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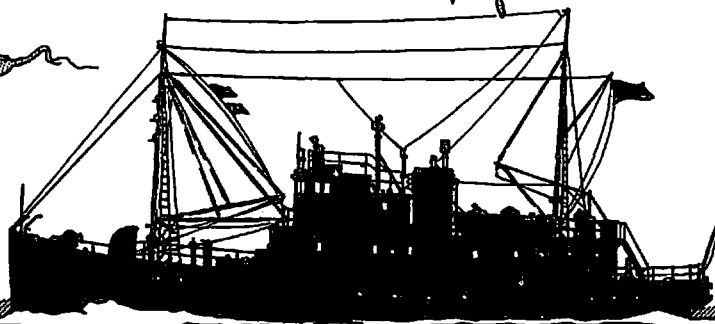
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RICHARD H. FLEMING  
Chairman

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Technical Report No. 162

QUASIGEOSTROPHIC FREE OSCILLATIONS IN ENCLOSED BASINS, by Maurice Rattray, Jr. and Robert L. Charnell. Journal of Marine Research, 24(1):82-103. 1966.

Technical Report No. 163

PALAGONITE TUFFS (HYALOCLASTITES) AND THE PRODUCTS OF POST-ERUPTIVE PROCESSES, by Y. Rammohanroy Nayudu. Bulletin Volcanologique, 27:391-410 + 5 plates. 1964.

Technical Report No. 164

CARBONATE DEPOSITS ON PLANTAGENET BANK NEAR BERMUDA, by M. Grant Gross. Geological Society of America Bulletin, 76:1283-1290. 1965.

Technical Report No. 165

DIGGING CHARACTERISTICS AND SAMPLING EFFICIENCY OF THE 0.1 m<sup>2</sup> VAN VEEN GRAB, by Ulf Lie and Mario M. Pamatmat. Limnology and Oceanography, 10(3):379-384. 1965.

Technical Report No. 166

GRAVITATIONAL CIRCULATION IN STRAITS AND ESTUARIES, by Donald V. Hansen and Maurice Rattray, Jr. Journal of Marine Research, 23(2):104-122. 1965.

Technical Report No. 167

OXYGEN AND CARBON ISOTOPIC COMPOSITION OF LIMESTONES AND DOLOMITES, BIKINI AND ENIWETOK ATOLLS, by M. Grant Gross and Joshua I. Tracy, Jr. Science, 151(3714):1082-1084. 1966.

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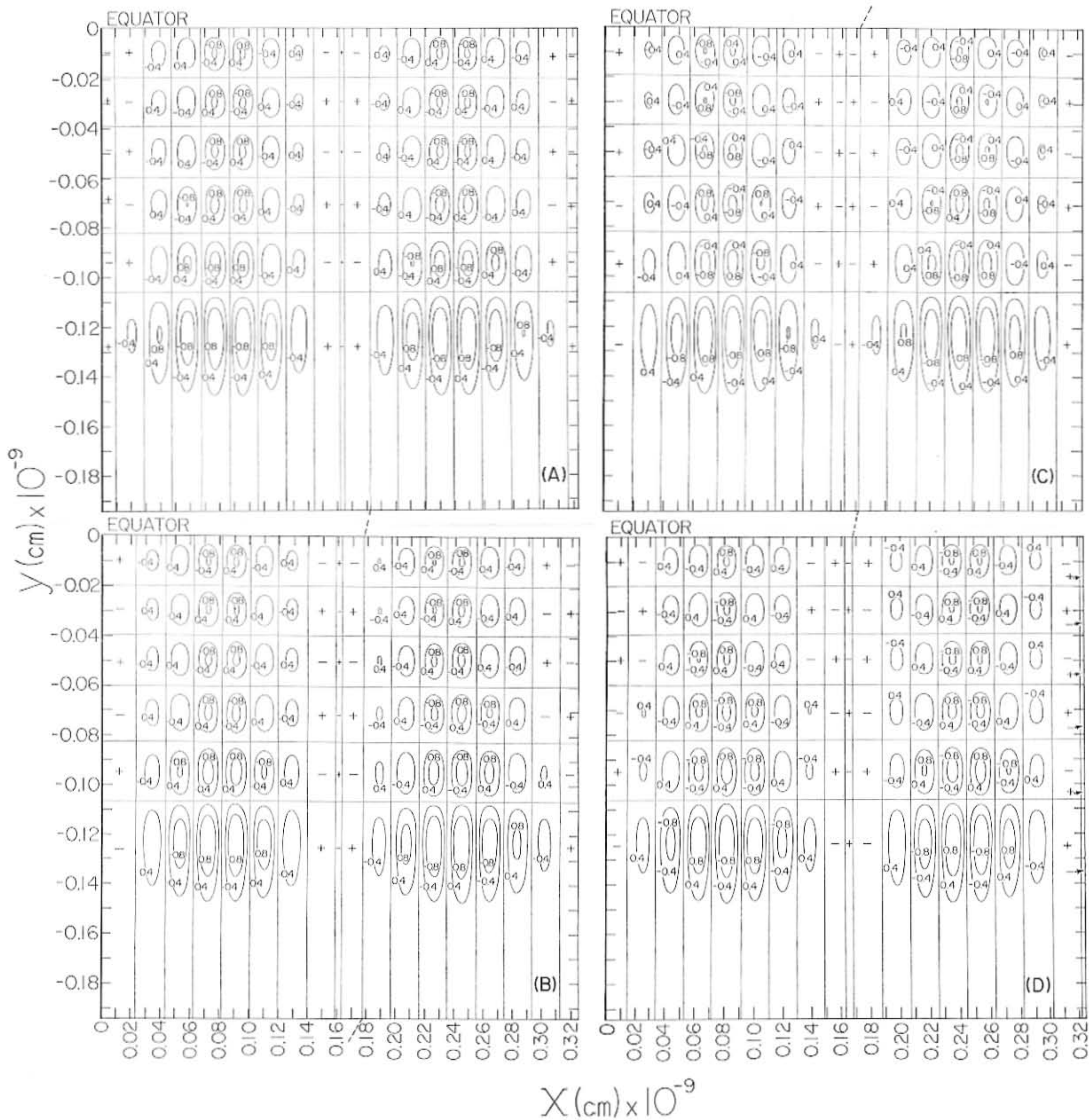


Figure 9. Time history of the streamlines for baroclinic free oscillations of the Pacific Ocean; eigenvalues:  $\sigma = 8$ ,  $\nu = 11.0$ .  
 (A) time  $t = 0$  (B)  $t = 1/4 T$ , 12.9 days later (C)  $t = 1/2 T$ , 25.8 days later (D)  $t = 3/4 T$ , 38.7 days later.

# *Quasigeostrophic Free Oscillations in Enclosed Basins<sup>1</sup>*

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## ABSTRACT

Solutions are obtained for quasigeostrophic free oscillations in enclosed basins of dimensions comparable to the Pacific, Atlantic, and Indian oceans. The lower-order baroclinic oscillations are restricted to the regions adjacent to the equator or the ocean boundaries, and for each basin the oscillations have an estimated minimum period of about 20 days. The barotropic oscillations fill the complete basin and have minimum periods varying from an estimated 2.7 days in the Pacific Ocean (wave length 14,900 km) to 3.8 days in the Indian Ocean (wave length 10,400 km).

Streamlines for representative modes are presented as progressive carrier waves with stationary amplitude-modulated envelopes. A time history, illustrating the progression of current cells toward the west, is presented for one barotropic and one baroclinic mode.

Observations of oceanic periodicities with approximately 4-day periods may be explained in terms of the present theory. A possibility exists that observed fluctuations with periods in the order of three to four months may also be due to waves of this type.

*Introduction.* Time variability is an important oceanographic feature that has proved difficult to investigate either observationally or theoretically. The observational difficulties are twofold: (i) the need for long records from many sensors with the attendant problems of data reduction; (ii) lack of understanding of the phenomena, which makes it impossible to design experiments that make good use of the facilities available. Present knowledge of time

1. Contribution No. 313 from the Department of Oceanography, University of Washington. This investigation was supported in part by the Office of Naval Research, Contract NONR-477(10) and Contract NONR-477(37), Project NR 083 012, and in part by the National Science Foundation under grant GP-3549. The solutions for free oscillations in oceans of small and large meridional extent were presented at the 13th General Assembly, I.U.G.G., 19-31 August 1963, Berkeley, California. Some of the solutions for free oscillations in rectangular oceans comparable to the Pacific, Atlantic, and Indian oceans were presented in a thesis submitted in partial fulfillment of the requirements for an M.S. degree (Charnell 1964).

2. Present address: Biological Laboratory, U.S. Bureau of Commercial Fisheries, Honolulu, Hawaii. Accepted for publication and submitted to press 18 November 1965.

variability in the ocean comes mostly from scattered observations of isolated phenomena from which it has not yet been possible to deduce an integrated picture.

The theoretical approach to the problem likewise has not been developed in a unified manner. Various approximations have been used in studies of quasigeostrophic or planetary waves, but in many cases these are not suitable for the conditions found in the real oceans. A more complete equation for time-dependent motion has been obtained by Rattray (1964), using the beta-plane and the conventional hydrostatic pressure distribution—long wavelength approximations.

In the present paper, solutions for that equation are obtained for free oscillations in enclosed basins comparable in size to the actual oceans. The oscillations considered have periods that range from a few days to more than a month. The effects of continuous density stratification are included in order to obtain waves corresponding to each of an infinite number of vertical modes. The relative magnitudes found for the vertical velocities justify the use of the hydrostatic, long-wave approximations. Similarly, the terms in the equations of motion containing the horizontal component of the Coriolis acceleration are relatively small everywhere, including the equatorial regions. As shown by Veronis (1963), a consistent beta-plane approximation is obtained by replacing the trigonometric functions of latitude with the first terms in a McLaurin series for these functions. It is important to note that this method differs from a commonly used beta-plane that is applicable for systems of small meridional extent only. Although the restricted form of beta-plane has found favor for investigation of ocean-wide phenomena, at least since 1948 (Stommel 1948), the more general case has not been considered. It is shown that this consistent beta-plane approximation extends the range of latitudes for which the approximation is adequate.

The free oscillations treated in this paper, with nearly integral meridional eigenvalues, are relatively insensitive to conditions at high latitudes. Thus, the periods, particularly for the baroclinic mode, are not greatly different for the three oceans, and certainly their order of magnitude is not effected by the beta-plane approximation used. Additional confirmation of this behavior is provided by Longuet-Higgins (1964), who has found that a beta-plane approximation gives good results for the periods of free oscillations in a basin centered on the equator and having a radius as great as one quadrant.

The effects of bathymetry, friction, and steady currents are not included. A brief discussion of their possible importance was included with the derivation of the normal-mode equation (Rattray 1964). It seems unlikely that these effects seriously modify the gross features of the oscillations described in this paper, but in other connections they will be of considerable interest and need further investigation.

*Normal-mode Equations.* Fjeldstad (1933) has shown that the eigenfunctions,  $\varphi_i$ , for small-amplitude oscillations in a vertically stratified incompressible ocean are given by the solutions to the equation

$$\frac{d}{dz} \left[ \rho_0 \frac{d\varphi_i}{dz} \right] - \frac{g}{c_i^2} \frac{d\rho_0}{dz} \varphi_i = 0, \quad (1)$$

subject to the boundary conditions that

$$\text{at the bottom, } z = 0, \varphi_i = 0, \quad (2)$$

$$\text{at the surface, } z = h, \frac{d\varphi_i}{dz} - \frac{g}{c_i^2} \varphi_i = 0, \quad (3)$$

where the eigenvalues  $c_i$  are the phase velocities of the long gravitational waves for each mode. The mean density,  $\rho_0$ , is a function of depth only, and the total depth,  $h$ , is taken to be constant. The horizontal velocities,  $u$  and  $v$ , and the vertical displacement,  $\zeta$ , of the particles are given in terms of the eigenfunctions as follows:

$$u(x, y, z, t) = \sum_i U_i(x, y, t) \frac{d\varphi_i(z)}{dz}, \quad (4)$$

$$v(x, y, z, t) = \sum_i V_i(x, y, t) \frac{d\varphi_i(z)}{dz}, \quad (5)$$

$$\zeta(x, y, z, t) = \sum_i Z_i(x, y, t) \varphi_i(z). \quad (6)$$

The normal-mode equations for the coefficients of the eigenfunctions obtained by Rattray (1964) with the beta-plane approximation for a two-layer ocean are equally applicable to the continuously stratified case.

For free oscillations these are:

$$\left[ \frac{\partial^3}{\partial t^3} - c_i^2 \frac{\partial}{\partial t} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + f^2 \frac{\partial}{\partial t} - c_i^2 \beta \frac{\partial}{\partial x} \right] V_i = 0, \quad (7)$$

$$\left[ \frac{\partial^2}{\partial t^2} - c_i^2 \frac{\partial^2}{\partial x^2} \right] U_i = \left[ f \frac{\partial}{\partial t} + c_i^2 \frac{\partial^2}{\partial x \partial y} \right] V_i, \quad (8)$$

and

$$\left\{ \begin{array}{l} \frac{\partial Z_i}{\partial t} = - \left( \frac{\partial U_i}{\partial x} + \frac{\partial V_i}{\partial y} \right) \\ \text{or} \\ c_i^2 \frac{\partial Z_i}{\partial x} = f V_i - \frac{\partial U_i}{\partial t} \end{array} \right\}. \quad (9)$$

*Free Oscillations in a Rectangular Basin.* Consider a rectangular basin with boundaries given by  $y = y_0, y_0 + L$ ;  $x = 0, W$ . The boundary conditions to be satisfied are

$$U_t = 0 \text{ at } x = 0, W \tag{10}$$

$$V_t = 0 \text{ at } y = y_0, y_0 + L. \tag{11}$$

With  $f = \beta y$ , the solutions to (7) for  $V_t$  can be written as

$$V_t = Y_v(y) e^{t([\beta x/2\sigma] + \sigma t)} [A_{v,n} e^{tR_{iv}x} + B_{v,n} e^{-tR_{iv}x}], \tag{12}$$

where

$$R_{iv}^2 = \{ \alpha_t + (\beta/2\sigma)^2 \} > 0, \tag{13}$$

and where the  $y$ -dependence is given by

$$\frac{d^2 Y}{dy^2} + \left( \frac{\sigma^2}{c_t^2} - \alpha_t - \frac{\beta^2 y^2}{c_t^2} \right) Y = 0, \tag{14}$$

with the separation constant  $\alpha_t$ . This is the Weber equation with solutions

$$Y_v(y) = A_{v,n}^* D_v^{(\nu)} \left( \left[ \frac{2\beta}{c_t} \right]^{1/2} y \right) + B_{v,n}^* D_v^{(0)} \left( \left[ \frac{2\beta}{c_t} \right]^{1/2} y \right), \tag{15}$$

where

$$2\nu + 1 = \frac{c_t}{\beta} \left( \frac{\sigma^2}{c_t^2} - \alpha_t \right). \tag{16}$$

The two independent solutions are even and odd functions, respectively, defined by:

$$D_v^{(\nu)}(\eta) = e^{-\eta^2/4} \left\{ 1 - \frac{\nu}{2!} \eta^2 + \frac{\nu(\nu-2)}{4!} \eta^4 - \dots \right\}, \tag{17}$$

$$D_v^{(0)}(\eta) = e^{-\eta^2/4} \left\{ \eta - \frac{(\nu-1)}{3!} \eta^3 + \frac{(\nu-3)(\nu-1)}{5!} \eta^5 - \dots \right\}, \tag{18}$$

where  $\nu$  can take on any value. When

$$\nu = r = \text{odd integer, } D_v^{(0)}(\eta) = D_r(\eta),$$

$$\text{and when } \nu = r = \text{even integer, } D_v^{(\nu)}(\eta) = D_r(\eta),$$

where

$$D_r \left( \left[ \frac{\beta}{c_t} \right]^{1/2} y \right) = e^{-\beta v^{1/2} c_t} H_r \left( \left[ \frac{\beta}{c_t} \right]^{1/2} y \right), \quad (19)$$

and  $H_r \left( \left[ \frac{\beta}{c_t} \right]^{1/2} y \right)$  is the Hermite polynomial of degree  $r$ .

For these waves the solution to (8) is

$$U_t = i e^{t((\beta x/2\sigma) + \sigma t)} \left\{ \frac{\left[ \frac{\sigma\beta y}{c_t^2} Y_v + \frac{dY_v}{dy} \left( \frac{\beta}{2\sigma} + R_{tv} \right) \right]}{\left[ \left( \frac{\beta}{2\sigma} + R_{tv} \right)^2 - \sigma^2/c_t^2 \right]} A_{vn} e^{tR_{tv}x} + \frac{\left[ \frac{\sigma\beta y}{c_t^2} Y_v + \frac{dY_v}{dy} \left( \frac{\beta}{2\sigma} - R_{tv} \right) \right]}{\left[ \left( \frac{\beta}{2\sigma} - R_{tv} \right)^2 - \sigma^2/c_t^2 \right]} B_{vn} e^{-tR_{tv}x} \right\}, \quad (20)$$

and when the quasigeostrophic approximation,

$$\frac{\sigma^2 \beta^2 y^2}{c_t^4} \ll R_{tv}^2 \frac{\beta}{c_t} (2v+1) < \frac{\beta^2}{4\sigma^2} \frac{\beta}{c_t} (2v+1), \quad (21)$$

is valid within the region of appreciable motion, (20) becomes

$$U_t \approx i e^{t((\beta x/2\sigma) + \sigma t)} \left\{ \frac{\frac{dY_v}{dy} \left( \frac{\beta}{2\sigma} + R_{tv} \right)}{\left( \frac{\beta}{2\sigma} + R_{tv} \right)^2 - \sigma^2/c_t^2} A_{vn} e^{tR_{tv}x} + \frac{\frac{dY_v}{dy} \left( \frac{\beta}{2\sigma} - R_{tv} \right)}{\left( \frac{\beta}{2\sigma} - R_{tv} \right)^2 - \sigma^2/c_t^2} B_{vn} e^{-tR_{tv}x} \right\}. \quad (22)$$

The boundary conditions (10) require that

$$\frac{\left( \frac{\beta}{2\sigma} + R_{tv} \right) A_{vn}}{\left( \frac{\beta}{2\sigma} + R_{tv} \right)^2 - \sigma^2/c_t^2} + \frac{\left( \frac{\beta}{2\sigma} - R_{tv} \right) B_{vn}}{\left( \frac{\beta}{2\sigma} - R_{tv} \right)^2 - \sigma^2/c_t^2} = 0, \quad (23)$$

and

$$\sin R_{1\nu} W = 0. \tag{24}$$

The latter is satisfied by

$$R_{1\nu} W = n\pi, \tag{25}$$

where  $n$  is an integer. The velocity components are therefore

$$U_{1t} \approx e^{t([\beta x/2\sigma] + \sigma t)} \sin \frac{n\pi x}{W} \cdot \frac{d}{dy} \left[ A_{\nu n}^* D_{\nu}^{(\sigma)} \left( \left[ \frac{2\beta}{c_t} \right]^{1/2} y \right) + B_{\nu n}^* D_{\nu}^{(\sigma)} \left( \left[ \frac{2\beta}{c_t} \right]^{1/2} y \right) \right], \tag{26}$$

$$V_{1t} \approx -e^{t([\beta x/2\sigma] + \sigma t)} \left[ \frac{n\pi}{W} \cos \frac{n\pi x}{W} + i \frac{\beta}{2\sigma} \sin \frac{n\pi x}{W} \right] \cdot \left[ A_{\nu n}^* D_{\nu}^{(\sigma)} \left( \left[ \frac{2\beta}{c_t} \right]^{1/2} y \right) + B_{\nu n}^* D_{\nu}^{(\sigma)} \left( \left[ \frac{2\beta}{c_t} \right]^{1/2} y \right) \right], \tag{27}$$

where the ratio  $A_{\nu n}^*/B_{\nu n}^*$  and the value of  $\nu$  are determined by the boundary conditions (11),

$$A_{\nu n}^* D_{\nu}^{(\sigma)} \left( \left[ \frac{2\beta}{c_t} \right]^{1/2} y \right) + B_{\nu n}^* D_{\nu}^{(\sigma)} \left( \left[ \frac{2\beta}{c_t} \right]^{1/2} y \right) = 0 \text{ at } y = y_0, y_0 + L. \tag{28}$$

The frequency is then given by

$$\sigma_{1,\nu n} = \frac{\beta}{2} \left[ \frac{n^2 \pi^2}{W^2} + (2\nu + 1) \frac{\beta}{c_t} \right]^{-1/2}. \tag{29}$$

The elevation is

$$Z_{1t} \approx -e^{t([\beta x/2\sigma] + \sigma t)} \left\{ \frac{\beta y}{c_t^2} \sin \frac{n\pi x}{W} \left[ A_{\nu n}^* D_{\nu}^{(\sigma)} \left( \left[ \frac{2\beta}{c_t} \right]^{1/2} y \right) + B_{\nu n}^* D_{\nu}^{(\sigma)} \left( \left[ \frac{2\beta}{c_t} \right]^{1/2} y \right) \right] + (2\nu + 1)^{-1} \left( \frac{1}{2c_t} \sin \frac{n\pi x}{W} + i \frac{n\pi}{W} \frac{\sigma}{\beta c_t} \cos \frac{n\pi x}{W} \right) \cdot \frac{d}{dy} \left[ A_{\nu n}^* D_{\nu}^{(\sigma)} \left( \left[ \frac{2\beta}{c_t} \right]^{1/2} y \right) + B_{\nu n}^* D_{\nu}^{(\sigma)} \left( \left[ \frac{2\beta}{c_t} \right]^{1/2} y \right) \right] \right\}. \tag{30}$$

These motions can be described as amplitude-modulated waves, with the carrier traveling west with velocity  $c^* = 2\sigma^2/\beta$  and wave number  $k^* = \beta/2\sigma$ , while the amplitude modulation is stationary, with the normal velocity components zero at the boundaries. They appear as lines of current cells drifting westward, each cell growing and decaying according to its position in the

stationary modulation pattern. Examples of the carrier and modulation are shown in Figs. 2-7 (pp. 94-99). Examples of the drifting cell pattern are shown in Figs. 8 and 9 (pp. 102, 103).

*Special Cases.* Three special cases, for which the above results take particularly simple forms, are considered. The first case applies to a basin of small meridional extent in which the Coriolis parameter,  $f$ , occurring in (7), (8), and (9), can be approximated by a constant. The second arises for a basin of large meridional extent in which the waves are confined to the equatorial regions. The third occurs in a basin symmetrical about the equator, where the solution contains only the even or the odd Weber function. The solution for this case can be used to evaluate the degree of approximation inherent in the two previous cases.

(i) **BASIN OF SMALL MERIDIONAL EXTENT.** For basins of small meridional extent, the change in Coriolis parameter over the basin is small and can be neglected. A constant Coriolis parameter yields a constant value for the  $y$ -component of the wave number, and the even and odd Weber functions become cosines and sines, respectively. Under these conditions the velocity components are:

$$U_t \approx \frac{m\pi}{L} e^{i([\beta x/2\sigma] + \sigma t)} \sin \frac{n\pi x}{W} \cos \frac{m\pi}{L} (y - y_0), \quad (31)$$

$$V_t \approx -e^{i([\beta x/2\sigma] + \sigma t)} \left( \frac{n\pi}{W} \cos \frac{n\pi x}{W} + i \frac{\beta}{2\sigma} \sin \frac{n\pi x}{W} \right) \sin \frac{m\pi}{L} (y - y_0); \quad (32)$$

and the elevation becomes

$$Z_t = -e^{i([\beta x/2\sigma] + \sigma t)} \left\{ \frac{f}{c_t^2} \sin \frac{n\pi x}{W} \sin \frac{m\pi}{L} (y - y_0) + \frac{m\pi/L}{\frac{f^2}{c_t^2} + \frac{m^2\pi^2}{L^2}} \left( \frac{\beta}{2c_t^2} \sin \frac{n\pi x}{W} + i \frac{n\pi}{W} \frac{\sigma}{c_t^2} \cos \frac{n\pi x}{W} \right) \cos \frac{m\pi}{L} (y - y_0) \right\}, \quad (33)$$

where both  $n$  and  $m$  are integers. The frequency is obtained by replacing  $(2\nu + 1)(\beta/c_t)$  with  $(f^2/c_t^2) + (m^2\pi^2/L^2)$  in (29) so that

$$\sigma_{t, mn} = \frac{\beta}{2} \left[ \frac{f^2}{c_t^2} + \pi^2 \left( \frac{m^2}{L^2} + \frac{n^2}{W^2} \right) \right]^{-1/2}. \quad (34)$$

Equation (34) reduces to the result of Arons and Stommel (1956) for  $L \rightarrow \infty$  (where it is inapplicable), and to the result of Longuet-Higgins (1964) for  $c_t \rightarrow \infty$ .

(ii) **BASINS OF LARGE MERIDIONAL EXTENT.** With  $\nu = r =$  an integer, the exponential term in  $D_r$  will cause the solution to approach zero for large values of  $y$ . Thus waves confined to the equatorial region are possible for conditions where the zonal boundaries occur at high latitudes. For these waves, the frequency is given by substituting  $r$  for  $\nu$  in (29). The velocity components corresponding to these frequencies are:

$$U_t \approx e^{i((\beta x/2\sigma) + \sigma t)} \sin \frac{n\pi x}{W} \cdot \frac{d}{dy} \left[ D_r \left( \left[ \frac{2\beta}{c_t} \right]^{1/2} y \right) \right], \quad (35)$$

$$V_t \approx -e^{i((\beta x/2\sigma) + \sigma t)} \left[ \frac{n\pi}{W} \cos \frac{n\pi x}{W} + i \frac{\beta}{2\sigma} \sin \frac{n\pi x}{W} \right] \cdot D_r \left( \left[ \frac{2\beta}{c_t} \right]^{1/2} y \right); \quad (36)$$

and the elevation then becomes:

$$Z_t \approx -e^{i((\beta x/2\sigma) + \sigma t)} \left\{ \frac{\beta y}{c_t^2} \sin \frac{n\pi x}{W} D_r \left( \left[ \frac{2\beta}{c_t} \right]^{1/2} y \right) + (2r + 1)^{-1} \left( \frac{1}{2c_t} \sin \frac{n\pi x}{W} + i \frac{n\pi}{W} \frac{\sigma}{\beta c_t} \cos \frac{n\pi x}{W} \right) \cdot \frac{d}{dy} \left[ D_r \left( \left[ \frac{2\beta}{c_t} \right]^{1/2} y \right) \right] \right\}. \quad (37)$$

(iii) **BASIN SYMMETRICAL ABOUT THE EQUATOR.** To obtain insight into the applicability of the approximations of small and large meridional extent, simple solutions are considered for symmetrical basins for which either  $A_{\nu n}^* = 0$  or  $B_{\nu n}^* = 0$ . The first few zeros of the even and odd Weber functions, obtained with the help of Jahnke and Emde (1945), Miller (1955), and Slater (1960), are used to determine the eigenvalues shown in Fig. 1. The zeros of  $D_{\nu^{(e)}}(\eta)$  are given by the ordinate and the curves for the even-numbered modes; the zeros of  $D_{\nu^{(o)}}(\eta)$  are given by the curves for the odd-numbered modes. For example,  $\eta = 0$  is a root of  $D_{\nu^{(o)}}(\eta)$  for all values of  $\nu$ , and

$$\begin{aligned} \text{for } 0 \leq \nu < 1, & \begin{cases} D_{\nu^{(e)}}(\eta) \text{ has one root} \\ D_{\nu^{(o)}}(\eta) \text{ has one root,} \end{cases} \\ 1 \leq \nu < 2, & \begin{cases} D_{\nu^{(e)}}(\eta) \text{ has one root} \\ D_{\nu^{(o)}}(\eta) \text{ has two roots,} \end{cases} \\ 2 \leq \nu < 3, & \begin{cases} D_{\nu^{(e)}}(\eta) \text{ has two roots} \\ D_{\nu^{(o)}}(\eta) \text{ has two roots,} \end{cases} \end{aligned}$$

and so forth. These curves are most useful for determining oscillations in oceans for which either  $y_0 = 0$  or  $y_0 = -L/2$ . Oscillations for which  $\nu \gg [2\beta/c_t]^{1/2} L/2$  correspond to conditions in an ocean of small meridional extent, and those for which  $\nu \ll [2\beta/c_t]^{1/2} L/2$  correspond to conditions of no zonal

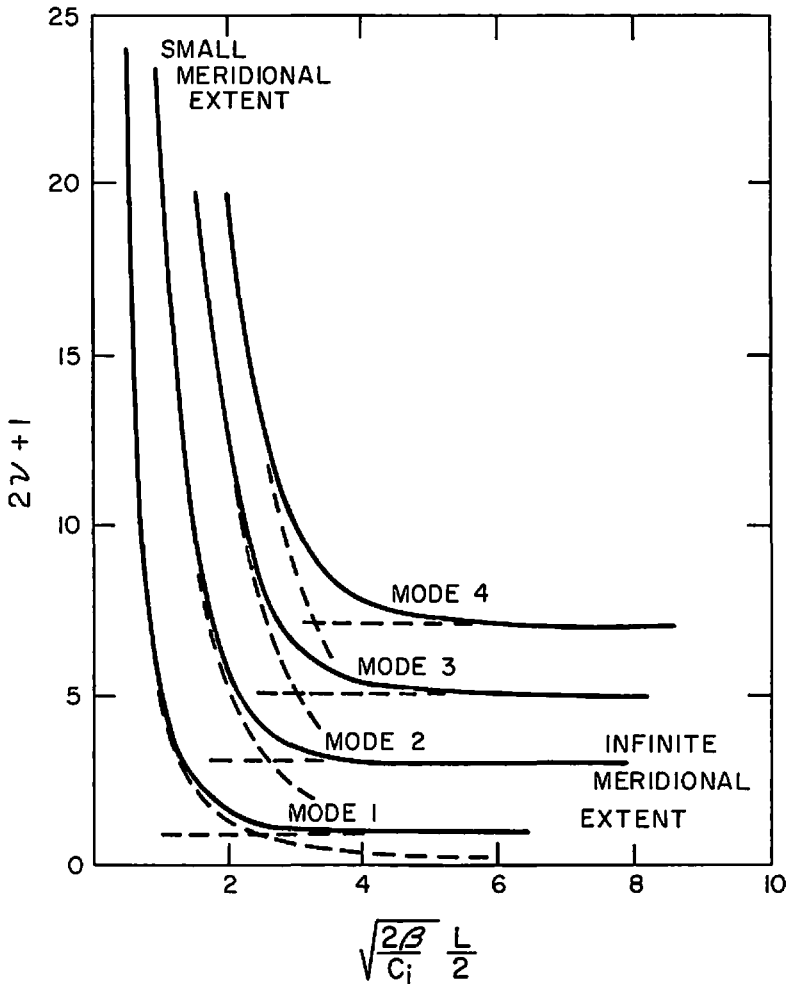


Figure 1. Dependence of meridional eigenvalues upon length of basin,  $L$ . Approximations of small and infinite meridional extent, dashed line; exact values, solid line.

boundaries or equatorial waves. This situation can be elucidated by considering oceans extending from  $y = -L/2$  to  $L/2$  and by comparing the exact value of  $2\nu + 1$  with the approximate values determined from the formulae appropriate to oceans of small and of infinite meridional extent. Fig. 1 shows also the results of such calculations and illustrates the ranges over which the simpler approximations will give satisfactory results. The actual value of  $\nu$  is greater than the values obtained by the approximations of either small or large meridional extent. Thus these approximations always give an upper limit for the frequency of a quasigeostrophic oscillation.

(iv) **GENERAL APPLICABILITY OF APPROXIMATIONS OF SMALL AND LARGE MERIDIONAL EXTENT.** The foregoing conclusions have been obtained for an ocean centered on the equator. For an ocean otherwise situated but of the same dimensions, variation in the Coriolis parameter is decreased and the approximation of small meridional extent is improved. However, without symmetry, the choice of a suitable average value of the Coriolis parameter becomes difficult, and, in practice, the approximation may be applied easily to only those cases in which the relative change of Coriolis parameter is small. For an ocean that straddles the equator but extends farther on one side, the approximation of infinite meridional extent is better than it is for a smaller, symmetrically placed ocean having both of its boundaries at the lesser of the two latitudes. However, an asymmetrical ocean has an eigenvalue  $\nu$  that is larger than the one corresponding to a symmetrically placed ocean of the same meridional extent. Under these conditions, Fig. 1 can be used as a guide to the magnitude of the meridional eigenvalue  $\nu$  and to the applicability of the simpler approximations for the conditions usually encountered.

*Free Oscillations in Rectangular Oceans Comparable to the Pacific, Atlantic, and Indian Oceans.* The beta-plane approximation derived by Veronis (1963) is used, in which

$$x = R\lambda, y = R\mu, \tan \varphi = \sinh \mu, \beta = \frac{2\Omega}{R},$$

where  $R$  = earth's radius  $\approx 6.4 \times 10^8$  cm;  $\Omega$  = angular velocity of rotation of the earth =  $.73 \times 10^{-4} s^{-1}$ ;  $\beta$  = rate of change of Coriolis parameter with latitude =  $2.3 \times 10^{-13} cm^{-1} s^{-1}$ ;  $\lambda$  = longitude;  $\varphi$  = latitude. Representative values for the surface and first internal gravity-wave modes are

$$c_0 = 2 \times 10^4 \text{ cm } s^{-1} \text{ and } c_1 = 2 \times 10^2 \text{ cm } s^{-1}.$$

These values correspond to ocean depths of 4000 m, with density distributions typical of the three oceans.

The ocean boundaries are approximated by:

	Pacific	Atlantic	Indian
$W$ (in $10^9$ cm)	1.3	0.65	0.78
$y_0$ (in $10^9$ cm)	-1.1	-1.1	-1.1
$L$ (in $10^9$ cm)	1.7	2.0	1.3

Numerical solutions to (14), subject to the boundary conditions of (11), have been obtained for lower modes in each of the oceans. Table I gives the resulting periods and the phase velocity and wave length of the carrier wave for each mode. The approximation given by (21) does not hold well for the

Table I. Properties of some quasigeostrophic free oscillations in ocean basins.

$n$	$c_j$	Atlantic Ocean				Pacific Ocean				Indian Ocean			
		$\nu$	$T$	$L^*$	$c^*$	$\nu$	$T$	$L^*$	$c^*$	$\nu$	$T$	$L^*$	$c^*$
		days	days	$10^8$ cm	m sec <sup>-1</sup>	days	days	$10^8$ cm	m sec <sup>-1</sup>	days	days	$10^8$ cm	m sec <sup>-1</sup>
1	$c_0$	0.00015	3.7	10.6	32.9	[0.016	2.7	14.9	64.8]*	0.406	3.8	10.4	31.2
1	↓	1.0024	4.8	8.3	19.9	1.102	4.1	9.6	27.0	2.025	5.4	7.3	15.6
1		2.016	5.7	7.0	14.1	2.301	5.3	7.5	16.4	3.769	6.7	5.9	10.1
2		0.00015	6.5	6.1	10.9	0.016	3.8	10.6	32.6	0.406	5.9	6.8	13.4
2		1.0024	7.1	5.6	9.0	1.102	4.9	8.1	19.1	2.025	7.0	5.7	9.4
2		2.016	7.8	5.1	7.6	2.301	5.9	6.7	13.1	3.769	8.1	4.9	7.1
3		0.00015	9.4	4.2	5.2	0.016	5.1	7.8	17.8	0.406	8.2	4.9	6.9
3		1.0024	9.8	4.0	4.7	1.102	6.0	6.6	12.9	2.025	9.0	4.4	5.7
3		2.016	10.4	3.8	4.3	2.301	6.9	5.8	9.8	3.769	9.8	4.0	4.7
6	$c_1$	11.0	104.5	.380	.0421	11.0	103.2	.385	.0431	11.0	104.0	.382	.0425
7	↓	11.0	105	.378	.0417	11.0	103.4	.384	.0430	11.0	104.4	.381	.0422
8		11.0	105.7	.376	.0411	11.0	103.6	.384	.0429	11.0	104.9	.379	.0418

\* Does not satisfy conditions of eq. (21).

lowest modes, and indeed, the inequality is reversed for the lowest-order barotropic mode in the Pacific Ocean and for the low-order baroclinic modes in all oceans. These lower modes require the addition of a Kelvin wave to satisfy the boundary conditions at the higher latitudes. The resulting motion is modified primarily in the boundary regions, with some distortion of the interior cell pattern. Since the period of oscillation depends primarily on the number and size of these cells, (29) may be used to give magnitude estimates of the minimum periods. Using (29), the minimum periods for the barotropic modes fall in the range of 3 to 4 days while the minimum baroclinic periods would be in the order of 20 days. The higher modes have longer periods but shorter wave lengths and smaller carrier-wave velocity. These long-period waves have particle velocities in approximate geostrophic balance at the higher latitudes, but in a significant region about the equator, such a simple relation is invalid.

The motion can be represented by a streamfunction,  $\psi_t$ , such that

$$U_t = -\frac{\partial \psi_t}{\partial y}, \quad V_t = \frac{\partial \psi_t}{\partial x}.$$

Streamlines are given for a number of illustrative examples in Figs. 2-9. Figs. 2, 3, and 4 show the contours of the standing modulation and of the progressive carrier for representative types of barotropic modes in the Pacific Ocean. It is apparent that the carrier has a wave length roughly one-half that of the modulation for these waves. The zonal eigenvalue,  $n$ , gives the number of stationary cells occurring in the zonal direction. The meridional eigenvalue,  $\nu$ , is related to the number of current cells in the meridional direction in the same manner as to the number of zeros in  $D_\nu(\eta)$ . Figs. 5 and 6, respectively, show comparable results for the Atlantic and Indian oceans.

The baroclinic oscillations shown are confined to the equatorial regions and have similar patterns for all three oceans. One mode of oscillation for the Pacific Ocean is illustrated in Fig. 7. The carrier wave length is in the order of one-eighth of the wave length of the modulation.

The streamfunction given by the product of the carrier and modulation amplitudes can be represented also by a number of cells that appear at the boundary or node on the east, travel toward the west, and disappear at the boundary or node on the west. Figs. 8 and 9 are time histories of these cells in the Pacific Ocean for a typical barotropic and baroclinic mode, respectively.

*Discussion.* The foregoing solutions, though obtained under certain simplifying assumptions, give a clear illustration of the nature of quasigeostrophic free oscillations that can occur in the real oceans. Some distortion of the regular cell pattern must inevitably occur due to geometrical and other effects, but the periods of the oscillations will be relatively insensitive to such changes.

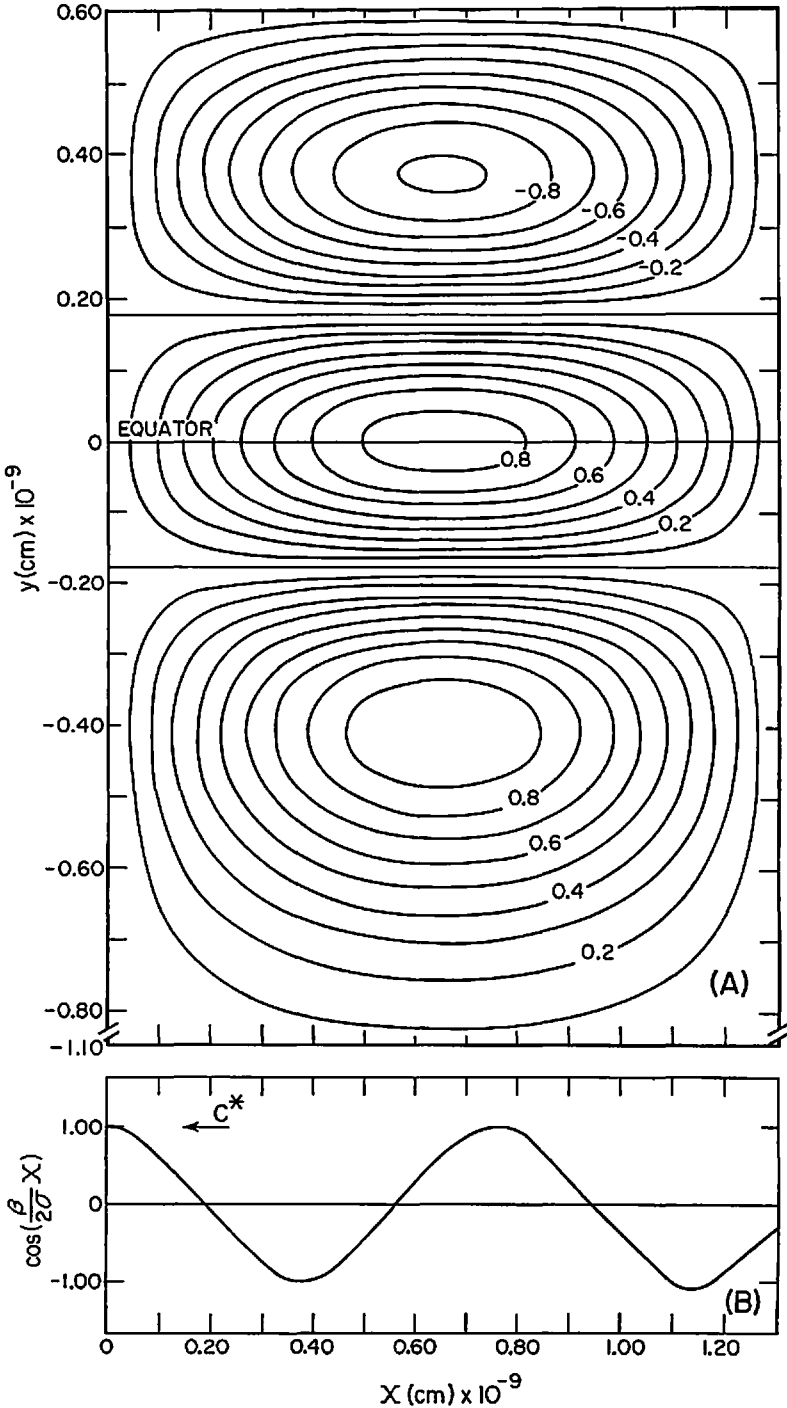


Figure 2. Normalized streamlines for the Pacific Ocean; barotropic mode; eigenvalues:  $n = 1$ ,  $\nu = 2.301$ . (A) stationary modulation (B) time-dependent carrier,  $c^* = 16.4 \text{ m sec}^{-1}$ .

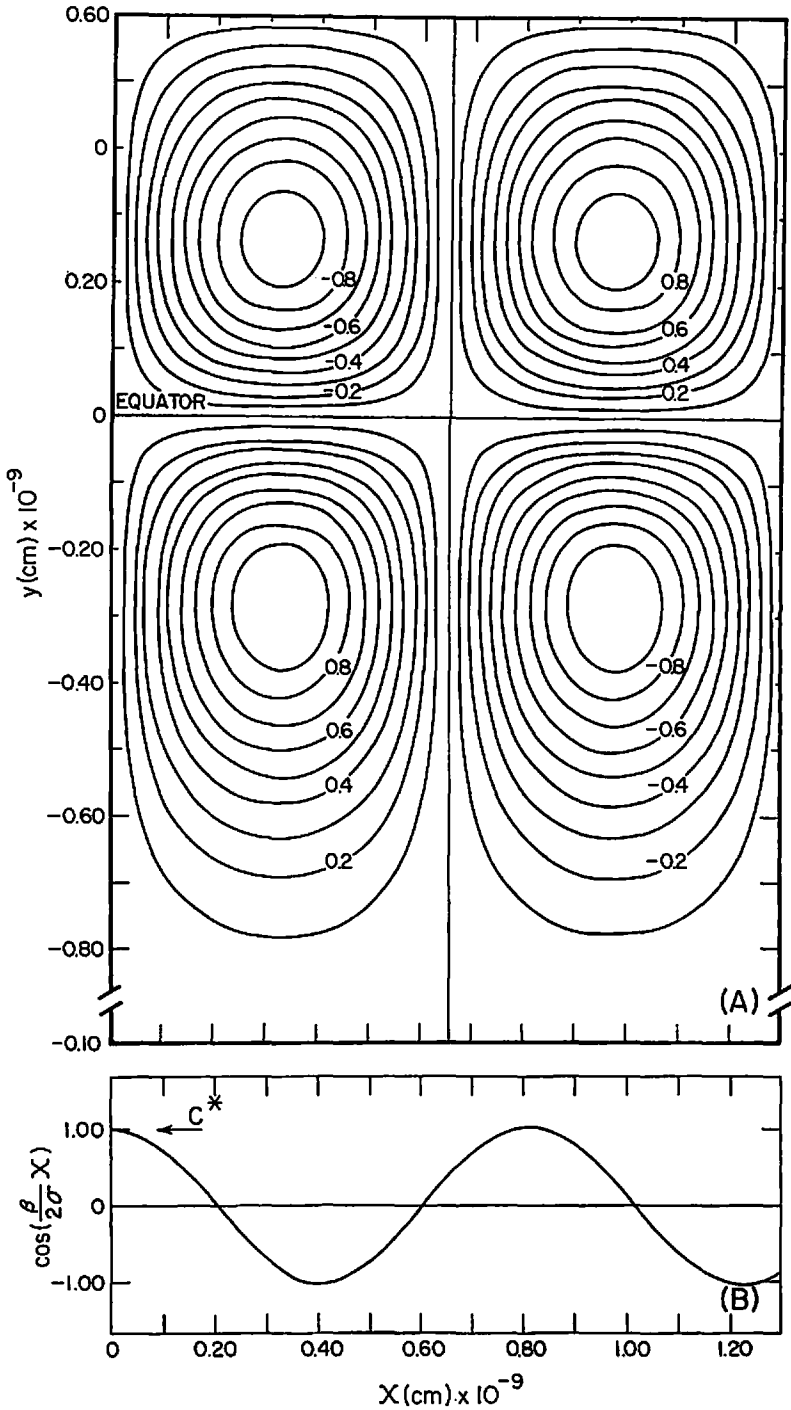


Figure 3. Normalized streamlines for the Pacific Ocean; barotropic mode; eigenvalues:  $n = 2$ ,  $\nu = 1.102$ . (A) stationary modulation (B) time-dependent carrier,  $c^* = 19.1 \text{ m sec}^{-1}$ .

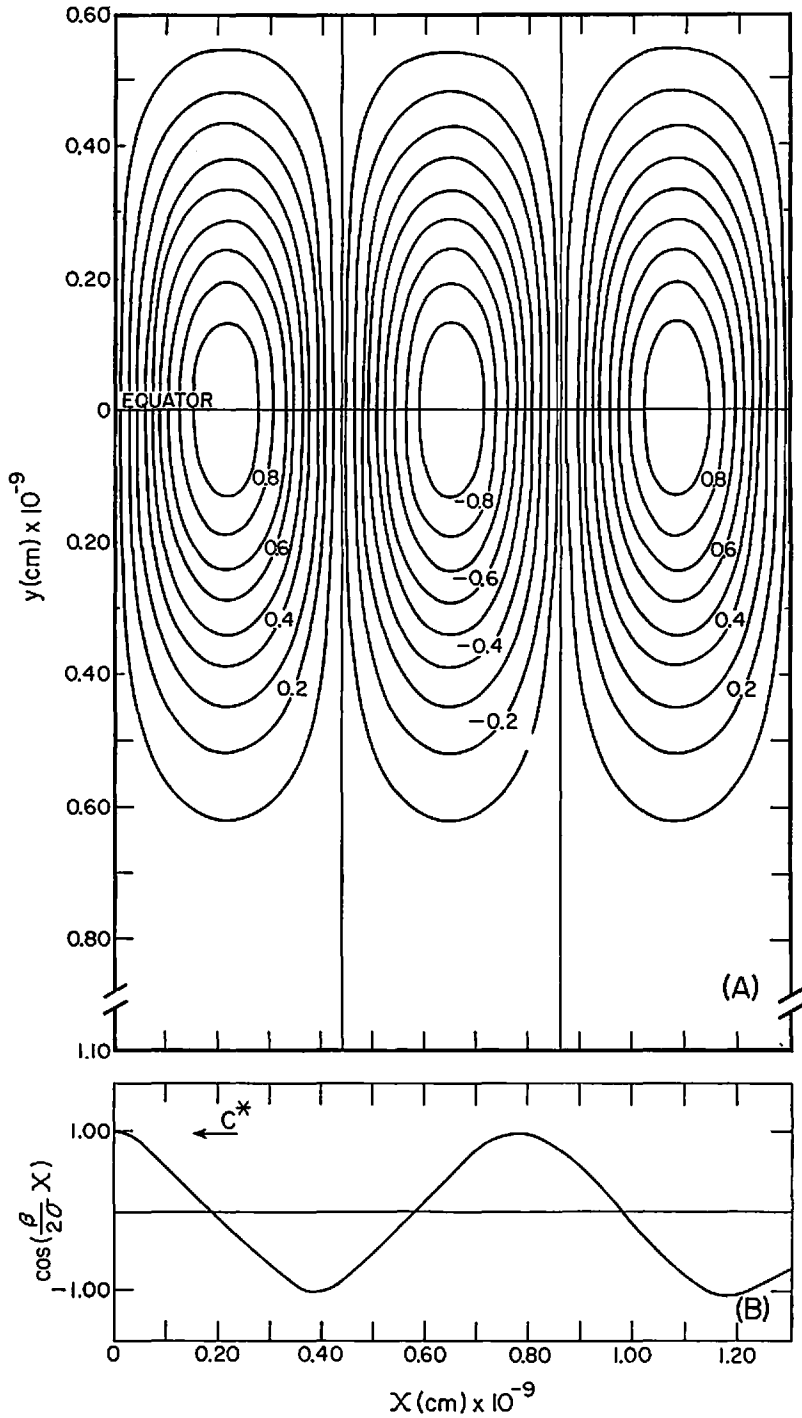


Figure 4. Normalized streamlines for the Pacific Ocean; barotropic mode; eigenvalues:  $n = 3$ ,  $\nu = 0.016$ . (A) stationary modulation (B) time-dependent carrier,  $c^* = 17.8 \text{ m sec}^{-1}$ .

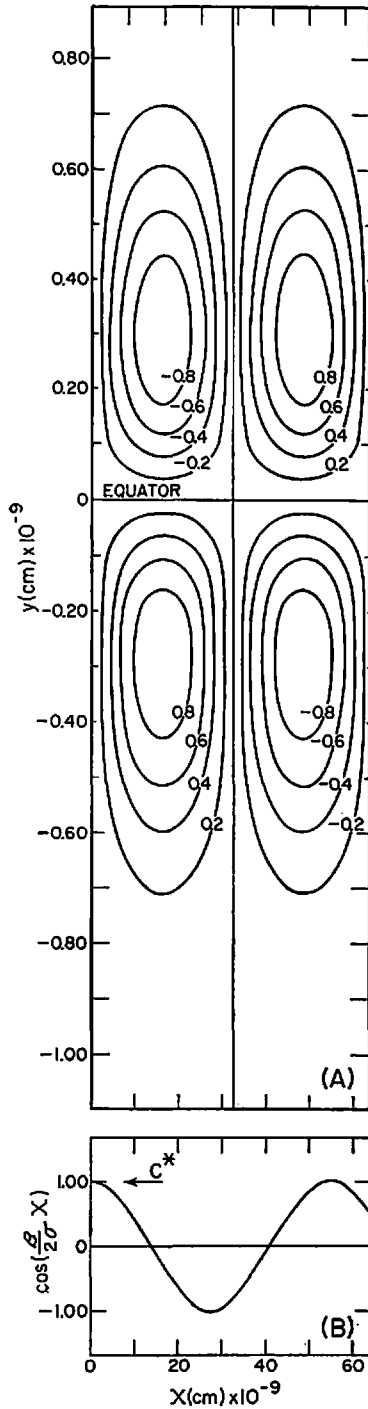


Figure 5. Normalized streamlines for the Atlantic Ocean; barotropic mode; eigenvalues:  $n = 2$ ,  $\nu = 1.002$ . (A) stationary modulation (B) time-dependent carrier,  $c^* = 9.0 \text{ m sec}^{-1}$ .

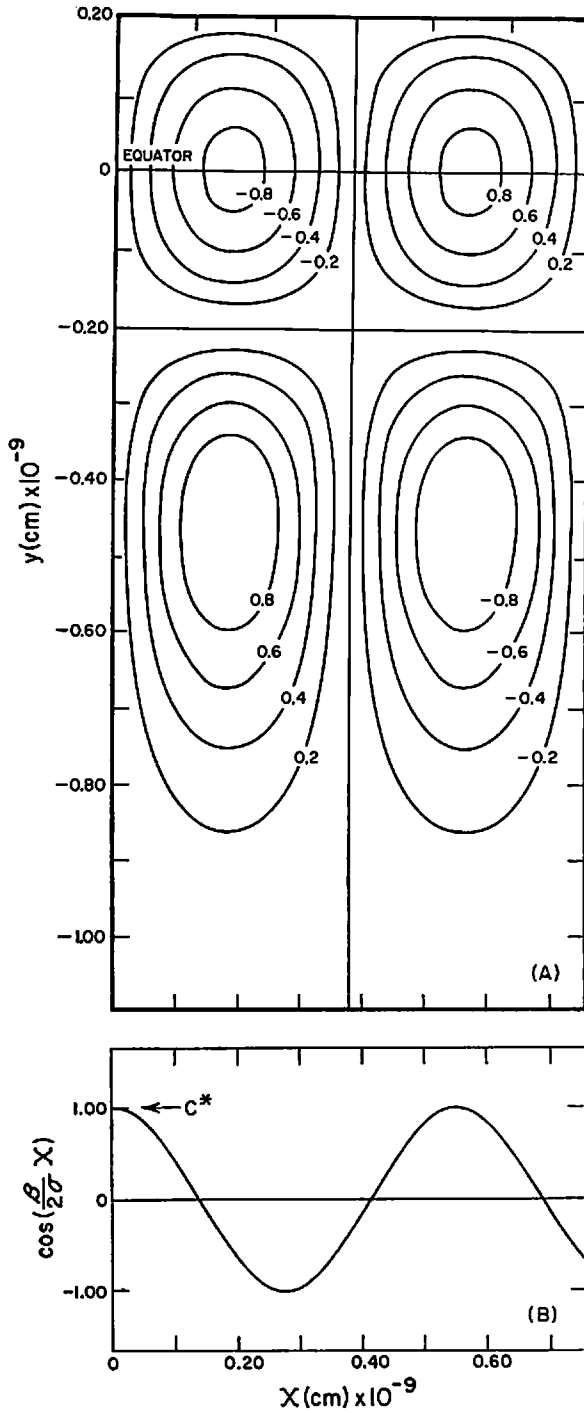


Figure 6. Normalized streamlines for the Indian Ocean; barotropic mode; eigenvalues:  $n = 2$ ,  $\nu = 2.025$ . (A) stationary modulation (B) time-dependent carrier,  $c^* = 9.4 \text{ m sec}^{-1}$ .

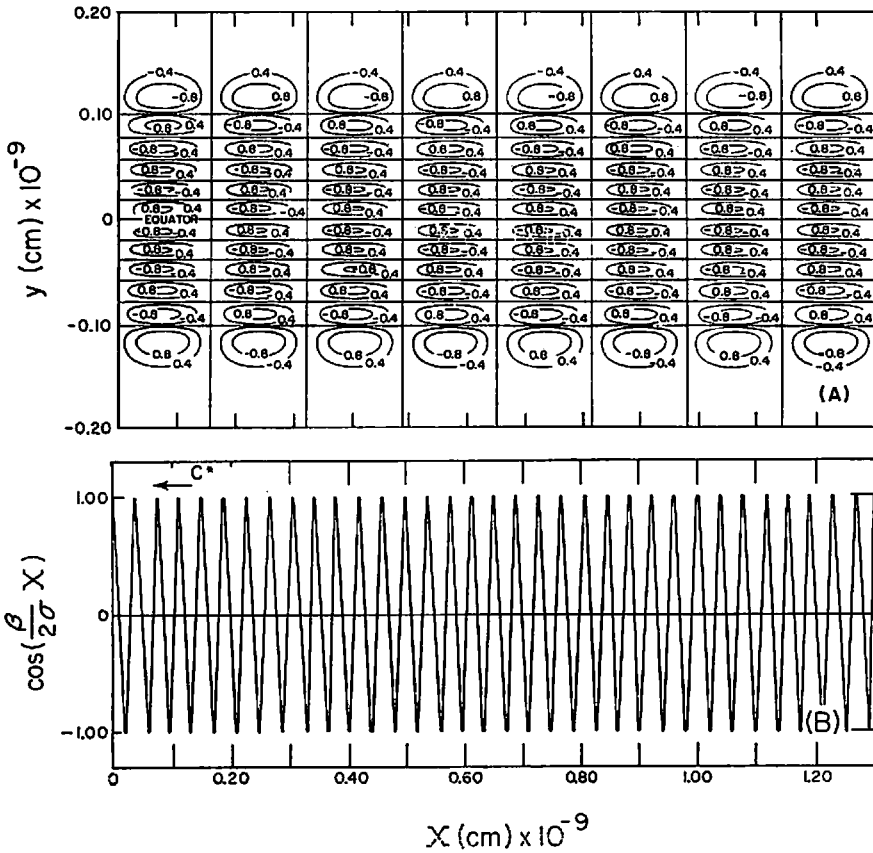


Figure 7. Normalized streamlines for the Pacific Ocean; baroclinic mode; eigenvalues:  $n = 8$ ,  $\nu = 11.0$ . (A) stationary modulation (B) time-dependent carrier,  $c^* = 4.29 \text{ cm sec}^{-1}$ .

The barotropic oscillations differ considerably from the baroclinic; the lower modes of barotropic oscillations fill the whole ocean basin, whereas lower-order baroclinic modes are restricted to equatorial and boundary regions. Previous investigations have not recognized this fundamental difference. Fofonoff (1962) has suggested minimum periods of 3.6 days and 6.8 months, respectively, for barotropic and baroclinic waves. The minimum periods found herein are comparable for the barotropic waves, but for the baroclinic waves they are considerably smaller than Fofonoff's values.

It is possible that certain otherwise unexplained sea-level and current fluctuations are manifestations of quasigeostrophic free oscillations. For example, Groves (1956) measured a fluctuation in sea level around Canton Island and speculated that it might be a result of a resonance phenomenon of the ocean excited by variation in the easterlies. The 3.84-day period he found agrees

closely with the 3.8-day period for the  $m = 2$ ,  $\nu = 0.016$  barotropic mode in the Pacific Ocean. This oscillation was not found at some adjacent stations, and a certain amount of selectivity needs to be accounted for. Canton Island is near an elevation node and current antinode for the 3.8-day oscillation. Thus, a variation in sea level could possibly be explained as a response to the fluctuating current and would therefore depend on the particular bathymetry of the local system.

Welsh (1964) reported negligible net motion, strong tidal components, and relatively large nontidal fluctuation in the currents over Agulhas Bank. For the nontidal component, he found a period of about 4.5 days and a maximum magnitude of about 20 cm sec. These currents could well be associated with quasigeostrophic free oscillations in adjacent oceans that have periods close to this observed value.

In another series of current measurements, Swallow and Hamon (1960) found significant movement in the deep water at about  $42^{\circ}\text{N}$  and  $15^{\circ}\text{W}$ . The observed currents were not steady, but the study was too limited in space and time to adequately define a period or wave length for the fluctuation. It is therefore uncertain whether these currents can be accounted for by the present theory. The barotropic modes of orders near one hundred have periods of 90–100 days and wave length at 100–200 km that are compatible with the reported data.

These periods also approach those for sea-level fluctuations at Bermuda, reported by Shaw and Donn (1964). Although the idea of significant energy in such high modes of oscillation is not a pleasant thought for those engaged in the study of ocean currents, discarding such a possibility, *a priori*, does not appear justified. Longuet-Higgins (1965) also has suggested that Swallow float measurements could be explained in terms of planetary waves.

The above examples are suggestive only. Although they can be explained in terms of the foregoing results, they do not demonstrate the existence of the oscillations presented in this paper. The theory presented here serves as a guide to the location and type of measurements required in order to unscramble the behavior of quasigeostrophic oscillations as they actually occur in the oceans. Only when these observations are made can it be ascertained whether free oscillations are as important for the oceans as they are for other dynamical systems.

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## ADDED IN PROOF

Longuet-Higgins (personal communication 1965) has drawn the authors' attention to recent results of Munk and Cartwright; from the spectrum of sea level at Honolulu, there is evidence of an oscillation corresponding to the fundamental barotropic mode of the Pacific Ocean.

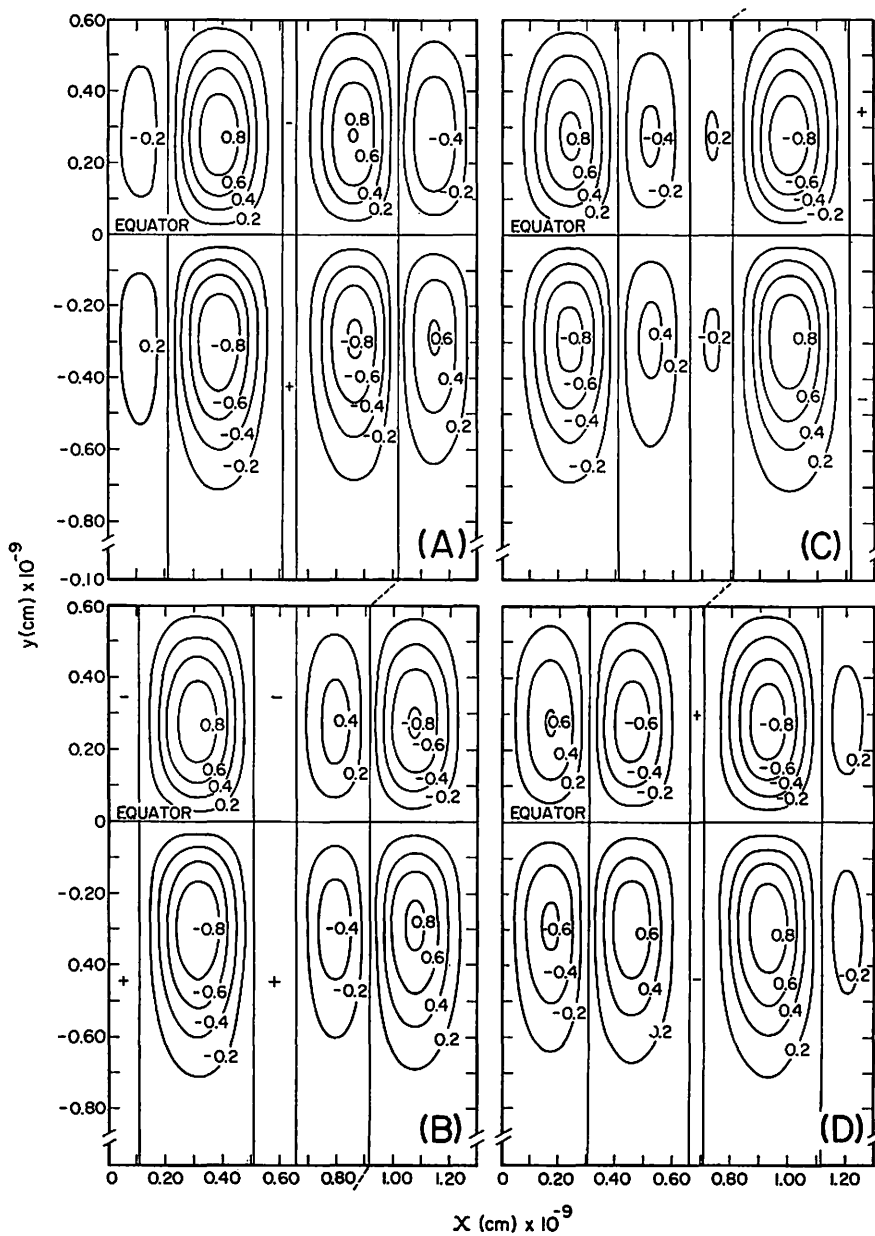


Figure 8. Time history of the streamlines for barotropic free oscillations of the Pacific Ocean; eigenvalues:  $n = 2$ ,  $\nu = 1.102$ . (A) time  $t = 0$  (B)  $t = 1/8 T$ , 14.8 hours later (C)  $t = 1/4 T$ , 29.6 hours later (D)  $t = 3/8 T$ , 44.4 hours later.