Classical and Computational Algebraic Geometry in Computer Vision

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Abstract

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A camera is a linear projective map $\mathbb{P}^3 \to \mathbb{P}^2$ which can be represented by a full rank matrix $A \in \mathbb{P}(\mathbb{R}^{3\times 4})$. Within the context of computer vision, we refer to points $q \in \mathbb{P}^3$ as world points and points $p \in \mathbb{P}^2$ as image points. A multi-view arrangement is a collection of camera A_i, world points q_j and image points p_{ij} satisfying $A_i q_j = p_{ij}$. Within computer vision we are of the concerned with the problem of reconstructability. That is, given partial information for a multi-view arrangement, can the rest of the arrangement be reconstructed and, if so, is that reconstruction unique?

The 7-point algorithm is a classical 3-D image reconstruction algorithm for two-view geometry, using 7 paired points $(x_i, y_i) \in \mathbb{P}^2 \times \mathbb{P}^2$ in two images to reconstruct the original cameras which produced them. This is done by constructing the 7×9 matrix Z with rows $(x_i^\top \otimes y_i^\top)$ and producing the rank 2 matrices in its nullspace. Each choice of rank 2 matrix determines a possible arrangement of cameras, unique up to change of coordinates in $\mathbb{P}^3$. Generically, there will be exactly three rank 2 matrices in this nullspace, but this will not always be the case. In particular, this algorithm will be ill-posed if the 7 tensors $(x_i^\top \otimes y_i^\top) \in \mathbb{P}^8$ are linearly dependent. We fully characterize the geometric conditions on $\{(x_i, y_i)\}_{i=1}^k$ under which k tensors $(x_i^\top \otimes y_i^\top) \in \mathbb{P}^8$ will be linearly dependent for $2 \leq k \leq 9$. For low values of k we use computational software to analyze the conditions under which linear dependence occurs. For $k = 6$, the answer is in terms of the geometry of cubic surfaces and the blowing up of $\mathbb{P}^2$ in 6 points. For $k = 7$ and 8 the answer is in terms of Cremona

transformations and cubic curves; this utilizes a special correspondence we discover between possible 3-D reconstructions of our images, lines normal to the span of the tensors $\{(x_i^\top \otimes y_i^\top)\}_{i=1}^k$, and Cremona transformations sending $x_i \mapsto y_i$ for all i. For all values of k barring $k = 6$, the geometry can be characterized as the existence of some special morphism sending $x_i \mapsto y_i$ for all i. For $k = 6$ no morphism exists, but the two pointsets $\{x_i\}_{i=1}^6$ and $\{y_i\}_{i=1}^6$ can be seen as duals in a highly geometric sense via the blowups.

This thesis also considers the related problem of resectioning: that of reconstructing the original cameras $A_i : \mathbb{P}^3 \to \mathbb{P}^2$ given the world points $q_j \in \mathbb{P}^3$ and the image points $p_{ij} \in \mathbb{P}^2$. This problem is in some sense dual to the more well-studied problem of triangulation: reconstructing the world points $q_j \in \mathbb{P}^3$ given the cameras $A_i : \mathbb{P}^3 \to \mathbb{P}^2$ and the image points p_{ij}. We make use of Carlson-Weinshall duality, a framework for interchanging the roles of cameras A_i and world points q_j to adapt the methodology for studying triangulation in [1] to prove similar results for the dual problem of resectioning. In particular, we find a universal Gröbner basis for the vanishing ideal of the associated resectioning variety. We also use Carlsson-Weinshall duality to produce a coordinate-free view of the atlas for the pinhole camera in [1]. This atlas is reduced in the formal sense of taking quotients and we find that the reduced resectioning and reduced triangulation varieties are isomorphic.

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DEDICATION

To my late grandmother Pat, for always watching out for me, even after the Alzheimer's developed. Our bear has kept me company during many late nights of work. I love you so much.

Chapter 1

INTRODUCTION

The problem of 3D image reconstruction is one of increasing relevance in the modern world, with use in self-driving cars, virtual reality, and a wide variety of assorted artificial intelligence applications. These problems vary mathematically based on both the number of cameras involved and the amount of detail known about the cameras. With autonomous vehicles, the images are taken by cameras with known specifications and fixed positions and orientations relative to each other; this is a much simpler problem than that of reconstructing 3-dimensional models of landmarks from tourist photos, where no such information is given. In the former case, the problem of reconstructing the world data is known as *triangulation*. In the latter case, the problem of reconstructing the world data requires us to first reconstruct the relative positions and orientations of the cameras; this is known as the problem of *3D image reconstruction*. Dual to the problem of triangulation is that of *resectioning*, where both the world and image data are known and the positions of the cameras must be reconstructed. In either case, we are interested in the problem of recovering the full mutli-view geometry from partial information.

All three of these problems may be either under- or over-determined from the data given, so questions of existence and uniqueness are both of interest. In Section 1.1 we review the fundamentals of mutli-view geometry necessary for these problems. In Section 1.2 we discuss the problem of triangulation and the conditions under which this problem will be under-determined. By generalizing this problem, we fully characterize the geometric conditions under which $2 \leq k \leq 9$ rank one tensors $x_i^\top \otimes y_i^\top$ will be linearly dependent. This is an overview of the content in Chapters 2 and 3. In Section 1.3 we discuss the problems of triangulation and resectioning, summarizing the content of Chapter 4. Comprehensive results for triangulation have been proven in work such as [1]; by studying the dual nature of these two problems, we are able to prove many equivalent results for the problem of resectioning.

1.1 Overview

1.1.1 Projective space

Projective space $\mathbb{P}^n$ can be described topologically as the space of all lines passing through the origin in $\mathbb{R}^{n+1}$, or equivalently, as the quotient space obtained from imposing the equivalence relation $x \sim \lambda x$, $\lambda \neq 0$ on $\mathbb{R}^{n+1} \backslash \{0\}$. Intuitively, we might instead view $\mathbb{P}^n$ as the compactification of $\mathbb{R}^n$, created by taking the existing *finite points* $(x_1, \ldots, x_n, 1)$ and adding in the *points at infinity* $(x_1, \ldots, x_n, 0)$. In projective space, formerly parallel lines $\ell_1 = vt + p_1$ and $\ell_2 = vt + p_2$, where $v, p_1, p_2 \in \mathbb{R}^n$, will intersect at the infinite point $(v, 0)$. To distinguish between $\mathbb{P}^n$ and $\mathbb{R}^{n+1}$, we will write $x = (x_1 : \ldots : x_{n+1})$ to indicate points in projective space and $x = (x_1, \ldots, x_{n+1})$ to indicate points in affine space. Complex projective space is defined similarly, using $\mathbb{C}^n$ rather than $\mathbb{R}^n$ and is denoted either $\mathbb{P}(\mathbb{C}^n)$ or simply $\mathbb{P}^n$ when contextually appropriate.

Similar to affine space, we may define linear projective maps from $\mathbb{P}^n \to \mathbb{P}^m$ using *projective matrices* of size $(m+1) \times (n+1)$. These can be represented using affine matrices of the same size, but with the understanding that they are only defined up to scale. Similarly, invertible projective transformations $\mathbb{P}^n \to \mathbb{P}^n$ are given by the elements of PGL(n); intuitively, this group is exactly GL(n) with the equivalence relation $M \sim \lambda M$, $\lambda \neq 0$. A *projective basis* or *projective frame* for $\mathbb{P}^n$ is a set of $n+2$ points such that no $n+1$ of them lie on a hyperplane. The standard example for $\mathbb{P}^2$ consists of the points $(1:0:0), (0:1:0), (0:0:1), (1:1:1)$. Given any two projective frames $\{a_i\}, \{b_i\}$, there exists a unique invertible projective transformation H such that $Ha_i \sim b_i$. In particular, if $v_4 \sim av_1 + bv_2 + cv_3$ then

$$H = \begin{bmatrix} | & | & | \\ av_1 & bv_2 & cv_3 \\ | & | & | \end{bmatrix} \tag{1.1}$$

is a change of basis from the standard basis on $\mathbb{P}^2$ to the basis v_1, v_2, v_3, v_4. We will often refer to invertible projective transformations as *homographies* for short.

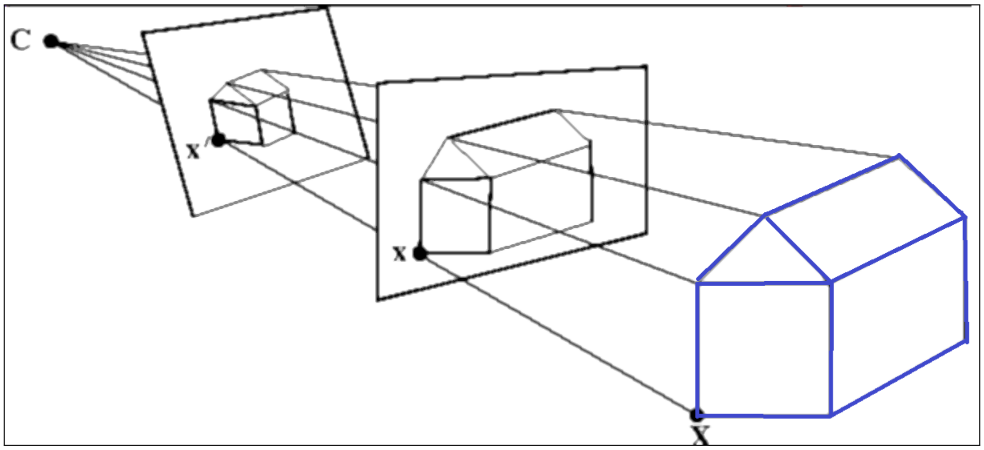

Figure 1.1: Two cameras, each with the same center, capture different images of an object in $\mathbb{P}^3$.[1]

1.1.2 Multi-view geometry

In multi-view geometry, a *camera* is a full rank linear map $A : \mathbb{P}^3 \to \mathbb{P}^2$. The domain of this map is the *world space* and the codomain is the *image plane*. Similarly, points $q \in \mathbb{P}^3$ are known as *world points* and points $p = A(q) \in \mathbb{P}^2$ are known as *image points*. The nullspace of the matrix A is the *camera center*, a single point $c \in \mathbb{P}^3$. The image plane can be represented by a single plane in $\mathbb{P}^3$ not passing through the camera center; then the image of a world point q under the camera A is the intersection of the line $\overline{cq}$ with the image plane. Similarly, the preimage of an image point p is a line passing through the camera center c. We refer to a collection of cameras $\{A_i\}_{i=1}^m$, world points $\{q_j\}_{j=1}^n$ and image points $\{p_{ij}\}_{i,j=1}^{m,n} = A_m q_n$ as a *multi-view arrangement* or *multi-view system*; we say such a system is *valid* if $A_i(q_j) = p_{ij}$ for all i, j. We are concerned with the following question:

Given partial information for a multi-view arrangement, does there exist a valid reconstruction of the full system? Further, if a valid reconstruction does exist, is that reconstruction unique?

Given only the image points, we refer to this question as the problem of 3D image reconstruction.

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[1] https://www.robots.ox.ac.uk/ vgg/hzbook/HZfigures.html

Problem 1.1.1 (3D Image Reconstruction)**.** *Given image points* $\{p_{ij}\}_{i,j=1}^{m,n}$*, does there exist a set of cameras* $A_1, \ldots, A_m$ *and world points* $q_1, \ldots, q_n$ *such that* $A_i q_j = p_{ij}$ *for all* i, j*? If so, is this set unique (up to* PGL(4) *action)?*

Similarly, starting from either the image points and the cameras, or the image points and the world points yields the problems, of triangulation and resectioning respectively.

Problem 1.1.2 (Triangulation)**.** *Given image points* $\{p_{ij}\}_{i,j=1}^{m,n}$ *and cameras* $A_1, \ldots, A_m$*, does there exist world points* $q_1, \ldots, q_n$ *such that* $A_i q_j = p_{ij}$ *for all* i, j*? If so, is this set unique?*

Problem 1.1.3 (Resectioning)**.** *Given image points* $\{p_{ij}\}_{i,j=1}^{m,n}$ *and world points* $q_1, \ldots, q_n$*, does there exist (unique) cameras* $A_1, \ldots, A_m$ *such that* $A_i q_j = p_{ij}$ *for all* i, j*? If so, is this set unique?*

We discuss Problem 1.1.1 in Chapters 2 and 3 and Problems 1.1.2 and 1.1.3 in Chapter 4.

1.2 The Rank Drop Problem

1.2.1 The Fundamental Matrix

When working in two-view geometry, the scenario in which we have exactly two cameras A_1, A_2, we will refer to our image points as $A_1 q_i = x_i$ and $A_2 q_i = y_i$ for the sake of simplicity. Given a pair of cameras A_1, A_2 and a pair of image points $(x, y) \in \mathbb{P}^2 \times \mathbb{P}^2$ such that $A_1 q = x$ and $A_2 q = y$, we can solve for the world point q by intersecting the preimages $A_1^{-1}(x) \cap A_2^{-1}(y)$. See Figure 1.2.

If we are given only the paired image point data (x_i, y_i), then our first step in 3D image reconstruction is to solve for the original cameras A_1, A_2. This can only be done up to projective equivalence, that is, up to PGL(4) action on (A_1, A_2). As a short proof, suppose we are given a pair of cameras A_1, A_2 and a set of world points $q_1, \ldots, q_n$. Then, given any homography $H \in$ PGL(4) we can replace the cameras A_i with $A_i H^{-1}$ and the world points q_i with $H q_i$; it is then clear that $A_i q_j = (A_i H^{-1}) H q_j$; thus, starting from only the image data, $(A_1 H^{-1}, A_2 H^{-1})$ would also be a valid candidate for the original cameras. This general principle, that $3D$ image reconstruction can only be done up to projective equivalence, extends to more general n-view reconstruction problems as well.

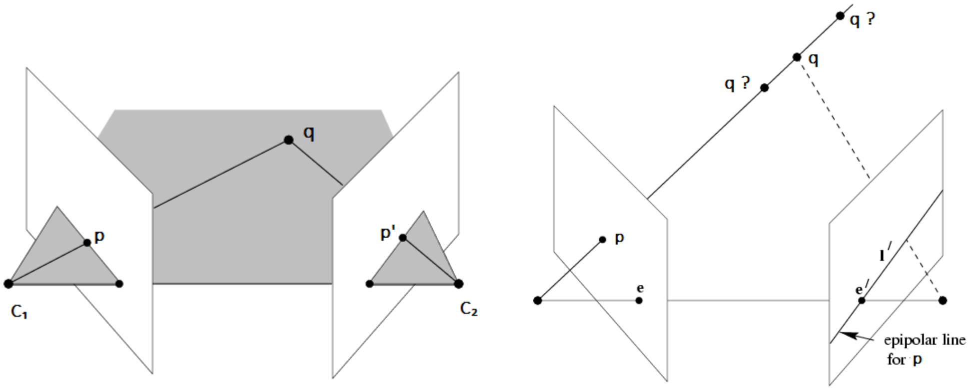

Figure 1.2: Left: A world point can be reconstructed by intersecting the preimage lines from two distinct cameras. Right: Points in one image correspond to lines in the other.[2]

The relationship between a pair of cameras is characterized via the *epipolar geometry*. Given two distinct cameras A_1, A_2 with centers c_1, c_2 we refer to the image points $e_x = A_1(c_2)$ and $e_y = A_2(c_1)$ as the *epipoles* of the cameras. We can observe that, given any other point x in the first image, the composition $A_2(A_1^{-1}(x))$ will yield a line in the second image, passing through the epipole e_y. This correspondence $x \mapsto \ell$ is given by a unique 3×3 matrix F of rank 2, which we refer to as the *fundamental matrix* of (A_1, A_2) [37]. Specifically, given a point x in the first image, the corresponding line can be defined $\ell = \{y \ : \ y^\top(Fx) = 0\}$. Similarly, given a point y in the second image, the corresponding line in the first image is defined $\ell = \{x \ : \ (y^\top F)x = 0\}$. We can observe that this implies $Fe_x = 0 = e_y^\top F$. In particular F can be defined as the unique 3×3 matrix of rank 2 satisfying the *epipolar equation*

$$y^\top Fx = 0 \qquad \text{for all } (x, y) \text{ corresponding to the same world point.} \tag{1.2}$$

Definition 1.2.1. *Given a pair of cameras A_1, A_2 with distinct centers, the fundamental matrix of A_1, A_2 is the unique 3×3 matrix F of rank 2 satisfying* (1.2).

Alternatively, we may define F explicitly by $F = A_2 A_1^\dagger$, where $A_1^\dagger$ is the pseudo-inverse of A_1 [37].

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[2] https://www.robots.ox.ac.uk/ vgg/hzbook/HZfigures.html

Conversely, given any rank 3×3 matrix F of rank 2 there exists a unique pair of cameras (A_1, A_2), up to PGL(4) action, such that F is the fundamental matrix of A_1, A_2. In computer vision, we traditionally work exclusively over real projective space, but this machinery extends naturally to complex projective space as well.

1.2.2 The 7-point Algorithm

Given 7 point pairs (x_i, y_i), each pair corresponding to the same world point q_i, we can make use of the epipolar equation to calculate the fundamental matrix in a process known as the *7-point algorithm*. First, observe that we can rewrite the equation $y_i^\top M x_i = 0$ as $(x_i^\top \otimes y_i^\top)\operatorname{vec}(M) = 0$, where $\operatorname{vec}(M)$ is the 9-dimensional vector obtained by concatenating the columns of M. Then the nullspace of the 7×9 matrix $Z = (x_i^\top \otimes y_i^\top)_{i=1}^7$ consists exactly of the matrices M satisfying (1.2) for all (x_i, y_i). If Z has 7 linearly independent rows we will be left with a 1-dimensional projective nullspace. Generically, imposing the condition $\det(M) = 0$ on this nullspace will result in three solutions, two of which may be complex.

Therefore, the first step of this algorithm is to construct the matrix $Z = (x_i^\top \otimes y_i^\top)_{i=1}^7$ and determine its nullspace Null(Z) = span(M_1, M_2). The second step is then to find the solutions to the cubic equation $\det(M_2 + \lambda M_2) = 0$. We note that this is a relaxation of the condition the fundamental matrix requires, which is rank$(M_1 + \lambda M_2) = 2$; it is possible for the 7-point algorithm to return rank 1 matrices, which will not be fundamental matrices. However, it can be shown that a rank 1 matrix is obtained as a solution if and only if we can reorder our point pairs such that $x_1, \dots, x_n$ lie on a line and $y_{n+1}, \dots, y_7$ lie on a line, for some $1 \leq n \leq 7$.

More generally, there are certain sets of point pairs, known as *degenerate configurations*, for which we will obtain either infinitely many solutions or a solution with multiplicity. To understand why this is an issue, we need to discuss the theory of *numerical conditioning*.

1.2.3 Numerical Conditioning

In applications, we use finite image data given in affine coordinates $u_i = (u_{i1}, u_{i2})$ corresponding to the pixel position of the point, which we then homogenize as $\widetilde{u}_i = (u_{i1}, u_{i2}, 1)$. Algebraically, this is equivalent to imposing the condition that none of our points $x_i, y_i \in \mathbb{P}^2$ as described above have last coordinate 0; this can always be assumed to be true after a change of coordinates.

Because of the finite resolution of real-world cameras, this recorded data will be slightly perturbed from the 'true' image data $\bar{u}_i = (\bar{u}_{i1}, \bar{u}_{i2})$. Let $\mathbf{x} = (u_i, v_i)_{i=1}^7$ and define $\bar{\mathbf{x}}$ similarly. If g is the map which sends $\mathbf{x}$ to its fundamental matrix F under the 7-point algorithm, then $d(g(\mathbf{x}), g(\bar{\mathbf{x}}))$ is the *output error* of the algorithm. We say that the *condition number* of g at $\mathbf{x}$ is

$$\kappa_g(\mathbf{x}) = \lim_{\epsilon \to 0} \sup_{d(\mathbf{x}, \bar{\mathbf{x}}) < \epsilon} \frac{d(g(\mathbf{x}), g(\bar{\mathbf{x}}))}{d(\mathbf{x}, \bar{\mathbf{x}})} \tag{1.3}$$

We say that a problem is *ill-posed* at an input $\mathbf{x}$ if the condition number is infinite; this occurs exactly when $\mathbf{x}$ is a degenerate configuration.

Moreover, it is a well-known meta-theorem of numerical analysis that the condition number at a point $\mathbf{x}$ is inversely proportional to the distance to the nearest degenerate configuration $(u_i', v_i')_{i=1}^7$ [9][19][20]. As such, describing these degenerate configurations is not only mathematically interesting, but is of practical application as well.

In particular, if Z is rank deficient, then the 7-point algorithm must be ill-posed. In chapters 2 and 3, we choose to generalize this to the problem of rank drop for any number of k point pairs, where $0 \leq k \leq 9$, rather than solving only the case where $k = 7$. Across these two chapters, we obtain complete solutions for the rank drop problem.

Problem 1.2.2. *Given k point pairs $(x_i, y_i) \in \mathbb{P}^2 \times \mathbb{P}^2$ where $2 \leq k \leq 9$, under what conditions will the matrix $Z = (x_i^\top \otimes y_i^\top)_{i=1}^k$ be rank deficient?*

The reader may observe that a basic solution to problem 1.2.2 is that Z will be rank deficient exactly when its maximal minors are all 0. However, this offers us very little insight about the point pairs themselves, so we do not consider it a good solution to the problem. Instead, we seek geometric characterizations of the point pairs $(x_i, y_i)_{i=1}^k$ which result in rank deficiency.

We can observe that acting on either image by $\mathrm{PGL}(3)$ will act on the nullspace of Z in the same manner. That is, the following two equations are equivalent:

$$(H_2 y_i)^\top H_2^{-\top} M H_1^{-1}(H_1 x_i) = 0 \quad \leftrightarrow \quad y_i^\top M x_i = 0. \tag{1.4}$$

It follows that the property of rank drop is a *projective invariant*: it is preserved under invertible projective transformations on the two images. Across Chapter 2 and Chapter 3, we use classical projective invariants to characterize the geometry of rank drop for $k \leq 7$. We also describe the geometric conditions under which rank drop occurs for all values of $2 \leq k \leq 9$.

For a fixed value of k, we say that rank drops due to an *inherited condition* if a $(k-1) \times 9$ submatrix of Z is rank-deficient. Equivalently, an inherited condition holds for a given set of k point pairs if and only if there exists a subset of $k-1$ point pairs such that the $(k-1) \times 9$ matrix Z' is rank deficient. If rank drops without an inherited condition, we say that it is an instance of *novel rank drop*. With this in mind, we present our results starting from $k = 2$ and then work our way up, considering only novel rank drop at each level.

1.2.4 Rank Drop for $k \leq 6$

This subsection summarizes the results in Chapter 2. For low values of k we are able to use a computational algebra package, specifically Macaulay2 [34], to analyze the conditions under which rank drops. We say that the *rank drop variety* for k point pairs is the subvariety of $(\mathbb{P}^2 \times \mathbb{P}^2)^k$ consisting of all $(x_i, y_i)_{i=1}^k$ such that $Z = (x_i^\top \otimes y_i^\top)_{i=1}^k$ is rank deficient. As we mentioned above, Z is rank deficient exactly when its maximal minors are all 0. Therefore, let I be the ideal generated by the maximal minors of Z. We can use Macaulay2 to decompose the radical of this ideal into its prime components

$$\sqrt{I} = P_1 \cap \ldots \cap P_n \tag{1.5}$$

each of which corresponds to an irreducible component of the rank drop variety. For low values of k, we can analyze the generators of these ideals to understand the geometric constraints under which rank drops. We now summarize our results; for complete technical detail see the corresponding theorems in Chapter 2.

Result 1.2.3 (Theorem 2.4.2)**.** *Suppose* $(x_i, y_i) \in \mathbb{P}^2 \times \mathbb{P}^2$ *and* $Z_k = (x_i^\top \otimes y_i^\top)_{i=1}^k$ *where* $2 \leq k \leq 4$. *Then:*

1. $\operatorname{rank}(Z_2) < 2$ if and only if $x_1 = x_2$ and $y_1 = y_2$.
2. $\operatorname{rank}(Z_3) < 3$ if and only if either an inherited condition holds or $x_1 = x_2 = x_3$ and y_1, y_2, y_3 are on a line, or x_1, x_2, x_3 are on a line and $y_1 = y_2 = y_3$.
3. $\operatorname{rank}(Z_4) < 4$ if and only if an inherited condition holds or one of the following is true:
   (a) $x_1 = x_2 = x_3 = x_4$ or $y_1 = y_2 = y_3 = y_4$.
   (b) The points $x_1, \ldots, x_4 \in \mathbb{P}^2_x$ are in a line and $y_1, \ldots, y_4 \in \mathbb{P}^2_y$ are in a line, and

$$(x_1, x_2; x_3, x_4) = (y_1, y_2; y_3, y_4) \tag{1.6}$$

Generically, the collinear points $x_1, \ldots, x_4$ and $y_1, \ldots, y_4$ are distinct and then (1.6) holds if and only if there is a homography $H \in \mathrm{PGL}(3)$ such that $Hx_i = y_i$ for $i = 1, \ldots, 4$.

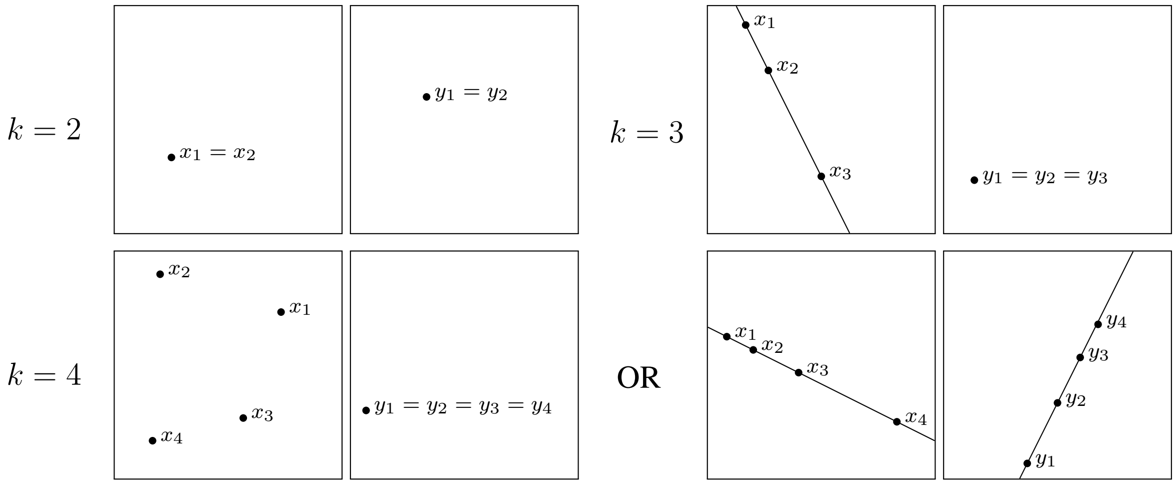

Figure 1.3: The conditions under which rank drop occurs.

After this point, Macaulay2 becomes less immediately useful due to both the rapidly rising computational complexity, which restricts our ability to obtain prime decompositions, as well as the explosion in degree and size of the polynomials which generate the prime components. However, Macaualy2 can still be used to some extent for $k = 5$ and $k = 6$. That is, we can verify that the ideal of maximal minors I is equal, up to taking the radical, to an ideal generated by certain geometrically meaningful polynomials.

Definition 1.2.4. *Given* 5 *points in* $p_1, \dots, p_5 \in \mathbb{P}^2$*, let* A *be any full rank linear projection* $\mathbb{P}^2 \to \mathbb{P}^1$ *with center* p_5*. We define the* ***planar cross-ratio*** *of the points* $p_1, \dots, p_4$ *around* p_5 *be the cross-ratio of* $Ap_1, \dots, Ap_4 \in \mathbb{P}^1$

$$(p_1, p_2; p_3, p_4; p_5) := (Ap_1, Ap_2; Ap_3, Ap_4). \tag{1.7}$$

For any other point p_5' *in the conic* ω *spanned by the* $\{p_i\}$ *we have* $(p_1, p_2; p_3, p_4; p_5) = (p_1, p_2; p_3, p_4; p_5')$*. For this reason, we define the* ***conic cross-ratio*** *of* $p_1, \dots, p_4$ *with respect to a conic* ω *as*

$$(p_1, p_2; p_3, p_4)_\omega := (p_1, p_2; p_3, p_4; c) \tag{1.8}$$

where $c \in \omega$ *is any point distinct form* $p_1, \dots, p_4$*.*

This definition allows us to obtain the following result:

Result 1.2.5 (Theorem 2.5.1, Theorem 2.5.2)**.** *Given* 5 *point pairs* $(x_i, y_i) \in \mathbb{P}^2 \times \mathbb{P}^2$ *novel rank drop occurs if and only if either all* 5 *points are collinear in both images OR one image has all* 5 *points in a line* ℓ *(say, the* x_i*'s) and the cross-ratio equations*

$$(x_i, x_j; x_k, x_r)_\omega = (y_i, y_J; y_k, y_r) \tag{1.9}$$

hold for all distinct i, j, k, r*, where* ω *is the conic passing through the* $\{x_i\}$*; by the action of permutations there are only* 5 *such equations to consider. Generically, these* 5 *equations hold if and only if there exists an isomorphism* $\omega \to \ell$ *such that* $x_i \mapsto y_i$ *for all* i*.*

For $k = 6$, the result relies on the theory of cubic surfaces and Schläfli double sixes. In particular, it is known that every cubic surface S can be obtained as the blowup of $\mathbb{P}^2$ in a set of 6 distinct points $p_1, \dots, p_6$. That is, there exists a morphism $\pi : S \to \mathbb{P}^2$ which is $1:1$ everywhere except on 6 exceptional lines $\ell_1, \dots, \ell_6 \mapsto p_1, \dots, p_6$, which are necessarily mutually skew.

Definition 1.2.6. *Given a set of* 12 *lines* $\{\ell_i\}_{i=1}^6, \{\ell^j\}_{j=1}^6$*, we say that they form a* ***Schläfli double six*** *if the* $\{\ell_i\}$ *are mutually skew, the* $\{\ell^j\}$ *are mutually skew, and* $\ell_i \cap \ell^j \neq \emptyset$ *if and only if* $i \neq j$*.*

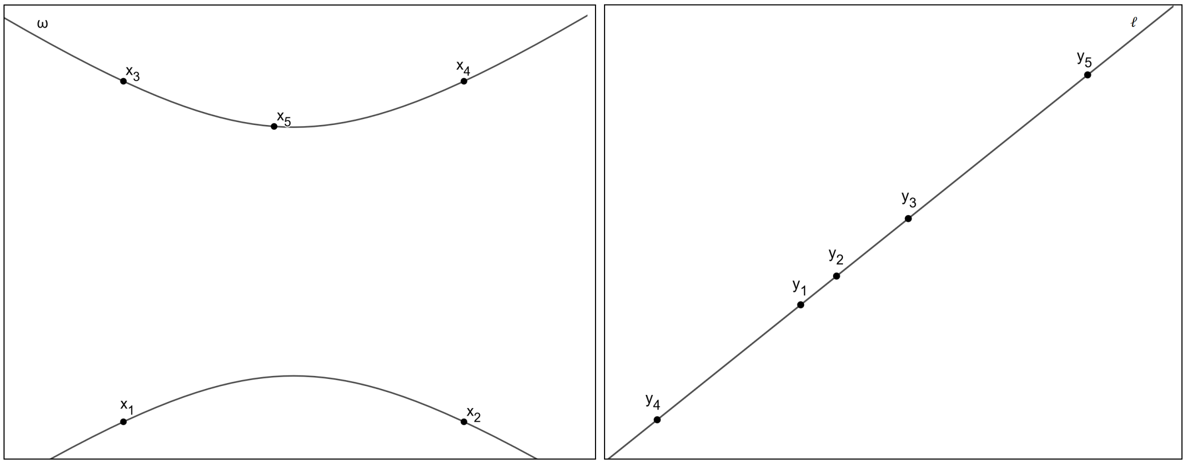

Figure 1.4: Rank drops exactly when there exists a conic-line isomorphism taking points in the first image to the corresponding points in the other.

It is a fact of cubic surfaces that any set of six mutually skew lines $\{\ell_i\}_{i=1}^6$ define a unique set of 6 other mutually skew lines $\{\ell^j\}_{j=1}^6$ such that they form a *Schläfli double six*. We can use this fact to obtain the following geometric result:

Result 1.2.7 (Theorem 2.6.9)**.** *Given* 6 *point pairs* $(x_i, y_i) \in \mathbb{P}^2 \times \mathbb{P}^2$*, novel rank drop occurs if and only if there exists a cubic surface* S *which can be expressed as the blowup of both* $\mathbb{P}^2$ *in* $\{x_i\}$ *and* $\mathbb{P}^2$ *in* $\{y_i\}$ *such that the corresponding exceptional lines form a Schläfli double six.*

We also prove the equivalence of a more algebraic characterization.

Result 1.2.8 (Theorem 2.6.4)**.** *Given* 6 *point pairs* $(x_i, y_i) \in \mathbb{P}^2 \times \mathbb{P}^2$*, let* ω_x^j, ω_y^j *refer to the conics passing through* $\{x_i\}_{i \neq j}, \{y_i\}_{i \neq j}$ *respectively. Then novel rank drop occurs if and only if the following cross-ratio equations*

$$(x_i, x_j; x_k, x_r)_{\omega_x^s} = (y_i, y_j; y_k, y_r)_{\omega_y^t} \tag{1.10}$$

hold for all distinct i, j, k, r, s, t*. Excluding permutations of the first four indices, this yields* 30 *distinct equations, each corresponding to one of the* 30 *intersections of the Schläfli double six.*

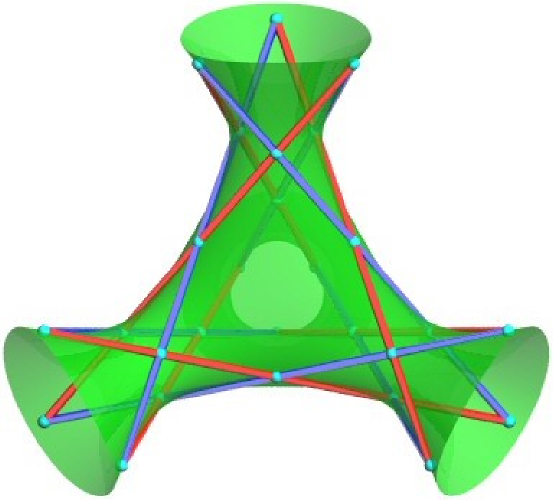

Figure 1.5: A cubic surface with the Schläfli double six highlighted. Rank drop occurs exactly when there exists a pair of blowup morphisms $\pi_x, \pi_y : S \to \mathbb{P}^2$ such that $\pi_x(\ell_i) = x_i$ and $\pi_y(\ell^j) = y_j$[3]

1.2.5 Rank Drop for $k \geq 7$: Lines, Quadrics, and Cremona Transformations

There are now only three values of k left to discuss. However, the solution for $k = 9$ is largely trivial: Rank drop occurs if and only if there exists a 3×3 matrix M satisfying the epipolar equation $y_i^\top M x_i = 0 \; i = 1, \ldots, 9$. As such, the only two interesting cases remaining are $k = 7$ and $k = 8$. Both of these cases are explained through a special relationship that we have termed *the trinity correspondence*: a three-way correspondence between lines in the nullspace of Z, quadric surfaces passing through potential 3D reconstructions, and *quadratic Cremona transformations* taking $x_i \mapsto y_i$.

A quadratic Cremona transformation is birational automorphism of $\mathbb{P}^2$ defined using quadratic forms, such as $(u_1, u_2, u_3) \mapsto (u_2u_3, u_1u_3, u_1u_2)$. Such a transformation f is undefined at 3 distinct, non-collinear points e_1, e_2, e_3 in the domain, known as the *base points* of f. The exceptional lines $\overline{e_{ij}}$ are the three lines in the domain on which f is not $1:1$; their images $f(\overline{e_{ij}}) = e'_k$ are exactly the base points of f^{-1}. We can define f geometrically as the transformation obtained by blowing up $\mathbb{P}^2$ in the points e_1, e_2, e_3, yielding a degree six surface S, and then blowing down the three exceptional

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[3] https://mathcurve.com/surfaces.gb/clebsch/doublesix.shtml

lines to points; composing the blowup morphisms in the reverse order yields f^{-1}. In light of this symmetry, we say that e_1, e_2, e_3 are the base points in the domain of f and e'_1, e'_2, e'_3 are the base points in the codomain of f.

Result 1.2.9 (Theorem 3.3.16; The Trinity Correspondence)**.** *Given a pointset $P = (x_i, y_i)_{i=1}^k \in \mathbb{P}^2 \times \mathbb{P}^2$, we define the following three sets:*

- *$\mathcal{L}_P$: the set of all lines in Null(Z).*
- *$\mathcal{Q}_P$: the set, up to projective equivalence, of all quadrics $Q \subset \mathbb{P}^3$ passing through a reconstruction $p_1, \ldots, p_k, c_1, c_2$ of P, consisting of both world points and camera centers.*
- *$\mathcal{C}_P$: the set of all quadratic Cremona transformations $\mathbb{P}^2 \to \mathbb{P}^2$ such that $x_i \mapsto y_i$ $\forall i$.*

Then there is a $1:1$ correspondence between the elements of $\mathcal{L}_P$ and $\mathcal{C}_P$, a $1:3$ correspondence between the elements of $\mathcal{L}_P$ and $\mathcal{Q}_P$, and a $3:1$ correspondence between the elements of $\mathcal{Q}_P$ and $\mathcal{C}_P$ as in:

$$\begin{tikzcd}
& \mathcal{Q}_P \arrow[dr, "3:1"] & \\
\mathcal{L}_P \arrow[ur, "1:3"] \arrow[rr, leftrightarrow] & & \mathcal{C}_P
\end{tikzcd} \tag{1.11}$$

Moreover, the three base point pairs of a Cremona transformation f are exactly the three epipole pairs of the corresponding $3D$ reconstructions under this relationship.

Remark 1.2.10. When we define the sets as above, each correspondence is only defined generically. In the full version, Theorem 3.3.16, we take slightly more narrow definitions of the sets $\mathcal{L}_P$, $\mathcal{Q}_P$, and $\mathcal{C}_P$ so that the maps will be defined everywhere; this approach requires more terminology than is practical for this introduction.

Given this result, the problem of characterizing $k = 8$ rank drop becomes rather simple. The 8×9 matrix Z will be rank deficient exactly when its nullspace contains a projective line. The trinity correspondence tells us that this will occur exactly when there is a quadratic Cremona transformation $\mathbb{P}^2 \to \mathbb{P}^2$ such that $x_i \mapsto y_i$ for all i.

Result 1.2.11. *For $k = 8$, rank drop occurs exactly when there exists a quadratic Cremona transformation $f : \mathbb{P}^2 \to \mathbb{P}^2$ such that $x_i \mapsto y_i$ for all i.*

Similarly, for $k = 7$ the 7×9 matrix Z will be rank deficient exactly when its nullspace contains a projective plane, a 2-parameter family of lines. We would therefore expect that rank drop occurs exactly when there exists a 2-parameter family of quadratic Cremona transformations $\mathbb{P}^2 \to \mathbb{P}^2$ such that $x_i \mapsto y_i$. This is in fact the case; however, by utilizing the $\mathcal{Q}_{\mathcal{P}}$ legs of the diagram (1.11) we are able to obtain an even stronger result. In order to state this result, we need to introduce some preliminary statements. In particular, we need to introduce the *Cremona hexahedral invariants* for 6 points in $\mathbb{P}^2$. Given a set of 6 ordered points $p_i \in \mathbb{P}^2$, we can define 6 cubics $a_p(u), \dots, f_p(u)$ and six scalars $\bar{a}_p, \dots, \bar{f}_p$. These invariants have a number of important properties relating to the blowup of $\mathbb{P}^2$ in 6 points, which are discussed in more detail in Chapter 2. However, they also have a close connection to rank drop in the case for $k = 7$.

Result 1.2.12 (Lemma 3.4.3, Lemma 3.4.7)**.** *Given any 6 generic point pairs $(x_i, y_i) \in \mathbb{P}^2 \times \mathbb{P}^2$, there exists unique cubic curves C_1, C_2 such that there exists as an isomorphism $C_1 \to C_2$ sending $x_i \mapsto y_i$. This isomorphism extends to a 2-parameter family of Cremona transformations $\mathbb{P}^2 \to \mathbb{P}^2$. Moreover, the curves C_1, C_2 contain exactly the base points of all members of this family, in addition to the $\{x_i\}$ and $\{y_i\}$ respectively. Furthermore, we can obtain explicit equations for these curves using the Cremona hexahedral invariants: if*

$$\begin{aligned} g_x(u) &:= \bar{a}_y a_x(u) + \bar{b}_y b_x(u) + \bar{c}_y c_x(u) + \bar{d}_y d_x(u) + \bar{e}_y e_x(u) + \bar{f}_y f_x(u), \\ g_y(v) &:= \bar{a}_x a_y(v) + \bar{b}_x b_y(v) + \bar{c}_x c_y(v) + \bar{d}_x d_y(v) + \bar{e}_x e_y(v) + \bar{f}_x f_y(v), \end{aligned} \tag{1.12}$$

then C_1 is the curve cut out by $g_x(u)$ and C_2 is the curve cut out by $g_y(v)$.

Given 7 point pairs $(x_i, y_i)_{i=1}^7$, we know that these cubic curves exist for any subset of 6 point pairs; rank drop occurs when the curves, and the isomorphism between them, exist for all 7 point pairs together.

Result 1.2.13 (Theorem 3.4.1)**.** *Given 7 point pairs $(x_i, y_i) \in \mathbb{P}^2 \times \mathbb{P}^2$, rank drop occurs exactly when there exist cubic curves C_1 through $\{x_i\}_{i=1}^7$ and C_2 through $\{y_i\}_{i=1}^7$ as well as an isomorphism*

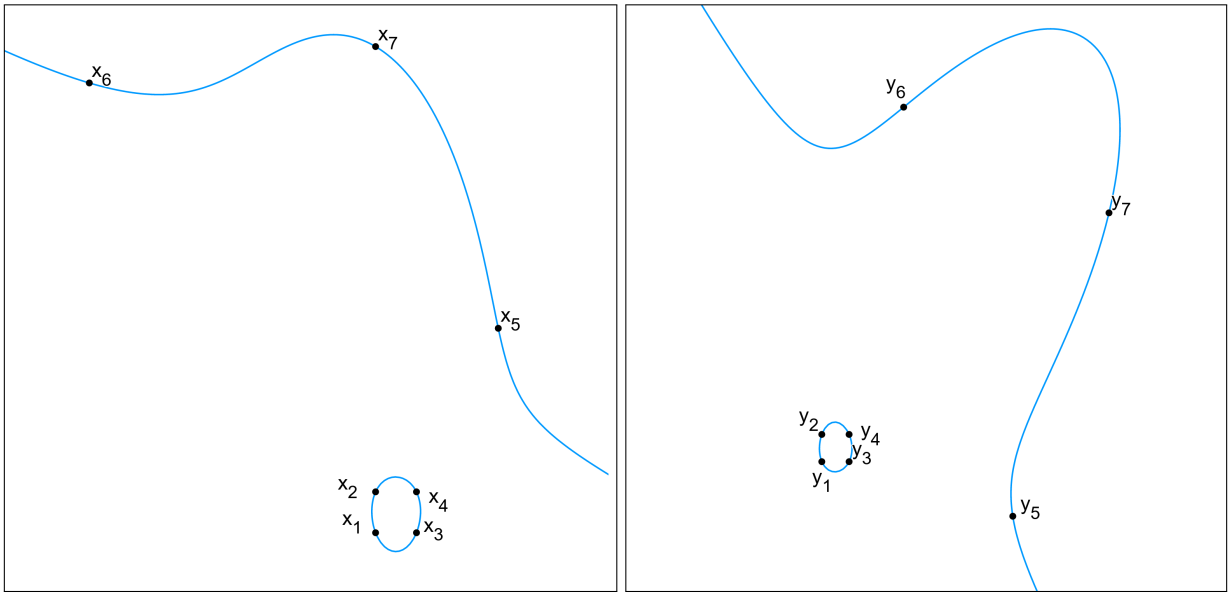

Figure 1.6: The cubic curves C_1 and C_2

$C_1 \to C_2$ such that $x_i \mapsto y_i$ for all i. Moreover, when this does hold the isomorphism extends to a 2-parameter family of quadratic Cremona transformations $\mathbb{P}^2 \to \mathbb{P}^2$.

The trinity can be applied to the case of $k = 6$ as well. We can see from (1.11) that rank will drop if and only if the original six world points and two cameras lie on three distinct quadrics. The intersection of 3 generic quadrics is known as a Cayley Octad, a geometric object studied in papers such as [57] consisting of exactly 8 points in $\mathbb{P}^3$ with special properties relating to the 28 bitangents of a quartic plane curve and the 27 lines on a cubic surface. As such, the application of the trinity can be seen as an alternate path towards proving Result 1.2.7. However, the trinity cannot be as easily applied to the cases where $k < 6$, due to the significant restrictions on the pointset geometry, such as the requirement that at least one side is entirely contained in a line.

To see this issue in more detail, we note that the trinity can be used to answer the question: Given $k-1$ generic point pairs $(x_i, y_i)_{i=1}^{k-1} \in \mathbb{P}^2 \times \mathbb{P}^2$, under what conditions will the addition of a kth point pair $(x_k, y_k) \in \mathbb{P}^2 \times \mathbb{P}^2$ result in rank drop? Beginning from a reconstruction, the world point q_k which gives rise to the pair of image points must lie on: a quadric surface for $k = 8$, the intersection of 2 quadric surfaces (a quartic space curve) for $k = 7$, or a certain Cayley Octad for $k = 6$. As k decreases, the dimension of this solution space in $\mathbb{P}^3$ decreases as well; for $k \leq 5$ it will cease to exist. This reflects the fact that given $k-1$ generic point pairs $(x_i, y_i)_{i=1}^{k-1} \in \mathbb{P}^2 \times \mathbb{P}^2$, where $k \leq 5$, there does not exist a pair $(x_k, y_k) \in \mathbb{P}^2 \times \mathbb{P}^2$ which will result in rank drop.

1.3 Algebra and Geometry of Camera Resectioning

Similar to 3D image reconstruction (1.1.1) are the problems of triangulation (1.1.2) and resectioning (1.1.3). As laid out in [1], we can consider the set of all valid arrangements of cameras $\{A_i\}_{i=1}^m$, world points $\{q_i\}_{i=1}^n$, and image points $\{p_{ij}\}_{i,j=1}^{m,n}$ as a variety by defining the *image formation correspondence*

$$\Gamma^{m,n}_{\mathbf{A},\mathbf{q},\mathbf{p}} = \overline{\{(\mathbf{A}, \mathbf{q}, \mathbf{p}) \in (\mathbb{P}^{11})^m \times (\mathbb{P}^3)^n \times (\mathbb{P}^2)^{mn} \mid A_i q_j \sim p_{ij} \quad \forall i \in [m],\, j \in [n]\}}. \tag{1.13}$$

We can then obtain new varieties from $\Gamma^{m,n}_{\mathbf{A},\mathbf{q},\mathbf{p}}$ via specialization and projection. The diagram of all such varieties, illustrated in Figure 1.7, is known as the *atlas* and was studied extensively in [1]. Two varieties of note are the *multiview variety* and the *resectioning variety*, used for the problems of triangulation and resectioning respectively. Given a generic camera arrangement $\bar{\mathbf{A}} = (\bar{A}_1, \ldots, \bar{A}_m) \in (\mathbb{P}^{11})^m$ we may define the multiview variety as

$$\Gamma^{m,n}_{\bar{\mathbf{A}},\mathbf{p}} = \{\mathbf{p} \in (\mathbb{P}^2)^{mn} \mid (\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}) \in \Gamma^{m,n}_{\mathbf{A},\mathbf{q},\mathbf{p}} \text{ for some } \mathbf{q} \in \left(\mathbb{P}^3\right)^n\}. \tag{1.14}$$

Similarly, given a generic arrangement of world points $\bar{\mathbf{q}} \in (\mathbb{P}^3)^n$ we may define the resectioning variety as

$$\Gamma^{m,n}_{\bar{\mathbf{q}},\mathbf{p}} = \left\{\mathbf{p} \in \left(\mathbb{P}^2\right)^{mn} \mid (\mathbf{A}, \bar{\mathbf{q}}, \mathbf{p}) \in \Gamma^{m,n}_{\mathbf{A},\mathbf{q},\mathbf{p}} \text{ for some } \mathbf{A} \in \left(\mathbb{P}^{11}\right)^m\right\}. \tag{1.15}$$

The multiview variety was the main object of study in [1]. In particular, the authors obtain a universal Gröbner basis for its vanishing ideal. They do so by characterizing the existence of points q_j satisfying $A_i q_j = p_{ij}$ via the rank deficiency of a certain matrix. Consider a generic point

$$(\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}) = (\bar{A}_1, \ldots, \bar{A}_m, q_1, \ldots, q_n, p_{11}, \ldots, p_{mn}) \in \Gamma^{m,n}_{\mathbf{A},\mathbf{q},\mathbf{p}}.$$

$$\begin{tikzcd}
 & \Gamma^{m,n}_{\bar{\mathbf{A}},\mathbf{q},\mathbf{p}} \arrow[dl] & \Gamma^{m,n}_{\mathbf{A},\mathbf{q},\bar{\mathbf{p}}} \arrow[dl] \arrow[dr] & \Gamma^{m,n}_{\mathbf{A},\bar{\mathbf{q}},\mathbf{p}} \arrow[dr] & \\
\Gamma^{m,n}_{\bar{\mathbf{A}},\mathbf{p}} & \Gamma^{m,n}_{\mathbf{A},\bar{\mathbf{p}}} & \Gamma^{m,n}_{\mathbf{A},\mathbf{q},\mathbf{p}} \arrow[ul, red] \arrow[u, red] \arrow[ur, red] \arrow[dl] \arrow[dr] & \Gamma^{m,n}_{\mathbf{q},\bar{\mathbf{p}}} & \Gamma^{m,n}_{\bar{\mathbf{q}},\mathbf{p}} \\
 & \Gamma^{m,n}_{\mathbf{A},\mathbf{p}} \arrow[ul, red] \arrow[u, red] \arrow[dr] & & \Gamma^{m,n}_{\mathbf{q},\mathbf{p}} \arrow[u, red] \arrow[ur, red] \arrow[dl] & \\
 & & \Gamma^{m,n}_{\mathbf{p}} & &
\end{tikzcd}$$

Figure 1.7: [1] The full atlas for the pinhole camera. The relationships between each variety are defined via specialization, in red, and projection.

It follows that there must exist nonzero scalars $\lambda_{11}, \dots, \lambda_{mn}$ such that $\bar{A}_i q_j = \lambda_{ij} p_{ij}$ for all i, j. In particular, for each $j = 1, \dots, n$ and any subset $\sigma = \{\sigma_1, \dots, \sigma_k\} \subset [m]$ of size ≥ 2, the matrix

$$\begin{bmatrix} \bar{A}_{\sigma_1} & p_{\sigma_1 j} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{A}_{\sigma_k} & 0 & \cdots & p_{\sigma_k j} \end{bmatrix} \tag{1.16}$$

must be rank-deficient. We say that the maximal $(4+k) \times (4+k)$ minors of these matrices are the *k-focals* associated with the camera arrangement $\bar{\mathbf{A}}$. Intuitively, we might expect the set of all k-focals for $2 \leq k \leq m$ would generate the vanishing ideal of the multiview variety. However, the authors of [1] are able to obtain a much stronger result: that the $2-, 3-$, and $4-$focals form a universal Gröbner basis for the multiview variety.

Our goal was to obtain similar results for the problem of resectioning. To adapt the notion of k-focals result to this context, we transform the world points $\bar{q}_j$ into 'hypercameras' $\bar{Q}_j : \mathbb{P}^{11} \to \mathbb{P}^2$ by defining

$$\bar{Q}_j := I_{3\times 3} \otimes \bar{q}_j^\top = \begin{bmatrix} \bar{q}_j^\top & 0 & 0 \\ 0 & \bar{q}_j^\top & 0 \\ 0 & 0 & \bar{q}_j^\top \end{bmatrix}. \tag{1.17}$$

As with the multiview variety, we find that if $(\mathbf{A}, \bar{\mathbf{q}}, \mathbf{p}) \in \Gamma_{\bar{\mathbf{A}},\mathbf{q},\mathbf{p}}$ then there exists λ_{ij} such that

$$\begin{bmatrix} \bar{Q}_1 & p_{i1} & \cdots & 0 \\ \vdots & \ddots & \vdots & \\ \bar{Q}_n & 0 & \cdots & p_{in} \end{bmatrix} \begin{bmatrix} \mathrm{vec}(A_i^\top) \\ -\lambda_{i1} \\ \vdots \\ -\lambda_{in} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{1.18}$$

In particular, for each $i = 1, \ldots, m$ and any subset $\sigma = \{\sigma_1, \ldots, \sigma_k\} \subset [n]$ of size ≥ 6, the matrix

$$\begin{bmatrix} \bar{Q}_{\sigma_1} & p_{i\sigma_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{Q}_{\sigma_k} & 0 & \cdots & p_{i\sigma_k} \end{bmatrix} \tag{1.19}$$

must be rank deficient. We refer to the maximal $(12 + k) \times (12 + k)$ minors of these matrices as the *k-focals* associated with the world point arrangement $\bar{\mathbf{q}}$. Letting $I_m(\bar{\mathbf{q}})$ be the ideal generated by $k = 6, \ldots, m$ focals, we are able to prove the following result:

Result 1.3.1. *Let $m, n \geq 1$ be integers. For any point arrangement $\bar{\mathbf{q}} \in (\mathbb{P}^3)^n$ such that no four points are coplanar, we have*

$$I(\Gamma^{m,n}_{\bar{\mathbf{q}},\mathbf{p}}) = I_m(\bar{\mathbf{q}}),$$

and the set of all k-focals for $6 \leq k \leq 12$ forms a universal Gröbner basis for this ideal.

Because of the specialized structure of our particular hypercameras $\bar{Q}_j$, we cannot directly adapt the methods and results of [1] to our problem. Instead, we start by considering generic 3×12 projective matrices B_j, without any assumed structure. These act as generic 'hypercameras' $\mathbb{P}^1 1 \to \mathbb{P}^2$ and it is relatively straightforward to adapt the results of [1] to this context. We then specialize these matrices in stages, proving that the desired properties carry over at each stage of the specialization, until we obtain matrices Q_j of the desired form.

The exchange of cameras and world points in the definition of the k-focals for resectioning is inspired by *Carlsson-Weinshall duality*, a framework for exchanging the roles of world points $q \in \mathbb{P}^3$ and cameras $A : \mathbb{P}^3 \to \mathbb{P}^2$ [64][37, Chapter 20.1]. We use this duality to study the relationship between the triangulation and resectioning varieties.

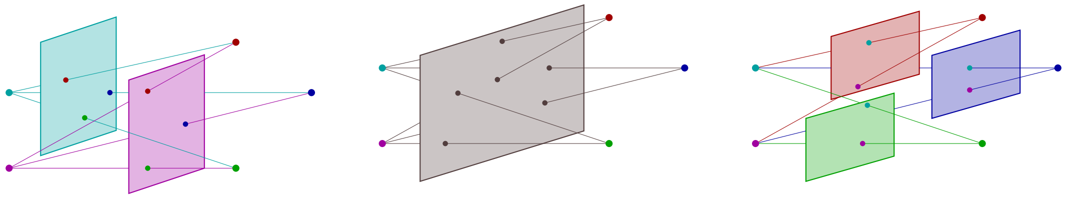

Figure 1.8: Two cameras viewing three points (left) are Carlsson-Weinshall dual to three cameras viewing two points (right).

Given a collection of cameras $\{A_i\}$ and world points $\{q_i\}$, we may assume up to a change of coordinates in the world space $\mathbb{P}^3$ and in the image planes $(\mathbb{P}^2)^m$ that the cameras are of the form

$$A_i = A(a_i) = \begin{bmatrix} a_{i1} & 0 & 0 & a_{i4} \\ 0 & a_{i2} & 0 & a_{i4} \\ 0 & 0 & a_{i3} & a_{i4} \end{bmatrix}. \tag{1.20}$$

Carlsson-Weinshall duality stems from the observation that $A(a_i)q_j = A(q_j)a_i$, allowing us to essentially swap the roles of cameras and world points. More formally, we can assume up to change of coordinates in $\mathbb{P}^3 \times (\mathbb{P}^2)^m$ that our cameras are of the form $A_i = A(a_i)$. We then say that the Carlsson-Weinshall dual of an arrangement $(A(a_1), \ldots, A(a_m), q_1, \ldots, q_n, p_{11}, p_{12}, \ldots, p_{mn})$ with m cameras and n world points is the arrangement $(A(q_1), \ldots, A(q_m), a_1, \ldots, a_n, p_{11}, p_{21}, \ldots, p_{mn})$ with n cameras and m world points. This exchange is illustrated geometrically in Figure 1.8.

By constructing a coordinate-free view of the atlas, we can apply Carlson-Weinshall duality more directly. In particular, we are interested in the *reduced image formation correspondence*

$$\mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}} = \overline{\{(\mathbf{a}, \mathbf{q}, \mathbf{p}) \in (\mathbb{P}^3)^m \times (\mathbb{P}^3)^n \times (\mathbb{P}^2)^{mn} \mid A(a_i) \cdot q_j \sim p_{ij} \quad \forall i \in [m],\, j \in [n]\}}, \tag{1.21}$$

which we show is reduced in the sense that

$$\Gamma^{m,n+4}_{\mathbf{A},\mathbf{q},\mathbf{p}} / \mathcal{G}_m \cong_{\mathbf{Bir}} \mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}}. \tag{1.22}$$

where $\mathcal{G}_m = (\text{PGL}_3)^m \times \text{Stab}_{\text{PGL}_4}(1,1,1,1)$. This quotient can be seen as a change of coordinates in the world space and the image planes which fixes $q_1, \ldots, q_4$ as the first 4 standard basis vectors in $\mathbb{P}^3$ and fixes $p_{i1} \ldots, p_{i4}$ as the 4 standard basis vectors in $\mathbb{P}^2$ for each i; this is discussed in more detail in Theorem 4.3.4. See the center image of Figure 1.8 for a visual representation of how this change of coordinates lines up the image planes. From the reduced image formation correspondence we can define the *reduced atlas* via specialization and projection, as before in 1.7. By applying Carlson-Weinshall duality in this setting, we are able to prove that $\mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}} \cong \mathrm{P}^{n,m}_{\mathbf{a},\mathbf{q},\mathbf{p}}$, formalizing the interchangeability of cameras and world points.

Finally, we find that by applying these coordinate changes to our resectioning k-focals we can transform them into well-known multi-view objects discovered by Carlsson and Weinshall. In particular, the 6-focals yield the dual fundamental matrix and the 7- and 8-focals yield the dual trifocal and quadrifocal tensors. This yields a nice symmetry with the case of triangulation, where the 2-, 3-, and $4-$focals yield the fundamental matrix and the trifocal and quadrifocal tensors respectively.

Chapter 2

RANK DROP FOR $K \leq 6$

This chapter is the content of [14], written with Sameer Agarwal, Alperen Ergur, and Rekha Thomas.

2.1 Introduction

Let $\otimes$ denote the Kronecker product [51]. We are interested in solving the following problem:

Problem 2.1.1. *Given k points* $(x_i, y_i) \in \mathbb{R}^3 \times \mathbb{R}^3$, $k \leq 9$, *consider the* $k \times 9$ *matrix* Z_k *whose rows are* $x_i^\top \otimes y_i^\top$ *for* $i = 1, \ldots, k$, *i.e.,*

$$Z_k = \begin{bmatrix} x_1^\top \otimes y_1^\top \\ \vdots \\ x_k^\top \otimes y_k^\top \end{bmatrix}$$

Delineate the geometry of point configurations $\{x_i\}$ *and* $\{y_i\}$ *for which* $\operatorname{rank}(Z_k) < k$.

In signal processing, the matrix Z_k is known as the *face-splitting product* [63] of the two matrices $X_k \in \mathbb{R}^{k\times 3}$ with rows $x_1^\top, \ldots, x_k^\top$ and $Y_k \in \mathbb{R}^{k\times 3}$ with rows $y_1^\top, \ldots, y_k^\top$. The face-splitting product is a special case of the *Khatri-Rao matrix product* in linear algebra and statistics [44]. However, our interest Problem 2.1.1 comes from a pair of estimation problems in 3-d computer vision known as the 7 and 5 point problems respectively [37, 46, 55].

Problem 2.1.2 (7-point problem)**.** *Given 7 points* $(\hat{x}_i, \hat{y}_i) \in \mathbb{R}^2 \times \mathbb{R}^2$ *find a matrix* $F \in \mathbb{R}^{3\times 3}$ *such that*

$$y_i^\top F x_i = 0, \ \forall i = 1, \ldots, 7 \ \textit{and} \ \det(F) = 0. \tag{2.1}$$

where $x_i = \begin{bmatrix} \hat{x}_i \\ 1 \end{bmatrix}$ *and* $y_i = \begin{bmatrix} \hat{y}_i \\ 1 \end{bmatrix}$.

Problem 2.1.3 (5-point problem)**.** *Given 5 points* $(\hat{x}_i, \hat{y}_i) \in \mathbb{R}^2 \times \mathbb{R}^2$ *find a matrix* $E \in \mathbb{R}^{3\times 3}$ *such that*

$$y_i^\top E x_i = 0,\ \forall i = 1, \dots, 5 \ \textit{ and } \ 2EE^\top E - \operatorname{Tr}(EE^\top)E = 0. \tag{2.2}$$

where $x_i = \begin{bmatrix} \hat{x}_i \\ 1 \end{bmatrix}$ *and* $y_i = \begin{bmatrix} \hat{y}_i \\ 1 \end{bmatrix}$.

A solution F to the 7 point algorithm that has rank 2 is known as a *fundamental matrix*, and a solution E to the 5 point algorithm is known as an *essential matrix* [37]. Observe that in both problems, the quantity of interest is a 3×3 matrix M that satisfies equations of the form

$$y_i^\top M x_i = 0,\ \forall i = 1, \dots, k,$$

which can be re-written as [1]

$$(x_i^\top \otimes y_i^\top)\operatorname{vec}(M) = 0,\ \forall i = 1, \dots, k$$

where $\operatorname{vec}(M)$ is the 9-dimensional vector obtained by concatenating the columns of M. Thus the 7 point algorithm involves computing the intersection of the null space of a 7×9 face splitting product with a determinantal variety. Similarly, the 5 point algorithm involves finding the intersection of the null space of a 5×9 face splitting product with the so called *essential variety* [18].

We wish to understand the point configurations $\{x_i\}$ and $\{y_i\}$ for which Problem 2.1.2 and Problem 2.1.3 are ill-posed. In both cases, it can be shown that a sufficient condition for ill-posedness is that the corresponding face-splitting product matrix be rank deficient and thus our interest in Problem 2.1.1

A first answer to Problem 2.1.1 is that $\operatorname{rank}(Z_k) < k$ if and only if all the maximal minors of Z_k are zero. These minors are bi-homogeneous polynomials in x_i and y_i, and typically do not shed much light on the geometry of the point configurations $\{x_i\}$ and $\{y_i\}$ that cause Z_k to drop rank. Our goal is to characterize the rank deficiency of Z_k in terms of the geometry of the x and y points.

---

[1] Here $\operatorname{vec}(\cdot)$ is the so called *vectorization* operator that concatenates the columns of a matrix into a vector.

To this end, we begin by observing that the rank of Z_k is a projective invariant. First, note that multiplying the x_i's and y_i's by non-zero scalars scales the rows of Z_k and does not change its rank. Second, if H_1, H_2 are invertible matrices, then from the mixed-product property of Kronecker products we have that:

$$Z'_k = \begin{bmatrix} (H_1x_1)^\top \otimes (H_2y_1)^\top \\ \vdots \\ (H_1x_k)^\top \otimes (H_2y_k)^\top \end{bmatrix} = \begin{bmatrix} (x_1^\top \otimes y_1^\top)(H_1^\top \otimes H_2^\top) \\ \vdots \\ (x_k^\top \otimes y_k^\top)(H_1^\top \otimes H_2^\top) \end{bmatrix} = Z_k(H_1^\top \otimes H_2^\top). \tag{2.3}$$

The matrix $H_1^\top \otimes H_2^\top$ has full rank since H_1 and H_2 are invertible, and hence, $\operatorname{rank}(Z'_k) = \operatorname{rank}(Z_k)$. These observations imply that $\operatorname{rank}(Z_k)$ is independent of scaling and choice of coordinates in $\mathbb{R}^3 \times \mathbb{R}^3$. Therefore, we should be studying the problem over projective space. While our computer vision applications care about real input, mathematically there is no loss in assuming that the input points (x_i, y_i) are complex because $\mathbb{R} \subset \mathbb{C}$. Therefore, letting $\mathbb{P}$ denote projective space over $\mathbb{C}$ we pass to the following formulation of Problem 2.1.1 which will be the main subject of study in this paper.

Problem 2.1.4. *Given k points $(x_i, y_i) \in \mathbb{P}^2 \times \mathbb{P}^2$, $k \leq 9$, consider the $k \times 9$ matrix Z_k whose rows are $x_i^\top \otimes y_i^\top$ for $i = 1, \dots, k$, i.e.,*

$$Z_k = \begin{bmatrix} x_1^\top \otimes y_1^\top \\ \vdots \\ x_k^\top \otimes y_k^\top \end{bmatrix}.$$

Delineate the geometry of point configurations $\{x_i\}$ and $\{y_i\}$ for which $\operatorname{rank}(Z_k) < k$.

Let $Z_k\{l\}$ denote a submatrix of Z_k with l of the k rows. If $\operatorname{rank}(Z_k\{l\}) < l$, then $\operatorname{rank}(Z_k) < k$, and any condition on the l points (x_i, y_i) that contribute the rows of $Z_k\{l\}$ and cause $Z_k\{l\}$ to drop rank will be a condition on $\{(x_i, y_i), i = 1, \dots, k\}$ such that Z_k drops rank. We will call such a condition an *inherited condition* for rank drop. For each value of k we will be most interested in the *non-inherited conditions* on the input that make Z_k rank deficient.

As an illustration of the type of answers we will provide, consider the version of Problem 2.1.4 in $\mathbb{P}^1 \times \mathbb{P}^1$. Suppose $\{x_i\}$ and $\{y_i\}$ are each sets of distinct points in $\mathbb{P}^1$. In Theorem 2.4.1 we prove that the 4×4 matrix Z_4 is rank deficient if and only if the cross-ratio of x_1, x_2, x_3, x_4 equals that of y_1, y_2, y_3, y_4. The cross-ratio is the fundamental projective invariant of four points in $\mathbb{P}^1$.

It is a classical fact that the geometry of a finite set of points in $\mathbb{P}^2$, that remains invariant under the action of PGL(3), can be expressed via polynomials in the 3×3 determinants (brackets) of the matrices whose rows are the points. Many of our answers to Problem 2.1.4 will be phrased in terms of these bracket expressions. The results have analogs in terms of cross-ratios when the points are in general position.

Even though the motivation for this paper are the 5 and 7-point problems, we will only address the cases $2 \leq k \leq 6$. This is because a great deal of classical geometry emerges for $k = 6$, and we expect a similarly rich story for the cases $k \geq 7$. In the interest of keeping the paper at a reasonable length we will address the cases $7 \leq k \leq 9$ in future work.

Summary of results and organization of the paper.

In Section 2.2 we show that one can make some simplifying assumptions on the input to Problem 2.1.4 without any loss of generality. While these assumptions are not needed for the theory, they are helpful and sometimes critical for computations. In Section 2.3 we review the tools from invariant theory that are needed in this paper.

In Section 2.4 we present our results for $k \leq 4$. **Theorem 2.4.2** solves Problem 2.1.4 for $k = 2, 3, 4$. Section 2.5 considers $k = 5$ and presents two theorems. **Theorem 2.5.1** characterizes the new conditions for rank drop in Z_5 in terms of bracket equations (2.24). A consequence is that when $k = 5$, a necessary condition for rank drop is that one set of points (say the y_i's) is on a line ℓ. The second result is **Theorem 2.5.2** which gives a geometric characterization of rank deficiency: when the y_i's on the line ℓ are distinct and the x_i's are in general position, then the rank drop of Z_5 is equivalent to the existence of an isomorphism from the conic ω through the x_i's to the line ℓ, taking x_i to y_i.

Finally, we consider the case $k = 6$ in Section 2.6 where we will see that Problem 2.1.4 is intimately related with the theory of cubic surfaces in $\mathbb{P}^3$. An important new fact here is that given 5 points (x_i, y_i) in general position there is a unique 6th point (x_6, y_6) that make Z_6 rank deficient.

This 6th point admits a rational expression in terms of the 5 points (**Lemma 2.6.1**).

The results in this section are organized in two parts. In Section 2.6.1 we characterize the rank deficiency of Z_6 using brackets; **Theorem 2.6.4** covers the case of neither the x_i's nor the y_i's being in a line, and **Theorem 2.6.7** addresses the situation in which one set of points is in a line. When the points are in general position, we identify a simple algebraic check (**Theorem 2.6.6**) for rank deficiency in terms of a vector of classical invariants (2.46) of 6 points in $\mathbb{P}^2$.

In Section 2.6.2, we explain the origins of the algebraic theorems in Section 2.6.1 using geometry. The central result of this section is **Theorem 2.6.9** which says that when the points are in general position, $\operatorname{rank}(Z_6) < 6$ if and only if there is a smooth cubic surface $\mathcal{S}$ that arises as the blow up of $\mathbb{P}^2$ in both the x points and the y points such that the two sets of exceptional curves form a *Schläfli double six* on $\mathcal{S}$. This result allows for a simple determinantal representation of $\mathcal{S}$ using the null space of Z_6.

We then interpret Theorem 2.6.9 in two different ways. The first is **Theorem 2.6.20** that identifies the unique 6th point, given 5 points in general position, that forces Z_6 to drop rank. The construction in this theorem relies on a recipe due to Sturm via conics in $\mathbb{P}^2 \times \mathbb{P}^2$, and this result proves the algebraic Theorem 2.6.4.

Theorem 2.6.9 can be rephrased in yet another way using the *Cremona hexahedral form* of a cubic surface (**Theorem 2.6.21**). This result explains the algebraic Theorem 2.6.6. Using the invariants needed in this theorem along with the classical *Joubert invariants* of 6 points in a line, we complete the answer to Problem 2.1.4 when $k = 6$ by proving Theorem 2.6.7.

A note about computations.

All of the algebraic statements in this paper are proven using the computational algebra software package Macaulay2 [34]. The codes that we used can be found at

`github.com/rekharthomas/rankdrop`

A basic strategy we employ to understand rank drop is to use Macaulay2 to decompose the ideal of maximal minors of Z_k over $\mathbb{Q}$ into its associated primes (over $\mathbb{Q}$). Then we interpret these ideals geometrically to state the conditions for rank drop. These geometric conditions will hold over $\mathbb{R}$

and $\mathbb{C}$ as well since the prime decomposition over $\mathbb{Q}$ can only refine over the larger fields.

2.2 Preprocessing the input

We begin by showing that one can make simplifying assumptions on the input to Problem 2.1.4 which is a collection of $k \leq 6$ points $(x_i, y_i) \in \mathbb{P}^2 \times \mathbb{P}^2$.

We sometimes refer to (x_i, y_i) as a *point pair*, especially when we need to decouple the components and think of the configurations x_i and y_i separately. It will also be convenient to refer to the $\mathbb{P}^2$ containing $\{x_i\}$ as $\mathbb{P}^2_x$ and the $\mathbb{P}^2$ containing $\{y_i\}$ as $\mathbb{P}^2_y$. Let PGL(3) be the group of invertible 3×3 matrices over $\mathbb{C}$ up to scale. An element $H \in$ PGL(3) induces a homography (projective isomorphism) on $\mathbb{P}^2$.

In the introduction we informally proved that the rank of $Z_k = (x_i^\top \otimes y_i^\top)_{i=1}^k$ is invariant under PGL$(3) \times$ PGL(3) acting on $\mathbb{P}^2_x \times \mathbb{P}^2_y$ by left multiplication. This allows us to change bases in $\mathbb{P}^2_x$ and $\mathbb{P}^2_y$ independently. We state the result formally below:

Lemma 2.2.1. *Let $(x_i, y_i) \in \mathbb{P}^2 \times \mathbb{P}^2$ for $i = 1, \ldots, k$ and let $H_1, H_2 \in$ PGL(3) be homographies acting on $\mathbb{P}^2_x$ and $\mathbb{P}^2_y$ respectively (i.e., $(x, y) \mapsto (H_1 x, H_2 y)$). Then $Z_k = \left(x_i^\top \otimes y_i^\top\right)_{i=1}^k$ has the same rank as $Z'_k = \left((H_1 x_i)^\top \otimes (H_2 y_i)^\top\right)_{i=1}^k$.*

Lemma 2.2.1 allows us to make certain assumptions on the configurations $\{x_i\}$ and $\{y_i\}$ which we phrase as corollaries to the lemma. While our proofs do not need these assumptions, they become critical for computations in Macaulay2. The first assumption is that the x_i's and y_i's in the input to Problem 2.1.4 are finite points. With a view towards computations, in the following lemma and in several parts of the paper, we fix representatives of points in $\mathbb{P}^2$ and write them out as vectors in $\mathbb{C}^3$.

Corollary 2.2.2. *Without loss of generality, we can assume that the input to Problem 2.1.4 is of the form $x_i = (x_{i1}, x_{i2}, 1)^\top$ and $y_i = (y_{i1}, y_{i2}, 1)^\top$ for all $i = 1, \ldots, k$, where $x_{i1}, x_{i2}, y_{i1}, y_{i2} \in \mathbb{C}$. Hence*

$$x_i^\top \otimes y_i^\top = (x_{i1}y_{i1}, x_{i1}y_{i2}, x_{i1}, x_{i2}y_{i1}, x_{i2}y_{i2}, x_{i2}, y_{i1}, y_{i2}, 1). \tag{2.4}$$

Proof. Let $x_i \in \mathbb{P}^2_x$ and $y_i \in \mathbb{P}^2_y$ be k pairs of points where k is finite, and suppose that some of these points lie on the line at infinity in their $\mathbb{P}^2$. Since we have only finitely many points, there exists a line ℓ, common to both $\mathbb{P}^2$, which contains none of these points. Let H be a homography that sends ℓ to the line at infinity in both $\mathbb{P}^2$. Then Hx_i and Hy_i are finite points for all i. By Lemma 2.2.1, the rank of $Z_k = (x_i^\top \otimes y_i^\top)_{i=1}^k$ equals the rank of $Z'_k = ((Hx_i)^\top \otimes (Hy_i)^\top)_{i=1}^k$ and hence we can work with the transformed points to understand the rank deficiency of Z_k. □

The other assumption we can make is that sufficiently generic configurations allow some of the points to be fixed, which will prove useful for computations.

Corollary 2.2.3. *When no three of the x_i's or y_i's are collinear in their respective $\mathbb{P}^2$, we may fix the first four in each set to be:*

$$\begin{aligned} &x_1 = y_1 = (0,0,1), \quad x_2 = y_2 = (1,0,1), \\ &x_3 = y_3 = (0,1,1), \quad x_4 = y_4 = (1,1,1). \end{aligned} \tag{2.5}$$

At certain times, for the sake of symmetry, we will instead fix the first four points as:

$$\begin{aligned} &x_1 = y_1 = (1,0,0), \quad x_2 = y_2 = (0,1,0), \\ &x_3 = y_3 = (0,0,1), \quad x_4 = y_4 = (1,1,1). \end{aligned} \tag{2.6}$$

Since the two fixings are related by a homography, results from one fixing transfer to the other.

2.3 Cross-ratios and brackets

By Lemma 2.2.1, the rank deficiency of Z_k is invariant under the action of PGL(3) $\times$ PGL(3) on $\mathbb{P}^2 \times \mathbb{P}^2$ given by $(H_1, H_2) \cdot (x, y) \mapsto (H_1 x, H_2 y)$. Therefore, the input to Problem 2.1.4 can be thought of as the PGL(3)-orbits of k points in each $\mathbb{P}^2$. The intrinsic geometry of k points $p_1, \ldots, p_k \in \mathbb{P}^2$ are the properties of the configuration that remain intact/invariant under the action of PGL(3). Since an element of PGL(3) can be assumed to have determinant 1, the above geometry is precisely the invariant properties, under the action of SL(3), of the configuration obtained by replacing each $p_i \in \mathbb{P}^2$ with a representative in $\mathbb{C}^3$. We now introduce the basic tools from the invariant theory of point sets needed to formulate our results. There are numerous references for this material such as [12], [21], [25], [24] and [66].

Let $\mathbb{C}[\mathbf{p}] := \mathbb{C}[p_{i1}, p_{i2}, p_{i3}, i = 1, \ldots, k]$ be the polynomial ring in the variables p_{ij} for $i = 1, \ldots, k$, $j = 1, 2, 3$, that represent the coordinates of k points $p_1, \ldots, p_k$ in $\mathbb{P}^2$ written via representatives in $\mathbb{C}^3$. A polynomial $f \in \mathbb{C}[\mathbf{p}]$ is *invariant* under SL(3) if $f(p_1, \ldots, p_k) = f(Hp_1, \ldots, Hp_k)$ for all $H \in$ SL(3). It is a well-known classical result that the invariant ring $\mathbb{C}[\mathbf{p}]^{\mathrm{SL}(3)}$, which is the collection of all polynomials in $\mathbb{C}[\mathbf{p}]$ that are invariant under SL(3), is generated by the 3×3 minors of the matrix whose rows are the symbolic $p_1^\top, \ldots, p_k^\top$. In other words, $\mathbb{C}[\mathbf{p}]^{\mathrm{SL}(3)}$ is generated as an algebra by the *bracket polynomials*

$$[ijl] := \det \begin{bmatrix} p_{i1} & p_{i2} & p_{i3} \\ p_{j1} & p_{j2} & p_{j3} \\ p_{l1} & p_{l2} & p_{l3} \end{bmatrix} \tag{2.7}$$

for all distinct choices of $i, j, l \in \{1, \ldots, k\}$. In our situation, we will write $[ijl]_x$ (respectively, $[ijl]_y$) when the input points come from $\{x_i\}$ (respectively, $\{y_i\}$). Equations in brackets can be used to express geometry. For example, three points $p_i, p_j, p_l \in \mathbb{P}^2$ are collinear if and only if $[ijl] = 0$, and 6 points lie on a conic if and only if they satisfy the bracket equation (2.17) in Lemma 2.3.5.

Often we consider points in a line ℓ in $\mathbb{P}^2$ as points in $\mathbb{P}^1$ where we can compute the bracket $[ij]$ for $p_i, p_j \in \ell$ by identifying ℓ with the line parameterized as say $(*, *, 0)$, and then dropping the 0 coordinate to get

$$[ij] := \det \begin{bmatrix} p_{i1} & p_{i2} \\ p_{j1} & p_{j2} \end{bmatrix}. \tag{2.8}$$

When we have many points $p_1, \ldots, p_k$ on a line and a point u not on that line, the $\mathbb{P}^2$ brackets $[iju]$ will be closely related to the $\mathbb{P}^1$ brackets $[ij]$.

Lemma 2.3.1. *For points $p_1, \ldots, p_k$ on a line and a point u not on the line, there exists a nonzero scalar λ depending only on the choice of coordinate representative of u such that $[iju] = \lambda[ij]$ for all i, j.*

Proof. We can assume without loss of generality that $p_i = (p_{i1}, p_{i2}, 0)$ for all i. We then calculate

$$[iju] = \det \begin{bmatrix} p_{i1} & p_{i2} & 0 \\ p_{j1} & p_{j2} & 0 \\ u_1 & u_2 & u_3 \end{bmatrix} = u_3[ij]. \tag{2.9}$$

□

Definition 2.3.2. *The* **cross-ratio** *of an ordered list of* 4 *points* $p_1, \dots, p_4 \in \mathbb{P}^1$ *is*

$$(p_1, p_2; p_3, p_4) := \frac{[13][24]}{[14][23]}. \tag{2.10}$$

The cross-ratio is the only projective invariant of 4 points in $\mathbb{P}^1$ in the following sense.

Lemma 2.3.3. *If* $p_1, \dots, p_4 \in \mathbb{P}^1$ *and* $q_1, \dots, q_4 \in \mathbb{P}^1$ *are two sets of points then* $(p_1, p_2; p_3, p_4) = (q_1, q_2; q_3, q_4)$ *if and only if there exists a homography* $H : \mathbb{P}^1 \to \mathbb{P}^1$ *such that* $Hp_i = q_i$ *for all* i*.*

Permuting the points on the line changes the cross-ratio in a systematic way, i.e., if

$$(p_1, p_2; p_3, p_4) = (q_1, q_2; q_3, q_4) \tag{2.11}$$

then

$$(p_{\sigma(1)}, p_{\sigma(2)}; p_{\sigma(3)}, p_{\sigma(4)}) = (q_{\sigma(1)}, q_{\sigma(2)}; q_{\sigma(3)}, q_{\sigma(4)}) \tag{2.12}$$

for all $\sigma \in S_4$. Thus we do not need to consider multiple orderings. The cross-ratio is $0, 1, \infty$, or $\frac{0}{0}$ if and only if the points are not distinct. See [61, III, §4-5].

Example 1. We now give an example to illustrate that equations in brackets can be more robust than equalities of cross-ratios. Consider the following points in $\mathbb{P}^1$ where the first set has repetition:

$$\begin{aligned} x_1 &= (1,1) \quad x_2 = (1,1) \qquad\qquad & y_1 &= (8,1) \quad y_2 = (4,1) \\ x_3 &= (1,1) \quad x_4 = (3,1) & y_3 &= (2,1) \quad y_4 = (5,1). \end{aligned}$$

Then

$$(x_1, x_2; x_3, x_4) = \frac{0}{0}, \qquad (y_1, y_2; y_3, y_4) = -1 \tag{2.13}$$

and therefore the cross-ratios are not equal. However, if we write the cross-ratio equality

$$(x_1, x_2; x_3, x_4) = (y_1, y_2; y_3, y_4) \tag{2.14}$$

in bracket form as

$$[13]_x[24]_x[14]_y[23]_y = [13]_y[24]_y[14]_x[23]_x \tag{2.15}$$

then equality does hold.

In Example 1, the points (x_i, y_i) lie on the variety defined by the bracket equation (2.15) but do not satisfy the cross ratio equality (2.14). The discrepancy is because cross-ratios are well-defined only on an open set, while the corresponding bracket equation holds on the Zariski closure of this open set. Since our interest is in describing the algebraic variety of input points that make Z_4 rank deficient, the bracket expressions are the correct choice. However, whenever the points are in sufficiently general position, we can pass to cross-ratio expressions which, besides being elegant, have the advantage that they can be visualized easily.

We will also need planar and conic cross-ratios which we now define.

Definition 2.3.4. *The* 5 **point cross-ratio** *of* $p_1, \ldots, p_5 \in \mathbb{P}^2$ *is*

$$(p_1, p_2; p_3, p_4; p_5) := \frac{[135][245]}{[145][235]}. \tag{2.16}$$

This is also called a **planar cross-ratio** *or the* **cross-ratio around** p_5.

Note that planar cross-ratios are preserved under a homography of $\mathbb{P}^2$. For 5 distinct points, the cross-ratio around x_5 can be obtained geometrically by drawing the 4 lines $\overline{p_i p_5}$ for $i = 1, 2, 3, 4$, cutting them with a transversal, and then computing the cross-ratio of the intersection points. See Figure 2.1. Also, the planar cross-ratio is transformed by permutations of $p_1, \ldots, p_4$ similarly to the line cross-ratio.

Lemma 2.3.5. *A collection of* 6 *points* $p_1, \ldots, p_6 \in \mathbb{P}^2$ *lie on a conic if and only if*

$$[135][245][146][236] = [136][246][145][235], \tag{2.17}$$

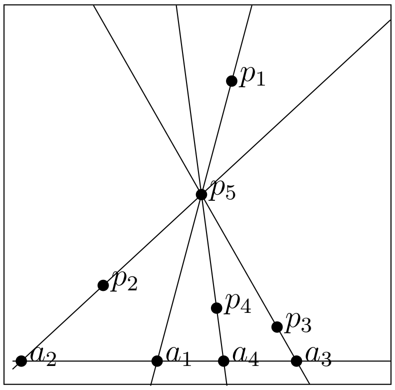

Figure 2.1: For distinct points $p_1, \ldots, p_5 \in \mathbb{P}^2$, the planar cross-ratio $(p_1, p_2; p_3, p_4; p_5)$ equals the 4 point cross-ratio $(a_1, a_2; a_3, a_4)$

.

or generically, the following cross-ratio equality holds:

$$(p_1, p_2; p_3, p_4; p_5) = (p_1, p_2; p_3, p_4; p_6). \tag{2.18}$$

Lemma 2.3.5 makes the following definition of a conic cross-ratio well-defined.

Definition 2.3.6. *The* **conic cross-ratio** *of 4 points $p_1, \ldots, p_4$ on a (non-degenerate) conic ω is*

$$(p_1, p_2; p_3, p_4)_\omega := (p_1, p_2; p_3, p_4; p) \tag{2.19}$$

where $p \in \omega$ is any new point.

Any non-degenerate conic ω is isomorphic to $\mathbb{P}^1$. If $\iota : \omega \to \mathbb{P}^1$ is one such isomorphism then

$$(p_1, p_2; p_3, p_4)_\omega = (\iota(p_1), \iota(p_2); \iota(p_3), \iota(p_4)). \tag{2.20}$$

2.4 $k \leq 4$

We will now answer Problem 2.1.4 in the cases $k \leq 4$. As a warm up to Problem 2.1.4 we first consider the analogous question in $\mathbb{P}^1 \times \mathbb{P}^1$. Recall that any condition on a proper subset of $\{(x_i, y_i)\}$ that make the corresponding rows of Z_k dependent is called an inherited condition for the rank drop of Z_k.

Theorem 2.4.1. *Consider the $k \times 4$ matrix $Z_k = (x_i \otimes y_i)_{i=1}^{k}$ where $(x_i, y_i) \in \mathbb{P}^1 \times \mathbb{P}^1$ and $k \leq 4$. Then,*

1. $\operatorname{rank}(Z_2) < 2$ *if and only if $x_1 = x_2$ and $y_1 = y_2$,*
2. $\operatorname{rank}(Z_3) < 3$ *if and only if either an inherited condition holds or $x_1 = x_2 = x_3$ or $y_1 = y_2 = y_3$,*
3. $\operatorname{rank}(Z_4) < 4$ *if and only if the following bracket equation holds:*

$$[13]_x[24]_x[14]_y[23]_y = [14]_x[23]_x[13]_y[24]_y. \tag{2.21}$$

Generically, $x_1, \ldots, x_4$ and $y_1, \ldots, y_4$ are distinct and then (2.21) *holds if and only if there is a homography $H \in$* PGL(2) *such that $Hx_i = y_i$ for $i = 1, \ldots, 4$.*

Proof. In each of the above cases, Z_k will be rank deficient if and only if all its maximal minors vanish. Hence we compute the ideal of $k \times k$ minors of the symbolic Z_k in each case and decompose it to understand rank deficiency. While computations in Macaulay2 are done over $\mathbb{Q}$, our results hold over $\mathbb{C}$ as explained at the end of the Introduction. By Corollary 2.2.2 we can fix the last coordinates of x_i and y_i to 1.

If $k = 2$ the ideal of 2×2 minors of Z_2 is the prime ideal $\langle x_{11} - x_{21}, y_{11} - y_{21} \rangle$ which says that rank drops if and only if $x_1 = x_2$ and $y_1 = y_2$. We will call this the *repeated point condition*.

If $k = 3$, the ideal of 3×3 minors of the symbolic Z_3 matrix is radical and decomposes into 5 prime ideals. Three of them correspond to repeated points $x_i = x_j$ and $y_i = y_j$ for $i \neq j$. These are inherited for rank drop from the $k = 2$ case. The remaining 2 prime ideals say that $x_1 = x_2 = x_3$ and $y_1 = y_2 = y_3$ respectively.

When $k = 4$, we can use Macaulay2 to check that

$$\det(Z_4) = [14]_x[23]_x[13]_y[24]_y - [13]_x[24]_x[14]_y[23]_y$$

which yields (2.21). In this case, the ideal of maximal minors is principal and prime, generated by $\det(Z_4)$, and does not decompose. However, note that the bracket expression for $\det(Z_4)$ vanishes under all of the conditions inherited from $k = 2$ and $k = 3$. Furthermore, when the x_i's and y_i's are distinct then (2.21) is equivalent to $(x_1, x_2; x_3, x_4) = (y_1, y_2; y_3, y_4)$ and the homography follows by Lemma 2.3.3. $\square$

We now consider Problem 2.1.4 for $k \leq 4$ points $(x_i, y_i) \in \mathbb{P}^2 \times \mathbb{P}^2$. Again all of the statements can be checked in Macaulay2 by following the same general strategy as in the proof of Theorem 2.4.1.

Theorem 2.4.2. *Suppose* $(x_i, y_i) \in \mathbb{P}^2 \times \mathbb{P}^2$ *and* $Z_k = (x_i^\top \otimes y_i^\top)_{i=1}^k$ *for* $k \leq 4$. *Then*

1. $\operatorname{rank}(Z_2) < 2$ *if and only if* $x_1 = x_2$ *and* $y_1 = y_2$.
2. $\operatorname{rank}(Z_3) < 3$ *if and only if either an inherited condition holds or* $x_1 = x_2 = x_3$ *and* y_1, y_2, y_3 *are on a line, or* x_1, x_2, x_3 *are on a line and* $y_1 = y_2 = y_3$.
3. $\operatorname{rank}(Z_4) < 4$ *if and only if an inherited condition holds or one of the following is true:*
   - *(a)* $x_1 = x_2 = x_3 = x_4$ *or* $y_1 = y_2 = y_3 = y_4$.
   - *(b) The points* $x_1, \dots, x_4 \in \mathbb{P}^2_x$ *are in a line and* $y_1, \dots, y_4 \in \mathbb{P}^2_y$ *are in a line, and*

$$[13]_x[24]_x[14]_y[23]_y = [14]_x[23]_x[13]_y[24]_y. \tag{2.22}$$

Generically, the collinear points $x_1, \dots, x_4$ *and* $y_1, \dots, y_4$ *are distinct and then* (2.22) *holds if and only if there is a homography* $H \in \mathrm{PGL}(3)$ *such that* $Hx_i = y_i$ *for* $i = 1, \dots, 4$.

Proof. The proof of (1) and (2) can be done using Macaulay2 by computing the ideal of maximal minors of Z_k and then decomposing the radical of the ideal to get all possible conditions for rank drop. The matrix Z_k is written symbolically as in (2.4) where, by Corollary 2.2.2 we assume that the last coordinate of all x_i's and y_i's are 1. For instance, the ideal of 3×3 minors of Z_3 is radical and of dimension 8 and degree 3. It is the intersection of 5 prime ideals of which 3 correspond to the repeated point conditions inherited from $k = 2$. The fourth ideal provides the new condition that

$x_1 = x_2 = x_3$ and y_1, y_2, y_3 are in a line. The fifth ideal provides the analogous condition with the roles of x and y switched. These ideals appear in the form:

$$\langle x_{11}-x_{21}, x_{12}-x_{22}, x_{11}-x_{31}, x_{12}-x_{32}, [123]_y\rangle \quad \text{and} \quad \langle y_{11}-y_{21}, y_{12}-y_{22}, y_{11}-y_{31}, y_{12}-y_{32}, [123]_x\rangle.$$

The ideal of maximal minors of Z_4 is radical of dimension 12 and degree 6. It decomposes into 17 primes ideals:

- 6 of the primes correspond to repeated points (inherited from $k = 2$). All of these have dimension 12 and degree 1.
- 8 of the primes correspond to 3 points in one side coinciding and the corresponding points on the other side on a line (inherited from $k = 3$). All of these ideals have dimension 11 and degree 2.
- 2 of the primes say $x_1 = x_2 = x_3 = x_4$ and $y_1 = y_2 = y_3 = y_4$ respectively (new conditions for $k = 4$). These ideals have dimension 10 and degree 1.
- The last prime component, call it J, has dimension 11 and degree 24. It implies that the x_i and y_i are in a line (new condition for $k = 4$). One way to verify this claim is to check that the ideal generated by the brackets $[123]_x, [124]_x$ and $[123]_y, [124]_y$ is contained in J. These brackets being 0 encode the collinearity of x_i's and the collinearity of y_i's.

One can understand J more precisely by making the following calculation. If the x and y points are both in a line, then we can use homographies to fix these lines, and assume that $x_i = (a_i, 0, 1)$ and $y_i = (b_i, 0, 1)$ for $i = 1, \dots, 4$. This means that we can identify x_i with $(a_i, 1) \in \mathbb{P}^1$ and y_i with $(b_i, 1) \in \mathbb{P}^1$. Under this substitution, Z_4 has exactly one non-zero minor

$$\det(M) = \det \begin{bmatrix} a_1b_1 & a_1 & b_1 & 1 \\ a_2b_2 & a_2 & b_2 & 1 \\ a_3b_3 & a_3 & b_3 & 1 \\ a_4b_4 & a_4 & b_4 & 1 \end{bmatrix} = [14]_x[23]_x[13]_y[24]_y - [13]_x[24]_x[14]_y[23]_y. \tag{2.23}$$

The second equality in (2.23) can be checked in Macaulay2. The bracket expression set to 0 is the equality of cross-ratios:

$$(x_1, x_2; x_3, x_4) = (y_1, y_2; y_3, y_4)$$

which becomes well-defined when the x_i's are distinct and the y_i's are distinct. □

2.5 $k = 5$

We now consider the rank deficiency of Z_5. In this case, we present two theorems. Theorem 2.5.1 is a characterization of rank deficiency in terms of brackets. We will see that a necessary condition for rank drop is that one set of points (say $\{y_i\}$) is on a line ℓ. Theorem 2.5.2 gives a geometric characterization of rank deficiency when the points are sufficiently generic, i.e., when the y_i's on the line ℓ are distinct and the x_i's are in general position. In this case the rank drop of Z_5 is equivalent to the existence of an isomorphism from the conic ω through the x_i's to the line ℓ taking x_i to y_i. This is analogous to Theorem 2.4.2 (3)(b) where there is a homography taking x_i to y_i under sufficient genericity. We will illustrate both results using Example 2.

Theorem 2.5.1. *Given a configuration* $\{(x_i, y_i)\}_{i=1}^5$ *in* $\mathbb{P}^2 \times \mathbb{P}^2$*, the matrix* Z_5 *is rank deficient if and only if either, one of the inherited conditions hold, or the points in one* $\mathbb{P}^2$ *(say in* $\mathbb{P}^2_y$*) are on a line and the following bracket equations hold for all distinct* $i_1, i_2, i_3, i_4, j \in \{1, 2, 3, 4, 5\}$*:*

$$[i_1 i_3 j]_x [i_2 i_4 j]_x [i_1 i_4]_y [i_2 i_3]_y = [i_1 i_4 j]_x [i_2 i_3 j]_x [i_1 i_3]_y [i_2 i_4]_y. \tag{2.24}$$

By the action of permutations, there are only 5 *equations to consider in* (2.24)*, one for each choice of* $j \in \{1, 2, 3, 4, 5\}$*. In the case that both the* x_i*'s and the* y_i*'s are contained in lines then* (2.24) *holds automatically. Generically,* (2.24) *is equivalent to the cross-ratio equalities:*

$$(x_{i_1}, x_{i_2}; x_{i_3}, x_{i_4}; x_j) = (y_{i_1}, y_{i_2}; y_{i_3}, y_{i_4}) \tag{2.25}$$

for all distinct $i_1, i_2, i_3, i_4, j \in \{1, 2, 3, 4, 5\}$.

Proof. The proof splits into two cases. For the first case, we assume that the points in neither side are entirely in a line and show that rank will drop only under inherited conditions. For the second case, we are then free to assume that one side is contained in a line, which allows us to simplify the Macaulay2 calculations.

1. Suppose neither the x_i's nor the y_i's are in a line, and none of the inherited conditions hold. Then we can assume without loss of generality that $x_1 \neq x_2$ and $y_1 \neq y_2$ because no side has four coincident points (due to the absence of inherited conditions). Since neither side is contained entirely in a line, we can also assume that x_1, x_2, x_3 are not collinear. Then either y_1, y_2, y_3 are also not collinear, or y_1, y_2, y_3 are in a line which means that we can assume y_1, y_2, y_4 are not collinear. Applying homographies, we therefore need to work with two different fixings of points:

   $$x_1 = (1,0,0) = y_1 \qquad \text{and} \qquad x_2 = (0,1,0) = y_2 \qquad \text{and} \qquad x_3 = (0,0,1) = y_3 \tag{2.26}$$

   and

   $$x_1 = (1,0,0) = y_1 \qquad \text{and} \qquad x_2 = (0,1,0) = y_2 \qquad \text{and} \qquad x_3 = (0,0,1) = y_4. \tag{2.27}$$

   Note that these fixings are not to finite points as we have done so far, but are equivalent to the fixing in (2.5) as remarked after Corollary 2.2.2. They allow for faster computations in Macaulay2.

   In (2.26), the ideal generated by the 5×5 minors of Z_5 is the intersection of 20 prime ideals. Of these, 4 correspond to invalid inputs $x_i, y_i = (0,0,0)$; 7 correspond to the inherited repeated point condition from Theorem 2.4.2(1); 6 correspond to the inherited condition from Theorem 2.4.2(2); and the remaining 3 correspond to the inherited bracket/cross-ratio condition from Theorem 2.4.2(3)(b).

   In (2.27), the ideal generated by the 5×5 minors of Z_5 is the intersection of 17 prime ideals. Of these, 4 correspond to invalid inputs $x_i, y_i = (0,0,0)$; 7 correspond to repeated points; 4 correspond to the inherited condition from Theorem 2.4.2(2); and the remaining 2 correspond to the inherited bracket/cross-ratio condition from Theorem 2.4.2(3)(b).

Thus, there are no new conditions for rank drop if neither the x_i's nor the y_i's are in a line.

2. Suppose $y_1, \ldots, y_5$ are on a line. Applying homographies, we may assume that $y_i = (m_i, 0, 1)$ for $i = 1, \ldots, 5$ and all x_i are finite. We may further assume that $x_1 = (0, 0, 1)$ and that $y_1 = (0, 0, 1)$. Then rearranging the columns of Z_5 we get

$$Z_5' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ x_{21}m_2 & x_{22}m_2 & x_{21} & x_{22} & m_2 & 1 & 0 & 0 & 0 \\ x_{31}m_3 & x_{32}m_3 & x_{31} & x_{32} & m_3 & 1 & 0 & 0 & 0 \\ x_{41}m_4 & x_{42}m_4 & x_{41} & x_{42} & m_4 & 1 & 0 & 0 & 0 \\ x_{51}m_5 & x_{52}m_5 & x_{51} & x_{52} & m_5 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{2.28}$$

Let I be the ideal generated by the 6 non-zero 5×5 minors of Z_5' and let J be the ideal generated by the 5 differences of left and right hand sides in the bracket equations (2.24) under the same fixings of points. Then we can verify using Macaulay2 that $I = \sqrt{J}$. A symmetric argument will switch the roles of the x and y points.

$\square$

Example 2. The following example illustrates Theorem 2.5.1 and motivates Theorem 2.5.2. Consider

$$\begin{array}{llll} x_1 = (0,0,1) & y_1 = (0,0,1) & x_4 = (1,1,1) & y_4 = (-4,0,1) \\ x_2 = (1,0,1) & y_2 = (1,0,1) & x_5 = (50,98,113) & y_5 = (8,0,1) \\ x_3 = (0,1,1) & y_3 = (3,0,1) & & \end{array}$$

Here the y_i's are on the line $y_i = (m_i, 0, 1)$ and we can verify that (2.24) is satisfied. The matrix Z_5 shown below, has rank 4.

$$Z_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 3 & 0 & 1 & 3 & 0 & 1 \\ -4 & 0 & 1 & -4 & 0 & 1 & -4 & 0 & 1 \\ 400 & 0 & 50 & 784 & 0 & 98 & 904 & 0 & 113 \end{bmatrix}.$$

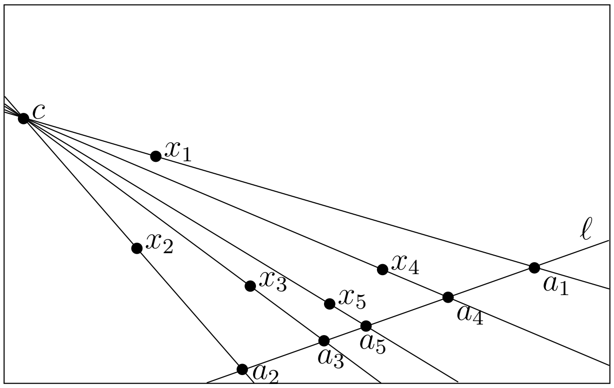

Figure 2.2: We construct the projection with center c.

Note that in this example, the y_i's (which are on a line) are distinct, and that no 3 of the x_i's are in a line. In this case, there is a 1-parameter family of rank 2 projective transformations $\{T_\lambda\}$ such that $T_\lambda(x_i) = y_i$ for $i = 1, \dots, 5$ except for 5 unique values of λ, one for each of $x_1, \dots, x_5$. In this example, this is the family

$$T_\lambda = \begin{bmatrix} 0 & 12 & 0 \\ -7 & -3 & 7 \end{bmatrix} + \lambda \begin{bmatrix} 7 & -15 & 0 \\ 7 & -5 & 0 \end{bmatrix}. \tag{2.29}$$

For each $i = 1, \dots, 5$ there exists unique λ_i such that T_{λ_i} has nullvector (center) x_i. These are

$$\lambda_1 = \infty \quad \lambda_2 = 0 \quad \lambda_3 = \frac{4}{5} \quad \lambda_4 = \frac{3}{2} \quad \lambda_5 = \frac{21}{20}. \tag{2.30}$$

Since $T_{\lambda_i}(x_i) = (0,0,0) \not\sim y_i$, these 5 transformations can be seen as degenerate members of the family.

We now formalize the second half of Example 2 as a theorem. As seen in Theorem 2.5.1, for Z_5 to drop rank, either the x_i's or the y_i's have to be in a line. Suppose the y_i's are distinct and in a line, and the x_i's are in general position. Then the x_i's lie on a conic ω and we will see that there is an isomorphism taking ω to the line containing the y_i's so that each x_i maps to y_i.

Theorem 2.5.2. *Suppose we have a configuration $\{(x_i, y_i)\}_{i=1}^5$ such that no 3 of the x_i's are collinear and the y_i's are distinct and on a line ℓ_y. Let ω be the conic through $x_1, \dots, x_5$. Suppose further that no inherited condition for rank drop holds. Then the following are equivalent:*

1. *The matrix* Z_5 *is rank deficient.*
2. *For all* $c \in \omega \setminus \{x_1, \dots, x_5\}$ *there exists a projective transformation* $T_c : \mathbb{P}^2 \to \mathbb{P}^1$ *centered at* c *such that* $T_c(x_i) \sim y_i$ *for all* i.
3. *For some* $c \in \omega \setminus \{x_1, \dots, x_5\}$ *there exists a projective transformation* $T_c : \mathbb{P}^2 \to \mathbb{P}^1$ *centered at* c *such that* $T_c(x_i) \sim y_i$ *for all* i.
4. *There exists a cross-ratio preserving isomorphism* $F : \omega \to \ell_y$ *such that* $F(x_i) = y_i$ *for all* i.

Proof. To prove (1) implies (2), let $c \in \omega \setminus \{x_1, \dots, x_5\}$. Then for all distinct $i_1, i_2, i_3, i_4, j \in \{1, \dots, 5\}$

$$(y_{i_1}, y_{i_2}; y_{i_3}, y_{i_4}) = (x_{i_1}, x_{i_2}; x_{i_3}, x_{i_4}; x_j) = (x_{i_1}, x_{i_2}; x_{i_3}, x_{i_4})_\omega = (x_{i_1}, x_{i_2}; x_{i_3}, x_{i_4}; c) \tag{2.31}$$

for any $c \in \omega$. The first equation is from (2.25) in Theorem 2.5.1 and the other two equalities follow by Definition 2.3.6. Let $\overline{cx_i}$ be the line through c and x_i and let ℓ be any line such that $c \notin \ell$. Then if we define a_i to be the unique intersection of ℓ and $\overline{cx_i}$ it follows that

$$(a_{i_1}, a_{i_2}; a_{i_3}, a_{i_4}) = (x_{i_1}, x_{i_2}; x_{i_3}, x_{i_4}; c) = (y_{i_1}, y_{i_2}; y_{i_3}, y_{i_4}) \tag{2.32}$$

for all $i_1, i_2, i_3, i_4 \in \{1, \dots, 5\}$. See Figure 2.2. It follows that there exists homography H such that $Ha_i \sim y_i$ for all i. Therefore if T'_c is the projection in $\mathbb{P}^2_x$, centered at c, onto the line ℓ then $T_c := HT'_c$ is the desired projective transformation with $T_c(x_i) \sim y_i$ for all i.

Clearly (2) implies (3). To show that (3) implies (1), suppose such a projection exists and is centered at c. Then

$$(x_{i_1}, x_{i_2}; x_{i_3}, x_{i_4}; x_j) = (x_{i_1}, x_{i_2}; x_{i_3}, x_{i_4})_\omega = (x_{i_1}, x_{i_2}; x_{i_3}, x_{i_4}; c) = (y_{i_1}, y_{i_2}; y_{i_3}, y_{i_4}) \tag{2.33}$$

for all distinct $i_1, i_2, i_3, i_4, j \in \{1, \dots, 5\}$. The first two equalities follow by Definition 2.3.6 and the last equality follows by hypothesis. It immediately follows that Z_5 is rank deficient by Theorem 2.5.1.

To show that (1) implies (4) we may assume that the cross-ratio equalities (2.25) hold. Then define an isomorphism $F : \omega \to \ell_y$ by $F(x_1) = y_1, F(x_2) = y_2, F(x_3) = y_3$. Then for all

$p \in \omega \setminus \{x_1, x_2, x_3\}$, $F(p) = q$ is such that $(x_1, x_2; x_3, p)_\omega = (y_1, y_2; y_3, q)$. It then follows that $F(x_4) = y_4$ and $F(x_5) = y_5$ because

$$(x_1, x_2; x_3, x_4)_\omega = (y_1, y_2; y_3, y_4) \qquad (x_1, x_2; x_3, x_5)_\omega = (y_1, y_2, y_3; y_5) \tag{2.34}$$

by hypothesis (Theorem 2.5.1).

Finally, to show that (4) implies (1), suppose such an isomorphism F exists. Then

$$(x_{i_1}, x_{i_2}; x_{i_3}, x_{i_4}; x_j) = (x_{i_1}, x_{i_2}; x_{i_3}, x_{i_4})_\omega = (F(x_{i_1}), F(x_{i_2}); F(x_{i_3}), F(x_{i_4})) = (y_{i_1}, y_{i_2}; y_{i_3}, y_{i_4}) \tag{2.35}$$

for all $i_1, i_2, i_3, i_4, j \in \{1, \dots, 5\}$. The first equality follows by Definition 2.3.6 and the second and third equalities follow by hypothesis. It then follows that Z_5 is rank deficient by Theorem 2.5.1. □

2.6 $k = 6$

The final case we will consider in this paper is a set of 6 inputs $\{(x_i, y_i)\}_{i=1}^{6} \in \mathbb{P}^2 \times \mathbb{P}^2$. We organize our results in two parts. In the first part we state algebraic characterizations of the rank deficiency of Z_6. These are analogous to the results in the previous sections in that the results are in terms of invariant theory. In the second part, we explain the geometry that leads to these algebraic statements. This relies on the classical theory of cubic surfaces and its combinatorics. The geometry works in an open region where the input is sufficiently generic. However, as we saw in previous sections, the algebra that it translates to describes the Zariski closure of these open regions and hence the set of all inputs that make Z_6 rank deficient.

We say that $p_1, \dots, p_6 \in \mathbb{P}^2$ are in **general position** if at most 2 points are in a line and at most 5 points are on a conic. In several results below we will need that the x points and/or the y points are in general position. We will also need that a collection of point pairs $\{(x_i, y_i)\} \subset \mathbb{P}^2 \times \mathbb{P}^2$ is in **general position**, by which we will mean that

1. the x points and y points are in general position, and
2. no more than 4 pairs (x_i, y_i) are related by a homography $Hx_i = y_i$.

Note that if (1) holds then Z_5 has full rank because all of our previous conditions for rank drop require at least the x_i's or the y_i's to be in a line.

The variety $\mathbb{P}^2 \times \mathbb{P}^2$ can be embedded in $\mathbb{P}^8$ as the image of the **Segre map**

$$\mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^8 \quad \text{such that} \quad (x, y) \mapsto x^\top \otimes y^\top. \tag{2.36}$$

In what follows it will be helpful to identify $\mathbb{P}^8$ with the set of all 3×3 complex matrices up to scale via the bijection $M \leftrightarrow \text{vec}(M)$. Then the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ in $\mathbb{P}^8$ (which we will also call $\mathbb{P}^2 \times \mathbb{P}^2$), can be identified with the set of rank one 3×3 matrices up to scale since

$$x^\top \otimes y^\top = (\text{vec}(yx^\top))^\top. \tag{2.37}$$

The Segre variety $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ has dimension 4 and degree 6. It is the zero locus of the determinantal ideal generated by the 2×2 minors of a symbolic 3×3 matrix whose entries denote the coordinates of $\mathbb{P}^8$.

We first show that given 5 points $\{(x_i, y_i)\}_{i=1}^5$ in general position, there exists a unique 6th point $(x_6, y_6) \in \mathbb{P}^2 \times \mathbb{P}^2$ such that $\{(x_i, y_i)\}_{i=1}^6$ is rank deficient.

Lemma 2.6.1. *Let $\{(x_i, y_i)\}_{i=1}^5$ be in general position. Then there is a unique new 6th point $(x_6, y_6) \in \mathbb{P}^2 \times \mathbb{P}^2$ such that the matrix Z_6 is rank deficient. This 6th point is a rational function of the first 5; if we use homographies to fix the first 4 points as*

$$x_1 = (1,0,0) = y_1 \qquad x_2 = (0,1,0) = y_2$$
$$x_3 = (0,0,1) = y_3 \qquad x_4 = (1,1,1) = y_4,$$

then the 6th point has the formula:

$$\begin{aligned} x_6 =& \Big(\frac{y_{53} - y_{52}}{y_{53}x_{52} - x_{53}y_{52}}, \frac{y_{53} - y_{51}}{y_{53}x_{51} - x_{53}y_{51}}, \frac{y_{51} - y_{52}}{y_{51}x_{52} - x_{51}y_{52}}\Big) \\ y_6 =& \Big(\frac{x_{53} - x_{52}}{y_{53}x_{52} - x_{53}y_{52}}, \frac{x_{53} - x_{51}}{y_{53}x_{51} - x_{53}y_{51}}, \frac{x_{51} - x_{52}}{y_{51}x_{52} - x_{51}y_{52}}\Big). \end{aligned} \tag{2.38}$$

Proof. Let $\mathcal{R}(Z_5)$ denote the row span of $Z_5 = (x_i \otimes y_i)_{i=1}^5$. By general position, $\text{rank}(Z_5) = 5$ which means that $\mathcal{R}(Z_5) \cong \mathbb{P}^4$. Therefore, generically, $\mathcal{R}(Z_5)$ intersects the 4-dimensional, degree

6 Segre variety $\mathbb{P}^2 \times \mathbb{P}^2$ transversally in 6 distinct points. Each of these points corresponds to a rank one matrix $yx^\top$ which we have identified with $x^\top \otimes y^\top$. Five of these intersection points are the matrices $x_i^\top \otimes y_i^\top$ for $i = 1, \ldots, 5$ and the 6th point $x_6^\top \otimes y_6^\top$ corresponds to a unique point $(x_6, y_6) \in \mathbb{P}^2 \times \mathbb{P}^2$.

To prove the second part of the Lemma, we first use homographies to fix the first 4 points as mentioned. We can then use Macaulay2 to verify that we can solve for (x_6, y_6) in terms of (x_5, y_5) as in equations (2.38). $\square$

The existence of, and constructions for, this 6th point were noted in both [58] and [68], but not the rational formula. The rational formula shows that if $(x_i, y_i)_{i=1}^5$ are real then (x_6, y_6) will be real as well. Note that if $\{x_i\}_{i=1}^5$ and $\{y_i\}_{i=1}^5$ were related by homography then the denominators in (2.38) would all be 0. Here is an illustration of Lemma 2.6.1.

Example 3. We begin with 5 point pairs

$$x_1 = (1,0,0) \quad y_1 = (1,0,0) \qquad x_2 = (0,1,0) \quad y_2 = (0,1,0)$$
$$x_3 = (0,0,1) \quad y_3 = (0,0,1) \qquad x_4 = (1,1,1) \quad y_4 = (1,1,1)$$
$$x_5 = (3,5,1) \quad y_5 = (8,2,1)$$

If we leave (x_6, y_6) symbolic and construct the matrix Z_6, we can verify with Macaulay2 that the unique new point pair that will result in rank deficiency is $x_6 \sim (-\frac{1}{3}, \frac{7}{5}, \frac{3}{17}),\ y_6 \sim (-\frac{4}{3}, \frac{2}{5}, -\frac{1}{17})$, which is exactly what we get from the formula above.

In Lemma 2.6.15 we will prove a counterpart to the above result, where we instead begin with $x_1, \ldots, x_6$ and calculate (up to homography) the points $y_1, \ldots, y_6$ that will result in rank deficiency.

2.6.1 Algebraic Characterizations of Rank Deficiency

In this section we will write down various algebraic characterizations of rank deficiency. While these statements are by no means obvious or intuitive, once stated, they are straightforward to check using Macaulay2. In the next section we will explain the geometric origins of these statements.

Our first result is a characterization of rank drop using brackets and cross-ratios as in the previous sections.

Definition 2.6.2. *Given* 6 *input points* $(x_i, y_i) \in \mathbb{P}^2 \times \mathbb{P}^2$*, let* $\mathcal{B}_{rackets}(Z_6)$ *be the ideal generated by the differences of the two sides in the following bracket equations:*

$$[ikp]_x[jrp]_x[irs]_y[jks]_y = [irp]_x[jkp]_x[iks]_y[jrs]_y \tag{2.39}$$

for all distinct indices $i, j, k, r, p, s \in \{1, 2, 3, 4, 5, 6\}$.

Up to permutations, there are 30 equalities in (2.39): one for each choice of p and s. Under sufficient genericity, the bracket equations are the planar cross-ratio equalities:

$$(x_i, x_j; x_k, x_r; x_p) = (y_i, y_j; y_k, y_r; y_s). \tag{2.40}$$

Next we define a specific kind of highly degenerate configuration for which Z_6 does not drop rank.

Definition 2.6.3. *We say that* $\{(x_i, y_i)\}_{i=1}^6$ *is an* ***asymmetric double triangle*** *if the* $\{x_i\}$ *and* $\{y_i\}$ *each consist of* 3 *distinct points such that, up to reordering,* $y_1 = y_2$*,* $y_3 = y_4$*,* $y_5 = y_6$*,* $x_1 = x_4$*,* $x_2 = x_5$*, and* $x_3 = x_6$*.*

Let $\mathcal{M}_{inors}(Z_6)$ denote the ideal of maximal minors of Z_6. The following statement can be checked using Macaulay2.

Theorem 2.6.4. *Let* $\{(x_i, y_i)\}_{i=1}^6$ *be such that neither the* x_i*'s nor the* y_i*'s are in a line. Then* Z_6 *is rank deficient if and only if one of the inherited conditions hold or the* 30 *bracket equations* (2.39) *hold and the configuration is not in an asymmetric double triangle.*

Proof. As in the proof of Theorem 2.5.1 we split into two cases. Applying homographies we can assume either that

$$x_1 = (1, 0, 0) = y_1 \qquad x_2 = (0, 1, 0) = y_2 \qquad x_3 = (0, 0, 1) = y_3.$$

or

$$x_1 = (1,0,0) = y_1 \qquad x_2 = (0,1,0) = y_2 \qquad x_3 = (0,0,1) = y_4.$$

In the first case the ideal $\mathcal{M}_{inors}(Z_6)$ decomposes into 19 components. The first 6 correspond to invalid inputs $x_i, y_i = (0,0,0)$, the next 12 correspond to repeated point pairs, and the 19th component we refer to as M_{19}. The ideal $\mathcal{B}_{rackets}(Z_6)$ decomposes into 13 components. One component is M_{19} and the other 12 components correspond to asymmetric double triangles.

In the second case the ideal $\mathcal{M}_{inors}(Z_6)$ decomposes into 17 components. The first 6 correspond to invalid inputs $x_i, y_i = (0,0,0)$, the next 10 correspond to repeated point pairs, and the 17th component we refer to as M_{17}. The ideal $\mathcal{B}_{rackets}(Z_6)$ decomposes into 21 components. One component is M_{17} and the other 20 components correspond to asymmetric double triangles. $\square$

In Theorem 2.6.4 the only assumption is that neither the x_i's nor the y_i's are in a line. In Theorem 2.6.7 we will tackle the case of one side in a line, and together the two theorems cover all possibilities.

Before we can do that, we consider an important subcase of Theorem 2.6.4, namely the generic case. The main result here is Theorem 2.6.6 which uses tools from invariant theory that we have not encountered thus far. Besides being a key result of this paper, it will also lead to a proof of Theorem 2.6.7, completing the story.

Suppose $x_1, \ldots, x_6 \in \mathbb{P}^2_x$ and $y_1, \ldots, y_6 \in \mathbb{P}^2_y$ are in general position. Then we may fix

$$\begin{aligned} x_1 = y_1 = (1,0,0), \qquad x_2 = y_2 = (0,1,0), \\ x_3 = y_3 = (0,0,1), \qquad x_4 = y_4 = (1,1,1) \end{aligned} \tag{2.41}$$

and consider the rank deficiency of Z_6 whose 5th and 6th rows are left symbolic as:

$$(x_{51}, x_{52}, x_{53}) \otimes (y_{51}, y_{52}, y_{53}), (x_{61}, x_{62}, x_{63}) \otimes (y_{61}, y_{62}, y_{63}). \tag{2.42}$$

Recall that the general position assumption guarantees that no submatrix of Z_6 with 5 rows is rank deficient. Under these assumptions, one can use Macaulay2 to check that $\mathcal{M}_{inors}(Z_6)$ is a radical ideal of dimension 9 and degree 4. Its prime decomposition consists of 14 components. The first 4

of these components correspond to invalid inputs $x_i, y_i = (0,0,0)$, the next 9 correspond to repeated points, and the 14th component, denoted as M_{14}, encodes a new set of conditions, and has dimension 8 and degree 35. Since Z_6 is rank deficient if and only if all its maximal minors vanish, we conclude that the new conditions for rank drop are exactly that (x_5, y_5) and (x_6, y_6) lie in the variety of M_{14}.

We will now interpret M_{14} in terms of the invariants of $x_1, \dots, x_6$ and $y_1, \dots, y_6$. For a set of 6 points $p_1, \dots, p_6 \in \mathbb{P}^2$ define

$$[(ij)(kl)(rs)] := [ijr][kls] - [ijs][klr]. \tag{2.43}$$

This is a classical invariant of points in $\mathbb{P}^2$ whose vanishing expresses that the three lines $\overline{p_i p_j}$, $\overline{p_k p_l}$ and $\overline{p_r p_s}$ meet in a point [12, pp 169]. Using these invariants, Coble defines the following 6 scalars [12, pp 170]:

$$\begin{aligned}
\bar{a} &= [(25)(13)(46)] + [(51)(42)(36)] + [(14)(35)(26)] + [(43)(21)(56)] + [(32)(54)(16)] \\
\bar{b} &= [(53)(12)(46)] + [(14)(23)(56)] + [(25)(34)(16)] + [(31)(45)(26)] + [(42)(51)(36)] \\
\bar{c} &= [(53)(41)(26)] + [(34)(25)(16)] + [(42)(13)(56)] + [(21)(54)(36)] + [(15)(32)(46)] \\
\bar{d} &= [(45)(31)(26)] + [(53)(24)(16)] + [(41)(25)(36)] + [(32)(15)(46)] + [(21)(43)(56)] \\
\bar{e} &= [(31)(24)(56)] + [(12)(53)(46)] + [(25)(41)(36)] + [(54)(32)(16)] + [(43)(15)(26)] \\
\bar{f} &= [(42)(35)(16)] + [(23)(14)(56)] + [(31)(52)(46)] + [(15)(43)(26)] + [(54)(21)(36)]
\end{aligned} \tag{2.44}$$

Let $\bar{a}_x, \bar{b}_x, \bar{c}_x, \bar{d}_x, \bar{e}_x, \bar{f}_x$ be the polynomials obtained by computing the expressions in (2.44) using 6 symbolic vectors $x_1, \dots, x_6$ where each $x_i = (x_{i1}, x_{i2}, x_{i3})^\top$. Similarly, let $\bar{a}_y, \bar{b}_y, \bar{c}_y, \bar{d}_y, \bar{e}_y, \bar{f}_y$ be obtained from symbolic $y_1, \dots, y_6$ where each $y_i = (y_{i1}, y_{i2}, y_{i3})^\top$.

Definition 2.6.5. *Let $\mathcal{I}_{nvariants}(Z_6)$ denote the ideal of 2×2 minors of*

$$\begin{bmatrix} \bar{a}_x & \bar{b}_x & \bar{c}_x & \bar{d}_x & \bar{e}_x & \bar{f}_x \\ \bar{a}_y & \bar{b}_y & \bar{c}_y & \bar{d}_y & \bar{e}_y & \bar{f}_y \end{bmatrix} \tag{2.45}$$

after specializing the first 4 points (x_i, y_i) as in (2.41).

The ideal $\mathcal{I}_{nvariants}(Z_6)$ is radical of dimension 9 and degree 4. Its prime decomposition consists of 14 components. The first 4 of these components correspond to invalid inputs $x_i, y_i = (0,0,0)$, the next 8 correspond to 3 repeated points in either $\mathbb{P}^2_x$ or $\mathbb{P}^2_y$, the 13th component corresponds to $x_5 \sim y_5$ and $x_6 \sim y_6$ (which encodes that $\{x_i\}$ and $\{y_i\}$ are related by a homography). The 14th component, denoted as I_{14}, provides a new condition. This ideal has dimension 8 and degree 35.

Theorem 2.6.6. *Given a configuration $\{(x_i, y_i)\}_{i=1}^6$ such that the $\{x_i\}$ and $\{y_i\}$ are in general position,*

$$\begin{pmatrix} \bar{a}_x & \bar{b}_x & \bar{c}_x & \bar{d}_x & \bar{e}_x & \bar{f}_x \end{pmatrix} \sim \begin{pmatrix} \bar{a}_y & \bar{b}_y & \bar{c}_y & \bar{d}_y & \bar{e}_y & \bar{f}_y \end{pmatrix} \tag{2.46}$$

if and only if either the matrix Z_6 is rank deficient or $y_i = Hx_i$ for some homography $H \in$ PGL(3).

Proof. It can be checked using Macaulay2 that $M_{14} = I_{14}$. ☐

The statement of Theorem 2.6.6 begs the question of how homography ties in with rank drop, if at all. We give an example to show that the existence of a homography such that $y_i = Hx_i$ for 6 points (x_i, y_i) is not sufficient for Z_6 to be rank deficient. In contrast, it does suffice for 7 point pairs.

Example 4. By Lemma 2.2.1, if there is a $H \in$ PGL(3) such that $y_i = Hx_i$ for $i = 1, \ldots, 6$, then we may assume that $x_i = y_i$ for $i = 1, \ldots, 6$. Then the null space of Z_6 consists exactly of the vectors vec(F) for the 3×3 matrices F such that

$$x_i^\top F x_i = 0, \ \forall\, i = 1, \ldots, 6. \tag{2.47}$$

Generically, this is exactly the set of 3×3 skew-symmetric matrices, Skew$_3$ $\cong \mathbb{P}^2$. Hence $\operatorname{rank}(Z_6) = 6$. For example, consider

$$\begin{array}{lll} x_1 = y_1 = (1,0,0) & x_2 = y_2 = (0,1,0) & x_3 = y_3 = (0,0,1) \\ x_4 = y_4 = (1,1,1) & x_5 = y_5 = (2,3,1) & x_6 = y_6 = (3,7,1) \end{array}$$

for which the nullspace of Z_6 is Skew$_3$. In general, if the homography is given by the matrix H then the null space of Z_6 is obtained by left action of $H^\top$ on Skew$_3$. We note that this null space is unusual in that will always consist entirely of rank 2 matrices.

To finish our algebraic discussion, we consider the case of one $\mathbb{P}^2$ having all input points in a line. In order to state the result, we introduce the classical **Joubert invariants** of 6 points in $\mathbb{P}^1$ [12] which are:

$$\begin{aligned}
A_y &= [25][13][46] + [51][42][36] + [14][35][26] + [43][21][56] + [32][54][16] \\
B_y &= [53][12][46] + [14][23][56] + [25][34][16] + [31][45][26] + [42][51][36] \\
C_y &= [53][41][26] + [34][25][16] + [42][13][56] + [21][54][36] + [15][32][46] \\
D_y &= [45][31][26] + [53][24][16] + [41][25][36] + [32][15][46] + [21][43][56] \\
E_y &= [31][24][56] + [12][53][46] + [25][41][36] + [54][32][16] + [43][15][26] \\
F_y &= [42][35][16] + [23][14][56] + [31][52][46] + [15][43][26] + [54][21][36].
\end{aligned} \tag{2.48}$$

Theorem 2.6.7. *Suppose $y_1, \dots, y_6 \in \mathbb{P}^2_y$ are in a line. Then Z_6 is rank deficient if and only if*

$$\bar{a}_x A_y + \bar{b}_x B_y + \bar{c}_x C_y + \bar{d}_x D_y + \bar{e}_x E_y + \bar{f}_x F_y = 0 \tag{2.49}$$

Furthermore, if $x_1, \dots, x_6$ are in general position and $y_1, \dots, y_6$ are distinct, then (2.49) *holds if and only if there exists a projection $T : \mathbb{P}^2_x \to \mathbb{P}^1_y$ such that $T(x_i) \sim y_i$ for all i.*

This theorem is analogous to Theorems 2.4.2 (3)(b) and 2.5.2. The proof of this theorem will follow from the geometry that we develop in the next section.

2.6.2 *Geometric Characterizations of Rank Deficiency*

We next explain where the algebraic results (and ideals) from the previous section come from by providing a geometric characterization of rank deficiency of Z_6. This relies on the classical theory of smooth cubic surfaces in $\mathbb{P}^3$. For the convenience of the reader, we have collected the needed facts about cubic surfaces in Appendix 2.7.

Cubic surfaces and Schäfli double sixes

A cubic surface in $\mathbb{P}^3$ is a hypersurface defined by a single homogenous polynomial of degree 3 in $\mathbb{C}[z_0, z_1, z_2, z_3]$ where $(z_0, \dots, z_3)$ are the coordinates of $\mathbb{P}^3$.

Definition 2.6.8. *A pair of 6 lines* $\{\ell_1, \ldots, \ell_6\}$ *and* $\{\ell_1', \ldots, \ell_6'\}$ *in* $\mathbb{P}^3$ *is called a* **Schläfli double six** *if*

1. *the* 6 *lines in each set are mutually skew (i.e., they don't intersect pairwise), and*
2. ℓ_i *intersects* ℓ_j' *if and only if* $i \neq j$.

A smooth cubic surface has 36 Schläfli double sixes. The rich and highly symmetric properties of these line configurations are listed in Appendix 2.7 and lie at the heart of all of our major results in this section. Our main (geometric) theorem in this section is the following.

Theorem 2.6.9. *Suppose* $\{(x_i, y_i)\}_{i=1}^6 \subset \mathbb{P}^2 \times \mathbb{P}^2$ *is in general position. Then* Z_6 *is rank deficient if and only if there exists a cubic surface* $\mathcal{S}$ *that can be obtained as the blow up of both* $\mathbb{P}_x^2$ *in* $x_1, \ldots, x_6$ *and* $\mathbb{P}_y^2$ *in* $y_1, \ldots, y_6$ *such that their respective exceptional curves* $\{\ell_1, \ldots, \ell_6\}, \{\ell_1', \ldots, \ell_6'\}$ *form a Schläfli double six on* $\mathcal{S}$*. When* Z_6 *is rank deficient, this cubic surface has the determinantal representation* $\mathcal{N}(Z_6) \cap \mathcal{D}$*, where* $\mathcal{N}(Z_6)$ *denotes the right null space of* Z_6 *and* $\mathcal{D}$ *is the cubic hypersurface of all rank deficient* 3×3 *matrices.*

To prepare for the proof of this theorem, we begin by constructing a cubic surface from 5 points $\{(x_i, y_i)\}_{i=1}^5$ in general position. Recall that we can identify a point in $\mathbb{P}^8$ with a 3×3 matrix M via its vectorization vec(M). Under this identification, $\mathcal{D} \subset \mathbb{P}^8$ is the cubic hypersurface $\mathcal{D} = \{M : \det(M) = 0\}$. Let $\mathcal{N}(Z_5) \subset \mathbb{P}^8$ be the null space of Z_5. Since $\{(x_i, y_i)\}_{i=1}^5$ is in general position, $\operatorname{rank}(Z_5) = 5$ and so, $\mathcal{N}(Z_5) \cong \mathbb{P}^3$. Therefore,

$$S := \mathcal{N}(Z_5) \cap \mathcal{D} \tag{2.50}$$

is a cubic surface in $\mathcal{N}(Z_5) \cong \mathbb{P}^3 \subset \mathbb{P}^8$. This surface is smooth by the general position assumption. (We note that if $\{x_i\}$ and $\{y_i\}$ were related by a homography then S would be degenerate (see Example 4).)

The surface S has a natural *determinantal representation* (see Section 2.7 for the definition): Pick a basis $M_0, \ldots, M_3$ of $\mathcal{N}(Z_5)$ and set

$$M(z) = z_0 M_0 + \cdots + z_3 M_3 \tag{2.51}$$

where $z_i \in \mathbb{C}$. If $(x_i, y_i), i = 1, \dots, 5$ are real, we can choose $M_0, \dots, M_3$ to be real matrices. Since $M(z)$ is a parameterization of $\mathcal{N}(Z_5)$, S is cut out by $\det(M(z)) = 0$, and $M(z)$ is a determinantal representation of S. Generically, the 3-dimensional $\mathcal{N}(Z_5)$ does not intersect the 4-dimensional $\mathbb{P}^2 \times \mathbb{P}^2$ of rank one matrices, and so a point $M(z)$ on S has rank exactly 2. This also follows from S being smooth.

Next we construct a Schläfli double six on the cubic surface $S = \mathcal{N}(Z_5) \cap \mathcal{D}$, from $\{(x_i, y_i)\}_{i=1}^5$.

Definition 2.6.10. *Given $\{(x_i, y_i)\}_{i=1}^5$ in general position, define*

$$\ell_{x_i} := \{M(z) \,:\, M(z)x_i = 0\} \text{ for } i = 1, \dots, 5, \text{ and} \tag{2.52}$$

$$\ell'_{y_j} := \{M(z) \,:\, y_j^\top M(z) = 0\} \text{ for } j = 1, \dots, 5, \tag{2.53}$$

where $M(z)$ is of the form (2.51).

For each $i = 1, \dots, 5$, $\ell_{x_i} \subset \mathcal{N}(Z_5)$ by the definition of $M(z)$ in (2.51). Further, since $M(z)x_i = 0$, $M(z)$ has a non-trivial right null space and hence, $\det(M(z)) = 0$. Therefore, $\ell_{x_i} \subset S = \mathcal{N}(Z_5) \cap \mathcal{D}$. The condition that $M(z) \in \mathcal{N}(Z_5)$ is equivalent to $M(z)$ satisfying the 5 linear constraints

$$\langle x_i^\top \otimes y_i^\top, M(z) \rangle = y_i^\top M(z) x_i = 0 \text{ for } i = 1, \dots, 5. \tag{2.54}$$

Therefore, ℓ_{x_i} is cut out by 8 linear constraints including the 3 constraints given by $M(z)x_i = 0$. However one of them is redundant since $M(z)x_i = 0$ implies that $y_i^\top M(z)x_i = 0$. Therefore, ℓ_{x_i} is cut out by 7 independent linear constraints and is hence a line on the cubic surface S. Similarly, each $\ell'_{y_j}, j = 1, \dots, 5$ is a line on S.

Lemma 2.6.11. *The lines ℓ_{x_i} and ℓ'_{y_j} satisfying the following incidence relations:*

1. *The lines $\{\ell_{x_i}, i = 1, \dots, 5\}$ are mutually skew.*

2. *The lines $\{\ell'_{y_j}, j = 1, \dots, 5\}$ are mutually skew.*

3. *ℓ_{x_i} intersects ℓ'_{y_j} if and only if $i \neq j$.*

Therefore, $\{\ell_{x_i}, i = 1, \dots, 5\}$ *and* $\{\ell'_{y_j}, j = 1, \dots, 5\}$ *form part of a unique Schläfli double six on* S*.*

Proof. The proof of the incidence relations follows from dimension counting and can be found in [58, Lemma 6.1] The two sets of lines are part of a unique Schläfli double six on S since each double six is already uniquely defined by three of its line pairs, and we have 5 line pairs here with the correct incidences (see Lemma 2.7.1 (3)). □

By Lemma 2.7.1, given a Schläfli double six on S, there are two sets of 6 points $\{p_i\} \subset \mathbb{P}^2_p$ and $\{q_j\} \subset \mathbb{P}^2_q$ such that S is the blow up of $\mathbb{P}^2_p$ at $\{p_i\}$ with blow up morphism $\pi_p : S \to \mathbb{P}^2_p$, and also the blow up of $\mathbb{P}^2_q$ at $\{q_j\}$ with blow up morphism $\pi'_q : S \mapsto \mathbb{P}^2_q$. Each set of skew lines in the given double six is the set of exceptional curves of these point sets under π_p and π'_q. The blow up morphism $\pi_p : S \to \mathbb{P}^2_p$ sends $M(z) \in S$ to the unique $p \in \mathbb{P}^2$ in its right null space and the blow up morphism $\pi'_q : S \to \mathbb{P}^2_q$ sends $M(z) \in S$ to the unique $q \in \mathbb{P}^2$ in its left null space. See the second half of Appendix 2.7.

We are now ready to prove the forward direction of Theorem 2.6.9.

Definition 2.6.12. *Let* ℓ_6, ℓ'_6 *denote the* 6*th pair of lines in the Schläfli double six on* S *consisting of* $\{\ell_{x_i}\}_{i=1}^5$ *and* $\{\ell'_{y_j}\}_{j=1}^5$ *from Lemma 2.6.11.*

Lemma 2.6.13. *Suppose* $\{(x_i, y_i)\}_{i=1}^5$ *is in general position. Then the addition of a new row* $x_6^\top \otimes y_6^\top$ *makes* Z_6 *rank deficient if and only if* $\{\ell_{x_i}\}_{i=1}^6$ *and* $\{\ell'_{y_j}\}_{j=1}^6$ *form a Schläfli double six on* S *where*

$$\ell_6 = \ell_{x_6} = \{M(z) : M(z)x_6 = 0\} \quad \text{and} \quad \ell'_6 = \ell'_{y_6} = \{M(z) : y_6^\top M(z) = 0\}. \tag{2.55}$$

Proof. To prove the forward direction, suppose Z_6 is rank deficient. Then $\mathcal{N}(Z_5) = \mathcal{N}(Z_6)$ and $S = \mathcal{N}(Z_6) \cap \mathcal{D}$. Check that ℓ_{x_6} is a line on S by the same argument as for $\ell_{x_i}, i \leq 5$. Similarly, ℓ'_{y_6} is a line on S. One can also check that the collection $\{\ell_{x_i}\}_{i=1}^6$ and $\{\ell'_{y_j}\}_{j=1}^6$ satisfies the incidence relations of a Schläfli double six. This must mean that $\ell_6 = \ell_{x_6}$ and $\ell'_6 = \ell'_{y_6}$ since the first 5 line pairs determine a unique double six on S.

To prove the reverse direction, suppose we have a Schläfli double six on S as in the statement of the lemma. Then by the dicussion before Definition 2.6.12, S is the blow up of $\mathbb{P}^2_x$ at $\{x_i\}$ under a morphism π_x and S is also the blow up of $\mathbb{P}^2_y$ at $\{y_i\}$ under a morphism π'_y. Further, $\pi_x(\ell_{x_i}) = x_i$ for $i = 1, \ldots, 6$ and $\pi'_y(\ell'_{y_j}) = y_j$ for $j = 1, \ldots, 6$. We need to argue that $Z_6 = (x_i^\top \otimes y_i^\top)_{i=1}^6$ is rank deficient. Recall Lemma 2.6.1 and let (x_0, y_0) be the unique new point pair such that Z' with rows $x_i^\top \otimes y_i^\top$ for $i = 0, \ldots, 5$ is rank deficient. Then, by the forwards direction, we must have ℓ_6 be the exceptional curve of x_0 under π_x and hence $x_0 = \pi_x(\ell_6) = x_6$. Similarly, $y_0 = \pi_y(\ell'_6) = y_6$. We conclude that $(x_0, y_0) = (x_6, y_6)$ and therefore, Z_6 is rank deficient. □

The following corollary to Lemma 2.6.13 and Theorem 2.7.5 concludes the proof of the forward direction of Theorem 2.6.9.

Corollary 2.6.14. *If $\{(x_i, y_i)\}_{i=1}^6$ is in general position and Z_6 is rank deficient then the cubic surface $S = \mathcal{N}(Z_6) \cap \mathcal{D}$ is the blow up of $\mathbb{P}^2_x$ at $x_1, \ldots, x_6$ with exceptional curves $\ell_{x_i}, i = 1, \ldots, 6$, and also the blow up of $\mathbb{P}^2_y$ at $y_1, \ldots, y_6$ with exceptional curves $\ell'_{y_j}, j = 1, \ldots, 6$. The two sets of lines form a Schläfli double six on S.*

Note that Lemma 2.6.13 gives a second construction for the unique point pair (x_6, y_6) in Lemma 2.6.1. Namely, as the images under the blow up morphisms, of the final pair of lines in the unique Schläfli double six on S determined by the lines $\{\ell_{x_i}\}_{i=1}^5$ and $\{\ell'_{y_j}\}_{j=1}^5$. This was also noted in [58].

To prove the backward direction of Theorem 2.6.9, we will first prove a counterpart to Lemma 2.6.1.

Lemma 2.6.15. *If $x_1, \ldots, x_6$ are in general position then there is a unique set of 6 points $y_1, \ldots, y_6$ in general position (up to homography) such that Z_6 is rank deficient. If we assume that*

$$x_1 = (1,0,0) = y_1 \qquad x_2 = (0,1,0) = y_2$$
$$x_3 = (0,0,1) = y_3 \qquad x_4 = (1,1,1) = y_4.$$

then we can obtain y_5, y_6 via the rational function

$$y_5 = \left(\frac{x_{63} - x_{62}}{x_{53}x_{62} - x_{63}x_{52}}, \frac{x_{63} - x_{61}}{x_{53}x_{61} - x_{63}x_{51}}, \frac{x_{62} - x_{61}}{x_{61}x_{52} - x_{51}x_{62}}\right) \tag{2.56}$$

$$y_6 = \left(\frac{x_{53} - x_{52}}{x_{53}x_{62} - x_{63}x_{52}}, \frac{x_{53} - x_{51}}{x_{53}x_{61} - x_{63}x_{51}}, \frac{x_{52} - x_{51}}{x_{61}x_{52} - x_{51}x_{62}}\right). \tag{2.57}$$

Proof. We first prove that this construction will result in rank deficiency; the uniqueness will come afterwards as a consequence of Corollary 2.6.14. We construct the matrix $Z_6 = (x_i^\top \otimes y_i^\top)_{i=1}^6$ and fix $x_1, \dots x_4, y_1, \dots y_6$ as in the statement of the Lemma. We can then verify with Macaulay2 that all 6×6 minors of Z_6 are 0 and therefore Z_6 is rank deficient.

To see that this choice is unique, construct $S = \mathcal{N}(Z_6) \cap \mathcal{D}$. Then by Corollary 2.6.14, S is both the blow up of $\mathbb{P}_x^2$ at $x_1, \dots, x_6$ and the blow up of $\mathbb{P}_y^2$ at $y_1, \dots, y_6$ and the two sets of exceptional curves $\{\ell_{x_1}, \dots, \ell_{x_6}\}, \{\ell'_{y_1}, \dots, \ell'_{y_6}\}$ form a Schläfli double six. Now suppose there was some other $y'_1, \dots, y'_6$ in general position such that $Z'_6 = (x_i^\top \otimes y_i'^\top)_{i=1}^6$ was rank deficient. Then $S' = \mathcal{N}(Z'_6) \cap \mathcal{D}$ is also the blow up of $\mathbb{P}_x^2$ at $x_1, \dots, x_6$. Moreover, it is also the blow up of $\mathbb{P}_y^2$ at $y'_1, \dots, y'_6$ and the exceptional curves $\ell_{x_1}, \dots, \ell_{x_6}, \ell'_{y'_1}, \dots, \ell'_{y'_6}$ form a Schläfli double six. It follows by Lemma 2.7.2 that $\ell'_{y_i} = \ell'_{y'_i}$ for all i. If we compose the blow up morphisms $\pi_{y'} \circ \pi_y^{-1} : \mathbb{P}_y^2 \to \mathbb{P}_y^2$ we obtain an invertible projective transformation; thus this composition is a homography taking $y_i \mapsto y'_i$. This concludes the proof. $\square$

The formula shows that if $x_1, \dots, x_6$ are real then, up to homography, $y_1, \dots, y_6$ will also be real.

Example 5. Consider Example 3 again and now fix the points

$$x_1 = (1,0,0) \quad x_2 = (0,1,0) \qquad\qquad y_1 = (1,0,0) \quad y_2 = (0,1,0)$$

$$x_3 = (0,0,1) \quad x_4 = (1,1,1) \qquad\qquad y_3 = (0,0,1) \quad y_4 = (1,1,1)$$

$$x_5 = (3,5,1) \quad x_6 = (-\frac{1}{3}, \frac{7}{5}, \frac{3}{17}).$$

If we leave y_5, y_6 symbolic and construct the matrix Z_6, we can verify with Macaulay2 that the unique new points that will result in rank deficiency are $y_5 \sim (8,2,1),\ y_6 \sim (-\frac{4}{3}, \frac{2}{5}, -\frac{1}{17})$, which is exactly what we get from the formula above. Moreover, check that this matches up exactly with the points in Example 3.

The following corollary now completes the reverse direction of Theorem 2.6.9.

Corollary 2.6.16. *Given a smooth cubic surface $\mathcal{S}$ with a Schläfli double six $\{\ell_i\}_{i=1}^6$ and $\{\ell'_j\}_{i=1}^6$ that arises as the exceptional lines of $x_1, \ldots, x_6 \in \mathbb{P}_x^2$ and $y_1, \ldots y_6 \in \mathbb{P}_y^2$, the matrix $Z_6 = (x_i^\top \otimes y_i^\top)_{i=1}^6$ is rank deficient.*

Proof. Lemma 2.6.15 provides us with unique (up to homography) points $y'_1, \ldots, y'_6 \in \mathbb{P}_y^2$ such that $Z'_6 = (x_i^\top \otimes {y'}_i^\top)_{i=1}^6$ is rank deficient. Then, following along the same lines as the proof of Lemma 2.6.15, $S = \mathcal{N}(Z'_6) \cap \mathcal{D}$ and the points $\{y_i\}$ and $\{y'_i\}$ must be related by homography. It follows by Lemma 2.2.1 that Z_6 is rank deficient. □

Combining Corollaries 2.6.14 and 2.6.16 we obtain a method to compute a determinantal representation of a smooth cubic surface starting with a Schläfli double six on it.

Corollary 2.6.17. *Let $\mathcal{S}$ be a smooth cubic surface with a Schläfli double six, $\{\ell_i\}$ and $\{\ell'_j\}$, and let $\{(x_i, y_i)\}_{i=1}^6 \subset \mathbb{P}^2 \times \mathbb{P}^2$ be such that ℓ_i are the exceptional lines of the blow up of $\mathbb{P}_x^2$ at $x_1, \ldots, x_6$ and ℓ'_6 are the exceptional lines of the blow up of $\mathbb{P}_y^2$ at $y_1, \ldots y_6$. Then a determinantal representation of $\mathcal{S}$ is given by $M(z) = \sum z_i M_i$ where $M_0, \ldots, M_3$ is a basis of the null space of $Z_6 = (x_i^\top \otimes y_i^\top)_{i=1}^6$.*

This gives us the final statement of Theorem 2.6.9, concluding the proof.

Remark 2.6.18. In the classical literature one finds a determinantal representation of a cubic surface from 6 points in a $\mathbb{P}^2$ (whose blow up at these points is the surface), via the Hilbert-Burch theorem (see Appendix 2.7). It is worthwhile to contrast that method with that in Corollary 2.6.17 where the starting point is 6 point pairs in $\mathbb{P}^2 \times \mathbb{P}^2$. Then a determinantal representation (2.51) of the surface can be obtained from a basis of $\mathcal{N}(Z_6)$ using simple linear algebra.

Conic intersections and the bracket ideal $\mathcal{B}_{rackets}(Z_6)$

We now use Theorem 2.6.9 to explain the ideal $\mathcal{B}_{rackets}(Z_6)$ used in the algebraic characterization of rank deficiency in Theorem 2.6.4.

Consider a Schläfli double six $\{\ell_i\}_{i=1}^6, \{\ell'_j\}_{j=1}^6$ on a cubic surface $\mathcal{S}$ where the lines ℓ_i are the exceptional lines under the blow up π_x of points $x_i \in \mathbb{P}_x^2$ and the lines ℓ'_j are the exceptional

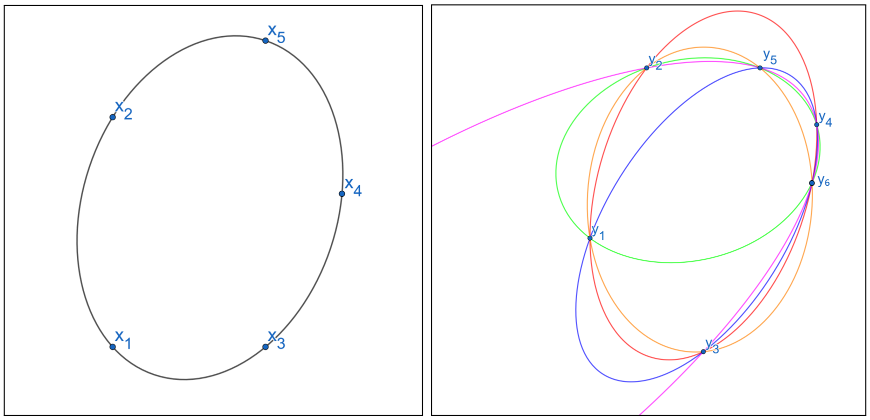

Figure 2.3: The point y_6 is the unique intersection point of the 5 conics C'_i in $\mathbb{P}^2_y$.

lines under the blow up π'_y of points $y_j \in \mathbb{P}^2_y$. Let C_i be the unique conic passing through $\{x_1, \ldots, x_6\} \backslash \{x_i\}$ and C'_i the unique conic passing through $\{y_1, \ldots, y_6\} \backslash \{y_i\}$. By Lemma 2.7.1, the lines ℓ_i are the strict transforms of the conics $C'_i \subset \mathbb{P}^2_y$ and and the lines ℓ'_j are the strict transforms of the conics $C_j \subset \mathbb{P}^2_x$. There is then a birational automorphism $\mathbb{P}^2_x \to S \to \mathbb{P}^2_y$ such that $x_i \mapsto C'_i$ and $C_i \mapsto y_i$. This map is a degree 5 Cremona transformation and can be formalized using the following recipe due to Rudolf Sturm [65], mentioned by Werner in [68]. These conics can be used to give a third construction of the unique 6th point pair (x_6, y_6) that can be added to $\{(x_i, y_i)\}_{i=1}^5$ in general position so that Z_6 is rank deficient. We state the result without proof in Lemma 2.6.19 and illustrate it in Figure 2.3. The data for this figure was chosen to make the conics easily visible.

Lemma 2.6.19. [65] *Let C be the unique conic through $x_1, \ldots, x_5$ in general position, and let H_i be the unique homography such that $y_j = H_i x_j$ for all $j \neq i$. Define $C'_i := H_i(C)$ for $i = 1, \ldots, 5$. Then $y_6 \in \bigcap_{i=1}^5 C'_i$ is the unique point in the intersection of the conics C'_i.*

Similarly, let C' be the unique conic through $y_1, \ldots, y_5$, and let G_j be the unique homography such that $x_i = G_j y_i$ for all $i \neq j$. Define $C_j := H_j(C')$ for $j = 1, \ldots, 5$. Then $x_6 \in \bigcap_{j=1}^5 C_j$ is the unique point in the intersection of the conics C_j.

We can now restate Theorem 2.6.9 using Lemma 2.6.19 to yield another characterization of minimal rank deficiency of Z_6.

Theorem 2.6.20. *Let $\{(x_i, y_i)\}_{i=1}^{5}$ be in general position. Then (x_6, y_6) is the unique new point under which Z_6 is rank deficient if and only if y_6 is the unique intersection point $\cap_{j=1}^{5} C_i'$ and x_6 is the unique intersection point $\cap_{j=1}^{5} C_j$.*

Theorem 2.6.20 explains the bracket and cross-ratio conditions that appear in Theorem 2.6.4. Let C be the conic through $x_1, \ldots, x_5$ and let H be the homography such that $Hx_i = y_i$ for all $i = 1, 2, 3, 4$. Then $C' := HC$ is a conic that passes through $y_1 = Hx_1, y_2 = Hx_2, y_3 = Hx_3, y_4 = Hx_4$ and Hx_5. By Lemma 2.3.5, $y_6 \in C'$ if and only if

$$\det(H)[135]_x \det(H)[245]_x[146]_y[236]_y = [136]_y[246]_y \det(H)[145]_x \det(H)[235]_x. \qquad (2.58)$$

This can be rewritten as the bracket equation (2.39) by eliminating $\det(H)$:

$$[135]_x[245]_x[146]_y[236]_y = [136]_y[246]_y[145]_x[235]_x \qquad (2.59)$$

The other 29 bracket equations follow by selecting different conics and homographies.

The Cremona hexahedral form of a cubic surface

We now give a third characterization of minimal rank deficiency of Z_6 via the *Cremona hexahedral form* of a cubic surface. This will explain the statement of Theorem 2.6.6.

Recall that every smooth cubic surface is the blow up of $\mathbb{P}^2$ at 6 points in general position. The Cremona hexahedral form of the surface provides explicit equations for it in terms of the points being blown up. We briefly describe these equations and the properties we need. More details can be found in [12, Section 4], [24, Section 9.4.3], [56, Section 5.6].

Given 6 points $p_1, \ldots, p_6 \in \mathbb{P}^2$, consider the following list of 6 cubic polynomials in $u =$

(u_0, u_1, u_2), the coordinates of $\mathbb{P}^2$, that vanish on them.

$$\begin{aligned}
a(u) &= [25u][13u][46u] + [51u][42u][36u] + [14u][35u][26u] + [43u][21u][56u] + [32u][54u][16u]\\
b(u) &= [53u][12u][46u] + [14u][23u][56u] + [25u][34u][16u] + [31u][45u][26u] + [42u][51u][36u]\\
c(u) &= [53u][41u][26u] + [34u][25u][16u] + [42u][13u][56u] + [21u][54u][36u] + [15u][32u][46u]\\
d(u) &= [45u][31u][26u] + [53u][24u][16u] + [41u][25u][36u] + [32u][15u][46u] + [21u][43u][56u]\\
e(u) &= [31u][24u][56u] + [12u][53u][46u] + [25u][41u][36u] + [54u][32u][16u] + [43u][15u][26u]\\
f(u) &= [42u][35u][16u] + [23u][14u][56u] + [31u][52u][46u] + [15u][43u][26u] + [54u][21u][36u]
\end{aligned} \tag{2.60}$$

In fact, each summand in each cubic is the product of the lines through 3 disjoint pairs of the 6 points and already vanishes on the points. Formally these expressions are similar to the scalars in (2.44). These cubic polynomials are *covariants* of $p_1, \ldots, p_6$ under the action of PGL(3), and appear in the work of Coble [12].

The 6 cubics in (2.60) span the 4-dimensional vector space of cubics that vanish on $p_1, \ldots, p_6$ and can be used to blow up $\mathbb{P}^2$ at $p_1, \ldots, p_6$ to obtain a cubic surface S. This surface is the closure of

$$\{(a(u), b(u), c(u), d(u), e(u), f(u)) \ : \ u \in \mathbb{P}^2\}) \subset \mathbb{P}^5, \tag{2.61}$$

and the zero set of the **Cremona hexahedral equations**:

$$\begin{aligned}
z_1^3 + z_2^3 + z_3^3 + z_4^3 + z_5^3 + z_6^3 &= 0\\
z_1 + z_2 + z_3 + z_4 + z_5 + z_6 &= 0\\
\bar{a} z_1 + \bar{b} z_2 + \bar{c} z_3 + \bar{d} z_4 + \bar{e} z_5 + \bar{f} z_6 &= 0
\end{aligned} \tag{2.62}$$

where the scalars $\bar{a}, \ldots, \bar{f}$ are as in (2.44). Here, $z_1, \ldots, z_6$ as the coordinates of $\mathbb{P}^5$.

The **Cremona hexahedral form** of S can be used to find the 27 lines on S explicitly via expressions for the 45 tritangent planes of S, see [12, Section 4]. In particular the 15 lines that are not part of the Schläfli double six corresponding to the blow up are obtained by permuting the equations:

$$z_1 + z_4 = z_2 + z_5 = z_3 + z_6 = 0. \tag{2.63}$$

Under the blow up morphism $\pi : S \to \mathbb{P}^2$, these lines correspond to specific lines $\overline{p_i p_j}$; the one above corresponds to the line between p_1 and p_2.

We can characterize the rank deficiency of Z_6 via the Cremona hexahedral form.

Theorem 2.6.21. *Let $x_1, \ldots, x_6 \in \mathbb{P}^2_x$ and $y_1, \ldots, y_6 \in \mathbb{P}^2_y$ be in general position and let S_x (respectively, S_y) be the blow up of $\mathbb{P}^2_x$ (respectively, $\mathbb{P}^2_y$) at $x_1, \ldots, x_6$ (respectively, $y_1, \ldots, y_6$) in Cremona hexahedral form. Then $S_x = S_y$ if and only if either there exists homography H such that $Hx_i = y_i$ for all $i = 1, \ldots, 6$ or the matrix Z_6 is rank deficient.*

Proof. If $S_x = S_y$, then they both contain the 15 lines in equation (2.63), none of which correspond to the proper transforms of either set of 6 points under their respective blow up morphisms. It follows that either the exceptional lines of $x_1, \ldots, x_6$ are exactly the exceptional lines of $y_1, \ldots, y_6$ or that the two sets are entirely distinct. Moreover, since each of the 15 lines in (2.63) have specific intersection conditions, either $\ell_{x_i} = \ell_{y_i}$ for all i, or $\{\ell_{x_1}, \ldots, \ell_{x_6}\}$ and $\{\ell_{y_1}, \ldots, \ell_{y_6}\}$ form a Schläfli double six.

In the first case, the exceptional lines of $x_1, \ldots, x_6$ are exactly the exceptional lines of $y_1, \ldots, y_6$. Then we can define an invertible projective transformation $\pi_y \circ \pi_x^{-1} : \mathbb{P}^2_x \to \mathbb{P}^2_y$, where π_x and π_y again refer to the blow up morphisms, such that $x_i \mapsto y_i$ for all i. This means that for each $i = 1, \ldots, 6$, x_i and y_i are related by a homography H.

In the second case the exceptional lines of $x_1, \ldots, x_6$ are distinct from the exceptional of $y_1, \ldots, y_6$. In this case they form a Schläfli double six and by Theorem 2.6.9, Z_6 is rank deficient.

To prove the reverse direction, suppose $Hx_i = y_i$ for some homography H. Then $S_x = S_y$ because $(\bar{a}_x, \ldots \bar{f}_x) \sim (\bar{a}_y, \ldots, \bar{f}_y)$ by the formula in (2.44) which means that S_x and S_y have the same Cremona hexahedral form. Similarly, if Z_6 is rank deficient then by Theorem 2.6.9, $\mathbb{P}^2_x$ blown up at $\{x_i\}$ agrees with $\mathbb{P}^2_y$ blown up at $\{y_i\}$ and hence $S_x = S_y$. $\square$

Theorem 2.6.21 provides a geometric proof of the algebraic characterization of rank deficiency in Theorem 2.6.6. As before, we subscript the cubics in (2.60) with x or y depending on whether they come from $\{x_i\}$ or $\{y_i\}$. The Cremona hexahedral equations (2.62) imply that the two cubic surfaces S_x and S_y are equal if and only if the following polynomials in u (coordinates on $\mathbb{P}^2_x$) and v (coordinates on $\mathbb{P}^2_y$) are identically zero.

$$\begin{aligned} \bar{a}_x a_y(v) + \bar{b}_x b_y(v) + \bar{c}_x c_y(v) + \bar{d}_x d_y(v) + \bar{e}_x e_y(v) + \bar{f}_x f_y(v) &\equiv 0 \\ \bar{a}_y a_x(u) + \bar{b}_y b_x(u) + \bar{c}_y c_x(u) + \bar{d}_y d_x(u) + \bar{e}_y e_x(u) + \bar{f}_y f_x(u) &\equiv 0. \end{aligned} \tag{2.64}$$

In other words,

$$\begin{aligned} \bar{a}_x z_1 + \bar{b}_x z_2 + \bar{c}_x z_3 + \bar{d}_x z_4 + \bar{e}_x z_5 + \bar{f}_x z_6 = 0 \qquad \forall z \in S_y \\ \bar{a}_y z_1 + \bar{b}_y z_2 + \bar{c}_y z_3 + \bar{d}_y z_4 + \bar{e}_y z_5 + \bar{f}_y z_6 = 0 \qquad \forall z \in S_x \end{aligned} \tag{2.65}$$

or equivalently, the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{a}_x & \bar{b}_x & \bar{c}_x & \bar{d}_x & \bar{e}_x & \bar{f}_x \\ \bar{a}_y & \bar{b}_y & \bar{c}_y & \bar{d}_y & \bar{e}_y & \bar{f}_y \end{bmatrix} \tag{2.66}$$

is rank deficient. One can check using Macaulay2 that the ideal of 3×3 minors of (2.66) coincides with the ideal $\mathcal{I}_{nvariants}(Z_6)$ of 2×2 minors of (2.45). Therefore, we conclude that S_x and S_y coincide if and only if (2.46) holds, which is the same as

$$\begin{pmatrix} \bar{a}_x & \bar{b}_x & \bar{c}_x & \bar{d}_x & \bar{e}_x & \bar{f}_x \end{pmatrix} = \lambda \begin{pmatrix} \bar{a}_y & \bar{b}_y & \bar{c}_y & \bar{d}_y & \bar{e}_y & \bar{f}_y \end{pmatrix}$$

for some non-zero scalar λ.

This concludes the discussion about 6 points $\{(x_i, y_i)\}_{i=1}^6$ in general position.

The Proof of Theorem 2.6.7

We can now use the tools developed in the previous subsection to prove Theorem 2.6.7 which is the last statement left to prove. This is the case of one $\mathbb{P}^2$ having all input points in a line; suppose this is $\mathbb{P}^2_y$. We first show that under this assumption, (2.64) can be simplified using Lemma 2.3.1. Note that if $y_1, \ldots, y_6$ are in a line ℓ then for all points $v \in \ell$

$$(a_y, b_y, c_y, d_y, e_y, f_y)(v) = (0, 0, 0, 0, 0, 0), \tag{2.67}$$

and for all points $v \in \mathbb{P}^2_y \backslash \ell$, using Lemma 2.3.1,

$$(a_y, b_y, c_y, d_y, e_y, f_y)(v) \sim (A_y, B_y, C_y, D_y, E_y, F_y) \tag{2.68}$$

where $A_y, \ldots, F_y$ are the Joubert Invariants introduced in (2.48). The Joubert invariants are known to satisfy

$$\begin{aligned} A + B + C + D + E + F &= 0 \\ A^3 + B^3 + C^3 + D^3 + E^3 + F^3 &= 0 \end{aligned} \tag{2.69}$$

(see [24, Theorem 9.4.10]), and any 5 of them give a complete basis for the space of invariants for 6 ordered points in $\mathbb{P}^1$. In particular, if $p_1, \ldots, p_6 \in \mathbb{P}^1$ and $q_1, \ldots, q_6 \in \mathbb{P}^1$ are each sets of 6 distinct points, then there exists a homography H sending $p_i \mapsto q_i$ if and only if

$$(A_p, \ldots, F_p) \sim (A_q, \ldots, F_q). \tag{2.70}$$

We further note that if $\{y_i\}_{i=1}^6$ are on a line then $\bar{a}_y = \cdots = \bar{f}_y = 0$ by looking at the definition of $[(ij)(kl)(rs)]$. We can then simplify (2.64) to the single equation (2.49) which was:

$$\bar{a}_x A_y + \bar{b}_x B_y + \bar{c}_x C_y + \bar{d}_x D_y + \bar{e}_x E_y + \bar{f}_x F_y = 0.$$

The following lemma proves the first part of Theorem 2.6.7.

Lemma 2.6.22. *Let* $\{(x_i, y_i)\}_{i=1}^6 \subset \mathbb{P}^2 \times \mathbb{P}^2$ *be such that the points in one* $\mathbb{P}^2$ *are in a line. Then* Z_6 *is rank deficient if and only if* (2.49) *holds.*

Proof. The proof is relatively straightforward. Suppose $y_1, \ldots, y_6$ are in a line. Applying homographies to each side we may assume that the x_i's are all finite and that $y_i = (y_{i1}, 1, 0)$. Then

$$Z_6 = \begin{bmatrix} x_{11}y_{11} & x_{11} & 0 & x_{12}y_{11} & x_{12} & 0 & y_{11} & 1 & 0 \\ x_{21}y_{21} & x_{21} & 0 & x_{22}y_{21} & x_{22} & 0 & y_{21} & 1 & 0 \\ x_{31}y_{31} & x_{31} & 0 & x_{32}y_{31} & x_{32} & 0 & y_{31} & 1 & 0 \\ x_{41}y_{41} & x_{41} & 0 & x_{42}y_{41} & x_{42} & 0 & y_{41} & 1 & 0 \\ x_{51}y_{51} & x_{51} & 0 & x_{52}y_{51} & x_{52} & 0 & y_{51} & 1 & 0 \\ x_{61}y_{61} & x_{61} & 0 & x_{62}y_{61} & x_{62} & 0 & y_{61} & 1 & 0 \end{bmatrix} \tag{2.71}$$

and Z_6 is rank deficient if and only if its only obviously non-zero 6×6 minor $\det(M) = 0$. Furthermore, a quick computation will give

$$\bar{a}_x A_y + \bar{b}_x B_y + \bar{c}_x C_y + \bar{d}_x D_y + \bar{e}_x E_y + \bar{f}_x F_y = 24 \det(M) \tag{2.72}$$

yielding the desired result. $\square$

Before we prove the second part of Theorem 2.6.7, we compute an example to illustrate the statement.

Example 6. Consider the following points in which the x points are in general position and the y points are all distinct and contained in a line. This can be seen as a configuration in $\mathbb{P}^2 \times \mathbb{P}^1$ so we fix

$$\begin{array}{llcll}
x_1 = (0,0,1) & y_1 = (0,1) & \qquad & x_4 = (1,1,1) & y_4 = (-4,1) \\
x_2 = (1,0,1) & y_2 = (1,1) & & x_5 = (3,5,1) & y_5 = (8,1) \\
x_3 = (0,1,1) & y_3 = (3,1) & & &
\end{array}$$

and leave x_6, y_6 as symbolic finite points. This creates a 6×6 matrix

$$Z_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 3 & 1 & 3 & 1 \\ -4 & 1 & -4 & 1 & -4 & 1 \\ 24 & 3 & 40 & 5 & 8 & 1 \\ x_{61}y_{61} & x_{61} & x_{62}y_{61} & x_{62} & y_{61} & 1 \end{bmatrix}. \tag{2.73}$$

(The reader may also view the y_i as points on the line at infinity in $\mathbb{P}^2$ where the third coordinate is 0; the non-zero part of the corresponding 6×9 matrix Z_6 is (2.73).)

The matrix Z_6 in (2.73) is rank deficient exactly when its determinant, a quadratic polynomial in the coordinates of x_6 and y_6, is 0. If we arbitrarily fix $x_6 = (2, 11, 1)$, then $\det(Z_6) = -918y_{61} + 2942$, and $y_6 = (2942, 918)$ is the unique point y_6 such that the configuration is rank deficient.

If we consider only the first 5 point pairs we fixed initially, there exists a unique projective transformation

$$T = \begin{bmatrix} 146 & -294 & 0 \\ 135 & -109 & 11 \end{bmatrix} \tag{2.74}$$

such that $Tx_i \sim y_i$ for all $i = 1, \dots, 5$. Moreover, we can check that $Tx_6 = (-2942, -918) \sim y_6$. This transformation T has center $u = (-1617, -803, 11888)$. This point actually has some further significance, because we can check that

$$(a_x(u), \dots, f_x(u)) \sim (48079, -55599, -88559, -17265, 22529, 90815) \sim (A_y, \dots, F_y) \tag{2.75}$$

$\square$

The following now finishes the proof of Theorem 2.6.7.

Proof of Theorem 2.6.7 (second part). Let S be the blow up of $\mathbb{P}^2_x$ in $x_1, \dots, x_6$ in Cremona hexahedral form, and recall (2.62) and (2.69). It follows that (2.49) holds if and only if $(A_y, \dots, F_y) \in S$, which holds if and only if there exists $u \in \mathbb{P}^2_x$ such that

$$(a_x(u), \dots, f_x(u)) \sim (A_y, \dots, F_y). \tag{2.76}$$

We first show that if Z_6 is rank deficient then a projection exists. Let $\ell \subset \mathbb{P}^2_x$ be a line such that $u \notin \ell$. We then construct a projection $T' : \mathbb{P}^2_x \to \ell$ with center u as in the proof of Theorem 2.5.2 and Figure 2.2. This projection has $T'(x_i) = a_i$ for all i, where the a_i are defined as in Figure 2.2. Then

$$(A_a, \dots, F_a) \sim (a_x(u), \dots, f_x(u)) \sim (A_y, \dots, F_y) \tag{2.77}$$

Since the Joubert Invariants generate the space of all invariants for 6 ordered points on a line it follows that there exists a homography $H : \ell \to \mathbb{P}^1_y$ such that $Ha_i = y_i$. We conclude that $T = HT'$ is the desired projection $T : \mathbb{P}^2_x \to \mathbb{P}^1_y$.

To prove the reverse direction, suppose such a projection T exists. We can use homography on each side to fix $u = (0, 0, 1)$ and to fix the image line at infinity. Equivalently, pick homographies on each side such that

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{2.78}$$

It then follows that $x_i = (y_{i1}, y_{i2}, b_i)$ for all i, where b_i are unknown scalars. The matrix Z_6 then becomes

$$\begin{bmatrix} y_{11}^2 & y_{11}y_{12} & y_{11}y_{12} & y_{12}^2 & y_{11}b_1 & y_{12}b_1 \\ y_{21}^2 & y_{21}y_{22} & y_{21}y_{22} & y_{22}^2 & y_{21}b_2 & y_{22}b_2 \\ y_{31}^2 & y_{31}y_{32} & y_{31}y_{32} & y_{32}^2 & y_{31}b_3 & y_{32}b_3 \\ y_{41}^2 & y_{41}y_{42} & y_{41}y_{42} & y_{42}^2 & y_{41}b_4 & y_{42}b_4 \\ y_{51}^2 & y_{51}y_{52} & y_{51}y_{52} & y_{52}^2 & y_{51}b_5 & y_{52}b_5 \\ y_{61}^2 & y_{61}y_{62} & y_{61}y_{62} & y_{62}^2 & y_{61}b_6 & y_{62}b_6 \end{bmatrix} \tag{2.79}$$

which is clearly rank deficient. This concludes the proof of Theorem 2.6.7. □

2.7 Appendix: Facts about cubic surfaces

This is a collection of classical facts about cubic surfaces needed in Section 2.6.2. See [39, V. Chapter 4] for a modern reference and [12] or [42] for classical references.

Lemma 2.7.1. *Let $\mathcal{S} \subset \mathbb{P}^3$ be a smooth cubic surface. Then,*

1. *$\mathcal{S}$ can be obtained by blowing up 6 points $p_1, \ldots, p_6 \in \mathbb{P}^2$ in general position. The exceptional curves of $p_1, \ldots, p_6$ in the blow up morphism $\pi : \mathcal{S} \to \mathbb{P}^2$ are 6 lines $\ell_1, \ldots, \ell_6$ on $\mathcal{S}$.*
2. *The cubic surface $\mathcal{S}$ has exactly 27 (complex) lines. They are:*
    - *(a) the exceptional lines $\ell_1, \ldots, \ell_6$,*
    - *(b) the strict transforms ℓ_{ij} of the lines in $\mathbb{P}^2$ passing through p_i, p_j for $1 \leq i < j \leq 6$ (15 of them), and*
    - *(c) the strict transforms ℓ'_j of the conics in $\mathbb{P}^2$ passing through five of the p_i for $i \neq j, j = 1, \ldots, 6$ (6 of them).*
3. *The two sets of lines $\{\ell_1, \ldots, \ell_6\}$ and $\{\ell'_1, \ldots, \ell'_6\}$ form a Schäfli double six on $\mathcal{S}$. There are 36 Schläfli double sixes on $\mathcal{S}$. Each double six is uniquely determined by three of the line pairs (ℓ_i, ℓ'_i).*
4. *Any set of 6 mutually skew lines among the 27 lines on $\mathcal{S}$ can play the role of $\ell_1, \ldots, \ell_6$. Precisely, if $\{g_1, \ldots, g_6\}$ is a set of 6 mutually skew lines among the 27 lines, then there is another morphism $\pi' : \mathcal{S} \to \mathbb{P}^2$, making $\mathcal{S}$ isomorphic to $\mathbb{P}^2$ with six points $q_1, \ldots, q_6$ in general position blown up such that $g_1, \ldots, g_6$ are the exceptional curves of $q_1, \ldots, q_6$ under π'.*

We now note several facts about double sixes in $\mathbb{P}^3$ (or more generally line configurations in $\mathbb{P}^3$).

Lemma 2.7.2. *1. Each double six in $\mathbb{P}^3$ is contained in a unique non-singular cubic surface and thus forms part of the set of 27 lines on that surface.*

2. *Any set of* 6 *mutually skew lines among* 27 *lines that satisfy the incidence relations is one half of a unique Schläfli double six and determines the other lines and the cubic surface.*

Next we talk about determinantal representations of a cubic surface.

Definition 2.7.3. *A* **determinantal representation** *of a cubic surface* $\mathcal{S}$ *defined by a cubic equation* $F(z) = F(z_0, z_1, z_2, z_3) = 0$ *is a* 3×3 *matrix of linear forms*

$$M(z) = M(z_0, z_1, z_2, z_3) = z_0 M_0 + z_1 M_1 + z_2 M_2 + z_3 M_3 \tag{2.80}$$

such that $\det(M(z)) = \lambda F(z)$ *for some* $\lambda \neq 0$*. Two representations* $M(z)$ *and* $M'(z)$ *are equivalent if* $M'(z) = AM(z)B$ *for* $A, B \in \mathrm{GL}(3)$.

Note that if $M(z)$ is a determinantal representation of $\mathcal{S}$, then so is $M(z)^\top$. We present the following facts about determinantal representations of a cubic surface [8].

Lemma 2.7.4. *1. Every smooth cubic surface* $\mathcal{S}$ *has a determinantal representation.*

2. *The equivalence classes of determinantal representations of* $\mathcal{S}$ *are in bijection with sets of* 6 *mutually skew lines on* $\mathcal{S}$ *that are the exceptional curves of the blow up of* $\mathbb{P}^2$ *at the* 6 *points that gives* $\mathcal{S}$*.*

3. *If the determinantal representation* M *of* $\mathcal{S}$ *corresponds to the skew lines* $\ell_1, \ldots, \ell_6$ *on* $\mathcal{S}$*, then* $M^\top$ *corresponds to another* 6 *skew lines* $\ell'_1, \ldots, \ell'_6$ *on* $\mathcal{S}$ *such that the pair of* 6 *lines form a Schläfli double six on* $\mathcal{S}$*. Thus there are* 72 *equivalence classes of determinantal representations of* $\mathcal{S}$ *one for each half of the* 36 *Schläfli double sixes on* $\mathcal{S}$*.*

4. *The choice of a Schlafli double six* $\ell_1, \ldots, \ell_6, \ell'_1, \ldots, \ell'_6$ *on* $\mathcal{S}$ *is equivalent to the choice of a pair of equivalence classes of determinantal representations of* $\mathcal{S}$*, namely*$[M]$ *and* $[M^\top]$*. (A smooth cubic surface cannot have a symmetric determinantal representation and hence* $[M] \neq [M^\top]$*.)*

Here are various recipes to explicitly realize some of the above assertions.

1. If $\mathcal{S}$ is the blow up of $\mathbb{P}^2$ at the 6 points $p_1, \ldots, p_6$, then $\mathcal{S}$ can be constructed as follows. Pick a basis $\{f_0, f_1, f_2, f_3\}$ of the 4-dimensional vector space of cubics through $p_1, \ldots, p_6$ and map
$$u \in \mathbb{P}^2 \mapsto (f_0(u) : f_1(u) : f_2(u) : f_3(u)) \in \mathbb{P}^3. \tag{2.81}$$

2. The variety of $\langle f_0, \ldots, f_3 \rangle$ is $\{p_0, \ldots, p_6\}$. By the *Hilbert-Burch theorem* [31, Section 20.4] there is a 3×4 matrix $L(u)$ of linear forms in u_0, u_1, u_2 whose maximal minors are $f_0, \ldots, f_3$. The matrix $L(u)$ generically has rank 3 and has rank 2 exactly when $u = p_i$ for some $i = 1, \ldots, 6$. Define a 3×3 matrix $M(z)$ of linear forms in z_0, z_1, z_2, z_3 by the relation
$$M(z) \cdot \begin{pmatrix} u_0 \\ u_1 \\ u_2 \end{pmatrix} = L(u) \cdot \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix}. \tag{2.82}$$
Then $M(z)$ is a determinantal representation of $\mathcal{S}$.

3. The blow up of $\mathbb{P}^2$ at $p_1, \ldots, p_6$ can be seen via (2.82). If $\bar{u} \in \mathbb{P}^2$ is not one of $p_1, \ldots, p_6$, then some $f_i(\bar{u}) \neq 0$ and $\operatorname{rank}(L(\bar{u})) = 3$. Therefore, the null space of $L(\bar{u})$ is a unique point $\bar{z} \in \mathbb{P}^3$ and $M(\bar{z})\bar{u} = L(\bar{u})\bar{z} = 0$. This means that $\det(M(\bar{z})) = 0$ and $\bar{z} \in \mathcal{S}$. If $\bar{z}$ is a generic point on $\mathcal{S}$, then $\operatorname{rank}(M(\bar{z})) = 2$, and the blow up morphism π sends $\bar{z}$ to the unique point $\bar{u} \in \mathbb{P}^2$ that generates the right null space of $M(\bar{z})$.

4. The 6 exceptional curves of the blow up are precisely
$$\ell_i := \{z \in \mathbb{P}^3 \,:\, M(z)p_i = 0\} \text{ for } i = 1, \ldots, 6. \tag{2.83}$$
By what is stated above, $\operatorname{rank}(L(u)) = 2$ if and only if $u = p_i$ for some $i = 1, \ldots, 6$ if and only if (via (2.82)) $M(z)u = L(u)z = 0$ is a system of 2 linearly independent equations in z (i.e., defines a line in $\mathbb{P}^3$). This line is entirely on $\mathcal{S}$ since $\det(M(z)) = 0$ if $M(z)p_i = 0$.

5. Corresponding results hold for the determinantal representation $M(z)^\top$ of $\mathcal{S}$.

(a) To obtain the points $q_1, \ldots, q_6 \in \mathbb{P}^2$ whose blow up is $\mathcal{S}$, compute $L'(u)$ via

$$M(z)^\top u = L'(u)z. \tag{2.84}$$

The variety of the maximal minors of $L'(u)$ is $\{q_1, \ldots, q_6\}$.

(b) A point $\bar{y} \in \mathbb{P}^2$ not equal to any q_i is sent to the unique point $\bar{z}$ in $\mathbb{P}^3$ in the right nullspace of $L'(\bar{y})$. This means $L'(\bar{y})\bar{z} = M(\bar{z})^\top \bar{y} = 0$ and $\bar{z} \in \mathcal{S}$ since $\det(M(\bar{z}))^\top = 0$.

(c) The morphism $\pi' : \mathcal{S} \to \mathbb{P}^2$ sends $\bar{z}$ to the unique point $\bar{y} \in \mathbb{P}^2$ that generates the right nullspace of $M(\bar{z})^\top$, or equivalently, the left nullspace of $M(\bar{z})$.

(d) The exceptional curves of the blow up are the lines

$$\ell'_j := \{z \in \mathbb{P}^3 \ : \ q_j^\top M(z) = 0\} \text{ for } j = 1, \ldots, 6.$$

The following theorem follows from the above facts.

Theorem 2.7.5. *Given a determinantal representation $M(z)$ of a smooth cubic surface $\mathcal{S} \subset \mathbb{P}^3$, there exists 6 points $(p_i, q_i) \in \mathbb{P}^2 \times \mathbb{P}^2$ such that $\mathcal{S}$ is the blow up of the first $\mathbb{P}^2$ at $\{p_i\}_{i=1}^6$ and also the blow up of the second $\mathbb{P}^2$ at $\{q_j\}_{j=1}^6$. The exceptional curves of the two blow ups form a Schläfli double six on $\mathcal{S}$:*

$$\ell_i := \{z \in \mathbb{P}^3 \ : \ M(z)p_i = 0\} \text{ for } i = 1, \ldots, 6, \tag{2.85}$$

$$\ell'_j := \{z \in \mathbb{P}^3 \ : \ q_j^\top M(z) = 0\} \text{ for } j = 1, \ldots, 6. \tag{2.86}$$

Chapter 3

LINES, QUADRICS, AND CREMONA TRANSFORMATIONS

The following is the content of [16], written with Rekha Thomas and Cynthia Vinzant.

3.1 Introduction

Given vectors $x, y \in \mathbb{R}^3$, let $x^\top \otimes y^\top = (x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_2y_2, x_2y_3, x_3y_1, x_3y_2, x_3y_3)$ denote the Kronecker product [51] of $x^\top$ and $y^\top$. We are interested in solving the following problem:

Problem 3.1.1. *Given k points $(x_i, y_i) \in \mathbb{R}^3 \times \mathbb{R}^3$, $k \leq 9$, consider the $k \times 9$ matrix Z_k whose rows are $x_i^\top \otimes y_i^\top$ for $i = 1, \dots, k$, i.e.,*

$$Z_k = \begin{bmatrix} x_1^\top \otimes y_1^\top \\ \vdots \\ x_k^\top \otimes y_k^\top \end{bmatrix}.$$

Delineate the geometry of point configurations $\{x_i\}$ and $\{y_i\}$ for which $\operatorname{rank}(Z_k) < k$.

In signal processing, the matrix Z_k is known as the *face-splitting product* [63] of the two matrices $X_k \in \mathbb{R}^{k\times 3}$ with rows $x_1^\top, \dots, x_k^\top$ and $Y_k \in \mathbb{R}^{k\times 3}$ with rows $y_1^\top, \dots, y_k^\top$. The face-splitting product is a special case of the *Khatri-Rao matrix product* in linear algebra and statistics [44]. However, our interest Problem 3.1.1 comes from a pair of estimation problems in 3D computer vision known as the 7- and 5- point problems respectively [37, 46, 55].

Problem 3.1.2 (7-point problem). *Given 7 points $(\hat{x}_i, \hat{y}_i) \in \mathbb{R}^2 \times \mathbb{R}^2$ find a matrix $F \in \mathbb{R}^{3\times 3}$ such that*

$$y_i^\top F x_i = 0,\ \forall i = 1, \dots, 7 \ \text{ and } \ \det(F) = 0. \tag{3.1}$$

where $x_i = \begin{bmatrix} \hat{x}_i \\ 1 \end{bmatrix}$ and $y_i = \begin{bmatrix} \hat{y}_i \\ 1 \end{bmatrix}$.

Problem 3.1.3 (5-point problem)**.** *Given 5 points* $(\hat{x}_i, \hat{y}_i) \in \mathbb{R}^2 \times \mathbb{R}^2$ *find a matrix* $E \in \mathbb{R}^{3\times 3}$ *such that*

$$y_i^\top E x_i = 0,\ \forall i = 1, \ldots, 5 \ \text{ and } \ 2EE^\top E - \operatorname{Tr}(EE^\top)E = 0, \tag{3.2}$$

where $x_i = \begin{bmatrix} \hat{x}_i \\ 1 \end{bmatrix}$ *and* $y_i = \begin{bmatrix} \hat{y}_i \\ 1 \end{bmatrix}$.

A solution F to the 7-point problem that has rank two is known as a *fundamental matrix*, and a solution E to the 5-point problem is known as an *essential matrix* [37]. Observe that in both problems, the quantity of interest is a 3×3 matrix M that satisfies equations of the form

$$y_i^\top M x_i = 0,\ \forall i = 1, \ldots, k,$$

which can be re-written as

$$(x_i^\top \otimes y_i^\top) \operatorname{vec}(M) = 0,\ \forall i = 1, \ldots, k$$

where $\operatorname{vec}(M)$ is the 9-dimensional vector obtained by concatenating the columns of M. Thus the algorithm underlying the 7-point problem involves computing the intersection of the nullspace of a 7×9 face splitting product with a determinantal variety. Similarly, the 5-point algorithm involves finding the intersection of the nullspace of a 5×9 face splitting product with the so called *essential variety* [18].

We wish to understand the point configurations $\{x_i\}$ and $\{y_i\}$ for which Problem 3.1.2 and Problem 3.1.3 are ill-posed. In both cases, it can be shown that a sufficient condition for ill-posedness is that the corresponding face-splitting product matrix be rank deficient and thus our interest in Problem 3.1.1.

A first answer to Problem 3.1.1 is that $\operatorname{rank}(Z_k) < k$ if and only if all the maximal minors of Z_k are zero. These minors are bi-homogeneous polynomials in x_i and y_i and typically do not shed much light on the geometry of the point configurations $\{x_i\}$ and $\{y_i\}$ that cause Z_k to drop rank. Our goal is to characterize the rank deficiency of Z_k in terms of the geometry of the x and y points.

As pointed out in the introduction of [14], the rank of Z_k is a projective invariant and is unaffected by scaling and the choice of coordinates in $\mathbb{R}^3 \times \mathbb{R}^3$. Therefore, we can study Problem 3.1.1 in the product of projective spaces $\mathbb{P}^2 \times \mathbb{P}^2$. While our computer vision applications involve real inputs, the mathematical theory developed here also works when the input points (x_i, y_i) are complex and so for convenience we work over $\mathbb{C}$. Throughout the paper, we use $\mathbb{P}^k$ to denote k-dimensional projective space over $\mathbb{C}$ and pass to the following formulation of Problem 3.1.1, which will be the main subject of study in this paper.

Problem 3.1.4. *Given* $k \leq 9$ *points* $(x_i, y_i) \in \mathbb{P}^2 \times \mathbb{P}^2$*, consider the* $k \times 9$ *matrix* Z_k *whose rows are* $x_i^\top \otimes y_i^\top$ *for* $i = 1, \ldots, k$*, i.e.,*

$$Z_k = \begin{bmatrix} x_1^\top \otimes y_1^\top \\ \vdots \\ x_k^\top \otimes y_k^\top \end{bmatrix}.$$

Delineate the geometry of point configurations $\{x_i\}$ *and* $\{y_i\}$ *for which* $\operatorname{rank}(Z_k) < k$*.*

In [14], Problem 3.1.4 was answered for $k \leq 6$. The results relied on the classical invariant theory of points in $\mathbb{P}^2$ and the theory of cubic surfaces. In this paper we complete the characterization for the remaining cases of $k = 7, 8, 9$. Once again, the results can be phrased in terms of classical algebraic geometry and invariants.

We note that Problem 3.1.4 can be rephrased geometrically and generalized to any algebraic variety.

Problem 3.1.5. *Given* $k \leq 9$ *points* $(x_i, y_i) \in \mathbb{P}^2 \times \mathbb{P}^2$*, delineate the geometry of the point configurations* $\{x_i\}$ *and* $\{y_i\}$ *for which the subspace spanned by the images of these points under the Segre embedding of* $\mathbb{P}^2 \times \mathbb{P}^2$ *in* $\mathbb{P}^8$ *has dimension less than* $k - 1$*.*

Summary of results and organization of the paper.

Throughout this paper, we will concern ourselves with point configurations that are *semi-generic*; a configuration of k point pairs (x_i, y_i) is *semi-generic* if every subset of $k - 1$ point pairs is fully generic. That is, we say that a property holds for a semi-generic choice of $(x_i, y_i) \in (\mathbb{P}^2 \times \mathbb{P}^2)^k$

if there is a nonempty Zariski open set $\mathcal{U} \subseteq (\mathbb{P}^2 \times \mathbb{P}^2)^{k-1}$ so that the property holds whenever $\{(x_i, y_i) : i \neq j\}$ lies in $\mathcal{U}$ for all $j = 1, \ldots, k$.

In [14] we studied Problem 3.1.4 algebraically by decomposing the ideal generated by the maximal minors of Z_k into its prime components and examining only those components that did not correspond to rank drop conditions for a submatrix of Z_k with at most $k-1$ rows, called *inherited conditions*, for the rank deficiency of Z_k. Through this we obtained both algebraic conditions that completely characterized rank drop, and geometric conditions that characterized rank drop under mild genericity assumptions. This method cannot be applied to the cases of $k = 7, 8, 9$ due to computational limitations. Additionally, in these cases, the novel component of rank drop has a greater dimension than all the components of inherited conditions. Previously, for $k \leq 5$ the novel component had a strictly lower dimension than the variety of inherited conditions, and for $k = 6$ the novel component had equal dimension to that of the inherited conditions variety. For this reason, we largely concern ourselves only with the geometric characterization of rank drop for semi-generic configurations with $k = 7, 8, 9$, rather than an algebraic characterization beyond the vanishing of the maximal minors of Z_k.

In Section 3.2 we establish a number of facts about Cremona transformations, cubic curves, and projective reconstructions that we will use throughout the paper. In Section 3.3 we study the problem for $k = 8$ and prove that Z_k is rank-deficient exactly when there is a quadratic Cremona transformation $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $f(x_i) = y_i$ for all i (Theorem 3.3.1). To do so, we establish a correspondence between three sets: lines in the nullspace of Z_k, quadrics passing through a projective reconstruction of the input point pairs, and Cremona transformations sending $x_i \mapsto y_i$ (Theorem 3.3.16 which depends on Theorem 3.3.2). We refer to this as the *trinity correspondence* and it is the foundation for all of our results in this paper. In Section 3.4 we study the problem for $k = 7$ and prove that Z_k is rank-deficient exactly when there are cubic curves in each copy of $\mathbb{P}^2$, passing through all seven points, and an isomorphism between these curves that sends $x_i \mapsto y_i$ (Theorem 3.4.1). We further prove that this occurs exactly when seven particular cubic curves in each copy of $\mathbb{P}^2$ are coincident and we provide an algebraic characterization for when this occurs (Theorem 3.4.9). In Section 3.5 we answer Problem 3.1.4 for $k = 9$, which is largely straight-forward (Theorem 3.5.1). We summarize our results in Section 3.6 and state a geometric consequence about reconstructions of semi-generic point pairs of size six, seven and eight.

3.2 Background and Tools

3.2.1 Quadratic Cremona Transformations and Cubic Curves

Definition 3.2.1. *A* **quadratic Cremona transformation** *of* $\mathbb{P}^2$ *is a birational automorphism* $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ *defined as* $f(x) = (f_1(x) : f_2(x) : f_3(x))$ *where* f_1, f_2, f_3 *are homogeneous quadratic polynomials in* $x = (x_1, x_2, x_3)$.

We will drop the word "quadratic" from now on as all the Cremona transformations we consider will be quadratic. Each Cremona transformation can be obtained by blowing up three points a_1, a_2, a_3 in the domain (called *base points*) at which the transformation is not defined and collapsing three lines $\gamma_1, \gamma_2, \gamma_3$ (called *exceptional lines*) which contain pairs of base points: for distinct i, j, k, the line γ_i contains a_j, a_k. Generically, the base points and exceptional lines of a Cremona transformation will all be distinct; when they are not all distinct, the transformation is said to be degenerate. In this paper we will only consider non-degenerate Cremona transformations.

The inverse of a Cremona transformation f is also a Cremona transformation with base points b_1, b_2, b_3 and exceptional lines τ_1, τ_2, τ_3 in the codomain of f. The map f sends $\gamma_i \mapsto b_i$ while f^{-1} sends $\tau_i \mapsto a_i$. For simplicity we will often refer to both the base points in the domain and the base points in the codomain (i.e. the base points of f^{-1}) as the base points of f. The standard Cremona transformation is

$$f(x_1, x_2, x_3) = (x_2x_3 : x_1x_3 : x_1x_2) \tag{3.3}$$

which has base points $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)$ and exceptional lines $x_i = 0$ for $i = 1, 2, 3$. This transformation is an involution since it is its own inverse, and the base points and exceptional lines of f^{-1} are again $(1 : 0 : 0), (0 : 1 : 0), (0 : 0 : 1)$ and $x_i = 0$ for $i = 1, 2, 3$. All Cremona transformations differ from the standard one only by projective transformations as stated below.

Lemma 3.2.2. *Let g be a Cremona transformation and f be the standard Cremona involution. Then there are projective transformations H_1, H_2 such that $g = H_1 \circ f \circ H_2$.*

Proof. Let $a_1, a_2, a_3 \in \mathbb{P}^2$ denote the base points of g. The coordinates (g_1, g_2, g_3) of g form a basis for the three-dimensional vector space of quadratics vanishing on the points a_1, a_2, a_3. Another basis is $h = (\ell_2\ell_3, \ell_1\ell_3, \ell_1\ell_2)$ where $\ell_i \in \mathbb{C}[x, y, z]_1$ defines the line joining a_j and a_k for every labeling $\{i, j, k\} = \{1, 2, 3\}$. Therefore there is some invertible linear transformation H_1 for which $g = H_1 h$. Similarly, (ℓ_1, ℓ_2, ℓ_3) is a basis for $\mathbb{C}[x, y, z]_1$ and so there is a linear transformation H_2 for which $H_2(x, y, z) = (\ell_1, \ell_2, \ell_3)$. The map h is given by $f \circ H_2$ and so $g = H_1 \circ f \circ H_2$. □

Throughout this paper we will be interested in $\mathbb{P}^2 \times \mathbb{P}^2$ and we typically denote points in the first $\mathbb{P}^2$ by x and those in the second $\mathbb{P}^2$ by y. The notation $\mathbb{P}^2_x$ and $\mathbb{P}^2_y$ will help keep this correspondence clear.

Lemma 3.2.3. *Let $f : \mathbb{P}^2_x \dashrightarrow \mathbb{P}^2_y$ be a Cremona transformation such that f and f^{-1} have base points $e^x_1 = e^y_1 = (1 : 0 : 0)$, $e^x_2 = e^y_2 = (0 : 1 : 0)$, $e^x_3 = e^y_3 = (0 : 0 : 1)$ in the domain and codomain. Then f has the form*

$$f(x_1, x_2, x_3) = (ax_2x_3 : bx_1x_3 : cx_1x_2) \tag{3.4}$$

where $a, b, c \in \mathbb{C}\backslash\{0\}$.

Proof. Suppose $f = (f_1, f_2, f_3)$ where f_1, f_2, f_3 are quadratic polynomials. Since f is undefined at the three base points in the domain, it follows that f_1, f_2, f_3 contain only the monomials x_1x_2, x_1x_3, x_2x_3. Moreover, we know that $f(x_1, x_2, 0) = (0 : 0 : 1)$. It follows that f_1, f_2 do not contain the monomials x_1x_2. In examining the other two exceptional lines, we find that f_1, f_2, f_3 contain only one monomial each and that f has the desired form. □

We note that the choice of (a, b, c) is equivalent to specifying a single point correspondence $p \mapsto q$, where neither p nor q lie on an exceptional line. It follows that a Cremona transformation has 14 degrees of freedom: six from the base points in the domain, six from the base points in the codomain, and two from the choice of a single point correspondence.

Next we prove some facts about Cremona transformations and isomorphisms of cubic curves.

Definition 3.2.4. *Let f be a Cremona transformation with base points $B(f)$. For a curve $C \subset \mathbb{P}^2$, define $f(C) := \overline{f(C \backslash B(f))}$, and for a given point p, let $\nu_p(C)$ be the multiplicity of the curve C at the point p.*

Lemma 3.2.5. [22] *Let $C \subset \mathbb{P}^2$ be a plane curve of degree n and let f be a Cremona transformation. Then*

$$deg(f(C)) = 2n - \sum_{p \in B(f)} \nu_p(C). \tag{3.5}$$

In particular, if C is a smooth cubic curve then $f(C)$ is also a cubic curve if and only if the base points of f lie on C. In this case, $f^{-1}(f(C)) = C$ implies that the base points of f^{-1} lie on $f(C)$.

Using this, we can prove the following result.

Lemma 3.2.6. *Let C be a smooth cubic curve and let f be a Cremona transformation with base points $a_1, a_2, a_3 \in C$ in the domain and b_1, b_2, b_3 in the co-domain. Then $f(C)$ is a smooth cubic curve and $\bar{f} : C \to f(C)$, defined by taking the closure of $f|_{C \backslash B(f)}$, is an isomorphism.*

Proof. By Lemma 3.2.5, $f(C)$ is a cubic curve. Moreover, since $f^{-1}(f(C)) = C$ is a cubic curve, it also follows that $b_1, b_2, b_3 \in f(C)$. The fact that $\bar{f}$ is an isomorphism follows from the corollary after [62, §1.6, Theorem 2] which says that a birational map between nonsingular projective plane curves is regular at every point, and is a one-to-one correspondence. □

Given a smooth cubic curve C, any automorphism $g : C \to C$ is of the form $u \mapsto au + b$, where $a = \pm 1$, $b \in C$, where addition is defined via the group law on C. Theorem 1.3 in [22] states that given a smooth cubic curve C and an automorphism $g : C \to C$ defined by some multiplier $a = \pm 1$ and translation $b \in C$, then g is induced by a Cremona transformation with base points a_1, a_2, a_3 if and only if $a(a_1 + a_2 + a_3) = 3b$, where again, addition is with respect to the group law on C. In particular, every automorphism of C is induced by a two-parameter family of Cremona transformations, which we obtain by picking the first two base points arbitrarily and then letting the third base point be determined by the equation $a_3 = a(3b - a_1 - a_2)$.

We can use this to prove a converse to Lemma 3.2.6.

Lemma 3.2.7. *Let $f : C \to C'$ be an isomorphism of smooth cubic plane curves. Then there is a two-parameter family of Cremona transformations $f'_\sigma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $f'_\sigma|_C = f$. The base points of these Cremona transformations will lie on the cubic curves.*

Proof. Since C and C' are isomorphic, they have the same Weirstrass form C_0. There are therefore homographies $H_1, H_2 \in \mathrm{PGL}(3)$ such that $H_1(C) = C_0 = H_2(C')$ and therefore $H_1^{-1}H_2(C') = C$. Then $H_1^{-1}H_2 \circ f : C \to C$ is an automorphism of C and it follows by [22, Theorem 1.3] that this is induced by some two-parameter family of Cremona transformations g_σ; the members of this family are obtained by picking the first two base points arbitrarily on C and then letting the third base point be determined by the equation $a_3 = a(3b - a_1 - a_2)$. Then $f'_\sigma := H_2^{-1}H_1 \circ g_\sigma$ is the desired family of Cremona transformations. By Lemma 3.2.5 the base points of each of these Cremona transformations lie on the cubic curves. $\square$

3.2.2 Fundamental Matrices and Projective Reconstruction

In this paper we will be concerned with pairs of linear projections $\pi_1, \pi_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ with non-coincident centers c_1, c_2. In the context of computer vision, these arise as *projective cameras* which are linear projections from $\mathbb{P}^3(\mathbb{R}) \dashrightarrow \mathbb{P}^2(\mathbb{R})$, represented by (unique) matrices $A_1, A_2 \in \mathbb{P}(\mathbb{R}^{3\times 4})$ of rank three, such that $\pi_i(p) \sim A_i p$ for all *world points* $p \in \mathbb{P}^3(\mathbb{R})$. The notation $\sim$ indicates equality in projective space. The centers c_i are the unique points in $\mathbb{P}^3(\mathbb{R})$ such that $A_i c_i = 0$ for $i = 1, 2$. The projections we consider in this paper are slightly more general in that they work over $\mathbb{C}$; they are represented by rank three matrices $A_i \in \mathbb{P}(\mathbb{C}^{3\times 4})$ and send $p \in \mathbb{P}^3$ to $A_i p \in \mathbb{P}^2$.

In the vision setting, the image formation equations $A_i p = \lambda_i \pi_i(p)$ for $i = 1, 2$ and some $\lambda_i \in \mathbb{R}$, imply that for all $p \in \mathbb{P}^3(\mathbb{R})$,

$$0 = \det \begin{bmatrix} A_1 & \pi_1(p) & 0 \\ A_2 & 0 & \pi_2(p) \end{bmatrix} = \pi_2(p)^\top F \pi_1(p) \tag{3.6}$$

for a unique matrix $F \in \mathbb{P}(\mathbb{R}^{3\times 3})$ of rank two, determined by (A_1, A_2) [37, Chapter 9.2]. This matrix F is called the *fundamental matrix* of the cameras/projections (A_1, A_2) / (π_1, π_2). It defines the bilinear form $B_F(x, y) = y^\top F x$ such that $B_F(\pi_1(p), \pi_2(p)) = \pi_2(p)^\top F \pi_1(p) = 0$ for all

$p \in \mathbb{P}^3(\mathbb{R})$. The entries of F are certain 4×4 minors of the 6×4 matrix obtained by stacking A_1 on top of A_2. The points $e^x := \pi_1(c_2)$ and $e^y := \pi_2(c_1)$ are called the *epipoles* of F. It is well-known [37, Chapter 9.2] that e^x and e^y are the unique points in $\mathbb{P}^2$ such that $Fe^x = 0 = (e^y)^\top F$. Conversely, for every rank-two matrix $F \in \mathbb{P}(\mathbb{R}^{3\times 3})$ there exists, up to projective transformation, a unique pair of cameras (A_1, A_2) / linear projections $\pi_1, \pi_2 : \mathbb{P}^3(\mathbb{R}) \dashrightarrow \mathbb{P}^2(\mathbb{R})$ with fundamental matrix F, see [37, Theorem 9.10]. All of these facts extend verbatim over $\mathbb{C}$ and we call a rank two matrix $F \in \mathbb{P}(\mathbb{C}^{3\times 3})$ a *fundamental matrix* of (π_1, π_2) if it satisfies (3.6).

Equation (3.6) is a constraint on the images of a world point in two cameras. Going the other way, given k point pairs $(x_i, y_i) \in \mathbb{P}^2(\mathbb{R}) \times \mathbb{P}^2(\mathbb{R})$, one can ask if they admit a *projective reconstruction*, namely a pair of real cameras A_1, A_2 and real world points $p_1, \ldots, p_k$ such that $A_1 p_i \sim x_i$ and $A_2 p_i \sim y_i$ for $i = 1, \ldots, k$. A necessary condition for a reconstruction is the existence of a rank-two matrix $F \in \mathbb{P}(\mathbb{R}^{3\times 3})$ such that $y_i^\top F x_i = 0$ for $i = 1, \ldots, k$, called a *fundamental matrix* of the point pairs $(x_i, y_i)_{i=1}^k$. Note that $\mathrm{vec}(F)$ lies in the nullspace of $Z_k = (x_i^\top \otimes y_i^\top)_{i=1}^k$. The necessary and sufficient conditions for the existence of a projective reconstruction of $(x_i, y_i)_{i=1}^k$ are 1) the existence of a fundamental matrix F and 2) for each i, either $Fx_i = 0$ and $y_i^\top F = 0$, or neither x_i nor y_i lie in the right and left nullspaces of F [49]. In this paper, we extend the above definition to $\mathbb{C}$ and call any rank-two matrix $F \in \mathbb{P}(\mathbb{C}^{3\times 3})$ that lies in the nullspace of Z_k, a *fundamental matrix* of the point pairs $(x_i, y_i)_{i=1}^k$.

3.3 $k = 8$

In this section we characterize the rank deficiency of $Z = Z_8 = (x_i^\top \otimes y_i^\top)_{i=1}^8$ when the point pairs (x_i, y_i) are semi-generic. When k is fixed we often write Z instead of Z_k.

Theorem 3.3.1. *For eight semi-generic point pairs $(x_i, y_i)_{i=1}^8$, the matrix Z drops rank if and only if there exists a Cremona transformation $f : \mathbb{P}^2_x \dashrightarrow \mathbb{P}^2_y$ such that $f(x_i) = y_i$ for all i.*

Proof. ($\Leftarrow$) Suppose we have a Cremona transformation $f : \mathbb{P}^2_x \dashrightarrow \mathbb{P}^2_y$ such that $f(x_i) = y_i$ for $i = 1, \ldots, 8$. After homographies we can assume that f is the basic quadratic involution

$(x_1, x_2, x_3) \mapsto\dashrightarrow (x_2x_3, x_1x_3, x_1x_2)$. Then

$$Z = \begin{bmatrix} x_{11}x_{12}x_{13} & x_{11}^2x_{13} & x_{11}^2x_{12} & x_{12}^2x_{13} & x_{11}x_{12}x_{13} & x_{11}x_{12}^2 & x_{12}x_{13}^2 & x_{11}x_{13}^2 & x_{11}x_{12}x_{13} \\ x_{21}x_{22}x_{23} & x_{21}^2x_{23} & x_{21}^2x_{22} & x_{22}^2x_{23} & x_{21}x_{22}x_{23} & x_{21}x_{22}^2 & x_{22}x_{23}^2 & x_{21}x_{23}^2 & x_{21}x_{22}x_{23} \\ x_{31}x_{32}x_{33} & x_{31}^2x_{33} & x_{31}^2x_{32} & x_{32}^2x_{33} & x_{31}x_{32}x_{33} & x_{31}x_{32}^2 & x_{32}x_{33}^2 & x_{31}x_{33}^2 & x_{31}x_{32}x_{33} \\ x_{41}x_{42}x_{43} & x_{41}^2x_{43} & x_{41}^2x_{42} & x_{42}^2x_{43} & x_{41}x_{42}x_{43} & x_{41}x_{42}^2 & x_{42}x_{43}^2 & x_{41}x_{43}^2 & x_{41}x_{42}x_{43} \\ x_{51}x_{52}x_{53} & x_{51}^2x_{53} & x_{51}^2x_{52} & x_{52}^2x_{53} & x_{51}x_{52}x_{53} & x_{51}x_{52}^2 & x_{52}x_{53}^2 & x_{51}x_{53}^2 & x_{51}x_{52}x_{53} \\ x_{61}x_{62}x_{63} & x_{61}^2x_{63} & x_{61}^2x_{62} & x_{62}^2x_{63} & x_{61}x_{62}x_{63} & x_{61}x_{62}^2 & x_{62}x_{63}^2 & x_{61}x_{63}^2 & x_{61}x_{62}x_{63} \\ x_{71}x_{72}x_{73} & x_{71}^2x_{73} & x_{71}^2x_{72} & x_{72}^2x_{73} & x_{71}x_{72}x_{73} & x_{71}x_{72}^2 & x_{72}x_{73}^2 & x_{71}x_{73}^2 & x_{71}x_{72}x_{73} \\ x_{81}x_{82}x_{83} & x_{81}^2x_{83} & x_{81}^2x_{82} & x_{82}^2x_{83} & x_{81}x_{82}x_{83} & x_{81}x_{82}^2 & x_{82}x_{83}^2 & x_{81}x_{83}^2 & x_{81}x_{82}x_{83} \end{bmatrix} \tag{3.7}$$

which one can see is rank deficient because its first, fifth and ninth columns are the same. □

In order to prove the '*only-if*' direction of Theorem 3.3.1, we develop a number of tools in § 3.3.1. The proof of Theorem 3.3.1 will then be completed in § 3.3.2.

3.3.1 The Trinity of Lines, Quadrics and Cremona Transformations

In order to establish the *trinity correspondence*, we need to introduce some genericity conditions for our main objects of interest. We say that a line $\ell \subset \mathbb{P}(\mathbb{C}^{3\times 3})$ is *generic* if it contains exactly three rank-two matrices. These lines are generic in the usual sense, since almost all lines in $\mathbb{P}(\mathbb{C}^{3\times 3})$ intersect the degree-three determinantal variety $\mathcal{D} := \{X \in \mathbb{P}(\mathbb{C}^{3\times 3}) : \det(X) = 0\}$ in three distinct points. Furthermore, given a pair of linear projections $\pi_1, \pi_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ with distinct centers c_1, c_2 we say that a smooth quadric Q through c_1, c_2 is *permissible* if it does not contain the line $\overline{c_1c_2}$ connecting the two centers.

Theorem 3.3.2 (Trinity correspondence)**.** *Consider the following three sets:*

1. *$\mathcal{L}$: the set of all generic lines ℓ in $\mathbb{P}(\mathbb{C}^{3\times 3})$,*

2. *$\mathcal{Q}$: the set (up to projective equivalence) of pairs of linear projections $\pi_1, \pi_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ with non-coincident centers c_1, c_2, along with a permissible quadric $Q \subset \mathbb{P}^3$ through c_1, c_2,*

3. $\mathcal{C}$: the set of (non-degenerate) Cremona transformations from $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$.

Then there is a $1:1$ *correspondence between $\mathcal{L}$ and $\mathcal{C}$, a* $1:3$ *correspondence between $\mathcal{L}$ and $\mathcal{Q}$, and a* $3:1$ *correspondence between $\mathcal{Q}$ and $\mathcal{C}$, such that diagram* (3.8) *commutes.*

$$\begin{tikzcd}
 & \mathcal{Q} \arrow[dr, "3:1"] & \\
\mathcal{L} \arrow[ur, "1:3"] \arrow[rr, leftrightarrow] & & \mathcal{C}
\end{tikzcd} \tag{3.8}$$

A similar theorem holds for lines which pass through exactly two rank-two matrices; however, we do not prove it here.

We first show that for fixed linear projections π_1, π_2 with centers $c_1 \neq c_2 \in \mathbb{P}^3$, there is a bijection between the quadrics that contain c_1, c_2 and lines in $\mathbb{P}(\mathbb{C}^{3\times 3})$ through the fundamental matrix F of (π_1, π_2). This result is well-known in the context of computer vision ([5],[36]), but we write an independent proof below.

Lemma 3.3.3. *Fix a pair of linear projections $\pi_1, \pi_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ with non-coincident centers c_1, c_2, and let F be its fundamental matrix. There is a* $1:1$ *correspondence between the quadrics $Q \subset \mathbb{P}^3$ through c_1, c_2 and lines $\ell \subset \mathbb{P}(\mathbb{C}^{3\times 3})$ through F.*

Proof. After projective transformations, we can assume that $c_1 = (1:0:0:0)$, $c_2 = (0:1:0:0)$, $\pi_1(u_1:u_2:u_3:u_4) = (u_2:u_3:u_4)$ and $\pi_2(u_1:u_2:u_3:u_4) = (u_1:u_3:u_4)$. If $F = (F_{ij})$ is the fundamental matrix of (π_1, π_2), then for all $u \in \mathbb{P}^3$,

$$0 = \pi_2(u)^\top F \pi_1(u) = \langle F, \pi_2(u)\pi_1(u)^\top \rangle = \left\langle \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}, \begin{pmatrix} u_1u_2 & u_1u_3 & u_1u_4 \\ u_2u_3 & u_3^2 & u_3u_4 \\ u_2u_4 & u_3u_4 & u_4^2 \end{pmatrix} \right\rangle. \tag{3.9}$$

Since the entries in position $(2,3)$ and $(3,2)$ of $\pi_2(u)\pi_1(u)^\top$ are the same, F is a scalar multiple of

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

and $B_F(x,y) = x_3y_2 - x_2y_3$. In particular, there exists some $p \in \mathbb{P}^3$ with $\pi_1(p) = x$ and $\pi_2(p) = y$ if and only if $x_3y_2 = x_2y_3$.

Consider the image of $\varphi : \mathbb{P}^3 \dashrightarrow \mathbb{P}(\mathbb{C}^{3\times 3})$ where $\varphi(u) = \pi_2(u)\pi_1(u)^\top$. By (3.9), $\varphi(\mathbb{P}^3)$ is contained in the hyperplane $F^\perp \subset \mathbb{P}(\mathbb{C}^{3\times 3})$. Any matrix in $\mathbb{P}(\mathbb{C}^{3\times 3})$ can be written as $sF + M$ for some scalar s and $M \in F^\perp$. Therefore,

$$\langle sF + M, \pi_2(u)\pi_1(u)^\top \rangle = \pi_2(u)^\top M \pi_1(u) \tag{3.10}$$

since $\pi_2(u)^\top F \pi_1(u) = 0$, and any linear function on the image of φ can be identified with its image in $F^\perp$. On the other hand, a line ℓ in $\mathbb{P}(\mathbb{C}^{3\times 3})$ through F is of the form $\{sF + tM : (s:t) \in \mathbb{P}^1\}$, where $M \in F^\perp$. Therefore, lines through F are in bijection with linear functions on $\varphi(\mathbb{P}^3)$, up to scaling.

The monomials $\{u_1u_2, u_1u_3, u_1u_4, u_2u_3, u_2u_4, u_3^2, u_3u_4, u_4^2\}$ form a basis for the 7-dimensional vector space of homogeneous quadratic polynomials that vanish on c_1, c_2. Therefore any quadratic polynomial in $\mathbb{C}[u_1, u_2, u_3, u_4]_2$ vanishing at c_1 and c_2 can be written as $\langle M, \pi_2(u)\pi_1(u)^\top \rangle$ for a unique matrix $M \in F^\perp$. This gives a linear isomorphism between linear functions on the image of φ, up to global scaling, (which have been identified with lines through F) and quadrics passing though c_1 and c_2. □

Corollary 3.3.4. *Let* $\pi_1, \pi_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ *be two linear projections with centers* $c_1 \neq c_2$ *and fundamental matrix* F*. Let* ℓ_F *be a line in* $\mathbb{P}(\mathbb{C}^{3\times 3})$ *through* F*. The correspondence* $\ell_F \mapsto Q$*, where* $Q \subset \mathbb{P}^3$ *is a quadric passing through* c_1, c_2*, is as follows. Let* $M \in \ell_F$ *be any* $M \neq F$*. Then* Q *is cut out by the bilinear form*

$$B_M(\pi_1(p), \pi_2(p)) = \pi_2(p)^\top M \pi_1(p) = 0. \tag{3.11}$$

The following result is well-known and can be proven by writing a comprehensive list of the equivalence classes, under projective transformation, of quadrics through a pair of distinct points and then testing an example from each class.

Lemma 3.3.5. *(*[5], [36], [37, Result 22.11]*) Under the* $1:1$ *correspondence in Lemma 3.3.3, the line* ℓ *corresponds to a permissible quadric* Q *through* c_1, c_2 *if and only if* ℓ *is a generic line.*

Next we prove that permissible quadrics through c_1, c_2 give rise to quadratic Cremona transformations from $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. Recall that all Cremona transformations we consider are assumed to be non-degenerate.

Lemma 3.3.6. *Fix $\pi_i : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ to be linear projections with non-coincident centers c_i for $i = 1, 2$. A permissible quadric Q through c_1, c_2 defines a Cremona transformation $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ so that for any point $p \in Q$, $f(\pi_1(p)) = \pi_2(p)$. The base points of f are $\pi_1(c_2)$ and the image under π_1 of the two lines contained in Q passing through c_1. Similarly, the base points of f^{-1} are $\pi_2(c_1)$ and the image under π_2 of the two lines contained in Q passing through c_2.*

Proof. Since $c_1, c_2 \in Q$, the restriction of π_1 (and π_2) to Q is generically $1 : 1$. Therefore, $\pi_1(Q)$ and $\pi_2(Q)$ are each birational to a $\mathbb{P}^2$. The map f will be $\pi_2 \circ (\pi_1|_Q)^{-1}$. Let us check that this is a quadratic Cremona transformation.

As before, we can take $\pi_1(u) = (u_2 : u_3 : u_4)$ and $\pi_2(u) = (u_1 : u_3 : u_4)$. Then $c_1 = (1 : 0 : 0 : 0)$ is the kernel of π_1, and we are given that it lies on Q. As we saw already, these assumptions imply that Q is defined by the vanishing of a polynomial of the form $q(u) = \alpha u_1 u_2 + \beta u_1 + \gamma u_2 + \delta$ where $\alpha \in \mathbb{C}$ is a scalar, $\beta, \gamma \in \mathbb{C}[u_3, u_4]$ are of degree 1, and $\delta \in \mathbb{C}[u_3, u_4]$ is of degree 2. We can then write q as

$$q(u) = a u_1 + b \tag{3.12}$$

where $a = (\alpha u_2 + \beta)$, $b = (\gamma u_2 + \delta) \in \mathbb{C}[u_2, u_3, u_4]$ with $\deg(a) = 1$, $\deg(b) = 2$. The map $(\pi_1|_Q)^{-1}$ is then given by

$$x \mapsto (-b(x) : x_1 a(x) : x_2 a(x) : x_3 a(x)) =: (u_1 : u_2 : u_3 : u_4). \tag{3.13}$$

To verify this, first check that $\pi_1(u) = a(x) \cdot x$ where $\cdot$ denotes scalar multiplication. To see that $u \in Q$

$$\begin{aligned} q(u) &= u_1 \cdot a(u_2, u_3, u_4) + b(u_2, u_3, u_4) \\ &= u_1 \cdot a(\pi_1(u)) + b(\pi_1(u)) \\ &= -b(x) \cdot a(a(x) \cdot x) + b(a(x) \cdot x) \\ &= -b(x)a(x)a(x) + a(x)^2 b(x) = 0 \end{aligned} \tag{3.14}$$

where the last equality comes from the homogeneity of a, b with $\deg(a) = 1$, $\deg(b) = 2$.

Composing with π_2 we have

$$\pi_2 \circ (\pi_1|_Q)^{-1}(x) = (-b(x) : x_2 a(x) : x_3 a(x)), \tag{3.15}$$

whose coordinates are indeed quadratic. Since $f = \pi_2 \circ (\pi_1|_Q)^{-1}$ is defined by quadratics and generically $1 : 1$, it is a quadratic Cremona transformation.

To show that this transformation is non-degenerate, we must demonstrate that it has three unique base points. To understand the base points of f, recall that on a smooth quadric surface there are two distinct (possibly complex) lines passing through each point. The images of the two lines passing through c_1 under the projection π_1 will each be a single point. Therefore f is not well-defined on these image points in $\mathbb{P}^2$. Similarly, f is undefined on $\pi_1(c_2)$ since $\pi_2(\pi_1^{-1}(\pi_1(c_2))) = \pi_2(c_2) = 0$. Therefore these three points are exactly the base points of f in the domain. Finally, because $\overline{c_1c_2} \not\subset Q$, these base points are all distinct. The base points in the codomain can be found symmetrically. $\square$

Thus far we have shown that if we fix linear projections $\pi_1, \pi_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ with centers $c_1 \neq c_2 \in \mathbb{P}^3$, then there is a bijection between permissible quadrics through c_1, c_2 and generic lines through the fundamental matrix F of (π_1, π_2). Furthermore, there is a map sending each generic line through F (permissible quadric through c_1, c_2) to the Cremona transformation from $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by $\pi_2 \circ (\pi_1|_Q)^{-1}$. These correspondences are summarized in (3.16), where $\mathcal{L}_F$ is the set of all generic lines through F and $\mathcal{Q}_F$ is the set of all permissible quadrics through c_1, c_2.

$$\begin{tikzcd}
& \mathcal{Q}_F \arrow[dl] \arrow[dr] & \\
\mathcal{L}_F \arrow[ur] \arrow[rr, dashed] & & \mathcal{C}
\end{tikzcd} \tag{3.16}$$

We can make the correspondence between generic lines through F and Cremona transformations even more explicit.

Lemma 3.3.7. *Given a generic line* $\ell \subset \mathbb{P}(\mathbb{C}^{3\times 3})$, *the set of points* $(x, y) \in \mathbb{P}^2 \times \mathbb{P}^2$ *satisfying* $y^T M x = 0$ *for all* $M \in \ell$ *coincides with the closure of the graph* $\{(x, f(x)) : x \in \mathbb{P}^2 \backslash B(f)\}$ *of a* *unique Cremona transformation* $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$*. This gives a* $1:1$ *correspondence between generic lines* $\ell \subset \mathbb{P}(\mathbb{C}^{3\times 3})$ *and Cremona transformations* $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$*. Moreover, when* $F \in \ell$ *has rank two, this Cremona transformation agrees with that induced by the maps* $\mathcal{L}_F \to \mathcal{Q}_F \to \mathcal{C}$*.*

Proof. Since ℓ is generic, we may assume without loss of generality that $\ell = \operatorname{span}\{F, M\}$ where F has rank two. This gives a pair of linear projections $\pi_1, \pi_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ with non-coincident centers c_1, c_2 with fundamental matrix F. In the 1:1 correspondence $\mathcal{L}_F \leftrightarrow \mathcal{Q}_F$ given in Corollary 3.3.4, the line ℓ corresponds to the permissible quadric Q given by the zero set of $q(u) = \pi_2(u)^\top M \pi_1(u)$. By Lemma 3.3.6, the Cremona transformation $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ corresponding to $q(u)$ in the correspondence $\mathcal{Q}_F \to \mathcal{C}$ satisfies $f(\pi_1(p)) = \pi_2(p)$ for all $p \in Q \backslash \{c_1, c_2\}$. Since $\pi_1(Q)$ is dense in $\mathbb{P}^2$, the graph of f and the set $\{(\pi_1(p), \pi_2(p)) : p \in Q\backslash\{c_1, c_2\}\} \subset \mathbb{P}^2 \times \mathbb{P}^2$ are both two-dimensional, as is their intersection. Each is the image of an irreducible variety under a rational map and so the Zariski-closures of these two sets are equal. By construction, this is contained in the zero sets of $y^T F x$ and $y^T M x$, as $\pi_2(p)^T F \pi_1(p) = 0$ for all $p \in \mathbb{P}^3$ and $\pi_2(p)^T M \pi_1(p) = 0$ for all $p \in Q$. Since F, M are linearly independent, the variety $\{(x, y) : y^T F x = y^T M x = 0\}$ in $\mathbb{P}^2 \times \mathbb{P}^2$ is two-dimensional. It therefore coincides with the Zariski-closure of the graph of f.

Conversely, suppose $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is a Cremona transformation. We claim that $\{f(x)x^\top : x \in \mathbb{C}^3\}$ spans a 7-dimensional linear space $V \subset \mathbb{C}^{3\times 3}$. Up to projective transformations on $\mathbb{P}^2_x$ and $\mathbb{P}^2_y$, we can take f to be the standard Cremona involution, giving

$$f(x)x^\top = \begin{pmatrix} x_1x_2x_3 & x_1^2x_3 & x_1^2x_2 \\ x_2^2x_3 & x_1x_2x_3 & x_1x_2^2 \\ x_2x_3^2 & x_1x_3^2 & x_1x_2x_3 \end{pmatrix}. \tag{3.17}$$

One can check explicitly that seven distinct monomials appear in this matrix and so the span of all such matrices is 7-dimensional. Projectively, the orthogonal complement gives a line $\ell = V^\perp$ in $\mathbb{P}(\mathbb{C}^{3\times 3})$. By definition, ℓ is exactly the set of all matrices M such that $y^\top M x = 0$ for all (x, y) in the graph of f. Under the assumption that f is the standard Cremona transformation, ℓ is the span of the diagonal matrices $F_1 = \operatorname{diag}(1, -1, 0)$ and $F_2 = \operatorname{diag}(0, 1, -1)$; in general ℓ will be projectively equivalent to this line. We can verify that this line contains exactly the three rank-two matrices, $F_1, F_2, F_1 + F_2$, and is therefore generic. $\square$

Remark 3.3.8. Given $\ell = \operatorname{span}\{F, M\}$ we can solve for the coordinates of the corresponding Cremona transformation $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ as follows. Given $x \in \mathbb{P}^2$, the corresponding point $y = f(x)$ will be the left kernel of the 3×2 matrix $\begin{pmatrix} Fx & Mx \end{pmatrix}$. The coordinates of y can be written explicitly in terms of the 2×2 minors of this matrix, which are quadratic in x. Note that, up to scaling, this formula for y is independent of the choice of basis $\{F, M\}$ for ℓ. Any point $x \in \mathbb{P}^2$ for which $\begin{pmatrix} Fx & Mx \end{pmatrix}$ has rank ≤ 1 will be a base point of this Cremona transformation. In particular, if $Fx = 0$, then x is a base point of f. As we will see below, there are three such points when ranging over all rank-two matrices in ℓ.

The next two results finish off the proof of the trinity correspondence (3.8) and proof of Theorem 3.3.2.

Lemma 3.3.9. *Let ℓ be a generic line in $\mathbb{P}(\mathbb{C}^{3\times 3})$, i.e., ℓ contains three rank-two matrices F_1, F_2, F_3.*

1. *Then ℓ gives rise to three permissible quadrics $Q_1, Q_2, Q_3 \subset \mathbb{P}^3$, each containing the centers of a pair of linear projections with fundamental matrices F_1, F_2, F_3 respectively.*
2. *The quadrics Q_1, Q_2, Q_3, in conjunction with their distinguished linear projections, all induce the same Cremona transformation f. The base points of f are e_1^x, e_2^x, e_3^x in the domain and e_1^y, e_2^y, e_3^y in the codomain, where e_i^x and e_i^y generate the right and left nullspaces of F_i respectively.*

Proof. A generic line $\ell \subset \mathbb{P}(\mathbb{C}^{3\times 3})$ intersects the determinantal variety $\mathcal{D}$ cut out by $\det X = 0$ in three rank-two matrices F_1, F_2, F_3. Each F_i is the fundamental matrix of a pair of linear projections $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ with noncoincident centers and, by Lemma 3.3.3 and Lemma 3.3.5 there is a unique permissible quadric Q_i through these centers corresponding to the line ℓ. By Lemma 3.3.7, each of these quadrics induce the same Cremona transformation $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$.

To conclude, we show that the base points of f and f^{-1} are e_1^x, e_2^x, e_3^x and e_1^y, e_2^y, e_3^y, respectively. We show that e_1^x, e_2^x, e_3^x are the base points of f and the argument for the base points of f^{-1} follows symmetrically. First, note that each e_i^x is a base point of f. This follows from Remark 3.3.8, since each $F_i \in \ell$ has rank two. Since the Cremona transformation f has three base points, it only remains to show that these points are distinct. If $e_1^x = e_2^x$, then by linearity $Fe_1^x = 0$ for all $F \in \ell = \operatorname{span}\{F_1, F_2\}$. This would imply that $\operatorname{rank}(F) \leq 2$ for all $F \in \ell$, contradicting genericity of the line ℓ. □

Corollary 3.3.10. *The correspondence* $\mathcal{Q} \to \mathcal{C}$ *is* $3:1$.

Proof. Let $\mathbf{Q} = (Q, \pi_1, \pi_2) \in \mathcal{Q}$ be a permissible quadric along with a pair of linear projections that correspond to $f \in \mathcal{C}$. If F is the fundamental matrix associated to (π_1, π_2), then there exists a unique generic line ℓ through F corresponding to Q by Lemma 3.3.3 and Lemma 3.3.5. With the full trinity correspondence, this line ℓ contains three fundamental matrices F_1, F_2, F_3 corresponding to $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3 \in \mathcal{Q}$ that each produce the Cremona transformation f. Moreover, by Lemma 3.3.7, this line ℓ is the unique line in $\mathbb{P}(\mathbb{C}^{3\times 3})$ corresponding to f. Therefore if $\mathbf{Q}' \in \mathcal{Q}$ is such that $\mathbf{Q}' \mapsto f$ it follows that π_1', π_2' have one of F_1, F_2, F_3 as their fundamental matrix and that the quadric Q' is produced by the line ℓ. We conclude that $\mathbf{Q}'$ is, up to projective equivalence, one of $\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3$. □

This completes the proof of Theorem 3.3.2. A consequence of Theorem 3.8 is the following generalization of Problem 3.1.5.

Theorem 3.3.11. *Given a generic codimension-two subspace* $V \subset \mathbb{P}(\mathbb{C}^{3\times 3})$*, the intersection of* V *with* R_1*, the Segre embedding of* $\mathbb{P}^2 \times \mathbb{P}^2$*, is a del Pezzo surface of degree six, and can be described explicitly via the trinity correspondence. Specifically, if* $g : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ *is the Cremona transformation corresponding to the line* $V^\perp$*, then*

$$V \cap R_1 = \{g(x)x^\top : x \in \mathbb{P}^2\} \cup \{xg^{-1}(x)^\top : x \in \mathbb{P}^2\}.$$

Proof. For convenience, we denote

$$V_1 := \{g(x)x^\top : x \in \mathbb{P}^2\} \cup \{xg^{-1}(x)^\top : x \in \mathbb{P}^2\}.$$

To see that this is a degree-six del Pezzo surface, we show that V_1 can be obtained as the blow up of $\mathbb{P}^2$ in three non-collinear points, specifically, at the base points of g: e_1^x, e_2^x, e_3^x. Let $\pi_x : V_1 \dashrightarrow \mathbb{P}^2$ be the morphism defined by $\pi_x(vu^\top) = u$. Let ℓ_i^y be the exceptional lines of g such that $g^{-1}(\ell_i^y) = e_i^x$. Then π_x is $1:1$ except on three mutually skew lines $\{y(e_i^x)^\top : y \in \ell_i^y\}$ which are taken to the points $\{e_i^x\}$. Therefore V_1 is the blow up of $\mathbb{P}^2$ in three non-collinear points and is a del Pezzo surface of degree six.

In particular, V_1 must be Zariski closed and it follows by Lemma 3.3.7 that $V \cap R_1 = V_1$. □

3.3.2 Back to the proof of Theorem 3.3.1

Before we can adapt the trinity correspondence to the reconstruction of point pairs, we need to address a certain kind of degeneracy. Given a configuration of point pairs $P = (x_i, y_i)_{i=1}^k$ consider the matrix $Z = (x_i^\top \otimes y_i^\top)_{i=1}^k$ and its right nullspace $\mathcal{N}_Z$.

Lemma 3.3.12. *Suppose $P = (x_i, y_i)_{i=1}^k$ admits a generic line $\ell \subseteq \mathcal{N}_Z$ (passing through three rank-two matrices F_1, F_2, F_3). Then for all $j = 1, 2, 3$, there is no i such that $y_i^\top F_j = 0 = F_j x_i$.*

Proof. Suppose, without loss of generality, $y_1^\top F_1 = 0 = F_1 x_1$. From the matrix F_1 and the line ℓ through it we obtain a pair of projections π_1, π_2 with centers c_1, c_2, and a smooth permissible quadric Q passing through them. Then $\pi_2(c_1)$ and $\pi_1(c_2)$ are the left and right epipoles of F_1, but since $y_1^\top F_1 = 0 = F_1 x_1$, it must be that $y_1 \sim \pi_2(c_1)$ and $x_1 \sim \pi_1(c_2)$. On the other hand, for any point p on the line connecting c_1, c_2,

$$\pi_2(p)^\top F_2 \pi_1(p) = \pi_2(c_1) F_2 \pi_1(c_2) = y_1^\top F_2 x_1 = 0$$

since $F_2 \in \mathcal{N}_Z$. Therefore, $p \in Q$ and thus $\overline{c_1 c_2} \subset Q$, which is a contradiction since Q is permissible. □

Even though a rank-two matrix F on a generic line in $\mathcal{N}_Z$ cannot have $y_i^\top F = 0 = F x_i$, it might be that one of the equations hold. We name this type of degeneracy in the following definition.

Definition 3.3.13. *A generic line $\ell \subseteq \mathcal{N}_Z$ is **P-degenerate** if there exists a rank-two matrix $F \in \ell$ such that either $F x_i = 0$ or $y_i^\top F = 0$ for some i. We call a generic line that is not P-degenerate a **P-generic** line.*

Any rank-two matrix F in a P-generic line will give a reconstruction $c_1, c_2, p_1, \ldots, p_k$ of the point pairs P. That is, there will be linear projections $\pi_1, \pi_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ with centers c_1, c_2 so that $\pi_2^\top(p) F \pi_1(p) = 0$ for all $p \in \mathbb{P}^3$ and $(x_i, y_i) = (\pi_1(p_i), \pi_2(p_i))$ for all $i = 1, \ldots, k$. A smooth quadric Q will contain two lines through any of its points.

Definition 3.3.14. *A quadric $Q \subset \mathbb{P}^3$ passes **degenerately** through a reconstruction c_1, c_2, $\{p_i\}_{i=1}^k$ of P if it passes through these $k+2$ points and contains the line through a center point c_i and reconstructed point p_j.*

Definition 3.3.15. *A Cremona transformation $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ maps $x_i \mapsto y_i$ **degenerately** if x_i is a base point of f and y_i lies on the corresponding exceptional line, or symmetrically, y_i is a base point of f^{-1} and x_i lies on the corresponding exceptional line.*

Generically, the trinity correspondence specializes to the reconstruction of point pairs in an intuitive way.

Theorem 3.3.16. *Given a configuration of point pairs $P = (x_i, y_i)_{i=1}^k$ and the matrix $Z = (x_i^\top \otimes y_i^\top)_{i=1}^k$, define the following subsets of $\mathcal{L}$, $\mathcal{Q}, \mathcal{C}$:*

1. *$\mathcal{L}_P$: the set of all P-generic lines $\ell \subseteq \mathcal{N}_Z := \operatorname{nullspace}(Z)$,*
2. *$\mathcal{Q}_P$: the set (up to projective equivalence) of all permissible quadrics passing non-degenerately through some reconstruction $c_1, c_2, p_1, \ldots, p_k$ of P,*
3. *$\mathcal{C}_P$: the set of all Cremona transformations $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ mapping $x_i \mapsto y_i$ non-degenerately for all $i = 1, \ldots, k$.*

Then there is a $1:1$ correspondence between the elements of $\mathcal{L}_P$ and $\mathcal{C}_P$, a $1:3$ correspondence between the elements of $\mathcal{L}_P$ and $\mathcal{Q}_P$, and a $3:1$ correspondence between the elements of $\mathcal{Q}_P$ and $\mathcal{C}_P$ as in:

$$\begin{tikzcd}
& \mathcal{Q}_P \arrow[dr, "3:1"] & \\
\mathcal{L}_P \arrow[ur, "1:3"] \arrow[rr, leftrightarrow] & & \mathcal{C}_P
\end{tikzcd} \tag{3.18}$$

Proof. We need to show that the trinity correspondence (3.8) can be restricted to the sets $\mathcal{L}_P, \mathcal{Q}_P, \mathcal{C}_P$. We will therefore examine each leg of this diagram.

($\mathcal{L}_P \to \mathcal{Q}_P$) We begin by considering a P-generic line $\ell = \operatorname{span}\{F, M\} \subseteq \mathcal{N}_Z$. Without loss of generality, we can take F to be one of the three fundamental matrices in ℓ with corresponding projections $\pi_1, \pi_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ with non-coincident centers c_1, c_2 that give reconstructions $p_1, \ldots, p_k \in \mathbb{P}^3$ of the point pairs P. By Lemma 3.3.3, the line ℓ corresponds to a smooth permissible quadric Q defined by the vanishing of $q(u) = \pi_2(u)^T M \pi_1(u)$. For any point p_i in the reconstruction, we have

$$q(p_i) = \pi_2(p_i)^\top M \pi_1(p_i) = y_i^\top M x_i = 0 \tag{3.19}$$

since $M \in \ell \subset \mathcal{N}_Z$. Therefore Q passes through the reconstruction $c_1, c_2, p_1, \ldots, p_k$. It remains to show that it does so non-degenerately. By Lemmas 3.3.6 and 3.3.9, a reconstructed point p_i lies on one of the lines through c_1 (or symmetrically through c_2) if and only if there exists $M \in \ell$ such that $Mx_i = 0$ (or symmetrically $y_i^\top M = 0$). Since ℓ is P-generic there is no such M, implying that the quadric passes through the reconstruction non-degenerately.

($\mathcal{Q}_P \to \mathcal{C}_P$) Consider a permissible quadric Q passing through a reconstruction $c_1, c_2, p_1, \ldots, p_k$ of P with linear projections π_1, π_2. As in Theorem 3.3.2, the tuple (Q, π_1, π_2) induces a Cremona transformation $f := \pi_2 \circ (\pi_1|_Q)^{-1}$. By Lemma 3.3.6, the base points of f are the images of the point c_2 and each of the lines in Q passing through c_1. Since $p_i \neq c_2$ and does not belong to these lines, the point $x_i = \pi_1(p_i)$ is not a base point of f. Similarly, the base points of f^{-1} are the images of the point c_1 and the lines in Q passing through c_2 under π_2, so a symmetric argument shows that $y_i = \pi_2(p_i)$ is not a base point of f^{-1}. Therefore f maps $x_i = \pi_1(p_i)$ to $y_i = \pi_2(p_i)$ non-degenerately.

($\mathcal{C}_P \to \mathcal{L}_P$) Consider a Cremona transformation $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ such that $x_i \mapsto y_i$ non-degenerately for all i. As in Lemma 3.3.7, f corresponds to a unique line $\ell \subset \mathbb{P}(\mathbb{C}^{3\times 3})$ defined by the property that $f(x)^\top Mx = 0$ for all $M \in \ell$ and $x \in \mathbb{P}^2$. In particular, $y_i^\top M x_i = 0$ for all $M \in \ell$ and $i = 1, \ldots, k$, implying that $\ell \subseteq \mathcal{N}_Z$. By assumption, no point x_i is a base point of f and no point y_i is a base point of f^{-1}. By Lemma 3.3.9, is then follows that $Mx_i \neq 0$ and $y_i^\top M \neq 0$ for all $M \in \ell$. Therefore ℓ is not P-degenerate. $\square$

Remark 3.3.17. The assumptions of non-degeneracy can be removed from the 1:1 correspondence between generic lines in $\mathcal{N}_Z$ and Cremona transformations mapping $x_i \mapsto y_i$. Extending this to quadrics is more subtle, as some rank-two matrices $F \in \ell \subset \mathcal{N}_Z$ may not give full reconstructions of the point pairs P.

Proof of Theorem 3.3.1(⇒). For $k = 8$ semi-generic point pairs, the matrix $Z = (x_i^\top \otimes y_i^\top)_{i=1}^8$ is rank deficient exactly when $\mathcal{N}_Z =: \ell$ is a line. This line ℓ is generic because it is also the nullspace of any submatrix of Z of size 7×9 and the corresponding seven point pairs are generic. Pick a subset of seven point pairs, say $(x_i, y_i)_{i=1}^7$, from the original eight pairs. Since these seven point pairs are generic, and ℓ is also generic, we can assume that $Fx_i \neq 0$ and $y_i^\top F \neq 0$ for any rank-two matrix $F \in \ell$ and all $i = 1, \ldots, 7$. On the other hand, if we pick a different set of seven point pairs, say $(x_i, y_i)_{i=2}^8$, then ℓ is also the nullspace of the corresponding Z_7 and by the same argument as before, $Fx_i \neq 0$ and $y_i^\top F \neq 0$ for any rank-two matrix $F \in \ell$ and all $i = 2, \ldots, 8$. Therefore, ℓ is P-generic.

Since ℓ is P-generic, by Theorem 3.3.16, ℓ gives rise to a Cremona transformation $f : \mathbb{P}_x^2 \dashrightarrow \mathbb{P}_y^2$ such that $f(x_i) = y_i$ for all $i = 1, \ldots, 8$. This finishes the proof of Theorem 3.3.1. $\square$

We end this section by demonstrating the trinity correspondence on an example, beginning with a single quadric through a reconstruction.

Example 7. Consider the quadric $Q \subset \mathbb{P}^3$ defined by the equation $x^2 + y^2 - z^2 - w^2 = 0$ and the following 10 points $p_1, \ldots, p_8, c_1, c_2 \in Q$:

$$c_1 = (1 : 0 : 0 : 1) \qquad c_2 = (0 : 1 : 0 : 1)$$

$$p_1 = (5 : 12 : 13 : 0) \qquad p_2 = (13 : 0 : 5 : 12)$$

$$p_3 = (12 : 5 : 13 : 0) \qquad p_4 = (3 : 4 : 5 : 0)$$

$$p_5 = (4 : 3 : 5 : 0) \qquad p_6 = (3 : 4 : 0 : 5)$$

$$p_7 = (4 : 3 : 0 : 5) \qquad p_8 = (5 : 0 : 4 : 3)$$

The two projections (cameras) with centers c_1, c_2 have matrices

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

and we can calculate the image points and epipoles

$$\begin{aligned}
e_x &= (-1 : 1 : 0) & \qquad e_y &= (1 : -1 : 0) \\
x_1 &= (5 : 12 : 13) & y_1 &= (5 : 12 : 13) \\
x_2 &= (1 : 0 : 5) & y_2 &= (13 : -12 : 5) \\
x_3 &= (12 : 5 : 13) & y_3 &= (12 : 5 : 13) \\
x_4 &= (3 : 4 : 5) & y_4 &= (3 : 4 : 5) \\
x_5 &= (4 : 3 : 5) & y_5 &= (4 : 3 : 5) \\
x_6 &= (-2 : 4 : 0) & y_6 &= (3 : -1 : 0) \\
x_7 &= (-1 : 3 : 0) & y_7 &= (4 : -2 : 0) \\
x_8 &= (2 : 0 : 4) & y_8 &= (5 : -3 : 4).
\end{aligned}$$

The point pairs (x_i, y_i) give us the matrix

$$Z_8 = \begin{bmatrix}
25 & 60 & 65 & 60 & 144 & 156 & 65 & 156 & 169 \\
13 & -12 & 5 & 0 & 0 & 0 & 65 & -60 & 25 \\
144 & 60 & 156 & 60 & 25 & 65 & 156 & 65 & 169 \\
9 & 12 & 15 & 12 & 16 & 20 & 15 & 20 & 25 \\
16 & 12 & 20 & 12 & 9 & 15 & 20 & 15 & 25 \\
-6 & 2 & 0 & 12 & -4 & 0 & 0 & 0 & 0 \\
-4 & 2 & 0 & 12 & -6 & 0 & 0 & 0 & 0 \\
10 & -6 & 8 & 0 & 0 & 0 & 20 & -12 & 16
\end{bmatrix}$$

which we can check is rank deficient and has nullspace spanned by the vectors

$$m_1 = (-1, 1, 0, -1, -1, 0, 0, 0, 1), \qquad m_2 = (0, 0, -1, 0, 0, -1, 1, 1, 0).$$

The reconstruction we started with has fundamental matrix

$$F = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

and if we take a different matrix

$$M = \begin{bmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

in the nullspace of Z_8 we can verify that $A_2^\top M A_1$ yields the original quadric Q

$$(x,y,z,w)A_2^\top M A_1 (x,y,z,w)^\top = (x,y,z,w)\begin{bmatrix} -1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 \end{bmatrix}(x,y,z,w)^\top = -x^2 - y^2 + z^2 + w^2.$$

The other two possible choices for fundamental matrices in the nullspace of Z_8 are

$$F_2 = \begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \quad \text{and} \quad F_3 = \begin{bmatrix} -1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{bmatrix},$$

which have epipoles

$$e_x^2 = (0:1:1) \qquad\qquad e_y^2 = (-1:0:1)$$
$$e_x^3 = (0:-1:1) \qquad\qquad e_y^3 = (1:0:1).$$

Moreover, we can verify that there is a unique Cremona transformation

$$f(x_1,x_2,x_3) = (x_1^2 - x_2^2 + x_3^2, x_1^2 + 2x_1x_2 + x_2^2 - x_3^2, 2x_1x_3)$$

such that $f(x_i) = y_i$ for all i. This Cremona transformation has base points exactly matching the epipoles. Finally, we can check that each camera center lies on two real lines on the quadric Q, parameterized by $(a:b) \in \mathbb{P}^1$ as

$$\ell_x^2 = (a:b:b:a), \quad \ell_x^3 = (a:-b:b:a), \quad \ell_y^2 = (-b:a:b:a), \quad \text{and} \quad \ell_y^3 = (b:a:b:a)$$

whose images are exactly the other two possible pairs of epipoles/base points (e_x^2, e_y^2) and (e_x^3, e_y^3).

3.4 $k = 7$

We now come to the case of $k = 7$ point pairs. In order to understand the case of seven point pairs, we will first need to understand six generic point pairs $(x_i, y_i)_{i=1}^6$. In this case, the nullspace $\mathcal{N}_Z$ of the matrix $Z = (x_i^\top \otimes y_i^\top)_{i=1}^6$ is projectively a plane and $\mathcal{N}_Z \cap \mathcal{D} =: C$ is a cubic curve in $\mathbb{P}(\mathbb{C}^{3\times 3})$ lying in the plane $\mathcal{N}_Z$. By our genericity assumption, C misses all rank-one matrices in $\mathcal{D}$ and hence every point on C is a fundamental matrix of $(x_i, y_i)_{i=1}^6$. Let κ_x and κ_y denote the quadratic maps that take a rank-two matrix $M \in \mathbb{P}(\mathbb{C}^{3\times 3})$ to its right and left nullvectors respectively. As a consequence of the classical theory of blowups and cubic surfaces as discussed in [14], the maps $C \mapsto \kappa_x(C) =: C_x \subset \mathbb{P}_x^2$ and $C \mapsto \kappa_y(C) =: C_y \subset \mathbb{P}_y^2$ are isomorphisms when $(x_i, y_i)_{i=1}^6$ is generic; we will go into more detail on the nature of these isomorphism in Section 4.2.1.

$$\begin{tikzcd} & C \arrow[dl, "\kappa_x"'] \arrow[dr, "\kappa_y"] & \\ \mathbb{P}_x^2 \supset C_x & & C_y \subset \mathbb{P}_y^2 \end{tikzcd} \tag{3.20}$$

By the composition $\kappa_y \circ \kappa_x^{-1}$, we get that C_x and C_y are isomorphic cubic curves. However, this isomorphism is not particularly useful; for instance, it does not take $x_i \mapsto y_i$. By construction, the curves C_x and C_y consist exactly of all possible epipoles of the fundamental matrices of $(x_i, y_i)_{i=1}^6$ in $\mathbb{P}_x^2$ and $\mathbb{P}_y^2$. We therefore call C_x and C_y the right and left *epipolar curves* of $(x_i, y_i)_{i=1}^6$. We will see that these cubic curves are closely tied to both rank drop and the trinity relationship established in Theorem 3.3.16.

Example 8. Consider the following six point pairs:

$$\begin{aligned} &x_1 = (0:0:1) \quad y_1 = (0:0:1) && x_2 = (1:0:1) \quad y_2 = (1:0:1) \\ &x_3 = (0:1:1) \quad y_3 = (0:1:1) && x_4 = (1:1:1) \quad y_4 = (1:1:1) \\ &x_5 = (3:5:1) \quad y_5 = (7:-2:1) && x_6 = (-7:11:1) \quad y_6 = (3:13:1) \end{aligned}$$

Figure 3.1 shows the curves C_x and C_y. Observe that $x_i \in C_x$ and $y_i \in C_y$ for all $i = 1, \dots, 6$,

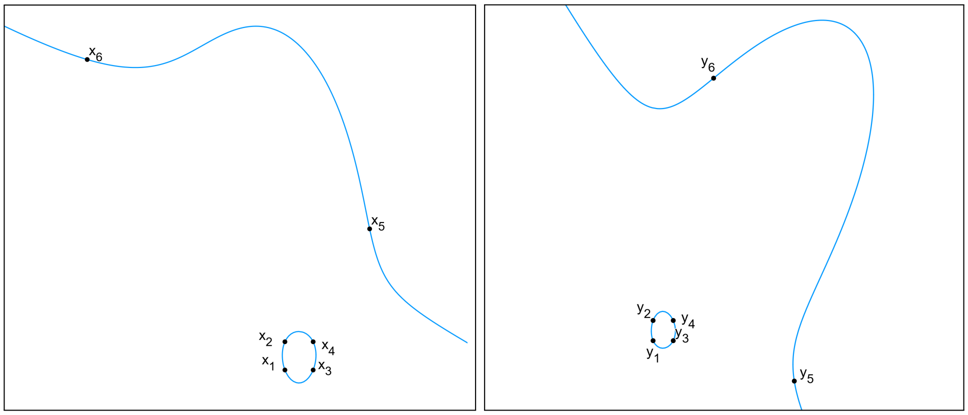

Figure 3.1: The cubic curves C_x and C_y from Example 8, with x_i and y_i labeled.

a fact we will prove in Section 3.4.1. The curves C_x and C_y are cut out by $g_x(u) = 0$ and $g_y(v) = 0$ in $\mathbb{P}^2_x$ and $\mathbb{P}^2_y$ where

$$\begin{aligned} g_x(u) = \quad & 447u_1^3 + 775u_1^2u_2 + 113u_1u_2^2 + 118u_2^3 - 4083u_1^2u_3 - 888u_1u_2u_3 \\ & -1521u_2^2u_3 + 3636u_1u_3^2 + 1403u_2u_3^2, \text{ and} \end{aligned}$$

$$\begin{aligned} g_y(v) = \quad & 447v_1^3 - 136v_1^2v_2 - 12v_1v_2^2 + 118v_2^3 - 3608v_1^2v_3 + 148v_1v_2v_3 \\ & -1478v_2^2v_3 + 3161v_1v_3^2 + 1360v_2v_3^2. \end{aligned}$$

In Section 3.4.2 we use classical invariant theory to derive the polynomials g_x and g_y. □

Given seven point pairs $(x_i, y_i)_{i=1}^7$, denote the epipolar curves obtained by excluding the ith point pair as $C_x^{\hat{i}}$ and $C_y^{\hat{i}}$. In the event that these curves are equal for all choices of i, we denote $C_x := C_x^{\hat{1}} = \ldots = C_x^{\hat{7}}$ and $C_y := C_y^{\hat{1}} = \ldots = C_y^{\hat{7}}$. We will see that this equality is necessary (Theorem 3.4.1) and sufficient (Theorem 3.4.9) for $Z_7 = (x_i^\top \otimes y_i^\top)_{i=1}^7$ to be rank deficient.

The maps κ_x, κ_y are not the only way to derive the epipolar curves C_x, C_y; it is also possible to obtain them via the trinity correspondence (3.18). This will be the subject of Section 4.1 and will allow us to prove the following result:

Theorem 3.4.1. *For $(x_i, y_i)_{i=1}^7$ semi-generic point pairs, Z_7 is rank deficient if and only if there exists cubic curves C_1 through $x_1, \ldots, x_7$ and C_2 through $y_1, \ldots, y_7$ as well as an isomorphism*

$f : C_1 \to C_2$ such that $x_i \mapsto y_i$. Moreover, if this holds then $C_1 = C_x$ and $C_2 = C_y$.

This is the first of the two main results in this section and is the more geometric theorem; we will prove it at the end of Section 4.1. In Section 4.2.1 we use the theory of cubic surfaces as in [14] to obtain explicit equations for the epipolar curves. In Section 4.2.2 we use these explicit equations to characterize rank deficiency of Z_7 using 14 algebraic equations and prove our second main result, Theorem 3.4.9, which is the more algebraic theorem. Finally, in Section 4.3 we collect some further results outside the assumption of semi-genericity.

3.4.1 Rank Drop and Cubic Curves

Before addressing the case of six generic point pairs and seven semi-generic point pairs, we establish an analogue of Lemma 3.3.7 to show how general projective planes in $\mathbb{P}(\mathbb{C}^{3\times 3})$ give rise to Cremona transformations of cubic curves.

Lemma 3.4.2. *Let $\mathcal{P} \subset \mathbb{P}(\mathbb{C}^{3\times 3})$ be a projective plane not containing any rank-one matrix. The set of points $(x, y) \in \mathbb{P}^2 \times \mathbb{P}^2$ satisfying $y^T M x = 0$ for all $M \in \mathcal{P}$ coincides with the closure of the graph $\{(x, f(x)) : x \in C_x^{\mathcal{P}}\}$ of the restriction of a Cremona transformation $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ to a cubic curve $C_x^{\mathcal{P}}$. Moreover there is a two-dimensional family of Cremona transformations $f_\ell : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, indexed by generic lines $\ell \subset \mathcal{P}$ as in Lemma 3.3.7, with the same restriction to $C_x^{\mathcal{P}}$.*

Proof. The curve $C_x^{\mathcal{P}}$ consists of the set of points $x \in \mathbb{P}^2$ for which there exists an $M \in \mathcal{P}$ with $Mx = 0$. When $\mathcal{P} = \mathcal{N}_Z$, this is the epipolar curve C_x described above. By choosing a basis $\{M_1, M_2, M_3\}$ for $\mathcal{P}$ we can write any $M \in \mathcal{P}$ as $aM_1 + bM_2 + cM_3$. Given $x \in \mathbb{P}^2$ there exists $(a : b : c) \in \mathbb{P}^2$ with $(aM_1 + bM_2 + cM_3)x = 0$ if and only if $\det \begin{pmatrix} M_1x & M_2x & M_3x \end{pmatrix} = 0$. Therefore $C_x^{\mathcal{P}}$ is defined by the vanishing of this determinant, which is a cubic form in x_1, x_2, x_3. Symmetrically the cubic curve $C_y^{\mathcal{P}}$ defined by the vanishing of the determinant of the matrix with rows $y^\top M_j$ coincides with C_y when $\mathcal{P} = \mathcal{N}_Z$.

Let $\ell = \operatorname{span}\{M_1, M_2\} \subset \mathcal{P} \subset \mathbb{P}(\mathbb{C}^{3\times 3})$ be a generic line. By Lemma 3.3.7, there is a Cremona transformation $f_\ell : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ whose graph is the set of points $(x, y) \in \mathbb{P}^2 \times \mathbb{P}^2$ satisfying $y^T M x = 0$ for all $M \in \ell$. As in Remark 3.3.8, the map f_ℓ is given by $x \mapstochar\dashrightarrow \ker \begin{pmatrix} M_1x & M_2x \end{pmatrix}$.

For $x \in C_x^{\mathcal{P}}$ except the three base points of f_ℓ, the left kernel of $\begin{pmatrix} M_1x & M_2x \end{pmatrix}$ is also the left kernel of the rank-two 3×3 matrix $\begin{pmatrix} M_1x & M_2x & M_3x \end{pmatrix}$, which is independent of the choice of $\ell = \operatorname{span}\{M_1, M_2\} \subset \mathcal{P}$.

Note that the graph $\{(x, f_\ell(x)) : x \in C_x^{\mathcal{P}}\}$ and the set of points $(x, y) \in \mathbb{P}_x^2 \times \mathbb{P}_y^2$ satisfying $y^T M x = 0$ for all $M \in \mathcal{P}$ have the same projection onto $\mathbb{P}_x^2$, namely $C_x^{\mathcal{P}}$. For any $x \in C_x^{\mathcal{P}}$, the corresponding point y is given by $f_\ell(x) = \ker \begin{pmatrix} M_1x & M_2x & M_3x \end{pmatrix}$. □

Six point pairs

Let $(x_i, y_i)_{i=1}^6$ be a set of six generic point pairs, $Z = (x_i, y_i)_{i=1}^6$ and F be any choice of fundamental matrix (i.e., a rank-two matrix on the projective plane $\mathcal{N}_Z$). Genericity guarantees a reconstruction $p_1, \ldots, p_6, c_1, c_2 \in \mathbb{P}^3$, of $(x_i, y_i)_{i=1}^6$ from F. Recall that c_1, c_2 are the centers of camera projections π_1, π_2 and $p_1, \ldots, p_6$ are world points such that $\pi_1(p_j) = x_j$ and $\pi_2(p_j) = y_j$ for all $j = 1, \ldots, 6$.

Since $\mathcal{N}_Z$ is a two-dimensional plane, it contains a pencil of lines through F (see (3.16) and (3.18)) which corresponds to a pencil of quadrics Q_λ, each passing through the reconstruction. The intersection of these quadrics, also obtainable as the intersection of any two distinct quadrics in the pencil, is a quartic space curve $W \subset \mathbb{P}^3$ that must also pass through the reconstruction. Since c_1, c_2 are on W, $\pi_1(W) \subset \mathbb{P}_x^2$ and $\pi_2(W) \subset \mathbb{P}_y^2$ are cubic curves. We will see that these cubic curves are independent of the choice of F, and that they are exactly the epipolar curves C_x and C_y. We will use this derivation to study their special properties arising from the trinity relationship. The following lemma assumes the set up just described.

Lemma 3.4.3. *For six generic point pairs $(x_i, y_i)_{i=1}^6$, we have the following:*

1. *The cubic curves $\pi_1(W)$ and $\pi_2(W)$ are the right and left epipolar curves C_x, C_y, respectively; In particular, they are independent of the choice of F;*
2. *The points x_i lie on C_x and y_i lie on C_y for $i = 1, \ldots, 6$.*
3. *There exists a two-parameter family of Cremona transformations $f_\ell : \mathbb{P}_x^2 \dashrightarrow \mathbb{P}_y^2$, indexed by lines ℓ in the projective plane $\mathcal{N}_Z$, such that the following holds:*

- *$f_\ell(x_i) = y_i$ for all $i = 1, \dots, 6$,*
- *the restriction of f_ℓ to a map $C_x \to C_y$ is independent of ℓ, and*
- *the base points of all the Cremona transformations f_ℓ lie in C_x, C_y.*

Proof. Let F be a fundamental matrix in $\mathcal{N}_Z$. Since $(x_i, y_i)_{i=1}^6$ is generic, F can be any element of the cubic curve $C = \mathcal{N}_Z \cap \mathcal{D}$, and we can use F to obtain a reconstruction consisting of world points $p_1, \dots, p_6$ and cameras corresponding to linear projections $\pi_1, \pi_2 : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$.

The quartic space curve W is defined by quadrics of the form $q(u) = \pi_2(u)^\top M \pi_1(u)$ where $M \in \mathcal{P} \cap F^\perp$. Therefore $\pi_1(W)$ contains the cubic plane curve C_x defined by $\{x \in \mathbb{P}^2 : \exists M \in \mathcal{N}_Z \text{ s.t. } Mx = 0\}$. Since $c_1 \in W$, $\pi_1(W)$ is a cubic plane curve and so these must be equal. A symmetric argument shows that $\pi_2(W) = C_y$. Since W contains each point p_i, this also implies that $x_i = \pi_1(p_i)$ belongs to C_x and $y_i = \pi_2(p_i)$ belongs to C_y for all $i = 1, \dots, 6$.

By Lemma 3.4.2, for any generic line $\ell \subset \mathcal{N}_Z$, the restriction of the Cremona transformation $f_\ell : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ to the cubic C_x is independent of the choice of ℓ. By Theorem 3.3.16, $f_\ell(x_i) = y_i$ for all i. As in Lemma 3.3.9, the base points of f_ℓ are the right kernels of the three rank-two matrices $F_1, F_2, F_3 \in \ell$ and therefore belong to C_x. Similarly, the base points of f_ℓ^{-1} are the left kernels of these matrices and so belong to C_y. $\square$

Remark 3.4.4. Given a rank two matrix $F \in \mathcal{N}_Z$, it may be the case that $Fx_i = 0$ (or $y_i^\top F = 0$) for some i. However, even in this case we can still apply the trinity (3.8) to obtain a pencil of quadrics (and a pencil of Cremona transformations), and from them the cubic curves C_x, C_y with the isomorphism between them. Therefore, even if ℓ is such that x_i is a base point of f_ℓ, the restriction of f_ℓ to a map $C_x \to C_y$, as in Lemma 3.2.6, would still satisfy $x_i \mapsto y_i$

From six points to seven

The trinity correspondence has allowed us to prove a number of properties of the epipolar curves corresponding to six generic point pairs. In particular, we know that there is an isomorphism $f : C_x \to C_y$ that sends $x_i \mapsto y_i$ for all $i = 1, \dots, 6$, which is induced by a two-parameter family of

Cremona transformations $\mathbb{P}^2_x \dashrightarrow \mathbb{P}^2_y$. For seven generic point pairs, the following corollary holds.

Lemma 3.4.5. *Let $(x_i, y_i)_{i=1}^7$ be seven semi-generic point pairs. Then the rank of $Z = (x_i^\top \otimes y_i^\top)_{i=1}^7$ drops if and only if there are cubic curves C_1, C_2, through $x_1, \ldots, x_7$ and $y_1 \ldots, y_7$ respectively, as well as a two-parameter family of Cremona transformations $f_\ell : \mathbb{P}^2_x \dashrightarrow \mathbb{P}^2_y$ such that $f(x_i) = y_i$ for all i and the family is well-defined on the restriction $C_1 \to C_2$. Furthermore, if this holds then $C_1 = C_x$ and $C_2 = C_y$.*

Proof. ($\Rightarrow$) Under semi-genericity, Z is rank deficient if and only if the nullspace of Z and the nullspaces of each of its 6×9 submatrices are identical. In particular, if P_i is the subset of 6 point pairs obtained by excluding the ith, then, using the notation from Theorem 3.3.16, $\mathcal{L}_{P_1} = \ldots = \mathcal{L}_{P_7}$. Applying Lemma 3.4.3, we find that the pairs of curves $C_x^{\hat{i}}, C_y^{\hat{i}}$ are identical for all i. Accordingly, we exclude the indexing and identify them as C_x and C_y respectively. Similarly, the family of Cremona transformations $\mathcal{C}_{P_1} = \ldots = \mathcal{C}_{P_7}$, and, as in Lemma 3.4.3, restricting this family to the map $C_x \to C_y$ yields a well-defined isomorphism with the property $x_i \mapsto y_i$ for all i.

($\Leftarrow$) For this direction, we use Theorem 3.3.16. In particular, the existence of such a family of Cremona transformations implies $\dim(\mathcal{L}_P) = \dim(\mathcal{C}_P) = 2$ as illustrated in (3.18). Since there is a two-dimensional family of lines ℓ in the projective nullspace of Z, we must have $\operatorname{rank}(Z) < 7$. We now need to verify that $C_1 = C_x$ and $C_2 = C_y$. It follows by Lemma 3.2.5 that the curves C_1, C_2 contain all possible base points of the Cremona transformations f_ℓ. Furthermore, by Lemma 3.3.9 the sets of all such base points in the domain and codomain is exactly the set of all possible right and left epipoles. It follows that $C_x \subset C_1$ and $C_y \subset C_2$ and therefore the curves are equal. □

Proof of Theorem 3.4.1. ($\Rightarrow$) This direction follows from Lemma 3.4.5. In particular, the isomorphism is exactly that obtained by restricting the family of Cremona transformations to the map $C_x \to C_y$.

($\Leftarrow$) Assume such curves C_1, C_2 exist, as well as the desired isomorphism $C_1 \to C_2$. By Lemma 3.2.7 there is a two-parameter family of Cremona transformations $\mathbb{P}^2_x \dashrightarrow \mathbb{P}^2_y$ whose restriction $C_1 \to C_2$ yields this isomorphism. It follows from Lemma 3.4.5 that Z is rank deficient and that $C_1 = C_x$ and $C_2 = C_y$. □

3.4.2 *The Cremona Hexahedral form of C_x and C_y*

In this section we return to the original characterization of the cubic curves C_x and C_y as the images under the quadratic maps κ_x and κ_y of the curve C as in (3.20). We will see that it is possible to derive explicit equations for these curves using the classical theory of cubic surfaces and a special invariant-theoretic representation of them called the Cremona hexahedral form. These ideas intersect substantially with the characterization of rank drop of Z_6 in [14]; in particular, we draw on the connection between six generic points pairs $(x_i, y_i)_{i=1}^6$ and cubic surfaces. We will begin by explicitly characterizing the curve $C = \mathcal{N}_Z \cap \mathcal{D}$ as the planar section of a cubic surface; we will then use this characterization in conjunction with material from [14] to find explicit equations for the curves C_x and C_y.

Six generic point pairs again

Suppose we have six generic point pairs $(x_i, y_i)_{i=1}^6$; in particular, $Z = (x_i^\top \otimes y_i^\top)_{i=1}^6$ has full rank. Let $Z_{\hat{j}}$ denote the 5×9 matrix obtained by deleting the jth row of Z. Then $\mathcal{N}_{Z_{\hat{j}}} \cong \mathbb{P}^3$ and $S_{\hat{j}} := \mathcal{N}_{Z_{\hat{j}}} \cap \mathcal{D}$ is a smooth cubic surface in $\mathcal{N}_{Z_{\hat{j}}}$ by the genericity assumption, and hence all points on it have rank-two. It was shown in [14] that $S_{\hat{j}}$ is the blowup of $\mathbb{P}_x^2$ at $(\{x_i\}_{i=1}^6 \setminus \{x_j\}) \cup \{\bar{x}_j\}$ where $\bar{x}_j$ is a new point that arises from $\{x_i\}_{i=1}^6 \setminus \{x_j\}$, see Lemma 6.1 of [14] for its derivation and formula. Symmetrically, $S_{\hat{j}}$ is also the blowup of $(\{y_i\}_{i=1}^6 \setminus \{y_j\}) \cup \{\bar{y}_j\}$ in $\mathbb{P}_y^2$ where $\bar{y}_j$ is a new point determined by $\{y_i\}_{i=1}^6 \setminus \{y_j\}$. The quadratic maps $\kappa_x^{\hat{j}} : S_{\hat{j}} \to \mathbb{P}_x^2$ and $\kappa_y^{\hat{j}} : S_{\hat{j}} \to \mathbb{P}_y^2$ are $1:1$ except on the exceptional lines of the blowup. The curve C is given by

$$C = \mathcal{N}_Z \cap \mathcal{D} = \mathcal{N}_{Z_{\hat{j}}} \cap \mathcal{D} \cap (x_j^\top \otimes y_j^\top)^\perp = S_{\hat{j}} \cap (x_j^\top \otimes y_j^\top)^\perp.$$

Therefore, C cuts each of the exceptional lines of the blowup in one point and therefore the restrictions of κ_x, κ_y to C are isomorphisms.

For a set of six points $u_1, \ldots, u_6 \in \mathbb{P}^2$, setting $[ijk] := \det[u_i\, u_j\, u_k]$, define

$$[(ij)(kl)(rs)] := [ijr][kls] - [ijs][klr]. \tag{3.21}$$

This is a classical invariant of $u_1, \dots, u_6$ under the action of PGL(3) whose vanishing expresses that the lines $\overline{u_i u_j}$, $\overline{u_k u_l}$ and $\overline{u_r u_s}$ meet in a point [12, pp 169]. Using these invariants, Coble defines the following six scalars [12, pp 170]:

$$\begin{aligned}
\bar{a} &= [(25)(13)(46)] + [(51)(42)(36)] + [(14)(35)(26)] + [(43)(21)(56)] + [(32)(54)(16)]\\
\bar{b} &= [(53)(12)(46)] + [(14)(23)(56)] + [(25)(34)(16)] + [(31)(45)(26)] + [(42)(51)(36)]\\
\bar{c} &= [(53)(41)(26)] + [(34)(25)(16)] + [(42)(13)(56)] + [(21)(54)(36)] + [(15)(32)(46)]\\
\bar{d} &= [(45)(31)(26)] + [(53)(24)(16)] + [(41)(25)(36)] + [(32)(15)(46)] + [(21)(43)(56)]\\
\bar{e} &= [(31)(24)(56)] + [(12)(53)(46)] + [(25)(41)(36)] + [(54)(32)(16)] + [(43)(15)(26)]\\
\bar{f} &= [(42)(35)(16)] + [(23)(14)(56)] + [(31)(52)(46)] + [(15)(43)(26)] + [(54)(21)(36)]
\end{aligned} \tag{3.22}$$

Coble also defines the following six cubic polynomials that vanish on $u_1, \dots, u_6$:

$$\begin{aligned}
a(u) &= [25u][13u][46u] + [51u][42u][36u] + [14u][35u][26u] + [43u][21u][56u] + [32u][54u][16u]\\
b(u) &= [53u][12u][46u] + [14u][23u][56u] + [25u][34u][16u] + [31u][45u][26u] + [42u][51u][36u]\\
c(u) &= [53u][41u][26u] + [34u][25u][16u] + [42u][13u][56u] + [21u][54u][36u] + [15u][32u][46u]\\
d(u) &= [45u][31u][26u] + [53u][24u][16u] + [41u][25u][36u] + [32u][15u][46u] + [21u][43u][56u]\\
e(u) &= [31u][24u][56u] + [12u][53u][46u] + [25u][41u][36u] + [54u][32u][16u] + [43u][15u][26u]\\
f(u) &= [42u][35u][16u] + [23u][14u][56u] + [31u][52u][46u] + [15u][43u][26u] + [54u][21u][36u]
\end{aligned} \tag{3.23}$$

These cubic polynomials are *covariants* of $u_1, \dots, u_6$ under the action of PGL(3).

It is a well-known result in algebraic geometry that every smooth cubic surface is the blowup of six points in $\mathbb{P}^2$. The blowup procedure furnishes an algorithm to find a determinantal representation of the surface. However, these representations do not directly reflect the six points that were blown up. The *Cremona hexahedral form* of a smooth cubic surface provides explicit equations for the surface in terms of the points being blown up. It consists of the following polynomials:

$$\begin{aligned}
z_1^3 + z_2^3 + z_3^3 + z_4^3 + z_5^3 + z_6^3 &= 0\\
z_1 + z_2 + z_3 + z_4 + z_5 + z_6 &= 0\\
\bar{a}z_1 + \bar{b}z_2 + \bar{c}z_3 + \bar{d}z_4 + \bar{e}z_5 + \bar{f}z_6 &= 0
\end{aligned} \tag{3.24}$$

Furthermore, the cubic surface can also be parameterized by

$$\overline{\{(a(u):b(u):c(u):d(u):e(u):f(u))\ :\ u\in\mathbb{P}^2\}}. \tag{3.25}$$

We will now use the above facts to obtain explicit equations (that depend on $(x_i, y_i)_{i=1}^6$), of the epipolar curves C_x and C_y. In what follows, we subscript $\bar{a}, \ldots, \bar{f}$ and $a(u), \ldots, f(u)$ with x (respectively, y) when $u_i = x_i$ (respectively, $u_i = y_i$).

Definition 3.4.6. *Given six point pairs $(x_i, y_i)_{i=1}^6$, define the following cubic polynomials:*

$$\begin{aligned} g_x(u) &:= \bar{a}_y a_x(u) + \bar{b}_y b_x(u) + \bar{c}_y c_x(u) + \bar{d}_y d_x(u) + \bar{e}_y e_x(u) + \bar{f}_y f_x(u), \\ g_y(v) &:= \bar{a}_x a_y(v) + \bar{b}_x b_y(v) + \bar{c}_x c_y(v) + \bar{d}_x d_y(v) + \bar{e}_x e_y(v) + \bar{f}_x f_y(v). \end{aligned} \tag{3.26}$$

Given seven point pairs $(x_i, y_i)_{i=1}^7$, let $g_x^{\hat{i}}$ and $g_y^{\hat{i}}$ denote the above cubic polynomials obtained from the point pairs $(x_j, y_j)_{j\neq i}$.

The polynomials g_x, g_y played a prominent role in the rank drop of Z_6 in [14].

Lemma 3.4.7. *Given generic point pairs $(x_i, y_i)_{i=1}^6$, let $C = \mathcal{N}_Z \cap \mathcal{D}$, $C_x = \kappa_x(C) \subset \mathbb{P}_x^2$ and $C_y = \kappa_y(C) \subset \mathbb{P}_y^2$. Also let S_x be the blowup of $\mathbb{P}_x^2$ at $x_1, \ldots, x_6$ and let S_y be the blowup of $\mathbb{P}_y^2$ at $y_1, \ldots, y_6$, each expressed in Cremona hexahedral form. Then the following hold true:*

1. *The plane cubic curves C_x and C_y have defining equations $g_x(u) = 0$ and $g_y(v) = 0$ respectively.*

2. *The cubic curve $C \cong S_x \cap S_y$ which has equations:*

$$\begin{aligned} z_1^3 + z_2^3 + z_3^3 + z_4^3 + z_5^3 + z_6^3 &= 0 \\ z_1 + z_2 + z_3 + z_4 + z_5 + z_6 &= 0 \\ \bar{a}_x z_1 + \bar{b}_x z_2 + \bar{c}_x z_3 + \bar{d}_x z_4 + \bar{e}_x z_5 + \bar{f}_x z_6 &= 0 \\ \bar{a}_y z_1 + \bar{b}_y z_2 + \bar{c}_y z_3 + \bar{d}_y z_4 + \bar{e}_y z_5 + \bar{f}_y z_6 &= 0 \end{aligned} \tag{3.27}$$

3. *The cubic curve $S_x \cap S_y$ is the image of C_x under the blowup of $\mathbb{P}_x^2$ at $x_1, \ldots, x_6$ and also the image of C_y under the blowup of $\mathbb{P}_y^2$ at $y_1, \ldots, y_6$.*

Proof. We begin with the first item. By Lemma 3.4.3, $x_i \in C_x$ for all i and by Definition 3.4.6, $g_x(x_i) = 0$ for all i since the cubic polynomials in (3.23) vanish on the x_i. For fixed $i = 1, \ldots, 6$, consider the 5 point pairs left after excluding (x_i, y_i) and let (u_i, v_i) be the unique new point pair (cf. Lemma 6.1 in [14]) such that the configuration

$$\Big\{(x_1, y_1), \ldots, \widehat{(x_i, y_i)}, \ldots, (x_6, y_6), (u_i, v_i)\Big\} \tag{3.28}$$

is rank deficient. For convenience, we assume without loss of generality that $i = 6$. In other words, if $Z_{\hat{6}} = (x_i \otimes y_i)_{i=1}^5$ then (u_6, v_6) is the unique point pair such that $S_{\hat{6}} = \mathcal{N}_{Z_{\hat{6}}} \cap \mathcal{D}$ can be obtained both by blowing up $\mathbb{P}^2_x$ in the points $x_1, \ldots, x_5, u_6$ and by blowing up $\mathbb{P}^2_y$ in the points $y_1, \ldots, y_5, v_6$. It follows that the curve $C \subset S_{\hat{6}}$ cuts the exceptional lines corresponding to u_6, v_6 exactly once each and therefore $u_6 \in C_x$ and $v_6 \in C_y$; it follows symmetrically that $u_i \in C_x$ and $v_i \in C_y$ for all $i = 1, \ldots, 6$. One can check using a computer algebra package that $g_x(u_6) = 0$ and $g_y(v_6) = 0$ after fixing points as in Lemma 6.1 in [14]; it follows symmetrically that $g_x(u_i) = 0$ and $g_y(v_i) = 0$ for all i. Finally, since C_x and the curve cut out by g_x share 12 distinct points, they must be the same cubic curve; similarly we can conclude that C_y is cut out by g_y. This finishes the proof of the first claim.

To prove the second and third claims, recall that $\kappa_x : C \to C_x$ is an isomorphism. Let $\kappa'_x : S_x \to \mathbb{P}^2_x$ and $\kappa'_y : S_y \to \mathbb{P}^2_y$ be the blow down morphisms. The Cremona hexahedral forms of S_x and S_y give

$$S_x \cap S_y = \{z \in S_x \ : \ \bar{a}_y z_1 + \ldots + \bar{f}_y z_6 = 0\}. \tag{3.29}$$

By (3.25),

$$S_x = \overline{\{(a_x(u) : b_x(u) : c_x(u) : d_x(u) : e_x(u) : f_x(u)) \ : \ u \in \mathbb{P}^2\}} \tag{3.30}$$

and since C_x is cut out by $g_x(u) = 0$, we get that

$$\begin{aligned} S_x \cap S_y &= \overline{\{(a_x(u) : \ldots : f_x(u)) \ : \ \bar{a}_y a_x(u) + \ldots + \bar{f}_y f_x(u) = 0, u \in \mathbb{P}^2_x\}} \\ &= \overline{\{(a_x(u) : \ldots : f_x(u)) \ : \ u \in C_x\}}. \end{aligned} \tag{3.31}$$

Therefore, $S_x \cap S_y$ is exactly the image of C_x under the blowup of $\mathbb{P}^2_x$ at $x_1 \ldots, x_6$. Restricting κ_x to $\kappa'_x|_{S_x \cap S_y}$: $S_x \cap S_y \to C_x$ we obtain an isomorphism, and we have $S_x \cap S_y \cong C_x \cong C$, which

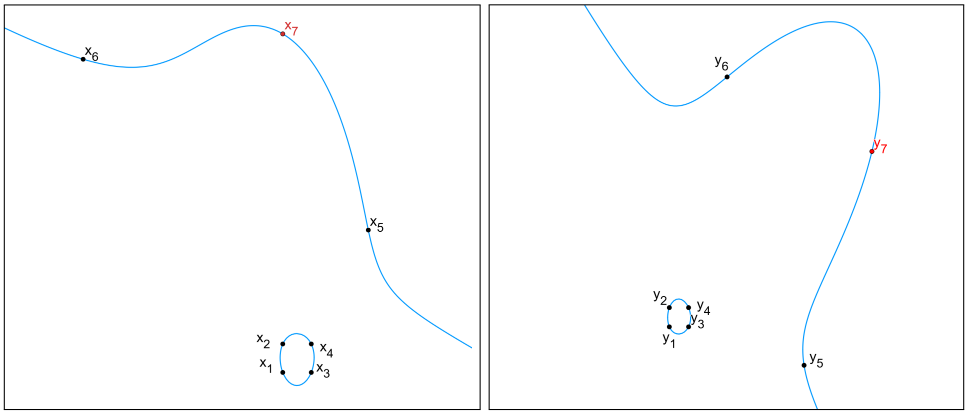

Figure 3.2: The cubic curves C_x and C_y, with x_7 and y_7 highlighted.

proves the second claim. Finally, we note that by a symmetric argument, $S_x \cap S_y$ is also exactly the image of C_y under the blowup of $\mathbb{P}^2_y$ at $y_1, \ldots, y_6$ proving the third claim as well. $\square$

Example 9 (Example 8, continued). One can verify that the polynomials (3.26) define the same cubic curves as those in Example 8. We then pick a specific point $x_7 = (0 : 1403 : 118) \in C_x$. Using a computer algebra package, one can compute the unique point $y_7 = (1802855 : 1562942 : 171287)$ such that the $Z = (x_i, y_i)_{i=1}^7$ is rank deficient. It is straight-forward to verify that $y_7 \in C_y$. Moreover, there is a two-parameter family of Cremona transformations f_ℓ such that $x_i \mapsto y_i$ for $i = 1, \ldots, 6$ and for all members of this family $f_\ell(x_7) = (1802855 : 1562942 : 171287)$, which lines up with Lemma 3.4.5. These points can be seen on the cubic curve in Figure 3.2.

Algebraic Conditions for the rank deficiency of Z_7

We are now ready to present our main algebraic result for rank drop given $k = 7$ point pairs. We begin with a basic lemma that will connect all of our results in the main theorem.

Lemma 3.4.8. *Let $(x_i, y_i)_{i=1}^7$ be seven semi-generic points. Then $Z = (x_i^\top \otimes y_i^\top)_{i=1}^7$ is rank deficient if and only if $C^{\hat{1}} = \cdots = C^{\hat{7}}$ where $C^{\hat{i}}$ is the cubic curve $\mathcal{N}_{Z_{\hat{i}}} \cap \mathcal{D}$.*

Proof. By semi-genericity, Z is rank deficient if and only if $\mathcal{N}_Z = \mathcal{N}_{Z_{\hat{1}}} = \cdots = \mathcal{N}_{Z_{\hat{7}}}$ for each 6×9 submatrix $Z_{\hat{i}}$ of Z. Since $C^{\hat{i}} = \mathcal{N}_{Z_{\hat{i}}} \cap \mathcal{D}$, Z is rank deficient if and only if $C = C^{\hat{1}} = \cdots = C^{\hat{7}}$. $\square$

The following theorem, which is the main result of this subsection, allows us to check for rank drop without computing Cremona transformations.

Theorem 3.4.9. *For $(x_i, y_i)_{i=1}^7$ semi-generic, the following are equivalent:*

1. *$Z = (x_i^\top \otimes y_i^\top)_{i=1}^7$ is rank deficient.*
2. *We have $x_i \in C_x^{\hat{i}}$ and $y_i \in C_y^{\hat{i}}$ for all $i = 1, \ldots, 7$.*
3. *We have $g_x^{\hat{i}}(x_i) = 0$ and $g_y^{\hat{i}}(y_i) = 0$ for all $i = 1, \ldots, 7$.*
4. *All 7 cubic curves in $\mathbb{P}_x^2$ are equal: $C_x^{\hat{7}} = \ldots = C_x^{\hat{1}}$.*
5. *All 7 cubic curves in $\mathbb{P}_y^2$ are equal: $C_y^{\hat{7}} = \ldots = C_y^{\hat{1}}$.*

Proof. By Lemma 3.4.7, $(2) \iff (3)$. We next prove that $(1) \implies (4), (5)$. If Z is rank deficient, then by Lemma 3.4.8, $C^{\hat{1}} = \cdots = C^{\hat{7}}$. Applying the quadratic maps κ_x and κ_y, we obtain (4) and (5). To prove the reverse direction we will show $(4) \implies (1)$; the proof that $(5) \implies (1)$ is symmetric. In particular, we will show that $C_x^{\hat{i}} = C_x^{\hat{j}}$ if and only if $C^{\hat{i}} = C^{\hat{j}}$. For ease of notation, we will assume $i = 6$ and $j = 7$. Consider the five point pairs $(x_i, y_i)_{i=1}^5$ and the matrix $Z_5 = (x_i^\top \otimes y_i^\top)_{i=1}^5$. Then $S = \mathcal{N}_{Z_5} \cap \mathcal{D}$ is a cubic surface and $\kappa_x : S \to \mathbb{P}_x^2$ and $\kappa_y : S \to \mathbb{P}_y^2$ are $1:1$ except on the six exceptional lines in each case. Moreover, we can obtain the cubic curves $C^{\hat{6}}$ and $C^{\hat{7}}$ by intersecting this surface with a plane. We can conclude that $\kappa_x(C^{\hat{6}}) = \kappa_x(C^{\hat{7}})$ only if $C^{\hat{6}} = C^{\hat{7}}$. It then follows that $(4) \implies (1)$ and, symmetrically, $(5) \implies (1)$.

We now prove $(1) \implies (2)$. Fix $i \in \{1, \ldots, 7\}$. Then $x_j \in C_x^{\hat{i}}$ for all $j \neq i$ by Lemma 3.4.3. Moreover, since $C_x^{\hat{i}} = C_x^{\hat{j}}$ by hypothesis it follows that $x_i \in C_x^{\hat{i}}$. The other equalities follow symmetrically.

Finally, we prove $(2) \implies (1)$. Since $x_j \in C_x^{\hat{i}}$ and $y_j \in C_y^{\hat{i}}$ for $j \neq i$ by construction, the additional hypothesis (2) gives that $x_1, \ldots, x_7 \in \cap_{i=1}^7 C_x^{\hat{i}}$ and $y_1, \ldots, y_7 \in \cap_{i=1}^7 C_y^{\hat{i}}$. We fix the first five point pairs $(x_i, y_i)_{i=1}^5$ and consider the 5×9 matrix $Z_5 = (x_i^\top \otimes y_i^\top)_{i=1}^5$. Consider the cubic surface $S = \mathcal{N}_{Z_5} \cap \mathcal{D}$ paired with the maps κ_x and κ_y. The cubic curves $C^{\hat{6}}$ and $C^{\hat{7}}$ are obtained by intersecting S with a plane. By genericity, the four matrices $\kappa_x^{-1}(x_6), \kappa_x^{-1}(x_7), \kappa_y^{-1}(y_6), \kappa_y^{-1}(y_7)$ are all distinct. Moreover, they are all contained in

$$C^{\hat{6}} \cap C^{\hat{7}} = (\mathcal{N}_{Z^{\hat{6}}} \cap \mathcal{D}) \cap (\mathcal{N}_{Z^{\hat{7}}} \cap \mathcal{D}) = \mathcal{N}_Z \cap \mathcal{D} \tag{3.32}$$

which can also be realized as the intersection of the cubic surface S with two planes. If $\mathcal{N}_Z$ were one-dimensional, it would intersect $\mathcal{D}$ is at most three points. Since we have found $4 > 3$ distinct points in $\mathcal{N}_Z \cap \mathcal{D}$, $\mathcal{N}_Z$ must have projective dimension ≥ 2, implying (1).

□

3.4.3 Beyond semi-genericity

Given seven semi-generic point pairs $(x_i, y_i)_{i=1}^7$, we have now fully characterized the conditions under which the matrix Z_7 will be rank deficient. This characterization was given geometrically (Theorem 3.4.1) and then algebratized using 14 polynomials (Theorem 3.4.9). We now move away from the assumptions of semi-genericity. We will first examine how Z_7 becomes rank deficient without these assumptions and, to some extent, generalize our algebraic condition (Theorem 3.4.9) to this case. We will also consider configurations where $(x_i, y_i)_{i=1}^7$ are fully generic, and therefore Z_7 must have full rank; in this case, we can use the cubic curves $C_x^{\hat{i}}, C_y^{\hat{i}}$ and their associated polynomials to characterize the epipoles of the possible fundamental matrices in terms of classical invariants.

We begin by presenting two relatively simple, but highly degenerate, conditions for the rank deficiency of Z_7. One of these conditions is that Z_7 will be rank deficient if $\{x_i\}$ and $\{y_i\}$ are equal up to a change of coordinates.

Lemma 3.4.10. *Suppose we have point pairs $(x_i, y_i)_{i=1}^7$ and an invertible projective transformation H such that $Hx_i = y_i$ for all i. Then $Z = (x_i^\top \otimes y_i^\top)_{i=1}^7$ is rank deficient.*

Proof. Since rank drop is a projective invariant, we can assume $x_i = y_i$ for all i. Then the equations $y_i^\top F x_i = x_i^\top F x_i = 0, i = 1, \ldots, 7$ hold for all 3×3 skew-symmetric matrices $F \in$ Skew$_3$. Since Skew$_3$ is a three-dimensional vector space, $\dim(\mathcal{N}_Z) \geq 3$ and rank$(Z) \leq 9 - 3 = 6$. □

The second simple condition is that the rank of Z will drop if the points in either $\mathbb{P}^2$ lie in a line.

Lemma 3.4.11. *Suppose* $(x_i, y_i)_{i=1}^7$ *is such that either* $\{x_i\}$ *or* $\{y_i\}$ *are in a line. Then* $Z = (x_i^\top \otimes y_i^\top)_{i=1}^7$ *is rank deficient.*

Proof. Suppose the y_i's are in a line. Then we may assume that $y_i = (m_i, 0, 1)$ after a change of coordinates. Then simple column operations on Z show that it is rank deficient. □

Remark 3.4.12. We note that the existence of such configurations does not necessarily imply that the rank drop variety is reducible. We suspect that these configurations are in the Zariski closure of the generic rank drop component.

It is simple to check that in both of the above cases we will have $g_x^{\hat{i}}(x_i) = 0 = g_y^{\hat{i}}(y_i)$ for all $i = 1, \ldots, 7$, suggesting a possible generalization of Theorem 3.4.9(3). This is possible to some extent. In particular, even without any genericity assumptions, if Z_7 is rank deficient then these 14 polynomial equations hold.

Lemma 3.4.13. *If* $Z = (x_i^\top \otimes y_i^\top)_{i=1}^7$ *is rank deficient then* $g_x^{\hat{i}}(x_i) = 0$ *and* $g_y^{\hat{i}}(y_i) = 0$ *for all* i.

Proof. Let I be the ideal generated by the 14 polynomials $g_x^{\hat{i}}(x_i)$ and $g_y^{\hat{i}}(y_i)$ for $i = 1, \ldots, 7$ in the polynomial ring $\mathbb{C}[x_{ij}, y_{ij} : i = 1, \ldots, 7, j = 1, 2, 3]$, treating $(x_i, y_i)_{i=1}^7$ as symbolic. If Z is the appropriate symbolic 7×9 matrix then it can be verified using Macaulay2 that I is contained in the ideal generated by the maximal minors of Z. □

However, the converse does not hold in general. We present two examples of highly degenerate configurations where the 14 equations hold, but Z_7 is not rank deficient.

Example 10. Take x_i to be the columns of the matrix X and y_i to be the columns of the matrix Y

$$X = \begin{bmatrix} 0 & 1 & 3 & 4 & 0 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & 1 & 4 & 0 & 9 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \tag{3.33}$$

where x_1, x_2, x_3, x_4, x_7 are on a line and $x_5 = x_6$. Similarly, y_1, y_2, y_3, y_5, y_6 are on a line and $y_5 = y_7$. We can verify that $g_x^{\hat{i}}(x_i) = 0 = g_y^{\hat{i}}(y_i)$ for all $i = 1, \ldots, 7$ and that the matrix Z is not rank deficient. In particular, $\mathcal{N}_Z$ is spanned by the singular matrices

$$\begin{bmatrix} 0 & 0 & -3 \\ 0 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

the latter of which has rank one.

Example 11. Take x_i to be the columns of the matrix X and y_i to be the columns of the matrix Y

$$X = \begin{bmatrix} 1 & 2 & 5 & 1 & 2 & 3 & 7 \\ 0 & 0 & 0 & 0 & 1 & 2 & 6 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 3 & 4 \\ 1 & 5 & 1 & 0 & 2 & 6 & 8 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{3.34}$$

where $\{x_i\}_{i=1}^4, \{y_i\}_{i=1}^4$ and $\{x_i\}_{i=5}^7, \{y_i\}_{i=5}^7$ are on distinct lines in each image. We can verify that $g_x^{\hat{i}}(x_i) = 0 = g_y^{\hat{i}}(y_i)$ for all $i = 1, \ldots, 7$ and that the matrix Z is not rank deficient. In particular, $\mathcal{N}_Z$ is spanned by the rank one matrices

$$\begin{bmatrix} 0 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

□

While the focus of this paper has been on the conditions under which Z drops rank, the tools we have developed have applications beyond rank drop. In particular, for a fully generic configuration of seven point pairs we can use the cubic curves $C_x^{\hat{i}}$ and $C_y^{\hat{i}}$ to find the possible epipoles of fundamental matrices. While this has minimal practical application, it is significant in that the characterization is entirely in terms of classical projective invariants.

Lemma 3.4.14. *Let $(x_i, y_i)_{i=1}^7$ be generic point pairs. In particular, we assume $\mathcal{N}_Z$ is one dimensional and contains three rank-two matrices F_1, F_2, F_3, two of which may be complex. Then the epipoles of these fundamental matrices, e_1^x, e_2^x, e_3^x and e_1^y, e_2^y, e_3^y, can be obtained as the unique three points in the intersections $\cap_{i=1}^7 C_x^{\hat{i}} \subset \mathbb{P}_x^2$ and $\cap_{i=1}^7 C_y^{\hat{i}} \subset \mathbb{P}_y^2$.*

Proof. Consider the two cubic curves $C_x^{\hat{7}}, C_x^{\hat{6}}$. The intersection $A_{6,7} = C_x^{\hat{7}} \cap C_x^{\hat{6}}$ will contain exactly nine points. We know that $x_1, \ldots, x_5 \in A_{6,7}$. Additionally, let (u_6, v_6) be the pair of rank drop points, as in Lemma 5.1 of [14], associated to $(x_i, y_i)_{i=1}^5$. Then, by Lemma 3.4.13, $u_6 \in A_{6,7}$ as well. There should be three more points in the intersection. Let f be the unique Cremona transformation such $f : \mathbb{P}_x^2 \dashrightarrow \mathbb{P}_y^2$ such that $x_i \mapsto y_i$ for $i = 1, \ldots, 7$. This f is contained in the two parameter family of Cremona transformations $\mathbb{P}_x^2 \dashrightarrow \mathbb{P}_y^2$ such that $x_i \mapsto y_i$ for all $i = 1, \ldots, 6$. By Lemma 3.4.3 the base points of f are contained in $C_x^{\hat{7}}$. By a symmetric argument these base points are also contained in $C_x^{\hat{6}}$ and we can conclude that these three base points are the last three points in the intersection. By Lemma 3.3.9 these base points are exactly the epipoles of the fundamental matrices, and it follows by symmetry that $e_x^1, e_x^2, e_x^3 \in \cap_{i=1}^7 C_x^{\hat{i}}$. Clearly the points $x_1, \ldots, x_5, u_6 \notin \cap_{i=1}^7 C_x^{\hat{i}}$ generically, and thus these three base points are the unique points in the intersection of all seven cubic curves. Symmetrically, e_1^y, e_2^y, e_3^y are the unique points in $\cap_{i=1}^7 C_y^{\hat{i}}$. $\square$

Example 12. Take x_i to be the columns of the matrix X and y_i to be the columns of the matrix Y.

$$X = \begin{bmatrix} 3 & 2 & 5 & 0 & 4 & -20 & -4 \\ 0 & 7 & 3 & 3 & 2 & 25 & 7 \\ 1 & 1 & 2 & 1 & 5 & 12 & 2 \end{bmatrix} \qquad Y = \begin{bmatrix} 0 & -49 & -15 & -3 & -5 & 5 & 7 \\ -1 & 14 & 25 & 0 & 10 & 4 & 4 \\ 1 & 9 & 4 & 1 & 6 & 2 & 1 \end{bmatrix} \tag{3.35}$$

We can then construct the seven cubic curves $C_x^{\hat{i}}$ and $C_y^{\hat{i}}$ in each $\mathbb{P}^2$. See Figure 3.3. Each set of seven cubic curves has three common intersection points. If we compute $\mathcal{N}_Z$ we find that there are exactly three possible real fundamental matrices. These matrices have epipoles

$$\begin{aligned} e_x^1 &= (0:0:1) & e_y^1 &= (0:0:1) \\ e_x^2 &= (-2:3:1) & e_y^2 &= (-3:4:1) \\ e_x^3 &= (4:3:4) & e_y^3 &= (3:2:2) \end{aligned} \tag{3.36}$$

and we can see that these are exactly the three common intersection points.

3.5 $k = 9$

We finish by characetrizing the rank deficiency of $Z = (x_i^\top \otimes y_i^\top)_{i=1}^9$, and this time we make no assumptions on the point pairs $(x_i, y_i)_{i=1}^9$. A simple algebraic characterization of rank drop in this

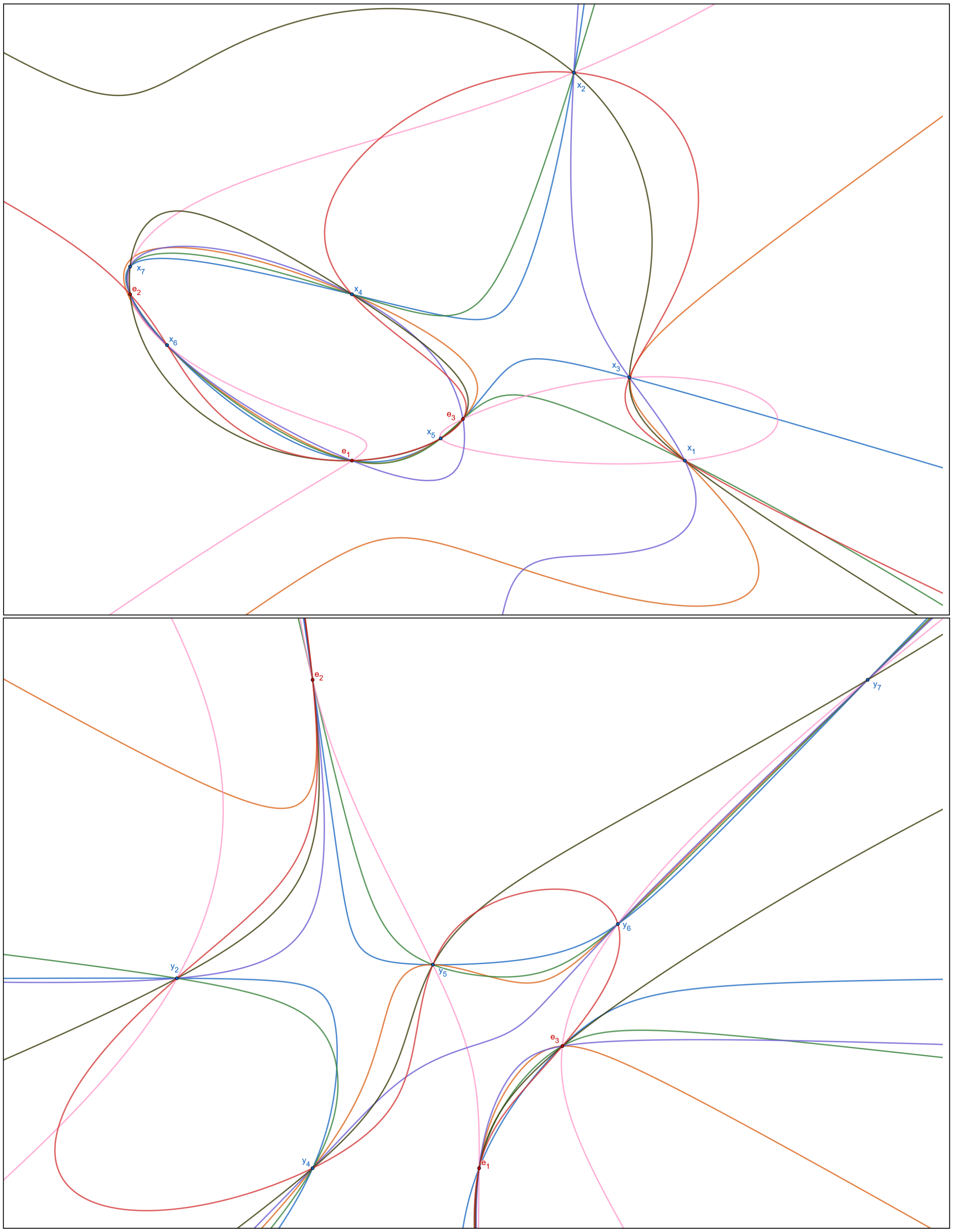

Figure 3.3: The cubic curves $C_x^{\hat{i}}$ and $C_y^{\hat{i}}$. The intersection points are exactly the three possible epipoles associated to the fundamental matrices.

case is that $\det(Z) = 0$. This is a single polynomial equation but as mentioned already, typically this equation does not shed much light on the geometry of the points $\{x_i\}$ and $\{y_i\}$ that makes Z rank deficient. By the methods of invariant theory, it is possible to write $\det(Z)$ as a polynomial in the brackets $[ijk]_x$ and $[ijk]_y$ constructed from $\{x_i\}$ and $\{y_i\}$ which may or may not offer geometric insight. Below we provide a geometric characterization of rank drop in terms of the two point sets in $\mathbb{P}^2_x$ and $\mathbb{P}^2_y$. The result is straight-forward.

Recall that if a, b are distinct points in $\mathbb{P}^2$, then $a \times b \in \mathbb{P}^2$ is the normal of the line containing a and b, i.e., $u \in \text{Span}\{a, b\}$ if and only if $u^\top(a \times b) = 0$. In what follows we let ℓ_{ab} denote the line spanned by a, b. Its normal $a \times b = [a]_\times b$ where $[a]_\times$ is the 3×3 skew symmetric matrix that expresses cross products with a as a matrix-vector multiplication.

Theorem 3.5.1. *The matrix $Z = (x_i^\top \otimes y_i^\top)_{i=1}^9$ is rank deficient if and only if there is a projective transformation $T : \mathbb{P}^2_x \dashrightarrow \mathbb{P}^2_y$ such that $y_i^\top(Tx_i) = 0$ for all $i = 1, \dots, 9$, or equivalently, y_i lies on the line with normal vector Tx_i for all $i = 1, \dots, 9$.*

This manifests in three possible ways depending on the rank of T:

1. *There exists a line $\ell \subset \mathbb{P}^2_x$ and a line $\ell' \subset \mathbb{P}^2_y$ such that for each i, either $x_i \in \ell$ or $y_i \in \ell'$ (both may happen for a given i).*
2. *There are two points $e \in \mathbb{P}^2_x$ and $e' \in \mathbb{P}^2_y$ and a $\mathbb{P}^1$-homography sending the pencil of lines through e to the pencil of lines through e' such that $\ell_{ex_i} \mapsto \ell_{e'y_i}$ for each i.*
3. *There is some $T \in \text{PGL}(3)$ such that y_i lies on the line with normal vector Tx_i for each i.*

Proof. The first statement is trivial. The matrix Z is rank deficient if and only if $\mathcal{N}_Z \subset \mathbb{P}^8$ contains at least one point. Representing such a point by $T \in \mathbb{P}(\mathbb{C}^{3\times 3})$ we have $(x_i^\top \otimes y_i^\top)\text{vec}(T) = y_i^\top(Tx_i) = 0$ for all $i = 1, \dots, 9$.

1. If $\text{rank}(T) = 1$, then $T = uv^\top$ for some $u, v \in \mathbb{C}^3$. Therefore, $(y_i^\top u)(v^\top x_i) = 0$ for all $i = 1, \dots 9$ which is equivalent to saying that for each i, at most one of $u^\top y_i$ or $v^\top x_i$ can be non-zero. Therefore there exists lines ℓ (with normal v) and ℓ' (with normal u) such that for each i, either $x_i \in \ell$ or $y_i \in \ell'$.

2. Suppose $\operatorname{rank}(T) = 2$. Let $e \in \mathbb{P}^2_x$ be the unique point in the right nullspace of T and $e' \in \mathbb{P}^2_y$ be the unique point in the left nullspace of T. The pencil of all lines through e (respectively e') can be identified with $\mathbb{P}^1$.

   Pick any line ℓ not passing through e and suppose its normal is n. Then the projective transformation $T[n]_\times$ is a $\mathbb{P}^1$-homography that takes $\ell_{ex_i} \to \ell_{e'y_i}$ [37, Result 9.5]. Indeed, suppose the intersection of ℓ and ℓ_{ex_i} is u_i. Since u_i is orthogonal to both n and $e \times x_i$, we have that $u_i \sim n \times (e \times x_i) = [n]_\times(e \times x_i)$. Since u_i lies on ℓ_{ex_i}, we have that $u_i = \lambda e + \mu x_i$ for some scalars λ, μ, and since ℓ does not contain e, $u_i \neq e$ which implies that $\mu \neq 0$. Therefore,

   $$T[n]_\times(e \times x_i) = Tu_i = \lambda Te + \mu Tx_i = 0 + \mu Tx_i \sim Tx_i$$

   which says that the normal of ℓ_{ex_i} is mapped to Tx_i by $T[n]_\times$. We just need to argue that Tx_i is the normal of $\ell_{e'y_i}$ to finish the proof. For this check that $(e')^\top Tx_i = 0$ since $(e')^\top T = 0$ and $y_i^\top Tx_i = 0$ by assumption. Therefore the line spanned by e' and y_i has normal Tx_i.

3. If $\operatorname{rank}(T) = 3$ then T is a homography (invertible projective transformation). Then $y_i^\top Tx_i = 0$ for all $i = 1, \ldots, 9$ implies that y_i lies on the line with normal Tx_i for each i.

□

Remark 3.5.2. In the proof of (2), if $x_i = e$ for some i then $[e]_\times e = 0$ and similarly, if $y_j = e'$ for some j then $[e']_\times y_j = 0$. Therefore, the $\mathbb{P}^1$-homography will not work for the indices i, j where $x_i = e$ or $y_j = e'$.

Remark 3.5.3. As we saw, if seven of the nine points on either side are in a line then the rank of Z_9 will drop. Condition (1) allows for the situations where $3 \leq s \leq 6$ points on one side are in a line and the $9 - s$ complementary y points are in a line.

Example 13. 1. Take x_i to be the columns of the matrix X and y_i to be the columns of the matrix

Y:

$$X = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 & 1 & 1 & 0 & 1 & -1 \\ 0 & 1 & 2 & 1 & 0 & 1 & 1 & 1 & -1 \end{bmatrix} \qquad Y = \begin{bmatrix} -1 & 1 & 0 & 0 & 1 & 1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 1 & 0 & 1 & 2 & 1 & 3 \end{bmatrix}. \tag{3.37}$$

One can check that all 8×9 submatrices of Z have rank 8. If the coordinates of $\mathbb{P}^2$ are u_1, u_2, u_3 then $x_1, \ldots, x_4$ lie on the line $u_1 = 0$ and $y_5, \ldots, y_9$ lie on the line $u_2 = 0$ and Z must drop rank by Condition (1). Indeed, the unique element in the nullspace of Z is the rank-one matrix

$$T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.38}$$

2. Take x_i to be the columns of the matrix X and y_i to be the columns of the matrix Y:

$$X = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad Y = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 3 \end{bmatrix}. \tag{3.39}$$

Again, Z and all its 8×9 submatrices have rank 8. The unique element in $\mathcal{N}_Z$ is the rank-two matrix

$$T = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}. \tag{3.40}$$

The points $e = e' = (1,1,0)^\top$ are generators of the right and left nullspaces of T. Note that $x_5 = e$ and $y_7 = e'$. Pick $\bar{\ell} = (1,2,3)^\top$. Then $e^\top \bar{\ell} \neq 0$. Now check that $[e']_\times Y = (T[\bar{\ell}]_\times)[e]_\times X$. Indeed,

$$[e]_\times = [e']_\times = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}, \qquad [\bar{\ell}]_\times = \begin{bmatrix} 0 & -3 & 2 \\ 3 & 0 & -1 \\ -2 & 1 & 0 \end{bmatrix}, \text{ and}$$

$$
\begin{aligned}
[e']_\times Y &= \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 3 \\ 0 & 0 & -1 & -1 & -1 & -1 & 0 & -1 & -3 \\ -1 & 1 & 0 & 0 & -1 & 1 & 0 & -1 & 3 \end{bmatrix} \\
&\sim \begin{bmatrix} 0 & 0 & 3 & 3 & 0 & 3 & 3 & 3 & 3 \\ 0 & 0 & -3 & -3 & 0 & -3 & -3 & -3 & -3 \\ 3 & -3 & 0 & 0 & 0 & 3 & -3 & -3 & 3 \end{bmatrix} = (T[\bar{\ell}]_\times)[e]_\times X
\end{aligned}
\tag{3.41}
$$

except in the columns of X and Y where $x_i = e$ and $y_j = e'$.

Here is another example where the epipoles do not appear among the x_i's or y_j's. Take x_i to be the columns of the matrix X and y_i to be the columns of the matrix Y:

$$
X = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & 1 & 0 & 2 & 1 & 1 & -1 \end{bmatrix}, \qquad Y = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 2 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 & 2 & 1 & 2 & -1 & -1 \end{bmatrix}.
\tag{3.42}
$$

The unique element in $\mathcal{N}_Z$ is the rank-two matrix

$$
T = \begin{bmatrix} 0 & 2 & 1 \\ -1 & -1 & 0 \\ -2 & 0 & 1 \end{bmatrix}.
\tag{3.43}
$$

The points $e = (-1, 1, -2)^\top$ and $e' = (1, 2, -1)^\top$ generate the right and left nullspaces of T. Pick $\bar{\ell} = e$. Then $e^\top e \neq 0$. Now check that $[e']_\times Y = (T[e]_\times)[e]_\times X$. Indeed,

$$
\begin{aligned}
[e']_\times Y &= \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 4 & 5 & -1 & -4 \\ -1 & -1 & -2 & -2 & -3 & -2 & -3 & 0 & 0 \\ -2 & -1 & -2 & -1 & -2 & 0 & -1 & -1 & -4 \end{bmatrix} \\
&\sim \begin{bmatrix} 0 & -12 & -6 & -18 & -24 & -12 & -30 & 6 & 18 \\ 6 & 12 & 6 & 12 & 18 & 6 & 18 & 0 & 0 \\ 12 & 12 & 6 & 6 & 12 & 0 & 6 & 6 & 18 \end{bmatrix} = (T[e]_\times)[e]_\times X.
\end{aligned}
\tag{3.44}
$$

3. Take x_i to be the columns of the matrix X and y_i to be the columns of the matrix Y:

$$X = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad Y = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 & 2 & 15 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & -5 \end{bmatrix}. \tag{3.45}$$

The unique element in $\mathcal{N}_Z$ is the rank-three matrix

$$T = \begin{bmatrix} 0 & 1 & -4 \\ 1 & 0 & 3 \\ -4 & 3 & 0 \end{bmatrix}. \tag{3.46}$$

By construction, $y_i^\top T x_i = 0$ for all $i = 1, \dots, 9$.

3.6 Conclusion

In combination with [14], we now have a complete characterization of how rank deficiency of the matrix $Z = (x_i^\top \otimes y_i^\top)_{i=1}^k$ occurs for all values of $k = 2, \dots, 9$. We have also demonstrated a strong correspondence between lines in $\mathbb{P}(\mathbb{C}^{3\times 3})$, quadric surfaces in $\mathbb{P}^3$, and quadratic Cremona transformations of $\mathbb{P}^2$ under appropriate genericity assumptions, which we have named the *trinity correspondence*. We conclude with a simple corollary of our work that highlights the geometry of reconstructions of semi-generic point pairs of sizes six, seven and eight.

Corollary 3.6.1. *Let $(x_i, y_i)_{i=1}^k \subset \mathbb{P}^2 \times \mathbb{P}^2$ be semi-generic. Then we get the following:*

- *When $k = 6$, Z_6 is rank deficient exactly when a reconstruction $p_1, \dots, p_6, c_1, c_2$ is a Cayley octad (eight points in the intersection of three generic quadrics).*
- *When $k = 7$, Z_7 is rank deficient exactly when the points $p_1, \dots, p_7, c_1, c_2$ of any reconstruction lie on a quartic curve that arises as the intersection of two quadrics.*
- *When $k = 8$, Z_8 is rank deficient exactly when the points $p_1, \dots, p_8, c_1, c_2$ of any reconstruction lie on a quadric.*

Proof. When $k = 8$, the matrix Z_8 is rank deficient exactly when $\mathcal{N}_{Z_8}$ is a line. By the semi-genericity of the point pairs, this line is P-generic and does not contain any rank one matrices. Any reconstruction of the point pairs corresponds to a fundamental matrix F on this line and by Lemma 3.3.3, the reconstruction lies on a quadric. Similarly, if the point pairs have a reconstruction, given by some fundamental matrix F, which lies on a quadric then there is a corresponding line through F in $\mathcal{N}_{Z_8}$ and Z_8 is rank deficient.

When $k = 7$, Z_7 is rank deficient exactly when $\mathcal{N}_{Z_7}$ is a plane. Given any reconstruction $p_1, \ldots, p_7, c_1, c_2$ of the point pairs, let F be the corresponding fundamental matrix. By semi-genericity of the point pairs, $\mathcal{N}_{Z_7}$ is a generic plane that intersects $\mathcal{D}$ in a curve C of rank-two matrices. If we take any two lines through F in $\mathcal{N}_{Z_7}$ then as in Lemma 3.3.3 we obtain two quadrics Q_1, Q_2 whose intersection is a quartic curve through the reconstruction. Similarly, if any reconstruction corresponding to a fundamental matrix F' lies on two distinct quadrics then there are two distinct lines through F' in $\mathcal{N}_{Z_7}$ and Z_7 is rank deficient.

For $k = 6$, Z_6 is rank deficient if and only if $\mathcal{N}_{Z_6}$ is a 3-dimensional plane. Equivalently, every rank- two matrix $F \in \mathcal{N}_{Z_6}$ lies on a net of lines in $\mathcal{N}_{Z_6}$, which corresponds to a net of quadrics containing the reconstruction corresponding to F. It follows that if the reconstruction lies on a Cayley octad $Q_1 \cap Q_2 \cap Q_3$ then Z_6 is rank deficient. For the other direction, suppose Z_6 is rank deficient. Then the reconstruction lies on a net of quadrics $Q_1 \cap Q_2 \cap Q_3$ and we need to show that this intersection contains exactly the 8 points $\{p_i\}_{i=1}^{6}, c_1, c_2$. If $p' \in Q_1 \cap Q_2 \cap Q_3$ is any point distinct from c_1, c_2, then $\pi_2(p')^\top M \pi_1(p') = 0$ for all $M \in \mathcal{N}_{Z_6}$. Due to semi-genericity, the hypothesis of [14][Lemma 6.1] holds for any subset of 5 point pairs, and it follows that $(\pi_1(p'), \pi_2(p')) = (x_i, y_i)$ for some i. We can conclude that $p' = p_i$ and the intersection is indeed a Cayley octad. $\square$

Chapter 4

ALGEBRA AND GEOMETRY OF CAMERA RESECTIONING

The following is the content of [15], written with Timothy Duff and Jessie Loucks-Tavitas.

4.1 Introduction

The *dramatis personae* of the classical pinhole camera model are a full-rank 3×4 matrix A representing a camera, a 4×1 matrix q representing a world point, and a 3×1 matrix p representing its projection into an image. Image formation may be understood via the projective-linear map

$$\begin{aligned} A : \mathbb{P}^3 &\dashrightarrow \mathbb{P}^2 \\ q &\mapsto Aq, \end{aligned} \tag{4.1}$$

and we write $Aq \sim p$ if these two vectors represent the same point in $\mathbb{P}^2$. The *center* of the camera A is the unique point where the map (4.1) is undefined.

The pinhole camera, despite its simplicity, remains a good model of physical cameras. This explains its importance in modern computer vision applications such as structure-from-motion (SfM) and Simultaneous Localization and Mapping (SLAM). On the other hand, classical problems associated with 3D reconstruction have been studied long before the advent of computers, and the role played by algebraic methods in their solution has long been apparent. For instance, Hesse in 1863 formulated the problem of constructing two homographic configurations of 7 lines in space, each prescribed to pass through a configuration of 7 points in a plane [40]. Hesse's reduction of this problem to computing the roots of a cubic equation may be understood as an early instance of the so-called 7 point algorithm. Similarly, Grunert's 1841 "3D Pothenot problem" [35] is known nowadays as the perspective 3-point (P3P) problem, and his general strategy reducing the problem to a quartic equation remains in use today.

In recent years, the name *algebraic vision* [45] has been coined to describe a body of interdisciplinary research in which notions from algebra and vision flow freely. To date, algebraic vision has largely focused on problems which we refer to as the *full reconstruction problem* and *triangulation*.

In the full reconstruction problem, we are given a collection of image points $\tilde{p}_{11}, \dots, \tilde{p}_{mn}$, and our task is to recover a set of cameras $A_1, \dots, A_m$ and world points $q_1, \dots, q_n$ that is consistent with these observations. Hesse's solution treats the "minimal" case $(m, n) = (2, 7)$. Today, there are many works which solve analogous minimal problems which can be used effectively in SfM pipelines (see eg. [28,29,47,48,67].)

In triangulation, we are given not only image points, but also the cameras that produced them, $\bar{A}_1, \dots, \bar{A}_m$. We need only recover one or more unknown world points. Already for $m = 2$, an exact solution to this problem will typically not exist, due to the fact that the lines in $\mathbb{P}^3$ projecting to generic image points under $\bar{A}_1, \bar{A}_2$ will be skew. Nevertheless, algebraic methods have led to a wealth of knowledge about the triangulation problem. For example, the *multiview ideal* associated to $\bar{A}_1, \dots, \bar{A}_m$ gives rise to a complete set of algebraic constraints on *any* m-tuple of image points they produce. There is a considerable literature related to multiview ideals [2,3,32,41]. Theorem 4.2.3 collects some important previous results.

Often regarded as being "dual" to triangulation is the problem of camera *resectioning*. Here, we assume n image points are given along with the configuration of world points $\bar{\mathbf{q}} = (\bar{q}_1, \dots, \bar{q}_n) \in (\mathbb{P}^2)^n$ from which they were produced by a single unknown camera A. Grunert's 1841 paper gives a minimal solution for $(m, n) = (1, 3)$ under the assumption that A is *Euclidean*. Without this assumption, A is a general 3×4 matrix, and we need $n \geq 6$.

4.1.1 Results and Organization

In this paper, we aim to bring the general resectioning problem up-to-speed with the latest developments in algebraic vision. In Section 4.2, after recalling some previous results about multiview varieties, we state our first main result, Theorem 4.2.6. This characterizes a complete set of algebraic constraints for the resectioning problem, under the genericity assumption that no four of the given world points are coplanar. These constraints are given by k-linear polynomials for

$6 \leq k \leq 12$ which generate the *resectioning ideal* $I(\Gamma^{m,n}_{\bar{\mathbf{q}},\mathbf{p}})$ (Definition 4.2.1). Our work is a natural continuation of recent work by Agarwal et al. [1], and we resolve three of its open questions. For instance, Theorem 4.2.6 resolves [1, §8.1, Q4] for generic $\bar{\mathbf{q}}$ by determining a universal Gröbner basis for $I(\Gamma^{m,n}_{\bar{\mathbf{q}},\mathbf{p}})$.

We note that resectioning ideals have several pleasant properties from the point of view of commutative algebra: namely, for generic $\bar{\mathbf{q}} \in (\mathbb{P}^3)^n$,

1. For fixed m and n, resectioning ideals are homogeneous with respect to a natural $\mathbb{Z}^{mn}$-grading, and have the same $\mathbb{Z}^{mn}$-graded Hilbert function as long as no four points are coplanar. Proposition 4.2.7 implies that this Hilbert function may be obtained by specializing a combinatorial formula of Li [50, Theorem 1.1], based on the inclusion-exclusion rule. Our ideal-theoretic result also considerably strengthens Li's set-theoretic description, and reduces the degrees of the equations that are needed.

2. The multidegrees of resectioning ideals are always equal to 1. A geometric explanation of this phenomenon follows along the lines explained in [6, §4]. See also [10, Theorem 4.2] for an explanation using multigraded Rees algebras.

3. For any monomial order $<$, the initial ideal $\mathrm{in}_<(I(\Gamma^{m,n}_{\bar{\mathbf{q}},\mathbf{p}}))$ and the multigraded generic initial ideal $\mathrm{gin}_<(I(\Gamma^{m,n}_{\bar{\mathbf{q}},\mathbf{p}}))$, although not equal as in the case of multiview ideals [3], are both radical. In particular, $I(\Gamma^{m,n}_{\bar{\mathbf{q}},\mathbf{p}})$ belongs to the class of *Cartwright-Sturmfels ideals*, recently surveyed by Conca, De Negri, and Gorla [13].

Our first basic insight is that the projection of a point $q \in \mathbb{P}^3$ under a pinhole camera $A : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ may be viewed as the projection of a point $\mathrm{vec}(A) \in \mathbb{P}^{11}$ under what we call a "hypercamera" $Q : \mathbb{P}^{11} \dashrightarrow \mathbb{P}^2$. This is reminiscent, and in fact a generalization, of a well-studied principle in computer vision known as *Carlsson-Weinshall duality* [64]. This is the subject of Section 4.3. Our Theorem 4.3.4 develops a reduced analogue of the "atlas" of algebraic varieties proposed in [1]. This addresses [1, §8.2, Q2]. The *reduced joint image* and its dual, recently studied by Trager, Ponce, and Hebert, are two members of this atlas. Carlsson-Weinshall duality amounts to a simple linear isomorphism between these two varieties. In Example 4.2.8 and Section 4.3.2, we explain how our perspective unifies previous approaches to resectioning constraints in the computer vision literature [52, 59, 60, 64], which can all be obtained from the ideal $I(\Gamma^{m,n}_{\bar{\mathbf{q}},\mathbf{p}})$ by specialization.

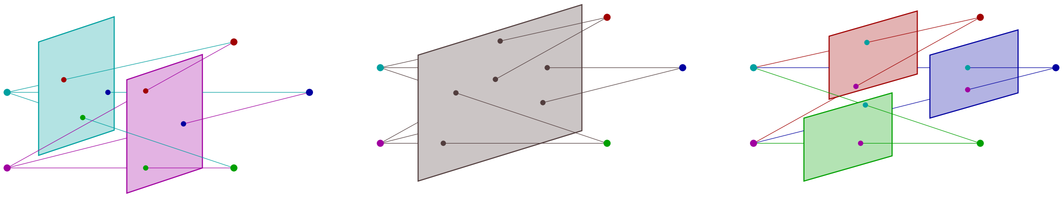

Figure 4.1: Two reduced cameras viewing three 3D points (left) are Carlsson-Weinshall dual to three reduced cameras viewing two 3D points (right). See Section 4.3.1 for details.

Theorem 4.3.5 in Section 4.3.2 shows that reduced resectioning varieties for generic point configurations are scheme-theoretically cut out by bilinear forms. This stands in stark contrast to the high degree polynomials in Theorem 4.2.6, whose proof we complete in Section 4.4. Finally, in Section 4.5, we address [1, §8.1, Q6] by investigating the *Euclidean distance degree* of the resectioning variety in affine pixel coordinates. This is a number that quantifies the algebraic complexity of a natural Euclidean distance optimization formulation of the camera resectioning problem. Our main contribution, based on evidence supplied by computational experiments, is Conjecture 4.5.1, giving a formula for this quantity as a cubic polynomial in n. The statement is analogous to, and inspired by, the *multiview conjecture*, recently resolved by Maxim, Rodriguez, and Wang [53]. We conclude with a short discussion in Section 4.6.

4.1.2 Notation and conventions

Our notation largely follows that established in [1]. Our basic algebro-geometric objects are affine and projective varieties over the field of complex numbers $\mathbb{C}$. The symbol $\mathbb{P}^n$ denotes complex n-dimensional projective space, which we may also identify with the projectivization $\mathbb{P}(V)$ of any

$(n+1)$-dimensional complex vector space V. As in the introduction, known quantities will usually be designated with a bar $\bar{\bullet}$. This bar is also used to denote the *Zariski closure* of a set: its usage will be clear from the context. If we wish to emphasize that given quantities in certain scenarios may be “noisy" due to deviations from the pinhole model or erroneous measurements, we instead use $\widetilde{\bullet}$.

4.2 Resectioning vs Triangulation

Let us recall a “universal" version of the imaging map (4.1). This is the map which sends m cameras $A_1, \dots, A_m \in \mathbb{P}\left(\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^4, \mathbb{C}^3)\right) \cong \mathbb{P}^{11}$ and n points $q_1, \dots, q_n \in \mathbb{P}^3$ to mn points in $\mathbb{P}^2$. The graph of this rational map is an incidence correspondence, dubbed the *image formation correspondence* in [1],

$$\Gamma^{m,n}_{\mathbf{A},\mathbf{q},\mathbf{p}} = \overline{\{(\mathbf{A}, \mathbf{q}, \mathbf{p}) \in (\mathbb{P}^{11})^m \times (\mathbb{P}^3)^n \times (\mathbb{P}^2)^{mn} \mid A_i q_j \sim p_{ij} \quad \forall i \in [m],\, j \in [n]\}}. \tag{4.2}$$

Given a generic camera arrangement $\bar{\mathbf{A}} = (\bar{A}_1, \dots, \bar{A}_m) \in \left(\mathbb{P}^{11}\right)^m$, one may also consider the associated *multiview variety*. In the notation of [1], this may be defined as

$$\Gamma^{m,n}_{\bar{\mathbf{A}},\mathbf{p}} = \{\mathbf{p} \in (\mathbb{P}^2)^{mn} \mid (\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}) \in \Gamma^{m,n}_{\mathbf{A},\mathbf{q},\mathbf{p}} \text{ for some } \mathbf{q} \in \left(\mathbb{P}^3\right)^n\}. \tag{4.3}$$

Multiview varieties and their vanishing ideals are well-understood objects. Our present study of camera resectioning is based on the following definition, which parallels (4.3) in that the role of cameras and 3D points are switched.

Definition 4.2.1. *The m-camera* resectioning variety *associated to a given point arrangement $\bar{\mathbf{q}} \in \left(\mathbb{P}^3\right)^n$ is the multiprojective variety*

$$\Gamma^{m,n}_{\bar{\mathbf{q}},\mathbf{p}} = \left\{\mathbf{p} \in \left(\mathbb{P}^2\right)^{mn} \mid (\mathbf{A}, \bar{\mathbf{q}}, \mathbf{p}) \in \Gamma^{m,n}_{\mathbf{A},\mathbf{q},\mathbf{p}} \textit{for some } \mathbf{A} \in \left(\mathbb{P}^{11}\right)^m\right\}. \tag{4.4}$$

The vanishing ideal $I(\Gamma^{m,n}_{\bar{\mathbf{q}},\mathbf{p}})$ is the resectioning ideal *of* $\bar{\mathbf{q}}$.

Remark 4.2.2. *It turns out that $\Gamma^{m,n}_{\bar{\mathbf{q}},\mathbf{p}} = (\mathbb{P}^2)^{mn}$ if and only if $n < 6$, assuming $\bar{\mathbf{q}} \in \left(\mathbb{P}^3\right)^n$ is sufficiently generic. Thus we assume $n \geq 6$ throughout this section.*

To better explain the analogy between resectioning and triangulation, we collect several previous results about the multiview ideals $I(\Gamma^{m,n}_{\bar{\mathbf{A}},\mathbf{p}})$ in Theorem 4.2.3 below. Our first main result, Theorem 4.2.6, involves certain multilinear *focal polynomials* which belong to the resectioning ideal $I(\Gamma^{m,n}_{\bar{\mathbf{q}},\mathbf{p}})$. These are structurally very similar to the classically-known focal polynomials belonging to $I(\Gamma^{m,n}_{\bar{\mathbf{A}},\mathbf{p}})$. We briefly recall a derivation of these constraints. Suppose we are given a camera arrangement $\bar{\mathbf{A}} \in (\mathbb{P}^{11})^n$. Consider a generic point

$$(\bar{\mathbf{A}}, \mathbf{q}, \mathbf{p}) = (\bar{A}_1, \dots, \bar{A}_m, q_1, \dots, q_n, p_{11}, \dots, p_{mn}) \in \Gamma^{m,n}_{\mathbf{A},\mathbf{q},\mathbf{p}}.$$

Fixing representatives for this point in homogeneous coordinates, there exist nonzero scalars $\lambda_{11}, \dots, \lambda_{mn} \in \mathbb{C}$ which satisfy the equations

$$\bar{A}_i q_j = \lambda_{ij} p_{ij}, \quad 1 \leq i \leq m,\ 1 \leq j \leq n. \tag{4.5}$$

From these conditions, one may obtain certain multilinear polynomials in $\bar{A}_i, p_{ij}$ alone, known in various sources as k-focals or k-multilinearities. Specifically, for each $j = 1, \dots, n$ and any subset $\sigma = \{\sigma_1, \dots, \sigma_k\} \subset [m]$ of size ≥ 2, the matrix

$$\begin{bmatrix} \bar{A}_{\sigma_1} & p_{\sigma_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{A}_{\sigma_k} & 0 & \cdots & p_{\sigma_k} \end{bmatrix} \tag{4.6}$$

must be rank-deficient. The maximal $(4+k) \times (4+k)$ minors of these matrices are the *k-focals* associated with the camera arrangement $\bar{\mathbf{A}}$.

In Theorem 4.2.3, we collect several previous results which make the relationship between $\Gamma^{m,n}_{\bar{\mathbf{A}},\mathbf{q},\mathbf{p}}$ and the k-focals more precise. These results impose progressively stronger genericity assumptions on the camera arrangement $\bar{\mathbf{A}}$.

Theorem 4.2.3. *Let $\bar{\mathbf{A}} = (\bar{A}_1, \dots, \bar{A}_m)$, for $m \geq 2$, be a fixed camera arrangement.*

1. [3, Theorem 2.1] *If all maximal 4×4 minors of the matrix $\left[\bar{A}_1^T \,\middle|\, \cdots \,\middle|\, \bar{A}_m^T\right]$ are nonzero, then the k-focals for $k \in \{2, 3, 4\}$ form a universal Gröbner basis for $I(\Gamma^{m,n}_{\bar{\mathbf{A}},\mathbf{p}})$.*

2. [2, Theorem 3.7] *If* $\bar{\mathbf{A}}$ *is such that the camera centers are distinct, then the* k*-focals for* $k \in \{2, 3\}$ *generate the vanishing ideal* $I(\Gamma^{m,n}_{\bar{\mathbf{q}},\mathbf{p}})$.
3. [2, Theorem 5.6] *If* $\bar{\mathbf{A}}$ *is such that the camera centers are distinct and do not lie in a common plane, then the* 2*-focals determine* $\Gamma^{m,n}_{\bar{\mathbf{q}},\mathbf{p}}$ *as a subscheme of* $(\mathbb{P}^2)^{mn}$.

Turning now to camera resectioning, suppose we are instead given $\bar{\mathbf{q}} \in (\mathbb{P}^3)^n$. Similar to (4.5), we wish to obtain conditions involving only $\bar{q}_j$ and p_{ij} from

$$A_i \bar{q}_j = \lambda_{ij} p_{ij}, \quad 1 \leq i \leq m,\ 1 \leq j \leq n. \tag{4.7}$$

To obtain these conditions, we may apply a well-known identity involving the matrix Kronecker product, denoted $\otimes$, and the vectorization operator $\operatorname{vec}(\bullet)$, which stacks the columns of a matrix vertically.

Proposition 4.2.4 (See eg. [43, p252, Exercise 22])**.** *For any* $M \in \mathbb{C}^{q \times r}$, $N \in \mathbb{C}^{r \times s}$,

$$\operatorname{vec}(MN) = (I_{s \times s} \otimes M) \operatorname{vec}(N), \tag{4.8}$$

where $I_{s \times s} \in \mathbb{C}^{s \times s}$ *is the identity matrix.*

We apply this identity with $M = \bar{q}_j^\top$ and $N = A_i^\top$. For the 3×12 matrix $I_{3\times 3} \otimes \bar{q}_j^\top$, we introduce the notation

$$\bar{Q}_j := I_{3\times 3} \otimes \bar{q}_j^\top = \begin{bmatrix} \bar{q}_j^\top & 0 & 0 \\ 0 & \bar{q}_j^\top & 0 \\ 0 & 0 & \bar{q}_j^\top \end{bmatrix}. \tag{4.9}$$

Combining (4.7) and Proposition 4.2.4, we deduce that

$$\bar{Q}_j \operatorname{vec}\left(A_i^\top\right) = \lambda_{ij} p_{ij}, \quad 1 \leq i \leq m,\ 1 \leq j \leq n.$$

Equivalently, for each $i = 1, \ldots, m$ we have

$$\begin{bmatrix} \bar{Q}_1 & p_{i1} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \bar{Q}_n & 0 & \cdots & p_{in} \end{bmatrix} \begin{bmatrix} \operatorname{vec}(A_i^\top) \\ -\lambda_{i1} \\ \vdots \\ -\lambda_{in} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Thus, if $\mathbf{p} \in \Gamma^{m,n}_{\bar{\mathbf{q}},\mathbf{p}}$, then we have the rank constraints

$$\operatorname{rank} \begin{bmatrix} \bar{Q}_1 & p_{i1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{Q}_n & 0 & \cdots & p_{in} \end{bmatrix} < 12 + n. \tag{4.10}$$

We observe that this rank constraint is equivalent to the vanishing of all maximal $(12+n) \times (12+n)$ minors. These minors are homogeneous polynomials in the entries of each $\bar{Q}_i$ and p_{ij}; indeed, for any nonzero scalars $c_1, \ldots, c_n, c_1', \ldots, c_n'$,

$$\begin{aligned} \operatorname{rank} \begin{bmatrix} c_1\bar{Q}_1 & c_1' p_{i1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_n\bar{Q}_n & 0 & \cdots & c_n' p_{in} \end{bmatrix} &= \operatorname{rank} \begin{bmatrix} \bar{Q}_1 & c_1^{-1} p_{i1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{Q}_n & 0 & \cdots & c_n^{-1} p_{in} \end{bmatrix} \\ &= \operatorname{rank} \begin{bmatrix} \bar{Q}_1 & p_{i1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{Q}_n & 0 & \cdots & p_{in} \end{bmatrix}. \end{aligned} \tag{4.11}$$

One may of course consider such rank constraints not only for 3×12 matrices of the form (4.9), but for any given arrangement of surjective linear maps,

$$\bar{B}_j : \mathbb{P}^{11} \dashrightarrow \mathbb{P}^2, \quad j = 1, \ldots, n,$$

represented by generic 3×12 matrices. To prevent confusion with cameras A_i, we refer to each $\bar{B}_j$ as a *hypercamera*. We denote a general arrangement of hypercameras by $\bar{\mathbf{B}} = (\bar{B}_1, \ldots, \bar{B}_n) \in (\mathbb{P}^{35})^n$. However, we instead write $\bar{\mathbf{Q}}$ to denote the special hypercamera arrangement associated to a point arrangement $\bar{\mathbf{q}} \in (\mathbb{P}^3)^n$ by the rule (4.9).

Let us also note that rank constraints analogous to (4.10) hold for any subset of at least 6 world points and their corresponding images. This motivates the following definition, as well as the statement of our first result.

Definition 4.2.5. *Fix a hypercamera arrangement* $\bar{\mathbf{B}} = (\bar{B}_1, \ldots, \bar{B}_n) \in (\mathbb{P}^{35})^n$. *For any set* $\{\sigma_1, \ldots, \sigma_k\} \subset [n]$ *of size* $k \geq 6$ *and an index* $i \in [m]$, *a* k-focal *polynomial is any maximal*

$(12+k) \times (12+k)$ *minor of the* $3k \times (12+k)$ *matrix*

$$\begin{bmatrix} \bar{B}_{\sigma_1} & p_{i\sigma_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \bar{B}_{\sigma_k} & 0 & \cdots & p_{i\sigma_k} \end{bmatrix}. \tag{4.12}$$

From context, it will be clear whether "focals" refers to the polynomials in Definition 4.2.5 or their triangulation counterparts. The ideal in $\mathbb{C}[\mathbf{p}]$ *generated by all* k*-focals,* $6 \leq k \leq m$*, is the* m*-camera* focal ideal $I_m(\bar{\mathbf{B}})$*. For a given point arrangement* $\bar{\mathbf{q}}$*, we define its focal ideal* $I_m(\bar{\mathbf{q}})$ *to be the focal ideal* $I_m(\bar{\mathbf{Q}})$ *for the associated hypercamera arrangement* $\bar{\mathbf{Q}}$*.*

Theorem 4.2.6. *Let* $m, n \geq 1$ *be integers. For any point arrangement* $\bar{\mathbf{q}} \in (\mathbb{P}^3)^n$ *such that no four points are coplanar, we have*

$$I(\Gamma^{m,n}_{\bar{\mathbf{q}},\mathbf{p}}) = I_m(\bar{\mathbf{q}}),$$

and the set of all k*-focals for* $6 \leq k \leq 12$ *forms a universal Gröbner basis for this ideal.*

Theorem 4.2.6 is the resectioning analogue of Theorem 4.2.3 part (1). Directly adapting the proof of this result is not straightforward. This is because $\bar{\mathbf{Q}}$ is a very special hypercamera arrangement. Nevertheless, the noncoplanarity hypothesis in Theorem 4.2.6 ensures that $\bar{\mathbf{Q}}$ is generic enough for Gröbner basis arguments to be applied.

In the setting of triangulation, we note that the range of interesting focals $2 \leq k \leq 4$ is much smaller than in Theorem 4.2.6, and in this setting the k-focals correspond to well-understood objects in multiview geometry—namely, fundamental matrices, trifocal tensors, and quadrifocal tensors [38, cf. Ch. 17]. It would seem that the k-focals for resectioning are less well-understood. Nevertheless, in Example 4.2.8, Section 4.3.2, we observe that they do specialize to "dual" multiview constraints appearing in the literature.

As a warm-up, we establish a set-theoretic variant of Theorem 4.2.6. By analogy with (4.3), let us define for any hypercamera arrangement $\bar{\mathbf{B}} \in (\mathbb{P}^{35})^n$ the variety $\Gamma^{n,m}_{\bar{\mathbf{B}},\mathbf{p}}$ to be the closed image of the associated imaging map $(\mathbb{P}^{11})^m \dashrightarrow (\mathbb{P}^2)^{mn}$. In other words, $\Gamma^{n,m}_{\bar{\mathbf{B}},\mathbf{p}}$ is a hypercamera version of the multiview variety. When $\bar{\mathbf{B}} = \bar{\mathbf{Q}}$, we have the following result.

Proposition 4.2.7. *Fix $\bar{\mathbf{q}} = (\bar{q}_1, \dots, \bar{q}_n) \in (\mathbb{P}^3)^n$ with no four $\bar{q}_i$ coplanar. Then*

$$\Gamma^{n,m}_{\bar{\mathbf{Q}},\mathbf{p}} = \Gamma^{m,n}_{\bar{\mathbf{q}},\mathbf{p}} = \mathrm{V}(I_m(\bar{\mathbf{q}})).$$

Proof. It is relatively straightforward to prove the inclusions

$$\Gamma^{n,m}_{\bar{\mathbf{Q}},\mathbf{p}} \subset \Gamma^{m,n}_{\bar{\mathbf{q}},\mathbf{p}} \subset \mathrm{V}(I_m(\bar{\mathbf{q}})),$$

so we focus on the harder inclusion $\mathrm{V}(I_m(\bar{\mathbf{q}})) \subset \Gamma^{n,m}_{\bar{\mathbf{Q}},\mathbf{p}}$. This is also where we need the noncoplanarity assumption. Consider any point

$$\mathbf{p} \in \mathrm{V}(I_m(\bar{\mathbf{q}})).$$

We will construct a sequence of points $(\mathbf{p}^{(k)}) \in \Gamma^{n,m}_{\bar{\mathbf{Q}},\mathbf{p}}$ converging to $\mathbf{p}$. To simplify notation in what follows, we consider the case $m = 1$. When $m > 1$, the same construction applies component-wise. We write p_i in place of p_{1i}, so that the kernel of the matrix

$$\begin{bmatrix} \bar{Q}_1 & p_1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \bar{Q}_n & 0 & \cdots & p_n \end{bmatrix}$$

contains a point $v = [v_1 : \cdots : v_{12+n}] \in \mathbb{P}^{11+n}$. Let us fix homogeneous coordinates for $p_1, \dots, p_n, \bar{q}_1, \dots, \bar{q}_n, v$. We define

$$A = \begin{bmatrix} v_1 & \cdots & v_4 \\ \vdots & \ddots & \vdots \\ v_9 & \cdots & v_{12} \end{bmatrix}, \quad \lambda_j = -v_{12+j}.$$

Let us first observe that the matrix A is nonzero, for otherwise we would have

$$\lambda_j p_j = A\bar{q}_j = \bar{Q}_j \operatorname{vec}(A^\top) = 0 \quad \Rightarrow \quad \lambda_j = 0$$

for all j, contradicting the fact that $v \neq 0$. Next, observe that at most three of the λ_j can be zero: otherwise, four of the points $\bar{q}_j$ would lie in some plane containing the kernel of A, contradicting our hypothesis that $\bar{\mathbf{q}}$ is noncoplanar. It follows that we can find a nonzero 3×4 matrix A' with

$A'\bar{q}_j = p_j$ for each j with $\lambda_j = 0$. Fix such a matrix A'. We now construct the desired sequence $(\mathbf{p}^{(k)})_{k\geq 1}$. Set

$$\begin{aligned} A^{(k)} &= A + (1/k)A', \\ p_j^{(k)} &= A^{(k)}\bar{q}_j, \quad j = 1, \ldots, n. \end{aligned}$$

To show convergence, note that when $\lambda_j = 0$ we have

$$p_j^{(k)} = (1/k)A'\bar{q}_j \sim A'\bar{q}_j = p_j.$$

For $\lambda_j \neq 0$ we attain p_j in the limit as $k \to \infty$, since

$$p_j^{(k)} = A\bar{q}_j + (1/k)A'\bar{q}_j \to A\bar{q}_j \sim p_j.$$

In summary, we have found for any point $\mathbf{p} \in \mathrm{V}(I_m(\bar{\mathbf{q}}))$ a sequence of points $(\mathbf{p}^{(k)}) \in \Gamma^{n,m}_{\bar{\mathbf{Q}},\mathbf{p}}$ converging to $\mathbf{p}$. Since $\Gamma^{n,m}_{\bar{\mathbf{Q}},\mathbf{p}}$ is closed in the Euclidean topology, we deduce the needed inclusion: $\mathrm{V}(I_m(\bar{\mathbf{q}})) \subset \Gamma^{n,m}_{\bar{\mathbf{Q}},\mathbf{p}}$. $\square$

We conclude this section with the simplest interesting example of a resectioning variety.

Example 4.2.8. *For $m = 1$ camera and a generic point arrangement $\bar{\mathbf{q}} = (\bar{q}_1, \ldots, \bar{q}_6) \in (\mathbb{P}^3)^n$, the resectioning variety $\Gamma^{1,6}_{\bar{\mathbf{q}},\mathbf{p}}$ is a hypersurface in $(\mathbb{P}^2)^6$. Applying a suitable permutation to the rows of the 6-focal matrix, this hypersurface has an 18×18 determinantal representation,*

$$\det \begin{bmatrix} \bar{q}_1^T & & & p_1[1] & & \\ \vdots & & & & \ddots & \\ \bar{q}_6^T & & & & & p_6[1] \\ & \bar{q}_1^T & & p_1[2] & & \\ & \vdots & & & \ddots & \\ & \bar{q}_6^T & & & & p_6[2] \\ & & \bar{q}_1^T & p_1[3] & & \\ & & \vdots & & \ddots & \\ & & \bar{q}_6^T & & & p_6[3] \end{bmatrix} = 0. \tag{4.13}$$

Out of the $3^6 = 729$ *possible terms of a sextilinear form on* $(\mathbb{P}^2)^6$*, the special structure of the* 6*-focal determinant dictates that only* $\binom{6}{2}\binom{4}{2} = 90$ *can be nonzero. On the other hand, Lemma 4.4.3 below shows that applying a general linear change of coordinates to* (4.13) *has the effect that all* 729 *possible terms become nonzero. This highlights an important distinction between resectioning and multiview ideals—the initial ideal for generic data* $\bar{\mathbf{q}}$ *is not the same as the* $\mathbb{Z}^6$*-graded Borel-fixed generic initial ideal (cf.* [3, §3].*) Letting* $<$ *denote the lexicographic order with* $p_6[3] < p_6[2] < p_6[1] < p_5[3] < \cdots < p_1[1]$*, we have*

$$\begin{aligned} \mathrm{in}_<(I(\Gamma^{1,6}_{\bar{\mathbf{q}},\mathbf{p}}) &= \langle p_1[1]p_2[1]p_3[2]p_4[2]p_5[3]p_6[3]\rangle, \\ \mathrm{gin}_<(I(\Gamma^{1,6}_{\bar{\mathbf{q}},\mathbf{p}}) &= \langle p_1[1]p_2[1]p_3[1]p_4[1]p_5[1]p_6[1]\rangle. \end{aligned} \tag{4.14}$$

Interestingly, (4.13) *also has several smaller determinantal representations. Many of these may be obtained from* (4.13) *using Schur complements. For example, we have the* 12×12 *determinantal representation*

$$(p_1[3]\cdots p_6[3])^{-1} \det \begin{bmatrix} p_1[3]\bar{q}_1^T & & -p_1[1]\bar{q}_1^T \\ \vdots & & \vdots \\ p_6[3]\bar{q}_6^T & & -p_6[1]\bar{q}_6^T \\ & p_1[3]\bar{q}_1^T & -p_1[2]\bar{q}_1^T \\ & \vdots & \vdots \\ & p_6[3]\bar{q}_6^T & -p_6[2]\bar{q}_6^T \end{bmatrix} = 0. \tag{4.15}$$

The 6*-focal determinant also has a* 6×6 *determinantal representation, which specializes to* (4.26) *below after fixing* $\bar{q}_1, \ldots, \bar{q}_4, p_1, \ldots, p_4$*. This is the classical form of the constraint appearing in works such as* [59, 64]*. Finally, we note that Schaffilitzky et al.* [60] *derive a* 3×3 *determinantal constraint relating 3D points and their 2D projections that is linear in a distinguished image point* p_6*. Theorem 4.2.6 implies that their determinant is a multiple of the* 6*-focal determinant. Notably, Schaffilitzky et al. use their constraint to solve the minimal problem of reconstructing* 6 *points from* 3 *views. Earlier works, eg.* [64]*, had already observed that this problem is equivalent to the classical* 7 *point problem in* 2 *views. This equivalence follows from the principle of Carlsson-Weinshall duality, which we revisit in the next section.*

4.3 Carlsson-Weinshall duality revisited

Recall the image formation variety $\Gamma^{m,n}_{\mathbf{A},\mathbf{q},\mathbf{p}}$ from (4.2). Previous work of Agarwal et al. [1] explains how the problems of reconstruction, triangulation, and resectioning may all be understood in terms of slicing and projection operations on this variety. The relationships between the varieties produced by these operations are summarized in a diagram designated as an *atlas* for the pinhole camera. One striking feature of the atlas's appearance is the apparently symmetric roles of cameras in $\mathbb{P}^{11}$ and world points in $\mathbb{P}^3$. A simple explanation for this phenomenon is as follows: for a given camera center $c \in \mathbb{P}^3$, world point $q \in \mathbb{P}^3$, and image plane $L \in \mathrm{Gr}(\mathbb{P}^2, \mathbb{P}^3)$, we obtain the same projected point on L whether we project c through q or project q through c. If we want to express this symmetry in terms of camera matrices instead of camera centers, one approach is to introduce coordinates on the image plane. Indeed, there are an additional $\dim \mathrm{PGL}_3 = 8 = 11 - 3$ degrees of freedom in choosing projective coordinates on L. A particular choice of coordinates leads directly to the framework of *Carlsson-Weinshall (CW) duality* from the multiview geometry literature.

In this section, we point out that several world-to-image point constraints which were previously discovered using CW duality arise naturally as specializations of our focal constraints. We also show in Theorem 4.3.4 that Carlsson-Weinshall duality gives rise to a rational quotient of the image formation correspondence, and develop a *reduced* version of the atlas that better explains the symmetry between cameras and world points—see Figure 4.3.

A direct application of the focal constraints described in Section 4.2 arises naturally in the setting of Carlsson-Weinshall (CW) duality. In the eponymous authors' celebrated work, CW duality is described as the notion that "*problems of [resectioning] and [triangulation] from image data are... dual in the sense that they can be solved with the same algorithm depending on the number of [world] points and cameras*" [64].

In this section, we develop CW duality in the context of a *reduced atlas*, analogous to that of [1], which makes the symmetry between cameras and 3D points evident. Theorems 4.3.4 and 4.3.5 explain how nodes in this atlas arise as rational quotients of their non-reduced counterparts. The latter result also includes an analogue of Theorem 4.2.6: the reduced resectioning variety is cut out scheme-theoretically by bilinear forms for a sufficiently generic point configuration.

Remark 4.3.1. *Recent work by Trager, Hebert, and Ponce* [52] *demonstrates that the exact coordinates of the camera centers and world points are not essential features of CW duality, contrary to the original setup. For simplicity, we state the main results of this section with respect to the conventional projective frame defined in* (4.20).

4.3.1 Geometric formulation

For m cameras and n world points, we define the *reduced image formation correspondence* to be the variety

$$\mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}} = \overline{\{(\mathbf{a},\mathbf{q},\mathbf{p}) \in (\mathbb{P}^3)^m \times (\mathbb{P}^3)^n \times (\mathbb{P}^2)^{mn} \mid A(a_i)\cdot q_j \sim p_{ij} \quad \forall i \in [m],\, j \in [n]\}}, \tag{4.16}$$

where for $a_i = [a_{i1} : a_{i2} : a_{i3} : a_{i4}] \in \mathbb{P}^3$ we define

$$A(a_i) = \begin{bmatrix} a_{i1} & 0 & 0 & a_{i4} \\ 0 & a_{i2} & 0 & a_{i4} \\ 0 & 0 & a_{i3} & a_{i4} \end{bmatrix}. \tag{4.17}$$

When $A(a_i)$ is of full rank, we call it the *reduced camera matrix* associated to the point a_i. The center of a reduced camera matrix $A(a_i)$ is $\mathcal{C}(a_i)$, where $\mathcal{C}$ is the quadratic Cremona involution

$$\begin{aligned} \mathcal{C} : \mathbb{P}^3 &\dashrightarrow \mathbb{P}^3 \\ [a_1 : a_2 : a_3 : a_4] &\mapsto [1/a_1 : 1/a_2 : 1/a_3 : -1/a_4]. \end{aligned} \tag{4.18}$$

Note that $\mathcal{C}(a_i)$ is defined exactly when at most one a_{ij} is zero, or equivalently, when $A(a_i)$ is a full-rank camera matrix with a well-defined center.

The key observation of Carlsson-Weinshall duality is expressed by the symmetric roles of a 3D point q_j and a reduced camera $A(a_i)$ in image formation:

$$A(a_i)q_j = A(q_j)a_i \quad \forall i = 1,\ldots,m,\ j = 1,\ldots,n. \tag{4.19}$$

The special form of the reduced camera matrix arises from fixing a projective basis in each image and a partial projective basis in the world. We adopt the notation of [38, Ch. 16]:

$$\begin{aligned}
&E_1 = [1:0:0:0], \quad e_1 = [1:0:0],\\
&E_2 = [0:1:0:0], \quad e_2 = [0:1:0],\\
&E_3 = [0:0:1:0], \quad e_3 = [0:0:1],\\
&E_4 = [0:0:0:1], \quad e_4 = [1:1:1],\\
&E_5 = [1:1:1:1].
\end{aligned} \tag{4.20}$$

Each set of four points $E_1, \dots E_4 \in \mathbb{P}^3$, $e_1, \dots, e_4 \in \mathbb{P}^3$ is said to span a *reference tetrahedron* in $\mathbb{P}^3$. The geometry relating these points and the Cremona transformation $\mathcal{C}$ can be appreciated in Figure 4.2.

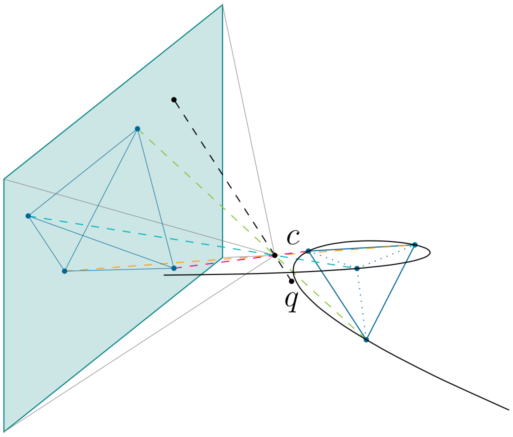

Figure 4.2: Four fixed points $E_1, \dots, E_4 \in \mathbb{P}^3$ determine a reference tetrahedron. They project through the camera center $c \in \mathbb{P}^3$ to the four points $e_1, \dots e_4 \in \mathbb{P}^2$. The Cremona transformation $\mathcal{C}$ maps the line through c and q to the unique twisted cubic passing through $\mathcal{C}(q), \mathcal{C}(c), E_1, \dots, E_4$.

Given a camera of the form $A = A(a), a \in \mathbb{P}^3$, we have that $A(a)E_i = e_i$ for $i = 1, \dots, 4$. The converse is true as well; that is, a camera matrix takes the reduced form (4.17) if and only if it sends E_i to e_i for each $i = 1, \dots, 4$. As we will soon demonstrate, there is a rational group action by an algebraic group $\mathcal{G}_m$ on $\Gamma^{m,n+4}_{\mathbf{A},\mathbf{q},\mathbf{p}}$ for which each $\mathcal{G}_m$-orbit, where defined, contains a unique element of $\mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}}$. That is, $\mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}}$ can be thought of as a kind of quotient of the general image formation correspondence. Theorem 4.3.4 makes this precise using the notion of a *rational quotient.* In this reduced setting, the roles of camera centers and world points are manifestly symmetric: a point $(\mathbf{a}, \mathbf{q}, \mathbf{p}) \in \mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}}$ is also a point in $\mathrm{P}^{n,m}_{\mathbf{a},\mathbf{q},\mathbf{p}}$ after swapping the $\mathbf{a}$ and $\mathbf{q}$ factors. By this observation, we then get the isomorphism $\mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}} \simeq \mathrm{P}^{n,m}_{\mathbf{a},\mathbf{q},\mathbf{p}}$.

Just as a point in $\Gamma^{m,n}_{\mathbf{A},\mathbf{q},\mathbf{p}}$ can be thought of as a configuration of cameras and points, a point in $\mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}}$ can be thought of as such a configuration up to certain coordinate changes. More precisely, points in $\mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}}$ correspond to orbits in $\Gamma^{m,n}_{\mathbf{A},\mathbf{q},\mathbf{p}}$ under the action of a group $\mathcal{G}_m$ consisting of coordinate changes in the world and each of the m images. Up to this group action, we may assume the image planes $L_1, \ldots, L_m \in \mathrm{Gr}(\mathbb{P}^2, \mathbb{P}^3)$ are all equal, ie. $L_1 = \ldots = L_m$. This explains the center image in Figure 4.1.

We now transition into a formal treatment of the notions described above. Define $\mathcal{G}_m = (\mathrm{PGL}_3)^m \times \mathrm{Stab}_{\mathrm{PGL}_4}(E_5)$, an algebraic group of dimension $8m + 12$ which acts rationally on $\Gamma^{m,n}_{\mathbf{A},\mathbf{q},\mathbf{p}}$ as follows:

$$\begin{aligned} \mathcal{G}_m \times \Gamma^{m,n}_{\mathbf{A},\mathbf{q},\mathbf{p}} &\dashrightarrow \Gamma^{m,n}_{\mathbf{A},\mathbf{q},\mathbf{p}} \\ (T_1, \ldots, T_m, S) \cdot (A_1, \ldots, A_m, q_1, \ldots q_n, p_{11}, \ldots, p_{mn}) & \\ = (T_1 A_1 S^{-1}, \ldots, T_m A_m S^{-1}, &Sq_1, \ldots, Sq_n, T_1 p_{11}, \ldots, T_m p_{mn}). \end{aligned} \tag{4.21}$$

To formalize the intuition that $\mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}}$ is a quotient of $\Gamma^{m,n}_{\mathbf{A},\mathbf{q},\mathbf{p}}$ by $\mathcal{G}_m$, we recall the definition of a *rational quotient* as follows.

Definition 4.3.2. *(cf.* [23, §6.2].*) Let X and Y be irreducible algebraic varieties and G an algebraic group acting rationally on X. We say Y is a* rational quotient *of X by G, and write $X/G \cong_{\boldsymbol{Bir}} Y$, if Y is a model for the field of G-invariant rational functions on X: that is, if there exists an isomorphism $\mathbb{C}(Y) \cong \mathbb{C}(X)^G$.*

A classical result due to Rosenlicht states that rational quotients always exist over any algebraically closed field (cf. [23, Theorem 6.2].) The following simple lemma provides sufficient conditions for recognizing a particular class of rational quotients in which the action yields a birational equivalence of X with $G \times Y$.

Lemma 4.3.3. *Let G be an algebraic group acting rationally on a variety X. For a subvariety $Y \subset X$, we have $X/G \cong_{\textbf{\textit{Bir}}} Y$ if there exists a rational map*

$$\begin{aligned} \mu_G : X &\dashrightarrow G \\ x &\mapsto \mu_G(x) \end{aligned}$$

such that $\mu_G(y) = \mathrm{id}_G$ for all y in a dense open subset of Y, and such that $\mu_G(g \cdot x) = \mu_G(x)\, g^{-1}$ for all (g, x) in dense open subset of $G \times X$. Moreover, these assumptions imply that

$$\begin{aligned} X &\dashrightarrow G \times Y \\ x &\mapsto (\mu_G(x), \mu_G(x) \cdot x) \end{aligned}$$

is a birational equivalence, with a rational inverse given by

$$\begin{aligned} G \times Y &\dashrightarrow X \\ (g, y) &\mapsto g^{-1} \cdot y. \end{aligned}$$

Proof. A function $f \in \mathbb{C}(Y)$ pulls back to a function $h \in \mathbb{C}(X)$ defined by $h(x) = f(\mu_G(x) \cdot x)$ on X. Our assumptions imply h is G-invariant, since

$$h(g \cdot x) = f(\mu_G(g \cdot x) \cdot (g \cdot x)) = f\left((\mu_G(x) g^{-1}) \cdot (g \cdot x)\right) = h(x).$$

Let us write $\varphi^* : \mathbb{C}(Y) \to \mathbb{C}(X)^G$. Since $Y \subset X$, we also have the induced map $\iota^* : \mathbb{C}(X)^G \to \mathbb{C}(Y)$. We show that φ^* and ι^* are mutual inverses. Taking any function $f \in \mathbb{C}(Y)$ and $y \in Y$ in its domain of definition, we calculate

$$\iota^* \varphi^* f(y) = f(\mu_G(y) \cdot y) = f(y).$$

Similarly, for any fixed $f \in \mathbb{C}(X)^G$, the values $f(x)$ and $f(\mu_G(x) \cdot x)$ are defined for a dense open subset of $x \in X$, for which we compute

$$\varphi^* \iota^* f(x) = f(\mu_G(x) \cdot x) = f(x).$$

This proves $X/G \cong_{\mathbf{Bir}} Y$. The birational equivalence of $G \times Y$ and X follows similarly. $\square$

Theorem 4.3.4 (CW duality)**.** *For any $m, n \geq 0$, we have a birational equivalence of varieties*

$$\Gamma^{m,n+4}_{\mathbf{A},\mathbf{q},\mathbf{p}} \cong_{Bir} \mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}} \times \mathcal{G}_m,$$

which yields following commutative diagram (in which each arrow labeled $\sim$ is a birational or biregular isomorphism.)

$$\begin{tikzcd}
\Gamma^{m,n+4}_{\mathbf{A},\mathbf{q},\mathbf{p}} \times (\mathrm{PGL}_3)^n \arrow[r, dashed, "\sim"] \arrow[d, dashed, "\wr"] & \Gamma^{n,m+4}_{\mathbf{A},\mathbf{q},\mathbf{p}} \times (\mathrm{PGL}_3)^m \arrow[d, dashed, "\wr"] \\
(\mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}} \times \mathcal{G}_m) \times (\mathrm{PGL}_3)^n \arrow[r, dashed, "\sim"] \arrow[d] & (\mathrm{P}^{n,m}_{\mathbf{a},\mathbf{q},\mathbf{p}} \times \mathcal{G}_n) \times (\mathrm{PGL}_3)^m \arrow[d] \\
\mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}} \arrow[r, "\sim"] & \mathrm{P}^{n,m}_{\mathbf{a},\mathbf{q},\mathbf{p}}
\end{tikzcd}$$

This diagram has the following additional properties:

1. *If $\nu_{m,n}$ denotes any of the horizontal maps, we have $\nu_{n,m} \circ \nu_{m,n} = \mathrm{id}$ wherever both maps are defined.*
2. *The vertical maps express the reduced image formation variety as a rational quotient of the image formation correspondence,*

$$\Gamma^{m,n+4}_{\mathbf{A},\mathbf{q},\mathbf{p}} / \mathcal{G}_m \cong_{\boldsymbol{Bir}} \mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}}. \tag{4.22}$$

3. *The duality between the problems of exact resectioning and triangulation may be expressed in terms of this commutative diagram and certain projections: eg., for the bottom row, if $\pi'_{\mathbf{a}}, \pi'_{\mathbf{q}}$ denote the projections from $\mathrm{P}_{\mathbf{a},\mathbf{q},\mathbf{p}}$ that forget the $\mathbf{a}$ and $\mathbf{q}$ factors, then the diagram below commutes.*

$$\begin{tikzcd}
 & (\mathbb{P}^3)^n \times (\mathbb{P}^2)^{mn} & \\
\mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}} \arrow[ur, "\pi'_{\mathbf{a}}"] \arrow[rr, "\sim"] \arrow[dr, "\pi'_{\mathbf{q}}"'] & & \mathrm{P}^{n,m}_{\mathbf{a},\mathbf{q},\mathbf{p}} \arrow[ul, "\pi'_{\mathbf{q}}"'] \arrow[dl, "\pi'_{\mathbf{a}}"] \\
 & (\mathbb{P}^3)^m \times (\mathbb{P}^2)^{mn} &
\end{tikzcd}$$

Proof. We begin by constructing the maps that yield the rational quotient (4.22). This part follows by applying Lemma 4.3.3 with $X = \Gamma^{m,n+4}_{\mathbf{A},\mathbf{q},\mathbf{p}}$, $Y = \mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}}$, and $G = \mathcal{G}_m$. To obtain the inclusion $\mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}} \subset \Gamma^{m,n+4}_{\mathbf{A},\mathbf{q},\mathbf{p}}$ we define

$$\begin{aligned} \iota : \mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}} &\to \Gamma^{m,n+4}_{\mathbf{A},\mathbf{q},\mathbf{p}} \\ (a_1, \ldots, a_m, q_1, \ldots, q_n, p_{11}, \ldots, p_{mn}) &\mapsto \\ &(A(a_1), \ldots, A(a_m), E_1, E_2, E_3, E_4, q_1, \ldots, q_n, e_1, e_2, e_3, e_4, p_{11}, \ldots, p_{mn}). \end{aligned}$$

To construct the map $\mu_{\mathcal{G}_m} : \Gamma^{m,n+4}_{\mathbf{A},\mathbf{q},\mathbf{p}} \dashrightarrow \mathcal{G}_m$, consider first the map

$$\begin{aligned} S : (\mathbb{P}^3)^4 &\dashrightarrow \mathrm{Stab}_{\mathrm{PGL}_4}(E_5) \\ (q_1, \ldots, q_4) &\mapsto \left(\left[\begin{array}{c|c|c|c} q_1 & q_2 & q_3 & q_4 \end{array} \right] \cdot \mathrm{diag}\left([5234]_{\mathbf{q}}, [1534]_{\mathbf{q}}, [1254]_{\mathbf{q}}, [1235]_{\mathbf{q}}\right) \right)^{-1}, \end{aligned}$$

where each $[5234]_{\mathbf{q}}, \ldots, [1235]_{\mathbf{q}}$ is the determinant of a matrix obtained by replacing $q_1, \ldots, q_4$ with E_5 in the 4×4 matrix whose columns are $q_1, \ldots, q_4$. We verify that $S(q_1, \ldots, q_4)$ is well-defined and contained in $\mathrm{Stab}_{\mathrm{PGL}_4}(E_5)$ using linear algebra. To ease notation, we write $Q = \left[\begin{array}{c|c|c|c} q_1 & q_2 & q_3 & q_4 \end{array} \right]$ and $D = \mathrm{diag}\left([5234]_{\mathbf{q}}, [1534]_{\mathbf{q}}, [1254]_{\mathbf{q}}, [1235]_{\mathbf{q}}\right)$. Rescaling any of the $q_1, \ldots, q_4$ then rescales the matrix product QD. Using Cramer's rule, we calculate that

$$\begin{aligned} S(q_1, \ldots, q_4)E_5 &= D^{-1} \cdot Q^{-1} E_5 \\ &= D^{-1}[[5234]_{\mathbf{q}} : [1534]_{\mathbf{q}} : [1254]_{\mathbf{q}} : [1235]_{\mathbf{q}}] \\ &= E_5. \end{aligned}$$

An analagous calculation can be used to verify that for any $S_0 \in \mathrm{PGL}_4$ we have

$$S(S_0 q_1, \ldots, S_0 q_4) = S(q_1, \ldots, q_4)\, S_0^{-1}. \tag{4.23}$$

Similar to our definition of S above, we may define a map

$$\begin{aligned} T : (\mathbb{P}^2)^4 &\dashrightarrow \mathrm{PGL}_3 \\ (p_1, \ldots, p_4) &\mapsto \left(\left[\begin{array}{c|c|c} p_1 & p_2 & p_3 \end{array} \right] \cdot \mathrm{diag}\left([423]_{\mathbf{p}}, [143]_{\mathbf{p}}, [124]_{\mathbf{p}}\right) \right)^{-1}, \end{aligned}$$

but we replace $p_1, \ldots, p_3$ by p_4 (rather than e_4) when forming the expressions $[423]_{\mathbf{p}}, \ldots, [124]_{\mathbf{p}}$. Once again, for $T_0 \in \mathrm{PGL}_3$ we have

$$T(T_0 p_1, \ldots, T_0 p_4) = T(p_1, \ldots, p_4)\, T_0^{-1}. \tag{4.24}$$

Finally, we define

$$\begin{gathered} \mu_{\mathcal{G}_m} : \Gamma^{m,n+4}_{\mathbf{A},\mathbf{q},\mathbf{p}} \dashrightarrow \mathcal{G}_n \\ (A_1, \ldots, A_m, q_1, \ldots, q_{n+4}, p_{11}, \ldots, p_{mn}) \mapsto \\ (T(p_{11}, \ldots, p_{14}),\ \ldots,\ T(p_{11}, \ldots, p_{m4}),\ S(q)). \end{gathered}$$

We check that the two assumptions of Lemma 4.3.3 are satisfied. The map $\mu_{\mathcal{G}_m}$ fixes $\mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}}$ pointwise since $T(e_1, e_2, e_3, e_4)$ and $S(E_1, E_2, E_3, E_4)$ both act as the identity. Similarly, for sufficiently generic $g \in \mathcal{G}_m$ and $x \in \Gamma^{m,n+4}_{\mathbf{A},\mathbf{q},\mathbf{p}}$ the assumption that $\mu_{\mathcal{G}_m}(g \cdot x) = \mu_{\mathcal{G}_m}(x)\, g^{-1}$ follows from (4.23) and (4.24).

Thus, we may conclude from Lemma 4.3.3 that we have the rational quotient (4.22), giving property 2 in the statement of the theorem. Moreover, the lemma implies that $\Gamma^{m,n+4}_{\mathbf{A},\mathbf{q},\mathbf{p}}$ is birationally equivalent to $\mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}} \times \mathcal{G}_m$, which allows us to define the vertical maps in the main diagram. To complete the diagram, it suffices to define the bottom-most map, which is

$$\begin{gathered} \nu_{m,n} : \mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}} \to \mathrm{P}^{n,m}_{\mathbf{a},\mathbf{q},\mathbf{p}} \\ (a_1, \ldots, a_m, q_1, \ldots, q_n, p_{11}, \ldots p_{mn}) \mapsto (q_1, \ldots, q_n, a_1, \ldots, a_m, p_{11}, \ldots p_{nm}). \end{gathered}$$

Now, to show that $\nu_{m,n}$ is an isomorphism, we use the symmetric equations (4.19). The remaining parts of the theorem now follow easily. □

The reduced image formation correspondence $\mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}}$ sits at the center of the *reduced atlas* depicted in Figure 4.3. Following [1], we may define the remaining entities in this figure using slices and projections of the reduced image formation correspondence. For instance, the varieties $\mathrm{P}^{m,n}_{\mathbf{a},\mathbf{p}}$ and $\mathrm{P}^{m,n}_{\mathbf{q},\mathbf{p}}$ are defined, respectively, as the image under the coordinate projections $\pi'_{\mathbf{q}} : \mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}} \to (\mathbb{P}^3)^m \times (\mathbb{P}^2)^{mn}$, $\pi'_{\mathbf{a}} : (\mathbb{P}^3)^n \times (\mathbb{P}^2)^{mn}$ appearing in Theorem 4.3.4. Slicing the variety $\mathrm{P}^{m,n}_{\mathbf{q},\mathbf{p}}$ with

$$\begin{tikzcd}
 & \mathrm{P}^{m,n}_{\bar{\mathbf{a}},\mathbf{q},\mathbf{p}} \arrow[dl] & \mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\bar{\mathbf{p}}} \arrow[dl] \arrow[dr] & \mathrm{P}^{m,n}_{\mathbf{a},\bar{\mathbf{q}},\mathbf{p}} \arrow[dr] & \\
\mathrm{P}^{m,n}_{\bar{\mathbf{a}},\mathbf{p}} & \mathrm{P}^{m,n}_{\mathbf{a},\bar{\mathbf{p}}} & \mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}} \arrow[ul, red] \arrow[u, red] \arrow[ur, red] \arrow[dl] \arrow[dr] & \mathrm{P}^{m,n}_{\mathbf{q},\bar{\mathbf{p}}} & \mathrm{P}^{m,n}_{\bar{\mathbf{q}},\mathbf{p}} \\
 & \mathrm{P}^{m,n}_{\mathbf{a},\mathbf{p}} \arrow[ul, red] \arrow[u, red] \arrow[dr] & & \mathrm{P}^{m,n}_{\mathbf{q},\mathbf{p}} \arrow[u, red] \arrow[ur, red] \arrow[dl] & \\
 & & \mathrm{P}^{m,n}_{\mathbf{p}} & &
\end{tikzcd}$$

Figure 4.3: An atlas for the reduced pinhole camera, cf. [1, Figure 1].

the coordinate planes defined by $\mathbf{q} = \bar{\mathbf{q}}$, we obtain the *reduced resectioning variety* $\mathrm{P}^{m,n}_{\bar{\mathbf{q}},\mathbf{p}}$. Up to Zariski closure, this is the *dual reduced joint image* introduced in [52]. Theorem 4.3.4 above and Theorem 4.3.5 below illustrate the correspondence between these varieties and their non-reduced counterparts in [1], and provide an explanation for the vertical symmetry present in both atlases.

4.3.2 Algebraic consequences

Readers familiar with multiview geometry will no doubt wonder how the focal constraints of Theorem 4.2.6 relate to various “dual multiview constraints", derived by Carlsson, Weinshall, and others. All of these previously-studied constraints may be interpreted as polynomials vanishing on $\mathrm{P}^{m,n}_{\bar{\mathbf{q}},\mathbf{p}}$. Specializing the 6-focal constraints to $\mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}}$, we obtain

$$\det \begin{bmatrix} E_1^\top \otimes I & e_1 & & & & & \\ E_2^\top \otimes I & & e_2 & & & & \\ E_3^\top \otimes I & & & e_3 & & & \\ E_4^\top \otimes I & & & & e_4 & & \\ \bar{Q}_{j_1} & & & & & p_{ij_1} & \\ \bar{Q}_{j_2} & & & & & & p_{ij_2} \end{bmatrix} = 0, \quad \forall i = 1, \ldots, m, \;\; 1 \le j_1 < j_2 \le n. \qquad (4.25)$$

Permuting the rows, (4.25) implies that

$$\det \left[\begin{array}{c|cccccc} & E_1 & & & E_4 & & \\ I_{12\times 12} & & E_2 & & E_4 & & \\ & & & E_3 & E_4 & & \\ \hline \bar{Q}_{j_1} & & & & & p_{ij_1} & \\ \bar{Q}_{j_2} & & & & & & p_{ij_2} \end{array}\right] = 0,$$

and taking the Schur complement, we find

$$\det\left(\begin{bmatrix} 0_{3\times 4} & p_{ij_1} & \\ 0_{3\times 4} & & p_{ij_2} \end{bmatrix} - \begin{bmatrix} \bar{Q}_{j_1} \\ \bar{Q}_{j_2} \end{bmatrix} \begin{bmatrix} E_1 & & & E_4 \\ & E_2 & & E_4 \\ & & E_3 & E_4 \end{bmatrix}\right) =$$

$$\det \begin{bmatrix} A(\bar{q}_{j_1}) & p_{ij_1} & \\ A(\bar{q}_{j_2}) & & p_{ij_2} \end{bmatrix} = 0. \tag{4.26}$$

Equation (4.26) is a bilinear form in p_{ij_1} and p_{ij_2}, which may be represented by Carlsson and Weinshall's 3×3 *dual fundamental matrix* (cf. [64, eq. 18]),

$$\begin{bmatrix} 0 & \bar{q}_{j_1}[2]\bar{q}_{j_2}[1] \det\begin{bmatrix} \bar{q}_{j_1}[4] & \bar{q}_{j_1}[3] \\ \bar{q}_{j_2}[4] & \bar{q}_{j_2}[3] \end{bmatrix} & \bar{q}_{j_1}[3]\bar{q}_{j_2}[1] \det\begin{bmatrix} \bar{q}_{j_1}[2] & \bar{q}_{j_1}[4] \\ \bar{q}_{j_2}[2] & \bar{q}_{j_2}[4] \end{bmatrix} \\ \bar{q}_{j_1}[1]\bar{q}_{j_2}[2] \det\begin{bmatrix} \bar{q}_{j_1}[3] & \bar{q}_{j_1}[4] \\ \bar{q}_{j_2}[3] & \bar{q}_{j_2}[4] \end{bmatrix} & 0 & \bar{q}_{j_1}[3]\bar{q}_{j_2}[2] \det\begin{bmatrix} \bar{q}_{j_1}[4] & \bar{q}_{j_1}[1] \\ \bar{q}_{j_2}[4] & \bar{q}_{j_2}[1] \end{bmatrix} \\ \bar{q}_{j_1}[1]\bar{q}_{j_2}[3] \det\begin{bmatrix} \bar{q}_{j_1}[4] & \bar{q}_{j_1}[2] \\ \bar{q}_{j_2}[4] & \bar{q}_{j_2}[2] \end{bmatrix} & \bar{q}_{j_1}[2]\bar{q}_{j_2}[3] \det\begin{bmatrix} \bar{q}_{j_1}[1] & \bar{q}_{j_1}[4] \\ \bar{q}_{j_2}[1] & \bar{q}_{j_2}[4] \end{bmatrix} & 0 \end{bmatrix}. \tag{4.27}$$

This construction yields a total of $m\binom{n}{2}$ bilinear equations vanishing on the reduced joint image $\mathrm{P}^{m,n}_{\bar{\mathbf{q}},\mathbf{p}}$. A similar application of this Schur complement trick to suitably-chosen 7- and 8-focals leads to the *dual trifocal and quadrifocal tensors* (cf. [64, §6.3–6.4], [52, §3, 4]).

For a sufficiently generic point configuration $\bar{\mathbf{q}} \in (\mathbb{P}^3)^n$, it turns out that the reduced 2-focals (4.26) determine $\mathrm{P}^{m,n}_{\bar{\mathbf{q}},\mathbf{p}}$ as a subscheme of $(\mathbb{P}^2)^{mn}$. Theorem 4.3.5 states precise genericity conditions such that this occurs. Thus, while equations needed to cut out $\Gamma^{m,n}_{\mathbf{A},\mathbf{q},\mathbf{p}}$ in a strong sense have very high degree, only bilinear equations are needed to cut out its quotient $\mathrm{P}^{m,n}_{\mathbf{a},\mathbf{q},\mathbf{p}}$ in a weaker sense. The essential insight is, via Carlsson-Weinshall duality, that $\mathrm{P}^{m,n}_{\bar{\mathbf{q}},\mathbf{p}}$ is simply the direct product of “ordinary" multiview varieties,

$$\mathrm{P}^{m,n}_{\bar{\mathbf{q}},\mathbf{p}} = \Gamma^{n,m}_{A(\bar{\mathbf{q}}),\mathbf{p}} \cong \Gamma^{n,1}_{A(\bar{\mathbf{q}}),\mathbf{p}} \times \cdots \times \Gamma^{n,1}_{A(\bar{\mathbf{q}}),\mathbf{p}} \quad \text{where } A(\bar{\mathbf{q}}) = (A(\bar{q}_1), \ldots, A(\bar{q}_n))\,. \tag{4.28}$$

A previous result, namely part (3) of Theorem 4.2.3, states that the multiview variety of a sufficiently generic camera arrangement is cut out by the bilinear forms in its vanishing ideal. The Cremona transformation $\mathcal{C}$ allows us to translate these genericity conditions on the cameras $A(\bar{\mathbf{q}})$ into conditions on the point arrangement $\bar{\mathbf{q}} \in (\mathbb{P}^2)$. A 4-nodal cubic surface in $\mathbb{P}^3$ containing the points $E_1, \ldots E_4$ is given by an equation of the form

$$a_1x_2x_3x_4 + a_2x_1x_3x_4 + a_3x_1x_2x_4 + a_4x_1x_2x_3 = 0, \quad [a_1 : a_2 : a_3 : a_4] \in \mathbb{P}^3. \tag{4.29}$$

If all a_i are nonzero, then such a surface is projectively equivalent to *Cayley’s nodal cubic surface*, for which $a_1 = a_2 = a_3 = a_4 = 1$ and the points $E_1, \ldots, E_4$ comprise the singular locus. We also allow degenerate cases where one or more $a_i = 0$ in (4.29), in which case the surface degenerates to the union of a plane and a quadric, or the union of three planes.

Theorem 4.3.5. *Fix n distinct points $\bar{q}_1, \ldots, \bar{q}_n \in \mathbb{P}^3 \setminus \{E_1, E_2, E_3, E_4\}$, $n \geq 2$, such that no four $\bar{q}_j$ lie on a common 4-nodal cubic surface through $E_1, \ldots E_4$. Write*

$$\bar{\mathbf{q}} = (\bar{q}_1, \ldots, \bar{q}_n, E_1, \ldots, E_4),$$

and $\bar{\mathbf{q}}'$ for the sub-arrangement of $\bar{\mathbf{q}}$ obtained by deleting the $E_1, \ldots, E_4$. We have a birational equivalence of varieties

$$\Gamma^{m,n+4}_{\bar{\mathbf{q}},\mathbf{p}} \simeq_{\mathbf{Bir}} \mathrm{P}^{m,n}_{\bar{\mathbf{q}}',\mathbf{p}} \times (\mathrm{PGL}_3)^m,$$

which realizes the reduced resectioning variety $\mathrm{P}^{m,n}_{\bar{\mathbf{q}}',\mathbf{p}}$ as a rational quotient of $\Gamma^{m,n}_{\bar{\mathbf{q}},\mathbf{p}}$ by $(\mathrm{PGL}_3)^m$. Additionally, $\mathrm{P}^{m,n}_{\bar{\mathbf{q}}',\mathbf{p}}$ is cut out scheme-theoretically by the $m\binom{n}{2}$ bilinear equations (4.26).

Proof. The statements involving rational quotients follow similarly as in Theorem 4.3.4. Under the isomorphism (4.28), the bilinear constraints in the theorem statement are the usual 2-focals vanishing on the multiview variety. We recall from part (3) of Theorem 4.2.3 that the 2-focals cut out the multiview variety scheme-theoretically whenever the camera centers are distinct and do not lie on a common plane. Now, since $\bar{q}_i$ is not in the span of any three E_j, the center of the camera $A(\bar{q}_i)$ is given by the Cremona transformation $\mathcal{C}(\bar{q}_i)$. Since $\mathcal{C}$ maps any plane in $\mathbb{P}^3$ to a 4-nodal cubic surface, and vice-versa, we are done. □

From the practitioner's point of view, the genericity assumptions of Theorem 4.3.5, as well as the implicit assumption that we can fix four fiducial 3D points and their images to the standard positions (4.20), may be quite reasonable. This is supported by the experiments of [52, §5], suggesting some potential uses of Carlsson-Weinshall duality in SfM settings.

4.4 Proof of Theorem 4.2.6

Our proof of Theorem 4.2.6 follows the general strategy used in the proof of [1, Theorem 3.2], but requires some nontrivial modifications.

Remark 4.4.1. *Unlike triangulation, resectioning is an interesting problem even for* $m = 1$ *camera. In fact, most of the work needed to prove Theorem 4.2.6 involves the special case* $m = 1$*. As in the proof of Proposition 4.2.7, we fix* $m = 1$ *and write* p_i *in place of* p_{1i}*. We also write* $I(\bullet)$ *in place of* $I_1(\bullet)$*, and* $\Gamma_{\bar{\mathbf{q}},\mathbf{p}}$ *instead of* $\Gamma^{1,n}_{\bar{\mathbf{q}},\mathbf{p}}$*. Finally, let us recall the variety* $\Gamma^{n,m}_{\bar{\mathbf{Q}},\mathbf{p}} \subset (\mathbb{P}^2)^{mn}$ *introduced in Proposition 4.2.7. In place of* $\Gamma^{n,1}_{\bar{\mathbf{Q}},\mathbf{p}}$*, we simply write* $\Gamma_{\bar{\mathbf{Q}},\mathbf{p}}$*.*

4.4.1 Proof outline and preliminary facts

To begin, we describe our proof strategy at a high level. The main steps of our proof can be understood via the diagram in Figure 4.4, with each of the steps (1)–(4) explained below.

(1) **Coordinate change to obtain a generic hypercamera arrangement from the structured one.** For $\bar{q}_1, \ldots, \bar{q}_n \in \mathbb{P}^3$ with no four coplanar, we establish in Lemmas 4.4.3 and 4.4.4 that there exist 3×3 invertible matrices $H_1, \ldots, H_n$ such that the transformed hypercamera arrangement

$$\bar{\mathbf{B}} := (H_1\bar{Q}_1, \ldots, H_n\bar{Q}_n)$$

$$\begin{tikzcd}
 & \textit{structured} & \textit{generic} \\
\textit{focal ideals} & I(\bar{\mathbf{Q}}) \arrow[r, "\mathbf{H}\cdot", "(1)"'] \arrow[d, equal, "(4)"'] & I(\bar{\mathbf{B}}) \arrow[d, equal, "(3)"] \\
\textit{vanishing ideals} & I(\Gamma_{\bar{\mathbf{q}},\mathbf{p}}) & I(\Gamma_{\bar{\mathbf{B}},\mathbf{p}}) \arrow[l, "(2)"', "\mathbf{H}^{-1}\cdot"]
\end{tikzcd}$$

Figure 4.4: Schematic outline of the proof of Theorem 4.2.6.

is *minor-generic* in the sense of Definition 4.4.2. Applying the coordinate change $\mathbf{H} = (H_1, \ldots, H_n)$ to $(\mathbb{P}^2)^n$ reduces the study of $I(\bar{\mathbf{q}})$ for the structured arrangement $\bar{\mathbf{Q}}$ to that of $I(\bar{\mathbf{B}})$ for the generic $\bar{\mathbf{B}}$.

(2) If we apply the inverse of the coordinate change $\mathbf{H}^{-1} = (H_1^{-1}, \ldots, H_n^{-1})$ from step (1) to $(\mathbb{P}^2)^n$, we can specialize from $I(\Gamma_{\bar{\mathbf{B}},\mathbf{p}})$ to $I(\Gamma_{\bar{\mathbf{q}},\mathbf{p}})$:

$$\bar{B}_j(A) = p_j \quad \Longleftrightarrow \quad H_j^{-1}\bar{B}_j(A) = H_j^{-1}p_j \quad \Longleftrightarrow \quad \bar{Q}_j(A) = H_j^{-1}p_j.$$

(3) **Show equality of focal and vanishing ideals in the generic case.** By the previous two steps, it is sufficient to show $I_{\text{foc}}(\bar{\mathbf{B}}) = I(\Gamma_{\bar{\mathbf{B}},\mathbf{p}})$. We establish this using Gröbner bases, as described in Section 4.4.2.

(4) **Show equality of focal and vanishing ideals in the structured case.** Combine steps (1)–(3).

For the first step in the proof outline, we need the following definition.

Definition 4.4.2. *We say the hypercamera arrangement* $\bar{\mathbf{B}} = (\bar{B}_1, \ldots, \bar{B}_n) \in (\mathbb{P}^{35})^n$ *is* minor-generic *if all* 12×12 *minors of the* $12 \times 3n$ *matrix* $\left(\bar{B}_1^\top \mid \cdots \mid \bar{B}_n^\top\right)$ *are nonzero.*

This is a direct analogue of the genericity condition in Theorem 4.2.3, part (1). We also need the following result. Let $\mathbb{F}$ be a field, and consider s matrices $A_1, \ldots, A_s \in \mathbb{F}^{M \times N}$. We say $A_1, \ldots, A_s$ are *rowspan-uniform* if, for any subset $S \subset [s]$ of size at least M/N, we have

$$\sum_{i \in S} \operatorname{rowspan}(A_i) = \mathbb{F}^N. \tag{4.30}$$

Lemma 4.4.3. *If $A_1, \dots, A_s \in \mathbb{F}^{M \times N}$ are rowspan-uniform, then there exists a dense Zariski-open set of matrices $(H_1, \dots, H_s) \in \mathrm{GL}(\mathbb{F}^M)^s$ such that the maximal $n \times n$ minors of the the $sM \times N$ matrix*

$$\begin{pmatrix} H_1 A_1 \\ \hline \vdots \\ \hline H_s A_s \end{pmatrix} \tag{4.31}$$

are all nonzero.

We leave the proof of this result to Section 4.7. This result is a direct generalization of [2, Lemma 3.6], in the setting of triangulation. In our setting of resectioning, we take $(M, N) = (3, 12)$, and deduce that we can transform the arrangement $\bar{\mathbf{Q}}$ for suitably generic $\bar{\mathbf{q}}$ to a minor-generic arrangement $\bar{\mathbf{B}}$ using the following result.

Lemma 4.4.4. *Suppose that $\bar{\mathbf{q}} \in (\mathbb{P}^3)^n$ is a point arrangement such that no four points are coplanar. Then $\bar{\mathbf{Q}}$ is rowspan-uniform.*

Proof. For any subset $S \subset [n]$ of size at least 4, we must show

$$\sum_{j \in S} \mathrm{rowspan}(\bar{Q}_j) = \mathbb{C}^{12}.$$

Noting the compatible direct-sum decompositions

$$\mathbb{C}^{12} \simeq \mathbb{C}^4 \oplus \mathbb{C}^4 \oplus \mathbb{C}^4,$$
$$\mathrm{rowspan}(\bar{Q}_j) \simeq \mathrm{rowspan}(\bar{q}_j^\top) \oplus \mathrm{rowspan}(\bar{q}_j^\top) \oplus \mathrm{rowspan}(\bar{q}_j^\top),$$

it suffices to observe that any set of four elements from the set $\{\bar{q}_j\}_{j \in S}$ span $\mathbb{P}^3$, from our assumption that such a set is noncoplanar. □

Lemma 4.4.4 gives us a geometric interpretation of when $\bar{\mathbf{Q}}$ is minor-generic. Algebraically, this condition is precisely what we need to obtain the Gröbner basis of Theorem 4.2.6 via a standard specialization argument. This is the focus of the next subsection.

4.4.2 Gröbner basis tools

To realize $I(\bar{\mathbf{q}})$ as the specialization of an ideal that is independent of $\bar{\mathbf{q}}$, we could replace the arrangement $\bar{\mathbf{Q}}$ with $\mathbf{B} = (B_1, \dots, B_n)$, where

$$B_i = \begin{bmatrix} B_i[1,1] & B_i[1,2] & B_i[1,3] & B_i[1,4] & B_i[1,5] & B_i[1,6] & B_i[1,7] & B_i[1,8] & B_i[1,9] & B_i[1,10] & B_i[1,11] & B_i[1,12] \\ B_i[2,1] & B_i[2,2] & B_i[2,3] & B_i[2,4] & B_i[2,5] & B_i[2,6] & B_i[2,7] & B_i[2,8] & B_i[2,9] & B_i[2,10] & B_i[2,11] & B_i[2,12] \\ B_i[3,1] & B_i[3,2] & B_i[3,3] & B_i[3,4] & B_i[3,5] & B_i[3,6] & B_i[3,7] & B_i[3,8] & B_i[3,9] & B_i[3,10] & B_i[3,11] & B_i[3,12] \end{bmatrix}, \tag{4.32}$$

thereby introducing $36n$ new indeterminates. Alternatively, we could replace $\bar{\mathbf{Q}}$ with the symbolic arrangement $\mathbf{B}^\star = (B_1^\star, \dots, B_n^\star)$, where

$$B_i^\star = \begin{bmatrix} B_i^\star[1,1] & B_i^\star[1,2] & B_i^\star[1,3] & B_i^\star[1,4] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_i^\star[2,1] & B_i^\star[2,2] & B_i^\star[2,3] & B_i^\star[2,4] & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B_i[3,1]^\star & B_i^\star[3,2] & B_i^\star[3,3] & B_i^\star[3,4] \end{bmatrix}, \tag{4.33}$$

for a total of $12n$ new indeterminates. For either $\mathbf{B}$ or $\mathbf{B}^\star$ and for each k with $6 \le k \le m$, we are interested in the determinants of the matrices

$$(\mathbf{B} \,|\, \mathbf{p})[\mathbf{r}]_\sigma = \begin{bmatrix} B_{\sigma_1}[\mathbf{r}_1, :] & p_{\sigma_1}[\mathbf{r}_1] & 0 & \dots & 0 \\ B_{\sigma_2}[\mathbf{r}_2, :] & 0 & p_{\sigma_2}[\mathbf{r}_2] & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ B_{\sigma_k}[\mathbf{r}_k, :] & 0 & \dots & 0 & p_{\sigma_k}[\mathbf{r}_k] \end{bmatrix}, \tag{4.34}$$

$$(\mathbf{B}^\star \,|\, \mathbf{p})[\mathbf{r}]_\sigma = \begin{bmatrix} B^\star_{\sigma_1}[\mathbf{r}_1, :] & p_{\sigma_1}[\mathbf{r}_1] & 0 & \dots & 0 \\ B^\star_{\sigma_2}[\mathbf{r}_2, :] & 0 & p_{\sigma_2}[\mathbf{r}_2] & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ B^\star_{\sigma_k}[\mathbf{r}_k, :] & 0 & \dots & 0 & p_{\sigma_k}[\mathbf{r}_k] \end{bmatrix}, \tag{4.35}$$

where $\mathbf{r} = (\mathbf{r}_1, \dots, \mathbf{r}_n) \subset [3]^n$,

$\#\mathbf{r}_1 + \dots + \#\mathbf{r}_k = k$,

$\sigma = \{\sigma_1, \dots, \sigma_k\} \subset [n]$.

Upon specializing $\mathbf{B} \to \bar{\mathbf{Q}}$, or $\mathbf{B}^\star \to \bar{\mathbf{Q}}$, the respective determinants of (4.34) or (4.35) specialize to the same k-focal.

Proposition 4.4.5. *Equip either ring* $\mathbb{C}[\mathbf{B}, \mathbf{p}]$ *or* $\mathbb{C}[\mathbf{B}^\star, \mathbf{p}]$ *with the* $\mathbb{Z}^{2n}$*-grading defined on generators by* $\deg(B_i[j,k]) = \deg(B_i^\star[j,k]) = e_i$ *and* $\deg(p_i[j]) = e_{n+i}$. *The polynomial* $\det(\mathbf{B} \,|\, \mathbf{p})[\mathbf{r}]_\sigma$

is homogeneous of multidegree

$$\sum_{i=1}^{k} (\#\mathbf{r}_i - 1) e_{\sigma_i} + e_{n+\sigma_i}. \tag{4.36}$$

The same is true for $\det(\mathbf{B}^\star \,|\, \mathbf{p})[\mathbf{r}]_\sigma$, *provided it is nonzero. When* $k > 12$, *we have* $\mathbf{r}_j = \{l\}$ *for some* $j \in [k]$, $l \in [3]$, *and hence*

$$\begin{aligned} \det(\mathbf{B} \,|\, \mathbf{p})[\mathbf{r}]_\sigma &= p_{\sigma_j}[l] \cdot \det(\mathbf{B} \,|\, \mathbf{p})[\mathbf{r}_1, \ldots, \widehat{\mathbf{r}_j}, \ldots, \mathbf{r}_k]_{\sigma \setminus \{j\}}, \\ \det(\mathbf{B}^\star \,|\, \mathbf{p})[\mathbf{r}]_\sigma &= p_{\sigma_j}[l] \cdot \det(\mathbf{B}^\star \,|\, \mathbf{p})[\mathbf{r}_1, \ldots, \widehat{\mathbf{r}_j}, \ldots, \mathbf{r}_k]_{\sigma \setminus \{j\}}. \end{aligned} \tag{4.37}$$

Proof. The argument is nearly identical to [1, Proposition 3.3]. The multidegree formula (4.36) follows from a calculation analagous to that already given in (4.11). Since $12 = \sum_{i=1}^{k} (\#\mathbf{r}_i - 1)$, it follows that at most 12 of the sets $\mathbf{r}_i$ can contain more than one element. If some $\mathbf{r}_j$ is a singleton, the factorization (4.37) follows by Laplace expansion. □

We now define four auxiliary ideals.

Definition 4.4.6. *The ideals* $I_{6..12}(\mathbf{B}), I_m(\mathbf{B}) \subset \mathbb{C}[\mathbf{B}, \mathbf{p}]$ *are those which are generated by all determinants of* (4.34) *for all* k, *respectively, in the ranges* $6 \le k \le 12$, *and* $k = m$. *Similarly,* $I_{6..12}(\mathbf{B}^\star), I_m(\mathbf{B}^\star) \subset \mathbb{C}[\mathbf{B}^\star, \mathbf{p}]$ *are generated by all determinants of* (4.35).

Each of the four auxiliary ideals in Definition 4.4.6 is useful for different reasons. For example, $I_m(\mathbf{B})$ and $I_m(\mathbf{B}^\star)$ are the ideals of maximal minors of a *sparse generic matrix* whose nonzero entries are distinct indeterminates. Thus, $I_m(\mathbf{B})$ and $I_m(\mathbf{B}^\star)$ belong to the class of *sparse determinantal ideals*, whose structure has been analyzed in several previous works [4, 33]. Most relevant to our work is the result of [4] which directly implies that the m-focals form a universal Gröbner basis for either of these ideals.

For the other two ideals $I_{6..12}(\mathbf{B})$, $I_{6..12}(\mathbf{B}^\star)$, we do not know whether or not the focals form universal Gröbner bases. However, Proposition 4.4.7 shows that they *do* form Gröbner bases for a class of product orders that allows us to make the necessary specialization argument.

We recall that a *product order* on $\mathbb{C}[\mathbf{B}, \mathbf{p}]$ with $\mathbf{B} < \mathbf{p}$ is a monomial order defined by comparing monomials first with some fixed monomial order in $\mathbf{p}$, then breaking any ties with some other monomial order in $\mathbf{B}$.

Proposition 4.4.7.

1. *The set* $G = \{\det(\mathbf{B} \,|\, \mathbf{p})[\mathbf{r}]_\sigma \mid 6 \le \#\sigma \le 12\}$ *forms a Gröbner basis for the ideal* $I_{6..12}(\mathbf{B})$ *for any product order with* $\mathbf{B} < \mathbf{p}$.
2. *The set* $G^\star = \{\det(\mathbf{B}^\star \,|\, \mathbf{p})[\mathbf{r}]_\sigma \mid 6 \le \#\sigma \le 12\}$ *forms a Gröbner basis for the ideal* $I_{6..12}(\mathbf{B}^\star)$ *for any product order with* $\mathbf{B}^\star < \mathbf{p}$.

We provide a proof in Section 4.7. A specialization argument applied to the two parts of Proposition 4.4.7 gives, respectively, the two parts of Lemma 4.4.8 below.

Lemma 4.4.8. *For any monomial order on* $\mathbb{C}[\mathbf{p}]$, *the following hold:*

1. *Let* $\bar{\mathbf{B}}$ *be a specialization of* $\mathbf{B}$ *such that the* $\bar{\mathbf{B}}$ *is minor generic. Then the specialized* $6 - 12$ *focals form a Gröbner basis for the ideal they generate.*
2. *Let* $\bar{\mathbf{Q}}$ *be a specialization of* $\mathbf{B}^\star$ *derived from a point arrangement* $\bar{\mathbf{q}} \in (\mathbb{P}^3)$ *with no four points coplanar. Then the specialized* $6 - 12$ *focals form a Gröbner basis.*

Proof. For part (1), let $<$ be any product order with $\mathbf{B} < \mathbf{p}$. Then

$$\text{in}_< \left(\sum_{p^{\alpha_1} < \ldots < p^{\alpha_k}} g_{\alpha_i}(\mathbf{B}) \mathbf{p}^{\alpha_i} \right) = \text{in}_<(g_{\alpha_k}(\mathbf{B})) p^{\alpha_k}. \tag{4.38}$$

Standard specialization results for Gröbner bases with respect to product orders [17, Theorem 2, §4.7] imply that the $\bar{\mathbf{B}}$ specialized $6 - 12$ focals form a Gröbner basis if each coefficient $g_{\alpha_k}(\bar{\mathbf{B}})$ is nonzero. Each of these coefficients is a 12×12 minor of $(\bar{B}_1^\top \mid \cdots \mid \bar{B}_n^\top)$. By minor-genericity, none of these coefficients vanish.

Similarly, for part (2), consider any product order $<$ with $\mathbf{B}^* < \mathbf{p}$. In this case, the nonzero coefficients $g_{\alpha_k}(\mathbf{B}^\star)$ are always products of three 4×4 determinants,

$$\prod_{i=1}^{3} \det \begin{bmatrix} B^*_{j_{i,1}}[i,1] & B^*_{j_{i,1}}[i,2] & B^*_{j_{i,1}}[i,3] & B^*_{j_{i,1}}[i,4] \\ B^*_{j_{i,2}}[i,1] & B^*_{j_{i,2}}[i,2] & B^*_{j_{i,2}}[i,3] & B^*_{j_{i,2}}[i,4] \\ B^*_{j_{i,3}}[i,1] & B^*_{j_{i,3}}[i,2] & B^*_{j_{i,3}}[i,3] & B^*_{j_{i,3}}[i,4] \\ B^*_{j_{i,4}}[i,1] & B^*_{j_{i,4}}[i,2] & B^*_{j_{i,4}}[i,3] & B^*_{j_{i,4}}[i,4] \end{bmatrix}. \tag{4.39}$$

Our noncoplanarity assumption implies that the $\mathbf{B}^\star \to \bar{\mathbf{Q}}$ specialization of (4.39) is nonzero. $\square$

Finally, we have the following result on the vanishing ideal of $\Gamma_{\bar{\mathbf{B}},\mathbf{p}}$ when $\bar{\mathbf{B}}$ is a minor-generic hypercamera arrangement. See Section 4.7 for the proof.

Proposition 4.4.9. *For a minor-generic hypercamera arrangement* $\bar{\mathbf{B}}$, *we have that*

$$I(\Gamma_{\bar{\mathbf{B}},\mathbf{p}}) = I(\bar{\mathbf{B}}).$$

4.4.3 Completing the proof

Using the results of the previous sections, we may complete the proof of Theorem 4.2.6, following the overall structure presented in Figure 4.4.

We first prove the statement for $m = 1$ camera. Let $\bar{\mathbf{q}} \in (\mathbb{P}^3)^m$ be a point arrangement with no four points coplanar. Lemma 4.4.4 then implies that the hypercamera arrangement $\bar{\mathbf{Q}}$ is rowspan-uniform, and thus Lemma 4.4.3 implies there exist coordinate changes in the images, $\mathbf{H} = (H_1, \ldots, H_m) \in (\mathrm{PGL}_3)^m$, such that the arrangement $\bar{\mathbf{B}} = (H_1\bar{Q}_1, \ldots, H_m\bar{Q}_m)$ is minor-generic. Noting

$$\begin{pmatrix} \bar{Q}_1 & p_1 & & \\ \vdots & & \ddots & \\ \bar{Q}_n & & & p_n \end{pmatrix} = \begin{pmatrix} H_1^{-1} & & \\ & \ddots & \\ & & H_n^{-1} \end{pmatrix} \begin{pmatrix} \bar{B}_1 & H_1p_1 & & \\ \vdots & & \ddots & \\ \bar{B}_n & & & H_np_n \end{pmatrix}, \tag{4.40}$$

we define the isomorphism of multigraded rings

$$\begin{aligned} L_{\mathbf{H}} : \mathbb{C}[\mathbf{p}] &\to \mathbb{C}[\mathbf{p}] \\ p_i &\to H_i p_i, \end{aligned}$$

and observe that

$$I(\bar{\mathbf{q}}) = L_{\mathbf{H}}(I(\bar{\mathbf{B}})).$$

To see this, take any focal $f \in I(\bar{\mathbf{q}})$. Just as in the proof of Lemma 4.4.3, corresponding minor of the focal matrix on the left of Equation (4.40) may written as a $\mathbb{C}$-linear combination of focals for the arrangement $\bar{\mathbf{B}}$. Hence the inclusion $I(\bar{\mathbf{q}}) \subset L_{\mathbf{H}}(I(\bar{\mathbf{B}}))$ holds, and the reverse follows similarly. Thus, we have

$$\begin{aligned} I(\bar{\mathbf{q}}) &= L_{\mathbf{H}}(I(\bar{\mathbf{B}})) \\ &= L_{\mathbf{H}}(I(\Gamma_{\bar{\mathbf{B}},\mathbf{p}})) && \text{(Proposition 4.4.9)} \\ &= I(\Gamma_{\bar{\mathbf{Q}},\mathbf{p}}) \\ &= I(\Gamma_{\bar{\mathbf{q}},\mathbf{p}}). && \text{(Proposition 4.2.7)} \end{aligned}$$

Thus the focals generate $I(\Gamma_{\bar{\mathbf{q}},\mathbf{p}})$. Moreover, Lemma 4.4.8 part (2) implies that they form a universal Gröbner basis, which completes the proof when $m = 1$.

Finally, if $m > 1$, it suffices to observe that $\Gamma^{m,n}_{\bar{\mathbf{q}},\mathbf{p}}$ is the direct product of varieties $\Gamma_{\bar{\mathbf{q}},\mathbf{p}}$, and hence the vanishing ideals sum. Moreover, two k-focals corresponding to different factors have disjoint support in $\mathbb{C}[\mathbf{p}]$, so their S-polynomials reduce to zero for any term order, and we may conclude that the focals form a universal Gröbner basis for any number of cameras.

4.5 Optimal single-camera resectioning

The results of Section 4.3 express a duality principle for the exact versions of the camera resectioning and triangulation problems. A consequence of this duality is that, in a certain sense, resectioning and triangulation are equivalent problems. However, we should be mindful that this equivalence holds in an idealized setting which assumes that the pinhole camera is exact and there is no measurement noise. In practice, neither of these assumptions hold.

In this section, we fix a generic point arrangement $\bar{\mathbf{q}}$ and consider $\Gamma^{1,n}_{\bar{\mathbf{q}},\mathbf{p}} \subset (\mathbb{P}^2)^n$ intersected with the affine chart where $p_i[3] \neq 0$ for all $1 \leq i \leq n$. We denote this affine variety by $X_{\bar{\mathbf{q}},n}$. In other words, for a point arrangement $\bar{\mathbf{q}} \in (\mathbb{P}^3)^n$ such that no four points are coplanar, the affine variety $X_{\bar{\mathbf{q}},n}$ is, by Proposition 4.2.7, equal to the closed image of the rational map

$$\begin{aligned} \psi_{\bar{\mathbf{q}},n} : \mathbb{P}^{11} &\dashrightarrow \mathbb{C}^{2n} \\ A &\mapsto \left(\frac{A[1,:]\bar{q}_1}{A[3,:]\bar{q}_1}, \frac{A[2,:]\bar{q}_1}{A[3,:]\bar{q}_1}, \ldots, \frac{A[1,:]\bar{q}_n}{A[3,:]\bar{q}_n}, \frac{A[2,:]\bar{q}_n}{A[3,:]\bar{q}_n} \right). \end{aligned} \tag{4.41}$$

In the resectioning problem, we are given world points $\bar{\mathbf{q}} = (\bar{q}_1, \ldots, \bar{q}_n) \in (\mathbb{P}^3)^n$ and pixel values of n corresponding image points, $(\tilde{u}_i, \tilde{v}_i)$ for $i = 1, \ldots, n$. We denote the vector of image measurement data by $\tilde{d}_{uv} = (\tilde{u}_1, \ldots, \tilde{v}_n) \in \mathbb{C}^{2n}$. In practice, $\tilde{d}_{uv}$ and $\bar{\mathbf{q}}$ are both defined over the real numbers. Our task is to recover a camera A such that

$$\psi_{\bar{\mathbf{q}},n}(A) = \tilde{d}_{uv}. \tag{4.42}$$

In an idealized setting, the pinhole model is exact and there is no measurement noise. Hence, we can recover A by computing the kernel of the n-focal matrix, and we expect a unique solution as soon as $n \geq 6$. This is the basis of the so-called “5.5-point" minimal solver.

In practice, the pinhole model is *not* exact and there *is* measurement noise. Thus, for $n \geq 6$, we should expect $\tilde{d}_{uv} \notin X_{\bar{\mathbf{q}},n}$, meaning that no solution to (4.42) can exist. However, we can still consider the following optimization problem:

$$L_{\tilde{d}_{uv}}(u_1, v_1, \ldots, u_n, v_n) = \sum_{i=1}^{n} (u_i - \tilde{u}_i)^2 + (v_i - \tilde{v}_i)^2 \quad \text{s.t.} \quad (u_1, \ldots, v_n) \in X_{\bar{\mathbf{q}},n}. \tag{4.43}$$

This is essentially the formulation of the optimal resectioning problem that is used in Hartley and Zisserman’s classic text [38][§7.2]. The only minor difference, implicit in their formulation, is that our feasible set $X_{\bar{\mathbf{q}},n}$ differs from theirs by a set of measure zero picked up through Zariski closures. Similar formulations, which make the camera matrix explicit, appear in other sources, eg. in work of Cifuentes [11, Example 6.5] who studied sums-of-squares relaxations of this problem. Hartley and Zisserman refer to the squared Euclidean loss function $L_{\tilde{d}_{uv}}$ as the *geometric error*, and suggest using local methods like Levenberg-Marquardt to optimize it. Here, we address the complexity of computing the *global minimum* of (4.43).

We recall the notion of the *Euclidean distance degree* of an affine variety, [26, §2]. For $X_{\bar{\mathbf{q}},n}$, we denote this quantity by $\mathrm{ED}(X_{\bar{\mathbf{q}},n})$. Given a generic data point $\tilde{d}_{uv}$, this is the number of critical points of the squared Euclidean loss $L_{\tilde{d}_{uv}}$ restricted to the smooth locus of $X_{\bar{\mathbf{q}},n}$.

Conjecture 4.5.1. *For all $n \geq 6$ and generic $\bar{\mathbf{q}} \in (\mathbb{P}^3)^n$, we have*

$$\mathrm{ED}(X_{\bar{\mathbf{q}},n}) = (80/3)n^3 - 368n^2 + (5068/3)n - 2580. \tag{4.44}$$

We return to Example 4.2.8, to verify the simplest case of this conjecture.

Example 4.5.2. *Consider the resectioning hypersurface* $H(u_1, \ldots, v_6) = 0$*, ie.* (4.13) *in the chart*

$$p_1[3] = p_2[3] = p_3[3] = p_4[3] = p_5[3] = p_6[3] = 1. \tag{4.45}$$

The affine variety $X_{\bar{\mathbf{q}},6} \subset \mathbb{C}^{12}$ *is in fact the cone over a projective variety in* $\mathbb{P}^{11}$*. This can be seen from the determinantal representation of* H *in* (4.15)*. It follows that* $X_{\bar{\mathbf{q}},n}$ *is singular. More precisely, the singular locus of* $X_{\bar{\mathbf{q}},n}$ *has dimension* 9*.*

Working over the finite field $\mathbb{F} = \mathbb{Z}_{32003}$*, we may verify Conjecture 4.5.1 with symbolic computation using the computer algebra system Macaulay2* [34]*. To do so, we draw a* $\mathbb{F}$*-valued point configuration* $\bar{\mathbf{q}} \in (\mathbb{P}^3)^6$ *and data vector* $\tilde{d}_{uv} \in \mathbb{F}^{12}$ *uniformly at random. The critical points of* (4.43) *correspond to points* $(u_1, \ldots, v_6) \in X_{\bar{\mathbf{q}},6}$ *such that*

$$\operatorname{rank} \begin{bmatrix} u_1 - \tilde{u}_1 & \cdots & v_6 - \tilde{v}_6 \\ \frac{\partial H}{\partial u_1} & \cdots & \frac{\partial H}{\partial v_6} \end{bmatrix} \leq 1. \tag{4.46}$$

To remove the singular points on $X_{\bar{\mathbf{q}},6}$ *which cause rank-deficiency in* (4.46)*, it is sufficient take the ideal generated by the* 2×2 *minors of this matrix and* $H(u_1, \ldots, v_6)$ *and compute its ideal quotient with respect to the ideal* $\langle \frac{\partial H}{\partial u_1}, \frac{\partial H}{\partial v_1} \rangle$*. The result of this operation is a zero-dimensional ideal of degree* 68*. Moreover, we may compute that the vanishing locus of this ideal consists of* 68 *distinct, nonsingular points on* $X_{\bar{\mathbf{q}},6}$*. The number* 68 *may be seen as quantifying the intrinsic algebraic difficulty of solving the constrained optimization problem* (4.43)*. This is further reinforced by heuristically computing the Galois/monodromy group of this problem, as in* [30]*, which reveals the full symmetric group* S_{68}*.*

Our conjectural formula (4.44) is reminiscent of recent results characterizing the Euclidean distance degree of the *affine multiview variety* $X_{\bar{\mathbf{A}},m}$. This can be defined by taking analagous affine charts on the multiview variety $\Gamma^{m,1}_{\bar{\mathbf{A}},\mathbf{p}}$. Using a topological formula for the ED-degree of a smooth variety, Maxim, Rodriguez, and Wang [53] proved

$$\operatorname{ED}(X_{\bar{\mathbf{A}},m}) = (9/2)m^3 - (21/2)m^2 + 8m - 4. \tag{4.47}$$

We compare this formula with ours in Table 4.1. We confirmed the entries of this table using numerical monodromy heuristics [27], using both the implementations provided in Macaulay2 [34] and Julia [7]. For these computations, it is advantageous to use the rational parametrization (4.41) instead of the implicit focal constraints in Theorem 4.2.6.

A surprising aspect of Conjecture 4.5.1 is that $\mathrm{ED}(X_{\bar{\mathbf{q}},n})$ is a polynomial of degree 3 in n. On the other hand, if we were to apply the methods of [53] to computing the affine ED-degree of the variety $\Gamma^{n,1}_{\mathbf{B},\mathbf{q}}$ associated to a *generic* hypercamera arrangement $\bar{\mathbf{B}} \in (\mathbb{P}^{11})$, this would give instead a polynomial of degree 11. This highlights some special properties of the hypercamera arrangement $\bar{\mathbf{Q}}$, and provides contrast with the results of previous sections. One explanation for this contrast is the fact that the projective coordinate changes used in Theorems 4.3.4 and 4.3.5 do not preserve the Euclidean distance. For similar reasons, the affine ED degree of the reduced resectioning variety, which is the same as $\mathrm{ED}(X_{\bar{\mathbf{A}},m})$, appears to be unrelated to that of the general resectioning variety.

We close this section by noting one immediate obstacle to proving Conjecture 4.5.1. As already seen in Example 4.5.2, the variety $X_{\bar{\mathbf{q}},n}$ for generic data $\bar{\mathbf{q}}$ is *not smooth* for any $n \geq 6$. This contrasts with the case of $X_{\bar{\mathbf{A}},m}$, which is smooth for a sufficiently generic arrangement of $m \geq 3$ cameras $\bar{\mathbf{A}}$. Thus, to prove (4.47) with similar techniques, the basic Euler characteristic formulas valid in the smooth case would need to be replaced by their singular counterparts involving Euler obstruction functions, eg. [54, Theorem 1.3].

4.6 Conclusion

In summary, our work takes several first steps in studying the resectioning problem for general projective cameras from the algebro-geometric perspective, with a focus on Gröbner bases, Carlsson-Weinshall duality, and Euclidean distance optimization. Our discoveries provide many parallels with the already well-studied multiview ideals associated with the triangulation problem. Still, many open questions remain.

In this paper, we considered resectioning in the setting of general projective cameras. Returning to the classical P3P problem [35], it would be worthwhile to carry out a parallel study in the setting of *Euclidean cameras*, as proposed in [1, §8.3, Q1]. In view of Theorem 4.2.3 parts (2)–

m / n	$\mathrm{ED}(X_{\bar{\mathbf{A}},m})$	$\mathrm{ED}(X_{\bar{\mathbf{q}},n})$
2	6	—
3	47	—
4	148	—
5	336	—
6	638	68
7	1081	360
8	1692	1036
9	2498	2256
10	3526	4180
11	4803	6968
12	6356	10780
13	8212	15776
14	10398	22116
15	12941	29960

Table 4.1: Euclidean distance degrees for optimal triangulation from m generic cameras (middle column) and optimal resectioning from n 3D points (right.)

(3), it is natural to ask: are all k-focals for $6 \leq k \leq 12$ are needed to generate $I_m(\bar{\mathbf{q}})$ under the noncoplanarity assumption of Theorem 4.2.6? What can we say about $I_m(\bar{\mathbf{q}})$ if this noncoplanarity assumption is relaxed? Using the reduced atlas developed Section 4.3 to answer more of the open questions in [1, §8] is yet another interesting avenue to pursue. Our focus on resectioning for linear maps $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ was motivated by computer vision. However, it would make just as much sense to study resectioning varieties in the context of general projections $\mathbb{P}^N \dashrightarrow \mathbb{P}^M$, [50], or even matrix multiplication maps as in [1, §8.3, Q3]. Finally, we offer Conjecture 4.5.1 as a challenge in Euclidean distance degree computation.

4.7 Appendix: Miscellaneous Proofs

First, we prove Lemma 4.4.3, justifying the coordinate change $\mathbf{H}$ used to prove Theorem 4.2.6.

Proof of Lemma 4.4.3. Consider some maximal minor of the matrix (4.31). We fix the set of indices $\{\sigma_1, \ldots, \sigma_k\}$, where $1 \leq \sigma_1 < \sigma_2 < \cdots < \sigma_k \leq s$, such that at least one row is taken from the submatrix $H_{\sigma_j}A_{\sigma_j}$ when forming this minor, and let $1 \leq i_{j1}, \ldots, i_{jr_j} \leq N$ index the rows that are taken from this submatrix. We compute this minor using the multilinearity of the determinant:

$$\det\left(\begin{array}{c} H_{\sigma_1}A_{\sigma_1}[i_{11},:] \\ \hline \vdots \\ \hline H_{\sigma_k}A_{\sigma_k}[i_{kr_k},:] \end{array}\right) = \det\left(\begin{array}{c} \displaystyle\sum_{l=1}^{M} H_{\sigma_1}[i_{11},\ell]A_{\sigma_1}[\ell,:] \\ \hline \vdots \\ \hline \displaystyle\sum_{l=1}^{M} H_{\sigma_k}[i_{kr_k},\ell]A_{\sigma_k}[\ell,:] \end{array}\right)$$

$$= \sum_{1 \leq \ell_1, \ldots, \ell_M \leq M} \det\left(\begin{array}{c} A_{\sigma_1}[\ell_1,:] \\ \hline \vdots \\ \hline A_{\sigma_k}[\ell_M,:] \end{array}\right) \cdot \left(H_{\sigma_1}[i_{11},\ell_1] \cdots H_{\sigma_k}[i_{kr_k},\ell_M]\right).$$

We think of this minor as a polynomial in the entries of $(H_1, \ldots, H_s)$. Our assumption of rowspan-uniformity implies that $A_{\sigma_1}[\ell_1,:], \ldots, A_{\sigma_k}[\ell_M,:]$ form a basis of $\mathbb{F}^N$ for some choice of indices in the sum above, and hence one of the coefficients of this polynomial is nonzero. Thus, the equation

$$\det \left(\begin{array}{c} H_{\sigma_1} A_{\sigma_1}[i_{11}, :] \\ \hline \vdots \\ \hline H_{\sigma_k} A_{\sigma_k}[i_{kr_k}, :] \end{array} \right) = 0 \tag{4.48}$$

defines a hypersurface in the affine space of all k-tuples of $M \times M$ matrices $(H_1, \dots, H_s)$. Taking the union over all such hypersurfaces and those defined by $\det H_i = 0$ gives us a proper Zariski-closed set Z in this affine space. The complement of Z is an open set which satisfies the desired conclusion. $\square$

Next, to prove Proposition 4.4.7, we recall [1, Definition 3.6].

Definition 4.7.1. *Consider a polynomial $f \in \mathbb{C}[\mathbf{B}, \mathbf{p}_{\sigma_1}, \dots, \mathbf{p}_{\sigma_k}]$ which is homogeneous of degree 1 in each group of variables $\mathbf{p}_{\sigma_i} = \{p_{\sigma_i}[1], p_{\sigma_i}[2], p_{\sigma_i}[3]\}$. We say f is* well-supported *with respect to $\mathbf{p}_{\sigma_1}, \dots, \mathbf{p}_{\sigma_k}$ if, for every choice of variables $p_{\sigma_1}[i_1] \in \mathbf{p}_{\sigma_1}, \dots, p_{\sigma_k}[i_k] \in \mathbf{p}_{\sigma_k}$ such that each $p_i[i_j]$ appears in some term of f, the monomial $\prod_{j=1}^{k} p_{\sigma_j}[i_j]$ also appears in f (with nonzero coefficient in $\mathbb{C}[\mathbf{B}]$.)*

More intuitively, f is well-supported if its monomial support in $\mathbf{p}$ is as large as possible given its variable support, or if it has a dense coefficient tensor in $\mathbb{C}[\mathbf{B}]$. For a product order with $\mathbf{B} < \mathbf{p}$, well-supportedness implies that the leading terms of k-focals depend only on the relative orderings of variables within each of the groups $\mathbf{p}_1, \dots, \mathbf{p}_n$ and the ordering on $\mathbf{B}$ (cf. [1, Lemma 3.8].) When $6 \le k \le m$, an argument using Laplace expansion can be used to show that the k-focal $\det(\mathbf{B} \mid \mathbf{p})[\mathbf{r}]_\sigma$ is well-supported with respect to $\mathbf{p}_{\sigma_1}, \dots, \mathbf{p}_{\sigma_k}$.

In contrast, the k-focal $\det(\mathbf{B}^\star \mid \mathbf{p})[\mathbf{r}]_\sigma$ is *not* well-supported with respect to $\mathbf{p}_{\sigma_1}, \dots, \mathbf{p}_{\sigma_k}$, since the nonzero coefficients of $\mathbf{p}$-monomials must have the form (4.39).

Proof of Proposition 4.4.7. For part (1), we construct an ascending chain of ideals

$$J_0 \subset J_1 \subset \cdots \subset J_n,$$

where $J_k = \langle G_k \rangle$, and G_k is an inductively-defined Gröbner basis with respect to the appropriate class of product orders. We take G_0 to be the set of m-focals, so that J_0 is a sparse determinantal ideal. As previously noted, [4, Proposition 5.4] implies that G_0 is a *universal* Gröbner basis. Having defined G_k for some $k \ge 0$, we define G_{k+1} to be the set consisting of all polynomials g such that either $g \in G_k$ and is not divisible by any entry of p_k or such that $g \notin G_k$ with $p_k[l] \cdot g \in G_k$ for some $l \in [3]$. Proposition 4.4.5 implies that each G_k may be obtained from G_0 by dividing out any entry of the matrices $p_1, \ldots, p_k$ from any m-focal containing it as a factor. When $k = n$, we obtain $G = G_n$ as the set of all k-focals for $6 \le k \le 12$. We claim that each G_k is a Gröbner basis for the appropriate class of product orders, and moreover that J_k can be expressed in terms of ideal quotients as

$$J_k = J_{k-1} : \langle p_k[1] \rangle = J_{k-1} : \langle p_k[2] \rangle = J_{k-1} : \langle p_k[3] \rangle. \tag{4.49}$$

The elements of each G_k are well-supported, and hence by [1, Corollary 3.9] the Gróbner basis property for product orders is preserved.

Having established part (1), we can prove part (2) via an argument used in the proof of [4, Proposition 5.4]. If $<$ is any product order with $\mathbf{B}^\star < \mathbf{p}$ then we can extend this to a product order with $\mathbf{B} < \mathbf{p}$ where the entries of $\mathbf{B}$ which are zero in $\mathbf{B}^\star$ are weighted last.

Let $f \in I_{6..12}(\mathbf{B}^\star)$ be nonzero, so that $f = \sum c_{\sigma,\mathbf{r}} \det(\mathbf{B}^\star \,|\, \mathbf{p})[\mathbf{r}]_\sigma$ for some coefficients $c_{\sigma,\mathbf{R}} \in \mathbb{C}[\mathbf{B}^\star, \mathbf{p}] \subset \mathbb{C}[\mathbf{B}, \mathbf{p}]$. Consider the lifted polynomial

$$\bar{f} = \sum c_{\sigma,\mathbf{r}}(\mathbf{B}, \mathbf{p}) \ \det(\mathbf{B} \,|\, \mathbf{p})[\mathbf{r}]_\sigma \in I_{6..12}(\mathbf{B}).$$

Our chosen weighting implies that $\text{in}(f) = \text{in}(\bar{f})$. Part (1) implies $\text{in}(\bar{f})$ is divisible by the leading monomial $\bar{m}_{\sigma,\mathbf{r}} = \text{in}(\det(\mathbf{B} \,|\, \mathbf{p})[\mathbf{r}]_\sigma)$ corresponding to some summand above. It follows that $\text{in}(f)$ is divisible by $m_{\sigma,\mathbf{r}} = \text{in}(\det(\mathbf{B}^\star \,|\, \mathbf{p})[\mathbf{r}]_\sigma)$. This gives part (2). $\square$

Proof of Proposition 4.4.9. Having shown the set-theoretic statement in Proposition 4.2.7, it is enough to show that the focal ideal is radical and saturated with respect to the irrelevant ideal of $(\mathbb{P}^2)^n$. Radicality follows from Lemma 4.4.8, since the initial ideal is squarefree. For saturatedness, we note that, in the notation of the previous proof, the focal ideal $I(\bar{\mathbf{B}})$ is the specialization of the ideal J_m. Using (4.49) with $k = m$ and the fact that specialization $\mathbf{B} \to \bar{\mathbf{B}}$ preserves the Gröbner basis property, it follows that $I(\bar{\mathbf{B}}) : \langle p_k[i] \rangle$ for all i and j. This in turn implies saturatedness with respect to the irrelevant ideal. $\square$

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