Strichartz estimates for wave equations with coefficients of Sobolev regularity

Matthew D. Blair

A dissertation submitted in partial fulfillment
of the requirements for the degree of

Doctor of Philosophy

University of Washington

2005

Program Authorized to Offer Degree: Mathematics

University of Washington

Graduate School

This is to certify that I have examined this copy of a doctoral dissertation by

Matthew D. Blair

and have found that it is complete and satisfactory in all respects, and that any and all revisions required by the final examining committee have been made.

Chair of the Supervisory Committee:

Hart Smith

Reading Committee:

Hart Smith

Robin Graham

Gunther Uhlmann

Date: 5/10/2005

University of Washington

Abstract

Strichartz estimates for wave equations with coefficients of Sobolev regularity

by Matthew D. Blair

Chair of the Supervisory Committee:

Professor Hart Smith
Mathematics

Wave packet techniques provide an effective method for proving Strichartz estimates on solutions to wave equations whose coefficients are not smooth. In this work, such methods are used to show Strichartz inequalities for wave equations with coefficients lying in an L^r Sobolev space of order strictly greater than $\frac{n-1}{r}+2$, n denoting the dimension of the spatial variables. In addition, a weaker family of weighted Strichartz type estimates are developed for wave equations with coefficients in an L^r Sobolev space of order $\frac{n-1}{r}+1+\alpha$, where $0<\alpha<1$.

TABLE OF CONTENTS

ACKNOWLEDGMENTS

I would like to thank the Department of Mathematics at the University of Washington for all of the opportunities they have extended to me. In particular, I am indebted to Professor Hart Smith for his introduction to the thesis problem as well as his guidance and inspiration throughout the research. Special thanks go to my family, Mom, Dad, and Marie, for all of their support and encouragement.

Chapter 1

INTRODUCTION

1.1 Overview

Strichartz estimates refer to a family of dispersive estimates on solutions to the wave equation. That is, estimates on functions $u : [-t_0, t_0] \times \mathbb{R}^n \to \mathbb{C}$ that are solutions to the following Cauchy problem

$$\begin{aligned} P(x,D)u(t,x) :&= \big(\rho(x)\partial_t^2 - A(x,D)\big)u(t,x) = F(t,x) \in L^1([t_0,t_0]; H^z(\mathbb{R}^n)) \\ u(0,x) &= f(x) \in H^{z+1}(\mathbb{R}^n) \\ \partial_t u(0,x) &= g(x) \in H^z(\mathbb{R}^n) \end{aligned} \tag{1.1.1}$$

where $P(x, D)$ denotes the wave operator and $A(x, D)$ is a second order elliptic differential operator whose second order terms are in the form $\sum_{1\leq i,j\leq n} a_{ij}(x)\partial_{x_i}\partial_{x_j}$. $H^z(\mathbb{R}^n)$ denotes the L^2 Sobolev space of order z on $\mathbb{R}^n$.

In their original formulation, Strichartz estimates appear as the following

$$\|u\|_{L_t^p([-t_0,t_0];L_x^q(\mathbb{R}^n))} \leq C\Big(\|f\|_{H^{s+1}(\mathbb{R}^n)} + \|g\|_{H^s(\mathbb{R}^n)} + \|F\|_{L_t^1([-t_0,t_0];H^s(\mathbb{R}^n))}\Big) \tag{1.1.2}$$

where $2 \leq p, q \leq \infty$, $n \geq 2$, and

$$\frac{1}{p} + \frac{n}{q} = \frac{n}{2} - (s+1), \qquad \frac{2}{p} + \frac{n-1}{q} \leq \frac{n-1}{2} \tag{1.1.3}$$

with the exception of the endpoint $(s, p, q) = (0, 2, \infty)$ in dimension $n = 3$. Such estimates are well established in the case where the coefficients of the wave operator $P(x, D)$ belong to $C^\infty(\mathbb{R}^n)$. The first estimates of this type were proved by R. Strichartz in [14], [15].

See [7], [4], and [6] and references therein for the full range of estimates.

When the assumption of C^∞ coefficients is removed the issue becomes much more delicate. The techniques used to prove Strichartz estimates for equations with smooth coefficients usually employ asymptotic methods to construct a suitable parametrix for the operator. As such, they are inapplicable when the coefficients are of limited differentiability. Therefore, one of the main challenges in dealing with equations with non-smooth coefficients lies in finding an effective parametrix which will yield the desired inequalities.

One such approach is to employ the technology of "wave packets" to construct a parametrix for the wave operator This was used by H. Smith in [10] to show Strichartz estimates for wave equations with coefficients in $C^{1,1}(\mathbb{R}^n)$ in dimensions $n = 2, 3$. More recently in [16], [17], and [18], D. Tataru was able to use the methods of the "FBI transform" to show that the full range of Strichartz inequalities hold under the weaker assumption that the coefficients of the wave operator P are time dependent and satisfy

$$\partial_t^k \rho(t,x), \partial_t^k a_{ij}(t,x) \in L^1_t([-t_0,t_0]; C^{2-k}(\mathbb{R}^n)) \qquad 0 \le k \le 2. \tag{1.1.4}$$

$C^{1,1}(\mathbb{R}^n)$ is the actually the lowest degree of continuity in x that the coefficients of P must possess to guarantee that the Strichartz estimates hold. Indeed, it was shown by counterexamples in [11] that for each $\alpha \in [0,1)$, there exist wave equations with coefficients in $C^{1,\alpha}(\mathbb{R}^n)$ for which the inequalities are not valid. This does not mean, however, that weaker estimates of this type are not available. In [18], Tataru showed that when the coefficients of the wave operator are time dependent and satisfy

$$\partial_t^k \rho(t,x), \partial_t^k a_{ij}(t,x) \in L^1_t([-t_0,t_0]; C^{1+\alpha-k}(\mathbb{R}^n)) \qquad \text{where } 0 \le \alpha \le 1 \text{ and } k = 0, 1$$

then a weaker family of Strichartz inequalities hold. Specifically, Tataru's results imply that

$$\|\langle D\rangle^{-s-\frac{\sigma}{p}} u\|_{L^p_t L^q_x([-t_0,t_0]\times\mathbb{R}^n)} \le C\big(\|f\|_{H^1(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} + \|F\|_{L^1_t L^2_x([-t_0,t_0]\times\mathbb{R}^n)}\big) \tag{1.1.5}$$

where $\sigma = \frac{1-\alpha}{3+\alpha}$ and $\langle D\rangle^z$ denotes the Fourier multiplier operator on $\mathbb{R}^n$ with symbol $\langle\xi\rangle^z := (1+|\xi|^2)^{\frac{z}{2}}$, . Such estimates are indeed weaker as they ask for more regularization of u to get an estimate on the $L^p_t L^q_x$ norm, meaning that $\frac{\sigma}{p}$ derivatives are "lost" in the estimate. However, we remark that when $\alpha = 1$ and $\sigma = 0$ this estimate actually implies the original estimate (1.1.2).

Our main purpose here is to explore opportunities for improving such results for wave operators with time independent coefficients that lie in a L^r Sobolev space of sufficiently high order with $r \in (1, \infty)$. Specifically, we aim to prove that estimates of the form (1.1.2) and (1.1.5) hold (the latter with $\sigma = 0$) when the coefficients lie in the space $L^{r,\kappa}(\mathbb{R}^n)$ (the L^r Sobolev space of order κ) with $\kappa > \frac{n-1}{r} + 2$. Thus by making some assumptions on the Sobolev regularity of the coefficients, we can obtain Strichartz estimates with no loss of derivatives even though the continuity of the coefficients is below $C^{1,1}$. We also intend to show that when the coefficients lie in the space $L^{r,\kappa}(\mathbb{R}^n) \cap Lip(\mathbb{R}^n)$ with $\kappa = \frac{n-1}{r} + 1 + \alpha$, $\alpha \in (0, 1)$, then inequalities of the type (1.1.5) hold with $\sigma = \frac{1-\alpha}{3+\alpha}$ and α taken as in the definition of κ. In both cases, this gives us estimates with a loss of fewer derivatives when compared to what would be obtained by using Sobolev embedding and applying the results above.

1.2 *The dispersive effect*

The standard family of Strichartz estimates as in (1.1.2) exploit the dispersive nature of solutions to the wave equation in order to control their norms in a mixed-L^p space. To demonstrate this, we fix a nonzero, radial Schwartz class function on $\mathbb{R}^n$, $\psi(x)$ and define $\psi_\lambda(x) = \lambda^n \psi(\lambda x)$. Consider the solution u_λ to the homogeneous Cauchy problem (1.1.1) with data $f = \psi_\lambda$, $g = 0$ and $F = 0$. Suppose also that s, p, q satisfy (1.1.3) with $p < \infty$. As λ gets large, ψ_λ becomes highly concentrated in a ball of radius λ^{-1} about the origin and

$$\lim_{\lambda \to \infty} \frac{\|\psi_\lambda\|_{L^q}}{\|\psi_\lambda\|_{\dot{H}^{s+1}}} = \lim_{\lambda \to \infty} \lambda^{\frac{1}{p}} = \infty$$

where $\dot{H}^{s+1}$ denotes the homogeneous L^2 Sobolev space of order $s+1$. Hence the estimate (1.1.2) depends on the fact that the high concentration of energy in $u_\lambda(t, \cdot)$ will disperse as time evolves, so that when $\|u_\lambda(t, \cdot)\|_{L^q}$ is averaged over time, it is possible to control the quantity by $\|\psi_\lambda\|_{H^{s+1}}$. What happens is that as t increases, the L^∞-norm of

$u_\lambda(t,\cdot)$ decreases and the function will be concentrated in a set of larger volume. This is what contributes to smaller L^q-norms of the solution in x.

A crucial matter in determining the dispersive qualities of solutions to the wave equation is the regularity of the Hamiltonian flows induced by the wave operator. That is, the flows on the cotangent bundle of $\mathbb{R}^n$, $T^*\mathbb{R}^n$, determined by

$$\frac{dx^\pm}{dt} = -H^\pm_\xi(t,x,\xi) \qquad\qquad \frac{d\xi^\pm}{dt} = H^\pm_x(t,x,\xi) \tag{1.2.1}$$

where (x,ξ) denotes coordinates on $T^*\mathbb{R}^n$ and

$$H^\pm(t,x,\xi) = \pm\sqrt{(\sum_{ij} a_{ij}(x)\xi_i\xi_j)/\rho(x)}$$

are the roots of the principal symbol of P as a quadratic in τ. Loosely speaking, the energy contained in the Cauchy data travels along these curves. Hence the dispersive qualities of solutions to the wave equation are depend largely on the geometry of the flow induced by (1.2.1).

Assume for the moment that we are working with the constant coefficient wave operator so that $\rho \equiv 1$ and $a_{ij} \equiv \delta_{ij}$. Then the solution of the system (1.2.1) with initial data (x_0,ξ_0) takes the form

$$x^\pm(t) = x_0 \mp t\frac{\xi_0}{|\xi_0|} \qquad\qquad \xi^\pm(t) \equiv \xi_0.$$

and the union of all such $x^\pm(t)$ as ξ_0 ranges over $\mathbb{R}^n \setminus \{0\}$ forms a sphere of radius t. Under this assumption, we consider u_λ, ψ_λ as above. Since the energy in ϕ_λ travels along these curves, the energy in $u_\lambda(t,\cdot)$ will be spread out evenly around the annulus

$$\{x : \big|\,|x| - t\big| \le \lambda^{-1}\} \tag{1.2.2}$$

and the function will decrease rapidly outside this set. Conservation of energy then suggests that since $u_\lambda(t,\cdot)$ is concentrated in a larger set, $\|u_\lambda(t,\cdot)\|_{L^\infty}$ should be smaller than $\|\psi_\lambda\|_{L^\infty}$. In practice, one needs much more than this to show Strichartz estimates. However, this example does shed some light on the dispersion that should occur in order for the inequalities to hold.

We now return the variable coefficient case and assume a_{ij}, ρ satisfy (1.1.4). In this case, we are able to use Gronwall's inequality to argue that for $t \in [-t_0, t_0]$ and t_0 chosen sufficiently small, the time slices of the light cone centered at 0

$$E_t := \{x(t) : (x(t), \xi(t)) \text{ solves (1.2.1) with } x(0) = 0,\ \xi(0) \in \mathbb{R}^n \setminus \{0\}\} \tag{1.2.3}$$

are convex C^1 embeddings of $\mathbb{S}^{n-1}$ with volume $\approx t^{n-1}$. In this case, $u_\lambda(t, \cdot)$ is now concentrated in the set

$$\{x : d(x, E_t) \leq \lambda^{-1}\}.$$

It can be shown that the geometry of this set is well-behaved and that the energy contained in the initial data disperses sufficiently. Hence the desired Strichartz estimates hold under these assumptions as well.

When coefficients of the wave operator lie in $L^{r,\kappa}$ with $\kappa > \frac{n-1}{r} + 2$, there is also a high degree of control over the geometry of the light cone. By Sobolev embedding, such coefficients have the property that for any $|\beta| \leq 2$,

$$\partial_x^\beta (a_{ij} \circ T)(x), \partial_x^\beta (\rho \circ T)(x) \in L^r(\mathbb{R}_{x_1}; L^\infty(\mathbb{R}^{n-1}_{x'})) \subset L^1_{loc}(\mathbb{R}_{x_1}; L^\infty(\mathbb{R}^{n-1}_{x'})) \tag{1.2.4}$$

for all $T \in O(n)$ and $x = (x_1, x')$. Furthermore, for any isometry $Sx = Tx + b$ where $T \in O(n)$, $b \in \mathbb{R}^n$

$$\|\partial_x^\beta (a_{ij} \circ S)\|_{L^1_{x_1}((-1,1);L^\infty_{x'}(-1,1)^{n-1})}, \|\partial_x^\beta (\rho \circ S)\|_{L^1_{x_1}((-1,1);L^\infty_{x'}(-1,1)^{n-1})} \leq C \tag{1.2.5}$$

for some constant C independent of S.

With this in mind, we reconsider the integral curves defined by (1.2.1). As we will see, it suffices to assume that $\|a_{ij} - \delta_{ij}\|_{Lip}, \|\rho - 1\|_{Lip} \leq \frac{1}{2}$ meaning that $A(x, D)$ is a perturbation of the Laplacian on flat space. Suppose our $(x(t), \xi(t))$ solves (1.2.1) with H^+ and the initial data $(x(0), \xi(0))$ is such that $\xi_1(0) \gg |\xi(0)'|$ with $\xi = (\xi_1, \xi') \in \mathbb{R}^n$. That is, $\xi(0)$ lies in a small cone about the positive ξ_1 axis. Since $H(\cdot, \xi) \in Lip$, we can conclude that, for t_0 chosen sufficiently small (with choice based on the Lipschitz norm of the coefficients), $\xi(t)$ lies in a slightly larger cone about the ξ_1 axis for $t \in [-t_0, t_0]$. This implies that $\frac{dx_1}{dt} = -H^+_{\xi_1}(x(t), \xi(t)) \approx -\frac{\xi_1(t)}{|\xi(t)|} < 0$ and hence the curve segment $(t, x(t), \xi(t))$, $-t_0 \leq t \leq t_0$ admits a reparametrization in x_1 satisfying the differential equations

$$\frac{dt}{dx_1} = -\frac{1}{H^+_{\xi_1}}, \qquad \frac{dx'}{dx_1} = \frac{H^+_{\xi'}}{H^+_{\xi_1}}, \qquad \frac{d\xi}{dx_1} = -\frac{H^+_x}{H^{+\prime}{}_{\xi_1}}.$$

But in light of (1.2.4) (with T taken to be the identity) and (1.2.5), we should be able to apply Gronwall's Lemma in a similar fashion to control the geometry of the set

$$\{x(t) : (x(t), \xi(t)) \text{ solves (1.2.1) with } x(0) = 0,\ \xi(0) \gg |\xi(0)'|\}$$

for $t \in [-t_0, t_0]$ and $t_0 > 0$ sufficiently small. By rotating coordinates, this line of reasoning holds equally well for any cone of comparable aperture. Therefore, we should be able to control the geometry of the entire time slice of the light cone (1.2.3). Since this suggests that solutions to the wave equation have good dispersive properties, we expect that Strichartz estimates of the form (1.1.2) and (1.1.5) should hold, the latter with $\sigma = 0$, for wave operators with these assumptions.

1.3 Weighted estimates

One of the key reasons that the standard Strichartz estimates fail to hold when the continuity of our coefficients in the x-variable drops below $C^{1,1}$ is because control over the geometry of the light cone is not guaranteed. Indeed, uniqueness of solutions to (1.2.1) is not clear and the time slices E_t may exhibit pathological geometry. These issues lie at the heart of the fore mentioned counterexamples for $C^{1,\alpha}$ coefficients. The integral curves of (1.2.1) in these situations tend to focus the energy contained in the initial data into a small set, contributing to disproportionately large $L^p_t L^q_x$ norms of the solution. In other words, there exist solutions to the wave equation in this instance which do not exhibit enough dispersion to allow the inequalities to hold.

In order to get any results for coefficients with less continuity than $C^{1,1}$, we will need to smooth the coefficients of our equation. Consider the case where $a_{ij}, \rho \in C^1(\mathbb{R}^n)$ and are independent of time. Fix a function $\phi \in \mathcal{S}(\mathbb{R}^n)$ such that $\widehat{\phi}$ is a compactly supported bump function that is identically 1 on a neighborhood of the unit ball. Set $\phi_\mu(x) = \phi(\mu x)\mu^n$ and define the following smooth approximations to a_{ij}, ρ

$$a^\mu_{ij} := a_{ij} * \phi_\mu \qquad\qquad \rho^\mu = \rho * \phi_\mu.$$

These regularized coefficients have C^1 norm uniformly bounded in μ, but the second derivatives satisfy

$$|\partial^\beta a^\mu_{ij}(x)|, |\partial^\beta \rho^\mu(x)| \leq C\mu, \qquad\qquad |\beta| = 2.$$

This now means that we can only control the geometry of the flow determined by the smoothed out Hamiltonian $H_\mu(x,\xi) = \sqrt{(\sum_{ij} a^\mu_{ij}(x)\xi_i\xi_j)/\rho^\mu(x)}$ up to time $t = \mu^{-\frac{1}{2}}$. One way to see this is to observe that by scaling the coefficients by $\mu^{-\frac{1}{2}}$ we have that $a^\mu_{ij}(\mu^{-\frac{1}{2}}x)$, $\rho^\mu(\mu^{-\frac{1}{2}}x)$ are bounded functions in the C^2 norm, and to recall that our control over the geometry of the light cone holds only up to time 1.

To prove (1.1.5) what is actually done is to localize the partial Fourier transform in x of u so that we have functions u_λ where $\operatorname{supp}(\widehat{u}_\lambda(t,\cdot)) \subset \{\xi : \frac{1}{2}\lambda \leq |\xi| \leq 2\lambda\}$ and apply our Strichartz estimates to the smoothed out wave operator $P^\mu(x,D)$ the coefficients of P^μ being a^μ_{ij}, ρ^μ, where $\mu = \lambda^{\frac{2}{3}}$. Since we can only control the geometry of flow induced by P^μ up to time $\mu^{-\frac{1}{2}}$, we partition the interval $[-t_0, t_0]$ into $\mu^{\frac{1}{2}}$ subintervals of size $2\mu^{-\frac{1}{2}}t_0$ and prove Strichartz estimates on each subinterval. This now yields an estimate of the form

$$\begin{aligned}
&\|u_\lambda\|_{L^p([-t_0,t_0];L^q(\mathbb{R}^n_x))} \\
&\qquad \leq C_1\lambda^{\frac{1}{3p}+s}\big(\|u_\lambda(0,\cdot)\|_{H^1} + \|\partial_t u_\lambda(0,\cdot)\|_{L^2} + \|P^\mu u_\lambda\|_{L^1([-t_0,t_0];L^2)}\big) \\
&\qquad\qquad \leq C_2\lambda^{\frac{1}{3p}+s}\big(\|u_\lambda(0,\cdot)\|_{H^1} + \|\partial_t u_\lambda(0,\cdot)\|_{L^2} + \|P u_\lambda\|_{L^1([-t_0,t_0];L^2)}\big),
\end{aligned}$$

the last inequality following from the fact that

$$\|P^\mu u_\lambda - P u_\lambda\|_{L^1_t([-\mu^{\frac{1}{2}}t_0,\mu^{\frac{1}{2}}t_0];L^2_x)} \leq \tilde{C}\left(\|u_\lambda(0,\cdot)\|_{H^1} + \|\partial_t u_\lambda(0,\cdot)\|_{L^2}\right). \tag{1.3.1}$$

We can now use Littlewood-Paley theory to sum over all $\lambda = 2^k$, $k \geq 0$ and get (1.1.5) in the case $\alpha = 0, \sigma = \frac{1}{3}$. The reason the coefficients are chosen to be truncated to frequencies

$\mu = \lambda^{\frac{2}{3}}$ is that it maximizes $\mu^{-\frac{1}{2}}$ under the constraint that the approximation a^{μ}_{ij} to a_{ij} need be close enough to get the error estimate (1.3.1).

The $\alpha \in (0,1)$ case is handled similarly. When the coefficients lie in the space $C^{1,\alpha}(\mathbb{R}^n)$, we lose fewer derivatives as we can truncate to fewer frequencies (this time choosing $\mu = \lambda^{\frac{2}{3+\alpha}}$) and still get a suitable estimate on $\|P_\mu u_\lambda - Pu_\lambda\|_{L^1_t L^2_x}$. In addition, one is able to use the extra regularity of the coefficients to get control over the light cone up to time $\mu^{-\frac{1}{2}(1-\alpha)}$, rather than $\mu^{-\frac{1}{2}}$. The ideas behind this "truncation/rescaling" approach appear in the work of Bahouri and Chemin (see [1] and [2]), though Tataru was the first to use this procedure to show weighted Strichartz estimates such as (1.1.5) (see [17] and [18]).

When the coefficients are in $L^{r,\kappa}$, $\kappa = \frac{n-1}{r} + 1 + \alpha$ we now have the related property that

$$\partial_x^\beta (a_{ij} \circ T)(x), \partial_x^\beta (\rho \circ T)(x) \in L^1_{loc}(\mathbb{R}_{x_1}; C^{1+\alpha-|\beta|}(\mathbb{R}^{n-1}_{x'})) \qquad |\beta| \leq 1. \qquad (1.3.2)$$

Thus if we can obtain Strichartz estimates in the case where the coefficients satisfy (1.2.4), we expect that we can use elements of the "truncation/rescaling" approach to prove estimates of the form (1.1.5) with $\sigma = \frac{1-\alpha}{3+\alpha}$ and α as in (1.3.2).

1.4 Representing the solution

In §1.2 we examined the geometric motivation behind the problem, discussing why we expect that the energy contained in the Cauchy data should disperse sufficiently to allow Strichartz estimates to hold in the case where the coefficients of P satisfy (1.2.4). However, in order to prove the estimates in this case, we need to find an effective way to represent solutions to the wave equation.

Until recently, this same issue was also a barrier to proving estimates for wave equations with $C^{1,1}$ coefficients. As alluded to above, the techniques used to represent solutions when the coefficients of P are C^∞ rely on asymptotic methods that use the smoothness explicitly and hence are of little use in the non-smooth setting. A significant breakthrough on this problem came in [10], when H. Smith was able to use the technology of "wave packets" to construct a suitable parametrix for P and then use it prove estimates under the assumption of $C^{1,1}$ coefficients in dimensions $n = 2, 3$.

Wave packets are a frame of functions on L^2 that are suitably concentrated in both space and in frequency. Their concentration properties mean that when the coefficients of P satisfy (1.1.4), the action of the wave group on each frame element is well-approximated by translating the center of the packet in space and frequency along the integral curves determined by (1.2.1). H. Smith manipulated these qualities to construct an approximate solution operator $\mathbf{e}(t)$ for P whose matrix in the wave packet frame is essentially a permutation matrix that has the effect of translating each packet along these curves up to time t. The key feature of this approximate solution operator is that it maps functions $h(x)$ to functions $(\mathbf{e}(t)h)(x)$ that lose only 1 derivative when P is applied to them instead of two. An iteration procedure can then be used to eliminate the error and represent solutions to the wave equation precisely.

This approach amounts to representing solutions to the wave equation as a superposition of these frame elements, each translated along curves determined by (1.2.1) that are parameterized by t. However, there are some crucial points in the construction of the parametrix where the boundedness of the second derivatives of the coefficients play a crucial role. Therefore, representing solutions to the wave equation when the coefficients lie in the space (1.2.4) is substantially more involved.

We thus consider a different approach to the problem. Suppose once again that the coefficients of P are Lipschitz functions with $\|a_{ij} - \delta_{ij}\|_{Lip}, \|\rho - 1\|_{Lip} \leq \frac{1}{2}$, which also satisfy (1.2.4). By rewriting the second order terms in $-P(x, D)$ as

$$a_{11}(x)\partial_1^2 + 2\sum_{j=2}^{n} a_{1j}(x)\partial_1\partial_j - (\rho(x)\partial_t^2 - \sum_{2\leq i,j\leq n} a_{ij}(x)\partial_i\partial_j)$$

we can view the wave operator as an operator that is hyperbolic in x_1 when restricted to frequencies $\{(\tau, \xi) : |\tau| \gg |(\xi_2, ..., \xi_n)|\}$ (with (τ, ξ) denoting Fourier transform variables in

t and x) as this means that the symbol of the operator is a quadratic in ξ_1 with 2 distinct real roots. Given that Strichartz estimates hold for equations hyperbolic in t when the coefficients satisfy (1.1.4) (a condition analogous to (1.2.4) with $T = I$) and that we have effective methods for representing solutions in that case, we expect that the estimates (1.1.2) should hold for u with Fourier transform supported in $\{(\tau, \xi) : |\tau| >> |(\xi_2, ..., \xi_n)|\}$.

But since we can consider any $T \in O(n)$ in (1.2.4), there is nothing special about the x_1-axis in this line of reasoning–we can repeat the argument above for any coordinate axis or for any direction in $\mathbb{R}^n$. Hence the estimates (1.1.2) should hold as long as the Fourier transform of the solution is sufficiently localized near the τ-axis. By representing solutions to the wave equation as a sum of such localized pieces we then expect that we should be able to establish Strichartz estimates for arbitrary solutions to our wave equation.

This idea will guide us in constructing a parametrix for our operator. We will write u as a finite sum of functions, with each function microlocally supported where either P is elliptic or P is "hyperbolic in x_1" under a rotation of coordinates in x. Since elliptic regularity and Sobolev embedding will provide estimates for components of the solution microlocally supported where P is elliptic, most of the work involved will be in effectively representing solutions to the wave equation whose full Fourier transform is supported in cones of the form

$$\{(\tau, \xi) : |\tau| >> |(\xi_2, ..., \xi_n)|\}.$$

This begins by constructing a pseudodifferential operator $\widetilde{P}$ that agrees with P for functions whose full Fourier transform is supported in the set above, yet is hyperbolic in x_1 rather than in t. We can then adapt the wave packet techniques in [10] to represent solutions to hyperbolic pseudodifferential equations. But since $\widetilde{P}$ is hyperbolic in x_1, we will be representing such solutions as a superposition of wave packets parameterized by x_1 rather than t.

Once this is accomplished, we are then left to use the representation to prove the desired estimates. This will boil down to showing that the wave packets parameterized by x_1 behave much like they are wave packets parameterized by t. We will then be able to use methods similar to those used in [10] again to obtain the Strichartz estimates for wave operators whose coefficients satisfy (1.2.4).

1.5 Characterizations of the Strichartz inequalities

So far we have discussed 2 forms of Strichartz estimates. They originally appeared in the style of (1.1.2), asking for a certain degree of regularity of the Cauchy data to obtain an estimate on the $L_t^p L_x^q$ norm of the solution u. In recent work concerning weighted Strichartz estimates in the low regularity setting, estimates in the form of (1.1.5) have appeared. These inequalities measure the $L_t^p L_x^q$ norm of a certain number of fractional derivatives of u and always assume the Cauchy data lies in the space $H^1 \times L^2 \times L_t^1 L_x^2$.

One is then lead to ask when weighted estimates of the form (1.1.5) imply estimates that are closer to the original style

$$\|u\|_{L_t^p([-t_0,t_0];L^q(\mathbb{R}^n))} \leq C\Big(\|f\|_{H^{s+1+\frac{\sigma}{p}}(\mathbb{R}^n)} + \|g\|_{H^{s+\frac{\sigma}{p}}(\mathbb{R}^n)} + \|F\|_{L_t^1([-t_0,t_0];H^{s+\frac{\sigma}{p}}(\mathbb{R}^n))}\Big), \quad (1.5.1)$$

which measure the $L_t^p L_x^q$ norms of u itself and require a certain degree of regularity in the Cauchy data. However, complications arise in the low regularity setting that often preclude showing estimates in this format, and hence inequalities in the style of (1.1.5) were used to avoid such problems. In short, there are some key commutator estimates that can break down when the coefficients are very rough.

We will thus explore how much regularity is needed in the coefficients for estimates of the form (1.1.5) to imply the inequalities (1.5.1). As a result, we will see that when the coefficients lie in the space $L^{r,\kappa}$, $\kappa > \frac{n-1}{r}+2$, then the inequality (1.1.5) with $\sigma = 0$ implies (1.1.2).

We will also consider these issues when u is a solution to the wave equation on a compact Riemannian manifold. That is, when $\mathbb{R}^n$ is replaced by a compact Riemannian manifold (M, g) in (1.1.1) with g a $L^{r,\kappa}$ metric with $\kappa = \frac{n-1}{r} + 1 + \alpha$, $0 < \alpha < 1$ and $A(x, D)$ is replaced by Δ_g, the Laplace-Beltrami operator induced by g. In this case, we can use the fact that Δ_g admits an eigenbasis on $L^2(M)$ and characterize the norms on the Sobolev spaces $H^z(M)$ in terms of eigenfunction expansions. This allows us to take advantage of spectral multiplier theorems for the Laplace-Beltrami operator to show that the estimates (1.1.5) on the solution in coordinate charts imply the estimates

$$\|u\|_{L_t^p([-t_0,t_0];L^q(M))} \leq C\Big(\|f\|_{H^{s+1+\frac{\sigma}{p}}(M)} + \|g\|_{H^{s+\frac{\sigma}{p}}(M)} + \|F\|_{L_t^1([-t_0,t_0];H^{s+\frac{\sigma}{p}}(M))}\Big).$$

1.6 Main Results

As discussed above, our approach here will be to use wave packet techniques as pioneered in [10], which provide an effective method for representing solutions, but do not provide the desired Strichartz estimates in dimensions $n \geq 4$. For simplicity, we will consider 3 sets of instances of (1.1.3). In dimension 2, we consider the case $p = q = 6, s = -\frac{1}{2}$. In dimension 3, we consider the case $p = q = 4, s = -\frac{1}{2}$ and the cases $p = \frac{2q}{q-6}, 6 \leq q < \infty, s = 1$. However, the arguments here should be generalizable enough to show the full range of possibilities (1.1.3) in dimensions $n = 2, 3$.

Theorem 1.6.1. *Suppose $\rho - 1, a_{ij} - \delta_{ij} \in L^{r,\kappa}(\mathbb{R}^n)$, with $\kappa > \frac{n-1}{r} + 2$. Then we have the following estimate for solutions u to the Cauchy problem* (1.1.1) *with $z = 0$*

$$\|\langle D\rangle^{-s}u\|_{L_t^pL_x^q([-t_0,t_0]\times\mathbb{R}^n)} \leq C\Big(\|f\|_{H^1(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} + \|F\|_{L_t^1L_x^2([-t_0,t_0]\times\mathbb{R}^n)}\Big) \qquad (1.6.1)$$

where one of the following hold

$$s = -\frac{1}{2} \qquad p = q = \frac{2(n+1)}{n-1} \qquad n = 2,3 \qquad (1.6.2)$$

$$s = 0 \qquad p = \frac{2q}{q-6},\ \ 6 \leq q < \infty \qquad n = 3. \qquad (1.6.3)$$

Theorem 1.6.2. *Suppose $\rho - 1, a_{ij} - \delta_{ij} \in L^{r,\kappa}(\mathbb{R}^n) \cap Lip(\mathbb{R}^n)$, with $\kappa = \frac{n-1}{r} + 1 + \alpha$, $0 < \alpha < 1$. Then we have the following estimate for solutions u to the Cauchy problem* (1.1.1) *with $z = 0$*

$$\|\langle D\rangle^{-s-\frac{\sigma}{p}}u\|_{L_t^pL_x^q([-t_0,t_0]\times\mathbb{R}^n)} \leq C\Big(\|f\|_{H^1(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} + \|F\|_{L_t^1L_x^2([-t_0,t_0]\times\mathbb{R}^n)}\Big) \quad (1.6.4)$$

where $\sigma = \frac{1-\alpha}{3+\alpha}$ and s, p, q, n either satisfy (1.6.2) *or* (1.6.3).

As stated before, we will be also able to show that Theorem 1.6.1 implies that estimates of the form (1.1.2) hold.

Corollary 1.6.3. *Suppose* $\rho - 1, a_{ij} - \delta_{ij} \in L^{r,\kappa}(\mathbb{R}^n)$*, with* $\kappa > \frac{n-1}{r} + 2$*. Then we have the following estimate for solutions* u *to the Cauchy problem* (1.1.1) *with* $z = -\frac{1}{2}$ *in dimensions* $n = 2, 3$

$$\|u\|_{L^{\frac{2(n+1)}{n-1}}([-t_0,t_0]\times\mathbb{R}^n)} \leq C\Big(\|f\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} + \|g\|_{H^{-\frac{1}{2}}(\mathbb{R}^n)} + \|F\|_{L^1([-t_0,t_0];H^{-\frac{1}{2}}(\mathbb{R}^n))}\Big).$$

Additionally, for solutions to (1.1.1) *with* $z = 0$ *we have*

$$\|u\|_{L^{\frac{2q}{q-6}}([-t_0,t_0];L^q(\mathbb{R}^n))} \leq C\Big(\|f\|_{H^1(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} + \|F\|_{L^1([-t_0,t_0];L^2(\mathbb{R}^n))}\Big)$$

with $6 \leq q < \infty$ *in dimension* $n = 3$.

Unfortunately, our methods do not handle coefficients in $L^{r,\kappa}$ with $\kappa = \frac{n-1}{r} + 1 + \alpha$ and $\alpha = 0$ or $\alpha = 1$. Sobolev embedding does not provide for the estimate (1.3.2) with $\alpha = 0$ in the former case and in the latter case it does not give us (1.2.4). When $\alpha = 0$ we are thus unable to provide any results, and when $\alpha = 1$, the best we can provide is Strichartz estimates with an arbitrarily small loss of derivatives.

We will consider 2 types of wave operators, with $A(x, D)$ either taking the form

$$A(x, D) = \sum_{1\leq i,j\leq n} \partial_i(a_{ij}(x)\partial_j) \qquad \text{or} \qquad A(x, D) = \sum_{1\leq i,j\leq n} a_{ij}(x)\partial_i\partial_j.$$

The former set of operators being called divergence operators in contrast to the latter nondivergence form. For convenience, we will assume that $\rho \equiv 1$ whenever P is in nondivergence form. We also assume the following bounds on the coefficients

$$c^{-1}|\xi|^2 \leq \sum_{i,j} a_{ij}(x)\xi_i\xi_j \leq c|\xi|^2 \qquad\qquad \rho(x) \geq c^{-1} > 0 \tag{1.6.5}$$

with c independent of $(x, \xi) \in \mathbb{R}^{2n}$.

We will occasionally write our wave operator in the form

$$P(x, D) = \rho(x)\partial_t^2 - \sum_{1\leq i,j\leq n} a_{ij}(x)\partial_i\partial_j - \sum_{j=1}^{n} b_j(x)\partial_j.$$

The convention here will be that $b_j(x) = \sum_i (\partial_i a_{ij})(x)$ when P is in divergence form and $\rho \equiv 1$, $b_j \equiv 0$ when P is in nondivergence form.

The organization is as follows, Chapter 2 details the main strategy of the paper, which focuses on obtaining estimates on components of the solution whose Fourier transform is dyadically localized and supported in regions where P can be considered a hyperbolic operator in x_1. Chapter 3 then constructs a pseudodifferential operator $\widetilde{P}$ that agrees with P on the desired components of the solution, yet is hyperbolic in x_1. Wave packet techniques are then used to represent solutions to pseudodifferential equations involving $\widetilde{P}$. Chapter 4 then uses this representation to provide the desired estimates for coefficients satisfying (1.2.4), eventually leading to a proof of Theorem 1.6.1. Chapter 5 expounds on the truncation/rescaling arguments needed to prove Theorem 1.6.2. In Chapter 6, we begin the discussion on the difference between estimates of the form (1.1.5) and (1.5.1), examining when the former imply the latter. In particular, it is in this chapter that we prove Corollary 1.6.3. We conclude in Chapter 7 with a discussion of how Strichartz estimates of the form (1.1.5) imply the estimates (1.5.1) when u is a solution to the wave equation on a compact manifold.

1.7 Notation

Unless otherwise stated, the expression $X \lesssim Y$ means that $X \leq CY$ for some constant C depending only on the dimension n and on the norms of ρ, a_{ij} as elements in the Lipschitz and $L^{r,\kappa}$ function spaces. Elements $y \in \mathbb{R}^n$ will often be denoted as $y = (y_1, y')$, so that y' denotes the last $n-1$ components of y. Also, for a function $v(t,x)$, and a subinterval $I \subset \mathbb{R}$ and a measurable set $U \subset \mathbb{R}^n$, $\|v\|_{L_t^{r_1} L_x^{r_2}}, \|v\|_{L_t^{r_1} L_x^{r_2}(I \times U)}$ will both abbreviate the norm on the Banach space $L^{r_1}(I; L^{r_2}(U))$ the former abbreviation being used when it is well understood from the context what I is.

Chapter 2

LOCALIZING THE SOLUTION

In this chapter we show the key reductions in the strategy for proving Theorems 1.6.1 and 1.6.2. After a few technical lemmas, we show that we can restrict our attention to showing estimates on compactly supported solutions to wave equations that are perturbations of the constant coefficient case. We then proceed to the main result of this chapter, which states that we can reduce the theorems to showing estimates on components of the solution dyadically localized and microlocally supported where P is "hyperbolic in x_1". That is, we reduce the problem to showing estimates on components of u where $\tau >> |(\xi_2, ..., \xi_n)|$ and $|(\tau, \xi')| \approx 2^k$.

Before we proceed with this approach we need a few technical lemmas.

Lemma 2.0.1. *Let $h \in Lip(\mathbb{R}^n)$ and R be a pseudodifferential operator determined by a symbol $R(t, x, \tau, \xi) \in S^z_{1,0}(\mathbb{R}^{n+1})$. Then for $1 < p < \infty$ the linear transformation $[R, h]$ (the commutator of R with the multiplication operator $v(x) \mapsto h(x)v(x)$) satisfies the following continuous mapping properties*

$$[R, h] : \begin{cases} L^p \to L^{p,1-z} & \text{when} \quad 0 \le z \le 1, \\ L^{p,z} \to L^{p,1} & \text{when} \quad -1 \le z \le 0 \end{cases} \tag{2.0.1}$$

$$[R, h] : \begin{cases} L^{p,z-1} \to L^p & \text{when} \quad 0 \le z \le 1, \\ L^{p,-1} \to L^{p,-z} & \text{when} \quad -1 \le z \le 0. \end{cases} \tag{2.0.2}$$

Proof. The commutator theorem of Coifman-Meyer states that if R is of order $z = 1$ then $[R, h] : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ continuously (see Section 3.6 of [19] for a discussion of this result). Thus when $z = 0$, the following map is bounded

$$\langle D \rangle^1 [R, h] = [\langle D \rangle^1 R, h] + [h, \langle D \rangle^1] R : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n).$$

This allows us to conclude that when $z = -1$,

$$\langle D\rangle^1[R,h]\langle D\rangle^1 = \langle D\rangle^1 R[h,\langle D\rangle^1] + \langle D\rangle^1[R\langle D\rangle^1, h] : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n).$$

An interpolation argument now allows us to conclude (2.0.1). By observing that

$$[R,h]^* = -[R^*, \overline{h}]$$

(2.0.2) is now an straightforward consequence of (2.0.1). □

As an immediate corollary we have that if R is a pseudodifferential operator with symbol $R(t,x,\tau,\xi) \in S^0_{1,0}(\mathbb{R}^{n+1})$ and $H^1_{t,x}$ denotes the L^2 Sobolev space of order 1 in the (t,x) variables on $\mathbb{R}^{n+1}$, then

$$[P,R] : H^1_{t,x} \to L^2_{t,x},$$

continuously with operator norm depending on the symbol estimates and on the Lipschitz norm of the coefficients of P. To see this, observe that the lemma implies

$$[\rho,R],[a_{ij},R] : H^{-1}_{t,x} \to L^2_{t,x} \qquad \text{and} \qquad [\rho,R],[a_{ij},R] : L^2_{t,x} \to H^1_{t,x}$$

with the desired operator norm properties. By continuity of differential operators in Sobolev spaces, the former mapping easily provides the result for nondivergence operators and the latter mapping does the same for divergence operators.

Lemma 2.0.2. *Let $\{T_k\}_{k=0}^{\infty}$ be a sequence of linear operators mapping $\mathcal{S}(\mathbb{R}^n)$ (the Schwartz class functions on $\mathbb{R}^n$) to $\mathcal{S}'(\mathbb{R}^n)$ (tempered distributions on $\mathbb{R}^n$). Suppose for any $\theta \in \mathbb{R}$, $\sum_k e^{ik\theta}T_k$ is a well defined mapping from $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ that extends to a continuous map from $H^z(\mathbb{R}^n)$ to $H^w(\mathbb{R}^n)$ with operator norm uniformly bounded in θ. Then there exists C depending on the T_k such that*

$$\sum_{k=0}^{\infty} \|T_k h\|^2_{H^w(\mathbb{R}^n)} \leq C\|h\|^2_{H^z(\mathbb{R}^n)}$$

for all $h \in H^z(\mathbb{R}^n)$.

Proof. Replacing the T_k by $\langle D\rangle^w T_k \langle D\rangle^{-z}$ if necessary, it suffices to proof the lemma in the case where $z = w = 0$. By Plancherel's equality we have

$$\sum_{k=0}^{\infty} \|T_k h\|^2_{L^2(\mathbb{R}^n)} \approx \int_0^{2\pi} \int_{\mathbb{R}^n} \Big|\sum_{k=0}^{\infty} e^{ik\theta}(T_k h)(x)\Big|^2 dx d\theta \leq 2\pi C\|h\|^2_{L^2(\mathbb{R}^n)}$$

as C can be taken independent of θ. This proves the claim. □

2.1 Preliminary reductions

Before localizing the solution in frequency, we show that it suffices to prove the main theorems for compactly supported solutions u and for P whose coefficients are a perturbation of the constant coefficient wave operator.

Proposition 2.1.1. *Suppose Theorems 1.6.1 and 1.6.2 hold with $t_0 = 1$ under the additional condition that $supp(u(t,\cdot)) \subset (-\frac{3}{2},\frac{3}{2})^n$ for all $t \in (-\frac{5}{4},\frac{5}{4})$ and that*

$$\sum_{1\leq i,j\leq n} \|a_{ij}-\delta_{ij}\|_{Lip} + \|\rho - 1\|_{Lip} < \varepsilon \tag{2.1.1}$$

where $\varepsilon > 0$ is sufficiently small but fixed. Then Theorem 1.6.1 and 1.6.2 hold without restriction for arbitrary $\infty > t_0 > 0$.

Proof. We first reduce to the case where the solution is compactly supported. Suppose u is a solution to (1.1.1), the support of $u(t,\cdot)$ is arbitrary, and that the coefficients of P satisfy (2.1.1) with Lipschitz norms taken over the entirety of $\mathbb{R}^n$. Let $\{\chi_k(x)\}_{k=1}^{\infty}$ be a smooth partition of unity on $\mathbb{R}^n$ such that each χ_k is supported in a cube with sidelength 3. We can take the sequence of functions so that

$$\|\partial_x^\beta \chi_k\|_\infty \leq C_\beta$$

where C_β is independent of k and each function in the sequence is a translation of the other functions. Thus by our hypotheses,

$$\|\langle D\rangle^{-s-\frac{\sigma}{p}} u\|_{L_t^p([-1,1];L^q(\mathbb{R}^n))} \leq C(\sum_{k=1}^{\infty} \|\langle D\rangle^{-s-\frac{\sigma}{p}} \chi_k u\|^2_{L_t^p([-1,1];L^q(\mathbb{R}^n))})^{\frac{1}{2}} \tag{2.1.2}$$

$$\leq C\Big(\sum_k(\|\chi_k u(0,\cdot)\|^2_{H^1(\mathbb{R}^n)} + \|\chi_k\partial_t u(0,\cdot)\|^2_{L^2(\mathbb{R}^n)} + \|P(\chi_k u)\|^2_{L_t^1([-1,1];L^2(\mathbb{R}^n))})\Big)^{\frac{1}{2}}.$$

To handle the last term we write

$$P(\chi_k u) = [P,\chi_k]u + \chi_k P u.$$

For any θ, $[P,\sum_k e^{ik\theta}\chi_k]$ is first order differential operator with coefficients in L^∞ and hence maps $H^1(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ continuously with operator norm uniform in θ. By Lemma 2.0.2,

$$\sum_k \|[P,\chi_k]u(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \lesssim \|u(t,\cdot)\|^2_{H^1(\mathbb{R}^n)}.$$

Applying Lemma 2.0.2 again along with energy estimates allows us to dominate (2.1.2) by

$$\|f\|_{H^1(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} + \|F\|_{L^1_t([-1,1];L^2(\mathbb{R}^n))}.$$

Before reducing the general problem to the case where the coefficients are a perturbation of the constant coefficient equation, we show that the inhomogeneous estimates in the conclusion of the main theorems imply some homogeneous, scale-invariant estimates. Let β_0 be a smooth compactly supported bump function identically 1 on the unit ball and let $\beta_0(D)$ denote the Fourier multiplier of order 0 determined by β_0. Also, for $z \in \mathbb{C}$, let $|D|^z$ denote the Fourier multiplier with symbol $|\xi|^z$ and let $\dot{H}^1(\mathbb{R}^n) = |D|^{-1}(L^2(\mathbb{R}^n))$ be the homogeneous Sobolev space of order 1. Observe that by Sobolev embedding, we have that

$$|D|^{-s-\frac{\sigma}{p}}\beta_0(D) : \dot{H}^1(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$$

as $s + \frac{\sigma}{p} < n(\frac{1}{2} - \frac{1}{q}) - 1$. In addition, by energy inequalities,

$$\|u\|_{L^\infty([-1,1];\dot{H}^1(\mathbb{R}^n))} \lesssim \|u(0,\cdot)\|_{\dot{H}^1(\mathbb{R}^n)} + \|\partial_t u(0,\cdot)\|_{L^2(\mathbb{R}^n)} + \|Pu\|_{L^1_t([-1,1];L^2(\mathbb{R}^n))}.$$

The conclusion of main theorems now imply the following scale invariant estimate involving homogeneous spaces with $t_0 = 1$

$$\begin{aligned}\||D|^{-s-\frac{\sigma}{p}}u\|_{L^p_tL^q_x([-t_0,t_0]\times\mathbb{R}^n)} &\lesssim \|\langle D\rangle^{-s-\frac{\sigma}{p}}(I-\beta_0)u\|_{L^p_tL^q_x} + \||D|^{-s-\frac{\sigma}{p}}\beta_0(D)u\|_{L^p_tL^q_x} \\ &\lesssim \|u(0,\cdot)\|_{\dot{H}^1} + \|\partial_t u(0,\cdot)\|_{L^2} + \|Pu\|_{L^1_t([-t_0,t_0];L^2(\mathbb{R}^n))}. \quad (2.1.3)\end{aligned}$$

We now reduce to the assumptions in (2.1.1). Suppose a_{ij}, ρ satisfy the hypotheses of Theorem 1.6.1 or Theorem 1.6.2 as well as the bounds in (1.6.5), but do not necessarily satisfy (2.1.1). By a linear change of coordinates we can assume that $\rho(0) = 1$, $a_{ij}(0) = \delta_{ij}$. Thus for $x \in (-\frac{3}{2}, \frac{3}{2})^n$ we have the following estimate for $R \leq 1$

$$\sum_{1\leq i,j\leq n}\sum_{|\alpha|\leq 1}|\partial^\alpha(a_{ij}(Rx) - \delta_{ij})| + \sum_{|\alpha|\leq 1}|\partial^\alpha(\rho(Rx) - 1)| < CR|x|.$$

where C is independent of R, x. Thus by choosing R sufficiently small, we have that the coefficients $\rho(R\cdot), a_{ij}(R\cdot)$ satisfy (2.1.1), implying that the inequalities in the conclusion of Theorem 1.6.1/Theorem 1.6.2 hold for $u(R\cdot)$. By a rescaling argument, this implies the estimate (2.1.3) above with $t_0 = R$ for compactly supported u such that $\mathrm{supp}(u(t,\cdot)) \subset (-\frac{3}{2}R, \frac{3}{2}R)^n$ for $t \in (-\frac{5}{4}R, \frac{5}{4}R)$.

Estimating $\beta_0(D)u, (I - \beta_0(D))u$ separately as before now yields the inhomogeneous estimates (1.6.1) and (1.6.4) for $t_0 = R$ and u supported in a cube of sidelength $3R$. Using a partition of unity as above now allows us to remove the assumption that u need be supported in small sets. Finally, the wave group property establishes the estimate for arbitrary $t_0 > 0$. □

Thus from now on we will assume that the coefficients satisfy

$$\sum_{1 \le i,j \le n} \|a_{ij} - \delta_{ij}\|_{Lip} + \|\rho - 1\|_{Lip} < \varepsilon \tag{2.1.4}$$

for some sufficiently small but fixed $\varepsilon > 0$. Additionally, we will assume that $t_0 = 1$ and that our solution u is supported in the cube $(-\frac{3}{2}, \frac{3}{2})^n$ for all $t \in (-\frac{5}{4}, \frac{5}{4})$. We will also assume that u solves the homogeneous Cauchy problem with $F \equiv 0$. By Duhamel's principle it is not difficult to see that the general case reduces to such an assumption.

Before we begin localizing our solution in frequency, we make a small alteration of our solution for technical reasons. Replace $u(t,x)$ by $\phi(t)u(t,x)$ where $\phi(t)$ is a smooth bump function equal to 1 on $[-1,1]$ and supported in $(-\frac{5}{4}, \frac{5}{4})$. It now makes sense to look at u as an element of a Sobolev space in $n+1$ variables over all of $\mathbb{R}^{n+1}$. Hence $H^w_{t,x}$ will denote the L^2 Sobolev space of order w on $\mathbb{R}^{n+1}$ in the (t,x) variables and H^w will denote the Sobolev space on $\mathbb{R}^n$ in only the x variables. This modification of u means that it is no longer a solution to the homogeneous problem. However, energy estimates allow us to control $[P, \phi]u$ so that we have the following inequality for all $t \in \mathbb{R}$

$$\|(Pu)(t,\cdot)\|_{L^2} \lesssim (\|f\|_{H^1} + \|g\|_{L^2}). \tag{2.1.5}$$

2.2 *Localization in frequency*

Here we discuss our main approach to the problem, which involves localizing our solution in frequency to dyadic pieces where P should be hyperbolic in x_1. Take a sequence of smooth Littlewood-Paley cutoffs $\{\beta_k\}_{k\geq 0}$, so that $\beta_k(\xi)$ is a smooth function defined on $\mathbb{R}^n$ with $\sum_k \beta_k \equiv 1$ and $\mathrm{supp}(\beta_k) \subset \{\xi \in \mathbb{R}^n : 2^{k-\frac{1}{2}} \leq |\xi| \leq 2^{k+\frac{3}{2}}\}$ when $k \neq 0$ and $\mathrm{supp}(\beta_0) \subset B_1(0)$. We can also take the sequence so that $\beta_k(\cdot) = \beta(2^{-k}\cdot)$ for $k \geq 1$ for some function $\beta \in C_0^\infty(\mathbb{R}^n)$.

Next, multiply β_0 by another smooth cutoff in τ to extend β_0 so that it is a smooth cutoff supported in the unit ball in $\mathbb{R}^{n+1}$ and one in a neighborhood of the origin in $\mathbb{R}^{n+1}$. That is, there exists $\bar{\beta}_0(\tau, \xi)$ with these properties such that $\bar{\beta}_0(0, \xi) = \beta_0(\xi)$. Treating $\bar{\beta}_0(D)$ as a Fourier multiplier on u in the (t, x) variables, $\bar{\beta}_0(D)$ is a smoothing operator. Hence by Sobolev embedding and energy estimates

$$\|\langle D\rangle^{-s}\bar{\beta}_0(D)u\|_{L_t^p L_x^q([-1,1]\times(\mathbb{R}^n))} \lesssim \|u\|_{H^1_{t,x}} \lesssim (\|f\|_{H^1} + \|g\|_{L^2}).$$

Now choose a cutoff $\Lambda^+ \in C^\infty(\mathbb{R}^{n+1} \setminus \{0\})$ such that

$$\mathrm{supp}(\Lambda^+) \subset \{(\tau, \xi) : 9^{-1}|\xi| \leq \tau \leq 9|\xi|\}$$

and is identically 1 on the slightly smaller cone $\{(\tau, \xi) : 8^{-1}|\xi| \leq \tau \leq 8|\xi|\}$. Since we already have estimates on $\bar{\beta}_0 u$ we will replace Λ^+ by $\Lambda^+(I - \bar{\beta}_0)$ so that Λ^+ is smooth on all of $\mathbb{R}^{n+1}$ and vanishes on in a neighborhood of the origin. Also, set $\Lambda^-(\tau, \xi) = \Lambda^+(-\tau, \xi)$. Our approach will be to get estimates on each of the following:

$$\Lambda^+(D)u, \Lambda^-(D)u, \text{ and } (I - \Lambda^+(D) - \Lambda^-(D) - \bar{\beta}_0(D))u.$$

The Strichartz estimates for $\Lambda^+ u, \Lambda^- u$ will require further localization. Consider a cutoff $\Omega(\xi) \in C^\infty(\mathbb{R}^n) \setminus \{0\}$ such that $\mathrm{supp}(\Omega) \subset \{\xi : \xi_1 > 200|\xi'|\}$ and is identically 1 on a slightly smaller cone. Set $\Gamma^\pm(\tau, \xi) = \Omega(\xi)\Lambda^\pm(\tau, \xi)$. By employing a family of similar cutoffs, we can represent $\Lambda^\pm u$ as a finite sum of functions supported in cones in ξ of comparable aperture, so it will suffice to show estimates for the localized $\Gamma^\pm(D)u$. Similarly, we will show them for $\Gamma^+ u$ as estimates for $\Gamma^- u$ will follow by an identical argument, so from now on we will suppress the '+' in Γ^+ and refer to it as Γ.

Set $\Gamma_k(D) = \beta_k(D) \circ \Gamma(D)$ so that $\Gamma_k u = \mathcal{F}^{-1}\{\beta_k(\xi)\Gamma(\tau, \xi)\widehat{u}(\tau, \xi)\}$. Let ψ be a smooth cutoff in x_1 that is identically 1 on a neighborhood of $[-\frac{3}{2}, \frac{3}{2}]$ and supported in $(-2, 2)$ so that $\psi(x_1)u(t,x) = u(t,x)$. We pause to mention that for any $x_1 \in \mathbb{R}$, the partial Fourier Transform of $\psi\Gamma_k u$ in t, x' is supported in the set

$$\operatorname{supp}(\widehat{\psi\Gamma_k u})(\cdot, x_1, \cdot) \subset \{(\tau,\xi') : \tau \geq 22|\xi'|\} \cap \{(\tau,\xi') : 2^{k-4-\frac{1}{2}} \leq |(\tau,\xi')| \leq 2^{k+4+\frac{3}{2}}\}$$

The rest of this chapter will now be devoted to proving the following theorem:

Theorem 2.2.1. *Let $\rho - 1, a_{ij} - \delta_{ij} \in L^{r,\kappa} \cap Lip$ with $\kappa = \frac{n-1}{r} + 1 + \alpha$ where either $0 < \alpha < 1$ or $\alpha > 1$. Suppose the following estimates hold uniformly over k*

$$\|\psi\Gamma_k u\|_{L_t^p L_x^q([-1,1]\times\mathbb{R}^n)} \lesssim 2^{k(s+\frac{\sigma}{p})}\big(2^k\|\Gamma_k u\|_{L_t^\infty L_x^2([-1,1]\times\mathbb{R}^n)} + \|\nabla_{t,x}(\psi\Gamma_k u)\|_{L_{x_1}^\infty L_{t,x'}^2(\mathbb{R}^{n+1})} + \|P(\psi\Gamma_k u)\|_{L_{x_1}^1 L_{t,x'}^2(\mathbb{R}^{n+1})}\big) \quad (2.2.1)$$

where σ is taken to be $\frac{1-\alpha}{3+\alpha}$ when $0 < \alpha < 1$ and $\sigma = 0$ otherwise. Then Theorem 1.6.1 holds if $\alpha > 1$ and Theorem 1.6.2 holds if $0 < \alpha < 1$.

2.3 Flux estimates

In order to control the $L_{x_1}^\infty L_{t,x'}^2(\mathbb{R}^{n+1})$ norm of the $\psi\Gamma_k u$, we need a family of estimates called "flux estimates" that exploit the fact that the Fourier transform of $\psi\Gamma_k u$ is supported in a region where P can be viewed as a hyperbolic operator in x_1.

Proposition 2.3.1. *Suppose $w(t,x) \in C^2(\mathbb{R}^{n+1})$ satisfies $\partial_{t,x}^\beta w \in L_{t,x}^2$ for $|\beta| \leq 2$ and the coefficients of P satisfy (2.1.4) for some $\varepsilon > 0$ sufficiently small. In addition, suppose that for all $y_1 \in \mathbb{R}$, $(R_{y_1}w)(t,x') := w(t,y_1,x')$, the restriction of w to the $x_1 = y_1$ hyperplane has Fourier Transform satisfying*

$$supp(\widehat{R_{y_1}w}) \subset \{(\tau,\xi') : |\xi'|^2 \leq \frac{1}{3}|\tau|^2\} \setminus B_{\frac{1}{2}}(0).$$

Then given a subinterval $I \subset \mathbb{R}$ and any $y_1, r_1 \in I$, there exists some constant C_I depending only on I, n, and on the Lipschitz norm of the coefficients of P such that

$$\|\nabla_{t,x}w(\cdot,y_1,\cdot)\|_{L_{t,x'}^2} \leq C_I\Big(\|\nabla_{t,x}w(\cdot,r_1,\cdot)\|_{L_{t,x'}^2} + \int_{r_1}^{y_1} \|Pw(\cdot,z_1,\cdot)\|_{L_{t,x'}^2}\,dz\Big).$$

Proof. Define

$$E(y_1)^2 = \int_{x_1=y_1} \frac{1}{2}a_{11}(\partial_1 w)^2 + \frac{1}{2}\rho(\partial_t w)^2 - \frac{1}{2}\sum_{i,j\geq 2} a_{ij}(\partial_i w)(\partial_j w)\, dtdx'$$

Differentiating this expression under the integral sign and integrating by parts in t and x' allows us to obtain the following estimate

$$\frac{dE^2}{dy_1} \lesssim \int_{x_1=y_1} |\partial_1 w||Pw| + (|\partial_t w|^2 + \sum_1^n |\partial_i w|^2)\, dtdx'. \tag{2.3.1}$$

We now use the support condition on $\widehat{R_{y_1} w}$ and Plancherel's equality to get that when $\varepsilon > 0$ is chosen sufficiently small in (2.1.4),

$$\begin{aligned}
2E(y_1)^2 &\geq (1-\varepsilon)\int_{x_1=y_1} (\partial_t w)^2 + (\partial_1 w)^2\, dtdx' - 2\sum_{i=2}^n \int_{x_1=y_1} (\partial_i w)^2 dtdx' \\
&\geq \int_{x_1=y_1} (1-\varepsilon)(\tau^2 - 2|\xi'|^2)|\widehat{R_{y_1} w}|^2\, d\tau d\xi' + (1-\varepsilon)\int_{x_1=y_1} (\partial_1 w)^2\, dtdx' \\
&\geq \frac{1}{9}\int_{x_1=y_1} (\tau^2 + |\xi'|^2)|\widehat{R_{y_1} w}|^2\, d\tau d\xi' + \frac{1}{9}\int_{x_1=y_1} (\partial_1 w)^2\, dtdx' \\
&\geq \frac{1}{9}(\|\partial_t w(\cdot, y_1, \cdot)\|^2_{L^2_{t,x'}} + \sum_{i=1}^n \|\partial_i w(\cdot, y_1, \cdot)\|^2_{L^2_{t,x'}}).
\end{aligned} \tag{2.3.2}$$

This inequality, along with (2.3.1) allows us to conclude that

$$E(y_1)\frac{dE}{dy_1} = \frac{1}{2}\frac{dE^2}{dy_1} \lesssim E(y_1)\|Pw(\cdot, y_1, \cdot)\|_{L^2_{t,x'}} + E(y_1)^2$$

Given (2.3.2), we can divide through by $E(y_1)$ and apply Gronwall's inequality to prove the result.

□

2.4 *Local estimates to global estimates*

Proof of Theorem 2.2.1. We first reduce matters to estimates on Γu by handling the term $(I - \Lambda^+ - \Lambda^-)u$. Since $n(\frac{1}{2} - \frac{1}{q}) + (\frac{1}{2} - \frac{1}{p}) \leq 2 + s$, we can use the Sobolev embedding estimate $\langle D \rangle^{-s} : H^2_{t,x} \to L^p_t L^q_x(\mathbb{R}^{n+1})$ to get that

$$\|\langle D \rangle^{-s}(I - \Lambda^+ - \Lambda^-)u\|_{L^p_t L^q_x(\mathbb{R}^{n+1})} \lesssim \|(I - \Lambda^+ - \Lambda^-)u\|_{H^2_{t,x}}.$$

The full Fourier transform of $(I - \Lambda^+ - \Lambda^-)u$ is supported in a region where

$$P(x, \tau, \xi) \gg |(\tau, \xi)|.$$

Hence by elliptic regularity results for pseudodifferential operators with rough symbols (see Theorem 2.2B in [19], which also holds in the Sobolev space setting) we have

$$\begin{aligned} \|(I - \Lambda^+ - \Lambda^-)u\|_{H^2_{t,x}} &\lesssim \|P(I - \Lambda^+ - \Lambda^-)u\|_{L^2_{t,x}} \\ &\lesssim \|[P, I - \Lambda^+ - \Lambda^-]u\|_{L^2_{t,x}} + \|Pu\|_{L^2_{t,x}} \\ &\lesssim \|f\|_{H^1} + \|g\|_{L^2}, \end{aligned}$$

the last inequality following by Lemma 2.0.1, energy estimates, and (2.1.5). This provides the desired estimate on $(I - \Lambda^+ - \Lambda^-)u$.

Turning our attention to Γu, we observe that by Littlewood-Paley theory

$$\begin{aligned} \|\langle D\rangle^{-s-\frac{\sigma}{p}}\Gamma u\|_{L^p_t L^q_x} &\approx \|(\sum_k |\langle D\rangle^{-s-\frac{\sigma}{p}}\beta_k \Gamma u|^2)^{\frac{1}{2}}\|_{L^p_t L^q_x} \\ &\lesssim \Big(\sum_k 2^{-2k(s+\frac{\sigma}{p})}\|\Gamma_k u\|^2_{L^p_t L^q_x}\Big)^{\frac{1}{2}} \end{aligned}$$

so it suffices to dominate the latter sum by $\|f\|_{H^1} + \|g\|_{L^2}$.

We now write $\Gamma_k u = \psi\Gamma_k u + (1-\psi)\Gamma_k u$. To control the terms involving $\psi\Gamma_k u$, first observe that as a consequence of the flux estimates we have

$$\begin{aligned} \|\nabla_{t,x}(\psi\Gamma_k u)\|^2_{L^\infty_{x_1} L^2_{t,x'}} &\lesssim \|\nabla_{t,x}(\psi\Gamma_k u)(\cdot, -2, \cdot)\|^2_{L^2_{t,x'}} + \|P(\psi\Gamma_k u)\|^2_{L^1_{x_1} L^2_{t,x'}(\mathbb{R}^{n+1}_{t,x})} \\ &\lesssim \|P(\psi\Gamma_k u)\|^2_{L^2_{t,x}} \end{aligned}$$

with the last inequality following from the compact support of ψ in x_1. Hence by (2.2.1), the sum over $\psi\Gamma_k u$ is now dominated by

$$\sum_k 2^{-2k(s+\frac{\sigma}{p})}\|\psi\Gamma_k u\|^2_{L^p_t L^q_x} \lesssim \sum_k \Big(\|\Gamma_k u\|^2_{L^\infty_t([-1,1];H^1)} + \|P(\psi\Gamma_k u)\|^2_{L^2_{t,x}}\Big)$$

To estimate the sum involving the $P(\psi\Gamma_k u)$ we begin by writing

$$P(\psi\Gamma_k u) = [P, \psi]\Gamma_k u + \psi[P, \Gamma_k]u + \psi\Gamma_k P u. \tag{2.4.1}$$

Consider the middle term. ψ is a bounded function, so it suffices to control the sum $\sum_k \|[P,\Gamma_k]u\|^2_{L^2_{t,x}}$. Indeed, the estimate

$$\sum_k \|[P,\Gamma_k]u\|^2_{L^2_{t,x}} \lesssim \|u\|^2_{H^1_{t,x}} \lesssim \|f\|^2_{H^1} + \|g\|^2_{L^2} \tag{2.4.2}$$

follows by Lemma 2.0.2 once it is observed that Lemma 2.0.1 implies

$$[P, \sum_k e^{ik\theta}\Gamma_k] : H^1_{t,x} \to L^2_{t,x}$$

continuously with operator norm bounded by some constant independent of θ. Given energy estimates and (2.1.5), the remaining terms in (2.4.1) are also easily controlled

$$\sum_k \left(\|[P,\psi]\Gamma_k u\|^2_{L^2_{t,x}} + \|\psi\Gamma_k Pu\|^2_{L^2_{t,x}}\right) \lesssim \sum_k \|\Gamma_k u\|^2_{H^1_{t,x}} + \|Pu\|^2_{L^2_{t,x}} \lesssim \|f\|^2_{H^1} + \|g\|^2_{L^2}.$$

To estimate the sum involving $\|\Gamma_k u\|_{L^\infty_t([-1,1];H^1)}$, let $\phi(t)$ be the smooth cutoff function in t defined above that is identically 1 on $[-1,1]$ and zero outside $(-\frac{5}{4},\frac{5}{4})$. Energy estimates then yield

$$\|\Gamma_k u\|_{L^\infty_t([-1,1];H^1)} \lesssim \|\phi\Gamma_k u(-2,\cdot)\|_{H^1} + \|P(\phi\Gamma_k u)\|_{L^2_{t,x}} \lesssim \|P(\phi\Gamma_k u)\|_{L^2_{t,x}}.$$

Inequalities to (2.4.1) and (2.4.2) now control $\sum_k \|\Gamma_k u\|^2_{L^\infty_t([-1,1];H^1_x)}$.

We are now left to control the sum $\sum_k 2^{-2k(s+\frac{\sigma}{p})}\|(1-\psi)\Gamma_k u\|^2_{L^p_t L^q_x}$. Since $\Gamma_k u$ is given by convolution against a Schwartz function decaying on the scale of 2^{-k} and by Proposition 2.1.1 u is supported in $(-\frac{3}{2},\frac{3}{2})$, we have that for any $N > 0$

$$\|(1-\psi)\Gamma_k u\|_{L^p_t L^q_x} \le C_N 2^{-kN}\|u\|_{L^2_{t,x}}$$

with C_N independent of k. Choosing N sufficiently large, this yields the inequality

$$\sum_k 2^{-2k(s+\frac{\sigma}{p})}\|(1-\psi)\Gamma_k u\|^2_{L^p_t L^q_x} \lesssim \|u\|^2_{L^2_t([-1,1];H^1_x)} \lesssim \|f\|_{H^1} + \|g\|_{L^2}.$$

□

Chapter 3

PARAMETRICES FOR THE WAVE OPERATOR

Now that we have reduced the main theorems to showing estimates on the $\psi\Gamma_k u$, we need to represent such functions in a manner that is suitable for proving Strichartz inequalities. In this chapter, we will use wave packet techniques as inspired by the work in [10] to represent the solution. Once this is done Chapters 4 and 5 will show how to use the representation to prove the estimates in the hypotheses of Theorem 2.2.1.

Throughout this chapter and in Chapter 4 we will assume that the coefficients of P are Lipschitz functions satisfying (2.1.4) and that

$$\partial_x^\beta a_{ij}(x), \partial_x^\beta \rho(x) \in L^r_{x_1} L^\infty_{x'}(\mathbb{R}^n) \qquad |\beta| \leq 2, \tag{3.0.1}$$

for some $1 < r < \infty$. This condition holds when the coefficients of the wave operator satisfy the hypotheses of Theorem 1.6.1 and a smoothing procedure will ensure that this holds when a_{ij}, ρ satisfy the conditions of Theorem 1.6.2. As we will see this condition is well suited for representing solutions to the wave equation whose partial Fourier transforms in t, x' are supported in a region where $\tau >> |\xi'|$.

As discussed in the introduction, we begin by constructing a pseudodifferential operator with rough symbol $\widetilde{P}(x, \tau, \xi)$ such that $\widetilde{P}(w) = \frac{1}{a_{11}} P(w)$ for functions w whose partial Fourier transforms are supported in a region where $\tau >> |\xi'|$, yet is hyperbolic in the x_1 variable rather than in t. Hence the symbol $\widetilde{P}(x_1, x', \tau, \xi_1, \xi')$ will be a quadratic in ξ_1. This will allow us to represent such w as solutions to equations involving P, while providing enough flexibility to construct a wave packet parametrix.

After introducing the fundamental properties of wave packets, we follow the approach in [10], constructing an approximate solution operator for $\widetilde{P}$. The matrix of this operator in the wave packet frame will behave much like a permutation matrix that has the effect of translating the center of one frame element along the Hamiltonian flow induced by $\widetilde{P}$. In our case the Hamiltonian will be one of the roots the symbol of $\widetilde{P}$ as a quadratic in ξ_1. This means that our approximate solution operator will be a superposition of wave packets travelling along curves parameterized by x_1, rather than in t. When this operator is composed with $\widetilde{P}$, the result is then an operator that loses only one derivative instead of two. Once the approximate solution operator is constructed, we will see how to use it to represent solutions precisely.

3.1 The operator $\widetilde{P}$

To construct $\widetilde{P}$, let $\Psi \in C^\infty(\mathbb{R}^n \setminus \{0\})$ be a smooth, homogeneous of degree 0 function in (τ, ξ') such that $\mathrm{supp}(\Psi) \subset \{(\tau,\xi') : \tau \geq 5|\xi'|\}$ and is identically 1 on the slightly smaller region $\{(\tau,\xi') : \tau \geq 6|\xi'|\}$. Next we pick a smooth radial cutoff Ω such that Ω is identically 1 on a neighborhood of $B_{\frac{1}{2}}(0) \subset \mathbb{R}^n$ and zero outside $B_{\frac{3}{4}}(0)$ and replace Ψ by $\Psi(1-\Omega)$ so that Ψ is smooth on all of $\mathbb{R}^n$. Additionally,

$$\Psi(\tau,\xi') = \Psi((\tau,\xi')/|(\tau,\xi')|) \quad \text{ for } |(\tau,\xi')| \geq \frac{3}{4}.$$

We now set $\tilde{a}_{ij} = a_{ij}/a_{11}, \tilde{\rho} = \rho/a_{11}$ and define $\widetilde{P}$ as the pseudodifferential operator with rough symbol given by

$$\begin{aligned}\widetilde{P}(x,\tau,\xi) = \xi_1^2 + \Big(2\sum_{j=2}^n \tilde{a}_{1j}(x)\xi_1\xi_j - (\tilde{\rho}(x)\tau^2 - \sum_{2\leq i,j\leq n} \tilde{a}_{ij}(x)\xi_i\xi_j)\Big)\Psi(\tau,\xi') \\ - (\tau^2 + \sum_{j=2}^n \xi_j^2)(1-\Psi(\tau,\xi')) + i\sum_1^n \tilde{b}_j(x)\xi_j\end{aligned}$$

Here either $\rho(x) \equiv 1$ and $\tilde{b}_i \equiv 0$ for nondivergence operators or $\tilde{b}_i = a_{11}^{-1}\sum_{j=1}^n \partial_j a_{ij}$ for divergence operators.

We will also need to smooth the coefficients of $\widetilde{P}$ by truncating their Fourier transform in x to frequencies less than $2^{\frac{k}{2}}$. Let $\widehat{\phi} = \Omega$, define $\phi_k(x) = 2^{\frac{nk}{2}}\phi(2^{\frac{k}{2}}x)$, and then set

$\tilde{a}_{ij}^k = \phi_k * \tilde{a}_{ij}$, $\tilde{\rho}^k = \phi_k * \tilde{\rho}$. We need a few estimates on the $\tilde{a}_{ij}^k$ and $\tilde{\rho}^k$. We first show the existence of an $L^r(\mathbb{R})$ function ϱ such that $\|(\tilde{a}_{ij} - \tilde{a}_{ij}^k)(x_1, \cdot)\|_{L^\infty_{x'}} \lesssim 2^{-k}\varrho(x_1)$. Set

$$\|D_x^2 A(x_1, \cdot)\|_{L^\infty_{x'}} := \sum_{|\alpha|=2} \sum_{i,j} \|\partial_x^\alpha a_{ij}(x_1, \cdot)\|_{L^\infty_{x'}} + \sum_{|\alpha|=2} \|\partial_x^\alpha \rho(x_1, \cdot)\|_{L^\infty_{x'}}.$$

and let $\varrho(x_1)$ denote the Hardy-Littlewood maximal function associated to the $L^r(\mathbb{R})$ function $x_1 \mapsto \|D_x^2 A(x_1, \cdot)\|_{L^\infty_{x'}}$. Note that since $1 < r < \infty$, $\varrho(x_1) \in L^r(\mathbb{R})$ as well. We next observe that for any Schwartz function $h \in \mathcal{S}(\mathbb{R})$ and $\lambda > 0$

$$\int \|D_x^2 A(x_1 - z, \cdot)\|_{L^\infty_{x'}} h(\lambda z)\lambda dz \leq C_h \varrho(x_1) \tag{3.1.1}$$

for some constant C_h depending only on h. This is a consequence of Theorem 2 in Chapter III, Section 2 of [13].

Since ϕ is radial and hence $\int y_i \phi(y) dy = 0$, a Taylor expansion of $\tilde{a}_{ij}$ with second order error yields the estimate

$$\begin{aligned} |\tilde{a}_{ij}(x) - \tilde{a}_{ij}^k(x)| &= |\int (\tilde{a}_{ij}(x) - \tilde{a}_{ij}(x-y))\phi(2^{\frac{k}{2}} y) 2^{\frac{kn}{2}} dy| \\ &\lesssim 2^{-k} \int_0^1 \int \|D_x^2 A(x_1 - (1-t)y_1, \cdot)\|_{L^\infty_{x'}} (2^{\frac{k}{2}} y_1)^2 \Phi(2^{\frac{k}{2}} y_1) 2^{\frac{k}{2}} dy_1 dt. \end{aligned}$$

with $\Phi(y_1) = \int |\phi(y_1, y')| dy'$. For $0 < t < 1$ we make a change variables $z_t = (1-t)y_1$ in the last integral and apply (3.1.1) with $h(z) = z^2\Phi(z)$ to get that

$$\|\tilde{a}_{ij}(x_1, \cdot) - \tilde{a}_{ij}^k(x_1, \cdot)\|_{L^\infty_{x'}} \lesssim 2^{-k}\varrho(x_1).$$

Note that we also get identical results for ρ^k.

A second application of the maximal function result above allows us to conclude

$$\|\partial_x^\beta \tilde{a}_{ij}^k(x_1, \cdot)\|_{L^\infty_{x'}}, \|\partial_x^\beta \tilde{\rho}^k(x_1, \cdot)\|_{L^\infty_{x'}} \leq C_\beta 2^{\frac{k}{2}(|\beta|-2)} \varrho(x_1) \qquad \text{for } |\beta| \geq 2.$$

Since $a_{ij}, \rho \in Lip(\mathbb{R}^n)$ we also have that for $|\beta| \leq 1$, $\|\partial_x^\beta \tilde{a}_{ij}^k\|_{L^\infty} \leq C$ for some C independent of k and that

$$\|\partial_x^\beta \tilde{a}_{ij}^k\|_{L^\infty}, \|\partial_x^\beta \tilde{\rho}^k\|_{L^\infty} \leq C_\beta 2^{\frac{k}{2}(|\beta|-1)} \qquad \text{for } |\beta| \geq 1.$$

Define $\widetilde{P}^k(x,D)$ as the pseudodifferential operator with smooth symbol obtained by replacing $\tilde{a}_{ij}$, $\tilde{\rho}$ by $\tilde{a}^k_{ij}$, $\tilde{\rho}^k$ in the definition of $\widetilde{P}$. By choosing $\varepsilon > 0$ in (2.1.4) sufficiently small, we see that the principal symbols of $\widetilde{P}, \widetilde{P}^k$ are quadratics in ξ_1 with 2 real roots, that is, they are operators that are hyperbolic in x_1.

We will employ wave packet techniques to construct a parametrix for $\widetilde{P}$, so to provide a more intuitive exposition we will reverse the roles of t and x_1 and assume that $\tilde{P}$ is hyperbolic in t. So for the rest of this chapter we will prove results about pseudodifferential hyperbolic operators $\widetilde{P}(t,x,\tau,\xi)$, $\widetilde{P}^k(t,x,\tau,\xi)$ obtained by switching the roles of the variables (t,τ) and (x_1,ξ_1). This means the estimates on the smoothed out coefficients of $\widetilde{P}^k$ take the form

$$\|(\tilde{a}_{ij}-\tilde{a}^k_{ij})(t,\cdot)\|_{L^\infty_x}, \|(\tilde{\rho}-\tilde{\rho}^k)(t,\cdot)\|_{L^\infty_x} \le C2^{-k}\varrho(t), \tag{3.1.2}$$

$$\|\partial^\beta_{t,x}\tilde{a}^k_{ij}(t,\cdot)\|_{L^\infty_x}, \|\partial^\beta_{t,x}\tilde{\rho}^k(t,\cdot)\|_{L^\infty_x} \le C_\beta 2^{\frac{k}{2}(|\beta|-2)}\varrho(t) \qquad \text{for } |\beta|\ge 2, \tag{3.1.3}$$

$$\|\partial^\beta_{t,x}\tilde{a}^k_{ij}\|_{L^\infty_{t,x}}, \|\partial^\beta_{t,x}\tilde{\rho}^k\|_{L^\infty_{t,x}} \le C_\beta 2^{\frac{k}{2}(|\beta|-1)} \qquad \text{for } |\beta|\ge 1. \tag{3.1.4}$$

Define Q, Q^k as the principal symbols of $\widetilde{P}, \widetilde{P}^k$ respectively. Let $H^\pm(t,x,\xi)$, $H^\pm_k(t,x,\xi)$ denote the roots of the symbols of Q, Q^k respectively as a quadratic in τ. These functions satisfy the following estimates:

$$\begin{aligned}
&|\partial^\alpha_{t,x}\partial^\beta_\eta H_k(t,x,\eta)| \le C_{\alpha,\beta}(1+|\eta|)^{1-|\beta|} && \text{for } |\alpha|\le 1,\\
&|\partial^\alpha_{t,x}\partial^\beta_\eta H_k(t,x,\eta)| \le C_{\alpha,\beta}(\varrho(t)+1)2^{\frac{k}{2}(|\alpha|-2)}(1+|\eta|)^{1-|\beta|} && \text{for } |\alpha|\ge 2,\\
&|\partial^\alpha_{t,x}\partial^\beta_\eta H_k(t,x,\eta)| \le C_{\alpha,\beta}2^{\frac{k}{2}(|\alpha|-1)}(1+|\eta|)^{1-|\beta|} && \text{for } |\alpha|\ge 1.
\end{aligned} \tag{3.1.5}$$

We can now write the operator $Q^k(t,x,D)$ as

$$Q^k(t,x,D) = (i\partial_t + H^-_k(t,x,D))\circ(i\partial_t + H^+_k(t,x,D)) + E^+_k(t,x,D) \tag{3.1.6}$$

with error term $E^+_k(t,x,D)$ of lower order. To create an approximate solution operator for $\widetilde{P}_k$ it should suffice to construct an evolution operator which loses no derivatives when composed with $(i\partial_t + H^+_k(t,x,D))$. Indeed, we will write out $E^+_k(t,x,D)$ below and argue that the error brought about by the this term and the first order terms will contribute to the loss of only one derivative.

3.2 Wave Packets

We now introduce "wave packets", which are a frame of functions on $L^2(\mathbb{R}^n)$ with properties that are well suited for the analysis of hyperbolic operators. This section is essentially a summary of the results in Section 2 of [10].

To construct this frame we begin by taking a partition of unity

$$|h_0(\xi)|^2 + \sum_{k=1}^{\infty} \sum_{\omega} |h_k^\omega(\xi)|^2 \equiv 1$$

on $\mathbb{R}^n$ where ω ranges over a set of $\approx 2^{\frac{k(n-1)}{2}}$ vectors on the unit sphere, equally spaced by a distance of $\approx 2^{-\frac{k}{2}}$. These functions can be taken so that

$$\mathrm{supp}(h_k^\omega) \subset \{\xi \in \mathbb{R}^n : 2^{k-\frac{1}{2}} \le |\xi| \le 2^{k+\frac{3}{2}}\} \cap \{\xi \in \mathbb{R}^n : |\omega - \xi/|\xi|| \le 2^{-\frac{k}{2}}\}$$

with derivatives that satisfy

$$|\langle \omega, \partial_\xi \rangle^j \partial_\xi^\beta h_k^\omega(\xi)| \le C_{\beta,j} 2^{-kj-\frac{k}{2}|\beta|}.$$

Such a partition of unity is constructed in Chapter 9, Section 4.4 of [12].

For each pair ω, k let Ξ_k^ω be the rectangular lattice in $\mathbb{R}^n$ evenly spaced by distance 2^{-k} in the ω direction and spaced $2^{-\frac{k}{2}}$ is the directions orthogonal to ω. Given any triple $\gamma = (x_0, k, \omega)$ where $x_0 \in \Xi_k^\omega$, set

$$\widehat{\varphi_\gamma}(\xi) = (2\pi)^{-\frac{n}{2}} 2^{-\frac{k(n+1)}{4}} e^{-i\langle x_0, \xi \rangle} h_k^\omega(\xi).$$

By using Fourier series, it can be shown that the collection $\{\varphi_\gamma(x)\}_\gamma$ forms a frame of functions for L^2 in the sense that if $f \in L^2(\mathbb{R}^n)$, and $c_\gamma = \int f(y)\overline{\varphi_\gamma(y)}\, dy$, then

$$f(y) = \sum_\gamma c_\gamma \varphi_\gamma(y), \qquad \int |f(y)|^2\, dy = \sum_\gamma |c_\gamma|^2.$$

However, these frame elements, which we will refer to as "wave packets", are not mutually orthogonal or even linearly independent. In space, these wave packets are now localized to a region of width 2^{-k} in the ω direction and width $2^{-\frac{k}{2}}$ orthogonally in the sense that

$$|(y-x_0)^\alpha \langle \omega, y-x_0 \rangle^i \langle \omega, \partial_y \rangle^j \langle \omega^\perp, \partial_y \rangle^\beta \varphi_\gamma(y)| \le C_{\alpha,\beta,i,j}\, 2^{\frac{k(n+1)}{4} + \frac{k}{2}(|\beta|-|\alpha|) + k(j-i)} \qquad (3.2.1)$$

where $\langle \omega^\perp, \partial_y \rangle^\beta$ denotes differentiation with respect to any family of $|\beta|$ directions orthogonal to ω. We also note that this frame also allows us to represent tempered distributions in an L^2 Sobolev space with norm approximated by

$$\|f\|^2_{H^s(\mathbb{R}^n)} \approx \sum_\gamma 2^{2ks} |c_\gamma|^2.$$

We next observe the correspondence between matrices on $\cup_{\omega,k} \Xi_k^\omega \times \cup_{\omega,k} \Xi_k^\omega$ and operators on $L^2(\mathbb{R}^n)$. Indeed given any operator T on $L^2(\mathbb{R}^n)$ we can associate a matrix to it given by

$$b(\gamma, \gamma') = \int \overline{\varphi_\gamma(y)} (T\varphi_{\gamma'})(y) dy.$$

Conversely, given any matrix $\{b(\gamma, \gamma')\}_{\gamma,\gamma'}$ we can formally associate an operator to it determined by

$$Tf = \sum_{\gamma,\gamma'} b(\gamma, \gamma') c_{\gamma'} \varphi_\gamma.$$

Below we will discuss conditions on matrices on $\cup_{\omega,k} \Xi_k^\omega \times \cup_{\omega,k} \Xi_k^\omega$ for which its associated operator is bounded on $L^2(\mathbb{R}^n)$.

This frame of functions is well-suited for analysis on the cosphere bundle of $\mathbb{R}^n$, $S^*(\mathbb{R}^n) = \mathbb{R}^n \times \mathbb{S}^{n-1}$. We place a natural pseudodistance on the cosphere bundle given by

$$d(x, \omega; \tilde{x}, \tilde{\omega}) = |\langle \omega, x - \tilde{x} \rangle| + |\langle \tilde{\omega}, x - \tilde{x} \rangle| + \min(|x - \tilde{x}|, |x - \tilde{x}|^2) + |\omega - \tilde{\omega}|^2.$$

We also have the following useful estimates on d

$$d(x, \omega; \tilde{x}, \tilde{\omega}) \approx |\langle \omega, x - \tilde{x} \rangle| + \min(|x - \tilde{x}|, |x - \tilde{x}|^2) + |\omega - \tilde{\omega}|^2$$
$$d(x, \omega; \tilde{x}, \tilde{\omega}) \le 6\big(d(x, \omega; x', \omega') + d(x', \omega'; \tilde{x}, \tilde{\omega})\big).$$

This pseudodistance can now be used to define a weight function on $\cup_{\omega,k} \Xi_k^\omega \times \cup_{\omega,k} \Xi_k^\omega$. For $\gamma = (x, \omega, k), \gamma' = (x', \omega', k') \in \cup_{\omega,k} \Xi_k^\omega$ we define

$$\mu_\delta(\gamma, \gamma') = (1 + |k - k'|^2)^{-1} 2^{-(\delta + \frac{n}{2})|k - k'|} \left(1 + \frac{d(x, \omega; x', \omega')}{2^{-k} + 2^{-k'}}\right)^{-n-\delta}$$

The weight function now allows us to give sufficient conditions on a matrix in $\cup_{\omega,k} \Xi_k^\omega \times \cup_{\omega,k} \Xi_k^\omega$ so that its associated operator is bounded on $L^2(\mathbb{R}^n)$ or is continuous between L^2 Sobolev spaces. Given a mapping χ on $S^*(\mathbb{R}^n)$, we say that the matrix $b(\gamma,\gamma')$ belongs to the class $\mathcal{M}^r_\delta(\chi)$ if

$$|b(\gamma,\gamma')| \le C_b 2^{rk'} \mu_\delta(\gamma, \chi(\gamma')) \tag{3.2.2}$$

where $\chi(\gamma') = (\chi(x',\omega'), k')$. We also define $\mathcal{M}^r(\chi) = \bigcap_{\delta>0} \mathcal{M}^r_\delta(\chi)$. An operator T defined as a map from Schwartz class functions on $\mathbb{R}^n$ to tempered distributions, is said to be of class $\mathcal{I}^r(\chi)$ if its associated matrix in the wave packet frame belongs to $\mathcal{M}^r(\chi)$. As a result of Theorem 2.7 in [10] we have that whenever χ_1, χ_2 are invertible mappings on $S^*(\mathbb{R}^n)$ that satisfy

$$C^{-1} d(x,\omega;x',\omega') \le d(\chi_l(x,\omega);\chi_l(x',\omega')) \le C d(x,\omega;x',\omega'), \qquad l = 1,2 \tag{3.2.3}$$

then

$$\mathcal{I}^r(\chi_1) : H^z(\mathbb{R}^n) \to H^{z-r}(\mathbb{R}^n) \qquad \text{and} \qquad \mathcal{I}^{r_1}(\chi_1) \circ \mathcal{I}^{r_2}(\chi_2) \subset \mathcal{I}^{r_1+r_2}(\chi_1 \circ \chi_2). \tag{3.2.4}$$

Furthermore, if the matrix estimates of an operator in $\mathcal{I}^r(\chi_1)$ satisfy

$$|b(\gamma,\gamma')| \le M \cdot C_\delta 2^{rk'} \mu_\delta(\gamma, \chi_1(\gamma'))$$

for some constant M independent of δ, then its norm as an operator $H^z \to H^{z-r}$ is bounded by $M \cdot C_z$ where C_z depends on z and on C_δ for some $\delta > 0$, but not on M. This result will play a crucial role in the analysis below as we will frequently need to show that certain maps are continuous between Sobolev spaces, and showing that the operator belongs to $\mathcal{I}^r(\chi)$ for some suitable transformation χ will often be the most effective way to accomplish this.

3.3 The approximate solution operator

Here we construct an approximate solution operator for $(i\partial_t + H_k^+(t,x,D))$ when $t \in [-2,2]$ in the sense that when it is composed with $(i\partial_t + H_k^+(t,x,D))$ the result is an operator that loses no derivatives. It is then shown that this implies that the operator composed with $\widetilde{P}$ yields a mapping that loses only one derivative. The first portion of the discussion in this section actually holds for either H^+ or H^- so we will often drop the superscript when there is no need to distinguish between the two.

Since $H(t,x,\xi) = |\xi|H(t,x,\xi/|\xi|)$ for $|\xi| \geq \frac{3}{4}$ we can redefine H (and similarly H_k) whenever $|\xi| < \frac{3}{4}$ so that H is homogeneous of degree 1 in ξ. This redefinition will not effect the results we prove concerning the operators $H(t,x,D), H_k(t,x,D)$ as they will almost always be applied to data at frequencies supported away from $B_{\frac{3}{4}}(0)$. Allowing H and H_k to be homogeneous of degree one now allows us to take the flow on $T^*(\mathbb{R}^n)$, the cotangent bundle on $\mathbb{R}^n$, determined by the vector field $(-H_\xi(t,x,\xi), H_x(t,x,\xi))$ and project it down to a flow on the cosphere bundle $S^*(\mathbb{R}^n) = \mathbb{R}^n \times \mathbb{S}^{n-1}$.

Given any $\gamma = (x,\omega,k)$, let $(x_\gamma(s), \omega_\gamma(s), \Theta_\gamma(s))$ be the solution to

$$\begin{aligned}
\frac{dx}{ds} &= -(H_k)_\xi(s,x,\omega) \\
\frac{d\omega}{ds} &= (H_k)_x(s,x,\omega) - \langle \omega, (H_k)_x(s,x,\omega)\rangle \omega \\
\frac{d\Theta}{ds} &= \Theta[\omega \cdot (H_k)_x(s,x,\omega)^T - (H_k)_x(s,x,\omega)\cdot \omega^T]
\end{aligned} \tag{3.3.1}$$

with initial conditions $(x_\gamma(0), \omega_\gamma(0), \Theta_\gamma(0)) = (x_\gamma, \omega_\gamma, I)$. This flow preserves $S^*(\mathbb{R}^n) \times O(n)$ and has the property that $\Theta_\gamma(t)\omega_\gamma(t) = \omega_\gamma$ for all time t as $\frac{d}{dt}(\Theta_\gamma(t)\omega_\gamma(t)) = 0$. Set $\varphi_\gamma(s,y) = \varphi_\gamma(\Theta_\gamma(s)(y - x_\gamma(s)) + x_\gamma)$ and define the operator

$$\mathbf{e}_k(t)(\sum_\gamma c_\gamma \varphi_\gamma)(y) = \sum_{\gamma : k_\gamma = k} c_\gamma \varphi_\gamma(t,y).$$

The key result of this chapter will be to prove that $\sum_k \mathbf{e}_k(t)$ is indeed an approximate solution operator for $\widetilde{P}$.

Before proceeding, we need to collect a few facts concerning the flows induced by H and H_k. For each $k \geq 1$, let $\chi_t^k : S^*(\mathbb{R}^n) \to S^*(\mathbb{R}^n)$ denote the mapping $\chi_t^k(x,\omega) = (x(t), \omega(t))$, where the curves $(x(s), \omega(s))$ are solutions to the first 2 equations in (3.3.1) with initial values (x,ω). Also let χ_t be the analogous mapping induced by the nonsmooth Hamiltonian H (that is, the mapping determined by replacing H_k by H in (3.3.1)).

First observe that we can adapt Lemma 2.2 in [10] to our slightly weaker conditions on H and conclude that

$$C^{-1}d(x,\omega;\tilde{x},\tilde{\omega}) \le d(\chi_t(x,\omega);\chi_t(\tilde{x},\tilde{\omega})) \le Cd(x,\omega;\tilde{x},\tilde{\omega}) \tag{3.3.2}$$

where C is independent of $t \in [-2,2]$. Hence the results (3.2.4) of the previously mentioned Theorem 2.7 in [10] apply, and will be used implicitly, in what follows below to imply that certain linear transformations are continuous as a maps between Sobolev spaces.

In the same vein, Lemma 3.6 of [10] is also easily adapted to allow us to conclude that

$$d(\chi_t^k(x,\omega);\chi_t(x,\omega)) \le C2^{-k} \tag{3.3.3}$$

where C is again independent of $t \in [-2,2]$. The proofs of both of these lemmas from [10] rely on an application of Gronwall's lemma which work just as well under our slightly weaker hypotheses.

This first theorem shows that when $\mathbf{e}_k(t)$ is composed with $i\partial_t + H_k(t,x,D)$ the result is indeed an operator that loses no derivatives. The key here is that

$$\sum_k (i\partial_t + H_k(t,x,D)) \circ \mathbf{e}_k(t) \in \mathcal{I}^0(\chi_t).$$

Theorem 3.3.1. *Let $\{H_k(t,x,D)\}_{k=1}^{\infty}$ be the family of pseudodifferential operators with symbols $H_k(t,\cdot,\cdot) \in S^1_{1,0}(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ defined above. Then for each fixed t we have*

$$\sum_k \mathbf{e}_k(t) \in \mathcal{I}^0(\chi_t),$$

$$\partial_t \circ \sum_k \mathbf{e}_k(t) \in \mathcal{I}^1(\chi_t),$$

$$\sum_k (i\partial_t + H_k(t,x,D))\mathbf{e}_k(t) \in \mathcal{I}^0(\chi_t),$$

$$\sum_k \partial_t \circ (i\partial_t + H_k(t,x,D))\mathbf{e}_k(t) \in \mathcal{I}^1(\chi_t).$$

Furthermore, if $\tilde{b}_l(t,\gamma',\gamma)$ denotes the matrix of $\sum_k \partial_t^l \circ (i\partial_t + H_k(t,x,D))\mathbf{e}_k(t)$ for $l = 0,1$, at a fixed time t we then have the following estimates on the matrices for any $t \in [-2,2]$

and $N, \delta > 0$

$$\tilde{b}_l(t,\gamma',\gamma) \leq \begin{cases} C_\delta 2^{kl}(1+\varrho(t))\mu_\delta(\gamma',\chi_t(\gamma)) & |k_{\gamma'}-k_\gamma| \leq 2, \\ C_{\delta,N}2^{-\max(k_\gamma,k_{\gamma'})N}(1+\varrho(t))\mu_\delta(\gamma',\chi_t(\gamma)) & |k_{\gamma'}-k_\gamma| \geq 3. \end{cases}$$

The constants $C_\delta, C_{\delta,N}$ *are independent of* $t \in [-2,2]$.

Proof. Computing $\partial_t\varphi_\gamma(t,y)$ yields

$$\partial_t\varphi_\gamma(t,y) = L(t,x(t),\omega(t),y,\partial_y)\varphi_\gamma(t,y) \tag{3.3.4}$$

where

$$L(t,x,\omega,y,\partial_y) = \langle (H_k)_\eta(t,x,\omega),\partial_y\rangle + \langle (H_k)_x(t,x,\omega), y-x\rangle\langle\omega,\partial_y\rangle \\ - \langle\omega, y-x\rangle\langle (H_k)_x(t,x,\omega),\partial_y\rangle. \tag{3.3.5}$$

Given this and (3.2.1) it is not hard to see that an an integration by parts argument gives us

$$|\partial_t^l\varphi_\gamma(t,y)| \leq C_N 2^{kl}(1+2^k|\langle\omega_\gamma(t), y-x_\gamma(t)\rangle| + 2^k|y-x_\gamma(t)|^2)^{-N}$$

for $l=0,1$. Next observe that $\int \varphi_\gamma(t,y)\overline{\varphi_{\gamma'}(y)}dy$ vanishes whenever $|\omega_\gamma(t)-\omega_{\gamma'}|^2 > 2^{-k+4}$ or $|k-k'| \geq 2$. Thus the claim that $\partial_t^l \circ \sum_k \mathbf{e}_k(t) \in \mathcal{I}^l(\chi_t)$ for $l=0,1$ is a consequence of the following estimates

$$|\int \partial_t^l\varphi_\gamma(t,y)\varphi_{\gamma'}(y)dy| \leq C_N 2^{kl}\big(1+2^k d(x_\gamma(t),\omega_\gamma(t);x_{\gamma'},\omega_{\gamma'})\big)^{-N} \tag{3.3.6}$$

which are straightforward when $|\omega_\gamma(t)-\omega_{\gamma'}|^2 \leq 2^{-k+4}$ and $|k-k'| \leq 2$.

We now turn our attention to the other claims. Consider the operator $i\partial_s + H_k(s,x,D)$ applied to $\varphi_\gamma(s,y)$ and abbreviate the curve $(x_\gamma(s),\omega_\gamma(s),\Theta_\gamma(s))$ defining the function by (x_s,ω_s,Θ_s). By employing a Taylor expansion of $H_k(s,y,\eta)$ about the point $(x_s,|\eta|\omega_s)$ with second order error term $R(\gamma,s,y,\eta)$ it is not hard to see that the operator

$$iL(s,x_s,\omega_s,y,\partial_y) + (H_k)(s,y,D)$$

has the symbol

$$\langle (H_k)_x(s, x_s, \omega_s), y - x_s\rangle |\eta| (\langle \omega_s, \omega_s - \eta/|\eta|\rangle) \\ + \langle \omega_s, y - x_s\rangle \langle (H_k)_x(s, x_s, \omega_s), \eta\rangle + R(\gamma, s, y, \eta)$$

Call the operator determined by the first 2 terms in the sum $T_{\gamma,s}$, and let $\partial_s T_{\gamma,s}$ be the operator obtained by differentiating its symbol with respect to s. Also, abbreviate the operator determined by $R(\gamma, s, \cdot)$ by $R_{\gamma,s}$ and take a similar convention for $\partial_s R_{\gamma,s}$. By using the Fourier transform we see that for $l = 0, 1$, $\int (\partial_s^l T_{\gamma,s}\varphi_\gamma)(s, y)\overline{\varphi_{\tilde{\gamma}}(y)}\, dy$ vanishes unless both $|\omega_{\tilde{\gamma}} - \omega_s| \leq 2^{-k+4}$ and $|k_\gamma - k_{\tilde{\gamma}}| \leq 1$. Employing arguments similar to those used in establishing (3.3.6) we have that for any $\delta > 0$ there exists C_δ such that

$$\left| \int (\partial_s^l T_{\gamma,s})\varphi_\gamma \overline{\varphi_{\tilde{\gamma}}}\, dy \right| \leq C_\delta 2^{kl} \mu_\delta(\chi_s^k(\gamma), \tilde{\gamma}) \approx C_\delta 2^{kl} \mu_\delta(\chi_s(\gamma), \tilde{\gamma}) \tag{3.3.7}$$

with the latter approximation following from (3.3.3).

We now turn our attention to estimates on the R term. Setting

$$F_{11}^\alpha(\gamma, s, y, \eta) = \int_0^1 (\partial_y^\alpha H_k)\big(s, ty + (1-t)x_s, t\frac{\eta}{|\eta|} + (1-t)\omega_s\big)\, dt,$$
$$F_{22}^\alpha(\gamma, s, y, \eta) = \int_0^1 (\partial_\eta^\alpha H_k)\big(s, ty + (1-t)x_s, |\eta|^{-1}(t\eta + (1-t)|\eta|\omega_s)\big)\, dt,$$
$$F_{12}^{ij}(\gamma, s, y, \eta) = \int_0^1 (\partial_{y_i}\partial_{\eta_j} H_k)\big(s, ty + (1-t)x_s, t\eta + (1-t)|\eta|\omega_s\big)\, dt$$

allows us to write:

$$R(\gamma, s, y, \eta) = \sum_{|\alpha|=2} (y - x_s)^\alpha |\eta| F_{11}^\alpha(\gamma, s, y, \eta) + \sum_{|\alpha|=2} (\eta - |\eta|\omega_s)^\alpha |\eta|^{-1} F_{22}^\alpha(\gamma, s, y, \eta) \\ + \sum_{1 \leq i,j \leq n} (y - x_s)_i (\eta - |\eta|\omega_s)_j F_{12}^{ij}(\gamma, s, y, \eta).$$

Also, observe that (3.1.5) implies the following estimates on $F = F_{11}^\alpha, F_{22}^\alpha$, or F_{12}^{ij} for $\eta \in \mathrm{supp}(\varphi_\gamma)$

$$|\partial_s^j \partial_y^{\beta_1} \partial_\eta^{\beta_2} F(\gamma, s, y, \eta)| \leq C_{\beta_1,\beta_2}(\varrho(s) + 1) 2^{\frac{k}{2}(|\beta_1| + j)} |\eta|^{-|\beta_2|}.$$

Writing

$$\partial_s\big((T_{\gamma,s}+R_{\gamma,s})\varphi_\gamma(s,\cdot)\big)=(\partial_s T_{\gamma,s}+\partial_s R_{\gamma,s})\varphi_\gamma(s,\cdot)+(T_{\gamma,s}+R_{\gamma,s})\partial_s\varphi_\gamma(s,\cdot)$$

we see that the theorem is now a consequence of (3.3.7), (3.3.4), and the following Lemma concerning the mapping properties of pseudodifferential operators in the wave packet frame.

□

Lemma 3.3.2. *Let $B(y,\eta)\in S^l_{1,\frac{1}{2}}$ with symbol estimates of the form*

$$|\partial_y^{\beta_1}\partial_\eta^{\beta_2}B(y,\eta)|\leq M\cdot C_{\beta_1,\beta_2}\langle\eta\rangle^{l+\frac{|\beta_1|}{2}-|\beta_2|}$$

where $M>0$ is some constant independent of β_1,β_2. Suppose also that ψ_γ is a function of the form

$$\widehat{\psi_\gamma}(\xi)=e^{-i\langle x_\gamma,\xi\rangle}2^{k(i-j+\frac{|\beta_1|}{2}-\frac{|\beta_2|}{2}-l+\frac{|\beta_3|}{2}-\frac{(n+1)}{4})}\cdot$$
$$\langle\omega_\gamma,\partial_\xi\rangle^i\langle\omega_\gamma^\perp,\partial_\xi\rangle^{\beta_1}\langle\omega_\gamma,\xi\rangle^j\langle\omega_\gamma^\perp,\xi\rangle^{\beta_2}|\xi|^l(\xi-|\xi|\omega_\gamma)^{\beta_3}h_k^{\omega_\gamma}(\xi)\quad(3.3.8)$$

so that ψ_γ is concentrated in a rectangle of width 2^{-k} in the ω direction and width $2^{-\frac{k}{2}}$ orthogonally, much like a wave packet. Then for $\gamma=(x_\gamma,\omega_\gamma,k_\gamma)$, $\tilde\gamma=(x_{\tilde\gamma},\omega_{\tilde\gamma},k_{\tilde\gamma})$ we have the following inequality:

$$|\int\overline{\varphi_{\tilde\gamma}(y)}B(y,D)\psi_\gamma(y)dy|\leq\begin{cases}M\cdot C_\delta 2^{k_\gamma l}\mu_\delta(\gamma,\tilde\gamma) & |k_\gamma-k_{\tilde\gamma}|\leq 2\\ M\cdot C_{\delta,N}2^{-\max(k_\gamma,k_{\tilde\gamma})N}\mu_\delta(\gamma,\tilde\gamma) & |k_\gamma-k_{\tilde\gamma}|\geq 3\end{cases}\quad(3.3.9)$$

where $C_\delta, C_{\delta,N}$ depend on δ, N, the constants C_{β_1,β_2}, and on the sequence of indices taken in (3.3.8)*, but not on M or the choice of $\gamma,\tilde\gamma$.*

Proof. For the purposes of this proof, given a sequence K of indices and/or quantities, C_K will denote some constant depending only on K and the constants C_{β_1,β_2}. We will also often abbreviate k_γ by k.

Integrating by parts in the integral defining $B(y,D)\psi_\gamma(y)$ yields estimates of the form

$$|\langle\omega_\gamma,y-x_\gamma\rangle^j(y-x_\gamma)^{\alpha_1}\langle\omega_\gamma^\perp,\partial_y\rangle^{\alpha_2}\partial_y^{\alpha_3}B(y,D)\psi_\gamma(y)|$$
$$\leq MC_{j,\alpha_1,\alpha_2,\alpha_3}2^{\frac{k(n+1)}{4}+kl+\frac{k}{2}(|\alpha_2|+2|\alpha_3|-|\alpha_1|-2j)}\quad(3.3.10)$$

where $C_{j,\alpha_1,\alpha_2,\alpha_3}$ is independent of M. This shows that such $B(\cdot,D)\psi_\gamma$ is concentrated in space to a rectangle of width 2^{-k} in the ω_γ direction and width $2^{-\frac{k}{2}}$ orthogonally. Let $B\psi_\gamma$ abbreviate $B(\cdot,D)\psi_\gamma$. As a result of the estimates above, integration by parts in the integral defining $\widehat{B\psi_\gamma}(\xi)$ gives us

$$|\widehat{B\psi_\gamma}(\xi)| \le MC_N 2^{kl-\frac{k(n+1)}{4}}(1+2^{-k}|\pi_\gamma^\perp(\xi)|^2+2^{-2k}|\xi|^2)^{-N}. \tag{3.3.11}$$

where $\pi_\gamma^\perp$ denotes projection on to the subspace orthogonal to ω_γ. Hence $\widehat{B\psi_\gamma}(\xi)$ is concentrated in a rectangle of width $2^{\frac{k}{2}}$ in directions orthogonal to ω_γ and length 2^k in directions parallel to ω_γ.

Another important estimate on $\widehat{B\psi_\gamma}(\xi)$ is obtained through integration by parts with respect to η and y in the integral

$$\widehat{B\psi_\gamma}(\xi) = \int\int e^{i\langle y,\eta-\xi\rangle}B(y,\eta)\widehat{\psi_\gamma}(\eta)d\eta dy. \tag{3.3.12}$$

This yields the following estimates for any N

$$|\widehat{B\psi_\gamma}(\xi)| \le MC_N 2^{kl-\frac{3k(n+1)}{4}}\int_{\text{supp}(\widehat{\varphi}_\gamma)}(1+2^{-k}|\xi-\eta|^2)^{-N}d\eta. \tag{3.3.13}$$

When $|k_\gamma - k_{\tilde\gamma}| \le 2$ and $\langle\omega_\gamma,\omega_{\tilde\gamma}\rangle \ge -\frac{1}{2}$ we use the inequalities (3.3.11) after observing that for $\xi \in \text{supp}(\widehat{\varphi}_{\tilde\gamma})$ we have

$$|\pi_\gamma^\perp(\xi)| \ge c(2^k|\omega_\gamma-\omega_{\tilde\gamma}|-2^{\frac{k}{2}})$$

with c uniform over all such $\gamma,\tilde\gamma$. This yields

$$\begin{aligned}|\int \widehat{B\psi_\gamma}\overline{\widehat{\varphi_{\tilde\gamma}}}| &\le MC_N 2^{-\frac{k(n+1)}{2}+kl}\int_{\text{supp}(\widehat{\varphi}_{\tilde\gamma})}(1+2^{-k}|\langle\omega_\gamma,\xi\rangle|+2^{-\frac{k}{2}}|\pi_\gamma^\perp(\xi)|)^{-2N}d\xi\\ &\le M\tilde C_N(1+2^k|\omega_\gamma-\omega_{\tilde\gamma}|^2)^{-N}.\end{aligned}$$

Now consider the case where either (a) $|k_\gamma - k_{\tilde\gamma}| \le 2$ with $\langle\omega_{\tilde\gamma},\omega_\gamma\rangle \le -\frac{1}{2}$ or (b) $|k_\gamma - k_{\tilde\gamma}| \ge 3$ with no restriction on $\omega_\gamma,\omega_{\tilde\gamma}$. In both (a) and (b), there exists c independent of choice of such $\gamma,\tilde\gamma$ such that $|\xi-\eta| \ge c2^{\max(k_\gamma,k_{\tilde\gamma})}$ for $\xi \in \text{supp}(\varphi_{\tilde\gamma})$, $\eta \in \text{supp}(\varphi_{\tilde\gamma})$. We can thus apply (3.3.13) to get

$$|\int \widehat{B\psi_\gamma}\overline{\widehat{\varphi_{\tilde\gamma}}}d\xi| \le M\tilde C_N 2^{\max(k_\gamma,k_{\tilde\gamma})(n+1)+k_\gamma l}(1+2^{\max(k_\gamma,k_{\tilde\gamma})})^{-N}$$

These inequalities, along with the results established in the previous case, yield the following estimates for arbitrary $\gamma, \tilde{\gamma}$

$$\left|\int \overline{\varphi_{\tilde{\gamma}}} B\psi_\gamma\right| \leq MC_N\zeta(N, k_{\tilde{\gamma}}, k_\gamma)2^{k_\gamma l - |k_\gamma - k_{\tilde{\gamma}}|N}(1 + 2^{k_\gamma}|\omega_\gamma - \omega_{\tilde{\gamma}}|^2)^{-N}$$

where

$$\zeta(N, k_{\tilde{\gamma}}, k_\gamma) \leq \begin{cases} 2^{-k_{\tilde{\gamma}}N} & k_{\tilde{\gamma}} \geq k_\gamma + 3 \\ 1 & |k_{\tilde{\gamma}} - k_\gamma| \leq 2 \\ 2^{-k_\gamma N} & k_{\tilde{\gamma}} \leq k_\gamma - 3 \end{cases}$$

To obtain the conclusion of the lemma when $|k_{\tilde{\gamma}} - k_\gamma| \leq 2$, we incorporate the estimates (3.3.10) on $B\psi_\gamma$ as well as the standard inequalities (3.2.1) on $\varphi_{\tilde{\gamma}}$ to get that

$$(1 + 2^k|\omega_\gamma - \omega_{\tilde{\gamma}}|^2)^{-2N}\left|\int \overline{\varphi_{\tilde{\gamma}}} B\psi_\gamma\right| \leq MC_N 2^{kl}\left(1 + \frac{d(x_\gamma, \omega_\gamma, x_{\tilde{\gamma}}, \omega_{\tilde{\gamma}})}{2^{-k} + 2^{-k_{\tilde{\gamma}}}}\right)^{-N}.$$

The result follows for $|k_{\tilde{\gamma}} - k_\gamma| \leq 2$ by observing that this implies

$$\begin{aligned}\left|\int \overline{\varphi_{\tilde{\gamma}}} B\psi_\gamma\right|^2 &\leq C_N 2^{2kl} M^2 2^{-2|k_\gamma - k_{\tilde{\gamma}}|N}(1 + 2^{k+1}|\omega_\gamma - \omega_{\tilde{\gamma}}|^2)^{-2N}\left|\int \overline{\varphi_{\tilde{\gamma}}} B\psi_\gamma\right| \\ &\leq C_N 2^{2kl} M^2 \mu_{N-n}(\gamma, \tilde{\gamma})^2\end{aligned}$$

A similar calculation establishes the desired estimate when $|k_{\tilde{\gamma}} - k_\gamma| \geq 3$. □

We are now able to show the following:

Theorem 3.3.3. *Suppose $H_k = H_k^+$ so that $\mathbf{e}_k(t)$ is defined by the integral curves of H_k^+. Then*

$$\sum_k (i\partial_t + H_k^-(t, x, D)) \circ (i\partial_t + H_k^+(t, x, D))\mathbf{e}_k(t) \in \mathcal{I}^1(\chi_t)$$

with the matrix of the operator $b(t, \gamma, \gamma')$ satisfying

$$|b(t, \gamma, \gamma')| \leq C_\delta(1 + \varrho(t))\mu_\delta(\gamma, \chi_t(\gamma')).$$

Proof. As a consequence of Theorem 3.3.1 we have that

$$\sum_k i\partial_t \circ (i\partial_t + H_k^+(t, x, D))\mathbf{e}_k(t) \in \mathcal{I}^1(\chi_t).$$

So it remains to show that

$$\sum_k H_k^-(t,x,D)\circ(i\partial_t + H_k^+(t,x,D))\mathbf{e}_k(t) \in \mathcal{I}^1(\chi_t). \tag{3.3.14}$$

Let $\tilde{w}_k(\tilde{\gamma},\gamma), w_k(\tilde{\gamma},\gamma)$ denote the matrices associated to

$$H_k^-(t,x,D), \quad \big(i\partial_t + H_k^+(t,x,D)\big)\mathbf{e}_k(t)$$

respectively. Our strategy will be to show that for any 3 triples $\tilde{\gamma},\gamma,\gamma'$ with $|k_{\gamma'}-k|\le 1$ and $\delta>0$ that there exists C_δ independent of choice of triples such that

$$|\tilde{w}_k(\tilde{\gamma},\gamma)w_k(\gamma,\gamma')| \le C_\delta(1+\varrho(t))2^{k_\gamma}\mu_\delta(\tilde{\gamma},\gamma)\mu_\delta(\gamma,\chi_t(\gamma')).$$

This in turn implies that for any $\delta>0$

$$|\tilde{w}_k(\tilde{\gamma},\gamma)w_k(\gamma,\gamma')| \le C_\delta(1+\varrho(t))2^{k_{\gamma'}}\mu_{\delta-1}(\tilde{\gamma},\gamma)\mu_{\delta-1}(\gamma,\chi_t(\gamma')).$$

By Lemma 2.5 in [10] we have that there exists a constant $\tilde{C}_\delta$ depending only on $\delta>0$ and n such that

$$\sum_\gamma \mu_\delta(\tilde{\gamma},\gamma)\mu_\delta(\gamma,\chi_t(\gamma')) \le \tilde{C}_\delta\mu_\delta(\tilde{\gamma},\chi_t(\gamma')).$$

Since the matrix of the operator in (3.3.14) is given by $\sum_\gamma \tilde{w}_k(\tilde{\gamma},\gamma)w_k(\gamma,\gamma')$, the theorem will then follow.

When $k_\gamma \ge k-2$,

$$H_k^-(s,y,\eta)(\beta_{k_\gamma-1}(\eta)+\beta_{k_\gamma}(\eta)+\beta_{k_\gamma+1}(\eta))$$

is a symbol of order $S^1_{1,\frac{1}{2}}$, with symbol estimates that can be taken to be uniform in k. The desired inequalities are then a result of Lemma 3.3.2 and Theorem 3.3.1.

The case $k_\gamma < k-2$ is more involved. Integration by parts in the integral

$$(T_k\varphi_\gamma)(y) := 2^{-\frac{k(n+1)}{4}}\int e^{i\langle y-x,\eta\rangle}H_k^-(s,y,\eta)h^\omega_{k_\gamma}(\eta)d\eta$$

yields the estimates

$$|\partial_y^\alpha T_k\varphi_\gamma(y)| \le C_{\alpha,N}2^{k_\gamma+\frac{k_\gamma(n+1)}{4}+\max(k_\gamma,\frac{k}{2})|\alpha|}(1+2^{k_\gamma}|\langle\omega,y-x\rangle|+2^{k_\gamma}|y-x|^2)^{-N}. \tag{3.3.15}$$

This result, along with the estimates on the $\widehat{T_k\varphi_\gamma}$ it induces, gives us the following matrix estimates which are much weaker than that of Lemma 3.3.2

$$|\int (T_k\varphi_\gamma)\overline{\varphi_{\tilde\gamma}}| \le C_\delta 2^{k(2\delta+2n+3)}\mu_\delta(\tilde\gamma,\gamma).$$

However, due to the estimates of Theorem 3.3.1 when $k_\gamma < k-2$ we have

$$|\tilde w_k(\tilde\gamma,\gamma)w_k(\gamma,\gamma')| \\ \le C_\delta(1+\varrho(t))2^{k_\gamma}(2^{k(2\delta+2n+3)}\mu_\delta(\tilde\gamma,\gamma))(2^{-k(2\delta+2n+3)}\mu_\delta(\gamma,\chi_t(\gamma)')).$$

As noted above, this completes the proof. □

By handling the error terms, we are now able show that $\sum_k \mathbf{e}_k(t)$ is indeed an approximate solution operator for $\widetilde{P}$.

Theorem 3.3.4. *Let $\mathbf{e}_k(t)$ be defined by the integral curves of H_k^+ as in Theorem 3.3.3. Then for $-1\le z\le 1$,*

$$\sum_k \widetilde{P}(t,x,D)\mathbf{e}_k(t) : H^{z+1}(\mathbb{R}^n)\to H^z(\mathbb{R}^n),$$

with operator norm bounded by $C(\varrho(t)+1)$ for $t\in[-2,2]$.

Proof. Begin by writing $\sum_k \widetilde{P}(t,x,D)\mathbf{e}_k(t)$ as

$$\sum_k \widetilde{P}(t,x,D)\mathbf{e}_k(t) = \sum_k Q^k(t,x,D)\mathbf{e}_k(t) + \Big(\sum_2^n \tilde b_i(t,x')\partial_i + \tilde b_1(t,x')\partial_t\Big)\circ\sum_k \mathbf{e}_k(t) \\ + \sum_k (Q(t,x,D)-Q^k(t,x,D))\mathbf{e}_k(t). \quad (3.3.16)$$

We begin by considering the first term here. By equation (3.1.6) and Theorem 3.3.3 it suffices to control the sum $\sum_k E_k(t,x,D)\mathbf{e}_k(t)$ where

$$E_k(t,x,D) = -i(\partial_t H_k^+)(t,x,D) - i(\partial_t H_k^-)(t,x,D) \\ + (H_k^+\circ H_k^-)(t,x,D) - (H_k^+H_k^-)(t,x,D). \quad (3.3.17)$$

Here the symbol of $(\partial_t H_k^\pm)(t,x,D)$ is $(\partial_t H_k^\pm)(t,x,\xi)$ and the operator $(H_k^+H_k^-)(t,x,D)$ has symbol $H_k^+(t,x,\xi)H_k^-(t,x,\xi)$. By splitting the sum up into groups of integers that are equal modulo 4, it suffices to show that the following sum is a map that loses one derivative:

$$\sum_k E_{4k}(t,x,D)\mathbf{e}_{4k}(t) = \Big(\sum_k E_{4k}(t,x,D)(\beta_{4k-1}+\beta_{4k}+\beta_{4k+1})\Big)\circ\Big(\sum_k \mathbf{e}_{4k}(t)\Big).$$

Since $\sum_k \mathbf{e}_{4k}(t) \in \mathcal{I}^0(\chi_t)$, the sum is now handled by observing that the symbol of (3.3.17) satisfies

$$\sum_k E_{4k}(t,x,D)(\beta_{4k-1}+\beta_{4k}+\beta_{4k+1}) \in S^1_{1,\frac{1}{2}}$$

with symbol estimates bounded by $C(\varrho(t)+1)$.

Next observe that since $b_i(t,\cdot) \in Lip(\mathbb{R}^n)$ with Lipschitz norm bounded by $C(\varrho(t)+1)$ we have that multiplication by $b_i(t,\cdot)$ is a bounded map on $H^z(\mathbb{R}^n)$. Thus since

$$\partial_t^l \circ \sum_k \mathbf{e}_k(t) \in \mathcal{I}^l(\chi_t)$$

for $l = 0, 1$ we have that

$$(\sum_2^n \tilde{b}_i(t,\cdot)\partial_i + \tilde{b}_1(t,\cdot)\partial_t) \circ \sum_k \mathbf{e}_k(t) : H^{z+1}(\mathbb{R}^n) \to H^z(\mathbb{R}^n)$$

continuously with operator norm bounded by $C(\varrho(t)+1)$.

Controlling the last term in (3.3.16) reduces to showing that if $a = \tilde{a}_{ij}$ for some i,j or $a = \tilde{\rho}$, then the sum

$$\sum_k (a(t,x') - a^k(t,x'))\beta_k(D) : H^{z-1}(\mathbb{R}^n) \to H^z(\mathbb{R}^n) \tag{3.3.18}$$

for $-1 \leq z \leq 1$. However, this result follows by the same methods used in Theorem 4.5 of [10] except that in our case the norm of this operator is now bounded by $C(\varrho(t)+1)$ for some uniform constant C. □

We pause to remark that if $\mathbf{e}_k(t)$ is instead defined by the integral curves of H_k^-, then a symmetric line of reasoning shows that it is also an approximate solution operator for $\widetilde{P}$. This fact will be used in the following section.

3.4 Representing the solution

We now show how to represent the solutions to the pseudodifferential equation $\widetilde{P}w = F$. The main idea here is to use the approximate solution operator of the previous section to construct operators $\mathbf{c}(t,s)$, $\mathbf{s}(t,s)$ that have the following properties when $t = s$

$$\begin{aligned} \mathbf{c}(t,s)|_{t=s} &= I, & \partial_t \mathbf{c}(t,s)|_{t=s} &= 0, \\ \mathbf{s}(t,s)|_{t=s} &= 0, & \partial_t \mathbf{s}(t,s)|_{t=s} &= I. \end{aligned} \tag{3.4.1}$$

In addition, these operators should enjoy the following properties as continuous operators between Sobolev spaces

$$\begin{aligned} \mathbf{c}(t,s) : H^{z+1}(\mathbb{R}^n) &\to H^{z+1}(\mathbb{R}^n) & \mathbf{s}(t,s) : H^{z}(\mathbb{R}^n) &\to H^{z+1}(\mathbb{R}^n) \\ \widetilde{P} \circ \mathbf{c}(t,s) : H^{z+1}(\mathbb{R}^n) &\to H^{z}(\mathbb{R}^n) & \widetilde{P} \circ \mathbf{s}(t,s) : H^{z}(\mathbb{R}^n) &\to H^{z}(\mathbb{R}^n) \end{aligned} \tag{3.4.2}$$

for $t, s \in [-2,2]$ and $-1 \le z \le 1$.

Once this is accomplished we can show that given Cauchy data

$$(w_0, w_1, F) \in H^{z+1} \times H^z \times L^1_t([-2,2]; H^z)$$

there exists a function $G(t,x) \in L^1_t([-2,2]; H^z(\mathbb{R}^n))$ such that

$$w(t,x) = \mathbf{c}(t,0)w_0(x) + \mathbf{s}(t,0)w_1(x) + \int_0^t \mathbf{s}(t,s)G(s,x)ds$$

so that $w(t,x)$ satisfies $\widetilde{P}w = F$ and $w(0,x) = w_0(x)$, $\partial_t w(0,x) = w_1(x)$.

We begin by defining operators $\mathbf{c}_k(t,s), \mathbf{s}_k(t,s)$ as in Section 4 of [10]. For a given $\gamma = (x_\gamma, \omega_\gamma, k)$, let $(x^\pm_\gamma(t,s), \omega^\pm_\gamma(t,s), \Theta^\pm_\gamma(t,s))$ be the solution to

$$\begin{aligned} \frac{dx^\pm}{dt} &= -(H^\pm_k)_\xi(t,x,\omega) \\ \frac{d\omega^\pm}{dt} &= (H^\pm_k)_x(t,x,\omega) - \langle \omega, (H^\pm_k)_x(t,x,\omega)\rangle\omega \\ \frac{d\Theta^\pm}{dt} &= \Theta[\omega \cdot (H^\pm_k)_x(t,x,\omega)^T - (H^\pm_k)_x(t,x,\omega)\cdot\omega^T] \end{aligned} \tag{3.4.3}$$

with the initial conditions $(x^{\pm}_\gamma(t,s), \omega^{\pm}_\gamma(t,s), \Theta^{\pm}_\gamma(t,s))\big|_{t=s} = (x_\gamma, \omega_\gamma, I)$. For each γ with $k_\gamma > 0$, define the function ϑ_γ by $\widehat{\vartheta}_\gamma(\xi) = -i2^k \langle \omega_\gamma, \xi\rangle^{-1}\widehat{\varphi}_\gamma(\xi)$. We now set

$$\varphi^{\pm}_\gamma(t,s,y) = \varphi_\gamma(\Theta^{\pm}_\gamma(t,s)(y - x^{\pm}_\gamma(t,s)) + x_\gamma)$$
$$\vartheta^{\pm}_\gamma(t,s,y) = \frac{2}{H^+_k(s,x_\gamma,\omega_\gamma) - H^-_k(s,x_\gamma,\omega_\gamma)}\vartheta_\gamma(\Theta^{\pm}_\gamma(t,s)(y - x^{\pm}_\gamma(t,s)) + x_\gamma).$$

This leads us to define the operators $\mathbf{c}_k(t,s), \mathbf{s}_k(t,s)$ on a function $f = \sum_\gamma c_\gamma \varphi_\gamma$ by

$$(\mathbf{c}_k(t,s)f)(y) = \frac{1}{2}\sum_{k_\gamma = k} c_\gamma(\varphi^+_\gamma(t,s,y) + \varphi^-_\gamma(t,s,y))$$
$$(\mathbf{s}_k(t,s)f)(y) = \frac{1}{2}\sum_{k_\gamma = k} 2^{-k} c_\gamma(\vartheta^+_\gamma(t,s,y) - \vartheta^-_\gamma(t,s,y))$$

Now set $\tilde{\mathbf{c}}(t,s) = \sum_{k=0}^{\infty}\mathbf{c}_k(t,s)$ and $\tilde{\mathbf{s}}(t,s) = \sum_{k\geq k_0}\mathbf{s}_k(t,s) + (t-s)\sum_{k<k_0}\Delta_k$ where Δ_k is defined by

$$\Delta_k(\sum_\gamma c_\gamma\varphi_\gamma) = \sum_{k_\gamma = k} c_\gamma\varphi_\gamma.$$

The proof Lemma 4.4 of [10] uses only the fact that $\|(H^{\pm}_k)_x(\cdot,\cdot,\xi)\|_{L^\infty_{t,x}} \leq C$ for some C independent of choice of $\xi \in \mathbb{S}^{n-1}$ and frequency index k and is easily adapted to our circumstances. It states the following:

Lemma 3.4.1. *For k_0 sufficiently large and depending on the estimates on the coefficients $\{a_{ij}(t,x)\}_{ij}$, the operator $\partial_t\tilde{\mathbf{s}}(t,s)|_{t=s}$ admits a bounded inverse on $L^2(\mathbb{R}^n)$. This inverse extends to a bounded operator on $H^z(\mathbb{R}^n)$ and is continuous in s in the norm topology on $H^z(\mathbb{R}^n)$.*

We thus choose k_0 to be as large as required in the lemma and set

$$\mathbf{s}(t,s) = \tilde{\mathbf{s}}(t,s) \circ (\partial_t\tilde{\mathbf{s}}(t,s)|_{t=s})^{-1}.$$

Also, define

$$\mathbf{c}(t,s) = \tilde{\mathbf{c}}(t,s) - \mathbf{s}(t,s) \circ (\partial_t\tilde{\mathbf{c}}(t,s)|_{t=s}).$$

It is now observed that $\mathbf{c}$ and $\mathbf{s}$ satisfy the desired properties in (3.4.1) and (3.4.2).

We now define operators T_0, T_1 by

$$T_0(t,s) = P(t,x,D)\mathbf{c}(t,s) \qquad\qquad T_1(t,s) = P(t,x,D)\mathbf{s}(t,s).$$

A proof similar to that in Theorem 3.3.4 now shows the following:

Theorem 3.4.2. *$T_0(t,s), T_1(t,s)$ enjoy the following mapping properties for $-1 \leq z \leq 2$*

$$T_0(t,s) : H^{z+1}(\mathbb{R}^n) \to H^z(\mathbb{R}^n)$$
$$T_1(t,s) : H^z(\mathbb{R}^n) \to H^z(\mathbb{R}^n)$$

with operator norm bounded by $C(\varrho(t)+1)$ for $t \in [-2,2]$.

By an iteration procedure, we are now able to represent solutions to the desired hyperbolic pseudodifferential equation.

Theorem 3.4.3. *Suppose $w_0 \in H^{z+1}(\mathbb{R}^n), w_1 \in H^z(\mathbb{R}^n), F \in L^1_t([-2,2]; H^z(\mathbb{R}^n))$ for some $-1 \leq z \leq 1$. Then there exists $G \in L^1_t([-2,2]; H^z)$ with norm bounded by*

$$\|G\|_{L^1_t([-2,2];H^z)} \leq C(\|w_0\|_{H^{z+1}} + \|w_1\|_{H^z} + \|F\|_{L^1_t([-2,2];H^z)})$$

such that

$$w(t,x) = (\mathbf{c}(t,0)w_0)(x) + (\mathbf{s}(t,0)w_1)(x) + \int_0^t (\mathbf{s}(t,s)F(s,\cdot))(x)\, ds$$

is a solution to the Cauchy Problem

$$\begin{aligned} w(t,x)|_{t=0} &= w_0(x) \\ \partial_t w(t,x)|_{t=0} &= w_1(x) \\ (\widetilde{P}(t,x,D)w)(t,x) &= F(t,x) \end{aligned} \tag{3.4.4}$$

Proof. This is also a slight modification of the techniques in Theorem 4.6 of [10]. As in that theorem, we set $v(t,x) = \int_0^t \mathbf{s}(t,s)G(s,x)\, ds$ for some $G(s,x) \in L^1_t H^\alpha_x$ to be determined. By a computation in that Theorem we have that

$$v(t,x) \in C([-2,2]; H^{\alpha+1}) \cap C^1([-2,2]; H^\alpha).$$

Also, $\partial_t v(t,x) = \int_0^t \partial_t \mathbf{s}(t,s)G(s,x)\,ds$ so that $v(0,x)=0$, $\partial_t v(0,x)=0$. We thus want v to satisfy

$$w(t,x) = (\mathbf{c}(t,0)w_0)(x) + (\mathbf{s}(t,0)w_1)(x) + v(t,x)$$

as this means that w takes on the desired initial values at $t=0$. Hence G should satisfy the equation

$$F(t,x) - (T_0(t,0)w_0)(x) + (T_1(t,0)w_1)(x) = G(t,x) + \int_0^t T_1(t,s)G(s,x)\,ds.$$

In order to solve this equation, set

$$\tilde{F}(t,x) = F(t,x) - (T_0(t,0)w_0)(x) + (T_1(t,0)w_1)(x)$$

and then define $G_n(t,x)$ to be

$$G_n(t,x) = \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} T_1(t,s_1)T_1(s_1,s_2)\cdots T_1(s_{n-1},s_n)\tilde{F}(s_n,x)ds_n \cdots ds_1$$

and $G(t,x) = \sum_{n=1}(-1)^n G_n(t,x) + \tilde{F}(t,x)$. To see that this sum converges absolutely in $L^1_t([-2,2];H^z)$ to a solution to the integral equation first observe that

$$\|G_n(t,\cdot)\|_{H^z} \leq \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} C^n(\varrho(t)+1)(\varrho(s_1)+1)\cdots(\varrho(s_{n-1})+1)\|\tilde{F}(s_n,\cdot)\|_{H^z} ds_n \cdots ds_1.$$

Since

$$\int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} (\varrho(s_1)+1)\cdots(\varrho(s_{n-1})+1)ds_{n-1}\cdots ds_1 = \frac{1}{(n-1)!}\Big(\int_0^t (\varrho(s)+1)ds\Big)^{n-1} \leq \frac{1}{(n-1)!}(4+\|\varrho\|_{L^1(-2,2)})^{n-1}$$

we have that

$$\|G_n(t,\cdot)\|_{H^z} \leq \|\tilde{F}\|_{L^1_t([-2,2];H^z(\mathbb{R}^n))}(\varrho(t)+1)C^n(1+\|\varrho\|_{L^1(-2,2)})^{n-1}/(n-1)!.$$

Hence the sum does indeed converge absolutely and we have the desired bounds

$$\begin{aligned}\|G\|_{L^1_t([-2,2];H^z(\mathbb{R}^n))} &\leq C_1\|\tilde{F}\|_{L^1_t([-2,2];H^z(\mathbb{R}^n))}\\ &\leq C_2(\|w_0\|_{H^{z+1}(\mathbb{R}^n)} + \|w_1\|_{H^z(\mathbb{R}^n)} + \|F\|_{L^1_t([-2,2];H^z(\mathbb{R}^n))})\end{aligned}$$

□

Chapter 4

ESTIMATES VIA THE PARAMETRIX

We now have the tools to represent solutions w to the wave equation over the domain $\mathbb{R} \times (-2,2) \times \mathbb{R}^{n-1}$ whose partial Fourier transform in t, x' is supported in the set

$$\operatorname{supp}(\widehat{w}(\tau, x_1, \xi')) \subset \{(\tau, \xi') : \tau > 11|\xi'|\} \setminus B_4(0). \tag{4.0.1}$$

In this chapter, given s, p, q as in Theorem 1.6.1 we will thus discuss how to show the following estimate for such w that can be represented by the wave packet construction of Chapter 3:

$$\|w\|_{L^p_t L^q_x([-1,1]\times(-2,2)\times\mathbb{R}^{n-1}_{x'})} \lesssim \|w_0\|_{H^{s+1}_{t,x'}} + \|w_1\|_{H^s_{t,x'}} + \|\widetilde{P}w\|_{L^1_{x_1}((-2,2);H^s_{t,x'})} \tag{4.0.2}$$

Here $w_0(t,x') = w(t,0,x')$ and $w_1(t,x') = \partial_{x_1} w(t,0,x')$. In practice, $\widehat{w}(\tau, x_1, \xi')$ will always be localized to a dyadic region in (τ, ξ') and hence (4.0.2) allows us to show the inequalities in the hypotheses of Theorem 2.2.1.

Now that the parametrix has been constructed we go back to the original roles of t, x_1 so that $\widetilde{P}$ is an operator that is hyperbolic in x_1. This also means that the our evolution operators $\mathbf{c}, \mathbf{s}$ are parameterized by x_1 so that our w takes the form

$$w(t,x) = (\mathbf{c}(x_1,0)w(\cdot,0,\cdot))(t,x') + (\mathbf{s}(x_1,0)\partial_{x_1} w(\cdot,0,\cdot))(t,x') + \int_0^{x_1} (\mathbf{s}(x_1,y_1)G(\cdot,y_1,\cdot))(t,x')\, dy_1.$$

Thus in order to show the estimates (4.0.2), it suffices to establish that

$$\|\bar{\mathbf{c}}(x_1,0)h\|_{L^p_t L^q_x([-1,1]\times[-2,2]\times\mathbb{R}^{n-1}_{x'})} \le C\|h\|_{H^{s+1}(\mathbb{R}^n_{t,x'})}, \tag{4.0.3}$$

$$\|\bar{\mathbf{s}}(x_1,y_1)h\|_{L^p_t L^q_x([-1,1]\times[-2,2]\times\mathbb{R}^{n-1}_{x'})} \le C\|h\|_{H^s(\mathbb{R}^n_{t,x'})}.$$

In this chapter, $\{\psi_\nu(t,x')\}_\nu$ will denote the family of wave packets constructed in section 3.2 except taken as functions of (t,x') and indexed by ν rather than γ. Hence ν ranges over triples of the form $\nu = (k_\nu, (t_\nu, x'_\nu), \omega_\nu)$. We will reserve the notation $\{\varphi_\gamma(x)\}_\gamma$ to denote the family of wave packets taken to be functions in x.

Let $\widetilde{H}^{\pm}(x,\tau,\xi'), \widetilde{H}^{\pm}_k(x,\tau,\xi')$ denote the roots of the principal symbols of $\widetilde{P}$, $\widetilde{P}^k$ as quadratics in ξ_1. The flow on the cosphere bundle is thus determined by solutions to the ODE

$$\begin{aligned}\frac{dt}{dx_1} &= -(\widetilde{H}^+_k)_\tau(x_1,x',\omega)\\ \frac{dx'}{dx_1} &= -(\widetilde{H}^+_k)_{\xi'}(x_1,x',\omega)\\ \frac{d\omega}{dx_1} &= (\widetilde{H}^+_k)_{t,x'}(x_1,x',\omega) - \langle\omega, (\widetilde{H}^+_k)_{t,x'}(x_1,x',\omega)\rangle\omega\\ \frac{d\Theta}{dx_1} &= \Theta[\omega\cdot(\widetilde{H}^+_k)_{t,x'}(x_1,x',\omega)^T - (\widetilde{H}^+_k)_{t,x'}(x_1,x',\omega)\cdot\omega^T].\end{aligned} \tag{4.0.4}$$

with a similar definition of the flow when we consider $\widetilde{H}^-_k$, $\widetilde{H}^{\pm}$.

Thus given a triple $\nu = (k_\nu, (t_\nu, x'_\nu), \omega_\nu)$, set

$$\psi_\nu(t,x) = \psi_\nu(\Theta_\nu(x_1)(t - t_\nu(x_1), x' - x_\nu(x_1)') + (t_\nu, x'_\nu))$$

where $((t_\nu(x_1), x_\nu(x_1)'), \omega_\nu(x_1), \Theta_\nu(x_1))|_{x_1=0} = ((t_\nu, x'_\nu), \omega_\nu, I)$ and is a solution to (4.0.4), with $k = k_\nu$. These are the traveling wave packets constructed in Chapter 3 now parameterized by x_1 rather than t.

We first want to show that we need only consider wave packets whose Fourier transforms lie in a small cone about the τ-axis. Let $\Phi, \widetilde{\Phi}$ be smooth functions in (τ,ξ') such that $\text{supp}(\Phi) \subset \{(\tau,\xi') : \tau \geq 10|\xi'|\} \setminus B_3(0)$, $\text{supp}(\widetilde{\Phi}) \subset \{(\tau,\xi') : \tau \geq 8|\xi'|\} \setminus B_{\frac{1}{2}}(0)$ and identically 1 on the sets $\{(\tau,\xi') : \tau \geq 11|\xi'|\} \setminus B_4(0)$ and $\{(\tau,\xi') : \tau \geq 9|\xi'|\} \setminus B_{\frac{2}{3}}(0)$ respectively.

For any integral curve $(t_\nu(z), x_\nu(z)', \omega_\nu(z))$ of $\widetilde{H}^{\pm}$ on the cosphere bundle, we have that $|\omega_\nu(z_1) - \omega_\nu(z_2)| \leq \mathcal{O}(\varepsilon)$ for $z_1, z_2 \in (-2,2)$ as $\|a_{ij} - \delta_{ij}\|_{Lip}, \|\rho - 1\|_{Lip} < \varepsilon$ by our assumption in (2.1.4). Thus by choosing $\varepsilon > 0$ to be sufficiently small, we can fix things so that given any wave packet with $\omega_\nu(z_0) \notin \{(\tau,\xi') : \tau \geq 8|\xi'|\}$ for some $z_0 \in (-2,2)$, then $\omega_\nu(z) \notin \{(\tau,\xi') : \tau \geq 10|\xi'|\}$ for any $z \in (-2,2)$. This implies that $\Phi(D)\bar{\mathbf{c}}(x_1,y_1) = \Phi(D)\bar{\mathbf{c}}(x_1,y_1)\widetilde{\Phi}(D)$ and $\Phi(D)\bar{\mathbf{s}}(x_1,y_1) = \Phi(D)\bar{\mathbf{s}}(x_1,y_1)\widetilde{\Phi}(D)$, for any $x_1, y_1 \in (-2,2)$.

We pause to remark that a similar line of reasoning also implies that

$$\bar{\mathbf{c}}(x_1,y_1)\Phi(D) = \widetilde{\Phi}(D)\bar{\mathbf{c}}(x_1,y_1)\Phi(D) \text{ and } \bar{\mathbf{s}}(x_1,y_1)\Phi(D) = \widetilde{\Phi}(D)\bar{\mathbf{s}}(x_1,y_1)\Phi(D),$$

for any $x_1, y_1 \in (-2,2)$, though this fact will be used later on.

Thus if w satisfies the condition in (4.0.1) we have that

$$\begin{aligned} w(t,x) = (\Phi w)(t,x) = (\Phi\mathbf{c}(x_1,0)\widetilde{\Phi}w(\cdot,0,\cdot))(t,x') + (\Phi\mathbf{s}(x_1,0)\widetilde{\Phi}\partial_{x_1}w(\cdot,0,\cdot))(t,x') \\ + \int_0^{x_1} (\Phi\mathbf{s}(x_1,y_1)\widetilde{\Phi}G(\cdot,y_1,\cdot))(t,x')\, dy_1. \end{aligned}$$

Hence in order to get the estimates (4.0.3) for such w it suffices to show

$$\begin{aligned} \|\Phi\bar{\mathbf{c}}(x_1,0)\widetilde{\Phi}h\|_{L^p_t L^q_x([-1,1]\times[-2,2]\times\mathbb{R}^{n-1}_{x'})} &\leq C\|h\|_{H^{s+1}(\mathbb{R}^n_{t,x'})}, \\ \|\Phi\bar{\mathbf{s}}(x_1,y_1)\widetilde{\Phi}h\|_{L^p_t L^q_x([-1,1]\times[-2,2]\times\mathbb{R}^{n-1}_{x'})} &\leq C\|h\|_{H^{s}(\mathbb{R}^n_{t,x'})} \end{aligned} \tag{4.0.5}$$

with C independent of choice of $x_1, y_1 \in [-2,2]$. For simplicity, we will consider the estimates on $\bar{\mathbf{c}}(x_1,0)$ as the estimates on $\bar{\mathbf{s}}(x_1,y_1)$ will follow by similar reasoning. Furthermore, we will only consider the half of $\bar{\mathbf{c}}(x_1,0)$ that translates packets along the curves of $\widetilde{H}^+$ as an identical argument establishes dispersive estimates for the other half of the operator.

The crucial step in the proof of the estimates (4.0.5) is to fix t and show that the family

$$\{\psi_\nu(t,\cdot) : \omega_\nu \in \operatorname{supp}(\widetilde{\Phi}), |t_\nu| < 4\}$$

behaves much like a frame of functions on $L^2(\mathbb{R}^n)$ in the x-variable. We will show that this family shares 2 key properties with the wave packet frame $\{\varphi_\gamma(x)\}_\gamma$. The first is that a function $\psi_\nu(t,\cdot)$ is highly concentrated in space and in frequency much like a single wave packet (though not strictly localized in frequency). To make this rigorous, we will use the weight functions μ_δ to show that $\psi_\nu(t,\cdot)$ is centered at a certain point in $\mathbb{N}\times S^*(\mathbb{R}^n_x)$ in the sense that its representation in the wave packet frame $\{\varphi_\gamma(x)\}_\gamma$ is rapidly decreasing as γ moves away from that point. The second property is that these points, the centers of the $\{\psi_\nu(t,\cdot)\}_\nu$, are spread out, similar to the way the collection of indices $\gamma=(k_\gamma,x_\gamma,\omega_\gamma)$ are evenly spaced. This allows us to prove a key technical lemma analogous to Lemma 2.5 in [10].

Once this is accomplished, we will see that the center of a function $\psi_\nu(t,\cdot)$ in the frame $\{\varphi_\gamma(x)\}_\gamma$ is the image of the center of the function $\psi_\nu(0,\cdot)$ under the transformation at time t determined by a Hamiltonian flow. Specifically, the flow parameterized by t and determined by the Hamiltonian $(\tilde{\rho}^k)^{-\frac{1}{2}}\langle\tilde{A}^k(x)\xi,\xi\rangle^{\frac{1}{2}}$, with $\tilde{A}^k(x)$ denoting the matrix of coefficients $\{\tilde{a}^k_{ij}(x)\}_{ij}$ and $k=k_\nu$. This entire approach amounts to arguing that the family of $\{\psi_\nu(t,\cdot)\}_\nu$ behave roughly as if they were wave packets in x translated along characteristics in t. We are then able to adapt the techniques used in Lemmas 6.2 and 6.3 of [10] to complete the proof.

We first reduce the matter to obtaining estimates on wave packets at a fixed frequency shell. Let S_k be the set of all ν such that $\omega_\nu\in\text{supp}(\widetilde{\Phi})$ and $k_\nu=k$. We consider estimates on the operator B_k mapping $l^2(S_k)$ to measurable functions on $\mathbb{R}\times[-2,2]\times\mathbb{R}^{n-1}$ given by

$$B_k\{c_\nu\}_{\nu\in S_k}=\sum_{\nu\in S_k}c_\nu\psi_\nu(t,x).$$

We wish to show that there exists C independent of k such that

$$\|B_k\{c_\nu\}_{\nu\in S_k}\|_{L^p_tL^q_x([-1,1]\times(-2,2)\times\mathbb{R}^{n-1}_{x'})}\leq C2^{k(s+1)}\|\{c_\nu\}_{\nu\in S_k}\|_{l^2(S_k)} \tag{4.0.6}$$

for p,q,s as in Theorem 1.6.1. Thus if $h=\sum_\nu c_\nu\psi_\nu$ for some $h\in L^2(\mathbb{R}^n_{t,x})$ this implies the stronger estimate

$$\|B_k\{c_\nu\}_{\nu\in S_k}\|_{L^p_tL^q_x([-1,1]\times(-2,2)\times\mathbb{R}^{n-1}_{x'})}\leq C2^{k(s+1)}\|h_k\|_{L^2(\mathbb{R}^n_{t,x'})}$$

where $\{h_k\}_{k\geq 0}$ is an appropriate Littlewood-Paley decomposition of h. Since the $B_k\{c_\nu\}_{\nu\in S_k}$ are dyadically localized to frequencies near 2^k in the t,x' variables, Littlewood-Paley theory will then imply (4.0.5) as

$$\|\sum_k B_k\{c_\nu\}_{\nu\in S_k}\|_{L^p_tL^q_x} \lesssim (\sum_k \|B_k\{c_\nu\}_{\nu\in S_k}\|^2_{L^p_tL^q_x})^{\frac{1}{2}} \lesssim (\sum_k 2^{2k(s+1)}\|h_k\|^2_{L^2(\mathbb{R}^n_{t,x'})})^{\frac{1}{2}}$$
$$\lesssim \|h\|_{H^{s+1}(\mathbb{R}^n_{t,x'})}$$

and $\sum_k B_k$ represents the half of $\bar{\mathbf{c}}(x_1,0)$ that translates packets along $\widetilde{H}^+$.

Let $\tilde{\psi} \in C_0^\infty(\mathbb{R})$ be a smooth bump function equal to 1 on the interval $[-2,2]$ and supported in $(-\frac{5}{2},\frac{5}{2})$. Define $\widetilde{B}_k$ as a map from $l^2(S_k)$ to measurable functions on $\mathbb{R}^{n+1}$ by

$$\widetilde{B}_k\{c_\nu\}_{\nu\in S_k} = \sum_{\nu\in S_k} c_\nu\tilde{\psi}(x_1)\psi_\nu(t,x)$$

since $\tilde{\psi}$ is compactly supported this is well-defined for any $x_1\in\mathbb{R}$. We now focus on proving a stronger inequality than (4.0.6):

$$\|\widetilde{B}_k\{c_\nu\}_{\nu\in S_k}\|_{L^p_tL^q_x([-1,1]\times\mathbb{R}^n)} \le C2^{k(s+1)}\|\{c_\nu\}_{\nu\in S_k}\|_{l^2(S_k)}. \tag{4.0.7}$$

Hence for the rest of this chapter we will replace ψ_ν by $\tilde{\psi}\psi_\nu$ as we will find the property that ψ_ν is defined on all of $\mathbb{R}^{n+1}$ useful in the analysis below.

A duality argument reduces (4.0.7) to showing

$$\|\widetilde{B}_k\widetilde{B}_k^*G\|_{L^p_tL^q_x([-1,1]\times\mathbb{R}^n)} \lesssim 2^{2k(s+1)}\|G\|_{L^{p'}_tL^{q'}_x([-1,1]\times\mathbb{R}^n)}. \tag{4.0.8}$$

We now characterize $\widetilde{B}_k\widetilde{B}_k^*$ by

$$(\widetilde{B}_k\widetilde{B}_k^*G)(t,x) = \int (W^k_{t,s}G(s,\cdot))(x)\,ds$$

where

$$(W^k_{t,s}G(s,\cdot))(x_1,x') = \sum_\nu \psi_\nu(t,x)\int_{-1}^1 \overline{\psi_\nu(s,y)}\,G(s,y)\,dy_1\,dy'.$$

The inequality (4.0.8) will now come from interpolation on the pair of estimates in Theorem 4.0.4 below (often called the "dispersive estimates") followed by an application of the Hardy-Littlewood-Sobolev inequality. This latter inequality states that if $1+\frac{1}{r}=\frac{1}{q}+\frac{1}{p}$, with $1<r<p<\infty$, then $\|\tilde{h}*|y|^{-1/q}\|_{L^p(\mathbb{R})} \le C_{p,r}\|\tilde{h}\|_{L^r(\mathbb{R})}$ (see Chapter 8, Section 4.2 of [12] for a discussion of this result). We are now led to the main result of this chapter.

Theorem 4.0.4. *In dimensions $n = 2, 3$, the operator $W^k_{t,s}$ enjoys the following mapping properties:*

$$\|W^k_{t,s} f\|_{L^2(\mathbb{R}^n)} \leq C\|f\|_{L^2(\mathbb{R}^n)}$$
$$\|W^k_{t,s} f\|_{L^\infty(\mathbb{R}^n)} \leq C2^{kn}(1 + 2^k|t - s|)^{-\frac{n-1}{2}}\|f\|_{L^1(\mathbb{R}^n)}$$

with C independent of $t, s \in [-1, 1]$.

Begin by setting $K := \{\nu \in S_k : |t_\nu| < 4\}$ and split the integral kernel of $W^k_{t,s}$ into 2 parts

$$\sum_\nu \psi_\nu(s, y)\psi_\nu(t, x) = \sum_{\nu \in K^C} \psi_\nu(s, y)\psi_\nu(t, x) + \sum_{\nu \in K} \psi_\nu(s, y)\psi_\nu(t, x)$$

Observe that if $|t_\nu| \geq 4$, and $\varepsilon > 0$ is chosen to be sufficiently small in (2.1.4), then when $x_1 \in (-\frac{5}{2}, \frac{5}{2})$, $|t_\nu(x_1)| \geq |t_\nu| - |t_\nu(x_1) - t_\nu| \geq 4 - \frac{11}{4}$. Hence $|t_\nu(x_1)|$ stays a uniform distance of $\frac{1}{4}$ away from the interval $[-1, 1]$ so by modifying the standard estimates on the localization of ψ_ν in space, we have that

$$|\psi_\nu(t, x)| \leq C_{N_1,N_2} 2^{-kN_1}(1 + 2^k|t - t_\nu| + 2^{\frac{k}{2}}|x' - x'_\nu|)^{-N_2}.$$

for any $N_1, N_2 > 0$. By choosing N_1, N_2 sufficiently large it is not difficult to obtain C independent of k such that

$$|\sum_{\nu \in K^C} \psi_\nu(s, y)\psi_\nu(t, x)| \leq C$$
$$\int_{\mathbb{R}^n} |\sum_{\nu \in K^C} \psi_\nu(s, y)\psi_\nu(t, x)|\, dx'\, dx_1 \leq C$$

This shows that the first part of the kernel has the desired mapping properties. It thus suffices to show the estimates assuming that the kernel is sums only over $\nu \in K$.

Since we are looking at wave packets at a fixed frequency 2^k translated along a fixed flow in what follows we abbreviate $\widetilde{H}^+_k$ as $\widetilde{H}$ and specify whenever we need to work with the rough Hamiltonian. Also, we will always assume implicitly that $\Psi = 1$ in the definition of $\widetilde{P}, \widetilde{P}^k$ as we are restricting our attention to a sufficiently small cone about the τ-axis.

This first lemma uses the weight functions to show that for $\nu \in K$, $\psi_\nu(t,\cdot)$ is indeed concentrated in space and in frequency much like a wave packet.

Lemma 4.0.5. *Suppose $(t_\nu(x_1), x_\nu(x_1)', \omega_\nu(x_1))$ is a solution of (4.0.4) with initial conditions $(t_\nu, x_\nu', \omega_\nu)$, such that $t_\nu(z) = t$ for some $z \in [-16, 16]$ and $\omega_\nu \in supp(\widetilde{\Phi})$. Then for any wave packet $\varphi_\gamma(x)$ and integer N, we have that*

$$\left|\int \psi_\nu(t,x)\varphi_\gamma(x)\,dx\right| \le \begin{cases} C_N \mu_N(\gamma; k_\nu, (z, x_\nu(z)'), \pi(\zeta_\nu(z), \omega_\nu(z)')) & |k_\nu - k_\gamma| \le 2 \\ C_N 2^{-\max(k_\nu,k_\gamma)N} \mu_N(\gamma; k_\nu, (z, x_\nu(z)'), \pi(\zeta_\nu(z), \omega_\nu(z)')) & |k_\nu - k_\gamma| \ge 3 \end{cases}$$

where $\zeta_\nu(z) = \widetilde{H}(z, x_\nu(z)', \omega_\nu(z))$, $\pi(\eta) = \eta/|\eta|$ denotes projection onto the unit sphere, and C_N is independent of γ, ν.

This lemma therefore implies that the center of $\psi_\nu(t,x)$ in the frame $\{\varphi_\gamma(x)\}_\gamma$ is

$$\big(k_\nu, (z, x_\nu(z)'), \pi(\zeta_\nu(z), \omega_\nu(z)')\big).$$

Proof. Throughout this proof t will remain fixed. Let $\eta \mapsto \pi_\nu^\perp(\eta)$ denote projection onto the subspace orthogonal to $(\zeta_\nu(z), \omega_\nu(z)')$. The key idea in this proof is to show the following inequalities:

$$|\langle(\zeta_\nu(z),\omega_\nu(z)')^\perp, \partial_x\rangle^{\beta_1}\partial_x^{\beta_2}\psi_\nu(t,x)| \le \tag{4.0.9}$$
$$C_N 2^{\frac{k}{2}(2|\beta_1|+|\beta_2|)}(1 + 2^k|\langle x - (z, x_\nu(z)'), (\zeta_\nu(z), \omega_\nu(z)')\rangle| + 2^k|x - (z, x_\nu(z)')|^2)^{-N}.$$

$$|\widehat{\psi_\nu}(t,\eta)| \le C_N 2^{-\frac{k(n+1)}{4}}(1 + 2^{-k}|\pi_\nu^\perp(\eta)|^2 + 2^{-2k}|\eta|^2)^{-N} \tag{4.0.10}$$

$$|\widehat{\psi_\nu}(t,\eta)| \le C_N 2^{-\frac{3k(n+1)}{4}} \int_{\mathrm{supp}(\widehat{\psi_\nu})} \left(1 + 2^{-k}|\eta - |(\tau,\xi')|\,(\zeta_\nu(z),\omega_\nu(z)')\,|^2\right)^{-N} d\tau d\xi'. \tag{4.0.11}$$

Indeed, by using the techniques of Lemma 3.3.2, the lemma follows from these inequalities once it is observed that (4.0.9), (4.0.10), and (4.0.11) are analogues of (3.3.10), (3.3.11) and (3.3.13) respectively.

Set $\Phi_\nu(t,x) = \Theta_\nu(x_1)(t - t_\nu(x_1), x' - x_\nu(x_1)')$ so that $\psi_\nu(t,x) = \psi_\nu(\Phi_\nu(t,x) + (t_\nu, x'_\nu))$. Since Θ_ν is orthogonal and $t_\nu(z) = t$, standard wave packet estimates give us that

$$|\psi_\nu(t,x)| \le C_N(1 + 2^k|\langle \omega_\nu, \Phi_\nu(t,x)\rangle| + 2^{\frac{k}{2}}|t_\nu(z) - t_\nu(x_1)| + 2^{\frac{k}{2}}|x' - x_\nu(x_1)'|)^{-N} \quad (4.0.12)$$

By performing a Taylor expansion about z on the curve $(t_\nu(x_1), x_\nu(x_1)')$ with second order error term $r_\nu(x_1)$ we can use the fact that $\Theta_\nu(x_1)^T\omega_\nu = \omega_\nu(x_1)$ to write

$$\begin{aligned}\langle \Phi_\nu(t,x), \omega_\nu\rangle &= \langle (t - t_\nu(x_1), x' - x_\nu(x_1)'), \omega_\nu(x_1)\rangle \\ &= \langle (t - t_\nu(z), x' - x_\nu(z)') - (\dot{t}_\nu(z), \dot{x}_\nu(z)')(x_1 - z) + r_\nu(x_1), \omega_\nu(x_1)\rangle.\end{aligned}$$

We can now use the fact that $(\dot{t}_\nu(z), \dot{x}_\nu(z)') = -\widetilde{H}_{\tau,\xi'}(x_1, x_\nu(z)', \omega_\nu(z))$ and homogeneity to get

$$\langle \Phi_\nu(t,x), \omega_\nu\rangle = \langle (x_1 - z, x' - x_\nu(z)'), (\zeta_\nu(z), \omega_\nu(z)')\rangle + e_\nu(x) \quad (4.0.13)$$

where

$$\begin{aligned}e_\nu(x) = \langle r_\nu(x_1), \omega_\nu(x_1)\rangle &+ \langle (t - t_\nu(z), x' - x_\nu(z)'), \omega_\nu(x_1) - \omega_\nu(z)\rangle \\ &+ \langle (t_\nu(z) - t_\nu(x_1), x_\nu(z) - x_\nu(x_1)), \omega_\nu(x_1) - \omega_\nu(z)\rangle\end{aligned}$$

and hence $|e_\nu(x)| \lesssim |(x_1 - z, x' - x_\nu(z)')|^2$ as $(t_\nu(x_1), x_\nu(x_1)')$ has bounded first and second derivatives. By choosing ε sufficiently small in (2.1.4), we also have $|t_\nu(z) - t_\nu(x_1)| \ge \frac{1}{2}|z - x_1|$ implying that

$$|t_\nu(z) - t_\nu(x_1)| + |x' - x_\nu(x_1)'| \gtrsim |(x_1 - z, x' - x_\nu(z)')|.$$

Hence as a result of (4.0.12) we have that for any $N > 0$

$$|\psi_\nu(t,x)| \le C_N(1 + 2^k|\langle (x_1, x') - (z, x_\nu(z)'), (\zeta_\nu(z), \omega_\nu(z)')\rangle| + 2^k|(x_1, x') - (z, x_\nu(z)')|^2)^{-N}.$$

In addition, observe that $\partial_x^\alpha e_\nu(x)$ is a finite sum of functions of the form $h_0^{k,\alpha}$, $h_1^{k,\alpha}$, and $h_2^{k,\alpha}$ that have following estimates

$$|h_l^{k,\alpha}(x)| \le C_\alpha 2^{\frac{k}{2}(|\alpha| + l - 2)}|(x_1 - z, x' - x_\nu(z)')|^l$$

with C_α independent of ν, k. We now write the phase of the defining integral

$$\psi_\nu(t,x) = 2^{-\frac{k(n+1)}{4}} \int e^{i\langle \Phi_\nu(t,x), (\tau, \xi')\rangle} h_k^{\omega_\nu}(\tau, \xi') d\tau d\xi'$$

as

$$i\langle \Phi_\nu(t,x),(\tau,\xi')\rangle = i\langle (x_1 - z, x' - x_\nu(z)'),(\zeta_\nu(z),\omega_\nu(z)')\rangle |(\tau,\xi')| \\ + ie_\nu(x)|(\tau,\xi')| + i\langle \Phi_\nu(t,x),(\tau,\xi') - |(\tau,\xi')|\omega_\nu\rangle.$$

By differentiating under the integral and applying (4.0.12), the estimates (4.0.9) follow. We are now able to integrate by parts in (x_1, x') in the integral defining $\widehat{\psi_\nu}(t,\eta)$ as in Lemma 3.3.2 to obtain (4.0.10) and (4.0.11). □

To see that the centers of the $\psi_\nu(t,\cdot)$ in the frame $\{\varphi_\gamma\}$ are indeed spread out, we show that the pseudodistance between the centers of 2 of these functions at a fixed time slice in the frame $\{\varphi_\gamma(x)\}_\gamma$ is bounded below by the pseudodistance between their centers as functions lying on the hyperplane $x_1 = 0$.

Lemma 4.0.6. *Let*

$$x_1 \mapsto (t_{x_1}, x'_{x_1}, \omega_{x_1}) \qquad \text{and} \qquad x_1 \mapsto (s_{x_1}, y'_{x_1}, \upsilon_{x_1})$$

denote integral curves of $\widetilde{H}$ projected onto the cosphere with initial values lying in the set

$$\{(t,x',\tau,\xi') \in (-4,4)\times\mathbb{R}^{n-1}\times\mathbb{R}^n : \tau \geq 8|\xi'|\}.$$

Suppose that in addition $t = t(z) = s(w)$ for $z, w \in (-16,16)$. Then there exists a uniform constant c such that

$$d(z, x'_z, \pi(\widetilde{H}(z,x'_z,\omega_z),\omega'_z); w, y'_w, \pi(\widetilde{H}(w,y'_w,\upsilon_w),\upsilon'_w)) \\ \gtrsim d(t_w, x'_w, \omega_w; s_w, y'_w, \upsilon_w) \approx d(t_0, x'_0, \omega_0; s_0, y'_0, \upsilon_0)$$

where π denotes projection onto the unit sphere as before.

Proof. Consider the map F defined over $(-16,16)\times\mathbb{R}^{n-1}\times\{(\tau,\xi')\neq 0 : \tau \geq 2|\xi'|\}$ given by $F(z,x',\tau,\xi') = (z,x',\widetilde{H}(z,x',\tau,\xi'),\xi')$ and let $\tilde{D}$ denote the submanifold

$$(-16,16)\times\mathbb{R}^{n-1}\times\{(\tau,\xi')\in\mathbb{S}^{n-1} : \tau \geq 2|\xi'|\}$$

Since $\widetilde{H}_\tau$ is bounded from below on this set, it can be shown that F descends to a map $\widetilde{F} = (\mathrm{Id}\times\pi)\circ F|_{\tilde{D}}$ that is a diffeomorphism onto its image with upper bounds on the pushforward $(\widetilde{F}^{-1})_*$ under the natural coordinates inheirited by $\mathbb{R}^{2n}$. This allows us to conclude there exists c uniform such that

$$|z-w|^2 + |x'_z - y'_w|^2 + \left|\pi(\widetilde{H}(z, x'_z, \omega_z), \omega'_z) - \pi(\widetilde{H}(w, y'_w, \upsilon_w), \upsilon'_w)\right|^2 \\ \geq c(|z-w|^2 + |x'_z - y'_w|^2 + |\omega_z - \upsilon_w|^2).$$

From here it is not hard to see that the quantity on the right is greater than

$$\tilde{c}(|z-w|^2 + |(t_w, x'_w) - (s_w, y'_w)|^2 + |\omega_w - \upsilon_w|^2)$$

with uniform constant $\tilde{c}$. We can now apply an argument similar to that used in establishing (4.0.13) to get

$$\left\langle \left(\widetilde{H}(z, x'_z, \omega_z), \omega'_z\right), \left(w - z, y'_w - x'_z\right)\right\rangle = \langle \omega_w, (t - t_w, y'_w - x'_w)\rangle - e(w, y'_w)$$

with $|e(w, y'_w)| \lesssim |(t_z, x'_z) - (s_w, y'_w)|^2$. This now implies that

$$|z-w|^2 + |x'_z - y'_w|^2 + |\pi(\widetilde{H}(z, x'_z, \omega_z), \omega'_z) - \pi(\widetilde{H}(w, y'_w, \upsilon_w), \upsilon'_w)|^2 \\ + \left|\left\langle \left(\widetilde{H}(z, x'_z, \omega_z), \omega'_z\right), \left(w - z, y'_w - x'_z\right)\right\rangle\right| \\ \gtrsim |\omega_w - \upsilon_w|^2 + |(t_w, x'_w) - (s_w, y'_w)|^2 + |\langle \omega_w, (t_w, x'_w) - (s_w, y'_w)\rangle|$$

The first inequality in the conclusion now follows as the estimate

$$|z - w| \approx |t_w - s_w|$$

implies that

$$|(t_w, x'_w) - (s_w, y'_w)| \approx |(z, x'_z) - (w, y'_w)|.$$

The last inequality in the lemma now follows by (3.3.2). □

This result now allows us to prove a key technical lemma that relies on the set of centers being sufficiently sparse.

Lemma 4.0.7. *For any $\nu \in K = \{\nu \in S_k : |t_\nu| < 4\}$, let $S\nu$ denote the index*

$$(k, (z_0, x_\nu(z_0)'), \pi(H(z_0, x_\nu(z_0)', \omega_\nu(z_0)), \omega_\nu(z_0)'))$$

where z_0 is such that $t_\nu(z_0) = 0$. For any $\gamma, \gamma', \gamma_0$ we have the following inequalities

$$\sum_{\nu \in K} \mu_\delta(S\nu, \gamma) \leq C_\delta(1 + 2^{n(k_\nu - k_\gamma)}) \qquad \textit{and} \qquad \sum_{\nu \in K} \mu_\delta(\gamma', S\nu)\mu_\delta(S\nu, \gamma_0) \leq C_\delta \mu_\delta(\gamma', \gamma_0)$$

for some C_δ depending only on n and $\delta > 0$.

Proof. The collection $K \cap \{\nu : d(S\nu, \gamma) \leq \inf_\nu d(S\nu, \gamma) + 1\}$ is finite, so there exists ν_0 such that $d(S\nu_0, \gamma) = \min_{\nu \in K} d(S\nu, \gamma)$. Therefore,

$$d(S\nu, \gamma) \geq \frac{1}{2} d(S\nu, \gamma) + \frac{1}{12} d(S\nu_0, S\nu) - \frac{1}{2} d(S\nu_0, \gamma) \geq c\, d(\nu_0, \nu).$$

The first inequality in the claim is now a straightforward adaptation the arguments in Lemmas 2.4 in [10]. By the reasoning in Lemma 2.5 of [10], this inequality is enough to show the second one. □

We are now ready to prove the main Theorem:

Proof of Theorem 4.0.4. Recall that we need only consider the part of the kernel that sums over $\nu \in K$. As such, given an index ν, there always exists $w \in (-16, 16)$ such that $t_\nu(w) = s$ as it implied by the conditions $|t_\nu| \leq 4$ and $|t| \leq 1$ for $\varepsilon > 0$ sufficiently small. Let $\tilde{b}(s, \nu, \gamma) = \int \psi_\nu(s, x)\overline{\varphi_\gamma(x)} dx$ and let

$$x_1 \mapsto (t_\nu(x_1), x_\nu(x_1), \tau_\nu(x_1), \xi_\nu(x_1)')$$

be the solution to the following ODE on $\mathbb{R}^{2n}$

$$\begin{aligned} \frac{dt}{dx_1} &= -\widetilde{H}^+_\tau(x, \tau, \xi') & \qquad \frac{d\tau}{dx_1} &= \widetilde{H}^+_t(x, \tau, \xi') \\ \frac{dx'}{dx_1} &= -\widetilde{H}^+_\xi(x, \tau, \xi') & \qquad \frac{d\xi'}{dx_1} &= \widetilde{H}^+_{x'}(x, \tau, \xi') \end{aligned} \tag{4.0.14}$$

with initial conditions $(t_\nu, x'_\nu, (\omega_\nu)_1, \omega'_\nu)$. Set

$$(\xi_\nu)_1(x_1) = \widetilde{H}^+(x_1, x_\nu(x_1)', \tau_\nu(x_1), \xi_\nu(x_1)')$$

so that $(t_\nu(x_1), x_1, x_\nu(x_1)', \tau_\nu(x_1), \xi_\nu(x_1))$ is a null bicharacteristic of $\widetilde{P}^k$ (with $\xi_\nu(x_1) = ((\xi_\nu)_1(x_1), \xi_\nu(x_1)'))$. It is not difficult to show that for $x_1 \in (-16, 16)$ and $\varepsilon > 0$ chosen sufficiently small, this curve admits a reparameterization $t \mapsto (t, x_\nu(t), \tau_\nu(t), \xi_\nu(t))$ for

$t \in [t(-16), t(16)]$ satisfying

$$\frac{dx}{dt} = -\nabla_\xi\Big((\tilde{\rho}^k)^{-\frac{1}{2}}\langle\widetilde{A}^k(x)\xi,\xi\rangle^{\frac{1}{2}}\Big) = (H_k)_\xi(x,\tau,\xi), \qquad (4.0.15)$$
$$\frac{d\xi}{dt} = \nabla_x\Big((\tilde{\rho}^k)^{-\frac{1}{2}}\langle\widetilde{A}^k(x)\xi,\xi\rangle^{\frac{1}{2}}\Big) = -(H_k)_x(x,\tau,\xi)$$

where $H_k(x,\tau,\xi) = \tau - (\tilde{\rho}^k)^{-\frac{1}{2}}\langle\widetilde{A}^k(x)\xi,\xi\rangle^{\frac{1}{2}}$.

By homogeneity of $\widetilde{H}$ and the fact that $((\omega_\nu)_1(x_1), \omega_\nu(x_1)') = \pi(\tau_\nu(x_1), \xi_\nu(x_1)')$, we have

$$\pi(\widetilde{H}(z, x_\nu(z)', \omega_\nu(z)), \omega_\nu(z)') = \pi(\xi_\nu(z))$$

for any z. Now let χ_t denote the transformation on $S^*(\mathbb{R}^n_x) = \mathbb{R}^n_x \times \mathbb{S}^{n-1}$ induced by projecting the flow (4.0.15) down to the cosphere bundle. We now have that if w is such that $t_\nu(w) = s$, then

$$\big(k_\nu, (w, x(w)'), \pi(\xi(w))\big) = \chi_s(S\nu).$$

This, in conjunction with Lemma 4.0.5, yields

$$|\tilde{b}(s,\nu,\gamma)| \leq \begin{cases} C_N\mu_N(\gamma,\chi_s(S\nu)) & |k_\nu - k_\gamma| \leq 2, \\ C_N 2^{-\max(k_\nu,k_\gamma)N}\mu_N(\gamma,\chi_s(S\nu)) & |k_\nu - k_\gamma| \geq 3. \end{cases} \qquad (4.0.16)$$

Hence the $\psi_\nu(t,x)$ do indeed behave roughly as if they were wave packets originating on the $t = 0$ hyperplane and translated along characteristics parameterized by t.

We thus have

$$(W^k_{t,s}\varphi_\gamma)(x) = \sum_\nu \psi_\nu(t,x)\int \psi_\nu(s,y)\overline{\varphi_\gamma(y)}\,dy = \sum_\nu \psi_\nu(t,x)\tilde{b}(s,\nu,\gamma)$$

This means the matrix of the operator $W^k_{t,s}$ satisfies

$$\left|\int \overline{\varphi_{\gamma'}(x)}(W^k_{t,s})\varphi_\gamma(t,x)\,dx\right| \leq \sum_\nu |\tilde{b}(s,\nu,\gamma)\tilde{b}(t,\nu,\gamma')|$$
$$\leq C_N\sum_\nu \mu_N(S\nu,\chi_{-t}(\gamma))\mu_N(S\nu,\chi_{-s}(\gamma')) \leq \tilde{C}_N\mu_N(\gamma,\chi_{t-s}(\gamma'))$$

by Lemma 4.0.7. Thus $W^k_{t,s}$ is an operator with matrix in $\mathcal{M}^0(\chi_{t-s})$ and the constant appearing in (3.2.2) can be taken to be independent of k, t, s. Hence $W^k_{t,s}$ has the desired L^2

mapping properties.

To show the second estimate, we wish to show the following bound on the kernel of $W_{t,s}^k$

$$|\sum_{\nu\in K}\psi_\nu(t,x)\psi_\nu(s,y)|\leq C2^{kn}(1+2^k|t-s|)^{-\frac{n-1}{2}}$$

with C independent of x,y. Begin by observing the upper bound

$$\left|\sum_{\nu\in K}\psi_\nu(t,x)\psi_\nu(s,y)\right|\leq\sum_{\nu\in K}\sum_{\gamma,\gamma'}|\varphi_\gamma(x)\varphi_{\gamma'}(y)||\tilde{b}(t,\nu,\gamma)||\tilde{b}(s,\nu,\gamma')|$$

Let $\Omega=\{(\omega_1,\omega')\in\mathbb{S}^{n-1}\subset\mathbb{R}^n:\omega_1\geq 6|\omega'|\}$ and

$$\tilde{E}_k=\{\gamma\in\cup_{\omega,k}\Xi_k^\omega:|k_\gamma-k|\leq 2,\omega_\gamma\in\Omega\}.$$

Observe that by choosing $\varepsilon>0$ sufficiently small, we have that for any $\gamma\notin\Omega$,

$$d(\gamma,\chi_t(S\nu))\geq c$$

with c independent of choice of such ν,γ. By (4.0.16) this implies the following estimates for any γ such that $\omega_\gamma\notin\Omega$ and $|k_\gamma-k_\nu|\leq 2$

$$|\tilde{b}(t,\nu,\gamma)|\leq C_{N,\delta}2^{-k_\nu N}\mu_\delta(\gamma,\chi_t(S\nu))$$

where $C_{N,\delta}$ can be taken independently of ν and $\gamma\notin\Omega$. In addition, recall that when $|k_\gamma-k_\nu|\geq 3$, we also have the estimate

$$|\tilde{b}(t,\nu,\gamma)|\leq C_{N,\delta}2^{-\max(k_\gamma,k_\nu)N}\mu_\delta(\gamma,\chi_t(S\nu)).$$

We can thus use the extra order of decay when $\gamma\notin\tilde{E}_k$ to get the upper bound

$$\sum_{\nu\in K}\sum_{(\gamma,\gamma')\notin\tilde{E}_k\times\tilde{E}_k}|\varphi_\gamma(x)\varphi_{\gamma'}(y)||\tilde{b}(t,\nu,\gamma)||\tilde{b}(s,\nu,\gamma')|\leq C.$$

It now suffices to restrict our attention to the sum

$$\sum_{\nu\in K}\sum_{\gamma,\gamma'\in\tilde{E}_k}|\varphi_\gamma(x)\varphi_{\gamma'}(y)||\tilde{b}(t,\nu,\gamma)||\tilde{b}(s,\nu,\gamma')|.$$

Using Lemma 4.0.7, this sum is dominated by

$$C_N \sum_{\gamma,\gamma' \in \tilde{E}_k} |\varphi_\gamma(x)\varphi_{\gamma'}(y)| \mu_N \left(\gamma, \chi_{t-s}(\gamma')\right).$$

Since the Hamiltonian in (4.0.15) is independent of time, we can assume that without loss of generality, $s = 0$. Then by the arguments in Lemma 6.2 of [10], this sum is bounded by the integral

$$C_N 2^{kn} \int_\Omega (1 + 2^k |\langle \omega(t), y - x(t)\rangle| + 2^k |y - x(t)|^2)^{-N} d\omega$$

where $(x(t), \omega(t))$ solves (4.0.15) with initial data (x, ω). The desired $L^1 \to L^\infty$ estimate then follows when $2^k|t| \leq 1$, so from now on we assume that $2^k|t| > 1$.

By the scaling arguments in Lemma 6.2, it now suffices to show that Lemma 6.3 of [10] holds under the alternate assumption that the Hamiltonian $H(x,\xi)$ is continuously differentiable in (x,ξ), homogeneous of degree 1 in ξ with $H(x,\cdot) \in C^\infty(\mathbb{R}^n \setminus \{0\})$, and satisfies

$$\sum_{|\alpha|+|\beta|\leq 2, |\alpha|\leq 1} \sup_{\frac{1}{2}\leq|\xi|\leq 2} |\partial_x^\alpha \partial_\xi^\beta (H(x,\xi) - |\xi|)| + \sum_{|\alpha|=2} \int_{-16}^{16} \sup_{\frac{1}{2}\leq|\xi|\leq 2} |\partial_x^\alpha H(x_1,\cdot,\cdot)|\, dx_1 \leq \tilde{\varepsilon} \tag{4.0.17}$$

for some sufficiently small but fixed $\tilde{\varepsilon} > 0$ with the slightly weaker conclusion that

$$\int_\Omega (1 + \lambda |\langle \tilde{\omega}(\omega), y - \tilde{x}(\omega)\rangle|)^{-2} d\omega \leq C\lambda^{-\frac{n-1}{2}}.$$

Indeed, (4.0.17) can be arranged by scaling the coefficients by a factor of R as

$$\int_{-2}^{2} \sup_{x' \in \mathbb{R}^{n-1}} \sum_{i,j} |\partial_x^\beta (a_{ij}(R(x_1,x')) - \delta_{ij})| + |\partial_x^\beta (\rho(R(x_1,x')) - 1)| dx_1 \leq CR^{1-\frac{1}{r}}.$$

for multi-indices $|\beta| = 2$.

Consider the Hamiltonian flow for η such that $\frac{\eta}{|\eta|} \in \Omega$

$$\begin{cases} \frac{dx}{ds}(s,\eta) = \nabla_\xi H(x(s,\eta), \xi(s,\eta)) \\ \frac{d\xi}{ds}(s,\eta) = -\nabla_x H(x(s,\eta), \xi(s,\eta)) \\ x(0,\eta) = 0, \qquad \xi(0,\eta) = \eta \end{cases}$$

The crucial step in the proof of Lemma 6.3 in [10] lies in showing that for $\frac{1}{2} \leq |\eta| \leq 2$, and $|s| \leq 1$ we have the estimates

$$|\xi(s,\eta) - \eta| \leq \mathcal{O}(\tilde{\varepsilon})$$
$$\left| x(s,\eta) - s\frac{\eta}{|\eta|} \right| \leq \mathcal{O}(\tilde{\varepsilon})$$
$$\left| \frac{\partial x}{\partial \eta}(1,\eta) - \left(\frac{\mathbf{I}}{|\eta|} - \frac{\eta \cdot \eta^T}{|\eta|^3} \right) \right| + \left| \frac{\partial \xi}{\partial \eta}(1,\eta) - \mathbf{I} \right| \leq \mathcal{O}(\tilde{\varepsilon})$$

The first 2 estimates follow easily by simple estimates on $\partial_s \xi(s,\eta)$ and $\partial_s x(s,\eta)$. To get the third estimate, observe that differentiation with respect to η in the equations above allows us to conclude that

$$\left| \frac{\partial x}{\partial \eta}(1,\eta) - \left(\frac{\mathbf{I}}{|\eta|} - \frac{\eta \cdot \eta^T}{|\eta|^3} \right) \right| + \left| \frac{\partial \xi}{\partial \eta}(1,\eta) - \mathbf{I} \right|$$
$$\leq \mathcal{O}(\tilde{\varepsilon}) + \int_0^1 C(\|\partial_x^2 (A(x_1(s), \cdot)\|_{L^\infty} + 1) \left| \frac{\partial x}{\partial \eta}(s,\eta) - \left(\frac{\mathbf{I}}{|\eta|} - \frac{\eta \cdot \eta^T}{|\eta|^3} \right) \right| \, ds$$

with $\|\partial_x^2 A(x_1, \cdot)\|_{L^\infty_{x'}} = \sum_{|\alpha|=2} \sum_{ij} \|\partial_x^\alpha a_{ij}(x_1, \cdot)\|_{L^\infty_{x'}}$. Since $\frac{dx_1}{ds}$ can be uniformly bounded from below whenever $\eta/|\eta| \in \Omega$, the third estimates now follows by Gronwall's inequality. The remaining arguments in Lemma 6.3 now follow. □

We are now able to prove Theorem 1.6.1.

Proof of Theorem 1.6.1. We now show that the hypotheses of Theorem 2.2.1 are satisfied with $\sigma = 0$. To do this, we will actually apply the parametrix and the estimates of this chapter to the pseudodifferential operator $\widetilde{P}^k$, the operator defined in Chapter 3 with coefficients truncated to frequency $2^{\frac{k}{2}}$.

Let $w(t,x)$ denote the function produced by Theorem 3.4.3 as a solution to the pseudodifferential equation (3.4.4) with data

$$w_0 = \psi \Gamma_k u(\cdot, 0, \cdot), \qquad w_1 = \partial_1 \psi \Gamma_k u(\cdot, 0, \cdot), \qquad F = \widetilde{P}^k(\psi \Gamma_k u) \tag{4.0.18}$$

(recall that $\psi(x_1)$ is the smooth cutoff such that $\psi|_{[-\frac{3}{2},\frac{3}{2}]} \equiv 1$ defined in § 2.2). By the observations at the beginning of this chapter that

$$\widetilde{\Phi}(D)\tilde{\mathfrak{c}}\Phi(D) = \tilde{\mathfrak{c}}\Phi(D) \qquad \text{and} \qquad \widetilde{\Phi}(D)\tilde{\mathfrak{s}}\Phi(D) = \tilde{\mathfrak{s}}\Phi(D)$$

as well as the fact that the Cauchy data determining w all have partial Fourier transforms in t, x' localized to a sufficiently small cone about the τ-axis, we can conclude that

$$\mathrm{supp}(\widehat{w})(\cdot, x_1, \cdot) \subset \{(\tau, \xi') : \tau \geq 6|\xi'|\}.$$

Hence an easy application of the flux estimates of Section 2.3 shows uniqueness of solutions to the PDE with the given data (4.0.18), implying that $w = \psi\Gamma_k u$. Therefore, the dispersive estimates of this chapter and the fact that $P^k(\psi\Gamma_k u)$ is localized to frequency 2^k in t, x' imply that

$$\begin{aligned}
&\|\psi\Gamma_k u\|_{L^p_t L^q_x([-1,1]\times(-2,2)\times\mathbb{R}^{n-1})} \\
&\lesssim 2^{ks}\big(2^k\|\psi\Gamma_k u(\cdot,0,\cdot)\|_{H^1_{t,x'}} + \|\partial_{x_1}\psi\Gamma_k u(\cdot,0,\cdot)\|_{L^2_{t,x'}} + \|\widetilde{P}^k(\psi\Gamma_k u)\|_{L^1_{x_1}L^2_{t,x'}(\mathbb{R}^n)}\big).
\end{aligned}$$

The estimate of Theorem 2.2.1 now follows by observing that

$$\sup_{x'\in\mathbb{R}^{n-1}} |\tilde{a}_{ij}(x_1,\cdot) - \tilde{a}^k_{ij}(x_1,\cdot)| \lesssim 2^{-k}\varrho(x_1)$$

implies the following inequality

$$\begin{aligned}
\|\widetilde{P}^k(\psi\Gamma_k u)\|_{L^1_{x_1}L^2_{t,x'}} &\lesssim \|(\widetilde{P}^k - \widetilde{P})(\psi\Gamma_k u)\|_{L^1_{x_1}L^2_{t,x'}} + \|\widetilde{P}(\psi\Gamma_k u)\|_{L^1_{x_1}L^2_{t,x'}} \\
&\lesssim \|\nabla_{t,x}(\psi\Gamma_k u)\|_{L^\infty_{x_1}L^2_{t,x'}} + \|P(\psi\Gamma_k u)\|_{L^1_{x_1}L^2_{t,x'}}.
\end{aligned}$$

□

We now conclude this chapter with a corollary stating that we can replace $[-1,1]$ by $\mathbb{R}$ in (4.0.2) as the domain of integration in the t variable. This will be of great importance in the next chapter as we will need to get estimates on dilations of $\psi\Gamma_k u$ and the $L^p_t L^q_x$ norms of such functions will have be taken over domains in t larger than $[-1,1]$.

Corollary 4.0.8. *Suppose $w \in C^\infty(\mathbb{R}^{n+1})$ satisfies*

$$supp(\widehat{w}(\cdot, x_1, \cdot)) \subset \{(\tau,\xi') : \tau \geq 12|\xi'|\} \setminus B_{\frac{3}{4}}(0) \cap \{(\tau,\xi') : \frac{1}{64}\lambda \leq |(\tau,\xi')| \leq 64\lambda\}.$$

for all $x_1 \in \mathbb{R}$. Assume also $\widetilde{P}^\lambda$ is the operator obtained by regularizing the coefficients of $\widetilde{P}$ and truncating them to frequencies less than $2\lambda^{1-\delta}$, for some $\delta > 0$. Then for λ sufficiently *large, there exists C_δ depending on $\delta > 0$ and on the norms of the coefficients of $\widetilde{P}^\lambda$ in the function spaces given by* (3.0.1) *and by $Lip(\mathbb{R}^n)$ such that*

$$\|w\|_{L_t^p L_x^q(\mathbb{R}\times(-2,2)\times\mathbb{R}^{n-1})} \leq C_\delta \lambda^s \big(\|w\|_{L^\infty_{x_1}((-2,2);H^1_{t,x'})} + \|\partial_1 w\|_{L^\infty_{x_1}((-2,2);L^2_{t,x'})} + \|\widetilde{P}^\lambda w\|_{L^1_{x_1}((-2,2);L^2_{t,x'}(\mathbb{R}^n))}\big),$$

with s, p, q as in Theorem 1.6.1.

Proof. Let $\{\phi_l(t)\}_{l\in\mathbb{Z}}$ be a smooth partition of unity on $\mathbb{R}$ such that $\phi_l(t)$ is supported in $(l-1, l+1)$ and $\phi_l(t) = \phi_0(t-l)$. A straightforward argument using Lemma 2.0.2 and Minkowski's inequality shows that

$$\|w\|_{L_t^p L_x^q(\mathbb{R}\times(-2,2)\times\mathbb{R}^{n-1})} \lesssim \big(\sum_l \|\phi_l w\|^2_{L_t^p L_x^q(\mathbb{R}\times(-2,2)\times\mathbb{R}^{n-1})}\big)^{\frac{1}{2}}.$$

Let $\widetilde{\Omega}(\tau,\xi')$ be a smooth function on $\mathbb{R}^n$ supported in $\{(\tau,\xi') : \tau > 11|\xi'|\} \setminus B_{\frac{1}{2}}(0)$ and identically 1 on the set $\{(\tau,\xi') : \tau \geq 12|\xi'|\} \setminus B_{\frac{3}{4}}(0)$. Also, let $\tilde{\theta}(\tau,\xi')$ be a smooth function supported in $\{(\tau,\xi') : \frac{1}{65} \leq |(\tau,\xi')| \leq 65\}$ and identically one on the slightly smaller set $\{(\tau,\xi') : \frac{1}{64} \leq |(\tau,\xi')| \leq 64\}$. Now define $\widetilde{\Omega}_\lambda(\tau,\xi') = \widetilde{\Omega}(\tau,\xi')\tilde{\theta}(\lambda^{-1}(\tau,\xi'))$. Since $(I - \widetilde{\Omega}_\lambda(D))w = 0$, we now have that

$$\|\phi_l w\|_{L_t^p L_x^q(\mathbb{R}\times(-2,2)\times\mathbb{R}^{n-1})} \leq \|\widetilde{\Omega}_\lambda \phi_l w\|_{L_t^p L_x^q([l-1,l+1]\times(-2,2)\times\mathbb{R}^{n-1})} + \|[I - \widetilde{\Omega}_\lambda, \phi_l]w\|_{L_t^p L_x^q([l-1,l+1]\times(-2,2)\times\mathbb{R}^{n-1})} \quad (4.0.19)$$

By applying flux estimates as in the proof of Theorem 1.6.1 it can be shown that when λ is sufficiently large, $\widetilde{\Omega}_\lambda \phi_l w$ is indeed represented by the parametrix of Chapter 3 as a solution to an equation involving $\widetilde{P}^\lambda$, yielding the inequality

$$\|\widetilde{\Omega}_\lambda(D)\phi_l w\|_{L_t^p L_x^q([l-1,l+1]\times(-2,2)\times\mathbb{R}^{n-1})} \leq C_\delta \lambda^s \big(\|\widetilde{\Omega}_\lambda \phi_l w(\cdot,0,\cdot)\|_{H^1_{t,x'}} + \|\widetilde{\Omega}_\lambda \phi_l \partial_1 w(\cdot,0,\cdot)\|_{L^2_{t,x'}} + \|\phi_l \widetilde{P}^\lambda w\|_{L^1_{x_1}((-2,2);L^2_{t,x'}(\mathbb{R}^n))} + \|[\widetilde{P}^\lambda, \widetilde{\Omega}_\lambda \phi_l]w\|_{L^1_{x_1}((-2,2);L^2_{t,x'}(\mathbb{R}^n))}\big).$$

Consider the sum over l of the squares of the first line on the right hand side of the inequality. It is straightforward to see that this sum is dominated by

$$\|w\|_{L^\infty_{x_1}((-2,2);H^1_{t,x'})} + \|\partial_1 w\|_{L^\infty_{x_1}((-2,2);L^2_{t,x'})} + \|\widetilde{P}^\lambda w\|_{L^1_{x_1}((-2,2);L^2_{t,x'}(\mathbb{R}^n))}.$$

It now suffices to consider the commutator terms. Let $R_\theta = \sum_l e^{i\theta l}\widetilde{\Omega}_\lambda \phi_l$ so that R_θ is a pseudodifferential operator in t, x' with uniform symbol estimates in $S^0_{1,0}$ over θ. Since the partial derivatives with respect to x_1 involved $[\widetilde{P}^\lambda, R_\theta]$ are of order at most 1, we can apply Lemma 2.0.1 to get that

$$\|[\widetilde{P}^\lambda, R_\theta]w(x_1,\cdot)\|_{L^2_{t,x'}} \lesssim \|w(x_1,\cdot)\|_{H^1_{t,x'}} + \|\partial_1 w(x_1,\cdot)\|_{L^2_{t,x'}}.$$

Applying Lemma 2.0.2 and Minkowski's inequality now yields

$$\sum_l \|[\widetilde{P}^\lambda, \widetilde{\Omega}_\lambda \phi_l]w\|^2_{L^1_{x_1}L^2_{t,x'}((-2,2)\times\mathbb{R}^n)} \lesssim \|w\|^2_{L^\infty_{x_1}((-2,2);H^1_{t,x'})} + \|\partial_1 w\|^2_{L^\infty_{x_1}((-2,2);L^2_{t,x'})}.$$

By Sobolev embedding and similar considerations,

$$\sum_l \|[I - \widetilde{\Omega}_\lambda, \phi_l]w\|^2_{L^p_t L^q_x([-l-1,-l+1]\times(-2,2)\times\mathbb{R}^{n-1})}$$
$$\lesssim \|w\|^2_{L^\infty_{x_1}((-2,2);H^1_{t,x'})} + \|\partial_1 w\|^2_{L^\infty_{x_1}((-2,2);L^2_{t,x'})}$$

completing the proof. $\square$

Chapter 5

ESTIMATES VIA SCALING

In this chapter, we show the truncation/rescaling arguments needed to prove Theorem 1.6.2. In Section 1.3, we discussed weighted Strichartz estimates for C^1 coefficients and saw that when u was dyadically localized in frequency, we could get the appropriate weighted estimate by decomposing the time interval $[-t_0, t_0]$ into smaller subintervals and then scaling the problem to get estimates over each subinterval. However, our situation is slightly different. First, the reductions of Theorem 2.2.1 mean that we want to consider components whose Fourier transform is dyadically localized in with the respect to partial Fourier transform in t, x'. Second, the nature of our parametrix permits us to prove estimates only over bounded intervals in x_1. This suggests that to obtain estimates on dyadic components of $\psi\Gamma_k u$, we need to decompose the interval $(-2, 2)$ into subintervals in x_1 and then rescale the problem to get estimates over each subinterval.

We begin this approach by smoothing the coefficients of the operator $\widetilde{P}$ as constructed in Chapter 3, comprising the "truncation" portion of our approach. Let $\{\beta_k\}_k$ be a Littlewood Paley decomposition in x as before. Set $\varsigma = \frac{2}{3+\alpha}$ with α as in Theorem 1.6.2 and recall that $\sigma = \frac{1-\alpha}{3+\alpha}$. For each $\tilde{a}_{ij} = a_{ij}/a_{11}$, $\tilde{\rho} = \rho/a_{11}$ define

$$\tilde{a}_{ij}^k = \sum_{l<k\varsigma} \beta_l(D)\tilde{a}_{ij}, \qquad \tilde{\rho}^k = \sum_{l<k\varsigma} \beta_l(D)\tilde{\rho}.$$

Let a denote an arbitrary function of the form $\tilde{a}_{ij} - \delta_{ij}$ or $\tilde{\rho} - 1$ and a^k its regularized counterpart $\tilde{a}_{ij}^k - \delta_{ij}$ or $\tilde{\rho}^k - 1$. Observe that we can write $\beta_l(D)a = \beta_l(D)\langle D\rangle^{-\kappa}(\langle D\rangle^{\kappa}a)$. Set $\tilde{\phi}_l = \mathcal{F}^{-1}\{\beta_l(\xi)\langle\xi\rangle^{-\kappa}\}$ so that $\beta_l(D)a = (\langle D\rangle^{\kappa}a) * \tilde{\phi}_l$. Using integration by parts we have the following estimates on $\tilde{\phi}_l$

$$|y^{\beta_1}\partial_y^{\beta_2}\tilde{\phi}_l(y)| \leq C_{\beta_1,\beta_2} 2^{l(n-|\beta_1|+|\beta_2|-\kappa)}$$

This yields the following estimates on the $\beta_l(D)a$

$$
\begin{aligned}
|\partial_x^\beta(\beta_l(D)a)(x)| &= |(\langle D\rangle^\kappa a) * (\partial_x^\beta \tilde{\phi}_l)(x)| \\
&\leq C_{\beta,n} 2^{l(n-1+|\beta|-\kappa)} \cdot \\
&\qquad \int \|(\langle D\rangle^\kappa a)(x_1 - y_1, \cdot)\|_{L^r_{x'}(\mathbb{R}^{n-1})} (1 + |2^l y_1|^2)^{-1} 2^l dy_1 \; \|(1 + |2^l y'|^2)^{-n}\|_{L^{r'}} \\
&\leq C_{\beta,n} 2^{l(|\beta|-1-\alpha)} M(\|(\langle D\rangle^\kappa a)(w_1, \cdot)\|_{L^r_{x'}})(x_1)
\end{aligned}
$$

Here $M(\|(\langle D\rangle^\kappa a)(w_1, \cdot)\|_{L^r})(x_1)$ denotes the maximal function associated to the $L^r(\mathbb{R})$ function $w_1 \mapsto \|(\langle D\rangle^\kappa a)(w_1, \cdot)\|_{L^r}$. The last inequality follows by the result on maximal functions cited in Chapter 3. Let ϱ be the $L^r(\mathbb{R})$ function given by

$$
\varrho(x_1) = \sum_{ij} M(\|\langle D\rangle^\kappa (\tilde{a}_{ij} - \delta_{ij})(w_1, \cdot)\|_{L^r_{x'}})(x_1) + M(\|\langle D\rangle^\kappa (\tilde{\rho} - 1)(w_1, \cdot)\|_{L^r_{x'}})(x_1).
$$

Note that this definition of ϱ is different from the one used in Chapter 3. This now yields the following inequalities for a in the form above

$$
|a(x) - a^k(x)| \leq \sum_{l \geq k\varsigma} |\beta_l(D)a(x)| \leq C\varrho(x_1) \sum_{l \geq k\varsigma} 2^{-l(1+\alpha)} \leq C' 2^{-k\varsigma(1+\alpha)} \varrho(x_1),
$$

$$
|\partial_x^\beta a^k(x)| \leq \sum_{l < k\varsigma} |\partial_k^\beta \beta_l(D)a(x)| \leq C_\beta 2^{k\varsigma(|\beta|-1-\alpha)} \varrho(x_1) \qquad \text{for } |\beta| \geq 2
$$

In addition, the C^1 norm of $\tilde{a}_{ij}^k$, $\tilde{\rho}^k$ is uniformly bounded as $\tilde{a}_{ij}$, $\tilde{\rho}$ are Lipschitz functions.

We now must be delicate about decomposing $(-2, 2)$ into subintervals in x_1, as in general, $\varrho(x_1)$ is unbounded, meaning that we cannot take the subintervals to be of equal size. Let $\{I_m^k\}$ be a maximal collection of disjoint subintervals of $(-2, 2)$ indexed by m ranging over positive integers such that $\cup_m I_m^k = (-2, 2)$, $|I_m^k| \leq 2^{-k\sigma}$, and $\int_{I_m^k} \varrho(x_1) dx_1 \leq 2^{-k\sigma}$. Let K_1 be the cardinality of the set $\{I_m^k : \int_{I_m^k} \varrho(x_1) dx_1 = 2^{-k\sigma}\}$ and K_2 the cardinality of the set $\{I_m^k : |I_m^k| = 2^{-k\sigma}\}$. By maximality, $K_1 + K_2$ is at least the cardinality of the collection $\{I_m^k\}$. Observe that

$$
2^{-k\sigma} K_1 \leq \int_{-2}^{2} \varrho(x_1)\, dx_1 \leq C \qquad\qquad 2^{-k\sigma} K_2 \leq 4
$$

and hence there is at most $\mathcal{O}(2^{k\sigma})$ subintervals in the collection. Now let $\chi_m^k(x_1)$ be the characteristic function of the interval I_m^k. This yields the following estimate

$$\int \sup_{x'}(\chi_m^k(x_1)|\tilde{a}_{ij}^k(x) - \tilde{a}_{ij}(x)|)dx_1 \leq C \int \chi_m^k(x_1) 2^{-\varsigma(1+\alpha)} \varrho(x_1) dx_1$$
$$\leq C 2^{(-\sigma-\varsigma(1+\alpha))k} = C2^{-k} \tag{5.0.1}$$

as a simple computation shows that $\sigma + \varsigma(1+\alpha) = 1$. Next observe that for any multi-index β such that $|\beta| = 2$

$$\int \chi_m^k(2^{-k\sigma}x_1) \sup_{x'} |\partial_x^\beta(\tilde{a}_{ij}^k(2^{-k\sigma}x) - \delta_{ij})| \, dx_1$$
$$\leq C2^{-2k\sigma+k\varsigma(1-\alpha)} \int \chi_m^k(2^{-k\sigma}x_1)\varrho(2^{-k\sigma}x_1)dx_1 \leq C \tag{5.0.2}$$

as $\varsigma(1-\alpha) = \sigma$. Also, the first order partials of the dilated coefficients are uniformly bounded and analogous inequalities hold when a_{ij} is replaced by ρ.

Let $\widetilde{P}^k(x, D)$ denote the operator $\widetilde{P}(x, D)$ with rough coefficients $\tilde{\rho}, \tilde{a}_{ij}$ replaced by smooth coefficients $\tilde{\rho}^k, \tilde{a}_{ij}^k$. By (5.0.2), we can now apply the results of Chapters 3 and 4 to equations involving the operator $\widetilde{P}^k(2^{-k\sigma}x, D)$. Set $v_k(t, x) = (\psi\Gamma_k u)(2^{-k\sigma}(t, x))$ so that $\widehat{v_k}(\tau, x_1, \xi')$ is localized to frequencies $\approx 2^{k(1-\sigma)}$ and the dilated coefficients of $\widetilde{P}$ are localized to frequencies less than $2^{k(\varsigma-\sigma)}$. Corollary 4.0.8 now allows us to conclude that

$$\|v_k\|_{L_t^p L_x^q(\mathbb{R}\times 2^{k\sigma}I_m^k\times\mathbb{R}^{n-1})} \lesssim 2^{k(1-\sigma)s}\Big(\|v_k\|_{L_{x_1}^\infty(2^{k\sigma}I_m^k;\dot{H}_{t,x'}^1)} + \|\partial_1 v_k\|_{L_{x_1}^\infty(2^{k\sigma}I_m^k;L_{t,x'}^2)}$$
$$+ \|\widetilde{P}^k(2^{-k\sigma}x, D)v_k\|_{L_{x_1}^1(2^{k\sigma}I_m^k;L_{t,x'}^2)}\Big)$$

where $2^{k\sigma}I_m^k$ denotes the set $\{2^{k\sigma}x_1 : x_1 \in I_m^k\}$ and $\dot{H}^1$ denotes the homogeneous L^2 Sobolev space of order 1. We now rescale the estimate to get

$$\|\chi_m\psi\Gamma_k u\|_{L_t^p L_x^q} \lesssim 2^{ks}(2^k\|\psi\Gamma_k u\|_{L_{x_1}^\infty L_{t,x'}^2}$$
$$+ \|\partial_1\psi\Gamma_k u\|_{L_{x_1}^\infty L_{t,x'}^2} + \|\chi_m \widetilde{P}^k(\psi\Gamma_k u)\|_{L_{x_1}^1 L_{t,x'}^2}). \tag{5.0.3}$$

Observe that the partial derivatives with respect to x_1 involved in $\widetilde{P} - \widetilde{P}^k$ are of order at most 1. We can thus use (5.0.1) to obtain the estimate

$$\|\chi_m(\widetilde{P}^k - \widetilde{P})(\psi\Gamma_k u)\|_{L_{x_1}^1 L_{t,x'}^2} \leq C(2^k\|\psi\Gamma_k u\|_{L_{x_1}^\infty L_{t,x'}^2} + \|\partial_1\psi\Gamma_k u\|_{L_{x_1}^\infty L_{t,x'}^2})$$

This now yields

$$\|\chi_m \psi \Gamma_k u\|_{L_t^p L_x^q} \lesssim 2^{ks}\big(2^k \|\psi\Gamma_k u\|_{L_{x_1}^\infty L_{t,x'}^2} + \|\partial_1 \psi \Gamma_k u\|_{L_{x_1}^\infty L_{t,x'}^2} + \|\chi_m \widetilde{P}(\psi\Gamma_k u)\|_{L_{x_1}^1 L_{t,x'}^2}\big).$$

Taking the sum over m in this inequality as it ranges over all subintervals $\{I_m^k\}$ now yields the following estimate when $p \leq q$

$$\begin{aligned} \|\psi\Gamma_k u\|_{L_t^p L_x^q} &\leq \left(\int \left(\sum_m \int |\chi_m^k \psi \Gamma_k u|^q dx\right)^{\frac{p}{q}} dt\right)^{\frac{1}{p}} \leq \left(\sum_m \|\chi_m^k \psi\Gamma_k u\|_{L_t^p L_x^q}^p\right)^{\frac{1}{p}} \\ &\leq C 2^{k(s+\frac{\sigma}{p})}\Big(2^k \|\psi\Gamma_k u\|_{L_{x_1}^\infty L_{t,x'}^2} + \|\partial_1\psi\Gamma_k u\|_{L_{x_1}^\infty L_{t,x'}^2} + \|P(\psi\Gamma_k u)\|_{L_{x_1}^1 L_{t,x'}^2}\Big) \qquad (5.0.4) \end{aligned}$$

To obtain a suitable estimate when $p \geq q$, we first observe that this implies that $n = 3$, $s = 0$ and $p = \frac{2q}{q-6}$ with $q \leq 8$. $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ by Sobolev embedding and hence

$$\|\psi\Gamma_k u\|_{L_t^\infty L_x^6} \leq C\|\Gamma_k u\|_{L_t^\infty H_x^1}.$$

We can now use interpolate this estimate and (5.0.4) with $s = 0$, $p = q = 8$ to get that

$$\begin{aligned} \|\psi\Gamma_k u\|_{L_t^p L_x^q} \lesssim 2^{k(s+\frac{\sigma}{p})}\Big(&\|\Gamma_k u\|_{L_t^\infty H_x^1} + 2^k\|\psi\Gamma_k u\|_{L_{x_1}^\infty L_{t,x'}^2} + \|\partial_1\psi\Gamma_k u\|_{L_{x_1}^\infty L_{t,x'}^2} \\ &+ \|P(\psi\Gamma_k u)\|_{L_{x_1}^1 L_{t,x'}^2}\Big) \end{aligned}$$

when $p > q$. Applying Theorem 2.2.1, Theorem 1.6.2 is now shown.

Chapter 6

ALTERNATE CHARACTERIZATIONS OF THE STRICHARTZ ESTIMATES

As discussed in the introduction, Strichartz estimates have appeared in 2 different forms. The more original style of (1.5.1) poses an estimate on $L_t^pL_x^q$ norm of u in contrast to the more recent method of controlling the $L_t^pL_x^q$ norm of $\langle D\rangle^{-z}u$ as in (1.1.5). The latter characterization is often necessary as the coefficients of P are in some cases too rough to show weighted estimates that are closer to the original family of inequalities. This is because when the coefficients are very rough, difficulties arise when summing over estimates on the dyadic components of the solution.

In the proof of Theorem 2.2.1, Lemma 2.0.1 and its corollaries played a crucial role in handling commutator terms, allowing us to sum over estimates on dyadically localized components to obtain global estimates. These arguments relied heavily on the fact that the initial data was posed in $H^1 \times L^2$, which meant that the mapping $[P, R] : H^1_{t,x} \to L^2_{t,x}$ for Fourier multipliers R of order 0 provided an effective way to control the sums (as energy estimates allow us to dominate $\|u\|_{H^1_{t,x}}$ by the initial data). However, if we pose estimates on u rather than $\langle D\rangle^{-z}u$, we are forced to deal with initial data in spaces $H^{z+1} \times H^z$ with $z \neq 0$ leading us to consider commutator estimates on $[P, R]$ for R of nonzero order. This creates difficulties as obtaining the mapping $[P, R] : H^{z+1+w}_{t,x} \to H^z_{t,x}$ when $z \neq 0$ and/or $w \neq 0$ is a delicate issue and is sensitive to whether P is in divergence or nondivergence form. Indeed, we do not know that this mapping holds in all relevant cases.

The purpose of this chapter is to discuss conditions under which

$$[P, R] : H^{z+1+w}_{t,x} \to H^z_{t,x}$$

is a continuous mapping, and as a result we are able to determine when estimates in the newer format of (1.1.5) imply estimates in the style of (1.5.1). In particular, we will see this always happens when the coefficients lie in the space $L^{r,\kappa}$, $\kappa > \frac{n-1}{r} + 2$, thus proving Corollary 1.6.3.

In the following theorem, a "multiplier" on a Sobolev space H^z refers to a function $v(x)$ such that the mapping $h(x) \mapsto v(x) \cdot h(x)$ defined on $\mathcal{S}(\mathbb{R}^n)$ extends to a continuous operation on H^z.

Theorem 6.0.9. *Suppose $\rho(x), a_{ij}(x) \in Lip(\mathbb{R}^n)$ and the following estimates hold for solutions v to the Cauchy problem* (1.1.1) *with $z = 0$, $p, q \geq 2$, $q < \infty$, and $|w| \leq 1$*

$$\|\langle D\rangle^{-w} v\|_{L^p([-t_0,t_0];L^q(\mathbb{R}^n))} \lesssim \|v(0,\cdot)\|_{H^1} + \|\partial_t v(0,\cdot)\|_{L^2} + \|Pv\|_{L^1([-t_0,t_0];L^2(\mathbb{R}^n))} \tag{6.0.1}$$

In addition, assume that the first partial derivatives of ρ, a_{ij} are multipliers on $H^w(\mathbb{R}^n)$ when $w > 0$ and $P(x, D)$ is in divergence form, and that first partials are multipliers on H^w when $w < 0$ and P is in nondivergence form. Then we have the following estimates on solutions u to (1.1.1) *with $z = w$*

$$\|u\|_{L^p([-t_0,t_0];L^q(\mathbb{R}^n))} \lesssim \|f\|_{H^{w+1}} + \|g\|_{H^w} + \|F\|_{L^1([-t_0,t_0];H^w(\mathbb{R}^n))}. \tag{6.0.2}$$

Proof. By Duhamel's principle it suffices to show the theorem in the case where $F = 0$ and u is a solution to the homogeneous problem on $(-t_0, t_0)$. However, by multiplying u by a smooth cutoff in t as in Chapter 2 we can actually assume u is compactly supported in t and that

$$\|Pu(t,\cdot)\|_{H^w} \leq C(\|f\|_{H^{w+1}} + \|g\|_{H^w})$$

with C independent of $t \in \mathbb{R}$.

Take a sequence of smooth Littlewood-Paley cutoffs $\{\beta_k\}_{k\geq 0}$ as before. It is straightforward to see that (6.0.1) is equivalent to (6.0.2) for $\beta_k u$, $k \geq 0$. Therefore,

$$\|u\|_{L^p_t L^q_x} \approx \|(\sum_k |\beta_k u|^2)^{\frac{1}{2}}\|_{L^p_t L^q_x} \lesssim (\sum_k \|\beta_k u\|^2_{L^p_t L^q_x})^{\frac{1}{2}}$$
$$\lesssim \Big(\sum_k 2^{2kw}(\|\beta_k f\|^2_{H^1} + \|\beta_k g\|^2_{L^2} + \|\beta_k Pu\|^2_{L^1_t L^2_x} + \|[P, \beta_k]u\|^2_{L^1_t L^2_x})\Big)^{\frac{1}{2}}$$

with $L^1_t L^2_x$ abbreviating $L^1([-t_0, t_0]; L^2(\mathbb{R}^n))$. A straightforward argument using the Fourier transform and Minkowski's inequality yields

$$\sum_k 2^{2kw}(\|\beta_k f\|^2_{H^1} + \|\beta_k g\|^2_{L^2} + \|\beta_k Pu\|^2_{L^1_t L^2_x}) \lesssim \|f\|^2_{H^{w+1}} + \|g\|^2_{H^w},$$

so it suffices to control the terms involving the commutator. Define the function h_j to be $h_j = -\sum_{i=1}^n \partial_i a_{ij}$ when P is in nondivergence form and $h_j = 0$ otherwise. As suggested by Lemma 2.0.2 we consider

$$\sum_k e^{ik\theta}[P, 2^{kw}\beta_k]$$
$$= \partial_t[\rho, \sum_k e^{ik\theta} 2^{kw}\beta_k]\partial_t - \sum_{ij} \partial_i[a_{ij}, \sum_k e^{ik\theta}2^{kw}\beta_k]\partial_j + \sum_1^n [h_j, \sum_k e^{ik\theta}2^{kw}\beta_k]\partial_j.$$

When $w \leq 0$, Lemma 2.0.1 tells us that

$$[\rho, \sum_k e^{ik\theta}2^{kw}\beta_k], [a_{ij}, \sum_k e^{ik\theta}2^{kw}\beta_k] : H^w(\mathbb{R}^{n+1}_{t,x}) \to H^1(\mathbb{R}^{n+1}_{t,x})$$

continuously with operator norm independent of θ. Also, since h_j is a multiplier on $H^w(\mathbb{R}^n)$ we have that

$$[h_j, \sum_k e^{ik\theta}2^{kw}\beta_k] : H^w(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

is a bounded operator with norm independent of θ. An application of Lemma 2.0.2 now yields

$$\sum_k 2^{2kw}\|[P, \beta_k]u\|^2_{L^1_t L^2_x} \lesssim \|u\|^2_{H^w_{t,x}} + \|u\|^2_{L^2_t H^w_x} \lesssim \|f\|^2_{H^{w+1}} + \|g\|^2_{H^w}.$$

When $w \geq 0$, we use the definition of b_j in the introduction to get

$$\sum_k e^{ik\theta}[P, 2^{kw}\beta_k]$$
$$= [\rho, \sum_k e^{ik\theta}2^{kw}\beta_k]\partial_t^2 - \sum_{ij}[a_{ij}, \sum_k e^{ik\theta}2^{kw}\beta_k]\partial^2_{ij} - \sum_1^n [b_j, \sum_k e^{ik\theta}2^{kw}\beta_k]\partial_j.$$

This time Lemma 2.0.1 and the fact that b_j is a multiplier on H^w gives us that

$$[\rho, \sum_k e^{ik\theta} 2^{kw} \beta_k], [a_{ij}, \sum_k e^{ik\theta} 2^{kw} \beta_k] : H^{w-1}(\mathbb{R}^{n+1}_{t,x}) \to L^2(\mathbb{R}^{n+1}_{t,x})$$

$$[b_j, \sum_k e^{ik\theta} 2^{kw} \beta_k] : H^w(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$$

The result for $w \geq 0$ now follows in a similar fashion. □

As mentioned above, we are now able to use this result to prove Corollary 1.6.3.

Proof of Corollary 1.6.3. The corollary follows immediately once we show that the first partial derivatives of a_{ij}, ρ are multipliers on $H^0(\mathbb{R}^n)$ and $H^{-\frac{1}{2}}(\mathbb{R}^n)$. The H^0 case is easy as $\partial_l a_{ij}, \partial_l \rho \in L^\infty(\mathbb{R}^n)$.

For the $H^{-\frac{1}{2}}$ case we observe that $\partial_l a_{ij}, \partial_l \rho \in L^{r,\kappa-1}(\mathbb{R}^n)$ and that $\kappa - 1 > \frac{n}{r} + 1 - \frac{1}{r}$. This means that when $r > 2$, $L^{r,\kappa-1} \hookrightarrow C^{\frac{1}{2}+\delta}$ for some $\delta > 0$ and hence the space is a multiplier on $H^{-\frac{1}{2}}$. When $1 < r \leq 2$, $L^{r,\kappa-1} \hookrightarrow L^{2,\frac{n}{2}+1-\frac{1}{r}}$ and so that the functions are once again a multiplier on $H^{-\frac{1}{2}}$. □

Chapter 7

ESTIMATES ON COMPACT MANIFOLDS

In the previous chapter, we saw that when our operator is in divergence form, the inequalities

$$\begin{aligned}\|u\|_{L^{\frac{2(n+1)}{n-1}}([-t_0,t_0]\times\mathbb{R}^n)} \\ \lesssim \|f\|_{H^{\frac{1}{2}+\frac{\sigma(n-1)}{2(n+1)}}(\mathbb{R}^n)} + \|g\|_{H^{-\frac{1}{2}+\frac{\sigma(n-1)}{2(n+1)}}(\mathbb{R}^n)} + \|F\|_{L^1_t([-t_0,t_0];H^{-\frac{1}{2}+\frac{\sigma(n-1)}{2(n+1)}}(\mathbb{R}^n))}\end{aligned} \tag{7.0.1}$$

can be derived from the estimates in Theorems 1.6.1 and 1.6.2. However, to show the estimates

$$\|u\|_{L^p_t([-t_0,t_0];L^q(\mathbb{R}^3))} \lesssim \|f\|_{H^{1+\frac{\sigma}{p}}(\mathbb{R}^3)} + \|g\|_{H^{\frac{\sigma}{p}}(\mathbb{R}^3)} + \|F\|_{L^1_t([-t_0,t_0];H^{\frac{\sigma}{p}}(\mathbb{R}^3))} \tag{7.0.2}$$

with $6 \leq q < \infty$, $p = \frac{2q}{q-6}$, and $\sigma \neq 0$ required additional regularity not necessarily satisfied by coefficients lying in $L^{r,\kappa}(\mathbb{R}^n) \cap Lip(\mathbb{R}^n)$, $\kappa = \frac{n-1}{r} + 1 + \alpha$ with $\alpha \in (0,1)$.

In this chapter, we show how to derive estimates of the form (7.0.2) from Theorem 1.6.2 in the case where the space $\mathbb{R}^n$ is replaced by a C^∞ compact manifold M. We assume that M is equipped with a rough metric $g_{ij} \in L^{r,\kappa} \cap Lip$, that is, in each coordinate chart, $g_{ij}(x) \in L^{r,\kappa}_{loc}(\mathbb{R}^n) \cap Lip(\mathbb{R}^n)$ and $\kappa = \frac{n-1}{r} + 1 + \alpha$, $\alpha \in (0,1)$.

We will denote the Laplace-Beltrami induced by the metric as Δ_g. Take the convention that Δ_g is a positive operator so that in coordinates

$$\Delta_g u = -\frac{1}{\sqrt{\det g_{ij}}}\partial_i(g^{ij}\sqrt{\det g_{ij}}\partial_j u).$$

This means that the wave operator on $[-t_0,t_0] \times M$ takes the form $\partial_t^2 + \Delta_g$.

We thus wish to show that

$$\|u\|_{L^p([-t_0,t_0];L^q(M))} \lesssim \|f\|_{H^{s+1+\frac{\sigma}{p}}(M)} + \|g\|_{H^{s+\frac{\sigma}{p}}(M)} + \|F\|_{L^1([-t_0,t_0];H^{s+\frac{\sigma}{p}}(M))} \tag{7.0.3}$$

for solutions u to

$$\begin{aligned}(\partial_t^2 + \Delta_g)u(t,x) &= F(t,x) \in L^1([-t_0,t_0];H^{s+\frac{\sigma}{p}}(M)) \\ u(0,x) &= f \in H^{s+\frac{\sigma}{p}+1}(M) \\ \partial_t u(0,x) &= g \in H^{s+\frac{\sigma}{p}}(M).\end{aligned} \tag{7.0.4}$$

Since the estimates (7.0.3) follow easily from (7.0.1) when $s = -\frac{1}{2}$, we will restrict our attention to the case $s = 0$. Additionally, we will only consider the homogeneous problem with $F = 0$ as a variation of parameters argument will show estimates on the inhomogeneous problem.

By setting $\rho = \sqrt{\det g_{ij}}$, $a_{ij} = g^{ij}\sqrt{\det g_{ij}}$ in any local coordinate chart, solutions $v(t,x)$ to homogeneous wave equation are also solutions to

$$\rho(x)\partial_t^2 v(t,x) - A(x,D)v(t,x) = 0.$$

where $A(x,D)$ is in divergence operator. Hence Theorem 1.6.2 will be applicable for functions v supported in a coordinate chart.

Δ_g admits an countable orthonormal eigenbasis for $L^2(M)$, $\{\varphi_j\}_{j=0}^{\infty}$ such that $\Delta_g\varphi_j = \lambda_j^2\varphi_j$ for all j. This is a well known result in Riemannian geometry when g is a smooth metric, but the theorem still holds under our weaker assumptions on g. See Theorem 3.7 of [9] for an approach that is suitable under these milder assumptions. We will also take the convention that the eigenvalues $\{\lambda_j\}_{j=0}^{\infty}$ are listed in increasing order

$$0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_j \leq \lambda_{j+1} \cdots .$$

and let E_j denote the projection operators given by $E_j(h) = \varphi_j \int_M h\overline{\varphi_j}$.

In order to show the estimates (7.0.3) when $s = 0$, we employ a Littlewood-Paley type decomposition of functions in $h \in L^2(M)$, $h \mapsto \{\beta_k h\}_{k \geq 0}$ where

$$\beta_k h = \sum_{\lambda_j \approx 2^k} E_j(h).$$

Much like before, our strategy will be to get estimates on $\beta_k u$ (as before, β_k is only applied to u in the x-variable) and then use Littlewood-Paley theory to sum over k, obtaining global estimates. Working with this decomposition has two advantages. First, much like before, we can use the decomposition to characterize Sobolev norms. That is,

$$\|h\|^2_{H^z(M)} \approx \sum_k 2^{2kz} \|\beta_k h\|^2_{L^2(M)}. \tag{7.0.5}$$

Second, and more importantly, β_k as an operator commutes with Δ_g, eliminating the problem of estimating commutator terms.

However, it is not immediate that if η is a smooth cutoff supported in a coordinate chart, then the coordinate representation of $\eta\beta_k h$ satisfies

$$\|\langle D\rangle^z (\eta\beta_k h)\|_{L^q(\mathbb{R}^n)} \approx 2^{kz} \|\eta\beta_k h\|_{L^q(\mathbb{R}^n)}. \tag{7.0.6}$$

That is, it is not trivial that the action of $\langle D\rangle^z$ on the coordinate representation of $\eta\beta_k h$ is well-approximated by multliplication by 2^{kz}. This is because localization with respect to the spectrum of the Laplacian on M is not the same as localization with respect to the spectrum of the Laplacian on Euclidean space. Hence, in order to apply the results of Theorem 1.6.2, we will at least need to show that $\|\eta\beta_k h\|_{L^q} \lesssim 2^{-kz} \|\langle D\rangle^z \eta\beta_k h\|$. This issue will be the crucial step in the proof of (7.0.3).

7.1 *Sobolev norms and eigenfunction expansions*

One of the key features of the eigenbasis is that it allows us to express L^2 Sobolev norms of functions on M in terms of their eigenfunction expansions. It is this property that will allow us to obtain (7.0.5).

We will define the L^2 Sobolev space of order s on M as follows: fix a C^∞, finite, partition of unity on M, $\{\eta_l\}_{l=1}^m$ such that $\mathrm{supp}(\eta_l) \subset U_l$, where (U_l, χ_l) is a coordinate chart on M with $\chi_l(U_l) = \mathbb{R}^n$ and let

$$H^s(M) = \{f \in \mathcal{D}'(M) : (\eta_l f) \circ \chi_l^{-1} \in H^s(\mathbb{R}^n) \text{ for all } 1 \leq l \leq m\}.$$

A natural norm on $H^s(M)$ is thus determined by

$$\|f\|_{H^s(M)}^2 = \sum_{l=1}^{m} \|\eta_l f \circ \chi_l^{-1}\|_{H^s(\mathbb{R}^n)}^2.$$

We now provide an equivalent norm based on the eigenfunction expansion of functions in $L^2(M)$.

Proposition 7.1.1. *Let $f \in H^s(M)$ for some $s \in [0,2]$. There exists a constant C_s independent of f such that*

$$C_s^{-1}\|f\|_{H^s(M)}^2 \leq \sum_{j=0}^{\infty}(1+\lambda_j^2)^s\|E_j(f)\|_{L^2(M)}^2 \leq C_s\|f\|_{H^s(M)}^2. \tag{7.1.1}$$

Proof. For a complex number z, let T_z denote the spectral multiplier defined by

$$T_z(f) = \sum_{j=0}^{\infty}(1+\lambda_j^2)^{\frac{z}{2}} E_j(f).$$

T_z is thus always well defined on any finite linear combination of eigenfunctions. For such functions we also have $T_{z+w} = T_z \circ T_w = T_w \circ T_z$ for $z, w \in \mathbb{C}$. T_z is also bounded on $L^2(M)$ with operator norm equal to 1 whenever $\mathrm{Re}(z) \leq 0$. Observe also that $T_2(f) = (I + \Delta_g)(f)$ for all f in the domain of Δ_g.

Elliptic regularity (see Theorem 9.11 in [5]) gives us the following estimate for any function h in $\mathbb{R}^n$ compactly supported in the image of a coordinate chart

$$\|h\|_{L^{p,2}(\mathbb{R}^n)} \leq C(\|\Delta_g h\|_{L^p(\mathbb{R}^n)} + \|h\|_{L^p(\mathbb{R}^n)}).$$

Furthermore, applying this inequality to the function $\langle D\rangle^{-1}h$ implies

$$\|h\|_{L^{p,1}(\mathbb{R}^n)} \leq C(\|\Delta_g h\|_{L^{p,-1}(\mathbb{R}^n)} + \|h\|_{L^p(\mathbb{R}^n)})$$

as $[\Delta_g, \langle D\rangle^{-1}] : L^p \to L^p$ continuously. These elliptic regularity estimates allow us to conclude

$$\|\eta_l f \circ \chi_l^{-1}\|_{H^2(\mathbb{R}^n)} \leq C\|T_2(f)\|_{L^2(M)}$$

and hence

$$\|f\|_{H^2(M)} \leq \tilde{C}\|T_2(f)\|_{L^2(M)}.$$

Let $W \subset L^2(M)$ denote the subpace of all finite linear combinations of the eigenfunctions $\{\varphi_j\}_{j\in\mathbb{N}}$, which is dense in $L^2(M)$. Since $1 + \lambda_j^2 > 0$ for any j, T_2 is an invertible map on W with inverse T_{-2} (though not continuous with respect to the $L^2(M)$ topology). Hence given any $g \in W$ there exists $f \in W$ such that $T_2(f) = g$

$$\|T_{-2}(g)\|_{H^2(M)} = \|f\|_{H^2(M)} \leq C\|T_2(f)\|_{L^2(M)} = C\|g\|_{L^2(M)}.$$

Additionally, T_{it} is an isometry on $L^2(\mathbb{R}^n)$ for $t \in \mathbb{R}$ so we can extend the estimate above to $z \in \mathbb{C}$ with $\mathrm{Re}(z) = 2$

$$\|T_{-2(1+it)}(g)\|_{H^2(M)} \leq C\|g\|_{L^2(M)}.$$

A complex interpolation argument now allows us to conclude that

$$\|T_{-s}(g)\|_{H^s(M)} \leq C\|g\|_{L^2(M)} \tag{7.1.2}$$

for $0 \leq s \leq 2$. Since W is dense in $L^2(M)$ this estimate extends to all $g \in L^2(M)$. In particular, when $g = T_s f \in L^2$, for some $s \in [0, 2]$ a limiting argument provides the estimate

$$\|f\|_{H^s(M)} \leq C\|T_s f\|_{L^2(M)}.$$

We now consider the second inequality in (7.1.1). We begin by establishing the inequality when $0 \leq s \leq 1$. A consequence of the approach taken in Theorem 3.7 of [9] is that

$$\int_M |\varphi_k|^2 dV_g + \int_M |\nabla\varphi_k|^2 dV_g = (1 + \lambda_k^2)$$

where $|\nabla\varphi_k|^2 = g^{ij}(\partial_i\varphi_k)(\partial_j\varphi_k)$ in coordinates and dV_g denotes Riemannian density. Next observe that

$$\int_M \langle\nabla\varphi_j, \nabla\varphi_k\rangle_g dV_g = \int_M \varphi_j \Delta_g \varphi_k dV_g = \lambda_k^2 \int_M \varphi_j\varphi_k dV_g = \lambda_k^2 \delta_{jk}.$$

Now define a norm $\|\cdot\|$ on $C^\infty(M)$ determined by

$$\|f\|^2 := \int_M |f|^2 dV_g + \int_M |\nabla f|^2 dV_g$$

for any $f \in C^\infty(M)$. The norm $\|\cdot\|$ is well-defined on $H^1(M)$ and is actually equivalent to the one already defined on the space. The Pythagorean theorem now gives us that

$$\begin{aligned}\|f\|^2_{H^1(M)} \approx \|f\|^2 &= \sum_{j=0}^{\infty} \int_M |E_j(f)|^2 dV_g + \int_M |\nabla E_j(f)|^2 dV_g \\ &= \sum_{j=0}^{\infty} (1+\lambda_j^2) \|E_j(f)\|^2_{L^2(M)}.\end{aligned}$$

This implies the desired inequality when $s = 1$.

Now let $\tilde{\eta}_l$ be a smooth function supported in U_l such that $\tilde{\eta}_l \equiv 1$ on a neighborhood of $\mathrm{supp}(\eta_l)$. In this and future calculations, given a function h defined on M and $X(\mathbb{R}^n)$ a Banach space of functions on $\mathbb{R}^n$, $\|\eta_l h\|_{X(\mathbb{R}^n)}$ will denote the $X(\mathbb{R}^n)$ norm of $(\eta_l h) \circ \chi_l^{-1}$ as a function in its coordinate representation on U_l. Similar considerations will apply for $\|\tilde{\eta}_l h\|_{X(\mathbb{R}^n)}$. In addition, the expression $\langle D \rangle^z \eta_l h$ will denote the action of $\langle D \rangle^z$ on $\eta_l h \circ \chi_l^{-1}$.

We will now actually prove that $T_s : L^2(M) \to H^{-s}(M)$. This will imply the second half of (7.1.1) by self adjointness of T_s. To see that this is sufficient, let $\rho_l = \sqrt{\det g_{ij}}$ in coordinates on U_l and observe that for any $\|g\|_{L^2(M)} = 1$

$$\begin{aligned}|\int_M T_s f \overline{g}\, dV_g| = |\int_M f \overline{T_s g}\, dV_g| &\leq (\sum_l \|\rho_l \tilde{\eta}_l f\|^2_{H^s(\mathbb{R}^n)})^{\frac{1}{2}} (\sum_l \|\eta_l T_s g\|^2_{H^{-s}(\mathbb{R}^n)})^{\frac{1}{2}} \\ &\leq C \|f\|_{H^s(M)} \|g\|_{L^2(M)}\end{aligned} \tag{7.1.3}$$

where the last inequality follows from the fact that Lipschitz functions are multipliers on H^w whenever $-1 \leq w \leq 1$ and by invariance of local Sobolev spaces under coordinate changes. Taking the supremum over all such g then implies the desired inequality. This reflects a more general principle that when $|s| \leq 1$, a linear operator S maps $H^s(M) \to L^2(M)$ if and only if S^* maps $L^2(M) \to H^{-s}(M)$.

By considering the adjoint and the equivalence of $\|\cdot\|_{H^1(M)}$ and $\|\cdot\|$, we have that $T_1 : L^2(M) \to H^{-1}(M)$ continuously. Since $T_{it} : L^2(M) \to L^2(M)$ is an isometry, a simple complex interpolation shows that indeed $T_s : L^2(M) \to H^{-s}(M)$ is bounded.

To see that the second inequality in (7.1.1) holds for $1 < s \leq 2$, set $w = s - 2$ and write $T_s f = T_w(I + \Delta_g)f$. Since $-1 < w \leq 0$, (7.1.2) shows that $T_w : L^2(M) \to H^{-w}(M)$. Applying an adjoint estimate as in (7.1.3) then shows that $T_w : H^w(M) \to L^2(M)$. Hence we have the following inequality by using the continuity of differential operators in Sobolev spaces and the previously mentioned fact that Lipschitz functions are multipliers on H^w, H^{w+1}

$$\|T_s(f)\|^2_{L^2(M)} \leq C \sum_l \|\eta_l(I + \Delta_g)f\|^2_{H^w(\mathbb{R}^n)} \leq C'\|f\|^2_{H^s(M)}.$$

□

7.2 *The spectral multiplier theorem*

An important theorem here will be the spectral multiplier theorem of Duong, Ouhabaz, and Sikora proved in [3]. They give a set of sufficient conditions for functions of any self adjoint positive operator on a metric space (X, d) equipped with a measure μ to be bounded on $L^p(X, \mu)$ for $1 < p < \infty$. This result will allow us to show the standard Littlewood-Paley approximation

$$\|h\|_{L^p(M)} \approx \|\big(\sum_k |\beta_k h|^2\big)^{\frac{1}{2}}\|_{L^p(M)} \tag{7.2.1}$$

and will be useful in establishing one of the inequalities in (7.0.6).

In order to apply the theorem, the measure in discussion must satisfy a doubling condition and the heat kernel generated by the positive operator must satisfy a Gaussian-type upper bound. Additionally, simple estimates on compactly supported functions of the operator must be satisfied. See assumptions 2.1 and 2.2 of that paper as well as the formulas 3.4 and 3.5 in their statement of Theorem 3.2.

We will now proceed to verify the necessary hypotheses. Begin by verifying the doubling condition (the previously mentioned assumption 2.1) that says there should exist a constant C such that

$$\mu(B(x,2r)) \leq C\mu(B(x,r)) \text{ for all } x \in M \text{ and all } r > 0.$$

Indeed, we can take a finite open covering $\{U_i\}_{i=1}^m$ of M such that each U_i is a precompact open set in M whose closure lies in the domain of a coordinate chart. In coordinates on U_i there exists constants m_i, M_i such that $0 < m_i \leq \sqrt{\det(g_{ij}(x))} \leq M_i$ for all $x \in U_i$. Thus if ω_n denotes the Euclidean volume of the unit ball in $\mathbb{R}^n$ we have

$$\mu(B(x,2r)) \leq M_i\omega_n 2^n r^n \leq m_i^{-1} M_i 2^n \mu(B(x,r)).$$

Let δ be the Lebesgue number of the open cover $\{U_i\}_i$ and set

$$C_1 = max_{1\leq i\leq n} m_i^{-1} M_i 2^n$$

so that the doubling condition holds with constant C_1 for any $x \in M$ and any $r \in (0,\delta)$. Since $\mu(x,r)$ is jointly continuous in x and r and M there exists C_2 such that

$$\frac{\mu(B(x,2r))}{\mu(B(x,r))} \leq C_2 \text{ for all } x \in M \text{ and all } r \in [\delta, \operatorname{diam}(M)]. \tag{7.2.2}$$

Choosing $C = \max(C_1, C_2)$ now shows the doubling condition for all x, r. We pause to mention here that an analogous line of reasoning shows that

$$\tilde{C}^{-1} r^n \leq \mu(B(x,r)) \leq \tilde{C} r^n$$

for some $\tilde{C}$ independent of $(x,r) \in M \times [0, \operatorname{diam}(M)]$.

Next we verify assumption 2.2 of [3] which says that the kernel $p(t,x,y)$ of the heat semigroup generated by $-\Delta_g$ should satisfy the following Gaussian upper bound, with C, b constants

$$|p(t,x,y)| \leq C\mu(B(y,t^{\frac{1}{2}}))^{-1} \exp(-b\, d_g(x,y)^2/t) \text{ for all } t > 0, x, y \in M. \tag{7.2.3}$$

This is a consequence of an estimate of Norris (see Theorem 2.1 in [8]), proven under the milder assumption that M is a Lipschitz Riemannian manifold. By this we mean a topological manifold equipped with an atlas whose transition maps between charts are Lipschitz homeomorphisms and the coordinate representation of the metric consists of Lipschitz functions. However, since Norris assumes only that the functions $g_{ij}(x)$ are measurable in each coordinate chart, he actually proves (7.2.3) with d_g replaced by the distance function

$$d_L(x,y) = \sup\{w(x) - w(y) : w \in Lip(M), |\nabla w| \le 1\}.$$

Since our metric is Lipschitz, this distance is comparable to Riemannian distance in any coordinate chart, that is, if d_g denotes Riemannian distance there exists a constant C_l depending on the choice of coordinate chart (U_l, χ_l) such that

$$C_l^{-1} d_g(x,y) \le d_L(x,y) \le C_l d_g(x,y)$$

for all $x, y \in U_l$. Since M is compact, it can then be shown that d_L and d_g generate the same topology on M and hence are equivalent distance functions on M. Thus the heat kernel satisfies (7.2.3) with standard Riemannian distance as the distance function, implying that assumption 2.2 holds.

We now turn our attention to verifying the hypotheses in Theorem 3.2 of [3]. To see that formula 3.4 in that paper holds with $\kappa = 1$ and $p = \infty$ observe that for a Borel function F with $\text{supp}(F) \subset [-1, N+1]$ the kernel of $F(\sqrt{\Delta_g})$ can be characterized in terms of the eigenfunctions $\{\varphi_j\}_{j=0}^{\infty}$ of Δ_g on M

$$K_{F(\sqrt{\Delta_g})}(x,y) = \sum_{\lambda_j \le N+1} F(\lambda_j)\varphi_j(x)\varphi_j(y)$$

Thus by the Pythagorean theorem,

$$\int_M |K_{F(\sqrt{\Delta_g})}(\cdot, y)|^2 \, dx = \sum_{\lambda_j \le N+1} |F(\lambda_j)\varphi_j(y)|^2 \le \|F\|_{N,\infty}^2 \sum_{\lambda_j \le N+1} |\varphi_j(y)|^2$$

But

$$\sum_{\lambda_j \le N+1} |\varphi_j(y)|^2 \le e \sum_{\lambda_j \le N+1} e^{-\frac{\lambda_j^2}{(N+1)^2}} |\varphi_j(y)|^2 \le e \cdot p((N+1)^{-2}, y, y)$$

where $p(t,x,y)$ denotes the kernel of the heat semigroup generated by Δ_g. The estimate in

Theorem 2.1 of [8] cited above now yields

$$\int_M |K_{F(\sqrt{\Delta_g})}(\cdot,y)|^2\,dx \le \|F\|^2_{N,\infty} \sum_{\lambda_j \le N+1} |\varphi_j(y)|^2$$
$$\le C\|\delta_N F\|_{N,\infty} N^n \le \tilde{C}\mu(B(y,1/N))^{-1}\|\delta_N F\|_{N,\infty}$$

where δ_N denotes the dilation operator defined by $\delta_N F(\cdot) = F(N\cdot)$. To verify the hypothesis in formula 3.5 of Theorem 3.2 of [3] we note that the computation above shows that $F(\sqrt{\Delta_g})$ maps $L^1(M)$ to $L^2(M)$ with operator norm bounded above by $C\|F\|_{N,\infty}N^{n/2}$ and that the inclusion map of $L^2(M)$ into $L^1(M)$ is continuous with norm bounded by $\mathrm{vol}(M)^{\frac{1}{2}}$.

Theorem 3.2 of [3] now tells us that if $\zeta \in C_c^\infty(\mathbb{R}_+)$ is a fixed function with

$$\mathrm{supp}(\zeta) \subset [\frac{1}{2}, 1]$$

and F is any bounded Borel function satisfying

$$\sup_{t>1} \|\zeta\delta_t F\|_{L^{\infty,s}} < \infty \tag{7.2.4}$$

for some $s > n/2$, then $F(\Delta_g)$ is bounded on $L^q(M)$ for any $1 < q < \infty$.

7.3 The Littlewood-Paley decomposition

In this section, we define the Littlewood-Paley decomposition and use the spectral multiplier theorem to show (7.2.1). Let $\{\alpha_m\}_{m=0}^\infty \subset C_c^\infty(\mathbb{R})$, be a sequence of functions such that $\alpha_m(\cdot) = \alpha_1(2^{-m-1}\cdot)$ for $m \ge 1$, $\mathrm{supp}(\alpha_m) \subset \{\lambda \in \mathbb{R} : 2^{m-\frac{1}{2}} \le |\lambda| \le 2^{m+\frac{3}{2}}\}$, and $\sum_0^\infty \alpha_m \equiv 1$ on $[-\frac{1}{2}, \infty)$. Let β_m denote the operator $\alpha_m(\sqrt{\Delta_g})$.

Let $\{r_m\}_{m=0}^\infty$ be the Rademacher functions. These are defined by first setting

$$r_0(\theta) = \begin{cases} +1 & \theta \in [0, \frac{1}{2}] \\ -1 & \theta \in (\frac{1}{2}, 1). \end{cases}$$

on the unit interval. Extend r_0 outside the unit interval by periodicity, setting $r_0(\theta+1) = r_0(\theta)$ and then define $r_m(\theta) = r_0(2^m\theta)$. It is shown in Appendix D of Stein [13] that for any sequence $\{a_m\}_{m=0}^{\infty} \in l^2$ the function $G(\theta) = \sum_{m=0}^{\infty} a_m r_m(\theta)$ enjoys the following $L^p([0,1])$ bounds for $1 < p < \infty$

$$c_p \|G\|_{L^p([0,1])} \leq \|G\|_{L^2([0,1])} = (\sum_{m=0}^{\infty} |a_m|^2)^{\frac{1}{2}} \leq C_p \|G\|_{L^p([0,1])}$$

where c_p, C_p depend only on the value of p.

Consider the function

$$F_\theta(\lambda) = \sum_{m=0}^{\infty} r_m(\theta)\alpha_m(\sqrt{\lambda}).$$

We verify that F_θ satisfies (7.2.4). Computing the derivatives of $\zeta(\lambda) r_m(\theta)\alpha_m(t\sqrt{\lambda})$ with respect to lambda for $t > 0$ yields the estimates

$$\frac{d^l}{d\lambda^l}(\zeta(\lambda) r_m(\theta)\alpha_m(t\sqrt{\lambda})) \leq C_l \lambda^{-l} \leq C_l(\frac{1}{2})^{-l}$$

where C_l depends only on the derivatives of α_0, α_1, and ζ of up to order l. Thus since $\text{supp}(\zeta) \subset [\frac{1}{2}, 2]$, $\zeta\delta_t F$ is a finite sum involving at most 3 functions and hence $\|\zeta\delta_t F_\theta\|_{L^{\infty,k}}$ is uniformly bounded in t and in θ for even integers k. Hence $F_\theta(\Delta_g)$ is continuous on $L^q(M)$ with operator norm uniformly bounded in θ. This now gives us that for any $h \in L^q(M)$, $1 < q < \infty$,

$$\|(\sum_k |\beta_k h|^2)^{\frac{1}{2}}\|_{L^q(M)}^q \leq C_q \int_M \int_0^1 |\sum_k r_k(\theta)\alpha_k(\sqrt{\Delta_g})h|^q \, d\theta dV_g \leq \tilde{C}_q \|h\|_{L^q(M)}.$$

A similar line of reasoning applied to the function

$$\widetilde{F}_\theta(\lambda) = \sum_{m=0}^{\infty} r_m(\theta)(\alpha_{m-1}(\sqrt{\lambda}) + \alpha_m(\sqrt{\lambda}) + \alpha_{m+1}(\sqrt{\lambda}))$$

shows that if $\tilde{\beta}_k = \beta_{k-1} + \beta_k + \beta_{k+1}$, then there exists C_q such that

$$\|(\sum_k |\tilde{\beta}_k h|^2)^{\frac{1}{2}}\|_{L^q(M)} \leq C_q \|h\|_{L^q(M)}$$

for any q, $1 < q < \infty$. Combining this with the observations that $\tilde{\beta}_k$ is self-adjoint and

$\tilde{\beta}_k \circ \beta_k = \beta_k$ we have that for any $h_1, h_2 \in C^\infty(M)$

$$\begin{aligned}\left|\int_M h_1 h_2 \, dV_g\right| &= \left|\int_M \sum_k (\beta_k h_1) h_2 \, dV_g\right| = \left|\int_M \sum_k (\beta_k h_1)(\tilde{\beta}_k h_2) \, dV_g\right| \\ &\leq \|(\sum_k |\beta_k h_1|^2)^{\frac{1}{2}}\|_{L^q(M)} \, \|(\sum_k |\tilde{\beta}_k h_2|^2)^{\frac{1}{2}}\|_{L^{q'}(M)} \\ &\leq C\|(\sum_k |\beta_k h_1|^2)^{\frac{1}{2}}\|_{L^q(M)} \, \|h_2\|_{L^{q'}(M)}.\end{aligned}$$

which allows us to conclude that

$$\|h_1\|_{L^q(M)} \leq C_q \|(\sum_k |\beta_k h_1|^2)^{\frac{1}{2}}\|_{L^q(M)}$$

and hence

$$\|h\|_{L^q(M)} \approx \|(\sum_k |\beta_k h|^2)^{\frac{1}{2}}\|_{L^q(M)}$$

for any $h \in L^q(M)$.

Note that as a consequence, we are now able to show the approximation (7.0.5) as

$$\|h\|^2_{H^s(M)} \approx \|T_s(h)\|^2_{L^2(M)} \approx \sum_k \|\beta_k T_s(h)\|^2_{L^2(M)} \approx \sum_k 2^{2ks} \|\beta_k h\|^2_{L^2(M)}.$$

7.4 *Proof of the weighted Strichartz estimate*

We are now able to prove the inequality (7.0.3). Assume u is a solution to the following homogeneous wave equation on $[-t_0, t_0] \times M$

$$\begin{aligned}&(\partial_t^2 + \Delta_g) u(t,x) = 0 \\ &(u(0,\cdot), \partial_t u(0,\cdot)) = (f,g) \in H^{1+\sigma/p}(M) \times H^{\sigma/p}(M).\end{aligned}$$

The Littlewood-Paley decomposition along with Minkowski's inequality gives us that

$$\begin{aligned}\|u\|_{L^p([-t_0,t_0];L^q(M))} &\approx \|(\sum_{k=0}^{\infty} |\beta_k u|^2)^{\frac{1}{2}}\|_{L^p([-t_0,t_0];L^q(M))} \\ &\leq C\left(\sum_{k=0}^{\infty} \|\beta_k u\|^2_{L^p([-t_0,t_0];L^q(M))}\right)^{\frac{1}{2}}\end{aligned}$$

The approach will be to justify the following calculation:

$$\begin{aligned}
\|\beta_k u\|_{L^p([-t_0,t_0];L^q(M))} &\lesssim 2^{\frac{k\sigma}{p}} \sum_{l=1}^{m} \|\langle D\rangle^{-\frac{\sigma}{p}} \eta_l \beta_k u\|_{L^p([-t_0,t_0];L^q(\mathbb{R}^n))} \\
&\lesssim 2^{\frac{k\sigma}{p}} \sum_{l=1}^{m} \Big(\|\eta_l \beta_k f\|_{H^1(\mathbb{R}^n)} + \|\eta_l \beta_k g\|_{L^2(\mathbb{R}^n)} + \|[\Delta_g, \eta_l]\beta_k u\|_{L^1([-t_0,t_0];L^2(\mathbb{R}^n))} \Big) \\
&\lesssim 2^{\frac{k\sigma}{p}} \sum_{l=1}^{m} \Big(\|\eta_l \beta_k f\|_{H^1(\mathbb{R}^n)} + \|\eta_l \beta_k g\|_{L^2(\mathbb{R}^n)} + \|\tilde{\eta}_l \beta_k u\|_{L^1([-t_0,t_0];H^1(\mathbb{R}^n))} \Big) \\
&\lesssim \|\beta_k f\|_{H^{1+\frac{\sigma}{p}}(M)} + \|\beta_k g\|_{H^{\frac{\sigma}{p}}(M)} + \|\beta_k u\|_{L^\infty([-t_0,t_0];H^{1+\frac{\sigma}{p}}(M))} \\
&\lesssim \|\beta_k f\|_{H^{1+\frac{\sigma}{p}}(M)} + \|\beta_k g\|_{H^{\frac{\sigma}{p}}(M)}
\end{aligned}$$

Taking the squares of both sides and then summing over k will then yield the result. Most of the work involved will be in showing the first inequality, which is the content of the proposition below. The second inequality then follows from the previously established Strichartz estimates in Theorem 1.6.2. The third inequality is a result of simple commutator estimates and the fact that $\tilde{\eta}_l$ is identically one on the support of $[\Delta_g, \eta_l]\beta_k u$. The fourth inequality follows from the characterization of H^s by the eigenfunctions and the last inequality is just energy conservation.

The following proposition thus provides key inequality in the computation.

Proposition 7.4.1. *Suppose $h \in L^q(M)$ for some $1 < q < \infty$. Then the following estimate holds for some constant C_s depending only on the value of $s \in [0,1]$, q, and the choice of partition of unity and open cover given above.*

$$\|\beta_k h\|_{L^q(M)} \le C_s 2^{ks} \sum_{l=1}^{m} \|\langle D\rangle^{-s} \eta_l \beta_k h\|_{L^q(\mathbb{R}^n)}$$

Proof. Since β_k is self-adjoint, we will actually show an adjoint estimate that

$$\left(\sum_{l} \|\langle D\rangle^{s} \eta_l \tilde{\beta}_k f\|^q_{L^q(\mathbb{R}^n)} \right)^{\frac{1}{q}} \le C 2^{ks} \|\tilde{\beta}_k f\|_{L^q(M)} \tag{7.4.1}$$

for arbitrary f and arbitrary $1 < q < \infty$, $s \in [0,1]$. Indeed, if this estimate holds we have that for any $f, g \in C^\infty(M)$ such that $\|f\|_{L^q(M)} = 1$ and ρ_l denoting $\sqrt{\det g_{ij}}$ in coordinates on U_l

$$\begin{aligned}|\int f\beta_k g\, dV_g| &\leq \left(\sum_l \|\langle D\rangle^s \rho_l \eta_l \tilde{\beta}_k f\|^q_{L^q(\mathbb{R}^n)}\right)^{\frac{1}{q}} \left(\sum_l \|\langle D\rangle^{-s} \tilde{\eta}_l \beta_k g\|^{q'}_{L^{q'}(\mathbb{R}^n)}\right)^{\frac{1}{q'}} \\ &\leq C2^{ks} \|f\|_{L^q(M)} \sum_l \|\langle D\rangle^{-s} \eta_l \beta_k g\|_{L^{q'}(\mathbb{R}^n)}.\end{aligned}$$

Here the second inequality follows from (7.4.1) and the fact that multiplication by a Lipschitz function is bounded operation on the Sobolev space $L^{q,s}$ for $|s| \leq 1$. Taking the supremum over all such functions f then yields the inequality

$$\|\beta_k g\|_{L^{q'}(M)} \leq C2^{ks} \sum_l \|\langle D\rangle^{-s} \eta_l \beta_k g\|_{L^{q'}(M)}.$$

Taking limits of smooth functions now establishes the inequality for any $g \in L^{q'}(M)$, $1 < q' < \infty$.

We now establish (7.4.1). Recall the elliptic regularity estimates established above.

$$\begin{aligned}\|h\|_{L^{q,2}(\mathbb{R}^n)} &\leq C(\|\Delta_g(h)\|_{L^q(\mathbb{R}^n)} + \|h\|_{L^q(\mathbb{R}^n)}) \\ \|h\|_{L^{q,1}(\mathbb{R}^n)} &\leq C(\|\Delta_g(h)\|_{L^{q,-1}(\mathbb{R}^n)} + \|h\|_{L^q(\mathbb{R}^n)}).\end{aligned}$$

Since $[\Delta_g, \eta_l] : L^{q,1} \to L^q$, we apply both of these regularity estimates to get

$$\begin{aligned}\|\eta_l \tilde{\beta}_k f\|_{L^{q,2}(\mathbb{R}^n)} &\leq C(\|\Delta_g \eta_l \tilde{\beta}_k f\|_{L^q(\mathbb{R}^n)} + \|\eta_l \tilde{\beta}_k f\|_{L^q(\mathbb{R}^n)}) \\ &\leq C(\|[\Delta_g, \eta_l]\tilde{\beta}_k f\|_{L^q(\mathbb{R}^n)} + \|\eta_l \Delta_g \tilde{\beta}_k f\|_{L^q(\mathbb{R}^n)} + \|\eta_l \tilde{\beta}_k f\|_{L^q(\mathbb{R}^n)}) \\ &\leq C(\|\tilde{\eta}_l \tilde{\beta}_k f\|_{L^{q,1}(\mathbb{R}^n)} + \|\Delta_g \tilde{\beta}_k f\|_{L^q(M)} + \|\tilde{\beta}_k f\|_{L^q(M)}) \\ &\leq C(\|\Delta_g \tilde{\beta}_k f\|_{L^q(M)} + \|\tilde{\beta}_k f\|_{L^q(M)}) \qquad (7.4.2)\end{aligned}$$

Let $G_k(\lambda) = 2^{-2k}\lambda^2 \sum_{j=k-2}^{k+2} \alpha_j(\lambda)$. A second application of the spectral multiplier theorem in [3] shows that the operator

$$G_k(\sqrt{\Delta_g}) = 2^{-2k}\Delta_g \circ (\beta_{k-2} + \tilde{\beta}_k + \beta_{k+2})$$

is continuous on $L^q(M)$ with operator norm uniformly bounded in k. This yields a constant $\tilde{C}$ independent of k such that $\|\Delta_g \tilde{\beta}_k f\|_{L^q(M)} \leq \tilde{C} 2^{2k} \|\tilde{\beta}_k f\|_{L^q(M)}$. This combined with (7.4.2) now yields (7.4.1) in the case where $s = 2$. The conclusion of the proposition for any $s \in [0,2]$ now follows by complex interpolation.

To see this, let $f \in C^\infty(M)$ and $g \in C_c^\infty(\mathbb{R}^n)$. Consider the function

$$F(z) = \int_{\mathbb{R}^n} 2^{-2kz}(\langle D\rangle^{2z}\eta_l\tilde{\beta}_k f)g\,dx.$$

Since f, g are smooth and compactly supported we can differentiate under the integral sign so that $F(z)$ is an analytic function. We also observe that it is uniformly bounded on the strip $\{z \in \mathbb{C} : \mathrm{Re}(z) \in [0,1]\}$ since

$$\begin{aligned}|F(z)| &\leq \|\langle D\rangle^{2\mathrm{Re}(z)}\eta_l\tilde{\beta}_k f\|_{L^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)} \leq \|\eta_l\tilde{\beta}_k f\|_{H^2(\mathbb{R}^n)}\|g\|_{L^2(\mathbb{R}^n)} \\ &\leq \|\tilde{\beta}_k f\|_{H^2(M)}\|g\|_{L^2(\mathbb{R}^n)} \leq C2^{2k}\|f\|_{L^2(M)}\|g\|_{L^2(\mathbb{R}^n)}\end{aligned}$$

For $t \in \mathbb{R}$ we have the stronger estimate

$$|F(1+it)| \leq 2^{-2k}\|\eta_l\tilde{\beta}_k f\|_{L^{q,2}(\mathbb{R}^n)}\|g\|_{L^{q'}(\mathbb{R}^n)} \leq C\|\tilde{\beta}_k f\|_{L^q(M)}\|g\|_{L^{q'}(\mathbb{R}^n)}.$$

Similar arguments yield $|F(it)| \leq C\|f\|_{L^q(M)}\|g\|_{L^{q'}(\mathbb{R}^n)}$. The constant C in both estimates can be taken to be independent of k. The 3 lines lemma now yields the estimate

$$\left|\int 2^{-ks}(\langle D\rangle^s\eta_l\tilde{\beta}_k f)g\right| \leq C\|\tilde{\beta}_k f\|_{L^q(M)}\|g\|_{L^{q'}(\mathbb{R}^n)}$$

Taking the supremum over all g with $\|g\|_{L^{q'}(\mathbb{R}^n)} = 1$ yields the desired adjoint estimate for smooth functions f, so that a limiting argument establishes the general case. □

BIBLIOGRAPHY

[1] Bahouri, H.; Chemin, J.-Y. Equations d'ondes quasilineares et effet dispersif. Int. Math. Res. Not. **1999**, 21, 1141-1178.

[2] Bahouri, H.; Chemin, J.-Y. Equations d'ondes quasilineares et estimations de Strichartz. Amer. J. Math. **1999**, 121 (6), 1337-1377.

[3] Duong, X.T.; Ouhabaz, E.M.; A. Sikora. Plancherel-type estimates and sharp spectral multipliers. J. Funct. Anal. **2002**, 196 (2), 443-485.

[4] Genibre, J.; Velo, G. Generalized Strichartz inequalities for the wave equation. J. Funct. Anal. **1995**, 133 (1), 50-68.

[5] Gilbarg, D.; Trudinger, N. *Elliptic Partial Differential Equations of Second Order*, 2nd Ed.; Springer-Verlag: New York, 1983.

[6] Keel, M.; Tao, T. Endpoint Strichartz Estimates. Amer. J. Math. **1998**, 120, 955-980.

[7] Lindblad, H.; Sogge, C. D. On existence and scattering with minimal regularity for semilinear wave equations. J. Funct. Anal. **1995**, 130 (2), 357-426.

[8] Norris, J.R. Heat kernel asymptotics and the distance function in Lipschitz Riemannian manifolds. Acta Math. **1997**, 179, 79-103.

[9] Sakai, T. *Riemannian Geometry*; American Mathematical Society: Providence, Rhode Island, 1996.

[10] Smith, H. A Parametrix Construction for Wave Equations with $C^{1,1}$ Coefficients. Ann. Inst. Fourier (Grenoble) **1998**, 48 (3), 797-835.

[11] Smith, H.; Sogge, C. On Strichartz and Eigenfunction estimates for low regularity metrics. Math. Res. Lett. **1994** 1 (6), 729-737.

[12] Stein, E. M. *Harmonic Analysis: Real Variable Methods, Orthogonality, and Oscillatory Integrals*; Princeton University Press: Princeton, N.J., 1993.

[13] Stein, E. M. *Singular Integrals and Differentiability Properties of Functions*; Princeton University Press: Princeton, N.J., 1970.

[14] Strichartz, R. A priori estimates for the wave equation and some applications. J. Funct. Analysis **1970**, 5, 218-235.

[15] Strichartz, R. Restriction of Fourier Transform to quadratic surfaces and decay of solutions to the wave equation. Duke Math J. **1977**, 44 (3), 705-714.

[16] Tataru, D. Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients and the nonlinear wave equation. Amer. J. Math **2000**, 122 (2), 349-376.

[17] Tataru, D. Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients II. Amer. J. Math **2001**, 123 (3), 385-423.

[18] Tataru, D. Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients III. J. Amer. Math. Soc. **2002**, 15 (2), 419-442.

[19] Taylor, M.E. *Pseudodifferential Operators and Nonlinear PDE*; Birkhhäuser: Boston, 1991.

VITA

Matthew Blair was born and raised in East Lansing, Michigan. It was there that he earned a Bachelor of Science in mathematics at Michigan State University. After completing his Ph.D. at the University of Washington, he plans to do postdoctoral work at Johns Hopkins University.