The geometry of measures with density bounds in a Hölder anisotropic setting

Ignacio José Tejeda Iriondo

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Reading Committee:

Tatiana Toro, Chair

Bobby Wilson

Steffen Rohde

David Rahbee

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University of Washington

Abstract

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Ignacio José Tejeda Iriondo

Chair of the Supervisory Committee:
Tatiana Toro
Department of Mathematics

Geometric measure theory provides tools for the study of the geometry of sets that are rough, in the sense that they are not smooth enough to support the methods of differential geometry. One approach that has been successfully exploited in this area is based on the idea that the regularity of a set can be characterized by the behavior of quantities that capture small-scale geometric properties. An important example of one such quantity is the notion of density of a set or Radon measure. Originally introduced by Besicovitch and later developed by several authors, this concept turns out to be closely tied to the local geometric properties of sets and measures, and has been a source of research problems in geometric measure theory until the present day. In this dissertation we study a problem in this direction, which aims to characterize regularity in terms of a quantitative density condition in a certain anisotropic setting.

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Dedication

To my mom, my dad, and my grandmother. I am forever grateful for your unconditional support and love, and for helping me cultivate a love of learning and joy in life.

Chapter 1

Introduction

The study of the geometric properties of subsets of Euclidean space is a problem that spans many different branches of classical and modern mathematics. From the point of view of analysis, a key question is whether these sets have a structure that is good enough to support the machinery of calculus. If the set in question is a smooth manifold, then the full strength of those tools is available within the framework of differential geometry. However, there are many interesting problems in which sets arise that are not regular enough to make the tools of differential geometry fully applicable. For example, this is the case with the Plateau problem, which aims to find a surface of least area among those surfaces with a prescribed boundary. Certain solutions to this problem have singular points, in the sense that they admit no neighborhood where the surface is locally diffeomorphic to a plane. At such points, the minimizing surface does not have a tangent plane, and therefore different techniques are needed in order to understand its local structure.

Geometric measure theory provides a framework to study the geometry of such sets, or more generally, the geometry of Radon measures. Its origins can be traced back to a 1919 paper of F. Hausdorff, where he introduced a measure that generalized the notion of length or area captured by Lebesgue measure. Known today as Hausdorff measure, this tool provided a natural way to capture the size of objects of dimension smaller than their ambient space without assumptions on their regularity, allowing, for example, non-smooth surfaces or fractal sets.

In the mid-1900s, A. S. Besicovitch carried out a detailed study of the local geometric properties of sets of finite "linear measure," which today we call 1-dimensional Hausdorff measure. His work laid down the foundations of many of the twentieth century developments of geometric measure theory, including the study of rectifiable sets, and the notion of sets of finite perimeter introduced by De Giorgi in the 1960's, which played a key role in the solution to the Plateau problem in higher dimensions.

An important line of thought was introduced in this context by E. Reifenberg. Also motivated by the Plateau problem, he introduced a geometric quantity that captures how well a set can be approximated by planes at small scales. Known today as Reifenberg's Topological Disk Theorem, his celebrated result states that if a set can be suitably approximated by planes, then it is locally a bi-Hölder image of a disk. This result suggested the idea that the regularity of a set can be characterized by the behavior of quantities that capture small-scale geometric properties, opening up a line of research that has been fruitfully developing over the last decades (see Figure 1.1). This dissertation considers a particular instance of this line of thought. Namely the characterization of the regularity of sets, or more generally Radon measures, by the behavior of their densities, a notion originated in the work of Besicovitch that has proven to play a key role in the study of rectifiable sets or measures, and also in higher regularity settings.

1.1 Density and rectifiability

Throughout this work, we let μ denote a Radon measure on $\mathbb{R}^{n+1}$. For $m \in \mathbb{N}$, $1 \leq m \leq n+1$, we denote by ω_m the m-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^m$, and by $B(X, r)$ the closed ball of radius r and center $X \in \mathbb{R}^{n+1}$ with

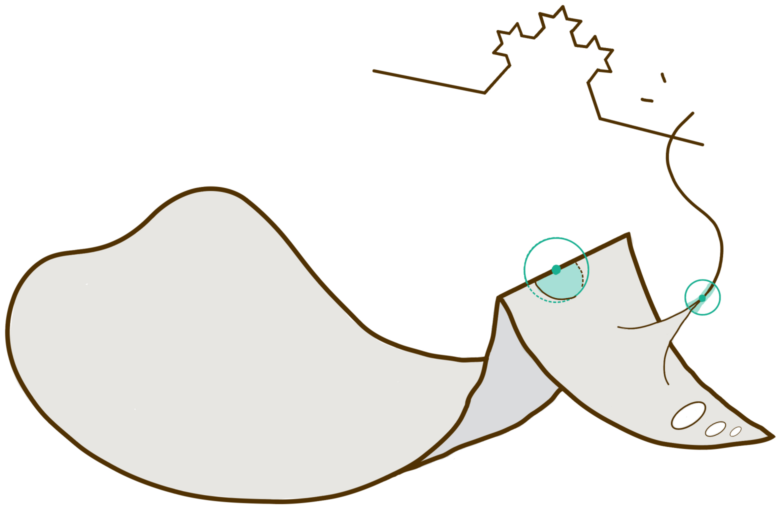

Figure 1.1: Capturing small-scale properties of a set; for a fixed point in the set, a small ball centered at that point is considered, and some geometric measurement is performed inside it.

respect to the Euclidean norm,

$$|Y| = \left(y_1^2 + y_2^2 + \cdots + y_{n+1}^2\right)^{1/2}, \quad Y = (y_1, y_2, \ldots, y_{n+1}) \in \mathbb{R}^{n+1}.$$

Definition 1.1.1 (Density)**.** *Let μ be a Radon measure on $\mathbb{R}^{n+1}$. Its m-*dimensional density *at a point $X \in \mathbb{R}^{n+1}$ is the quantity*

$$\Theta^m(\mu, X) = \lim_{r \searrow 0} \frac{\mu(B(X,r))}{\omega_m r^m}, \tag{1.1.1}$$

whenever the limit exists.

Remark 1. If the limit in (1.1.1) does not exist, one can consider the lower and upper densities of μ, $\Theta_*^m(\mu, \cdot)$ and $\Theta^{*m}(\mu, \cdot)$, obtained by replacing the limit with $\liminf$ or $\limsup$ as $r \searrow 0$, respectively, both of which always exist.

This concept goes back to the foundational work of Besicovitch in [1], [2] and [3], where he studied the geometry of 1-dimensional sets in $\mathbb{R}^2$ in terms of the existence of 1-dimensional densities. Intuitively, since $\omega_m r^m$ is the m-dimensional Lebesgue measure of a ball of radius r in $\mathbb{R}^m$, the quantity $\Theta^m(\mu, X)$ captures the extent to which the infinitesimal behavior μ at X resembles that of m-dimensional Lebesgue measure, from the point of view of their action on open balls. Depending on whether the limit defining $\Theta^m(\mu, X)$ exists, and if it does, on how close it is to 1, that similarity can be more or less accurate, which can be considered a parameter of the regularity of μ around X.

It is worth mentioning that even though $\Theta^m(\mu, X)$ can be defined even if m is not an integer, a result of Marstrand in [19] states that the existence of $\Theta^m(\mu, X)$ on a set of positive measure forces m to be an integer. The measures that we consider in this work shall be assumed to have a density at every point, so we may restrict ourselves without loss of generality to the case where $m \in \mathbb{N}$.

The main model examples considered in this work are constructed using Hausdorff measure.

Definition 1.1.2 (Hausdorff Measure)**.** *The m-dimensional Hausdorff measure of a set $E \subset \mathbb{R}^{n+1}$ is given by*

$$\mathcal{H}^m(E) = \lim_{\delta \searrow 0} \mathcal{H}^m_\delta(E), \quad \textit{where} \quad \mathcal{H}^m_\delta(E) = \inf \sum_{i=1}^{\infty} \omega_m \left(\frac{\operatorname{diam}(E_i)}{2} \right)^m,$$

and the infimum is taken over all countable collections of sets $E_i \subset \mathbb{R}^{n+1}$, $i = 1, 2, \ldots$, such that $E \subset \bigcup_{i=1}^{\infty} E_i$ and $\operatorname{diam}(E_i) \le \delta$ for every i, where $\operatorname{diam}(E_i) = \sup_{X,Y \in E_i} |X - Y|$.

Example 1. Suppose $P \subset \mathbb{R}^{n+1}$ is an m-dimensional plane (or m-plane), and $\mu = \mathcal{H}^m \llcorner P$. Here the symbol " $\llcorner$ " denotes restriction of a measure. Then for every $X \in P$ and $r > 0$,

$$\frac{\mu(B(X,r))}{\omega_m r^m} = \frac{\mathcal{H}^m(P \cap B(X,r))}{\omega_m r^m} = 1.$$

In particular, $\Theta^m(\mu, X) = 1$. More generally, if Σ is a C^1 m-dimensional submanifold

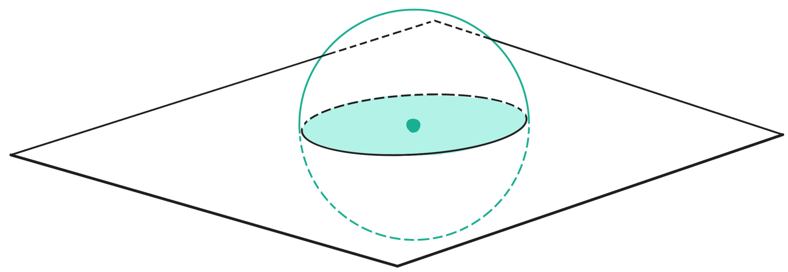

Figure 1.2: Computing the density of $\mu = \mathcal{H}^m \llcorner P$ when P is an m-plane; in this case, $P \cap B(X, r)$ is an m-dimensional ball of radius r, and $\mathcal{H}^m(P \cap B(X, r))$ is its m-dimensional Lebesgue measure.

of $\mathbb{R}^{n+1}$ and $\mu = \mathcal{H}^m \llcorner \Sigma$, then for every $X \in \Sigma$,

$$\frac{\mu(B(X,r))}{\omega_m r^m} = 1 + o(1) \quad \text{as } r \searrow 0.$$

In particular, $\Theta^m(\mu, X) = 1$.

This example confirms that sets or measures with nice geometry should have good density behavior. Surprisingly, however, there exist measures with notoriously different geometry whose density ratio still behaves optimally, as shown by Preiss in [25] and Kowalski-Preiss in [14].

Example 2 (Kowalski-Preiss cone)**.** Let

$$\mathcal{C} = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + x_3^2 = x_4^2\}.$$

If $\mu = \mathcal{H}^n \llcorner \mathcal{C}$, then there exists a constant $c > 0$ such that for every $X \in \mathcal{C}$ and $r > 0$,

$$\frac{\mu(B(X,r))}{\omega_n r^n} = c.$$

However, no point with $x_1 = x_2 = x_3 = x_4 = 0$ admits a neighborhood where $\mathcal{C}$ can be parametrized by a C^1 image of $\mathbb{R}^n$ (see Figure 1.3).

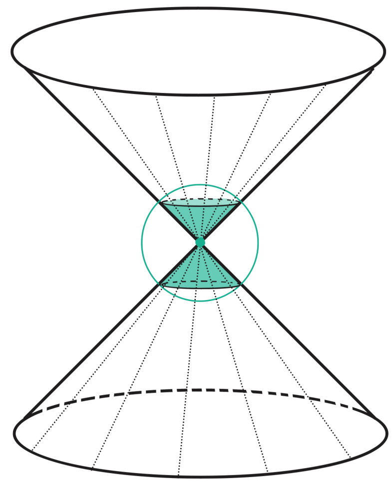

Figure 1.3: Density at a singular point of the Kowalski-Preiss cone $\mathcal{C}$; even though its center is a singularity, the density of $\mu = \mathcal{H}^n \llcorner \mathcal{C}$ at that point still behaves nicely.

This example shows that the existence, positivity and finiteness of the density of a Radon measure does not guarantee that the measure is infinitesimally flat. However, such a density condition is tied to a weaker, measure-theoretic notion of infinitesimal flatness:

Definition 1.1.3 (Rectifiable sets and measures)**.** *A set $E \subset \mathbb{R}^{n+1}$ is m-rectifiable if there exist Lipschitz maps $f_i : \mathbb{R}^m \to \mathbb{R}^{n+1}$, $i = 1, 2, \ldots$, such that*

$$\mathcal{H}^m \left(E \setminus \bigcup_{i=1}^{\infty} f_i(\mathbb{R}^m) \right) = 0.$$

A Radon measure μ on $\mathbb{R}^{n+1}$ is m-rectifiable if $\mu \ll \mathcal{H}^m$, i.e. μ is absolutely continuous with respect to $\mathcal{H}^m$, and there exists an m-rectifiable Borel set $E \subset \mathbb{R}^{n+1}$ such that

$$\mu(\mathbb{R}^{n+1} \setminus E) = 0.$$

By Rademacher's Theorem, Lipschitz mappings are differentiable almost everywhere, so Lipschitz images of $\mathbb{R}^m$ have m-dimensional tangent planes $\mathcal{H}^m$-almost

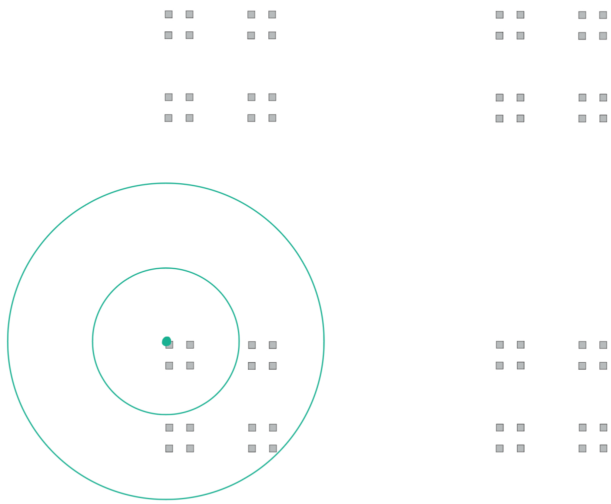

Figure 1.4: The 4-corner Cantor set; a 1-dimensional purely unrectifiable set and a point in it where density fails to exists due to an oscillatory behavior of the density ratio.

everywhere. Rectifiable sets enjoy a similar property if one replaces tangent planes with the weaker, measure theoretic notion of approximate tangent plane (see [21] for definitions).

A characterization of rectifiable sets and measures was established over the last century in terms of the existence, positivity and finiteness of densities. The first results in this direction were produced by Besicovitch ([1], [2], [3]), who showed that if $\Sigma \subset \mathbb{R}^2$ and $0 < \mathcal{H}^1(\Sigma) < \infty$, then the existence, positivity and finiteness $\mathcal{H}^1-$almost everywhere of $\Theta^1(\mathcal{H}^1 \llcorner \Sigma, \cdot)$ on Σ is equivalent to the $1-$rectifiability of Σ (see Figure 1.4). After several decades, work of various authors including Marstrand [18], Mattila [20] and Preiss [25] culminated in a deep result of Preiss, stating that given any integer $1 \leq m \leq n+1$ and any Radon measure μ on $\mathbb{R}^{n+1}$, the $\mu-$almost everywhere existence, positivity and finiteness of $\Theta^m(\mu, \cdot)$ is equivalent to the $m-$rectifiability of μ (see also notes by De Lellis in [16]).

1.2 Density and higher order regularity

Both the existence of a density and the rectifiability of a set or measure are qualitative conditions, in the sense that they do not contain any quantitative information about the behavior of the ratio $\frac{\mu(B(X,r))}{\omega_m r^m}$ (e.g. its rate of convergence) or the features of the individual Lipschitz mappings in the definition of rectifiability (e.g. their Lipschitz constant, or how big a portion of the set under consideration they cover). It then becomes natural to ask whether quantitative conditions on the behavior of densities correspond to quantitative conditions on the geometry of a set or measure, and vice versa.

A number of authors have worked in this direction over the last few decades. For example, Tolsa showed in [29] that the so-called weak density condition implies uniform rectifiability for Ahlfors-David regular measures, extending a result of David and Semmes ([5], [6]) to arbitrary dimensions. In a slightly different direction, higher order rectifiability and parametrization results have been obtained by David, Kenig and Toro [7], Ghinassi [9], Del Nin and Idu [8], Hoffman [10] and Lewis [17], where the authors study densities and other geometric quantities, and the mappings under consideration are $C^{1,\gamma}$ for some $\gamma \in (0,1)$, where $C^{1,\gamma}$ denotes the space of vector-valued functions on Euclidean space with γ-Hölder continuous first order derivatives.

The main subject of this work is the density behavior of certain measures supported on $C^{1,\gamma}$ submanifolds of $\mathbb{R}^{n+1}$. Recall that the support of a measure μ on $\mathbb{R}^{n+1}$ is the set

$$\operatorname{spt}(\mu) = \{X \in \mathbb{R}^{n+1} : \mu(B(X,r)) > 0, \text{ for all } r > 0\}.$$

A Radon measure μ is said to be supported on a set $\Sigma \subset \mathbb{R}^{n+1}$ if $\operatorname{spt}(\mu) = \Sigma$. We consider the following motivating example:

Example 3. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = |x|^{1+\gamma}$, where $\gamma \in (0,1)$. Let Σ be the graph of f, i.e. the set $\{(x,y) \in \mathbb{R}^2 : x \in \mathbb{R},\ y = f(x)\}$. Notice that Σ is a $C^{1,\gamma}$ submanifold of dimension 1 of $\mathbb{R}^2$. Consider the measure $\mu = \mathcal{H}^1 \llcorner \Sigma$. We estimate the 1-dimensional

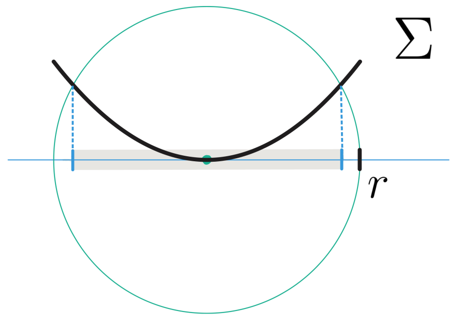

Figure 1.5: Computing density on the graph of $f(x) = |x|^{1+\gamma}$, $x \in \mathbb{R}$. The projection of $\Sigma \cap B((0,0), r)$ differs from the interval $(-r, r) \times \{0\}$ by a set of $\mathcal{H}^1$ measure $\sim r^{1+\gamma}$.

density ratio of μ at the origin (see Figure 1.5):

$$\begin{aligned}\mu(B((0,0),r)) &= \int_{-r}^{r} \sqrt{1 + f'(x)^2}\mathrm{d}x + O(r^{1+\gamma}) \\ &= \int_{-r}^{r} 1 + \frac{1}{2}x^{2\gamma} + O(x^{4\gamma})\mathrm{d}x + O(r^{1+\gamma}) \\ &= 2r + O(r^{1+\gamma}).\end{aligned}$$

Thus, since $\omega_1 = 2$,

$$\left| \frac{\mu(B((0,0),r))}{\omega_1 r} - 1 \right| \lesssim r^{\gamma},$$

which not only implies that $\Theta^1(\mu, (0,0)) = 1$, but also gives information on the rate at which this limit is attained.

In [7], the authors showed that under an additional mild geometric assumption, the density behavior in the example above in fact implies the $C^{1,\gamma}$ regularity of the support of a Radon measure, possibly upon adjusting γ:

Theorem 1.2.1 ([7]). *For each $\alpha \in (0,1)$ there exists $\gamma = \gamma(\alpha) \in (0,1)$ with the following property: Suppose $\Sigma = \operatorname{spt}(\mu) \subset \mathbb{R}^{n+1}$ for some positive Radon measure μ, and that for each compact set $K \subset \mathbb{R}^{n+1}$ there is a constant $C_K > 0$ such that*

$$\left| \frac{\mu(B(X,r))}{\omega_n r^n} - 1 \right| \leq C_K r^{\alpha}, \tag{1.2.1}$$

for $X \in \Sigma \cap K$ and $0 < r < 1$. If $n \geq 3$, suppose in addition that Σ is Reifenberg-flat with small constant. Then Σ is a $C^{1,\gamma}$ n-dimensional submanifold of $\mathbb{R}^{n+1}$.

For details on the meaning of the flatness assumption in Theorem 1.2.1, see Section 2. This assumption helps to ensure that Σ does not have many holes [27], as well as rule out the existence of singular points of the type described in Example 2.

More generally, it is shown in [7] that Theorem 1.2.1 remains valid if (1.2.1) is replaced with a suitable doubling condition, namely, μ being α-Hölder asymptotically optimally doubling:

Definition 1.2.1 (Asymptotic optimal doubling)**.** *A Radon measure μ on $\mathbb{R}^{n+1}$ with support Σ is* asymptotically optimally doubling of dimension n *if for every compact set $K \subset \mathbb{R}^{n+1}$,*

$$\lim_{r \searrow 0} \sup_{\substack{X \in \Sigma \cap K \\ t \in [\frac{1}{2}, 1]}} \left| \frac{\mu(B(X, tr))}{\mu(B(X, r))} - t^n \right| = 0. \tag{1.2.2}$$

Additionally, given $\alpha \in (0, 1)$, μ is α-Hölder asymptotically optimally doubling of dimension n *if for every compact set $K \subset \mathbb{R}^{n+1}$ there exist constants $C_K > 0$ and $r_K > 0$ such that for every $r \in (0, r_K]$,*

$$\sup_{\substack{X \in \Sigma \cap K \\ t \in [\frac{1}{2}, 1]}} \left| \frac{\mu(B(X, tr))}{\mu(B(X, r))} - t^n \right| \leq C_K r^{\alpha}. \tag{1.2.3}$$

A direct computation shows that condition (1.2.3) is weaker than (1.2.1). However, as shown in [7], the validity of (1.2.3) implies the existence of a measure that satisfies (1.2.1) and has the same support as μ, which explains why assumption (1.2.1) can be relaxed in this way. It is important to mention that Theorem 1.2.1 and its doubling counterpart were originally proven in codimension 1. Preiss, Tolsa and Toro later generalized these results to arbitrary codimension in [26].

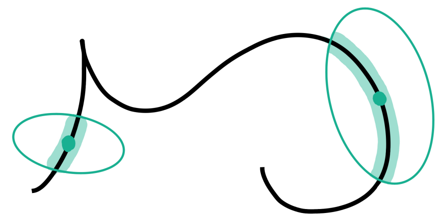

Figure 1.6: An example of a measure with 1-dimensional support and two Λ-balls centered on it.

1.3 Main results

In this work we consider measures that satisfy anisotropic analogues of (1.2.1) and (1.2.3), in the sense that the balls used in both conditions are now determined by a rough Riemannian metric on Euclidean space. This approach is motivated by recent work of Casey, Goering, Toro and Wilson in [4], where the authors characterize rectifiability of measures in terms of the existence of anisotropic densities and principal values.

More precisely, we consider a matrix-valued function $X \mapsto \Lambda(X)$, $X \in \mathbb{R}^{n+1}$, such that $\Lambda(X)$ is symmetric, positive definite for every X. The corresponding balls, that we shall refer to as Λ-balls or ellipses, are given by

$$B_\Lambda(X,r) = X + \Lambda(X)B(0,r), \quad r > 0. \tag{1.3.1}$$

Definition 1.3.1 (Λ-density). *Let μ be a Radon measure on $\mathbb{R}^{n+1}$. The m-dimensional Λ-density of μ is defined as*

$$\Theta^m_\Lambda(\mu, X) = \lim_{r \searrow 0} \frac{\mu(B_\Lambda(X,r))}{\omega_m r^m}, \quad X \in \Sigma = \mathrm{spt}(\mu), \tag{1.3.2}$$

whenever the limit exists.

Our main goal is to establish the $C^{1,\gamma}$ regularity of the support of a Radon measure μ under quantitative density and doubling assumptions formulated in terms of $\Theta^m_\Lambda(\mu, \cdot)$. We focus on the codimension 1 case.

The statements of some of our main results make reference to a flatness condition as well, formulated in terms of the n-dimensional bilateral β-numbers $b\beta^n_\Sigma(X,r)$ (see Section 2.3 for definitions), where $\Sigma = \operatorname{spt}(\mu)$. Given a compact set $K \subset \mathbb{R}^{n+1}$ and a radius $r_0 > 0$, we denote

$$b\beta_\Sigma(K, r_0) = b\beta^n_\Sigma(K, r_0) = \sup_{r\in(0,r_0]} \sup_{X\in\Sigma\cap K} b\beta^n_\Sigma(X,r).$$

The flatness condition we consider is the following:

$$\text{For every compact set } K \subset \mathbb{R}^{n+1}\text{, there exists } r_K > 0 \text{ depending on } K \text{ and } \Lambda\text{, such that } b\beta_\Sigma(K, r_K) < \delta_K\text{, where } \delta_K > 0 \text{ is a small number determined by } K \text{ and } \Lambda. \tag{1.3.3}$$

Note that any set Σ satisfies (1.3.3) if $\delta_K \geq 1$. On the other hand, if $\delta_K < 1$, then (1.3.3) gives information on the flatness of Σ at points in $\Sigma\cap K$. Condition (1.3.3) is a replacement for the flatness assumption of Theorem 1.2.1, and it means that locally, a certain amount of flatness is required, and this amount is determined only by location and the mapping Λ.

Our first main result is the following:

Theorem 1.3.1. *Let $\alpha, \beta \in (0,1)$. Suppose $X \mapsto \Lambda(X)$ is locally Hölder continuous on $\mathbb{R}^{n+1}$ with exponent β. Let μ be a Radon measure on $\mathbb{R}^{n+1}$ with support Σ such that the following holds: For every compact set $K \subset \mathbb{R}^{n+1}$ with $\Sigma \cap K \neq \varnothing$ there exists a constant $C_K > 0$ such that for every $X \in \Sigma \cap K$ and $r \in (0,1]$,*

$$\left| \frac{\mu(B_\Lambda(X,r))}{\omega_n r^n} - 1 \right| \leq C_K r^\alpha. \tag{1.3.4}$$

If $n \geq 3$, suppose additionally that Σ satisfies (1.3.3). Then Σ is a $C^{1,\gamma}$ n-dimensional submanifold of $\mathbb{R}^{n+1}$, for some $\gamma \in (0,1)$ depending on α and β.

Remark 2. The Hölder continuity condition above and (1.3.4) will be often referred to as the continuity and density assumptions of Theorem 1.3.1. For a precise formulation of the continuity assumption see Section 2.2.

We also prove a version of Theorem 1.3.1 with a doubling assumption:

Theorem 1.3.2. *Let $\alpha, \beta \in (0,1)$. Suppose $X \mapsto \Lambda(X)$ is locally Hölder continuous on $\mathbb{R}^{n+1}$ with exponent β. Let μ be a Radon measure on $\mathbb{R}^{n+1}$ with support Σ such that the following holds: For every compact set $K \subset \mathbb{R}^{n+1}$ with $\Sigma \cap K \neq \varnothing$ there exists a constant $C_K > 0$ such that for every $r \in (0,1]$,*

$$\sup_{\substack{X \in \Sigma \cap K \\ t \in [\frac{1}{2},1]}} \left| \frac{\mu(B_\Lambda(X,tr))}{\mu(B_\Lambda(X,r))} - t^n \right| \leq C_K r^\alpha. \tag{1.3.5}$$

If $n \geq 3$, suppose additionally that Σ satisfies (1.3.3). Then Σ is a $C^{1,\gamma}$ n-dimensional submanifold of $\mathbb{R}^{n+1}$, for some $\gamma \in (0,1)$ depending on α and β.

These are analogues of the corresponding results in [7]. The main novelty here is the ability to replace round balls with ellipses that change from point to point. This type of question lies in the framework of studying densities or other related analytic quantities determined by norms other than the Euclidean one. In our case, the associated norm depends on the point, and is given by $\|Z\|_X = |\Lambda(X)^{-1}Z|$, so that

$$B_\Lambda(X,r) = \{Y \in \mathbb{R}^{n+1} : \|Y - X\|_X < r\}.$$

The proof of Theorem 1.3.2 is based on Theorem 1.3.1. On the other hand, the proof of Theorem 1.3.1 uses the following result in [7].

Proposition 1.3.1 ([7] - Proposition 9.1)**.** *Let $\gamma \in (0,1)$. Suppose Σ is a Reifenberg-flat set with vanishing constant of dimension m in $\mathbb{R}^{n+1}$, $m \leq n+1$, and that for each compact set $K \subset \mathbb{R}^{n+1}$ there exist constants $C_K, r_K > 0$ such that*

$$\beta_\Sigma(X,r) \leq C_K r^\gamma, \tag{1.3.6}$$

for all $X \in K \cap \Sigma$ and $r \in (0, r_K]$. Then Σ is a $C^{1,\gamma}$ submanifold of dimension m of $\mathbb{R}^{n+1}$.

Remark 3. It can be seen from the proof of this result that Σ only needs to be assumed to be Reifenberg-flat with small constant.

By virtue of this result, the proof of Theorem 1.3.1 consists of the following steps:

Step 1. Prove that (1.3.6) holds under the assumptions of Theorem 1.3.1.

Step 2. Show that under the assumptions of Theorem 1.3.1, Σ is Reifenberg flat with vanishing constant.

Our last main result describes the case in which (1.3.5) is satisfied but no flatness assumption is made on $\Sigma = \operatorname{spt}(\mu)$. This is an anisotropic analogue of [26, Theorem 1.7] in codimension 1:

Theorem 1.3.3. *Let $\alpha, \beta \in (0,1)$. Suppose $X \mapsto \Lambda(X)$ is locally Hölder continuous on $\mathbb{R}^{n+1}$ with exponent β. Let μ be a Radon measure on $\mathbb{R}^{n+1}$ with support Σ such that the following holds: For every compact set $K \subset \mathbb{R}^{n+1}$ with $\Sigma \cap K \neq \varnothing$ there exists a constant $C_K > 0$ such that for every $r \in (0,1]$,*

$$\sup_{\substack{X \in \Sigma \cap K \\ t \in [\frac{1}{2}, 1]}} \left| \frac{\mu(B_\Lambda(X, tr))}{\mu(B_\Lambda(X, r))} - t^n \right| \leq C_K r^\alpha.$$

Then $\Sigma = \mathcal{R} \cup \mathcal{S}$, where $\mathcal{S}$ is a closed set of Hausdorff dimension at most $n-3$ if $n \geq 3$, or $\mathcal{S} = \varnothing$ if $n \leq 2$, and $\mathcal{R}$ is a $C^{1,\gamma}$ submanifold of dimension n of $\mathbb{R}^{n+1}$, for some $\gamma \in (0,1)$ depending on α and β.

This dissertation is organized as follows. Chapter 2 contains preliminary definitions and facts that are needed later on in the proofs of the main results. Steps 1 and 2 of the proof of Theorem 1.3.1 are contained in Chapter 3 and Chapter 4, respectively. Finally, Chapter 5 contains the proofs of Theorem 1.3.2 and Theorem 1.3.3.

Chapter 2

Preliminaries

We will adopt the convention that any local constants depending on a compact set $K \subset \mathbb{R}^{n+1}$ may be denoted by C_K. Moreover, we may allow C_K to depend on the matrix-valued function Λ, and any updates to the value of C_K may be incorporated without changing notation.

2.1 Measure theoretic background

We start with a summary of some of the properties of uniform and flat measures, as well as their relevance in our setting via the notion of tangent measure.

Definition 2.1.1 (Uniform measure)**.** *Given $0 < m \leq n+1$, a Radon measure ν on $\mathbb{R}^{n+1}$ is m-*uniform *if there exists a constant $C > 0$ such that for all $X \in \mathrm{spt}(\nu)$ and $r > 0$,*

$$\nu(B(X,r)) = Cr^m. \tag{2.1.1}$$

Uniform measures resemble, in particular, the asymptotic behavior as $r \to 0$ of a measure μ that satisfies (1.2.1). As mentioned above, a well-known result of Marstrand [19] implies that if ν is m-uniform, then m is an integer. An important example of m-uniform measure is when ν is *flat*:

Definition 2.1.2 (Flat measure). *Given $m \in \mathbb{N}$, $m \leq n+1$, a Radon measure ν on $\mathbb{R}^{n+1}$ is* flat of dimension m *if there exists an m-plane P and a constant $C > 0$ such that $\nu = C\mathcal{H}^m \llcorner P$.*

Flat measures play a key role in the study of rectifiability (see for example [21] for a good summary). However, not all uniform measures are flat, as shown by the Kowalski-Preiss cone in Example 2. Even though a complete classification of uniform measures is still an open problem, the following results summarize some of the main facts we need to know about uniform measures in this work:

Theorem 2.1.1 ([25]). *Let ν be an m-uniform measure on $\mathbb{R}^{n+1}$, $m \in \mathbb{N}$, $m \leq n+1$. If $m \leq 2$, then ν is flat of dimension m.*

Theorem 2.1.2 ([14]). *Let ν be an n-uniform measure in $\mathbb{R}^{n+1}$. Then after a translation, rotation and multiplication by a constant, either*

$$\nu = \mathcal{H}^n \llcorner \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} = 0\}, \tag{2.1.2}$$

or $n \geq 3$ and

$$\nu = \mathcal{H}^n \llcorner \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + x_2^2 + x_3^2 = x_4^2\}. \tag{2.1.3}$$

Theorem 2.1.2 characterizes n-uniform measures in codimension 1. In higher codimension, other families of examples of non-flat 3-uniform measures have been found by A. Dali Nimer [24], but our arguments will not rely on those directly.

The way in which uniform measures arise in our context is as tangent measures of a measure that satisfies the assumptions of Theorems 1.3.2, 1.3.1 or 1.3.3. Given a Radon measure μ on $\mathbb{R}^{n+1}$, $X \in \mathrm{spt}(\mu)$ and $r > 0$, consider the map $T_{X,r}$ and the measure $\mu_{X,r}$ given by

$$T_{X,r}(Z) = \frac{Z - X}{r}, \quad \mu_{X,r} = \frac{1}{\mu(B(X,r))} T_{X,r}[\mu], \tag{2.1.4}$$

where $Z \in \mathbb{R}^{n+1}$, and $T_{X,r}[\mu]$ is the push-forward measure of μ via $T_{X,r}$.

Definition 2.1.3 (Tangent measure). *Let μ, ν be nonzero Radon measures on $\mathbb{R}^{n+1}$. We say that ν is a tangent measure of μ at $X \in \operatorname{spt}(\mu)$ if $\mu_{X,r_k} \rightharpoonup \nu$ for some sequence $r_k > 0$ with $r_k \to 0$, where "$\rightharpoonup$" denotes weak convergence of Radon measures.*

Remark 4. Note that if μ is m-uniform and ν is a tangent measure of μ as defined above, then $\nu(B(0,1)) = 1$.

The following is a standard notion of distance between Radon measures that we will need later in connection with tangent measures. For $r > 0$, let $\mathcal{L}(r)$ denote the set of non-negative Lipschitz functions φ on $\mathbb{R}^{n+1}$ with $\operatorname{spt}(\varphi) \subset B(0,r)$ and $\operatorname{Lip}(\varphi) \leq 1$. Given Radon measures α, β on $\mathbb{R}^{n+1}$, let

$$\mathcal{F}_r(\nu, \tilde{\nu}) = \sup_{\varphi \in \mathcal{L}(r)} \left| \int \varphi \mathrm{d}\nu - \int \varphi \mathrm{d}\tilde{\nu} \right|, \quad \mathcal{F}(\nu, \tilde{\nu}) = \sum_{k \geq 1} 2^{-k} \mathcal{F}_{2^k}(\nu, \tilde{\nu}). \tag{2.1.5}$$

As the following result shows (see [25, Proposition 1.12]), these functionals metrize the weak convergence of Radon measures:

Proposition 2.1.1 ([25]). *Suppose ν, ν_k are radon measures on $\mathbb{R}^{n+1}$, $k \in \mathbb{N}$. Then the following conditions are equivalent:*

$$\text{(i) } \nu_k \rightharpoonup \nu, \quad \text{(ii)} \lim_k \mathcal{F}(\nu_k, \nu) = 0, \quad \text{(iii) } \textit{For all } r > 0,\ \lim_k \mathcal{F}_r(\nu_k, \nu) = 0.$$

We will also need a result of Preiss based on the notion of tangent measure at infinity:

Definition 2.1.4 (Tangent measure at infinity). *Let μ, ν be nonzero Radon measures on $\mathbb{R}^{n+1}$. Then ν is a tangent measure of μ at infinity (of dimension m) if for every $X \in \mathbb{R}^{n+1}$,*

$$\frac{1}{r^m} T_{X,r}[\mu] \rightharpoonup \nu,$$

as $r \to \infty$.

Note that *a priori*, different sequences of radii may lead to different limits. However, Preiss showed that this is not the case if μ is m-uniform. Recall the notion of conical measure:

Definition 2.1.5 (Conical measure)**.** *An m-uniform measure ν is conical if for every $r > 0$,*

$$\frac{1}{r^m} T_{0,r}[\nu] = \nu. \tag{2.1.6}$$

Theorem 2.1.3 ([25])**.** *Let ν be an m-uniform measure in $\mathbb{R}^{n+1}$. Then ν has a unique tangent measure $\tilde{\nu}$ at ∞, where $\tilde{\nu}$ is m-uniform and conical. Moreover, for every $X \in \mathrm{spt}(\nu)$, ν has a unique tangent measure at X, which is also m-uniform and conical.*

The question of whether an m-uniform measure is flat, is only relevant when $m \geq 3$, by Theorem 2.1.1. A key result of Preiss in this direction says that for an m-uniform measure ν, with $m \geq 3$, if ν is close enough to being flat at infinity, then it is flat:

Theorem 2.1.4 ([25])**.** *If $3 \leq m \leq n+1$, there exists a constant $\varepsilon_0 > 0$ depending only on n and m such that if ν is an m-uniform measure on $\mathbb{R}^{n+1}$ with $\nu(B(X,1)) = 1$ for $X \in \mathrm{spt}(\nu)$, for which its tangent measure $\tilde{\nu}$ at ∞ satisfies*

$$\tilde{F}(\tilde{\nu}) := \min_P \int_{B(0,1)} \mathrm{dist}(X,P)^2 \mathrm{d}\tilde{\nu}(X) \leq \varepsilon_0^2, \tag{2.1.7}$$

then ν is flat. Here, the minimum is taken over all m-planes P in $\mathbb{R}^{n+1}$ with $0 \in P$.

De Lellis introduced in [16, Lemma 6.20] a smooth version of $\tilde{F}$ that is convenient to work with. Let $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$ be a radially non-increasing function such that $\chi_{B(0,1)} \leq \varphi \leq \chi_{B(0,2)}$. Given a Radon measure ν with $0 \in \mathrm{spt}(\nu)$, let

$$F(\nu) = \min_P \frac{1}{\nu(B(0,1))} \int \varphi(Z) \mathrm{dist}(Z,P)^2 \mathrm{d}\nu(Z), \tag{2.1.8}$$

where the minimum is taken over all m-planes P in $\mathbb{R}^{n+1}$ with $0 \in P$. In contrast with $\tilde{F}$, F has the property of being continuous with respect to sequential weak convergence of Radon measures. Moreover, $\nu(B(0,1))F(\nu) \geq \tilde{F}(\nu)$ for any Radon measure ν, so Theorem 2.1.4 can be reformulated as follows:

Corollary 2.1.1 ([16]). *Let ν be an m-uniform measure on $\mathbb{R}^{n+1}$ with $\nu(B(0,1)) = 1$. If $3 \leq m \leq n+1$, there exists $\varepsilon_0 > 0$, depending only on m and n, such that if*

$$\limsup_{R\to\infty} F(\nu_{0,R}) \leq \varepsilon_0^2, \tag{2.1.9}$$

then ν is flat.

2.2 The matrix-valued function Λ

This section contains all the necessary definitions and basic facts about the mapping Λ considered in Theorems 1.3.1, 1.3.2 and 1.3.3. We denote by $\mathrm{GL}(n+1,\mathbb{R})$ the space of $(n+1)\times(n+1)$ real invertible matrices, endowed with the operator norm

$$\|A\| = \sup_{\substack{V\in\mathbb{R}^{n+1}\\ |V|\leq 1}} |AV|, \quad A \in \mathrm{GL}(n+1,\mathbb{R}),$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^{n+1}$. We consider a mapping $\Lambda : \mathbb{R}^{n+1} \to \mathrm{GL}(n+1,\mathbb{R})$ with the property that $\Lambda(X)$ is a symmetric positive definite matrix for each $X \in \mathbb{R}^{n+1}$. In particular, all the eigenvalues of $\Lambda(X)$ are real and positive. We also assume that Λ is locally Hölder continuous with exponent $\beta \in (0,1)$, in the sense that for each compact set $K \subset \mathbb{R}^{n+1}$ there exists a constant $H_K > 0$ such that for all $X, Y \in K$,

$$\|\Lambda(X) - \Lambda(Y)\| \leq H_K |X-Y|^\beta. \tag{2.2.1}$$

Important properties of Λ will be encoded in the smallest and largest eigenvalues of $\Lambda(X)$ at a given point X, which we will denote by $\lambda_{\min}(X)$ and $\lambda_{\max}(X)$, respectively.

Lemma 2.2.1 (Regularity of eigenvalues). *For all $X, Y \in \mathbb{R}^{n+1}$ we have*

$$|\lambda_{\min}(X) - \lambda_{\min}(Y)| \le \|\Lambda(X) - \Lambda(Y)\|,$$

$$|\lambda_{\max}(X) - \lambda_{\max}(Y)| \le \|\Lambda(X) - \Lambda(Y)\|.$$

These estimates and the continuity assumption (2.2.1) imply that the functions $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ are locally Hölder continuous with exponent β. From this and from the fact that $\Lambda(X)$ is an invertible matrix for every $X \in \mathbb{R}^{n+1}$, it follows that $\lambda_{\min}(\cdot)$, $\lambda_{\min}(\cdot)^{-1}$, $\lambda_{\max}(\cdot)$ and $\lambda_{\max}(\cdot)^{-1}$ are locally bounded from above and below by positive constants, and $\lambda_{\min}(\cdot)^{-1}$ and $\lambda_{\max}(\cdot)^{-1}$ are also locally Hölder continuous with exponent β. These considerations will be used in many of our estimates.

Proof of Lemma 2.2.1. For $\lambda_{\max}$ we can write $\lambda_{\max}(X) = \|\Lambda(X)\|$, so the second estimate in the statement follows from triangle inequality. As for $\lambda_{\min}$, notice that $1/\lambda_{\min}(X)$ is the largest eigenvalue of $\Lambda(X)^{-1}$, so $1/\lambda_{\min}(X) = \|\Lambda(X)^{-1}\|$. Therefore

$$\begin{aligned}
|\lambda_{\min}(X) - \lambda_{\min}(Y)| &= \left| \frac{1}{\|\Lambda(X)^{-1}\|} - \frac{1}{\|\Lambda(Y)^{-1}\|} \right| = \frac{|\|\Lambda(X)^{-1}\| - \|\Lambda(Y)^{-1}\||}{\|\Lambda(X)^{-1}\|\|\Lambda(Y)^{-1}\|} \\
&\le \frac{\|\Lambda(X)^{-1} - \Lambda(Y)^{-1}\|}{\|\Lambda(X)^{-1}\|\|\Lambda(Y)^{-1}\|} = \frac{|\Lambda(X)^{-1}(I - \Lambda(X)\Lambda(Y)^{-1})\|}{\|\Lambda(X)^{-1}\|\|\Lambda(Y)^{-1}\|} \\
&= \frac{\|\Lambda(X)^{-1}(\Lambda(Y) - \Lambda(X))\Lambda(Y)^{-1}\|}{\|\Lambda(X)^{-1}\|\|\Lambda(Y)^{-1}\|} \le \|\Lambda(X) - \Lambda(Y)\|.
\end{aligned}$$

□

The main role of the mapping Λ in our context is to determine the ellipses $B_\Lambda(X, r)$ in (1.3.1). In particular, the regularity of Λ ensures that these ellipses enjoy some compatibility, as the next lemma shows.

Lemma 2.2.2 (Nested nonconcentric ellipses). *Suppose Λ is Hölder continuous as in* (2.2.1). *Let $K \subset \mathbb{R}^{n+1}$ be compact. If $X, Y \in K$, $r > 0$ and $|X - Y| < C_K r$ for some constant $C_K > 0$ depending on K, then*

$$B_\Lambda(X, r) \subset B_\Lambda(Y, r + \lambda_{\min}(X)^{-1}|X - Y| + C_K r^{1+\beta}). \tag{2.2.2}$$

If in addition $|X-Y| \le \lambda_{\min}(X)r/2$ *and* r *is small enough depending on* K *and* Λ, *then*

$$r - \lambda_{\min}(X)^{-1}(X)|X-Y| - C_K r^{1+\beta} > 0$$

and

$$B_\Lambda(X,r) \supset B_\Lambda(Y, r - \lambda_{\min}(X)^{-1}|X-Y| - C_K r^{1+\beta}). \tag{2.2.3}$$

Proof. Let $Z \in B_\Lambda(X,r)$. Write $Z = X + \Lambda(X)W$, where $W \in B(0,r)$. Then

$$Z = Y + X - Y + \Lambda(X)W = Y + \Lambda(Y)[\Lambda(Y)^{-1}(X - Y + \Lambda(X)W)].$$

Estimating the term in the brackets and keeping in mind the continuity of Λ, we get

$$\begin{aligned}
|\Lambda(Y)^{-1}(X-Y+\Lambda(X)W)| &\le |\Lambda(Y)^{-1}(X-Y)| + |\Lambda(Y)^{-1}\Lambda(X)W| \\
&\le \lambda_{\min}(Y)^{-1}|X-Y| + |(\Lambda(Y)^{-1}(\Lambda(X)-\Lambda(Y)) + I)W| \\
&\le \lambda_{\min}(Y)^{-1}|X-Y| + H_K\lambda_{\min}(Y)^{-1}|X-Y|^\beta|W| + |W| \\
&\le (\lambda_{\min}(X)^{-1} + H_K|X-Y|^\beta)|X-Y| \\
&\qquad + H_K\lambda_{\min}(Y)^{-1}|X-Y|^\beta|W| + |W| \\
&\le \lambda_{\min}(X)^{-1}|X-Y| + C_K r^{1+\beta} + |W| \\
&\le \lambda_{\min}(X)^{-1}|X-Y| + C_K r^{1+\beta} + r.
\end{aligned}$$

This implies that

$$Z \in Y + \Lambda(Y)B(0, r + \lambda_{\min}(X)^{-1}|X-Y| + C_K r^{1+\beta}),$$

which proves (2.2.2). To prove (2.2.3), let $Z \in B_\Lambda(Y,\rho)$, with $\rho > 0$ to be determined. Write

$$Z = Y + \Lambda(Y)W = X + \Lambda(X)[\Lambda(X)^{-1}(Y - X + \Lambda(Y)W)],$$

where $W \in B(0,\rho)$, and estimate similarly as before

$$\begin{aligned}
|\Lambda(X)^{-1}(Y - X + \Lambda(Y)W)| &\leq |\Lambda(X)^{-1}(Y - X)| + |\Lambda(X)^{-1}\Lambda(Y)W| \\
&\leq |\Lambda(X)^{-1}(Y - X)| + |\Lambda(X)^{-1}(\Lambda(Y) - \Lambda(X))W| + |W| \\
&\leq \lambda_{\min}(X)^{-1}|X - Y| + H_K\lambda_{\min}(X)^{-1}|X - Y|^{\beta}|W| + |W| \\
&< \lambda_{\min}(X)^{-1}|X - Y| + C_K r^{1+\beta} + \rho.
\end{aligned} \tag{2.2.4}$$

We would like this upper bound not to exceed r, which can be achieved by choosing

$$\rho = r - \lambda_{\min}(X)^{-1}|X - Y| - C_K r^{1+\beta}.$$

Notice that by our assumptions, if r is small enough depending on K and Λ, we have

$$\rho \geq \frac{r}{2} - C_K r^{1+\beta} > 0.$$

With this choice of ρ, it follows from (2.2.4) that $Z \in X + \Lambda(X)B(0,r)$, which completes the proof of the lemma. □

2.3 Flatness notions

This section contains the necessary definitions and some basic facts concerning bilateral β-numbers and Reifenberg-flat sets. We let $D[A,B]$ denote the Hausdorff distance between two closed sets A and B, i.e.

$$D[A,B] = \max\left\{\sup_{X\in A}\operatorname{dist}(X,B), \sup_{Y\in B}\operatorname{dist}(Y,A)\right\},$$

where $\operatorname{dist}(X,B) = \inf_{Y\in B}|X - Y|$ and similarly for $\operatorname{dist}(Y,A)$.

Definition 2.3.1 (Bilateral β-numbers)**.** *Given a closed set $\Sigma \subset \mathbb{R}^{n+1}$, the m-dimensional bilateral β-numbers of Σ are*

$$b\beta_{\Sigma}^{m}(X,r) = \inf_{P}\left\{\frac{1}{r}D[\Sigma \cap B(X,r); P \cap B(X,r)]\right\}, \tag{2.3.1}$$

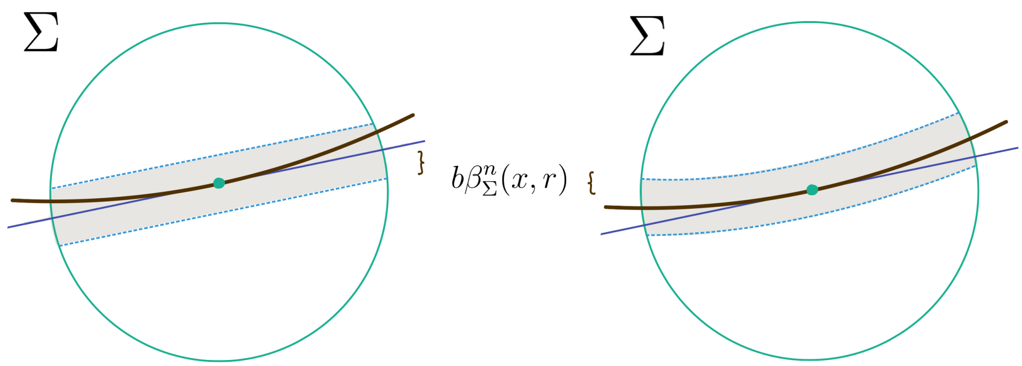

Figure 2.1: Bilateral β-numbers and the two ε-neighborhood inclusions they determine.

where $X \in \Sigma$, $r > 0$, *and the infimum is taken over all* m*-planes* P *through* X.

Remark 5. Whenever the value of m is clear from the context, we may simplify the notation and write $b\beta_\Sigma(X,r)$. Most of the time, this will occur under the assumption that $m = n$.

We denote the closed ε-neighborhood of a set $E \subset \mathbb{R}^{n+1}$ by

$$(E;\varepsilon) = \{Z \in \mathbb{R}^{n+1} : \operatorname{dist}(Z,E) \leq \varepsilon\}. \tag{2.3.2}$$

The numbers $b\beta^m_\Sigma$ measures flatness of Σ in a bilateral way, in the sense that if $P(X,r)$ is an m-plane attaining the infimum in (2.3.1), then (see Figure 2.1)

$$\Sigma \cap B(X,r) \subset (P(X,r) \cap B(X,r); b\beta^m_\Sigma(X,r)r),$$

and

$$P(X,r) \cap B(X,r) \subset (\Sigma \cap B(X,r); b\beta^m_\Sigma(X,r)r).$$

From this point of view, flatness of Σ is determined by how close $b\beta^m_\Sigma(X,r)$ is to 0. The following notions of flatness based on $b\beta^m_\Sigma$ are standard in the literature (see [27]), and they play an important role in some of our arguments:

Definition 2.3.2 (Reifenberg-flat set)**.** *A closed set $\Sigma \subset \mathbb{R}^{n+1}$ is:*

1. *δ-Reifenberg flat (or Reifenberg flat with constant δ) of dimension m, where $\delta > 0$, if for every compact set $K \subset \mathbb{R}^{n+1}$ with $\Sigma \cap K \neq \varnothing$ there exists $r_K > 0$ such that*
$$b\beta_\Sigma^m(K, r_K) = \sup_{r \in (0, r_K]} \sup_{X \in \Sigma \cap K} b\beta_\Sigma^m(X, r) \leq \delta.$$
2. *Reifenberg flat with vanishing constant of dimension m if it is δ-Reifenberg flat of dimension m for all $\delta > 0$, or equivalently, for every compact set $K \subset \mathbb{R}^{n+1}$ with $\Sigma \cap K \neq \varnothing$,*
$$\lim_{r \to 0} b\beta_\Sigma^m(K, r) = 0.$$

We will also need the following anisotropic version of $b\beta_\Sigma^m$,

$$b\beta_{\Sigma,\Lambda}^m(X, r) = \inf_P \left\{ \frac{1}{r} D[\Sigma \cap B_\Lambda(X, r); P \cap B_\Lambda(X, r)] \right\},$$

where the infimum is again taken over all m-planes P through X. The only difference between this quantity and $b\beta_\Sigma^m(X, r)$ is that $B(X, r)$ is replaced by $B_\Lambda(X, r)$. The following lemma provides a way to compare $b\beta_\Sigma^m$ with $b\beta_{\Sigma,\Lambda}^m$. Its statement and many estimates later on make reference to the following eigenvalue bounds, associated with any compact set $K \subset \mathbb{R}^{n+1}$,

$$\lambda_{\min}(K) = \min_{X \in (\Sigma \cap K; 1)} \lambda_{\min}(X), \quad \lambda_{\max}(K) = \sup_{X \in (\Sigma \cap K; 1)} \lambda_{\max}(X), \tag{2.3.3}$$

as well as a local notion of eccentricity of Λ,

$$e_\Lambda(K) = \frac{\lambda_{\max}(K)}{\lambda_{\min}(K)}. \tag{2.3.4}$$

The fact that some of these quantities consider a neighborhood of $\Sigma \cap K$ as opposed to just $\Sigma \cap K$ will become relevant in later sections.

Lemma 2.3.1 (Euclidean and anisotropic flatness)**.** *Let $\Sigma \subset \mathbb{R}^{n+1}$ be closed and let $K \subset \mathbb{R}^{n+1}$ be compact with $\Sigma \cap K \neq \varnothing$. Then there exists a constant $\delta_K > 0$*

depending on K and Λ with the following property. Let $\delta \in (0, \delta_K)$, $X, \overline{X} \in \Sigma \cap K$, $r > 0$, $r' = \lambda_{\max}(K)r$, $r'' = \lambda_{\min}(K)r$, and let P be an m-plane through X, $m \leq n$. Let $B_\Lambda(X, \overline{X}, r) = X + \Lambda(\overline{X})B(0, r)$.

1. *If*
$$D[\Sigma \cap B(X, r'); P \cap B(X, r')] \leq \delta r', \tag{2.3.5}$$
then
$$D[\Sigma \cap B_\Lambda(X, \overline{X}, r); P \cap B_\Lambda(X, \overline{X}, r))] \leq (2 + e_\Lambda(K))\delta r'. \tag{2.3.6}$$

2. *If*
$$D[\Sigma \cap B_\Lambda(X, \overline{X}, r); P \cap B_\Lambda(X, \overline{X}, r))] \leq \delta r, \tag{2.3.7}$$
then
$$D[\Sigma \cap B(X, r''); P \cap B(X, r'')] \leq 2\delta r. \tag{2.3.8}$$

Moreover, δ_K can be taken to be
$$\delta_K = \min\{\lambda_{\min}(K), e_\Lambda(K)^{-1}\}. \tag{2.3.9}$$

The following corollary is a direct consequence of this lemma in the case $X = \overline{X}$.

Corollary 2.3.1. *Let $\Sigma \subset \mathbb{R}^{n+1}$ be closed and let $K \subset \mathbb{R}^{n+1}$ be compact with $\Sigma \cap K \neq \varnothing$. Then there exists a constant $\delta_K > 0$ depending on K, Λ and n with the following property. Let $\delta \in (0, \delta_K)$, $X \in \Sigma \cap K$, $r > 0$, $r' = \lambda_{\max}(K)r$, $r'' = \lambda_{\min}(K)r$, and let P be an m-plane through X, $m \leq n$.*

1. *If*
$$D[\Sigma \cap B(X, r'); P \cap B(X, r')] \leq \delta r', \tag{2.3.10}$$
then
$$D[\Sigma \cap B_\Lambda(X, r); P \cap B_\Lambda(X, r))] \leq (2 + e_\Lambda(K))\delta r'. \tag{2.3.11}$$

2. *If*

$$D[\Sigma \cap B_\Lambda(X,r); P \cap B_\Lambda(X,r))] \leq \delta r, \tag{2.3.12}$$

then

$$D[\Sigma \cap B(X,r''); P \cap B(X,r'')] \leq 2\delta r. \tag{2.3.13}$$

Proof of Lemma 2.3.1. The proof makes repeated use of the fact that with $X, \overline{X}, r, r'$ and r'' as in the statement, we have

$$B(X,r'') \subset B_\Lambda(X,\overline{X},r) \subset B(X,r'). \tag{2.3.14}$$

Let us first prove (2.3.11) under the assumption that (2.3.10) holds. We proceed in two steps.

1. First we show that

$$\Sigma \cap B_\Lambda(X,\overline{X},r) \subset (P \cap B_\Lambda(X,\overline{X},r); (1+e_\Lambda(K))\delta r'). \tag{2.3.15}$$

Let $Y \in \Sigma \cap B_\Lambda(X,\overline{X},r)$, and write $Y = X + Y_{||} + Y_\perp$, where $Y_{||}$ and $Y_\perp$ are parallel and orthogonal, respectively, to P. We consider two cases.

Case (i): $X + Y_{||} \in P \cap B_\Lambda(X,\overline{X},r)$. In this situation, using that $B_\Lambda(X,\overline{X},r) \subset B(X,r')$ and (2.3.10), we see that

$$\text{dist}(Y, P \cap B_\Lambda(X,\overline{X},r)) = |Y_\perp| = \text{dist}(Y, P \cap B(X,r')) \leq \delta r',$$

which implies (2.3.11).

Case (ii): $X + Y_{||} \notin P \cap B_\Lambda(X,\overline{X},r)$, or equivalently, $X + Y_{||} \notin B_\Lambda(X,\overline{X},r)$. Now in addition to $|Y_\perp|$, we also need to control the distance from $X + Y_{||}$ to $P \cap B_\Lambda(X,\overline{X},r)$. Write

$$X + Y_{||} = X + \Lambda(X)W,$$

and notice that $X + Y_{||} \notin B_\Lambda(X, \overline{X}, r)$ implies $|W| = |\Lambda(\overline{X})^{-1}(Y_{||})| > r$. Let $Y' = X + \Lambda(\overline{X})W'$, where $W' = rW/|W|$. Note that $|\Lambda(\overline{X})^{-1}(Y' - X)| = |W'| = r$, so $Y' \in \partial B_\Lambda(X, \overline{X}, r)$. Moreover, by construction Y' belongs to the line through X and $X + Y_{||}$, so in particular $Y' \in P$. Therefore

$$\operatorname{dist}(X + Y_{||}, P \cap B_\Lambda(X, \overline{X}, r)) \leq |X + Y_{||} - Y'|. \tag{2.3.16}$$

Denote $\rho = |X + Y_{||} - Y'|$. Then

$$|W - W'| = |\Lambda(\overline{X})^{-1}(X + Y_{||} - Y')| \geq \lambda_{\max}(K)^{-1}\rho.$$

Therefore, taking into account that $W - W' = \left(1 - \frac{r}{|W|}\right) W$ and $W' = \frac{r}{|W|}W$, so that both $W - W'$ and W' are colinear and point in the same direction, we get

$$|W| = |W - W'| + |W'| \geq r + \lambda_{\max}(K)^{-1}\rho.$$

In particular, we have $B(W, \lambda_{\max}(K)^{-1}\rho) \subset \mathbb{R}^{n+1} \backslash B(0, r)$, and applying $X + \Lambda(\overline{X})(\cdot)$ we obtain

$$X + \Lambda(\overline{X})B(W, \lambda_{\max}(K)^{-1}\rho) \subset \mathbb{R}^{n+1} \backslash B_\Lambda(X, \overline{X}, r). \tag{2.3.17}$$

Now, notice that

$$\begin{aligned} X + \Lambda(\overline{X})B(W, \lambda_{\max}(K)^{-1}\rho) &= X + \Lambda(\overline{X})W + \Lambda(\overline{X})B(0, \lambda_{\max}(K)^{-1}\rho) \\ &\supset X + \Lambda(\overline{X})W + B\left(0, \frac{\lambda_{\min}(K)}{\lambda_{\max}(K)}\rho\right) \\ &= B\left(X + Y_{||}, e_\Lambda(K)^{-1}\rho\right). \end{aligned} \tag{2.3.18}$$

Combining (2.3.17) with (2.3.18) we get

$$B\left(X + Y_{||}, e_\Lambda(K)^{-1}\rho\right) \cap B_\Lambda(X, \overline{X}, r) = \varnothing. \tag{2.3.19}$$

In particular, since $Y \in B_\Lambda(X, \overline{X}, r)$, (2.3.16) and (2.3.19) imply that

$$|Y_\perp| = |Y - (X + Y_{||})| \geq e_\Lambda(K)^{-1}\rho \geq e_\Lambda(K)^{-1}\operatorname{dist}(X + Y_{||}, P \cap B_\Lambda(X, \overline{X}, r)),$$

which gives

$$\operatorname{dist}(X+Y_{||}, P\cap B_\Lambda(X,\overline{X},r)) \le e_\Lambda(K)|Y_\perp|.$$

From this estimate and (2.3.10), which ensures that $|Y_\perp| \le \delta r'$, we deduce that

$$\begin{aligned}\operatorname{dist}(Y, P\cap B_\Lambda(X,\overline{X},r)) &\le |Y_\perp| + \operatorname{dist}(X+Y_{||}, P\cap B_\Lambda(X,\overline{X},r))\\ &\le (1+e_\Lambda(K))|Y_\perp| \le (1+e_\Lambda(K))\delta r',\end{aligned}$$

which proves (2.3.15) .

2. Next, we show that

$$P\cap B_\Lambda(X,\overline{X},r) \subset (\Sigma\cap B_\Lambda(X,\overline{X},r); (2+e_\Lambda(K))\delta r'). \tag{2.3.20}$$

Let $Y \in P\cap B_\Lambda(X,\overline{X},r)$. We would like to use (2.3.10) to obtain a point in Σ which is close to Y. However, if we do this directly at Y, the resulting point in Σ may not necessarily be contained in $B_\Lambda(X,\overline{X},r)$. We compensate for this by adjusting Y in the following way. Write $Y = X + \Lambda(\overline{X})W$, where $|W| < r$. Let $W' = (1-\rho)W$, where $\rho \in (0,1)$ will be chosen later, and let $Y' = X + \Lambda(\overline{X})W'$.

We will first find a ball with center Y' that is contained in $B_\Lambda(X,\overline{X},r)$. To do this, note that because $|W'| < (1-\rho)r$, we have $B(W',\rho r) \subset B(0,r)$. Therefore

$$X + \Lambda(\overline{X})B(W',\rho r) \subset X + \Lambda(\overline{X})B(0,r) = B_\Lambda(X,\overline{X},r). \tag{2.3.21}$$

Now, note that

$$\begin{aligned}X + \Lambda(\overline{X})B(W',\rho r) &= X + \Lambda(\overline{X})W' + \Lambda(\overline{X})B(0,\rho r)\\ &= Y' + \Lambda(\overline{X})B(0,\rho r)\\ &\supset Y' + B(0,\lambda_{\min}(K)\rho r).\end{aligned} \tag{2.3.22}$$

Combining this with (2.3.21) gives

$$B(Y',\lambda_{\min}(K)\rho r) \subset B_\Lambda(X,\overline{X},r). \tag{2.3.23}$$

Next, note that since $Y \in P \cap B_\Lambda(X, \overline{X}, r)$, by construction we have $Y' \in P \cap B_\Lambda(X, \overline{X}, r)$ as well, so in particular,

$$Y' \in P \cap B(X, r').$$

We can now use (2.3.10) to deduce that there exists $Q \in \Sigma \cap B(X, r')$ such that

$$|Y' - Q| \leq \delta r'. \tag{2.3.24}$$

We will use Q to approximate Y. We would like to ensure that $Q \in B_\Lambda(X, \overline{X}, r)$. To do this, notice that by (2.3.23) it suffices to show that

$$|Y' - Q| < \lambda_{\min}(K)\rho r. \tag{2.3.25}$$

But from (2.3.24), we see that this holds as long as $e_\Lambda(K)\delta < \rho$. To ensure that this is the case, we assume that $\delta_K < e_\Lambda(K)^{-1}$ and

$$\rho \in (e_\Lambda(K)\delta_K, 1). \tag{2.3.26}$$

In this scenario (2.3.25) holds, which implies that $Q \in B_\Lambda(X, \overline{X}, r)$. Moreover, since $Q \in \Sigma$, we have $Q \in \Sigma \cap B_\Lambda(X, \overline{X}, r)$. To conclude, we estimate

$$|Q - Y| \leq |Q - Y'| + |Y' - Y| \leq \lambda_{\max}(K)\delta r + |\Lambda(X)(W' - W)| \leq \delta r' + \rho r'. \tag{2.3.27}$$

We now assume, in addition to (2.3.26), that

$$\rho < (1 + e_\Lambda(K))\delta_K. \tag{2.3.28}$$

Then (2.3.27) implies $|Q - Y| \leq (2 + e_\Lambda(K))\delta r'$, proving (2.3.20). Now (2.3.11) follows from (2.3.15) and (2.3.20).

Next we assume (2.3.12) and prove (2.3.13). We proceed in two steps as before.

1. First we claim that

$$P \cap B(X, r'') \subset (\Sigma \cap B(X, r''); 2\delta r). \tag{2.3.29}$$

To prove this, let $Y \in P \cap B(X, r'')$. Consider

$$Y' = Y - \frac{\delta}{\lambda_{\min}(K)}(Y - X).$$

Notice that $Y' \in P$. Moreover, if $\delta_K < \lambda_{\min}(K)$, then since $Y \in B(X, r'')$,

$$|Y' - X| = \left(1 - \frac{\delta}{\lambda_{\min}(K)}\right)|Y - X| < r'',$$

so $Y' \in B(X, r'')$ as well. Now, since $B(X, r'') \subset B_\Lambda(X, \overline{X}, r)$, by (2.3.12) there exists $Z \in \Sigma \cap B_\Lambda(X, \overline{X}, r)$ such that

$$|Y' - Z| \leq \delta r. \tag{2.3.30}$$

This implies

$$\begin{aligned} |X - Z| &\leq |X - Y'| + |Y' - Z| \\ &\leq \left(1 - \frac{\delta}{\lambda_{\min}(K)}\right)|X - Y| + \delta r < r'' - \delta r + \delta r = r'', \end{aligned} \tag{2.3.31}$$

so $Z \in \Sigma \cap B(X, r'')$. Moreover, by (2.3.30) Z satisfies

$$|Y - Z| \leq |Y - Y'| + |Y' - Z| \leq \frac{\delta}{\lambda_{\min}(K)}|X - Y| + \delta r = 2\delta r. \tag{2.3.32}$$

This proves (2.3.29).

2. Now we show that

$$\Sigma \cap B(X, r'') \subset (P \cap B(X, r''); \delta r). \tag{2.3.33}$$

Let $Y \in \Sigma \cap B(X, r'')$. Let Z be the orthogonal projection of Y onto P. Then because P contains X, we have $Z \in B(X, r'')$, so $Z \in P \cap B(X, r'')$. Moreover, using that $B(X, r'') \subset B_\Lambda(X, \overline{X}, r)$ and (2.3.12), we get

$$|Y - Z| = \operatorname{dist}(Y, P \cap B(X, r'')) = \operatorname{dist}(Y, P \cap B_\Lambda(X, \overline{X}, r)) \leq \delta r. \tag{2.3.34}$$

This gives (2.3.33). Combining equations (2.3.29) and (2.3.33) we obtain (2.3.13), which completes the proof of Lemma 2.3.1. $\square$

Chapter 3

Density and decay of $\beta-$numbers

3.1 Moment estimates

Here we start deriving geometric information about a measure μ under the assumption that μ and Λ satisfy the density and continuity conditions of Theorem 1.3.1, i.e. equations (1.3.4) and (2.2.1) (no flatness assumption needs to be made at this point). To accomplish this we consider certain *moments*, an idea that has already been successfully exploited in the literature, most notably in the study of uniform measures (see [25], [14]), as well as in the case of measures that are not necessarily uniform but rather asymptotically uniform in a sense, such as those satisfying (1.2.1). *A priori*, an appropriate notion of moment in our setting would incorporate suitable Λ terms. However, we will instead consider a transformation $\tilde{\mu}$ of μ for which the standard notion of moment will suffice.

From now on $K \subset \mathbb{R}^{n+1}$ will be a fixed compact set with $K \cap \Sigma \neq \varnothing$, and X_0 will denote an arbitrary point in $K \cap \Sigma$. We will study the regularity of Σ near X_0 by considering the following transformation. Let

$$\tilde{K} = \Lambda(X_0)^{-1}K, \quad \tilde{\Sigma} = \Lambda(X_0)^{-1}(\Sigma) = \mathrm{spt}(\tilde{\mu}), \tag{3.1.1}$$

$$\tilde{\mu} = \Lambda(X_0)^{-1}[\mu], \quad \tilde{\Lambda}(Y) = \Lambda(X_0)^{-1}\Lambda(\Lambda(X_0)Y), \tag{3.1.2}$$

where $Y \in \tilde{\Sigma} \cap \tilde{K}$ and $\Lambda(X_0)^{-1}[\cdot]$ denotes push-forward via $\Lambda(X_0)^{-1}$. As we will see, the regularity of Σ near X_0 will be determined by that of $\tilde{\Sigma}$ near $Y_0 = \Lambda(X_0)^{-1}X_0$. The main benefits of performing this transformation come from the fact that

$$\tilde{\Lambda}(Y_0) = \mathrm{Id}. \tag{3.1.3}$$

Let us start by using the density assumption on μ in Theorem 1.3.1 to derive a corresponding estimate for $\tilde{\mu}$. If $X \in \Sigma \cap K$ and $Y = \Lambda(X_0)^{-1}X \in \tilde{\Sigma} \cap \tilde{K}$ (notice that a generic point of $\tilde{\Sigma} \cap \tilde{K}$ can always be written in this way), then

$$\begin{aligned}
\mu(B_\Lambda(X,r)) &= \mu(X + \Lambda(X)B(0,r)) \\
&= \mu(\Lambda(X_0)[\Lambda(X_0)^{-1}X + \Lambda(X_0)^{-1}\Lambda(X)B(0,r)]) \\
&= \tilde{\mu}(\Lambda(X_0)^{-1}X + \tilde{\Lambda}(\Lambda(X_0)^{-1}X)) = \tilde{\mu}(B_{\tilde{\Lambda}}(\Lambda(X_0)^{-1}X,r)) = \tilde{\mu}(B_{\tilde{\Lambda}}(Y,r)).
\end{aligned}$$

Thus, (1.3.4) implies that for every $Y \in \tilde{\Sigma} \cap \tilde{K}$ and $r \in (0,1]$,

$$\left| \frac{\tilde{\mu}(B_{\tilde{\Lambda}}(Y,r))}{\omega_n r^n} - 1 \right| \leq C_K r^\alpha. \tag{3.1.4}$$

We will often use this estimate in the form

$$\omega_n r^n - C_K r^{n+\alpha} \leq \tilde{\mu}(B_{\tilde{\Lambda}}(Y,r)) \leq \omega_n r^n + C_K r^{n+\alpha}. \tag{3.1.5}$$

Remark 6. By our assumptions on Λ, we have for every $Y, Y' \in \tilde{\Sigma} \cap \tilde{K}$,

$$\|\tilde{\Lambda}(Y) - \tilde{\Lambda}(Y')\| \leq e_\Lambda(K) H_K |Y - Y'|^\beta,$$

with H_K as in (2.2.1) and $e_\Lambda(K)$ as in (2.3.4). This guarantees that as we work with $\tilde{\mu}$ and $\tilde{\Lambda}$ throughout the rest of this section, any local constants that arise from (3.1.4) and the continuity of $\tilde{\Lambda}$ (including the lemmas in Section 2) can be taken to depend on K and Λ, but not on the particular choice of X_0 (or equivalently $\tilde{K}$). This will become important later on.

We consider the following moments of $\tilde{\mu}$ at Y_0:

$$b = \frac{n+2}{2\omega_n r^{n+2}} \int_{B(Y_0,r)} (r^2 - |Z - Y_0|^2)(Z - Y_0)\mathrm{d}\tilde{\mu}(Z), \tag{3.1.6}$$

$$Q(Y) = \frac{n+2}{\omega_n r^{n+2}} \int_{B(Y_0,r)} \langle Y, Z - Y_0\rangle^2 \mathrm{d}\tilde{\mu}(Z), \tag{3.1.7}$$

as well as the trace of the quadratic form Q,

$$\mathrm{tr}(Q) = \frac{n+2}{\omega_n r^{n+2}} \int_{B(Y_0,r)} |Z|^2 \mathrm{d}\tilde{\mu}(Z).$$

The fact that these quantities are well suited to $\tilde{\mu}$ is a consequence of (3.1.3). As in [7], we will use b and Q to show that near Y_0, $\tilde{\Sigma}$ is close to the zero set of a quadratic polynomial. This is the content of the main result of this section:

Proposition 3.1.1. *Suppose Λ and μ satisfy the continuity and density assumptions of Theorem 1.3.1. Let $X_0 \in \Sigma \cap K$, where $K \subset \mathbb{R}^{n+1}$ is compact, and let $\tilde{\mu}$, $\tilde{\Lambda}$, $\tilde{\Sigma}$, $\tilde{K}$ be as in* (3.1.2)*, and $Y_0 = \Lambda(X_0)^{-1}X_0$. Then with b and Q as defined in* (3.1.6) *and* (3.1.7)*, there exist $C_K > 0$ and $r_K > 0$ depending only on K, Λ and n, such that*

$$|\mathrm{tr}(Q) - n| \leq C_K r^\alpha, \tag{3.1.8}$$

$$|2\langle b, Y - Y_0\rangle + Q(Y - Y_0) - |Y - Y_0|^2| \leq C_K \left(\frac{|Y - Y_0|^3}{r} + r^{2+\min\{\alpha,\beta\}} \right), \tag{3.1.9}$$

whenever $r \in (0, r_K]$ and $Y \in \tilde{\Sigma} \cap B(Y_0, r/2)$.

Remark 7. It will be useful to keep in mind that even though $\tilde{\mu}$, $\tilde{\Sigma}$ and $\tilde{K}$ depend on X_0, the constants C_K and r_K in this result are independent of the particular choice of $X_0 \in \Sigma \cap K$.

Proof of Proposition 3.1.1. Let us assume without loss of generality that $Y_0 = 0$, and record for later use the fact that $\tilde{\Lambda}(0) = \mathrm{Id}$. We start by proving (3.1.8). Here and in what follows we will make repeated use of the following consequence of Fubini's theorem, valid for any measurable set E and any non-negative measurable function f:

$$\int_E f(Z)\mathrm{d}\tilde{\mu}(Z) = \int_0^\infty \tilde{\mu}(\{Z \in E : f(Z) > t\})\mathrm{d}t.$$

We see that

$$\begin{aligned}\int_{B(0,r)} |Z|^2\mathrm{d}\tilde{\mu}(Z) &= \int_{B(0,r)} |Z|^2\mathrm{d}\tilde{\mu}(Z)\\ &= \int_0^{r^2} \tilde{\mu}(\{Z \in B(0,r) : |Z|^2 > t\})\mathrm{d}t = \int_0^{r^2} \tilde{\mu}(\{B(0,r)\backslash B(0,\sqrt{t})\})\mathrm{d}t.\end{aligned}$$

Now by (3.1.5) and because $\tilde{\Lambda}(0) = \mathrm{Id}$, we have for $0 < t < r^2$,

$$|\mu(B(0,r)\backslash B(0,\sqrt{t})) - \omega_n(r^n - t^{n/2})| \leq C_K(r^{n+\alpha} + t^{(n+\alpha)/2}) \leq C_K r^{n+\alpha}.$$

Therefore,

$$\begin{aligned}\left|\int_{B(0,r)} |Z|^2\mathrm{d}\tilde{\mu}(Z) - \int_0^{r^2} \omega_n(r^n - t^{n/2})\mathrm{d}t\right| &\leq \int_0^{r^2} |\tilde{\mu}(B(0,r)\backslash B(0,\sqrt{t})) - \omega_n(r^n - t^{n/2})|\mathrm{d}t\\ &\leq C_K r^{n+\alpha+2},\end{aligned}$$

which gives

$$\left|\frac{n+2}{\omega_n r^{n+2}}\int_{B(0,r)} |Z|^2\mathrm{d}\tilde{\mu}(Z) - n\right| \leq \left|\frac{n+2}{\omega_n r^{n+2}}\int_0^{r^2} \omega_n(r^n - t^{n/2})\mathrm{d}t - n\right| + C_K r^\alpha \leq C_K r^\alpha,$$

proving (3.1.8).

We now prove (3.1.9). Assume $0 < r < 1/2$, and let $Y \in \tilde{\Sigma} \cap B(0, r/2)$. We consider some ellipses that will help us obtain the necessary estimates. Let

$$\begin{aligned} D_1 &= B_{\tilde{\Lambda}}(Y, r - |Y| - C_K r^{1+\beta}), \quad & D_3 &= B_{\tilde{\Lambda}}(Y, r),\\ D_2 &= B_{\tilde{\Lambda}}(0, r) = B(0,r), & D_4 &= B_{\tilde{\Lambda}}(Y, r + |Y| + C_K r^{1+\beta}).\end{aligned}$$

If r is small enough depending on Λ and K, all four radii above are positive, and Lemma 2.2.2 ensures that

$$D_1 \subset D_2 \subset D_4, \quad D_1 \subset D_3 \subset D_4. \tag{3.1.10}$$

Let, for each $j \in \{1, 2, 3, 4\}$,

$$J_i = \int_{D_j} (r^2 - |\tilde{\Lambda}(Y)^{-1}(Z - Y)|^2)^2 \mathrm{d}\tilde{\mu}(Z).$$

Notice that (3.1.10) implies

$$J_1 \leq J_2 \leq J_4, \quad J_1 \leq J_3 \leq J_4,$$

so

$$|J_2 - J_3| \leq J_4 - J_1. \tag{3.1.11}$$

We first estimate the right hand side of this inequality. If $Z \in D_4 \backslash D_1$, we can write $Z = Y + \tilde{\Lambda}(Y)W$, where $|\tilde{\Lambda}(Y)^{-1}(Z - Y)| = |W|$ satisfies

$$r - |Y| - C_K r^{1+\beta} < |W| < r + |Y| + C_K r^{1+\beta}.$$

Using this and the fact that $|Y| \leq r/2$ and $r \leq 1/2$,

$$\begin{aligned}|r^2 - |\tilde{\Lambda}(Y)^{-1}(Z - Y)|^2| &= |r - |W||(r + |W|) \leq (|Y| + C_K r^{1+\beta}))(r + r + |Y| + C_K r^{1+\beta}) \\ &\leq 2|Y|r + |Y|^2 + C_K|Y|r^{1+\beta} + C_K r^{2+\beta} \leq C_K(r|Y| + r^{2+\beta}).\end{aligned}$$

Therefore,

$$\begin{aligned}J_4 - J_1 &= \int_{D_4 \backslash D_1} (r^2 - |\tilde{\Lambda}(Y)^{-1}(Z - Y)|^2)^2 \mathrm{d}\tilde{\mu}(Z) \\ &\leq C_K(r|Y| + r^{2+\beta})^2 \tilde{\mu}(D_4 \backslash D_1).\end{aligned} \tag{3.1.12}$$

Now, by (3.1.5) we have

$$\tilde{\mu}(D_4 \backslash D_1) \leq \omega_n \left[(r + |Y| + C_K r^{1+\beta})^n - (r - |Y| - C_K r^{1+\beta})^n\right] + C_K(r^{n+\alpha} + r^{n+\alpha+\beta}). \tag{3.1.13}$$

To estimate the term in brackets we use the fact that if $r > 0$ and $\rho \leq Cr$, then

$$(r + \rho)^n - (r - \rho)^n \leq C r^{n-1} \rho. \tag{3.1.14}$$

We use (3.1.14) with r as in (3.1.13) and $\rho = |Y| + C_K r^{1+\beta}$. Recall that $|Y| \leq r/2$, so if r is small enough depending on K and Λ, then $\rho \leq \frac{3}{4}r$. It follows that

$$(r + |Y| + C_K r^{1+\beta})^n - (r - |Y| - C_K r^{1+\beta})^n \leq C r^{n_1}(|Y| + C_K r^{1+\beta}),$$

which we combine with (3.1.13) to deduce that for r small depending on K and Λ,

$$\tilde{\mu}(D_4 \backslash D_1) \leq C r^{n-1}(|Y| + C_K r^{1+\beta}) + C_K(r^{n+\alpha} + r^{n+\alpha+\beta}). \tag{3.1.15}$$

Thus, by (3.1.12),

$$\begin{aligned} J_4 - J_1 &\leq C_K(r|Y| + r^{2+\beta})^2 \left[r^{n-1}(|Y| + C_K r^{1+\beta}) + (r^{n+\alpha} + r^{n+\alpha+\beta}) \right] \\ &\leq C_K r^{n+1}|Y|^3 + C_K r^{n+4+\min\{\alpha,\beta\}}. \end{aligned} \tag{3.1.16}$$

We now estimate J_3. Write

$$\begin{aligned} J_3 &= \int_{B_{\tilde{\Lambda}}(Y,r)} (r^2 - |\tilde{\Lambda}(Y)^{-1}(Z-Y)|^2)^2 \mathrm{d}\tilde{\mu}(Z) \\ &= \int_0^{r^4} \tilde{\mu}(\{Z \in B_{\tilde{\Lambda}}(Y,r) : (r^2 - |\tilde{\Lambda}(Y)^{-1}(Z-Y)|^2)^2 > t\})\mathrm{d}t \\ &= \int_0 \tilde{\mu}(\{Z \in B_{\tilde{\Lambda}}(Y,r) : |\tilde{\Lambda}(Y)^{-1}(Z-Y)| < (r^2 - \sqrt{t})^{1/2}\})\mathrm{d}t \\ &= \int_0^{r^4} \tilde{\mu}(\{B_{\tilde{\Lambda}}(Y, (r^2 - \sqrt{t})^{1/2})\})\mathrm{d}t. \end{aligned}$$

Let $h(t) = (r^2 - \sqrt{t})^{1/2}$. Equation (3.1.5) implies

$$|\mu(B_{\tilde{\Lambda}}(Y, h(t))) - \omega_n h(t)^n| \leq C_K h(t)^{n+\alpha} \leq C_K r^{n+\alpha},$$

so if we let

$$I(r) = \int_0^{r^4} \omega_n h(t)^n \mathrm{d}t,$$

then

$$|J_3 - I(r)| \leq C_K \int_0^{r^4} r^{n+\alpha} \mathrm{d}t \leq C_K r^{n+4+\alpha}. \tag{3.1.17}$$

Similarly,

$$\left| \int_{B(0,r)} (r^2 - |Z|^2)^2 \mathrm{d}\tilde{\mu}(Z) - I(r) \right| \leq C_K r^{n+4+\alpha}. \tag{3.1.18}$$

Combining (3.1.17) and (3.1.18), we get

$$\left| J_3 - \int_{B(0,r)} (r^2 - |Z|^2)^2 \mathrm{d}\tilde{\mu}(Z) \right| \leq C_K r^{n+4+\alpha}. \tag{3.1.19}$$

Set now

$$\begin{aligned} I &= J_2 - \int_{B(0,r)} (r^2 - |Z|^2)^2 \mathrm{d}\tilde{\mu}(Z) \\ &= \int_{B(0,r)} (r^2 - |\tilde{\Lambda}(Y)^{-1}(Z-Y)|^2)^2 - (r^2 - |Z|^2)^2 \mathrm{d}\tilde{\mu}(Z). \end{aligned} \tag{3.1.20}$$

By (3.1.16) and (3.1.19),

$$\begin{aligned} |I| &\leq |J_2 - J_3| + C_K r^{n+4+\alpha} \\ &\leq J_4 - J_1 + C_K r^{n+4+\alpha} \leq C_K (r^{n+1}|Y|^3 + r^{n+4+\min\{\alpha,\beta\}}). \end{aligned} \tag{3.1.21}$$

We would now like to replace the term $\tilde{\Lambda}(Y)$ in the definition of I with $\tilde{\Lambda}(0) = \mathrm{Id}$. Using that $Z \in B_{\tilde{\Lambda}}(0,r)$, $|Y| \leq r/2$ and the continuity of $\tilde{\Lambda}$,

$$\begin{aligned} ||\tilde{\Lambda}(Y)^{-1}(Z-Y)|^2 - |Z-Y|^2| &\leq (|\tilde{\Lambda}(Y)^{-1}(Z-Y)| + |Z-Y|)||\tilde{\Lambda}(Y)^{-1}(Z-Y)| - |Z-Y|| \\ &\leq C_K r |(\tilde{\Lambda}(Y)^{-1} - \mathrm{Id})(Z-Y)| \leq C_K r^{2+\beta}. \end{aligned}$$

Therefore

$$\begin{aligned} |(r^2 - |\tilde{\Lambda}(Y)^{-1}(Z-Y)|^2)^2 - (r^2 - |Z-Y|^2)^2| &\leq C_K r^2 ||\tilde{\Lambda}(Y)^{-1}(Z-Y)|^2 - |Z-Y|^2| \\ &\leq C_K r^{4+\beta}. \end{aligned} \tag{3.1.22}$$

If we now let

$$I'' = \int_{B(0,r)} (r^2 - |Z-Y|^2)^2 - (r^2 - |Z|^2)^2 \mathrm{d}\tilde{\mu}(Z),$$

then (3.1.22) and (3.1.5) imply

$$\begin{aligned} |I - I''| &\leq \int_{B(0,r)} |(r^2 - |\tilde{\Lambda}(Y)^{-1}(Z-Y)|^2)^2 - (r^2 - |Z-Y|^2)^2| \mathrm{d}\tilde{\mu}(Z) \\ &\leq C_K \tilde{\mu}(B(0,r)) r^{4+\beta} \leq C_K(\omega_n r^n + C_K r^{n+\alpha}) r^{4+\beta} \leq C_K r^{n+4+\beta}. \end{aligned} \tag{3.1.23}$$

The integral I'' will help us transition to the following integral, which as we will see is almost the quadratic polynomial in (3.1.9),

$$I' = \int_{B(0,r)} \left\{-2|Y|^2(r^2 - |Z|^2) + 4(r^2 - |Z|^2)\langle Z, Y\rangle + 4\langle Y, Z\rangle^2\right\} \mathrm{d}\tilde{\mu}(Z). \tag{3.1.24}$$

We will show that I and I' are close using I''. To this end, note that by (3.1.23),

$$|I - I'| \leq |I' - I''| + C_K r^{n+4+\beta}. \tag{3.1.25}$$

Now we need to estimate $|I' - I''|$. Notice first that

$$\begin{aligned}
I'' &= \int_{B(0,r)} (r^2 - |Z - Y|^2)^2 - (r^2 - |Z|^2)^2 \mathrm{d}\tilde{\mu}(Z) \\
&= \int_{B(0,r)} \left\{r^4 - 2r^2|Z - Y|^2 + |Z - Y|^4 - r^4 + 2r^2|Z|^2 - |Z|^4\right\} \mathrm{d}\tilde{\mu}(Z) \\
&= \int_{B(0,r)} \{-2r^2(|Z|^2 - 2\langle Y, Z\rangle + |Y|^2) + (|Z|^2 - 2\langle Y, Z\rangle + |Y|^2)^2 \\
&\qquad\qquad + 2r^2|Z|^2 - |Z|^4\}\mathrm{d}\tilde{\mu}(Z) \\
&= \int_{B(0,r)} \{4r^2\langle Y, Z\rangle - 2r^2|Y|^2 + |Z|^4 + 4\langle Y, Z\rangle^2 + |Y|^4 \\
&\qquad\qquad - 4|Z|^2\langle Y, Z\rangle + 2|Y|^2|Z|^2 - 4|Y|^2\langle Y, Z\rangle - |Z|^4\}\mathrm{d}\tilde{\mu}(Z) \\
&= \int_{B(0,r)} \{-2(r^2 - |Z|^2)|Y|^2 + 4(r^2 - |Z|^2)\langle Y, Z\rangle + 4\langle Y, Z\rangle^2 \\
&\qquad\qquad + |Y|^4 - 4|Y|^2\langle Y, Z\rangle\}\mathrm{d}\tilde{\mu}(Z).
\end{aligned}$$

Therefore, recalling the definition of I' and using that $|Y| \leq r/2$ and (3.1.5),

$$\begin{aligned}
|I' - I''| &\leq \int_{B(0,r)} ||Y|^4 - 4|Y|^2\langle Y, Z\rangle|\mathrm{d}\tilde{\mu}(Z) \leq \tilde{\mu}(B(0,r))\left(\frac{|Y|^3 r}{2} + 4|Y|^3 r\right) \\
&\leq C(r^n + C_K r^{n+\alpha})|Y|^3 r \leq C r^{n+1}|Y|^3 + C_K r^{n+4+\alpha}.
\end{aligned}$$

Combining this with (3.1.25) we get

$$\begin{aligned}
|I - I'| &\leq \frac{9}{2}\omega_n r^{n+1}|Y|^3 + C_K r^{n+4+\alpha} + C_K r^{n+4+\beta} \\
&\leq C r^{n+1}|Y|^3 + C_K r^{n+4+\min\{\alpha,\beta\}}.
\end{aligned} \tag{3.1.26}$$

To conclude, we obtain the desired quadratic polynomial from I'. Observe first that

$$\begin{aligned}\int_{B(0,r)}(r^2-|Z|^2)\mathrm{d}\tilde{\mu}(Z) &= \int_0^{r^2}\tilde{\mu}(\{Z\in B(0,r): r^2-|Z|^2>t\})\mathrm{d}t\\ &= \int_0^{r^2}\tilde{\mu}\left(B(0,\sqrt{r^2-t})\right)\mathrm{d}t\\ &= \int_0^{r^2}\left(\omega_n(r^2-t)^{n/2}+C_K(r^2-t)^{\frac{n+\alpha}{2}}\right)\mathrm{d}t\\ &= \frac{2\omega_n r^{n+2}}{n+2}+O(r^{n+2+\alpha}),\end{aligned}$$

where $|O(r^{n+2+\alpha})|/r^{n+2+\alpha}\le C_K$. From here it follows, by multiplying by $(n+2)|Y|^2/(2\omega_n r^{n+2})$, that

$$\left||Y|^2-\frac{n+2}{2\omega_n r^{n+2}}|Y|^2\int_{B(0,r)}(r^2-|Z|^2)\mathrm{d}\tilde{\mu}(Z)\right|\le C_K|Y|^2r^\alpha. \tag{3.1.27}$$

Combining (3.1.27) with (3.1.6), (3.1.7) and (3.1.24), we get

$$\begin{aligned}\left|\frac{(n+2)}{4\omega_n r^{n+2}}I'-\{-|Y|^2+2\langle b,Y\rangle+Q(Y)\}\right| &= \left||Y|^2-\frac{n+2}{2\omega_n r^{n+2}}|Y|^2\int_{B(0,r)}(r^2-|Z|^2)\mathrm{d}\tilde{\mu}(Z)\right|\\ &\le C_K|Y|^2r^\alpha.\end{aligned}$$

Finally, combining this estimate with (3.1.21) and (3.1.25), and keeping in mind that $|Y|\le r/2$, we get

$$\begin{aligned}\left|2\langle b,Y\rangle+Q(Y)-|Y|^2\right| &\le \frac{n+2}{4\omega_n r^{n+2}}|I'|+C_K|Y|^2r^\alpha\\ &\le \frac{n+2}{4\omega_n r^{n+2}}(|I|+Cr^{n+1}|Y|^3+C_Kr^{n+4+\min\{\alpha,\beta\}})+C_Kr^{2+\alpha}\\ &\le \frac{C_K}{r^{n+2}}(r^{n+1}|Y|^3+r^{n+4+\min\{\alpha,\beta\}})+C_Kr^{2+\alpha}\\ &\le C_K\left(\frac{|Y|^3}{r}+r^{2+\min\{\alpha,\beta\}}\right).\end{aligned}$$

This shows that (3.1.9) holds and completes the proof of Proposition 3.1.1. □

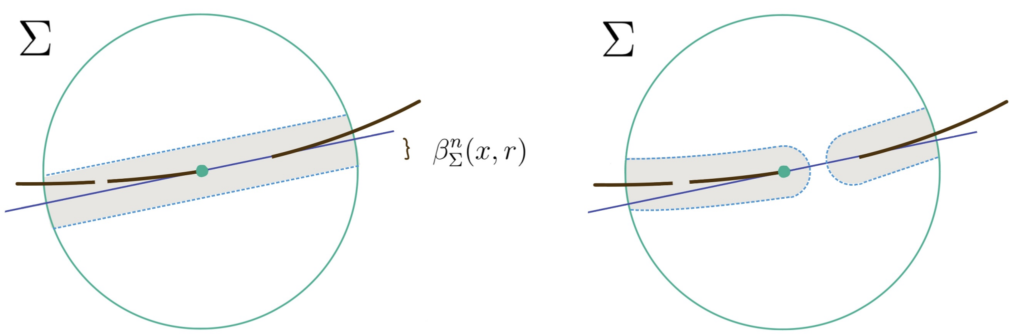

Figure 3.1: β-numbers capture approximation by planes in a unilateral fashion.

3.2 Estimate on β_Σ

In this section, we continue to assume that μ and Λ satisfy the density and continuity assumptions of Theorem 1.3.1. The main goal here is to obtain an estimate on the rate of decay of the n-dimensional β-numbers of Σ,

$$\beta_\Sigma(X,r) = \beta^n_\Sigma(X,r) = \inf_P \left\{ \sup_{Y\in\Sigma\cap B(X,r)} \frac{\operatorname{dist}(Y,P)}{r} \right\}, \tag{3.2.1}$$

where $X \in \Sigma = \operatorname{spt}(\mu)$, $r > 0$, and the infimum is taken over all n-planes $P \subset \mathbb{R}^{n+1}$ such that $X \in P$.

The quantity in (3.2.1) is a centered version of P. Jones' β_∞ numbers introduced in [11], as the planes in (3.2.1) all go through X. The numbers β_Σ can be considered a unilateral version of $b\beta_\Sigma$, in that they capture if Σ is locally close to a plane, but not the converse (see Figure 3.1).

Consider a compact set $K \subset \mathbb{R}^{n+1}$ such that $\Sigma \cap K \neq \varnothing$. We will show that under an additional flatness condition, the density and continuity assumptions of Theorem 1.3.1 imply that $\beta_\Sigma(\cdot,r)$ decays as a power of r as $r \to 0$, uniformly on $\Sigma \cap K$. To prove this, we resort to Proposition 3.1.1 and show that as in [7], moment estimates can be used to control $\beta_\Sigma(\cdot,r)$, provided that Σ is flat enough.

Before we state the main result of this section, let us recall the quantities $\lambda_{\min}(K)$,

$\lambda_{\max}(K)$ and $e_\Lambda(K)$ defined in (2.3.3) and (2.3.4), as well as the transformation introduced in (3.1.1) and (3.1.2). Let us notice the following fact, which is a consequence of the continuity of Λ: for each compact set $K \subset \mathbb{R}^{n+1}$, there exists a number $d_K > 0$ depending only on K and Λ such that for every $X_0 \in \Sigma \cap K$,

$$\Lambda(X_0)((\tilde{\Sigma} \cap \tilde{K}; d_K)) \subset (\Sigma \cap K, 1), \tag{3.2.2}$$

where $\tilde{\Sigma}$ and $\tilde{K}$ are as in (3.1.1). In fact, we may take $d_K = \lambda_{\max}(K)^{-1}$. The main result of this section is the following:

Proposition 3.2.1. *Suppose that μ and Λ satisfy the density and continuity assumptions of Theorem 1.3.1. If $n \geq 3$, suppose also that Σ is Reifenberg-flat with vanishing constant. Then, for every compact set $K \subset \mathbb{R}^{n+1}$ with $\Sigma \cap K \neq \varnothing$ there exist $C_K > 0$ and $r_K > 0$, both depending only on K, Λ and n, such that for all $X_0 \in \Sigma \cap K$ and $r \in (0, r_K]$,*

$$\beta_\Sigma(X_0, r) \leq C_K r^\gamma, \tag{3.2.3}$$

where $\gamma \in (0, 1)$ depends on α and β.

Remark 8. Note that the assumption that Σ is Reifenberg-flat with vanishing constant when $n \geq 3$ is *a priori* stronger than the flatness assumption of Theorem 1.3.1. However, as we will see in Chapter 4, the assumptions of Theorem 1.3.1 imply that Σ is in fact Reifenberg flat with vanishing constant. This will make Proposition 3.2.1 applicable in the proof of Theorem 1.3.1.

Proof of Proposition 3.2.1. Let K and X_0 be as in the statement. We consider the transformation $\tilde{\mu}$ of μ introduced in (3.1.1) and (3.1.2), as well as $Y_0 = \Lambda(X_0)^{-1} X_0$. It will be important to keep in mind that $\tilde{\mu}$ depends on X_0. The proof has two main steps.

Step 1: Bounding $\beta_{\tilde{\Sigma}}(Y_0, r)$. This will rely on Proposition 3.1.1 and arguments in connection with [7, Proposition 8.6], which proves something similar in the Euclidean case, and whose statement we include below.

Step 2: Bounding $\beta_\Sigma(X_0, r)$. This will be a consequence of our estimate on $\beta_{\tilde{\Sigma}}(Y_0, r)$ from Step 1 and particular features of the transformation $\mu \mapsto \tilde{\mu}$.

Proposition 3.2.2 ([7], Proposition 8.6)**.** *Let $\tilde{\mu}$ be a Radon measure in $\mathbb{R}^{n+1}$ with support $\tilde{\Sigma}$. Assume that for each compact set $\tilde{K} \subset \mathbb{R}^{n+1}$ there is a constant $C_{\tilde{K}} > 0$ such that*

$$\left| \frac{\tilde{\mu}(B(Y,r))}{\omega_n r^n} - 1 \right| \leq C_{\tilde{K}} r^\alpha, \tag{3.2.4}$$

for all $Y \in \tilde{K} \cap \tilde{\Sigma}$ and $0 < r < 1$. If $n \geq 3$, assume also that $\tilde{\Sigma}$ is Reifenberg-flat with vanishing constant. Then for each compact set $\tilde{K} \subset \mathbb{R}^{n+1}$ there exists $r_{\tilde{K}} > 0$ depending on n, α and $\tilde{K}$, so that for all $Y \in \tilde{K}$ and $0 < r \leq r_{\tilde{K}}$,

$$\beta_{\tilde{\Sigma}}(Y, r) \leq C_{\tilde{K}} r^\gamma, \tag{3.2.5}$$

where $\gamma \in (0,1)$ depends on α and β.

Remark 9. It is worth mentioning, although not necessary for our arguments, that the proof of this result remains valid if $\tilde{\Sigma}$ is only assumed to be Reifenberg-flat with constant δ_n, where $\delta_n > 0$ is small enough depending only on n.

It should be noted that the transformation $\tilde{\mu}$ of μ from (3.1.1) and (3.1.2) does not satisfy the assumptions on the measure $\tilde{\mu}$ in the above proposition. However, we can still draw a parallel and show that both measures in fact satisfy similar conclusions. Let us briefly recall the main elements in the transformation $\mu \mapsto \tilde{\mu}$:

$$\tilde{\mu} = \Lambda(X_0)^{-1}[\mu], \quad \tilde{\Lambda}(X) = \Lambda(X_0)^{-1}\Lambda(\Lambda(X_0)X), \quad \tilde{K} = \Lambda(X_0)^{-1}(K), \quad \tilde{\Sigma} = \Lambda(X_0)^{-1}(\Sigma). \tag{3.2.6}$$

3.2.1 Step 1: Bound for $\beta_{\tilde{\Sigma}}(Y_0, r)$

The first observation we need to make is that, as mentioned above, we cannot directly apply Proposition 3.2.2 to the transformation $\tilde{\mu}$ of μ given by (3.2.6), the reason being that such $\tilde{\mu}$ only satisfies (3.2.4) at Y_0, whereas at other points $Y \neq Y_0$, $B(Y, r)$ needs to be replaced with $B_{\tilde{\Lambda}}(Y,r)$. Therefore, instead of applying Proposition 3.2.2, we will argue that its proof can still be adapted in our setting to obtain a somewhat weaker conclusion:

$$\text{There exist } C_{\tilde{K}} > 0 \text{ and } r_{\tilde{K}} > 0, \text{ both depending on } \tilde{K}, \text{ and there exists } \gamma \in (0,1) \text{ depending only on } \min\{\alpha,\beta\}, \text{ such that for every } r \in (0, r_{\tilde{K}}],\ \beta_{\tilde{\Sigma}}(Y_0,r) \le C_{\tilde{K}} r^{\gamma}. \tag{3.2.7}$$

Note that this condition is only different from the conclusion of Proposition 3.2.2 in that the β-number estimate in (3.2.7) only holds at Y_0, as opposed to an arbitrary point of $\tilde{\Sigma} \cap \tilde{K}$. Therefore, what we need to discuss is the extent to which the arguments in [7] carry over when proving not the full conclusion of Proposition 3.2.2 in our setting, but rather its validity at Y_0. By an inspection of [7], we see that those arguments rely only on two components:

(i) A density estimate and two moment estimates for $\tilde{\mu}$ at Y_0; and

(ii) $\tilde{\Sigma}$ being Reifenberg with vanishing or small constant.

We will show that both components are still available in our setting, only with minor differences that do not interfere with the proof of Proposition 3.2.2, from which the validity of (3.2.7) will follow.

(i) *Density and moment estimates.* These are inequalities whose corresponding analogues have been established in the previous section. We first recollect them for the sake of convenience. By (3.1.4) and because $\tilde{\Lambda}(Y_0) = \mathrm{Id}$, we have for all $r \in (0,1]$,

$$\left| \frac{\tilde{\mu}(B(Y_0,r))}{\omega_n r^n} - 1 \right| \le C_K r^{\alpha}. \tag{3.2.8}$$

Also, by Proposition 3.1.1 we know that with b and Q as defined in (3.1.6) and (3.1.7), we have

$$|\mathrm{tr}(Q) - n| \le C_K r^{\alpha}, \tag{3.2.9}$$

and

$$||Y - Y_0|^2 - 2\langle b, Y - Y_0\rangle - Q(Y - Y_0)| \leq C_K \left(\frac{|Y - Y_0|^3}{r} + r^{2+\min\{\alpha,\beta\}} \right), \quad (3.2.10)$$

for all $Y \in \tilde{\Sigma} \cap B(Y_0, r/2)$ and $r \in (0, r_K]$. These estimates are very similar to the ones required in the argument of [7] for the proof of Proposition 3.2.2. There are only a few differences, but we can see why none of them interfere with their argument.

- The first difference is that the exponent on the last term in (3.2.10) is $\min\{\alpha, \beta\}$, as opposed to α as in [7]. This is not a problem, since we can adjust all three estimates above by replacing α with $\min\{\alpha, \beta\}$.

- The second one is that, as mentioned above, (3.2.8) implies that (3.2.4) holds at $Y = Y_0$, but not necessarily at other points Y. However, an inspection of the arguments in [7] shows that the validity of (3.2.4) at points $Y \neq Y_0$ is only needed in order to ensure that two moment estimates analogous to (3.2.9) and (3.2.10) hold. In our case, the validity of both moment estimates has already been established in Section 3.1.

- The third difference is that while (3.2.9) and (3.2.10) hold with constants C_K and r_K that do not depend on X_0, the analogous moment estimates for $\tilde{\mu}$ needed in [7] hold with constants that depend on $\tilde{K}$, and therefore also on X_0 (see (3.2.6)). This is also not a problem, since it only means that the constants in (3.2.9) and (3.2.10) enjoy extra uniformity.

- The fourth one is that the analogues of (3.2.9) and (3.2.10) in [7] hold with $r \in (0, 1/2]$, as opposed to $r \in (0, r_K]$ as in our setting. But this is also not a problem since the threshold radius in (3.2.7) can be adjusted accordingly.

- The last one is that the constant C_K in (3.2.10) multiplies the entire right hand side, as opposed to just the last term as in [7]. However, an inspection of their argument shows that this does not interfere either, since the only difference is that some of the absolute constants that arise in their setting will now depend on K.

(ii) *Flatness of* $\tilde{\Sigma}$. The statement of Proposition 3.2.2 assumes that $\tilde{\Sigma}$ is Reifenberg flat with vanishing or small constant. However, all that the proof of Proposition 3.2.2 in [7] requires is that this condition holds near $\tilde{K}$, in the following sense:

There exist $d_{\tilde{K}} > 0$ and $t_{\tilde{K}} > 0$ such that for all $r \in (0, t_{\tilde{K}}]$ and $Y \in \tilde{\Sigma} \cap (\tilde{K}; d_{\tilde{K}})$,

$$b\beta_{\tilde{\Sigma}}(Y, r) \leq \delta_n, \tag{3.2.11}$$

where $\delta_n > 0$ is small enough, depending only on n. We will show that this holds in our setting as a consequence of Σ being Reifenberg flat with vanishing constant. To see this, let $\varepsilon > 0$ and take $d_{\tilde{K}} = d_K$, with d_K as in (3.2.2), let $Y \in \tilde{\Sigma} \cap (\tilde{K}; d_K)$, and write $Y = \Lambda(X_0)^{-1}X$ for some $X \in \Sigma \cap (K, 1)$. Since Σ is Reifenberg flat with vanishing constant, there exists $r_K > 0$ such that if $0 < r \leq r_K$, then

$$b\beta_{\Sigma}(X, r) \leq \varepsilon. \tag{3.2.12}$$

Let $t_K = r_K \lambda_{\max}(K)^{-1}$ and suppose $0 < r \leq t_K$. Assume also that ε is small enough so that the assumptions of Lemma 2.3.1 are satisfied. Let P be an n-plane through X that attains the infimum in the definition of $b\beta_{\Sigma}(X, \lambda_{\max}(K)r)$, and denote $\tilde{P} = \Lambda(X_0)^{-1}P$. Then, by Lemma 2.3.1 and (3.2.12),

$$\begin{aligned} D\left[\tilde{\Sigma} \cap B(Y, r); \tilde{P} \cap B(Y, r)\right] &\leq \lambda_{\min}(X_0)^{-1} D\left[\Sigma \cap \Lambda(X_0)B(Y, r); \Lambda(X_0)(\tilde{P} \cap B(Y, r))\right] \\ &\leq \lambda_{\min}(K)^{-1} D\left[\Sigma \cap \{X + \Lambda(X_0)B(0, r)\}; P \cap \{X + \Lambda(X_0)B(0, r)\}\right] \\ &\leq \lambda_{\min}(K)^{-1}(2 + e_{\Lambda}(K))\lambda_{\max}(K)\varepsilon r \leq C_K \varepsilon r. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows that (3.2.11) holds with $t_{\tilde{K}} = t_K$, and in fact $\tilde{\Sigma}$ is Reifenberg flat with vanishing constant too.

Remark 10. It will be important to notice that the value of $t_{\tilde{K}}$ found above depends on K, but not on the particular choice of X_0, so it enjoys extra uniformity.

This completes our justification that the proof of Proposition 3.2.2 is applicable to $\tilde{\mu}$ at Y_0, and as a consequence, (3.2.7) holds, concluding step 1.

3.2.2 Step 2: Bound for $\beta_\Sigma(X_0, r)$

We now use (3.2.7) and translate it into a estimate for $\beta_\Sigma(X_0, \cdot)$. The main aspect we need to deal with is the fact that the constants $C_{\tilde{K}}$ and $r_{\tilde{K}}$ in (3.2.7) depend *a priori* on $\tilde{K}$, and therefore on the choice of $X_0 \in \Sigma \cap K$. However, as some of the above arguments suggest, these constants should in fact depend on K, but not on the particular choice of X_0. We will justify that this is the case, and then use this information to estimate β_Σ as follows:

(i) $C_{\tilde{K}}$ can be taken to be independent of X_0;

(ii) $r_{\tilde{K}}$ can be taken to be independent of X_0;

(iii) β_Σ satisfies (3.2.3).

(i) $C_{\tilde{K}}$ *is independent of* $X_0 \in \Sigma \cap K$. An examination of the proof of Proposition 3.2.2 shows that the constant $C_{\tilde{K}}$ in (3.2.7) comes from its occurrence in the density and moment estimates discussed in Step 1 (i) above, and subsequent multiplication by various absolute constants. However, as noted before, the constants in these density and moment estimates can be taken to depend on K only. Therefore, the same applies to $C_{\tilde{K}}$ in (3.2.7), and we may write $C_{\tilde{K}} = C_K$.

(ii) $r_{\tilde{K}}$ *is independent of* $X_0 \in \Sigma \cap K$. First note that the way the threshold $r_{\tilde{K}}$ of equation (3.2.5) is chosen in [7] (r_0 in their notation), is as $r_{\tilde{K}} = \frac{1}{4} t_{\tilde{K}}^{1+\tau}$, $\tau \in (0, 1)$, where $t_{\tilde{K}}$ is a threshold radius for which (3.2.11) holds. But as noted in Remark 10, such threshold can be taken to be independent of X_0, so the same is true about $\tilde{r}_K$. Thus we may write $r_{\tilde{K}} = r_K$.

(iii) *Decay of* $\beta_\Sigma(X_0, r)$. To estimate $\beta_\Sigma(X_0, r)$, first notice that by points (i) and (ii), the estimate in (3.2.7) becomes

$$\beta_{\tilde{\Sigma}}(Y_0, r) \leq C_K r^\gamma, \tag{3.2.13}$$

for all $r \in (0, r_K]$, where $C_K > 0$ and $r_K > 0$ depend only on K, Λ and n. Let $r \in (0, r_K]$ and let $\tilde{P}$ be an n-plane through Y_0 attaining the infimum in the definition of $\beta_{\tilde{\Sigma}}(Y_0, r)$. As before, we can write $\tilde{P} = \Lambda(X_0)^{-1}P$, where P is an n-plane through X_0. We will estimate $\beta_\Sigma(X_0, \lambda_{\min}(K)r)$. Notice that

$$B(X_0, \lambda_{\min}(K)r) \subset B(X_0, \lambda_{\min}(X_0)r) \subset B_\Lambda(X_0, r) = \Lambda(X_0)B(Y_0, r). \tag{3.2.14}$$

Thus given any $W \in \Sigma \cap B(X_0, \lambda_{\min}(K)r)$, we can write $W = \Lambda(X_0)Z$, with $Z \in \tilde{\Sigma} \cap B(Y_0, r)$. Then by (3.2.13),

$$\mathrm{dist}(W, P) = \mathrm{dist}(\Lambda(X_0)Z, \Lambda(X_0)\tilde{P}) \leq \lambda_{\max}(K)\mathrm{dist}(Z, \tilde{P}) \leq C_K r^{1+\gamma}.$$

This implies that $\beta_\Sigma(X_0, \lambda_{\min}(K)r) \leq C_K r^\gamma$, for all $r \in (0, r_K]$, or equivalently,

$$\beta_\Sigma(X_0, r) \leq C_K(\lambda_{\min}(K)^{-1}r)^\gamma \leq C_K r^\gamma, \tag{3.2.15}$$

for all $r \in (0, \lambda_{\min}(K)r_K]$. This shows that (3.2.3) holds and completes the proof of Proposition 3.2.1. $\square$

Chapter 4

Vanishing flatness condition and proof of Theorem 1.3.1

4.1 Λ-pseudo tangents of μ and proof of Theorem 1.3.1

In order to prove Theorem 1.3.1 it is essentially enough to combine Proposition 3.2.1 and the following result.

Proposition 4.1.1 ([7] - Proposition 9.1)**.** *Let $\gamma \in (0,1]$. Suppose Σ is a Reifenberg-flat set with vanishing constant of dimension m in $\mathbb{R}^{n+1}$, $m \le n+1$, and that for each compact set $K \subset \mathbb{R}^{n+1}$ with $\Sigma \cap K \neq \varnothing$ there is a constant $C_K > 0$ such that*

$$\beta_{\Sigma}^{m}(X,r) \le C_K r^{\gamma}, \tag{4.1.1}$$

for all $X \in K \cap \Sigma$ and $r \in (0, r_K]$. Then Σ is a $C^{1,\gamma}$ submanifold of dimension m of $\mathbb{R}^{n+1}$.

The caveat is that the assumption that Σ is Reifenberg-flat with vanishing constant is stronger than the flatness assumption in Theorem 1.3.1. However, the following result ensures that the assumptions of the latter suffice.

Proposition 4.1.2. *Suppose μ and Λ satisfy the density and continuity assumptions of Theorem 1.3.1, and let $\Sigma = \mathrm{spt}(\mu)$. If $n \geq 3$, suppose also that μ satisfies* (1.3.3), *i.e. for every compact set $K \subset \mathbb{R}^{n+1}$ there exists $r_K > 0$ such that*

$$b\beta_\Sigma(K, r_K) = \sup_{r\in(0,r_K]} \sup_{X\in\Sigma\cap K} b\beta^n_\Sigma(X, r) \leq \delta_K, \tag{4.1.2}$$

where $\delta_K > 0$ is small enough depending on K and Λ. Then Σ is Reifenberg-flat with vanishing constant.

The assumptions of this proposition are just those of Theorem 1.3.1; we choose to write them down explicitly to keep them handy. We first use this result to complete the proof of Theorem 1.3.1.

Proof of Theorem 1.3.1. Let μ and Λ be as in the assumptions of Theorem 1.3.1. By Proposition 4.1.2, Σ is Reifenberg-flat with vanishing constant. Therefore, Proposition 3.2.1 ensures that (4.1.1) holds, and the conclusion of Theorem 1.3.1 follows from Proposition 4.1.1.

$\square$

To prove Proposition 4.1.2 we follow an approach based on that of [13] in the Euclidean setting, with two main steps:

Step 1. Show that all Λ-pseudo tangent measures of μ are uniform (see Definition 4.1.1); and

Step 2. Prove, via Theorem 2.1.2, that (4.1.2) implies that those Λ-pseudo tangents are flat, and use this to conclude.

This section is devoted to the first step, which happens to be independent of the smallness of δ_K. We first consider some relevant definitions and facts that will be needed later.

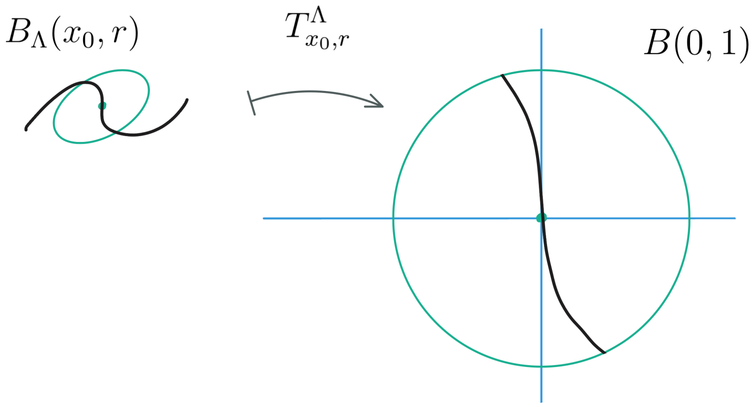

Figure 4.1: The blow-up maps $T^\Lambda_{\cdot,\cdot}$ send Λ-balls to Euclidean balls.

4.1.1 Λ-pseudo tangent measures

Given a point $X \in \Sigma$ and a radius $r > 0$, consider the mapping

$$T^\Lambda_{X,r}(Z) = \Lambda(X)^{-1}\left(\frac{Z - X}{r}\right), \quad Z \in \mathbb{R}^{n+1}, \tag{4.1.3}$$

and the measure

$$\mu_{X,r} = \frac{1}{\mu(B_\Lambda(X,r))} T^\Lambda_{X,r}[\mu]. \tag{4.1.4}$$

Here $T^\Lambda_{X,r}[\cdot]$ denotes push-forward via $T^\Lambda_{X,r}$, so for $E \subset \mathbb{R}^{n+1}$,

$$\mu_{X,r}(E) = \frac{\mu(X + r\Lambda(X)E)}{\mu(B_\Lambda(X,r))}.$$

Definition 4.1.1 (Λ-pseudo tangent measure)**.** *A measure $\nu \not\equiv 0$ is a* Λ-pseudo tangent measure *of μ at $Q \in \Sigma$ if there exists a sequence of points $Q_i \in \Sigma$ and radii $\rho_i > 0$ with $Q_i \to Q$ and $\rho_i \to 0$ as $i \to \infty$, such that*

$$\mu_{Q_i,\rho_i} \rightharpoonup \nu.$$

Here the symbol $\rightharpoonup$ denotes weak convergence of Radon measures. In particular, if $\mathcal{O}, \mathcal{C} \subset \mathbb{R}^{n+1}$ are open and compact, respectively, then

$$\nu(\mathcal{O}) \leq \liminf_{i\to\infty} \mu_{Q_i,\rho_i}(\mathcal{O})$$

and

$$\nu(\mathcal{C}) \geq \limsup_{i\to\infty} \mu_{Q_i,\rho_i}(\mathcal{C}).$$

Additionally, if $f \in C_c(\mathbb{R}^{n+1})$, then

$$\int f \mathrm{d}\nu = \lim_{i\to\infty} \int f \mathrm{d}\mu_{Q_i,\rho_i} = \lim_{i\to\infty} \int f\left(\frac{1}{\rho_i}\Lambda(Q_i)^{-1}(Z - Q_i)\right) \mathrm{d}\mu(Z).$$

When the points Q_i in Definition 4.1.1 satisfy $Q_i = Q$ for all i, the resulting measure ν is a Λ-tangent measure of μ (see [4]). If $\Lambda(Q_i) = \mathrm{Id}$, then ν is a (pseudo) tangent measure of μ (see [13]). The following are well-known facts about tangent measures in the Euclidean setting (see [21]).

Lemma 4.1.1 (Existence of Λ-pseudo tangent measures)**.** *Let μ be a Radon measure with support $\Sigma \subset \mathbb{R}^{n+1}$, such that for each compact set $K \subset \mathbb{R}^{n+1}$ with $\Sigma \cap K \neq \varnothing$,*

$$\sup_{\substack{0<r\leq 1\\ X\in\Sigma\cap K}} \frac{\mu(B_\Lambda(X, 2r))}{\mu(B_\Lambda(X, r))} < \infty.$$

Then every sequence of numbers $r_i > 0$ with $r_i \searrow 0$ and points $Q_i \in \Sigma$ contains a subsequence r_{i_l}, Q_{i_l} such that the measures $\mu_{Q_{i_l},r_{i_l}}$ converge to a Λ-pseudo tangent measure of μ at X.

Proof. Let K be a compact set with $Q_i \in K$ for all i, and denote by c the supremum in the statement of the lemma. Then for every $k \in \mathbb{N}$ we have

$$\begin{aligned} \limsup_{i\to\infty} \mu_{Q_i,r_i}(B(0, 2^k)) &= \limsup_{i\to\infty} \frac{1}{\mu(B_\Lambda(Q_i, r_i))} T^\Lambda_{Q_i,r_i}[\mu](B(0, 2^k)) \\ &= \limsup_{i\to\infty} \frac{\mu(B_\Lambda(Q_i, 2^k r_i))}{\mu(B_\Lambda(Q_i, r_i))} \leq c^k < \infty. \end{aligned} \tag{4.1.5}$$

It follows that the sequence $\mu_{Q_i,r_i}(F)$ is bounded for every compact set $F \subset \mathbb{R}^{n+1}$, and the conclusion of the lemma follows by a standard compactness result for Radon measures (see [21, Theorem 1.23]). $\square$

Lemma 4.1.2. *If μ satisfies the assumptions of Theorem 1.3.1 and ν is a Λ-pseudo tangent measure of μ, then $0 \in \operatorname{spt}(\nu)$.*

Proof. Recall that under the assumptions of Theorem 1.3.1, for every $X \in \Sigma \cap K$ and $r \in (0,1]$ we have

$$\omega_n r^n - C_K r^{n+\alpha} \leq \mu(B_\Lambda(X,r)) \leq \omega_n r^n + C_K r^{n+\alpha}. \tag{4.1.6}$$

Thus, if $R > 0$, $X \in \Sigma \cap K$ and $r > 0$ is small enough,

$$\mu_{X,r}(B(0,R)) = \frac{\mu(B_\Lambda(X,rR))}{\mu(B_\Lambda(X,r))} \geq \frac{(rR)^n - C_K(rR)^{n+\alpha}}{r^n + C_K r^{n+\alpha}} \geq \frac{R^n}{2}. \tag{4.1.7}$$

Now, since ν is a Λ-pseudo tangent measure of μ, we have $\mu_{P_i,\rho_i} \rightharpoonup \nu$ for some $P_i \in \Sigma \cap K$, where $K \subset \mathbb{R}^{n+1}$ is a compact set, $\rho_i > 0$ and $\rho_i \to 0$. Therefore, applying (4.1.7) with $X = P_i$ and $r = \rho_i$, we get

$$\begin{aligned}\nu(B(0,2R)) \geq \nu(\overline{B(0,R)}) &\geq \limsup_{i\to\infty} \mu_{P,\rho_i}(\overline{B(0,R)}) \\ &\geq \limsup_{i\to\infty} \mu_{P,\rho_i}(B(0,R)) \geq \frac{R^n}{2} > 0,\end{aligned}$$

from which the desired conclusion follows. □

One of the main results in this section states that if Λ and μ satisfy the continuity and density assumptions of Theorem 1.3.1, then all Λ-pseudo tangent measures of μ are n-uniform. We will state this result in a more general form by considering a more relaxed assumption on μ.

Definition 4.1.2. *A Radon measure μ in $\mathbb{R}^{n+1}$ with support Σ is called* Λ-asymptotically optimally doubling of dimension n *if for every compact set $K \subset \mathbb{R}^{n+1}$,*

$$\lim_{r\searrow 0} \sup_{\substack{X\in\Sigma\cap K \\ \tau\in[\frac{1}{2},1]}} \left| \frac{\mu(B_\Lambda(X,\tau r))}{\mu(B_\Lambda(X,r))} - \tau^n \right| = 0. \tag{4.1.8}$$

Compare this notion with (1.2.2). We summarize a couple of facts about this condition and its connection with measures that satisfy the density condition (1.3.4) in Theorem 1.3.1.

Proposition 4.1.3. *Let μ be a Radon measure with support Σ in $\mathbb{R}^{n+1}$.*

1. *If μ satisfies* (1.3.4), *then it also satisfies* (4.1.8).

2. *If μ satisfies* (4.1.8), *then for every $K \subset \mathbb{R}^{n+1}$ compact,*

$$\lim_{r\to 0} \sup_{\substack{X\in\Sigma\cap K \\ t\in(0,1)}} \left| \frac{\mu(B_\Lambda(X,tr))}{\mu(B_\Lambda(X,r))} - t^n \right| = 0. \tag{4.1.9}$$

Proof. For the proof of the first statement, note that by (1.3.4), if $r > 0$ is small enough then

$$\left| \frac{\mu(B_\Lambda(X,\tau r))}{\omega_n(\tau r)^n} - 1 \right| \le C_K(\tau r)^\alpha \quad \text{and} \quad \left| \frac{\omega_n r^n}{\mu(B_\Lambda(X,r))} - 1 \right| \le C_K r^\alpha.$$

Therefore,

$$\left| \frac{\mu(B_\Lambda(X,\tau r))}{\mu(B_\Lambda(X,r))} - \tau^n \right| \le \tau^n \left\{ \left| \frac{\omega_n r^n}{\mu(B_\Lambda(X,r))} \left(\frac{\mu(B_\Lambda(X,\tau r))}{\omega_n(\tau r)^n} - 1 \right) \right| + \left| \frac{\omega_n r^n}{\mu(B_\Lambda(X,r))} - 1 \right| \right\}$$

$$\le \tau^n \left\{ C_K(\tau r)^\alpha + C_K r^\alpha \right\} \le C_K \tau^n r^\alpha \le C_K r^\alpha. \tag{4.1.10}$$

This gives (4.1.8). For the second statement, (4.1.8) implies that given $\varepsilon > 0$, there exists $R > 0$ so that for $r \in (0,R)$, $X \in \Sigma \cap K$ and $\tau \in [\frac{1}{2}, 1]$,

$$|\mu(B_\Lambda(X,\tau r)) - \tau^n \mu(B_\Lambda(X,r))| \le \varepsilon \mu(B_\Lambda(X,r)). \tag{4.1.11}$$

Let $t \in (0,1]$, and let $j \ge 1$ be such that $\frac{1}{2^j} \le t < \frac{1}{2^{j-1}}$, so that $\tau := t^{1/j} \in [\frac{1}{2}, \frac{1}{\sqrt{2}})$. Then by (4.1.11), we have for $X \in \Sigma \cap K$, $r \in (0,R)$ and $k \ge 1$,

$$|\mu(B_\Lambda(X,\tau^k r)) - \tau^n \mu(B_\Lambda(X,\tau^{k-1} r))| \le \varepsilon \mu(B_\Lambda(X,r)).$$

Therefore,

$$\begin{aligned}|\mu(B_\Lambda(X,tr)) - t^n\mu(B_\Lambda(X,r))| &\leq \sum_{k=0}^{j-1}\tau^{nk}|\mu(B_\Lambda(X,\tau^{j-k}r)) - \tau^n\mu(B_\Lambda(X,\tau^{j-k-1}r))| \\ &\leq \varepsilon\mu(B_\Lambda(X,r))\sum_{k=0}^{j-1}\tau^{nk} \\ &\leq \varepsilon\mu(B_\Lambda(X,r))\sum_{k=0}^{\infty}\frac{1}{(\sqrt{2})^{nk}} = C\varepsilon\mu(B_\Lambda(X,r)),\end{aligned}$$

where $C > 0$ depends only on n. This implies

$$\left|\frac{\mu(B_\Lambda(X,tr))}{\mu(B_\Lambda(X,r))} - t^n\right| \leq C\varepsilon,$$

for all $r \in (0,R)$, from which the desired conclusion follows. □

The following is the main result of this section.

Proposition 4.1.4. *Suppose that Λ satisfies the continuity assumption of Theorem 1.3.1 and μ is Λ-asymptotically optimally doubling of dimension m in $\mathbb{R}^{n+1}$, $m \leq n$. If ν is a Λ-pseudo tangent measure of μ, then ν is m-uniform. Moreover,* (2.1.1) *holds with $C = 1$.*

To prove this we start with a description of the support of any given Λ-pseudo tangent measure of μ.

Lemma 4.1.3. *Suppose μ is a Λ-asymptotically optimally doubling measure of dimension m in $\mathbb{R}^{n+1}$ with support Σ, $m \leq n$. Let $\rho_i > 0$ and $Q_i \in \Sigma$ be such that $\rho_i \to 0$, $Q_i \to Q \in \Sigma$ and $\mu_{Q_i,\rho_i} \rightharpoonup \nu$ as $i \to \infty$, where ν is a Λ-pseudo tangent measure of μ. If $T^\Lambda_{Q_i,\rho_i}$ is defined as in* (4.1.3) *and $X \in \mathbb{R}^{n+1}$, then $X \in \operatorname{spt}(\nu)$ if and only if there exist $X_i \in T^\Lambda_{Q_i,r_i}(\Sigma)$ such that $X_i \to X$ as $i \to \infty$.*

Proof of Lemma 4.1.3. For the forward direction, let $X \in \operatorname{spt}(\nu)$. Suppose for a contradiction that there exist $\varepsilon_0 > 0$ and $i_k \in \mathbb{N}$ with $i_k \to \infty$, and for every i_k

$$\operatorname{dist}(X, T^\Lambda_{Q_{i_k},r_{i_k}}(\Sigma)) \geq \varepsilon_0. \tag{4.1.12}$$

If $\varphi \in C_c(\mathbb{R}^{n+1})$ and $\chi_{B(X,\varepsilon_0/4)} \leq \varphi \leq \chi_{B(X,\varepsilon_0/2)}$, by (4.1.12) we have $\varphi\left(T^{\Lambda}_{Q_{i_k},r_{i_k}}(Y)\right) = 0$ for every $Y \in \Sigma$. Therefore,

$$\nu(B(X,\varepsilon_0/4)) \leq \int \varphi \mathrm{d}\nu = \lim_{k\to\infty} \frac{1}{\mu(B_\Lambda(Q_{i_k},\rho_{i_k}))} \int_\Sigma \varphi\left(T^{\Lambda}_{Q_{i_k},r_{i_k}}(Y)\right) \mathrm{d}\mu(Y) = 0,$$

which contradicts the assumption that $X \in \operatorname{spt}(\nu)$.

To prove the converse, let $X_i \in T^{\Lambda}_{Q_i,\rho_i}$ be such that $X_i \to X$ as $i \to \infty$, and write

$$X_i = \Lambda(Q_i)^{-1}\left(\frac{Z_i - Q_i}{\rho_i}\right), \quad Z_i \in \Sigma.$$

Given $r > 0$,

$$\begin{aligned}
\mu_{Q_i,\rho_i}(B(X,r)) &= \frac{\mu(\rho_i\Lambda(Q_i)B(X,r)+Q_i)}{\mu(B_\Lambda(Q_i,\rho_i))} \\
&= \frac{\mu(\rho_i\Lambda(Q_i)(B(0,r)+X)+Q_i)}{\mu(B_\Lambda(Q_i,\rho_i))} \\
&= \frac{\mu(\Lambda(Q_i)B(0,r\rho_i)+Q_i+\rho_i\Lambda(Q_i)X)}{\mu(B_\Lambda(Q_i,\rho_i))} \\
&= \frac{\mu(B_\Lambda(Q_i+\rho_i\Lambda(Q_i)X, r\rho_i))}{\mu(B_\Lambda(Q_i,\rho_i))}.
\end{aligned} \tag{4.1.13}$$

To get a lower bound, we need to shift the center $Q_i + \rho_i\Lambda(Q_i)X$ in the numerator to a point in Σ so that we can use the doubling assumption. Notice that

$$\begin{aligned}
|(Q_i + \rho_i\Lambda(Q_i)X) - Z_i| &= |(Q_i + \rho_i\Lambda(Q_i)X) - (Q_i + \rho_i\Lambda(Q_i)X_i)| \\
&= \rho_i|\Lambda(Q_i)(X - X_i)|,
\end{aligned} \tag{4.1.14}$$

so by Lemma 2.2.2,

$$B_\Lambda(Q_i+\rho_i\Lambda(Q_i)X, r\rho_i) \supset B_\Lambda(Z_i, r\rho_i - \lambda_{\min}(Q_i+\rho_i\Lambda(Q_i)X)^{-1}\rho_i|\Lambda(Q_i)(X-X_i)| - C_K(r\rho_i)^{1+\beta}).$$

Assuming i is large enough depending on r, K and Λ, we have

$$\lambda_{\min}(Q_i + \rho_i\Lambda(Q_i)X)^{-1}|\Lambda(Q_i)(X - X_i)| \leq \frac{r}{4}, \quad C_K(r\rho_i)^{1+\beta} \leq \frac{r}{4}.$$

It follows from the last inclusion above that for all i large enough,

$$B_\Lambda(Q_i + \rho_i \Lambda(Q_i) X, r\rho_i) \supset B_\Lambda(Z_i, r\rho_i/4).$$

From this and (4.1.13) we get

$$\mu_{Q_i\rho_i}(B(X,r)) \geq \frac{\mu(B_\Lambda(Z_i, r\rho_i/4))}{\mu(B_\Lambda(Q_i,\rho_i))}. \tag{4.1.15}$$

Next, we proceed similarly as above to change the center once more, so that both centers coincide. Note that

$$|Q_i - Z_i| = \rho_i |\Lambda(Q_i) X_i| \leq C_K \rho_i,$$

where $C_K > 0$ is a constant depending on X, K and Λ. Thus by an application of Lemma 2.2.2, equation (2.2.2), we get

$$B_\Lambda(Q_i,\rho_i) \subset B_\Lambda(Z_i, \rho_i + \lambda_{\min}(Q_i)^{-1}\rho_i |\Lambda(Q_i)X_i| + C_K \rho_i^{1+\beta}) \subset B_\Lambda(Z_i, C_K\rho_i).$$

Combining this with (4.1.15) and using the doubling assumption on μ, if i is large enough depending on r and C_K,

$$\mu_{Q_i,\rho_i}(B(X,r)) \geq \frac{\mu(B_\Lambda(Z_i, r\rho_i/4))}{\mu(B_\Lambda(Z_i, C_K\rho_i))} \geq \frac{1}{2}\left(\frac{r}{4C_K}\right)^m.$$

Therefore, since $\mu_{Q_i,\rho_i} \rightharpoonup \nu$,

$$\nu(B(X,2r)) \geq \nu(\overline{B(X,r)}) \geq \limsup_{i\to\infty} \mu_{Q_i,\rho_i}(\overline{B(X,r)}) \geq \frac{1}{2}\left(\frac{r}{4C_K}\right)^m.$$

This implies that $\nu(B(X,r)) > 0$ for every $r > 0$, which in turn shows that $X \in \operatorname{spt}(\nu)$ as desired. $\square$

4.1.2 Proof of Proposition 4.1.4

Proof. Let ν be a Λ-pseudo tangent of μ, and let $Q_i \in \Sigma$ and $\rho_i > 0$ be such that $Q_i \to Q$, $\rho_i \to 0$ and $\mu_{Q_i,\rho_i} \rightharpoonup \nu$ as $i \to \infty$. By Lemma 4.1.3, there exist $X_i \in T^\Lambda_{Q_i,\rho_i}(\Sigma)$ such that $X_i \to X$ as $i \to \infty$. Write

$$X_i = \Lambda(Q_i)^{-1}\left(\frac{Z_i - Q_i}{\rho_i}\right), \quad Z_i \in \Sigma.$$

Let $r > 0$. We need to get lower and upper bounds for

$$\mu_{Q_i,\rho_i}(B(X,r)) = \frac{\mu(B_\Lambda(Q_i + \rho_i\Lambda(Q_i)X), r\rho_i)}{\mu(B_\Lambda(Q_i,\rho_i))}.$$

We start with an upper bound. Let $\varepsilon > 0$. As in the proof of Lemma 4.1.3, by (4.1.14),

$$|(Q_i + \rho_i\Lambda(Q_i)X) - Z_i| = \rho_i|\Lambda(Q_i)(X - X_i)|.$$

So an application of Lemma 2.2.2, equation (2.2.2), gives

$$B_\Lambda(Q_i+\rho_i\Lambda(Q_i)X, r\rho_i) \subset B_\Lambda(Z_i, r\rho_i+\lambda_{\min}(Q_i+\rho_i\Lambda(Q_i)X)^{-1}\rho_i|\Lambda(Q_i)(X-X_i)|+C_K(r\rho_i)^{1+\beta}).$$

If i is large enough depending on r, X, K and Λ, we can guarantee that

$$\lambda_{\min}(Q_i + \rho_i\Lambda(Q_i)X)^{-1}\rho_i|\Lambda(Q_i)(X - X_i)| \le \varepsilon r\rho_i, \quad C_K(r\rho_i)^{1+\beta} \le \varepsilon r\rho_i.$$

It then follows from the inclusion above that for such i,

$$B_\Lambda(Q_i + \rho_i\Lambda(Q_i)X, r\rho_i) \subset B_\Lambda(Z_i, r\rho_i(1 + 2\varepsilon)), \tag{4.1.16}$$

and consequently,

$$\mu_{Q_i,\rho_i}(B(X,r)) \le \frac{\mu(B_\Lambda(Z_i, r\rho_i(1 + 2\varepsilon)))}{\mu(B_\Lambda(Q_i,\rho_i))}. \tag{4.1.17}$$

Write

$$\frac{\mu(B_\Lambda(Z_i, r\rho_i(1 + 2\varepsilon)))}{\mu(B_\Lambda(Q_i,\rho_i))} = \frac{\mu(B_\Lambda(Z_i, r\rho_i(1 + \varepsilon)))}{\mu(B_\Lambda(Q_i, r\rho_i(1 + 2\varepsilon)))} \cdot \frac{\mu(B_\Lambda(Q_i, r\rho_i(1 + 2\varepsilon)))}{\mu(B_\Lambda(Q_i,\rho_i))}. \tag{4.1.18}$$

Assume without loss of generality that $Q_i \in \Sigma \cap B(Q, 1)$. If i is large enough depending on r, ε and Λ, then by the doubling assumption on μ, the second factor above satisfies

$$\frac{\mu(B_\Lambda(Q_i, r\rho_i(1 + 2\varepsilon)))}{\mu(B_\Lambda(Q_i,\rho_i))} \le (1 + \varepsilon)[r(1 + 2\varepsilon)]^m. \tag{4.1.19}$$

To deal with the first factor, we would like to move the center Q_i to Z_i. However, doing so directly would introduce an error comparable to ρ_i, which is a larger order of magnitude than what we can allow if r is small. The following estimate avoids this obstacle. Let $\kappa > 0$ be a large constant to be determined. Then for i large depending on κ and ε,

$$\begin{aligned}\frac{\mu(B_\Lambda(Z_i, r\rho_i(1+\varepsilon)))}{\mu(B_\Lambda(Q_i, r\rho_i(1+2\varepsilon)))} &= \frac{\mu(B_\Lambda(Z_i, r\rho_i(1+2\varepsilon)))}{\mu(B_\Lambda(Z_i, \kappa r\rho_i(1+2\varepsilon)))} \cdot \frac{\mu(B_\Lambda(Z_i, \kappa r\rho_i(1+2\varepsilon)))}{\mu(B_\Lambda(Q_i, \kappa r\rho_i(1+2\varepsilon)))} \\ &\qquad \cdot \frac{\mu(B_\Lambda(Q_i, \kappa r\rho_i(1+2\varepsilon)))}{\mu(B_\Lambda(Q_i, r\rho_i(1+2\varepsilon)))} \\ &\leq (1+\varepsilon)^2 \cdot \frac{\mu(B_\Lambda(Z_i, \kappa r\rho_i(1+2\varepsilon)))}{\mu(B_\Lambda(Q_i, \kappa r\rho_i(1+2\varepsilon)))}.\end{aligned} \tag{4.1.20}$$

We can now make the centers coincide. Recall that

$$|Q_i - Z_i| = \rho_i |\Lambda(Q_i) X_i| \leq C_K \rho_i,$$

where $C_K > 0$ is a constant that depends on X, K and Λ. Therefore, by Lemma 2.2.2,

$$\begin{aligned} B_\Lambda(Z_i, \kappa r\rho_i(1+2\varepsilon)) &\subset B_\Lambda(Q_i, \kappa r\rho_i(1+2\varepsilon)) + \lambda_{\min}(Z_i)^{-1}\rho_i|\Lambda(Q_i)X_i| + C_K(\kappa r\rho_i(1+2\varepsilon))^{1+\beta}) \\ &\subset B_\Lambda(Q_i, \kappa r\rho_i(1+2\varepsilon) + C_K\rho_i + C_K(\kappa r\rho_i(1+2\varepsilon))^{1+\beta}) \\ &\subset B_\Lambda\left(Q_i, \kappa r\rho_i\left[1+2\varepsilon+\frac{C_K}{\kappa r} + C_K(\kappa r\rho_i)^\beta(1+2\varepsilon)^{1+\beta}\right]\right). \end{aligned} \tag{4.1.21}$$

We now take κ to be sufficiently large, depending on X, K, Λ, r and ε, so that $\frac{C_K}{\kappa r} < \varepsilon$. In addition, we assume that i is sufficiently large, depending on X, K, Λ, r and ε, so that

$$C_K(\kappa r\rho_i)^\beta(1+2\varepsilon)^{1+\beta} \leq \varepsilon.$$

In this scenario, (4.1.21) implies

$$B_\Lambda(Z_i, \kappa r\rho_i(1+2\varepsilon)) \subset B_\Lambda(Q_i, \kappa r\rho_i(1+4\varepsilon)).$$

It follows from this inclusion and the doubling assumption on μ,

$$\begin{aligned}\frac{\mu(B_\Lambda(Z_i,\kappa r\rho_i(1+2\varepsilon)))}{\mu(B_\Lambda(Q_i,\kappa r\rho_i(1+2\varepsilon)))} &\leq \frac{\mu(B_\Lambda(Q_i,\kappa r\rho_i(1+4\varepsilon)))}{\mu(B_\Lambda(Q_i,\kappa r\rho_i(1+2\varepsilon)))}\\ &\leq \frac{\mu(B_\Lambda(Q_i,\kappa r\rho_i(1+4\varepsilon)))}{\mu(B_\Lambda(Q_i,\kappa r\rho_i))} \leq (1+\varepsilon)(1+4\varepsilon)^m.\end{aligned}$$

Combining this with (4.1.20) we get

$$\frac{\mu(B_\Lambda(Z_i,r\rho_i(1+\varepsilon)))}{\mu(B_\Lambda(Q_i,r\rho_i(1+2\varepsilon)))} \leq (1+\varepsilon)^3(1+4\varepsilon)^m. \tag{4.1.22}$$

Putting this together with (4.1.19) and coming back to (4.1.17), we obtain

$$\mu_{Q_i,\rho_i}(B(X,r)) \leq (1+\varepsilon)^3(1+4\varepsilon)^m(1+\varepsilon)[r(1+2\varepsilon)]^m \leq r^m(1+4\varepsilon)^{2m+4},$$

for all i large depending on X, K, Λ, r and ε. This shows that

$$\limsup_{i\to\infty} \mu_{Q_i,\rho_i}(B(X,r)) \leq r^m. \tag{4.1.23}$$

A similar argument gives

$$\liminf_{i\to\infty} \mu_{Q_i,\rho_i}(B(X,r)) \geq r^m. \tag{4.1.24}$$

We just outline the corresponding estimates in this case. We first apply Lemma 2.2.2, equation (2.2.3), to obtain

$$\begin{aligned}B_\Lambda(Q_i+\rho_i\Lambda(Q_i)X,r\rho_i) &\supset B_\Lambda(Z_i,r\rho_i-\lambda_{\min}(Q_i+\rho_i\Lambda(Q_i)X)^{-1}|\Lambda(Q_i)(X-X_i)|-C_K(r\rho_i)^{1+\beta})\\ &\supset B_\Lambda(Z_i,r\rho_i(1-2\varepsilon)),\end{aligned}$$

and

$$\mu_{Q_i,\rho_i}B(X,r) \geq \frac{\mu(B_\Lambda(Z_i,r\rho_i(1-2\varepsilon)))}{\mu(B_\Lambda(Q_i,\rho_i))}. \tag{4.1.25}$$

We then decompose the right hand side of this estimate as in (4.1.18). The corresponding estimate for the second factor in the decomposition is

$$\frac{\mu(B_\Lambda(Q_i,r\rho_i(1+2\varepsilon)))}{\mu(B_\Lambda(Q_i,\rho_i))} \geq (1-\varepsilon)[r(1-2\varepsilon)]^m. \tag{4.1.26}$$

The estimate for the first factor now becomes

$$\frac{\mu(B_\Lambda(Z_i, r\rho_i(1-\varepsilon)))}{\mu(B_\Lambda(Q_i, r\rho_i(1-2\varepsilon)))} \geq (1-\varepsilon)^2 \frac{\mu(B_\Lambda(Z_i, \kappa r\rho_i(1-2\varepsilon)))}{\mu(B_\Lambda(Q_i, \kappa r\rho_i(1-2\varepsilon)))}. \tag{4.1.27}$$

Then by Lemma 2.2.2, equation (2.2.3),

$$B_\Lambda(Z_i, \kappa r\rho_i(1-2\varepsilon)) \supset B_\Lambda\left(Q_i, \kappa r\rho_i\left[1-2\varepsilon - \frac{C_K}{\kappa r} - C_K(\kappa r\rho_i)^\beta(1-2\varepsilon)^{1+\beta}\right]\right),$$

giving

$$B_\Lambda(Z_i, \kappa r\rho_i(1-2\varepsilon)) \supset B_\Lambda(Q_i, \kappa r\rho_i(1-4\varepsilon)).$$

Therefore

$$\begin{aligned}\frac{\mu(B_\Lambda(Z_i, \kappa r\rho_i(1-2\varepsilon)))}{\mu(B_\Lambda(Q_i, \kappa r\rho_i(1-2\varepsilon)))} &\geq \frac{\mu(B_\Lambda(Q_i, \kappa r\rho_i(1-4\varepsilon)))}{\mu(B_\Lambda(Q_i, \kappa r\rho_i(1-2\varepsilon)))} \\ &\geq \frac{\mu(B_\Lambda(Q_i, \kappa r\rho_i(1-4\varepsilon)))}{\mu(B_\Lambda(Q_i, \kappa r\rho_i)} \\ &\geq (1-\varepsilon)(1-4\varepsilon)^m,\end{aligned}$$

and combining this with (4.1.25), (4.1.26) and (4.1.27) we obtain

$$\mu_{Q_i,\rho_i}(B(X,r)) \geq r^m(1-4\varepsilon)^{2m+4},$$

for all i large enough, which shows that (4.1.24) holds as desired.

Combining (4.1.23) and (4.1.24) we can show that ν satisfies the desired conclusion. In fact, using that $\mu_{Q_i,\rho_i} \rightharpoonup \nu$ we get

$$\nu(B(X,r)) \leq \liminf_{i\to\infty} \mu_{Q_i,\rho_i}(B(X,r)) \leq r^m, \tag{4.1.28}$$

and given any $\varepsilon \in (0,1)$,

$$\begin{aligned}\nu(B(X,r)) \geq \nu(\overline{B(X,(1-\varepsilon)r)}) &\geq \limsup_{i\to\infty} \mu_{Q_i,\rho_i}(\overline{B(X,(1-\varepsilon)r)}) \\ &\geq \limsup_{i\to\infty} \mu_{Q_i,\rho_i}(B(X,(1-\varepsilon)r)) \geq [(1-\varepsilon)r]^m.\end{aligned} \tag{4.1.29}$$

Since this holds for every $\varepsilon > 0$, we conclude from (4.1.28) and (4.1.29) that

$$\nu(B(X,r)) = r^m,$$

completing the proof of Proposition 4.1.4. □

4.2 Flatness of measures with uniform Λ-pseudo tangents

In this section we complete Step 2 of the proof of Proposition 4.1.2. We do this by proving the more general statement that if all Λ-pseudo tangent measures of μ are n-uniform, and if $\Sigma = \operatorname{spt}(\mu)$ satisfies flatness condition (4.1.2) when $n \geq 3$, then Σ is Reifenberg flat with vanishing constant. This does not require density estimate (1.3.4) to be satisfied. However, when proving Theorem 1.3.1, the fact that all Λ-pseudo tangent measures of μ are n-uniform will be a consequence of (1.3.4), as discussed in the previous section.

Except for δ_K, all other local constants that arise will eventually be denoted by C_K as before. It may be convenient to recall the quantities associated with Λ and any compact set $K \subset \mathbb{R}^{n+1}$, $\lambda_{\min}(K)$, $\lambda_{\max}(K)$ and $e_\Lambda(K)$, introduced in (2.3.3) and (2.3.4). We will also consider the quantity

$$M_K = (2 + e_\Lambda(K))\lambda_{\max}(K). \tag{4.2.1}$$

Proposition 4.2.1. *Let μ be a Radon measure on $\mathbb{R}^{n+1}$ such that all its Λ-pseudo tangent measures are n-uniform, where Λ satisfies the continuity assumption of Theorem 1.3.1, and let $K \subset \mathbb{R}^{n+1}$ be compact with $\Sigma \cap K \neq \varnothing$. If $n \geq 3$, suppose also that there exists $r_K > 0$ such that*

$$b\beta_\Sigma(K, r_K) = \sup_{r \in (0, r_K]} \sup_{X \in \Sigma \cap K} b\beta_\Sigma^n(X, r) \leq \delta_K, \tag{4.2.2}$$

where is $\delta_K > 0$ is small enough depending on K and Λ. Then

$$\lim_{r \searrow 0} b\beta_\Sigma(K, r) = 0.$$

In particular, if $n \leq 2$, or $n \geq 3$ and (4.2.2) holds for every compact $K \subset \mathbb{R}^{n+1}$ with $\Sigma \cap K \neq \varnothing$, then Σ is Reifenberg-flat with vanishing constant.

Assuming this result momentarily, the proof of Proposition 4.1.2 is short.

Proof of Proposition 4.1.2. By Proposition 4.1.3, μ is Λ-asymptotically optimally doubling of dimension n, so by Proposition 4.1.4 all its Λ-pseudo tangent measures are n-uniform. Proposition 4.2.1 then implies that Σ is Reifenberg-flat with vanishing constant. $\square$

At the core of the proof of Proposition 4.2.1 is Theorem 2.1.2. In our case, the measure ν in that theorem will be a suitable Λ-pseudo tangent measure of μ that captures how flat μ is. The key point is that the cone in (2.1.3) is not δ-Reifenberg flat if for example $\delta < 1/\sqrt{2}$. This implies that if ν inherits (4.2.2), then by Theorem 2.1.2, ν must be flat. Such information can then be used to show that Σ is Reifenberg flat with vanishing constant. This approach follows ideas developed by Kenig and Toro in [13] in the Euclidean setting.

Remark 11. Before proceeding with the proof, we record for later use the following compactness property of Hausdorff distance: if $\Gamma_i \subset \mathbb{R}^{n+1}$ contains the origin for all $i \in \mathbb{N}$, then there exists a subsequence i_k and a set $\Gamma \subset \mathbb{R}^{n+1}$ such that

$$\Gamma_{i_k} \to \Gamma,$$

with respect to Hausdorff distance, uniformly on compact subsets of $\mathbb{R}^{n+1}$.

Proof of Proposition 4.2.1. Let $K \subset \mathbb{R}^{n+1}$ be compact. Consider

$$\ell = \lim_{\tau \searrow 0} b\beta_\Sigma(K, \tau),$$

where $b\beta_\Sigma(K,\tau)$ is as in (4.2.2). We will show that $\ell = 0$. Let $\tau_i > 0$ be such that $\tau_i \searrow 0$ and $b\beta_\Sigma(K, \tau_i) \to \ell$. Let $Q_i \in \Sigma \cap K$ be points for which

$$b\beta_\Sigma(Q_i, \tau_i) \to \ell. \tag{4.2.3}$$

Since $\Sigma \cap K$ is compact, we may assume without loss of generality that $Q_i \to Q \in \Sigma \cap K$. We will need to work with the auxiliary scales

$$\rho_i = \lambda_{\min}(K)^{-1}\tau_i. \tag{4.2.4}$$

Recall the map

$$T^{\Lambda}_{Q_i,\rho_i}(X) = \Lambda(Q_i)^{-1}\left(\frac{X - Q_i}{\rho_i}\right), \quad X \in \mathbb{R}^{n+1}.$$

Notice first that $0 \in T^{\Lambda}_{Q_i,\rho_i}(\Sigma)$ for all i. Thus, by Remark 11 we may assume modulo passing to a subsequence that there exists $\Sigma_\infty \subset \mathbb{R}^{n+1}$ such that

$$T^{\Lambda}_{Q_i,\rho_i}(\Sigma) \to \Sigma_\infty, \tag{4.2.5}$$

with respect to Hausdorff distance D, uniformly on compact sets. We may also assume upon taking a further subsequence that $\mu_{Q_i,\rho_i} \rightharpoonup \nu$, where μ_{Q_i,ρ_i} is as in (4.1.4) and ν is a Λ-pseudo tangent measure of μ. Moreover, we know by Proposition 4.1.4 that ν is n-uniform, and we may assume without loss of generality, upon multiplying ν by a suitable constant, that (2.1.1) is satisfied with with $C = \omega_n$, so that the assumptions of Theorem 2.1.2 are satisfied. Note that by Lemma 4.1.3 and (4.2.5), we have

$$\operatorname{spt}(\nu) = \Sigma_\infty.$$

Thus, by Theorem 2.1.2, we know that Σ_∞ must be an n-plane or a light cone as in (2.1.3).

We will now use the fact that $T^{\Lambda}_{Q_i,\rho_i} \to \Sigma_\infty$ with respect to D and (4.2.2) to rule out the case in which Σ_∞ is a light cone. Let $X \in \Sigma_\infty$. By Lemma 4.1.3, there exist points $Z_i \in \Sigma$ such that if

$$X_i = T^{\Lambda}_{Q_i,\rho_i}(Z_i),$$

then $X_i \to X$ as $i \to \infty$. Notice that this implies $|Z_i - Q_i| \to 0$. Assume without loss of generality that $|X - X_i| \leq 1/2$, $|Q - Q_i| \leq 1/2$ and $|Q_i - Z_i| \leq 1/2$. Observe that then $Z_i \in (\Sigma \cap K; 1)$. We consider two auxiliary radii that will help us compare Σ with Σ_∞,

$$r_i = \rho_i(1 + |X - X_i|), \quad s_i = \rho_i(1 - |X - X_i|).$$

We start with a compatibility statement about minimizing planes for $b\beta_\Sigma(Z_i,\cdot)$ at certain scales. For each i, let

$$r_i' = \lambda_{\max}(K) r_i, \quad s_i' = \lambda_{\max}(K) s_i.$$

Since $\rho_i \to 0$ as $i \to \infty$, we can assume that r_i', $s_i' \leq r_K$ if i is large enough depending on K and Λ, where r_K is as in the statement of Proposition 4.2.1. First, by (4.2.2) there are n-planes $P(Z_i, r_i')$, $P(Z_i, s_i')$ such that

$$D[\Sigma \cap B(Z_i, r_i'); P(Z_i, r_i') \cap B(Z_i, r_i')] \leq \delta_K r_i', \tag{4.2.6}$$

$$D[\Sigma \cap B(Z_i, s_i'); P(Z_i, s_i') \cap B(Z_i, s_i')] \leq \delta_K s_i'. \tag{4.2.7}$$

Note that by (4.2.6), (4.2.7) and Corollary 2.3.1, if $\delta_K < \min\{\lambda_{\min}(K), e_\Lambda(K)^{-1}\}$, then

$$D[\Sigma \cap B_\Lambda(Z_i, r_i); P(Z_i, r_i') \cap B_\Lambda(Z_i, r_i)] \leq M_K \delta_K r_i, \tag{4.2.8}$$

$$D[\Sigma \cap B_\Lambda(Z_i, s_i); P(Z_i, s_i') \cap B_\Lambda(Z_i, s_i)] \leq M_K \delta_K s_i. \tag{4.2.9}$$

Claim: if $\delta_K < \lambda_{\min}(K) M_K^{-1}/3$, then

$$P(Z_i, r_i') \cap B_\Lambda(Z_i, s_i) \subset (P(Z_i, s_i') \cap B_\Lambda(Z_i, s_i); \sigma_K \delta_K (s_i + 2r_i)), \tag{4.2.10}$$

where $\sigma_K > 0$ depends only on K and Λ.

Proof of the claim. The proof of this is analogue to the one in [13] for round balls. Given $Y \in P(Z_i, r_i') \cap B_\Lambda(Z_i, s_i)$, write $Y = Z_i + \Lambda(Z_i) W$, where $|W| < s_i$. Consider

$$\overline{Y} = Z_i + \Lambda(Z_i) \left(\left[1 - \frac{M_K \delta_K r_i'}{\lambda_{\min}(K) s_i'} \right] W \right).$$

Using that $r_i'/s_i' \leq 3$ and our assumption on δ_K, we see that for all i

$$1 - \frac{M_K \delta_K r_i'}{\lambda_{\min}(K) s_i'} > 0. \tag{4.2.11}$$

Next, since $(1 - \frac{M_K \delta_K r_i'}{\lambda_{\min}(K) s_i'})|W| < |W| < s_i$, we have

$$\overline{Y} \in B_\Lambda(Z_i, s_i) \subset B_\Lambda(Z_i, r_i),$$

and

$$|\Lambda(Z_i)^{-1}(\overline{Y} - Y)| = \frac{M_K \delta_K r_i'}{\lambda_{\min}(K) s_i'}|W| \leq \frac{M_K \delta_K r_i}{\lambda_{\min}(K) s_i}|W| < \lambda_{\min}(K)^{-1} M_K \delta_K r_i. \quad (4.2.12)$$

Moreover, $Y \in P(Z_i, r_i')$ implies that $\overline{Y} \in P(Z_i, r_i')$ as well, by construction. Combining this with (4.2.12) and recalling that $\lambda_{\min}(K)^{-1} M_K \delta_K < 1/3$, we see that

$$\overline{Y} \in P(Z_i, r_i') \cap B_\Lambda(Z_i, r_i).$$

Thus we can apply (4.2.8) to obtain a point $Z \in \Sigma \cap B_\Lambda(Z_i, r_i)$ such that

$$|Z - \overline{Y}| \leq M_K \delta_K r_i. \quad (4.2.13)$$

Using (4.2.13) and the definition of $\overline{Y}$,

$$\begin{aligned} |\Lambda(Z_i)^{-1}(Z - Z_i)| &\leq |\Lambda(Z_i)^{-1}(Z - \overline{Y})| + |\Lambda(Z_i)^{-1}(\overline{Y} - Z_i)| \\ &\leq \lambda_{\min}(K)^{-1}|Z - \overline{Y}| + s_i - \lambda_{\min}(K)^{-1} M_K \delta_K r_i \\ &\leq \lambda_{\min}(K)^{-1} M_K \delta_K r_i + s_i - \lambda_{\min}(K)^{-1} M_K \delta_K r_i = s_i, \end{aligned}$$

so $Z \in B_\Lambda(Z_i, s_i)$. But we also know $Z \in \Sigma$, so $Z \in \Sigma \cap B_\Lambda(Z_i, s_i)$. Therefore, by (4.2.9) there exists $Y' \in P(Z_i, s_i') \cap B_\Lambda(Z_i, s_i)$ such that

$$|Y' - Z| \leq M_K \delta_K s_i. \quad (4.2.14)$$

Combining (4.2.12), (4.2.13) and (4.2.14), we obtain

$$\begin{aligned} |Y - Y'| &\leq |Y - \overline{Y}| + |\overline{Y} - Z| + |Z - Y'| \\ &\leq |\Lambda(Z_i)\Lambda(Z_i)^{-1}(\overline{Y} - Y)| + M_K \delta_K r_i + M_K \delta_K s_i \quad (4.2.15) \\ &\leq e_\Lambda(K) M_K \delta_K r_i + M_K \delta_K r_i + M_K \delta_K s_i \leq \sigma_K \delta_K (s_i + 2r_i), \end{aligned}$$

where $\sigma_K = M_K \max\{e_\Lambda(K), 1\}$. This completes the proof of the claim. $\square$

As a next step, we want to unravel (4.2.5) into estimates that capture how closely Σ can be approximated by an affine copy of Σ_∞ near Q. Let $\varepsilon > 0$. Equation (4.2.5) guarantees that if i is large enough depending on ε and K, then

$$D[\Sigma_\infty \cap B(X,1), T^\Lambda_{Q_i,\rho_i}(\Sigma) \cap B(X,1)] \le \varepsilon. \tag{4.2.16}$$

We will use this estimate to obtain inclusions in two directions.

1. On one hand, (4.2.16) implies

$$T^\Lambda_{Q_i,\rho_i}(\Sigma) \cap B(X_i, 1-|X-X_i|) \subset T^\Lambda_{Q_i,\rho_i}(\Sigma) \cap B(X,1) \subset (\Sigma_\infty \cap B(X,1);\varepsilon).$$

Applying $(T^\Lambda_{Q_i,\rho_i})^{-1}(\cdot) = Q_i + \rho_i\Lambda(Q_i)(\cdot)$, we get

$$\begin{aligned} \Sigma \cap [Q_i + \rho_i\Lambda(Q_i)B(X_i, 1-|X-X_i|)] &\subset ((T^\Lambda_{Q_i,\rho_i})^{-1}[\Sigma_\infty \cap B(X,1)]; \lambda_{\max}(Q_i)\rho_i\varepsilon) \\ &\subset ((T^\Lambda_{Q_i,\rho_i})^{-1}[\Sigma_\infty \cap B(X,1)]; \lambda_{\max}(K)\rho_i\varepsilon). \end{aligned} \tag{4.2.17}$$

We would like to adjust the left hand side in a way that it looks like the intersection of Σ with a suitable ellipse. We proceed as follows,

$$\begin{aligned} B_\Lambda(Z_i,s_i) &= Z_i + \rho_i\Lambda(Z_i)B(0,1-|X-X_i|) \\ &\subset Z_i + \rho_i\Lambda(Q_i)B(0,1-|X-X_i|) + \rho_i(\Lambda(Z_i)-\Lambda(Q_i))B(0,1-|X-X_i|) \\ &\subset Q_i + \rho_i\Lambda(Q_i)X_i + \rho_i\Lambda(Q_i)B(0,1-|X-X_i|) + B(0,\varepsilon\rho_i(1-|X-X_i|)) \\ &= Q_i + \rho_i\Lambda(Q_i)B(X_i,1-|X-X_i|) + B(0,\varepsilon\rho_i(1-|X-X_i|)) \\ &\subset (Q_i + \rho_i\Lambda(Q_i)B(X_i,1-|X-X_i|); \varepsilon\rho_i(1-|X-X_i|)), \end{aligned}$$

where the third line holds for all i large enough, depending on K and Λ, by continuity of Λ. Combining this with (4.2.17) and recalling that $\rho_i(1-|X-X_i|) = s_i$, we get

$$\begin{aligned} \Sigma \cap B_\Lambda(Z_i,s_i) &\subset (\Sigma \cap [Q_i + \rho_i\Lambda(Q_i)B(X_i,1-|X-X_i|)]; \varepsilon s_i) \\ &\subset ((T^\Lambda_{Q_i,\rho_i})^{-1}[\Sigma_\infty \cap B(X,1)]; \varepsilon s_i + \lambda_{\max}(K)\rho_i\varepsilon). \end{aligned} \tag{4.2.18}$$

2. The other inclusion we can extract from (4.2.16) is

$$\Sigma_\infty \cap B(X,1) \subset (T^\Lambda_{Q_i,\rho_i}(\Sigma) \cap B(X,1); \varepsilon) \subset (T^\Lambda_{Q_i,\rho_i}(\Sigma) \cap B(X_i, 1+|X-X_i|); \varepsilon).$$

Applying $(T^\Lambda_{Q_i,\rho_i})^{-1}(\cdot)$ as before, this inclusion gives

$$(T^\Lambda_{Q_i,\rho_i})^{-1}[\Sigma_\infty \cap B(X,1)] \subset (\Sigma \cap \{Q_i + \rho_i \Lambda(Q_i) B(X_i, 1+|X-X_i|)\}\,; \lambda_{\max}(K)\rho_i\varepsilon). \tag{4.2.19}$$

Proceeding similarly as above, we can make the right hand side look like the intersection of Σ with a suitable ellipse. Namely, if i is large enough depending on K and Λ,

$$\begin{aligned} Q_i+&\rho_i\Lambda(Q_i)(B(X_i, 1+|X-X_i|)) \\ &= Q_i + \rho_i\Lambda(Q_i)(X_i + B(0, 1+|X-X_i|)) \\ &= Z_i + \rho_i\Lambda(Q_i)B(0,1+|X-X_i|) \\ &\subset Z_i + \rho_i\Lambda(Z_i)B(0,1+|X-X_i|) + \rho_i(\Lambda(Q_i)-\Lambda(Z_i))B(0,1+|X-X_i|) \\ &= B_\Lambda(Z_i, r_i) + \rho_i(\Lambda(Q_i)-\Lambda(Z_i))B(0,1+|X-X_i|) \subset (B_\Lambda(Z_i,r_i); \varepsilon r_i). \end{aligned}$$

This and (4.2.19) give

$$(T^\Lambda_{Q_i,\rho_i})^{-1}[\Sigma_\infty \cap B(X,1)] \subset (\Sigma \cap B_\Lambda(Z_i, r_i); \lambda_{\max}(K)\rho_i\varepsilon + \varepsilon r_i). \tag{4.2.20}$$

We would now like to use (4.2.18) and (4.2.20) along with assumption (4.2.2) to show that Σ_∞ must be a plane. Recall the planes $P(Z_i, r_i')$, $P(Z_i, s_i')$ from (4.2.8) and (4.2.9). On one hand, using (4.2.10), (4.2.9) and (4.2.18), and keeping in mind that $\rho_i - s_i = \rho_i|X-X_i|$, $s_i \le \rho_i$ and $r_i \le 2\rho_i$, we see that

$$\begin{aligned} P(Z_i, r_i') \cap B_\Lambda(Z_i, \rho_i) &\subset (P(Z_i, r_i') \cap B_\Lambda(Z_i, s_i); \lambda_{\max}(K)\rho_i|X-X_i|) \\ &\subset (P(Z_i, s_i') \cap B_\Lambda(Z_i, s_i); C_K\rho_i|X-X_i| + \sigma_K\delta_K(s_i + 2r_i)) \\ &\subset (\Sigma \cap B_\Lambda(Z_i, s_i); C_K(\rho_i|X-X_i| + s_i + 2r_i) + M_K\delta_K s_i) \\ &\subset ((T^\Lambda_{q_i,\rho_i})^{-1}\left[\Sigma_\infty \cap B(X,1)\right]; C_K(\rho_i|X-X_i| + s_i + 2r_i + \delta_K\rho_i) + \varepsilon(s_i + \lambda_{\max}(\\ &\subset ((T^\Lambda_{q_i,\rho_i})^{-1}\left[\Sigma_\infty \cap B(X,1)\right]; C_K\rho_i(|X-X_i| + \delta_K + \varepsilon)). \end{aligned} \tag{4.2.21}$$

Similarly, from (4.2.20), (4.2.8) and (4.2.10) we get

$$
\begin{aligned}
(T^{\Lambda}_{Q_i,\rho_i})^{-1}[\Sigma_\infty \cap B(X,1)] &\subset (\Sigma \cap B_\Lambda(Z_i,r_i);\varepsilon(\lambda_{\max}(K)\rho_i + r_i))\\
&\subset (P(Z_i,r_i') \cap B_\Lambda(Z_i,r_i);\varepsilon(C_K\rho_i + r_i) + M_K\delta_K r_i)\\
&\subset (P(Z_i,r_i') \cap B_\Lambda(Z_i,\rho_i);\varepsilon(C_K\rho_i + r_i) + C_K\delta_K\rho_i + \lambda_{\max}(K)\rho_i|X - X_i|)\\
&\subset (P(Z_i,r_i') \cap B_\Lambda(Z_i,\rho_i);C_K\rho_i(|X - X_i| + \delta_K + \varepsilon)).
\end{aligned}
\tag{4.2.22}
$$

We now want to apply $T^{\Lambda}_{Q_i,\rho_i}(\cdot)$ on (4.2.21) and (4.2.22). Notice that

$$
\begin{aligned}
T^{\Lambda}_{Q_i,\rho_i}(B_\Lambda(Z_i,\rho_i)) &= \Lambda(Q_i)^{-1}\left(\frac{B_\Lambda(Z_i,\rho_i) - Q_i}{\rho_i}\right)\\
&= \Lambda(Q_i)^{-1}\left(\frac{Z_i - Q_i}{\rho_i}\right) + \Lambda(Q_i)^{-1}\Lambda(Z_i)B(0,1)\\
&= X_i + \Lambda(Q_i)^{-1}\Lambda(Z_i)B(0,1).
\end{aligned}
\tag{4.2.23}
$$

We would like to compare this set with $B(X_i,1)$. Notice that by continuity of Λ and because $|Z_i - Q_i| \to 0$, if i is large enough depending on K and Λ, we have

$$
\|\Lambda(Q_i)^{-1}\Lambda(Z_i) - \mathrm{Id}\| \le \varepsilon, \quad \|\Lambda(Z_i)^{-1}\Lambda(Q_i) - \mathrm{Id}\| \le \varepsilon.
$$

We claim that this implies

$$
B(0,1-\varepsilon) \subset \Lambda(Q_i)^{-1}\Lambda(Z_i)B(0,1) \subset B(0,1+\varepsilon). \tag{4.2.24}
$$

To see this, note that on one hand,

$$
\begin{aligned}
\Lambda(Q_i)^{-1}\Lambda(Z_i)B(0,1) &\subset (\Lambda(Q_i)^{-1}\Lambda(Z_i) - I)B(0,1) + B(0,1)\\
&\subset B(0,\varepsilon) + B(0,1) = B(0,1+\varepsilon).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
B(0,1-\varepsilon) &\subset (I-\Lambda(Q_i)^{-1}\Lambda(Z_i))B(0,1-\varepsilon)+\Lambda(Q_i)^{-1}\Lambda(Z_i)B(0,1-\varepsilon) \\
&\subset B(0,\varepsilon(1-\varepsilon))+\Lambda(Q_i)^{-1}\Lambda(Z_i)B(0,1-\varepsilon) \\
&\subset \Lambda(Q_i)^{-1}\Lambda(Z_i)\left[(\Lambda(Z_i)^{-1}\Lambda(Q_i)-I)B(0,\varepsilon(1-\varepsilon))+B(0,\varepsilon(1-\varepsilon))\right] \\
&\qquad\qquad\qquad\qquad +\Lambda(Q_i)^{-1}\Lambda(Z_i)B(0,1-\varepsilon) \\
&\subset \Lambda(Q_i)^{-1}\Lambda(Z_i)\left[B(0,\varepsilon^2)+B(0,\varepsilon(1-\varepsilon))\right]+\Lambda(Q_i)^{-1}\Lambda(Z_i)B(0,1-\varepsilon) \\
&\subset \Lambda(Q_i)^{-1}\Lambda(Z_i)B(0,1).
\end{aligned}
$$

These inclusions prove (4.2.24).

Now, combining (4.2.23) and (4.2.24) we obtain

$$
B(X_i,1-\varepsilon)\subset T^{\Lambda}_{Q_i,\rho_i}(B_\Lambda(Z_i,\rho_i))\subset B(X_i,1+\varepsilon). \tag{4.2.25}
$$

Denote by P_i the plane $T^{\Lambda}_{Q_i,\rho_i}(P(Z_i,r'_i))$, and notice that $X_i\in P_i$. Applying $T^{\Lambda}_{Q_i,\rho_i}$ on (4.2.21) and (4.2.22), we obtain

$$
P_i\cap T^{\Lambda}_{Q_i,\rho_i}(B_\Lambda(Z_i,\rho_i))\subset(\Sigma_\infty\cap B(X,1);C_K(|X-X_i|+\delta_K+\varepsilon)),
$$

$$
\Sigma_\infty\cap B(X,1)\subset(P_i\cap T^{\Lambda}_{Q_i,\rho_i}(B_\Lambda(Z_i,\rho_i));C_K(|X-X_i|+\delta_K+\varepsilon)).
$$

Taking now (4.2.25) into account, the last two inclusions above give, respectively,

$$
\begin{aligned}
P_i\cap B(X_i,1)\subset(P_i\cap B(X_i,1-\varepsilon);\varepsilon)&\subset(P_i\cap T^{\Lambda}_{Q_i,\rho_i}(B_\Lambda(Z_i,\rho_i));\varepsilon) \\
&\subset(\Sigma_\infty\cap B(X,1);C_K(|X-X_i|+\delta_K+\varepsilon)),
\end{aligned}
$$

and

$$
\begin{aligned}
\Sigma_\infty\cap B(X,1)&\subset(P_i\cap T^{\Lambda}_{Q_i,\rho_i}(B_\Lambda(Z_i,\rho_i));C_K(|X-X_i|+\delta_K+\varepsilon)) \\
&\subset(P_i\cap B(X_i,1+\varepsilon);C_K(|X-X_i|+\delta_K+\varepsilon)) \\
&\subset(P_i\cap B(X_i,1);C_K(|X-X_i|+\delta_K+\varepsilon)).
\end{aligned}
$$

These inclusions show that

$$
D[\Sigma_\infty\cap B(X,1);P_i\cap B(X_i,1)]\le C_K(|X-X_i|+\delta_K+\varepsilon). \tag{4.2.26}
$$

To conclude, we want to replace X_i with X in this estimate and use it to rule out the case in which Σ_∞ is a cone. Let $P_i' = P_i - X_i + X$. Since $X_i \in P_i$, we have $X \in P_i'$. Also, note that

$$D[P_i' \cap B(X,1); P_i \cap B(X_i,1)] \leq |X - X_i|.$$

Combining this with (4.2.26), we get

$$D[\Sigma_\infty \cap B(X,1); P_i' \cap B(X,1)] \leq C_K(|X - X_i| + \delta_K + \varepsilon). \tag{4.2.27}$$

By compactness of the space of n-planes through X in $\mathbb{R}^{n+1}$, we can assume upon passing to a subsequence, that there is an n-plane P_X through X such that $P_i' \to P_X$ with respect to D, uniformly on compact sets. By (4.2.27), P_X satisfies

$$D[\Sigma_\infty \cap B(X,1); P_X \cap B(X,1)] \leq C_K(\delta_K + \varepsilon). \tag{4.2.28}$$

Now, by Theorem 2.1.2, if Σ_∞ is not an n-plane, then $n \geq 3$ and there exist $X_\infty \in \Sigma_\infty$ and a rotation $\mathcal{R}$ of $\mathbb{R}^{n+1}$ such that $\mathcal{R}(\Sigma_\infty - X_\infty)$ is the light cone

$$\mathcal{C} = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : x_4^2 = x_1^2 + x_2^2 + x_3^2\}.$$

In such scenario, applying (4.2.28) with $X = X_\infty$ and denoting by L the plane $\mathcal{R}(P_{X_\infty} - X_\infty)$, we get

$$D[\mathcal{C} \cap B(0,1); L \cap B(0,1)] = D[\Sigma_\infty \cap B(X_\infty,1); P_{X_\infty} \cap B(X_\infty,1)] \leq C_K(\delta_K + \varepsilon).$$

Notice that since P_{X_∞} contains X_∞, L must contain the origin. Then if $\delta_K < C_K^{-1}/\sqrt{2}$ and ε is small enough,

$$D[\mathcal{C} \cap B(0,1); L \cap B(0,1)] < \frac{1}{\sqrt{2}}. \tag{4.2.29}$$

However, a quick calculation shows that this inequality fails for every plane L through the origin. It follows that Σ_∞ must be an n-plane. Moreover, since $\Sigma_\infty = \operatorname{spt}(\nu)$ and

ν is a Λ-pseudo tangent measure of μ, we have $0 \in \Sigma_\infty$ by Remark 4.1.2. So we can use (4.2.16) with $X = 0$ to get

$$D[\Sigma_\infty \cap B(0,1); T^{\Lambda}_{Q_i,\rho_i}(\Sigma) \cap B(0,1)] \leq \varepsilon.$$

Applying $T^{\Lambda}_{Q_i,\rho_i}(\cdot) = Q_i + \rho_i \Lambda(Q_i)(\cdot)$, we obtain

$$D[\Sigma^{(i)}_\infty \cap B_\Lambda(Q_i,\rho_i); \Sigma \cap B_\Lambda(Q_i,\rho_i)] \leq \lambda_{\max}(K)\rho_i\varepsilon, \tag{4.2.30}$$

where $\Sigma^{(i)}_\infty = T^{\Lambda}_{Q_i,\rho_i}(\Sigma_\infty)$. Notice that $\Sigma^{(i)}_\infty$ is an n-plane containing Q_i. Now recall that $\rho_i = \lambda_{\min}(K)^{-1}\tau_i$, so if we combine (4.2.30) with Corollary 2.3.1, we get

$$\begin{aligned} D[\Sigma^{(i)}_\infty \cap B(Q_i,\tau_i); \Sigma \cap B(Q_i,\tau_i)] &\leq 2\lambda_{\max}(K)\rho_i\varepsilon \\ &= 2e_\Lambda(K)\tau_i\varepsilon. \end{aligned}$$

Combining this with (4.2.3), we deduce that

$$\lim_{\tau\to 0} b\beta_\Sigma(K,\tau) = \lim_{i\to\infty} b\beta_\Sigma(Q_i,\tau_i) \leq \limsup_{i\to\infty} \frac{1}{\tau_i} D[\Sigma^{(i)}_\infty \cap B(Q_i,\tau_i); \Sigma \cap B(Q_i,\tau_i)] \leq 2e_\Lambda(K)\varepsilon.$$

Since this holds for every $\varepsilon > 0$, we conclude that

$$\lim_{\tau\to 0} b\beta_\Sigma(K,\tau) = 0,$$

completing the proof of Proposition 4.2.1. $\square$

Chapter 5

Doubling measures

5.1 Proof of Theorem 1.3.2

We recall the statement of Theorem 1.3.2 here for convenience.

Theorem 5.1.1. *Let $\alpha, \beta \in (0,1)$. Suppose $X \mapsto \Lambda(X)$ is locally Hölder continuous on $\mathbb{R}^{n+1}$ with exponent β. Let μ be a Radon measure on $\mathbb{R}^{n+1}$ with support Σ such that the following holds: For every compact set $K \subset \mathbb{R}^{n+1}$ with $\Sigma \cap K \neq \varnothing$ there exists a constant $C_K > 0$ such that for every $r \in (0,1]$,*

$$\sup_{\substack{X \in \Sigma \cap K \\ t \in [\frac{1}{2}, 1]}} \left| \frac{\mu(B_\Lambda(X, tr))}{\mu(B_\Lambda(X, r))} - t^n \right| \leq C_K r^\alpha.$$

If $n \geq 3$, suppose additionally that Σ satisfies (1.3.3). Then Σ is a $C^{1,\gamma}$ n-dimensional submanifold of $\mathbb{R}^{n+1}$, for some $\gamma \in (0,1)$ depending on α and β.

The key idea of the proof is that (1.3.5) can be used to obtain information about the density $\Theta_\Lambda(\mu, X)$ introduced in (1.3.2). More specifically, the assumptions of Theorem 1.3.2 imply that (1.3.4) holds when μ is replaced with a certain measure which has the same support as μ and α is replaced with a number that depends on α and β, making Theorem 1.3.1 applicable. These ideas are contained in the following lemma.

Lemma 5.1.1. *Let Λ and μ be as in the assumptions of Theorem 1.3.2. Then*

$$0 < \Theta_\Lambda(\mu, X) < \infty, \tag{5.1.1}$$

for every $X \in \Sigma = \operatorname{spt}(\mu)$. Also, for every compact set $K \subset \mathbb{R}^{n+1}$ there exists a constant $C_K > 0$ depending on K and Λ, such that

$$|\log \Theta_\Lambda(X) - \log \Theta_\Lambda(Y)| \leq C_K |X - Y|^{\frac{\gamma}{1+\alpha}}, \tag{5.1.2}$$

whenever $X, Y \in \Sigma \cap K$ and $|X - Y| \leq \Delta_K$, where $\Delta_K > 0$ is small enough depending on K and Λ, and $\gamma = \min\{\alpha, \beta\}$. Moreover, the measure

$$\mathrm{d}\mu_0(X) = \frac{1}{\Theta_\Lambda(\mu, X)} \mathrm{d}\mu(X)$$

is a Radon measure with $\operatorname{spt}(\mu_0) = \Sigma$, with the property that for every compact set $K \subset \mathbb{R}^{n+1}$ there exist $r_K > 0$ and $C_K > 0$ such that for every $X \in K \cap \Sigma$ and $r \in (0, r_K]$,

$$\left| \frac{\mu_0(B_\Lambda(X, r))}{\omega_n r^n} - 1 \right| \leq C_K r^{\gamma'}, \tag{5.1.3}$$

where $\gamma' = \frac{\min\{\alpha,\beta\}}{1+\alpha}$.

Proof. The proof is similar to that of [7, Proposition 6.1]. Let K be as in the statement and let $X \in \Sigma \cap K$. For $r_k = 2^{-k}$, $k \geq 0$, let

$$D_k(X) = \frac{\mu(B_\Lambda(X, r_k))}{\omega_n r_k^n}, \quad l_k = \log D_k(X),$$

and for any $t \in [\frac{1}{2}, 1]$, let

$$R_t(X, r) = \frac{\mu(B_\Lambda(X, tr))}{\mu(B_\Lambda(X, r))} - t^n.$$

Notice that

$$\frac{D_{k+1}(X)}{D_k(X)} - 1 = \frac{2^n \mu(B_\Lambda(X, r_{k+1}))}{\mu(B_\Lambda(X, r_k))} - 1 = 2^n R_{1/2}(X, r_k).$$

By (1.3.5), we have

$$2^n R_{1/2}(X, r_k) \leq C_K 2^{-k\alpha}. \tag{5.1.4}$$

Thus, if k_0 is large enough and $k \geq k_0$,

$$|l_{k+1} - l_k| = \left|\log \frac{D_{k+1}(X)}{D_k(X)}\right| \leq C_K 2^{-k\alpha}. \tag{5.1.5}$$

This implies that the sequence $\{l_k\}$ is Cauchy, so $l_\infty := \lim_{k\to\infty} l_k$ exists and is finite, and we have

$$\lim_{k\to\infty} D_k(X) = e^{l_\infty}. \tag{5.1.6}$$

It also follows from (5.1.5) that if k_0 is large enough,

$$|l_k - l_\infty| \leq C_K 2^{-k\alpha}. \tag{5.1.7}$$

We will show that

$$\Theta_\Lambda(\mu, X) = e^{l_\infty}. \tag{5.1.8}$$

Let $r \in (0,1)$, and write $r = tr_k$ for some $t \in [\frac{1}{2}, 1]$ and some $k \geq 0$. Then

$$\begin{aligned} \frac{\mu(B_\Lambda(X,r))}{\omega_n r^n} &= \frac{\mu(B_\Lambda(X,tr_k))}{\omega_n t^n r_k^n} \\ &= \frac{\mu(B_\Lambda(X,tr_k))}{t^n \mu(B_\Lambda(X,r_k))} D_k(X) = t^{-n}(R_t(X,r_k) + t^n) D_k(X). \end{aligned} \tag{5.1.9}$$

Letting $r \to 0$, we have $r_k \to 0$, $R_t(X,r_k) \to 0$ by (1.3.5), and $D_k(X) \to e^{l_\infty}$ by (5.1.6). Thus, (5.1.9) yields (5.1.8), and in particular

$$0 < \Theta_\Lambda(\mu, X) < \infty.$$

We will now prove (5.1.2). Let us denote $\delta = \log(1 + t^{-n} R_t(X, r_k))$. By (1.3.5), and keeping in mind that $t \geq 1/2$ and $r_k \leq 2r$, if r is small enough depending on K and Λ, we have

$$|\delta| \leq \log(1 + t^{-n} C_K r k^\alpha) \leq C_K r^\alpha. \tag{5.1.10}$$

Notice that if r is small enough, then by (5.1.7), (5.1.9) and (5.1.10),

$$\begin{aligned} \left|\log \frac{\mu(B_\Lambda(X,r))}{\omega_n r^n} - \log \Theta_\Lambda(\mu, X)\right| &\leq |\delta| + |\log D_k(X) - \log \Theta_\Lambda(\mu, X)| \\ &= |\delta| + |l_k - l_\infty| \leq C_K r^\alpha, \end{aligned} \tag{5.1.11}$$

where we have used that $2^{-k\alpha} = r_k^\alpha \leq 2^\alpha r^\alpha$. We will show that (5.1.2) holds when $|X-Y|$ is small, depending on K and Λ. Suppose $|X-Y|^{\frac{1}{1+\alpha}} < r_{k_0}$, with k_0 as in (5.1.5) and (5.1.7). Let $k \geq k_0$ be such that

$$r_{k+1} \leq |X-Y|^{\frac{1}{1+\alpha}} < r_k.$$

Choosing k_0 large enough depending on K and Λ, we have

$$|X-Y| \leq r_k^{1+\alpha} \leq \lambda_{\min}(K)\frac{r_k}{2},$$

where $\lambda_{\min}(K)$ is as in (2.3.3). In particular, we can apply Lemma 2.2.2 to ensure that

$$B_\Lambda(Y, r_k - \lambda_{\min}(X)^{-1}|X-Y| - C_K r_k^{1+\beta}) \subset B_\Lambda(X, r_k).$$

Now, using again that $|X-Y| < r_k^{1+\alpha}$, setting $\gamma = \min\{\alpha,\beta\}$ we obtain

$$\begin{aligned} r_k - \lambda_{\min}(X)^{-1}|X-Y| - C_K r_k^{1+\beta} &\geq r_k - C_K(r_k^{1+\alpha} + r_k^{1+\beta}) \geq r_k(1 - C_K r_k^\gamma) \\ &= r_k(1 - C_K r_{k+1}^\gamma) \geq r_k(1 - C_K|X-Y|^{\frac{\gamma}{1+\alpha}}). \end{aligned} \tag{5.1.12}$$

Denoting $\rho = r_k(1-C_K|X-Y|^{\frac{\gamma}{1+\alpha}})$, we see that (5.1.12) implies $B_\Lambda(Y,\rho) \subset B_\Lambda(X, r_k)$, and thus

$$\frac{\mu(B_\Lambda(Y,\rho))}{\omega_n \rho^n} \leq \frac{\mu(B_\Lambda(X,r_k))}{\omega_n \rho^n} = \frac{r_k^n}{\rho^n} D_k(X).$$

Therefore,

$$\begin{aligned} \log\left(\frac{\mu(B_\Lambda(Y,\rho))}{\omega_n\rho^n}\right) &\leq n\log\frac{r_k}{\rho} + l_k \leq n\log\frac{r_k}{\rho} + l_\infty + C_K 2^{-k\alpha} \\ &= l_\infty + C_K r_k^\alpha - n\log\frac{\rho}{r_k} \\ &\leq l_\infty + C_K|X-Y|^{\frac{\alpha}{1+\alpha}} - n\log\left(1 - C_K|X-Y|^{\frac{\gamma}{1+\alpha}}\right) \\ &\leq l_\infty + C_K|X-Y|^{\frac{\alpha}{1+\alpha}} + C_K|X-Y|^{\frac{\gamma}{1+\alpha}} \leq l_\infty + C_K|X-Y|^{\frac{\gamma}{1+\alpha}}. \end{aligned} \tag{5.1.13}$$

On the other hand, we know by (5.1.11) that

$$\left|\log \frac{\mu(B_\Lambda(Y,\rho))}{\omega_n \rho^n} - \log \Theta_\Lambda(\mu, Y)\right| \leq C_K \rho^\alpha \leq C_K |X-Y|^{\frac{\alpha}{1+\alpha}}. \tag{5.1.14}$$

Thus, combining (5.1.8), (5.1.13) and (5.1.14) we obtain

$$\log \Theta_\Lambda(\mu, Y) \leq l_\infty + C_K |X-Y|^{\frac{\gamma}{1+\alpha}} = \log \Theta_\Lambda(\mu, X) + C_K |X-Y|^{\frac{\gamma}{1+\alpha}}.$$

An analog argument can be used to show that

$$\log \Theta_\Lambda(\mu, X) \leq \log \Theta_\Lambda(\mu, Y) + C_K |X-Y|^{\frac{\gamma}{1+\alpha}},$$

from which (5.1.2) follows.

Now we continue with the measure μ_0 defined in the statement of Lemma 5.1.1. From (5.1.1) and (5.1.2), it follows that $\Theta_\Lambda(\mu, \cdot)$ is locally bounded above and below by positive constants. This implies that μ_0 is a Radon measure with support $\mathrm{spt}(\mu_0) = \mathrm{spt}(\mu) = \Sigma$. We will show that (5.1.3) holds. Let $X \in K \cap \Sigma$, and suppose that $0 < r \leq \lambda_{\max}(K)^{-1} r_K$ and $r_K < 1$. Then every $Y \in B_\Lambda(X, r)$ satisfies $|X-Y| < r_K$ and $Y \in \Sigma \cap (K; 1)$, so applying (5.1.2) to the compact set $(K; 1)$,

$$\eta := \sup_{Y \in \Sigma \cap B_\Lambda(X,r)} |\log \Theta_\Lambda(\mu, X) - \log \Theta_\Lambda(\mu, Y)| \leq C_K r^{\frac{\gamma}{1+\alpha}}. \tag{5.1.15}$$

From the definition of η, it follows that for every $Y \in \Sigma \cap B_\Lambda(X, r)$,

$$e^{-\eta} \leq \frac{\Theta_\Lambda(\mu, Y)}{\Theta_\Lambda(\mu, X)} \leq e^{\eta}.$$

If we integrate this inequality with respect to $\mathrm{d}\mu_0(Y)$ over $B_\Lambda(X, r)$, we get

$$\mu_0(B_\Lambda(X,r)) e^{-\eta} \leq \frac{\mu(B_\Lambda(X,r))}{\Theta_\Lambda(\mu, X)} \leq \mu_0(B_\Lambda(X,r)) e^{\eta},$$

or equivalently,

$$e^{-\eta} \leq \frac{\Theta_\Lambda(\mu, X) \mu_0(B_\Lambda(X,r))}{\mu(B_\Lambda(X,r))} \leq e^{\eta}.$$

This implies that

$$\left|\log \frac{\mu_0(B_\Lambda(X,r))}{\omega_n r^n}\right| = \left|\log\left(\frac{\Theta_\Lambda(X,r)\mu_0(B_\Lambda(X,r))}{\mu(B_\Lambda(X,r))}\cdot\frac{\mu(B_\Lambda(X,r))}{\Theta_\Lambda(\mu,X)\omega_n r^n}\right)\right| \tag{5.1.16}$$

$$\leq \eta + \left|\log \frac{\mu(B_\Lambda(X,r))}{\omega_n r^n} - \log \Theta_\Lambda(\mu,X)\right|.$$

It then follows from (5.1.11), (5.1.15) and (5.1.16) that

$$\left|\log \frac{\mu_0(B_\Lambda(X,r))}{\omega_n r^n}\right| \leq C_K(r^{\frac{\gamma}{1+\alpha}} + r^\alpha) \leq C_K r^{\frac{\gamma}{1+\alpha}},$$

or equivalently

$$e^{-C_K r^{\gamma'}} \leq \frac{\mu_0(B_\Lambda(X,r))}{\omega_n r^n} \leq e^{C_K r^{\gamma'}},$$

where $\gamma' = \frac{\gamma}{1+\alpha}$. Thus, for all $r > 0$ small enough depending on K and Λ,

$$1 - C_K r^{\gamma'} \leq \frac{\mu_0(B_\Lambda(X,r))}{\omega_n r^n} \leq 1 + C_K r^{\gamma'}, \tag{5.1.17}$$

completing the proof of the lemma. □

We are now ready to prove Theorem 1.3.2.

Proof of Theorem 1.3.2. Let μ be as in the assumptions of the Theorem and μ_0 as in Lemma 5.1.1. Lemma 5.1.1 implies that $\Sigma = \mathrm{spt}(\mu) = \mathrm{spt}(\mu_0)$, and by (5.1.3), μ_0 satisfies the density condition (1.3.4) with α replaced by $\frac{\min\{\alpha,\beta\}}{1+\alpha}$. Thus, by Theorem 1.3.1, Σ is a $C^{1,\gamma}$ submanifold of dimension n of $\mathbb{R}^{n+1}$, where $\gamma \in (0,1)$ depends on α and β. □

5.2 Proof of theorem 1.3.3: the regular set

We define the regular set of $\Sigma = \mathrm{spt}(\mu)$ as

$$\mathcal{R} = \{X \in \Sigma : \limsup_{r \searrow 0} b\beta^n_\Sigma(X,r) = 0\}, \tag{5.2.1}$$

and the singular set as $\mathcal{S} = \Sigma \backslash \mathcal{R}$. In this section we study $\mathcal{R}$. The techniques used here are based on ideas developed in [26] in the Euclidean setting.

Proposition 5.2.1 (The regular set). *Under the assumptions of Theorem 1.3.3, the set $\mathcal{R}$ defined in* (5.2.1) *is a $C^{1,\gamma}$ submanifold of dimension n of $\mathbb{R}^{n+1}$. In particular, $\mathcal{R}$ is an open subset of Σ.*

5.2.1 Technical results and proof of Proposition 5.2.1

Suppose μ and Λ satisfy the assumptions of Theorem 1.3.3. By Lemma 5.1.1, we may assume without loss of generality that μ satisfies (5.1.3). Note for later use that if $X_0 \in \Sigma \cap K$, where $K \subset \mathbb{R}^{n+1}$ is compact, and $r > 0$ is small enough depending on K and Λ, then (5.1.3) implies

$$C_K^{-1} r^n \leq \mu(B(X_0, r)) \leq C_K r^n. \tag{5.2.2}$$

For example, the upper bound in (5.2.2) can be obtained by noting that $B(X_0, r) \subset B_\Lambda(X_0, \lambda_{\min}(K)^{-1} r)$, and applying (5.1.3) to $B_\Lambda(X_0, \lambda_{\min}(K)^{-1} r)$. The lower bound can be obtained similarly.

As in [26], we need a smooth version of the β_2-numbers of μ, which are in turn an L^2 version of the β-numbers considered in Section 3.2. Let $\varphi \in C_c^\infty(\mathbb{R}^{n+1})$ be a radially non-increasing function such that $\chi_{B(0,2)} \leq \varphi \leq \chi_{B(0,3)}$. For $X_0 \in \Sigma = \mathrm{spt}(\mu)$ and $B = B(X_0, r)$, let

$$\tilde{\beta}_{2,\mu}(B) = \tilde{\beta}_{2,\mu}(X_0, r) = \min_P \left(\frac{1}{r^{n+2}} \int \varphi\left(\frac{|X - X_0|}{r} \right) \mathrm{dist}(X, P)^2 \mathrm{d}\mu(X) \right)^{1/2}, \tag{5.2.3}$$

where the minimum is taken over all n-planes P in $\mathbb{R}^{n+1}$. Note that if P is a minimizing n-plane for $b\beta_\Sigma(X_0, 3r)$ and r is small, then by (5.2.2),

$$\begin{aligned} \tilde{\beta}_{2,\mu}(X_0, r) &\leq \min_{P'} \left(\frac{1}{r^{n+2}} D[\Sigma \cap B(X_0, 3r); P' \cap B(x_0, 3r)]^2 \mu(B(X_0, 3r)) \right)^{1/2} \\ &\leq \frac{C_K}{r} D[\Sigma \cap B(X_0, 3r); P \cap B(x_0, 3r)] = C_K b\beta_\Sigma(X_0, 3r), \end{aligned} \tag{5.2.4}$$

for some constant $C_K > 0$ depending only on K and Λ, where D denotes Hausdorff distance as before.

It is also convenient to observe that the coefficients $\tilde{\beta}_{2,\mu}$ enjoy some regularity: if $X_0, X_0' \in \Sigma \cap K$, $B = B(X_0, r)$, $B' = B(X_0', r')$, $B' \subset B \subset K$, and $r' \geq cr$, then there exists a constant $C_K > 0$ depending on c, K and Λ such that

$$\tilde{\beta}_{2,\mu}(B') \leq C_K \tilde{\beta}_{2,\mu}(B). \tag{5.2.5}$$

In the same spirit as Lemma 2.3.1, we need to establish a comparison between the quantity on the right-hand side of (5.2.3) and the corresponding quantity obtained when the term $|X - X_0|/r$ is replaced with an anisotropic rescaling determined by Λ. Recall the numbers $\lambda_{\max}(K)$ and $\lambda_{\min}(K)$ associated with any compact set K, introduced in (2.3.3).

Lemma 5.2.1. *Let $r > 0$ and suppose $K \subset \mathbb{R}^{n+1}$ is compact. Denote $r' = \lambda_{\max}(K)r$ and $r'' = \lambda_{\min}(K)r$, where Λ satisfies the assumptions of Theorem 1.3.3. If P is any n-plane in $\mathbb{R}^{n+1}$ and μ is a Radon measure in $\mathbb{R}^{n+1}$ with support Σ, then for every $X_0, Z \in \Sigma \cap K$,*

$$\int \varphi\left(\frac{|\Lambda(Z)^{-1}(X - X_0)|}{r}\right) \operatorname{dist}(X, P)^2 \mathrm{d}\mu(X) \leq \int \varphi\left(\frac{|X - X_0|}{r'}\right) \operatorname{dist}(X, P)^2 \mathrm{d}\mu(X), \tag{5.2.6}$$

$$\int \varphi\left(\frac{|X - X_0|}{r''}\right) \operatorname{dist}(X, P)^2 \mathrm{d}\mu(X) \leq \int \varphi\left(\frac{|\Lambda(Z)^{-1}(X - X_0)|}{r}\right) \operatorname{dist}(X, P)^2 \mathrm{d}\mu(X). \tag{5.2.7}$$

Remark 12. The statement remains true if $\Lambda(\cdot)$ is replaced with $\Lambda(\cdot)^{-1}$, as long as r' and r'' are adjusted accordingly. More specifically, since the smallest and largest eigenvalues of $\Lambda(\cdot)^{-1}$ are $\lambda_{\max}(\cdot)^{-1}$ and $\lambda_{\min}(\cdot)^{-1}$, respectively, the lemma applies with $\Lambda(\cdot)^{-1}$ in place of $\Lambda(\cdot)$ if the scales r' and r'' are taken to be $r' = \lambda_{\min}(K)^{-1}r$ and $r'' = \lambda_{\max}(K)^{-1}r$.

Proof of Lemma 5.2.1. With K, X_0 and Z as in the assumptions, we have for any $r > 0$,

$$\frac{1}{r}|\Lambda(Z)^{-1}(X - X_0)| \geq \frac{1}{r}|\lambda_{\max}(K)^{-1}(X - X_0)| = \frac{|X - X_0|}{r'},$$

so (5.2.6) follows because φ is radially non-increasing. Equation (5.2.7) follows for the same reason, by observing that

$$\frac{1}{r}|\Lambda(Z)^{-1}(X-X_0)| \leq \frac{1}{r}|\lambda_{\min}(K)^{-1}(X-X_0)| = \frac{|X-X_0|}{r''}.$$

$\square$

The proof of Theorem 1.3.3 relies on the following two results, which are analogues of Theorem 4.2 and Theorem 4.3 in [26]. It is worth noticing that even though we state both results in codimension 1, the statements remain true in any codimension. Recall the notion of a Λ-asymptotically optimally doubling measure (see Definition 4.1.2).

Theorem 5.2.1. *Let μ be a Λ-asymptotically optimally doubling measure of dimension n on $\mathbb{R}^{n+1}$ with support Σ. Let $K \subset \mathbb{R}^{n+1}$ be a compact set, and suppose that*

$$C_K^{-1} r^n \leq \mu(B(X,r)) \leq C_K r^n, \tag{5.2.8}$$

for $X \in \Sigma \cap K$, $0 < r \leq \operatorname{diam}(K)$. For any $\eta > 0$, there exists $\delta > 0$ depending only on η, n, μ, K and Λ such that if B is a ball contained in K and centered at $\Sigma \cap K$ with $\tilde{\beta}_{2,\mu}(B) \leq \delta$, then $\tilde{\beta}_{2,\mu}(B') \leq \eta$ for any ball $B' \subset B$ centered at $\Sigma \cap \frac{1}{2}B$.

Theorem 5.2.2. *Let μ be a Λ-asymptotically optimally doubling measure of dimension n on $\mathbb{R}^{n+1}$ with support Σ. Assume that $0 \in \Sigma$. Let $K \subset \mathbb{R}^{n+1}$ be a compact set such that $B(0,2) \subset K$, and suppose that (5.2.8) holds for $X \in \Sigma \cap K$, $0 < r \leq \operatorname{diam}(K)$. Given $\varepsilon > 0$, there exists $\delta \in (0, \varepsilon_0)$, depending only on ε, n, μ, K and Λ such that if $\tilde{\beta}_{2,\mu}(B) \leq \delta$ for every ball B contained in $B(0,2)$ and centered at $\Sigma \cap K$, then there exists $R > 0$ such that $b\beta_\Sigma(X,r) < \varepsilon$ for all $X \in \Sigma \cap B(0,1)$ and $r \in (0,R)$.*

We will use these results combined in the form of the following corollary.

Corollary 5.2.1. *Let μ be a Λ-asymptotically optimally doubling measure in $\mathbb{R}^{n+1}$ with support Σ. Let $K \subset \mathbb{R}^{n+1}$ be compact, and suppose that* (5.2.8) *holds for $X \in \Sigma \cap K$, $0 < r \leq \operatorname{diam}(K)$. Given $\varepsilon > 0$, there exists $\delta \in (0, \varepsilon_0)$ depending only on ε, n, μ, K and Λ such that if $\tilde{\beta}_{2,\mu}(B(X_0, 4R_0)) \leq \delta$, where $X_0 \in \Sigma$ and $B(X_0, 4R_0) \subset K$, then there exists $R > 0$ such that $b\beta_\Sigma(X, r) < \varepsilon$ for all $X \in \Sigma \cap B(X_0, R_0)$ and $r \in (0, R)$. In particular, $\Sigma \cap B(X_0, R_0)$ is ε-Reifenberg flat.*

Before proving Theorem 5.2.1 and Theorem 5.2.2, we use Corollary 5.2.1 to derive Proposition 5.2.1.

Proof of Proposition 5.2.1. Let μ be as in the assumptions of Theorem 1.3.3. As before, by Lemma 5.1.1 we may assume without loss of generality that μ satisfies (5.1.3), so that (5.2.2) holds. By Proposition 4.1.3, we know that μ is Λ-asymptotically optimally doubling. Let $X_0 \in \mathcal{R}$ and $\sigma > 0$. By definition of $\mathcal{R}$, there exists $R_0 > 0$ such that $b\beta_\Sigma(X_0, r) \leq \sigma$ whenever $0 < r \leq 12R_0$. Let $K = \overline{B(X_0, 4R_0)}$. By (5.2.4), we have

$$\tilde{\beta}_{2,\mu}(B(X_0, 4R_0)) \leq C_K \sigma, \tag{5.2.9}$$

where $C_K > 0$ depends only on K and Λ. Let us assume without loss of generality that R_0 is small enough so that (5.1.3) and (5.2.2) hold for every $X \in \Sigma \cap K$ and $r < 8R_0 = \operatorname{diam}(K)$, ensuring that the assumptions of Corollary 5.2.1 are satisfied.

Given any $\varepsilon > 0$, let $\delta \in (0, \varepsilon_0)$ be as in the conclusion of Corollary 5.2.1. If σ is small enough so that $C_K \sigma < \delta$, then by (5.2.9) we have $\tilde{\beta}_{2,\mu}(B(X_0, 4R_0)) < \delta$, and Corollary 5.2.1 implies that $\Sigma \cap B(X_0, R_0)$ is ε-Reifenberg flat. We can assume without loss of generality that $\varepsilon < \delta_K$, where $K = \overline{B(X_0, R_0)}$ and δ_K is as in Proposition 4.2.1. Then, by Proposition 4.2.1,

$$\lim_{r \searrow 0} b\beta_\Sigma(K, r) = 0.$$

This and (5.1.3) ensure that $\mu \llcorner B(X_0, R_0)$ satisfies the assumptions of Theorem 1.3.1. Therefore, $\Sigma \cap B(X_0, R_0)$ is a $C^{1,\gamma}$-submanifold of $\mathbb{R}^{n+1}$ of dimension n for some $\gamma \in (0,1)$ depending on α and β. This also implies that $\mathcal{R}$ is an open subset of Σ, as desired. $\square$

5.2.2 Proof of technical results

We now turn to the proofs of Theorem 5.2.1 and Theorem 5.2.2. The main ingredient is the following lemma, where for any ball $B = B(X, r)$ and any positive number $c > 0$, we denote $r(B) = r$ and $cB = B(X, cr)$.

Lemma 5.2.2. *Let μ be a Λ-asymptotically optimally doubling measure of dimension n on $\mathbb{R}^{n+1}$. Let $K \subset \mathbb{R}^{n+1}$ be compact, and let δ_0 be any positive constant. Suppose that* (5.2.8) *holds for $X \in \Sigma \cap K$, $0 < r \leq \operatorname{diam}(K)$. Then there exists some constant ε_1 depending on ε_0 and C_0, but not on δ_0, and there exists an integer $N > 0$ depending only on μ, K, Λ and δ_0, such that if B is a ball centered at Σ with $2^k B \subset K$ and*

$$\tilde{\beta}_{2,\mu}(2^k B) \leq \varepsilon_1, \quad k \in \{1, \ldots, N\}, \tag{5.2.10}$$

then

$$\tilde{\beta}_{2,\mu}(B) \leq \delta_0.$$

Proof of Lemma 5.2.2. Suppose, for contradiction, that such an N does not exist. Then there is a sequence of points $\{X_j\} \subset \Sigma \cap K$ and balls $B_j = B(X_j, r_j)$ such that $2^j B_j \subset K$ and

$$\tilde{\beta}_{2,\mu}(2^k B_j) \leq \varepsilon_1, \quad k \in \{1, \ldots, j\}, \tag{5.2.11}$$

but $\tilde{\beta}_{2,\mu}(B_j) > \delta_0$. Note that since K is bounded and $2^j B_j \subset K$, we have $r_j \to 0$ as $j \to \infty$. For each $j \geq 1$, let

$$\mu_j = \frac{1}{\mu(B_\Lambda(X_j, r_j))} T^{\Lambda}_{X_j, r_j}[\mu].$$

Upon taking a subsequence, we may assume without loss of generality that $\mu_j \rightharpoonup \nu$, where ν is a Λ-pseudo tangent of μ, which we know is n-uniform by Proposition 4.1.4.

We will show that

$$\tilde{\beta}_{2,\nu}(B(0, 2^k \lambda_{\max}(K)^{-1})) \leq C_K \varepsilon_1, \quad k \geq 1, \tag{5.2.12}$$

and

$$\tilde{\beta}_{2,\nu}(B(0, \lambda_{\min}(K)^{-1})) \geq C_K^{-1} \delta_0. \tag{5.2.13}$$

To prove (5.2.12), fix $k \geq 1$. Let L_j^* be a minimizing plane for $\tilde{\beta}_{2,\mu}(2^k B_j)$, and let

$$L_j = \frac{1}{r_j} \Lambda(X_j)^{-1}(L_j^* - X_j).$$

Upon taking a subsequence, we may assume that $L_j \to L$ with respect to D, uniformly on compact sets, where L is an n-plane. Note that this implies that $\operatorname{dist}(\cdot, L_j) \to \operatorname{dist}(\cdot, L)$ uniformly on compact subsets of $\mathbb{R}^{n+1}$. Combining this with the fact that the function φ in the definition of $\tilde{\beta}_{2,\cdot}$ is continuous, $|\varphi| \leq 1$ and $\mu_j \rightharpoonup \nu$, it follows that

$$\begin{aligned}
&\left| \int \varphi \left(\frac{|X|}{2^k \lambda_{\max}(K)^{-1}} \right) \operatorname{dist}(X, L_j)^2 \mathrm{d}\mu_j(X) - \int \varphi \left(\frac{|X|}{2^k \lambda_{\max}(K)^{-1}} \right) \operatorname{dist}(X, L)^2 \mathrm{d}\nu(X) \right| \\
&\quad \leq \mu_j(B(0, 3 \cdot 2^k \lambda_{\max}(K)^{-1})) \|\operatorname{dist}(\cdot, L_j)^2 - \operatorname{dist}(\cdot, L)^2\|_{L^\infty(B(0, 2^k \lambda_{\max}(K)^{-1}))} \\
&\quad + \left| \int \varphi \left(\frac{|X|}{2^k \lambda_{\max}(K)^{-1}} \right) \operatorname{dist}(X, L)^2 \mathrm{d}\mu_j(X) - \int \varphi \left(\frac{|X|}{2^k \lambda_{\max}(K)^{-1}} \right) \operatorname{dist}(X, L)^2 \mathrm{d}\nu(X) \right| \\
&\quad \to 0,
\end{aligned} \tag{5.2.14}$$

as $j \to \infty$. On the other hand, by Remark 12, an application of Lemma 5.2.1, equation (5.2.7) with $\Lambda(\cdot)^{-1}$ in place of $\Lambda(\cdot)$ gives

$$
\begin{aligned}
\frac{1}{2^{k(n+2)}} \int \varphi\left(\frac{|X|}{2^k \lambda_{\max}(K)^{-1}}\right) & \operatorname{dist}(X, L_j)^2 \mathrm{d}\mu_j(X) \\
&\leq \frac{1}{2^{k(n+2)}} \int \varphi\left(\frac{|\Lambda(X_j)X|}{2^k}\right) \operatorname{dist}\left(X, L_j\right)^2 \mathrm{d}\mu_j(X) \\
&\leq \frac{C_K}{2^{k(n+2)}\mu(B_\Lambda(X_j, r_j))} \int \varphi\left(\frac{|X - X_j|}{2^k r_j}\right) \operatorname{dist}\left(\frac{\Lambda(X_j)^{-1}(X - X_j)}{r_j}, L_j\right)^2 \mathrm{d}\mu(X).
\end{aligned}
\tag{5.2.15}
$$

By the definition of L_j,

$$
\begin{aligned}
\operatorname{dist}\left(\frac{\Lambda(X_j)^{-1}(X - X_j)}{r_j}, L_j\right) &= \operatorname{dist}\left(\frac{\Lambda(X_j)^{-1}(X - X_j)}{r_j}, \frac{\Lambda(X_j)^{-1}(L_j^* - X_j)}{r_j}\right) \\
&\leq \frac{C_K}{r_j} \operatorname{dist}(X, L_j^*).
\end{aligned}
$$

Combining this with (5.2.15) and (5.2.11) we obtain

$$
\begin{aligned}
\frac{1}{2^{k(n+2)}} \int \varphi\left(\frac{|X|}{2^k \lambda_{\max}(K)^{-1}}\right) & \operatorname{dist}(X, L_j)^2 \mathrm{d}\mu_j(X) \\
&\leq \frac{C_K}{2^{k(n+2)} r_j^{n+2}} \int \varphi\left(\frac{|X - X_j|}{2^k r_j}\right) \operatorname{dist}(X, L_j^*)^2 \mathrm{d}\mu(X) \\
&= C_K \tilde{\beta}_{2,\mu}(2^k B_j) \leq C_K \varepsilon_1.
\end{aligned}
\tag{5.2.16}
$$

This estimate and (5.2.14) with a choice of j large enough give

$$
\frac{1}{(2^k \lambda_{\max}(K)^{-1})^{n+2}} \int \varphi\left(\frac{|X|}{2^k \lambda_{\max}(K)^{-1}}\right) \operatorname{dist}(X, L)^2 \mathrm{d}\nu(X) \leq C_K \varepsilon_1,
$$

from which (5.2.12) follows.

To prove (5.2.13), let L be any n-plane. Using Lemma 5.2.1 applied to $\Lambda(\cdot)^{-1}$, along with the definition of μ_j,

$$\begin{aligned}
\tilde{\beta}_{2,\nu}(0,\lambda_{\min}(K)^{-1}) &\geq C_K \int \varphi\left(\frac{|X|}{\lambda_{\min}(K)^{-1}}\right)\mathrm{dist}(X,L)^2 \mathrm{d}\mu_j(X) \\
&\geq C_K \int \varphi(|\Lambda(X_j)X|)\mathrm{dist}(X,L)^2 \mathrm{d}\mu_j(X) \\
&= \frac{C_K}{\mu(B_\Lambda(X_j,r_j))} \int \varphi\left(\frac{|X-X_j|}{r_j}\right)\mathrm{dist}\left(\frac{\Lambda(X_j)^{-1}(X-X_j)}{r_j}, L\right)^2 \mathrm{d}\mu(X) \\
&\geq \frac{C_K}{r_j^{n+2}} \int \varphi\left(\frac{|X-X_j|}{r_j}\right)\mathrm{dist}(X, X_j + r_j\Lambda(X_j)L)^2 \mathrm{d}\mu(X) \\
&\geq C_K \tilde{\beta}_{2,\mu}(B_j) > C_K\delta_0,
\end{aligned}$$

by our assumption on $\tilde{\beta}_{2,\mu}(B_j)$. This proves (5.2.13).

We are now ready to complete the proof of the lemma. We claim that ε_1 is small enough, then (5.2.11) implies that the tangent measure $\tilde{\nu}$ at ∞ of ν satisfies

$$\min_P \int_{B(0,1)} \mathrm{dist}(X,P)^2 \mathrm{d}\tilde{\nu}(X) \leq \varepsilon_0^2, \tag{5.2.17}$$

where the minimum is taken over all n-planes $P \subset \mathbb{R}^{n+1}$. To show this, notice first that by arguments similar to those leading up to (5.2.16) and by definition of $\tilde{\nu}$, we have

$$\tilde{\beta}_{2,\tilde{\nu}}(0,3) \leq C_K \tilde{\beta}_{2,\nu}(0, 2^k\lambda_{\max}(K)^{-1}),$$

for k large. Also by the estimates leading up to (5.2.16), we have

$$\tilde{\beta}_{2,\nu}(0, 2^k\lambda_{\max}(K)^{-1}) \leq C_K \tilde{\beta}_{2,\mu}(2^k B_j) \leq C_K\varepsilon_1.$$

It follows that if j and $k \in \{1,\ldots,j\}$ are large enough, then $\tilde{\beta}_{2,\tilde{\nu}}(0,3) \leq C_K\varepsilon_1$, which gives (5.2.17) by choosing ε_1 small enough depending on K, Λ and ε_0, and observing that the left hand side of (5.2.17) is upper bounded by $\tilde{\beta}_{2,\tilde{\nu}}(0,3)$.

To conclude, we combine (5.2.17) with Theorem 2.1.4 to deduce that ν is flat, which contradicts (5.2.13), completing the proof of the lemma. $\square$

With this lemma in hand, we can prove Theorems 5.2.1 and 5.2.2 essentially in the same way as [26].

Proof of Theorem 5.2.1. Let $\eta > 0$, let ε_1 and N be as in Lemma 5.2.2, and set $\delta_0 = \min\{\varepsilon_1, \eta\}$. Let $\delta > 0$ be a small number to be determined, and suppose B is a ball of radius $r(B)$ contained in K and centered at $\Sigma \cap K$ with $\tilde{\beta}_{2,\mu}(B) \leq \delta$. If δ is small enough depending on ε_1, η and N, and B' is any ball contained in B, centered at $\Sigma \cap B$, with radius $r(B') \geq 2^{-N-1} r(B)$, then by (5.2.5),

$$\tilde{\beta}_{2,\mu}(B') \leq \min\{\varepsilon_1, \eta\}. \tag{5.2.18}$$

Let now B' be any ball centered at $\Sigma \cap \frac{1}{2}B$ with $2^{-N-2} r(B) \leq r(B') < 2^{-N-1} r(B)$. Then $2^N B'$ is centered at $\Sigma \cap \frac{1}{2}B$ and $r(2^N B') < r(B)/2$, so $2^N B' \subset B$ and $\tilde{\beta}_{2,\mu}(2^k B') \leq \varepsilon_1$ for every $k \in \{1, \ldots, N\}$. Therefore, we can apply Lemma 5.2.2 to B' and deduce that B' satisfies (5.2.18). From this and the arguments above it follows that if B' is any ball centered on $\Sigma \cap \frac{1}{2}B$ with radius $r(B') \geq 2^{-N-2} r(B)$, then B' satisfies (5.2.18). Iterating this procedure, we deduce that for any $j \geq 2$, if B' is a ball centered at $\Sigma \cap \frac{1}{2}B$ with $r(B') \geq 2^{-N-j}$, then B' satisfies (5.2.18), which completes the proof.

$\square$

Proof of Theorem 5.2.2. Suppose, for contradiction, that there exists $\varepsilon_1 > 0$ such that for each $i \geq i_0$ for some $i_0 \geq 1$, and for each ball $B \subset B(0,2)$ centered at $\Sigma \cap K$, we have

$$\tilde{\beta}_{2,\mu}(B) \leq 2^{-i} \leq \varepsilon_0,$$

but there are $X_i \in \Sigma \cap B(0,1)$ and $r_i \searrow 0$ such that $b\beta_\Sigma(X_i, r_i) \geq \varepsilon_1$. Write $r_i = \lambda_{\min}(K)\tau_i$. Fix $i \geq 1$ momentarily, and let P be a minimizing plane for $b\beta_{\Sigma_i}(0,1)$. Write

$$P = \frac{1}{\tau_i}\Lambda(X_i)^{-1}(\tilde{P} - X_i),$$

for some n-plane $\tilde{P}$. Consider $\Sigma_i = \frac{1}{\tau_i}\Lambda(X_i)^{-1}(\Sigma - X_i)$, as well as the measures

$$\mu_i = \frac{1}{\mu(B_\Lambda(X_i, \tau_i))} T^{\Lambda}_{X_i,\tau_i}[\mu].$$

Note that $\Sigma_i = \mathrm{spt}(\mu_i)$. By Corollary 2.3.1,

$$\begin{aligned} b\beta_{\Sigma_i}(0,1) &= D[\Sigma_i \cap B(0,1); P \cap B(0,1)] \\ &\geq \frac{C_K}{\tau_i} D[\Sigma \cap B_\Lambda(X_i,\tau_i); \tilde{P} \cap B_\Lambda(X_i,\tau_i)] \\ &\geq \frac{C_K}{r_i} D[\Sigma \cap B(X_i,r_i); \tilde{P} \cap B(X_i,r_i)] \geq C_K b\beta_\Sigma(X_i,r_i) \geq \varepsilon_1. \end{aligned} \tag{5.2.19}$$

Note that this estimate holds for every $i \geq 1$. We also know that upon passing to a subsequence, we have $\mu_i \rightharpoonup \nu$, where ν is an n-uniform Λ-pseudo tangent of μ, and $\Sigma_i \to \Sigma_\infty = \mathrm{spt}(\nu)$ with respect to D, uniformly on compact sets, as before. This, combined with (5.2.19) implies that

$$b\beta_{\Sigma_\infty}(0,1) \geq \varepsilon_1/2. \tag{5.2.20}$$

On the other hand, similarly as in (5.2.14), we have for every $r > 0$, $\tilde{\beta}_{2,\mu}(0,r) \to \tilde{\beta}_{2,\nu}(0,r)$. But our initial assumptions imply that for every $r > 0$ there exists $i_r \geq 1$ such that if $i \geq i_r$, then $\tilde{\beta}_{2,\mu_i}(0,r) \leq 2^{-i}$. It then follows that $\tilde{\beta}_{2,\nu}(0,r) = 0$ for every $r > 0$, which implies that Σ_∞ is contained in an n-plane. This contradicts (5.2.20) and completes the proof. □

5.3 Proof of Theorem 1.3.3: the singular set

Our arguments involve several different measures along with their singular sets, so to avoid confusion we denote the singular set of any Radon measure ν, as defined in the previous section, by $\mathcal{S}_\nu$. We also let $\dim_{\mathcal{H}}$ denote Hausdorff dimension, and we continue to denote $\Sigma = \mathrm{spt}(\mu)$, with μ as in the assumptions of Theorem 1.3.3 or Proposition 5.3.1 below. To complete the proof of Theorem 1.3.3 it is enough to show the following.

Proposition 5.3.1 (The singular set). *Suppose μ is a Λ-asymptotically optimally doubling measure of dimension m on $\mathbb{R}^{n+1}$, $1 \leq m \leq n+1$. Then either $m \leq 2$ and $\mathcal{S}_\mu = \varnothing$, or $m \geq 3$ and*

$$\dim_{\mathcal{H}}(\mathcal{S}_\mu) \leq m - 3. \tag{5.3.1}$$

The Euclidean analogue of this result was proven by Nimer [23] under the assumption that μ be *uniformly asymptotically doubling*. In our setting, an anisotropic analogue of such notion would go beyond the scope of this paper, so we only consider the particular case in which μ is Λ-asymptotically optimally doubling. It is still worth mentioning that a detailed analysis shows that the proofs are very similar in both cases. We also point out that even though we prove Proposition 5.3.1 in any codimension, Theorem 1.3.3 only relies on its validity in codimension 1.

The dimension bound of Proposition 5.3.1 can be motivated by the case where μ is n-uniform in $\mathbb{R}^{n+1}$. In that scenario, it follows from Theorem 2.1.2 that $\mathcal{S}_\mu$ is either empty or, up to a rotation and translation, an $(n-3)$-dimensional plane. More generally, Nimer showed in [22] that in any codimension, the dimension of the singular set of an m-uniform measure μ is at most $m-3$. Since the measure μ in the assumptions of Theorem 1.3.3 behaves asymptotically as $r \to 0$ as an n-uniform measure in a certain sense, it is natural to expect that the same dimension bound should be preserved.

5.3.1 Technical results and proof of Proposition 5.3.1

Our arguments rely crucially on a dimension reduction result, and a statement about the singular set of conical 3-uniform measures in arbitrary codimension, both proven by Nimer.

Lemma 5.3.1 (Dimension reduction [23]). *Let ν be an m-uniform, conical measure on $\mathbb{R}^{n+1}$, $1 \leq m \leq n+1$. Let $X \in \operatorname{spt}(\nu)$, $X \neq 0$, and let λ be the tangent measure* *of ν at X (normalized so that $\lambda(B(0,1)) = 1$). Then, up to rotation,*

$$\lambda = c\mathcal{H}^m \llcorner (\mathbb{R} \times A),$$

where $c > 0$ is a constant and $A \subset \mathbb{R}^n$ is the support of an $(m-1)$-uniform measure.

Theorem 5.3.1 (Regularity of conical 3-uniform measures [22]). *Let ν be a conical 3-uniform measure on $\mathbb{R}^{n+1}$. Then there exists $\gamma > 0$ such that $\operatorname{spt}(\nu)\backslash\{0\}$ is a $C^{1,\gamma}$ submanifold of dimension 3 of $\mathbb{R}^{n+1}$.*

Recall, for $X \in \Sigma$ and $r > 0$, the mappings $T_{X,r}$ and $T^{\Lambda}_{X,r}$ defined in (2.1.4) and (4.1.3), as well as the corresponding re-scalings of a Radon measure ν,

$$\nu_{X,r} = \frac{1}{\nu(B(X,r))} T_{X,r}[\nu], \quad \nu^{\Lambda}_{X,r} = \frac{1}{\nu(B_{\Lambda}(X,r))} T^{\Lambda}_{X,r}[\nu].$$

We will need the following technical results, whose Euclidean analogues are in [23].

Lemma 5.3.2 (Connectedness property). *Suppose μ is a Λ-asymptotically optimally doubling measure of dimension m in $\mathbb{R}^{n+1}$, $1 \leq m \leq n+1$. Let $X_k \in \operatorname{spt}(\mu) \cap \overline{\mathrm{B}(0,1)}$ be such that $X_k \to X$ as $k \to \infty$. Let also $\tau_k > \sigma_k > 0$ be such that $\tau_k, \sigma_k \to 0$ and*

$$\mu^{\Lambda}_{X_k,\tau_k} \rightharpoonup \alpha, \quad \mu^{\Lambda}_{X_k,\sigma_k} \rightharpoonup \beta,$$

as $k \to \infty$, where α and β are non-zero Radon measures. If α is flat, then β is flat.

Lemma 5.3.3 (Λ-tangents at a singular point). *Suppose μ is a Λ-asymptotically optimally doubling measure of dimension m in $\mathbb{R}^{n+1}$, $1 \leq m \leq n+1$. Then:*

1. *If $X \in \mathcal{S}_\mu$, then μ has no flat Λ-tangent measures at X.*

2. *If μ has a Λ-tangent measure at X which is not flat, then $X \in \mathcal{S}_\mu$.*

One of the key insights of Nimer [23] in the Euclidean setting is that if a measure is regular enough, then its singularities are "preserved" under blow-ups. The following is the corresponding anisotropic version of this statement.

Proposition 5.3.2 (Conservation of singularities)**.** *Suppose μ is a Λ-asymptotically optimally doubling measure of dimension m on $\mathbb{R}^{n+1}$, $1 \le m \le n+1$. Let $X \in \operatorname{spt}(\mu)$ and $r_k > 0$ be such that $r_k \to 0$ and $\mu^{\Lambda}_{X,r_k} \rightharpoonup \nu$ as $k \to \infty$, where ν is a nonzero Radon measure. If $X_k \in \mathcal{S}_\mu$ and $Y_k = \Lambda(X)^{-1}\left(\frac{X_k - X}{r_k}\right) \to Y$ for some $Y \in \mathbb{R}^{n+1}$ as $k \to \infty$, then $Y \in \mathcal{S}_\nu$.*

An important corollary of the Euclidean version of this proposition is the following result.

Corollary 5.3.1 (Singularities of 3-uniform measures [23])**.** *Let ν be a 3-uniform measure in $\mathbb{R}^{n+1}$. Then for every compact set $K \subset \mathbb{R}^{n+1}$, $\mathcal{S}_\nu \cap K$ is finite. In particular, $\dim_{\mathcal{H}}(\mathcal{S}_\nu) = 0$.*

Before we proceed with the proofs of these results we use them to prove Proposition 5.3.1.

Proof of Proposition 5.3.1. First consider the case $m \le 2$. Suppose that there exists $X \in \mathcal{S}_\mu$. By Lemma 5.3.3, μ has a Λ-tangent measure ν at X that is not flat. By Lemma 4.1.4, we know that ν is 2-uniform, so by Theorem 2.1.1 we obtain a contradiction. This implies that $\mathcal{S}_\mu = \varnothing$.

Next we handle the case $m \ge 3$ by induction. For the base case, consider $m = 3$ and suppose, for contradiction, that there exists $s > 0$ such that $\mathcal{H}^s(\mathcal{S}_\mu) > 0$. We first find a point $X \in \mathcal{S}_\mu$ and a Λ-tangent measure ν of μ at X, such that

$$\mathcal{H}^s(\mathcal{S}_\nu \cap \overline{B(0,1)}) > 0. \tag{5.3.2}$$

The assumption that $\mathcal{H}^s(\mathcal{S}_\mu) > 0$ implies that $\mathcal{H}^s_\infty(\mathcal{S}_\mu) > 0$ (by the arguments in the proof of Lemma 2.1 in [15], for example). By sub-additivity of $\mathcal{H}^s_\infty$, there exists a compact set $K \subset \mathbb{R}^{n+1}$ such that $\mathcal{H}^s_\infty(\mathcal{S}_\mu \cap K) > 0$. Let $\mathcal{S}_\mu(K) = \mathcal{S}_\mu \cap K$. Then, according to the proof of Theorem 2.7 in [15], for $\mathcal{H}^s$-almost every $X \in \mathcal{S}_\mu(K)$ we have

$$\theta^{s,*}(\mathcal{H}^s_\infty \llcorner \mathcal{S}_\mu(K), X) \ge 2^{-s}.$$

Fix any such X. Then there exists a sequence of radii $r_k > 0$ such that $r_k \to 0$, and $\mathcal{S}_k := \Lambda(X)^{-1}\left(\frac{\mathcal{S}_\mu(K)-X}{r_k}\right)$ satisfies

$$\mathcal{H}^s_\infty\left(\mathcal{S}_k \cap \overline{B(0,1)}\right) \geq C_K^{-1}2^{-s}, \tag{5.3.3}$$

for all k, where $C_K > 0$ depends only on K and Λ. In particular, X is a limit point of $\mathcal{S}_\mu$, and since $\mathcal{S}_\mu$ is closed by Proposition (5.2.1), it follows that $X \in \mathcal{S}_\mu(K)$, or $0 \in \mathcal{S}_k$ for all k. Thus, upon passing to a subsequence, we may find a closed set $F \subset \mathbb{R}^{n+1}$ such that

$$\mathcal{S}_k \to F, \tag{5.3.4}$$

with respect to Hausdorff distance, uniformly on compact sets.

Upon passing to a further subsequence, we may also assume $\mu^\Lambda_{X,r_k} \rightharpoonup \nu$, where ν is m-uniform with $\nu(B(0,1)) = 1$ by Proposition 4.1.4. We claim that $F \subset \mathcal{S}_\nu$. In fact, if $Y \in F$, we can find $Y_k \in \mathcal{S}_k$ such that $Y_k \to Y$. By definition of $\mathcal{S}_k$, we have $Y_k = \Lambda(X)^{-1}\left(\frac{X_k - X}{r_k}\right)$ for some $X_k \in \mathcal{S}_\mu$, so Proposition 5.3.2 implies that $Y \in \mathcal{S}_\nu$, proving the claim. This, combined with (5.3.4), implies that for any $\varepsilon > 0$ and k large enough,

$$\mathcal{S}_k \cap \overline{B(0,1)} \subset (\mathcal{S}_\nu;\varepsilon) \cap \overline{B(0,1)} \subset (\mathcal{S}_\nu \cap \overline{B(0,1)};\varepsilon). \tag{5.3.5}$$

We use this information to estimate $\mathcal{H}^s_\infty(\mathcal{S}_\nu \cap \overline{B(0,1)})$. Let $\delta > 0$, $\mathcal{S}_\nu(B) = \mathcal{S}_\nu \cap \overline{B(0,1)}$, and let $\{E_\ell\}$ be a cover of $\mathcal{S}_\nu(B)$ such that

$$\mathcal{H}^s_\infty(\mathcal{S}_\nu(B)) > \sum_\ell \omega_s\left(\frac{\mathrm{diam}(E_\ell)}{2}\right)^s - \delta.$$

By incorporating small open neighborhoods of the sets E_ℓ if necessary we may assume that each E_ℓ is open. By Proposition 5.2.1, $\mathcal{S}_\nu$ is closed, so $\mathcal{S}_\nu(B)$ is compact, and we may assume that $\mathcal{S}_\nu(B) \subset E_1 \cup \cdots \cup E_L$ for some $L < \infty$. Therefore, by (5.3.5), if k is large enough we have $\mathcal{S}_k(B) := \mathcal{S}_k \cap \overline{B(0,1)} \subset E_1 \cup \cdots \cup E_L$, which implies

$$\mathcal{H}^s_\infty(\mathcal{S}_k(B)) \leq \omega_s \sum_{\ell=1}^L \left(\frac{\mathrm{diam}(E_\ell)}{2}\right)^s < \mathcal{H}^s_\infty(\mathcal{S}_\nu(B)) + \delta.$$

With this and (5.3.3) we obtain

$$\mathcal{H}^s_\infty(\mathcal{S}_\nu(B)) \geq \limsup_{k\to\infty} \mathcal{H}^s_\infty(\mathcal{S}_k(B)) > 0,$$

which implies (5.3.2). However, we know that ν is 3-uniform, so by Corollary 5.3.1, $\mathcal{S}_\nu$ is at most countable, which contradicts (5.3.2). This implies that $s = 0$, and completes the proof of the base case.

For the inductive step, assume that $m \geq 4$, and suppose that the theorem holds for measures of dimension $m - 1$. Let $s > 0$ be such that $\mathcal{H}^s(\mathcal{S}_\mu) > 0$. By the procedure described above, we can find a point $X \in \mathcal{S}_\mu$ and a Λ-tangent measure ν of μ at X, such that (5.3.2) holds. Now, by (5.3.2), we can apply the same procedure again with $\Lambda(X)^{-1}$ replaced by Id, to obtain a tangent measure ν' of ν at a point $X' \in \mathcal{S}_\nu \cap \overline{B(0,1)}$, such that $\mathcal{H}^s(\mathcal{S}_{\nu'} \cap \overline{B(0,1)}) > 0$. We know that ν is m-uniform, so by Theorem 2.1.4, ν' is m-uniform and conical, and in particular it satisfies the assumptions of Lemma 5.3.1. Moreover, since $\mathcal{H}^s(\mathcal{S}_{\nu'} \cap \overline{B(0,1)}) > 0$ and $s > 0$, we have that for some $\rho > 0$ small enough, $\mathcal{H}^s(\mathcal{S}_{\nu'} \cap \overline{B(0,1)} \backslash B(0,\rho)) > 0$. Thus, we can apply the above procedure once more to obtain a point $X'' \in \mathcal{S}_{\nu'}$, $X'' \neq 0$, and a tangent measure ν'' of ν' at X'' such that

$$\mathcal{H}^s(\mathcal{S}_{\nu''} \cap \overline{B(0,1)}) > 0. \tag{5.3.6}$$

By Lemma 5.3.1, upon a rotation, $\Sigma'' := \mathrm{spt}(\nu'')$ satisfies $\Sigma'' = \mathbb{R} \times A$, where $A \subset \mathbb{R}^n$ is the support of an $(m-1)$-uniform measure ν''_0. In particular, we have $\mathcal{S}_{\nu''} \subset \mathbb{R} \times \mathcal{S}_{\nu''_0}$, which implies

$$\dim_{\mathcal{H}}(\mathcal{S}_{\nu''}) \leq \dim_{\mathcal{H}}(\mathcal{S}_{\nu''_0}) + 1. \tag{5.3.7}$$

Since ν''_0 is $(m-1)$-uniform, our induction hypothesis implies that $\dim_{\mathcal{H}}(\mathcal{S}_{\nu''_0}) \leq m-4$, so (5.3.7) gives $\dim_{\mathcal{H}}(\mathcal{S}_{\nu''}) \leq m - 3$. Combining this with (5.3.6) we obtain

$$s \leq \dim_{\mathcal{H}}(\mathcal{S}_{\nu''}) \leq m - 3.$$

Since $\mathcal{H}^s(\mathcal{S}_\mu) > 0$ by assumption, it then follows that $\dim_{\mathcal{H}}(\mathcal{S}_\mu) \leq m - 3$, completing the inductive step and the proof of Proposition 5.3.1. $\square$

5.3.2 Proof of technical results

Now we turn to the proofs of Lemma 5.3.2, Lemma 5.3.3 and Proposition 5.3.2. Some of the arguments below rely on the following fact.

Lemma 5.3.4 (Scaling before and after blow-up)**.** *If ν, ν' are nonzero Radon measures on $\mathbb{R}^{n+1}$, ν is Λ-asymptotically optimally doubling of dimension m, $1 \le m \le n+1$, $X_k, X \in \mathrm{spt}(\nu)$, $R, R_k, r_k > 0$, $R_k \to R$, $r_k \to 0$ and $\nu^{\Lambda}_{X_k,r_k} \rightharpoonup \nu'$ as $k \to \infty$, then*

$$\nu^{\Lambda}_{X_k,R_k r_k} \rightharpoonup \nu'_{0,R}. \tag{5.3.8}$$

Proof. Let $\psi \in C_c(\mathbb{R}^{n+1})$. It suffices to show that

$$\int \psi \mathrm{d}\nu^{\Lambda}_{X_k,R_k r_k} = \frac{1}{\nu^{\Lambda}_{X_k,r_k}(B(0,R_k))} \int \psi \mathrm{d}(T_{0,R_k}[\nu^{\Lambda}_{X_k,r_k}]) \to \frac{1}{\nu'(B(0,R))} \int \psi \circ T_{0,R} \mathrm{d}\nu'. \tag{5.3.9}$$

Note that by the doubling property of ν, ν' is m-uniform, so for any $\rho > 0$, $\nu'(\partial B(0,\rho)) = 0$, which implies $\nu^{\Lambda}_{X_k,r_k}(B(0,\rho)) \to \nu'(B(0,\rho))$. We first show that

$$\nu^{\Lambda}_{X_k,r_k}(B(0,R_k)) \to \nu'(B(0,R)). \tag{5.3.10}$$

We write

$$\begin{aligned} &|\nu^{\Lambda}_{X_k,r_k}(B(0,R_k)) - \nu'(B(0,R))| \\ &\qquad \le |\nu^{\Lambda}_{X_k,r_k}(B(0,R_k)) - \nu^{\Lambda}_{X_k,r_k}(B(0,R))| + |\nu^{\Lambda}_{X_k,r_k}(B(0,R)) - \nu'(B(0,R))|. \end{aligned} \tag{5.3.11}$$

The last term goes to 0 by the above observation. For the first term, let $\varepsilon \in (0,R)$ and suppose that k is large enough so that $|R_k - R| < \varepsilon$. Then

$$\begin{aligned} |\nu^{\Lambda}_{X_k,r_k}(B(0,R_k)) - \nu^{\Lambda}_{X_k,r_k}(B(0,R))| &\le \nu^{\Lambda}_{X_k,r_k}(B(0,R+\varepsilon)\backslash B(0,R-\varepsilon)) \\ &\to \nu'(B(0,R+\varepsilon)\backslash B(0,R-\varepsilon)) \le C\varepsilon R^{m-1}, \end{aligned} \tag{5.3.12}$$

as $k \to \infty$. Since this holds for every $\varepsilon \in (0, R)$, (5.3.10) now follows from (5.3.11). Next we show that

$$\int \psi \mathrm{d}(T_{0,R_k}[\nu^{\Lambda}_{X_k,r_k}]) = \int \psi \circ T_{0,R_k} \mathrm{d}\nu^{\Lambda}_{X_k,r_k} \to \int \psi \circ T_{0,R} \mathrm{d}\nu'. \tag{5.3.13}$$

Note that $\psi \circ T_{0,R_k} \to \psi \circ T_{0,R}$ uniformly. Let $M > 0$ be such that $\operatorname{spt}(\psi \circ T_{0,R_k}) \subset B(0, M)$ for all k. Since $\nu^{\Lambda}_{X_k,r_k}$ converges weakly, the sequence $\nu^{\Lambda}_{X_k,r_k}(\overline{B(0,M)})$ is bounded, so

$$\left| \int \psi \circ T_{0,R_k} \mathrm{d}\nu^{\Lambda}_{X_k,r_k} - \int \psi \circ T_{0,R} \mathrm{d}\nu^{\Lambda}_{X_k,r_k} \right| \leq C \|\psi \circ T_{0,R_k} - \psi \circ T_{0,R}\|_{\infty} \to 0, \tag{5.3.14}$$

as $k \to \infty$. On the other hand, since $\nu^{\Lambda}_{X_k,r_k} \rightharpoonup \nu'$ we have

$$\int \psi \circ T_{0,R} \mathrm{d}\nu^{\Lambda}_{X_k,r_k} \to \int \psi \circ T_{0,R} \mathrm{d}\nu'.$$

Combining this with (5.3.14) we obtain (5.3.13), and from (5.3.10) and (5.3.13) we obtain (5.3.9).

□

Proof of Lemma 5.3.2. Recall the functional F defined in (2.1.8). Since μ is Λ-asymptotically optimally doubling of dimension m, we know that α and β are m-uniform. Suppose α is flat, but β is not. Then $F(\alpha) = 0$, but by Corollary 2.1.9, for every $R > 0$ we can find $R_0 > R$ such that $F(\beta_{0,R_0}) > \varepsilon_0^2$, with $\varepsilon_0 > 0$ as in (2.1.9). In particular, since $\mu^{\Lambda}_{X_k,\tau_k} \rightharpoonup \alpha$, we have $\mu^{\Lambda}_{X_k,R_0\sigma_k} \rightharpoonup \beta_{0,R_0}$ by Lemma 5.3.4, so by continuity of F with respect to weak convergence, it follows that for some $\kappa \in (0, \varepsilon_0^2)$ and for all k large enough,

$$F(\mu^{\Lambda}_{X_k,\tau_k}) < \kappa < F(\mu^{\Lambda}_{X_k,R_0\sigma_k}). \tag{5.3.15}$$

We will first show that

$$\tau_k/\sigma_k \to \infty, \tag{5.3.16}$$

as $k \to \infty$. Assume, by contradiction, that this is not the case. Since $\tau_k > \sigma_k$ by assumption, upon passing to a subsequence we have $\gamma_k := \tau_k/\sigma_k \to \gamma$, for some $\gamma \geq 1$. This implies

$$\frac{1}{\mu(B_\Lambda(X_k, \sigma_k))} T^\Lambda_{X_k,\tau_k}[\mu] = \frac{1}{\mu(B_\Lambda(X_k, \sigma_k))} T^\Lambda_{X_k,\gamma_k\sigma_k}[\mu] \rightharpoonup T_{0,\gamma}[\beta]. \tag{5.3.17}$$

On the other hand, since $\tau_k/\sigma_k \to \gamma$, (4.1.9) implies that $\mu(B(X_k, \tau_k))/\mu(B(X_k, \sigma_k)) \to \gamma^n$ as $k \to \infty$. Therefore,

$$\frac{1}{\mu(B_\Lambda(X_k, \sigma_k))} T^\Lambda_{X_k,\tau_k}[\mu] = \frac{\mu(B_\Lambda(X_k, \tau_k))}{\mu(B_\Lambda(X_k, \sigma_k))} \frac{T^\Lambda_{X_k,\tau_k}[\mu]}{\mu(B_\Lambda(X_k, \tau_k))} \rightharpoonup \gamma^n \alpha,$$

which contradicts (5.3.17) because $\gamma^n\alpha$ is flat but $T_{0,\gamma}[\beta]$ is not. This proves (5.3.16). In particular, if k is large enough, we have $R_0\sigma_k < \tau_k$.

Note that for each k, the function $r \mapsto F(\mu^\Lambda_{X_k,r})$ is continuous away from $r = 0$, by continuity of F. Therefore, (5.3.15) implies that there exist $\delta_k \in [R_0\sigma_k, \tau_k]$ such that $F(\mu^\Lambda_{X_k,\delta_k}) = \kappa$ and

$$F(\mu^\Lambda_{X_k,r}) \leq \kappa, \tag{5.3.18}$$

for every $r \in [\delta_k, \tau_k]$ (take $\delta_k = \max\{r \in [R_0\sigma_k, \tau_k] : F(\mu^\Lambda_{X_k,r}) \geq \kappa\}$). By the doubling property of μ, we may assume upon passing to a subsequence that $\mu^\Lambda_{X_k,\delta_k} \rightharpoonup \nu$, where ν is an m-uniform Λ-pseudo tangent measure of μ. Note that by continuity of F, we have

$$F(\nu) = \kappa. \tag{5.3.19}$$

We will now show that ν is flat, which will contradict (5.3.19) and complete the proof. Note that since $F(\mu^\Lambda_{X_k,\delta_k}) = \kappa > 0$ for all k large, the same proof that $\tau_k/\sigma_k \to \infty$ can be applied to show that $\tau_k/\delta_k \to \infty$. Let $R > 1$, and suppose k is large enough so that $R\delta_k < \tau_k$. By (5.3.18) we have $F(\mu^\Lambda_{X_k,R\delta_k}) \leq \kappa$ for all k large. But we also have $\mu^\Lambda_{X_k,R\delta_k} \rightharpoonup T_{0,R}[\nu]$. Therefore, by continuity of F, it follows that $F(\nu_{0,R}) \leq \kappa$, and letting $R \to \infty$ we get

$$\limsup_{R\to\infty} F(\nu_{0,R}) \leq \kappa < \varepsilon_0^2.$$

By Corollary 2.1.9, this implies that ν is flat, as desired. $\square$

Proof of Lemma 5.3.3. Let us denote $\Sigma = \mathrm{spt}(\mu)$. For the proof of 1, applying Lemma 5.3.2 with $X_k = X$ for all k, we see that it suffices to show that μ has at least one Λ-tangent measure at X that is not flat. Since $X \in \mathcal{S}_\mu$, there exists a sequence $r_k > 0$, $k \in \mathbb{N}$, with $r_k \searrow 0$ as $k \to \infty$, such that

$$b\beta^m_\Sigma(X, r_k) \geq c, \tag{5.3.20}$$

for all k. Let $\Sigma_k = T^\Lambda_{X,r_k}(\Sigma)$. Then by (5.3.20) and an argument as in Step 2 of Section 3.2, we have for $\rho_0 = \lambda_{\min}(X)^{-1}$,

$$b\beta_{\Sigma_k}(0, \rho_0) \geq c_1 b\beta_\Sigma(X, r_k) \geq c_2 > 0, \tag{5.3.21}$$

where c_1 and c_2 depend on X. Since $0 \in \Sigma_k$ for all k, we have as in Section 4.2 that upon passing to a subsequence, there exists a closed set $\Sigma_\infty \subset \mathbb{R}^{n+1}$ such that $0 \in \Sigma_\infty$ and $\Sigma_k \to \Sigma_\infty$ as $k \to \infty$ with respect to Hausdorff distance, uniformly on compact sets. Note that (5.3.21) implies that

$$b\beta_{\Sigma_\infty}(0, \rho_0) \geq c_2/2. \tag{5.3.22}$$

Let now $\mu_k = \mu^\Lambda_{X,r_k}$. Since μ is Λ-asymptotically optimally doubling, we may assume by Lemma 4.1.1 that upon passing to a further subsequence, we have $\mu_k \rightharpoonup \nu$, where ν is a Λ-tangent measure of μ. Moreover, since $\Sigma_k \to \Sigma_\infty$ with respect to Hausdorff distance, Lemma 4.1.3 implies that $\mathrm{spt}(\nu) = \Sigma_\infty$. But (5.3.22) implies that Σ_∞ cannot be a plane, so ν is not flat, as desired.

For the proof of 2, suppose $\mu^\Lambda_{X,r_k} \rightharpoonup \nu$, where ν is not flat. Then by a similar argument as above we may assume upon passing to a subsequence, that $\Sigma_k = \mu^\Lambda_{X,r_k}$ satisfies $\Sigma_k \to \Sigma_\infty$ for some $\Sigma_\infty \subset \mathbb{R}^{n+1}$ with respect to Hausdorff distance, and $\Sigma_\infty = \mathrm{spt}(\nu)$. Since ν is not flat, we have $b\beta_{\Sigma_\infty}(0,1) > 0$, so for k large enough and $\rho_1 = \lambda_{\max}(X)^{-1}$,

$$b\beta_\Sigma(X, \rho_1 r_k) \geq c_2 b\beta_{\Sigma_k}(0,1) \geq c_2 b\beta_{\Sigma_\infty}(0,1) > 0,$$

which implies that $X \in \mathcal{S}_\mu$. $\square$

Proof of Proposition 5.3.2. Note first that since $Y_k \to Y$, we have

$$|X_k - X| \leq \lambda_{\max}(X) r_k |Y_k| \to 0,$$

as $k \to \infty$, so $X_k \to X$. We start by constructing a sequence of radii $\sigma_k > 0$ such that $\sigma_k \to 0$, $r_k/\sigma_k \to \infty$, and $\mu^\Lambda_{X_k,\sigma_k} \rightharpoonup \nu^{(\infty)}$ as $k \to \infty$, where $\nu^{(\infty)}$ satisfies

$$\limsup_{R\to\infty} F(\nu^{(\infty)}_{0,R}) > \varepsilon_0^2/2. \tag{5.3.23}$$

Since μ is Λ-asymptotically optimally doubling, we can construct inductively a family of sequences $\{s_j^{(k)}\}_{i\in\mathbb{N}}$, where $\{s_j^{(1)}\}$ is a subsequence of $\{r_j\}$, $\{s_j^{(k+1)}\}$ is a subsequence of $\{s_j^{(k)}\}$ for $k \geq 1$, and for every k,

$$\mu^\Lambda_{X_k, s_j^{(k)}} \rightharpoonup \nu^{(k)},$$

as $j \to \infty$, where $\nu^{(k)}$ is an m-uniform Λ-tangent measure of μ at X_k with $\nu^{(k)}(B(0,1)) = 1$ by Proposition 4.1.4. For each k, note that $s_j^{(k)} \to 0$ as $j \to \infty$, and since $X_k \in \mathcal{S}_\mu$, Lemma 5.3.3 implies that $\nu^{(k)}$ is not flat. Therefore, by Corollary 2.1.9, for each $R > 0$ we can find $R' > R$, depending on k, such that

$$F(\nu^{(k)}_{0,R'}) > \varepsilon_0^2. \tag{5.3.24}$$

Let R_k be any such R'. Note that by uniformity of $\nu^{(k)}$, for every $r > 0$ we have

$$\nu^{(k)}_{0,R_k}(B(0,r)) = \frac{\nu^{(k)}(B(0,rR_k))}{\nu^{(k)}(B(0,R_k))} = r^m,$$

so upon passing to a subsequence we may assume that $\nu^{(k)}_{0,R_k} \rightharpoonup \nu^{(\infty)}$ as $k \to \infty$, for some Radon measure $\nu^{(\infty)}$. We will show that $\nu^{(\infty)}$ satisfies (5.3.23). Recall the functionals $\mathcal{F}_r$ and $\mathcal{F}$ introduced in (2.1.5). Note first that $\mu^\Lambda_{X_k, R_k s_j^{(k)}} \rightharpoonup \nu^{(k)}_{0,R_k}$ as $j \to \infty$, so by Proposition 2.1.1 and because $s_j^{(k)} \to 0$ as $j \to \infty$, we can find $j = j_k$ large enough so that

$$\mathcal{F}(\mu^\Lambda_{X_k, R_k s_{j_k}^{(k)}}, \nu^{(k)}_{0,R_k}) < 2^{-k}, \quad \sigma_k := R_k s_{j_k}^{(k)} \leq r_k^2. \tag{5.3.25}$$

Since $\nu^{(k)}_{0,R_k} \rightharpoonup \nu^{(\infty)}$, (5.3.25) implies that $\mu^{\Lambda}_{X_k,\sigma_k} \rightharpoonup \nu^{(\infty)}$, so $\nu^{(\infty)}$ is a Λ-pseudo tangent of μ at X. Note that $\frac{r_k}{\sigma_k} \geq \frac{1}{r_k} \to \infty$ as $k \to \infty$.

We now show that (5.3.23) holds. Suppose, for contradiction, that $F(\nu^{(\infty)}_{0,R}) \leq \varepsilon_0^2/2$ for all $R > 1$ large enough. For each such R, let P be a minimizing plane in the definition of $F(\nu^{(\infty)}_{0,R})$. Then with φ as in the definition of F, and keeping in mind that $\nu^{(k)}_{0,RR_k}(B(0,1)) = 1$ by Proposition 4.1.4,

$$\begin{aligned} F(\nu^{(k)}_{0,RR_k}) &\leq \int \varphi(Z)\mathrm{dist}(Z,P)^2 \mathrm{d}\nu^{(k)}_{0,RR_k}(Z) \\ &= \int \varphi(Z)\mathrm{dist}(Z,P)^2 \mathrm{d}\nu^{(k)}_{0,RR_k}(Z) - \int \varphi(Z)\mathrm{dist}(Z,P)^2 \mathrm{d}\nu^{(\infty)}_{0,R}(Z) + F(\nu^{(\infty)}_{0,R}) \\ &\leq C\mathcal{F}_3(\nu^{(k)}_{0,RR_k}, \nu^{(\infty)}_{0,R}) + \frac{\varepsilon_0^2}{2}. \end{aligned} \tag{5.3.26}$$

Now, by (5.3.8) we have $\nu^{(k)}_{0,RR_k} \rightharpoonup \nu^{(\infty)}_{0,R}$ as $k \to \infty$, so by Proposition 2.1.1 and (5.3.26) it follows that for all k large enough, $F(\nu^{(k)}_{0,RR_k}) < \varepsilon_0^2$. This contradicts (5.3.24), proving that (5.3.23) holds. This implies that $\nu^{(\infty)}$ is not flat.

Next, let ν, Y_k and Y be as in the statement of the proposition. We know by Lemma 4.1.3 and Proposition 4.1.4 that $Y \in \mathrm{spt}(\nu)$ and ν is m-uniform with $\nu(B(0,1)) = 1$. We show that

$$\mu^{\Lambda}_{X_k,r_k} \rightharpoonup \nu_{Y,1}. \tag{5.3.27}$$

Note first that

$$T^{\Lambda}_{X_k,r_k} = T_{Y_k,1} \circ \Lambda(X_k)^{-1} \circ \Lambda(X) \circ T^{\Lambda}_{X,r_k}.$$

Therefore, if $\varphi \in C_c(\mathbb{R}^{n+1})$ and $A_k = \Lambda(X_k)^{-1} \circ \Lambda(X)$, then

$$\begin{aligned} \int \varphi \mathrm{d}\mu^{\Lambda}_{X_k,r_k} &= \frac{1}{\mu(B_\Lambda(X_k,r_k))} \int \varphi \mathrm{d}(T^{\Lambda}_{X_k,r_k}[\mu]) \\ &= \frac{1}{\mu(B_\Lambda(X_k,r_k))} \int \varphi \circ T_{Y_k,1} \circ A_k \mathrm{d}(T^{\Lambda}_{X,r_k}[\mu]) \\ &= \frac{\mu(B_\Lambda(X,r_k))}{\mu(B_\Lambda(X_k,r_k))} \int \varphi \circ T_{Y_k,1} \circ A_k \mathrm{d}\mu^{\Lambda}_{X,r_k}. \end{aligned} \tag{5.3.28}$$

By an argument as the one leading up to (4.1.22), we have $\mu(B_\Lambda(X,r_k))/\mu(B_\Lambda(X_k,r_k)) \to 1$ as $k\to\infty$. On the other hand, since $A_k \to \mathrm{Id}$ and $T_{Y_k,1} \to T_{Y,1}$ uniformly on compact sets as $k\to\infty$, it follows that $\varphi\circ T_{Y_k,1}\circ A_k \to \varphi\circ T_{Y,1}$ uniformly as $k\to\infty$. Let $M>0$ be such that $\mathrm{spt}(\varphi\circ T_{Y_k,1}\circ A_k) \subset B(0,M)$ for all k. The fact that $\mu^\Lambda_{X,r_k} \rightharpoonup \nu$ as $k\to\infty$ implies that the sequence $\mu^\Lambda_{X,r_k}(\overline{B(0,M)})$ is bounded. Therefore, for $\varepsilon>0$, if k is large enough,

$$\begin{aligned}
&\left|\int \varphi\circ T_{Y_k,1}\circ A_k \mathrm{d}\mu^\Lambda_{X,r_k} - \int \varphi \mathrm{d}\nu_{Y,1}\right| \\
&\quad\le \left|\int \varphi\circ T_{Y_k,1}\circ A_k \mathrm{d}\mu^\Lambda_{X,r_k} - \int \varphi\circ T_{Y,1}\mathrm{d}\mu^\Lambda_{X,r_k}\right| + \left|\int \varphi\circ T_{Y,1}\mathrm{d}\mu^\Lambda_{X,r_k} - \int \varphi\circ T_{Y,1}\mathrm{d}\nu\right| \\
&\quad\le \varepsilon\mu^\Lambda_{X,r_k}(\overline{B(0,M)}) + \varepsilon \le C\varepsilon.
\end{aligned} \tag{5.3.29}$$

Thus

$$\int \varphi\circ T_{Y_k,1}\circ A_k \mathrm{d}\mu^\Lambda_{X,r_k} \to \int \varphi \mathrm{d}\nu_{Y,1},$$

as $k\to\infty$, and it then follows from (5.3.28) that $\int\varphi\mathrm{d}\mu^\Lambda_{X_k,r_k} \to \int\varphi\mathrm{d}\nu_{Y,1}$ as $k\to\infty$, proving (5.3.27).

Finally, we use $\nu^{(\infty)}$ to show that ν has a tangent measure at Y that is not flat, so that by Lemma 5.3.3

$$Y \in \mathcal{S}_\nu. \tag{5.3.30}$$

Let $\rho_k = \sigma_k/r_k$, and note that $\rho_k \to 0$ as shown above. We know that ν is m-uniform, so we may assume upon passing to a subsequence that $\nu_{Y,\rho_k} \rightharpoonup \alpha$, where α is a tangent measure of ν at Y with $\alpha(B(0,1)) = 1$. We will prove that α is not flat. By (5.3.27) and since $\rho_k \to 0$, for every k we can find ℓ_k such that if $\ell \ge \ell_k$, then

$$\mathcal{F}_1(\mu^\Lambda_{X_\ell,r_\ell},\nu_{Y,1}) < \frac{\rho_k}{k}\nu_{Y,1}(B(0,\rho_k)), \quad \rho_\ell < \rho_k. \tag{5.3.31}$$

We may assume without loss of generality that $\ell_{k+1} < \ell_k$ for all $k\ge 1$. Let $\tilde\tau_k = r_{\ell_k}\rho_k$ and $\tilde X_k = X_{\ell_k}$. We claim that

$$\mathcal{F}_R(\mu^\Lambda_{\tilde X_k,\tilde\tau_k},\nu_{Y,\rho_k}) < \frac{2}{k}. \tag{5.3.32}$$

In fact, if $\varphi \in \mathcal{L}(R)$ (see (2.1.5)),

$$
\begin{aligned}
&\left|\int \varphi \, \mathrm{d}\mu^{\Lambda}_{\tilde{X}_k,\tilde{\tau}_k} - \int \varphi \mathrm{d}\nu_{Y,\rho_k}\right| \\
&\quad \le \left| \frac{1}{\mu(B_\Lambda(\tilde{X}_k,\tilde{\tau}_k)))} \int \varphi \mathrm{d}(T^{\Lambda}_{\tilde{X}_k,\tilde{\tau}_k}[\mu]) - \frac{1}{\nu_{Y,1}(B(0,\rho_k))} \int \varphi \mathrm{d}(T_{0,\rho_k}[\nu_{Y,1}]) \right| \\
&\quad \le \left| \frac{\mu(B_\Lambda(\tilde{X}_k, r_{\ell_k}))}{\mu(B_\Lambda(\tilde{X}_k,\tilde{\tau}_k)))} \int \varphi \mathrm{d}(T_{0,\rho_k}[\mu^{\Lambda}_{\tilde{X}_k,r_{\ell_k}}]) - \frac{1}{\nu_{Y,1}(B(0,\rho_k))} \int \varphi \mathrm{d}(T_{0,\rho_k}[\nu_{Y,1}]) \right| \\
&\quad \le I_1 + I_2,
\end{aligned}
\tag{5.3.33}
$$

where

$$
\begin{aligned}
I_1 &= \left| \frac{\mu(B_\Lambda(\tilde{X}_k, r_{\ell_k}))}{\mu(B_\Lambda(\tilde{X}_k,\tilde{\tau}_k)))} - \frac{1}{\nu_{Y,1}(B(0,\rho_k))} \right| \int \varphi \mathrm{d}(T_{0,\rho_k}[\mu^{\Lambda}_{\tilde{X}_k,r_{\ell_k}}]), \\
I_2 &= \frac{1}{\nu_{Y,1}(B(0,\rho_k))} \left| \int \varphi \mathrm{d}(T_{0,\rho_k}[\mu^{\Lambda}_{\tilde{X}_k,r_{\ell_k}}]) - \int \varphi \mathrm{d}(T_{0,\rho_k}[\nu_{Y,1}]) \right|.
\end{aligned}
$$

By the doubling property of μ, we have

$$
\left| \frac{\mu(B_\Lambda(\tilde{X}_k, r_{\ell_k}))}{\mu(B_\Lambda(\tilde{X}_k,\tilde{\tau}_k)))} - \frac{1}{\nu_{Y,1}(B(0,\rho_k))} \right| = \left| \frac{\mu(B_\Lambda(\tilde{X}_k, r_{\ell_k}))}{\mu(B_\Lambda(\tilde{X}_k,\tilde{\tau}_k)))} - \frac{r_{\ell_k}}{\tilde{\tau}_k} \right| \to 0,
$$

as $k \to \infty$. Moreover, since $\varphi \in \mathcal{L}(R)$, we have $|\varphi \circ T_{0,\rho_k}| \le CR$ for some $C > 0$, and since $\varphi \circ T_{0,\rho_k}$ is continuous and supported in $B(0,R)$ for k large, we have

$$
\int \varphi \mathrm{d}(T_{0,\rho_k}[\mu^{\Lambda}_{\tilde{X}_k,r_{\ell_k}}]) = \int \varphi \circ T_{0,\rho_k} \mathrm{d}\mu^{\Lambda}_{\tilde{X}_k,r_{\ell_k}}
$$

is bounded, so $I_1 \to 0$ as $k \to \infty$. On the other hand, using (5.3.31) we have for all k large enough,

$$
I_2 \le \frac{1}{\rho_k \nu_{Y,1}(B(0,\rho_k))} \mathcal{F}_{R\rho_k}(\mu^{\Lambda}_{\tilde{X}_k,r_{\ell_k}}, \nu_{Y,1}) \le \frac{1}{\rho_k \nu_{Y,1}(B(0,\rho_k))} \mathcal{F}_1(\mu^{\Lambda}_{X_{\ell_k},r_{\ell_k}}, \nu_{Y,1}) \le \frac{1}{k},
\tag{5.3.34}
$$

so $I_2 \to 0$ as well.

Thus, (5.3.33) implies (5.3.32), and since $\nu_{Y,\rho_k} \rightharpoonup \alpha$, we deduce that $\mu^{\Lambda}_{\tilde{X}_k,\tilde{\tau}_k} \rightharpoonup \alpha$. But we also know that $\mu^{\Lambda}_{X_k,\sigma_k} \rightharpoonup \nu^{(\infty)}$, and $\nu^{(\infty)}$ is not flat. Therefore, by Lemma 5.3.2, α cannot be flat, which implies (5.3.30) and completes the proof. □

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