathcal{R}^n$ to represent the summation of efforts on the right-hand side, the Lagrangian DAE is given by

$$\begin{aligned} M\ddot{q} + C_n^T \lambda &= \Upsilon_0 \\ C &= 0 \\ \Gamma &= 0. \end{aligned} \tag{5.114}$$

This formulation is called *Lagrange's equation of the first kind.* It is characterized by second-order differential equations in terms of a single differential variable, displacement q. Matrix M is so called since in mechanics it represents a mass matrix. Here, M represents a matrix of inertial coefficients. In the general multidiscipline case, this matrix may be singular or zero.

Letting $\dot{q} = f$ the Lagrangian DAE is given in *descriptor form*[5] by

$$\begin{aligned} \dot{q} &= f \\ M\dot{f} + C_n^T\lambda &= \Upsilon_0 \\ C &= 0 \\ \Gamma &= 0. \end{aligned} \tag{5.115}$$

This is a set of $2n+m_o+m_3$ equations in the same number of unknowns $(q, f, \lambda, e^\gamma)$ with state variables $q(t)$ and $f(t)$.

The descriptor form of Lagrange's equation is characterized by first-order differential equations in terms of two differential variables displacement and flow (q, f). The descriptor form of the Lagrangian DAE is used henceforward as the basis of a systematic approach to modeling multidiscipline systems.

EXAMPLE 5.8 The two-link robot arm is used here to illustrate the descriptor form. The descriptor form for this example is given by

$$\begin{aligned} \dot{q} &= f \\ M\dot{f} + \Phi_q^T\lambda &= \Upsilon_0 \\ \Phi &= 0. \end{aligned} \tag{5.116}$$

From the multiplier form of the equations of motion (5.92) the following set of equations is readily obtained

$$\begin{aligned} \dot{q} &= f \\ \begin{bmatrix} m_1 & & & 0 \\ & m_1 & & \\ & & m_2 & \\ 0 & & & m_2 \end{bmatrix} \dot{f} + \begin{bmatrix} 2q_1 & 2(q_1-q_3) \\ 2q_2 & 2(q_2-q_4) \\ 0 & 2(q_3-q_1) \\ 0 & 2(q_4-q_2) \end{bmatrix} \lambda &= \begin{bmatrix} F_{q_1} \\ F_{q_2} \\ F_{q_3} \\ F_{q_4} \end{bmatrix} \end{aligned} \tag{5.117}$$

[5]This nomenclature is from Brenan *et al.* [6].

$$\begin{bmatrix} q_1^2 + q_2^2 - l_1^2 \\ (q_3 - q_1)^2 + (q_4 - q_2)^2 - l_2^2 \end{bmatrix} = 0.$$

The terms M, Φ_q^T, Υ_0 and Φ in (5.116) correspond to the bracketed matrix expressions in (5.117). $\diamond$

EXAMPLE 5.9 The electrical circuit in Fig. 5.6 contains a nonlinear resistor R_1 characterized by a constitutive law given by

$$e_\alpha(i) = k\,|i|\,i. \tag{5.118}$$

Thus the nonlinear resistance R_1 is given by $k\,|i|$. Since the resistance R_1 is nonlinear, the dissipation function of this element does not have the usual quadratic form, that is, $D(\dot{q}_1) \neq \frac{1}{2} R_1 \dot{q}_1^2$.

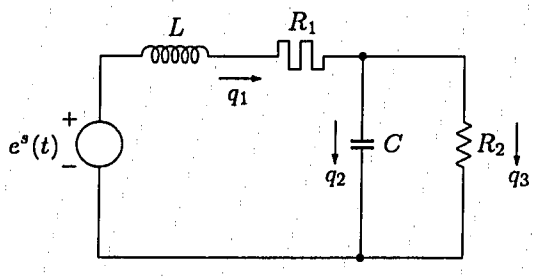


Figure 5.6. Nonlinear electrical circuit

One approach to modeling the contribution of this element to the dynamics of the circuit is to pose the constitutive law of the element as a dynamic constraint. In this cases, the implicit effort e^γ is assigned to the element as shown in Fig. 5.7, and the dynamic constraint is given by

$$\gamma := e^\gamma + k\,|\dot{q}_1|\,\dot{q}_1 = 0. \tag{5.119}$$

With this formulation and with the sign convention indicated, the effort e^γ always opposes the flow of current $\dot{q}_1$.

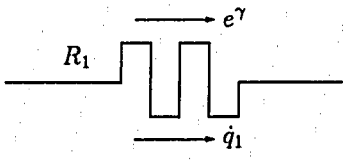


Figure 5.7. Voltage drop as an implicit effort.

The energy functions are given by:

$$\begin{aligned} T^* &= \tfrac{1}{2} L \dot{q}_1^2 \\ D &= \tfrac{1}{2} R_2 \dot{q}_3^2 \\ V &= \frac{q_2^2}{2C}. \end{aligned} \tag{5.120}$$

The electrical node imposes a holonomic flow constraint given ψ by

$$\psi := \dot{q}_1 - \dot{q}_2 - \dot{q}_3 = 0. \tag{5.121}$$

Since this constraint involves the variable q_2 which represents a physical displacement (the charge of the capacitor), the constraint must be integrated to obtain the holonomic form given by

$$\phi := q_1 - q_2 - q_3 + q_{2o} = 0. \tag{5.122}$$

The virtual work of source efforts and implicit efforts is given by

$$\begin{aligned} \delta W &= e^s \, \delta q_1 + e^\gamma \, \delta q_1 \\ &= (e^s + e^\gamma) \, \delta q_1. \end{aligned} \tag{5.123}$$

Thus $Q_{q_1} = e^s + e^\gamma$ and $Q_{q_2} = Q_{q_3} = 0$. Lagrange's equation is given by

$$\begin{aligned} \tfrac{d}{dt} \nabla_{\dot{q}} T^* - \nabla_q T^* + \nabla_{\dot{q}} D + \nabla_q V + \Phi_q^T \lambda &= Q_q \\ \Phi &= 0 \\ \Gamma &= 0, \end{aligned} \tag{5.124}$$

yielding

$$\begin{bmatrix} L\ddot{q}_1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ R_2\dot{q}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{q_2}{C} \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \lambda = \begin{bmatrix} e^s + e^\gamma \\ 0 \\ 0 \end{bmatrix}.$$

$$\begin{aligned} \left[q_1 - q_2 - q_3 + q_{2o} \right] &= 0 \\ \left[e^\gamma + k\,|\dot{q}_1|\,\dot{q}_1 \right] &= 0. \end{aligned} \tag{5.125}$$

Letting $\dot{q} = f$, the descriptor form of the DAE is given by

$$\begin{aligned} \dot{q} &= f \\ \begin{bmatrix} L & & \\ & 0 & \\ & & 0 \end{bmatrix} \dot{f} + \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \lambda &= \begin{bmatrix} e^s + e^\gamma \\ -\frac{q_2}{C} \\ -R_2 f_3 \end{bmatrix}. \\ \left[q_1 - q_2 - q_3 + q_{2o} \right] &= 0 \\ \left[e^\gamma + k\,|f_1|\,f_1 \right] &= 0. \quad \diamond \end{aligned} \tag{5.126}$$

EXAMPLE 5.10 An electromagnetic suspension is shown in Fig. 5.8. The voltage source $e^s(t)$ can be modulated in response to the disturbance force $F^s(t)$ such that a desired air gap u_1 can be maintained. Coil inductance L is a function of the air gap given by

$$L(u_1) = \frac{\gamma_o}{\gamma_1 + u_1}, \tag{5.127}$$

where γ_o and γ_1 are known parameters. Coordinates (u_1, u_2, u_3) are assigned as shown in Fig. 5.8.

The energy functions are given by

$$\begin{aligned} T^* &= \tfrac{1}{2}m\dot{u}_1^2 + \tfrac{1}{2}L\dot{u}_2^2 = \tfrac{1}{2}m\dot{u}_1^2 + \frac{\gamma_o \dot{u}_2^2}{2(\gamma_1 + u_1)} \\ D &= \tfrac{1}{2}R\dot{u}_3^2 \\ V &= 0. \end{aligned} \tag{5.128}$$

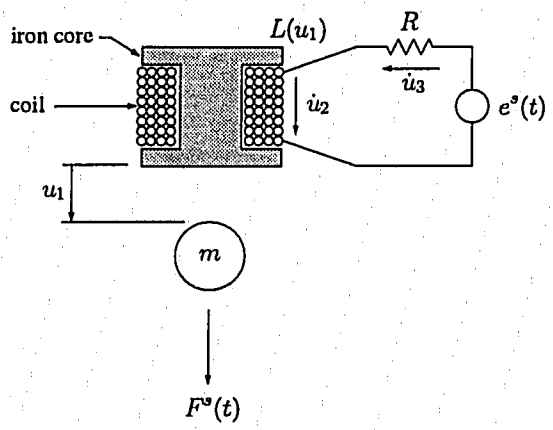


Figure 5.8. An electromagnetic suspension.

The single constraint is a holonomic flow constraint given by

$$\psi_1 := \dot{u}_2 - \dot{u}_3 = 0. \tag{5.129}$$

It is not necessary to integrate this flow constraint equation to impose a constraint on displacements (u_2, u_3) since neither of these two displacements represent a physical accumulation of charge. The virtual work of the source efforts is given by

$$\delta W = F^s(t)\,\delta u_1 + e^s(t)\,\delta u_3, \tag{5.130}$$

hence $Q_1 = F^s(t)$, $Q_2 = 0$ and $Q_3 = e^s(t)$. The Lagrangian DAE is given by

$$\begin{aligned} m\ddot{u}_1 + \frac{\gamma_o \dot{u}_2^2}{2(\gamma_1 + u_1)^2} &= F^s(t) \\ \frac{\gamma_o \ddot{u}_2}{\gamma_1 + u_1} - \frac{\gamma_o \dot{u}_1 \dot{u}_2}{(\gamma_1 + u_1)^2} + \lambda &= 0 \\ R\dot{u}_3 - \lambda &= e^s(t) \\ \dot{u}_2 - \dot{u}_3 &= 0. \end{aligned} \tag{5.131}$$

Letting $q = u$ and $f = \dot{u}$, the equations of motion are given in descriptor form by

$$\dot{q} = f$$

$$\begin{bmatrix} m & 0 & 0 \\ 0 & \dfrac{\gamma_o}{\gamma_1+q_1} & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{f} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \lambda = \begin{bmatrix} F^s - \dfrac{\gamma_o f_2^2}{2(\gamma_1+q_1)^2} \\ \dfrac{\gamma_o f_1 f_2}{(\gamma_1+q_1)^2} \\ e^s - Rf_3 \end{bmatrix} \tag{5.132}$$

$$0 = f_2 - f_3.$$

This is a set of seven equations with unknowns $(q_1, q_2, q_3, f_1, f_2, f_3, \lambda)$. This DAE is comprised of six ODE's and a single algebraic constraint equation. The first three ODE's are explicit. The second three ODE's are linearly implicit in the derivative $\dot{f}$. The coefficient of $\dot{f}$ is the inertial coefficient matrix M, which in this case is singular. $\diamond$

5.4 Underlying ODE

A disadvantage of the descriptor formulation is that a numerical solution may be difficult to obtain. The relevant measure of this difficulty is called the *index* of the DAE.[6] Generally, the higher the index, the more difficult the problem is numerically. The Lagrangian DAE is typically index–3 or higher. Index–1 and index–0 DAEs are generally simpler to solve than DAEs with a higher index [6]. In this section is presented a method of reducing the Lagrangian equations of motion to a set of ODE's, which are index–0 by definition.

If the system constraints are such that an explicit solution can be obtained for the multipliers λ and for the implicit efforts e^γ, and assuming that the inertial matrix M is invertible, then the equations of motion can be reduced to the form

$$\begin{aligned} \dot{q} &= f \\ \dot{f} &= \mathcal{F}(f, q, t), \end{aligned} \tag{5.133}$$

where $\mathcal{F}$ is a nonlinear function of the state variables, time and the parameters of the

[6]Index is defined in Appendix A.

system. In this section, the procedure for obtaining this ODE from the Lagrangian DAE is demonstrated for a holonomic system.

To eliminate the multipliers from the equations of motion, the constraints are differentiated twice with respect to time. Let the vector of holonomic constraints be given by $\Phi(q,t) = 0$. The first time derivative is given by

$$\begin{aligned} 0 = \frac{d}{dt}\Phi(q,t) &= \frac{\partial \Phi}{\partial q}\dot{q} + \frac{\partial \Phi}{\partial t} \\ &= \Phi_q f + \Phi_t. \end{aligned} \tag{5.134}$$

The second time derivative is given by

$$\begin{aligned} 0 = \frac{d^2}{dt^2}\Phi(q,t) &= \tfrac{d}{dt}\left(\Phi_q f + \Phi_t\right) \\ &= \left(\Phi_q f + \Phi_t\right)_f \dot{f} + \left(\Phi_q f + \Phi_t\right)_q \dot{q} + \left(\Phi_q f + \Phi_t\right)_t \\ &= \Phi_q \dot{f} + \left(\Phi_q f + \Phi_t\right)_q f + \left(\Phi_q f + \Phi_t\right)_t, \end{aligned} \tag{5.135}$$

yielding

$$\begin{aligned} \Phi_q \dot{f} &= \underbrace{-\left(\Phi_q f + \Phi_t\right)_q f - \left(\Phi_q f + \Phi_t\right)_t}_{\Upsilon_1(f,q,t)} \\ \Phi_q \dot{f} &= \Upsilon_1. \end{aligned} \tag{5.136}$$

From the descriptor form of the Lagrangian DAE (5.115) is obtained

$$M\dot{f} + \Phi_q^T \lambda = \Upsilon_0, \tag{5.137}$$

which can be solved for $\dot{f}$, assuming the inverse of M exists, to obtain

$$\dot{f} = M^{-1}\left(\Upsilon_0 - \Phi_q^T \lambda\right). \tag{5.138}$$

This expression for $\dot{f}$ is substituted into (5.136) to obtain

$$\Phi_q M^{-1}\left(\Upsilon_0 - \Phi_q^T \lambda\right) = \Upsilon_1, \tag{5.139}$$

which can be solved for λ, assuming the inverse of $\Phi_q M^{-1} \Phi_q^T$ exists, to obtain

$$\lambda = \left(\Phi_q M^{-1} \Phi_q^T\right)^{-1}\left(\Phi_q M^{-1}\Upsilon_0 - \Upsilon_1\right). \tag{5.140}$$

Substituting this expression for λ into (5.138) yields

$$\dot{f} = M^{-1}\left[\Upsilon_0 - \Phi_q^T\left(\Phi_q M^{-1}\Phi_q^T\right)^{-1}\left(\Phi_q M^{-1}\Upsilon_0 - \Upsilon_1\right)\right]. \tag{5.141}$$

Letting $\mathcal{F}$ denote the expression on the right, the equations of motion are given by

$$\begin{aligned} \dot{q} &= f \\ \dot{f} &= \mathcal{F}(f, q, t), \end{aligned} \tag{5.142}$$

which was to be shown.

This set of explicit, first-order, nonlinear ODE's is called *Lagrange's equation of the second kind,* or the *underlying ODE.* Given some initial conditions for (q, f), these equations are readily solved using standard ODE numerical solvers. Depending on the size of the system, a numerical solution may be obtained more quickly using this approach than is obtained using a direct approach based on the descriptor form of the DAE.

One drawback of the underlying-ODE formulation is that the inverse of the inertia matrix M and the inverse of the matrix $\Phi_q M^{-1}\Phi_q^T$ must exist and be well-conditioned. In electrical networks with no inductors, for example, the inertia matrix $M = 0$.

A second drawback of the underlying-ODE formulation is that only the derivatives of the constraint equation are guaranteed to be satisfied by the solution. The displacement constraint equation $\Phi = 0$ is not guaranteed to be satisfied. In such cases the numerical solution for displacements may drift from the true displacement trajectories. This is particularly noticeable for systems having closed kinematic chains, such as a four-bar mechanism or a slider-crank mechanism.

For these reasons, the underlying ODE formulation of the equations of motion is eschewed in favor of the descriptor form of the DAE, which is used henceforward as the basis for systematically formulating and numerically solving the equations of motion of multidiscipline systems. Strategies for integrating index-3 DAEs are outlined in Chapter 8.

5.5 Problem formulation

To obtain the equations of motion using the descriptor form of the Lagrangian DAE, it is necessary for the analyst to assign coordinates (q, f) to system elements. Expressions are developed in terms of these coordinates for the energy functions T^*, V and D, for the independent constraints ϕ, ψ and γ, and for the virtual work of the source efforts e^s and implicit efforts e^γ. The terms M, C, C_n, Γ and Υ_0 are then generated by manipulating these expressions according to the definitions associated with the descriptor form. This procedure is illustrated by the previous examples.

5.6 Interpretation of Lagrange's equation

Rearranging the terms of (5.106) yields

$$\sum_{j=1}^{n}\left(\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_j}\right)\delta q_j = \sum_{j=1}^{n}\left(\frac{\partial T^*}{\partial q_j} - \frac{\partial V}{\partial q_j} - \frac{\partial D}{\partial \dot{q}_j} - \sum_k \frac{\partial \phi_k}{\partial q_j}\kappa_k - \sum_k b_{kj}\mu_k + Q_j\right)\delta q_j. \qquad (5.143)$$

This formulation lends some insight into the physical meaning of each term. The left–hand parenthetical term is the time rate of change of momentum. The product of this quantity with a displacement δq represents a variation in the energy of motion.

Each term on the right–hand side represents a component of effort acting over a virtual displacement δq. The first parenthetical term represents kinetic efforts. The second term on the right represents potential efforts. The third term represent dissipative efforts. The two terms involving multipliers represent efforts imposed on the system due to the presence of constraints. And Q represents all nonpotential efforts not already accounted for. The product of all these efforts with a displacement δq represents virtual work.

Thus the physical interpretation of this form of Lagrange's equation is simply the equivalence of work and energy. This is not a surprising result, given that Lagrange's

equation is derivable from the first law of thermodynamics. Conceptually (5.143) is a statement of the following equivalence

$$\sum \dot{p}\,\delta q = \sum e\,\delta q, \tag{5.144}$$

which is a work–energy form of the dynamic requirement $\dot{p} = e$. This work–energy equation, written for mechanical systems, is the traditional starting point from which Lagrange's equation is derived. Pars [29] gives a detailed derivation from this perspective.

By the multiplier theorem (5.143) is true for arbitrary δq, hence Lagrange's equation is reduced to an equivalence of efforts represented by

$$\dot{p}_j = e_j \qquad j = 1, \ldots, n. \tag{5.145}$$

Redfield [31] exploits this interpretation in formulating bond graphs to represent Lagrange's equation for holonomic mechanical systems.

Chapter 6

THE HAMILTONIAN DAE OF MOTION

In this chapter is presented a formulation of the DAE using an alternate or non–Lagrangian pair of variables to describe the motion of a system. This is the first of three alternate variable pairs examined in this research.

In the Lagrangian formulation of the equations of motion for a multidiscipline system, the motion of the system describes a trajectory in state space, where the state variables are displacement and flow. Through a transformation of variables the motion can be described instead as a trajectory in *phase space*, where the phase variables are displacement and momentum.

The transformation is accomplished using the Legendre transform, in which the flows associated with kinetic stores are transformed into momenta. When applied to Lagrange's equation this transformation yields Hamilton's equation, without loss of information content regarding the dynamic behavior of the system.

6.1 The Legendre transform

Consider a multidiscipline system with a reduced–order configuration space of dimension n. In the Lagrangian formulation, the motion of a system is described by a trajectory in a $2n$–dimensional state space, comprised of n displacements $q(t)$ and n flows $\dot{q}(t)$. Let the number of these flows associated with kinetic stores be denoted by $s \leq n$. Thus flows are divided into two categories: kinetic flows $\dot{q}^s := (\dot{q}_1, \ldots, \dot{q}_s)$ that are associated with kinetic stores, and all other flows $\dot{q}^r := (\dot{q}_{s+1}, \ldots, \dot{q}_n)$ where the index $r := n - s$. Hamilton's equation is obtained from Lagrange's equation by applying the Legendre transform to the kinetic coenergy function $T^*(\dot{q}^s)$ such that these s kinetic flow variables $\dot{q}^s_j$ are

transformed into s momentum variables p_j. Nonkinetic flows $\dot{q}^r$ are not affected by the variable transformation.

The Legendre transform operates on a function of m variables $y^{(o)}(x_1, \ldots, x_m)$ such that the k^{th} Legendre transform of $y^{(o)}$ is the new function $y^{(k)}(\xi_1, \ldots, \xi_k, x_{k+1}, \ldots, x_m)$ where ξ is defined as the partial derivative of the original function with respect to the transformed variables, that is, $\xi_i := \partial y^{(o)}/\partial x_i$.[1]

Using this nomenclature, the kinetic coenergy function T^* is the function $y^{(o)}$, the kinetic flows $(\dot{q}_1, \ldots, \dot{q}_s)$ are the first s variables $(x_1, \ldots, x_s)$ to be transformed, and the momenta $(p_1, \ldots, p_s)$ are the transformed variables $(\xi_1, \ldots, \xi_s)$. The untransformed variables are displacements q and time t, as well as the nonkinetic flows, which by definition do not appear in T^*. In overview, the transformation proceeds as shown in Fig. 6.1.

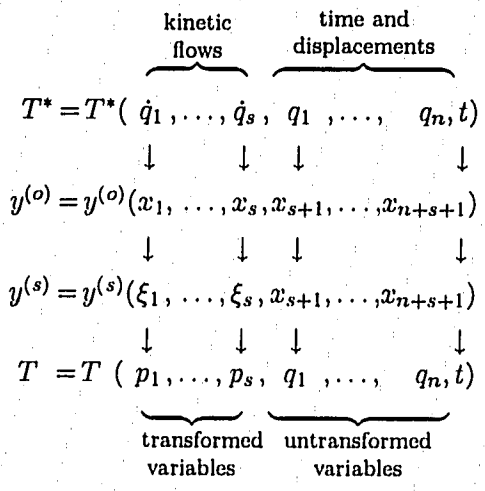


Figure 6.1. Overview of the Legendre transform for kinetic flows.

The definition of ξ_i for the transformed variables yields

$$\xi_i := \frac{\partial y^{(o)}}{\partial x_i} = \frac{\partial T^*}{\partial \dot{q}_i} = p_i \qquad i = 1, \ldots, s. \tag{6.1}$$

[1]The nomenclature and basic principles of the Legendre transform are given in Appendix A.

Thus the new variables ξ_i are components of momentum p_i. This relationship accords with the definition of the momenta of kinetic stores. The definition of $y^{(s)}$ is given by

$$y^{(s)} := y^{(o)} - \sum_{i=1}^{s} \xi_i x_i, \tag{6.2}$$

yielding

$$\begin{aligned} y^{(s)} &= T^*(\dot{q}) - \sum_{i=1}^{s} p_i \dot{q}_i \\ &= \underbrace{T^*(\dot{q}) + T(p) - \sum_{i=1}^{s} p_i \dot{q}_i}_{=0} - T(p) \\ &= -T(p). \end{aligned} \tag{6.3}$$

Thus the s^{th} transform of the kinetic coenergy function $T^*(\dot{q}_1 \ldots, \dot{q}_s)$ is the negative of the kinetic energy function $T(p_1 \ldots, p_s)$. For the untransformed variables, the transformation yields the general result that

$$\frac{\partial y^{(k)}}{\partial x_i} = \frac{\partial y^{(o)}}{\partial x_i} \qquad i = s+1, \ldots, n+s+1, \tag{6.4}$$

which yields

$$-\frac{\partial T}{\partial t} = \frac{\partial T^*}{\partial t} \quad \text{and} \quad -\frac{\partial T}{\partial q_i} = \frac{\partial T^*}{\partial q_i} \quad i = 1, \ldots, n. \tag{6.5}$$

And finally, for the transformed variables, the transform gives the general result that

$$\frac{\partial y^{(s)}}{\partial \xi_i} = -x_i \qquad i = 1, \ldots, s, \tag{6.6}$$

yielding

$$\dot{q}_i = \frac{\partial T}{\partial p_i} \qquad i = 1, \ldots, s. \tag{6.7}$$

This equation accords with the definition of the flows of kinetic stores.

Equations (6.1), (6.5) and (6.7) are the desired results of the Legendre transform that are used to transform Lagrange's equation into Hamilton's equation.

6.2 Derivation

Consider Lagrange's equation in multiplier form, given by

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_j} - \frac{\partial T^*}{\partial q_j} + \frac{\partial V}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} + \sum_{k=1}^{m_1} \frac{\partial \phi_k}{\partial q_j}\kappa_k + \sum_{k=1}^{m_2} b_{kj}\mu_k = Q_j, \tag{6.8}$$

where $j = 1, \ldots, n$. Among these n equations are s equations corresponding to the kinetic flow coordinates $\dot{q}^s = (\dot{q}_1, \ldots, \dot{q}_s)$. For these s equations, direct substitution from (6.1) and (6.5) into Lagrange's equation yields

$$\dot{p}_j + \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} + \sum_{k=1}^{m_1} \frac{\partial \phi_k}{\partial q_j}\kappa_k + \sum_{k=1}^{m_2} b_{kj}\mu_k = Q_j, \tag{6.9}$$

where $j = 1, \ldots, s$. For the r nonkinetic flow variables $\dot{q}^r = (\dot{q}_{s+1}, \ldots, \dot{q}_n)$ a kinetic momentum variable p_j is not defined since these flows are not transformed. Hence for the r equations corresponding to the nonkinetic flows $\dot{q}^r$ of a system, the result of the variable transformation is identical to (6.9) excluding the first term, that is,

$$\frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} + \sum_{k=1}^{m_1} \frac{\partial \phi_k}{\partial q_j}\kappa_k + \sum_{k=1}^{m_2} b_{kj}\mu_k = Q_j, \tag{6.10}$$

where $j = s+1, \ldots, n$. These two equations comprise a set of n equations with $s+n$ state variables, that is, $p \in \mathcal{R}^s$ and $q \in \mathcal{R}^n$. An additional set of s equations is given by (6.7). Together with the algebraic constraint equations, these transformed differential equations comprise the differential–algebraic form of Hamilton's equation called herein the *Hamiltonian DAE*, given by

$$\begin{aligned}
\dot{q}_j &= \frac{\partial T}{\partial p_j} \qquad j = 1, \ldots, s \\
\dot{p}_j + \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} + \sum_{k=1}^{m_1} \frac{\partial \phi_k}{\partial q_j}\kappa_k + \sum_{k=1}^{m_2} b_{kj}\mu_k &= Q_j \qquad j = 1, \ldots, s \\
\frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} + \sum_{k=1}^{m_1} \frac{\partial \phi_k}{\partial q_j}\kappa_k + \sum_{k=1}^{m_2} b_{kj}\mu_k &= Q_j \qquad j = s+1, \ldots, n \\
\phi_k(q,t) &= 0 \qquad k = 1, \ldots, m_1 \\
\psi_k(\dot{q}, q, t) &= 0 \qquad k = 1, \ldots, m_2 \\
\gamma_k(e^\gamma, \dot{q}, q, t) &= 0 \qquad k = 1, \ldots, m_3.
\end{aligned} \tag{6.11}$$

Using the following definitions, these equations can be written in vector form. The displacement vector q is partitioned into inertial displacements $q^s := (q_1, \ldots, q_s)$ and noninertial displacements $q^r := (q_{s+1}, \ldots, q_n)$ such that

$$q(t) = \begin{bmatrix} q^s(t) \\ q^r(t) \end{bmatrix}. \tag{6.12}$$

Let the multiplier vector λ and the constraint vector C be given by

$$\lambda(t) := \begin{bmatrix} \kappa(t) \\ \mu(t) \end{bmatrix} \qquad C(\dot{q}, q, t) := \begin{bmatrix} \Phi(q,t) \\ \Psi(\dot{q}, q, t) \end{bmatrix}. \tag{6.13}$$

Let C_s be the Jacobian of the kinematic constraint vector C with respect to the inertial state variables $(q^s, \dot{q}^s)$, and C_r be the Jacobian of C with respect to the noninertial state variables $(q^r, \dot{q}^r)$ given by

$$C_s(q,t) := \begin{bmatrix} \Phi_{q^s}(q,t) \\ \Psi_{\dot{q}^s}(q,t) \end{bmatrix} \qquad C_r(q,t) := \begin{bmatrix} \Phi_{q^r}(q,t) \\ \Psi_{\dot{q}^r}(q,t) \end{bmatrix}, \tag{6.14}$$

Then the Hamiltonian DAE is given in vector form by

$$\begin{aligned} \dot{q}^s &= \nabla_p T \\ \dot{p} + \nabla_{q^s} T + \nabla_{q^s} V + \nabla_{\dot{q}^s} D + C_s^T \lambda &= Q^s \\ \nabla_{q^r} T + \nabla_{q^r} V + \nabla_{\dot{q}^r} D + C_r^T \lambda &= Q^r \\ C &= 0 \\ \Gamma &= 0. \end{aligned} \tag{6.15}$$

The Hamiltonian DAE is characterized by first-order differential equations in terms of momentum and displacement variables (p, q). The differential equations are not necessarily explicit because of the dissipation term $\nabla_{\dot{q}} D$. Consequently the Hamiltonian DAE given by (6.15) is not in a convenient form for systematic formulation and numerical solution of the equations of motion.

6.3 Semi-explicit form

For systematic formulation and numerical solution of the equations of motion, it is useful to formulate the Hamiltonian DAE such that the differential equations are explicit in

$(\dot{p}, \dot{q})$, and that the constraints are strictly algebraic, that is, free of derivatives altogether. Such a formulation is presented in this section. The resulting equations are *semi-explicit nonlinear DAEs*, which have the general form

$$\begin{aligned} \dot{x}(t) &= \mathcal{F}_1(x(t), z(t), t) \\ 0 &= \mathcal{F}_2(x(t), z(t), t). \end{aligned} \tag{6.16}$$

The distinguishing characteristic of this DAE is the the differential equations are explicit in $\dot{x}$ and that $\dot{z}$ does not appear. Equations written in this form "have properties which may be exploited by some numerical algorithms" [6]. Moreover, the semi-explicit form of the Hamiltonian DAE has the added property that it is suitable for systematic formulation of the equations of motion.

Let $\dot{q} = f$, where flows f are partitioned into inertial flows $f^s := (f_1, \ldots, f_s)$ and noninertial flows $f^r := (f_{s+1}, \ldots, f_n)$ such that

$$f(t) = \begin{bmatrix} f^s(t) \\ f^r(t) \end{bmatrix}. \tag{6.17}$$

The Hamiltonian DAE is given by

$$\begin{aligned} \dot{q}_j &= f_j & j &= 1, \ldots, n \\ \dot{p}_j + \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} + \frac{\partial D}{\partial f_j} + \sum_{k=1}^{m_1} \frac{\partial \phi_k}{\partial q_j} \kappa_k + \sum_{k=1}^{m_2} b_{kj} \mu_k &= Q_j & j &= 1, \ldots, s \\ \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} + \frac{\partial D}{\partial f_j} + \sum_{k=1}^{m_1} \frac{\partial \phi_k}{\partial q_j} \kappa_k + \sum_{k=1}^{m_2} b_{kj} \mu_k &= Q_j & j &= s+1, \ldots, n \\ f_j &= \frac{\partial T}{\partial p_j} & j &= 1, \ldots, s \\ \phi_k(q, t) &= 0 & k &= 1, \ldots, m_1 \\ \psi_k(f, q, t) &= 0 & k &= 1, \ldots, m_2 \\ \gamma_k(e^\gamma, f, q, t) &= 0 & k &= 1, \ldots, m_3. \end{aligned} \tag{6.18}$$

Let the multiplier vector λ, the constraint vector C, and the Jacobians C_s and C_r be

given by

$$\begin{aligned} \lambda(t) &:= \begin{bmatrix} \kappa(t) \\ \mu(t) \end{bmatrix} & C_s(q,t) &:= \begin{bmatrix} \Phi_{q^s}(q,t) \\ \Psi_{f^s}(q,t) \end{bmatrix} \\ C(f,q,t) &:= \begin{bmatrix} \Phi(q,t) \\ \Psi(f,q,t) \end{bmatrix} & C_r(q,t) &:= \begin{bmatrix} \Phi_{q^r}(q,t) \\ \Psi_{f^r}(q,t) \end{bmatrix}. \end{aligned} \tag{6.19}$$

Then the Hamiltonian DAE is given in vector form by

$$\begin{aligned} \dot{q} &= f \\ \dot{p} + \nabla_{q^s} T + \nabla_{q^s} V + \nabla_{f^s} D + C_s^T \lambda &= Q^s \\ \nabla_{q^r} T + \nabla_{q^r} V + \nabla_{f^r} D + C_r^T \lambda &= Q^r \\ f^s &= \nabla_p T \\ C &= 0 \\ \Gamma &= 0. \end{aligned} \tag{6.20}$$

The first $n+s$ equations are ODE's explicit in $(\dot{p}, \dot{q})$ and the remaining $n+m_o+m_3$ equations are algebraic in $(p, q, f, \lambda, e^\gamma)$. Collecting the differential equations and algebraic equations into two groups yields

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} f \\ Q^s - \nabla_{q^s} T - \nabla_{q^s} V - \nabla_{f^s} D - C_s^T \lambda \end{bmatrix} \\ 0 &= \begin{bmatrix} Q^r - \nabla_{q^r} T - \nabla_{q^r} V - \nabla_{f^r} D - C_r^T \lambda \\ \nabla_p T - f^s \\ C \\ \Gamma \end{bmatrix}. \end{aligned} \tag{6.21}$$

This DAE is in semi-explicit form, that is,

$$\begin{aligned} \dot{x} &= \mathcal{F}_1(x, z, t) \\ 0 &= \mathcal{F}_2(x, z, t), \end{aligned} \tag{6.22}$$

where

$$
\begin{aligned}
x(t) &:= \begin{bmatrix} p(t) & q(t) \end{bmatrix}^T \\
z(t) &:= \begin{bmatrix} f(t) & \lambda(t) & e^{\gamma}(t) \end{bmatrix}^T,
\end{aligned} \tag{6.23}
$$

and $\mathcal{F}_1$ and $\mathcal{F}_2$ are given by the vectors on the right-hand side of (6.21). This DAE is comprised of $2n+s+m_o+m_3$ equations in the same number of unknowns, namely $(q, f, p, \lambda, e^{\gamma})$.

Like the descriptor form of the Lagrangian DAE, the semi-explicit form of the Hamiltonian DAE is suitable for systematic formulation of the equations of motion for multidiscipline systems. However, this DAE too is generally index-3 or higher. The same strategies outlined in Chapter 8 for integrating an index-3 Lagrangian DAE can be used to integrate an index-3 Hamiltonian DAE.

EXAMPLE 6.1 A two-link robot arm is shown in Fig. 6.2 with displacement coordinates q and momentum coordinates p assigned for a Hamiltonian formulation of the equations of motion.

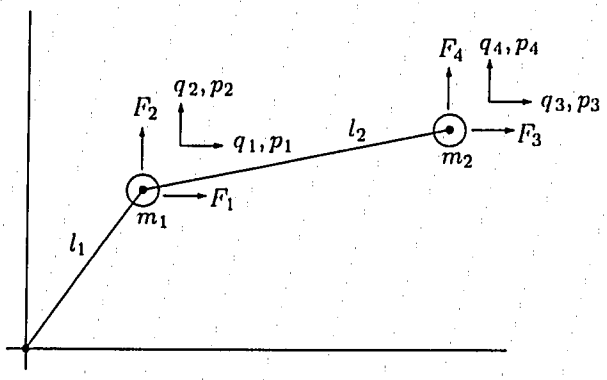


Figure 6.2. A two-link robot arm using a Hamiltonian formulation.

The energy functions are given by

$$
\begin{aligned}
T &= \frac{p_1^2 + p_2^2}{2m_1} + \frac{p_3^2 + p_4^2}{2m_2} \\
D &= V = 0.
\end{aligned} \tag{6.24}
$$

Two holonomic constraints are given by

$$\begin{aligned} \phi_1 &:= q_1^2 + q_2^2 - l_1^2 = 0 \\ \phi_2 &:= (q_3 - q_1)^2 + (q_4 - q_2)^2 - l_2^2 = 0. \end{aligned} \tag{6.25}$$

The virtual work of the force sources is given by

$$\delta W = F_1\,\delta q_1 + F_2\,\delta q_2 + F_3\,\delta q_3 + F_4\,\delta q_4, \tag{6.26}$$

hence $Q_1 = F_1$, $Q_2 = F_2$, $Q_3 = F_3$ and $Q_4 = F_4$. The Hamiltonian DAE for this example is given by

$$\begin{aligned} \dot{q} &= \nabla_p T \\ \dot{p} + \Phi_q^T \kappa &= Q_u \\ \Phi &= 0, \end{aligned} \tag{6.27}$$

which yields the following set of differential–algebraic equations of motion

$$\begin{aligned} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} &= \begin{bmatrix} p_1/m_1 \\ p_2/m_1 \\ p_3/m_2 \\ p_4/m_2 \end{bmatrix} \\ \begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_3 \\ \dot{p}_4 \end{bmatrix} + \begin{bmatrix} 2q_1 & 2(q_1 - q_3) \\ 2q_2 & 2(q_2 - q_4) \\ 0 & 2(q_3 - q_1) \\ 0 & 2(q_4 - q_2) \end{bmatrix} \begin{bmatrix} \kappa_1 \\ \kappa_2 \end{bmatrix} &= \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix} \\ \begin{bmatrix} q_1^2 + q_2^2 - l_1^2 \\ (q_3 - q_1)^2 + (q_4 - q_2)^2 - l_2^2 \end{bmatrix} &= 0. \end{aligned} \tag{6.28}$$

It happens that this DAE is semi-explicit. Nevertheless, for the sake of consistency with the idea of a systematic formulation of the equations of motion, the state variable $f = \dot{q}$

is introduced to obtain

$$
\begin{aligned}
\dot{q} &= f \\
\dot{p} &= \begin{bmatrix} F_1 - 2q_1\kappa_1 + 2(q_3 - q_1)\kappa_2 \\ F_2 - 2q_2\kappa_1 + 2(q_4 - q_2)\kappa_2 \\ F_3 + 2(q_1 - q_3)\kappa_2 \\ F_4 + 2(q_4 - q_2)\kappa_2 \end{bmatrix} \\
0 &= \begin{bmatrix} f_1 - p_1/m_1 \\ f_2 - p_2/m_1 \\ f_3 - p_3/m_2 \\ f_4 - p_4/m_2 \\ q_1^2 + q_2^2 - l_1^2 \\ (q_3 - q_1)^2 + (q_4 - q_2)^2 - l_2^2 \end{bmatrix}
\end{aligned}
\tag{6.29}
$$

This is a set of 14 equations in semi–explicit form with unknowns (q, f, p, κ). This DAE is comprised of eight explicit ODE's and six algebraic equations. $\diamond$

EXAMPLE 6.2 An electromagnetic suspension is shown in Fig. 6.3. The voltage source $e^s(t)$ can be modulated in response to the disturbance force $F^s(t)$ such that a desired air gap u_1 can be maintained. Coil inductance L is a function of the air gap given by

$$
L(u_1) = \frac{\gamma_o}{\gamma_1 + u_1}, \tag{6.30}
$$

where γ_o and γ_1 are known parameters. Coordinates $(u_1, u_2, u_3, \varrho_1, \varrho_2)$ are assigned as shown in Fig. 6.3.

The energy functions are given by

$$
\begin{aligned}
T &= \frac{\varrho_1^2}{2m} + \frac{\varrho_2^2}{2L} = \frac{\varrho_1^2}{2m} + \frac{\gamma_1 + u_1}{2\gamma_o}\varrho_2^2 \\
D &= \tfrac{1}{2}R\dot{u}_3^2 \\
V &= 0.
\end{aligned}
\tag{6.31}
$$

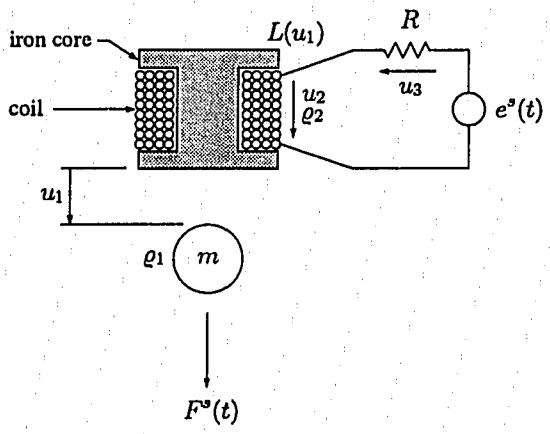


Figure 6.3. An electromagnetic suspension using a Hamiltonian formulation.

The single constraint is a holonomic flow constraint given by

$$\psi_1 := \dot{u}_2 - \dot{u}_3 = 0. \tag{6.32}$$

It is not necessary to integrate this flow constraint equation to impose a constraint on displacements (u_2, u_3) since neither of these two displacements represent a physical accumulation of charge. The virtual work of the source efforts is given by

$$\delta W = F^s(t)\,\delta u_1 + e^s(t)\,\delta u_3, \tag{6.33}$$

hence $Q_1 = F^s(t)$, $Q_2 = 0$ and $Q_3 = e^s(t)$. The Hamiltonian DAE is given by

$$\begin{aligned}
\dot{u}_1 &= \frac{\varrho_1}{m} \\
\dot{u}_2 &= \frac{\gamma_1 + u_1}{\gamma_o}\varrho_2 \\
\dot{\varrho}_1 + \frac{\varrho_2^2}{2\gamma_o} &= F^s(t) \\
\dot{\varrho}_2 + \lambda &= 0 \\
R\dot{u}_3 - \lambda &= e^s(t) \\
\dot{u}_2 - \dot{u}_3 &= 0.
\end{aligned} \tag{6.34}$$

Letting $q = u$, $p = \varrho$ and $f = \dot{u}$, the equations of motion are given in semi-explicit form

by

$$\begin{aligned}
\dot{q} &= f \\
\dot{p}_1 &= F^s - \frac{p_2^2}{2\gamma_o} \\
\dot{p}_2 &= -\lambda \\
0 &= e^s - Rf_3 + \lambda \\
0 &= f_1 - \frac{p_1}{m} \\
0 &= f_2 - \frac{\gamma_1 + q_1}{\gamma_o} p_2 \\
0 &= f_2 - f_3.
\end{aligned} \tag{6.35}$$

This is a set of nine equations with unknowns $(q_1, q_2, q_3, f_1, f_2, f_3, p_1, p_2, \lambda)$. This DAE is comprised of five ODE's and four algebraic equations. $\diamond$

6.4 Underlying ODE and the canonical form

If the system constraints are such that an explicit solution can be obtained for the multipliers λ and the implicit efforts e^γ, and assuming that the inertial coordinates (p, q^s) are sufficient to describe the motion of the system, then all algebraic equations can be eliminated from the DAE. The resulting differential equations are in implicit form, that is,

$$\mathcal{F}(\dot{x}, x, t) = 0, \tag{6.36}$$

where

$$x(t) := \begin{bmatrix} p(t) & q(t) \end{bmatrix}^T, \tag{6.37}$$

and $\mathcal{F}$ is a nonlinear function of the state variables, time and the parameters of the system. In this section, a procedure for obtaining this ODE from the Hamiltonian DAE is demonstrated for a holonomic system.

The holonomic constraint equation $\Phi(q, t) = 0$ is twice differentiated with respect to

time to obtain (5.136), repeated here

$$\Phi_q \dot{f} \;=\; \Upsilon_1, \tag{6.38}$$

where

$$\Upsilon_1(f,q,t) \;:=\; -\left(\Phi_q f + \Phi_t\right)_q f - \left(\Phi_q f + \Phi_t\right)_t. \tag{6.39}$$

Assuming that the vector of constituitive laws of the kinetic stores has the general form $f = J(p,q,t)$, then $\dot{f}$ is given by

$$\dot{f} = J_p \dot{p} + J_q \dot{q} + J_t, \tag{6.40}$$

which is substituted in (6.38) to obtain

$$\Phi_q \left(J_p \dot{p} + J_q \dot{q} + J_t\right) = \Upsilon_1. \tag{6.41}$$

Rearranging yields

$$\begin{aligned} \Phi_q J_p \dot{p} \;&=\; \underbrace{\Upsilon_1 - \Phi_q \left(J_q \dot{q} + J_t\right)}_{\Upsilon_2} \\ \Phi_q W \dot{p} \;&=\; \Upsilon_2, \end{aligned} \tag{6.42}$$

where $W := J_p$ takes the place here of M^{-1} in the Lagrangian underlying ODE. From the Hamiltonian DAE,

$$\dot{p} + \Phi_q^T \lambda \;=\; \underbrace{Q - \nabla_q T - \nabla_q V - \nabla_{\dot{q}} D}_{\Upsilon_3}. \tag{6.43}$$

Premultiplying by $\Phi_q W$ yields

$$\Phi_q W \dot{p} + \Phi_q W \Phi_q^T \lambda \;=\; \Phi_q W \Upsilon_3. \tag{6.44}$$

Substituting for $\Phi_q W \dot{p}$ from (6.42) yields

$$\Upsilon_2 + \Phi_q W \Phi_q^T \lambda = \Phi_q W \Upsilon_3. \tag{6.45}$$

Rearranging and assuming the inverse of $\Phi_q W \Phi_q^T$ exists, this equation can be solved for λ to obtain

$$\lambda = \left(\Phi_q W \Phi_q^T\right)^{-1} (\Phi_q W \Upsilon_3 - \Upsilon_2) . \tag{6.46}$$

Substituting this expression for λ into (6.43) yields the differential equations of motion given by

$$\begin{aligned} \dot{q} - \nabla_p T &= 0 \\ \dot{p} + \Phi_q^T \left(\Phi_q W \Phi_q^T\right)^{-1} (\Phi_q W \Upsilon_3 - \Upsilon_2) - \Upsilon_3 &= 0. \end{aligned} \tag{6.47}$$

This equation, which is the Hamiltonian analogue to the Lagrangian underlying ODE, is called herein the *Hamiltonian underlying ODE*. Letting $x = (p, q)$, this set of equations is an implicit ODE of the form

$$\mathcal{F}(\dot{x}, x, t) = 0, \tag{6.48}$$

which was to be shown.

In the special case that an independent set of inertial coordinates (p, q) can be selected to represent the motion of the system, the underlying ODE reduces to

$$\begin{aligned} \dot{q} &= \nabla_p T \\ \dot{p} &= Q - \nabla_q T - \nabla_q V - \nabla_{\dot{q}} D. \end{aligned} \tag{6.49}$$

Let the general nonpotential effort Q^n represent the efforts of sources and dissipators such that $Q^n = Q - \nabla_{\dot{q}} D$. Defining the Hamiltonian energy function $H(p, q, t) := T(p, q, t) + V(q, t)$, the underlying ODE is given by

$$\begin{aligned} \dot{q} &= \nabla_p H \\ \dot{p} &= -\nabla_q H + Q^n. \end{aligned} \tag{6.50}$$

In this form, Hamilton's equation is widely used in the study of control and stability of dynamic systems. In the absence of nonpotential efforts, that is, $Q^n = 0$, this equation

is reduced to

$$\begin{aligned} \dot{q} &= \nabla_p H \\ \dot{p} &= -\nabla_q H, \end{aligned} \tag{6.51}$$

which is Hamilton's equation in *canonical form* [25]. For modeling and simulation of multidiscipline systems, this form of the equations of motion has limited utility, since the selection of coordinates is limited to an independent set of displacements and momenta among the kinetic stores only, and nonpotential efforts are excluded. For these reasons, modeling procedures based on the canonical form are not in common use.

6.5 Problem formulation

To obtain the equations of motion using the semi–explicit form of the Hamiltonian DAE, it is necessary for the analyst to assign coordinates (p^s, q^s, f^s) to inertial elements and coordinates (q^r, f^r) to noninertial elements. Expressions are developed in terms of these coordinates for the energy functions T, V and D, for the independent constraints C and Γ, and for the virtual work of the nonpotential efforts Q. Through manipulation of these expressions, the semi–explicit form (6.22) is obtained. The examples given in this chapter illustrate the process.

6.6 Comparison of two formulations

Both the Lagrangian DAE and the Hamiltonian DAE have the same information content regarding the dynamic behavior of a multidiscipline engineering system. The Hamiltonian DAE in semi–explicit form has s more equations and unknowns than the Lagrangian DAE in descriptor form. But the Hamiltonian DAE contains fewer *differential* equations to be solved — Hamilton's equation comprises $n+s$ ODE's compared to the $2n$ ODE's of Lagrange's equation — and in Hamilton's equation, all the ODE's are explicit since the Hamiltonian DAE contains no M matrix to be inverted in the course of a numerical solution as does the Lagrangian DAE. These characteristics of the two formulations

suggest that the Hamiltonian DAE may be more suitable than the Lagrangian DAE for numerical implementation. To paraphrase Troutman [41] regarding the canonical form,

> *In general, Hamilton's equations cannot have more than one solution* $(q(t), p(t))$ *with* (q_o, p_o) *prescribed, so that with any given Hamiltonian, the solution curves in* (q, p) *space* cannot *intersect. This makes them much more stable with respect to numerical integration schemes than those of Lagrange.*

One disadvantage of the Hamiltonian approach is that the analyst must distinguish between inertial and noninertial state variables, adding an additional bit of bookkeeping to the analyst's tasks. No such distinction is required in the Lagrangian formulation. A second disadvantage of the Hamiltonian form is that the set of state variables includes generalized momentum. It is likely that most engineers have less experience using momentum trajectories to describe the motion of a system than they have with flow trajectories to describe the motion of a system. Hence the analyst is faced with the difficulty of assigning momentum coordinates and interpreting the numerical result. The analyst is less likely to trust his or her assessment of whether or not a solution "makes sense" using the Hamiltonian formulation compared to the Lagrangian formulation.

Nevertheless, the assignment of momentum coordinates and interpretation of numerical results is a skill that improves with practice. The structure of the Hamiltonian DAE may prove to be sufficiently advantageous with respect to its numerical properties compared to the Lagrangian DAE that the inconvenience of unfamiliar coordinates is more than compensated for by improved numerical performance. A conclusive assessment can be made only after continued experience with the two formulations and robust numerical algorithms.

Lastly, it is noted by comparing the Lagrangian DAE (5.112) and the Hamiltonian DAE (6.15) that the two formulations are identical in the case of a system with no kinetic stores. In such a case, both DAEs reduce to a set of first-order differential equations in

$\dot{q}$ given by

$$\begin{aligned} \nabla_q V + \nabla_{\dot{q}} D + C_n^T \lambda &= Q \\ C &= 0 \\ \Gamma &= 0. \end{aligned} \tag{6.52}$$

EXAMPLE 6.3 (Comparison of Lagrangian and Hamiltonian DAEs) The equations of motion for an electromagnetic suspension are obtained in Lagrangian form in Ex. 5.10 and again in Hamiltonian form Ex. 6.2. The Lagrangian DAE is given by

$$\begin{aligned} \dot{q} &= f \\ \begin{bmatrix} m & 0 & 0 \\ 0 & \dfrac{\gamma_o}{\gamma_1 + q_1} & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{f} + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \lambda &= \begin{bmatrix} F^s - \dfrac{\gamma_o f_2^2}{2(\gamma_1 + q_1)^2} \\ \dfrac{\gamma_o f_1 f_2}{(\gamma_1 + q_1)^2} \\ e^s - R f_3 \end{bmatrix} \\ 0 &= f_2 - f_3, \end{aligned}$$

and the Hamiltonian DAE is given by

$$\begin{aligned} \dot{q} &= f \\ \dot{p}_1 &= F^s - \frac{p_2^2}{2\gamma_o} \\ \dot{p}_2 &= -\lambda \\ 0 &= e^s - R f_3 + \lambda \\ 0 &= f_1 - \frac{p_1}{m} \\ 0 &= f_2 - \frac{\gamma_1 + q_1}{\gamma_o} p_2 \\ 0 &= f_2 - f_3. \end{aligned}$$

Compared to the Lagrangian formulation, the Hamiltonian formulation contains two more equations and two more unknowns, but one less differential equation. Moreover,

the differential equations in the Hamiltonian formulation are all explicit. As a result, there is no M matrix to invert in the course of a numerical solution. Furthermore, the nonlinearities in the Hamiltonian formulation have a simpler form.

This example also illustrates the drawbacks of the Hamiltonian formulation. First, it is necessary to distinguish between inertial and noninertial flows, adding an additional bit of bookkeeping to the analyst's tasks. No such distinction is required in the Lagrangian formulation. Second, the solution of the Hamiltonian DAE includes trajectories of momentum. $\diamond$

Chapter 7

COMPLEMENTARY FORMS OF THE DAE OF MOTION

In this chapter are presented formulations of the DAE using two alternate or non–Lagrangian pairs of variables to describe the motion of a system. The co–Lagrangian DAE is formulated in terms of effort and momentum and the co–Hamiltonian DAE, like the Hamiltonian DAE, is formulated in terms of momentum and displacement. These formulations are duals to the Lagrangian and Hamiltonian DAEs.

7.1 The Co–Lagrangian DAE

7.1.1 Derivation

In this section, the complementary form of Lagrange's equation for multidiscipline systems is derived from the first law of thermodynamics. The variational form of the first law is given by

$$\delta E = \delta W, \tag{7.1}$$

where $E(\varrho, u) = T(\varrho) + V(u)$ and virtual work is given by $\sum e^n \, \delta u$, or equivalently $\sum f^n \, \delta \varrho$. Applying the δ operator to T and V and utilizing the flow representation of virtual work yields the following differential–variational form of the first law,

$$\sum_{i=1}^{N} \frac{\partial T}{\partial \varrho_i} \, \delta \varrho_i + \sum_{i=1}^{N} \frac{\partial V}{\partial u_i} \, \delta u_i = \sum_{i=1}^{N} f_i^n \, \delta \varrho_i. \tag{7.2}$$

In the complementary form of Lagrange's equation, the variable pair $(\varrho, \dot{\varrho})$ is selected to represent the motion of the system. This choice requires that the term involving δu in the first law be replaced with an equivalent expression in terms of $\delta \varrho$. To this end, the work of the potential stores is considered in its two equivalent forms, namely

$$\sum_{i=1}^{N} f_i \, \delta \varrho_i = \sum_{i=1}^{N} e_i \, \delta u_i. \tag{7.3}$$

Substituting $f_i = \dot{u}_i$ and $e_i = -\partial V/\partial u_i$ yields

$$\sum_{i=1}^{N} \dot{u}\, \delta\varrho_i = -\sum_{i=1}^{N} \frac{\partial V}{\partial u_i}\, \delta u_i. \tag{7.4}$$

Potential displacements satisfy $u_i = -\partial V^*/\partial e_i$. Substituting this expression in (7.4) yields

$$-\sum_{i=1}^{N} \frac{d}{dt}\frac{\partial V^*}{\partial e_i}\, \delta\varrho_i = -\sum_{i=1}^{N} \frac{\partial V}{\partial u_i}\, \delta u_i, \tag{7.5}$$

or since $\dot{\varrho}_i = e_i$,

$$\sum_{i=1}^{N} \frac{d}{dt}\frac{\partial V^*}{\partial \dot{\varrho}_i}\, \delta\varrho_i = \sum_{i=1}^{N} \frac{\partial V}{\partial u_i}\, \delta u_i. \tag{7.6}$$

Substituting the left–hand term term into the first law and collecting terms yields

$$\sum_{i=1}^{N} \left(\frac{d}{dt}\frac{\partial V^*}{\partial \dot{\varrho}_i} + \frac{\partial T}{\partial \varrho_i} - f_i^n \right) \delta\varrho_i = 0. \tag{7.7}$$

If the virtual momenta $\delta\varrho_i$ are independent, then the coefficients vanish, yielding the *complement* (or dual) of the classical form of Lagrange's equation given by

$$\frac{d}{dt}\frac{\partial V^*}{\partial \dot{\varrho}_i} + \frac{\partial T}{\partial \varrho_i} = f_i^n \qquad i = 1, \ldots, N. \tag{7.8}$$

It is assumed that the nonpotential flows are known functions of time. This equation is given in vector form by

$$\tfrac{d}{dt}\nabla_{\dot{\varrho}} V^* + \nabla_{\varrho} T = f^n. \tag{7.9}$$

7.1.2 Reduced–order coordinates

If the set of configuration coordinates $\varrho = (\varrho_1, \ldots, \varrho_N)$ is transformed via a set of N transformation equations to obtain a reduced–order set of coordinates $p = (p_1, \ldots, p_n)$, then $\varrho_i = \varrho_i(p, t)$ and $\dot{\varrho}_i = \dot{\varrho}_i(\dot{p}, p, t)$. Consequently the potential coenergy function has the general form $V^* = V^*(\dot{\varrho}) = V^*(\dot{p}, p, t)$. The dependence of potential coenergy V^* on momentum p implies the dependence of potential energy V on momentum p, that is, $V^* = V^*(\dot{p}, p, t)$ implies $V = V(q, p, t)$.

The total energy stored in a system is given by $E(p,q,t) := T(p,t) + V(p,q,t)$. The variational form of the first law $\delta E = \delta W^{(n)}$ yields

$$\sum_{j=1}^{n} \frac{\partial T}{\partial p_j}\delta p_j + \sum_{j=1}^{n} \frac{\partial V}{\partial p_j}\delta p_j + \sum_{j=1}^{n} \frac{\partial V}{\partial q_j}\delta q_j = \sum_{j=1}^{n} f^n\, \delta p_j. \tag{7.10}$$

A term like $\partial E/\partial t$ does not appear since time t is not varied. As in (7.6), the work equivalence $\sum e\,\delta q = \sum f\,\delta p$ for potential stores yields

$$\sum_{j=1}^{n} \frac{\partial V}{\partial q_j}\delta q_j = \sum_{j=1}^{n} \frac{d}{dt}\frac{\partial V^*}{\partial \dot{p}_j}\delta p_j. \tag{7.11}$$

This expression is to be substituted for the third term in (7.10).

Potential energy and coenergy satisfy $V(q,p,t) + V^*(e,p,t) = -\sum eq$. Applying the variational operator to both sides of this equation yields

$$\sum_{j=1}^{n} \frac{\partial V}{\partial q_j}\delta q_j + \sum_{j=1}^{n} \frac{\partial V}{\partial p_j}\delta p_j + \sum_{j=1}^{n} \frac{\partial V^*}{\partial e_j}\delta e_j + \sum_{j=1}^{n} \frac{\partial V^*}{\partial p_j}\delta p_j = -\sum_{j=1}^{n} e_j\,\delta q_j - \sum_{j=1}^{n} q_j\,\delta e_j. \tag{7.12}$$

Collecting terms yields

$$\sum_{j=1}^{n} \left(\frac{\partial V}{\partial q_j} + e_j\right)\delta q_j + \sum_{j=1}^{n} \left(\frac{\partial V^*}{\partial e_j} + q_j\right)\delta e_j + \sum_{j=1}^{n} \left(\frac{\partial V}{\partial p_j} + \frac{\partial V^*}{\partial p_j}\right)\delta p_j = 0. \tag{7.13}$$

By definition, first two parenthetic terms vanish, yielding

$$\sum_{j=1}^{n} \left(\frac{\partial V}{\partial p_j} + \frac{\partial V^*}{\partial p_j}\right)\delta p_j = 0, \tag{7.14}$$

which is an energy equation of the form $\langle f, \delta p\rangle = 0$, implying that the partial derivative terms represent components of flow. These flows are designated potential flows f^v and f^{v^*} such that

$$\sum_{j=1}^{n} \left(f^v + f^{v^*}\right)\delta p_j = 0. \tag{7.15}$$

It follows that

$$\sum_{j=1}^{n} \frac{\partial V}{\partial p_j}\delta p_j = -\sum_{j=1}^{n} \frac{\partial V^*}{\partial p_j}\delta p_j. \tag{7.16}$$

This expression is to be substituted for the second term in (7.10).

Nonpotential flows can be classified as source flows f^s which are assumed to be known functions of time, dissipative flows f^d which satisfy the definition of co-content, that is,

$$f_j^d = -\frac{\partial G}{\partial \dot{p}_j}, \tag{7.17}$$

and all other nonpotential flows, called implicit flows f^γ, such that $f^n = f^s + f^\gamma + f^d$. Let the j^{th}-component of the sum of flows $f^s + f^\gamma$ be designated P_j, that is,

$$P_j := f_j^s + f_j^\gamma. \tag{7.18}$$

The vector P is a generalized flow vector in the same sense as Q is a generalized effort vector in Lagrange's equation. Then the virtual work of the nonpotential flows is given by

$$\delta W^{(n)} = \sum_{j=1}^{n} f_j^n \, \delta p_j = \sum_{j=1}^{n} \left(P_j - \frac{\partial G}{\partial \dot{p}_j} \right) \delta p_j, \tag{7.19}$$

which is to be substituted for the right-hand side of (7.10).

Making the aforementioned substitutions into (7.10) and collecting terms yields the reduced-order, virtual-work form of the complement of Lagrange's equation, given by

$$\sum_{j=1}^{n} \left(\frac{d}{dt} \frac{\partial V^*}{\partial \dot{p}_j} - \frac{\partial V^*}{\partial p_j} + \frac{\partial T}{\partial p_j} + \frac{\partial G}{\partial \dot{p}_j} - P_j \right) \delta p_j = 0. \tag{7.20}$$

If the virtual momenta δp_j are independent, then the coefficients of δp_j in (7.20) must vanish. In such a case the coordinates $(p_1, \ldots, p_n)$ are generalized coordinates, that is, a coordinate set of minimum dimension. The vanishing of the coefficients of δp yields the generalized-coordinate form of the complement of Lagrange's equation, given by

$$\frac{d}{dt} \frac{\partial V^*}{\partial \dot{p}_j} - \frac{\partial V^*}{\partial p_j} + \frac{\partial T}{\partial p_j} + \frac{\partial G}{\partial \dot{p}_j} = P_j \qquad j = 1, \ldots, n. \tag{7.21}$$

This equation is suitable for obtaining the equations of motion of unconstrained systems. If the nonpotential flow P contains m_3 unknown implicit flows f^γ, then constraint equations

$$\bar{\gamma}_k(f^\gamma, \dot{p}, p, t) = 0 \qquad k = 1, \ldots, m_3 \tag{7.22}$$

are required to solve the equations of motion. These equations are dual to the implicit effort constraints γ of Lagrange's equation. In vector form the set of constraints is given by

$$\tilde{\Gamma}(f^{\gamma}, \dot{p}, p, t) := \begin{bmatrix} \tilde{\gamma}_1(e^{\gamma}, \dot{p}, p, t) \\ \vdots \\ \tilde{\gamma}_{m_3}(f^{\gamma}, \dot{p}, p, t) \end{bmatrix}, \tag{7.23}$$

and the DAE is given in vector form by

$$\begin{aligned} \tfrac{d}{dt}\nabla_{\dot{p}}V^* - \nabla_p V^* + \nabla_p T + \nabla_{\dot{p}} G &= P_p \\ \tilde{\Gamma} &= 0. \end{aligned}$$

7.1.3 Multipliers

For systems subject to constraints on momentum p given by

$$\tilde{\phi}_k(p,t) = 0 \qquad k = 1, \ldots, m_1, \tag{7.24}$$

and constraints on effort $\dot{p}$ given by

$$\tilde{\psi}_k(\dot{p}, p, t) := \sum_{j=1}^{n} \tilde{b}_{kj}(p,t)\,\dot{p}_j + \tilde{b}_k(p,t) = 0 \qquad k = 1, \ldots, m_2. \tag{7.25}$$

the virtual momenta are not independent. The momentum constraints $\tilde{\phi} = 0$ are dual to the Lagrangian holonomic constraints and the effort constraints $\tilde{\psi} = 0$ are dual to the Pfaffian flow constraints. These constraints are given in vector form by

$$\tilde{\Phi}(p,t) := \begin{bmatrix} \tilde{\psi}_1(p,t) \\ \vdots \\ \tilde{\psi}_{m_1}(p,t) \end{bmatrix} = 0 \tag{7.26}$$

$$\tilde{\Psi}(\dot{p}, p, t) := \begin{bmatrix} \tilde{\psi}_1(\dot{p}, p, t) \\ \vdots \\ \tilde{\psi}_{m_2}(\dot{p}, p, t) \end{bmatrix} = \tilde{B}\,\dot{p} + \tilde{b} = 0. \tag{7.27}$$

The virtual momenta are required to comply with these constraints, that is,

$$\tilde{\Phi}_p \, \delta p = 0 \qquad \text{and} \qquad \tilde{B} \, \delta p = 0. \tag{7.28}$$

Invoking the multiplier theorem, the constraints are appended to the complement of Lagrange's equation to obtain

$$\sum_{j=1}^{n} \left(\frac{d}{dt}\frac{\partial V^*}{\partial \dot{p}_j} - \frac{\partial V^*}{\partial p_j} + \frac{\partial T}{\partial p_j} + \frac{\partial G}{\partial \dot{p}_j} + \sum_{k=1}^{m_1} \frac{\partial \tilde{\phi}_k}{\partial p_j}\kappa_k + \sum_{k=1}^{m_2} \tilde{b}_{kj}\mu_k - P_j \right) \delta p_j = 0, \tag{7.29}$$

where $j = 1, \ldots, n$. By the multiplier theorem, this equation is rendered true for *all* $\delta p_j \in \mathcal{R}^n$. Thus the coefficient of each δp_j must vanish, yielding

$$\frac{d}{dt}\frac{\partial V^*}{\partial \dot{p}_j} - \frac{\partial V^*}{\partial p_j} + \frac{\partial T}{\partial p_j} + \frac{\partial G}{\partial \dot{p}_j} + \sum_{k=1}^{m_1} \frac{\partial \tilde{\phi}_k}{\partial p_j}\kappa_k + \sum_{k=1}^{m_2} \tilde{b}_{kj}\mu_k = P_j, \tag{7.30}$$

where $j = 1, \ldots, n$. This is the *multiplier form* of the complement of Lagrange's equation for constrained systems. This is a set of n equations with $n+m_1+m_2+m_3$ unknowns $(p, \kappa, \mu, f^\gamma)$.

The momentum constraints, the Pfaffian effort constraints and the implicit flow constraints provide the additional equations necessary to determine the motion of a system. The constraint equations $\tilde{\phi} = 0$ and $\tilde{\psi} = 0$ are added to the equations of motion to solve for the unknown multipliers κ and μ. The implicit flow constraint equations $\tilde{\gamma} = 0$ are added to the equations of motion to solve for the implicit flows $f^\gamma \in P$. The complement of Lagrange's equation plus the algebraic constraint equations comprise an $n+m_1+m_2+m_3$ set of differential–algebraic equations called herein the *co–Lagrangian DAE*, given by

$$\begin{aligned}
\frac{d}{dt}\frac{\partial V^*}{\partial \dot{p}_j} - \frac{\partial V^*}{\partial p_j} + \frac{\partial T}{\partial p_j} + \frac{\partial G}{\partial \dot{p}_j} + \sum_{k=1}^{m_1} \frac{\partial \tilde{\phi}_k}{\partial p_j}\kappa_k \;+\; \sum_{k=1}^{m_2} \tilde{b}_{kj}\mu_k = P_j \quad j = 1, \ldots, n \\
\tilde{\phi}_k(p,t) \;=\; 0 \qquad k = 1, \ldots, m_1 \\
\tilde{\psi}_k(\dot{p}, p, t) \;=\; 0 \qquad k = 1, \ldots, m_2 \\
\tilde{\gamma}_k(f^\gamma, \dot{p}, p, t) \;=\; 0 \qquad k = 1, \ldots, m_3.
\end{aligned} \tag{7.31}$$

Let the multipliers κ and μ be adjoined into a single vector of multipliers λ such that

$$\lambda(t) := \begin{bmatrix} \kappa(t) \\ \mu(t) \end{bmatrix} \in \mathcal{R}^{m_o}, \tag{7.32}$$

and let the dynamic constraints $\tilde{\Phi}$ and $\tilde{\Psi}$ be adjoined into a single vector $\tilde{C}$ such that

$$\tilde{C}(\dot{p}, p, t) := \begin{bmatrix} \tilde{\Phi}(p, t) \\ \tilde{\Psi}(\dot{p}, p, t) \end{bmatrix} \in \mathcal{R}^{m_o}, \tag{7.33}$$

where the index $m_o = m_1 + m_2$ is the total number of dynamic constraints. Then the constraint Jacobian $\tilde{C}_n$ is defined as

$$\tilde{C}_n(p, t) := \begin{bmatrix} \tilde{\Phi}_p(p, t) \\ \tilde{B}(p, t) \end{bmatrix} \in \mathcal{R}^{m_o \times n}, \tag{7.34}$$

and the co-Lagrangian DAE is given by

$$\begin{aligned} \tfrac{d}{dt}\nabla_{\dot{p}} V^* - \nabla_p V^* + \nabla_p T + \nabla_{\dot{p}} G + \tilde{C}_n^T \lambda &= P \\ \tilde{C} &= 0 \\ \tilde{\Gamma} &= 0. \end{aligned} \tag{7.35}$$

7.1.4 Descriptor form

Carrying out the time derivative in the co-Lagrangian DAE and rearranging terms yields

$$\underbrace{\nabla_{\dot{p}}^2 V^*}_{A} \ddot{p} + \tilde{C}_n^T \lambda = \underbrace{P - (\nabla_{\dot{p}} V^*)_p \dot{p} - (\nabla_{\dot{p}} V^*)_t + \nabla_p T - \nabla_p V^* - \nabla_{\dot{p}} G}_{\Upsilon_0}. \tag{7.36}$$

Introducing $A(\dot{p}, p, t) \in \mathcal{R}^{n \times n}$ to represent the Hessian matrix $\nabla_{\dot{p}}^2 V^*$ and $\Upsilon_0(f^\gamma, \dot{p}, p, t) \in \mathcal{R}^n$ to represent the summation of flows on the right-hand side, the co-Lagrangian DAE is given by

$$\begin{aligned} A\ddot{p} + \tilde{C}_n^T \lambda &= \Upsilon_0 \\ \tilde{C} &= 0 \\ \tilde{\Gamma} &= 0. \end{aligned} \tag{7.37}$$

This formulation is the complement of Lagrange's equation of the first kind, characterized by second–order differential equations in terms of a single differential variable, momentum p. Matrix A is so called since in mechanics it represents a flexibility or compliance matrix. Here it represents a matrix of compliance coefficients. In the general multidiscipline case, this matrix may be singular.

Letting $\dot{p} = e$, the co–Lagrangian DAE is given in descriptor form by

$$\begin{aligned} \dot{p} &= e \\ A\dot{e} + \tilde{C}_n^T \lambda &= \Upsilon_0 \\ \tilde{C} &= 0 \\ \tilde{\Gamma} &= 0, \end{aligned} \tag{7.38}$$

This is a set of $2n+m_o+m_3$ equations in the same number of unknowns $(p, e, \lambda, f^\gamma)$. The descriptor form of the co–Lagrangian DAE is characterized by first–order differential equations with state variables $p(t)$ and $e(t)$.

EXAMPLE 7.1 (Electrical circuit, co–Lagrangian formulation) The circuit shown in Fig. 7.1 is comprised of linear elements and two current sources $f_1^s(t)$ and $f_2^s(t)$. This problem, from [24], is formulated as a co–Lagrangian DAE in terms of independent momentum (flux–linkage) coordinates (p_1, p_2, p_3).

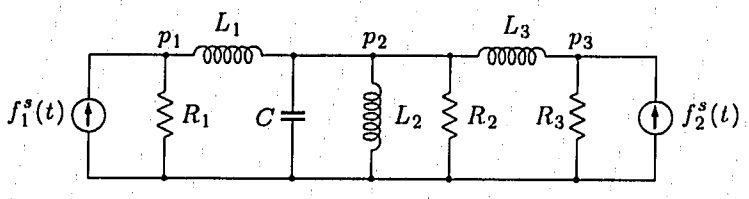


Figure 7.1. An electrical circuit with coordinates assigned for a co–Lagrangian formulation.

The state variables for the system are given by $(p_1, p_2, p_3, \dot{p}_1, \dot{p}_2, \dot{p}_3)$ where the $\dot{p}_j = e_j$

represent node voltages. Analysis yields

$$\begin{aligned} V^* &= \tfrac{1}{2}C_1\dot{p}_2^2 \\ G &= \frac{\dot{p}_1^2}{2R_1} + \frac{\dot{p}_2^2}{2R_2} + \frac{\dot{p}_3^2}{2R_3} \\ T &= \frac{(p_1-p_2)^2}{2L_1} + \frac{p_2^2}{2L_2} + \frac{(p_2-p_3)^2}{2L_3} \\ \delta W &= f_1^s(t)\,\delta p_1 + f_2^s(t)\,\delta p_2. \end{aligned} \tag{7.39}$$

Since the coordinates are independent, there are no constraint equations and multipliers are not used. The co-Lagrangian equations of motion for this problem are given by

$$\tfrac{d}{dt}\nabla_{\dot{p}}V^* + \nabla_{\dot{p}}G + \nabla_p T = P, \tag{7.40}$$

yielding the unconstrained equations of motion given by

$$\begin{aligned} \frac{\dot{p}_1}{R_1} + \frac{p_1-p_2}{L_1} &= f_1^s(t) \\ C_1\ddot{p}_2 + \frac{\dot{p}_2}{R_2} - \frac{p_1-p_2}{L_1} + \frac{p_2}{L_2} + \frac{p_2-p_3}{L_3} &= 0 \\ \frac{\dot{p}_3}{R_3} - \frac{p_2-p_3}{L_3} &= f_2^s(t). \end{aligned} \tag{7.41}$$

Letting $\dot{p} = e$, the equations of motion are given in descriptor form by

$$\begin{aligned} \dot{p} &= e \\ A\,\dot{e} &= \Upsilon_0, \end{aligned} \tag{7.42}$$

that is,

$$\begin{aligned} \dot{p} &= e \\ \begin{bmatrix} 0 & & \\ & C_1 & \\ & & 0 \end{bmatrix}\dot{e} &= \begin{bmatrix} f_1^s - \dfrac{e_1}{R_1} - \dfrac{p_1-p_2}{L_1} \\ -\dfrac{e_2}{R_2} + \dfrac{p_1-p_2}{L_1} - \dfrac{p_2}{L_2} - \dfrac{p_2-p_3}{L_3} \\ f_2^s - \dfrac{e_3}{R_3} + \dfrac{p_2-p_3}{L_3} \end{bmatrix}. \end{aligned} \tag{7.43}$$

This is a set of six equations with unknowns $(p_1, p_2, p_3, e_1, e_2, e_3)$. The compliance matrix A in this case is singular. The first three ODEs are explicit and the second three ODEs

are linearly implicit in the derivative $\dot{e}$. Thus the co–Lagrangian formulation has a structure similar to the Lagrangian formulation. See, for example, Ex. 5.10. $\diamond$

EXAMPLE 7.2 The mass–spring–damper system shown in Fig. 7.2 is excited by a velocity source $f^s(t)$. This problem, from [9], is formulated as a co–Lagrangian DAE. The state variables for the system are given by $(p_1, p_2, p_3, \dot{p}_1, \dot{p}_2, \dot{p}_3)$ where the $\dot{p}_j = e_j$ represent forces. The interconnections of the elements impose the constraints that the force acting on the mass, the spring force and the force of the damper are all equal.

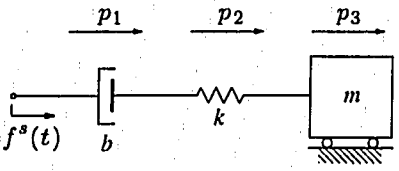


Figure 7.2. A mass–spring–damper system with coordinates assigned for a co–Lagrangian formulation.

Analysis yields

$$\begin{aligned} V^* &= \frac{\dot{p}_2^2}{2k} \\ G &= \frac{\dot{p}_1^2}{2b} \\ T &= \frac{p_3^2}{2m} \\ \tilde{\phi}_1 &:= p_1 - p_3 + p_{3o} = 0 \\ \tilde{\phi}_2 &:= p_2 - p_3 + p_{3o} = 0 \\ \delta W &= f^s(t)\,\delta p_1. \end{aligned} \tag{7.44}$$

The co–Lagrangian equations of motion for this problem are given by

$$\begin{aligned} \tfrac{d}{dt}\nabla_{\dot{p}} V^* + \nabla_p G + \nabla_p T + \tilde{C}_n^T \lambda &= P \\ \tilde{C} &= 0, \end{aligned} \tag{7.45}$$

yielding the co-Lagrangian DAE given by

$$\begin{aligned} \frac{\dot{p}_1}{b} + \lambda_1 &= f^s(t) \\ \frac{\ddot{p}_2}{k} + \lambda_2 &= 0 \\ \frac{p_3}{m} - \lambda_1 - \lambda_2 &= 0 \\ p_1 - p_3 + p_{3o} &= 0 \\ p_2 - p_3 + p_{3o} &= 0. \end{aligned} \tag{7.46}$$

Letting $\dot{p} = e$, the equations of motion are given in descriptor form by

$$\begin{aligned} \dot{p} &= e \\ \begin{bmatrix} 0 & & \\ & \frac{1}{k} & \\ & & 0 \end{bmatrix} \dot{e} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \lambda &= \begin{bmatrix} f^s - \frac{e_1}{b} \\ 0 \\ -\frac{p_3}{m} \end{bmatrix} \\ \begin{bmatrix} p_1 - p_3 + p_{3o} \\ p_2 - p_3 + p_{3o} \end{bmatrix} &= 0. \end{aligned} \tag{7.47}$$

This is a set of eight equations with unknowns $(p_1, p_2, p_3, e_1, e_2, e_3, \lambda_1, \lambda_2)$. This DAE is comprised of six ODE's and two algebraic constraint equations. The first three ODE's are explicit and the second three ODE's are linearly implicit in the derivative $\dot{e}$. The compliance matrix A is singular.

The co-Lagrangian formulation can be interpreted in physical terms as follows. In this example, an explicit solution for the multipliers is readily obtained, yielding

$$\begin{aligned} \lambda_1 &= f^s - \frac{\dot{p}_1}{b} \\ \lambda_1 &= -\frac{\ddot{p}_2}{k}, \end{aligned} \tag{7.48}$$

and from the constraints is obtained

$$p_3 = p_1 = p_2. \tag{7.49}$$

Letting $p = p_3$ represent the momentum of the mass and eliminating the multipliers from

the DAE yields a single second–order ODE given by

$$\frac{\ddot{p}}{k} + \frac{\dot{p}}{b} + \frac{p}{m} = f^s. \tag{7.50}$$

The physical interpretation of this equation is that the velocity of the mass p/m is the resultant of the velocity of the source less the velocities of the spring and damper.◇

EXAMPLE 7.3 The fluid system shown in Fig. 7.3 is comprised of fluid capacitors C_1 and C_2, fluid inertance I and resistance R. All elements have linear constituitive laws and the fluid is incompressible. The pump provides a prescribed flow rate and is modeled as a flow source $f^s(t)$. This problem, from [48], is formulated as a co–Lagrangian DAE. The state variables for the system are given by $(p_1, p_2, p_3, p_4, \dot{p}_1, \dot{p}_2, \dot{p}_3, \dot{p}_3)$ where the $\dot{p}_j = e_j$ represent pressures.

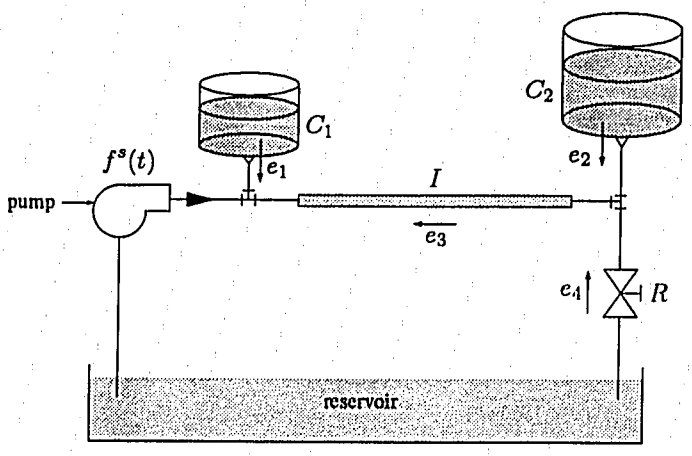


Figure 7.3. A fluid system with coordinates assigned for a co–Lagrangian formulation.

The energy functions are given by

$$\begin{aligned} V^* &= \tfrac{1}{2}C_1\dot{p}_1^2 + \tfrac{1}{2}C_2\dot{p}_2^2 \\ G &= \frac{\dot{p}_4^2}{2R} \\ T &= \frac{p_3^2}{2I}. \end{aligned} \tag{7.51}$$

The pressure drop e_3 is constrained such that $e_3 = e_1 - e_2$, that is to say, $\dot{p}_3 = \dot{p}_1 - \dot{p}_2$. The variable p_3 represents pressure momentum Γ since it is the coordinate assigned to a fluid kinetic store. Hence p_3 is a physically–based momentum variable and the rate constraint involving p_3 must be modeled in integrated form. Assuming zero initial conditions, this constraint is given by

$$\tilde{\phi} := p_1 - p_2 - p_3 = 0. \tag{7.52}$$

The second constraint, given by

$$\tilde{\psi} := \dot{p}_2 - \dot{p}_4 = 0, \tag{7.53}$$

can remain a rate equation since neither p_2 nor p_4 represents physical pressure momentum. Virtual work of the flow source is given by

$$\delta W = f^s(t)\,\delta p_1. \tag{7.54}$$

The co–Lagrangian equations of motion for this problem are given by

$$\begin{aligned} \tfrac{d}{dt}\nabla_{\dot{p}} V^* + \nabla_p G + \nabla_p T + \tilde{C}_n^T \lambda &= P \\ \tilde{C} &= 0, \end{aligned} \tag{7.55}$$

yielding the co–Lagrangian DAE given by

$$\begin{aligned} \begin{bmatrix} C_1\ddot{p}_1 \\ C_2\ddot{p}_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{\dot{p}_4}{R} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{p_3}{I} \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} &= \begin{bmatrix} f^s \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} p_1 - p_2 - p_3 \\ \dot{p}_2 - \dot{p}_4 \end{bmatrix} &= 0. \end{aligned} \tag{7.56}$$

Letting $\dot{p} = e$, the equations of motion are given in descriptor form by

$$\begin{aligned} \dot{p} &= e \\ \begin{bmatrix} C_1 & & & \\ & C_2 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} \dot{e} + \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \lambda &= \begin{bmatrix} f^s \\ 0 \\ -\frac{p_3}{I} \\ -\frac{e_4}{R} \end{bmatrix} \end{aligned} \tag{7.57}$$

$$\begin{bmatrix} p_1 - p_2 - p_3 \\ e_2 - e_4 \end{bmatrix} = 0.$$

This is a set of ten equations with unknowns $(p_1, p_2, p_3, p_4, e_1, e_2, e_3, e_4, \lambda_1, \lambda_2)$. This DAE is comprised of eight ODE's and two algebraic constraint equations. The first four ODE's are explicit and the second four ODE's are linearly implicit in the derivative $\dot{e}$. The compliance matrix A is singular. $\diamond$

7.1.5 Underlying ODE

If the system constraints are such that an explicit solution can be obtained for the multipliers λ and for the implicit flows f^γ, and assuming that the compliance matrix A is invertible, then the equations of motion can be reduced to the form

$$\begin{aligned} \dot{p} &= e \\ \dot{e} &= \mathcal{F}(p, e, t), \end{aligned} \tag{7.58}$$

where $\mathcal{F}$ is a nonlinear function of the state variables, time and the parameters of the system. This set of equations is the underlying ODE of the co-Lagrangian DAE.

The procedure for obtaining the underlying ODE is similar to the procedure used to obtain the underlying ODE of the Lagrangian DAE. (See Section 5.4.) The inverse of A and the inverse of $\tilde{\Phi}_p A^{-1} \tilde{\Phi}_p^T$ must exist. The constraints are differentiated twice, hence a numerical solution is not guaranteed to satisfy the momentum constraints $\tilde{\phi}_k$ or the effort constraints $\tilde{\psi}_k$. For these reasons, the underlying ODE formulation of the equations of motion is eschewed in favor of the descriptor form of the DAE for systematic formulation and numerical solution the equations of motion.

7.1.6 Problem formulation

To obtain the equations of motion using the descriptor form of the co-Lagrangian DAE, it is necessary for the analyst to assign coordinates (p, e) to system elements. Expressions are developed in terms of these coordinates for the energy functions V^*, T and G, for the

independent constraints $\tilde{\phi}$, $\tilde{\psi}$ and $\tilde{\gamma}$, and for the virtual work of the source flows f^s and implicit flows f^γ. The terms A, $\tilde{C}$, $\tilde{C}_n$, $\tilde{\Gamma}$ and Υ_0 are then generated by manipulating these expressions according to the definitions associated with the descriptor form. This procedure is illustrated by the previous examples.

7.2 The Co–Hamiltonian DAE

7.2.1 The Legendre transform

In the co–Lagrangian formulation, the motion of a system is described by a trajectory in a $2n$–dimensional state space, comprised of n momenta $p(t)$ and n efforts $\dot{p}(t)$. Let the number of these efforts associated with potential stores be denoted by $s \leq n$, dividing efforts into two categories: potential efforts $\dot{p}^s = (\dot{p}_1 \ldots, \dot{p}_s)$ that are associated with potential stores, and all other efforts $\dot{p}^r = (\dot{p}_{s+1} \ldots, \dot{p}_n)$, where the index $r := n - s$. The complementary form of Hamilton's equation is obtained from the complementary form of Lagrange's equation by applying the Legendre transformation the potential coenergy function $V^*(\dot{p})$ such that these s potential effort variables $\dot{p}_j$ are transformed into s displacement variables q_j. Nonpotential efforts are not affected by the variable transformation.

Using the nomenclature of the Legendre transform,[1] the potential coenergy function V^* is the function $y^{(o)}$, the potential efforts $(\dot{p}_1, \ldots, \dot{p}_s)$ are the first s variables $(x_1, \ldots, x_s)$ to be transformed, and the displacements $(q_1, \ldots, q_s)$ are the transformed variables $(\xi_1, \ldots, \xi_s)$. The untransformed variables are momenta p and time t, as well as the nonpotential efforts, which by definition do not appear in V^*. In overview, the transformation proceeds as shown in Fig. 7.4.

The definition of ξ_i for the transformed variables yields

$$\xi_i := \frac{\partial y^{(o)}}{\partial x_i} = \frac{\partial V^*}{\partial \dot{p}_i} = -q_i \qquad i = 1, \ldots, s. \tag{7.59}$$

[1] See Appendix A.

$$
\begin{array}{rl}
 & \overbrace{\phantom{\dot{p}_1,\ldots,\dot{p}_s}}^{\text{potential efforts}} \quad \overbrace{}^{\text{time and momenta}} \\
V^* = V^*(& \dot{p}_1,\ldots,\dot{p}_s,\ p_1\ ,\ldots,\ p_n, t) \\
 & \downarrow \quad \downarrow \quad \downarrow \qquad\qquad \downarrow \\
y^{(o)} = y^{(o)}(& x_1,\ \ldots, x_s, x_{s+1},\ldots,x_{n+s+1}) \\
 & \downarrow \quad \downarrow \quad \downarrow \qquad\qquad \downarrow \\
y^{(s)} = y^{(s)}(& \xi_1,\ \ldots, \xi_s, x_{s+1},\ldots,x_{n+s+1}) \\
 & \downarrow \quad \downarrow \quad \downarrow \qquad\qquad \downarrow \\
V\ = V\ (& q_1,\ldots,q_s,\ p_1\ ,\ldots,\ p_n, t) \\
 & \underbrace{}_{\text{transformed variables}} \quad \underbrace{}_{\text{untransformed variables}}
\end{array}
$$

Figure 7.4. Overview of the Legendre transform for potential efforts.

Thus the new variables ξ_i are negative components of displacement $-q_i$. This relationship accords with the definition of the displacement of potential stores. The definition of $y^{(s)}$ is given by

$$y^{(s)} := y^{(o)} - \sum_{i=1}^{s} \xi_i x_i, \tag{7.60}$$

yielding

$$
\begin{aligned}
y^{(s)} &= V^*(\dot{q}) + \sum_{i=1}^{s} q_i \dot{p}_i \\
&= \underbrace{V^*(\dot{p}) + V(q) + \sum_{i=1}^{s} q_i \dot{p}_i}_{=0} - V(q) \\
&= -V(q).
\end{aligned} \tag{7.61}
$$

Thus the s^{th} transform of the potential coenergy function $V^*(\dot{p}_1 \ldots, \dot{p}_s)$ is the negative of the potential energy function $V(q_1 \ldots, q_s)$. For the untransformed variables, the transformation yields the general result that

$$\frac{\partial y^{(k)}}{\partial x_i} = \frac{\partial y^{(o)}}{\partial x_i} \qquad i = s+1, \ldots, n+s+1, \tag{7.62}$$

which yields

$$-\frac{\partial V}{\partial t} = \frac{\partial V^*}{\partial t} \quad \text{and} \quad -\frac{\partial V}{\partial p_i} = \frac{\partial V^*}{\partial p_i} \quad i = 1, \ldots, n. \tag{7.63}$$

And finally, for the transformed variables, the transform gives the general result that

$$\frac{\partial y^{(s)}}{\partial \xi_i} = -x_i \qquad i = 1, \ldots, s, \tag{7.64}$$

yielding

$$\dot{p}_i = -\frac{\partial V}{\partial q_i} \qquad i = 1, \ldots, s. \tag{7.65}$$

This equation accords with the definition of the efforts of potential stores.

Equations (7.59), (7.63) and (7.65) are the desired results of the Legendre transform that are used to transform the complementary form of Lagrange's equation into the complementary form of Hamilton's equation.

7.2.2 Derivation

The complementary form of Lagrange's equation with multipliers is given by

$$\frac{d}{dt}\frac{\partial V^*}{\partial \dot{p}_j} - \frac{\partial V^*}{\partial p_j} + \frac{\partial T}{\partial p_j} + \frac{\partial G}{\partial \dot{p}_j} + \sum_{k=1}^{m_1} \frac{\partial \tilde{\phi}_k}{\partial p_j}\kappa_k + \sum_{k=1}^{m_2} \tilde{b}_{kj}\mu_k = P_j, \tag{7.66}$$

where $j = 1, \ldots, n$. Among these n equations are s equations corresponding to the potential effort coordinates $\dot{p}^s = (\dot{p}_1, \ldots, \dot{p}_s)$. For these s equations, direct substitution from (7.59) and (7.63) into the co-Lagrange's equation yields

$$-\dot{q}_j + \frac{\partial V}{\partial p_j} + \frac{\partial T}{\partial p_j} + \frac{\partial G}{\partial \dot{p}_j} + \sum_{k=1}^{m_1} \frac{\partial \tilde{\phi}_k}{\partial p_j}\kappa_k + \sum_{k=1}^{m_2} \tilde{b}_{kj}\mu_k = P_j, \tag{7.67}$$

where $j = 1, \ldots, s$. For the r nonpotential effort variables $\dot{p}^r = (\dot{p}_{s+1}, \ldots, \dot{p}_n)$ a potential displacement variable q_j is not defined since these efforts are not transformed. Hence for the r equations corresponding to the nonpotential efforts $\dot{p}^r$ of a system, the result of the variable transformation is identical to (7.67) excluding the first term, that is,

$$\frac{\partial V}{\partial p_j} + \frac{\partial T}{\partial p_j} + \frac{\partial G}{\partial \dot{p}_j} + \sum_{k=1}^{m_1} \frac{\partial \tilde{\phi}_k}{\partial p_j}\kappa_k + \sum_{k=1}^{m_2} \tilde{b}_{kj}\mu_k = P_j, \tag{7.68}$$

where $j = s+1, \ldots, n$. These two equations comprise a set of n equations with $s+n$ state variables, that is, $q \in \mathcal{R}^s$ and $p \in \mathcal{R}^n$. An additional set of s equations is given by (7.65). Together with the algebraic constraint equations, these transformed differential equations comprise the differential–algebraic form of the complement of Hamilton's equation called herein the *co–Hamiltonian DAE*, given by

$$\begin{aligned}
\dot{p}_j &= -\frac{\partial V}{\partial q_j} && j = 1, \ldots, s \\
-\dot{q}_j + \frac{\partial V}{\partial p_j} + \frac{\partial T}{\partial p_j} + \frac{\partial G}{\partial \dot{p}_j} + \sum_{k=1}^{m_1} \frac{\partial \tilde{\phi}_k}{\partial p_j}\kappa_k + \sum_{k=1}^{m_2} \tilde{b}_{kj}\mu_k &= P_j && j = 1, \ldots, s \\
\frac{\partial V}{\partial p_j} + \frac{\partial T}{\partial p_j} + \frac{\partial G}{\partial \dot{p}_j} + \sum_{k=1}^{m_1} \frac{\partial \tilde{\phi}_k}{\partial p_j}\kappa_k + \sum_{k=1}^{m_2} \tilde{b}_{kj}\mu_k &= P_j && j = s+1, \ldots, n \\
\tilde{\phi}_k(p,t) &= 0 && k = 1, \ldots, m_1 \\
\tilde{\psi}_k(\dot{p},p,t) &= 0 && k = 1, \ldots, m_2 \\
\tilde{\gamma}_k(f^\gamma,\dot{p},p,t) &= 0 && k = 1, \ldots, m_3. \quad (7.69)
\end{aligned}$$

Using the following definitions, these equations can be written in vector form. The momentum vector p is partitioned into potential momenta $p^s := (p_1, \ldots, p_s)$ and nonpotential momenta $p^r := (p_{s+1}, \ldots, p_n)$ such that

$$p(t) = \begin{bmatrix} p^s(t) \\ p^r(t) \end{bmatrix}. \qquad (7.70)$$

Let the multiplier vector λ and the constraint vector $\tilde{C}$ be given by

$$\lambda(t) := \begin{bmatrix} \kappa(t) \\ \mu(t) \end{bmatrix} \qquad \tilde{C}(\dot{p},p,t) := \begin{bmatrix} \tilde{\Phi}(p,t) \\ \tilde{\Psi}(\dot{p},p,t) \end{bmatrix}. \qquad (7.71)$$

Let $\tilde{C}_s$ be the Jacobian of the constraint vector $\tilde{C}$ with respect to the potential state variables $(p^s, \dot{p}^s)$, and $\tilde{C}_r$ be the Jacobian of $\tilde{C}$ with respect to the nonpotential state variables $(p^r, \dot{p}^r)$ given by

$$\tilde{C}_s(p,t) := \begin{bmatrix} \tilde{\Phi}_{p^s}(p,t) \\ \tilde{\Psi}_{\dot{p}^s}(p,t) \end{bmatrix} \qquad \tilde{C}_r(p,t) := \begin{bmatrix} \tilde{\Phi}_{p^r}(p,t) \\ \tilde{\Psi}_{\dot{p}^r}(p,t) \end{bmatrix}, \qquad (7.72)$$

Then the Hamiltonian DAE is given in vector form by

$$\begin{aligned} \dot{p}^s &= -\nabla_q V \\ -\dot{q} + \nabla_{p^s} V + \nabla_{p^s} T + \nabla_{\dot{p}^s} G + \tilde{C}_s^T \lambda &= P^s \\ \nabla_{p^r} V + \nabla_{p^r} T + \nabla_{\dot{p}^r} G + \tilde{C}_r^T \lambda &= P^r \\ \tilde{C} &= 0 \\ \tilde{\Gamma} &= 0. \end{aligned} \tag{7.73}$$

The co–Hamiltonian DAE is characterized by first–order differential equations in terms of momentum and displacement variables (p, q). The differential equations are not necessarily explicit because of the dissipation term $\nabla_{\dot{p}} G$. Consequently the co–Hamiltonian DAE given by (7.73) is not in a convenient form for systematic formulation and numerical solution of the equations of motion.

7.2.3 Semi–explicit form

Let $\dot{p} = e$, where efforts e are partitioned into potential efforts $e^s := (e_1, \ldots, e_s)$ and nonpotential efforts $e^r := (e_{s+1}, \ldots, e_n)$ such that[2]

$$e(t) = \begin{bmatrix} e^s(t) \\ e^r(t) \end{bmatrix}. \tag{7.74}$$

The co–Hamiltonian DAE is given by

$$\begin{aligned} \dot{p}_j &= e_j & j &= 1, \ldots, n \\ -\dot{q}_j + \frac{\partial V}{\partial p_j} + \frac{\partial T}{\partial p_j} + \frac{\partial G}{\partial e_j} + \sum_{k=1}^{m_1} \frac{\partial \tilde{\phi}_k}{\partial p_j} \kappa_k + \sum_{k=1}^{m_2} \tilde{b}_{kj} \mu_k &= P_j & j &= 1, \ldots, s \\ \frac{\partial V}{\partial p_j} + \frac{\partial T}{\partial p_j} + \frac{\partial G}{\partial e_j} + \sum_{k=1}^{m_1} \frac{\partial \tilde{\phi}_k}{\partial p_j} \kappa_k + \sum_{k=1}^{m_2} \tilde{b}_{kj} \mu_k &= P_j & j &= s+1, \ldots, n \\ e_j &= -\frac{\partial V}{\partial q_j} & j &= 1, \ldots, s \\ \tilde{\phi}_k(p, t) &= 0 & k &= 1, \ldots, m_1 \end{aligned}$$

[2]Here e^s represents a vector of efforts associated with potential stores, not an effort source.

$$\begin{aligned} \tilde{\psi}_k(e,p,t) &= 0 \qquad k=1,\ldots,m_2 \\ \tilde{\gamma}_k(f^\gamma,e,p,t) &= 0 \qquad k=1,\ldots,m_3. \end{aligned} \tag{7.75}$$

Let the multiplier vector λ, the constraint vector $\tilde{C}$, and the Jacobians $\tilde{C}_s$ and $\tilde{C}_r$ be given by

$$\begin{aligned} \lambda(t) &:= \begin{bmatrix} \kappa(t) \\ \mu(t) \end{bmatrix} & \tilde{C}_s(p,t) &:= \begin{bmatrix} \tilde{\Phi}_{p^s}(p,t) \\ \tilde{\Psi}_{e^s}(p,t) \end{bmatrix} \\ \tilde{C}(e,p,t) &:= \begin{bmatrix} \tilde{\Phi}(p,t) \\ \tilde{\Psi}(e,p,t) \end{bmatrix} & \tilde{C}_r(p,t) &:= \begin{bmatrix} \tilde{\Phi}_{p^r}(p,t) \\ \tilde{\Psi}_{e^r}(p,t) \end{bmatrix}. \end{aligned} \tag{7.76}$$

Then the co–Hamiltonian DAE is given in vector form by

$$\begin{aligned} \dot{p} &= e \\ -\dot{q} + \nabla_{p^s} V + \nabla_{p^s} T + \nabla_{e^s} G + \tilde{C}_s^T \lambda &= P^s \\ \nabla_{p^r} V + \nabla_{p^r} T + \nabla_{e^r} G + \tilde{C}_r^T \lambda &= P^r \\ e^s &= -\nabla_q V \\ \tilde{C} &= 0 \\ \tilde{\Gamma} &= 0. \end{aligned} \tag{7.77}$$

The first $n+s$ equations are ODE's explicit in $(\dot{p},\dot{q})$ and the remaining $n+m_o+m_3$ equations are algebraic in (p,q,e,λ,f^γ). Collecting the differential equations and algebraic equations into two groups yields

$$\begin{aligned} \begin{bmatrix} \dot{p} \\ \dot{q} \end{bmatrix} &= \begin{bmatrix} e \\ -P^s + \nabla_{p^s} V + \nabla_{p^s} T + \nabla_{e^s} G + \tilde{C}_s^T \lambda \end{bmatrix} \\ 0 &= \begin{bmatrix} -P^r + \nabla_{p^r} V + \nabla_{p^r} T + \nabla_{e^r} G + \tilde{C}_r^T \lambda \\ \nabla_q V + e^s \\ \tilde{C} \\ \tilde{\Gamma} \end{bmatrix}. \end{aligned} \tag{7.78}$$

This DAE is in semi–explicit form, that is,

$$\dot{x} = \mathcal{F}_1(x,z,t)$$

$$0 \;=\; \mathcal{F}_2(x, z, t), \tag{7.79}$$

where

$$\begin{aligned} x(t) &:= \begin{bmatrix} p(t) & q(t) \end{bmatrix}^T \\ z(t) &:= \begin{bmatrix} e(t) & \lambda(t) & f^\gamma(t) \end{bmatrix}^T, \end{aligned} \tag{7.80}$$

and $\mathcal{F}_1$ and $\mathcal{F}_2$ are given by the vectors on the right-hand side of (7.78). This DAE is comprised of $2n+s+m_o+m_3$ equations in the same number of unknowns, namely $(p, e, q, \lambda, f^\gamma)$.

Like the descriptor form of the co-Lagrangian DAE, the semi-explicit form of the co-Hamiltonian DAE is suitable for systematic formulation of the equations of motion for multidiscipline systems. However, this DAE too is generally index-3 or higher. The same strategies outlined in Chapter 8 for integrating an index-3 Lagrangian DAE can be used to integrate an index-3 co-Hamiltonian DAE.

EXAMPLE 7.4 The mass-spring-damper system of Ex. 7.2 is used to illustrate the co-Hamiltonian formulation. The same three momenta coordinates as in the co-Lagrangian formulation are assigned here and an additional coordinate q is assigned to the spring, a potential store. See Fig. 7.5.

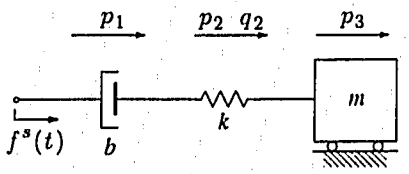


Figure 7.5. A mass-spring-damper system with coordinates assigned for a co-Hamiltonian formulation.

Analysis yields

$$\begin{aligned} V &= \tfrac{1}{2} k q_2^2 \\ T &= p_3^2/2m \end{aligned}$$

$$
\begin{aligned}
G &= \dot{p}_1^2/2b \\
\tilde{\phi}_1 &:= p_1 - p_3 + p_{3o} = 0 \\
\tilde{\phi}_2 &:= p_2 - p_3 + p_{3o} = 0 \\
\delta W &= f^s(t)\,\delta p_1.
\end{aligned}
\tag{7.81}
$$

The constraint matrix is given by

$$
\tilde{C} := \begin{bmatrix} p_1 - p_3 + p_{3o} \\ p_2 - p_3 + p_{3o} \end{bmatrix}. \tag{7.82}
$$

The constraint Jacobian is given by

$$
\tilde{C}_n := \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}, \tag{7.83}
$$

from which are obtained the potential Jacobian $\tilde{C}_s$ and nonpotential Jacobian $\tilde{C}_r$ given in transpose form by

$$
\begin{aligned}
\tilde{C}_s^T &:= \begin{bmatrix} 0 & 1 \end{bmatrix} \\
\tilde{C}_r^T &:= \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}.
\end{aligned}
\tag{7.84}
$$

The co–Hamiltonian DAE for this example is given in semi–explicit form by

$$
\begin{aligned}
\dot{p} &= e \\
\dot{q} &= -P^s + \nabla_{p^s} V + \nabla_{p^s} T + \nabla_{e^s} G + \tilde{C}_s^T \lambda \\
0 &= -P^r + \nabla_{p^r} V + \nabla_{p^r} T + \nabla_{e^r} G + \tilde{C}_r^T \lambda \\
0 &= \nabla_q V + e^s \\
0 &= \tilde{C}.
\end{aligned}
\tag{7.85}
$$

The equations of motion are

$$
\begin{aligned}
\begin{bmatrix} \dot{p} \\ \dot{q}_2 \end{bmatrix} &= \begin{bmatrix} e \\ \lambda_2 \end{bmatrix} \\
0 &= \begin{bmatrix} f^s - e_1/b - \lambda_1 \\ -p_3/m + \lambda_1 + \lambda_2 \\ kq_2 + e_2 \\ p_1 - p_3 + p_{3o} \\ p_2 - p_3 + p_{3o} \end{bmatrix}
\end{aligned}
\tag{7.86}
$$

This is a set of nine equations with unknowns $(p_1, p_2, p_3, e_1, e_2, e_3, q_2, \lambda_1, \lambda_2)$. This DAE is comprised of four explicit ODE's and five algebraic constraint equations. $\diamond$

7.2.4 Complementary canonical form

If the system constraints are such that an explicit solution can be obtained for the multipliers λ and the implicit flows f^γ, and assuming that the potential coordinates (q, p^s) are sufficient to describe the motion of the system, then all algebraic equations can be eliminated from the DAE. The resulting differential equations are in implicit form, that is,

$$\mathcal{F}(\dot{x}, x, t) = 0, \tag{7.87}$$

where

$$x(t) := \begin{bmatrix} q(t) & p(t) \end{bmatrix}^T, \tag{7.88}$$

and $\mathcal{F}$ is a nonlinear function of the state variables, time and the parameters of the system.

The procedure for obtaining the underlying ODE is similar to the procedure used to obtain the underlying ODE of the Hamiltonian DAE. (See Section 6.4.) The inverse of $\tilde{\Phi}_p W \tilde{\Phi}_p^T$ must exist. The constraints are differentiated twice, hence a numerical solution is not guaranteed to satisfy the momentum constraints $\bar{\phi}_k$ or the effort constraints $\bar{\psi}_k$.

In the special case that an independent set of potential coordinates (p, q) can be selected to represent the motion of the system, the underlying ODE reduces to

$$\begin{aligned} \dot{p} &= -\nabla_q V \\ \dot{q} &= -P + \nabla_p V + \nabla_p T + \nabla_{\dot{p}} G. \end{aligned} \tag{7.89}$$

Let the general nonpotential flow P^n represent the flows of sources and dissipators such that $P^n = P - \nabla_{\dot{p}} G$. Defining the Hamiltonian energy function $H := T + V$, the underlying ODE is given by

$$\dot{p} = -\nabla_q H$$

$$\dot{q} = \nabla_p H - P^n. \tag{7.90}$$

In the absence of nonpotential flows, that is, $P^n = 0$, this equation is reduced to

$$\begin{aligned} \dot{p} &= -\nabla_q H \\ \dot{q} &= \nabla_p H. \end{aligned} \tag{7.91}$$

This equation, called herein the *co-canonical form*, is the complement of Hamilton's equation in canonical form. For modeling and simulation of multidiscipline systems, this form of the equations of motion has limited utility, since the selection of coordinates is limited to an independent set of displacements and momenta among the potential stores only, and nonpotential flows are excluded. For these reasons, modeling procedures based on the canonical form are not in common use.

7.2.5 Problem formulation

To obtain the equations of motion using the semi-explicit form of the co-Hamiltonian DAE, it is necessary for the analyst to assign coordinates (q^s, p^s, e^s) to potential energy stores and coordinates (p^r, e^r) to nonpotential elements. Expressions are developed in terms of these coordinates for the energy functions T, V and G, for the independent constraints $\tilde{C}$ and $\tilde{\Gamma}$, and for the virtual work of the nonpotential flows P. Through manipulation of these expressions, the semi-explicit form of (7.79) is obtained.

7.2.6 Comparison of two formulations

The co-Lagrangian formulation and the co-Hamiltonian formulation have the same attributes with respect to one another as the Lagrangian and Hamiltonian formulations have with respect to one another. The co-Hamiltonian DAE in semi-explicit form has s more equations and unknowns than the co-Lagrangian DAE in descriptor form. But the co-Hamiltonian DAE contains fewer differential equations to be solved and all the ODE's are explicit since the co-Hamiltonian DAE contains no A matrix to be inverted as does

the co–Lagrangian DAE. Thus, like the Hamiltonian formulation, the co–Hamiltonian formulation may be more suitable than the co–Lagrangian DAE for numerical implementation.

Lastly, it is noted by comparing the co–Lagrangian DAE (7.35) and the co–Hamiltonian DAE (7.73) that the two formulations are identical in the case of a system with no potential stores. In such a case, both DAEs reduce to a set of first–order differential equations in $\dot{p}$ given by

$$\begin{aligned} \nabla_p T + \nabla_{\dot{p}} G + \tilde{C}_n^T \lambda &= P \\ \tilde{C} &= 0 \\ \tilde{\Gamma} &= 0. \end{aligned} \tag{7.92}$$

Chapter 8

MODELING AND SIMULATION

One of the primary goals of this research is to develop a systematic method of modeling multidiscipline systems. In this chapter is presented a process of analysis, function manipulation, numerical solution and automation that meets this goal. To make the modeling procedure systematically applicable to the general multidiscipline system, Lagrange's equation and its dual are formulated as a set of linearly implicit DAEs in descriptor form and Hamilton's equation and its dual are formulated as a set of semi-explicit DAEs. These forms are the basis of an automated modeling and simulation algorithm.

8.1 Analysis

The following outline comprises the extent of analysis using the analytical method. The brevity and clarity of these steps is one of the attractive features of the this approach. Following the outline is a brief explanation of each step illustrated by example.

8.1.1 Outline of the analytical method

1. Obtain a system *schematic.*
2. Select *coordinates* to represent the motion of the system.
3. Select a suitable *formulation*, that is, Lagrangian, Hamiltonian, co-Lagrangian or co-Hamiltonian.
4. Write expressions for the *energy* functions.
5. Write expressions for the system *constraints.*
6. Write expressions for the *virtual work* of sources and implicit efforts or flows.
7. Determine a set of *initial conditions* consistent with the constraints.
8. Select numerical values for system *parameters.*
9. *Solve* the differential-algebraic equations of motion.

8.1.2 Obtain a schematic

To begin the analysis of a multidiscipline system, a schematic of the system must be obtained. The schematic must be of sufficient detail to identify the necessary state variables and to illustrate the interconnections among system elements. Any method of signifying the elements and interconnections that is adequate for this purpose is acceptable. Thus a variety of discipline–specific conventions may be used in constructing such a schematic. No formal schematic–building algorithm is required as in the conventional linear graph or bond graph modeling techniques. Often the energy functions, constraint equations and virtual work expression may be written from inspection of the system schematic. The schematic of the system used in this example is shown in Fig. 8.1.

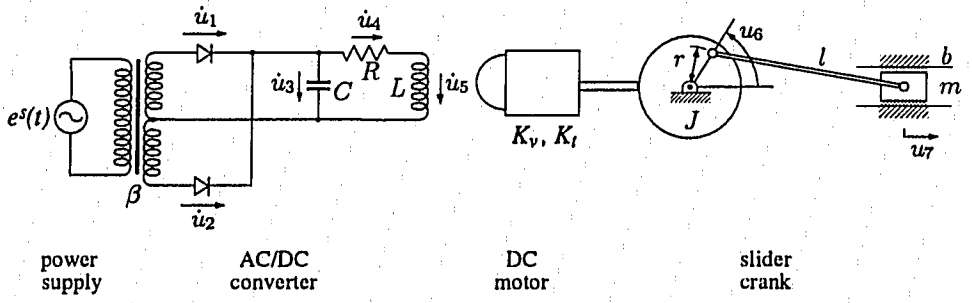


Figure 8.1. An electromechanical system.

8.1.3 Assign coordinates

It is necessary for coordinates to be assigned to system elements. For the Lagrangian DAE, these coordinates are the state variables $(u, \dot{u})$. These coordinates need not be independent, nor must they necessarily comprise a set of minimum dimension. This accommodation of redundant coordinates is one of the features of this approach that eases the mathematical burden on the analyst, since it is not imperative that the analyst manipulate equations to reduce a coordinate set to minimum dimension. Nor is it necessary that the analyst carefully or cleverly choose a particular set of independent state

variables in order to improve the efficiency of the solution algorithm. For the example at hand, the coordinates are assigned as shown in Fig. 8.1.

8.1.4 Select a formulation

In this example the Lagrangian formulation is used. The Hamiltonian, co–Hamiltonian or co–Lagrangian formulations may be used as well, as long as the formulation is suitable for the state variables selected by the analyst to represent the motion of the system. Thus the analyst is given the freedom to choose the representation of motion best suited for the problem at hand, and then select the DAE that can be solved most readily for this set of variables. For example, for a system with a significant number of kinetic stores the Hamiltonian DAE may be better suited than the Lagrangian DAE for efficient numerical solution.

In this research, a set of guidelines or "recipe" for selecting a suitable formulation is not given. A conclusive assessment can be made only after continued experience with the the four formulations and robust numerical algorithms. The development of such guidelines is left to the future. However, some general insights into formulation selection follow.

The differences among the four formulations arise from the different variable pairs used to represent the motion of a system. The Lagrangian DAE, which represents the motion of a system in terms of displacement q and flow f, is the formulation most familiar to dynamicists. Since displacement and flow constraints are common in engineering systems, solutions in terms of these variables are readily checked for constraint compliance. The analyst can quickly assess the reasonableness of a solution.

In constrast, the co–Lagrangian DAE represents the motion of a system in terms of momentum p and effort e. Constraints in terms of these variables usually are not discerned as readily as displacement and flow constraints, particularly for mechanical systems with complex geometries. Hence the type of constraints present in a system and

the ease with which they are discerned and formulated bears significantly on the choice of formulation.

The Hamiltonian DAE represents the motion of a system in terms of displacement q and momentum p, where the momenta are of kinetic stores only. The Hamiltonian DAE is in semi–explicit form, which may prove to be better suited to numerical solution in the general case than the descriptor form of the Lagrangian DAE. Moreover, the constraints of the Hamiltonian DAE are formulated in the same manner as in the Lagrangian DAE, retaining the ease of constraint formulation and solution assessment. One drawback of the Hamiltonian solution is the average analyst's unfamiliarity with assigning momentum variables to a systems and then assessing solutions posed in terms of momentum coordinates. This is not a serious difficulty, however, since facile interpretation of numerical results is a skill that improves with practice.

Like the Hamiltonian DAE, the co–Hamiltonian DAE represents the motion of a system in terms of displacement q and momentum p, but here the displacements are of potential stores only. Constraints are formulated in the same manner as in the co–Lagrangian DAE, that is, in terms of momentum and effort. Thus the same reservations expressed about constraints in co–Lagrangian form apply to the co–Hamiltonian form. Yet, like the Hamiltonian, the co–Hamiltonian is a semi–explicit DAE, and so may have superior numerical properties to the co–Lagrangian in descriptor form.

Clearly, the type of elements and the type of constraints present in a system are important guides to choosing the formulation best suited to that system.

8.1.5 Energy functions

Expressions for the energy state functions are written. Often these can be obtained by inspection of the system schematic. In other cases, the analyst may have to resort to the definitions of these functions. For the example at hand the energy functions are given

by

$$\begin{aligned} T^* &= \tfrac{1}{2}L\dot{u}_5^2 + \tfrac{1}{2}J\dot{u}_6^2 + \tfrac{1}{2}m\dot{u}_7^2 \\ V &= \frac{u_3^2}{2C} \\ D &= \tfrac{1}{2}R\dot{u}_4^2 + \tfrac{1}{2}b\dot{u}_7^2. \end{aligned} \tag{8.1}$$

Since the descriptor form of the Lagrangian DAE is to be used in formulating this model, a change in coordinates is made to conform to this formulation. The displacement variable u is renamed q and the flow variable $\dot{u}$ is renamed f. The energy functions are given in this nomenclature by

$$\begin{aligned} T^* &= \tfrac{1}{2}Lf_5^2 + \tfrac{1}{2}Jf_6^2 + \tfrac{1}{2}mf_7^2 \\ V &= \frac{q_3^2}{2C} \\ D &= \tfrac{1}{2}Rf_4^2 + \tfrac{1}{2}bf_7^2. \end{aligned} \tag{8.2}$$

8.1.6 Kinematic constraints

The slider–crank mechanism imposes a holonomic constraint given by

$$\phi_1(u) := (u_7 - r\cos u_6)^2 + (r\sin u_6)^2 - l^2 = 0, \tag{8.3}$$

and the electrical nodes impose holonomic flow constraints given by

$$\begin{aligned} \psi_1(\dot{u}) &:= \dot{u}_1 + \dot{u}_2 - \dot{u}_3 - \dot{u}_4 = 0 \\ \psi_2(\dot{u}) &:= \dot{u}_4 - \dot{u}_5 = 0. \end{aligned} \tag{8.4}$$

Since the flow constraint ψ_1 involves a capacitor, which accumulates physical charge, this flow constraint must be integrated to obtain a holonomic displacement constraint. Including this displacement constraint in the DAE ensures that the constraint on the charge of the capacitor is satisfied. The second flow constraint ψ_2 involves a resistor and inductor, which do not accumulate charge, hence this constraint can remain in flow variable form. The numerical solution need not ensure that the integral of this constraint

is satisfied. Thus, with a change in nomenclature, the kinematic constraint equations are given in final form by

$$\begin{aligned} \phi_1(u) &:= (q_7 - r\cos q_6)^2 + (r\sin q_6)^2 - l^2 = 0 \\ \phi_2(u) &:= q_1 + q_2 - q_3 - q_4 + q_{3_o} = 0 \\ \psi_1(f) &:= f_4 - f_5 = 0. \end{aligned} \tag{8.5}$$

8.1.7 Dynamic constraints

Each diode has a constituitive law given by $i = i_s(e^{\alpha v} - 1)$, where i is current, v is voltage, and i_s and α are known parameters. Voltage v is an implicit effort e^γ, and the diode constituitive law imposes the dynamic constraint given by $\gamma := i - i_s(e^{\alpha e^\gamma} - 1) = 0$. Denoting the implicit efforts for the two diodes e_1^γ and e_2^γ respectively, two dynamic constraints are given by

$$\begin{aligned} \gamma_1(e^\gamma, f) &= f_1 - i_s\left(e^{\alpha e_1^\gamma} - 1\right) = 0 \\ \gamma_2(e^\gamma, f) &= f_2 - i_s\left(e^{\alpha e_2^\gamma} - 1\right) = 0. \end{aligned} \tag{8.6}$$

Denoting the DC motor torque with e_3^γ and its back-emf with e_4^γ, the two dynamic constraints imposed by the DC motor are given by

$$\begin{aligned} \gamma_3(e^\gamma, f) &:= e_3^\gamma + K_t\, f_5 = 0 \\ \gamma_4(e^\gamma, f) &:= e_4^\gamma - K_v\, f_5 = 0. \end{aligned} \tag{8.7}$$

Lastly, the phase–inverting transformer is modeled as a device that acts as a dual voltage source, as shown in Fig. 8.2. Voltage source e^s acts in the u_1 direction such that $e_1^s = \beta e^s(t)$ and simultaneously acts as a voltage source in the u_2 direction such that $e_2^s = -\beta e^s(t)$. These relationships could be formulated as two dynamic constraints, but they are just as easily embedded in the virtual work equation without introducing two additional unknown implicit efforts.

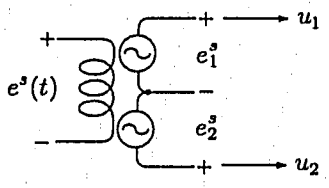


Figure 8.2. A phase–inverting transformer modeled as a voltage source.

8.1.8 Virtual work

A virtual work expression is written to account for the contribution of all source efforts and implicit efforts. In this example, the implicit efforts are the diode voltages e_1^γ and e_2^γ, the motor torque e_3^γ and back–emf e_4^γ, and the two phase–inverted sources e_1^s and e_2^s. The virtual work of these efforts is given by

$$\delta W = -e_1^\gamma\,\delta u_1 - e_2^\gamma\,\delta u_2 + e_3^\gamma\,\delta u_6 + e_4^\gamma\,\delta u_5 + e_1^s\,\delta u_1 + e_2^s\,\delta u_2, \tag{8.8}$$

where the signs indicate whether the effort acts in the direction of or in opposition to the assigned coordinate direction. Substituting $e_1^s = \beta e^s(t)$ and $e_2^s = -\beta e^s(t)$ accounts for the contribution of the AC voltage source and eliminates e_1^s and e_2^s from the equations. Collecting terms and changing nomenclature yields

$$\delta W = (\beta e^s - e_1^\gamma)\;\delta q_1 - (\beta e^s + e_2^\gamma)\;\delta q_2 + e_4^\gamma\,\delta q_5 + e_3^\gamma\,\delta q_6. \tag{8.9}$$

The coefficients of the δq_i are the Q_i of Lagrange's equation.

8.1.9 Determine initial conditions

The system unknowns are the set of variables given by

$$x := (q_1, \ldots, q_7, f_1, \ldots, f_7, \lambda_1, \ldots, \lambda_3, e_1^\gamma, \ldots, e_4^\gamma), \tag{8.10}$$

where $\lambda \in \mathcal{R}^3$ since there are three kinematic constraints. For a numerical solver to proceed, initial conditions $x_o := x(t_o)$ must be determined that are consistent with the constraints. This is not always a simple task. Some numerical algorithms contain

subroutines that automate the selection of initial conditions. A set of initial conditions are not required to symbolically formulate the DAE, but they are required to obtain a numerical solution.

8.1.10 Select parameters

Numerical constants or time–dependent functions are assigned to the parameters of the system. These include constituitive parameters such as mass, moments of inertia, capacitances, and other coefficients as well as functions such as sources. Any of these parameters may be nonlinear, time–dependent or state–dependent. Like the initial conditions, parameter selection is not required to formulate the DAE but is required for numerical integration.

This concludes the analysis of this system. The remaining steps of formulating and solving the equations of motion are steps that are readily automated. The time and effort of the analyst is devoted to developing an understanding of the physical system that is sufficient to perform the modeling steps outlined above. The set of equations from which a model is obtained is given by

$$
\begin{aligned}
T^* &= \tfrac{1}{2}Lf_5^2 + \tfrac{1}{2}Jf_6^2 + \tfrac{1}{2}mf_7^2 \\
V &= \frac{q_3^2}{2C} \\
D &= \tfrac{1}{2}Rf_4^2 + \tfrac{1}{2}bf_7^2 \\
\phi_1 &:= (q_7 - r\cos q_6)^2 + (r\sin q_6)^2 - l^2 = 0 \\
\phi_2 &:= q_1 + q_2 - q_3 - q_4 + q_{3_o} = 0 \\
\psi_1 &:= f_4 - f_5 = 0 \\
\gamma_1 &:= f_1 - i_s\left(e^{\alpha e_1^\gamma} - 1\right) = 0 \\
\gamma_2 &:= f_2 - i_s\left(e^{\alpha e_2^\gamma} - 1\right) = 0 \\
\gamma_3 &:= e_3^\gamma + K_t\, f_5 = 0 \\
\gamma_4 &:= e_4^\gamma - K_v\, f_6 = 0
\end{aligned}
\tag{8.11}
$$

$$\delta W \quad = \quad (\beta e^s - e_1^\gamma)\ \delta q_1 - (\beta e^s + e_2^\gamma)\ \delta q_2 + e_4^\gamma\ \delta q_5 + e_3^\gamma\ \delta q_6.$$

8.2 Function manipulation

From the results of the above analysis, the quantities M, C, C_n, Γ and Υ_0 of the Lagrangian DAE are formed. Prior to the advent of symbolic software capable of performing these manipulations, this was the step that, except for the simplest systems, modelers found most intimidating. With the advent of such software, however, the burden of these manipulations can be put where it belongs — on the computer — freeing the analyst from the drudgery of equation manipulation. However, for the simple example considered here, the manipulations are done manually to illustrate the method.

The M matrix is defined as the Hessian of the kinetic coenergy given by

$$M := \nabla_f^2 T^* = \text{diag}\{0\ 0\ 0\ 0\ L\ J\ m\}. \tag{8.12}$$

The vector C is defined as the vector of all kinematic constraints given by

$$C := \begin{bmatrix} \Phi \\ \Psi \end{bmatrix} = \begin{bmatrix} q_7^2 - 2rq_7\cos q_6 + r^2 - l^2 \\ q_1 + q_2 - q_3 - q_4 + q_{3_o} \\ f_4 - f_5 \end{bmatrix}. \tag{8.13}$$

The matrix C_n is defined as the Jacobian of C with respect to the state variables given by

$$C_n := \begin{bmatrix} \Phi_q \\ \Psi_f \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2rq_7\sin q_6 & 2(q_7 - r\cos q_6) \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix}. \tag{8.14}$$

The vector Γ is defined as the vector of all dynamic constraints given by

$$\Gamma := \begin{bmatrix} f_1 - i_s\left(e^{\alpha e_1^\gamma} - 1\right) \\ f_2 - i_s\left(e^{\alpha e_2^\gamma} - 1\right) \\ e_3^\gamma + K_t\ f_5 \\ e_4^\gamma - K_v\ f_6 \end{bmatrix}. \tag{8.15}$$

And the vector Υ_0 is defined as the collection of efforts given by

$$\Upsilon_0 := Q - (\nabla_f T^*)_q\, f + (\nabla_f T^*)_t + \nabla_q T^* - \nabla_q V - \nabla_f D. \tag{8.16}$$

For this example Υ_0 is given by

$$\Upsilon_0 = \begin{bmatrix} \beta e^s(t) - e_1^\gamma \\ -\beta e^s(t) - e_2^\gamma \\ -q_3/C \\ -Rf_4 \\ e_4^\gamma \\ e_3^\gamma \\ -bf_7 \end{bmatrix}. \tag{8.17}$$

Having these matrix expressions, determining a consistent set of initial conditions

$$x(t_o) := (q_{1_o}, \ldots, q_{7_o}, f_{1_o}, \ldots, f_{7_o}, \lambda_{1_o}, \ldots, \lambda_{3_o}, e_{1_o}^\gamma, \ldots, e_{4_o}^\gamma), \tag{8.18}$$

and selecting system parameters $(L, J, m, r, l, i_s, \alpha, K_t, K_v, \beta, C, R, b)$ concludes the modeling procedure. The model is given by

$$\dot{q} = f$$

$$\begin{bmatrix} 0 & & & & & & \\ & 0 & & & & & \\ & & 0 & & & & \\ & & & 0 & & & \\ & & & & L & & \\ & & & & & J & \\ & & & & & & m \end{bmatrix} \dot{f} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ 2rq_7 \sin q_6 & 0 & 0 \\ 2(q_7 - r\cos q_6) & 0 & 0 \end{bmatrix} \lambda = \begin{bmatrix} \beta e^s(t) - e_1^\gamma \\ -\beta e^s(t) - e_2^\gamma \\ -q_3/C \\ -Rf_4 \\ e_4^\gamma \\ e_3^\gamma \\ -bf_7 \end{bmatrix}$$

$$\begin{bmatrix} q_7^2 - 2rq_7\cos q_6 + r^2 - l^2 \\ q_1 + q_2 - q_3 - q_4 + q_{3_o} \\ f_4 - f_5 \\ f_1 - i_s\left(e^{\alpha e_1^\gamma} - 1\right) \\ f_2 - i_s\left(e^{\alpha e_2^\gamma} - 1\right) \\ e_3^\gamma + K_t\, f_5 \\ e_4^\gamma - K_v\, f_6 \end{bmatrix} = 0. \tag{8.19}$$

The first 14 equations of this DAE are linearly implicit ODE's and the remaining 7 equations are algebraic. The unknowns are $(q, f, \lambda, e^\gamma)$.

8.3 Numerical methods

Two algorithms are in common use for the direct solution of a DAE: Euler's method and backward-differentiation formula (BDF) methods. In both methods an approximation is made of the derivatives in the DAE, turning the DAE at each time step into a set of nonlinear algebraic equations of the form $F = 0$. This set of equations is solved at each time step.

To illustrate the basic concepts underlying this approach to numerically solving the DAE, Euler's method is presented below followed by an outline of the BDF method. For illustration purposes, the descriptor form of the Lagrangian DAE is used. A parallel development can be made for solving the Hamiltonian DAE and the complementary forms of Lagrange's and Hamilton's equations.

8.3.1 Euler's method

The Lagrangian DAE is given by

$$\begin{aligned} \dot{q} &= f \\ M\dot{f} + C_n^T\lambda &= \Upsilon_0 \\ C &= 0 \\ \Gamma &= 0, \end{aligned} \tag{8.20}$$

where

$$\begin{aligned} q &= (q_1, \ldots, q_n) \\ f &= (f_1, \ldots, f_n) \\ \lambda &= (\lambda_1, \ldots, \lambda_{m_o}), \end{aligned} \tag{8.21}$$

and the solution space also includes

$$e^\gamma = (e_1^\gamma, \ldots, e_{m_3}^\gamma). \tag{8.22}$$

The model is comprised of $2n+m_o+m_3$ equations in the same number of unknowns.

In Euler's method the derivatives $\dot{q}$ and $\dot{f}$ are replaced by finite-differences. The approximation of a time derivative $\dot{x}$ at time step $t^{\nu+1}$ is given by

$$\dot{x}(t^{\nu+1}) \approx \frac{x(t^{\nu+1}) - x(t^{\nu})}{t^{\nu+1} - t^{\nu}}. \tag{8.23}$$

where ν indicates a discrete time step. A shorthand notation is given by

$$\dot{x}^{\nu+1} \approx \frac{x^{\nu+1} - x^{\nu}}{h^{\nu+1}}, \tag{8.24}$$

where $h^{\nu+1} := t^{\nu+1} - t^{\nu}$ is the current time step of the numerical integration. Euler's method uses a fixed time step h.

Applying this method of approximation to the derivatives in the Lagrangian DAE yields the function F given by

$$F := \begin{bmatrix} h^{-1}\left(q^{\nu+1} - q^{\nu}\right) - f^{\nu+1} \\ h^{-1}M^{\nu+1}\left(f^{\nu+1} - f^{\nu}\right) + \left(C_n^T\right)^{\nu+1}\lambda^{\nu+1} - \Upsilon_0^{\nu+1} \\ C^{\nu+1} \\ \Gamma^{\nu+1} \end{bmatrix} = 0. \tag{8.25}$$

Let x be the vector of unknowns given by $x := (q, f, \lambda, e^{\gamma})$. Knowing the value of x at time step t^{ν}, this set of equations can be solved for the unknown values of x at time step $t^{\nu+1}$. In other words, knowing x^{ν}, $F = 0$ can be solved for $x^{\nu+1}$. The solution method most often used for solving this type of nonlinear algebraic equation set is Newton's method.

Newton's method requires knowledge of the Jacobian of F at each time step, given by $J^{\nu+1} := \partial F/\partial x^{\nu+1}$. In general this Jacobian can be determined either analytically or numerically. In the case of the descriptor form of the DAE, determining the Jacobian analytically can be a daunting task. Hence in the systematic approach given here, it is suggested that the Jacobian be determined numerically at each time step through the use

of finite–differences. This approach has been successfully implemented for classroom–style problems [13].

8.3.2 BDF method

Backward–differentiation formula methods for solving DAEs are presented in great detail in the literature. See for example [4,6,42]. The basic features of these methods are outlined in this section.

Backward–differentiation formula methods for solving DAEs are conceptually similar to Euler's method. In both methods the derivatives in the DAE are replaced with approximations and the resulting set of algebraic equations is solved at each time step for the system unknowns at the next time step. The BDF methods differ from Euler's method in the manner in which the derivatives are approximated and in the selection of step size. BDF methods use polynomial approximations of the derivatives and vary the step size depending on error estimates. BDF methods also incorporate error control schemes. These differences between Euler's method and BDF methods are outlined in Table 8.1.

Table 8.1. Comparison of Euler's method and BDF methods.

Characteristic	Euler	BDF
derivative approximation	finite–difference	polynomial
order of approximation	1^{st}	variable, usually 1^{st} to 5^{th}
time step	fixed	variable
error control	none	yes

The advantages of a BDF method compared to Euler's method are that BDF methods are typically faster at solving the DAE because the algorithm adjusts the step size depending on error estimate information. For a region of a system's trajectory for which the system is not dynamically stiff, the error estimates become small and the algorithm increases the step size. This reduces the overall time taken to solve the DAE. For stiff

regions, that is, regions of the trajectory subject to rapidly changing dynamics, the step size is reduced.

The fixed step size of Euler's method, on the other hand, must be fixed at a value small enough to successfully integrate the worst–case or stiffest region of the trajectory. This causes a much longer solution time than is necessary for accurate results in the nonstiff regions of a trajectory.

Second, the polynomial approximations to the derivative of the BDF method are usually more accurate than the first–order approximation of Euler's method. And the error–control schemes of the BDF methods allow the analyst to extract information from a simulation regarding the numerical behavior of the algorithm. This capability is lacking in Euler's method.

Nevertheless, the BDF methods have one serious drawback compared to Euler's method. Most BDF algorithms contain heuristically–determined parameters that must be specified by the user to obtain the best numerical performance from the algorithm. These parameters amount to "rules–of–thumb" that may require extensive experience on the user's part to manipulate successfully. In contrast, Euler's method is straightforward to implement and, given a small enough step size, usually successful. As computers become faster, the slower progress of Euler's method becomes less significant a drawback. (Of course as computers become faster, analysts tend to tackle more difficult problems. *C'est là le diable.*)

8.4 Numerical examples

In this section are presented the results of numerical simulations of several simple systems. The purpose of this section is to demonstrate the steps by which a Lagrangian DAE and a Hamiltonian DAE are solved using Euler's method or a BDF method. In all cases, the numerical algorithms are implemented modularly and are suitable for systematic application to the DAE of motion. Numerical solutions are implemented in Matlab

script.

EXAMPLE 8.1 (Simple pendulum, Lagrangian DAE) The configuration of the simple pendulum shown in Fig. 8.3 is given by the coordinate pair (u_1, u_2). The rod of fixed length l imposes a single kinematic holonomic constraint. The effort source acting on the mass is the force due to gravity. This problem is formulated as a Lagrangian DAE. The numerical solver is an implementation of Euler's method called `int_dae1`, based on a Matlab script by Fabien [13]. The script file is listed in Appendix F.

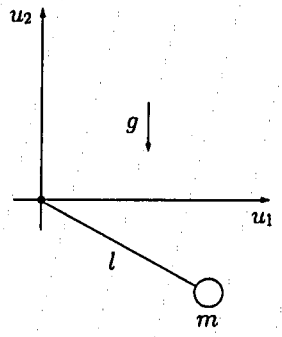


Figure 8.3. A simple pendulum.

Analysis yields $D = V = 0$ and

$$\begin{aligned} T^* &= \tfrac{1}{2} m \left(\dot{u}_1^2 + \dot{u}_2^2 \right) \\ \phi &:= u_1^2 + u_2^2 - l^2 = 0 \\ \delta W &= -mg\, \delta u_2. \end{aligned} \tag{8.26}$$

Letting $v = \dot{u}$, the Lagrangian DAE for this example is given by

$$\begin{aligned} \dot{u} &= v \\ M\dot{v} + C_n^T \lambda &= \Upsilon_0 \\ C &= 0, \end{aligned} \tag{8.27}$$

yielding

$$\begin{aligned} \dot{u} &= v \\ \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \dot{v} + \begin{bmatrix} 2u_1 \\ 2u_2 \end{bmatrix} \lambda &= \begin{bmatrix} 0 \\ -mg \end{bmatrix} \\ \begin{bmatrix} u_1^2 + u_2^2 - l^2 \end{bmatrix} &= 0. \end{aligned} \tag{8.28}$$

Initial conditions, consistent with the constraint, are given by

$$\begin{bmatrix} u_{1_o} & u_{2_o} & v_{1_o} & v_{2_o} \end{bmatrix} = \begin{bmatrix} l\sin(\pi/20) & -l\cos(\pi/20) & 0 & 0 \end{bmatrix}. \tag{8.29}$$

The system parameters are given in terms of consistent units by $l = 2.34$, $m = 1.13$ and $g = 9.81$.

To implement Euler's method using the `int_dae1` solver, a driver function is created to initialize the algorithm and invoke the solver. In the driver the initial conditions are stated, a fixed time step and the time domain of the simulation are denoted, the number of configuration coordinates and the number of constraints are stated, and the solver is invoked. The driver is a brief section of code written by the user. The driver for this example is written for implementation in Matlab, as shown in Fig. 8.4. This script file illustrates the simplicity of setting up a problem for numerical solution once an adequate numerical engine (in this case `int_dae1`) is developed.

As shown in Fig. 8.4, the first argument passed in the call to the solver is the filename `pend_func`. This file, also written by the user, contains the problem–specific components of the descriptor formulation, namely, the matrices M and C_n, the vectors C and Υ_0, and the system parameters. This function file is invoked by the solver at each time step, hence the current value of time t^ν, displacement u^ν and flow v^ν are input arguments. The quantities described in this function file are in general dependent on the state variables and time. The function file for the pendulum example is shown in Fig. 8.5. Comparison of this file with the equations of motion (8.28) illustrates that the file contains

```
% filename: pend_driver.m
% ----- initial conditions
Y0 = [2.34*sin(pi/20);-2.34*cos(pi/20);0;0];

% ----- set numerical parameters
h = 1.0e-3; % time increment
nstep = 4000; % number of time steps
n_disp = 2; % number of displacement coordinates
n_con = 1; % number of kinematic constraints

% ----- invoke the solver. Output Y = [u v] and T is time.
[T,Y] = int_dae1('pend_func',Y0,h,nstep,n_disp,n_con);

% ----- save results for plotting
save pend_results
% endfile
```

Figure 8.4. A driver file for the Lagrangian DAE solver `int_dae1`.

```
% filename: pend_func.m
function[M,C,C_n,Ups_0] = pend_func(u,v,t)
m = 1.13;
L = 2.34;
g = 9.81;
M = [m,0;0,m];
C = [u(1)^2 + u(2)^2 - L^2];
C_n = [2*u(1), 2*u(2)];
Ups_0 = [0;-m*g];
% endfile
```

Figure 8.5. A function evaluation file for the Lagrangian DAE called by the `int_dae1` solver.

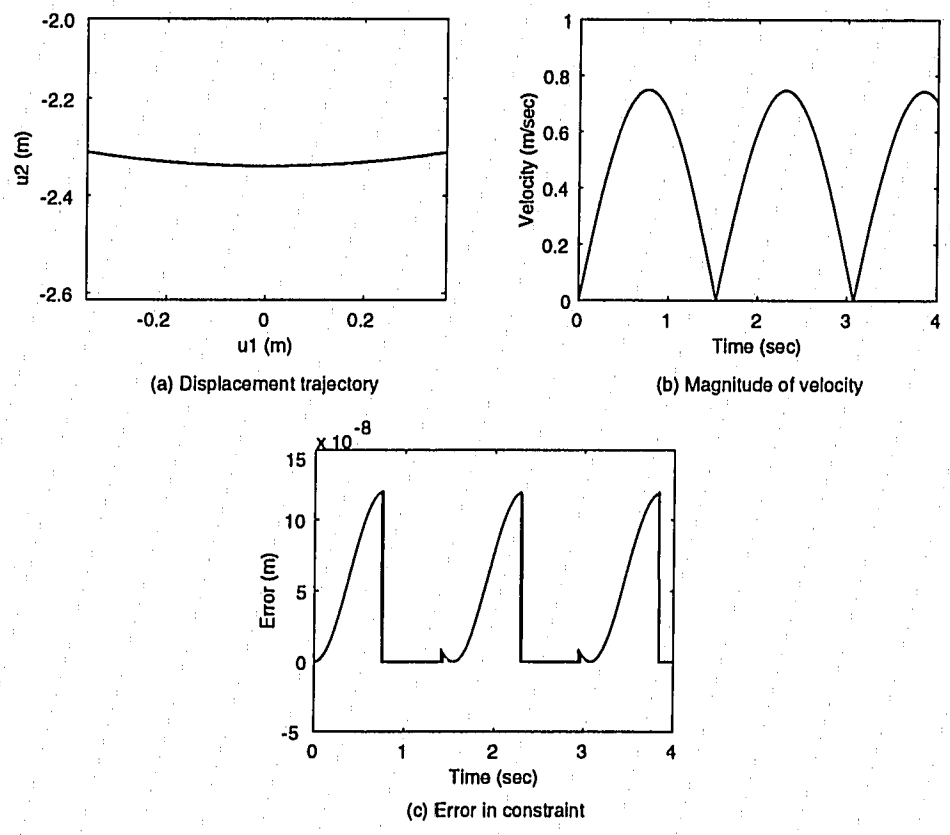


Figure 8.6. Simulation results for a pendulum modeled using a Lagrangian DAE.

straightforward representations of the system functions. This script file too illustrates the simplicity of setting up a problem for numerical solution using the descriptor form of the equations of motion.

The numerical solution of the DAE is obtained by executing the driver file. The output of the simulation is a time response of each of the four state variables (u_1, u_2, v_1, v_2). From these time responses are obtained three results of interest shown in Fig. 8.6. The plot of u_2 vs. u_1 in (a) indicates that the trajectory of the mass does indeed describe a circular arc. The magnitude of the velocity of the pendulum, given by $\sqrt{v_1^2 + v_2^2}$ and

plotted in (b), shows the expected periodic behavior of the pendulum as it oscillates between positions of zero velocity and positions of maximum velocity. The constraint error in (c) is a plot of the function ϕ given by $u_1^2 + u_2^2 - l^2$. In this example the numerical solver has satisfied the constraint to within $10^{-4}\%$ of the maximum displacement. A small constraint error is a necessary condition of accurate numerical results, although in the general case it is not a sufficient condition. The constraint error can be small even for a numerical solution that drifts from the true solution. $\diamond$

EXAMPLE 8.2 (Slider-crank mechanism, Lagrangian DAE) The slider-crank model shown in Fig. 8.7 consists of two point masses and two rigid, massless rods. Rod l_1 pivots about the origin and the mass m_2 is constrained to move only in the u_3 direction, that is, $u_4 = 0$. This problem is formulated as a Lagrangian DAE in terms of coordinates (u_1, u_2, u_3). The effort source F^s is assumed to act in a line perpendicular to link l_1.

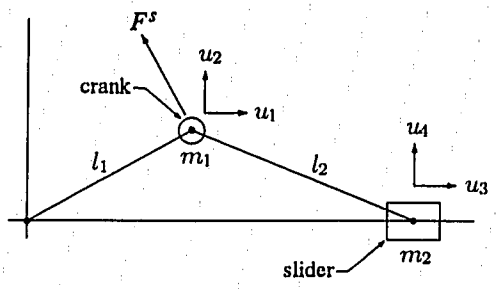


Figure 8.7. A slider-crank mechanism.

Analysis yields $D = V = 0$ and

$$\begin{aligned} T^* &= \tfrac{1}{2} m_1 \left(\dot{u}_1^2 + \dot{u}_2^2 \right) + \tfrac{1}{2} m_2 \dot{u}_3^2 \\ \phi_1 &:= u_1^2 + u_2^2 - l_1^2 = 0 \\ \phi_2 &:= (u_3 - u_1)^2 + u_2^2 - l_2^2 = 0 \\ \delta W &= -F^s \frac{u_2}{l_1} \delta u_1 + F^s \frac{u_1}{l_1} \delta u_2. \end{aligned} \tag{8.30}$$

Letting $v = \dot{u}$, the Lagrangian DAE for this example is given by

$$\begin{aligned} \dot{u} &= v \\ \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_1 & 0 \\ 0 & 0 & m_2 \end{bmatrix} \dot{v} + \begin{bmatrix} 2u_1 & 2(u_1 - u_3) \\ 2u_2 & 2u_2 \\ 0 & 2(u_3 - u_1) \end{bmatrix} \lambda &= \begin{bmatrix} -F^s \dfrac{u_2}{l_1} \\ F^s \dfrac{u_1}{l_1} \\ 0 \end{bmatrix} \\ \begin{bmatrix} u_1^2 + u_2^2 - l_1^2 \\ (u_3 - u_1)^2 + u_2^2 - l_2^2 \end{bmatrix} &= 0. \end{aligned} \tag{8.31}$$

Initial conditions, consistent with the constraints, are given by

$$\begin{bmatrix} u_o & v_o \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 & 0 & 0 & 0 \end{bmatrix}. \tag{8.32}$$

The system parameters are given in terms of consistent units by $m_1 = 2.26$, $m_2 = 1.13$, $l_1 = \sqrt{2}$, $l_2 = \sqrt{5}$ and $F^s = 10$.

A driver file and a function file similar to those given in the previous example are written to obtain a numerical solution using the `int_dae1` solver. The time responses of the state variables $(u_1, u_2, u_3, v_1, v_2, v_3)$ are used to obtain the following results of interest. The trajectory of the point mass m_1 describes a circular trajectory as shown in Fig. 8.8(a). The position of the slider is shown in (b), which clearly depicts the transient response as the system starts from rest. The error in the first constraint, given by $u_1^2 + u_2^2 - l_1^2$, is plotted in (c) and the error in the second constraint, given by $(u_3 - u_1)^2 + u_2^2 - l_2^2$, is plotted in (d). These errors are negligible, indicating at least that a necessary condition for an accurate simulation is satisfied.◇

EXAMPLE 8.3 (Lead–filter, Lagrangian DAE) The lead–filter model shown in Fig. 8.9 is a simple electrical circuit containing a capacitor and two resistors driven by a voltage source $e^s(t)$. The system variable of interest is the output voltage $e^o(t)$. This problem is formulated as a Lagrangian DAE in terms of coordinates (q_1, q_2, q_3) and a numerical

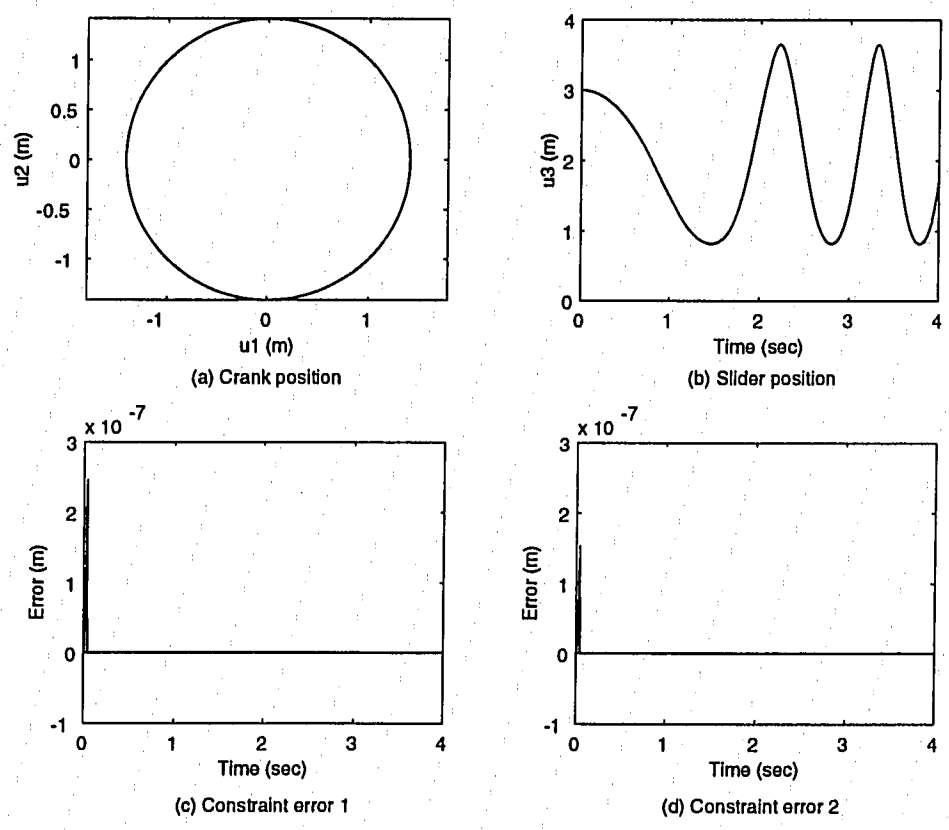


Figure 8.8. Simulation results for a slider-crank modeled using a Lagrangian DAE.

solution is obtained. Since there is no inductor in this circuit the kinetic coenergy function is zero and the inertial matrix M is singular.

Analysis yields $T^* = 0$ and

$$\begin{aligned} V &= \frac{q_2^2}{2C} \\ D &= \tfrac{1}{2}R_1\dot{q}_1^2 + \tfrac{1}{2}R_2\dot{q}_3^2 \\ \phi &:= q_1 + q_2 - q_3 = 0 \\ \delta W &= e^s\,\delta q_3. \end{aligned} \tag{8.33}$$

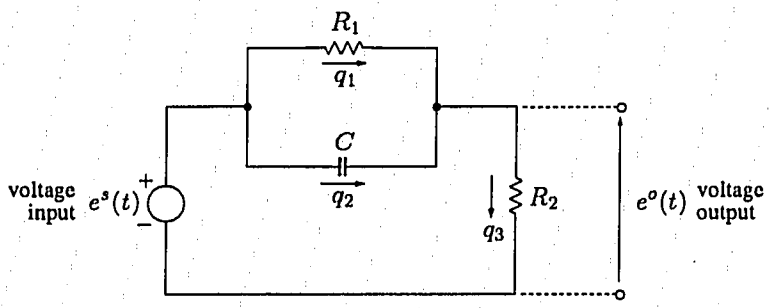


Figure 8.9. A lead-filter.

Here the kinematic constraint must be expressed in holonomic form since the constraint includes q_2 which represents a physical accumulation of displacement or charge of the capacitor. Letting $f = \dot{q}$, the Lagrangian DAE for this example is given by

$$\begin{aligned} \dot{q} &= f \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{f} + \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \lambda &= \begin{bmatrix} -R_1 f_1 \\ -\dfrac{q_2}{C} \\ e^s - R_2 f_3 \end{bmatrix} \qquad (8.34) \\ q_1 + q_2 - q_3 &= 0. \end{aligned}$$

Initial conditions, consistent with the constraint, are given by

$$\begin{bmatrix} q_o & f_o \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (8.35)$$

The system parameters are given in terms of consistent units by $R_1 = R_2 = C = 0.5$ and $e^s(t) = \sin 8\pi t$.

A driver file and a function file are written to obtain a numerical solution using the `int_dae1` solver. The time responses of the state variables $(q_1, q_2, q_3, f_1, f_2, f_3)$ are used to obtain the following results of interest. The desired output variable $e^o(t)$ is the difference between the effort source and the voltage drop across the parallel resistor and

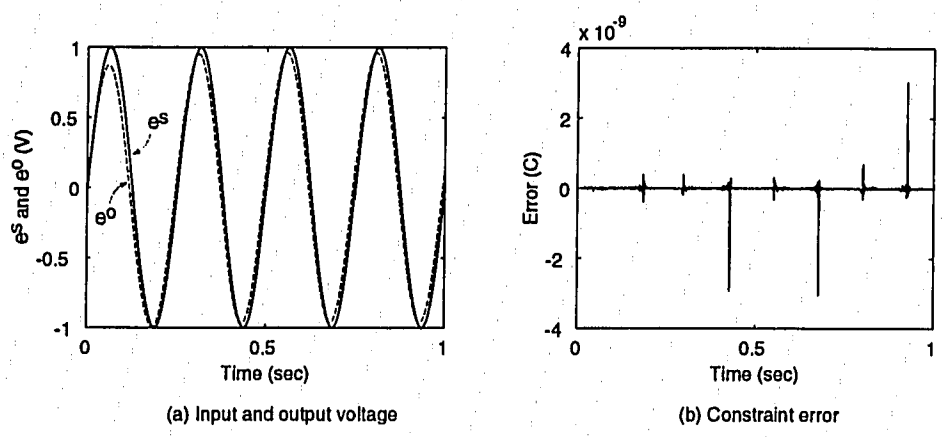


Figure 8.10. Simulation results for a lead–filter modeled using a Lagrangian DAE.

capacitor given by

$$e^o(t) = e^s(t) - R_1 f_1(t). \tag{8.36}$$

In Fig. 8.10(a) the voltage output $e^o(t)$ is plotted with the input voltage $e^s(t)$ shown for reference. The figure shows that the output voltage, for the given system parameters, has a small phase–shift and a small reduction in amplitude compared to the input voltage. This result is identical to the time response obtained from the lead–filter transfer function $G(s)$ given by

$$G(s) = \alpha \frac{\tau s + 1}{\alpha \tau s + 1}, \tag{8.37}$$

where

$$\alpha := \frac{R_2}{R_1 + R_2} \qquad \text{and} \qquad \tau := R_1 C, \tag{8.38}$$

subject to the input function $e^s(t)$

The error in the constraint, given by $q_1 + q_2 - q_3$ and plotted in (b), is negligible. The spikes in the constraint error trajectory arise due to changing numerical properties of the constraint Jacobian at each time step in the simulation. In particular, at a given time step certain columns of the Jacobian matrix are independent. At the next time

step certain other columns, not necessarily the same ones, are independent. The spikes in the constraint trajectory are associated with this phenomenon. ◇

EXAMPLE 8.4 (Fluid lead–filter, Lagrangian DAE) The fluid system shown in Fig. 8.11 is structurally similar to the electrical lead–filter of the previous example. The tank in this system is modeled as a fluid capacitor and the two valves are modeled as nonlinear resistors.This problem is formulated as a Lagrangian DAE in terms of coordinates (u_1, u_2, u_3) and a numerical solution is obtained.

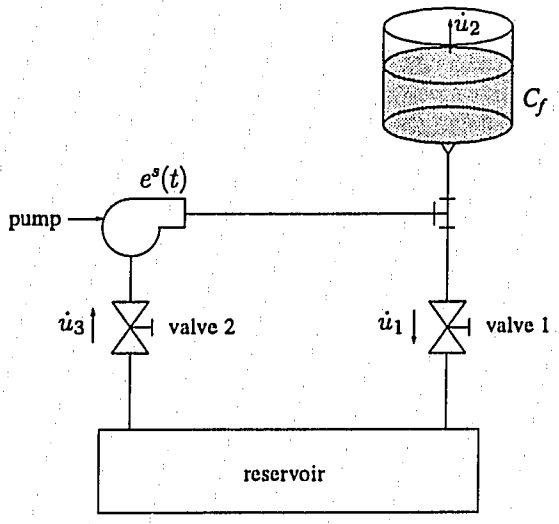


Figure 8.11. A fluid lead–filter.

The fluid is incompressible. The tank has fluid capacitance C_f given by

$$C_f = \frac{A_t}{\rho g}, \tag{8.39}$$

where A_t is the tank cross-sectional area, ρ is the fluid density and g is the acceleration due to gravity. The two valves have constituitive laws given by

$$e_\alpha(\dot{u}) = C_r \dot{u} \left| \sqrt{\dot{u}} \right|, \tag{8.40}$$

where e_{cl} is the pressure drop across the valve, C_r is a resistance coefficient and $\dot{u}$ is the flow through the valve. The pressure source is given by

$$e^s(t) = 10^6 \log(t+1). \tag{8.41}$$

Analysis yields $T^* = 0$ and

$$\begin{aligned} V &= \frac{u_2^2}{2C_f} \\ D &= -\sum_j \int e_j(\dot{u})\, d\dot{u}_j = \sum_j \int e_{cj}(\dot{u})\, d\dot{u}_j \\ &= \int C_{r1}\dot{u}_1 \left|\sqrt{\dot{u}_1}\right| d\dot{u}_1 + \int C_{r2}\dot{u}_3 \left|\sqrt{\dot{u}_3}\right| d\dot{u}_3 \\ \phi &:= u_1 + u_2 - u_3 - (u_{1_o} + u_{2_o} - u_{3_o}) = 0 \\ \delta W &= e^s\, \delta u_3. \end{aligned} \tag{8.42}$$

Here the kinematic constraint must be expressed in holonomic form since the constraint includes u_2 which represents a physical accumulation of displacement or volume of the capacitor. The dissipation function integrals need not be evaluated since it is the integrands, that is, the partial derivatives $\partial D/\partial \dot{u}_j$, which are used in Lagrange's equation. Letting $v = \dot{u}$ and assuming $u_{1_o} = u_{3_o} = 0$ the Lagrangian DAE for this example is given by

$$\begin{aligned} \dot{u} &= v \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{v} + \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \lambda &= \begin{bmatrix} -C_{r1} v_1 \left|\sqrt{v_1}\right| \\ -\dfrac{u_2}{C_f} \\ e^s - C_{r2} v_3 \left|\sqrt{v_3}\right| \end{bmatrix} \\ u_1 + u_2 - u_3 - u_{2_o} &= 0. \end{aligned} \tag{8.43}$$

Initial conditions, consistent with the constraint equation, are given by

$$\begin{bmatrix} u_o & v_o \end{bmatrix} = \begin{bmatrix} 0 & 5 & 0 & 1 & -1 & 0 \end{bmatrix}. \tag{8.44}$$

The system parameters, in consistent units, are given by $\rho = 500$, $g = 9.81$, $C_{r1} = 100$ and C_{r2} is given by

$$C_{r2} = \frac{\rho}{2C_d^2 A_o^2}, \tag{8.45}$$

where $C_d = 0.62$ is a discharge coefficient and A_o is an orifice area based on a diameter $d_o = 0.1$. The tank diameter A_t is based on a tank diameter $d_t = 17.7$.

A driver file and a function file are written to obtain a numerical solution using the `int_dae1` solver. In Fig. 8.12(a) is shown the time response of the tank volume u_2. From an initial positive value, the volume decreases initially due to the initial flow rate conditions. As the pressure output of the pump increases, the flow rate $\dot{u}_3$ increases and the tank is refilled and approaches a steady-state value. The flow rates $(\dot{u}_1, \dot{u}_2, \dot{u}_3)$ are shown in (b). The error in the constraint, given by $u_1 + u_2 - u_3 - u_{2_o}$ and plotted in (c), is negligible.◇

EXAMPLE 8.5 (Pendulum with spring, Hamiltonian DAE) The pendulum shown in Fig. 8.13 is free to pivot about point O at the end of a rod attached to a spring constrained to move in the vertical direction only. The spring has stiffness k, the vertical rod is subject to damping b and pendulum has mass m. The rods are rigid and massless. The effort source is the force due to gravity.

This problem is formulated as a Hamiltonian DAE in terms of inertial coordinates (x_1, x_2, p_1, p_2) and a noninertial displacement coordinate x_3. The numerical solver is an implementation of Euler's method based on `int_dae1` that is modified to solve the Hamiltonian DAE. A listing of the Hamiltonian solver `int_daeh` is listed in Appendix F.

Analysis yields

$$\begin{aligned} T &= \frac{p_1^2}{2m} + \frac{p_2^2}{2m} \\ D &= \tfrac{1}{2} b \dot{x}_3^2 \\ V &= \tfrac{1}{2} k x_3^2 \end{aligned} \tag{8.46}$$

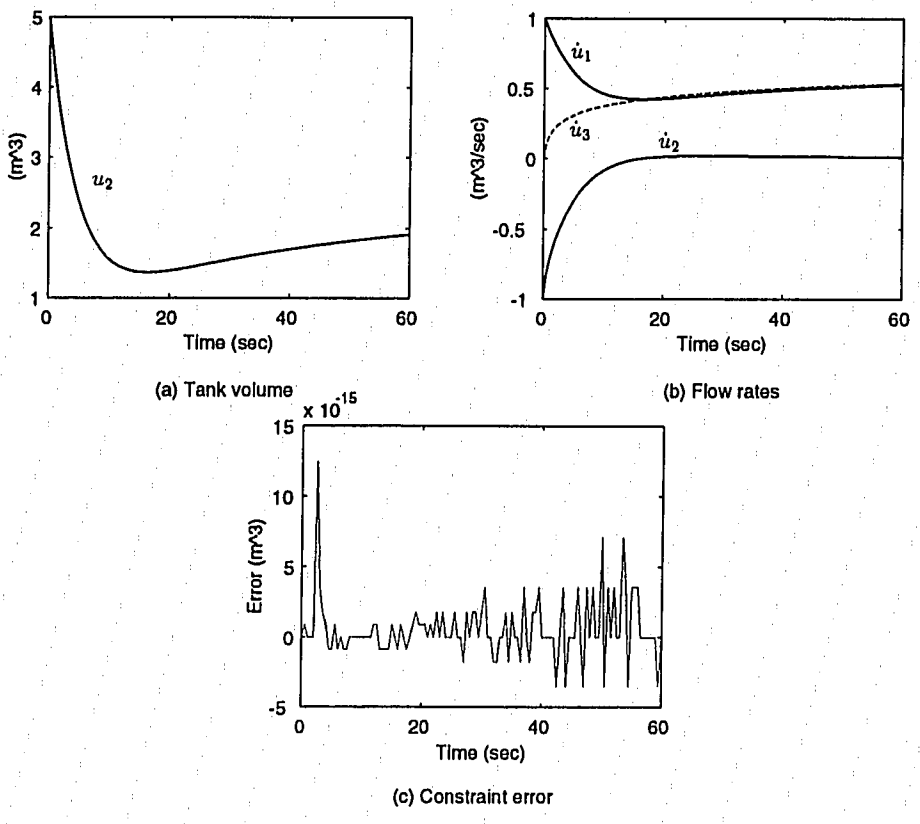


Figure 8.12. Simulation results for a fluid lead–filter modeled using a Lagrangian DAE.

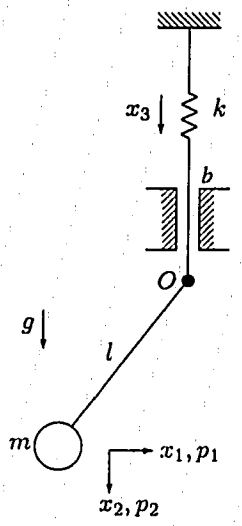


Figure 8.13. A pendulum and spring.

$$\begin{aligned} \phi &:= (x_2 - x_3)^2 + x_1^2 - l^2 = 0 \\ \delta W &= mg\,\delta x_2. \end{aligned}$$

Letting $v = \dot{x}$, the Hamiltonian DAE for this example is given by

$$\begin{aligned} \dot{x} &= v \\ \dot{p} + C_s^T \lambda &= \Upsilon_s \\ C_r^T \lambda &= \Upsilon_r \\ v^s &= \nabla_p T \\ C &= 0, \end{aligned} \tag{8.47}$$

yielding

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} + \begin{bmatrix} 2x_1 \\ 2(x_2 - x_3) \end{bmatrix} \lambda = \begin{bmatrix} 0 \\ mg \end{bmatrix}$$

$$
\begin{aligned}
[2(x_3 - x_2)]\,\lambda &= [-kx_3 - bv_3] \\
\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} \frac{p_1}{m} \\ \frac{p_2}{m} \end{bmatrix} \\
\left[(x_2 - x_3)^2 + x_1^2 - l^2\right] &= 0.
\end{aligned}
\tag{8.48}
$$

Initial conditions, consistent with the constraint, are given by

$$
\begin{bmatrix} x_{1_o} & x_{2_o} & v_{1_o} & v_{2_o} & p_{1_o} & p_{2_o} & x_{3_o} & v_{3_o} \end{bmatrix} = \begin{bmatrix} 3 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{8.49}
$$

The system parameters are given in terms of consistent units by $l = \sqrt{34}$, $m = 5$ and $g = 9.81$, $k = 100$ and $b = 50$.

To implement Euler's method using the `int_daeh` solver, a driver function is created to initialize the algorithm and invoke the solver. The driver for this example is written for implementation in Matlab, as shown in Fig. 8.14. This script file illustrates the simplicity of setting up a problem for numerical solution once an adequate numerical engine (in this case `int_daeh`) is developed.

As shown in Fig. 8.4, the first argument passed in the call to the solver is the filename `ham_func`. This file, also written by the user, contains the problem–specific components of the Hamiltonian formulation, namely, the matrices C_s and C_r, the vectors C, Υ_s, Υ_r and $\nabla_p T$, and the system parameters. The function file for the pendulum example is shown in Fig. 8.15. Comparison of this file with the equations of motion (8.48) illustrates that the file contains straightforward representations of the system functions. This script file too illustrates the simplicity of setting up a problem for numerical solution using the Hamiltonian DAE.

The numerical solution of the DAE is obtained by executing the driver file. The output of the simulation is a time response of the state variables $(x_1, x_2, x_3, v_1, v_2, v_3, p_1, p_2)$. From these time responses are obtained three results of interest shown in Fig. 8.16.

```
% filename: ham_driver.m
% ----- nomenclature
%       x: inertial displacements
%       y: inertial flows
%       p: inertial momenta
%       w: noninertial displacements
%       z: noninertial flows
%       lam: multipliers
%       egam: implicit efforts

% ----- initial conditions [x0,y0,p0,w0,z0,lam0,egam0]
Y0=[3;5;0;0;0;0;0;0;0];

% ----- set numerical parameters
h = 1; % step size
nstep = 10; % number of time steps
s = 2; % number of inertial coordinates
r = 1; % number of non--inertial coordinates
m0 = 1; % number of kinematic constraints
m3 = 0; % number of dynamic constraints

% ----- invoke the solver. Output Y = [x y p w z] and T is time.
[T,Y] = int_daeh('ham_func',Y0,h,nstep,s,r,m0,m3);

% ----- save results for plotting
save ham_results
```

Figure 8.14. A driver file for the Hamiltonian DAE solver `int_daeh`.

```
% filename: ham_func
function [C_s,C_r,Ups_s,Ups_r,grad_T,C,Gamma] ...
                    = ham_func(x,y,p,w,z,egam,t)
l = sqrt(34); % length of pendulum
m = 5; % mass of pendulum
g = 9.81; % gravity
k = 100; % spring constant
b = 50; % damping constant
C_s = [2*x(1) 2*(x(2)-w(1))];
C_r = [2*(w(1)-x(2))];
Ups_s = [0;m*g];
Ups_r = [-k*w(1)-b*z(1)];
grad_T = [1/m*p(1);1/m*p(2)];
C = [(x(2)-w(1))^2+x(1)^2-l^2];
Gamma = []; % null vector in this case
% endfile
```

Figure 8.15. A function evaluation file for the Hamiltonian DAE called by the int_daeh solver.

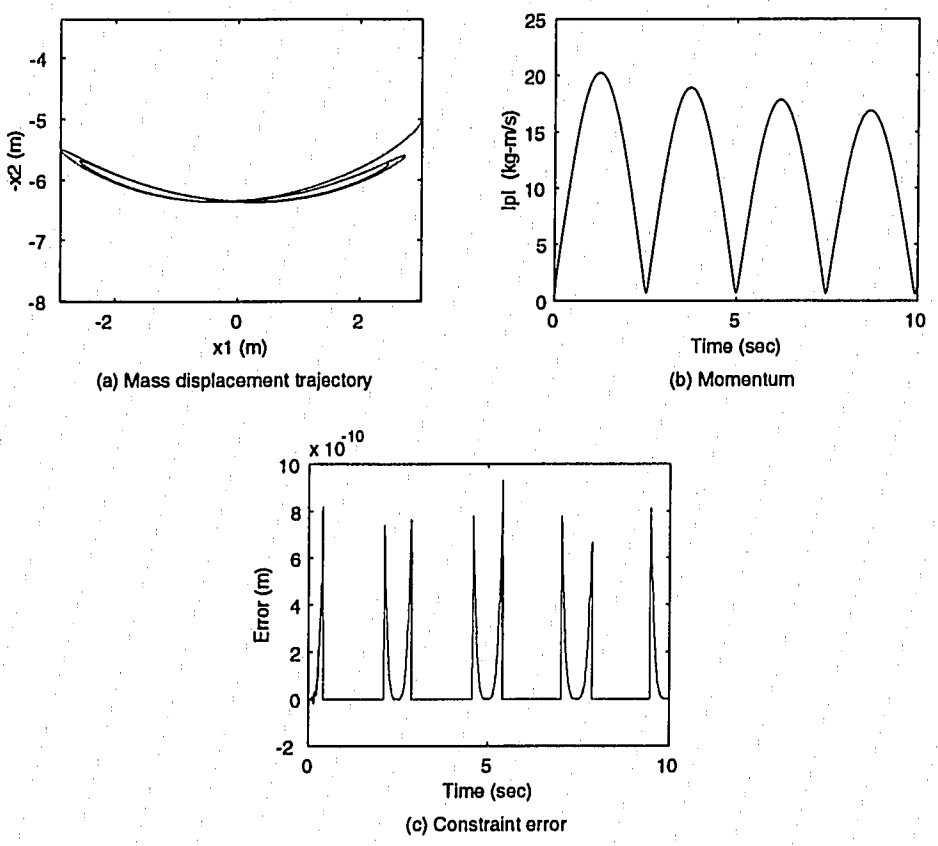


Figure 8.16. Simulation results for a pendulum and spring modeled using a Hamiltonian DAE.

The plot of x_2 vs. x_1 in (a) indicates that the trajectory of the mass describes an arc that also has a vertical component due to the spring. The magnitude of the momentum of the mass, given by $\sqrt{p_1^2 + p_2^2}$ and plotted in (b), shows the expected periodic behavior of the pendulum as it oscillates between positions of minimum momentum and positions of maximum momentum. Because of the motion due to the spring, the mass does not return to a state of zero momentum in the time interval shown. The constraint error in (c) is negligible. $\diamond$

This example illustrates also that the use of momentum coordinates need not be an impediment to obtaining physically recognizable results. The mass of the pendulum is constant, hence the momentum trajectory divided by m yields the velocity trajectory. The numerical result is more quickly assessed for reasonableness when expressed in terms of velocity than when expressed in terms of momentum. This example is simple, but the concept is applicable to the general problem.

EXAMPLE 8.6 (Lead filter, BDF method) In Ex. 8.3 an electrical lead–filter is modeled using a Lagrangian formulation and the numerical simulation is based on Euler's method as implemented in `int_dae1`. In this example the same Lagrangian formulation is solved using a BDF method as implemented in the solver called `int_bdf1`. This solver is based on the DASSL algorithm by Petzold [6]. The Matlab implementation of this solver is listed in Appendix F. The driver file and function file for invoking `int_bdf1` are modules similar to those used for invoking `int_dae1`.

The state variables $(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3)$ in this example represent electrical displacement and flow. The numerical result obtained using the BDF method is identical to the result obtained using Euler's method (see for example Fig. 8.10) except for the variable q_1 which is the displacement or integrated current of the resistor R_1. For this variable the BDF solution is inconsistent with the Euler solution. This inconsistency is correlated to the maximum step size used in the integration.

The basic BDF algorithm varies the integration step size h based on error estimates. The purpose of varying h is to improve the speed of the simulation. In theory the lower limit to h is machine epsilon and no upper limit is required. In practice an upper limit is desirable to control drift in the solution. As shown in Fig. 8.17, as the maximum h in the BDF method is reduced from 10^{-1} to 10^{-3}, the solution for q_1 approaches the Euler result obtained with a fixed step size of 10^{-5}.

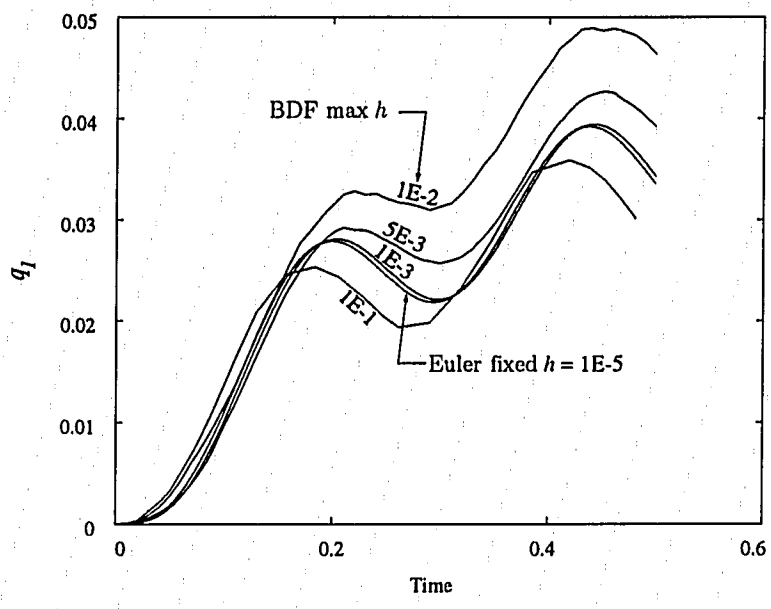


Figure 8.17. Drift in numerical result using the BDF method. The solution for the state variable q_1 of the electrical lead filter example is plotted for different maximum step sizes. The Euler–method result is shown for comparison.

This figure illustrates both an advantage and a disadvantage of the BDF method compared to Euler's method. The advantage of the BDF method is that a solution nearly identical to an Euler's–method solution is obtained with a maximum step size two orders of magnitude larger than the fixed step size of Euler's method. Thus the BDF simulation runs much faster than the Euler simulation. The disadvantage of the BDF method is that the selection of a maximum step size for a given problem is entirely arbitrary. The analyst must choose a maximum step size based on experience with the algorithm and the class of problem to be solved.

This "rule–of–thumb" aspect to selecting a step size is representative of numerous aspects of available numerical codes that implement BDF algorithms. These codes are efficient, but are rife with *ad hoc* parameters aimed at improving efficiency or gauging accuracy. The analyst must understand and adjust these numerical parameters to use

these codes with confidence. ◇

8.5 Automated modeling and simulation

The automation of the system modeling and simulation procedure can be accomplished by combining a set of existing software tools. This section briefly explains how such an automated modeling and simulation package can be implemented on a computer that uses a Unix based operating system.[1] The basic philosophy is to break the modeling and simulation procedure into a set of subtasks. Each subtask is solved by calling its own specialized routine. All these specialized routines are in turn coordinated by a driver program. This approach has been recently applied to the solution of finite time optimal control problems [12].

Having performed the analysis previously outlined, the analyst will have available all the equations necessary to evaluate the dynamic behavior of the system. An example of such a set of equations is given by (8.11). From these equations an automated software package can be used to symbolically formulate and solve the Lagrangian DAE given appropriate initial conditions.

8.5.1 Formulation of the Lagrangian DAE

The construction of the Lagrangian DAE involves the computation of several partial derivatives. A criticism of the Lagrangian approach to modeling mechanical systems is that these partial derivatives are tedious to compute and provide opportunities for the analyst to make errors. Recently however, a variety of software tools have become available for the efficient and accurate computation of these partial derivatives. In particular, symbolic manipulation tools such as Maple or Mathematica can be easily programmed to formulate the Lagrangian DAE [7,50].

Alternatively, the partial derivatives can be computed using Fortran or C callable

[1]This section is based on material provided by B. C. Fabien.

subroutines such as ADIFOR [1], Odyssée [36], ADIC [2] and ADOL-C [8,18]. These subroutines are particularly well suited for integration into a package for modeling and simulation since most are in the public domain and unlike Maple and Mathematica are available at no cost. Also, the source code for these subroutines can be modified to accommodate the formulation of the Lagrangian DAEs.

8.5.2 Solution of the Lagrangian DAE

The Lagrangian DAE is typically index–3. The numerical solution of an index–3 DAE can be accomplished by integrating the DAE directly or by reducing the index of the DAE. The first approach is a current area of research and only experimental software such as `int_dae1` [13] is available.

Differentiating the displacement constraints with respect to time will give a flow constraint and reduce the DAE from index–3 to index–2. If the system is a mechanical system then the index–2 DAE can be integrated using the subroutine MEXX [23] or HEM5 [3]. MEXX uses an extrapolation technique to solve the index–2 DAE arising from a constrained mechanical system, whereas HEM5 uses a half–explicit Runge–Kutta method to solve these systems. The use of MEXX and HEM5 for the integration of DAEs arising from multidiscipline systems has not been explored.

The index–3 DAE can be reduced to an index–1 DAE by differentiating the displacement constraints twice and the flow constraints once with respect to time. There are a number of subroutines that can be used to solve the resultant index–1 DAE. These include DASSL [6], LSODI [20], and LIMEX [10]. DASSL solves a fully implicit index–1 DAE using a BDF method. LSODI solves a linearly implicit index–1 DAE also using a BDF. LIMEX solves a linearly implicit index–1 DAE using the extrapolation method. All of these codes are in the public domain and are available at no cost.

Although these subroutines are efficient at solving the reduced index–1 DAE, there is no guarantee that the displacement and flow constraints will be satisfied at each time

step. To ensure that the displacement and flow constraints are satisfied at each time step the solutions produced by the subroutines must be projected onto the constraint surface [23]. This projection step can be most easily accomplished by modifying the existing subroutines.

In an automated approach to modeling and simulation the user would produce a text file that describes the system in terms of the displacement, flows, energies, constraints and applied efforts. The coordinating program will then call the differentiation routine (Maple, ADOL–C, *etc.*) in order to formulate the Lagrangian DAE. The DAE can then be written in a form that is appropriate for the numerical solver (DASSL, LIMEX, *etc.*). On computers that use a Unix based operating system such coordinating programs are simple to construct [12]. The overall structure of such an approach to automated modeling and simulation is outlined in the flow chart in Fig. 8.18.

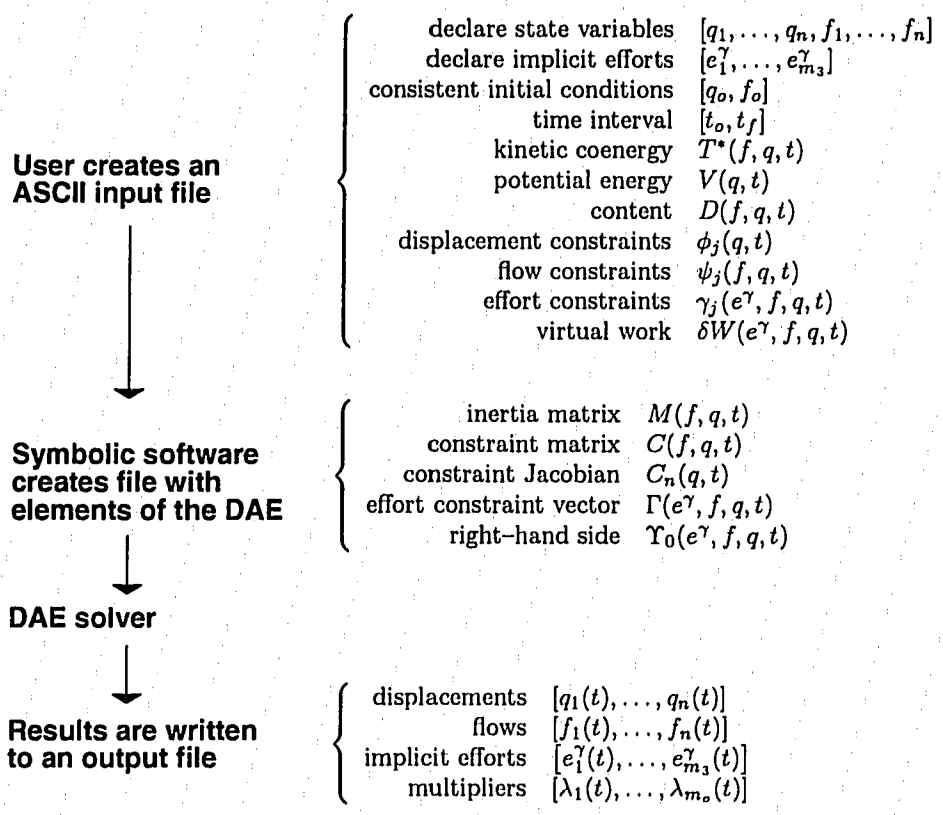


Figure 8.18. An approach to automated modeling and simulation. The variables indicated are for a Lagrangian DAE formulation.

Chapter 9

CONCLUSION

The method of analysis presented in this work results in continuous–time, discrete–element mathematical models of multidiscipline systems subject to deterministic excitations. Nonholonomic equality constraints and time–varying parameters are accommodated. Nonlinear equations of motion are formulated in terms of a unified set of variables. System elements are classified according to the manner in which they manipulate energy, and energy functions are represented by the complementary forms designated energy and coenergy. Lagrange's equation is derived for multidiscipline systems, undetermined multipliers are introduced and Lagrange's equation is formulated as a differential–algebraic equation (DAE). Through application of a partial Legendre transform, Hamilton's equation is newly formulated as a DAE. Complementary or dual forms of the Lagrangian DAE and the Hamiltonian DAE are derived.

The basic contribution of this work to the study of dynamic systems is the systematic exposition of the principles of analytical dynamics applied to the problem of modeling multidiscipline engineering systems in the time domain. The resulting modeling technique, combined with methods of solving DAEs numerically, eases the task of modeling complex multidiscipline systems. This result is accomplished by focusing the analyst's effort on the energy transactions and system geometry that underlie the dynamic behavior of a system. With the recent advent of computer software capable of reliable manipulation of symbolic equations, the task of systematically obtaining equations of motion is readily automated. The numerical solution of the resulting differential–algebraic equations of motion is an area of ongoing research.

In addition to this primary result of simplifying the task of modeling complex systems, this research has other features of interest. The concept of dynamic constraints is intro-

duced, augmenting the traditional constraint classifications from analytical dynamics. A new derivation of Lagrange's equation is given, starting from a differential–variational form of the first law of thermodynamics. The derivation fills a gap in the literature by producing a general, multidiscipline form of Lagrange's equation that is less restrictive than forms traditionally presented in the literature. In the traditional derivation it is common to assume 1) constant inertial coefficients, 2) quadratic kinetic coenergy, 3) catastatic energy functions (that is, energy functions that are not explicit functions of time) and 4) catastatic, holonomic constraints. These assumptions are not invoked in this derivation, nor does their absence complicate the exposition.

Another feature of interest of the new derivation is that even though the variational operator δ is used, neither the calculus of variations from which the operator arises, nor its usual handmaiden in mechanics — Hamilton's principle — is invoked in this work. Consequently, the equations of motion retain their physical significance as embodiments of the work–energy equivalence of the first law of thermodynamics. It also happens that the requisite mathematics of this approach is appropriate to the senior undergraduate, making the new approach accessible to the engineer who has only casual knowledge, if any, of analytical methods.

A primary objective of this research is to examine the consequences of selecting different variable pairs to represent the motion of a multidiscipline system. The result is a quartet of DAE formulations of the equations of motion: the Lagrangian DAE, the Hamiltonian DAE, the co–Lagrangian DAE and the co–Hamiltonian DAE. In spite of their differences, these formulations have two properties in common. First, each formulation is intended to be applicable to the general multidiscipline problem. Second, each formulation is structured for numerical solution.

To satisfy the first requirement, the equations of motion are formulated such that the class of systems to which the method can be applied (continuous–time, discrete elements, *etc.*) is subjected to as few restrictions as possible. Consequently, it is sometimes the

case that the DAE could be manipulated further to obtain a set of equations having a different form favored by a particular analyst. This possibility is especially likely when the system is a single–discipline system to which the mechanical engineer, electrical engineer, hydraulic engineer or thermodynamicist is predisposed to apply his or her customary set of analytical tools in formulating and solving the differential equations of motion. However, research in system dynamics is motivated in part by a strong desire to simplify multidiscipline analysis through the development of unified methods to augment, if not replace, the patchwork of single–discipline methods in common use. The DAE structures used in this work are consistent with this concept of general multidiscipline applicability.

Suitability for numerical solution is the second consideration used in selecting the structure of the DAE. The descriptor form of the Lagrangian DAE and its dual and the semi–explicit form of the Hamiltonian DAE and its dual are the forms best suited for numerical solution given the general applicability requirement. Using these formulations the DAE may be index–3 or higher, but the numerical solution of higher–index DAEs is a current research topic. Progress is being made in obtaining consistent, meaningful results using algorithms tailored for the descriptor form. The semi–explicit form too has a structure that can be exploited when developing a numerical solution algorithm. The continued development of numerical methods that reliably solve these DAEs, an important area of research in its own right, is necessary for the benefits of this analytical method to be fully realized. This is an important topic for future research.

The basic differences among the four formulations are due to the different variable pairs used to represent motion. The Lagrangian DAE, which represents the motion of a system in terms of displacement and flow, is the formulation most familiar to dynamicists. Displacement and flow variables are usually easy to assign to a system, displacement and flow constraints are usually more readily discerned than effort and momentum constraints, and numerical solutions in terms of displacement and flow are

readily checked for constraint compliance as well as physical significance. The analyst can quickly assess the reasonableness of a solution to the Lagrangian DAE.

In contrast, with the co-Lagrangian DAE the motion of a system is represented in terms of momentum and effort variables. Coordinate assignment, constraint formulation, and solution assessment in terms of these variables are likely to be tasks that are both unfamiliar and difficult for the analyst, particularly for systems with complex spatial geometries. With practice, the drawback of unfamiliarity can be overcome. But for systems subject to kinematic constraints for which the corresponding momentum constraints are not easily discerned, the co-Lagrangian DAE is not an appropriate formulation.

The Hamiltonian DAE represents the motion of a system in terms of displacement and the momenta of kinetic stores. The Hamiltonian DAE is in semi-explicit form, which may prove to have superior numerical properties compared to the descriptor form of the Lagrangian DAE. And since the constraints of the Hamiltonian DAE are formulated in the same manner as in the Lagrangian DAE, the Hamiltonian formulation retains the ease of constraint formulation and solution assessment that characterizes the Lagrangian formulation. One drawback of the Hamiltonian solution is the average analyst's unfamiliarity with assigning coordinates, expressing energy functions and interpreting results in terms of momentum coordinates. This difficulty too, however, can be overcome with practice.

The co-Hamiltonian DAE represents the motion of a system in terms of momentum and the displacements of potential stores. Constraints are formulated in the same manner as in the co-Lagrangian DAE, that is, in terms of momentum and effort. Thus the same reservations expressed about constraints in co-Lagrangian form apply to the co-Hamiltonian form. Yet, unlike the co-Lagrangian, the co-Hamiltonian is a semi-explicit DAE, and may prove to have superior numerical properties compared to the descriptor form of the co-Lagrangian DAE.

Considering these general characteristics of the four DAE formulations, it is apparent

that the type of elements and the type of constraints present in a particular system, and the ease with which they are discerned and formulated, bear significantly on the choice of an appropriate formulation for modeling the system. System elements determine the energy modes in a system. Constraints describe the interconnections among the elements. That these are the important factors to consider in choosing a formulation is not a surprising conclusion, since the basic premise of system dynamics is that the dynamic response of a system is the observable manifestation of a system's energy transactions. And energy, by the first law, is both a function of the state of the system elements and a function of the path between successive states, where the path is determined in part by the constraining relationships. Future work in prescribing a set of guidelines for matching a system to an appropriate formulation should be based on this fundamental relationship between energy and dynamic behavior.

The utility of an energy–based approach to multidiscipline dynamics does not end with the selection of a DAE formulation. There are two areas of research in particular, important in the analysis of dynamic systems, that are possibly tractable from the foundation developed in this research. The first topic is inequality constraints and the second topic is stability.

Inequality constraints as a class of nonholonomic constraints are briefly described in this work, but no technique is offered for accounting for them in the equations of motion. Yet inequality constraints abound in the physical world. Common examples are dry friction and the more complicated phenomenon called stiction. Both are nonlinear functions with discontinuities that can be represented in part using inequalities. The systematic inclusion of such constraints in the DAE would have far–reaching consequences for modeling in general.

The possibility that the energy method developed in this work may make inequality constraints tractable rests on the demonstrable relationships among energy functions, multiplier theory and optimization theory. It is known that satisfying Lagrange's equa-

tion (for holonomic systems only) is a necessary, though not sufficient, condition for finding a function that makes stationary an integral functional. This is the fundamental problem in the calculus of variations. Equality and inequality constraints specify a certain subset of function spaces in which a solution may lie. The method of undetermined multipliers is used to incorporate both equality and inequality constraints in this optimization problem, though the conditions of applying the multiplier theory are more complex than those cited in this work.

The problem to be overcome in making the transition from the optimization problem to the general multidiscipline dynamics problem is that the trajectory of a nonholonomic system does not satisfy a stationarity requirement or, as it is usually called, a least action principle. Hence the solution to the Lagrangian DAE, for example, does not necessarily make stationary any particular integral. To bring the advanced analytical tools of contemporary variational calculus to bear on the problem of inequality constraints in multidiscipline systems, a rigorous connection must be made between optimization theory and the DAEs of motion. This is a significant and difficult problem.

The second topic of importance related to this work is stability. Stability theory is well–developed for linear catastatic (or *autonomous*) systems, but the DAEs developed in this work are generally neither linear nor autonomous. The basic methods for investigation of stability of dynamic systems are the indirect method of linearizing a system about equilibrium points and examining small perturbations, and the direct method of Liapunov. Since Liapunov functions for nonautonomous systems are difficult to construct, such systems are usually linearized. Hence, a useful area of research concerning the stability of systems described by DAEs is the development of a systematic method of linearizing the DAE and formulating stability criteria specifically for such systems.

The Liapunov approach is appropriate for examining the stability of nonlinear systems, though there is no systematic way of producing a Liapunov function for a general dynamic system. Since much of the work in Liapunov's method concerns the stability of

canonical Hamiltonian systems, extending this work to the non–canonical Hamiltonian DAE is a natural starting point from which new stability criteria could be developed. Furthermore, the Liapunov approach can be sometimes interpreted in terms of energy functions, and the DAE is an energy–based representation of dynamics. Once again the concept of energy is a unifying theme. And, as in the case of inequality constraints, the problem of stability for the general multidiscipline system is a significant and difficult problem.

The stability problem leads naturally to the control problem. As in the case of stability, control theory is well–developed for linear catastatic systems and a large body of work exists concerning canonical Hamiltonian control. Again, the extension of linear control theory and Hamiltonian control theory to accommodate DAE representations of systems are likely starting points from which control design methodologies could be developed. Interestingly, Hamiltonian control theory is closely tied to optimization theory, which has already been mentioned in conjunction with inequality constraints. The unifying theme is energy.

Other areas of research that will extend the work presented here are briefly outlined below. These topics are somewhat less significant than the issues of inequalities and stability, yet are important to the problem of modeling multidiscipline systems nonetheless. The first of these secondary topics is the extension of the analytical method to the Eulerian or control volume point of view. This extension would be particularly effective in bringing the benefits of energy–based analysis to a broad class of thermo–fluid problems. This area of research is of ongoing interest in the bond–graph community, indicating that an unmet need exists to simplify the unified analysis of systems with control volumes.

The next secondary topic is the extension of this work to the study of distributed–parameter systems. Much work has been done in this area too, but a systematic exposition of the basic principles underlying the analytical approach to distributed–parameter, multidiscipline systems has not been developed. The energy functions for such systems

would have to be developed along the lines of Timoshenko's beam model, and the calculus of variations would have to be invoked to determine boundary conditions. This approach leads naturally to partial differential equations (PDEs), and so the prospect is raised of PDEs of motion subject to algebraic constraints. Happily, the numerical methods used to solve DAEs are also appropriate for the solution of certain classes of PDEs.

Another topic of secondary interest is the specialization of the analytical method for multibody mechanical systems. The practical utility of Lagrange's equation for mechanical systems is sometimes disputed, even though it was precisely such systems that Lagrange analyzed. The simplicity of analysis required to obtain the Lagrangian DAE (assuming that function manipulations are automated) and the prospect of robust numerical solution of multibody problems supports the idea of a fresh examination of analytical mechanics. A related topic is the relationship between Lagrange's equation and Kane's equation. It is known that the two formulations are functionally related, which raises the possibility of a Kane's DAE as well as a complimentary form of Kane's equation. Unfortunately, the advantages of Kane's formulation are only realized through the careful selection of particular coordinates to represent the motion of the system. The formulations presented in this work offer the analyst more freedom in the selection of representational variables than any other modeling method of which the author is aware.

A final extension of this work is the incorporation of feedback and control laws within the structure of the DAE. Since the equations of motion already contain both differential and algebraic equations, any feedback and control law that can be expressed as a set of differential and algebraic equations is easily accommodated within the formulation and numerical solution structure proposed herein. Future work in systematically formulating common control structures for the DAE formulation would not necessarily break new ground conceptually, but would certainly be useful for simulation purposes.

In this chapter is presented a summary of the major findings of this research as

well as an overview of significant and secondary research topics that will extend this work. The utility of the energy–based analytical approach to system dynamics is clearly demonstrated, both by the scope of the text and by the wide range of systems, dynamics and control issues that can be examined in terms of the fundamental concepts presented in this dissertation.

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Appendix A

MATHEMATICAL TECHNIQUES

Presented in this appendix are details of some of the mathematical concepts used in the dissertation. The sections are entitled as follows:

- The variational operator
- Integrability of the Pfaffian form
- The Legendre transform
- Cramer's rule
- Index of a DAE

A.1 The variational operator

This section is based on Rockafellar [32] and Wan [45]. The *fundamental problem* in the calculus of variations is to find a function $y(x) \in \mathcal{R}^N$ for $x \in [a, b]$ which minimizes the definite integral

$$J(y) = \int_a^b f\left(x, y(x), y'(x)\right)\, dx, \tag{A.1}$$

where the function f is continuously differentiable, and y is subject to end–point constraints. In the search for this minimizing function, certain perturbations of y are examined. These perturbations are the variations δy.

In obtaining the equations of motion of dynamic systems, the variable x represents time t. If the function y represents a displacement trajectory $u(t)$, then y' represents a flow trajectory $\dot{u}(t)$. Let $\hat{u}(t) \in \mathcal{R}^N$ represent the actual (and unknown) displacement trajectory of a system on the time interval $[t_o, t_f]$. A trajectory $u(t)$ that varies from the true trajectory without violating the constraints is given by

$$u(t) := \hat{u}(t) + \varepsilon\eta(t), \tag{A.2}$$

where $\delta u := \varepsilon\eta(t)$ is called the variation of $\hat{u}$. The function $\eta(t)$ is smooth on the given time interval, and ε is a small parameter. The function $\eta(t)$ is arbitrary except that it conforms to the constraints. A one–dimensional representation of $\hat{u}(t)$ and $u(t)$ is shown in Fig. A.1.

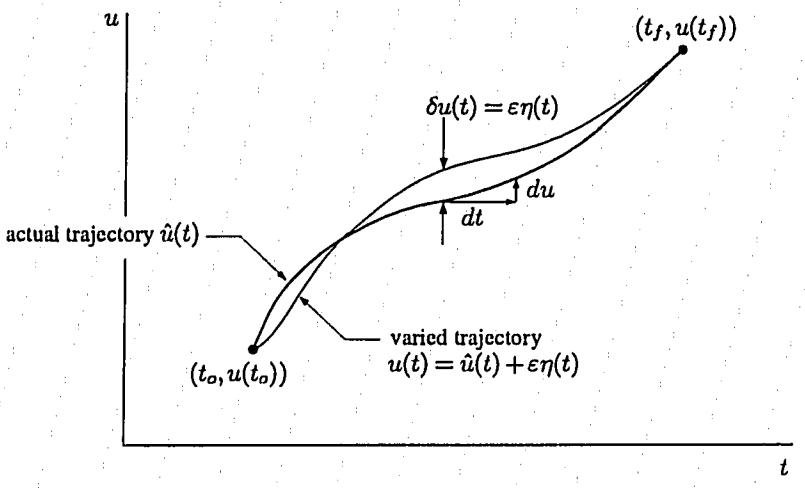

Figure A.1. Graphical interpretation of a variation.

This figure illustrates the distinction between the variation δu and the differential du. The variation δu is a small change in the entire trajectory $\hat{u}$ on the interval $[t_o, t_f]$. The differential du is a small change in the value of $\hat{u}$ at an instant due to a small change in time dt. Furthermore, the variation affects $\hat{u}$, not t. Thus at any instant in the trajectory of u, time is not varied. This give rise to the common statement that a virtual displacement δu occurs with time fixed, or in zero time. That time is not varied is a more accurate statement.

A.1.1 Properties of the variational operator

In this research, the basic dynamics problem for multidiscipline systems is not posed as a calculus of variations problem. In other words, the trajectory $\hat{u}(t)$ which is the solution to the dynamics problem does not necessarily minimize some particular definite integral.

Nevertheless, it is from the basic definitions of variational calculus that the properties of δ derive.

By definition, $\delta u_i = \varepsilon\eta_i$. From the following,

$$\dot{u} = \tfrac{d}{dt}(u) = \tfrac{d}{dt}(\hat{u} + \varepsilon\eta) = \dot{\hat{u}} + \varepsilon\dot{\eta}, \tag{A.3}$$

it is seen that

$$\delta\dot{u}_i = \varepsilon\dot{\eta}_i. \tag{A.4}$$

Since t is not varied, two operators δ and d/dt are interchangeable, as follows,

$$\delta\left(\frac{du}{dt}\right) = \delta\dot{u} = \varepsilon\dot{\eta} = \frac{d}{dt}(\varepsilon\eta) = \frac{d}{dt}(\delta u). \tag{A.5}$$

Similarly, the two operators δ and d are interchangeable,

$$\delta(du) = \delta(d\hat{u} + \varepsilon\, d\eta) = \varepsilon\, d\eta = d(\delta u). \tag{A.6}$$

In a similar manner, it can be shown that the variations of sums, products, quotients, powers, etc., are completely analogous to the corresponding laws of differentiation. For example,

$$\delta(a+b) = \delta a + \delta b, \tag{A.7}$$

$$\delta(ab) = a\,\delta b + b\,\delta a. \tag{A.8}$$

A.1.2 Variation of a function

Suppose a scalar function exists $f = f(t, u, \dot{u})$ with $u \in \mathcal{R}^N$. At a particular time t, the change in f along the varied path compared to f along the actual path is given by

$$\begin{aligned}\Delta f &= f_{\text{varied}} - f_{\text{actual}} \\ &= f(t, \hat{u} + \varepsilon\eta, \dot{\hat{u}} + \varepsilon\dot{\eta}) - f(t, \hat{u}, \dot{\hat{u}}).\end{aligned} \tag{A.9}$$

Expanding the first term on the right in a Taylor series about $\varepsilon = 0$ yields

$$\begin{aligned}f(t, \hat{u} + \varepsilon\eta, \dot{\hat{u}} + \varepsilon\dot{\eta}) &= f(t, \hat{u}, \dot{\hat{u}}) + \sum_{i=1}^{N} (\hat{u}_i + \varepsilon\eta_i - \hat{u}_i)\left.\frac{\partial f}{\partial u_i}\right|_{(\hat{u},\dot{\hat{u}})} \\ &\quad + \sum_{i=1}^{N} \left(\dot{\hat{u}}_i + \varepsilon\dot{\eta}_i - \dot{\hat{u}}_i\right)\left.\frac{\partial f}{\partial \dot{u}_i}\right|_{(\hat{u},\dot{\hat{u}})} + \mathcal{O}(\varepsilon^2).\end{aligned} \tag{A.10}$$

Then Δf is given by

$$\Delta f = \sum_{i=1}^{N} \frac{\partial f}{\partial u_i} \varepsilon \eta_i + \sum_{i=1}^{N} \frac{\partial f}{\partial \dot{u}_i} \varepsilon_i \dot{\eta} + \mathcal{O}(\varepsilon^2). \tag{A.11}$$

Substituting $\delta u_i = \varepsilon \eta_i$ and $\delta \dot{u}_i = \varepsilon \dot{\eta}_i$ into the expression for Δf and neglecting the higher–order terms, the first–order terms are defined as the variation of f, that is,

$$\delta f = \sum_{i=1}^{N} \frac{\partial f}{\partial u_i} \delta u_i + \sum_{i=1}^{N} \frac{\partial f}{\partial \dot{u}_i} \delta \dot{u}_i. \tag{A.12}$$

Compared to the differential df of this same function, given by

$$df = \sum_{i=1}^{N} \frac{\partial f}{\partial u_i} du_i + \sum_{i=1}^{N} \frac{\partial f}{\partial \dot{u}_i} d\dot{u}_i + \frac{\partial f}{\partial t} dt, \tag{A.13}$$

it is seen that the variation of f contains no term that looks like $\dfrac{\partial f}{\partial t}\delta t$, since t is not varied.

A.2 Integrability of the Pfaffian form

Methods are presented for determining whether or not a constraint is holonomic or, expressed differently, whether or not there exists integrals in holonomic form for constraints given in Pfaffian form.

A.2.1 A demonstration

In this section a demonstration is presented of a common method for determining if a constraint if Pfaffian form is an exact differential.

Consider the constraint on (dx, dy) from Ex. 3.9 given by

$$(\tan\theta)\, dx - dy = 0. \tag{A.14}$$

If this equation represented a differential form of a holonomic constraint f, then f would have the general form $f(t, x, y, \theta) = 0$, and the differential of f would be given by

$$df := \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial \theta} d\theta. \tag{A.15}$$

For this example, it follows that

$$\frac{\partial f}{\partial t} = 0, \quad \frac{\partial f}{\partial x} = \tan\theta, \quad \frac{\partial f}{\partial y} = -1, \quad \frac{\partial f}{\partial \theta} = 0. \tag{A.16}$$

If (A.14) is an exact differential, then from $\partial f/\partial x = \tan\theta$, it follows that f must have the form

$$f = (\tan\theta)\, x + g(t, y, \theta), \tag{A.17}$$

where $g(t, y, \theta) \neq g(x)$ is an undetermined function.

Taking the derivative of f with respect to θ yields

$$\begin{aligned} \frac{\partial f}{\partial \theta} &= \frac{\partial}{\partial \theta}(x\tan\theta + g) \\ &= x\sec^2\theta + \frac{\partial g}{\partial \theta} \\ &= 0, \text{ by (A.16).} \end{aligned} \tag{A.18}$$

This is true if and only if

$$\frac{\partial g}{\partial \theta} = -x\sec^2\theta. \tag{A.19}$$

Since $g = g(t, y, \theta)$, the derivative $\partial g/\partial\theta$ cannot be a function of x. Hence the last equation is false and (A.14) is not an exact differential. The constraint is nonintegrable and therefore nonholonomic.

A.2.2 Integrability conditions

This section is based on Rosenberg [35]. As shown in the previous section, determining that a Pfaffian constraint is not an exact differential is sufficient to demonstrate nonintegrability. This condition, however, is not a necessary condition for nonintegrability. The necessary condition is as follows.

For a single constraint vector in Pfaffian form given by

$$A\,dx + B\,dy + C\,dz = 0, \tag{A.20}$$

to be integrable, it is necessary and sufficient that the following equation be satisfied identically:

$$A\left(\frac{\partial B}{\partial z}-\frac{\partial C}{\partial y}\right)+B\left(\frac{\partial C}{\partial x}-\frac{\partial A}{\partial z}\right)+C\left(\frac{\partial A}{\partial y}-\frac{\partial B}{\partial x}\right)=0. \tag{A.21}$$

For a single constraint equation in N variables, that is,

$$\sum_{i=1}^{N} A_s(u_1,\ldots,u_N)\,du_s=0, \tag{A.22}$$

the necessary and sufficient condition for the existence of an integral of this constraint having the form

$$f(u_1,\ldots,u_N)=0, \tag{A.23}$$

is that the set of equations given by

$$A_\gamma\left(\frac{\partial A_\beta}{\partial u_\alpha}-\frac{\partial A_\alpha}{\partial u_\beta}\right)+A_\beta\left(\frac{\partial A_\alpha}{\partial u_\gamma}-\frac{\partial A_\gamma}{\partial u_\alpha}\right)+A_\alpha\left(\frac{\partial A_\gamma}{\partial u_\beta}-\frac{\partial A_\beta}{\partial u_\gamma}\right)=0, \tag{A.24}$$

where $\alpha,\beta,\gamma=1,\ldots,N$, are simultaneously and identically satisfied.

A.3 The Legendre transform

Two forms of the Legendre transform are presented, differing by a sign convention only. This difference in sign is described briefly for a univariate function, followed by a more detailed exposition of the nomenclature and principle results of the transform for both univariate and multivariate functions.

Given a function $f(y)$, a new variable $\xi := \partial f/\partial y$ is introduced. The Legendre transform of f is a new function $g(\xi)$ given by

$$g(\xi) := y\xi - f(y). \tag{A.25}$$

This is the traditional form of the transform used in mathematics.

The second form of the transform, popular with thermodynamicists, is the new function $\bar{g}(\xi)$ given by

$$\bar{g}(\xi) := f(y) - y\xi. \tag{A.26}$$

This transform is simply the negative of the first, that is $\bar{g}(\xi) = -g(\xi)$.

The utility of the second form of the transform is that it permits the transform to be interpreted geometrically. For each point $\hat{(y)}$, $f(\hat{y})$ is a point in the trajectory of the function, $\hat{\xi}$ is the tangent to the trajectory at that point and $\bar{g}(\hat{\xi})$ can be considered the intercept of the tangent at that instant. In these terms the transform can be written for all y

$$f(y) = \xi y + \bar{g}(\xi), \tag{A.27}$$

which is an equation in slope–intercept form describing the tangent line to the trajectory. In the multivariate case, therefore, the Legendre transform has a geometric interpretation as a collection of intercepts corresponding to a family of slopes. The transform results in a new function in which one or more independent variables is replaced by its slope.

In the first section below, the transform $g(\xi)$ is described for the univariate case in terms of the traditional mathematical form. In the subsequent section, the multivariate case is given in terms of the second form $\bar{g}(\xi)$. A change of nomenclature is introduced in the multivariate case, in which the original function is denoted $y^{(o)}$ and the transform is denoted $y^{(k)}$.

A.3.1 Transform of a univariate function

The Legendre transform, applied to a function of a single variable, proceeds as follows. Given a univariate function $f(y)$, the differential of f is given by

$$\begin{aligned} df &= \frac{\partial f}{\partial y} dy \\ &= \xi \, dy, \end{aligned} \tag{A.28}$$

where $\xi := \partial f / \partial y$. The Legendre transform consists of defining a new function $g(\xi)$ given by

$$g(\xi) := y\xi - f(y). \tag{A.29}$$

The differential of g is given by

$$\begin{aligned} dg &= d(y\xi) - df \\ &= y\,d\xi + \xi\,dy - \xi\,dy \\ &= y\,d\xi. \end{aligned} \tag{A.30}$$

This transformation yields two results of interest. From (A.29) is obtained the relationship

$$f(y) + g(\xi) = y\xi, \tag{A.31}$$

and integrating (A.30) yields

$$g(\xi) = \int y(\xi)\,d\xi. \tag{A.32}$$

A.3.2 Transform of multivariate functions

The nomenclature and development in this section is from Modell and Reid [26]. Consider a function of m independent variables $y^{(o)}(x_1, \ldots, x_m)$. For $k \leq m$, the k^{th} Legendre transform of this function is the new function $y^{(k)}$ given by

$$y^{(k)} := y^{(o)} - \sum_{i=1}^{k} \xi_i\, x_i, \tag{A.33}$$

where

$$\xi_i := \frac{\partial y^{(o)}}{\partial x_i}. \tag{A.34}$$

The transformation from $y^{(o)}$ to $y^{(k)}$ can be considered as a replacement of the first k variables by their slopes, that is,

$$\begin{aligned} y^{(o)} &= y^{(o)}(x_1, \ldots, x_k, x_{k+1}, \ldots, x_m) \\ &\qquad \downarrow\ \downarrow\ \downarrow \\ y^{(k)} &= y^{(k)}(\xi_1, \ldots, \xi_k, x_{k+1}, \ldots, x_m). \end{aligned} \tag{A.35}$$

Each variable x_i and its transform ξ_i form a *conjugate pair*. The variables $(x_{k+1}, \ldots, x_m)$ are unaffected by the transformation.

The differential of the original function is given by

$$dy^{(o)} = \sum_{i=1}^{m} \frac{\partial y^{(o)}}{\partial x_i} dx_i = \sum_{i=1}^{m} \xi_i \, dx_i. \tag{A.36}$$

The differential of the new function is given by

$$dy^{(k)} = dy^{(o)} - d\left(\sum_{i=1}^{k} \xi_i \, x_i \right). \tag{A.37}$$

Substituting for $dy^{(o)}$ and expanding the differential term on the right yields

$$\begin{aligned} dy^{(k)} &= \sum_{i=1}^{m} \xi_i \, dx_i - \sum_{i=1}^{k} \xi_i \, dx_i - \sum_{i=1}^{k} x_i \, d\xi_i \\ &= \underbrace{-\sum_{i=1}^{k} x_i \, d\xi_i}_{\substack{\text{transformed} \\ \text{variables}}} + \underbrace{\sum_{i=k+1}^{m} \xi_i \, dx_i}_{\substack{\text{untransformed} \\ \text{variables}}}. \end{aligned} \tag{A.38}$$

Thus, for the transformed variables, it follows that

$$\frac{\partial y^{(k)}}{\partial \xi_i} = -x_i \qquad i = 1, \ldots, k, \tag{A.39}$$

and for the untransformed variables

$$\frac{\partial y^{(k)}}{\partial x_i} = \xi_i. \tag{A.40}$$

Since $\xi_i = \partial y^{(o)} / \partial x_i$ by definition, it follows that for the untransformed variables

$$\frac{\partial y^{(k)}}{\partial x_i} = \frac{\partial y^{(o)}}{\partial x_i} \qquad i = k+1, \ldots, m. \tag{A.41}$$

A.4 Cramer's rule

This section is from Strang [40]. For vectors $x, b \in \mathcal{R}^n$ such that $Ax = b$, where $A \in \mathcal{R}^{n \times n}$, the j^{th} component of

$$x = A^{-1} b \tag{A.42}$$

is

$$x_j = \frac{\det B_j}{\det A}, \tag{A.43}$$

where

$$B_j := \begin{bmatrix} a_{11} & a_{12} & \cdots & b_1 & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n & \cdots & a_{nn} \end{bmatrix}. \tag{A.44}$$

EXAMPLE A.1 (from Strang) The solution of

$$\begin{aligned} x_1 + 3x_2 &= 0 \\ 2x_1 + 4x_2 &= 6 \end{aligned} \tag{A.45}$$

is

$$\begin{aligned} x_1 &= \frac{\begin{vmatrix} 0 & 3 \\ 6 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}} = \frac{-18}{-2} = 9 \\ x_2 &= \frac{\begin{vmatrix} 1 & 0 \\ 2 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}} = \frac{6}{-2} = -3. \quad \diamond \end{aligned} \tag{A.46}$$

A.5 Index of a DAE

This section is from Brenan *et al.* [6]. Given the general nonlinear DAE

$$F(t, y, y') = 0, \tag{A.47}$$

the minimum number of times that all or part of the DAE must be differentiated with respect to t in order to determine y' as a continuous function of (y, t), is the *index* of the DAE.

Consider the special case of a semi-explicit DAE

$$\begin{aligned} x' &= f(x, y, t) \\ 0 &= g(x, y, t). \end{aligned} \tag{A.48}$$

Differentiating the constraint equation $g = 0$ with respect to t yields

$$\begin{aligned} x' &= f(x,y,t) \\ g_x(x,y,t)\,x' + g_y(x,y,t)\,y' &= -g_t(x,y,t). \end{aligned} \tag{A.49}$$

If g_y is nonsingular, this system of equations is an implicit ODE. Since a single differentiation was performed, the original DAE is said to have index one.

If this is not the case, suppose that after a change in coordinates and some algebraic manipulation, (A.49) is written as a semi-explicit ODE in terms of the new coordinates. The new constraint equation is differentiated. If an implicit ODE results, the original problem has index two. If the new system is not an implicit ODE, the process is repeated. The number of differentiation steps in this procedure is the index.

Brenan states that a series of differentiations and coordinate changes is not recommended as a general solution procedure for DAE's. Rather "the number of such differentiation steps that would be required in theory turns out to be an important quantity in understanding the behavior of numerical methods."

EXAMPLE A.2 For a planar pendulum shown in Fig A.2, the Lagrangian DAE is given by

$$\begin{aligned} m\ddot{u}_1 + 2u_1\lambda &= 0 \\ m\ddot{u}_2 + 2u_2\lambda &= -mg \\ u_1^2 + u_2^2 - l^2 &= 0. \end{aligned} \tag{A.50}$$

Letting $\dot{u} = v$ yields

$$\begin{aligned} \dot{u}_1 &= v_1 \\ \dot{u}_2 &= v_2 \\ m\dot{v}_1 &= -2u_1\lambda \\ m\dot{v}_2 &= -2u_2\lambda - mg \end{aligned} \tag{A.51}$$

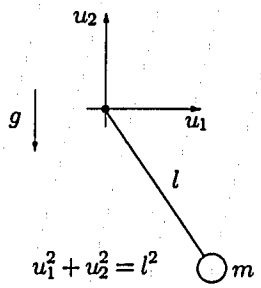

Figure A.2. A pendulum as an index–3 system.

$$0 \;=\; u_1^2 + u_2^2 - l^2.$$

This set of equations is index–3 since three differentiations are required to pose the problem as an implicit ODE, as follows. Differentiating the displacement constraint with respect to time yields

$$u_1 v_1 + u_2 v_2 = 0. \tag{A.52}$$

Differentiating a second time yields

$$v_1^2 + v_2^2 + u_1 \dot{v}_1 + u_2 \dot{v}_2 = 0. \tag{A.53}$$

Solving (A.51) for $\dot{v}_1$ and $\dot{v}_2$ and substituting in the differentiated constraint equation yields

$$\begin{aligned} 0 \;&=\; m v_1^2 + m v_2^2 - 2\lambda \Big(\underbrace{u_1^2 + u_2^2}_{l^2} \Big) - m g u_2 \\ &=\; m v_1^2 + m v_2^2 - 2 l^2 \lambda - m g u_2. \end{aligned} \tag{A.54}$$

Differentiating a third time yields

$$m v_1 \dot{v}_1 + m v_2 \dot{v}_2 - l^2 \dot{\lambda} = \tfrac{1}{2} m g v_2. \tag{A.55}$$

The complete set of equations is given by

$$\dot{u}_1 \;=\; v_1$$

$$\begin{aligned} \dot{u}_2 &= v_2 \\ m\dot{v}_1 &= -2u_1\lambda \\ m\dot{v}_2 &= -2u_2\lambda - mg \\ mv_1\dot{v}_1 + mv_2\dot{v}_2 - l^2\dot{\lambda} &= \tfrac{1}{2}mgv_2, \end{aligned} \tag{A.56}$$

which is an implicit ODE in (u, v, λ). Since three differentiations were required to obtain this implicit ODE, the original DAE is index–3. $\diamond$

Appendix B

ALTERNATE DERIVATIONS OF LAGRANGE'S EQUATION

In this appendix are presented two derivations of Lagrange's equation based on principles other than the first law of thermodynamics. The first derivation is based on what Meirovitch [25] calls the extended form of Hamilton's principle. The second derivation is based on what Pars [29] calls the *fundamental equation.* The second derivation closely follows the method used by Lagrange, generalized for the multidiscipline case.

B.1 From Hamilton's principle

The extended form of Hamilton's principle is given by

$$\int (\delta T^* + \delta W)\, dt = 0. \tag{B.1}$$

Kinetic energy and coenergy satisfy

$$T^* + T = \sum fp. \tag{B.2}$$

The variation of this equation is given by

$$\delta T^* + \delta T = \sum f\, \delta p + \sum p\, \delta f. \tag{B.3}$$

This equation can be solved for δT^* as follows, where $T = T(p, q, t)$,

$$\begin{aligned}
\delta T^* &= \sum f\, \delta p + \sum p\, \delta f - \delta T \\
&= \sum f\, \delta p + \sum p\, \delta f - \left(\sum \frac{\partial T}{\partial p} \delta p + \sum \frac{\partial T}{\partial q}\, \delta q \right) \\
&= \sum \underbrace{\left(f - \frac{\partial T}{\partial p} \right)}_{0} \delta p + \sum p\, \delta f - \sum \frac{\partial T}{\partial q}\, \delta q \\
&= \sum p\, \delta \dot{q} - \sum \frac{\partial T}{\partial q}\, \delta q.
\end{aligned} \tag{B.4}$$

The time derivative of the quantity $p\,\delta q$ is given by

$$\frac{d}{dt}(p\,\delta q) = \dot{p}\,\delta q + p\,\delta\dot{q}. \tag{B.5}$$

Solving this equation for $p\,\delta\dot{q}$ yields

$$p\,\delta\dot{q} = \frac{d}{dt}(p\,\delta q) - \dot{p}\,\delta q. \tag{B.6}$$

Substituting this expression into (B.4) yields

$$\delta T^* = \frac{d}{dt}(p\,\delta q) - \dot{p}\,\delta q - \sum \frac{\partial T}{\partial q}\,\delta q. \tag{B.7}$$

Substituting

$$\dot{p} = \frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}} \qquad \text{and} \qquad -\sum \frac{\partial T}{\partial q}\,\delta q = \sum \frac{\partial T^*}{\partial q}\,\delta q \tag{B.8}$$

yields

$$\delta T^* = \frac{d}{dt}(p\,\delta q) - \sum \frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}}\,\delta q + \sum \frac{\partial T^*}{\partial q}\,\delta q. \tag{B.9}$$

Virtual work is given by

$$\delta W = \sum e\,\delta q = \sum \left(e^g + e^\phi\right)\,\delta q. \tag{B.10}$$

By Lagrange's principle, constraint efforts e^ϕ do no virtual work, hence

$$\delta W = \sum e^g\,\delta q. \tag{B.11}$$

Substituting the expressions for δT^* and δW in Hamilton's principle yields

$$\int \left[\sum \frac{d}{dt}(p\,\delta q) - \sum \frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}}\,\delta q + \sum \frac{\partial T^*}{\partial q}\,\delta q + \sum e^g\,\delta q\right] dt = 0. \tag{B.12}$$

Rearranging yields

$$\int \left[\sum \left(\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}} - \frac{\partial T^*}{\partial q} - e^g\right)\,\delta q\right] dt = \sum \int \frac{d(p\,\delta q)}{dt}\,dt. \tag{B.13}$$

The right–hand side of this equation vanishes,

$$\sum \int \frac{d(p\,\delta q)}{dt}\,dt = \sum \left(p\,\delta q\Big|_{\delta q(t_o)}^{\delta q(t_f)}\right) = 0, \tag{B.14}$$

since the condition is imposed from the calculus of variations that δq vanishes at the beginning and end of its trajectory, that is, given the time interval $t \in [t_o, t_f]$, $\delta q(t_o) = \delta q(t_f) = 0$. Thus from (B.13) is obtained

$$\int \left[\sum \left(\frac{d}{dt} \frac{\partial T^*}{\partial \dot{q}} - \frac{\partial T^*}{\partial q} - e^g \right) \delta q \right] dt = 0. \tag{B.15}$$

For this equation to hold for arbitrary t, the integrand must vanish. This concept is often called the *fundamental lemma* of the calculus of variations [46]. The vanishing of the integrand yields

$$\sum \left(\frac{d}{dt} \frac{\partial T^*}{\partial \dot{q}} - \frac{\partial T^*}{\partial q} - e^g \right) \delta q = 0. \tag{B.16}$$

Given efforts e^g are classified such that $e^g = e^p + e^n = e^p + e^d + e^s + e^\gamma$. Potential efforts e^p and dissipation efforts e^d satisfy

$$e^p = -\frac{\partial V}{\partial q} \qquad \text{and} \qquad e^d = -\frac{\partial D}{\partial \dot{q}}. \tag{B.17}$$

Let the generalized effort Q represent the sum $e^s + e^\gamma$. Substitution yields

$$\sum \left(\frac{d}{dt} \frac{\partial T^*}{\partial \dot{q}} - \frac{\partial T^*}{\partial q} + \frac{\partial V}{\partial q} + \frac{\partial D}{\partial \dot{q}} - Q \right) \delta q = 0. \tag{B.18}$$

This is the virtual work form of Lagrange's equation. If the δq are independent, the parenthetical term vanishes, yielding Lagrange's equation

$$\frac{d}{dt} \frac{\partial T^*}{\partial \dot{q}_j} - \frac{\partial T^*}{\partial q_j} + \frac{\partial V}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} - Q_j = 0 \qquad j = 1, \ldots, n. \tag{B.19}$$

If the δq are dependent, Lagrange multipliers are introduced to produce the multiplier form of Lagrange's equation, which together with the constraint equations, comprise the Lagrangian DAE, given in vector form by

$$\begin{aligned} \tfrac{d}{dt} \nabla_{\dot{q}} T^* - \nabla_q T + \nabla_q V + \nabla_{\dot{q}} D + C_n^T \lambda &= Q \\ C &= 0 \\ \Gamma &= 0. \end{aligned} \tag{B.20}$$

B.2 From the fundamental equation

The generalized dynamic requirement for a system with an N–dimensional configuration space is written in component form as

$$\dot{p}_i = e_i \qquad i = 1, \ldots, N. \tag{B.21}$$

Efforts are classified as given efforts e^g and constraint efforts e^ϕ such that $e_i = e_i^g + e_i^\phi$. Thus the dynamic requirement is given by

$$\dot{p}_i = e_i^g + e_i^\phi \qquad i = 1, \ldots, N. \tag{B.22}$$

These efforts, including the momentum term, can be imagined to act at an instant through a virtual displacement δu.

$$\dot{p}_i \, \delta u_i = \left(e_i^g + e_i^\phi\right) \delta u_i \qquad i = 1, \ldots, N. \tag{B.23}$$

The products in this equation represent virtual work. By the equivalence of work among the engineering disciplines, the increments of virtual work can be summed over the entire configuration of the system to obtain

$$\underbrace{\sum_{i=1}^{N} \dot{p}_i \, \delta u_i}_{\substack{\text{virtual work of the} \\ \text{inertial efforts}}} = \underbrace{\sum_{i=1}^{N} e_i^g \, \delta u_i}_{\substack{\text{virtual work of the} \\ \text{given efforts}}} + \underbrace{\sum_{i=1}^{N} e_i^\phi \, \delta u_i.}_{\substack{\text{virtual work of the} \\ \text{constraint efforts}}} \tag{B.24}$$

By Lagrange's principle the virtual work of the constraint efforts vanishes. The remaining terms of this equation are rearranged to obtain

$$\sum_{i=1}^{N} (\dot{p}_i - e_i^g) \delta u_i = 0. \tag{B.25}$$

This equation is called (using Pars' nomenclature) the *fundamental equation* of analytical system dynamics, though in mechanics it is also known as "Lagrange's form of d'Alembert's principle." This equation is fundamental because it is the starting

point from which many analytical techniques for modeling dynamic systems are derived, including Hamilton's equation, the Boltzman-Hamel equations, Kane's equation, the Gibbs-Appel equation, and of course Lagrange's equation. Only Lagrange's equation is considered here.

B.2.1 Reduced–order coordinates

In the general case, each displacement component u_i is a function of the reduced–order coordinates q and possibly time $u = u(q, t)$. The differentials du_i and dq_j are related by

$$du_i = \sum_{j=1}^{n} \frac{\partial u_i}{\partial q_j} dq_j + \frac{\partial u_i}{\partial t} dt \qquad i = 1, \ldots, N, \tag{B.26}$$

and the virtual displacements δu_i and δq_j are related by

$$\delta u_i = \sum_{j=1}^{n} \frac{\partial u_i}{\partial q_j} \delta q_j \qquad i = 1, \ldots, N. \tag{B.27}$$

B.2.2 Two equalities

It is convenient at this point to establish two mathematical relationships. Starting with the equation $u_i = u_i(q_i, \ldots, q_n, t)$, the time derivative of u_i is given by

$$\dot{u}_i = \frac{\partial u_i}{\partial q_1} \dot{q}_1 + \cdots + \frac{\partial u_i}{\partial q_n} \dot{q}_n + \frac{\partial u_i}{\partial t}. \tag{B.28}$$

The derivative of this equation with respect to $\dot{q}_j$ is taken to obtain

$$\begin{aligned} \frac{\partial \dot{u}_i}{\partial \dot{q}_j} &= \frac{\partial}{\partial \dot{q}_j} \left(\frac{\partial u_i}{\partial q_1} \dot{q}_1 + \cdots + \frac{\partial u_i}{\partial q_n} \dot{q}_n + \frac{\partial u_i}{\partial t} \right) \\ &= \frac{\partial u_i}{\partial q_j}, \end{aligned} \tag{B.29}$$

which is the first of the desired equalities. Taking the derivative of (B.28) with respect to q_j yields

$$\begin{aligned} \frac{\partial \dot{u}_i}{\partial q_j} &= \frac{\partial}{\partial q_j} \left(\frac{\partial u_i}{\partial q_1} \dot{q}_1 + \cdots + \frac{\partial u_i}{\partial q_n} \dot{q}_n + \frac{\partial u_i}{\partial t} \right) \\ &= \frac{\partial}{\partial q_j} \left(\frac{\partial u_i}{\partial q_1} \dot{q}_1 \right) + \cdots + \frac{\partial}{\partial q_j} \left(\frac{\partial u_i}{\partial q_n} \dot{q}_n \right) + \frac{\partial}{\partial q_j} \left(\frac{\partial u_i}{\partial t} \right) \end{aligned}$$

$$= \frac{\partial}{\partial q_1}\left(\frac{\partial u_i}{\partial q_j}\right)\dot{q}_1 + \cdots + \frac{\partial}{\partial q_n}\left(\frac{\partial u_i}{\partial q_j}\right)\dot{q}_n + \frac{\partial}{\partial t}\left(\frac{\partial u_i}{\partial q_j}\right)$$
$$= \frac{d}{dt}\left(\frac{\partial u_i}{\partial q_j}\right), \tag{B.30}$$

which is the second of the desired equalities.

B.2.3 Derivation

The fundamental equation is restated

$$\sum_{i=1}^{N}(\dot{p}_i - e_i^g)\delta u_i = 0. \tag{B.31}$$

Substituting the generalized coordinate form of the virtual displacement δu_i from (B.27) yields

$$\sum_{i=1}^{N}(\dot{p}_i - e_i^g)\sum_{j=1}^{n}\frac{\partial u_i}{\partial q_j}\delta q_j = 0 \tag{B.32}$$

$$\sum_{j=1}^{n}\left\{\sum_{i=1}^{N}\left(\dot{p}_i\frac{\partial u_i}{\partial q_j} - e_i^g\frac{\partial u_i}{\partial q_j}\right)\right\}\delta q_j = 0. \tag{B.33}$$

Consider the time derivative of $p_i\,\partial u_i/\partial q_j$

$$\frac{d}{dt}\left(p_i\frac{\partial u_i}{\partial q_j}\right) = \dot{p}_i\frac{\partial u_i}{\partial q_j} + p_i\frac{d}{dt}\left(\frac{\partial u_i}{\partial q_j}\right). \tag{B.34}$$

Substituting the two equalities (B.29) and (B.30) yields

$$\frac{d}{dt}\left(p_i\frac{\partial \dot{u}_i}{\partial \dot{q}_j}\right) = \dot{p}_i\frac{\partial u_i}{\partial q_j} + p_i\frac{\partial \dot{u}_i}{\partial q_j}, \tag{B.35}$$

which can be rearranged to obtain

$$\dot{p}_i\frac{\partial u_i}{\partial q_j} = \frac{d}{dt}\left(p_i\frac{\partial \dot{u}_i}{\partial \dot{q}_j}\right) - p_i\frac{\partial \dot{u}_i}{\partial q_j}. \tag{B.36}$$

Substituting this expression for $\dot{p}_i\,\partial u_i/\partial q_j$ into (B.33) yields

$$\sum_{j=1}^{n}\left\{\sum_{i=1}^{N}\left[\frac{d}{dt}\left(p_i\frac{\partial \dot{u}_i}{\partial \dot{q}_j}\right) - p_i\frac{\partial \dot{u}_i}{\partial q_j} - e_i^g\frac{\partial u_i}{\partial q_j}\right]\right\}\delta q_j = 0. \tag{B.37}$$

Given efforts are classified as potential efforts e^p and nonpotential efforts e^n such that $e^g = e^p + e^n$. Substituting these efforts yields

$$\sum_{j=1}^{n}\left\{\sum_{i=1}^{N}\left[\frac{d}{dt}\left(p_i\frac{\partial \dot{u}_i}{\partial \dot{q}_j}\right) - p_i\frac{\partial \dot{u}_i}{\partial q_j} - e_i^p\frac{\partial u_i}{\partial q_j} - e_i^n\frac{\partial u_i}{\partial q_j}\right]\right\}\delta q_j = 0. \tag{B.38}$$

The remainder of the derivation consists of showing that the following identities hold.

$$\begin{aligned} \frac{\partial T^*}{\partial \dot{q}_j} &= \sum_{i=1}^{N} p_i\frac{\partial \dot{u}_i}{\partial \dot{q}_j} & -\frac{\partial V}{\partial q_j} &= \sum_{i=1}^{N} e_i^p\frac{\partial u_i}{\partial q_j} \\ \frac{\partial T^*}{\partial q_j} &= \sum_{i=1}^{N} p_i\frac{\partial \dot{u}_i}{\partial q_j} & Q_j - \frac{\partial D}{\partial \dot{q}_j} &= \sum_{i=1}^{N} e_i^n\frac{\partial u_i}{\partial q_j}. \end{aligned} \tag{B.39}$$

Kinetic coenergy

Each component of flow $\dot{u}_i$ is a function of the reduced-order coordinates, their derivatives, and time, that is, $\dot{u} = \dot{u}(\dot{q}, q, t)$. It follows, therefore, that kinetic coenergy has the form $T^* = T^*(\dot{u}(\dot{q}, q, t))$, and the differential of kinetic coenergy dT^* is given by

$$dT^* = \sum_{i=1}^{N}\frac{\partial T^*}{\partial \dot{u}_i}\left(\sum_{j=1}^{n}\frac{\partial \dot{u}_i}{\partial \dot{q}_j}d\dot{q}_j + \sum_{j=1}^{n}\frac{\partial \dot{u}_i}{\partial q_j}dq_j + \frac{\partial \dot{u}_i}{\partial t}dt\right). \tag{B.40}$$

By definition, the initial multiplier $\partial T^*/\partial \dot{u}_i$ is the i^{th}-component of momentum p_i, yielding

$$dT^* = \sum_{i=1}^{N} p_i\left(\sum_{j=1}^{n}\frac{\partial \dot{u}_i}{\partial \dot{q}_j}d\dot{q}_j + \sum_{j=1}^{n}\frac{\partial \dot{u}_i}{\partial q_j}dq_j + \frac{\partial \dot{u}_i}{\partial t}dt\right). \tag{B.41}$$

Since energy is a state function (that is, not a path-dependent function) it can be integrated along an arbitrary path. The path C is selected to occur at an instant in time. An initial state $(q_o, \dot{q}_o) = (0, 0)$ is chosen, corresponding to the origin of the inertial reference frame. Holding $\dot{q} = 0$, all displacements are brought to their final values q for the instant t. Then holding q fixed, all flows are brought to their final values $\dot{q}$ for the same instant. This procedure is expressed

$$T^* = \int_C dT^*$$

$$= \sum_{i=1}^{N} \left(\underbrace{\int_0^q \sum_{j=1}^{n} p_i \frac{\partial \dot{u}_i}{\partial q_j} dq_j}_{\substack{\dot{q}=0 \\ t \text{ fixed}}} + \underbrace{\int_0^{\dot{q}} \sum_{j=1}^{n} p_i \frac{\partial \dot{u}_i}{\partial \dot{q}_j} d\dot{q}_j}_{\substack{q \text{ fixed} \\ t \text{ fixed}}} + \underbrace{\int_t^t p_i \frac{\partial \dot{u}_i}{\partial t} dt}_{\substack{q \text{ fixed} \\ \dot{q} \text{ fixed}}} \right). \quad \text{(B.42)}$$

The last term vanishes since time is fixed. The remaining terms can be rearranged to obtain

$$T^* = \sum_{j=1}^{n} \left[\int_0^{\dot{q}} \left(\sum_{i=1}^{N} p_i \frac{\partial \dot{u}_i}{\partial \dot{q}_j} \right) d\dot{q}_j + \int_0^q \left(\sum_{i=1}^{N} p_i \frac{\partial \dot{u}_i}{\partial q} \right) dq_j \right]. \quad \text{(B.43)}$$

Taking the partial derivatives of this equation with respect to $\dot{q}_j$ and q_j respectively yields the following equalities

$$\frac{\partial T^*}{\partial \dot{q}_j} = \sum_{i=1}^{N} p_i \frac{\partial \dot{u}_i}{\partial \dot{q}_j} \qquad j = 1, \ldots, n \quad \text{(B.44)}$$

$$\frac{\partial T^*}{\partial q_j} = \sum_{i=1}^{N} p_i \frac{\partial \dot{u}_i}{\partial q_j} \qquad j = 1, \ldots, n, \quad \text{(B.45)}$$

which are two of the desired terms noted in (B.39).

Potential energy

Each component of displacement u_i is a function of the reduced–order coordinates q and time, hence the total differential of potential energy $V(u(q,t))$ is given by

$$dV = \sum_{i=1}^{N} \frac{\partial V}{\partial u_i} \left(\sum_{j=1}^{n} \frac{\partial u_i}{\partial q_j} dq_j + \frac{\partial u_i}{\partial t} dt \right). \quad \text{(B.46)}$$

By definition, the initial multiplier $\partial V / \partial u_i$ is the negative of the i^{th}–component of potential effort, $-e_i^p$, yielding

$$-dV = \sum_{i=1}^{N} e_i^p \left(\sum_{j=1}^{n} \frac{\partial u_i}{\partial q_j} dq_j + \frac{\partial u_i}{\partial t} dt \right). \quad \text{(B.47)}$$

Integrating along path C, as before, yields

$$V = \int_C dV$$

$$-V = \sum_{i=1}^{N} \left(\underbrace{\int_0^q \sum_{j=1}^{n} e_i^p \frac{\partial u_i}{\partial q_j} dq}_{t \text{ fixed}} + \underbrace{\int_t^t e_i^p \frac{\partial u_i}{\partial t} dt}_{q \text{ fixed}} \right). \quad \text{(B.48)}$$

The right–most term vanishes, leaving

$$-V = \sum_{j=1}^{n} \int_0^q \sum_{i=1}^{N} e_i^p \frac{\partial u_i}{\partial q_j} dq_j, \tag{B.49}$$

and the partial derivative of both sides of this equation is taken with respect to q_j to obtain

$$-\frac{\partial V}{\partial q_j} = \sum_{i=1}^{N} e_i^p \frac{\partial u_i}{\partial q_j} \qquad j = 1, \ldots, n, \tag{B.50}$$

which is the third term noted in (B.39).

Generalized effort

The virtual work of the nonpotential efforts can be written

$$\delta W^{(n)} = \sum_{j=1}^{n} \left(\sum_{i=1}^{N} e_i^n \frac{\partial u_i}{\partial q_j} \right) \delta q_j. \tag{B.51}$$

Nonpotential efforts are subclassified as source efforts, implicit efforts and dissipative efforts such that $e^n = e^s + e^\gamma + e^d$, where dissipative efforts satisfy the definition of the dissipation function, that is,

$$e_i^d = -\frac{\partial D}{\partial \dot{u}_i}. \tag{B.52}$$

Making this substitution yields

$$\delta W = \sum_{j=1}^{n} \left[\sum_{i=1}^{N} (e^s + e^\gamma) \frac{\partial u_i}{\partial q_j} - \sum_{i=1}^{N} \frac{\partial D}{\partial \dot{u}_i} \frac{\partial u_i}{\partial q_j} \right] \delta q_j. \tag{B.53}$$

Making use of the equality $\partial \dot{u}_i / \partial \dot{q}_j = \partial u_i / \partial q_j$, the dissipation term is given by

$$\sum_{i=1}^{N} \frac{\partial D}{\partial \dot{u}_i} \frac{\partial u_i}{\partial q_j} = \sum_{i=1}^{N} \frac{\partial D}{\partial \dot{u}_i} \frac{\partial \dot{u}_i}{\partial \dot{q}_j}. \tag{B.54}$$

Restricting the definition of the dissipation function to flows $\dot{u} = \dot{u}(\dot{q})$ such that $D = D(\dot{u}) = D(\dot{u}(\dot{q}))$, this last expression is identical to $\partial D / \partial \dot{q}$. Making this substitution and denoting the summation of source efforts and implicit efforts by Q_j yields

$$\delta W = \sum_{j=1}^{n} \left[\underbrace{\sum_{i=1}^{N} (e^s + e^\gamma) \frac{\partial u_i}{\partial q_j}}_{Q_j} - \underbrace{\sum_{i=1}^{N} \frac{\partial D}{\partial \dot{u}_i} \frac{\partial \dot{u}_i}{\partial \dot{q}_j}}_{\frac{\partial D}{\partial \dot{q}_j}} \right] \delta q_j. \tag{B.55}$$

Therefore, the j^{th}–component of nonpotential effort is given by

$$Q_j - \frac{\partial D}{\partial \dot{q}_j} = \sum_{i=1}^{N} e_i^n \frac{\partial u_i}{\partial q_j}, \tag{B.56}$$

which is the last of the desired terms noted in (B.39).

Lagrange's equation

Making the substitutions noted in (B.39) yields

$$\sum_{j=1}^{n} \left(\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_j} - \frac{\partial T^*}{\partial q_j} + \frac{\partial V}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} - Q_j \right) \delta q_j = 0. \tag{B.57}$$

This is the virtual work form of Lagrange's equation. If the δq are independent, the parenthetical term vanishes, yielding Lagrange's equation

$$\frac{d}{dt}\frac{\partial T^*}{\partial \dot{q}_j} - \frac{\partial T^*}{\partial q_j} + \frac{\partial V}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} - Q_j = 0 \qquad j = 1, \ldots, n. \tag{B.58}$$

If the δq are dependent, Lagrange multipliers are introduced to produce the multiplier form of Lagrange's equation, which together with the constraint equations, comprise the Lagrangian DAE, given in vector form by

$$\begin{aligned} \tfrac{d}{dt}\nabla_{\dot{q}} T^* - \nabla_q T + \nabla_q V + \nabla_{\dot{q}} D + C_n^T \lambda &= Q \\ C &= 0 \\ \Gamma &= 0. \end{aligned} \tag{B.59}$$

Appendix C

VIRTUAL MOMENTUM

In this section, virtual momentum is shown to be the infinitesimal quantity $\delta\varrho$ that satisfies the following relationship with virtual displacement δu.

$$\sum_{i=1}^{N} \dot{u}_i \, \delta\varrho_i = \sum_{i=1}^{N} \dot{\varrho}_i \, \delta u_i. \tag{C.1}$$

C.1 The general case

The variational form of the first law

$$\delta E = \delta W, \tag{C.2}$$

in terms of configuration coordinates, is given by

$$\sum_{i=1}^{N} \frac{\partial E}{\partial \varrho_i} \, \delta\varrho_i + \sum_{i=1}^{N} \frac{\partial E}{\partial u_i} \, \delta u_i = \sum_{i=1}^{N} e_i^n \, \delta u_i. \tag{C.3}$$

For $E := T(\varrho) + V(u)$, and $\delta W := \sum e_i^n \, \delta u_i$,

$$\frac{\partial E}{\partial \varrho_i} = \frac{\partial T}{\partial \varrho_i} \quad \text{and} \quad \frac{\partial E}{\partial u_i} = \frac{\partial V}{\partial u_i}. \tag{C.4}$$

Substituting these terms into the first law and rearranging yields

$$\sum_{i=1}^{N} \frac{\partial T}{\partial \varrho_i} \, \delta\varrho_i = \sum_{i=1}^{N} \left(e_i^n - \frac{\partial V}{\partial u_i} \right) \delta u_i. \tag{C.5}$$

Substituting the following terms

$$\frac{\partial T}{\partial \varrho_i} = f_i \qquad - \frac{\partial V}{\partial u_i} = e_i^p, \tag{C.6}$$

into the first law yields yields

$$\sum_{i=1}^{N} f_i \, \delta\varrho_i \;=\; \sum_{i=1}^{N} (e_i^n + e_i^p) \; \delta u_i \tag{C.7}$$

$$= \sum_{i=1}^{N} e_i^g \delta u_i \tag{C.8}$$

$$= \sum_{i=1}^{N} \left(e_i^g + e_i^\phi \right) \delta u_i \tag{C.9}$$

$$= \sum_{i=1}^{N} e_i \, \delta u_i, \tag{C.10}$$

where use has been made of the following: $e^n + e^p := e^g$; $e^g + e^\phi := e$; and the virtual work of the constraint efforts vanishes, that is $\delta W^\phi = \sum e_i^\phi \, \delta u_i = 0$. From Paynter's diagram, $f_i = \dot{u}_i$ and $e_i = \dot{\varrho}_i$. It follows that

$$\sum_{i=1}^{N} \dot{u}_i \, \delta \varrho_i = \sum_{i=1}^{N} \dot{\varrho}_i \, \delta u_i, \tag{C.11}$$

or in vector form

$$\langle \dot{u}, \delta \varrho \rangle = \langle \dot{\varrho}, \delta u \rangle, \tag{C.12}$$

which was to be shown.

EXAMPLE C.1 Consider the second-order, linear mass–spring–damper system shown in Fig. C.1. The energy stored in this system is given by

$$\begin{aligned} E \;=\; T(p) + V(q) \;&=\; \int_0^p \frac{p}{m} \, dp - \int_0^q (-kq) \, dq \\ &=\; \frac{1}{2m} p^2 + \frac{1}{2} k q^2. \end{aligned} \tag{C.13}$$

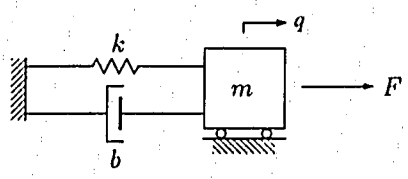

Figure C.1. A simple translational system.

The variation in stored energy is given by

$$\delta E = \frac{\partial E}{\partial p} \, \delta p + \frac{\partial E}{\partial q} \, \delta q = \frac{p}{m} \, \delta p + kq \, \delta q. \tag{C.14}$$

The virtual work of the nonpotential efforts is given by

$$\delta W = e^n \, \delta q = (F - b\dot{q}) \, \delta q, \tag{C.15}$$

Setting $\delta E = \delta W$ yields

$$\frac{p}{m} \, \delta p = (F - b\dot{q} - kq) \, \delta q. \tag{C.16}$$

For this example, $p/m = \dot{q}$, and by Newton's second law, the sum of forces on the right-hand side of the equation is the time rate of change of linear momentum $\dot{p}$. Thus

$$\dot{q} \, \delta p = \dot{p} \, \delta q, \tag{C.17}$$

which was to be shown. $\diamond$

C.2 A geometric interpretation

For the mass–spring–damper example (C.17) is given in vector form by

$$\begin{bmatrix} \dot{p} & -\dot{q} \end{bmatrix} \begin{bmatrix} \delta q \\ \delta p \end{bmatrix} = 0. \tag{C.18}$$

This vector equation can be interpreted geometrically with motion of the system described in two–dimensional (q, p) space, or phase space. Assuming undamped oscillatory motion of the mass, the trajectory of the system in phase space is elliptical as shown in Fig. C.2. Examination of one quadrant of this trajectory is instructive. The three points A, B and C are of interest.

The phase–space coordinates at A are given by $(q, p) = (q_{ex}, 0)$, where the subscript ex indicates an extreme value for a variable ξ such that $|\xi(t)| \leq \xi_{ex}$. In this example at A, $\dot{q} = 0$ and $\dot{p} = \dot{p}_{ex} \neq 0$. For (C.18) to be true δq must vanish. In other words,

$$\begin{bmatrix} \dot{p}_{ex} & 0 \end{bmatrix} \begin{bmatrix} \delta q \\ \delta p \end{bmatrix} = 0 \tag{C.19}$$

implies that

$$\delta q = 0. \tag{C.20}$$

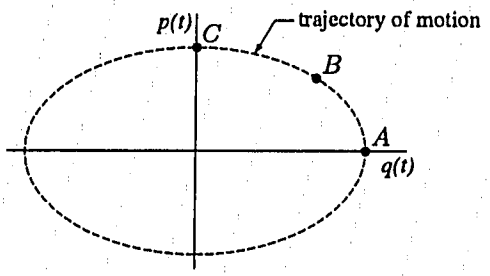

Figure C.2. Phase-space representation of steady-state oscillation.

The perturbation in momentum δp is arbitrary, but the phase–space perturbation vector $[0 \ \ \delta p]$ is not arbitrary. Let vectors r_A and δ_A be defined as follows

$$r_A := \begin{bmatrix} \dot{p}_{ex} & 0 \end{bmatrix}^T \quad \text{and} \quad \delta_A := \begin{bmatrix} 0 & \delta p \end{bmatrix}^T, \tag{C.21}$$

then

$$\langle r_A, \delta_A \rangle = 0. \tag{C.22}$$

The two vectors are orthogonal as shown in Fig. C.3.

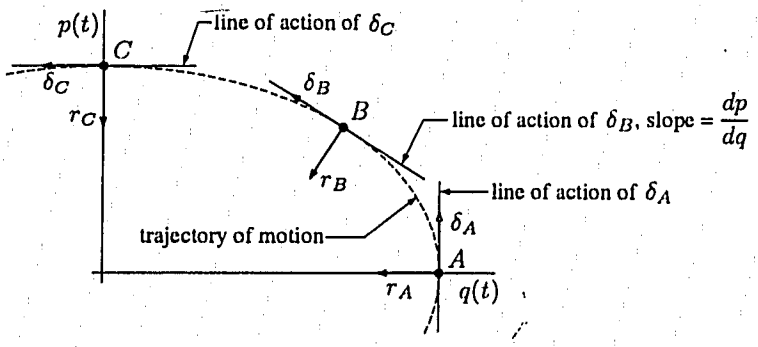

Figure C.3. A phase–space interpretation of δp.

At B, $(q, p, \dot{q}, \dot{p}) \neq 0$. In this case (C.17) yields the following expression for δp,

$$\delta p = \frac{\dot{p}}{\dot{q}} \delta q = \frac{dp/dt}{dq/dt} \delta q = \frac{dp}{dq} \delta q, \tag{C.23}$$

indicating not only that δp is proportional to δq, but that the proportionality ratio is the instantaneous slope dp/dq. Thus the perturbation vector at B, $\delta_B = [\delta q \;\; \delta p]^T$, lies on a line with slope dp/dq tangent to the trajectory of motion at B. Let $r_B = [\dot{p} \;\; -\dot{q}]^T$, then

$$\langle r_B, \delta_B \rangle = 0, \tag{C.24}$$

as shown in Fig. C.3.

At C, $(q,p) = (0, p_{ex})$, $\dot{p} = 0$, and $\dot{q} = \dot{q}_{ex} \neq 0$. For (C.17) to hold, δp must vanish. In other words,

$$\begin{bmatrix} 0 & -\dot{q}_{ex} \end{bmatrix} \begin{bmatrix} \delta q \\ \delta p \end{bmatrix} = 0, \tag{C.25}$$

implies that

$$\delta p = 0. \tag{C.26}$$

Let $r_C = [0 \;\; -\dot{q}_{ex}]^T$ and $\delta_C = [\delta q \;\; 0]^T$, then

$$\langle r_C, \delta_C \rangle = 0, \tag{C.27}$$

as shown in Fig. C.3.

It is concluded from Fig. C.3 that the general perturbation vector $\delta = [\delta q \;\; \delta p]^T$ always lies on a line tangent to the phase–space trajectory of motion. Furthermore, the vector $r = [\dot{p} \;\; -\dot{q}]^T$ is essentially an effort–flow vector $[e \;\; -f]^T$ that lies on a line normal to the trajectory of motion.

This geometric interpretation can be extended to the general system having a configuration space of dimension N, where $p \in \mathcal{R}^N$ and $u \in \mathcal{R}^N$. The trajectory of motion is described in a phase space of dimension $2N$, the perturbation vector $[\delta u \;\; \delta p]^T$ lies in a plane tangent to the trajectory of motion, and the effort–flow vector $[\dot{p} \;\; -\dot{u}]^T$ lies in a plane normal to the trajectory of motion.

The significance of this interpretation of δp is that the means by which virtual momentum is constrained becomes clear. The trajectory of motion in phase space defines

a tangent space at each instant in which the perturbation vector $[\delta u \ \delta p]^T$ lies. This trajectory occurs in the presence of kinematic constraints which impose conditions on δu only. Thus the kinematic constraints and the motion of the system jointly impose conditions on δp. Therefore, in the sense that kinematic constraints statically impose conditions on δu, it can be said that the system motion dynamically imposes conditions on δp.

Appendix D

NONCENTROIDAL ANGULAR MOMENTUM

The angular momentum of a system element that rotates about its center of mass, that rotates about a fixed point, or that rotates such that all forces act through its center of mass is accounted for in the equations of motion through the constituitive law of a kinetic store. See Table 2.7. The implicit effort and dynamic constraint concepts are used to account for the angular momenta of system elements that do not meet these criteria, as follows.

The angular momentum of a rigid body about an arbitrary point b is the sum of the centroidal angular momentum vector H_c plus the moment of the linear momentum vector P with respect to b when P is considered to act at the centroid,

$$H_b = H_c + (r_c \times P), \tag{D.1}$$

where r_c is the position vector from b to the mass center c of the system. The relationship between torque about b and angular momentum is given by

$$\begin{aligned} \tau_b &= \frac{dH_b}{dt} \\ &= \frac{dH_c}{dt} + \left(r_c \times \frac{dP}{dt}\right). \end{aligned} \tag{D.2}$$

Rearranging, and designating the cross product as an implicit effort τ^γ yields

$$\frac{dH_c}{dt} = \underbrace{\tau_b - \left(r_c \times \frac{dP}{dt}\right)}_{\tau^\gamma}, \tag{D.3}$$

which has the form

$$\dot{p} = e^s + e^\gamma. \tag{D.4}$$

Thus, in the equations of motion the centroidal angular momentum H_c contributes to the kinetic coenergy $T^*(\omega)$ of the element through the constituitive law given by

$$H_c(\omega) = I_c\omega, \tag{D.5}$$

the torque τ_b is a source effort and cross product term τ^γ is an implicit effort. The expression which defines τ^γ is expressed as a dynamic constraint given by

$$\gamma(\tau^\gamma, \omega, \theta, t) := \tau^\gamma + \left(r_c \times \frac{dP}{dt}\right) = 0. \tag{D.6}$$

Appendix E

POINT AND PATH FUNCTIONS

In this appendix are given additional comments on the differences between point or state functions and path functions. In this section, Van Wylen and Sonntag [43] are qutoed extensively.

Consider a system comprised of the gas contained in a cylinder and piston of cross–sectional area A. Let the piston move upward some distance dL. The initial state of the gas is state 1 and the final state of the gas is state 2. The states through which the system progresses in the course of the process are assumed to be equilibrium states, hence the process is called a quasiequilibrium process. The work done by the system during this process is given by

$$đW = P\,dV, \tag{E.1}$$

where P is pressure and dV is the change in the volume of the gas given by $A\,dL$. The work done at the moving boundary given a quasiequilibrium process is found by integrating (E.1), yielding

$$_1W_2 = \int_1^2 đW = \int_1^2 P\,dV, \tag{E.2}$$

where the symbol $_1W_2$ represents the work done during the process from state 1 to state 2. On a $P - V$ diagram this work would be represented by the area under the curve from a point representing state 1 to a point representing state 2.

Van Wylen and Sonntag state, "It is possible to go from state 1 to state 2 along many different equilibrium paths. Since the area under each curve represents the work for each process, it is evident that the amount of work involved in each case is a function not only of the end states of the process, but in addition is dependent on the path that is followed in going from one state to another. For this reason work is called a path function or, in mathematical parlance, $đW$ is an inexact differential.

This leads to a brief consideration of point and path functions or, to use another term, exact and inexact differentials. Thermodynamic properties are point functions, a name that arises from the fact that for a given point on a diagram or a surface, the state is fixed, and thus there is a definite value of each property corresponding to this point. The differentials of a point functions are exact differentials, and the integration is simply

$$\int_1^2 dV = V_2 - V_1. \tag{E.3}$$

Thus we can speak of the volume in state 2 and the volume in state 1, and the change in volume depends only on the initial and final states.

Work, on the other hand, is a path function for, as has been indicated, the work done in a quasiequilibrium process between two given states depends on the path followed. The differentials of path functions are inexact differentials."

In this work, the symbol đ is used to designate inexact differentials (in contrast to d for exact differentials). Thus for work we could write

$$\int_1^2 đW = {}_1W_2. \tag{E.4}$$

"Implied in this notation is that the process between states 1 and 2 has been specified. It should be noted that we never speak about the work in the system in state 1 or state 2, and thus we would never write $W_2 - W_1$."

The energy of kinetic and potential stores, as defined herein, are point functions. These energy functions depend only on the state at each particular point in the trajectory of the system, and the change in energy depends only on the initial and final states. Work is a path function, and depends on both the path and the endpoints. Thus the distinction between types of differentials embodied in the first law

$$dE = đW, \tag{E.5}$$

in which dE is an exact differential and $đW$ is an inexact differential, is maintained in

the variational form of the first law

$$\delta E = \delta W, \tag{E.6}$$

in which δE is a variation of a state or point function, and δW is not. Virtual work δW — defined as the work done by efforts acting through a virtual displacement δq — maintains the path–dependent character of work, even though the displacements in this case are virtual, not actual.

Appendix F

COMPUTER CODE FOR NUMERICAL EXAMPLES

In this appendix are listed the Matlab script files used to solve the numerical examples given in this dissertation. The appendix is organized as follows.

- A solver based on Euler's method for problems formulated as Lagrangian DAE's
- A solver based on Euler's method for problems formulated as Hamiltonian DAE's
- A solver based on a BDF method for problems formulated as Lagrangian DAE's

Each example includes a driver file, a function file and a plotting file. At the end of each section is given the code for the numerical solver and its subroutines.

F.1 Euler's method and the Lagrangian formulation

F.1.1 Pendulum

For the driver file and function file for the pendulum example, see Figs. 8.4 and 8.5. The plotting file is given below.

```
% pend_plot.m
load pend_results % T, Y
t = T;
x1 = Y(1,:);
x2 = Y(2,:);
v1 = Y(3,:);
v2 = Y(4,:);

% ----- compute constraint and velocity magnitude
L1err = sqrt(x1.^2+x2.^2) - 2.34*ones(size(t),1);
veloc = sqrt(v1.^2+v2.^2);

subplot(221),plot(x1,x2,'b')
  xlabel('u1 (m)'), ylabel('u2 (m)'), axis('equal')
  title('(a) Displacement trajectory')
subplot(222),plot(t,veloc,'b')
  xlabel('Time (sec)'), ylabel('Velocity (m/sec)'), axis([0 4 0 1])
  title('(b) Magnitude of velocity')
```

```
subplot(223),plot(t,L1err,'b'),
  xlabel('Time (sec)'), ylabel('Error (m)'), axis([0 4 -5e-8 15e-8])
  title('(c) Error in constraint')
% endfile
```

F.1.2 Slider–crank mechanism

```
% filename: slider_driver.m
clear
Y0=[1;1;3;0;0;0]; %initial condition
h =5.0e-3;  % time increment
nstep = 800;% number of time steps
n_disp = 3; % number of coordinates
n_con = 2;  % number of constraints
[T,Y]=int_dae1('slider_func',Y0,h,nstep,n_disp,n_con);
save slider_results
% endfile

% filename: slider_func.m
function [M,Phi,Phi_x,Psi_1] = slider_func(x,v,t)
mass1 = 2.26;
mass2 = 1.13;
L1 = 1.41421356237310;
L2 = 2.23606797749979;
f = 10.0;
M=[mass1,0,0;
  0,mass1,0;
  0,0,mass2];
Psi_1=[-f*x(2)/L1;
    f*x(1)/L1;
    0];
Phi = [x(1)^2+x(2)^2-L1^2;
    (x(3)-x(1))^2+x(2)^2-L2^2];
Phi_x = [2*x(1),2*x(2),0;
    -2*(x(3)-x(1)),2*x(2),2*(x(3)-x(1))];
% endfile

% filename: slider_plot.m
load slider_results
t = T;
u1 = Y(:,1);
u2 = Y(:,2);
u3 = Y(:,3);
v1 = Y(:,4);
v2 = Y(:,5);
v3 = Y(:,6);
err1 = u1.^2 + u2.^2 - 2;
err2 = (u3-u1).^2 + u2.^2 - 5;
subplot(221), plot(u1,u2)
```

```
  xlabel('u1 (m)'), ylabel('u2 (m)'),axis('equal')
  title('(a) Crank position')
subplot(222), plot(t,u3)
  xlabel('Time (sec)'), ylabel('u3 (m)')
  title('(b) Slider position')
subplot(223), plot(t,err1)
  xlabel('Time (sec)'), ylabel('Error (m)'),axis([0 4 -1e-7 3e-7])
  title('(c) Constraint error 1')
subplot(224), plot(t,err2)
  xlabel('Time (sec)'), ylabel('Error (m)'),axis([0 4 -1e-7 3e-7])
  title('(d) Constraint error 2')
% endfile
```

F.1.3 Electrical lead–filter

```
% filename: lead_driver.m
clear
Y0=[0;0;0;0;0;0];
h =1.0e-3;  % time increment
nstep = 1000;% number of time steps
n_disp = 3; % number of coordinates
n_con = 1;  % number of constraints
[T,Y]=int_dae1('lead_func',Y0,h,nstep,n_disp,n_con);
save lead_results
% endfile

% filename: lead_func.m
function [M,Phi,Phi_x,Psi_1] = lead_func(x,v,t)
M=zeros(3,3);
r1 = 0.5;
r2 = 0.5;
c1 = 0.5;
vin = sin(8*pi*t);
Psi_1=[-r1*v(1);
    -x(2)/c1;
    -r2*v(3)+vin];
Phi = [x(1)+x(2)-x(3)];
Phi_x = [1, 1, -1];
% endfile

% filename: lead_plot.m
load lead_results
r1 = 0.5;
t = T;
q1 = Y(:,1);
q2 = Y(:,2);
q3 = Y(:,3);
f1 = Y(:,4);
f2 = Y(:,5);
```

```
f3 = Y(:,5);
err = q1+q2-q3;
ein = sin(8*pi*t);
eout = ein - r1*f1;
subplot(221), plot(t,err)
   xlabel('Time'), ylabel('Error')
subplot(222), plot(t,ein,t,eout)
   xlabel('Time'), ylabel('Ein and Eout')
% endfile
```

F.1.4 Fluid lead–filter

```
% filename: fluid_lead_driver.m
clear,format short e,format compact
Y0=[0;5;0;1;-1;0];
h = 5e-1;   % time increment
tf = 60; % final time
nstep = tf/h; % number of time steps
n_disp = 3;  % number of coordinates
n_con = 1;   % number of constraints
[T,Y]=int_dae1('fluid_lead_func',Y0,h,nstep,n_disp,n_con);
save fluid_lead_results
% endfile

% filename: fluid_lead_func.m
function [M,Phi,Phi_x,Psi_1] = fluid_lead_func(u,v,t)
rho= 500; % density
g  = 9.81; % gravity
Cd = 0.62; % discharge coefficient
do = 0.1; % orifice diameter
dt = 17.7; % capacitive tank diameter
Ao = pi*do^2/4; % orifice area
At = pi*dt^2/4; % tank area
Cf = At/(rho*g); % tank capacitance
Cr1 = 100; % resistance coefficient
Cr2 = 0.5*(rho/(Cd^2*Ao^2)); % resistance coefficient
Ps = 1e+6*log(t+1); % pressure source
M = zeros(3,3);
Phi = [u(1)+u(2)-u(3)-5];
Phi_x = [1,1,-1];
Psi_1 = [ -Cr1*v(1)*sqrt(abs(v(1)));
    -u(2)/Cf;
    Ps-Cr2*v(3)*sqrt(abs(v(3)))];
% endfile

% filename: fluid_lead_plot.m
load fluid_lead_results  % contains T, Y

t = T;
```

```
u1 = Y(:,1);
u2 = Y(:,2);
u3 = Y(:,3);
v1 = Y(:,4);
v2 = Y(:,5);
v3 = Y(:,6);

err = u1+u2-u3-5*ones(size(v1)); % compute constraint, should equal zero

figure(1),clg
subplot(221), plot(t,u2,'b')
   xlabel('Time (sec)'), ylabel('u2 (m^3)')
   title('(a) Tank volume')
subplot(222), plot(t,v1,'b',t,v2,'b',t,v3,'b')
   xlabel('Time (sec)'),ylabel('v1, v2, v3 (m^3/sec)')
   title('(b) Flow rates')
subplot(223), plot(t,err,'b')
   xlabel('Time (sec)'), ylabel('Error (m^3)')
   title('(c) Constraint error')
% endfile
```

F.1.5 Solver int_dae1 and subroutines

```
% filename: int_dae1.m
function [Time,YOUT] = int_dae1(user_fun,int_con,h,nstep,n,m)
%
%  Integrate index 1 implicit DAE using Euler approximation
%  This is a very inefficient way to solve this problem
%  but it works most of the time
%
%  B. Fabien and R. Layton 2/17/95
%
Max_itt = 10;
Max_j = 15;
eps = 1.0e-6;
t = 0;
nu = zeros(m,1);
y1=[int_con;nu];
y0=y1;
YOUT=zeros(nstep+1,2*n);
Time=zeros(nstep+1,1);
Time(1) = 0.0;
YOUT(1,:)= int_con';
for t_itt = 1:nstep
  t = h*t_itt;
  Time(t_itt+1) = t;
  %
  % Newton's Iteration
```

```
  %
  for n_itt = 1:Max_itt
    F = eval_F1(user_fun,y1,y0,t,h,n,m);
    nF = norm(F);
    if( nF < eps )
      break;
    end;
    dF = eval_dF1(user_fun,y1,y0,t,h,n,m);
    dy = -dF\F;
    ndy = norm(dy);
    %
    % Damped correction
    %
    alpha = 1.0;
    for j = 1:Max_j
      yn = y1+alpha*dy;
      F1 = eval_F1(user_fun,yn,y0,t,h,n,m);
    nF1 = norm(F1);
    if( nF1 < nF )
      y1 = yn;
      break;
    else
      alpha = alpha*0.5;
    end;
    end;
    if( j >= Max_j)
      error('Too many damped corrections, ...
      (1) check constraints and I.C. (2) make h smaller');
      end;
    end;
    if( n_itt >= Max_itt )
      error('Too many Newton iterations, ...
      (1) check constraints and I.C. (2) make h smaller');
    end;
    YOUT(t_itt+1,:)=y1(1:2*n)';
    y0 = y1;
end;
% endfile

% filename: eval_F1.m
function F = eval_F1(u_Fun,y1,y0,t,h,n,m)
x0=y0(1:n);
v0=y0(n+1:2*n);
lam0=y0(2*n+1:2*n+m);
x1=y1(1:n);
v1=y1(n+1:2*n);
lam1=y1(2*n+1:2*n+m);
[M,Phi,Phi_x,Psi_1]=feval(u_Fun,x1,v1,t);
if lam1 == []
```

```
    F=[(x1-x0)./h-v1;
      M*(v1-v0)./h-Psi_1];
else
    F=[(x1-x0)./h-v1;
      M*(v1-v0)./h+Phi_x'*lam1-Psi_1;
    Phi];
end;
% endfile

% filename: eval_dF1.m
function dF = eval_dF1(u_Fun,y1,y0,t,h,n,m)
%
% Evaluate dF by finite difference
%
dtol = 1.0e-8;
dF = zeros(2*n+m,2*n+m);
F0 = eval_F1(u_Fun,y1,y0,t,h,n,m);
for i = 1:2*n+m
  yd1 = y1;
  if( abs(yd1(i)) < dtol )
    dy = dtol;
  else
    dy = dtol*yd1(i);
  end;
  yd1(i) = yd1(i) + dy;
  F1 = eval_F1(u_Fun,yd1,y0,t,h,n,m);
  dF(:,i)=(F1-F0)./dy;
end;
% endfile
```

F.2 Euler's method and the Hamiltonian formulation

F.2.1 Pendulum and spring

For the driver file and function file for the pendulum and spring example, see Figs. 8.14 and 8.15.

```
% filename: ham_plot_exam.m
%
% This plot subroutine plots the time response of the pendulum
%  an dspring example modeled using a Hamiltonian DAE simulation.
%
load ham_results
x1 = Y(:,1);
x2 = Y(:,2);
y1 = Y(:,3);
y2 = Y(:,4);
```

```
p1 = Y(:,5);
p2 = Y(:,6);
w1 = Y(:,7);
z1 = Y(:,8);
lam1 = Y(:,9);

const = (x2-w1).^2 + x1.^2-34; % constraint
p = sqrt(p1.^2 + p2.^2);
v = sqrt(y1.^2 + y2.^2);

subplot(221), plot(x1,-x2)
   xlabel('x1 (m)'), ylabel('-x2 (m)'),axis('equal')
   title('(a) Mass displacement trajectory')
subplot(222), plot(T,p)
   xlabel('Time (sec)'), ylabel('|p|  (kg-m/s)'),
   title('(b) Momentum')
subplot(223), plot(T,const)
   xlabel('Time (sec)'), ylabel('Error (m)'),
   title('(c) Constraint error')
% endfile
```

F.2.2 Solver int_daeh and subroutines

This solver is a version of the Lagrangian DAE solver int_dae1 modified for the Hamiltonian formulation. No attempt is made in this solver to take advantage of the semi-explicit form of the Hamiltonian DAE. Like the Lagrangian solver, int_daeh is set up for equations in descriptor form.

```
% filename: int_daeh.m
%
% This program integrates implicit Hamiltonian DAEs using
%  Euler approximation.
%
% T. Anderson and R. Layton 7/27/95
%
function [Time,YOUT] = int_daeh(user_fun,int_con,h,nstep,s,r,m0,m3)

Max_itt = 10; % maximum iterations allowed
Max_j = 15;
eps = 1.0e-6; % smallest time step allowed
t = 0;
Y1=[int_con]; % All initial conditions are to be set by the user
YO=Y1;
YOUT=zeros(nstep+1,3*s+2*r+m0+m3);
Time=zeros(nstep+1,1);
Time(1) = 0.0;
YOUT(1,:)= int_con';
for t_itt = 1:nstep
```

```
    t = h*t_itt;
    Time(t_itt+1) = t;

  % ----- Newton's Iteration
    for n_itt = 1:Max_itt
      F = eval_Fh(user_fun,Y1,Y0,t,h,s,r,m0,m3);
    nF = norm(F);
    if( nF < eps )
      break;
      end;
    dF = eval_dFh(user_fun,Y1,Y0,t,h,s,r,m0,m3);
    %cond_number_dF = cond(dF)
    dY = -dF\F;
    ndY = norm(dY);
    %
    % Damped correction
    %
    alpha = 1.0;
    for j = 1:Max_j
      Yn = Y1+alpha*dY;
      F1 = eval_Fh(user_fun,Yn,Y0,t,h,s,r,m0,m3);
      nF1 = norm(F1);
      if( nF1 < nF )
        Y1 = Yn;
        break;
      else
        alpha = alpha*0.5;
      end;
    end;
    if( j >= Max_j)
      error('Too many damped corrections, (1) check constraints ...
      and I.C. (2) make h smaller');
      end;
    end;
    if( n_itt >= Max_itt )
      error('Too many Newton iterations, (1) check constraints ...
      and I.C. (2) make h smaller');
    end;
    YOUT(t_itt+1,:)=Y1(1:3*s+2*r+m0+m3)';
    Y0 = Y1;
  % The following command is not required.  It displays each time interval
  %   on the screen if the user desires to track progress of simulation.
  disp(['t =  ' num2str(t) 'in ' num2str(n_itt) ' iterations']), end;
  end;
  % endfile

  % filename: eval_Fh.m
  %
  %  This subroutine is called by the program int_daeh.m.
```

```
%  It defines the initial conditions and the matrix analysis
%  for the Hamiltonian DAE.
%
%  7/27/95
%
function F = eval_Fh(u_Fun,Y1,Y0,t,h,s,r,m0,m3)
%
%  Terms x0, y0, p0, w0, z0, lam0, and egam0 all represent
%  the initial conditions of the individual terms.
%  If one of the given terms is non-existent for a
%  given simulation, the program will compensate.
%  The position of the I.C. in the input vector are
%  defined below.
%
%  The vector of initial conditions is Y0, and the vector
%  of current conditions is Y1.
%
x0=Y0(1:s);
x1=Y1(1:s);

y0=Y0(s+1:2*s);
y1=Y1(s+1:2*s);

p0=Y0(2*s+1:3*s);
p1=Y1(2*s+1:3*s);

w0=Y0(3*s+1:3*s+r);
w1=Y1(3*s+1:3*s+r);

z0=Y0(3*s+r+1:3*s+2*r);
z1=Y1(3*s+r+1:3*s+2*r);

if m0==0
  lam0=[];
  lam1=[];
else
  lam0=Y0(3*s+2*r+1:3*s+2*r+m0);
  lam1=Y1(3*s+2*r+1:3*s+2*r+m0);
end;

if m3==0
  egam0=[];
  egam1=[];
else
  egam0=Y0(3*s+2*r+m0+1:3*s+2*r+m0+m3);
  egam1=Y1(3*s+2*r+m0+1:3*s+2*r+m0+m3);
end;
```

```
% ----- matrix analysis of the Hamiltonian DAE.
[C_s,C_r,Ups_1,Ups_2,Ups_3,C,Gamma]=feval(u_Fun,x1,y1,p1,w1,z1,egam1,t);

if lam1 == [];
  F = [(x1-x0)./h-y1;
    (w1-w0)./h-z1;
    (p1-p0)./h-Ups_1;
    -Ups_2;
    y1-Ups_3];
else
  F= [(x1-x0)./h-y1;
    (w1-w0)./h-z1;
    (p1-p0)./h+C_s'*lam1-Ups_1;
    C_r'*lam1-Ups_2;
    y1-Ups_3;
    C];
end;

if egam1 == []
  F=[F];
else
  F=[F;
    Gamma];
end;
% endfile

% filename: eval_dFh.m
%
% 7/27/95
%
% This is the evaluation subroutine for solving the
%  Hamiltonian DAE.  This subroutine is called by
%  eval_Fh.m which, in turn, is called by int_daeh.m.
%
% This evaluation of dF is done using a finite difference
%  method.
%
function dF = eval_dFh(u_Fun,Y1,Y0,t,h,s,r,m0,m3)
dtol = 1.0e-8;
dF = zeros(3*s+2*r+m0+m3,3*s+2*r+m0+m3);
F0 = eval_Fh(u_Fun,Y1,Y0,t,h,s,r,m0,m3);
for i = 1:3*s+2*r+m0+m3
  Yd1 = Y1;
    if( abs(Yd1(i)) < dtol )
      dY = dtol;
    else
      dY = dtol*Yd1(i);
  end;
  Yd1(i) = Yd1(i) + dY;
```

```
  F1 = eval_Fh(u_Fun,Yd1,Y0,t,h,s,r,m0,m3);
  dF(:,i)=(F1-F0)./dY;
end;
% endfile
```

F.3 BDF method and the Lagrangian formulation

F.3.1 Electrical lead filter

```
% filename: lead_driver2.m
clear,clc,format compact,format short e

% ----- integration parameters
t0 = 0; % initial time
tf = 0.5; % final time
hmin = 1e-5; % smallest integration time step allowed
hmax = 1e-0; % largest integration time step allowed
eps1 = 1e-5; % tolerance for close to zero, solving F = 0

% ----- problem-specific parameters
user_fun ='lead_func2'; % user- function, computes [M,Phi,Phi_u,Psi]
data_fname ='lead_data2'; % user-provided name for data file
n = 3;  % number of coordinates
m = 1;   % number of kinematic constraints
y0 = zeros(2*n+m,1); % initial conditions (must be consistent)

% ----- initialization
N = 2*n+m; % number of unknowns
index = [n m N]; % vector for passing to subroutines
T = t0; % recent time values, most recent at bottom
Y = y0'; % recent solutions, row-wise, most recent at bottom
% Set up data file for accumulating solution iterates
     precision = 'float64';
fid = fopen(data_fname,'w'); % delete contents of data file, if any
% initial conditions: first row of data file
    fwrite(fid,[t0 y0'],precision);
% open file for appending data as solution proceeds
     fid = fopen(data_fname,'a');

% ----- invoke solver
int_bdf1(t0,tf,hmin,hmax,eps1,user_fun,data_fname,n,m,y0,N, ...
     index,T,Y,precision,fid);

% endfile

% filename: lead_func2
function [M,Phi,Phi_u,Psi] = lead_func2(t,u,v)
R1 = 0.5;
R2 = 0.5;
```

```
C = 1.0;
es = sin(8*pi*t);
M  = zeros(3,3);
Phi = [u(1)+u(2)-u(3)];
Phi_u = [1,1,-1];
Psi = [ -R1*v(1);
      -1/C*u(2);
      es-R2*v(3)];
% endfile

% filename: plot_lead2.m
nvar = 8;
data_f_name = 'lead_data2';

fid = fopen(data_f_name,'r');
A = fread(fid,inf,'float64');
nrow = floor(length(A)/nvar);

fid = fopen(data_f_name,'r');
A = fread(fid,[nvar,nrow],'float64')';
fclose(fid);

t = A(:,1);
u1 = A(:,2);
u2 = A(:,3);
u3 = A(:,4);
v1 = A(:,5);
v2 = A(:,6);
v3 = A(:,7);
lam1 = A(:,8);
err1 = u1+u2-u3; % compute constraint

figure(1),clg
   subplot(221), plot(t,u1),xlabel('Time'), ylabel('u1'),grid
   subplot(222), plot(t,u2),xlabel('Time'), ylabel('u2'),grid
   subplot(223), plot(t,u3),xlabel('Time'), ylabel('u3'),grid
figure(2),clg
   subplot(221), plot(t,v1),xlabel('Time'), ylabel('v1'),grid
   subplot(222), plot(t,v2),xlabel('Time'), ylabel('v2'),grid
   subplot(223), plot(t,v3),xlabel('Time'), ylabel('v3'),grid
figure(3),clg
   subplot(221), plot(t,lam1),xlabel('Time'), ylabel('lam1'),grid
   subplot(222),plot(t,err1),xlabel('Time'), ylabel('Error in constraint 1')
% endfile
```

F.3.2 Solver int_bdf1 and subroutines

```
%=========================================================================
% filename int_bdf1.m                                    Richard A. Layton
```

```
% 6/1/95
%
% Solving DAE systems using implicit BDF (similar to DASSL).
% Based on Brenan, Campbell, Petzold (1989).
%
% The set of DAE's to be solved is
% du/dt = v
% M*dv/dt + Phi_u ' * lam = Psi
% Phi = 0
%
% where the column vector of unknowns y = (u,v,lam) has length 2*n+m
% u := generalized displacements  vector of length n
% v := generalized velocities  vector of length n
% lam := Lagrange multipliers  vector of length m
%
%=====================================================================
function int_bdf1(t0,tf,hmin,hmax,eps1,user_fun,data_fname,n,m, ...
      y0,N,index,T,Y,precision,fid)
fprintf(1,'   %s\n',[' it   ' '   k   ' '     h   ' '        r   ' ...
      '      t   ']); % labels for screen output

% First step
k = 0; % k is the order of the interpolating polynomial
h = hmin; % start with smallest time interval
t = t0 + h;
psi_n = [];
psi_n1 = [h];
as = -1;
yp = y0; % predicted solution y(tn+1)
ypp = zeros(N,1); % predicted slope at y(tn+1)

[y,it_N] = newton(user_fun,t,yp,ypp,index,k,h,as,eps1);
T = [T;t];
Y = [Y;y'];
fwrite(fid,[t y' k],precision); % finish first step, psi(n+1) is 1-D

fprintf(1,'%6i %6i %11.5f %6.2f %7.3f\n',[it_N k h 1 t]); % screen display

% 2nd thru 4th step, increase order k each step
for j = 2:4,
k = j-1;
h = hmin;
t = t + h;
psi_n = psi_n1;
psi_n1= [h;h*ones(size(psi_n))+psi_n];
[alph,phi,sig,as,ao,yp ypp] = update1(k,h,N,psi_n1,psi_n,T,Y);
[y,it_N] = newton(user_fun,t,yp,ypp,index,k,h,as,eps1);
T = [T;t];
```

```
Y = [Y;y'];
fwrite(fid,[t y' k],precision);
fprintf(1,'%6i %6i %11.5f %6.2f %7.3f\n',[it_N k h 1 t]); % screen display
end

% 5th thru 8th step
% Stay at 3rd-order while building T&Y.  At the end of the 8th step, T and Y
% have 9 rows each.  For error control at kth-order, k+3 rows are required
% returning from DT subroutine which requires k+4 rows in T&Y. Therefore,
% for k=5 (max)  need 9 rows in T&Y.  Also requires 7 elements each in psi(n)
% and psi(n+1). for j = 5:8,
k = 3;
h = hmin;
t = t + h;
psi_n = psi_n1; % note: still building up psi(n) and psi(n+1) at this point
psi_n1 = [h;h*ones(size(psi_n))+psi_n];
[alph,phi,sig,as,ao,yp ypp] = update1(k,h,N,psi_n1,psi_n,T,Y);
[y,it_N] = newton(user_fun,t,yp,ypp,index,k,h,as,eps1);
T = [T;t];
Y = [Y;y'];
fwrite(fid,[t y' k],precision);
fprintf(1,'%6i %6i %11.5f %6.2f %7.3f\n',[it_N k h 1 t]); % screen display
end

%  Set up for standard iteration with error control and variable step-size and
% order. T&Y have been built up to their permanent dimension of 9 rows.
% Psi is built large enough for its permanent dimension of 7.

k = 3; % this is the last time k and h are preset
h = hmin;
psi_n  = psi_n(1:7);
psi_n1= psi_n1(1:7);

psi_n1_last = psi_n1; % suffix '-last' means 'last accepted'
t_last = t;
h_last = h;
nsch = 8; % no. of steps at constant stepsize h
no_rejects = 0;

term = 0;
while term == 0, % ============================================
t = t_last + h;
psi_n = psi_n1_last;
psi_n1 = h*ones(7,1)+[0;psi_n(1:6)];
[alph,phi,sig,as,ao,yp ypp] = update2(k,h,N,psi_n1,psi_n,T,Y);
[y,it_N] = newton(user_fun,t,yp,ypp,index,k,h,as,eps1);

% Step size acceptance
```

```
M = max(alph(k+1),abs(alph(k+1)+as-ao));
ERR = M*norm(y-yp);
if ERR <= 1 | h == hmin,
  accept_h = 1; % step is accepted if error is small or if h=hmin
else
  accept_h = 0; % otherwise step is rejected, compute new step
end
if accept_h == 1,% compute nsch for use by order-selection routine
  if h == h_last,
    nsch = nsch + 1;
  else
    nsch = 0; % reset if current accepted step is not equal to
  end %  last accepted step
end

% Select order of interpolating polynomial, regardless of step acceptance
% uses nsch and no_rejects, returns value of k and TERK
accept_k = 0;
while accept_k == 0,
if no_rejects >= 3,
  TERK      = norm((k+1)*sig(k+1)*phi(k+2,:));
elseif k == 1,
  TERKM1 = norm(k*sig(k)*phi(k+1,:));
  TERK      = norm((k+1)*sig(k+1)*phi(k+2,:));
  TERKP1  = norm((k+2)*sig(k+2)*phi(k+3,:));
  seq = [TERKM1;TERK;TERKP1];
  if seq(1) > seq(2) & seq(2) > seq(3) &  nsch >= k+1,
    k = min(k+1,3);
  end
  accept_k = 1;
else
  TERKM2 = norm((k-1)*sig(k-1)*phi(k,:));
  TERKM1 = norm(k*sig(k)*phi(k+1,:));
  TERK      = norm((k+1)*sig(k+1)*phi(k+2,:));
  TERKP1  = norm((k+2)*sig(k+2)*phi(k+3,:));
  seq = [TERKM2;TERKM1;TERK;TERKP1];
  if seq(1) > seq(2) & seq(2) > seq(3),
    accept_k = 1;
    if nsch >= k+1 & seq(3) > seq(4),
      k = min(k+1,3);
    end
  else
    k = max(1,k-1);
  end
  if k == 1,
    accept_k = 1;
  end
end
```

```
end % while

% Compute next step size for either accepted or rejected
EST = TERK;
r = (2*EST)^(-1/(k+1));

if accept_h == 1,% step just used was successful, or accepted
  no_rejects = 0; % reset
  T = [T(2:9);t]; % update 'window' on time sequence
  Y = [Y(2:9,:);y']; % update most recent solutions
  fwrite(fid,[t y' k],precision); % store accepted data

  h_last = h; % set n=n+1 for next step
  t_last = t;
  psi_n1_last = psi_n1;

  if r >= 2,% step is accepted, choose multiplier r for next step
    r = 2; % increase only if h can be doubled, then double it
  elseif r >= 1,
    r = 1;
  elseif r >= 0.9,
    r = 0.9; % decrease h by factor of r, where 0.5 <= r <= 0.9
  else
    r = max(0.5,r);
  end
else % step was unsuccessful since ERR > 1 and h > hmin
  no_rejects = no_rejects + 1;
  if no_rejects == 1,% first failure since last successful step
    r = 0.9*r; % start with a reduced value of r
    if r >= 0.9,
      r = 0.9; % decrease h by factor of r, where 0.25 <= r <= 0.9
    else
      r = max(0.25,r);
    end
  else % after second and subsequent failures
      r = 0.25; % since last successful step
  end
end

if accept_h == 0,
  fprintf(1,'%6i %6i %11.5f %6.2f\n',[it_N k h r]);
else
  fprintf(1,'%6i %6i %11.5f %6.2f %7.3f\n',[it_N k h r t]);
end

h = r*h;
if h >= hmax,
  h = hmax;
```

```
else
  h = max(hmin,h);
end

% ----- adjust final h such that final h + t -> tf
if t + h > tf, h = tf - t; end
if t >= tf, term = 1; break, end % break while loop
end
% endfile

% filename: plot_bdf1.m
nvar = 8;
data_f_name = 'bdf_data1';

fid = fopen(data_f_name,'r');
A = fread(fid,inf,'float64');
nrow = floor(length(A)/nvar);

fid = fopen(data_f_name,'r');
A = fread(fid,[nvar,nrow],'float64')';
fclose(fid);

t = A(:,1);
u1 = A(:,2);
u2 = A(:,3);
u3 = A(:,4);
v1 = A(:,5);
v2 = A(:,6);
v3 = A(:,7);
lam1 = A(:,8);

% compute constraint
err1 = u1+u2-u3;

figure(1),clg
   subplot(221), plot(t,u1),xlabel('Time'), ylabel('u1'),grid
   subplot(222), plot(t,u2),xlabel('Time'), ylabel('u2'),grid
   subplot(223), plot(t,u3),xlabel('Time'), ylabel('u3'),grid
figure(2),clg
   subplot(221), plot(t,v1),xlabel('Time'), ylabel('v1'),grid
   subplot(222), plot(t,v2),xlabel('Time'), ylabel('v2'),grid
   subplot(223), plot(t,v3),xlabel('Time'), ylabel('v3'),grid
figure(3),clg
   subplot(221), plot(t,lam1),xlabel('Time'), ylabel('lam1'),grid
   subplot(222),plot(t,err1),xlabel('Time'), ylabel('Error in constraint 1')
% endfile

% filename: eval_F2.m
function F = eval_F2(user_fun,t,y,yp,ypp,index,h,as)
```

```
n = index(1); m = index(2); N = index(3);

u = y(1:n);
v = y(n+1:2*n);
lam = y(2*n+1:N);

up = yp(1:n);
vp = yp(n+1:2*n);

upp = ypp(1:n);
vpp = ypp(n+1:2*n);

F = zeros(N,1);

[M,Phi,Phi_u,Psi] = feval(user_fun,t,u,v);

F = [upp-as/h*(u-up)-v;
M*(vpp-as/h*(v-vp))+Phi_u'*lam-Psi;
Phi];
% endfile

% filename: eval_dF2.m
function dF = eval_dF2(user_fun,t,y,yp,ypp,index,h,as)
%
% Evaluate dF by finite difference
%
dtol = 1e-8;
N = index(3);
dF = zeros(N,N);
F0 = eval_F2(user_fun,t,y,yp,ypp,index,h,as);
for i = 1:N
  yd = y;
  if( abs(yd(i)) < dtol )
    dy = dtol;
  else
    dy = dtol*yd(i);
  end;
  yd(i) = yd(i) + dy;
  F1 = eval_F2(user_fun,t,yd,yp,ypp,index,h,as);
  dF(:,i)=(F1-F0)./dy;
end;
% endfile

% filename: update1.m
function [alph,phi,sig,as,ao,yp,ypp] = update1(k,h,N,psi_n1,psi_n,T,Y)

alph = h*psi_n1.^(-1);
```

```
nn = length(psi_n1);
beta = ones(nn,1); % beta(1)=1
  for j = 2:nn,
  beta(j) = beta(j-1)*psi_n1(j-1)/psi_n(j-1);
  end

m = k;

phi = zeros(m+1,N); % result: col(phi(1),...,phi(k+3))
  % Compute divided difference table DT. DT has m rows and m*N columns.
  % We use only the mth row of DT
  DT = div_dif(T,Y,m,N);
  [nrow,ncol] = size(Y);
  phi(1,:) = Y(nrow,:); % phi(1) = yn' = last row of Y
  for j = 2:m+1,
  phi(j,:) = prod(psi_n(1:j-1))*DT(m,(j-2)*N+1:(j-1)*N);
  end

phi_star = zeros(k+1,N);
  for j = 1:k+1,
  phi_star(j,:) = beta(j)*phi(j,:);
  end

sig = ones(k+1,1); % sig(1)=1        % result: col(sig(1),...,sig(k+2))
  intgr = [1 2 3 4 5 6];
  for j = 2:k+1,
  sig(j) = h^j*prod(intgr(1:j-1))/prod(psi_n1(1:j));
  end

gam = zeros(k+1,1); % gam(1)=0
  for j = 2:k+1,
  gam(j) = sum(alph(1:j-1))/h;
  end

inv_intgr = [1/1 1/2 1/3 1/4 1/5 1/6];
as = -sum(inv_intgr(1:k));
ao = -sum(alph(1:k));

yp = (sum(phi_star))'; % sum of columns of phi_star
ypp = phi_star'*gam;
% endfile

% filename: update2.m
function [alph,phi,sig,as,ao,yp,ypp] = update2(k,h,N,psi_n1,psi_n,T,Y)

alph = h*psi_n1.^(-1);

nn = length(psi_n1);
beta = ones(nn,1); % beta(1)=1
```

```
  for j = 2:nn,
  beta(j) = beta(j-1)*psi_n1(j-1)/psi_n(j-1);
  end

m = k+2;

phi = zeros(m+1,N); % result: col(phi(1),...,phi(k+3))
  % Compute divided difference table DT. DT has m rows and m*N columns.
  % We use only the mth row of DT
  DT = div_dif(T,Y,m,N);
  [nrow,ncol] = size(Y);
  phi(1,:) = Y(nrow,:); % phi(1) = yn' = last row of Y
  for j = 2:m+1,
  phi(j,:) = prod(psi_n(1:j-1))*DT(m,(j-2)*N+1:(j-1)*N);
  end

phi_star = zeros(k+1,N);
  for j = 1:k+1,
  phi_star(j,:) = beta(j)*phi(j,:);
  end

sig = ones(k+2,1); % sig(1)=1        % result: col(sig(1),...,sig(k+2))
  intgr = [1 2 3 4 5 6];
  for j = 2:k+2,
  sig(j) = h^j*prod(intgr(1:j-1))/prod(psi_n1(1:j));
  end

gam = zeros(k+1,1); % gam(1)=0
  for j = 2:k+1,
  gam(j) = sum(alph(1:j-1))/h;
  end

inv_intgr = [1/1 1/2 1/3 1/4 1/5 1/6];
as = -sum(inv_intgr(1:k));
ao = -sum(alph(1:k));

yp = (sum(phi_star))'; % sum of columns of phi_star
ypp = phi_star'*gam;
% endfile

% filename: newton.m
function [y,it_N] = newton(user_fun,t,yp,ypp,index,k,h,as,eps1)

max_it_N = 10; % max iterations each Newton's method
max_it_ss = 15; % max iterations step size

term_N = 0; % termination flag
it_N = 0; % no. of Newton iterations
y = yp; % first guess at a solution y that satisfies the DAE
```

```
while (term_N == 0),  % while Newton termination criteria not met
  it_N = it_N +1;
  if it_N > max_it_N, term_N = 1; break; end; % too many Newton iterations

% only y changes during Newton iterations
  F = eval_F2(user_fun,t,y,yp,ypp,index,h,as);
  nF = norm(F);
  if (nF < eps1), % solved F = 0 at the present value of y
    term_N = 2; break
    end

  dF = eval_dF2(user_fun,t,y,yp,ypp,index,h,as);
  dy = -dF\F;

  alpha = 1; % Step size selection
  it_ss = 0;
  while (it_ss <= max_it_ss),
    it_ss = it_ss + 1;
    if it_ss > max_it_ss, term_N = 3; break; end; % too many damped corrections
      yn = y+alpha*dy;
      F1 = eval_F2(user_fun,t,yn,yp,ypp,index,h,as);
    if( norm(F1) < nF ),
      y = yn;  % yn gives a new |F| < old |F|
      break;
    end
    alpha = alpha*0.5;
    end % of step size selection
  end % while for Newton iteration

% Newton iteration termination statements
if term_N==1
    error('Error due to ==> too many Newton iterations'), end;
if term_N==2
    disp(['t =  ' num2str(t) 'in ' num2str(it_N) ' iterations; ...
    k = ' num2str(k) '; h = ' num2str(h)]), end;
if term_N==3, error('Error due to ==> too many damped corrections'), end;
% endfile

% filename: div_dif.m
function DT = div_dif(T,Y,k,N)

% Input T is a col vector (time) with at least k+1 elements given by
%          (...,tn-k,...,tn-1,tn)
%  Y is a matrix with N columns and at least k+1 rows given by
%          (...,yn-k,...,yn-1,yn)
%  k is the desired dimension of the difference table
%  N is the length of one y vector
%
```

```
% Output DT is a lower triangular matrix with the lower-most divided
%  differences in the bottom row.  For use in the interpolating
%  polynomials, the bottom row of DT contains k row vectors of length N
% containing ([yn,yn-1],[yn,yn-1,yn-2],...,[yn,yn-1,...,yn-k+1])

DT = zeros(k,k*N); % initialize
dT = diff(T); % create first set of time differences
dY = diff(Y); % create first set of y differences

% create first N columns of DT one row at a time
for j = 1:k,
  DT(j,1:N) = dY(j,:)./dT(j);
end

% create successive N columns of DT one row at a time
for i = 1:k-1,
  dY = diff(DT(:,(i-1)*N+1:i*N));
  for j = i+1:k,
    DT(j,i*N+1:(i+1)*N) = dY(j-1,:)./(T(j+1)-T(j-i));
end
end
% endfile
```

VITA

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Education

Ph.D. in Mechanical Engineering, 1995
University of Washington, Seattle WA
Dissertation title: Analytical system dynamics.
Advisor: Dr. B. C. Fabien

M.S. in Mechanical Engineering, 1993
University of Washington, Seattle WA
Thesis title: Batch mixing process improvements for composite solid propellants.
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B.S. in Engineering, 1991
California State University, Northridge CA

Honors and Awards

Louis and Katherine Marsh Memorial Fellowship, 1991
1990 Scholar of the Year, American Consulting Engineers Council
Litton Data Systems Management Club Scholarship 1989, 1990
Engineer–in–Training Certificate (EIT) 1985

Publications

Layton R. A. and Fabien B. C., Formulation of a Lagrangian DAE of motion for physical systems. (In preparation).

Fabien B. C. and Layton R. A., Numerical solution of a Lagrangian DAE of motion for physical systems. (In preparation).

Layton R. A., Murray W. R. and Garbini J. L., The control of power for efficient batch mixing. *Propellants, Pyrotechnics, Explosives* (Submitted).

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Technical Experience

Contract design engineer — 1992–1993
Rocket Research Company, Redmond, WA
Provided engineering services in conjunction with M.S. degree.

Technical summer hire — Summer 1991
Rocket Research Company, Redmond, WA
New product development and testing group.

Technical summer hire — Summer 1989
Rockwell International, Rocketdyne Division, Canoga Park, CA
Nonmetallic materials group, space shuttle main engine.

Project engineer — 1981–1988
Gary Walker & Associates, Consulting Engineers, Simi Valley, CA
HVAC and electrical design, contract administration and construction.

Designer, draftsman — 1979–1981
Hargis Engineers, Anchorage, AK

Academic Experience

Research assistant — 1994–1995
University of Washington, Seattle, WA
Developed course notes for advanced modeling and simulation course.

Lecturer — Summer 1993
University of Washington, Seattle, WA
Taught junior–level fluid mechanics course with lab.

Teaching assistant — 1991–1994
University of Washington, Seattle, WA
Controls, dynamics, system dynamics, microprocessor applications.

Senior–design group manager — 1990-1991
California State University, Northridge, CA
Structural design and fabrication of a high–altitude aircraft.

Activities & Affiliations

Elementary school reading and computer tutor
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