The topology of graph homomorphisms

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Abstract

The topology of graph homomorphisms

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In this thesis we consider topological aspects of graph homomorphisms. Our main object of study is the Hom complex of graphs, a space first introduced by Lovász to obtain lower bounds on chromatic number, and more recently studied by Babson and Kozlov and others in a series of papers. We prove structural results regarding the interaction of the Hom complex with several graph theoretical operations, including arbitrary products and exponentials. We introduce a notion of homotopy of graphs based on the right adjoint to the categorical product, and show that notions of graph homotopy equivalence are characterized by topological properties of the Hom complex. We relate graph homotopy to the notion of graph folding and in the process reprove a theorem of Kozlov regarding foldings preserving the homotopy type of Hom complexes.

We show that Hom complexes are 'universal' in the sense that for all connected graphs T, one can obtain an arbitrary homotopy type as $\mathrm{Hom}(T, G)$ for some graph G; this extends work of Csorba for $T = K_2$. We consider notions of discrete homotopy groups and show that they can be recovered as ordinary homotopy groups of a pointed version of the Hom complex. This is related to A-theory of graphs, where one seeks a space to encode similarly defined homotopy groups based on the Cartesian product of graphs. Finally, we combine these results with work of Schultz to provide a method of constructing new 'test graphs' for bounds on chromatic number. This extends work of Schultz, who showed that the Hom complexes involving the test graphs K_2 and the odd cycles C_{2r+1} were related in a nice way.

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Chapter 1

INTRODUCTION

In this thesis we discuss some applications of topology to the study of graph homomorphisms. Our main tool will be a poset enrichment of the category of graphs known as the Hom complex, a functorial way to associate a poset (and hence topological space) $\mathrm{Hom}(G, H)$ to a pair of graphs G and H. Spaces of this kind were first introduced in by Lászlo Lovász [Lov78] in his seminal 1978 paper proving the Kneser conjecture. In the eyes of many (see [dL03]) this paper marked the advent of 'topological combinatorics' as a branch of mathematics.

Although graphs certainly form a significant object of study in combinatorics with connections to many other branches of mathematics, the systematic study of graph homomorphisms has only recently begun to gain the attention of mathematicians (this according to the authors of [HN04]). One exception is of course the interest in the chromatic number $\chi(G)$ of a graph G (graph homomorphism to complete graphs), where numerous tools, devices, and generalizations have been developed to incorporate methods from algebra, geometry, and topology.

Interest in the chromatic number of a graph has enjoyed a long history, with several well-known solved and unsolved problems dating back over one hundred years. The famous *Four-Color Theorem* was first suggested by F. Guthrie in 1852, first published as a conjecture by Cayley in 1878, and finally proved in 1976 by Appel and Haken in a computer assisted case by case analysis. In 1943, Hadwiger stated his famous conjecture that every graph contains a $K_{\chi(G)}$ minor. (Here K_n is the *complete graph* on n vertices, and a *minor* of a graph G is by definition obtained by contracting and edges G/e, deleting edges $G\backslash e$, and deleting isolated vertices). In 1937 it was shown by Wagner that the $\chi(G) = 5$ case is equivalent to the Four Color Theorem.

A significant amount of research has addressed the *complexity* of graph colorings. The problem of computing the chromatic number of a graph in general is NP-complete (in the number of vertices); this holds true even for determining whether a given planar graph is 3-colorable. For this reason, many mathematicians and theoretical computer scientists are interested in bounds on chromatic number that are fast to compute.

One of the most famous gadgets for encoding the colorings of a graph G is via the *chromatic polynomial* $C_G(t)$, a specialization of the *Tutte polynomial.* By definition, $C_G(t)$ is the real-valued function whose value at an integer n is the number of n-colorings of the graph G. This turns out to be a polynomial in t by virtue of the contraction/deletion property: $C_{G\setminus e}(t) = C_G(t) + C_{G/e}(t)$. Stanley showed that $C_G(-1)$ is equal to the number of *acyclic orientations* of the graph G.

For our purposes, we view a coloring of a graph with n vertices as a particular graph homomorphism $G \rightarrow K_n$ to the complete graph on n vertices. Although colorings and chromatic number will be important examples and applications of our constructions and results, we will be more interested in the structure of the category in general. Interestingly, many applications to bounding chromatic number arise only when one considers the entire category.

One such example is the Hom complexes themselves. We will see that $\mathrm{Hom}(T, G)$ is a space that parametrizes the graph maps from T to G (definitions are given in the next chapter). In his original paper, Lovász showed that the chromatic number of a graph is bounded below by the connectivity of (a complex later shown to be homotopy equivalent to) the space $\mathrm{Hom}(K_2, G)$.

Theorem 1.0.1 (Lovász). *If the space* $\mathrm{Hom}(K_2, G)$ *is* k*-connected, then* $\chi(G) \geq k + 3$.

Some 25 years later, answering a conjecture of Lovász, Babson and Kozlov were able to show that the connectivity of $\mathrm{Hom}(C_{2r+1}, G)$ provided the next natural bound on the chromatic number of G (here C_{2r+1} is an odd cycle).

Theorem 1.0.2 (Babson and Kozlov). *If the space* $\mathrm{Hom}(C_{2r+1}, G)$ *is* k*-connected, then* $\chi(G) \geq k + 4$.

We will discuss the relevant definitions and proofs of these facts in the next chapter. For now, we would like to point out that bounds on chromatic number $\chi(G)$ are obtained from a consideration of the maps from a 'test graph' T to the graph G, with no mention of maps to complete graphs.

In fact, the Hom complex can be seen as a certain example of a general method of reducing the question of what (graphs) G *maps into* (in this case the complete graph K_n) to one of what *maps into* G. We write $G \rightarrow H$ if G admits a map to H, and $G \nrightarrow H$ if no such map exists. This consideration has some interesting consequences in terms of notions of duality in the category of graphs and other relational structures (see [NT00]). It turns out this question is also related to 'density' of the category of graphs (see [Neš99]): if $\chi(H) \geq 3$ and $G \rightarrow H$ but $G \nleftarrow H$, then there exists a graph G' such that $G \rightarrow G' \rightarrow H$ but $G \nleftarrow G'$ and $G' \nleftarrow H$. In terms of complexity issues, this is related to the fact that it is NP-complete to determine whether a graph admits a homomorphism to a fixed graph H if and only if H is not bipartite (see [HN90]).

The study of graph homomorphisms naturally leads to considerations of products and other monoidal structures. The two most popular such structures, the 'categorical' and the 'cartesian' product, will both be discussed in this thesis in relation to the Hom complex and other topological methods of study. We mention here a relevant (still open) question regarding the connection between the (categorical) product and colorings.

Conjecture 1.0.3 (Hedetniemi). *If G and H are graphs, then*

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}.$$

The inequality $\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$ is clear, and hence one must show that whenever $G \times H \rightarrow K_n$ then either $G \rightarrow K_n$ or $H \rightarrow K_n$. This is of course reminiscent of our discussion regarding duality, as the product $G \times H$ is an object that admits maps *from* other graphs. Another favorite conjecture of ours describes the chromatic number of a certain amalgamation of cliques (complete graphs).

Conjecture 1.0.4 (Erdös-Faber-Lovász). *If a graph G is the union of k copies of k-cliques intersecting in at most one vertex pairwise, then $\chi(G) = k$.*

The application of topological methods to the study of graph homomorphisms will rely heavily on equivariant (mostly $\mathbb{Z}_2$) methods. In fact, the original Lovász bound came from a Borsuk-Ulam type obstruction to equivariant maps between $\mathbb{Z}_2$-antipodal spheres. Although the original statements of bounds on $\chi(G)$ were in terms of the *connectivity* of the $\text{Hom}(T, G)$ complexes, a more careful consideration of the relevant group action has become an important aspect the study of Hom complexes.

It turns out that when G is a loopless graph (as is the case when we consider the chromatic number), and T carries a $\mathbb{Z}_2$-action that flips an edge, the space $\text{Hom}(T, G)$ is a free $\mathbb{Z}_2$-spaces, and one can consider a list of numerical invariants that measure the complexity of this action. As a way to take advantage of this $\mathbb{Z}_2$-topology, Babson and Kozlov introduced the use of *characteristic classes* into the study of Hom complexees. They proposed and partially (according to the parity of $\chi(G)$) proved the following result incorporating the $\mathbb{Z}_2$-action on the Hom complex.

Theorem 1.0.5 (Babson and Kozlov, Schultz). *For any graph G we have*

$$\text{ht}_{\mathbb{Z}_2}\text{Hom}(C_{2r+1}, G) \leq \chi(G) - 3.$$

Here, for a free $\mathbb{Z}_2$-space X, $\text{ht}_{\mathbb{Z}_2}(X)$ is the highest nonvanishing power of the first Stiefel-Whitney class of $X/\mathbb{Z}_2$ (definitions to be given below). Since the connectivity of X, denoted $\text{conn}(X)$, satisfies $\text{conn}(X) + 1 \leq \text{ht}_{\mathbb{Z}_2}\text{X}$, this implies the connectivity bound that they did succeed in proving completely. This statement is not only strictly stronger than that of Theorem 1.0.2 but also represents (at least in theory) a *computable* criterion, in the sense that the Stiefel Whitney class is an element of the simplicial cohomology and hence amenable to techniques from linear algebra.

Since the original papers of Babson and Kozlov regarding the structural theory of the Hom complexes and ultimately the proof of the Lovász conjecture (see [BK06], [BK03], and [BK07]), several others have investigated the Hom complexes from a variety of perspectives. These include [Koz06a], [Koza], and [Scha] regarding further understanding of odd cycle complexes and short proofs of the Lovásáz conjecture, [CLSW04] and [MZ04] regarding other notions of the original Lovász complex, [ČK05] and [Eng06] regarding connections to maximum degree of the 'test graph', [Cso] and [Docc] regarding homotopy types of Hom complexes, [Doca], [Docb], and [Schb] regarding connections to discrete homotopy, [Kah07] incorporating random graphs, [Koz06c] and [Koz06b] with an in depth discussion of 'foldings' in the context of Hom complexes, [Kozb] regarding homology test complexes, [CL] and [Schc] regarding 'graph coloring manifolds', [Pfe07] regarding connections to triangulations and the 'Cayley trick', [LZ] regarding hypergraphs, and [Živ05], [Ziva], and [Zivb] with a discussion of generalizations.

We also would like to mention that the Hom complex is one example of several approaches to incorporate the methods of other branches of mathematics into the study of graph homomorphisms (especially colorings). For example, in [Bre98], Brenti conjectured that the chromatic polynomial of a graph G could be realized as the Hilbert polynomial of some associated graded algebra; the first explicit construction of such an object was given by Steingrímsson in [Ste01]. Here the author constructs a monomial ideal I_G in the Stanley-Reisner ring A of the order complex of the Boolean poset on $|V(G)|$, whose monomials are in one-to-one correspondence with the proper colorings of G. The ideal is generated by square-free monomials, and hence A/I_G is the Stanley-Reisner ring of a simplicial complex C_G which Steingrímsson calls the 'coloring complex' of the graph G. In related work, the authors of [HGR05] and [Sto06] seek a 'categorization' of the chromatic polynomial by constructing a chain complex of graded modules over the ring of polynomials whose Euler characteristic is equal to the chromatic number of a given graph. This is similar in spirit to work of Khovanov [Kho00] regarding the categorization of the Jones polynomial.

We now say a few words about the organization of this thesis. In chapter 2 we collect together some of the basic objects of study from the theory of graph homomorphism and topological combinatorics. Here we discuss foundational aspects of the category of graphs, some definitions and tools from poset topology, as well as the basics of equivariant topology. We end the chapter with a discussion of an alternative approach to the category of graphs, as well as notions of group objects in the category. Although these last two sections are not used in the subsequent chapters of the thesis, we have not found them in the literature and have decided to include them here.

In Chapter 3 we introduce the main object of our study, the Hom complex, and discuss some of its properties relevant to our study. Along the way, we also establish several results that describe the interaction of the Hom complex with certain graph theoretical operations, including exponentials and arbitrary products. We also include a brief historical account of the origins and development of the Hom complex, including sketches of the Lovász and Babson-Kozlov results regarding bounds on chromatic number. We end this chapter with a discussion regarding the Hom complex associated with a more general notion of graphs.

Much of the material in Chapter 4 is taken from [Doca]. Here we investigate a notion of $\times$-homotopy of graph maps that is based on the internal hom associated to the categorical product in the category of graphs. It is shown that graph $\times$-homotopy is characterized by the topological properties of the Hom complex. Graph $\times$-homotopy naturally leads us to a notion of homotopy equivalence which we show has several equivalent characterizations. Here we see the importance of 'foldings' of graphs in the context of Hom complexes. We apply the notions of homotopy equivalence to the class of dismantlable graphs to get a list of conditions that again characterize these. We end with a discussion of graph homotopies arising from other internal homs, including the construction of the 'A-theory' associated to the cartesian product in the category of reflexive graphs.

In Chapter 5 we prove that Hom complexes are 'universal' in the sense that given a connected nontrivial graph T and an arbitrary finite simplicial complex X, there is a graph G such that the complex $\mathrm{Hom}(T, G)$ is homotopy equivalent to X. The proof is constructive, and uses a nerve lemma. Along the way we discuss connections between foldings of graphs and subdivisions of Hom complexes. Material in this chapter is taken from [Docc].

Chapter 6 is based on material from [Docb]. Here the notion of $\times$-homotopy from Chapter 4 is investigated in the context of the category of pointed graphs. The main result is a long exact sequence that relates the higher homotopy groups of the space $Hom_*(G, H)$ with the homotopy groups of $Hom_*(G, H^I)$. As a corollary it is shown that $\pi_i(\mathrm{Hom}_*(G, H)) \simeq [G, \Omega^i H]_\times$, where ΩH is the graph of closed paths in H; hence the (usual) homotopy groups of a pointed version of the Hom complex can be identified with the graph theoretically defined objects of discrete homotopy. This is similar in spirit to the results of [BBdLL06], where the authors seek a space whose homotopy groups encode a similarly defined homotopy theory for graphs. The categorical connections to those constructions are discussed.

Chapter 7 represents joint work with Carsten Schultz. Here we use some constructions in equivariant topology to produce new 'test graphs' for bounds on chromatic number. Bounds obtained from these new graphs are related to the original Lovász bounds in much the same way as the odd cycle results demonstrate. We discuss these connections and why we view our graphs as higher dimensional analogues of odd cycles.

Chapter 2

THE CATEGORY OF GRAPHS AND BASIC OBJECTS OF STUDY

2.1 The category of graphs

We will work in the category of graphs. A *graph* $G = (V(G), E(G))$ consists of a vertex set $V(G)$ and an edge set $E(G) \subseteq V(G) \times V(G)$ such that if $(v, w) \in E(G)$ then $(w, v) \in E(G)$. Hence our graphs are undirected and do not have multiple edges, but may have loops (if $(v, v) \in E(G)$). If $(v, w) \in E(G)$ we will often say that v and w are *adjacent* and denote this as $v \sim w$. Given a pair of graphs G and H, a *graph homomorphism* (or *graph map*) is a map of the vertex set $f : V(G) \to V(H)$ that preserves adjacency: if $v \sim w$ in G, then $f(v) \sim f(w)$ in H (equivalently $(v, w) \in E(G)$ implies $(f(v), f(w)) \in E(H)$). With these as our objects and morphisms we obtain a category of graphs which we will denote $\mathcal{G}$. If G and H are graphs, we will use $\mathcal{G}(G, H)$ to denote the set of graph maps between them.

In this section, we record some of the structure of $\mathcal{G}$. Of particular importance for us will be the existence of an *internal hom* associated to the categorical product. We start by reviewing related constructions (for undefined categorical terms, see [ML98]; for more about graph homomorphisms and the category of graphs, see [HN04] and [GR01]).

Definition 2.1.1. *If G and H are graphs, then the categorical coproduct $G \amalg H$ is the graph with vertex set $V(G) \amalg V(H)$ and with adjacency given by $(x, x') \in E(G \amalg H)$ if $(x, x') \in E(G)$ or $(x, x') \in E(H)$.*

Definition 2.1.2. *If G and H are graphs, then the categorical product $G \times H$ is a graph with vertex set $V(G) \times V(H)$ and adjacency given by $(g, h) \sim (g', h')$ in $G \times H$ if both $g \sim g'$ in G and $h \sim h'$ in H (see Figure 2.1).*

Definition 2.1.3. *For graphs G and H, the categorical exponential graph H^G is a graph with vertex set $\{f : V(G) \to V(H)\}$, the collection of all vertex set maps, with adjacency given by $f \sim f'$ if whenever $v \sim v'$ in G we have $f(v) \sim f'(v')$ in H (see Figure 2.2).*

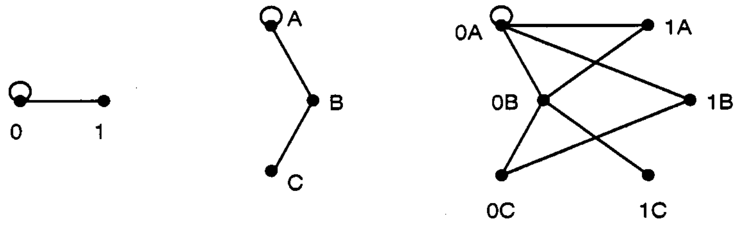

Figure 2.1: The graphs G, H, and $G \times H$.

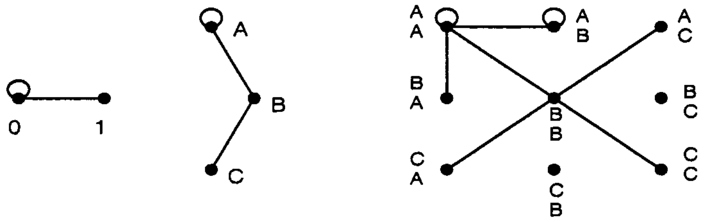

Figure 2.2: The graphs G, H, and H^G.

The next lemma shows that the exponential graph construction provides a right adjoint to the categorical product. This gives the category of graphs the structure of an *internal hom* associated with the (monoidal) categorical product.

Lemma 2.1.4. *For graphs A, B and C, we have a natural isomorphism of sets*

$$\varphi : \mathcal{G}(A \times B, C) \to \mathcal{G}(A, C^B)$$

given by $(\varphi(f)(v))(w) = f(v, w)$ for all $f \in \mathcal{G}(A \times B, C)$, $v \in V(A)$, $w \in V(B)$.

Proof. Let $f : A \times B \to C$ be an element of $\mathcal{G}(A \times B, C)$. To see that $\varphi(f) \in \mathcal{G}(A, C^B)$, suppose that $a \sim a'$ are adjacent vertices in A. We need $\varphi(f)(a)$ and $\varphi(f)(a')$ to be adjacent vertices in C^B. To check this, suppose $b \sim b'$ in B. Then we have $\varphi(f)(a)(b) = f(a, b)$ and $\varphi(f)(a')(b') = f(a', b')$, which are adjacent vertices of C since f is a graph map.

To check naturality, suppose $f : A \to A'$ and $g : C \to C'$ are graph maps. We need to verify that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{G}(A \times B, C) \arrow[d, "\varphi"'] & \mathcal{G}(A' \times B, C') \arrow[l, "(f\times B,g)"'] \arrow[d, "\varphi"] \\
\mathcal{G}(A, C^B) & \mathcal{G}(A', (C')^B) \arrow[l, "(f,g^B)"]
\end{tikzcd}$$

For this, let $\alpha \in \mathcal{G}(A' \times B, C)$. Then on the one hand we have $(\varphi(f \times B, g))(\alpha)(a)(b) = (f \times B, g)(\alpha)(a, b) = g(\alpha(f(a), b))$. In the other direction, we have $((f, g^B)(\varphi))(\alpha)(a)(b) = g(\varphi(\alpha)(f(a))(b)) = g(\alpha(f(a), b))$. Hence the diagram commutes, and so the isomorphism φ is natural. ☐

The category $\mathcal{G}$ is also closed under taking limits and colimits.

Lemma 2.1.5. *The category $\mathcal{G}$ has all finite limits and colimits.*

Proof. It suffices to show that the category $\mathcal{G}$ has equalizers and coequalizers, and finite products and coproducts.

We have seen that graphs have products and coproducts. For the others, suppose we have a pair of maps $f, g : G \to H$. Then the equalizer will be the inclusion $i : X \to G$, where $X \subseteq G$ is the induced subgraph of G on the vertex set $V(X) = \{v \in V(G) : f(x) = g(x)\}$.

The coequalizer will be the projection $p : H \to Y$. Here, Y is defined to be the graph with vertex set $V(Y) = V(H)/\sim$, where $\sim$ is the equivalence relation on $V(H)$ generated by $f(x) = g(x)$ for some $x \in V(G)$. Adjacency in Y is given by $[y] \sim [y']$ if $y \sim y'$ for some representatives of the equivalence classes. ☐

We close this section with a few additional definitions. The *terminal object* in $\mathcal{G}$ is the graph consisting of a single vertex and a single (looped) edge, a graph we will denote as $\mathbf{1}$. The *initial object* is the empty graph, which we denote as $\emptyset$.

A *reflexive* graph G is a graph with loops on all its vertices ($v \sim v$ for all $v \in V(G)$). A map of reflexive graphs will be a graph map on the underlying graph. We will use $\mathcal{G}^\circ$ to denote the category of reflexive graphs.

We see that $\mathcal{G}^\circ$ is a subcategory of $\mathcal{G}$, and we let $i : \mathcal{G}^\circ \to \mathcal{G}$ denote the inclusion functor. Let $S : \mathcal{G} \to \mathcal{G}^\circ$ denote the functor given by taking the subgraph induced by looped vertices, and $L : \mathcal{G} \to \mathcal{G}^\circ$ denote the functor given by adding loops to all vertices (see Figure 2.3).

One can check that i is a left adjoint to S, whereas i is a right adjoint to L. As functors $\mathcal{G} \rightarrow \mathcal{G}$, one can check that L (strictly speaking iL) is a left adjoint to S (strictly speaking iS). We will make some use of these facts in a later section.

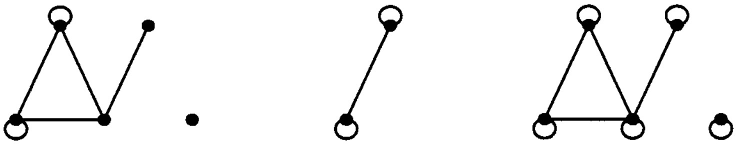

Figure 2.3: The graph G, and the reflexive graphs $S(G)$ and $L(G)$.

If v and w are vertices of a graph G, the *distance* $d(v, w)$ is the length of the shortest path in G from v to w. The *diameter* of a finite connected graph G, denoted $\mathrm{diam}(G)$ is the maximum distance between two vertices of G. The *neighborhood* of a vertex v, denoted $N_G(v)$ (or $N(v)$ if the context is clear), is the set of vertices that are adjacent to v (so that $v \in N(v)$ if and only if v has a loop).

There are several simplicial complexes one can associate with a given graph G. One such construction is the *clique complex* $\Delta(G)$, a simplicial complex (a notion defined in Section 2.2.1) with vertices given by all *looped* vertices of G, and with faces given by all cliques (complete subgraphs) on the looped vertices of G.

2.2 Some poset topology

In this section we briefly review some basics of poset topology that will be used throughout the thesis. Our main source will be Björner's excellent survey [Bjö95], as well as the topology texts [Spa66], [May99], and [Hat02]. Readers familiar with the material are invited to skip this section.

2.2.1 Basic objects

An *(abstract) simplicial complex* $\Delta = (V, \Delta)$ is a set V of vertices, together with a collection of finite subsets of V (called *simplicies* of *faces*) with the property that $v \in \Delta$ for all $v \in V$

and also $\sigma \subseteq \tau \in \Delta$ implies $\tau \in \Delta$. The *face poset* $F(\Delta)$ of a simplicial complex (or more generally a regular cell complex) Δ is the set of faces ordered by inclusion.

If $P = (P, \leq)$ is a poset (partially ordered set), a totally ordered subset $x_0 \leq x_1 \leq \cdots \leq x_k$ is called a *chain of length* k. If $x \in P$, we use $P_{\leq x}$ (respectively $P_{\geq x}$) to denote the subposet induced by elements y such that $y \leq x$ (respectively $y \geq x$). The collection of all chains of P forms a simplicial complex, called the *order complex* of P, denoted $\Delta(P)$. We use $|P|$ to denote the *geometric realization* of this simplicial complex, or sometimes simply P if the context is clear. A *poset map* $f : P \to Q$ (which is by definition an order-preserving map on the underlying sets) induces a simplicial map on the order complexes, and hence a topological (continuous) map $|f| : |P| \to |Q|$. Again, we will often denote this map as simply $f : P \to Q$.

For a complex Δ, we let $\mathrm{bd}(\Delta) := \Delta\big(F(\Delta)\big)$ denote the (first) barycentric subdivision of Δ. This process can be iterated, and we let $\mathrm{bd}^k(\Delta)$ denote the k^{th} barycentric subdivision of Δ. A basic fact is that Δ and $\mathrm{bd}(\Delta)$ are homeomorphic.

For completeness, we mention one other functorial way to assign a topological space to a poset P (this notion will not be used in the thesis): we let the *order-ideals* (by definition, subsets $A \subseteq P$ with the property that $x \leq y \in A$ implies $x \in A$) define open sets of a topological space $T(P)$. This gives the *ideal topology* on P, which has a surprisingly rich homotopy theory (see for instance [McC66]) and has relevance to the study of sheaf cohomology on posets.

The *direct product* $P \times Q$ of two posets is defined to be the Cartesian product of the underlying sets with the relation $(x, y) \leq (x', y')$ if $x \leq x'$ in P and $y \leq y'$ in Q. The *join* $P * Q$ of two posets is their disjoint union with the relation given by making each element of P less than each element of Q and otherwise maintaining the given ordering within P and Q. It is a basic fact (see [Qui78] and [Wal88]) that if P and Q are posets, there exist the following homeomorphisms:

$$|P \times Q| \approx |P| \times |Q|,$$

$$|P * Q| \approx |P| * |Q|.$$

If P and Q are posets, the *poset of maps* $\mathrm{Poset}(P, Q)$ is defined to be the poset whose elements are all order preserving (poset) maps $P \to Q$, and with the relation $f \leq q$ if $f(x) \leq g(x)$ for all $x \in P$. If P and Q are posets with actions by some group Γ, then we let $\mathrm{Poset}_\Gamma(P,Q)$ denote the subposet of $\mathrm{Poset}(P,Q)$ given by all Γ-equivariant poset maps.

We refer to [Hat02] for the definitions of homotopy and homology in the category of topological spaces. A space X is called *k-connected* if for all $0 \leq i \leq k$, every map from the i-sphere $f : \mathbb{S}^i \to X$ can be extended to a map from the $(i+1)$-ball $\bar{f} : \mathbb{B}^{i+1} \to X$. We define $(-1)-$ *connected* to mean 'nonempty'. We will use the notation $\mathrm{conn}(X)$ to denote the maximum integer k for which X is k-connected. We point out that a simplicial complex Δ is *contractible* (homotopy equivalent to a point) if and only if Δ is k-connected for every $k \geq 0$, and is k-connected if and only if its $(k+1)$-skeleton is k-connected. For $i \geq 0$, we let $\pi_i(X) = \pi_i(X, x)$ denote the *ith homotopy group* of X (with basepoint x), and let $\pi_0(X)$ denote the set of path components of X.

We let $H_i(X;G)$ (respectively $H^i(X;G)$) denote the *ith (co)homology group* of the space X with coefficients in the abelian group G, and use $H_i(X) := H_i(X, \mathbb{Z})$. The notation $\tilde{H}(X;G)$ will denote the *reduced* homology groups. We say a complex Δ is *k-acyclic over G* if $\tilde{H}_i(X;G) = 0$ for all $i \leq k$. We list several useful relations between homotopy and homology properties, some of which hold only for a simplicial complex Δ.

- Δ is k-connected if and only if Δ is k-acyclic and simply connected.

- Δ is contractible if and only if Δ is $\mathbb{Z}$-acyclic and simply connected.

- If Δ is simply connected, $\tilde{H}_i(\Delta) = 0$ for $i \neq d > 1$, and $\tilde{H}_d(\Delta) \simeq \mathbb{Z}^k$, then Δ is homotopy equivalent to a wedge of k d-spheres.

- If $\dim \Delta = d \geq 0$, then Δ is $(d-1)$-connected if and only if Δ is homotopy equivalent to a wedge of d-spheres.

2.2.2 Tools

In this section, we collect some tools from poset topology that will be useful for our subsequent work. Again, these will be taken from [Bjö95] and [Qui73]; one can refer to those references for proofs.

Theorem 2.2.1 (Quillen, 1978). *Let $f : P \to Q$ be a poset map. If for all $y \leq y' \in Q$, the induced inclusion map $f^{-1}(Q_{\leq y}) \to f^{-1}(Q_{\leq y'})$ is a homotopy equivalence, then for every $x \in f^{-1}(y)$ one obtains the following long exact sequence*

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{i+1}(Q, y) \arrow[r, "\delta"] & \pi_i(f^{-1}(Q_{\leq y}), x) \arrow[dl, "\iota_*"'] & \\
& \pi_i(P, x) \arrow[r, "f_*"] & \pi_i(Q, y) \arrow[r] & \cdots
\end{tikzcd}$$

Corollary 2.2.2. *Let $f : P \to Q$ be a poset map.*

- *If all fibers $f^{-1}(Q_{\leq y})$, $y \in Q$, are contractible, then f induces a homotopy equivalence between P and Q.*
- *If all fibers $f^{-1}(Q_{\geq y}$, $y \in Q$, are k-connected, then Q is k-connected if and only if P is k-connected.*

Recall that the *nerve* of a family of sets $(A_i)_{i \in I}$ is the simplicial complex $N(A_i)$ defined on the vertex set I such that $\sigma \in N(A_i)$ whenever $\bigcap_{i \in \sigma} A_i \neq \emptyset$. The following result is often called the 'Nerve Theorem'; we use the formulation given in [Bjö95].

Theorem 2.2.3 (Borsuk 1948, Björner et al. 1985). *Let Δ be a simplicial complex and $(\Delta_i)_{i \in I}$ a collection of subcomplexes such that $\Delta = \bigcup_{i \in I} \Delta_i$.*

- *If every nonempty finite intersection $\Delta_{i_1} \cap \cdots \cap \Delta_{i_t}$ is contractible then Δ and the nerve $N(\Delta_i)$ are homotopy equivalent.*
- *If every nonempty finite intersection $\Delta_{i_1} \cap \cdots \cap \Delta_{i_t}$ is $(k - t + 1)$-connected then Δ is k-connected if and only if $N(\Delta_i)$ is k-connected.*

The next theorem, also due to Quillen, is sometimes called the 'Order Homotopy Theorem'.

Theorem 2.2.4 (Quillen). *Let $f, g : P \to Q$ be maps of posets. If $f(x) \leq g(x)$ for every $x \in P$ then f and g are homotopic.*

Corollary 2.2.5. *If $f : P \to P$ is a poset map such that $f(x) \geq x$ for all $x \in P$ then f induces a homotopy equivalence between P and $f(P)$. If in addition $f^2(x) = f(x)$ for all $x \in P$, (in which case f is called a closure map on P) then $f(P)$ is a strong deformation retract of P.*

2.2.3 Collapsing and simple homotopy type

Certain combinatorial notions of homotopy (of posets and simplicial complexes) will also be used in parts of this thesis. We review these concepts next.

Let Δ be a simplicial complex (or more generally a regular cell complex) and suppose that $\sigma \in \Delta$ is a proper face of precisely one face $\tau \in \Delta$. Then the complex $\Delta' := \Delta \backslash \{\sigma, \tau\}$ is obtained from Δ by an *elementary collapse.* In this case $\Delta \simeq \Delta'$ and is said to be *simple homotopy equivalent.* It turns out that two simply connected finite complexes Δ and Δ' are homotopy equivalent if and only if one can find a sequence of elementary collapses and anticollapses that transforms Δ into Δ' (see [Coh73]). More generally, two complexes X and Y are simple homotopy equivalent if and only if there exists a homotopy equivalence $f : X \to Y$ whose Whitehead torsion (an element of the *Whitehead group* of $\pi_1(Y)$) vanishes; this provides an interesting connection between simple homotopy theory and higher algebraic K-theory.

One can define other recursive structures on simplicial complexes such as shellablity, vertex-decomposability, etc.; these are often natural ways to determine high connectivity of the space in question. For the most part these notions will not be used in the thesis.

2.3 Free $\mathbb{Z}_2$-spaces and Stiefel-Whitney classes

In this section we review some basics of group actions and equivariant topology. We will focus our attention on the group $\mathbb{Z}_2$ (the group of order 2), although many of these constructions make sense for arbitrary groups. The primary references here will be [Koza] and the textbooks [MS74] [May99], and [tD87]. Once again, readers familiar with these notions are invited to skip this section.

Our basic setup will be a simplicial complex (or regular cell complex) X with a given cellular action by some group Γ (usually $\mathbb{Z}_2$). In the case that Γ acts freely, X will be called a *Γ-space.* By the general theory of Γ-bundles, there exists a Γ-equivariant map $f : X \to E\Gamma$ that fits into the appropriate pullback square; the induced map on the quotient $X/\Gamma \to E\Gamma/\Gamma := B\Gamma$ is unique up to homotopy and is called the *classifying map* for this action. Here $E\Gamma$ is a contractible Γ-space, and $B\Gamma$ is the *classifying space* for Γ.

Passing to cohomology, we see that the induced map $f^* : H^*(B\Gamma) \to H^*(X/\Gamma)$ does not depend on the choice of f, and hence any element $z \in H^*(B\Gamma)$ has a well defined image $f^*(z) := X(z)$ in $H^*(X/\Gamma)$, called the *characteristic class* associated to z. If $g : X \to Y$ is an equivariant map of Γ-spaces, then the above discussion implies the very important *naturality* of the characteristic classes: $X(z) = g^*(Y(z))$.

We next specialize to the case when X is a free $\mathbb{Z}_2$-space. In this case we can take $E\mathbb{Z}_2 = \mathbb{S}^\infty_a$, the infinite-dimensional sphere with the antipodal action, and in the above discussion we get a map $f : X/\mathbb{Z}_2 \to \mathbb{S}^\infty/\mathbb{Z}_2 \simeq \mathbb{R}P^\infty$. We take cohomology with $\mathbb{Z}_2$ coefficients and get an algebra map $f^* : \mathbb{Z}_2[z] \simeq H^*(\mathbb{R}P^\infty; \mathbb{Z}_2) \to H^*(X/\mathbb{Z}_2; \mathbb{Z}_2)$ which is completely determined by the image of z. The element $f^*(z)$ is called the *first Stiefel-Whitney class* of X and will be denoted $w_1(X)$.

Stiefel-Whitney classes can be used to provide obstructions to the existence of equivariant maps between $\mathbb{Z}_2$-spaces. Consideration of Stiefel Whitney classes of spheres with the antipodal action can be used to prove the famous Borsuk-Ulam Theorem.

Theorem 2.3.1 (Borsuk-Ulam). *If $f : \mathbb{S}^m_a \to \mathbb{S}^n_a$ is an equivariant map of spheres with antipodal actions, then $m \leq n$.*

There are several numerical invariants one can associate to $\mathbb{Z}_2$-space X. The *height* of X, denoted $\mathrm{ht}_{\mathbb{Z}_2}(X)$ is defined to be the highest nonvanishing power of $w_1(X)$ in $H^*(X/\mathbb{Z}_2; \mathbb{Z}_2)$. We define the *index*, denoted $\mathrm{ind}_{\mathbb{Z}_2}(X)$, to be the minimum integer n for which there exists an equivariant map $X \to \mathbb{S}^n_a$. The *coindex*, denoted $\mathrm{coind}_{\mathbb{Z}_2}(X)$, is the maximum integer m such that $\mathbb{S}^m_a \to X$. One can check that these values satisfy the following string of inequalities:

$$\mathrm{conn}(X) + 1 \leq \mathrm{coind}_{\mathbb{Z}_2}(X) \leq \mathrm{ht}_{\mathbb{Z}_2}(X) \leq \mathrm{ind}_{\mathbb{Z}_2}(X).$$

One can also check that if $X \rightarrow Y$ is an equivariant map of $\mathbb{Z}_2$-spaces, then $\alpha(X) \leq \alpha(Y)$, where $\alpha(?)$ is any of the numerical invariant listed above. These inequalities will be important in our discussion of topological bounds on chromatic number.

2.4 Other approaches to the graph category

We take a bit of a detour in this section to discuss an alternative approach to the construction of the category of graphs. This section is not required for the main chapters of this thesis; it will only come up again in the last section of the next chapter. In fact one can turn to Chapter 3 of the thesis at this point and not sacrifice any continuity. We include this section here because we feel it is an important perspective that inspired many of our constructions.

Our basic viewpoint here will be that a graph object is a (contravariant) functor from a category $\mathcal{C}$ that is a certain truncation of the simplex category. Again, the important difference is that the category $\mathcal{C}$ is missing the analogous degeneracy maps; this provides a significant restriction on the collection of graph morphisms. To make our graphs 'undirected', we introduce an additional involution on the edge set. The restriction on the morphisms is that the edge maps must be equivariant.

Although this approach seems to complicate matters, there are in fact several advantages. First, the representation of graphs as a diagram category leads to immediate conclusions regarding its structure. Secondly, the similarity to the well-developed category of simplicial (sets) suggests constructions that one is familiar with in that context. Finally, as a functor from $\mathcal{C}$ we are naturality led to graphs in other categories. These will play some role in later chapters.

We let $\mathcal{DG}$ denote the category with two objects $V = \{0\}$ and $E = \{0, 1\}$, and with two non-identity morphisms $h, t : V \rightarrow E$.

Definition 2.4.1. *Let $\mathcal{C}$ be a category. A directed graph on $\mathcal{C}$ is a contravariant functor $G : \mathcal{DG} \rightarrow \mathcal{C}$. A directed graph is a directed graph on SET, the category of sets.*

More concretely, a directed graph $G = (E(G), V(G), h_G, t_G)$ on $\mathcal{C}$ is a pair of objects $E(G), V(G) \in \mathrm{Ob}(\mathcal{C})$ together with a pair of morphisms $h_G, t_G : E(G) \to V(G) \in \mathrm{Mor}(\mathcal{C})$.

$$\begin{tikzcd}
E(G) \arrow[r, shift left, "h_G"] \arrow[r, shift right, "t_G"'] & V(G)
\end{tikzcd}$$

A map $f = (f, \tilde{f}) : G \to H$ between directed graphs $G = (E(G), V(G), h_G, t_G)$ and $H = (E(H), V(H), h_H, t_H)$ (on the same category) is a pair of morphisms $f : V(G) \to V(H)$ and $\tilde{f} : E(G) \to E(H)$ such that the left and right squares commute in the following diagram:

$$\begin{tikzcd}
E(G) \arrow[r, "\tilde{f}"] \arrow[d, shift right, "t_G"'] \arrow[d, shift left, "h_G"] & E(H) \arrow[d, shift right, "t_H"'] \arrow[d, shift left, "h_H"] \\
V(G) \arrow[r, "f"'] & V(H)
\end{tikzcd}$$

This defines a category, which we call DIGRAPH($\mathcal{C}$).

Next, we let $\mathcal{G}$ denote the category with the same objects and morphisms of $\mathcal{DG}$, with the additional morphism $\sigma : E \to E$ such that $\sigma^2 = \mathrm{id}_E$, and $\sigma h = t$.

Definition 2.4.2. *An undirected graph on a category $\mathcal{C}$ is a contravariant functor $G : \mathcal{G} \to \mathcal{C}$.*

More concretely, this means that an undirected graph $G = (E(G), V(G), h_G, t_G, \sigma_G)$ on $\mathcal{C}$ is a directed graph $(E(G), V(G), h_G, t_G)$ on $\mathcal{C}$ together with a morphism $\sigma_G : E(G) \to E(G)$ such that $\sigma_G \circ \sigma_G = \mathrm{id}_{E(G)}$ and such that the following diagram commutes:

$$\begin{tikzcd}
E(G) \arrow[r, "h_G"] \arrow[d, "\sigma_G"] & V(G) \arrow[d, "\mathrm{id}_{V_G}"] \\
E(G) \arrow[r, "t_G"] & V(G)
\end{tikzcd}$$

A map $f = (f, \tilde{f}) : G \to H$ between undirected graphs is a map between the underlying directed graphs, with the extra condition that $\sigma_H \tilde{f} = \tilde{f} \sigma_G$. This defines a category GRAPH($\mathcal{C}$). The category GRAPH will denote the category of undirected graphs on SET.

A *graph* will be an object in this category. If G and H are graphs, we will denote the set of maps between them as $GRAPH(G, H)$.

This definition of the graph category is more along the lines of Serre's use in [Ser80], as well as that of Stallings in [Sta83].

2.4.1 Useful terminology

An *edge* in a graph G is a pair $\{E, \sigma_G(E)\} \subseteq E(G)$. A *loop* is an edge $\{E, \sigma_G(E)\} \subseteq E(G)$ such that $h_G(E) = t_G(E)$ (so that $h_G(\sigma_G(E)) = t_G(\sigma(E))$). A *semiloop* in a graph G is a single element $E \in E(G)$ such that $\sigma(E) = E$ (so that $h_G(E) = t_G(E)$). The semiloops are precisely the fixed points of σ_G.

We will often refer to an edge (loop, semiloop) of a graph G as a single element of the set $E(G)$. The following simple observation justifies this practice, and will be useful in our calculations.

Lemma 2.4.3. *Suppose $G = (V(G), E(G), h_G, t_G, \sigma_G)$ is a graph. If $E \in E(G)$, then there is an element $F \in E(G)$ such that $h_G(F) = t_G(E)$ and $t_G(F) = h_G(E)$.*

Proof. Recall the following diagram

$$\begin{tikzcd}
E(G) \arrow[r, "h"] \arrow[d, "\sigma"] & V(G) \arrow[d, "\mathrm{id}_V"] \\
E(G) \arrow[r, "t"] & V(G)
\end{tikzcd}$$

If $\sigma(E) = F$, then we have $h(E) = t\sigma(E) = t(F)$ and $h(F) = h(\sigma(E)) = t\sigma^2(E) = t(E)$. Note that if $\sigma(E) = E$, then we get $F = E$ (and hence E is a semiloop). ☐

If G is a graph and $u, v \in V(G)$, we will say that u and v are *adjacent in G* (also sometimes notated $u \sim v$ or $(u, v) \in E(G)$) if we have some $E \in E(G)$ such that $h_G(E) = u$ and $t_G(E) = v$.

Definition 2.4.4. *Let $G = (E(G), V(G), h_G, t_G, \sigma_G)$ be a graph, and let $v \in V(G)$. The neighborhood of v, denoted $N(v)$, is the collection of vertices adjacent to v. More symbolically, $N(v) = h_G(t_G^{-1}(v))$.*

Note that if $f = (f, \tilde{f}) : G \to H$ is a graph map, then $f(N(v)) \subseteq N(f(v))$ for all $v \in V(G)$. To see this, let $x \in f(N(v)) = f(h_G(t_G^{-1}(v)))$. If $t_G^{-1} = \emptyset$, then the claim is clear, otherwise take $E \in E(G)$ such that $t_G(E) = v$ and such that $x = f(h_G(E))$. Then we have $x = f(h_G(E)) = h_H(\tilde{f}(E)) \in h_H(t_H^{-1}(t_H(\tilde{f}(E))) = h_H(t_H^{-1}(f(t_G(E))) = h_H(t_H^{-1}(f(v)) = N(f(v))$.

2.5 *Group objects and actions*

We end this chapter with a somewhat tangential discussion of group objects in the category $\mathcal{G}$. Again, this section is more or less self-contained, and is not required for the rest of the thesis. We point out that our approach here is similar to the discussion in [Bro94].

2.5.1 Graph structures on groups

Definition 2.5.1. *Let G be a group. A graph structure on G is a graph (G, E) with elements of G as vertices, and a set E of edges including a loop on the identity vertex, and such that the multiplication $m : G \times G \to G$ and inversion $i : G \to G$ are both graph maps. With this assignment of edges, we will call G a graphed group.*

If the resulting graph is simple (no more than a single edge between a pair of vertices), then we will have a simple graph structure on the graphed group G.

Note that giving a graph structure on a group is the same as specifying a group object in the category of graphs (with $\mathbf{1}$ as the terminal object $\mathbf{1}$, and product as above) with $m : G \times G \to G$ the given group multiplication, $e : \mathbf{1} \to G$ the inclusion of the identity, and $i : G \to G$ the given inversion map. Given a group object G in the category of graphs, we get a group object in the category of sets (i.e. a group) via a forgetful functor.

Examples

- We can give any group G the *trivial* graph structure by including only the single edge $(e, e) \in E(G)$.

- We can give G the *discrete* graph structure by placing loops on each vertex. In this case each edge in $G \times G$ is of the form $((g, g), (g, g))$; this maps to (g^2, g^2) under m

so that m is indeed a graph map. Also, each edge $(g, g) \in G$ gets sent to (g^{-1}, g^{-1}) under i, and hence i is also a graph map.

- We can give G the *complete* graph structure by giving G all possible edges (including loops). In this case there are no obstructions to maps.

Note that if H is a subgroup of G then a graph structure on H can be extended to a graph structure on G via $(g, g') \in E(G)$ if and only if $g, g' \in H$ and $(g, g') \in E(H)$.

2.5.2 Graph structures on the automorphism group of a graph

For a graph G, we can use the exponential graph construction to associate a graph structure to the group of permutations of the vertices of G.

Definition 2.5.2. *For a graph G, $\mathrm{AUT}(G)$ is defined to be the induced subgraph of G^G with vertices given by the vertex bijections of $V(G)$.*

Note that this $\mathrm{AUT}(G)$ is not the typical automorphism group $\mathrm{Aut}(G)$ associated to graph G, whose elements are given by the invertible graph maps $\phi : G \to G$. However, we will see that $\mathrm{Aut}(G)$ naturally sits inside $\mathrm{AUT}(G)$ as a sub graphed group.

Lemma 2.5.3. $\mathrm{AUT}(G)$ *gives a natural graph structure on the group $S_{V(G)}$ (here S_n is the symmetric group on n letters).*

Proof. Since the vertices of $\mathrm{AUT}(G)$ are indexed by vertex bijections, we have a natural way to associate the vertices of $\mathrm{AUT}(G)$ to elements of the group $S_{V(G)}$. We first note that since the identity map $G \to G$ is a graph map, we get a loop on the vertex of $\mathrm{AUT}(G)$ that corresponds to the identity in $S_{V(G)}$. To check the multiplication, suppose $((\alpha, \beta), (\gamma, \delta)) \in E(\mathrm{AUT}(G) \times \mathrm{AUT}(G))$ so that $(\alpha, \gamma), (\beta, \delta) \in E(\mathrm{AUT}(G))$. We need to show that $(\alpha\beta, \gamma\delta) \in E(\mathrm{AUT}(G))$. For this suppose $(v, w) \in E(G)$. Since $(\beta, \delta) \in E(\mathrm{AUT}(G))$, we see that $(\beta(v), \delta(w))$ is an edge in G. But then since $(\alpha, \gamma) \in E(\mathrm{AUT}(G))$, we also have $(\alpha(\beta(v)), \gamma(\delta(w)))$ an edge in G, as desired. The inversion map is treated similarly. $\square$

For example, if $G = K_n$ is the complete graph on n vertices, then $\mathrm{AUT}(G)$ is the symmetric group S_n with the discrete graph structure.

Note that if $\varphi : G \to G$ is an invertible graph map (an element of the usual automorphism group), then φ is a bijection on the vertices of G and hence naturally a vertex of $\mathrm{AUT}(G)$. Since φ is a graph map, φ has a loop as a vertex of $\mathrm{AUT}(G)$. We then recover a discrete graph structure on $\mathrm{Aut}(G)$ as a sub grouped graph of $\mathrm{AUT}(G)$.

Lemma 2.5.4. *Let $\mathcal{F}$ denote the forgetful functor $\mathcal{F}$: GRAPH(GROUP) $\to$ GRAPH. If $G = (E, V, h, t, \sigma)$ is an object in GRAPH(GROUP) (i.e., an undirected graph on GROUP), then $\mathcal{F}(G)$ is a group object in GRAPH (more specifically a graph structure on the group V). Conversely, if H is a group with a graph structure, then there exists an object $G = (E, H, h, t, \sigma) \in$ GRAPH(GROUP) such that $\mathcal{F}(G) = H$.*

Proof. For the first statement, suppose $G = (E, V, h, t, \sigma) \in$ GRAPH(GROUP). V has a loop on the identity since if $e \in E$ is the identity we have $h(e) = t(e)$. Next we check the multiplication map on $V \times V$. Suppose (u, v) is adjacent to $(u', v') \in V \times V$; we want uv and $u'v'$ to be adjacent in V. Since (u, v) and (u', v') are both edges in V we have some $e, f \in E$ such that $h(e) = u, t(e) = u'$ and $h(f) = v, t(f) = v'$. Hence we have $h(ef) = h(e)h(f) = uv$ and $t(ef) = t(e)t(f) = u'v'$ so that uv and $u'v'$ are adjacent, as desired. For the inversion map, suppose u and v are adjacent in V. Again we have some $e \in E$ such that $h(e) = u$ and $t(e) = v$. But then $h(e^{-1}) = (h(e))^{-1} = u^{-1}$ and $t(e^{-1}) = (t(e))^{-1} = v^{-1}$, so that u^{-1} and v^{-1} are adjacent in V, as desired.

For the other direction, let H be a group with a graph structure. We define $E \subseteq H \times H$ (group product) according to $(h, h') \in E$ if and only if h and h' are adjacent in H. We claim that E is a group (a subgroup of $H \times H$). To see this, first note that $(e, e) \in E$ since e has a loop in H. Next, if $(a, a'), (b, b') \in E$, then $(a, a')(b, b') = (ab, a'b') \in E$ since (a, b) and (a', b') are adjacent in $H \times H$ (graph product) and $m : H \times H \to H$ is a graph map. Finally, if $(a, b) \in E$, then $(a, b)^{-1} = (a^{-1}, b^{-1}) \in E$ since $i : H \to H$ is a graph map.

For the maps, we define $h : E \to H$ by $h(a, b) = a$, $t : E \to H$ by $t(a, b) = b$, and $\sigma : E \to E$ by $\sigma(a, b) = (b, a)$. We take $G = (E, H, h, t, \sigma)$ and get $G \in$ GRAPH(GROUP) with $\mathcal{F}(G) = H$. ☐

Example: If G is a group with the discrete graph structure, we can consider the object Γ in GRAPH(GROUP) given by $E = V = G$, $h = t = \mathrm{id} : E \to V$, and $\sigma = \mathrm{id} : E \to E$. Then $\mathcal{F}(\Gamma) = G$.

2.5.3 Graph structures on $\mathbb{Z}_p$

Definition 2.5.5. *If G is a group, the horizontal graph on G is the graph (also written G) with vertices $V(G) = G$ and edges given by $(g,h) \in E(G)$ if and only if $h = g^{-1}$.*

Lemma 2.5.6. *If G is an abelian group, then the horizontal graph on G is a graph structure on G.*

Proof. First note that the identity $e \in G$ has a loop. Next, suppose (a,b) is adjacent to (c,d) in $G \times G$, so that (a,c) and (b,d) are edges in G, and hence $c = a^{-1}$ and $d = b^{-1}$. We have $cd = a^{-1}b^{-1} = (ba)^{-1} = (ab)^{-1}$ and hence ab is adjacent to cd, as desired. For the inversion map, suppose $(a,b) = (a,a^{-1})$ is an edge in G. Then $i(a) = a^{-1}$ is adjacent to $i(a^{-1}) = a$ and hence i is also a graph map. □

Note that the horizontal graph on the symmetric group S_3 is not a graph structure.

Lemma 2.5.7. *If $G = \mathbb{Z}_p$ is a cyclic group of prime order, then the only simple graph structures on G are trivial, discrete, complete, and horizontal.*

Proof. We have seen that these give graph structures on G. To see that these are all possible structures, suppose G has some graph structure and let (E, G, h, t, σ) be the corresponding element in GRAPH(GROUP). The group maps $h, t : E \to G$ determine a map $h \times t : E \to G \times G$ such that $\pi_1 \circ (h \times t) = h$ and $\pi_2 \circ (h \times t) = t$. This map must be injective since the graph is simple. Hence E is a subgroup of $G \times G$ and so the only possibilities are $E \simeq 0, E \simeq \mathbb{Z}_p, E \simeq \mathbb{Z}_p \times \mathbb{Z}$.

If $E \simeq \mathbb{Z}_p \times \mathbb{Z}_p$ then the projection maps give the complete graph structure. For the case $E \simeq \mathbb{Z}_p$, we claim that the only possible maps for $\sigma : E \to E$ are the maps $x \mapsto x$ (the identity, id) and $x \mapsto x^{-1}$ (call this inv). To see this, note that every automorphism $\theta : \mathbb{Z}_p \to \mathbb{Z}_p$ is of the form $x^i \mapsto x^{ji}$ for some fixed integer j (where x is some fixed generator).

Since we require $\sigma^2 = \mathrm{id}$ this means that $x^i = \sigma^2(x^i) = x^{j^2 i}$ so that $j^2 \equiv 1 \pmod{p}$. We conclude that $j = 1, -1$ as claimed. For $\sigma = \mathrm{id}$, the projection maps give the discrete graph structure on G. For $\sigma = \mathrm{inv}$ the projection maps give the horizontal graph structure. Finally, if $E = 0$ then the projection maps give the trivial graph structure. □

2.5.4 Extensions of graphed groups

Lemma 2.5.8. *Let A and B be graphed groups. If there is a split central extension of groups: $1 \to A \to G \to B \to 1$. (i.e., G is a semi-direct product of B and A), then the graph $A \times B$ determines a graph structure on the group G.*

Proof. Since the sequence splits, G has a subgroup $\tilde{B}$ isomorphic to B such that every element $g \in G$ is uniquely expressible in the form $g = i(a)(\tilde{b})$, with $a \in A, \tilde{b} \in \tilde{B}$. With this factorization we identify each element (vertex) of G with an element (vertex) (a, b) of $A \times B$ and give the group G the graph structure of the graph $A \times B$.

First note that the identity e_G has a loop in G, since $e_G = e_A e_B$. Also, with this edge assignment, the multiplication $G \times G \to G$ is a graph map. To see this, suppose (g_1, g_2) is adjacent to (g_3, g_4) in $G \times G$, so that (g_1, g_3) and (g_2, g_4) are both edges in G. We need to show that $(g_1 g_2, g_3 g_4)$ is an edge in G. For this, express each g_i as $a_i b_i$ for $1 \le i \le 4$. Then we have $(a_1 b_1, a_3 b_3), (a_2 b_2, a_4 b_4) \in E(G)$ and hence (a_1, a_3), $(a_2, a_4) \in E(A)$ and $(b_1, b_3), (b_2, b_4) \in E(B)$. Since both A and B are groups with graph structures, this means that $(a_1 a_2, a_3 a_4) \in E(A)$ and $(b_1 b_2, b_3 b_4) \in E(B)$. But then

$$E(G) = E(A \times B) \ni (a_1 a_2 b_1 b_2, a_3 a_4 b_3 b_4) = (a_1 b_1 a_2 b_2, a_3 b_3 a_4 b_4) = (g_1 g_2, g_3 g_4)$$

as desired. The penultimate equality from above follows from the fact that $A \subseteq Z(G)$.

The inversion map is treated similarly. □

The lemma does not hold for split non-central extensions. For example, the dihedral group D_4 is given as the noncentral split extension: $e \to \mathbb{Z}_4 \to D_4 \to \mathbb{Z}_2$.
If we take the horizontal graph structure on $\mathbb{Z}_4 :=< \alpha >$ and the complete graph structure on $\mathbb{Z}_2 :=< \tau >$, then the induced product structure on D_4 is not a graph structure. Indeed, we have $(1, \tau) \in E(\mathbb{Z}_2)$ and $(\alpha^2, \alpha^2), (\alpha^3, \alpha) \in E(\mathbb{Z}_4)$ and hence $(\alpha^2, \alpha^2\tau), (\alpha^3, \alpha\tau) \in E(D_4)$.

So then (α^2, α^3) and $(\alpha^2\tau, \alpha\tau)$ are adjacent in $D_4 \times D_4$. However, $(\alpha^5, \alpha^2\tau\alpha\tau) = (\alpha, \alpha)$ is not an edge (loop) in D_4 since α does not have a loop in $\mathbb{Z}_4$.

For a graph G with $|V(G)| = n$, we have seen that $\mathrm{AUT}(G)$ gives a graph structure on S_n. It would be interesting to know which graph structures arise in this way.

Question 2.5.9. *Which graph structures on S_n can be realized as $\mathrm{AUT}(G)$ for some graph G with $|V(G)| = n$?*

In Figure 2.4, we provide a complete list of (simple) graph structures on the groups S_3 and $\mathbb{Z}_6$ (hence all groupgraphs with 6 elements).

2.5.5 Graphed group actions on graphs and quotients

Definition 2.5.10. *Let H be a graph and G a graphed group. We say that G acts on H (on the left) if we there is a graph map $\alpha : G \times H \to H$ such that the following diagrams commute.*

$$\begin{tikzcd} H \arrow[r, "{(e,\mathrm{id})}"] \arrow[dr, "\mathrm{id}"'] & G \times H \arrow[d, "\alpha"] \\ & H \end{tikzcd} \qquad \begin{tikzcd} G \times G \times H \arrow[r, "\mathrm{id}\times\alpha"] \arrow[d, "m\times\mathrm{id}"] & G \times H \arrow[d, "\alpha"] \\ G \times H \arrow[r, "\alpha"] & H \end{tikzcd}$$

The graphed group $G = \mathrm{AUT}(H)$ (as defined above) with $\alpha : G \times H \to H$ given by $(\varphi, h) \mapsto \varphi(h)$ provides an example of such an action. Another example comes from $G = \mathrm{Aut}(H)$, the group of all invertible graph maps $\varphi : H \to H$. With the discrete graph structure, restricting this action gives the familiar action $\mathrm{Aut}(H) \times H \to H$ of the 'ordinary' automorphism group of H.

Given an action $\alpha : G \times H \to H$ we can now define Q, the quotient of H by the action of G, as the colimit of the diagram:

$$\begin{tikzcd} G \times H \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\pi_2"'] & H \end{tikzcd}$$

This broadens the notion of a group action on a graph.

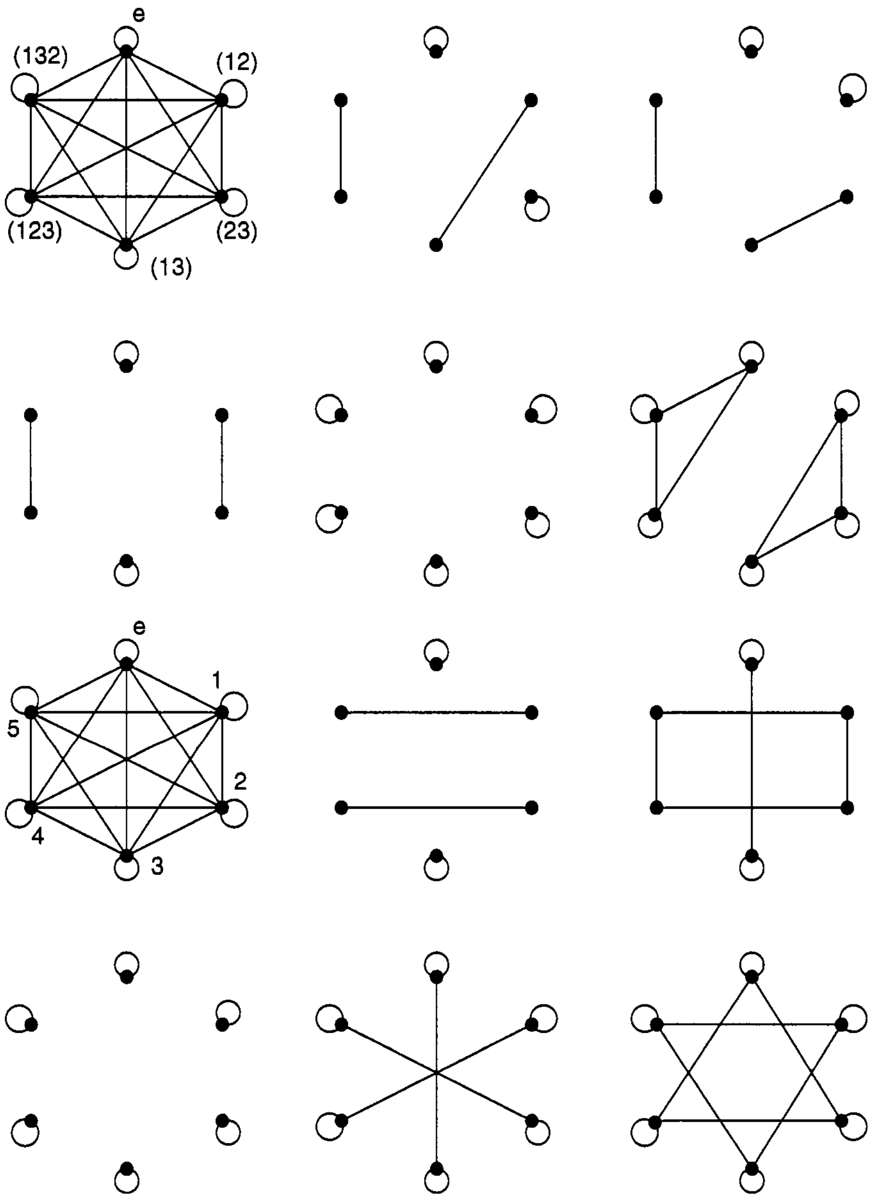

Figure 2.4: The graph structures on S_3 and $\mathbb{Z}_6$.

Chapter 3

HOM COMPLEXES

3.0.6 Introduction

In this chapter we introduce our main object of study, the Hom complex. We begin with some definitions, and then turn to a discussion of several important properties of the Hom complex. We describe its interaction with the exponential graph construction and also show that as a functor $\mathrm{Hom}(T, ?)$ preserves limits. We next turn to a discussion of 'foldings' of graphs, and describe how these relate to the Hom complexes. Much of this material is taken from the preprint [Doca].

(Versions of) the Hom complex were originally used by Lovász, Babson and Kozlov, and others to provide *topological* lower bounds on the chromatic numbers of graphs. We include a brief review of this history, in which we place these various results in a common 'modern' context.

Finally, we end the chapter with a discussion of (a candidate for) a notion of the Hom complex for general graphs (in the sense of the penultimate section of the previous chapter); the basic issue is how to deal with multiple edges. We show that the functor which takes a graph to its simplified version (collapsing edges) induces a homotopy equivalence of the relevant Hom complexes.

3.1 The Hom complex

We begin by recalling the construction of the Hom complex associated to a pair of graphs. We would like to think of this as an enrichment of the category of graphs over the category of posets, and hence offer the following definition.

Definition 3.1.1. *For graphs G, H, we define* $\mathrm{Hom}(G, H)$ *to be the poset whose elements are given by all functions* $\eta : V(G) \rightarrow 2^{V(H)} \backslash \{\emptyset\}$*, such that if* $(x, y) \in E(G)$*, then* $(\tilde{x}, \tilde{y}) \in E(H)$

for all $\tilde{x} \in \eta(x)$ *and* $\tilde{y} \in \eta(y)$. *The relation is given by containment, so that* $\eta \leq \eta'$ *if* $\eta(x) \subseteq \eta'(x)$ *for all* $x \in V$ *(see Figure 3.1).*

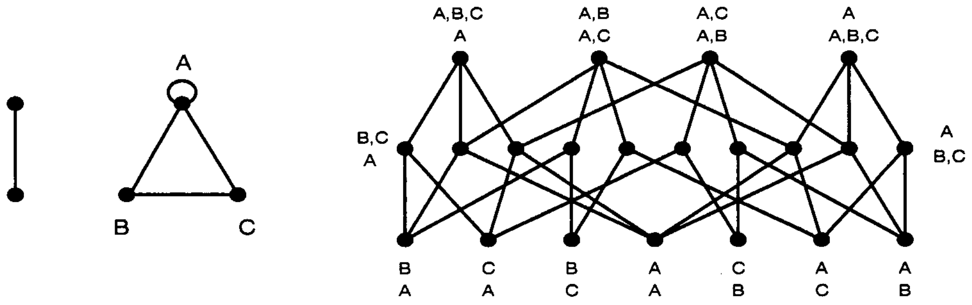

Figure 3.1: The graphs G and H, and the poset $\mathrm{Hom}(G, H)$

We will often refer to $\mathrm{Hom}(G, H)$ as a topological space; by this we mean the geometric realization of (the order complex of) the poset (see Figure 3.2). We refer to Chapter 2 of this thesis for all the relevant definitions.

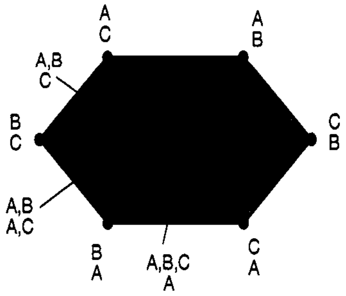

Figure 3.2: The realization of the poset $\mathrm{Hom}(G, H)$ (up to barycentric subdivision)

Note that if G and H are both finite, then (the geometric realization of the order complex of) this $\mathrm{Hom}(G, H)$ yields a simplicial complex that is isomorphic to the barycentric subdivision of the polyhedral Hom complex as defined in [BK06].

Fixing one of the coordinates of the Hom complexes provides a covariant functor $\mathrm{Hom}(T, ?)$, and a contravariant functor $\mathrm{Hom}(?, T)$, from $\mathcal{G}$ to the category of posets. If $f : G \to H$ is a graph map, we have in the first case an induced poset map $f_T : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, H)$

given by $f_T(\alpha)(t) = \{f(g) : g \in \alpha(t)\}$ for $\alpha \in \mathrm{Hom}(T, G)$ and $t \in V(T)$. In the other case, we have $f^T : \mathrm{Hom}(H,T) \to \mathrm{Hom}(G,T)$ given by $f^T(\beta)(g) = \beta(f(g))$ for $\beta \in \mathrm{Hom}(H,T)$ and $g \in V(G)$. Hence if $\mathrm{Aut}(G)$ and $\mathrm{Aut}(H)$ denote the automorphism groups of the respective graphs, we see that $\mathrm{Hom}(G, H)$ carries a natural $(\mathrm{Aut}(G) \times \mathrm{Aut}(H))$-action.

A graph map $f : G \to H$ induces a natural transformation $\bar{f} : \mathrm{Hom}(?, G) \to \mathrm{Hom}(?, H)$ in the following way. For every $T \in \mathrm{Ob}(\mathcal{G})$, we have $\bar{f}_T : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, H)$ given by $(\bar{f}_T(\alpha))(t) = \{f(g) : g \in \alpha(t)\}$ for each $\alpha \in \mathrm{Hom}(T, G)$ and $t \in V(T)$. If $g : S \to T$ is a graph map, the diagram

$$\begin{tikzcd}
\mathrm{Hom}(S,G) \arrow[d, "\bar{f}_S"] & \mathrm{Hom}(T,G) \arrow[l, "g^G"'] \arrow[d, "\bar{f}_T"] \\
\mathrm{Hom}(S,H) & \mathrm{Hom}(T,H) \arrow[l, "g^H"']
\end{tikzcd}$$

commutes since if $\alpha \in \mathrm{Hom}(T, G)$ and $s \in V(S)$ then on the one hand $((\bar{f}_S g^G)(\alpha))(s) = \{f(x) : x \in ((g^G(\alpha))(s)\} = \{f(x) : x \in \alpha(g(s))\}$, and on the other hand $((g^H \bar{f}_T)(\alpha))(s) = ((\bar{f}_T)(\alpha))(g(s)) = \{f(x) : x \in \alpha(g(s))\}$.

The function induced by composition $\mathrm{Hom}(G, H) \times \mathrm{Hom}(H, K) \to \mathrm{Hom}(G, K)$ is a poset map; see [Koza] for the details.

Many operations in the category of graphs interact nicely with the topology of the Hom complexes. We now gather together some of these results. The first observation comes from the paper of [BK06].

Lemma 3.1.2. *For graphs A, B, C there is an isomorphism of posets*

$$\mathrm{Hom}(A \amalg B, C) \cong \mathrm{Hom}(A, C) \times \mathrm{Hom}(B, C).$$

Also, if A is connected and not a single vertex, then

$$\mathrm{Hom}(A, B \amalg C) \cong \mathrm{Hom}(A, B) \amalg \mathrm{Hom}(A, C).$$

As we will see, other graph operations are preserved by the Hom complexes up to homotopy type.

3.2 Adjunctions and Limits

For graphs A, B, and C the exponential graph construction provides the adjunction $\mathcal{G}(A \times B, C) = \mathcal{G}(A, C^B)$, an isomorphism of sets. The next proposition shows that there is a homotopy equivalence of the associated Hom complexes.

Proposition 3.2.1. *For graphs* A, B, C, $\mathrm{Hom}(A \times B, C)$ *can be included in* $\mathrm{Hom}(A, C^B)$ *so that* $\mathrm{Hom}(A \times B, C)$ *is the image of a closure map on* $\mathrm{Hom}(A, C^B)$. *In particular, we have the inclusion of a strong deformation retract*

$$|\mathrm{Hom}(A \times B, C)| \underset{\simeq}{\hookrightarrow} |\mathrm{Hom}(A, C^B)|.$$

Proof. Let $P = \mathrm{Hom}(A \times B, C)$ and $Q = \mathrm{Hom}(A, C^B)$ be the respective posets. Our plan is to define an inclusion map $j : P \to Q$ and a closure map $c : Q \to Q$ such that $\mathrm{Im}(j) = \mathrm{Im}(c)$, from which the result would follow.

We define the map of posets $j : P \to Q$ according to

$$j(\alpha)(a) = \{f : V(B) \to V(C) | f(b) \in \alpha(a, b) \forall b \in B\}$$

for every vertex $a \in V(A)$ and $\alpha \in P$. To show that $j(\alpha)$ is in fact an element of Q, we need to verify that if $(a, a') \in E(A)$ then we $(f, f') \in E(C^B)$ for all $f \in j(\alpha)(a), f' \in j(\alpha)(a')$. For this, consider $(b, b') \in E(B)$. Then $((a, b), (a', b')) \in E(A \times B)$. Hence $(c, c') \in E(C)$ for all $c \in \alpha(a, b)$ and all $c' \in \alpha(a', b')$. In particular, $(f(b), f(b')) \in E(C)$, and we conclude that $(f, f') \in E(C^B)$ so that $j(\alpha)$ is indeed an element of $\mathrm{Hom}(A, C^B)$.

We claim that j is injective. To see this, let $\alpha \neq \alpha'$ be distinct elements of the poset P, with $\alpha(a, b) \neq \alpha'(a, b)$ for some $(a, b) \in V(A \times B)$. Without loss of generality, suppose $c \in \alpha(a, b) \backslash \alpha'(a, b)$. Then we have $f \in j(\alpha)(a)$ such that $f(b) = c$, and yet $f \notin j(\alpha')(a)$. We conclude that $j(\alpha) \neq j(\alpha')$, and hence j is injective.

Next we define the closure operator $c : Q \to Q$. If $\gamma : V(A) \to 2^{V(C^B)} \backslash \{\emptyset\}$ is an element of $Q = \mathrm{Hom}(A, C^B)$, define $c(\gamma) \in Q$ as follows: fix some $a \in V(A)$, and for every $b \in V(B)$ let $C^{\gamma}_{ab} = \{f(b) \in V(C) : f \in \gamma(a)\}$. Then define $c(\gamma)$ according to

$$c(\gamma)(a) = \{g : V(B) \to V(C) | g(b) \in C^{\gamma}_{ab} \forall b \in B\}.$$

We first verify that c maps into Q, i.e. $c(\gamma) \in \operatorname{Hom}(A, C^B)$. Suppose $(a, a') \in E(A)$ are adjacent vertices in A, and let $f \in c(\gamma)(a)$ and $g \in c(\gamma)(a')$. To show that $(f, g) \in E(C^B)$, consider $(b, b') \in E(B)$. Then by construction there is some $f' \in \gamma(a)$, $g' \in \gamma(a')$ such that $f(b) = f'(b)$ and $g(b') = g'(b')$. Hence $(f(b), g(b')) \in E(C)$, as desired.

It is clear that $c(q) \geq q$ and $(c \circ c)(q) = c(q)$ for all $q \in Q$. Thus c is a closure map.

Next we claim that $c(Q) \subseteq j(P)$. To see this, suppose $\tilde{\gamma} \in c(Q)$. We define $\alpha : V(A \times B) \to 2^{V(C)} \backslash \{\emptyset\}$ by $\alpha(a, b) = C^{\gamma}_{ab}$, where $\gamma \in \operatorname{Hom}(A, C^B)$ is such that $c(\gamma) = \tilde{\gamma}$ and $C^{\gamma}_{ab} \subseteq V(C)$ is as above. We claim that $\alpha \in \operatorname{Hom}(A \times B, C)$. Let $((a, b), (a', b')) \in E(A \times B)$ and $c \in \alpha(a, b) = C^{\gamma}_{ab}, c' \in \alpha(a', b') = C^{\gamma}_{a'b'}$. Since $(a, a') \in E(A)$ we have $(f, f') \in E(C^B)$ for all $f \in \gamma(a), f' \in \gamma(a')$, and hence since $(b, b') \in E(B)$ we get $(f(b), f'(b')) \in E(C)$. In particular we obtain $(c, c') \in E(C)$ as desired.

Finally, $j(P) \subseteq c(Q)$ since $j(P) \subset Q$ and $c(j(P)) = j(P)$. Thus $j(P) = c(Q)$, implying that $\operatorname{Hom}(A \times B, C) \simeq \operatorname{Hom}(A, C^B)$ via this inclusion. ☐

Remark 3.2.2. *As a result of the proposition, for all graphs G and H, there is a homotopy equivalence*

$$\operatorname{Hom}(G, H) = \operatorname{Hom}(\mathbf{1} \times G, H) \simeq \operatorname{Hom}(\mathbf{1}, H^G),$$

where $\mathbf{1}$ denotes a single looped vertex. The last of these posets is the face poset of the clique complex on the looped vertices of H^G, and hence its realization is the barycentric subdivision of the clique complex. Since the looped vertices in H^G are precisely the graph homomorphisms $G \to H$, we see that $\operatorname{Hom}(G, H)$ can be realized up to homotopy type as the clique complex of the subgraph of H^G induced by the graph homomorphisms.

By a *diagram of graphs* $D = \{D_i\}$, we mean a collection of graphs $\{D_i\}$ with a specified collection of maps between them (the image of a category D under some functor to $\mathcal{G}$). For a graph T, any such diagram of graphs gives rise to a diagram of posets $\operatorname{Hom}(T, D)$ obtained by applying the functor $\operatorname{Hom}(T, ?)$ to each object and each morphism (see Figure 3.3).

We can combine the facts from Lemma 2.1.4 and Proposition 3.2.1 to see that Hom complexes preserve (up to homotopy type) limits of such diagrams.

$$\begin{tikzcd}
 & \cdot \arrow[d] \\
\cdot \arrow[r] & \cdot
\end{tikzcd}
\qquad
\begin{tikzcd}
 & D_1 \arrow[d, "f"] \\
D_3 \arrow[r, "g"'] & D_2
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{Hom}(T, D_1) \arrow[d, "f_T"] \\
\mathrm{Hom}(T, D_3) \arrow[r, "g_T"'] & \mathrm{Hom}(T, D_2)
\end{tikzcd}$$

Figure 3.3: A category D, a diagram of graphs, and the induced diagram of posets

Proposition 3.2.3. *Let D be a diagram of graphs with $\lim(D)$ the limit. Then for every graph T we have a homotopy equivalence:*

$$\big|\mathrm{Hom}\big(T, \lim(D)\big)\big| \simeq \big|\lim\big(\mathrm{Hom}(T, D)\big)\big|.$$

Proof. Let T be a graph. We will express the functor $\mathrm{Hom}(T, ?)$ as a composition of functors that each preserve limits. First we note that the functor $(?)^T : \mathcal{G} \to \mathcal{G}$ given by $G \mapsto G^T$ preserves limits since it has the left adjoint given by the functor $? \times T$; this was the content of Proposition 3.2.1. Hence for any diagram of graphs D, we obtain $(\lim(D))^T = \lim(D^T)$.

Next we note that the functor $L : \mathcal{G} \to \mathcal{G}^\circ$ that takes the induced subgraph on the looped vertices (described above) also preserves limits since it has the left adjoint given by the inclusion functor $\mathcal{G}^\circ \to \mathcal{G}$. Thus $L(\lim(D)) = \lim(L(D))$.

Now we claim that the functor $\mathrm{Hom}(\mathbf{1}, ?)$ preserves limits up to homotopy type. To see this, we recall that $\mathrm{Hom}(\mathbf{1}, ?)$, as a functor from the category of reflexive graphs, associates to a given (reflexive) graph G the face poset of its *clique complex*, $\Delta(G)$. Hence, taking geometric realization, we get $|\mathrm{Hom}(\mathbf{1}, G)| \simeq |\Delta(G)|$, for a reflexive graph G. As a functor to flag simplicial complexes, the clique complex Δ has an inverse functor given by taking the 1-skeleton and adding loops to each vertex. In particular, this shows that $\Delta(?)$ preserves limits, and so $\Delta(\lim \tilde{D}) = \lim(\Delta(\tilde{D}))$ for any diagram of reflexive graphs $\tilde{D}$.

Finally, we put these observations together to obtain the following string of homotopy equivalences:

$$\begin{aligned}
\left|\mathrm{Hom}(T, \lim(D))\right| &\simeq \left|\mathrm{Hom}(\mathbf{1}, (\lim(D))^T)\right| \\
&= \left|\mathrm{Hom}(\mathbf{1}, \lim(D^T))\right| = \left|\mathrm{Hom}\Big(\mathbf{1}, L\big(\lim(D^T)\big)\Big)\right| = \left|\mathrm{Hom}\Big(\mathbf{1}, \lim\big(L(D^T)\big)\Big)\right| \\
&\simeq \left|\Delta\Big(\lim\big(L(D^T)\big)\Big)\right| = \left|\lim\Big(\Delta\big(L(D^T)\big)\Big)\right| \simeq \left|\lim\Big(\mathrm{Hom}\big(\mathbf{1}, L(D^T)\big)\Big)\right| \\
&= \left|\lim\Big(\big(\mathrm{Hom}(\mathbf{1}, (D^T)\big)\Big)\right| \simeq \left|\lim\big(\mathrm{Hom}(T, D)\big)\right|.
\end{aligned}$$

The first and last homotopy equivalences here are as in Proposition 3.2.1. □

Recall that the product $G \times H$ is a limit (pullback) of the diagram $G \to \mathbf{1} \leftarrow H$. Since $\mathrm{Hom}(T, \mathbf{1})$ is a point for every graph T, this implies that $|\mathrm{Hom}(T,G)| \times |\mathrm{Hom}(T,H)|$ is homotopy equivalent to $|\mathrm{Hom}(T, G \times H)|$, for graphs T, G, and H. In fact we can prove the following stronger result.

Proposition 3.2.4. *For graphs* $T, G,$ *and* H, *the poset* $\mathrm{Hom}(T,G) \times \mathrm{Hom}(T,H)$ *can be included into* $\mathrm{Hom}(T, G \times H)$ *so that* $\mathrm{Hom}(T,G) \times \mathrm{Hom}(T,H)$ *is the image of a closure map on* $\mathrm{Hom}(T, G \times H)$. *In particular, we have the inclusion of a strong deformation retract*

$$|\mathrm{Hom}(T,G)| \times |\mathrm{Hom}(T,H)| \xhookrightarrow[\simeq]{} |\mathrm{Hom}(T, G \times H)|.$$

Proof. We let $Q = \mathrm{Hom}(T,G) \times \mathrm{Hom}(T,H)$ and $P = \mathrm{Hom}(T, G \times H)$ be the respective posets. Once again, our plan is to define an inclusion $i : Q \to P$ and a closure map $c : P \to P$ such that $\mathrm{Im}(i) = \mathrm{Im}(c)$.

We define the map $i : Q \to P$ according to $i(\alpha, \beta)(v) = (\alpha(v) \times \beta(v))$, for a vertex $v \in V(T)$. Note that if v, w are adjacent vertices of T then $(\tilde{v}, \tilde{w}) \in E(G)$ and $(v', w') \in E(H)$ for all $\tilde{v} \in \alpha(v)$, $\tilde{w} \in \alpha(w))$, $v' \in \beta(v)$, and $w' \in \beta(w)$. Hence $((\tilde{v}, \tilde{w}), (v', w')) \in E(G \times H)$, so that $i(\alpha, \beta)$ is indeed an element of $\mathrm{Hom}(T, G \times H)$. Clearly, i is injective.

Next, we define the closure operator $c : P \to P$, whose image will coincide with that of the map i. If $\gamma \in P$ is an element of the poset, we define $c(\gamma) \in \mathrm{Hom}(T, G \times H)$ as follows: for every $v \in T$ we have minimal vertex subsets $A_v \subseteq V(G)$, $B_v \subseteq V(H)$ such that $\gamma(v) \subseteq \{(a,b) : a \in A_v, b \in B_v\}$; define $c(\gamma)(v) = \{(a,b)\} = A_v \times B_v$ to be this minimal set of vertices of $G \times H$.

We first verify that c maps into P, i.e. $c(\gamma) \in \operatorname{Hom}(T, G \times H)$. Suppose $(v, w) \in E(T)$ are adjacent vertices. If $(\tilde{a}, \tilde{b}) \in c(\gamma)(v)$ and $(a', b') \in c(\gamma)(w)$ then we have $(\tilde{a}, \tilde{y}), (\tilde{x}, \tilde{b}) \in \gamma(v)$ and $(a', y'), (x', b') \in \gamma(w)$ for some $\tilde{x}, x' \in G$ and $\tilde{y}, y' \in H$. Hence we must have $(\tilde{a}, a') \in E(G)$ and also $(\tilde{b}, b') \in E(H)$ so that $((\tilde{a}, \tilde{b}), (a', b')) \in E(G \times H)$, as desired.

Since $c(p) \geq p$ and $(c \circ c)(p) = c(p)$ for all $p \in P$, we see that $c : P \to P$ is a closure operator.

Next we claim that $c(P) \subseteq i(Q)$. Suppose $c(\gamma) \in c(P)$, so that for all $v \in T$, we have $c(\gamma)(v) = A_v \times B_v$, where $A_v \subset V(G)$ and $B_v \subset V(H)$. Define $\alpha : V(T) \to 2^{V(G)} \backslash \{\emptyset\}$ by $\alpha(v) = A_v$, and $\beta : V(T) \to 2^{V(H)} \backslash \{\emptyset\}$ by $\beta(v) = B_v$. We claim that $\alpha \in \operatorname{Hom}(T, G)$ and $\beta \in \operatorname{Hom}(T, H)$. Indeed, if $w \in T$ is a vertex adjacent to v and $\alpha(w) = A_w$, then if $a_i \in A_v$ and $a_{i'} \in A_w$, we have $(a_i, y) \in \gamma(v)$ and $(a_{i'}, y') \in \gamma(w)$ for some $y, y' \in H$. Hence (a_i, y) and $(a_{i'}, y')$ are adjacent vertices in $G \times H$ (since $\gamma \in \operatorname{Hom}(T, G \times H)$). This implies that a_i and $a_{i'}$ are adjacent in G, as desired.

Finally, $i(Q) \subseteq c(P)$ since $i(Q) \subseteq P$ and $c(i(Q)) = i(Q)$. Therefore $i(Q) = c(P)$ and hence $\operatorname{Hom}(T, G) \times \operatorname{Hom}(T, H) \simeq \operatorname{Hom}(T, G \times H)$ via this inclusion. ☐

3.3 *Foldings*

The Hom complexes also interact well with a graph operation known as *folding*. We review this construction next.

Definition 3.3.1. *Let u and v be vertices of a graph G satisfying $N(v) \subseteq N(u)$. Then the map $f : G \to G \backslash v$ given by $f(v) = u$ and $f(x) = x$ for all $x \neq v$ is called a folding of G at the vertex v. The inclusion $i : G \backslash v \to G$ is called an unfolding (see Figure 3.4).*

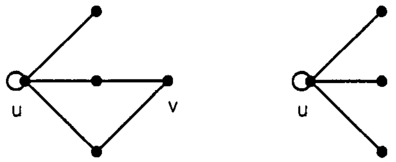

Figure 3.4: The graph G and the folded graph $G\backslash v$

In the original papers regarding Hom complexes (see for example [BK06]), it was shown that folds in the first coordinate of the Hom complex preserved homotopy type. For some time it was an open question whether the same was true in the second coordinate of the Hom complex. Kozlov investigated this question in the papers [Koz06d] and [Koz06b], and showed that indeed this was the case.

Proposition 3.3.2. *(Kozlov) If G and H are graphs, and u and v are vertices of G such that $N(v) \subseteq N(u)$, then the folding and unfolding maps induce inclusions of strong deformation retracts*

$$\mathrm{Hom}(G\backslash v, H) \xhookrightarrow[\simeq]{f^H} \mathrm{Hom}(G,H), \qquad \mathrm{Hom}(H, G\backslash v) \xhookrightarrow[\simeq]{i_H} \mathrm{Hom}(H,G).$$

In fact, Kozlov exhibits these deformation retracts as closure maps on the levels of the posets, which he shows preserve the *simple* homotopy type of the associated simplicial complex. Note that although Kozlov deals only with the situation of finite H, his proof extends to the case of arbitrary H. In Chapters 5 and 6 of this thesis we see the further importance of folds in the context of the Hom complex.

3.4 History and applications of the Hom complex

Here we briefly review some of the history and applications of the Hom complexes. The original version of such a complex was defined in [Lov78], where they were used to prove the Kneser conjecture. Although purely combinatorial proofs of this theorem were later published (see [Bár78]), the original methods employed by Lovász remain arguably the most important contribution from his paper.

Theorem 3.4.1. *(Kneser Conjecture] If n and k are positive integers satisfying $n \geq 2k$, then $\chi(K_{n,k}) = n - 2k + 2$.*

Here $K_{n,k}$ is the *Kneser graph*, defined to be the simple graph whose vertex set is the collection of all k-subsets of $[n] := \{1, \dots, n\}$, with adjaceny given by $X \sim Y$ if $X \cap Y = \emptyset$. To prove the inequality $\chi(K_{n,k}) \geq n - 2k + 2$, Lovász used a version of the Hom complex known as the *neighborhood complex* to provide topological obstructions to the existence of graph colorings. He determined that the complexes $\mathrm{Hom}(K_2, K_{n,k})$ have a certain high connectivity, and showed that this was sufficient to provide a lower bound on the chromatic number. In an equivalent reformulation, he proved the following theorem.

Theorem 3.4.2 (Lovász, 1978). *If* G *is a graph such that* $\mathrm{Hom}(K_2, G)$ *is* k*-connected for some integer* $k \geq -1$*, then* $\chi(G) \geq k+3$.

We discuss the proof of this last theorem in a more general context, as described in [Koza]. The main idea is to recognize a certain free group action on the relevant Hom complex, and to invoke the theory of characteristic classes to obtain topological obstructions to maps to smaller complete graphs. Free actions on the Hom complexes arise in the following way.

Lemma 3.4.3. *If for some vertex* $v \in V(T)$ *there exists an element* $\varphi \in \mathrm{Aut}(T)$ *such that* $(v, \varphi(v)) \in E(T)$*, then the induced map* $\varphi^G : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, G)$ *is fixed pointed free for a graph* G *without loops.*

In our context, we note that flipping the edge of K_2 induces a free $\mathbb{Z}_2$-action on the topological space $\mathrm{Hom}(K_2, G)$, for a graph G without loops.

Next, we consider the quotient space $X := \big(\mathrm{Hom}(K_2, G)\big)/\mathbb{Z}_2$ under this free $\mathbb{Z}_2$-action, and examine its characteristic classes. The assumption on the connectivity of $\mathrm{Hom}(K_2, G)$ gives us a $\mathbb{Z}_2$-equivariant map from the $(k+1)$-sphere with the antipodal action, $\mathbb{S}_a^{k+1} \to \mathrm{Hom}(K_2, G)$. This in turn implies that $w_1^{k+1}(X) \neq 0$, where w_1 is the *first Stiefel-Whitney class* of this quotient space (see Chapter 1 for a discussion of these implications). Now, if $\chi(G) < k+3$ then there exists a map $G \to K_{k+2}$ which induces an equivariant map $\mathrm{Hom}(K_2, G) \to \mathrm{Hom}(K_2, K_{k+2})$. The following proposition then implies that $w_1^{k+1}\big(\mathrm{Hom}(K_2, K_{k+2})/\mathbb{Z}_2\big) = 0$, and hence, by naturality of the Stiefel-Whitney classes, $w_1^{k+1}(X) = 0$. This is a contradiction.

Proposition 3.4.4 (Babson, Kozlov). *The cell complex* $\mathrm{Hom}(K_m, K_n)$ *is homotopy equivalent to a wedge of* $(n-m)$*-dimensional spheres. Moreover, the complex* $\mathrm{Hom}(K_2, K_{k+2})$ *with the* $\mathbb{Z}_2$*-action induced by flipping the edge of* K_2 *is* $\mathbb{Z}_2$*-homeomorphic to the* k*-sphere with the antipodal action. In particular,* $w_1^{k+1}\big(\mathrm{Hom}(K_2, K_{k+2})\big) = 0$.

Our discussion so far shows that the Lovász result is a special case of the following result.

Theorem 3.4.5 (Babson, Kozlov). *If G is a graph such that $\mathrm{Hom}(K_m, G)$ is k-connected for some integer $k \geq -1$, then $\chi(G) \geq k + m + 1$.*

In fact, we notice that lower bounds on chromatic numbers are provided by the following more general observation. The utility of this last formulation is that the relevant criteria is in terms of a computable parameter (vanishing of a class in cohomology). See [Kozb] for a further discussion of this.

Theorem 3.4.6. *Let G be a graph without loops, and let T be a graph with a $\mathbb{Z}_2$-action that flips some edge in T. If $w_1^k((\mathrm{Hom}(T, G))) \neq 0$, but $w_1^k(\mathrm{Hom}(T, K_m)) = 0$, then $\chi(G) \geq m + 1$.*

We have seen that taking $T = K_n$ gives us lower bounds on the chromatic number of G in terms of connectivity of the complex $\mathrm{Hom}(T, G)$. Lovász next considered taking the test graph T to be an odd cycle, $T = C_{2r+1}$. He made the following conjecture, later proved by Babson and Kozlov in [BK07].

Theorem 3.4.7 (Lovász conjecture). *If G is a graph such that $\mathrm{Hom}(C_{2r+1}, G)$ is k-connected for some integer $k \geq -1$, then $\chi(G) \geq k + 4$.*

Note that the case $r = 1$ is a special case of Theorem 3.4.5. The odd cycle C_{2r+1} has a $\mathbb{Z}_2$-action given by reflecting along an axis of symmetry (fixing a vertex); this action flips an edge in the graph C_{2r+1}. Hence, with the machinery established so far, the Lovász conjecture/theorem is a consequence of the following result.

Theorem 3.4.8 (Babson, Kozlov, Schultz). *For all $r \geq 1$ and $k \geq 2$, we have $w_1^{k-2}(Hom(C_{2r+1}, K_k)) = 0$.*

In their original proof of the Lovász conjecture, Babson and Kozlov proved this last statement only for odd k (and made the conjecture for all k), and used a separate argument for the case of even k in the proof of Theorem 3.4.7 . In both cases they used an auxiliary construction $\mathrm{Hom}_+(T, H)$ and a map to the simplex supp : $\mathrm{Hom}_+(T, H) \rightarrow \Delta^{V(T)}$. The $\mathrm{Hom}_+(T, G)$ construction is similar to the $\mathrm{Hom}(T, G)$ complex; the elements are again set maps $\alpha : V(T) \to 2^{V(G)}$, but this time the image of a vertex can be empty. The skeletal filtration of the simplex $\Delta^{[2r+1]}$ then induces a filtration on $\mathrm{Hom}_{+}(C_{2r+1}, K_n)$ via the supp map, and the resulting spectral sequence relates the cohomology of $\mathrm{Hom}_{+}(C_{2r+1}, K_n)$ to that of $\mathrm{Hom}(C_{2r+1}, K_n)$. In particular, $\mathrm{Hom}(C_{2r+1}, K_n)$ is isomorphic to the inverse image of the barycenter of the simplex $\Delta^{[2r+1]}$ under the supp map, and hence the last nonzero column in the E_1 page of the spectral sequence is $H^*\big(\mathrm{Hom}(C_{2r+1}, K_n)\big)$. The spectral sequence itself converges to $H^*\big(\mathrm{Hom}_{+}(C_{2r+1}, K_n)\big)$, which can be described completely.

In the more recent paper [Scha], Schultz uses an embedding of the complex $\mathrm{Hom}(C_{2r+1}, K_k)$ into a product of spheres to get $w_1^{k-2}\big(Hom(C_{2r+1}, K_k)\big) = 0$ for all k. In Chapter 7 we will say more about the methods employed by Schultz, and also discuss some generalizations of his work.

We end this section with a brief discussion of other results from the literature regarding Hom complexes. The first was originally proved in [ČK05] (see also [Eng06]), and relates the valency of a graph G to the connectivity of the complex $\mathrm{Hom}(G, K_n)$.

Theorem 3.4.9 (Cukic, Kozlov, 2004). *If G is a graph such that the maximum valency of G is equal to d, then the complex $\mathrm{Hom}(G, K_n)$ is k-connected, for all integers $k \geq -1$ and $n \geq d + k + 2$.*

Other results relate valency to chromatic numbers via connectivity of Hom complexes. The first is from the paper [BW04] (see also [BW00] for related work).

Theorem 3.4.10 (Brightwell and Winkler, 2004). *Let G be a graph. If for all graphs T with maximum valency at most d, the complex $\mathrm{Hom}(T, G)$ is either connected or empty, then $\chi(G) \geq d/2 + 2$.*

Lovász has suggested that this last result can be strengthened, as well as generalized to higher dimensions. Our results in Chapter 7 of the thesis are related to the following conjecture.

Conjecture 3.4.11. *(Lovász) Let G be a graph. If for all graphs T with maximum valency at most d, the complex $\mathrm{Hom}(T, G)$ is either k-connected or empty, then $\chi(G) \geq d + k + 2$.*

3.5 Hom complexes of general graphs

We end this chapter with a discussion of Hom complexes for general graphs (as defined in Chapter 2.4). We offer a definition for the $\mathrm{Hom}(G,H)$ in this context and prove that in fact up to homotopy these complexes are equivalent to the complexes obtained by 'simplifying' the graphs in each entry. The results and constructions discussed here are not used in the rest of the thesis.

Definition 3.5.1. *If G and H are graphs, we define $\mathrm{Hom}(G,H)$ to be the poset whose elements are indexed by pairs of functions (α,β) where $\alpha : V(G) \to 2^{V(H)}\backslash\{\emptyset\}$ and $\beta : E(G) \to 2^{E(H)}\backslash\{\emptyset\}$, such that the following conditions are satisfied:*

(1) If $E \in E(G)$ with $h_G(E) = v$, $t_G(E) = w$, then for all $\tilde{v} \in \alpha(v)$ and $\tilde{w} \in \alpha(w)$, there is some $F \in \beta(E)$ such that $h_H(F) = \tilde{v}$ and $t_H(F) = \tilde{w}$.

(2) If $F \in \beta(E)$ for $E \in E(G)$ then $h_H(F) \in \alpha(h_G(E))$ and $t_H(F) \in \alpha(t_G(E))$.

(3) For all $F \in \beta(E)$ and $\tilde{F} \in \beta(\sigma_G(E))$, we have $\sigma_H(F) = \tilde{F}$

The relation is given by containment in each factor, so that $(\alpha,\beta) \leq (\alpha',\beta')$ whenever $\alpha(v) \subseteq \alpha'(v)$ and $\beta(E) \subseteq \beta'(E)$ for all $v \in V(G)$ and $E \in E(G)$.

We will refer to the elements (α,β) of $\mathrm{Hom}(G,H)$ as *generalized graph maps.* If the graphs G and H are both simple, then the element (α,β) is completed determined by α, and this construction gives a poset that is isomorphic to the construction given in Section 1 of this chapter. We will refer to $\mathrm{Hom}(G,H)$ as a topological space; in this context we will mean the geometric realization of the poset.

Example 3.5.2. *Let $G = K_2$ and let H be the graph with vertices $V(H) = \{A,B\}$ and edges $E(H) = \{1,2,3,4\}$, with adjacency given by $h(1) = A$, $t(1) = B$; $h(2) = A$, $t(2) = B$; $h(3) = t(3) = A$; and $h(4) = t(4) = B$, and $\sigma(1) = 2$, $\sigma(3) = 3$, and $\sigma(4) = 4$. Then the realization of $\mathrm{Hom}(G,H)$ is homeomorphic to the 3-disc.*

Subdividing the edges 1 and 2 will give a simple graph $\tilde{H}$ (a square with loops on opposite vertices). Note that $\mathrm{Hom}(G,\tilde{H})$ is homotopy equivalent to a circle.

We next verify that this construction enjoys the desired functorial properties.

Lemma 3.5.3. *For every graph T,* $\mathrm{Hom}(T, ?)$ *is a covariant functor from GRAPH to POSET. If* $(f, \tilde{f}) : G \to H$ *is a graph map, then* $f_T : Hom(T, G) \to Hom(T, H)$ *is the poset map given by* $f_T(\alpha)(t) = \{f(g) : g \in \alpha(t)\}$ *and* $f_T(\beta)(E) = \{\tilde{f}(F) : F \in \beta(E)\}$*, for each* $(\alpha, \beta) \in Hom(T, G)$.

Proof. We claim $(f_T(\alpha), f_T(\beta)) \in \mathrm{Hom}(T, H)$. For this we need to check the conditions given in Definition 3.5.1. Recall that $(f, \tilde{f})$ is given by the following diagram.

$$\begin{tikzcd}
E(G) \arrow[r, "\tilde{f}"] \arrow[d, shift right, "t_G"'] \arrow[d, shift left, "h_G"] & E(H) \arrow[d, shift right, "t_H"'] \arrow[d, shift left, "h_H"] \\
V(G) \arrow[r, "f"'] & V(H)
\end{tikzcd}$$

We first check condition (1). For this, let $E \in E(T)$ such that $h(E) = v$ and $t(E) = w$, and suppose $\tilde{v} \in f_T(\alpha)(v)$ and $\tilde{w} \in f_T(\alpha)(w)$. Then $\tilde{v} = f(g)$ and $\tilde{w} = f(g')$ for some $g \in \alpha(v)$ and some $g' \in \alpha(w)$. Now, since $(\alpha, \beta) \in \mathrm{Hom}(T, G)$, there is some $F \in \beta(E)$ such that $h_G(F) = g$ and $t_G(F) = g'$. We claim $\tilde{f}(F) \in E(H)$ is an edge that satisfies condition (1). To see this, note that $h_H(\tilde{f}(F)) = f(h_G(F)) = f(g) = \tilde{v}$, and similarly $t_H(\tilde{f}(F)) = f(t_G(F)) = f(g') = \tilde{w}$.

Next we check condition (2). Let $F \in f_T(\beta)(E)$ for some $E \in E(T)$. Then $F = \tilde{f}(A)$ for some $A \in \beta(E)$, and hence $h_G(A) \in \alpha(h_T(E))$ and $t_G(A) \in \alpha(t_T(E))$. To satisfy the condition, we need $h_H(F) \in (f_T(\alpha))(h_T(E))$ and $t_H(F) \in (t_T(\alpha))(t_T(E))$. But $h_H(F) = h_H\tilde{f}(A) = f(h_G(A))$ and $h_G(A) \in \alpha(h_T(E))$ and hence $h_H(F) \in \{f(g) : g \in \alpha(h_T(E))\} = (f_T(\alpha))(h_T(E))$. Similarly, we see that $t_H(F) \in (t_T(\alpha))(t_T(E))$, so our condition is satisfied.

For condition (3), let $F \in f_T(\beta(E))$ and $\tilde{F} \in f_T(\beta(\sigma_T(E)))$. Then $F = \tilde{f}(X)$ for some $X \in \beta(E)$, and $\tilde{F} = \tilde{f}(\tilde{X})$ for some $\tilde{X} \in \beta(\sigma_T(E))$. Hence $\sigma_G(X) = \tilde{X}$ (since $\beta \in \mathrm{Hom}(T, G)$, and we have $\sigma_H(F) = \sigma(\tilde{f}(X) = \tilde{f}(\sigma_G(X)) = \tilde{f}(\tilde{X}) = \tilde{F}$.

We next show that $f_T : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, H)$ is a poset map. To this end, let $(\alpha, \beta) \leq (\alpha', \beta') \in \mathrm{Hom}(T, G)$, so that $\alpha(v) \subseteq \alpha'(v)$ and $\beta(E) \subseteq \beta'(E)$ for all $v \in V(T)$, $E \in E(T)$. If $\tilde{v} \in f_T(\alpha)(v)$, then $\tilde{v} = f(g)$ for some $g \in \alpha(v)$, and hence $g \in \alpha'(v)$. But then $\tilde{v} = f(g) \in \{f(x) : x \in \alpha'(v)\} = f_T(\alpha')(v)$. Similarly, if $\tilde{E} \in f_T(\beta)(E)$, then $\tilde{E} = \tilde{f}(F)$ for some $F \in \beta(E)$, and hence $F \in \beta'(E)$. This implies that $\tilde{E} = \tilde{f}(F) \in \{\tilde{f}(X) : X \in \beta'(E)\}$, and we conclude $f_T(\alpha, \beta) \leq f_T(\alpha', \beta')$.

□

Lemma 3.5.4. *For every graph G,* Hom$(?, T)$ *is a contravariant functor from GRAPH to POSET. If $(f, \tilde{f}) : G \to H$ is a graph map, $f^T : Hom(H,T) \to Hom(G,T)$ is the poset map given by $(f^T(\alpha))(g) = \alpha(f(g))$ and $(f^T(\beta))(E) = \beta(\tilde{f}(E))$, for each $\alpha \in Hom(T,G)$, $g \in V(G)$, and $E \in E(G)$.*

Proof. To show that $(f^T(\alpha), f^T(\beta)) \in \mathrm{Hom}(G,T)$ we need to check the conditions given in Definition 3.5.1. For condition (1), let $E \in E(G)$ with $h_G(E) = v$ and $t_G(E) = w$, and let $\tilde{v} \in f^T(\alpha)(v) = (\alpha(f(v))$ and $\tilde{w} \in f^T(\alpha)(w) = \alpha(f(w))$. Applying the map f, we have $\tilde{f}(E) \in E(H)$ with $h_H(\tilde{f}(E)) = f(h_G(E)) = f(v)$, and $t_H(\tilde{f}(E)) = f(t_G(E)) = f(w)$. Now, since $(\alpha, \beta) \in \mathrm{Hom}(H,T)$, there is some $F \in \beta(\tilde{(}f(E))$ such that $h_T(F) = \tilde{v}$ and $t_T(F) = \tilde{w}$, as required.

For condition (2), we let $F \in f^T(\beta)(E) = \beta(\tilde{f}(E))$. Again, since $(\alpha, \beta) \in \mathrm{Hom}(H,T)$, we have $h_T(F) \in \alpha(h_H(\tilde{f}(E))$ and $t_T(F) \in \alpha(t_H(\tilde{f}(E))$. But $t_H\tilde{f} = ft_G$ and hence $t_T(F) \in \alpha(f(t_G(E)) = f^T(\alpha)(t_G(E))$. Similarly, $h_T(F) \in f^T(\alpha)(h_G(E))$, so condition (2) is satisfied.

For condition (3), let $F \in f^T(\beta)(E)$ and $\tilde{F} \in f^T(\beta)(\sigma_G(E))$. Then $F \in \beta(\tilde{f}(E))$ and $\tilde{F} \in \beta(\tilde{f}(\sigma_G(E))) = \beta(\sigma_H(\tilde{f}(E))$. But $\beta \in \mathrm{Hom}(H,T)$, and hence $\sigma_T(F) = \tilde{F}$

Finally, to show that $f^T : \mathrm{Hom}(H,T) \to \mathrm{Hom}(G,T)$ is a poset map, suppose $(\alpha, \beta) \leq (\alpha', \beta')$, so that $\alpha(v) \subseteq \alpha'(v)$ and $\beta(E) \subseteq \beta'(E)$ for all $v \in V(H)$ and $E \in E(H)$. If $w \in V(G)$ then $F^T\alpha(w) = \alpha(f(w)) \subseteq \alpha'(f(w)) = f^T\alpha(w)$, and similarly if $F \in E(G)$ then $f^T\beta(F) = \beta(\tilde{f}(F)) \subseteq \beta'(\tilde{f}(F)) = f^T\beta'(F)$. We conclude that $f^T(\alpha, \beta) \leq f^T(\alpha', \beta')$.

□

There is a functor $\mathcal{S} : GRAPH \to SGRAPH$ from graphs to simple graphs given by consolidating edges, so that if $G \in GRAPH$, the edges of $\mathcal{S}(G)$ are given by ordered pairs $E(\mathcal{S}(G)) = \{(h_G(E), t_G(E)) : E \in E(G)\}$, with the obvious h, t, and σ maps. This functor is left adjoint to the forgetful functor $\mathcal{F} : SGRAPH \to GRAPH$ given by inclusion of objects and morphisms. We record this observation as the next lemma.

Lemma 3.5.5. *For every graph G and every simple graph S there is a natural isomorphism of sets $SGRAPH(\mathcal{S}G, S) \cong GRAPH(G, \mathcal{F}S)$.*

Proof. We define a map of sets $\varphi : SGRAPH(\mathcal{S}G, S) \to GRAPH(G, \mathcal{F}S)$ according to $\varphi(\alpha, \tilde{\alpha}) = (\alpha, \varphi(\tilde{\alpha}))$ for all $\alpha \in SGRAPH(\mathcal{S}G, S)$, where $\varphi(\tilde{\alpha})(E) = \tilde{\alpha}(h_G(E), t_G(E))$ for all $E \in E(G)$. We define an inverse map $\psi : GRAPH(G, \mathcal{F}S) \to SGRAPH(\mathcal{S}G, S)$ according to $\psi(\beta, \tilde{\beta}) = (\beta, \tilde{\beta})$ for all $\beta \in GRAPH(G, \mathcal{F}S)$, where $\tilde{\beta}(h_G(E), t_G(E)) = \beta(E)$ for all $(h_G(E), t_G(E)) \in E(\mathcal{S}G)$. Note that ψ is well-defined since if $E' \in E(G)$ with $h_G(E') = h_G(E)$ and $t_G(E') = t_G(E)$, then we have $h_{\mathcal{F}S}(\beta(E')) = \beta h_G(E') = \beta h_G(E) = h_{\mathcal{F}S}(\beta(E'))$. Similarly, $t_{\mathcal{F}S}(\beta(E')) = t_{\mathcal{F}S}(\beta(E))$, so that $\beta(E') = \beta(E)$. We leave the verification of the naturality of φ to the reader. The result follows. ☐

The consolidation functor $\mathcal{S}$ also preserves the homotopy type of the Hom complexes. More precisely we have the following result.

Proposition 3.5.6. *Let G and H be graphs. Then the poset maps $\mathcal{S}^H : \mathrm{Hom}(\mathcal{S}(G), H) \to \mathrm{Hom}(G, H)$ and $\mathcal{S}_G : \mathrm{Hom}(G, H) \to \mathrm{Hom}(G, \mathcal{S}(H))$ are both homotopy equivalences. Also, $\mathrm{Hom}(G, H)$ is homotopy equivalent to $\mathrm{Hom}(\mathcal{S}(G), \mathcal{S}(H))$.*

Proof. For the first statement, we define a closure operator $c : \mathrm{Hom}(G, H) \to \mathrm{Hom}(G, H)$ according to $(\alpha, \beta) \mapsto (\alpha, \beta')$, where $\beta' : E(G) \to 2^{E(H)} \backslash \{\emptyset\}$ is given by

$$\beta'(E) = \{\beta(E' : E' \in E(G), h_G(E') = h_G(E), t_G(E') = t_G(E)\}.$$

It is clear that $c(\alpha, \beta') \geq (\alpha, \beta')$ and also that $c^2 = c$. Hence c is a closure operator and we obtain $\mathrm{Hom}(G, H) \simeq c\big(\mathrm{Hom}(G, H)\big)$. Next we note that $\mathcal{S}^H : \mathrm{Hom}(\mathcal{S}(G), H) \to \mathrm{Hom}(G, H)$ is an inclusion whose image is equal to the image of c. We conclude that $\mathcal{S}^H$ is a homotopy equivalence

For the second statement, we define a closure operator $d : \mathrm{Hom}(G, H) \to \mathrm{Hom}(G, H)$ according to $(\alpha, \beta) \mapsto (\alpha, \tilde{\beta})$, where $\tilde{\beta} : E(G) \to 2^{E(H)} \backslash \{\emptyset\}$ is given by

$$\tilde{\beta}(E) = \{X \in E(H) : h_H(X) \in \alpha(h_G(E)), t_H(X) \in \alpha(t_G(E))\}.$$

One sees that $(\alpha, \tilde{\beta})$ is the largest element in $\mathrm{Hom}(G, H)$ with the property that the restriction to the vertices is α; hence $d(\alpha, \tilde{\beta}) \geq (\alpha, \beta)$. Also, $d^2 = d$, so that d is a closure operator and hence $\mathrm{Hom}(G, H) \simeq d(\mathrm{Hom}(G, H))$.

Next we see that $\mathcal{S}_G : \mathrm{Hom}(G, H) \to \mathrm{Hom}(G, \mathcal{S}(H))$ is given by $(\alpha, \beta) \to (\alpha, \tilde{\beta})$, where $\tilde{\beta}(E) = \{(h_H(F), t_H(F)) : F \in \beta(E)\}$. Hence $\mathcal{S}_G$, restricted to the image of d, defines an isomorphism onto $\mathrm{Hom}(G, \mathcal{S}(H))$. The result follows.

For the last statement, we note that $\mathcal{S}_G$ has a right inverse poset map given by f_G, where $f : \mathcal{S}(H) \to H$ is the (generalized) graph map given by $v \mapsto v$ and $(h_H(E), t_H(E)) \mapsto \{F \in E(H) : h_H(F) = h_H(E), t_H(F) = t_H(E)\}$. Hence the composition $f_G \circ \mathcal{S}^H : \mathrm{Hom}(\mathcal{S}(G), \mathcal{S}(H)) \to \mathrm{Hom}(G, H)$ is a homotopy equivalence.

□

In particular, a graph G is homotopy equivalent to its simplification $\mathcal{S}(G)$.

Chapter 4

GRAPH HOMOTOPY

4.1 Introduction

In this chapter, we investigate a notion of ×-homotopy of graph maps that is based on the internal hom associated to the categorical product in the category of graphs. We show that graph ×-homotopy is characterized by the topological properties of the Hom complex. Graph ×-homotopy naturally leads us to a notion of homotopy equivalence which we show has several equivalent characterizations. We describe the relationship between homotopy equivalence and the (un)foldings of graphs. In the process we recover a result of Kozlov from [Koz06d], and shed some light on the importance of foldings in the context of Hom complexes. We apply the notions of homotopy equivalence to a class of graphs called 'dismantlable' to obtain a list of conditions that again characterize these. We end with a discussion of graph homotopy arising from the internal hom associated with the Cartesian product; in the category of reflexive graphs this construction is known as A-theory in the literature. Most of the material in this chapter is taken from [Doca].

4.1.1 Homotopy in categories with a path object

In many categories, the notion of a pair of homotopic maps can be phrased in terms of a map from some specified object into an *exponential object* associated to an internal hom structure on that category (we will review these constructions below). The typical example is the category of (compactly generated) topological spaces, where a homotopy between maps $f : X \to Y$ and $g : X \to Y$ is nothing more than a map from the interval I into the topological space $\text{Map}(X, Y)$. Other examples include simplicial objects, as well as the category of chain complexes of R-modules. For the latter, a chain homotopy between chain maps $f : C \to D$ and $g : C \to D$ can be recovered as a map from the chain complex I (defined to be the complex consisting of 0 in all dimensions except R in dimensions 0 and

1, with the identity map between them) into the complex $\mathrm{Hom}(C, D)$.

In this chapter, we consider these constructions in the context of the category of graphs. In particular, we investigate a notion of what we call $\times$-homotopy that arises from consideration of the well known internal hom associated to the categorical product. We use the notion of (graph theoretic) connectivity to provide a notion of a 'path' in the exponential graph. It turns out that $\times$-homotopy classes of maps are related to the topology of the Hom-complexes, a functorial way to assign a poset $\mathrm{Hom}(G, H)$ (and hence topological space) to a pair of graphs G and H. Elements of the poset $\mathrm{Hom}(G, H)$ are graph *multi*-homomorphisms from G to H, with the set of graph homomorphisms the atoms. Fixing one of the two coordinates of the Hom complex in each case provides a functor from graphs to topological spaces, and we show that $\times$-homotopy of graph maps can be characterized by the topological homotopy type of the maps induced by these functors.

Graph $\times$-homotopy of maps naturally leads us to a notion of weak equivalence of graphs, which we show can again be characterized in terms of the topological properties of relevant Hom complexes. The graph operations known as 'folding' and 'unfolding' preserve homotopy type, and in fact we show that in some sense these operations generate the weak equivalence class of a given graph. In particular, a pair of *stiff* graphs are weakly equivalent if and only if they are isomorphic. One particular case of interest arises when the graph can be folded down to a single looped vertex, a class of graphs called *dismantlable* in the literature. The characterization of weak equivalence in terms of the topology of the Hom complex gives us a list of conditions equivalent to a graph being dismantlable.

4.2 Graph $\times$-homotopy and Hom *complexes*

In this section, we define a notion of homotopy for graph maps and describe its interaction with the Hom complexes. The motivation comes from the internal hom structure in the category $\mathcal{G}$ as described above.

Recall that a vertex set map $f : V(G) \to V(H)$ is a looped vertex in H^G if and only if f is a *graph* map $G \to H$. Hence the set of graph maps $\mathcal{G}(G, H)$ are precisely the looped vertices in the internal hom graph H^G. The (path) connected components of the graph H^G then provide a natural notion of 'homotopic' graph maps: two maps $f, g : G \to H$ will be considered ×-*homotopic* if we can find a path along the looped vertices H^G that starts at f and ends at g.

To make the notion of a path truly graph theoretic we want to think of it as a map from a path-like graph object into the graph H^G.

Definition 4.2.1. *We let* I_n *denote the graph with vertices* $\{0, 1, \ldots, n\}$ *and adjacency given by* $i \sim i$ *for all* i *and* $(i-1) \sim i$ *for all* $1 \leq i \leq n$ *(see Figure 4.1).*

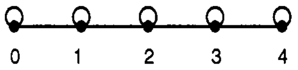

Figure 4.1: The graph I_4

Note that $N(n) = \{n, n-1\} \subset \{n, n-1, n-2\} = N(n-1)$, and hence we can fold the endpoint of I_n. This gives us the following property.

Lemma 4.2.2. $\mathrm{Hom}(T, I_n)$ *is contractible for all* $n \geq 0$ *and every graph* T.

Proof. We proceed by induction on n. For $n = 0$, we have that $\mathrm{Hom}(T, I_0) = \mathrm{Hom}(T, \mathbf{1})$ is a point. For $n > 0$, we use the fact that $N(n) \subset N(n-1)$ to get $\mathrm{Hom}(T, I_n) \simeq \mathrm{Hom}(T, I_{n-1})$ by Proposition 3.3.2. The latter complex is contractible by induction. ☐

Definition 4.2.3. *If* $f, g : G \to H$ *are graph maps, then we say that* f *and* g *are ×-homotopic if there exists an integer* $n \geq 1$ *and a map of graphs* $F : I_n \to H^G$ *such that* $F(0) = f$ *and* $F(n) = g$. *In this case we will also say that the maps are n-homotopic.*

We will denote ×-homotopic maps as $f \simeq_\times g$. Graph ×-homotopy determines an equivalence relation on the set of graph maps between G and H, and we let $[G, H]_\times$ denote the set of ×-homotopy classes of maps between graphs G and H.

Example 4.2.4. *As an example we can take* $G = K_2$ *and* $H = K_3$ *to be the complete graphs on 2 and 3 vertices. The graph* H^G *is displayed in Figure 4.2.*

We see that each of the six graph maps $f : G \to H$ *is represented by a looped vertex in the exponential graph* H^G. *In this case, any two maps* f *and* g *are connected by a path*

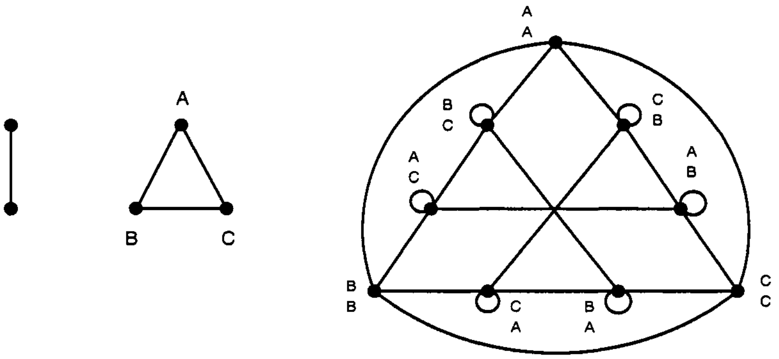

Figure 4.2: The graphs $G = K_2$, $H = K_3$, and H^G.

along other looped vertices, and hence in our setup all maps from $G = K_2$ to $H = K_3$ will be considered $\times$-homotopic (so that there is a single homotopy class of maps).

Example 4.2.5. *On the other hand, if we take $G = K_2$, and this time $H = K_2$, we get two distinct $\times$-homotopy classes of maps. The graph H^G is displayed in Figure 4.3.*

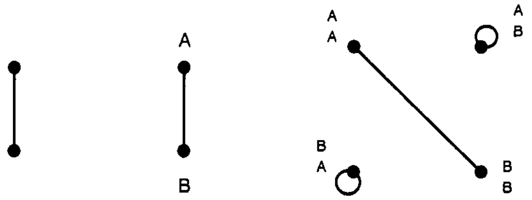

Figure 4.3: The graphs $G = K_2$, $H = K_2$, and H^G

We see that the two graph maps $G \to H$ are represented by looped vertices in H^G, but this time are disconnected from one another. Hence in this example, each of the two graph maps is in its own $\times$-homotopy class.

We can understand $\times$-homotopy in other ways by considering the adjoint properties available to us. Note that for all $m \leq n$, we have a map $\iota_m : G \to G \times I_n$ given by $v \mapsto (v, m)$, an isomorphism onto its image. A map $F : I_n \to H^G$ corresponds to a map $\tilde{F} : G \times I_n \to H$ with the property that $\tilde{F} \times 0 = f$ and $\tilde{F} \times n = g$. It is this formulation that we will most often use to check for $\times$-homotopy. We record this observation as a lemma.

Lemma 4.2.6. *If $f, g : G \to H$ are graph maps, then f and g are $\times$-homotopic if and only if there exists an integer n and a graph map $F : G \times I_n \to H$ such that $F_0 \equiv F \circ \iota_0 = f : G \to H$ and $F_n \equiv F \circ \iota_n = g : G \to H$.*

$$\begin{tikzcd}
G \arrow[d, "\iota_0"'] \arrow[dr, "f"] & \\
G \times I_n \arrow[r, "F"] & H \\
G \arrow[u, "\iota_n"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Next we investigate the interaction of $\times$-homotopy of graph maps with the Hom complex enrichment. It turns out that $\times$-homotopy equivalence classes of maps are characterized by the topology of the Hom complex in the following way.

Proposition 4.2.7. *Let G and H be graphs, and $f, g : G \to H$ be graph maps. Then f and g are $\times$-homotopic if and only if they are in the same path-connected component of* $\mathrm{Hom}(G, H)$*. In particular, the number of $\times$-homotopy classes of maps from G to H is equal to the number of path components in* $\mathrm{Hom}(G, H)$.

Proof. Suppose $f, g : G \to H$ are graph maps such that f and g are in the same component of $\mathrm{Hom}(G, H)$. Then we can find a path from f to g in $|\mathrm{Hom}(G, H)|$, which can be approximated as a finite walk on the 1-skeleton $f, x_1, x_2, \dots, g$. We claim that we can extend this to a walk $f = f_0, x_1, f_1, x_2, f_2, \dots, f_n = g$, where each $f_i : G \to H$ is a graph map (i.e., $f_i(v)$ consists of a single element for each $v \in V(G)$).

To see this, note that $f \leq x_1$ in $\mathrm{Hom}(G, H)$. First suppose that $x_1 \leq x_2$. Then for each $v \in V(G)$, we choose (by the choice axiom, say) a single element of $x_1(v)$ to get our map $f_1 : G \to H$ such that $f_1 \leq x_1 \leq x_2$. Next suppose $x_2 \leq x_1$. If x_2 is already a graph map, take $f_1 = x_2$, and otherwise for each $v \in V(G)$ choose a single element of $x_2(v)$ to get a map $f_1 : G \to H$.

Now, to get our homotopy, we define a map $F : G \times I_n \to H$ by $F(v, i) = f_i(v)$. Then F is indeed a graph map since we have an $x_i \in \mathrm{Hom}(G, H)$ such that $f_{i-1}, f_i \leq x_i$ for each $0 < i \leq n$. Hence the maps $f = f_0$ and $g = f_n$ are $\times$-homotopic.

For the other direction, suppose that $f, g : G \to H$ are distinct maps that are $\times$-homotopic for $n = 1$. We define a function $\xi : V(G) \to 2^{V(H)} \backslash \{\emptyset\}$ by $v \mapsto \{f(v), g(v)\}$. We claim that ξ is a cell in $\mathrm{Hom}(G, H)$. To see this, suppose $(v, w) \in E(G)$. Then both $(f(v), f(w))$ and $(g(v), g(w))$ are edges in H since f and g are graph maps. Also, $(0v, 1w)$ and $(0w, 1v)$ are edges in $G \times I_1$ and since f and g are 1-homotopic this implies that $(f(v), g(w))$ and $(f(w), g(v))$ are both edges in H. Hence vertices of $\xi(v)$ are adjacent to vertices of $\xi(w)$ as desired. It is clear that both f and g are vertices of ξ and hence we have a path from f to g. Now, suppose f and g are $\times$-homotopic for some choice of n and let $F : G \times I_n \to H$ be the homotopy. Let $f_i : G \mapsto H$ be the graph map given by $v \mapsto F(\iota_i(v))$. Then by induction we have a path in $\mathrm{Hom}(G, H)$ from f to f_{n-1} and the above construction gives a path from f_{n-1} to $f_n = g$. □

We end this section with another observation that will be useful later.

Lemma 4.2.8. *Let G be a graph, $k \leq n$ integers, and let $\iota_k : G \to G \times I_n$ denote the graph map given by $\iota(g) = (g, k)$. Then for a graph T, the induced map $\iota_{k_T} : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, G \times I_n)$ is a homotopy equivalence.*

Proof. Let $i : \mathrm{Hom}(T, G) \times \mathrm{Hom}(T, I_n) \hookrightarrow \mathrm{Hom}(T, G \times I_n)$ denote the inclusion, a homotopy equivalence by Proposition 3.2.4. Let $\phi_k : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, G) \times \mathrm{Hom}(T, I_n)$ denote the inclusion given by $x \mapsto (x, c_k)$, where $c_k \in \mathrm{Hom}(T, I_n)$ is the constant map sending all elements of $V(T)$ to k. We note that ϕ_k is a homotopy equivalence by Lemma 4.2.2. We then have the following commutative diagram showing that $\iota_{k_T} = i \circ \phi_k$ is a homotopy equivalence.

$$\begin{tikzcd}
\mathrm{Hom}(T, G) \arrow[r, "\iota_{k_T}"] \arrow[d, "\phi_k", "\simeq"'] & \mathrm{Hom}(T, G \times I_n) \\
\mathrm{Hom}(T, G) \times \mathrm{Hom}(T, I_n) \arrow[ur, "i", "\simeq"'] &
\end{tikzcd}$$

□

4.3 *Homotopy equivalence of graphs*

If $f, g : G \to H$ are graph maps, the functors obtained by fixing a graph T in one coordinate of the Hom complex in each case provides a pair of topological maps. For a fixed test graph T, the functor $\mathrm{Hom}(T, ?)$ provides the pair of maps $f_T, g_T : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, H)$, while $\mathrm{Hom}(?, T)$ provides the maps $f^T, g^T : \mathrm{Hom}(H, T) \to \mathrm{Hom}(G, T)$. If f and g are $\times$-homotopic, we can ask how these induced maps are related up to (topological) homotopy. It turns out that the induced maps are homotopic, and in fact provide a characterization of graph $\times$-homotopy in each case. More precisely, we have the following result.

Theorem 4.3.1. *Let $f, g : G \to H$ be graph maps. Then the following are equivalent.*

(1) f and g are $\times$-homotopic.

(2) For every graph T, the induced maps $f_T, g_T : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, H)$ are homotopic.

(3) The induced maps $f_G, g_G : \mathrm{Hom}(G, G) \to \mathrm{Hom}(G, H)$ are homotopic.

(4) For every graph T, the induced maps $f^T, g^T : \mathrm{Hom}(H, T) \to \mathrm{Hom}(G, T)$ are homotopic.

(5) The induced maps $f^H, g^H : \mathrm{Hom}(H, H) \to \mathrm{Hom}(G, H)$ are homotopic.

Proof. We first prove (1) $\Rightarrow$ (2). Suppose $f, g : G \to H$ are $\times$-homotopic via a graph map $F : G \times I_n \to H$. Then (with notation as above) we have a commutative diagram in $\mathcal{G}$ and, via the functor $\mathrm{Hom}(T, ?)$, the induced diagram in $\mathcal{TOP}$, the category of topological spaces and continuous maps, of the form:

$$\begin{tikzcd}
G \arrow[d, "\iota_0"'] \arrow[dr, "f"] & \\
G \times I_n \arrow[r, "F"] & H \\
G \arrow[u, "\iota_n"] \arrow[ur, "g"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Hom}(T, G) \arrow[d, "\iota_{0T}"'] \arrow[dr, "f_T"] & \\
\mathrm{Hom}(T, G \times I_n) \arrow[r, "F_T"] & \mathrm{Hom}(T, H) \\
\mathrm{Hom}(T, G) \arrow[u, "\iota_{nT}"] \arrow[ur, "g_T"'] &
\end{tikzcd}$$

Now, $\mathrm{Hom}(T, I_n)$ is path connected (contractible) by Lemma 4.2.2. Let $\gamma : I = [0, 1] \to \mathrm{Hom}(T, I_n)$ be a path such that $\gamma(0) = c_0$ and $\gamma(1) = c_n$ (where again $c_i \in \mathrm{Hom}(T, I_n)$ is the constant map sending all vertices of T to i). Let $j_i : \mathrm{Hom}(T,G) \to \mathrm{Hom}(T,G) \times I$ be the (topological) map given by (id, i). Then we have the following diagram in $\mathcal{TOP}$ (where $(T,G) = \mathrm{Hom}(T,G)$, etc.):

$$\begin{tikzcd}
 & (T,G) \arrow[ld, "j_0"'] \arrow[rd, "\iota_{0_T}"] & & \\
(T,G) \times I \arrow[r, "id\times\gamma"] & (T,G) \times (T,I_n) \arrow[r, hook] & (T,G \times I_n) \arrow[r, "F_T"] & (T,H) \\
 & (T,G) \arrow[lu, "j_1"] \arrow[ru, "\iota_{n_T}"'] & &
\end{tikzcd}$$

We claim that this diagram commutes. To see this, suppose $\alpha \in \mathrm{Hom}(T,G)$. Then for all $t \in V(T)$ we have $\iota_{0_T}(\alpha)(t) = \{\iota_0(x) : x \in \alpha(t)\} = \{(x,0) : x \in \alpha(t)\} \in \mathrm{Hom}(T,G) \times \mathrm{Hom}(T,I^n)$, so that $\iota_{0_T}(\alpha) = (\alpha, c_0)$. On the other hand, $(id \times \gamma)(j_0)(\alpha) = (id \times \gamma)(\alpha, 0) = (\alpha, c_0)$. The bottom square is similar.

Now, let $\Phi : \mathrm{Hom}(T,G) \times I \to \mathrm{Hom}(T,H)$ be the composition from above. We have that $\Phi \circ j_0 = F_T \circ \iota_{0_T} = f_T : \mathrm{Hom}(T,G) \to \mathrm{Hom}(T,H)$ and similarly $\Phi \circ j_1 = g_T$, so that f_T and g_T are homotopic.

The implication (2) $\Rightarrow$ (3) is clear, so we next turn to (3) $\Rightarrow$ (1). To prove the contrapositive, suppose $f, g : G \to H$ are not $\times$-homotopic. Then f and g are in different path components of $\mathrm{Hom}(G,H)$ by Proposition 4.2.7. We claim that the induced maps $f_G, g_G : \mathrm{Hom}(G,G) \to \mathrm{Hom}(G,H)$ are also not homotopic. To obtain a contradiction, suppose they are and let $\Phi : \mathrm{Hom}(G,G) \times I \to \mathrm{Hom}(G,H)$ be a (topological) homotopy between them. Note that if $\mathrm{id} \in \mathrm{Hom}(G,G)$ is the identity map, then $f_G(\mathrm{id}) = f$ and $g_G(\mathrm{id}) = g$ since, for instance, we have $f_G(\mathrm{id})(x) = \{f(y) : y \in \mathrm{id}(x)\} = \{f(y) : y \in \{x\}\} = \{f(x)\}$ for all $x \in V(G)$. So then the restriction $\Phi|_{\{\mathrm{id}\}\times I} : \mathrm{Hom}(G,G) \times I \to \mathrm{Hom}(G,H)$ gives a path in $\mathrm{Hom}(G,H)$ from f to g, a contradiction.

We next prove (1) $\Rightarrow$ (4). Again, suppose $f, g : G \to H$ are $\times$-homotopic via $F : G \times I_n \to H$. Then this time we have the commutative diagram in $\mathcal{G}$ and the induced diagram in $\mathcal{TOP}$ of the form:

$$\begin{tikzcd}
G \arrow[d, "\iota_0"'] \arrow[dr, "f"] & \\
G \times I_n \arrow[r, "F"] & H \\
G \arrow[u, "\iota_n"] \arrow[ur, "g"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathrm{Hom}(G,T) \\
\mathrm{Hom}(H,T) \arrow[ur, "f^T"] \arrow[r, "F^T"] \arrow[dr, "g^T"'] & \mathrm{Hom}(G \times I_n, T) \arrow[u, "\iota_0^T"'] \arrow[d, "\iota_n^T"] \\
 & \mathrm{Hom}(G,T)
\end{tikzcd}$$

To show that f^T and g^T are homotopic, we will find a map $\Psi : \mathrm{Hom}(H,T) \to \mathrm{Hom}(G,T)^I$ such that $p_0\Psi = f^T$ and $p_1\Psi = g^T$. First, we define a map $\varphi : \mathrm{Hom}(I_n, T^G) \times \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\} \to \mathrm{Hom}(\mathbf{1}, T^G)$ via $\varphi(\alpha, \frac{i}{n})(v) = \alpha(i)$ for $v \in \mathbf{1}$, $\alpha \in \mathrm{Hom}(I_n, T^G)$, and $0 \leq i \leq n$. This extends to a map $\varphi : \mathrm{Hom}(I_n, T^G) \times I \to \mathrm{Hom}(\mathbf{1}, T^G)$ since the maps $\varphi_j : \mathrm{Hom}(I_n, T^G) \to \mathrm{Hom}(\mathbf{1}, T^G)$ are all homotopic for $0 \leq j \leq n$ (recall $\iota_j : \mathbf{1} \to I_n$ induces a homotopy equivalence). Let $\tilde{\varphi} : \mathrm{Hom}(I_n, T^G) \to \mathrm{Hom}(\mathbf{1}, T^G)^I$ be the adjoint map. Next, from the above proposition, we have a map $\psi : \mathrm{Hom}(\mathbf{1}, T^G) \to \mathrm{Hom}(G,T)$ that is a homotopy inverse to the inclusion $\mathrm{Hom}(\mathbf{1} \times G, T) \to \mathrm{Hom}(\mathbf{1}, T^G)$. Let $\tilde{\psi} : \mathrm{Hom}(\mathbf{1}, T^G)^I \to \mathrm{Hom}(G,T)^I$ be the induced map on the path spaces. Define $\Phi : \mathrm{Hom}(I_n, T^G) \to \mathrm{Hom}(G,T)^I$ by the composition $\Phi = \tilde{\varphi}\tilde{\psi}$. Finally, we get the desired map Ψ as the horizontal composition in the commutative diagram below.

$$\begin{tikzcd}
 & & (G,T) & \\
(H,T) \arrow[r, "F^T"] & (G \times I_n, T) \arrow[ur, "\iota_{0_T}"] \arrow[dr, "\iota_{n_T}"'] \arrow[r, hook] & (I_n, T^G) \arrow[r, "\Phi"] & (G,T)^I \arrow[ul, "p_0"'] \arrow[dl, "p_1"] \\
 & & (G,T) &
\end{tikzcd}$$

The implication (4) $\Rightarrow$ (5) is again clear, and so we are left with only (5) $\Rightarrow$ (1). For this, suppose $f, g : G \to H$ are not $\times$-homotopic, so that f and g are in different path components of $\mathrm{Hom}(G,H)$. We claim that the induced maps $f^H, g^H : \mathrm{Hom}(H,H) \to \mathrm{Hom}(G,H)$ are not homotopic. Suppose not, so that we have $f^H, g^H : \mathrm{Hom}(H,H) \to \mathrm{Hom}(G,H)$ are homotopic via a (topological) map $\Phi : \mathrm{Hom}(H,H) \times I \to \mathrm{Hom}(G,H)$. Here note that if $\mathrm{id} \in \mathrm{Hom}(H,H)$ is the identity map, then $f^H(\mathrm{id}) = f$ and $g^H(\mathrm{id}) = g$ since, for instance, $f^H(\mathrm{id})(x) = \mathrm{id}(f(x)) = f(x)$. Hence the restriction $\Phi|_{\{\mathrm{id}\} \times I} : \mathrm{Hom}(H,H) \times I \to \mathrm{Hom}(G,H)$ gives a path in $\mathrm{Hom}(G,H)$ from f to g, a contradiction. The result follows. $\square$

The notion of $\times$-homotopy of graph maps provides a natural candidate for the notion of homotopy equivalence of graphs. Again, this has several equivalent formulations, which we discuss next.

Theorem 4.3.2. *Let $f : G \to H$ be maps of graphs. Then the following are equivalent.*

(1) There exists a map $g : H \to G$ such that $f \circ g \simeq_\times \mathrm{id}_H$ and $g \circ f \simeq_\times \mathrm{id}_G$ (call g a homotopy inverse to f).

(2) For every graph T, the induced map $f_T : \mathrm{Hom}(T, G) \to \mathrm{Hom}(T, H)$ is a homotopy equivalence.

(3) For every graph T, the induced map $(f_T)_0 : \pi_0(\mathrm{Hom}(T, G)) \to \pi_0(\mathrm{Hom}(T, H))$ is an isomorphism (bijection).

(4) For every graph T, the induced map $f_T : [T, G]_\times \to [T, H]_\times$ is an isomorphism (bijection).

(5) The maps $f_G : \mathrm{Hom}(G, G) \to \mathrm{Hom}(G, H)$ and $f_H : \mathrm{Hom}(H, G) \to \mathrm{Hom}(H, H)$ both induce isomorphisms on the path components.

(6) For every graph T, the induced map $f^T : \mathrm{Hom}(H, T) \to \mathrm{Hom}(G, T)$ is a homotopy equivalence.

(7) The maps $f^G : \mathrm{Hom}(H, G) \to \mathrm{Hom}(G, G)$ and $f^H : \mathrm{Hom}(H, H) \to \mathrm{Hom}(G, H)$ both induce isomorphisms on path components.

$$\begin{tikzcd}
& \mathrm{Hom}(G,H) & \\
\mathrm{Hom}(G,G) \arrow[ur, "f_G"] & & \mathrm{Hom}(H,H) \arrow[ul, "f^H"'] \\
& \mathrm{Hom}(H,G) \arrow[ul, "f^G"] \arrow[ur, "f_H"'] &
\end{tikzcd}$$

Proof. For (1) $\Rightarrow$ (2), g_T is a homotopy inverse by Theorem 4.3.1.

(2) $\Rightarrow$ (3) is clear.

(3) $\iff$ (4) is implied by Proposition 4.2.7.

(3) $\Rightarrow$ (5) is clear.

For (5) $\Rightarrow$ (1), we assume $(f_H)_0 : \pi_0(\mathrm{Hom}(H, G)) \to \pi_0(\mathrm{Hom}(H, H))$ is an isomorphism. Let ϕ be its inverse and let $(\mathrm{id}_H)_0$ denote the connected component of id_H in $\mathrm{Hom}(H, H)$.

Let $g \in \phi((\mathrm{id}_H)_0)$ be a vertex of $\mathrm{Hom}(H, G)$ (i.e., a graph map). We claim that g satisfies the conditions of (1). To see this note that $((f_H)_0\phi)((\mathrm{id}_H)_0) = (\mathrm{id}_H)_0$ and since $g \in \phi((\mathrm{id}_H)_0)$ we have that $fg = f_H(g)$ is in the same component as id_H in $\mathrm{Hom}(H, H)$. Hence $fg \simeq_\times \mathrm{id}_H$, as desired. A similar consideration of the isomorphism $(f_G)_0 : \pi_0(\mathrm{Hom}(G, G) \to \pi_0(\mathrm{Hom}(G, H))$ shows that $gf \simeq_\times \mathrm{id}_G$.

For (1) $\Rightarrow$ (6), g^T again provides the inverse by Theorem 4.3.1.

(6) $\Rightarrow$ (7) is clear.

Finally, we check (7) $\Rightarrow$ (1). For this we assume $(f^G)_0 : \pi_0(\mathrm{Hom}(H, G)) \to \pi_0(\mathrm{Hom}(G, G))$ is an isomorphism. Let ψ be the inverse and let $(\mathrm{id}_G)_0$ denote the connected component of id_G. Let $g \in \psi((\mathrm{id}_G)_0)$ be a graph map $g : H \to G$. We claim that g satisfies the conditions that we need. Note that $((f^G)_0\psi)((\mathrm{id}_G)_0) = (\mathrm{id}_G)_0$ and $f^G(g) = gf$, and hence $gf \simeq_\times \mathrm{id}_G$. Similarly we get $fg \simeq_\times \mathrm{id}_H$ and the result follows. □

Definition 4.3.3. *A graph map $f : G \to H$ is called a homotopy equivalence if it satisfies any of the above conditions. Homotopy equivalence of graphs is an equivalence relation, and we let $[G]$ denote the homotopy equivalence class of G.*

Aside from certain qualitative similarities, homotopy equivalences of graphs satisfy many of the formal properties enjoyed by equivalences in any abstract homotopy theory, see [Bau89], [Hov99], and [Qui67]. In the rest of this section we discuss some of these.

Definition 4.3.4. *Let $\mathcal{M}$ be a class of maps in a category $\mathcal{C}$. $\mathcal{M}$ is said to satisfy the 2 out of 3 property if, for all maps f and g, whenever any two of f, g, gf are in $\mathcal{M}$, then so is the third.*

Lemma 4.3.5. *Homotopy equivalences of graphs satisfy the 2 out of 3 property.*

Proof. Let $f : X \to Y$ and $g : Y \to Z$ be maps of graphs, and let T be a graph. We will be considering the following diagrams.

$$\begin{tikzcd}
\mathrm{Hom}(T, X) \arrow[r, shift left, "f_T"] & \mathrm{Hom}(T, Y) \arrow[l, shift left, "a_T"]
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}(T, Y) \arrow[r, shift left, "g_T"] & \mathrm{Hom}(T, Z) \arrow[l, shift left, "b_T"]
\end{tikzcd}$$

$$\begin{tikzcd}\mathrm{Hom}(T,X) \arrow[r, shift left, "gf_T"] & \mathrm{Hom}(T,Y) \arrow[l, shift left, "c_T"] .\end{tikzcd}$$

First suppose f and g are both homotopy equivalences, with homotopy inverse maps $a : Y \to X$ and $b : Z \to Y$ respectively. We claim ab is the homotopy inverse to fg. To see this, note that $(abgf)_T = a_T b_T g_T f_T \simeq a_T f_T \simeq \mathrm{id}_X$. Similarly, $(gfab)_T \simeq \mathrm{id}_Z$, so that gf is a homotopy equivalence.

Next suppose that f and gf are homotopy equivalences, and let $c : Z \to X$ be the homotopy inverse to gf. We claim $fc : Z \to Y$ is the homotopy inverse to g. For this we compute $(gfc)_T = g_T f_T c_T \simeq \mathrm{id}_Z$ and $(fcg)_T = f_T c_T g_T \simeq f_T c_T g_T f_T a_T \simeq f_T a_T \simeq \mathrm{id}_Y$. We conclude that g is a homotopy equivalence.

Finally, we claim that if g and gf are homotopy equivalences then $cg : Y \to X$ is the homotopy inverse to f. This follows from the fact that $(fcg)_T = f_T c_T g_T \simeq b_T g_T f_T c_T g_T \simeq b_T g_T \simeq \mathrm{id}_Y$ and also $(cgf)_T = c_T g_T f_T \simeq \mathrm{id}_Z$. ☐

Definition 4.3.6. *Let $g : G \to H$ be a map in a category $\mathcal{C}$. Recall that f is a retract of g if there is a commutative diagram of the following form,*

$$\begin{tikzcd}X \arrow[r, "\alpha"] \arrow[d, "f"'] & G \arrow[r, "\gamma"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y \arrow[r, "\beta"'] & H \arrow[r, "\delta"'] & Y\end{tikzcd}$$

where the horizontal composites are identities.

Lemma 4.3.7. *Homotopy equivalences of graphs are closed under retracts.*

Proof. Suppose g is a homotopy equivalence. Then for every graph T we have the diagram,

$$\begin{tikzcd}\mathrm{Hom}(T,X) \arrow[r, "\alpha_T"] \arrow[d, "f_T"'] & \mathrm{Hom}(T,G) \arrow[r, "\gamma_T"] \arrow[d, "g_T"', "\simeq"] & \mathrm{Hom}(T,X) \arrow[d, "f_T"] \\ \mathrm{Hom}(T,Y) \arrow[r, "\beta_T"'] & \mathrm{Hom}(T,H) \arrow[r, "\delta_T"'] & \mathrm{Hom}(T,Y)\end{tikzcd}$$

with $g_T : \mathrm{Hom}(T,G) \to \mathrm{Hom}(T,H)$ a homotopy equivalence. We consider the induced maps on homotopy groups. Since $\gamma_T\alpha_T = \mathrm{id}$, we have that $(\alpha_T)_*$ is injective and hence so is $(f_T)_*$, since $(\beta_T)_*(f_T)_* = (g_T)_*(\alpha_T)_*$ is injective. Similarly, since $\delta_T\beta_T = \mathrm{id}$, we have that $(\delta_T)_*$ is surjective and hence so is $(f_T)_*$. We conclude that f_T induces an isomorphism on all homotopy groups and hence f_T is a homotopy equivalence on the CW-type Hom complexes. $\square$

Remark 4.3.8. *We end this section with the observation that one can apply our Theorem 4.3.2 to obtain a (rather roundabout) proof of one part of Kozlov's result Proposition 3.3.2. As we mentioned, it was known that if* $G \to H = G \backslash \{v\}$ *is a folding, then* $f^T : \mathrm{Hom}(H,T) \to \mathrm{Hom}(G,T)$ *is a homotopy equivalence for all* T. *We can then apply* (6) $\Rightarrow$ (2) *in Theorem 4.3.2 to conclude that* $f_T : \mathrm{Hom}(T,G) \to \mathrm{Hom}(T,H)$ *is also a homotopy equivalence, and hence 'folds in the second coordinate' also preserve homotopy type of* Hom *complexes.*

4.4 Stiff graphs and dismantlable graphs

We next investigate some of the consequences of homotopy equivalence. Recall that if $f : G \to \tilde{G}$ is a map realized by a sequence of foldings and unfoldings, then $f_T : \mathrm{Hom}(T,G) \to \mathrm{Hom}(T,\tilde{G})$ is a homotopy equivalence for all T, and hence G and $\tilde{G}$ are weakly equivalent. We first consider the case when G has no more foldings available.

4.4.1 Stiff graphs

Definition 4.4.1. *A graph* G *will be called stiff if there does not exist a pair of distinct vertices* $u, v \in V(G)$ *such that* $N(v) \subseteq N(u)$.

Lemma 4.4.2. *Suppose* G *is a stiff graph. Then the identity map* id_G *is an isolated point in the realization of* $\mathrm{Hom}(G,G)$.

Proof. If not, then we have some $\alpha \in \mathrm{Hom}(G,G)$ such that $x \in \alpha(x)$ for all $x \in V(G)$, and such that $\{v,w\} \subseteq \alpha(v)$ for some $v \neq w$. Since G is stiff we have some vertex $x \in V(G)$ such that $x \in N(v) \backslash N(w)$. But then since $x \in \alpha(x)$ we need x to be adjacent to w (to satisfy the conditions of Hom), a contradiction. $\square$

Proposition 4.4.3. *If* G *and* H *are both stiff graphs, then* G *and* H *are weakly equivalent if and only if they are isomorphic.*

Proof. Sufficiency is clear. For the other direction, suppose $f : G \to H$ is a homotopy equivalence with inverse $g : H \to G$. Then gf is $\times$-homotopic to the identity id_G, so that gf and id_G are in the same component of $\mathrm{Hom}(G, G)$ by Proposition 4.2.7. But then $gf = \mathrm{id}_G$ since G is stiff. Similarly we get $fg = \mathrm{id}_H$, so that f is an isomorphism. □

From this it follows that if G and H are finite graphs and $f : G \to H$ is a homotopy equivalence, then one can fold both graphs to their unique (up to isomorphism) stiff subgraphs $\tilde{G}$ and $\tilde{H}$ and get an isomorphism $\tilde{G} = \tilde{H}$. However, it is not known if we can make these foldings commute with the map f.

Question 4.4.4. *Suppose G and H are (finite) graphs and $f : G \to H$ is a homotopy equivalence. Can f be factored as a sequence of foldings and unfoldings?*

Note that an affirmative answer to this question would yield another characterization of homotopy equivalence to the list in Theorem 4.3.2, at least under the condition that G and H are both finite.

(8) The graph map $f : G \to H$ can be factored as a sequence of foldings and unfoldings.

Here we point out that it is not enough to consider only foldings of G and H to their stiff subgraphs $\tilde{G}$ and $\tilde{H}$. From what we have discussed, we do know that $\tilde{G}$ and $\tilde{H}$ are isomorphic, but it is not true that this isomorphism will necessarily commute with the map $f : G \to H$. We present such an example.

Example 4.4.5. *Let G be the graph with 5 vertices $V(G) = \{1, 2, 3, 4, 5\}$ and edges $E(G) = \{11, 12, 15, 22, 23, 33, 35, 34, 44, 45\}$ (see Figure 4.4). Let $f : \mathbf{1} \to G$ be the map that maps $\mathbf{1} \mapsto 1$.*

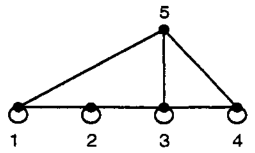

Figure 4.4: The graph G

We note that G is foldable to a looped vertex $\mathbf{1}$, but cannot be folded to $\operatorname{Im}(f)$ *by a sequence of solely foldings.*

4.4.2 Dismantlable graphs

We will call a finite graph G *dismantlable* if it can be folded down to $\mathbf{1}$. Note that G is dismantlable if *any* sequence of foldings of G down to its stiff subgraph results in the looped vertex $\mathbf{1}$. Dismantlable graphs have gained some attention in the recent papers of Brightwell and Winkler (see [BW00] and [BW04]), where they are related to the uniqueness of Gibbs measure on the set of homomorphisms between two graphs. We can apply the results of Theorem 4.3.2 to obtain the following characterizations of dismantlable graphs.

Proposition 4.4.6. *Suppose G is a finite graph, and let $f : G \to \mathbf{1}$ be the unique map. Then the following are equivalent:*

(0) G is dismantlable

(1) There exists a map $g : \mathbf{1} \to G$ such that $fg \simeq_\times \mathrm{id}_\mathbf{1}$ and $gf \simeq_\times \mathrm{id}_G$.

(2) For every graph T, the map $f_T : \operatorname{Hom}(T, G) \to \operatorname{Hom}(T, \mathbf{1})$ is a homotopy equivalence.

(2a) For every graph T, $\operatorname{Hom}(T, G)$ is contractible.

(3) For every graph T, $\operatorname{Hom}(T, G)$ is connected.

(4) For every graph T, the set $[T, G]_\times$ consists of a single homotopy class.

(5) G has at least one looped vertex and $\operatorname{Hom}(G, G)$ is connected.

(6) The map $f^G : \operatorname{Hom}(\mathbf{1}, G) \to \operatorname{Hom}(G, G)$ induces an isomorphism on path components.

Proof. (1) $\Rightarrow$ (2) is a special case of Theorem 4.3.2 (with $H = \mathbf{1}$), and (2) $\Rightarrow$ (2a) since $\operatorname{Hom}(T, \mathbf{1})$ is contractible for all T.

(2a) $\Rightarrow$ (3) is clear, and the sequence of equivalences (3) $\iff$ (4) $\iff$ (1) is again a special case of Theorem 4.3.2.

The implication (3) $\Rightarrow$ (5) is clear. For (5) $\Rightarrow$ (1), we assume that $v \in V(G)$ is a looped vertex, and that $\operatorname{Hom}(G, G)$ is connected. Let $g : \mathbf{1} \to G$ be the graph map given by $\mathbf{1} \to v$. We claim that g satisfies the conditions of (1). First, we have $fg = \mathrm{id}_\mathbf{1}$. Also, since

$\mathrm{Hom}(G, G)$ is path connected, we have that $gf : G \to G$ is in the same path component as the identity id_G. Hence $gf \simeq_\times \mathrm{id}_G$, and so g is the desired graph map.

Finally, (6) $\iff$ (1) is another special case of Theorem 4.3.2. Here note that $f^1 : \mathrm{Hom}(\mathbf{1}, \mathbf{1}) \to \mathrm{Hom}(G, \mathbf{1})$ is always an isomorphism.

It only remains to show (0) $\iff$ (3). If G is foldable to a looped vertex then we have seen that $\mathrm{Hom}(T, G) \simeq \mathrm{Hom}(T, \mathbf{1})$; the latter space is a point (and hence connected) for all T. For the other direction, we suppose $\mathrm{Hom}(T, G)$ is connected for all graphs T. The unique map $G \to \mathbf{1}$ gives a bijection $\pi_0\big((\mathrm{Hom}(T, G)\big) \to \pi_0\big(\mathrm{Hom}(T, \mathbf{1})\big)$ for all T, and hence G and $\mathbf{1}$ are weakly equivalent. So then if G is stiff, we have that G is isomorphic to $\mathbf{1}$ by Proposition 4.4.3. Otherwise we perform folds to reduce the number of vertices and use induction on $|V(G)|$. $\square$

4.5 Other internal homs and A-theory

In this last section we investigate other notions of graph homotopy that arise under considerations of different internal hom structures. One such homotopy theory (associated to the cartesian product) has seen some application in the literature.

Recall that in our construction of $\times$-homotopy, we relied on the fact that the categorical product has the looped vertex at its unit, and also possesses an internal hom (exponential) construction. This meant that graph maps from G to H were encoded by the looped vertices in the graph H^G, and two maps $f, g : G \to H$ were considered $\times$-homotopic if one could walk from f to g along a path composed of other graph maps.

Hence, in the general set-up we will be interested in monoidal (tensor) category structures on the category of graphs that have the looped vertex as the unit element, together with an internal hom for that structure. Recall that *having an internal hom* means that the set valued functor $T \mapsto \mathcal{C}(T \otimes G, H)$ is representable by an object of $\mathcal{C}$, which we will denote by H^G. We then have $T \mapsto \mathcal{C}(T \otimes G, H) = \mathcal{C}(T, H^G)$. Since we require the looped vertex (which we denote by $\mathbf{1}$) to be the unit we also get $\mathcal{C}(G, H) = \mathcal{C}(\mathbf{1} \otimes G, H) = \mathcal{C}(\mathbf{1}, H^G)$, so that H^G is a graph with the looped vertices as precisely the set of graph maps $G \to H$. A pair of graph maps f and g will then be considered homotopic in this context if, once again, we can find a (finite) path from f to g along looped vertices. One such product of interest is the *cartesian product*; we recall its definition below.

Definition 4.5.1. *For graphs A and B, the cartesian product $A \Box B$ is the graph with vertex set $V(A) \times V(B)$ and adjacency given by $(a, b) \sim (a', b')$ if either $a \sim a'$ and $b = b'$, or $a = a'$ and $b \sim b'$ (see Figure 4.5).*

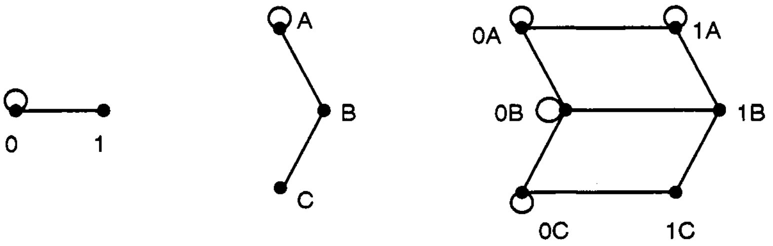

Figure 4.5: The graphs A, B, and $A \Box B$

One can check that the cartesian product gives the category of graphs the structure of a monoidal category with a (unlooped) vertex as the unit element. We next claim that the cartesian product also has an internal hom; we first define the functor that will serve as its right adjoint.

Definition 4.5.2. *For graphs A and B, the cartesian exponential graph B^A is the graph with vertex set $\{f : A \to B\}$ the set of all graph maps, with adjacency given by $f \sim f'$ if $f(a) \sim f'(a)$ for all $a \in A$ (see Figure 4.6).*

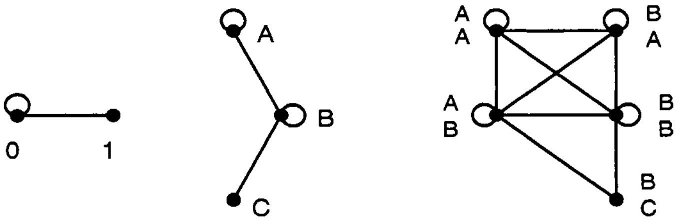

Figure 4.6: The graphs A, B, and B^A

Our next result shows that this exponential construction indeed provides the right adjoint for the cartesian product defined above.

Lemma 4.5.3. *For graphs A, B, C, there is a natural bijection $\Phi : \mathcal{G}(A\Box B, C) \to \mathcal{G}(A, C^B)$ given by the cartesian exponential graph.*

Proof. Given $f \in \mathcal{G}(A\Box B, C)$, and $a \in V(A)$, $b \in V(B)$, we define $\Phi(f)(a)(b) = f(a, b)$. We first verify that $\Phi(f)(a)$ is a graph map, so that $\Phi(f)(a) \in C^B$. For this, suppose $b \sim b'$ are adjacent vertices of B. Then we have $(a, b) \sim (a, b')$ in $A\Box B$ and hence $f(a, b) \sim f(a, b')$ as desired.

Next we verify that $\Phi(f)$ is a graph map. For this suppose $a \sim a'$ are adjacent vertices of A. Then, once again, $(a, b) \sim (a', b)$ in $A\Box B$ for all $b \in V(B)$. Hence $\Phi(f)(a)(b) = f(a, b)$ is adjacent to $\Phi(f)(a')(b) = f(a', b)$ for all $b \in V(B)$, so that $\Phi(f)(a) \sim \Phi(f)(a')$.

To see that Φ is a bijection, we construct an inverse $\Psi : \mathcal{G}(A, C^B) \to \mathcal{G}(A \times B, C)$ via $\Psi(g)(a, b) = g(a)(b)(g)$ for all $g \in \mathcal{G}(A, C^B)$. One checks that Ψ is well defined and an inverse to Φ. $\Box$

Recall that a *reflexive* graph is a graph with loops on each vertex, and that a map between reflexive graphs is just a map of the underlying graphs. The cartesian product of two reflexive graphs is once again reflexive, and hence the cartesian product gives the category $\mathcal{G}^\circ$ of reflexive graphs the structure of a monoidal category with the looped vertex $\mathbf{1}$ as the unit element.

Also, if A and B are both reflexive, then all vertices of B^A are looped (so that B^A is indeed a reflexive graph). Hence we have a graph B^A whose looped vertices are precisely the graph maps $B \to A$. The map Φ described above then gives a bijection $\mathcal{G}^\circ(A\Box B, C) \simeq \mathcal{G}^\circ(A, C^B)$.

In some recent papers (see for example [BBdLL06] and [BL05]), a homotopy theory called A-theory has been developed as a way to capture ‘combinatorial holes’ in simplicial complexes. The definition can be reduced to a construction in graph theory, applied to a certain graph associated to the simplicial complex in question. It turns out that A-theory of graphs fits nicely into the set-up that we have described, where the homotopy theory is associated to the cartesian product in the category of reflexive graphs. We recall the definition of A-homotopy of graph maps and A-homotopy equivalence of graphs (as in [BBdLL06]).

Definition 4.5.4. *Let $f, g : (G, x) \to (H, y)$ be a pair of based maps of reflexive graphs. Then f and g are said to be A-homotopic, denoted $f \simeq_A g$, if there is an integer $n \geq 1$ and a graph map $\varphi : G \Box I_n \to H$ such that $\varphi(?, 0) = f$ and $\varphi(?, n) = g$, and such that $\varphi(x, i) = y$ for all i.*

We call (G, x) and (H, y) A-homotopy equivalent if there exist based maps $f : G \to H$ and $g : H \to G$ such that $gf \simeq_A \mathrm{id}_G$ and $fg \simeq_A \mathrm{id}_H$.

Using the adjunction of Lemma 4.5.3, we see that an A-homotopy between two based maps of reflexive graphs $f, g : G \to H$ is the same thing as a map $\tilde{\varphi} : I_n \to H^G$ with $\tilde{\varphi}(0) = f$ and $\tilde{\varphi}(n) = g$, or in other words a path from f to g along looped vertices in the based version of the (cartesian) exponential graph H^G. This places the A-theory of graphs into the general set-up described above.

Chapter 5

THE UNIVERSALITY OF HOM COMPLEXES

5.1 Introduction

We have seen that the Hom complex is a functorial way to assign a poset (and hence topological space) $\mathrm{Hom}(T, G)$ to a pair of graphs T and G. In this chapter, we address the question of which spaces can arise as $\mathrm{Hom}(T, G)$ for fixed graphs T.

The automorphism group of T naturally acts on the space $\mathrm{Hom}(T, G)$, and in the case that $T = K_2$ is an edge and G is graph without loops, the complex $\mathrm{Hom}(T, G)$ is a space with a *free* $\mathbb{Z}_2$-action. In [Cso] Csorba shows that *any* free $\mathbb{Z}_2$-space can be realized (up to $\mathbb{Z}_2$-homotopy type) as $\mathrm{Hom}(K_2, G)$ for some suitably chosen graph G. His proof involves a simple and elegant construction in which one obtains a graph G whose vertices are precisely those of the given $\mathbb{Z}_2$-simplicial complex.

A natural question to ask is what homotopy types can be realized as $\mathrm{Hom}(T, ?)$ for other test graphs T. As Csorba points out, arbitrary homotopy types cannot be realized by Hom complexes of *loopless* graphs even with $T = K_2$ as the test graph; all such Hom complexes will be free $\mathbb{Z}_2$-spaces (see the discussion in Chapter 2) and hence will present topological obstructions (e.g. parity of the Euler characteristic). However, if we allow loops on our graphs, and do not concern ourselves with group actions, we are able to prove the following 'universality' of Hom complexes.

Theorem 5.1.1. *Let T be a connected graph with at least one edge, and let X be an arbitrary finite simplicial complex. Then there exists a graph $G_{k,X}$ (depending on X and the diameter of T) such that* $\mathrm{Hom}(T, G_{k,X})$ *is homotopy equivalence to X.*

The graph $G_{k,X}$ will be *reflexive* (that is, has loops on all the vertices), and hence the space $\mathrm{Hom}(T, G_{k,X})$ will no longer carry a free $\mathrm{Aut}(T)$ action. The idea behind our proof of this theorem will be to consider $X^k = \mathrm{bd}^k(X)$, a high enough (depending on the diameter of T) barycentric subdivision of the given simplicial complex X, and to define $G_{k,X}$ as the 1-skeleton of X^k with loops placed on each vertex. To show that $\mathrm{Hom}(T, G_{k,X})$ has the desired homotopy type, we will first replace it with a homotopy equivalent space X' (which will be the clique complex of some graph). We then determine the homotopy type of X' by covering it with a collection of contractible subcomplexes (with contractible intersections) and then employing a nerve lemma. Material in this chapter is taken from [Docc].

5.2 Proof of the main theorem

In this section we provide the proof of Theorem 5.1.1. Note that if $T = \mathbf{1}$ is a single looped vertex, we have $\mathrm{Hom}(\mathbf{1}, G) \simeq \Delta(G)$, the clique complex on the looped vertices of G (see Chapter 3) . Hence to obtain the result in this case, we define $G_{k,X}$ to be the graph obtained by taking the 1-skeleton of $bd(X) = X^1$, the first barycentric subdivision of the given complex X. Since the barycentric subdivision of a simplicial complex is a flag complex, we get that the 1-skeleton provides an inverse to the Δ functor in this case, and hence $X \simeq X^1 = \Delta(G_{k,X})$.

In the general case we will similarly obtain $G_{k,X}$ as the looped 1-skeleton of some iterated subdivision of X, but this time we have to take into account the diameter of the test graph T. Recall the setup: we are given a connected graph T with at least one edge, and a finite simplicial complex X. If $d = \mathrm{diam}(T)$ is the diameter of T, we fix an integer $k \geq 2$ such that $2^{k-1} - 1 \geq d$.

Next, we let $X^k = \mathrm{bd}^k(X)$ denote the k^{th} barycentric subdivision of the simplicial complex X. We define $G_{k,X}$ to be the graph given by the 1-skeleton of X^k, with loops placed at every vertex (see Figure 5.1).

We claim that $\mathrm{Hom}(T, G_{k,X}) \simeq X$. From Proposition 3.2.1, we have $\mathrm{Hom}(T, G_{k,X}) \simeq \mathrm{Hom}\big(\mathbf{1}, (G_{k,X})^T\big)$ (where $\mathbf{1}$ is the graph with one looped vertex). The latter space is homeomorphic to $\Delta\big((G_{k,X})^T\big)$, the clique complex on the (looped vertices of the) graph $\Delta\big((G_{k,X})^T\big)$. Hence to prove the main result (Theorem 5.1.1) it is enough to prove the following restatement.

Theorem 5.2.1. *Let T be an arbitrary connected graph with at least one edge, and let X be*

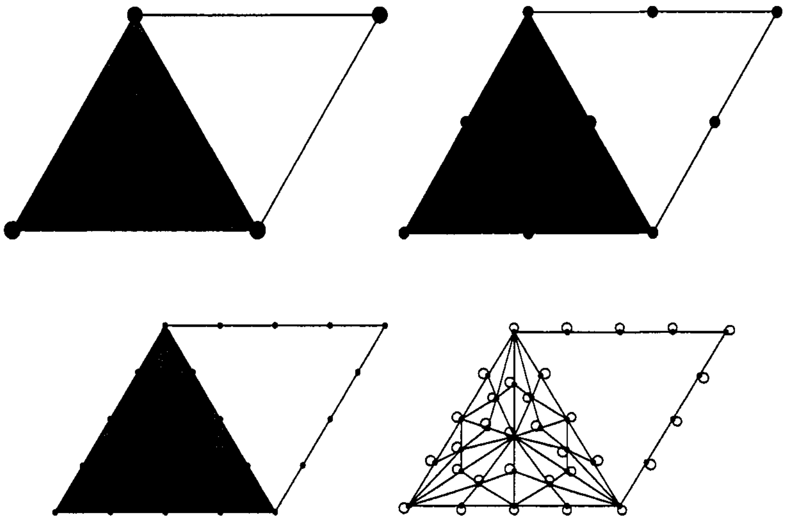

Figure 5.1: The complexes X, X^1, X^2, and the reflexive graph $G_{2,X}$.

a finite simplicial complex. Then for $k \geq \max\{2, \mathrm{diam}(T)\}$ *there is a homotopy equivalence*

$$X \simeq \Delta\big((G_{k,X})^T\big).$$

Proof. We consider subcomplexes of $\Delta\big((G_{k,X})^T\big)$ of the form $\Delta\big((G^x_{k,X})^T\big)$ (see Definition 5.2.2 below for the definition of the graph $G^x_{k,X}$). By Lemma 5.2.4 the collection of these subcomplexes form a cover of $\Delta\big((G_{k,X})^T\big)$, and by Lemma 5.2.5 the nerve of this cover is isomorphic to the simplicial complex X. By Lemma 5.2.7 and Lemma 5.2.8, these subcomplexes and nonempty intersections are contractible. The result follows from the nerve lemma of [Bjö95]. $\square$

We next turn to the definition of our subcomplexes and the proofs of the lemmas mentioned above. Recall that the simplicial complex $\Delta\big((G_{k,X})^T\big)$ is determined by its 1-skeleton $(G_{k,X})^T$, whose vertices are given by all graph maps $f : T \to G_{k,X}$, and with edges $\{f, f'\}$ whenever $f(t) \sim f'(t')$ for all $t \sim t'$ in T. We note that the vertices of the original complex X are naturally vertices of the graph $G_{k,X}$. We will work with certain graph theoretic 'open neighborhoods' of these vertices, as described in the following definition.

Definition 5.2.2. *For a fixed vertex x of the original complex X, define $G^x_{k,X}$ to be the subgraph of $G_{k,X}$ induced by the vertices $\{w \in G_{k,X} : d(x, w) \leq 2^k - 1\}$. Hence the vertices* *of* $G^x_{k,X}$ *are the vertices of* $G_{k,X}$ *that are distance at most* $2^k - 1$ *from the vertex* x *(see Figure 5.2).*

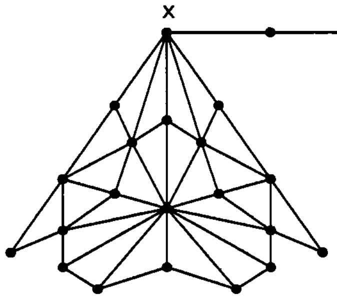

Figure 5.2: The graph $G^x_{2,X}$ (without the loops)

It is this collection of subcomplexes $\left\{\Delta\left((G^x_{k,X})^T\right)\right\}_{x \in V(X)}$ that we wish to show cover the complex $\Delta\left((G_{k,X})^T\right)$. For this we will need a general lemma regarding clique complexes of exponential graphs. For graphs T and G, and a simplex $\alpha = \{f_1, \ldots, f_a\} \in \Delta(G^T)$, define G_α to be the subgraph of G induced by the vertices $\{f_i(t) : 1 \leq i \leq a, t \in V(T)\}$. We then make the following observation.

Lemma 5.2.3. *Let* T *be a finite connected graph with diameter* $d = \operatorname{diam}(T)$*, and suppose* G *is a graph. Then* $\operatorname{diam}(G_\alpha) \leq \max\{2, d\}$ *for all* $\alpha \in \Delta(G^T)$.

Proof. Suppose T and G are as above, and suppose $\alpha = \{f_1, \ldots f_a\}$ is a face of $\Delta(G^T)$. Let $v = f_i(t)$ and $v' = f_{i'}(t')$ be two elements of G_α. We will find a path in G_α from v to v' of length $\leq d$. If $t \neq t'$, then by assumption we have a path in T from t to t' given by $(t = t_0, t_1, \ldots, t_j = t')$, with $j \leq d$. If $t = t'$, we take our path to be $(t, t_1, t_2 = t)$, where t_1 is any neighbor of t. So we have $j \leq \max\{2, d\}$

Now, since α is a clique in the graph G^T, we have that $f_i \sim f_j$ for all $1 \leq i, j \leq k$, and hence $f_i(t) \sim f_j(t')$ for all adjacent $t \sim t'$. Hence we can take our desired path to be $f_i(t) = f_i(t_0), f_i(t_1), \ldots, f_i(t_{j-1}), f_{i'}(t_j) = f_{i'}(t')$. □

We can now show that our subcomplexes indeed form a cover.

Lemma 5.2.4. *The collection of complexes* $\left\{\Delta\left((G^x_{k,X})^T\right)\right\}_{x \in V(X)}$ *covers the complex* $\Delta\left((G_{k,X})^T\right)$.

Proof. To simplify indices, in our notation for graphs we will suppress reference to the integer k and the simplicial complex X, so that for this proof $G = G_{k,X}$ and $G^x = G^x_{k,X}$. If α is a face of $\Delta(G^T)$ then by Lemma 5.2.3 we have either $k = 2$ and $\mathrm{diam}(G_\alpha) = 2$, or else $\mathrm{diam}(G_\alpha) \leq d \leq 2^{k-1} - 1$. We claim that $G_\alpha \subset G^x$ for some $x \in V(X)$, which would prove our claim.

Let $m = \min\{d(w, x) : w \in G_\alpha, x \in V(X)\}$. Note that $m \leq 2^{k-1}$ since every vertex of X^k is within distance 2^{k-1} of some vertex of the original complex X.

If $m = 0$ then we have $y \in G_\alpha$ for some vertex $y \in V(X)$. Hence $G_\alpha \subset G^y$ since G^y contains all vertices of distance at most $2^k - 1$ from y (this number is at least 2 since $k \geq 2$).

If $m > 0$ let w be a vertex of G_α such that $d(w, x) = m$ for some vertex $x \in X$, and choose w such that it is contained in the interior of a face of X of minimum dimension. We need to show that $G_\alpha \subset G^x$. To see this, first consider the case that $k > 2$. By Lemma 5.2.3, all vertices w' in G_α are distance at most $d \leq 2^{k-1} - 1$ from w. So all vertices of G_α are at most $m + d \leq 2^{k-1} + 2^{k-1} - 1 = 2^k - 1$ away from x, which implies $G_\alpha \subset G^x$.

If $k = 2$, then we have $m = 1$ or $m = 2$. If $m = 1$ then all vertices of G_α are distance at most $1 + 2 = 3 = 2^k - 1$ away from x, as desired. If $m = 2$, then all vertices of G_α are distance at least 2 from *every* vertex of X. Now, w is contained in the interior of some face $F_w = \{x, x_1, \ldots, x_j\}$ of the original complex X. If w' is any other vertex of G_α, then w' cannot be contained in any proper face of F_w since otherwise we would have taken $w = w'$. Hence w' is contained in the interior of F_w, so that $d(w', x) \leq 2^k - 1$, as desired. This shows that $G_\alpha \subset G^x$. □

We next turn to the combinatorics of this cover. Recall that the *nerve* of a covering by subcomplexes is the simplicial complex with vertices given by the subcomplexes and with faces corresponding to all non-empty intersections. We then have the following observation.

Lemma 5.2.5. *The nerve of the covering of $\Delta\big((G_{k,X})^T\big)$ given by the subcomplexes $\Delta\big((G^x_{k,X})^T\big)$ is isomorphic to the simplicial complex X.*

Proof. By construction, the vertices of the nerve determined by the $\Delta\big((G^x_{k,X})^T\big)$ are indexed by $V(X)$, the vertices of the simplicial complex X. A collection $I \subseteq V(X)$ of such subcomplexes has nonempty intersection if and only if there exists a vertex x within distance $2^k - 1$

from each $v \in I$ in X^k, the k^{th} barycentric subdivision of X. But this occurs if and only if the collection I of vertices form a face of X. $\square$

Next we wish to show that each subcomplex $\Delta\big((G^x_{k,X})^T\big)$ is contractible. To do this we will show that each graph $G^x_{k,X}$ is in fact dismantlable. Recall from Chapter 4 that a finite graph G is called *dismantlable* if it can be folded down to the looped vertex $\mathbf{1}$ (see Chapter 4 and also [BW04] for other characterizations). It follows from the results of [Koz06b] that if G is dismantlable, then $\mathrm{Hom}(S, G)$ is contractible for *every* graph S. Hence to show that the subcomplexes $\mathrm{Hom}(T, G^x_{k,X}) \simeq \Delta\big((G^x_{k,X})^T\big)$ are each contractible, it suffices to show that each graph $G^x_{k,X}$ is dismantlable.

For this we will describe a recursive folding procedure for the graph $G^x_{k,X}$. In our induction we will need the fact that barycentric subdivision preserves dismantlability, as described by the following lemma.

Lemma 5.2.6. *If G is a dismantlable graph and $\Delta(G)$ is its clique complex (on its looped vertices), then the one-skeleton of* $\mathrm{bd}(\Delta(G))$ *is again dismantlable.*

Proof. Suppose G is a dismantlable graph, and let G' denote the graph obtained by taking the looped one-skeleton of $\mathrm{bd}(\Delta(G))$. We can think of G' as the graph whose vertices are the elements of the poset $\mathrm{Hom}(\mathbf{1}, G)$, with adjacency given by $x \sim y$ if x and y are comparable.

To show that G' is dismantlable, we proceed by induction on n, the number of looped vertices of G. If $n = 1$ we have that $G = G'$ is a single looped vertex, and hence dismantlable.

Next suppose $n > 1$, and let v and w be distinct looped vertices of G such that $N_G(v) \subseteq N_G(w)$. For future reference, we let $N_G(v) = \{v, w, v_1, \dots, v_m\}$ denote the neighboring vertices of v in the graph G. We will use the running example depicted in Figure 5.3, in which the loops (present on all vertices) will be omitted for the sake of space.

For the inductive step, we need to fold away all vertices in G' that are barycenters of simplices that have v as a vertex (including the vertex v itself). But this is precisely $N_{G'}(v)$, the collection of neighboring vertices of v in the graph G'.

We will first fold away the vertices in $N_{G'}(v)$ that are furthest from w. We let S denote the collection of vertices in $N_{G'}(v)$ that are barycenters of simplices that do *not* contain w.

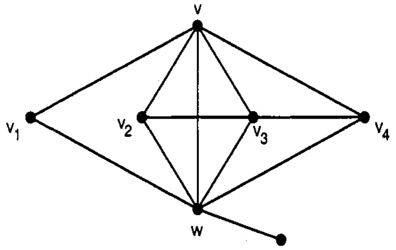

Figure 5.3: The containment $N_G(v) \subseteq N_G(w)$.

So S is the collection of vertices in $\mathrm{bd}(\Delta(G))$ that are barycenters of simplices with vertices among the set $\{v, v_1, \ldots, v_m\}$.

Each vertex $s \in S$ is the barycenter of a face of a certain dimension, and we will fold away the elements of S in descending order according to this dimension. If s is the barycenter of a face $\{v, v_{i_1}, \ldots, v_{i_r}\}$ of *maximal* dimension then we have $N_{G'}(s) \subseteq N_{G'}(y)$, where $y \in G'$ is the barycenter of the face $\{v, v_{i_1}, \ldots v_{i_r} w\}$; this collection forms a face of $\Delta(G)$ since $N_G(v) \subseteq N_G(w)$. Hence we can fold away s in this case.

In general, s is the barycenter of a face $F_s = \{v, v_{j_1}, \ldots, v_{j_\ell}\}$ and, as we have folded away the vertices of greater dimension in S (barycenters of faces that contain F_s), we have $N(s) \subseteq N(y)$ in the resulting graph, where again y is the barycenter of the face $\{v, v_{j_1}, \ldots, v_{j_\ell}, w\}$.

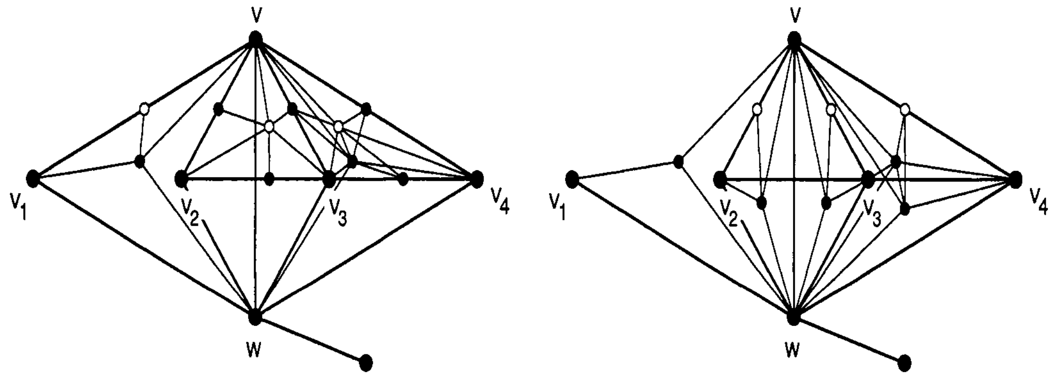

Figure 5.4: Folding away vertices of S

In the Figure 5.4 above, the first step is to fold away the barycenters of $\{v, v_1\}$, $\{v, v_2, v_3\}$, and $\{v, v_3, v_4\}$ (the vertices in white). In the second step we fold away the barycenters of $\{v, v_2\}$, $\{v, v_3\}$, and $\{v, v_4\}$.

Next we fold away the vertex v. If $u \in N(v)$ is a neighbor of v in the graph at this stage of the folding, then u is the barycenter of some face that contains both v and w, and hence we have $N(v) \subseteq N(z)$, where z is the barycenter of $\{v, w\}$. We fold away v and now have that all neighbors of z are barycenters of faces that contain the vertex w. Hence we now have $N(z) \subseteq N(w)$, and we proceed to fold away the vertex z.

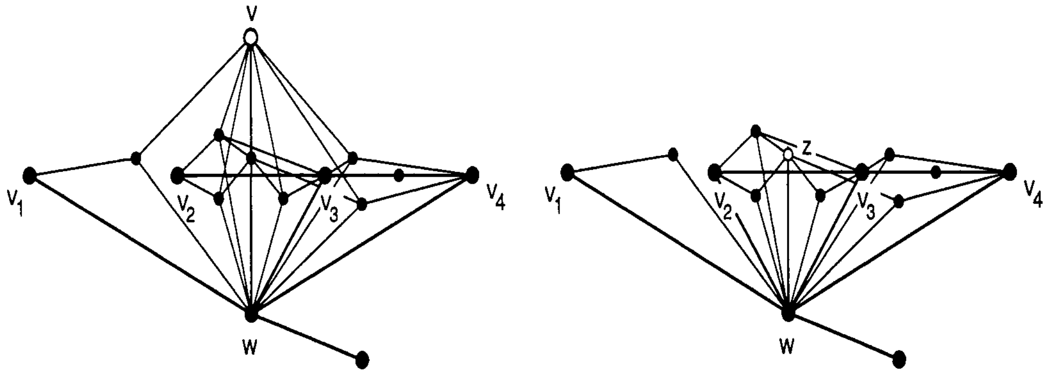

Figure 5.5: Folding away v and z

At this point, we are left with a subset $Y \subseteq N_{G'}(v)$ that consists of vertices that are barycenters of faces that contain v, w, and at least one vertex from $\{v_1, \ldots, v_m\}$. We fold away these vertices in *ascending* order according to their dimension. If $y \in Y$ is the barycenter of a face $\{v, w, v_i\}$ of *minimal* dimension, then $N(y) \subset N(z)$, where z is the barycenter of the face consisting of $\{w, v_i\}$ (since vertices that are barycenters of faces including v have been folded away). In the general case, y is the barycenter of a face $\{v, w, v_{i_1}, \ldots, v_{i_j}\}$ and, as we have folded away the vertices of smaller dimension in Y, we now have $N(y) \subset N(z)$, where again z is the barycenter of the face $\{w, v_{i_1}, \ldots, v_{i_j}\}$.

$\square$

We can now use Lemma 5.2.6 to prove the following result concerning the $G^x_{k,X}$ graphs.

Lemma 5.2.7. *For every vertex $x \in X$, the graph $G^x_{k,X}$ is dismantlable.*

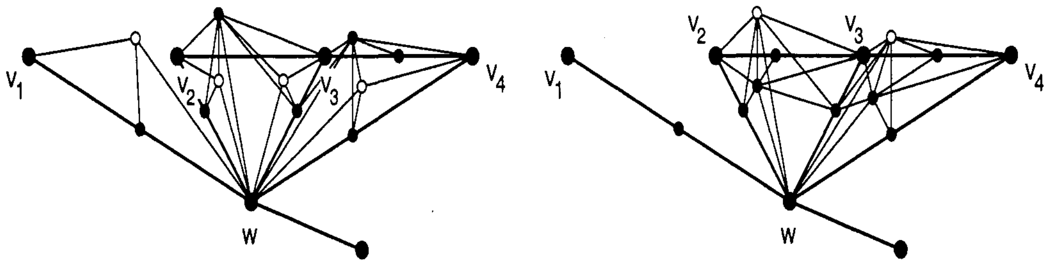

Figure 5.6: Folding away the remaining vertices of $N_{G'}(v)$.

Proof. Recall that $G^x_{k,X}$ is the subgraph of $G_{k,X}$ induced by the vertices that are distance at most $2^k - 1$ from x. We will prove the claim by induction on k. For $k = 1$ the graph $G^x_{k,X}$ consists of $N_{G_{k,X}}(x)$, the neighbors of the vertex x in $G_{k,X}$ (including x itself). Hence $G^x_{k,X}$ folds down to the single looped vertex x, as desired.

Next suppose $k > 1$. Our plan is to first fold away the vertices in $G^x_{k,X}$ that are distance *exactly* $2^k - 1$ from x. The resulting subgraph one obtains is the looped 1-skeleton of the barycentric subdivision of the clique complex $\Delta(G^x_{k-1,X})$ (this graph is called $\left(G^x_{k-1,X}\right)'$ in the notation of the proof of Lemma 5.2.6). By induction, together with Lemma 5.2.6, this graph is dismantlable and hence our claim will be proved.

Let V_x denote the collection of vertices in $G^x_{k,X}$ that are distance exactly $2^k - 1$ from x; it is this collection of vertices that we wish to fold away. First we set up some notation. Note that every vertex v in the graph $G_{k,X}$ has a pair of parameters $\alpha(v) = (i,j)$ associated with it, where i is the dimension of the face in X that v lies in, and where j is the dimension of the face of X^{k-1} that v is the barycenter of (note that $j \leq i$). We will say that v is *of type* (i,j) if $\alpha(v) = (i,j)$. See Figure 5.7 for an example.

We will fold away the vertices of $V_x \subset G^x_{k,X}$ in lexicographic order according to their type (i,j). First note that if $v \in V_x$ is of type (i,j) then $j \geq 1$, and hence our base case to consider is when $v \in V_x$ is of type $(1,1)$. In this case v is the barycenter of an edge $\{a,b\}$ in X^{k-1}, where b is a vertex of X, and a is distance $2^{k-1} - 2$ from x. Any neighbor $w \in N_{G^x_{k,X}}(v)$ of v is a barycenter of a simplex that has a as a vertex; hence we have $w \sim a$. We conclude that v can be folded onto the neighboring vertex a.

Next we consider the case v is of type (i,j), where $i > 1$ is fixed. We proceed by

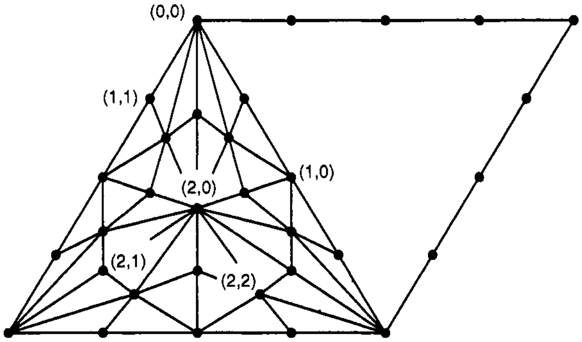

Figure 5.7: The types (i, j) of various vertices in the graph $G_{k,X}$

induction on j. If $j = 1$ then v is the barycenter of an edge $\{c, d\}$, where $c \notin V_x$ and d is of type $(i, 0)$ and is distance $2^k - 2$ from x. Any other neighbor $w \in N_{G^x_{k,X}}(v)$ is the barycenter of a simplex that has d as a vertex; we conclude that $w \sim d$. Hence in this case v can be folded onto d (see Figure 5.8).

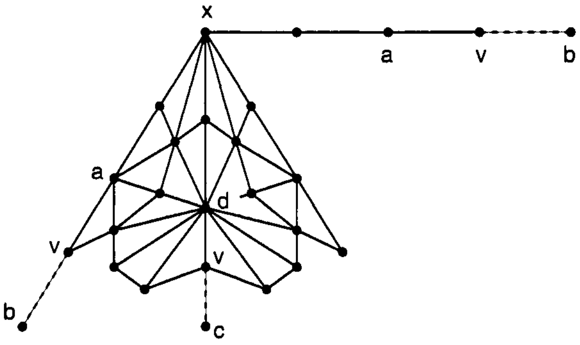

Figure 5.8: Folding away the vertex v when v is of type $(i, 1)$

For the same fixed $i > 1$, we next consider the case that v is of type (i, j), where $j > 1$. By induction, we have that all vertices in V_x of type (i', k) and of type (i, j') have been folded away, where $i' < i$ and $j' < j$. Pick a vertex $w \in N(v)$ in the neighborhood of v such that $w \in G^x_{k-1,X}$ and such that the type of w is largest in the lexicographic order - that is, of type (i, j) where j is maximum among maximum i.

We claim that $N_{G^x_{k,X}}(v) \subseteq N_{G^x_{k,X}}(w)$, so that the vertex v can be folded onto w. To see this, suppose $u \in N_{G^x_{k,X}}(v)$. If $u \in V_x$ (so that $d(u,x) = 2^k - 1$), then by induction we know that u is of type (i', j'), where either $i' > i$ or else $i' = i$ and $j' > j$. In either case we see that u is the barycenter of a simplex U that contains the vertex w, and hence $u \sim w$ as claimed. If $u \notin V_x$, so that $d(u,x) = 2^k - 2$, then either $u = w$ or else the type of u is lexicographically smaller than the type of w. In this latter case u is the barycenter of a simplex U' that contains the vertex w, and hence again $u \sim w$. We conclude that $u \in N_{G^x_{k,X}}(w)$ and hence $N_{G^x_{k,X}}(v) \subseteq N_{G^x_{k,X}}(w)$ as desired.

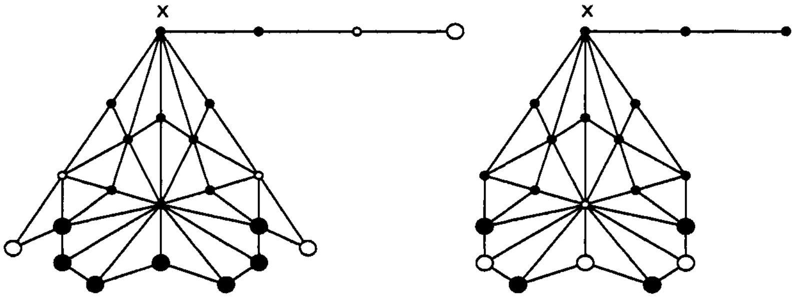

Figure 5.9: Folding away vertices of type $(1,1)$ and of type $(2,1)$ in $G^x_{2,X}$.

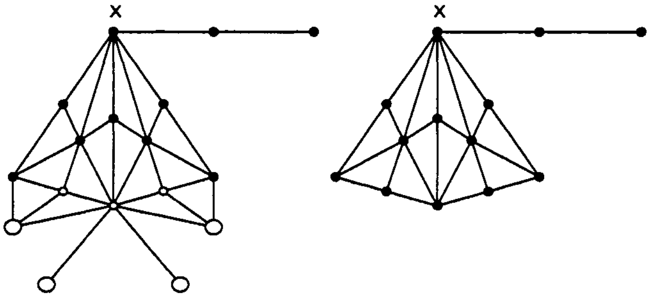

Figure 5.10: Folding away vertices of type $(2,2)$ and the resulting $(G^x_{1,X})'$

This completes the induction on j and hence we have now folded away all vertices of V_x that are of type $(i, ?)$. This in turn completes the induction on i and we conclude that all vertices in V_x can be folded away. As we noted above, the resulting graph is $\left(G^x_{k-1,X}\right)'$, the barycentric subdivision of $G^x_{k-1,X}$, which we conclude is dismantlable by induction on k and by applying Lemma 5.2.6. The result follows. □

The final step in proving our theorem is to consider the intersections of the subcomplexes $G^x_{k,X}$.

Lemma 5.2.8. *All nonempty intersections of the subcomplexes* $\{(G^x_{k,X})^T\}_{x \in V(X)}$ *are contractible.*

Proof. We prove this in much the same way as we handled the contractibility of the subcomplexes themselves. In particular it is enough to show that the subgraphs obtained as nonempty intersections of $\{G^x_{k,X}\}_{x \in V(X)}$ are dismantlable. A vertex of such a graph is, by definition, within a distance of $2^k - 1$ of every vertex $x \in V(X)$ in some index set $I \subseteq V(X)$.

Suppose $G^I_{k,X}$ is such a graph. Again, we will show that $G^I_{k,X}$ is dismantlable by induction on k. If $k = 1$ then the graph $G^I_{k,X}$ is a single looped vertex, the barycenter of the face of X defined by the index set I, which is of course dismantlable.

For the case $k > 1$ we will, as above, fold away the vertices of $G^I_{k,X}$ that are distance $2^k - 1$ from some vertex $x \in I$. We will refer to these vertices as V_I, so that $V_I = \{v \in G^I_{k,X} : d(v,x) = 2^k - 1 \text{ for some } x \in I\}$.

Again, we fold away the vertices of V_I in lexicographic order according to their type (i,j). Since $V_I \subset V_x$ (for any $x \in I$), we can follow the same procedure as we described in the proof of Lemma 5.2.7. In particular, to fold away a vertex $v \in V_I$ of type (i,j), we choose a vertex $w \in N_{V_x}(v)$ in the neighborhood of v such that $w \in G^I_{k-1,X}$ and such that the type of w is largest in the lexicographic order.

We just need to check that w is within $2^k - 1$ of *every* vertex $x' \in I$, so that indeed $w \in V_I$. But this follows from the choice of w: since v is in the interior of the face of X determined by the vertices I, any neighbor w' of v that lies outside of V_I will be of type (i',j), where $i' < i$. But v has neighbors in $G^I_{k-1,X}$ that are of type (i,j'), so that the choice of w will indeed lie in V_I.

Hence the double induction follows through in this case, and we are left with a graph $G^{I'}_{k-1,X}$, the barycentric subdivision of the graph $G^I_{k-1,X}$ (informally speaking). Once again we employ Lemma 5.2.6 and by induction we get that this graph is also dismantlable. □

5.3 Further questions

Having constructed our graph $G_{k,X}$ as the 1-skeleton of the k^{th} iterated subdivision of X, a natural question to ask is if this choice of k is best possible. We have a feeling that it is not, and in fact, for the case $\mathrm{diam}(T) = 1$ (so that T is a complete graph with possibly some loops) we conjecture that $k = 1$ will do the job.

Conjecture 5.3.1. *If X is a finite simplicial complex, and T is a finite connected graph with $\mathrm{diam}(T) = 1$, then there is a homotopy equivalence*

$$\mathrm{Hom}(K_2, G_{1,X}) \simeq X.$$

Another thing to consider would be simplicial complexes with a specified group action.

Question 5.3.2. *Given a graph T with automorphism group $\Gamma = Aut(T)$, and a Γ-simplicial complex X, can one find a graph G such that $\mathrm{Hom}(T, G)$ is Γ-homotopy equivalent to X?*

Chapter 6

HOMOTOPY GROUPS OF HOM COMPLEXES

6.1 Introduction

In several recent papers (see [BBdLL06], [BL05]), a homotopy theory of reflexive graphs termed A-theory has been developed in which graph theoretic homotopy groups are defined to measure 'combinatorial holes' in simplicial complexes. In [BBdLL06] the authors construct a cubical complex X_G (associated to the graph G), and a homomorphism from the homotopy groups of the geometric realization of X_G to the A-theory groups of G; modulo a (yet unproved) version of cubical approximation, they show that this map is in fact an isomorphism.

In Chapter 4 of this thesis, we defined a similar homotopy theory for general graphs called $\times$-homotopy. There we discussed how both A-theory and $\times$-homotopy can be placed in a common framework of exponential graph constructions associated to the relevant products (cartesian for A-theory, categorical for $\times$-homotopy). It was shown that $\times$-homotopy is characterized by topological properties of the Hom complex; in particular, the $\times$-homotopy class of maps from graphs G to H are seen to coincide with the path components of the space $\mathrm{Hom}(G, H)$.

In this chapter, we consider the graph theoretic notions of homotopy groups that arise in the context of $\times$-homotopy. We show that these groups are isomorphic to the usual homotopy groups of a pointed version of the Hom complex (denoted Hom_*). Our method for proving this is to construct a 'path graph' G^I associated to a (pointed) graph G and to show that the natural endpoint map induces a long exact sequence of homotopy groups of the Hom_* complexes. The material in this chapter is taken from [Docb].

The chapter is organized as follows. In Section 2, we describe the category of pointed graphs and the notions of both A-homotopy and $\times$-homotopy in this setting. In Section 3, we introduce the Hom_* functors and prove some basic facts about them, including interaction with the relevant adjunction and also a graph operation known as folding. In Section 4, we introduce the notion of a path graph, and state and prove our main result regarding the long exact sequence of the homotopy groups of the Hom_* complexes.

6.2 Pointed graphs

In this chapter, we work primarily in the category of pointed graphs. A *pointed graph* $G = (G, x)$ is a graph G together with a specified looped vertex x. A *map of pointed graphs* $f : (G, x) \to (H, y)$ is a map of graphs such that $f(x) = y$. The resulting category of pointed graphs will be denoted $\mathcal{G}_*$. The category $\mathcal{G}_*$ enjoys some useful properties that we discuss next.

Definition 6.2.1. *For pointed graphs $G = (G, x)$ and $H = (H, y)$, the smash product $G \wedge H$ is the pointed graph with vertex set given by the quotient of $(V(G) \times V(H))$ under the identifications $(x, h) = (g, y)$ for all $g \in V(G)$, $h \in V(H)$. Adjacency is given by $[(g, h)] \sim [(g', h')]$ if $g \sim g'$ and $h \sim h'$ for some representatives. The graph $G \wedge H$ is pointed by the vertex $[(x, y)]$ (see Figure 6.1).*

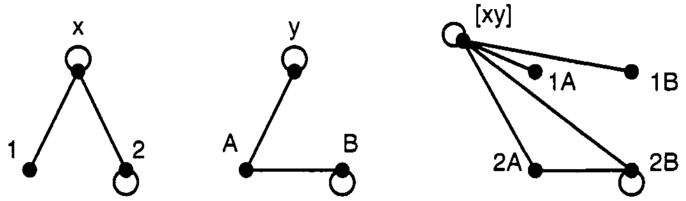

Figure 6.1: The graphs $G = (G, x)$, $H = (H, y)$, and $G \wedge H = (G \wedge H, [(x, y)])$

Definition 6.2.2. *For pointed graphs $G = (G, x)$ and $H = (H, y)$, the pointed internal hom graph, denoted H^G is the pointed graph with vertices given by all set maps $\{f : V(G) \to V(H) : f(x) = y\}$. Adjacency is given by $f \sim g$ if $f(v) \sim g(v')$ in H for all $v \sim v'$ in G. The graph H^G is pointed by the graph map that sends every vertex of G to the vertex $y \in H$.*

These last two constructions are adjoint to one another, as described in the following lemma. We let $\mathcal{G}_*(G, H)$ denote the set of pointed maps between the pointed graphs G and

H.

Lemma 6.2.3. *For pointed graphs $A = (A, x)$, $B = (B, y)$, and $C = (C, z)$ we have a natural bijection of sets $\varphi : \mathcal{G}_*(A \wedge B, C) \to \mathcal{G}_*(A, C^B)$, given by $\varphi(f)(a)(b) = f[(a, b)]$ for $a \in V(A)$ and $b \in V(B)$.*

Proof. First note that $\varphi(f)$ is well defined since $f(x, b) = f(a, y) = z$ for $a \in V(A)$, $b \in V(B)$. Next we see that $\varphi(f)(a) \in C^B$ since $\varphi(f)(a)(y) = f(a, y) = z$, and is pointed since $\varphi(f)(x)(b) = f(x, b) = z$. We claim that $\varphi(f)$ is a graph map. To see this, suppose $a \sim a'$; we need to check that $\varphi(f)(a) \sim \varphi(f)(a')$. Suppose $b \sim b'$. Then we have $\varphi(f)(a)(b) = f(a, b)$ and $\varphi(f)(a')(b') = f(a', b')$, and hence (the equivalence classes) are adjacent in $A \wedge B$.

Next, to show that φ is a bijection, we define a map $\psi : \mathcal{G}_*(A, C^B) \to \mathcal{G}_*(A \wedge B, C)$ via $\psi(g)[(a, b)] = g(a)(b)$. We note that $\psi(g)(x, b) = g(x)(b) = z$ and $\psi(g)(a, y) = g(a)(y) = z$, for $a \in V(A)$ and $b \in V(B)$, and hence $\psi(g)$ is well defined on the vertices of $A \wedge B$. Similarly, one checks that $\psi(g)$ is a pointed graph map. It is clear that ψ is the inverse to φ and the result follows. ☐

6.3 Homotopy of pointed graphs

We next turn to the definition of 'homotopy' in the context of the pointed category. The construction runs along the same lines of the ×-homotopy discussed in Chapter 3. Although the constructions for the pointed category are straightforward modifications of the notions for the unpointed category, we will for convenience record all the definitions here.

If G is a graph, then we define G_* to be the pointed graph obtained by adding a distinguished disjoint looped vertex to the graph G.

Definition 6.3.1. *The graph I_n is the (reflexive) graph with vertices $\{0, 1, \ldots, n\}$, and with adjacency given by $i \sim j$ if $|i - j| \leq 1$ (see Figure 6.2).*

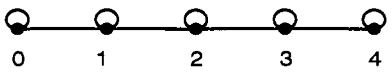

Figure 6.2: The graph I_4

For our definition of homotopy, we will want to consider path components of the exponential graph H^G. To capture this notion in the category of pointed graphs, the relevant graph to map from will be the pointed graph I_{n*}, since a pointed map $I_{n*} \to (G,x)$ will send $*$ to x, and 0 and n to the endpoints of a path in G.

Definition 6.3.2. *A pair of maps $f, g : G \to H$ between pointed graphs is called $\times$-homotopic if there is an integer n and a (pointed) graph map $F : I_{n*} \to H^G$ such that $F(0) = f$ and $F(n) = g$.*

This defines an equivalence relation on $\mathcal{G}_*(G,H)$, and the set of $\times$-homotopy classes of pointed maps between G and H will be designated $[G,H]_\times$. Applying the adjunction of Lemma 6.2.3 we note that a homotopy between f and g is the same as a pointed map $\tilde{F} : G \wedge I_{n*} \to H$ with $\tilde{F}(?,0) = f$ and $\tilde{F}(?,n) = g$.

Also in Chapter 4, we relate the construction of our $\times$-homotopy to the A-theory of [BBdLL06]. We briefly recall the discussion here.

Definition 6.3.3. *A pair of pointed graph maps $f, g : (G,x) \to (H,y)$ between reflexive graphs is called A-homotopic, denoted $f \simeq_A g$, if there is an integer n and a graph map*

$$\phi : G \Box I_n \to H,$$

such that $\phi(?,0) = f$ and $\phi(?,n) = g$, and such that $\phi(x,i) = y$ for all i.

One can check that f and g are A-homotopic if and only if there is a pointed graph map $\tilde{\phi} : I_{n*} \to H^G$ such that $\tilde{\phi}(0) = f$ and $\tilde{\phi}(n) = g$; here H^G is the internal hom graph that is right adjoint to the cartesian product. In other words, our definition of $\times$-homotopy is the 'same' as that of A-homotopy, with the categorical product playing the role of the cartesian product.

In the paper [BBdLL06], the authors consider the *A-homotopy groups* of a reflexive pointed graph (G,x), denoted $A_n(G,x)$. By definition, $A_n(G,x)$ is the set of A-homotopy classes of graph maps

$$f : (I^m, \partial I^m) \to (G,x).$$

Here $I^m = I_n \square I_n \square \ldots \square I_n$ is the m-fold cartesian product of I_n, and ∂I^m is the subgraph of I^m consisting of vertices with at least one coordinate equal to 0 or n.

The authors of [BBdLL06] seek to construct a topological space whose homotopy groups encode the A-homotopy groups of the graph G; this is seen as a generalization of a result from [BL05], where it is shown that $A_1(G, x)$ is isomorphic to the fundamental group of the space obtained by attaching 2-cells to all 3- and 4-cycles of the graph G. The main result of [BBdLL06] is the construction of a cubical complex $M_*(G)$ associated to the reflexive graph G, and a homomorphism from the geometric realization $X_G := |M_*(G)|$ to the A-homotopy groups of G. Here the i-cube $M_i(G)$ is the set $\mathcal{G}(I_1^m, G)$ of all graph maps from I_1^m to G.

In this paper we consider the analogous questions in the context of ×-homotopy. One can follow the procedure of [BBdLL06] and construct a cubical complex built from the sets $\mathcal{G}(I_1^m, G)$, where this time I_1^m denotes the m-fold *categorical* product. In this way we obtain a map from the realization of this space to the graph-theoretically defined '×-homotopy groups' of G. However, it turns out that we can follow a somewhat different route to obtain a space $\mathrm{Hom}_*(T, G)$ whose homotopy groups do in fact coincide with what we will call the 'T-homotopy groups of G'. We turn to a discussion of these spaces in the next section.

6.4 **Hom$_*$** *complexes*

There is a natural notion of the Hom complex in the pointed setting (which we will denote as Hom_*). As in the unpointed setting, this construction interacts well with the adjunction of Lemma 6.2.3, and the path components of this space characterize the set of ×-homotopy of pointed maps. We collect these facts next.

Definition 6.4.1. *For pointed graphs $G = (G, x), H = (H, y)$, we define $\mathrm{Hom}_*(G, H) \subseteq \mathrm{Hom}(G, H)$ to be the (pointed) poset whose elements are given by all functions $\eta : V(G) \rightarrow 2^{V(H)} \backslash \{\emptyset\}$, such that $\eta(x) = \{y\}$, and if $(v, w) \in E(G)$, then $(\tilde{v}, \tilde{w}) \in E(H)$ for all $\tilde{v} \in \eta(v)$ and $\tilde{w} \in \eta(w)$. The relation is given by containment, so that $\eta \leq \eta'$ if $\eta(v) \subseteq \eta'(v)$ for all $v \in V(G)$.*

Note that $\mathrm{Hom}_*(G, H)$ is itself pointed by the map that sends all vertices of G to the vertex $y \in V(H)$.

For pointed graphs $A = (A, x), B = (B, y), C = (C, z)$, we have seen that the exponential graph construction provides the adjunction $\mathcal{G}_*(A \wedge B, C) = \mathcal{G}_*(A, C^B)$, an isomorphism of sets. As is the case in the unpointed context (see Chapter 3 and also [Koza]), this extends to a homotopy equivalence of the analogous Hom_* complexes.

Proposition 6.4.2. *If* $A = (A, x), B = (B, y)$, *and* $C = (C, z)$ *are pointed graphs, then* $\mathrm{Hom}_*(A \wedge B, C)$ *can be included in* $\mathrm{Hom}_*(A, C^B)$ *so that* $\mathrm{Hom}_*(A \wedge B, C)$ *is a strong deformation retract of* $\mathrm{Hom}_*(A, C^B)$. *In particular,* $\mathrm{Hom}_*(A \wedge B, C) \simeq \mathrm{Hom}_*(A, C^B)$.

Proof. We follow the proof of the analogous statement in the unpointed context (see Chapter 4). We let $P = \mathrm{Hom}_*(A \wedge B, C)$ and $Q = \mathrm{Hom}_*(A, C^B)$ denote the respective posets. We define a map of posets $j : P \to Q$ given by $\alpha \mapsto j(\alpha)$ where $j(\alpha)(a) = \{f : (V(B), y) \to (V(C), z) | f(b) \in \alpha(a, b) \forall b \in B\}$ for every vertex $a \in A$. Note that $z \in \alpha(x, b)$ for all $b \in V(B)$ and hence the constant function $f_z : V(B) \to V(C)$, given by $b \mapsto z$ for all b, is an element of $j(\alpha)(x)$. Also, if $(a, a') \in E(A)$ then we have $(f, f') \in E(C^B)$ for all $f \in j(\alpha)(a), f' \in j(\alpha)(a')$. Hence $j(\alpha)$ is indeed an element of $\mathrm{Hom}_*(A, C^B)$.

Note that j is injective. To see this, suppose $\alpha \neq \alpha' \in \mathrm{Hom}_*(A \wedge B, C)$ with $\alpha(a, b) \neq \alpha'(a, b)$. Then we have $\{f(b) | f \in j(\alpha)(a)\} \neq \{g(b) | g \in j(\alpha')(a)\}$, so that $j(\alpha) \neq j(\alpha')$.

Next we define a closure operator $c : Q \to Q$. If $\gamma : V(A) \to 2^{V(C^B)} \backslash \{\emptyset\}$ is an element of Q, define $\tilde{\gamma} \in \mathrm{Hom}_*(A, C^B)$, as follows: fix some $a \in V(A)$, and for every $b \in V(B)$ let $C^{\gamma}_{ab} = \{f(b) \in V(C) | f \in \gamma(a)\}$; define $\tilde{\gamma}(a)$ to be the collection of functions $g : V(B) \to V(C)$ where $g(b)$ varies over all $c \in C^{\gamma}_{ab}$. Finally we define $c(\gamma) = \tilde{\gamma}$.

One can verify that $\tilde{\gamma} \in \mathrm{Hom}_*(A, C^B)$. Also, $c(p) \geq p$ and $(c \circ c)(p) = c(p)$ for all $p \in P$.

Next we claim that $c(Q) \subseteq j(P)$. To see this, suppose $\tilde{\gamma} \in c(Q)$. We define $\alpha : V(A \wedge B) \to 2^{V(C)} \backslash \{\emptyset\}$ by $\alpha(a, b) = C^{\gamma}_{ab}$, where $\gamma \in \mathrm{Hom}_*(A, C^B)$ such that $c(\gamma) = \tilde{\gamma}$ and $C^{\gamma}_{ab} \subseteq V(C)$ is as above. One can verify that $\alpha \in \mathrm{Hom}_*(A \wedge B, C)$.

Finally $j(P) \subseteq c(Q)$ since $j(P) \subset Q$ and $c(j(P)) = j(P)$. So then $j(P) = c(Q)$, and hence $P \simeq j(P)$ is the image of a closure operator on Q and hence a strong deformation retract of Q. ☐

We let $\mathbf{1}_*$ denote the graph consisting of a pair of disjoint, looped vertices, and note that $G \wedge \mathbf{1}_* = G$ for every pointed graph $G = (G, x)$. The above proposition gives us

$\mathrm{Hom}_*(G,H) = \mathrm{Hom}_*(\mathbf{1}_* \wedge G, H) \simeq \mathrm{Hom}_*(\mathbf{1}_*, H^G)$, for pointed graphs $G = (G,x)$ and $H = (H,y)$. The last of these complexes is simply $\Delta(H^G)$, the clique complex on the looped vertices of H^G, with a distinguished vertex given by the map that sends all vertices of G to y. Recall that the looped vertices in H^G are the (pointed) graph homomorphisms $(G,x) \to (H,y)$. Hence, for pointed graphs G and H, the complex $\mathrm{Hom}_*(G,H)$ can be realized up to homotopy type as the clique complex of the subgraph of H^G induced by the (pointed) graph homomorphisms.

With this observation we obtain the following characterization of ×-homotopy.

Lemma 6.4.3. *Suppose $G = (G,x)$ and $H = (H,y)$ are pointed graphs, and $f, g : G \to H$ are pointed graph maps. Then f and g are $\times$-homotopic (as pointed maps) if and only if they are in the same path-connected component of* $\mathrm{Hom}_*(G,H)$.

Proof. A $\times$-homotopy $F : I_{n*} \to H^G$ from f to g is a path in the 1-skeleton of $\Delta(H^G) \simeq \mathrm{Hom}_*(G,H)$. Conversely, a path $I \to |\mathrm{Hom}_*(G,H)|$ can be approximated as a simplicial map from some finite subdivision of I into $\mathrm{Hom}_*(G,H) \simeq \Delta(H^G)$. ☐

For a (not necessarily pointed) graph map $f : G \to H$, and a fixed graph T, the $\mathrm{Hom}(T,?)$ and $\mathrm{Hom}(?,T)$ functors provide maps f_T and f^T in the category of topological spaces. In Chapter 4 it was shown that ×-homotopy of graph maps is characterized by the homotopy properties of these induced maps. For our purposes, we will only need the following implication (from Chapter 4).

Lemma 6.4.4. *Let $f, g : G \to H$ be maps of graphs. If f and g are $\times$-homotopic, then the induced maps of posets $f_T, g_T :$ $\mathrm{Hom}(T,G) \to \mathrm{Hom}(T,H)$ are homotopic for every graph T.*

The pointed Hom_* complexes also interact well with a graph operation known as *folding.* We review this construction next. Recall that if v is a vertex of a graph G, then $N(v) = \{w \in V(G) : (v,w) \in E(G)\}$ is the *neighborhood* of v.

Definition 6.4.5. *Let u and v are vertices of a pointed graph $G = (G,x)$ with $v \neq x$ such that $N(v) \subseteq N(u)$. Then we have a (pointed) map $f : G \to G\backslash v$ given by $f(y) = y$, $y \neq v$, and $f(v) = u$. We call the map f a folding of G at the vertex v. Also, the inclusion $i : G\backslash v \to G$ is called an unfolding.*

Proposition 6.4.6. *Suppose* $G = (G, x)$ *and* $H = (H, y)$ *are pointed graphs, and* u *and* v *are vertices of* G *with* $v \neq x$ *such that* $N(v) \subseteq N(u)$. *Then* $i^H : \mathrm{Hom}_*(G, H) \to \mathrm{Hom}_*(G\backslash v, H)$ *and* $f_H : \mathrm{Hom}_*(H, G) \to \mathrm{Hom}_*(H, G\backslash v)$ *are both deformation retracts, where* i^H *and* f_H *are the poset maps induced by the graph unfolding and folding maps.*

Proof. We mimic the proof given in [Koz06d] of the analogous statement in the unpointed setting. For the first deformation retract, we identify $\mathrm{Hom}_*(G\backslash v, H)$ with the subposet of $\mathrm{Hom}_*(G, H)$ consisting of all α such that $\alpha(v) = \alpha(u)$. Define X to be the subposet of $\mathrm{Hom}_*(G, H)$ given by $X = \{\alpha \in \mathrm{Hom}_*(G, H) : \alpha(u) \subseteq \alpha(v)\}$. Next, define poset maps $\varphi : \mathrm{Hom}_*(G, H) \to X$ and $\psi : X \to \mathrm{Hom}_*(G\backslash v, H)$ according to:

$$\varphi(\alpha)(w) = \begin{cases} \alpha(u) \cup \alpha(v) & \text{if } w = v \\ \alpha(w) & \text{otherwise} \end{cases}$$

$$\psi(\alpha)(w) = \begin{cases} \alpha(u) & \text{if } w = v \\ \alpha(w) & \text{otherwise.} \end{cases}$$

We see that both ψ and φ are closure maps, and that $i^H = \psi\varphi$. Since $\mathrm{Im}(i^H) = \mathrm{Hom}_*(G\backslash v, H)$, the result follows.

For the other statement, we define Y to be the subposet of $\mathrm{Hom}_*(H, G)$ given by $Y = \{\beta \in \mathrm{Hom}_*(H, G\backslash v) : \beta(w) \cap \{u, v\} \neq \{v\} \text{ for all } w \in V(H)\}$. Define poset maps $\rho : \mathrm{Hom}_*(H, G) \to Y$ and $\sigma : Y \to \mathrm{Hom}_*(H, G\backslash v)$ according to:

$$\rho(\beta)(w) = \begin{cases} \beta(w) \cup \{u\} & \text{if } v \in \beta(w) \\ \beta(w) & \text{otherwise} \end{cases}$$

$$\sigma(\beta)(w) = \begin{cases} \beta(w)\backslash\{v\} & \text{if } v \in \beta(w) \\ \beta(w) & \text{otherwise.} \end{cases}$$

Once again, we see that both ρ and σ are closure maps, with $\sigma\rho = f_H$. Since $\mathrm{Im}(f_H) = \mathrm{Hom}_*(H, G\backslash v)$, the result follows. ☐

For each $n \geq 0$, the graph I_n is pointed by the vertex 0. We can use Lemma 6.4.6 to show that all Hom_* complexes involving the pointed graph I_n are contractible.

Lemma 6.4.7. *For every pointed graph $G = (G, x)$ and for every integer n, the complex $\mathrm{Hom}_*(I_n, G)$ is contractible.*

Proof. If $n = 0$, then I_0 is a single looped vertex, and $\mathrm{Hom}_*(I_0, G)$ is a vertex. For $n > 0$, we have $N(n) \subset N(n-1)$, and hence the unfolding map $i : I_{n-1} \to I_n$ induces a homotopy equivalence $f^G : \mathrm{Hom}_*(I_n, G) \to \mathrm{Hom}_*(I_{n-1}, G)$. The claim follows by induction. □

6.5 T-homotopy groups and the main result

We have seen that the path components of $\mathrm{Hom}_*(G, H)$ characterize the ×-homotopy groups of maps from G to H. To relate the higher homotopy groups to graph theoretic constructions, we will want to work with the 'path graph' of a given pointed graph G, which we define next. Recall that the graph I_n is pointed by the vertex 0.

Definition 6.5.1. *For a pointed graph $G = (G, x)$, we define G^I to be the pointed graph obtained as the colimit (union) of the diagram,*

$$\begin{tikzcd}
G^{I_0} \arrow[r] \arrow[drr] & \cdots \arrow[r] & G^{I_n} \arrow[r] \arrow[d] & G^{I_{n+1}} \arrow[r] \arrow[dl] & \cdots \\
 & & G^I & &
\end{tikzcd}$$

where the maps $j_n : G^{I_n} \to G^{I_{n+1}}$ are induced by the maps $I_{n+1} \to I_n$ given by $i \mapsto i$ $(i \neq n+1)$, and $n+1 \mapsto n$ (see Figure 6.3).

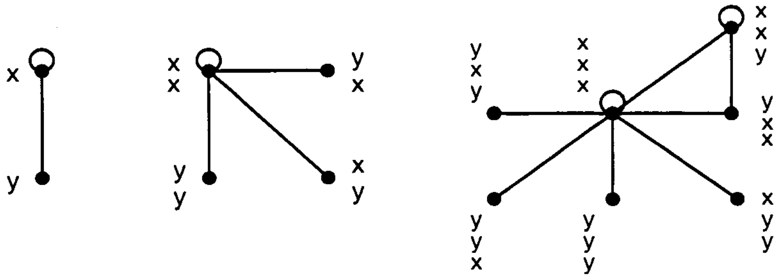

Figure 6.3: The graphs G, G^{I_2}, and G^{I_2} (with the images of 1, 2, and 3).

Note that a vertex of G^I is a (set) map $f : \mathbb{N} \to V(G)$ from the nonnegative integers into the vertices of G with $f(0) = x$ that is *eventually constant*, so that there is some integer N_f

such that $f(i) = f(j)$ for all $i, j \geq N_f$. Adjacency is given by $f \sim g$ if $f(i) \sim g(j)$, for all i, j with $|i - j| \leq 1$. We think of G^I as the graph that parameterizes the collection of paths (along looped vertices) in the graph $G = (G, x)$ that begin at the vertex $x \in G$. We have an *endpoint map* $\varphi : G^I \to G$ given by $\varphi(f) = f(N_f)$.

Definition 6.5.2. *For a pointed graph $G = (G, x)$, we define the loop space graph ΩG to be the (pointed) subgraph of G^I induced by elements that are eventually constant on the vertex $x \in G$.*

Hence ΩG is the graph with vertices given by closed paths along looped vertices in G that start at x and eventually end (and stabilize) at x. Note that ΩG is pointed by the closed path that is constant on the vertex $x \in G$. We will want to iterate the loop space construction, and will use $\Omega^n(G)$ to denote $\Omega(\Omega(\dots(\Omega(G))\dots))$ (n times). It is this Ω functor that will allow us to give a graph theoretical interpretation of the higher homotopy groups of the Hom_* complexes. First we collect some lemmas. The first shows that the path and loop space graph functors commute with exponentials of finite graphs.

Lemma 6.5.3. *If $G = (G, x)$ and $T = (T, y)$ are pointed graphs, with T finite, then $(G^I)^T = (G^T)^I$ and also $(\Omega G)^T = \Omega(G^T)$.*

Proof. Both equalities follow from identical arguments, and so we prove only the second of these claims. Define a map $\alpha : (\Omega G)^T \to \Omega(G^T)$ by $\alpha(f)(i)(t) = f(t)(i)$, for $f \in (\Omega G)^T$. To show that $\alpha(f) \in \Omega(G^T)$, pick an integer j such that each element of $\{f(t)\}_{t \in V(T)}$ stabilizes at j (this is possible since $V(T)$ is finite). So we have $f(t)(k) = f(t)(k') = x$ for all $t \in V(T)$ and $k, k' \geq j$. Hence $\alpha(f)(k) = \alpha(f)(k')$ so that $\alpha(f)$ stabilizes at j. It is easy to check that α is a pointed graph map. To show that it is an isomorphism, define a map $\beta : \Omega(G^T) \to (\Omega G)^T$ by $\beta(g)(t)(i) = g(i)(t)$. Suppose $g \in \Omega(G^T)$ stabilizes at the integer j. Then we have $g(k) = g(k')$ as elements of G^T, for all $k, k' \geq j$. Hence for every t, we have that $\beta(g)(t)(k) = \beta(g)(t)(k')$ so that in fact β maps to $(\Omega G)^T$. Again one can check that β is a pointed graph map and the inverse to α. The result follows. $\square$

Lemma 6.5.4. *For a pointed graph G and a finite pointed graph T, $\text{Hom}_*(T, G^I)$ is contractible.*

Proof. We prove the claim for $T = \mathbf{1}_*$, from which the result follows from the homotopy equivalence $\mathrm{Hom}_*(T, G^I) \simeq \mathrm{Hom}_*(\mathbf{1}_*, (G^I)^T) = \mathrm{Hom}_*(\mathbf{1}_*, (G^T)^I)$.

We first show that $\mathrm{Hom}_*(\mathbf{1}_*, G^{I_n})$ is contractible for every integer n. By Proposition 6.4.2 we have that $\mathrm{Hom}_*(\mathbf{1}_*, G^{I_n}) \simeq \mathrm{Hom}_*(\mathbf{1}_* \wedge I_n, G) = \mathrm{Hom}_*(I_n, G)$ (where I_n is pointed by the vertex 0). The latter space is contractible by Lemma 6.4.6.

Finally we prove that $\mathrm{Hom}_*(\mathbf{1}_*, G^I)$ is contractible. We have seen that $\mathrm{Hom}_*(\mathbf{1}_*, X) = \mathrm{Hom}(\mathbf{1}, X)$ is the clique complex on the looped vertices of the graph X. Hence, as a functor, $\mathrm{Hom}_*(\mathbf{1}_*, ?)$ preserves colimits. By definition, G^I is the colimit of the sequence of maps $\cdots \to G^{I_n} \to G^{I_{n+1}} \to \dots$, and hence $\mathrm{Hom}_*(\mathbf{1}_*, G^I) = \mathrm{colim}\big(\mathrm{Hom}_*(\mathbf{1}_*, G^{I_n})\big)$. We have seen that the sequence defining the last of these spaces is composed of all contractible spaces. Hence the colimit is contractible, and the result follows. ☐

Next we state and prove our main result, a long exact sequence in the homotopy groups of the Hom_* complexes induced by the endpoint map $\varphi : G^I \to G$. Our main tool will be the Quillen fiber Lemma B (see [Qui73]). If $\psi : P \to Q$ is a map of posets and $q \in Q$, we will let $\psi^{-1}(\leq q) := \{p \in P | \psi(p) \leq q\}$ denote the *Quillen fiber* of q. The lemma says that if for all $q \leq q'$ the induced map $\psi^{-1}(\leq q) \to \psi^{-1}(\leq q')$ is a homotopy equivalence, then the map ψ is a quasi-fibration.

Theorem 6.5.5. *Let G be a pointed graph, let T be a finite pointed graph, and let φ_T : $\mathrm{Hom}_*(T, G^I) \to \mathrm{Hom}_*(T, G)$ be the map induced by the endpoint map. Then for all $\gamma \in \mathrm{Hom}_*(T, G)$, and all $\beta \in \varphi_T^{-1}(\leq \gamma)$ there is a connecting homomorphism $\delta : \pi_{i+1}\big(\mathrm{Hom}_*(T, G), \gamma\big) \to \pi_i\big(\varphi_T^{-1}(\leq \gamma)\big)$ that fits into the following long exact sequence.*

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{i+1}\big(\mathrm{Hom}_*(T, G), \gamma\big) \arrow[r, "\delta"] & \pi_i\big(\varphi_T^{-1}(\leq \gamma), \beta\big) \arrow[dl, "\iota_*"'] & \\
 & \pi_i\big(\mathrm{Hom}_*(T, G^I), \beta\big) \arrow[r, "\varphi_*"] & \pi_i\big(\mathrm{Hom}_*(T, G), \gamma\big) \arrow[r] & \cdots
\end{tikzcd}$$

Here φ_* is the map induced by φ_T, and ι_* is induced by the inclusion $\iota : \big(\varphi_T^{-1}(\leq \gamma), \beta\big) \to \big(\mathrm{Hom}_*(T, G^I), \beta\big)$.

Proof. We first prove the claim for the case $T = \mathbf{1}_*$, and the map $\varphi_{\mathbf{1}_*} : \mathrm{Hom}(\mathbf{1}_*, G^I) \to \mathrm{Hom}(\mathbf{1}_*, G)$.

We use the Quillen fiber Lemma B, applied to the poset map $\varphi_{\mathbf{1}_*}$. Suppose $\gamma \leq \gamma' \in \mathrm{Hom}_*(\mathbf{1}_*, G)$. As elements of $\mathrm{Hom}_*(\mathbf{1}_*, G)$, each of γ and γ' can be identified with a collection of looped vertices of the graph G, each of which determines a clique (complete subgraph) of G. We will also use γ and γ' to denote these collections of vertices, and will distinguish an element $g \in \gamma \subseteq \gamma' \subseteq V(G)$. Note that g is adjacent to all other elements of γ' (see Figure 6.4).

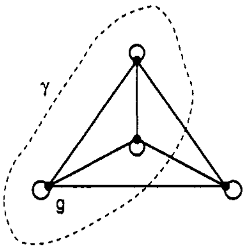

Figure 6.4: $g \in \gamma \subseteq \gamma'$

Next, we let $Y = \varphi_{\mathbf{1}_*}^{-1}(\leq \gamma)$ and $Y' = \varphi_{\mathbf{1}_*}^{-1}(\leq \gamma')$ denote the respective Quillen fibers. Here Y and Y' are both subposets of $\mathrm{Hom}_*(\mathbf{1}_*, G^I)$. We wish to show that the induced map $k : Y \to Y'$ is a homotopy equivalence, from which the result would follow. For this, we consider finite approximations of these spaces, in the following sense. We let $\varphi_n : G^{I_n} \to G$ denote the endpoint map of the finite path graph G^{I_n}, given by $\varphi_n(f) = f(n)$. We let $H_n \subseteq G^{I_n}$ denote the induced subgraph on the vertices $\varphi_n^{-1}(\gamma)$, and similarly $H'_n = \varphi_n^{-1}(\gamma')$. We can think of H_n (respectively H'_n) as the subgraph of G^{I_n} induced by maps from I_n that end at some vertex of γ (respectively γ'). We have the obvious inclusions $k_n : H_n \to H'_n$ and also the inclusions $i_n : H'_n \to h'_{n+1}$ and $j_n : H_n \to X_{n+1}$ given by $i_n(f)(n+1) = f(n)$ (and similarly for j_n).

We will let $Y_n = \mathrm{Hom}(\mathbf{1}, H_n)$ and $Y'_n = \mathrm{Hom}(\mathbf{1}, H'_n)$ denote the respective posets. The i_n and j_n maps determine directed systems for which

$$Y = \varphi^{-1}(\leq \gamma) = \mathrm{Hom}(\mathbf{1}, \mathrm{colim} H_n) = \mathrm{colim} Y_n,$$
$$Y' = \varphi^{-1}(\leq \gamma') = \mathrm{Hom}(\mathbf{1}, \mathrm{colim} H'_n) = \mathrm{colim} Y'_n.$$

The poset map $k : Y \to Y'$ is given by $\mathrm{colim}(k_n : H_n \to H'_n)$. We also need the graph map

$h_n : H'_n \to H_{n+1}$ given by

$$h_n(f)(i) = \begin{cases} f(i) & \text{if } i \leq n = 0 \\ g & \text{if } i = n+1. \end{cases}$$

These maps all fit into the following diagram of graphs.

$$\begin{tikzcd}
\cdots \arrow[r] & H'_n \arrow[r, "i_n"] \arrow[dr, "h_n"] & H'_{n+1} \arrow[r, "i_{n+1}"] \arrow[dr, "h_{n+1}"] & H'_{n+2} \arrow[r] & \cdots \\
\cdots \arrow[r] & H_n \arrow[r, "j_n"'] \arrow[u, "k_n"'] & H_{n+1} \arrow[r, "j_{n+1}"'] \arrow[u, "k_{n+1}"'] & H_{n+2} \arrow[r] \arrow[u, "k_{n+2}"'] & \cdots
\end{tikzcd}$$

We claim that this diagram commutes up to (graph) $\times$-homotopy. In particular, we have $k_{n+1}h_n \simeq_\times i_n$ and $h_{n+1}k_{n+1} \simeq_\times j_{n+1}$. For the first homotopy, define a map $A : H'_n \times I_1 \to H'_{n+1}$ according to:

$$A(f,i)(j) = \begin{cases} f(j) & \text{if } i = 0 \text{ and } j \leq n \\ f(n) & \text{if } i = 0 \text{ and } j = n+1 \\ g & \text{if } i = 1. \end{cases}$$

Recall that $g \in \alpha \subseteq V(G)$ is our distinguished vertex. Now, it is easy to see that $A(?,0) = i_n$ and $A(?,1) = k_{n+1}h_n$.

$$\begin{tikzcd}
H'_n \arrow[d, "\iota_0"'] \arrow[dr, "i_n"] & \\
H'_n \times I_1 \arrow[r, "A"] & H'_{n+1} \\
H'_n \arrow[u, "\iota_1"] \arrow[ur, "k_{n+1}h_n"'] &
\end{tikzcd}$$

It remains to check that A is a graph map. For this, suppose that f and f' are adjacent vertices in H'_n. We need $A(f,0)$ and $A(f',1)$ to be adjacent in X'_{n+1}. Note that $A(f,0)(n+1) = f(n) \in \gamma'$ and $A(f',1)(n+1) = g$. The element g is adjacent to all elements in the clique γ', and hence adjacent to $f(n)$. Also, $A(f,0)(n+1) = f(n)$ and $A(f',1)(n) = f'(n)$, which are adjacent since $f \sim f'$ in H'_n. Finally, we have $A(f,0)(n) = f(n)$ and $A(f,1)(n+1) = x$. Once again, these are adjacent since $f(n) \in \gamma'$.

To check the homotopy $h_{n+1}k_{n+1} \simeq_{\times} j_{n+1}$, we similarly define a map $B : H_n \times I_1 \to H_{n+1}$ according to:

$$B(f,i)(j) = \begin{cases} f(n) & \text{if } i = 0 \text{ and } j = n+1 \\ g & \text{if } i = 1 \text{ and } j = n+1 \\ f(j) & \text{if } j \leq n. \end{cases}$$

Once again, one can check that B is indeed a graph map and that $B(?,0) = j_{n+1}$ and $B(?,1) = h_{n+1}k_{n+1}$. We conclude that the diagram under consideration commutes up to $\times$-homotopy, and hence by Lemma 6.4.4 every diagram of posets induced by a $\mathrm{Hom}(T,?)$ functor also commutes up to (topological) homotopy.

Next, to show that $k : Y \to Y'$ is a homotopy equivalence we show that k induces an isomorphism on homotopy groups. Suppose $\rho, \sigma : S^m \to Y$ are pointed maps from the m-sphere into Y, and let $\rho' = k\rho$ and $\sigma' = k\sigma$ be the induced maps $S^m \to Y'$. Suppose that $\rho' \simeq \sigma'$ are homotopic as maps into Y' via a homotopy $\Psi : S^m \times I \to Y'$. We claim that in fact $\rho \simeq \sigma$ are also homotopic, so that k is injective on all homotopy groups. To see this, pick n big enough so that the image of Ψ sits inside the subcomplex $Y'_n := \mathrm{Hom}(\mathbf{1}, H_n) \subseteq Y'$ (this is possible since $S^m \times I$ is compact). Now, the composition $h_{n_1}\Psi : S^m \times I \to Y_{n+1}$ is a homotopy from $h_{n_1}\rho' = h_{n_1}k_{n_1}\rho$ to $h_{n_1}\sigma' = h_{n_1}k_{n_1}\sigma$. But $h_{n_1}k_{n_1} \simeq j_{n_1}$ and hence ρ and σ are homotopic as maps into Y_{n+1}, as desired.

Next, we claim that k induces a surjection on each homotopy group. To see this, suppose $\rho' : S^m \to Y'$ is a pointed map of the m-sphere into Y'. We wish to find a map $\rho : S^m \to Y$ such that $k\rho \simeq \rho'$. As above, choose n large enough so that the image of the map ρ' is contained in Y'_n, and let $\rho = h_{n_1}\rho' : S^m \to Y_{n+1}$. Then we have $k\rho = k_{(n+1)_1}h_{n_1}\rho' \simeq i_{n_1}\rho' \simeq \rho'$, as desired. We conclude that k induces an isomorphism on each homotopy group, so that k is a homotopy equivalence. Hence the conditions of the Quillen Lemma B are satisfied, and we get a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{i+1}(\mathrm{Hom}_*(\mathbf{1}_*, G), \gamma) \arrow[r] & \pi_i(\varphi_{\mathbf{1}*}^{-1}(\leq \gamma), \beta) \arrow[dl] & \\
 & \pi_i(\mathrm{Hom}_*(\mathbf{1}_*, G), \beta) \arrow[r] & \pi_i(\mathrm{Hom}_*(\mathbf{1}_*, G), \alpha) \arrow[r] & \cdots
\end{tikzcd}$$

It remains to prove the claim for general finite T. For this we note that $\mathrm{Hom}_*(T, G) \simeq$

$\mathrm{Hom}_*(\mathbf{1}_*, G^T)$, and also $\mathrm{Hom}_*(T, G^I) \simeq \mathrm{Hom}_*(\mathbf{1}_*, (G^I)^T) = \mathrm{Hom}_*(\mathbf{1}_*, (G^T)^I)$. Hence we take $G = G^T$ and make the appropriate substitutions in the above sequence. $\square$

Corollary 6.5.6. *If G and T and pointed graphs, with T finite, then* $\pi_i(\mathrm{Hom}_*(T,G),\gamma) \simeq [T, \Omega^i(G)]_\times$.

Since for a (pointed) topological space $\pi_i(X,x) = \pi_{i-1}(\Omega X, \tilde{x})$, and also $\pi_0(\mathrm{Hom}_*(T,G)) = [T,G]_\times$, we see that in some sense the loop space functor Ω commutes with the Hom_* complex, where it becomes the graph theoretic version within the arguments.

Proof. Suppose $T = (T,y)$ and $G = (G,x)$ are pointed graphs, with T finite. Let $\varphi : \mathrm{Hom}_*(T, G^I) \to \mathrm{Hom}_*(T,G)$ be the map of posets induced by the (pointed) endpoint map $G^I \to G$. We apply Theorem 6.5.5 and in the long exact sequence choose $\gamma \in \mathrm{Hom}_*(T,G)$ to be the basepoint (where γ is given by $\gamma(t) = x$ for all $t \in V(T)$). Hence γ is an atom in the poset $\mathrm{Hom}_*(T,G)$, so that $\varphi_T^{-1}(\leq \gamma) = \varphi_T^{-1}(\gamma) = \mathrm{Hom}_*(T, \Omega G)$. We choose β to be the basepoint in $\mathrm{Hom}_*(T, \Omega G)$. Hence our sequence becomes

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{i+1}(\mathrm{Hom}_*(T,G),\gamma) \arrow[r, "\delta"] & \pi_i(\mathrm{Hom}_*(T,\Omega G),\beta) \arrow[dll, "\iota_*"] \\
\pi_i(\mathrm{Hom}_*(T,G^I),\beta) \arrow[r, "\varphi_*"] & \pi_i(\mathrm{Hom}_*(T,G),\gamma) \arrow[r] & \cdots
\end{tikzcd}$$

From Lemma 6.5.4, we have that $\mathrm{Hom}_*(T, G^I)$ is contractible, so that $\pi_i(\mathrm{Hom}_*(T,G^I)) = 0$ for all i. Hence the δ maps are all isomorphisms, and we get $\pi_i(\mathrm{Hom}_*(T,G),\gamma) \simeq \pi_{i-1}(\mathrm{Hom}_*(T,\Omega G),\beta)$. Applying this isomorphism i times, we get

$$\pi_i(\mathrm{Hom}_*(T,G),\gamma) \simeq \pi_0(\mathrm{Hom}_*(T,\Omega^i(G)),\tilde{\beta}) \simeq [T,\Omega^i(G)]_\times,$$

where the last isomorphism is from Lemma 6.4.3. This proves the claim. $\square$

With ΩG as our (pointed) loop space associated to a graph G, we can define a graph theoretic notion of '×-homotopy' groups analogous to the A-homotopy groups from [BBdLL06], as discussed in Section 2. For a pointed graph $G = (G,x)$, one can interpret the path connected components of the graph $\Omega^n(G)$ as ×-homotopy classes of maps from the "n cube" $I_m \times I_m \cdots \times I_m$ (n times) into the graph G such that the boundary is mapped to the pointed vertex $x \in G$. This set is naturally a group under 'stacking' and, by the above result, is isomorphic to the group $\pi_n(\mathrm{Hom}_*(\mathbf{1}_*, G))$. The latter space is the clique complex on the subgraph of G induced by the looped vertices.

In the more general setting, we have a natural notion of the 'T-homotopy groups' of a pointed graph $G = (G, x)$ (for a fixed finite pointed graph T). These are defined according to $\pi_i^T(G, x) := [T, \Omega^i(G)]_\times$, so the groups described in the previous paragraphs are obtained by setting $T = \mathbf{1}_*$. Note that this definition makes sense in any category 'with a path object' (see [Bau89] for an in depth discussion). Our results show that $\pi_i^T(G, x) \simeq \pi_i\big(\mathrm{Hom}_*(T, G)\big)$.

Chapter 7

CONSTRUCTING NEW TEST GRAPHS

7.1 Introduction

In chapter 3 of this thesis, we discussed how the Hom complex can be used to obtain lower bounds on chromatic numbers of graphs. We saw that the original application involved the $\mathrm{Hom}(K_2, ?)$ complexes, where Lovász showed that the connectivity of $\mathrm{Hom}(K_2, G)$ provided a lower bound on $\chi(G)$. The next (collection of) 'test graphs' came in the form of odd cycles; Babson and Kozlov showed that the connectivity of $\mathrm{Hom}(C_{2r+1}, G)$ provided the next natural bound on $\chi(G)$. In both cases, obstructions to maps are obtained by utilizing the free $\mathbb{Z}_2$-action on the associated Hom complex induced by the relevant edge-flipping actions on the graphs (the nontrivial automorphism of K_2 and the reflection of C_{2r+1}).

As a way to take advantage of the $\mathbb{Z}_2$-topology, Babson and Kozlov introduced the use of characteristic classes into the study of Hom complexes. They proposed and partially (according to the parity of $\chi(G)$) proved the following result incorporating the $\mathbb{Z}_2$-action on the Hom complex.

Theorem 7.1.1. *For every graph G,*

$$\mathrm{ht}_{\mathbb{Z}_2}\mathrm{Hom}(C_{2r+1}, G) \leq \chi(G) - 3.$$

The first complete proof of Theorem 7.1.1 was given by Carsten Schultz [Scha]. More recently, in [Schb], the same author was able to prove the following statement which not only implies Theorem 7.1.1 but also provides insight into the structure of the Hom complexes, and suggests extensions that are the theme to this chapter.

Theorem 7.1.2 (Schultz). *For every graph G,*

$$\mathrm{colim}_r\mathrm{Hom}(C_{2r+1}, G) \simeq_{\mathbb{Z}_2} \mathrm{Map}_{\mathbb{Z}_2}\left(\mathbb{S}^1_b, \mathrm{Hom}(K_2, G)\right).$$

The directed system that defines the colimit is obtained by applying the $\mathrm{Hom}(?, G)$ functor to the system $\cdots \to C_{2r+3} \to C_{2r+1} \to \cdots$. Here $\mathbb{S}^1_b$ denotes a circle with the $\mathbb{Z}_2$ antipodal action on the left and the $\mathbb{Z}_2$ reflection action on the right (which in total can be considered an action of $\mathbb{Z}_2 \times \mathbb{Z}_2$).

In his paper, Schultz makes the observation that $\mathrm{Hom}(K_2, C_{2r+1}) \simeq_{\mathbb{Z}_2 \times \mathbb{Z}_2} \mathbb{S}^1_b$, where the action on the first space is induced by the nonidentity automorphism of K_2 and by the reflection of C_{2r+1} that flips an edge. The importance of $\mathbb{S}^1_b$ in this context is the following result, also from [Schb].

Proposition 7.1.3 (Schultz). *If X is a $\mathbb{Z}_2$-space, then* $\mathrm{ht}_{\mathbb{Z}_2}(\mathbb{S}^1_b \times_{\mathbb{Z}_2} X) \geq \mathrm{ht}_{\mathbb{Z}_2} X + 1$.

One can then combine these observations to obtain the following corollary which, when combined with Theorem 3.4.2, implies Theorem 7.1.1.

Corollary 7.1.4 (Schultz). *If G is a graph with at least one edge, then*

$$\mathrm{ht}_{\mathbb{Z}_2}\mathrm{Hom}(C_{2r+1}, G) + 1 \leq \mathrm{ht}_{\mathbb{Z}_2}\mathrm{Hom}(K_2, G).$$

Theorem 7.1.2 is similar in spirit to the results and constructions discussed in Chapter 6 of this thesis, where it is shown that the homotopy groups of a related space $\mathrm{Hom}_*(T, G)$ can be determined by certain graph theoretic closed paths in a graph G^T. In this context, the role of a closed path (a circle) is played by C'_m, the looped 1-skeleton of a triangulated circle, i.e., a cycle with loops on all the vertices.

In this chapter, we generalize the results of Schultz by showing that one can work directly with the graph C'_m to construct new test graphs. Here we consider the analogous $\mathbb{Z}_2 \times \mathbb{Z}_2$-action on the graph C'_m, and investigate its applications to Hom complexes in light of the constructions discussed above. Our main results show that taking the $\mathbb{Z}_2$-product of C'_m with a given $\mathbb{Z}_2$-graph T interacts well with the relevant Hom complexes. In particular, we show that one may recover the space of equivariant maps from the circle into the complex $\mathrm{Hom}(T, G)$ as a colimit of the complexes $\mathrm{Hom}(T \times_{\mathbb{Z}_2} C'_m, G)$ (details are below). As a consequence we see that if T is a *Stiefel Whitney test graph* (in the sense of [Koza]) with certain additional properties, then so is $T \times_{\mathbb{Z}_2} C'_m$; in addition, certain topological invariants (e.g. connectivity) of $\mathrm{Hom}(T, G)$ are closely related to those of $\mathrm{Hom}(T \times_{\mathbb{Z}_2} C'_m, G)$. We view this as a generalization of the connection between K_2 and the odd cycle C_{2r+1}, as described in [Schb]. The results in this chapter are based on joint work with Carsten Schultz.

7.2 Definitions and main results

Before stating our main results, we discuss some definitions and constructions that will be involved. If Γ is a group, and X and Y are spaces with (respectively) a right and a left Γ-action, then Γ acts diagonally on the product $X \times Y$ according to $\gamma \cdot (x, y) := (\gamma x, \gamma y)$. We then define $X \times_\Gamma Y$ to be the orbit space under this action, so that $X \times_\Gamma Y := (X \times Y) \backslash \sim$, where $(x, y) \sim (x', y')$ if $x' = \gamma x$ and $y' = \gamma y$ for some $\gamma \in \Gamma$. Similarly, if G is a graph with a left Γ-action and H is a graph with a right Γ-action, then $G \times_\Gamma H$ is defined to be the graph with vertices given by the orbits of the diagonal Γ-action on $G \times H$, and with adjacency given by $[(g, h)] \sim [(g', h')]$ if there exists representatives in $G \times H$ with $(g, g') \sim (h, h')$.

For our results we will also need the following construction as way to obtain a (reflexive) graph from a poset.

Definition 7.2.1. *Let P be a poset. We define $F(P)$ to be the reflexive graph with vertices given by the atoms of P, and with adjacency given by $x \sim y$ if there exists $z \in P$ with $z \geq x$ and $z \geq y$.*

We are now able to state and discuss our results. Our main theorem shows us that $\mathrm{Hom}(T, ?)$ has an 'approximate' right adjoint, provided by taking the 1-skeleton of a given poset P. This allows us to relate the topological properties of the Hom complexes obtained by applying the graph theoretical construction described above.

Theorem 7.2.2. *Suppose $\mathbb{Z}_2$ acts on the left on the poset P and on the right on the graph T. Then there is a homotopy equivalence*

$$\mathrm{Poset}_{\mathbb{Z}_2}\big(P, \mathrm{Hom}(T, G)\big) \simeq \mathrm{Hom}\big(T \times_{\mathbb{Z}_2} F(P), G\big).$$

We present the proof of this theorem in the next section. In the case the poset P is of the form $P = \mathrm{Hom}(\mathbf{1}, B^A) \simeq \mathrm{Hom}(A, B)$, then we have $F(P) = S(B^A)$, where S is the functor that takes the induced subgraph on looped vertices. In this case we obtain:

$$\begin{aligned}\mathrm{Poset}_{\mathbb{Z}_2}\big(\mathrm{Hom}(A,B),\mathrm{Hom}(T,G)\big) &\simeq \mathrm{Poset}_{\mathbb{Z}_2}\big(\mathrm{Hom}(\mathbf{1},B^A),\mathrm{Hom}(T,G)\big) \text{ (by Lemma 7.3.2)}\\ &\simeq \mathrm{Hom}\big(F(\mathrm{Hom}(\mathbf{1},B^A)\times_{\mathbb{Z}_2} T, G\big)\text{(by Theorem 7.2.2)}\\ &= \mathrm{Hom}\big(S(B^A)\times_{\mathbb{Z}_2} T, G\big).\end{aligned}$$

Our main application of Theorem 7.2.2 will involve replacing P with the face poset of appropriate subdivisions of the circle. As we have seen in previous chapters, a (reflexive) cycle with loops on all vertices will function as our graph theoretical $\mathbb{S}^1$. Accordingly, we define C'_m to be the cycle of length $2m$ with looped vertices labeled by $\{0, 1, \ldots, 2m-1\}$. The graph C'_m has a pair of $\mathbb{Z}_2$-actions: the antipodal left action given by $i \mapsto i + m \pmod{2m}$, and the reflection right action $i \mapsto 2m - 1 - i \pmod{2m}$. With these actions we have a homeomorphism $\mathrm{Hom}(\mathbf{1}, C'_m) \approx_{\mathbb{Z}_2\times\mathbb{Z}_2} \mathbb{S}^1_b$.

Our primary construction will involve taking quotients of products of graphs in the context of the left $\mathbb{Z}_2$-action on C'_m. Given a graph G with a (right) $\mathbb{Z}_2$-action, we will want to consider the graph $G \times_{\mathbb{Z}_2} C'_m$ (see Figure 7.1 for the case of $G = K_2$ with the nontrivial $\mathbb{Z}_2$-action).

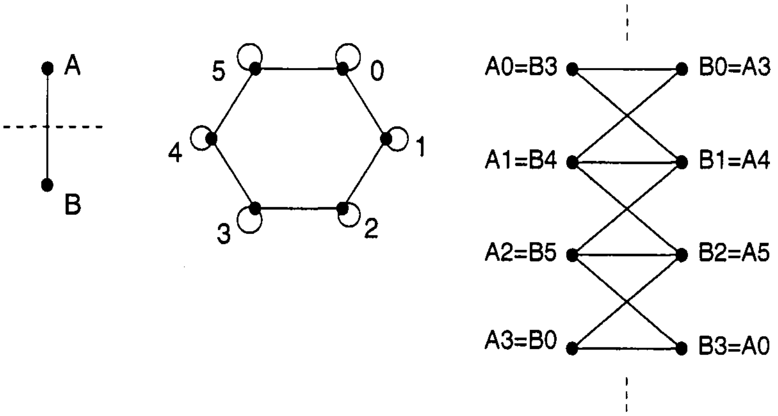

Figure 7.1: The graphs K_2, C'_3, and $K_2 \times_{\mathbb{Z}_2} C'_3$.

Our next result describes how graphs obtained via the $? \times_{\mathbb{Z}_2} C'_m$ construction relate to the original Lovász $\mathrm{Hom}(K_2, ?)$ complexes. The proof of this theorem will come in the next section.

Theorem 7.2.3. *Let T be a graph with a right $\mathbb{Z}_2$-action. Then for all $m \geq 3$ we have*

$$\mathrm{Hom}(K_2, T \times_{\mathbb{Z}_2} C'_m) \simeq_{\mathbb{Z}_2 \times \mathbb{Z}_2} \mathrm{Hom}(K_2, T) \times_{\mathbb{Z}_2} \mathbb{S}^1_b.$$

The graph C'_m has the right (reflection) $\mathbb{Z}_2$-action which extends to $G \times_{\mathbb{Z}_2} C'_m$, and hence we can consider iterations of the $? \times_{\mathbb{Z}_2} C'_m$ construction. In particular, if we start with an edge K_2 (with its nontrivial $\mathbb{Z}_2$-action), and repeat the construction k times we get the following 2-parameter family of graphs.

Definition 7.2.4. *For integers $k, m \geq 1$ we define the graph*

$$T_{k,m} := K_2 \underbrace{\times_{\mathbb{Z}_2} C'_m \times_{\mathbb{Z}_2} \cdots \times_{\mathbb{Z}_2} C'_m}_{k-times}.$$

The example in Figure 7.1 is then $T_{1,3}$. We point out that each $T_{k,m}$ is a graph without loops. In the last section of this chapter we will discuss the role of these graphs as 'test graphs' for bounds on chromatic number.

For these applications, we will need the following result which describes the specific role of C'_m in its interaction with the Hom complex.

Theorem 7.2.5. *Let T be a graph with a right $\mathbb{Z}_2$-action. Then we have*

$$\mathrm{colim}_m \mathrm{Hom}(T \times_{\mathbb{Z}_2} C'_m, G) \simeq_{\mathbb{Z}_2} \mathrm{Map}_{\mathbb{Z}_2}(\mathbb{S}^1_b, \mathrm{Hom}(T, G)),$$

where the directed system that defines the colimit is described below.

Proof. This follows from Theorem 7.2.2, with $P = \mathrm{Hom}(\mathbf{1}, C'_m)$, so that $F(P) = C'_m$. To complete the proof we need to describe the directed system involved in the colimit. For each integer $m \geq 1$ we have a map of graphs $f_m : C'_{3^m} \to C'_{3^{m-1}}$ given by $i \mapsto \lfloor \frac{i}{3} \rfloor$. This map respects each of the $\mathbb{Z}_2$-actions, and gives us a directed system:

$$\cdots \longrightarrow C'_{3^{m+1}} \xrightarrow{f_{m+1}} C'_{3^m} \xrightarrow{f_m} C'_{3^{m-1}} \longrightarrow \cdots$$

Given a graph G, and a $\mathbb{Z}_2$-graph T, this directed system of graphs in turn induces a directed system of posets $\mathrm{Hom}(T \times_{\mathbb{Z}_2} C'_{3^m}, G) \to \mathrm{Hom}(T \times_{\mathbb{Z}_2} C'_{3^{m-1}}, G)$. Applying our result to this system completes the proof. □

We can now apply Theorem 7.2.5 to investigate the behavior of the Stiefel-Whitney classes of the relevant Hom complexes. Specific results in terms of test graphs will be given in the last section, but here we collect together the following somewhat technical conclusions.

Proposition 7.2.6. *Let T be a graph with a (right) $\mathbb{Z}_2$-action and let G be a graph. Then for $m \geq 1$ we have*

$$\mathrm{ht}_{\mathbb{Z}_2}\big(\mathrm{Hom}(T \times_{\mathbb{Z}_2} C'_m), G)\big) + 1 \leq \mathrm{ht}_{\mathbb{Z}_2}\big(\mathrm{Hom}(T, G)\big).$$

Also, if $\mathrm{coind}_{\mathbb{Z}_2}\big(\mathrm{Hom}(T, G)\big) \geq 1$*, then*

$$\lim_{m \to \infty} \mathrm{coind}_{\mathbb{Z}_2}\big(\mathrm{Hom}(T \times_{\mathbb{Z}_2} C'_m)\big) + 1 \geq \mathrm{coind}_{\mathbb{Z}_2}\big(\mathrm{Hom}(T, G)\big).$$

Proof. We follow the argument given in [Schb]. From the naturality of Theorem 7.2.5 we have the horizontal poset map given by evaluation

$$\begin{tikzcd}
\mathrm{Hom}(\mathbf{1}, C'_m) \times_{\mathbb{Z}_2} \mathrm{Hom}(T \times_{\mathbb{Z}_2} C'_m, G) \arrow[d, "\simeq_{\mathbb{Z}_2}"] & \\
\mathrm{Hom}(\mathbf{1}, C'_m) \times_{\mathbb{Z}_2} \mathrm{Poset}_{\mathbb{Z}_2}\big(\mathrm{Hom}(\mathbf{1}, C'_m), \mathrm{Hom}(T, G)\big) \arrow[r, "\mathbb{Z}_2"'] & \mathrm{Hom}(T, G)
\end{tikzcd}$$

and hence a continuous map

$$\mathbb{S}^1_b \times_{\mathbb{Z}_2} |\mathrm{Hom}(T \times_{\mathbb{Z}_2} C'_m, G)| \to |\mathrm{Hom}(T, G)|.$$

The first inequality now follows from Proposition 7.1.3.

For the other inequality, we suppose $k \geq 0$ and assume $\mathrm{coind}_{\mathbb{Z}_2}\big(\mathrm{Hom}(T, G)\big) \geq k + 1$. Since $\mathbb{S}^1_b \times_{\mathbb{Z}_2} \mathbb{S}^k$ is a $(k+1)$-dimensional free $\mathbb{Z}_2$-space, there exists an equivariant map $\mathbb{S}^1_b \times_{\mathbb{Z}_2} \mathbb{S}^k \to_{\mathbb{Z}_2} |\mathrm{Hom}(T, G)|$. We apply simplicial approximation to this map and obtain, for some $m \gg 0$, a poset $\mathbb{Z}_2$-poset P with $|P| \approx_{\mathbb{Z}_2} \mathbb{S}_k$, and an equivariant poset map

$$\mathrm{Hom}(\mathbf{1}, C'_m) \times_{\mathbb{Z}_2} P \to_{\mathbb{Z}_2} \mathrm{Hom}(T, G)$$

and hence an equivariant poset map

$$P \to_{\mathbb{Z}_2} \mathrm{Poset}_{\mathbb{Z}_2}\big(\mathrm{Hom}(\mathbf{1}, C'_m), \mathrm{Hom}(T, G)\big) \simeq_{\mathbb{Z}_2} \mathrm{Hom}(T \times_{\mathbb{Z}_2} C'_m, G).$$

It follows that $\mathrm{coind}_{\mathbb{Z}_2}\big(\mathrm{Hom}(T \times_{\mathbb{Z}_2} C'_m)\big) \geq k$, and hence the desired inequality. □

7.3 Proofs of the main results

In this section we provide the proofs of Theorems 7.2.2 and 7.2.3. For the proof of Theorem 7.2.2, we first consider the case of trivial $\mathbb{Z}_2$-actions with $T = \mathbf{1}$. Recall from above that if P is a poset, $F(P)$ is defined to be the graph with vertices given by the atoms of P, with adjacency given by $x \sim y$ if there exists $z \in P$ with $z \geq x$ and $z \geq y$. Recall also that $\mathrm{Poset}(P, Q)$ denotes the poset of order preserving maps between posets P and Q, where $f \leq g$ if $f(p) \leq g(p)$ for all $p \in P$.

Proposition 7.3.1. *Let T and G be graphs, let P be a poset. Then there is a closure map on the level of posets that induces a homotopy equivalence*

$$\mathrm{Hom}\big(F(P), G\big) \simeq \mathrm{Poset}\big(P, \mathrm{Hom}(\mathbf{1}, G)\big).$$

Proof. We define a map of posets $\varphi : \mathrm{Hom}\big(F(P), G\big) \to \mathrm{Poset}\big(P, \mathrm{Hom}(\mathbf{1}, G)\big)$, according to

$$\varphi(\alpha)(x) = \bigcup_{y \leq x,\ y \text{ an atom}} \alpha(y),$$

for $x \in P$ and $\alpha \in \mathrm{Hom}\big(F(P), G\big)$ and $x \in P$.

We first check that $\varphi(\alpha)(x) \in \mathrm{Hom}(\mathbf{1}, G)$. For this we let $z \in \alpha(y)$ and $z' \in \alpha(y')$ where $y, y' \leq x$ are atoms. Then $y \sim y'$ in $F(P)$ and hence $z \sim z'$ in G. Next we see that $\varphi(\alpha)$ is a poset map since if $x \leq x'$ in P, then $y \leq x'$ for every atom y with $y \leq x$, and hence $\varphi(\alpha)(x) \leq \varphi(\alpha)(x')$. Finally, we check that φ itself is a poset map. For this, suppose $\alpha \leq \alpha'$ in $\mathrm{Hom}\big(F(P), G\big)$. We need to show that $\varphi(\alpha) \leq \varphi(\alpha')$. Since $\alpha \leq \alpha'$, we have $\alpha(y) \subseteq \alpha'(y)$ for any $y \in F(P)$. Hence we have

$$\varphi(\alpha)(x) = \bigcup_{y \leq x} \alpha(y) \subseteq \bigcup_{y \leq x} \alpha'(y) = \varphi(\alpha')(x).$$

For convenience, let $Q = \mathrm{Poset}\big(P, \mathrm{Hom}(\mathbf{1}, G)\big)$. We wish to define a closure map $c : Q \to Q$ whose image coincides with the image of the map φ defined above. For $\beta \in Q$, we define

$$c(\beta)(x) = \bigcup_{y \leq x,\ y \text{ an atom}} \beta(y).$$

We first verify that $c(\beta)(x) \in \mathrm{Hom}(\mathbf{1}, G)$. For this, let z and z' be elements of $c(\beta)(x)$, so that $z \in \beta(y)$ and $z' \in \beta(y')$ for atoms $y, y' \leq x$. We need to show that $z \sim z'$. Since β is a poset map, we have $\beta(y) \subseteq \beta(x)$ and $\beta(y') \subseteq \beta(x)$ so that $z, z' \in \beta(x)$ and hence $z \sim z'$. Also, $c(\beta) \in Q$ since if $x \leq x'$ in P then clearly $c(\beta)(x) \leq c(\beta)(x')$. Next, we verify that c itself is a poset map. For this, let $\beta \leq \beta'$ be elements of Q, so that $\beta(x) \leq \beta'(x)$ for all $x \in P$. Then in particular we have $\beta(y) \leq \beta'(y)$ for every atom y such that $y \leq x$. Hence $\beta(y) \subseteq \beta'(y)$ so that $c(\beta)(x) \leq c(\beta')(x)$ for all x.

We claim that $\mathrm{Im}(c) = \mathrm{Im}(\varphi)$. From the construction, it is clear that $c(\varphi(\alpha)) = \alpha$ for every $\alpha \in \mathrm{Hom}\big(F(P), G\big)$, and hence $\mathrm{Im}(\varphi) \subseteq \mathrm{Im}(c)$. To see that $\mathrm{Im}(c) \subseteq \mathrm{Im}(\varphi)$, let $c(\beta) \in imc$ and define $\alpha \in \mathrm{Hom}(F(P), G)$ by $\alpha(x) := \beta(x)$ for any $x \in F(P)$. Since $x \in P$ is an atom, we have $\alpha(x) = \beta(x) = c(\beta)(x)$. To verify that $\alpha \in \mathrm{Hom}\big(F(P), G\big)$, let $x \sim x'$ in $F(P)$, and let $z \in \alpha(x)$, $z' \in \alpha(x')$. Then we have some $y \in P$ with $y \geq x$ and $y \geq x'$. Hence $\beta(y) \geq \beta(x)$ and $\beta(y) \geq \beta(x')$, so that $z, z' \in \beta(y)$. We conclude that $z \sim z'$ in G, as desired. □

To prove theorem 7.2.2 for arbitrary T, we will need to following lemma.

Lemma 7.3.2. *Suppose Q is a poset, and $c : Q \to Q$ is a closure map (say with $c(q) \geq q$ for all $q \in Q$). Then given a poset P, the induced maps $c_* : \mathrm{Poset}(P, Q) \to \mathrm{Poset}(P, Q)$ and $c^* : \mathrm{Poset}(Q, P) \to \mathrm{Poset}(Q, P)$ are both closure maps.*

Proof. For the first case, suppose $\phi \in \mathrm{Poset}(P, Q)$. Then we have $c_*(\phi)(p) = c(\phi(p)) \geq \phi(p)$ since c is a closure map. Hence $c_*(\phi) \geq \phi$.

For the other case, suppose $\psi \in \mathrm{Poset}(Q, P)$. Then we have $c^*(\psi)(q) = \psi(c(q)) \geq \psi(q)$ since ψ is a poset map and $c(q) \geq q$. Hence $c^*(\psi) \geq \psi$. □

We can now use this to prove the following.

Proposition 7.3.3. *Given graphs T and G and a poset P, there is a closure map on the level of posets that induces a homotopy equivalence*

$$\mathrm{Poset}\big(P, \mathrm{Hom}(T, G)\big) \simeq \mathrm{Hom}\big(T \times F(P), G\big).$$

Proof. We have the closure map $\mathrm{Hom}\big(F(P), G^T\big) \to \mathrm{Hom}\big(F(P) \times T, G\big)$, thinking of $\mathrm{Hom}\big(F(P) \times T, G\big)$ as a subposet of $\mathrm{Hom}\big(F(P), G^T\big)$, and also the closure map $\mathrm{Hom}(\mathbf{1}, G^T) \to \mathrm{Hom}(T, G)$. The result follows from Theorem 7.3.1 and Lemma 7.3.2. ☐

If the poset P and the graph T are both endowed with $\mathbb{Z}_2$-actions, keeping track of these actions in the above arguments gives us the equivariant version in Theorem 7.2.2.

We next turn to the proof of Theorem 7.2.3. The result will follow from a pair of lemmas which we now establish. The first tells us that in some sense $\mathrm{Hom}(K_2, ?)$ preserves a $\mathbb{Z}_2$-bundle structure.

Lemma 7.3.4. *If T is a nontrivial graph with a $\mathbb{Z}_2$-action, then for all $m \geq 3$ there is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-homotopy equivalence of posets*

$$\mathrm{Hom}(K_2, T \times_{\mathbb{Z}_2} C'_m) \simeq_{\mathbb{Z}_2 \times \mathbb{Z}_2} \mathrm{Hom}(K_2, T) \times_{\mathbb{Z}_2} \mathrm{Hom}(K_2, C'_m),$$

where the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action is given by the nontrivial automorphism of K_2 and the reflection of C'_m. The $\mathbb{Z}_2$-action on the poset that defines the quotient on the right is the diagonal action given by the given action on T and the antipodal action on C'_m.

Proof. We first note that $\mathbb{Z}_2$ acts on $\mathrm{Hom}(K_2, T \times C'_m)$ according to $(\sigma \cdot \alpha)(v) = \{\sigma(x, y) = (\sigma(x), \sigma(y)) : (x, y) \in \alpha(v)\}$ for $\alpha \in \mathrm{Hom}(K_2, T \times C'_m)$, $v \in K_2$, and $\sigma \in \mathbb{Z}_2$. With this action we claim that there is an equality of posets:

$$\mathrm{Hom}(K_2, T \times_{\mathbb{Z}_2} C'_m) = \mathrm{Hom}(K_2, T \times C'_m)/\mathbb{Z}_2.$$

To see this, we note that the quotient map $T \times C'_m \to T \times_{\mathbb{Z}_2} C'_m$ induces a poset map

$$\varphi : \mathrm{Hom}(K_2, T \times C'_m) \to \mathrm{Hom}(K_2, T \times_{\mathbb{Z}_2} C'_m),$$

which descends to the quotient since φ is constant on the fibers.

To obtain an inverse map $\psi : \mathrm{Hom}(K_2, T \times_{\mathbb{Z}_2} C'_m) \to \mathrm{Hom}(K_2, T \times C'_m)/\mathbb{Z}_2$, we let x and y denote the vertices of K_2 and if $\beta \in \mathrm{Hom}(K_2, T \times_{\mathbb{Z}_2} C'_m)$, we define $\psi(\beta)$ in the following way. We pick representatives (t', i') for some $[(t', i')] \in \beta(x)$ and (s', j') for some $[(s', j')] \in \beta(y)$ such that $(t', i') \sim (s', j')$ in $T \times C'_m$. Then define

$$\psi(\beta)(y) = \{(s,j) : [(s,j)] \in \beta(y), (s,j) \sim (t',i')\}, \text{and}$$
$$\psi(\beta)(x) = \{(t,i) : [(t,i)] \in \beta(x), (t,i) \sim (s',j')\}.$$

The map ψ is defined up to the $\mathbb{Z}_2$-action on $\mathrm{Hom}(K_2, T \times C'_m)$ since two elements (t_1, i_1) and (t_2, i_2) of $\beta(x)$ will have $|i_1 - i_2| \leq 2$. From Proposition 3.2.4, the inclusion of posets

$$\mathrm{Hom}(K_2, T) \times \mathrm{Hom}(K_2, C'_m) \hookrightarrow \mathrm{Hom}(K_2, T \times C'_m)$$

is a homotopy equivalence, and the homotopy inverse exhibited there is a closure map from the latter poset into itself. This map is $\mathbb{Z}_2 \times \mathbb{Z}_2$ equivariant with respect to the action given by the flip of K_2 and the reflection of C'_m. The $\mathbb{Z}_2$-action on $\mathrm{Hom}(K_2, T \times C'_m)$ described above, when restricted to this subposet, recovers the diagonal action that defines the quotient in the statement of the lemma. Hence we obtain the desired $\mathbb{Z}_2 \times \mathbb{Z}_2$ homotopy equivalence,

$$\mathrm{Hom}(K_2, T) \times_{\mathbb{Z}_2} \mathrm{Hom}(K_2, C'_m) \simeq_{\mathbb{Z}_2 \times \mathbb{Z}_2} \mathrm{Hom}(K_2, T \times C'_m)/\mathbb{Z}_2 = \mathrm{Hom}(K_2, T \times_{\mathbb{Z}_2} C'_m).$$

□

Lemma 7.3.5. *The map $K_2 \to \mathbf{1}$ induces a $\mathbb{Z}_2 \times \mathbb{Z}_2$-homotopy equivalence*

$$\mathrm{Hom}(\mathbf{1}, C'_m) \to_{\mathbb{Z}_2 \times \mathbb{Z}_2} \mathrm{Hom}(K_2, C'_m)$$

where the action is given by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action on C'_m described above.

Proof. We will explicitly describe the homotopy equivalence between the spaces. For this we will work with the following descriptions of the relevant Hom complexes.

We can realize the space $|\mathrm{Hom}(K_2, C'_m)|$ as the space $X = ([0, 2m] \times [-1, 1])/\sim$, where $(0, y) \sim (2m, y)$. To see this, we identify vertices of the order complex of $\mathrm{Hom}(K_2, C'_m)$ with points of X according to the following.

$$(\{i\}, \{i\}) \mapsto (i, 0),\ (\{i, i+1\}, \{i, i+1\}) \mapsto (\tfrac{2i+1}{2}, 0),$$
$$(\{2i+1\}, \{2i\}) \mapsto (\tfrac{4i+1}{2}, 1),\ (\{2i+1\}, \{2i+2\}) \mapsto (\tfrac{4i+3}{2}, 1),$$

$$(\{2i+1\},\{2i,2i+2\}) \mapsto (2i+1,1),\ (\{2i+1,2i+3\},\{2i+2\}) \mapsto (2i+2,1),$$
$$(\{2i+1\},\{2i,2i+1,2i+2\} \mapsto (2i+1,\tfrac{1}{2}),\ (\{2i+1,2i+2,2i+3\},\{2i+2\}) \mapsto (2i+2,\tfrac{1}{2}),$$
$$(\{2i,2i+1\},\{2i\}) \mapsto (\tfrac{4i+1}{4},\tfrac{1}{2}),\ (\{2i+1\},\{2i,2i+2\}) \mapsto (\tfrac{4i+3}{4},\tfrac{1}{2}),$$
$$(\{2i+1\},\{2i+1,2i+2\}) \mapsto (\tfrac{4i+5}{4},\tfrac{1}{2}),\ (\{2i+1,2i+2\},2i+2\}) \mapsto (\tfrac{4i+7}{4},\tfrac{1}{2}).$$

Vertices of $\mathrm{Hom}(K_2, C'_m)$ obtained from these by the $\mathbb{Z}_2$-action will be given the same x-coordinate, and with the same but negative y-coordinate. The map $K_2 \to \mathbf{1}$ induces a inclusion $\mathrm{Hom}(\mathbf{1}, C'_m) \to \mathrm{Hom}(K_2, C'_m)$ whose image under this geometric realization is the circle along the $y = 0$ equator of X (see Figure 7.2).

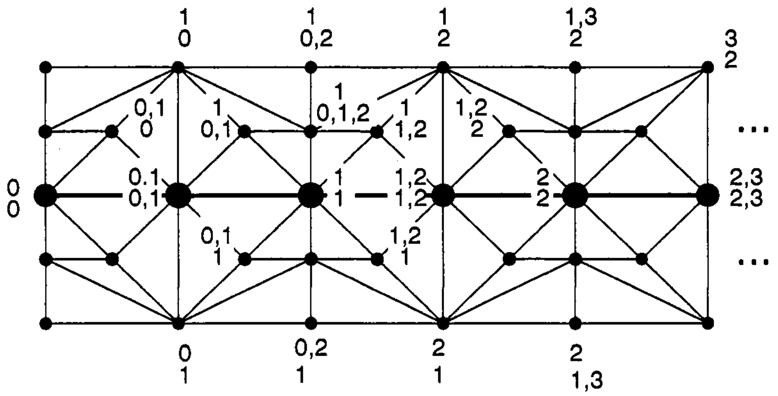

Figure 7.2: Retracting $|\mathrm{Hom}(K_2, C'_m)|$ onto $|\mathrm{Hom}(\mathbf{1}, C'_m)|$.

The map $X \to |\mathrm{Hom}(\mathbf{1}, C'_m)|$ given by $(x, y) \mapsto (x, 0)$ provides the homotopy inverse to this inclusion. This map respects the relevant group actions and proves our claim. ☐

We can combine the results of Lemma 7.3.4 and Lemma 7.3.5 to obtain the following corollary.

Corollary 7.3.6. $\mathrm{Hom}(K_2, T) \times_{\mathbb{Z}_2} \mathrm{Hom}(K_2, C'_m) \simeq_{\mathbb{Z}_2} \mathrm{Hom}(K_2, T) \times_{\mathbb{Z}_2} \mathrm{Hom}(\mathbf{1}, C'_m)$.

Proof of Theorem 7.2.3. Suppose T is a graph with a nontrivial $\mathbb{Z}_2$-action. Then applying our lemmas we get

$$\begin{aligned}\mathrm{Hom}(K_2, T \times_{\mathbb{Z}_2} C'_m) &\simeq_{\mathbb{Z}_2\times\mathbb{Z}_2} \mathrm{Hom}(K_2, T) \times_{\mathbb{Z}_2} \mathrm{Hom}(K_2, C'_m) \text{ (by Lemma 7.3.4)}\\ &\simeq_{\mathbb{Z}_2\times\mathbb{Z}_2} \mathrm{Hom}(K_2, T) \times_{\mathbb{Z}_2} \mathbb{S}^1_b \text{ (by Lemma 7.3.5).}\end{aligned}$$

Iterating this construction we also obtain the following corollary.

Corollary 7.3.7. $\mathrm{Hom}(K_2, T_{k,m}) \simeq \underbrace{\mathbb{S}^1_b \times_{\mathbb{Z}_2} \cdots \times_{\mathbb{Z}_2} \mathbb{S}^1_b}_{k\text{-times}}.$

7.4 Applications

7.4.1 Constructing new test graphs

In this section we discuss some applications of our main results. Our primary interest will be to construct new test graphs for bounding chromatic number. Recall from [Koza] that a graph T is called a *homotopy test graph* if for every graph G, we have the following inequality:

$$\chi(G) > \chi(T) + \mathrm{conn}\big(\mathrm{Hom}(T, G)\big).$$

The results of Lovász and Babson, Kozlov imply that K_2 and the odd cycles C_{2r+1} are homotopy test graphs. For some time it was an open question whether *all* graphs were homotopy test graphs, but Hoory and Linial showed that this was not the case in [HL] by constructing a graph H with $\chi(H) = 5$ such that $\mathrm{Hom}(H, K_5)$ is connected. In fact there are very few graphs that are known to be test graphs (see [Schb] and [Ziva] for some discussion regarding this).

Now suppose T is graph with a $\mathbb{Z}_2$-action that flips an edge. Also from [Koza], we say that a graph T is a *Stiefel-Whitney test graph* if $\mathrm{ht}_{\mathbb{Z}_2}\big(\mathrm{Hom}(T, K_n)\big) = n - \chi(T)$ for all n. We note that every Stiefel-Whitney test graph is also a homotopy test graph since $\mathrm{conn}(X) + 1 \geq \mathrm{ht}_{\mathbb{Z}_2}(X)$ for a $\mathbb{Z}_2$-space X; hence if T is a Stiefel-Whitney test graph, then

$$\chi(T) \leq \chi(G) - \mathrm{conn}\big(\mathrm{Hom}(T, K_{\chi(G)})\big) - 1.$$

Our first application in this section will be a method for constructing new test graphs with certain properties.

Proposition 7.4.1. *Let T be a Stiefel-Whitney test graph with the additional property*

(*) There exists a $\mathbb{Z}_2$-equivariant map $T \to K_{\chi(T)}$,

where the $\mathbb{Z}_2$-action on K_n is given by exchanging the vertices 1 *and* 2, *and leaving all other vertices fixed. Then for all m, $T \times_{\mathbb{Z}_2} C'_m$ is a Stiefel-Whitney test graph satisfying (*).*

Proof. Suppose T is a Stiefel-Whitney test graph satisfying the condition (*) with $\chi(T) = n+2$, and let $T' := T \times_{\mathbb{Z}_2} C'_m$. We first claim that $\chi(T') = n+3$. The inequality $\chi(T') \geq n+3$ follows from Proposition 7.2.6 and the fact that T is assumed to be a Stiefel-Whitney test graphs. Hence it remains to show that $\chi(T') \leq n+3$ via a $\mathbb{Z}_2$-equivariant map $T' \to K_{n+2}$. For this, note that there is a $\mathbb{Z}_2 \times \mathbb{Z}_2$-equivariant map of graphs $K_3^{K_2} \to K_{n+3}^{K_{n+2}}$ given by extending each map on the vertices according to $i+2 \mapsto i+3$. Now, assuming that T satisfies condition (*), there is an equivariant map $T \to K_{n+2}$ which then fits into the diagram

$$C_{2m+1}^{K_2} \to K_3^{K_2} \to K_{n+3}^{n+2} \to K_{n+3}^{T}.$$

This in turn provides an equivariant map $T \times_{\mathbb{Z}_2} C_{2m+1}^{K_2} \to K_{n+3}$, as desired. Finally, to see that $T' = T \times_{\mathbb{Z}_2} C'_m$ is a Stiefel-Whitney test graph, we apply Proposition 7.2.6 and obtain the desired inequality:

$$\begin{aligned}
\mathrm{ht}_{\mathbb{Z}_2}\big(\mathrm{Hom}(T \times_{\mathbb{Z}_2} C'_{2r+1})\big) + 1 &\leq \mathrm{ht}_{\mathbb{Z}_2}\big(\mathrm{Hom}(T, K_n)\big) \\
&= n - \chi(T) \text{ (since } T \text{ is a test graph)} \\
&= n - \chi(T \times_{\mathbb{Z}_2} C'_{2r+1}) + 1.
\end{aligned}$$

□

We point out that both K_2 and the odd cycles C_{2r+1} are Stiefel-Whitney test graphs that satisfy the condition (*) of Proposition 7.4.1. Hence one can take either K_2 or C_{2r+1} as the 'base' graph and apply iterations of the $? \times_{\mathbb{Z}_2} C'_m$ construction to obtain new Stiefel-Whitney test graphs (see Figure 7.3).

We chose K_2 in our definition of $T_{k,m}$, and our results show that these graphs provide lower bounds on the chromatic number. We state this as a corollary.

Corollary 7.4.2. *Let G be a graph and let $k, m \geq 1$ be integers. Then*

$$\mathrm{ht}_{\mathbb{Z}_2}\mathrm{Hom}(T_{k,m}, G) + \chi(T_{k,m}) \leq \mathrm{ht}_{\mathbb{Z}_2}\mathrm{Hom}(K_2, G) \leq \chi(G) - 2.$$

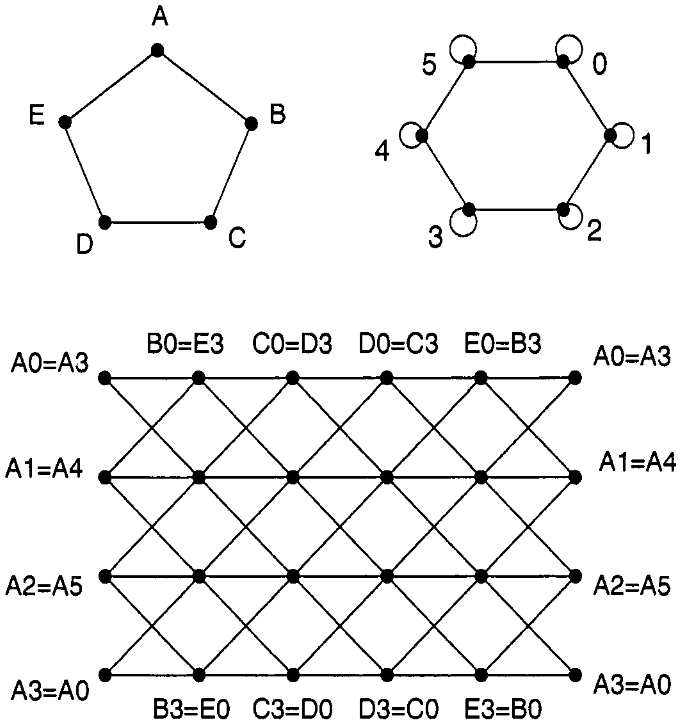

Figure 7.3: The graphs C_5, C_3', and $C_5 \times_{\mathbb{Z}_2} C_3'$.

In addition, we can apply Theorem 7.2.5 to see that maps from the graphs $T_{k,m}$ are implied by the original 'Lovász' bounds.

Corollary 7.4.3. *If* $\text{ht}_{\mathbb{Z}_2}\text{Hom}(K_2, G) \geq k$ *then there exists an* m *such that* $T_{k,m} \to G$.

7.4.2 A conjecture of Lovász

We end this chapter with a discussion regarding an application of our results to a conjecture of Lovász mentioned in Chapter 3. For convenience, we recall the statement.

Conjecture 7.4.4 (Lovàsz). *Let* G *be a graph with no loops. If* $\text{Hom}(T, G)$ *is empty or* k*-connected for all graphs* T *of maximum degree* $\leq d$*, then* $\chi(G) \geq k + d + 2$.

In [BW04] Brightwell and Winkler have managed to prove a weaker version of this conjecture for the case $j = 0$.

Theorem 7.4.5 (Brightwell and Winkler). *If* $\mathrm{Hom}(T, G)$ *is empty or connected for all graphs* T *of maximum degree* $\leq d$ *then* $\chi(G) \geq \frac{d}{2} + 2$.

Note that the $d = 2$ case of the Conjecture 7.4.4 follows from what we already know. If $\mathrm{Hom}(K_2, G)$ is k-connected, then $\mathrm{Hom}(C_{2r+1}, G)$ is nonempty for some r. Hence $\mathrm{Hom}(C_{2r+1}, G)$ is k-connected by assumption, and so $\chi(G) \geq k+4 = k+d+2$ by the Babson-Kozlov result. We can apply this same simple argument with the graphs $T_{k,m}$ to get the following (weaker) version of the original conjecture.

Proposition 7.4.6. *If* $\mathrm{Hom}(T, G)$ *is empty or* k*-connected for every graph* T *with maximum degree* $\leq d$*, then*

$$\chi(G) \geq \min\{k+1, \log_3 d\} + k + 3.$$

In particular, if $d = 3^{k+1}$*, we have* $\chi(G) \geq 2k + 4$.

Proof. We assume G has at least one edge, so that $\mathrm{Hom}(K_2, G)$ is nonempty and hence by assumption is k-connected. So then we have $\mathrm{coind}_{\mathbb{Z}_2}\mathrm{Hom}(K_2, G)$, and hence by Corollary 7.4.3 we have a graph map $T_{j,m} \to G$ for some m for all $j \leq k+1$. We see that $T_{j,m}$ has maximum degree $d_j = 3^j$. We pick the maximum $j \leq k+1$ such that $3^j \leq d$, and by assumption we get that $\mathrm{Hom}(T_{j,m}, G)$ is k-connected. Since the $T_{j,m}$ are Stiefel-Whitney test graphs (and hence homotopy test graphs) this implies that

$$\chi(G) \geq \chi(T_{j,m}) + k + 1 = j + 2 + k + 1 \geq \min\{k+1, \log_3 d\} + k + 3.$$

In the case $d = 3^{k+1}$ we take $j = k+1$ and get $\chi(g) \geq 2k + 4$. ☐

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