Bispectral Operator Algebras

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Abstract

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This dissertation is an amalgamation of various results on the structure of bispectral differential operator algebras, ie. algebras of differential operators with possibly noncommutative coefficients in a variable x satisfying the property of having a family $\psi(x, y)$ of joint eigenfunctions which are also eigenfunctions of another operator in the *spectral* parameter y. In this document, we extend the modern theory of commuting differential operators to differential operators with noncommutative coefficients. We prove under fairly general circumstances that such algebras are isomorphic to endomorphism rings of torsion-free modules on rational curves. We also classify all rank 1 noncommutative bispectral differential operator algebras and explore the role of Darboux transformations in the construction of bispectral differential operator algebras, particularly for the bispectral operator algebras associated to a weight matrix w.

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DEDICATION

to my parents and all of my family, including Sarah Juul,

and in memory of James H. Olsen, a mentor and friend

Chapter 1

INTRODUCTION

1.0.1 Algebras of Bispectral Operators

This thesis concerns the study of bispectrality, and its influence on the structure of operator algebras. A question originally posed by Grünbaum and Duistermaat in [15], inspired by questions in X-ray tomography, is to classify all bispectral triples (P, Q, ψ), where $P = P(x, \partial_x)$ and $Q = Q(y, \partial_y)$ are differential operators and $\psi(x, y)$ is a function of two variables satisfying

$$P \cdot \psi(x, y) = \psi(x, y)\lambda(y), \quad Q \cdot \psi(x, y) = \psi(x, y)\theta(x)$$

for some functions $\lambda(y)$ and $\theta(x)$ (here $\cdot$ denotes the action of the differential operator on the function). More generally, given P and $\psi(x, y)$ the collection of all Q such that (P, Q, ψ) is a bispectral triple forms an algebra $\mathfrak{D}(P, \psi)$, which we call an algebra of bispectral operators. The notion of a bispectral operator algebra can be naturally generalized in many ways.

Wilson proved in [70] that rank-1 bispectral algebras of differential operators correspond to tuples $(X, \mathcal{L}, \pi, \phi, p)$ with X a rational plane curve. Moreover, Duistermaat and Grünbaum classified all the bispectral operators of order 2 in terms of Darboux transformations of a finite collection of Schrödinger operators [15]. We are interested in extending classification results of this kind to noncommutative bispectral operator algebras.

Very recently, Grünbaum, Geiger, and others have considered bispectral algebras of matrix differential operators [36][44][24]. In this case, the bispectral triples are of the form (P, Q, ψ) for $P = P(x, \partial x)$ and $Q = Q(y, \partial_y)$ matrix differential operators and $\psi = \psi(x, y)$ a matrix-valued function. These considerations have shown interesting applications solutions of nonlinear PDEs, such as the AKNS system and the matrix Sine-Gordon equation [29][62].

Another generalization considered recently in the literature is to determine the structure of the algebra $\mathfrak{D}(w)$ associated to a weight matrix w [37][11][68][40][71][72]. This may be viewed as a bispectral problem through the relation of w to a block Jacobi matrix [12][23][21]. The algebra $\mathfrak{D}(w)$ is the set of matrix differential operators for which the orthogonal matrix polynomials of w are eigenfunctions. This bispectral problem is related to Bochner's problem for matrix differential operators, which is to determine for what weight matrices w the algebra $\mathfrak{D}(w)$ contains an operator of order 2 [17][5]. Solutions of Bochner's problem for matrix differential operators have been shown to have significant applications in other fields of mathematics including representation theory of Lie groups [55][46][45] and probability theory. They have also had interesting applications in science and engineering, including quasi birth and death processes [38], time and band limiting [10][19], and the numerical calculation of special functions [13][60].

1.0.2 Differential Operators and Algebraic Geometry

Our structural results will rely on the connection between differential operators and algebraic geometry. The pioneering work of Burchnall-Chaundy [8] and Schur [64] showed that two commuting differential operators $P = P(x, \partial_x)$ and $Q = Q(x, \partial_x)$ must in fact satisfy an algebraic relationship $F(P, Q) = 0$, drawing an unexpected connection between differential operators and algebraic curves. However the significance of this connection, and its application to the solution of nonlinear partial differential equations, was not discovered until much later by Krichever and others in the Russian school [49][48][14], and brought to its modern state by other authors including Mumford [54], Segal and Wilson [65], Mulase [52], and Sato [61].

Krichever's ideas formed from an attempt to resolve solutions to certain nonlinear partial differential equations (PDEs) such as the KdV equation, the KP equation, the Sine-Gordon equation, and the nonlinear Schrödinger equation with "algebraic" potentials not able to be handled by scattering theory [50]. The modern result is a correspondence between solutions of algebraically integrable nonlinear PDEs and tuples $(X, \mathcal{L}, \pi, \phi, p)$ consisting of an algebraic curve X, a line bundle $\mathcal{L}$ on X, a nonsingular point $p \in X$, a cover ϕ of X ramified at p, and a local trivialization π of $\mathcal{L}$ in a neighborhood of p. The tuples $(X, \mathcal{L}, \pi, \phi, p)$ in turn correspond to algebras of commuting differential operators. A nice account of this by Mulase may be found in [53].

1.1 A Reading Guide

In Chapter 2, we recall some basic theory of algebras of differential and pseudo-differential operators over a differential ring R. In Chapter 3, we describe the modern theory of commuting differential operators in terms of the Sato grassmannian, and we extend Sato's grassmannian to a vector-valued version usable in the noncommutative context. In Chapter 4, we construct a matrix-valued Baker-Alkheizer function and the Sato-Segal-Wilson tau function. We also construct a new, matrix-valued tau function which compares favorably to the Sato-Segal-Wilson tau function, and which satisfies a formula akin to Sato's formula for the Baker-Alkheizer function. Structural results for bispectral differential operator algebras are contained in Chapter 5 and beyond.

1.2 Conventions and Notations

Throughout this document, we will reserve the use of symbols of the form $\mathfrak{p}, \mathfrak{d}, \mathfrak{b}, \mathfrak{v}, \mathfrak{w}$, etc to refer to differential operators or more generally pseudo-differential operators. We will also use the following basic notation throughout the document:

- I the identity matrix (with size determined by context)
- $R[[x]]$ the ring of power series in x with coefficients in R
- $R((x))$ the ring of Laurent series in x with coefficients in R
- $\mathfrak{D}(R)$ the ring of differential operators with coefficients in a differential ring R
- $\mathfrak{P}(R)$ the ring of pseudo-differential operators with coefficients in a differential ring R

- $\ker(\delta)$ the kernel of a differential operator as a linear operator on the differential ring R
- $\mathfrak{v}^{-1}$ the inverse of a pseudo-differential operator $\mathfrak{v} \in \mathfrak{P}(R)$ (if it exists)
- $\mathrm{Gr}(R)$ the Sato grassmannian and $\mathrm{Gr}(R;\mu)$ its index μ subset
- $\overline{\mathrm{Gr}}(R)$ the Hilbert space grassmannian of R (assuming R is an $\mathbb{C}$-algebra) and $\overline{\mathrm{Gr}}(R;\mu)$ its index μ subset
- $\tau_H(\vec{t}, z)$ the Sato-Segal-Wilson tau funcion of a point $H \in \overline{\mathrm{Gr}}(R;0)_+$ in the big cell of the index 0 grassmannian
- $\mathcal{T}_H^g(\vec{t}, z)$ the g-twisted matrix tau function of a point $H \in \overline{\mathrm{Gr}}(R;0)_+$

Chapter 2
BACKGROUND

2.1 Differential Operator Algebras

Classically, differential operators are linear operators arising naturally from the structure of linear ordinary differential equations. Given a differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = f(x),$$

we can rewrite this as a linear operator equation $L \cdot y = f(x)$ with

$$L = a_n(x)\partial_x^n + a_{n-1}(x)\partial^{n-1} + \cdots + a_1(x)\partial + a_0(x)$$

where here ∂_x denotes the operation of differentiation $\partial_x \cdot y = y'$ and we are using $a_j(x)$ to denote both a function of x and the multiplication operator $a_j(x) \cdot y = a_j(x)y$. By studying the *algebra* of operators of this form, we can relate the solution of differential equations to the study of eigenvalue problems.

2.1.1 Basic Definitions

Our definition of differential rings, differential algebras, and the ring of constants follows Kaplansky [43] and similar sources related to differential galois theory. However, our focus will differ in that we will be more interested in the study of non-commutative differential algebras, rather than differential fields. Our definition of the ring of pseudo-differential operators is consistent with that found in [56], [31], and others motiviated by Schur's original work [63].

Definition 2.1.1.1. A **differential ring** is a pair (R, ∂) where R is a ring and $\partial : R \to R$

is a function satisfying the two properties

$$\partial(rs) = \partial(r)s + r\partial(s),$$

$$\partial(r + s) = \partial(r) + \partial(s),$$

for all $r, s \in R$. We call ∂ the **derivative** of the differential ring R. We will write R in place of (R, ∂) if there is no ambiguity. We will also write $\partial \cdot r$ in place of $\partial(r)$, and will sometimes write r', r'', or $r^{(m)}$ in place of $\partial \cdot r$, $\partial^2 \cdot r$ and $\partial^k \cdot r$. The **ring of constants** of a differential ring R is the subring $K \subseteq R$ consisting of elements which ∂ maps to 0. Elements of the ring of constants are called **constant**. Note in particular that the identity is always a constant. A differential ring R, which is also a k-algebra in which the k-values are all constant is called a **differential k-algebra**.

Example 2.1.1.2. Let U be an open subset of $\mathbb{C}$ and let R be the algebra of holomorphic functions on U. Then R is a differential $\mathbb{C}$-algebra with the usual derivative.

Example 2.1.1.3. Let (R, ∂) be a differential ring which is a commutative integral domain. Then there exists a unique extension of ∂ to the fraction field $F(R)$ of R, making $F(R)$ a differential ring.

Example 2.1.1.4. Let R be a ring and $a \in R$. Then the map $\partial : R \to R$ defined by $\partial(r) = ar - ra$ defines a derivative on R, making R a differential ring.

Example 2.1.1.5. Let R be a differential ring. Then the ring $M_N(R)$ of $N \times N$ matrices with coefficients in R is a differential ring, with the derivative operator ∂ acting coefficient-wise.

Definition 2.1.1.6. Let R be a differential ring. Then the **ring of differential operators** with coefficients in R is defined to be the set of expressions of the form

$$\mathfrak{D}(R) = \left\{ \sum_{j=0}^{n} r_j \partial^j : n \geq 0, \text{ and } r_j \in R \text{ for all } j \right\}.$$

This set has a ring structure with the obvious sum, and with the product satisfying

$$\left(\sum_{j=0}^{n} r_j \partial^j \right) \left(\sum_{i=0}^{m} q_i \partial^i \right) = \sum_{j=0}^{n} \sum_{\ell=0}^{j} \binom{j}{\ell} r_j (\partial^\ell \cdot q_i) \partial^{i+j-\ell}.$$

The **ring of pseudo-differential operators** is defined similarly as

$$\mathfrak{P}(R) = \left\{ \sum_{j=-\infty}^{n} r_j \partial^j : n \geq 0, \text{ and } r_j \in R \text{ for all } j \right\},$$

where the product satisfies

$$\left(\sum_{j=0}^{n} r_j \partial^j \right) \left(\sum_{i=0}^{m} q_i \partial^i \right) = \sum_{j=0}^{n} \sum_{\ell=0}^{\infty} \binom{j}{\ell} r_j (\partial^\ell \cdot q_i) \partial^{i+j-\ell}.$$

using the (generalized) binomial coefficient

$$\binom{j}{\ell} = \frac{j(j-1)(j-2)\dots(j-\ell+1)}{\ell!}.$$

Elements of $\mathfrak{D}(R)$ and $\mathfrak{P}(R)$ are called **differential operators** and **pseudo-differential operators**, respectively. Note that $\mathfrak{P}(R)$ contains $\mathfrak{D}(R)$.

Remark 2.1.1.7. The binomial coefficient $\binom{j}{\ell}$ is integer-valued for all integers j, ℓ. Therefore it may be viewed as an element of the ring of constants K of R as a repeated sum of 1. This works even if the characteristic of R is p.

Definition 2.1.1.8. Let R be a differential ring, and consider a pseudo-differential operator $\mathfrak{p} \in \mathfrak{P}(R)$, with

$$\mathfrak{p} = \sum_{j=-\infty}^{n} r_j \partial^j.$$

The highest value of j for which r_j is nonzero is called the **order** of $\mathfrak{p}$. If $\mathfrak{p}$ has order n, then r_n is called the **leading coefficient** of $\mathfrak{p}$ and r_{n-1} is called the **subleading coefficient** of $\mathfrak{p}$. We call $\mathfrak{p}$ **monic** if its leading coefficient is 1, and **normalized** if it is both monic and has subleading coefficient equal to 0. A **wave operator** is a pseudo-differential operator which is monic of order 0.

2.1.2 Kernels of Differential Operators and Reducibility

We next define the notions of the kernel of a differential operator, as well as what it means for a differential ring to be differentially closed. The main idea is that the kernel of a differential operator $\mathfrak{d}$ is closely tied to various factorizations of $\mathfrak{d}$. This in itself is related to the method of "reduction of order" taught in an undergraduate differential equations class [7]. In reduction of order, if we know a particular solution y_p of a second-order homogeneous ordinary differential equation

$$\mathfrak{d} \cdot y = y'' + by' + cy = 0,$$

for $\mathfrak{d} = \partial^2 + b\partial + c$, then by setting $y = vy_p$ we find

$$v'' + (b + y_p' y_p^{-1})v' = 0,$$

which is a first order differential equation in v'. Solving this, we obtain $v = \int \exp \int (b + y_p' y_p^{-1})$ and thereby a two-parameter family of solutions

$$y = y_p \int \exp \int (b + y_p' y_p^{-1}),$$

with the parameters coming from the constants of integration. The reason this works is that $y_p^{-1} \mathfrak{d} y_p = \mathfrak{b}\partial$ for some differential operator $\mathfrak{b}$ of order 1, and this relates the kernel of $\mathfrak{d}$ and $\mathfrak{b}$. This same line of pursuit leads us to the idea of a Darboux transformation, which we will employ later on in our study of the algebra $D(w)$ of differential operators associated to a weight matrix w.

Definition 2.1.2.1. Let R be a differential ring and $\mathfrak{d} \in \mathfrak{D}(R)$. We define the **kernel** of $\mathfrak{d}$ in R to be the set

$$\ker_R(\mathfrak{d}) = \{r \in R : \mathfrak{d} \cdot r = 0\}.$$

We will write $\ker(\mathfrak{d})$ in place of $\ker_R(\mathfrak{d})$ when there is no ambiguity.

Note of course that the kernel of a differential operator depends on the differential ring R over which it is defined. For example, the differential operator $\partial - 1$ has no kernel as an element of $\mathfrak{D}(\mathbb{C}[x])$, but does have a kernel when viewed as an element of $\mathfrak{D}(\mathbb{C}[[x]])$. In the case that $R = L$ is a field of characteristic 0, the differential map ∂ extends to any separable field extension of L, and we can extend up to a point where the kernel of a differential operator $\mathfrak{d}$ stabilizes. In particular, it grows until the dimension of $\ker(\mathfrak{d})$ as a vector space over the field of constants K of L has the same dimension as the order of $\mathfrak{d}$.

Proposition 2.1.2.2. *Let $\mathfrak{d} \in \mathfrak{D}(R)$ be a differential operator of order d, and suppose $r \in \ker(\mathfrak{d})$ is a unit in R. Then there exists $\mathfrak{b} \in \mathfrak{D}(R)$ which is monic of order $d-1$ such that*

$$\mathfrak{d} = \mathfrak{b}(\partial - r'r^{-1}).$$

Proof. Suppose that $r \in \ker(\mathfrak{d})$ is a unit, and write

$$r^{-1}\mathfrak{d}r = \sum_{j=0}^{d} a_j \partial^j,$$

for some $a_0, \ldots, a_d \in R$. Then clearly $r^{-1}\mathfrak{d}r \cdot 1 = a_0$. However, since $\mathfrak{d} \cdot r = 0$ we also have

$$r^{-1}\mathfrak{d}r \cdot 1 = r^{-1}\mathfrak{d} \cdot r = r^{-1}0 = 0,$$

and therefore $a_0 = 0$. Hence

$$r^{-1}\mathfrak{d}r = \left(\sum_{j=0}^{d-1} a_{j+1}\partial^j \right) \partial,$$

and setting $\mathfrak{b} = r \left(\sum_{j=0}^{d-1} a_{j+1}\partial^j \right) r^{-1}$ we find

$$\mathfrak{d} = \mathfrak{b} r \partial r^{-1} = \mathfrak{b}(\partial - r'r^{-1}).$$

□

The previous proposition motivates our definition of differentially closed and differentially reducible. Our definition of differentially closed is consistent with the usual definition from differential galois theory in the case that R is a differential field.

Definition 2.1.2.3. Let R be a differential ring. We say that R is **differentially reducible** if every monic differential operator $\mathfrak{d} \in \mathfrak{D}(R)$ factors as a product of monic differential operators of order 1. We say that R is **differentially closed** if the kernel of every monic differential operator $\mathfrak{d}$ contains a unit of R.

Proposition 2.1.2.4. *Let R be a differential ring. If R is differentially closed, then R is differentially reducible.*

Proof. Suppose that $\mathfrak{d} \in \mathfrak{D}(R)$ is monic of order r. Since R is differentially closed, there exists a unit $r \in R$ with $\mathfrak{d} \cdot r = 0$. It follows that $\mathfrak{b} := r^{-1}\mathfrak{d}r$ is a monic differential operator of order r with

$$\mathfrak{b} \cdot 1 = r^{-1}\mathfrak{d}r \cdot 1 = r^{-1}\mathfrak{d} \cdot r = 0.$$

Therefore the kernel of $\mathfrak{b}$ contains 1. Writing

$$\mathfrak{b} = \sum_{n=0}^{r} b_n \partial^n,$$

this says that $b_0 = 0$, and therefore $\mathfrak{b} = \mathfrak{b}_0\partial$. Hence

$$\mathfrak{d} = (r\mathfrak{b}_0 r^{-1})(r\partial r^{-1}) = \mathfrak{d}_0(\partial - r'r^{-1}),$$

for $\mathfrak{d}_0 = r\mathfrak{b}_0 r^{-1}$. Order arguments tell us that $\mathfrak{d}_0$ is monic of order $r - 1$. Thus by the obvious inductive argument, $\mathfrak{d}$ may be factored as a product of monic, order 1 differential operators. □

Example 2.1.2.5. The ring R of holomorphic functions on a simply connected open subset U of $\mathbb{C}$, along with the usual differential, is differentially closed.

Example 2.1.2.6. The ring $R = \mathbb{C}[x]$ of polynomials with coefficients in $\mathbb{C}$, along with the usual differential, is not differentially reducible. For example, the operator $\mathfrak{d} = \partial^2 - x$ does not factor in R.

Example 2.1.2.7. The ring $R = \mathbb{C}[[x]]$ of power series with coefficients in $\mathbb{C}$ is differentially closed.

Example 2.1.2.8. The ring $R = \mathbb{C}((x))$ of Laurent series with coefficients in $\mathbb{C}$ not differentially closed, since for example the operator $\partial - \frac{1}{2x}$ has no kernel in $\mathbb{C}((x))$.

Example 2.1.2.9. The ring $R = \mathbb{C}$ with the trivial derivative $\partial = 0$ is differentially reducible, but not differentially closed.

Remark 2.1.2.10. It is important that in our definition of differentially closed we are considering monic operators, even if R is an integral domain. For example, the operator

$$\mathfrak{d} = 2x^2\partial^2 - 1$$

has no kernel in $\mathbb{C}[[x]]$, even though by our definition $\mathbb{C}[[x]]$ is differentially closed.

Definition 2.1.2.11. Let R be a differential ring and $\mathfrak{d} \in \mathfrak{P}(R)$. We call $\mathfrak{d}$ **right almost monic** if there exists $\mathfrak{b} \in \mathfrak{D}(R)$ with $\mathfrak{d}\mathfrak{b}$ a monic. The notion of **left almost monic** is defined similarly, and an element which is both left and right almost monic is called **almost monic**.

Note that if $\mathfrak{b}$ is almost monic, then $\mathfrak{b}$ is a unit in $\mathfrak{P}(R)$.

Proposition 2.1.2.12. *Suppose that R is a differentially closed ring, and let $\mathfrak{d}, \mathfrak{b} \in \mathfrak{D}(R)$ with $\mathfrak{b}$ almost monic. Then there exists $\mathfrak{q} \in \mathfrak{D}(R)$ satisfying $\mathfrak{d} = \mathfrak{q}\mathfrak{b}$ (equivalently, the element $\mathfrak{d}\mathfrak{b}^{-1}$ of $\mathfrak{P}(R)$ lies in $\mathfrak{D}(R)$) if and only if* $\ker(\mathfrak{b}) \subseteq \ker(\mathfrak{d})$.

Proof. Let $\mathfrak{d}, \mathfrak{b} \in \mathfrak{D}(R)$ be as in the statement of the proposition. If there exists a $\mathfrak{q} \in \mathfrak{D}(R)$ satisfying $\mathfrak{q}\mathfrak{b} = \mathfrak{d}$, then clearly $\ker(\mathfrak{b}) \subseteq \ker(\mathfrak{d})$.

Conversely, suppose that $\mathfrak{b} \in \mathfrak{D}(R)$ satisfies $\ker(\mathfrak{b}) \subseteq \ker(\mathfrak{d})$. Choose $\mathfrak{p} \in \mathfrak{D}(R)$ with $\mathfrak{b}\mathfrak{p}$ monic. Then $r \in \ker(\mathfrak{b}\mathfrak{p})$ if and only if $\mathfrak{p} \cdot r \in \ker(\mathfrak{b})$. Therefore if $r \in \ker(\mathfrak{b}\mathfrak{p})$ then $\mathfrak{p} \cdot r \in \ker(\mathfrak{d})$, so that $r \in \ker(\mathfrak{d}\mathfrak{p})$. Hence $\ker(\mathfrak{b}\mathfrak{p}) \subseteq \ker(\mathfrak{d}\mathfrak{p})$. Now since R is differentially reducible, we can write

$$\mathfrak{b}\mathfrak{p} = \mathfrak{b}_m\mathfrak{b}_{m-1}\dots\mathfrak{b}_1,$$

where each $\mathfrak{b}_i$ is monic of order 1. Then Proposition 2.1.2.2 combined with the assumption that R is differentially closed tells us that there exists $\mathfrak{q}_1 \in \mathfrak{D}(R)$ with $\mathfrak{d}\mathfrak{p} = \mathfrak{q}_1\mathfrak{b}_1$. Repeating this argument with $\mathfrak{b}_2, \dots, \mathfrak{b}_m$, we may recursively define $\mathfrak{q}_2, \dots, \mathfrak{q}_m \in \mathfrak{D}(R)$ such that $\mathfrak{q}_{k-1} = \mathfrak{q}_k\mathfrak{b}_k$ for all $2 \le k \le m$. Consequently

$$\mathfrak{b}\mathfrak{p} = \mathfrak{q}_m\mathfrak{b}_m\mathfrak{b}_{m-1}\dots\mathfrak{b}_1 = \mathfrak{q}_m\mathfrak{b}\mathfrak{p}.$$

Since $\mathfrak{p}$ is a non-right zero divisor in $\mathfrak{D}(R)$, this implies $\mathfrak{q}_m\mathfrak{b} = \mathfrak{d}$. This proves the converse. □

Corollary 2.1.2.12.1. *Suppose that R is a differentially closed ring, and let $\mathfrak{d}, \mathfrak{b} \in \mathfrak{D}(R)$ with $\mathfrak{b}$ almost monic. Then $\mathfrak{b}\mathfrak{d}\mathfrak{b}^{-1} \in D(R)$ if and only if $\mathfrak{d} \cdot \ker(\mathfrak{b}) \subseteq \ker(\mathfrak{b})$.*

Proof. By the previous proposition, $\mathfrak{b}\mathfrak{d}\mathfrak{b}^{-1}$ is a differential operator if and only if $\ker(\mathfrak{b}) \subseteq \ker(\mathfrak{b}\mathfrak{d})$. The latter is true if and only if $\mathfrak{d} \cdot \ker(\mathfrak{b}) \subseteq \ker(\mathfrak{b})$. □

2.2 Bispectral Operators

The classical notion a bispectral differential operator is a differential operators $\mathfrak{d}$ in variable x with a family of eigenfunctions $\psi(x, y)$ satisfying

$$\mathfrak{d} \cdot \psi(x, y) = \psi(x, y)g(y)$$

for some non-constant function $g(y)$ which is also a family of eigenfunctions for some differential operator in the spectral parameter y. More specifically so that there exists a differential operator $\mathfrak{b}$ in parameter y, and a function $f(x)$ such that

$$\mathfrak{b} \cdot \psi(x, y) = \psi(x, y)f(x).$$

The classification of bispectral operators of order 2 was performed by Duistermaat and Grünbaum [15]. The classification of bispectral operators of prime order was completed sixteen years later by Horozov [42]. In general however, the classification of bispectral operators is currently incomplete.

Further studies have focused on the study of algebras of bispectral operators, ie. for fixed $\mathfrak{b}, \psi(x, y)$, and $f(x)$ the algebra of all differential operators $\mathfrak{d}$ in x for which $\psi(x, y)$ is a family of eigenfunctions. In [70], Wilson shows that the algebras of bispectral differential operators of rank 1 (ie containing two differential operators of relatively prime order) are determined by line bundles on rational plane curves. In particular, these algebras are commutative and their spectra are rational plane curves. More generally, algebras of bispectral operators have been used to construct new examples of bispectral operators from old ones [2][3].

More recently, authors have explored the idea of noncommutative bispectral differential operators, ie. bispectral differential operators whose differential ring is noncommutative [35][25][36][44]. However, to get around issues of non-commutativity it is useful to take the action of one of the differential operators to be a right action. The general setup in this case is as follows.

2.2.1 Basic Definitions

Definition 2.2.1.1. We define an **operator algebra** to be an algebra A with a distinguished subalgebra $\mathfrak{M}(A)$ such that the natural structure of $\mathfrak{M}(A)$ as a left and right $\mathfrak{M}(A)$-module extends to a left and right A-module structure on $\mathfrak{M}(A)$. The subalgebra $\mathfrak{M}(A)$ is called the **algebra of multiplication operators**.

Example 2.2.1.2. Let A be an algebra. Then A is an operator algebra with $\mathfrak{M}(A) = A$.

Example 2.2.1.3. Let R be a differential ring. Then $\mathfrak{D}(R)$ is an operator algebra with multiplicative subalgebra $\mathfrak{M}(A) = R$.

Example 2.2.1.4. Let A be an operator algebra. Then the opposite ring A^{op} is also an operator algebra with $\mathfrak{M}(A^{op}) = \mathfrak{M}(A)^{op}$.

Definition 2.2.1.5. We define a **bispectral setup** to be a triple (A, B, M) where A, B are operator algebras and $M =_A M_B$ is an A, B-bimodule. A **bispectral triple** in a bispectral setup (A, B, M) is a triple (a, b, m) with $a \in A, b \in B$ and $m \in M$ such that m has trivial left and right annihilator, and such that there exist $f \in \mathfrak{M}(A)$ and $g \in \mathfrak{M}(B)$ satisfying

$$a \cdot m = m \cdot g \quad \text{and} \quad m \cdot b = f \cdot m.$$

Example 2.2.1.6. Let U_1, U_2 be open, simply connected subsets of $\mathbb{C}$, and let R_i be the ring of holomorphic functions on U_i for $i = 1, 2$. Let ∂_x and ∂_y represent the derivatives on R_1 and R_2, respectively. Then set $A = \mathfrak{D}(R_1)$, $B = \mathfrak{D}(R_2)^{op}$, and $M = Holo(U \times V)$ the set of holomorphic functions on $U \times V$. Then M has a natural A, B-bimodule structure making (A, M, B) a bispectral setup. The following are all examples of bispectral triples for this bispectral setup for appropriate choices of U_1, U_2.

$$(\partial_x, \partial_y, e^{xy})$$

$$\left(\partial_x^2 - \frac{2}{x^2}, \partial_y^2 - \frac{2}{y}, e^{xy}\left(1 - \frac{1}{xy}\right)\right)$$

$$\left(\partial_x^2 - \frac{6}{x^2}, \partial_y^2 - \frac{6}{y}, e^{xy}\left(1 - \frac{3}{xy} + \frac{3}{x^2y^2}\right)\right)$$

With this in mind, the bispectral problem is the following.

Problem 2.2.1.7 (The Bispectral Problem)**.** The bispectral problem for a bispectral setup (A, B, M) is to find all bispectral triples (a, m, b).

Usually, the bispectral problem is too difficult to answer in general, and serves more as a motivation rather than a realistic goal. More typically, research is focused on finding all bispectral triples satisfying a certain property, and this has historically been much more successful. Often efforts are also directed toward finding methods of constructing new bispectral triples from old ones.

A related problem is to determine the algebraic structure of bispectral algebras of differential operators. Given a bispectral setup (A, B, M) and $m \in M$, we can construct an algebra

$$\text{Bis}(m) = \{(a, b) : (a, b, m) \text{ is a bispectral triple}\},$$

where multiplication is done coordinate-wise. This decomposes as the product of two algebras

$$\text{Bis}_L(m) = \{a : \text{there exists } b \text{ with } (a, b, m) \text{ a bispectral triple}\},$$

$$\text{Bis}_R(m) = \{b : \text{there exists } a \text{ with } (a, b, m) \text{ a bispectral triple}\}.$$

We refer to these as the left and right bispectral algebras of $m \in M$. Typically, both algebras are trivial. However, when they are nontrivial the structure of these algebras is very interesting. This leads us to the following problem.

Problem 2.2.1.8. Given $m \in M$, what is the algebraic structure of $\mathrm{Bis}_L(m)$ and $\mathrm{Bis}_R(m)$?

Wilson's result shows that in the case of the classical bispectral problem, when the left or right bispectral algebras are rank 1, then they must be commutative with the spectra of rational plane curves. One can actually readily show in the classical case that they have to be commutative with rational spectra. However, in general do they have to be plane curves? The situation becomes even more interesting in the noncommutative case. We will show later that rank 1 algebras of bispectral matrix differential operators must be endomorphism rings of torsion-free modules over rational plane curves.

2.2.2 The Eigenvalue Homomorphism

Consider a bispectral setup (A, B, M) and $m \in M$ with trivial left and right annihilator. Then for all $a \in \mathrm{Bis}_L(m)$, there exists $g \in \mathfrak{M}(B)$ such that $am = mg$. Moreover, since m has trivial right annihilator, the value of g is unique. Thus we have a function

$$\Lambda_L : \mathrm{Bis}_L(m) \to \mathfrak{M}(B), \quad am = m\Lambda_L(a).$$

One may verify that Λ_L is an injective anti-homomorphism. In a similar fashion, we may obtain an injective anti-homomorphism

$$\Lambda_R : \mathrm{Bis}_R(m) \to \mathfrak{M}(B), \quad mb = \Lambda_L(b)m.$$

Definition 2.2.2.1. We call the maps Λ_L and Λ_R defined above the left and right **eigenvalue homomorphisms** of m.

2.2.3 Darboux Transformations

One of the principal ways of generating new bispectral triples from old ones is by means of Darboux transformations.

Definition 2.2.3.1. Let A be an operator algebra, and let $a \in A$. A **Darboux transformation** of a is an element $\widetilde{a} \in A$ satisfying the property that $a = a_1a_2$ and $\widetilde{a} = a_2a_1$ for some $a_1, a_2 \in A$.

Note that assuming a_1 is invertible in some ring extension B of A (eg. the ring of pseudo-differential operators), this just says $\widetilde{a} = a_1^{-1}aa_1$, ie that a and b are conjugates. However, being a Darboux transformation is a stronger condition than being a conjugate, because it implies that the intermediate quantity $a_1^{-1}a$ is actually in A.

Example 2.2.3.2. The element

$$\partial^2 - \frac{2}{x^2}$$

is a bispectral transformation of ∂^2 because

$$\partial^2 = \left(\partial + \frac{1}{x}\right)\left(\partial - \frac{1}{x}\right)$$

$$\partial^2 - \frac{2}{x^2} = \left(\partial - \frac{1}{x}\right)\left(\partial + \frac{1}{x}\right)$$

Example 2.2.3.3. The element

$$\partial^2 - \frac{6}{x^2}$$

is a conjugate of ∂^2 because $\partial^2 - \frac{6}{x^2} = \mathfrak{b}\partial^2\mathfrak{b}^{-1}$ for

$$\mathfrak{b} = \left(\partial - \frac{1}{x}\right)\left(\partial - \frac{2}{x}\right),$$

However it is not a bispectral transformation of ∂^2, because $\mathfrak{b}$ is too large to factor ∂^2.

Definition 2.2.3.4. Let (A, B, M) be a bispectral setup with bispectral triple (a, b, m). A **bispectral Darboux transformation** of (a, b, m) is a triple $(\widetilde{a}, \widetilde{b}, \widetilde{m})$ with $\widetilde{a} \in A, \widetilde{b} \in B$, and $\widetilde{m} \in M$ satisfying

$$a = a_1a_2, \quad \widetilde{a} = a_2a_1, \quad b = b_1b_2, \quad \widetilde{b} = b_2b_1,$$

and

$$\widetilde{m} = a_2 \cdot m \cdot t = s \cdot m \cdot b_1,$$

for some $a_1, a_2 \in A$, $s \in \mathfrak{M}(A)$, $b_1, b_2, \in B$, and $t \in \mathfrak{M}(B)$ with s, t units.

Proposition 2.2.3.5. *Let (A, B, M) be a bispectral setup. If $(\widetilde{a}, \widetilde{b}, \widetilde{m})$ is a bispectral Darboux transformation of a bispectral triple (a, b, m), then $(\widetilde{a}, \widetilde{b}, \widetilde{m})$ is also a bispectral triple.*

Proof. Let a_1, a_2, b_1, b_2, s, and t be as in the previous definition. Since (a, b, m) is a bispectral triple, there exists $f \in \mathfrak{M}(A)$ and $g \in \mathfrak{M}(B)$ with $a \cdot m = m \cdot g$ and $m \cdot b = f \cdot m$. Therefore

$$\widetilde{a} \cdot \widetilde{m} = a_2 a_1 \cdot (a_2 \cdot m \cdot t) = a_2 \cdot (a \cdot m) \cdot t = (a_2 \cdot m) \cdot gt = \widetilde{m} \cdot t^{-1} gt.$$

Similarly,

$$\widetilde{m} \cdot \widetilde{b} = (s \cdot m \cdot b_1) \cdot b_2 b_1 = s \cdot (m \cdot b) \cdot b_1 = sf \cdot (m \cdot b_1) = sfs^{-1} \cdot \widetilde{m}.$$

This shows that $(\widetilde{a}, \widetilde{b}, \widetilde{m})$ is a bispectral triple. ☐

Example 2.2.3.6. The bispectral triple

$$\left(\partial_x^2 - \frac{2}{x^2}, \partial_y^2 - \frac{2}{y^2}, e^{xy}\left(1 - \frac{1}{xy}\right)\right)$$

is a bispectral Darboux transformation of $(\partial_x, \partial_y, e^{xy})$.

Chapter 3

ALGEBRAIC THEORY OF COMMUTING DIFFERENTIAL OPERATORS

3.1 Commuting Differential Operators and Schur's Theorem

3.1.1 Commuting Differential Operators

Lemma 3.1.1.1. *Let R be a differentially closed ring, and $\mathfrak{d} \in \mathfrak{P}(R)$ be monic. Then there exists a unit $r \in R$ such that $r^{-1}\mathfrak{d}r$ is normalized and monic.*

Proof. Since conjugation by a unit r of $\mathfrak{D}(R)$ preserves the order of operators, conjugating the positive-order part $\mathfrak{d}_+$ of $\mathfrak{d}$ to a normalized differential operator will also conjugate $\mathfrak{d}$ to a normalized pseudo-differential operator. Therefore without loss of generality, we may assume $\mathfrak{d}$ is a differential operator. In fact, by this argument it suffices to consider the case that $\mathfrak{d} = \partial^\ell + a\partial^{\ell-1}$ for some $a \in R$.

Consider the differential equation $r' - (a/\ell)r = 0$. This equation has a solution $r \in R$ which is a unit by the assumption that R is differentially closed. Using Leibniz rule, we calculate

$$r^{-1}\mathfrak{d}r = \partial^\ell I + \sum_{j=0}^{\ell-1} r^{-1}\left[\binom{\ell}{j} r^{(n-j)} + \binom{\ell-1}{j} a r^{(\ell-1-j)}\right]\partial^j.$$

The $\partial^{\ell-1}$'th coefficient in the above expression is $r^{-1}[\ell r' - ar] = 0I$, and thus $r^{-1}\mathfrak{d}r$ is monic and normalized. This proves our lemma. □

Lemma 3.1.1.2. *Let $\mathfrak{v} \in \mathfrak{P}(R)$ be almost monic. Then $\mathfrak{v}$ is a unit in $\mathfrak{P}(R)$.*

Proof. We first prove that if $\mathfrak{v}$ is monic, then $\mathfrak{v}$ is a unit. Since ∂^m is invertible for all m, it suffices to consider the case that $\mathfrak{v}$ is monic of order 0, eg. $\mathfrak{v} = 1 + \sum_{m=1}^{\infty} v_m \partial^{-m}$ for some

$v_1, v_2, \ldots, \in R$. Let $\mathfrak{w} = 1 + \sum_{n=1}^{\infty} w_n \partial^{-n}$ and consider the product:

$$\begin{aligned}\mathfrak{w}\mathfrak{v} &= 1 + \sum_{k=1}^{\infty}(w_k + v_k)\partial^{-k} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\sum_{j=0}^{m}\binom{m}{j} w_n v_m^{(j)} \partial^{-m-n-j} \\ &= 1 + [w_1 + v_1]\partial^{-1} \\ &+ \sum_{k=2}^{\infty}\left[w_k + v_k + \sum_{n=1}^{k-1}\sum_{m=1}^{k-n}\binom{m}{k-n-m} w_n v_m^{(j)}\right]\partial^{-k}.\end{aligned}$$

1f we define $w_1 = -v_1$ and w_k recursively for $k > 1$ by

$$w_k = -v_k - \sum_{n=1}^{k-1}\sum_{m=1}^{k-n}\binom{m}{k-n-m} w_n v_m^{(j)}.$$

Then $\mathfrak{w}\mathfrak{v} = 1$. This shows that every zeroth-order monic pseudo-differential operator has a left inverse which is also a zeroth-order monic pseudo-differential operator.

Let $\mathfrak{w}$ be a left inverse of $\mathfrak{v}$ and $\mathfrak{x}$ be a left inverse of $\mathfrak{w}$, so that $\mathfrak{w}\mathfrak{v} = 1$ and $\mathfrak{x}\mathfrak{w} = 1$. Then

$$w(\mathfrak{v} - \mathfrak{x})w = w\mathfrak{v}w - w\mathfrak{x}w = 1w - w1 = 0.$$

Since w is not a left or right zero divisor in $M_N(\mathfrak{P}(U))$, this shows that $\mathfrak{v} = \mathfrak{x}$. Hence $\mathfrak{v}$ is a left inverse of $\mathfrak{w}$, eg. $\mathfrak{v}\mathfrak{w} = I$. Thus $\mathfrak{w}$ is an inverse of $\mathfrak{v}$, and this shows that $\mathfrak{v}$ is a unit.

More generally, suppose that $\mathfrak{v}$ is almost monic. Then there exists $\mathfrak{b}_l$ and $\mathfrak{b}_r$ with $\mathfrak{b}_l\mathfrak{v}$ monic and $\mathfrak{v}\mathfrak{b}_r$ monic. By the previous argument, each of these is a unit, and therefore $\mathfrak{v}$ has a left inverse and a right inverse and is therefore a unit. □

Lemma 3.1.1.3. *Let $\mathfrak{p} \in \mathfrak{P}(R)$ be a normalized pseudo-differential operator of order* 1. *Then there exists a wave operator $\mathfrak{v}$ satisfying $\mathfrak{v}^{-1}\mathfrak{p}\mathfrak{v} = \partial$.*

Proof. Let $\mathfrak{d} = \partial + \sum_{n=1}^{\infty} a_n \partial^{-n}$ and consider an arbitrary zeroth-order, monic pseudo-differential operator $\mathfrak{v} = 1 + \sum_{m=1}^{\infty} v_m \partial^{-m}$ with $v_0 = 1$. We calculate

$$\begin{aligned}\mathfrak{v}\mathfrak{d} - \partial\mathfrak{v} &= \sum_{m=1}^{\infty} v_m' \partial^{-m} + \sum_{n=1}^{\infty} a_n \partial^{-n} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\binom{m}{j} v_m a_n^{(j)} \partial^{-m-n-j} \\ &= [v_1' + a_1]\partial^{-1} \\ &+ \sum_{k=2}^{\infty}\left[v_k' + a_k + \sum_{m=1}^{k-1}\sum_{n=1}^{k-m}\binom{m}{k-m-n} v_m a_n^{(k-m-n)}\right]\partial^{-k}\end{aligned}$$

Thus if we define $v_1, v_2, \dots \in R$ recursively with $v_1' = -a_1$ and for $k \geq 2$

$$v_k' = -a_k - \sum_{m=1}^{k-1} \sum_{n=1}^{k-m} \binom{m}{k-m-n} v_m a_n^{(k-m-n)}$$

then $\mathfrak{v}$ satisfies $\mathfrak{v}\mathfrak{d} = \partial\mathfrak{v}$. □

Theorem 3.1.1.4. *Let R be a differential algebra of characteristic p, and let $\mathfrak{d} \in \mathfrak{P}(R)$ be a monic, normalized pseudo-differential operator of positive order ℓ with p and ℓ relatively prime. Then there exists a monic, normalized pseudo-differential operator $\mathfrak{q}$ of order 1 satisfying $\mathfrak{q}^n = \mathfrak{d}$.*

Proof. Since $\mathfrak{d}$ is monic and normalized of order ℓ, there exists $\mathfrak{d}_{\ell-2}, d_{\ell-3}, \dots$ such that

$$\mathfrak{d} = \partial^\ell + \sum_{n=-\infty}^{\ell-2} d_n \partial^n.$$

Consider an arbitrary monic, normalized pseudo-differential operator of order 1

$$\mathfrak{q} = \partial + \sum_{n=1}^{\infty} a_n \partial^{-n}.$$

Also for each integer m, consider the truncation map $T_m : \mathfrak{P}(R) \to \mathfrak{P}(R)$ defined by

$$T_m : \sum_{n=-\infty}^{j} c_j \partial^j \mapsto \sum_{n=-m}^{j} c_j \partial^j.$$

Then since $\mathfrak{q}$ is order 1 order considerations imply

$$T_m(\mathfrak{q}^\ell) = T_m(T_{m+\ell-1}(\mathfrak{q})^\ell)$$

for all $m \geq 0$. Then since

$$T_{m+\ell}(\mathfrak{q}) = T_{m+\ell-1}(\mathfrak{q}) + a_{m+\ell}\partial^{-m-\ell},$$

we calculate

$$\begin{aligned} T_{m+1}(\mathfrak{q}^\ell) &= T_{m+1}(T_{m+\ell}(\mathfrak{q})^\ell) \\ &= T_{m+1}((T_{m+\ell-1}(\mathfrak{q}) + a_{m+\ell}\partial^{-m-\ell})^\ell) \\ &= T_{m+1}(T_{m+\ell-1}(\mathfrak{q})^\ell) + \ell a_{m+\ell}\partial^{-m-1} \\ &= T_m(\mathfrak{q}^\ell) + (b_{m+\ell} + \ell a_{m+\ell})\partial^{-m-1} \end{aligned}$$

for

$$b_m \partial^{-m-1} = T_{m+1}(T_{m+\ell-1}(\mathfrak{q})^\ell) - T_m(T_{m+\ell-1}(\mathfrak{q})^\ell).$$

Since ℓ and p are relatively prime the element ℓ is invertible in R. Also since the $b_{m+\ell}$ is defined in terms of $T_{m+\ell-1}(\mathfrak{q})$, it depends only on the value of a_j for $j \leq m+\ell-1$. Therefore we may define a_m recusively by $a_1 = d_1/\ell$ and more generally for $m \geq 1$

$$a_m = (d_{\ell-m-1} - b_m)/\ell.$$

Using this, it follows that $T_{m+1}(\mathfrak{q}^\ell) - T_m(\mathfrak{q}^\ell) = d_{-m-1}\partial^{-m-1}$ for all m. Hence $\mathfrak{q}^\ell = \mathfrak{d}$. □

Corollary 3.1.1.4.1. *Let R be a differential algebra of characteristic p, and let $\mathfrak{d} \in \mathfrak{P}(R)$ be a normalized pseudo-differential operator of order ℓ relatively prime to p. Then there exists a wave operator $\mathfrak{v}$ satisfying $\mathfrak{v}^{-1}\mathfrak{d}\mathfrak{v} = \partial^\ell$.*

Proof. By the previous theorem, we may choose $\mathfrak{q}$ of order 1 such that $\mathfrak{q}^\ell = \mathfrak{d}$. Furthermore, a previous proposition tells us that there exists a wave operator $\mathfrak{v}$ satisfying $\mathfrak{v}^{-1}\mathfrak{q}\mathfrak{v} = \partial$. Consequently $\mathfrak{v}^{-1}\mathfrak{d}\mathfrak{v} = \partial^\ell$. □

Lemma 3.1.1.5. *Let R be a differential algebra of characteristic p, and let K be the ring of constants of R. Then for any $\ell > 0$ relatively prime to p, the centralizer of ∂^ℓ in $\mathfrak{P}(R)$ is $K((\partial^{-1}))$, the ring of formal Laurent series in ∂^{-1} with coefficients in K.*

Proof. Suppose that $\mathfrak{d} \in$ commutes with ∂^ℓ and assume that $\mathfrak{d} \notin K((\partial^{-1}))$. Then writing

$$\mathfrak{d} = \sum_{n=-\infty}^{m} d_n \partial^n,$$

without loss of generality we may assume that $d_m \notin K$. Then we calculate

$$\partial^n \mathfrak{d} - \mathfrak{d}\partial^n = \sum_{n=-\infty}^{m} \sum_{k=1} \binom{\ell}{k} d_n^{(k)} \partial^{n+\ell-k}.$$

The leading coefficient of the above expression is $\ell d'_m$, which is nonzero. This is a contradiction, and it follows that the centralizer of ∂^m in $\mathfrak{P}(R)$ is $K((\partial^{-1}))$. □

Theorem 3.1.1.6. *Let R be a differential algebra of characteristic p, and let $\mathfrak{d} \in \mathfrak{D}(R)$ be a monic, normalized differential operator of order ℓ relatively prime to p. If the ring of constants K of R is a commutative ring, then the centralizer $C(\mathfrak{d})$ of $\mathfrak{d}$ in $\mathfrak{D}(R)$ is commutative.*

Proof. Choose a wave operator $\mathfrak{v} \in \mathfrak{P}(R)$ satisfying $\mathfrak{v}^{-1}\mathfrak{d}\mathfrak{v} = \partial^\ell$. Then by the previous theorem the centralizer of ∂^ℓ is $K((\partial^{-1}))$. Then since $\mathfrak{v}$ is a unit, conjugation by $\mathfrak{v}$ defines an embedding of the centralizer of $\mathfrak{d}$ in $\mathfrak{D}(R)$ into the ring $K((\partial^{-1}))$. In particular, it is commutative. □

3.2 Sato Grassmannian and Krichever Correspondence

The Sato grassmannian and Krichever's correspondence were originally developed in the context of differential operators whose coefficients are scalar-valued analytic functions. For the results of this thesis, we will have to extend these definitions to a broader collection of differential operator algebras. However, for the machinery we develop to work correctly we must restrict ourselves somewhat by considering the case of a differential k-algebra R whose ring of constants K is a simple Artinian k-algebra for a fixed field k. In this case K has a unique (up to isomorphism) simple right K-module V.

3.2.1 The Sato grassmannian

Consider the right K-module $\mathbb{V}$ defined by

$$\mathbb{V} = V((z^{-1})) = \left\{ \sum_{n=-\infty}^{\ell} v_j z^j : \ell \geq 0 \text{ and } v_j \in V \text{ for all } j \right\}.$$

This decomposes as a direct sum of right K-modules

$$\mathbb{V}_+ = V[z] = \left\{ \sum_{n=0}^{\ell} v_j z^j : \ell \geq 0,\ v_j \in V \right\} \text{ and } \mathbb{V}_- = V[[z^{-1}]] = \left\{ \sum_{n=-\infty}^{-1} v_j z^j \in V : v_j \in V \right\},$$

and we have a distinguished projection map

$$\pi_+ : \mathbb{V} \to \mathbb{V}/\mathbb{V}_- = \mathbb{V}_+.$$

Definition 3.2.1.1. We define the **Sato grassmannian** $\mathrm{Gr}(R)$ of R to be the collection of k-linear subspaces of $\mathbb{V}$ for which the restriction of the projection π_+ has finite kernel and cokernel, ie.

$$\mathrm{Gr}(R) = \{W \subseteq \mathbb{V} : \dim\ker(\pi_+|_W) < \infty \text{ and } \dim\operatorname{coker}(\pi_+|_W) < \infty\}.$$

The **index** of $W \in \mathrm{Gr}$ is defined to be $\dim\ker(\pi_+|_W) - \dim\operatorname{coker}(\pi_+|_W)$, and the Sato grassmannian decomposes as the union $\mathrm{Gr}(R) = \bigcup_{\mu\in\mathbb{Z}} \mathrm{Gr}(R;\mu)$ with $\mathrm{Gr}(R;\mu)$ being the collection of elements of index μ, which we call the **index-μ Sato grassmannian**. Of particular importance is the index-0 grassmannian $\mathrm{Gr}(R;0)$, and specifically the **big cell of** $\mathrm{Gr}(R;0)$, denoted $\mathrm{Gr}(R;0)_+$, which is the set of all $W \in \mathrm{Gr}(R;0)$ whose kernels and cokernels both have dimension 0.

Let $L = k((z^{-1}))$. To each $W \in \mathrm{Gr}(R)$, we associate an algebra A_W consisting of right L-module homomorphisms of $\mathbb{V}$ preserving W, ie.

$$A_W = \{\varphi \in \mathrm{End}_L(\mathbb{V}) : \varphi(W) \subseteq W\}.$$

Note that we have a natural identification of $\mathrm{End}_L(\mathbb{V})$ with $K((z^{-1}))$, the ring of Laurent series with coefficients in K, wherein an endomorphism $\varphi : \mathbb{V} \to \mathbb{V}$ corresponds to an element $a \in K((z^{-1}))$, acting on elements of $\mathbb{V}$ on the right in the natural way. With this in mind, we can and will identify A_W with the algebra $\{a \in K((z^{-1})) : Wa \subseteq W\}$.

If the center of A_W is large enough, we can use algebraic geometry to study the structure of A_W and W. One sufficiently robust notion of "largeness" of the center is the following. For any $W \in \mathrm{Gr}(R)$, we denote the center of A_W by Z_W.

Definition 3.2.1.2. Let $W \in \mathrm{Gr}(R)$. The pair (W, A) with $k \subseteq A \subseteq A_W$ is called a **Schur pair**. We call a Schur pair (W, A) **robust** if $A \cap k((z^{-1}))$ is nontrivial, ie. contains more than just k. We denote the intersection $A_W \cap k((z^{-1}))$ as S_W (the S standing for "scalar"). If (W, A) is robust, the **rank** of (W, A) is the greatest common divisor of the orders of the nonzero elements of $S_W \cap A$.

Proposition 3.2.1.3. *Let $W \in \mathrm{Gr}(R)$ and suppose that (W, A_W) is robust. Then W and A_W are finitely generated over S_W, and hence over Z_W. Furthermore, both are torsion-free Z_W-modules and W is a torsion-free right A_W-module.*

Proof. We will first show that W and A_W are finitely generated over S_W. Choose $a \in A_W \cap k((z^{-1}))\backslash k$. Since A_W is a subalgebra of $K((z^{-1}))$ and $k((z^{-1}))$ is the center of $K((z^{-1}))$, we know that $a \in Z_W$. Therefore it suffices to show that W and A_W are finitely generated $k[a]$-modules.

Note that since $K((z^{-1}))$ is free as a $k[a]$-module, so too is W. Therefore we may choose a $k[a]$-basis $\{w_j\}_{j\in J}$ for W. Suppose that $|J| > \dim(K)\ell$, where ℓ is the order of a, and let d be the maximum of the orders of the first $\dim(V)\ell + 1$ generators. Also let W_n denote the subspace of W consisting of elements of order at most n. Then for all $n \geq 0$ $\pi_+(W_{n\ell+d})$ maps to polynomials of degree $\leq n\ell + d$ with coefficients in V, so that we have

$$\dim(W_{n\ell+d}) - \dim(\ker(\pi_+|_W)) \leq \dim(\pi_+(W_{n\ell+d})) \leq \dim(V)(n\ell + d + 1),$$

and also

$$\dim(W_{n\ell+d}) \geq \dim \oplus_{i=1}^{\dim(V)\ell+1}(k[a]w_i)_{\leq n\ell+d} \geq (\dim(V)\ell + 1)n.$$

Thus for all $n \geq 0$, we have

$$\dim(V)n\ell + \dim(V)(d+1) + \dim\ker(\pi_+|_W) \geq \dim(V)\ell n + n,$$

and since $\dim\ker(\pi_+|_W) < \infty$ this is a contradiction. Therefore $|J|$ is finite, and W is finitely generated over $k[a]$. Then since each $a \in A_W$ defines a Z_A-module endomorphism of W, we have that

$$A_W \subseteq \mathrm{End}_{Z_A}(W) \subseteq \mathrm{End}_{k[a]}(W).$$

Since W is a free $k[a]$-module of finite rank, so too is $\mathrm{End}_{k[a]}(W)$. Therefore A_W is a submodule of a finitely generated module over a PID, and is therefore finitely generated. It follows that A_W is finitely generated over $k[a]$ and hence over S_W.

Next we must show that both W and A_W are torsion-free over Z_W, ie. that if $b \in Z_W$ annihilates some nonzero element of W or A_W, then b must be a zero divisor in Z_W. To do so, suppose that $wb = 0$ for some nonzero $w \in W$. Since A_W is algebraic over $k[a]$, there exists a monic $p(x) \in k[a][x]$ with $p(b) = 0$. Without loss of generality, we may assume that $p(x)$ is minimal among such choices (ie. of smallest degree). Then if $p(x) = x^n + \sum_{j=0}^{n-1} p_j(a)x^j$, we have

$$0 = w0 = wp(b) = wb^n + \sum_{j=0}^{n-1} wb^j p_j(a) = wp_0(a).$$

Since $k[a]$ does not annihilate any element of W, this implies $p_0(a) = 0$, and therefore $0 = q(b)b$ for $q(x) = x^{n-1} + \sum_{j=0}^{n-2} p_{j+1}(a)x^j$. Note that $q(b) \in Z_W$, but by minimality $q(b) \neq 0$, and therefore b is a zero divisor of Z_W. This shows that W is torsion-free over Z_W. The proof that A_W is torsion-free over Z_W and that W is a torsion-free right A_W-module work similarly. □

3.2.2 Krichever correspondence

For robust pairs (W, A_W), we can explore the algebraic structure of W, A_W and Z_W by viewing them as S_W-algebras and S_W-modules. In the case that (W, A_W) is rank 1, we have the surprising result that S_W and Z_W are actually equal, and that A_W is generically a matrix algebra. Using this, we can establish the so-called Krichever correspondence between rank 1 Schur pairs (A, W) and quintuples of algebro-geometric data $(X, \mathcal{W}, \infty, t, \varphi)$, where here: X is a projective curve; $\mathcal{W}$ is a torsion-free sheaf on X of rank $N = \dim_k(V)$; ∞ is a smooth point of X; t is a uniformizer of X at ∞; and φ is a local trivialization of $\mathcal{W}$ (ie $\mathcal{W}_\infty \cong V\mathcal{O}_{X,\infty}$) in a neighborhood of ∞.

To prove this, we first establish a key lemma.

Lemma 3.2.2.1. *Let $W \in \mathrm{Gr}(R)$, and suppose that (W, A_W) is robust of rank 1. Let F_W be the fraction field of S_W and $L = k((z^{-1}))$. Then the natural map*

$$WF_W \otimes_{F_W} L \to WF_W L = WL = \mathbb{V}$$

is an isomorphism.

Proof. Let $d = \dim_k(V)$. The result that $WF_WL = WL = \mathbb{V}$ follows from $F_W \subseteq L$ and the finite dimensionality of the cokernel of $\pi_+|_W$. Furthermore, the natural map $WF_W \otimes_{F_W} L \to WF_WL$ is surjective. Therefore to prove it is an isomorphism, we need only show that the dimension of $WF_W \otimes_{F_W} L$ over L is no greater than the dimension of $\mathbb{V}$ over L, the latter being d.

Since the cokernel of $\pi_+|_W$ is finite dimensional, for each $v \in V$ there exists $w \in W$ such that the leading coefficient of w is v. Then since S_W contains two elements of relatively prime order, F_W contains an element of order n for all $n \in \mathbb{Z}$. Hence WF_W contains an element of order n with leading coefficient v for all $n \in \mathbb{Z}$ and all $v \in V$. Let $\{v_1, \ldots, v_d\}$ be a basis for V over k, and for each i choose an element $w_i \in WF_W$ of order 0 with leading coefficient v_i. We claim that $\{w_1 \otimes 1, \ldots, w_d \otimes 1\}$ spans $WF_W \otimes_{F_W} L$ over L.

To see this, suppose $w \in WF_W$ has order 0, and let $u \in F_W$ be an element of order 1. Then recursively we may choose $a_{ij} \in k$ such that $\sum_{j=1}^{d} a_{ij}w_j$ is the leading coefficient of $w - \sum_{m=0}^{i-1}\sum_{j=1}^{d} a_{mj}w_ju^{-m}$. Then for $\alpha_j = \sum_{m=0}^{\infty} a_{mj}u^{-m} \in L$, the element $w \otimes 1 - \sum_{j=0}^{d}(w_j \otimes \alpha_j)$ has order less than i for all $i < 0$, and is therefore zero. We conclude that $w \otimes 1$ is in the span of the $w_j \otimes 1$. It follows that $w \otimes 1$ is in the span of the $w_j \otimes 1$ for all $w \in W$, and thus all simple tensors are included in the span of the $w_j \otimes 1$. Hence the $w_j \otimes 1$ span the tensor product, so the dimension of $WF_W \otimes_{F_W} L$ over L is at most d. This proves our lemma. □

Proposition 3.2.2.2. *Let $W \in \mathrm{Gr}(R)$, and suppose that (W, A_W) is robust of rank 1. Then $Z_W = S_W$ and*

$$A_W \cong \mathrm{End}_{Z_W}(W).$$

In particular, the center of A is an integral domain of Krull dimension 1, and A is isomorphic to the endomorphism ring of a torsion-free module over its center.

Proof. Note that each element of A_W defines a Z_W-endomorphism of W by right multiplication, and therefore we have natural maps

$$A_W \to \mathrm{End}_{Z_W}(W) \to \mathrm{End}_{S_W}(W).$$

We claim that this composition is an isomorphism. To prove this, we will define a morphism $\mathrm{End}_{S_W}(W) \to \mathrm{End}_L(\mathbb{V})$ which sends each element of $\mathrm{End}_{S_W}(W)$ to an element of $\mathrm{End}_L(\mathbb{V})$ preserving W, and thus equal to an element of A_W. The induced map $\mathrm{End}_{S_W}(W) \to A_W$ will be the inverse of $A_W \to \mathrm{End}_{S_W}(W)$.

Consider the surjective vector space homomorphisms $W \otimes_{S_W} F_W \to WF_W$ and $WF_W \otimes_{F_W} L \to WF_W L = WL = \mathbb{V}$. The first is simply localization, so it is an isomorphism. Furthermore, by the previous lemma the second map is also an isomorphism. Hence we have a natural map

$$\mathrm{End}_{S_W}(W) \to \mathrm{End}_{F_W}(W \otimes_{S_W} F_W) = \mathrm{End}_{F_W}(WF_W) \to \mathrm{End}_L(WF_W \otimes_{F_W} L) = \mathrm{End}_L(\mathbb{V}),$$

which sends an S_W-endomorphism $\varphi : W \to W$ to a point $a \in K((z^{-1}))$ such that $\varphi(w) = wa$ for all $a \in W$. In particular, $W \subseteq W$ and therefore $a \in A$. One may verify that this map $\mathrm{End}_{S_W}(W) \to A$, as well as the map $A \to \mathrm{End}_{S_W}(W)$ are inverses, and therefore $A \cong \mathrm{End}_{S_W}(W)$.

In particular, this implies that the inclusion $A \subseteq K((z^{-1}))$ factors through the map $\mathrm{End}_{F_W}(WF_W) \to K((z^{-1}))$, and this latter map sends the center of $\mathrm{End}_{F_W}(WF_W)$ to $F_W \in K((z^{-1}))$. Therefore the center of A must be mapped to F_W, under inclusion. In other words, $Z_W \subseteq F_W$, and it follows that $Z_W = S_W$. □

Given a robust Schur pair (W, A_W), we can use the multiplicative filtration by order to obtain a sheaf of algebras $\mathcal{A}$ along with a $\mathcal{A}$-module $\mathcal{W}$ over a projective curve X_W. Specifically, for every integer $d \in \mathbb{Z}$ let $(S_W)_d$, $(A_W)_d$, and W_d denote the k-linear subspace of elements with order at most d. Note that $(S_W)_d = 0$ for $d < 0$. We consider the Rees ring,

$$\mathrm{Rees}(S_W) = \sum_{d \geq 0} (S_W)_d t^d$$

along with

$$\mathrm{Rees}(A_W) = \sum_{d \in \mathbb{Z}} (A_W)_d t^d$$

and also

$$\mathrm{Rees}(W) = \sum_{d\in\mathbb{Z}} W_d t^d$$

Since S_W is a Dedekind domain, $\mathrm{Rees}(S_W)$ is a graded ring with Krull dimension 2. Thus defining $X_W = \mathrm{Proj}(\mathrm{Rees}(S_W))$ gives us a projective curve. Specifically X_W consists of $\mathrm{Spec}(S_W)$ plus an additional smooth "point at infinity" ∞, corresponding to the valuation of the fraction field F_W of X_W induced by the order valuation. The ring $\mathrm{Rees}(A_W)$ is a graded $\mathrm{Rees}(S_W)$ algebra, with graded $\mathrm{Rees}(A_W)$-module $\mathrm{Rees}(W)$, and therefore A_W and S_W define a coherent sheaf of algebras $\mathcal{A}$ and a $\mathcal{A}$-module $\mathcal{W}$ on X_W which is coherent and torsion-free over X_W.

Lemma 3.2.2.3. *Let X be a projective curve and $\mathcal{W}$ a coherent, torsion-free sheaf on X of rank N, and set $\mathcal{A} = \mathcal{H}om(\mathcal{W}, \mathcal{W})$. Then the following are equivalent.*

(a) if $f : \widetilde{X} \to X$ is a morphism of schemes such that $\mathcal{W} \cong f_*\widetilde{\mathcal{W}}$ for some torsion-free sheaf $\widetilde{\mathcal{W}}$ on $\widetilde{X}$ of rank N, then the natural map $\mathcal{O}_X \to f_*\mathcal{O}_{\widetilde{X}}$ is an isomorphism

(b) the natural map $\mathcal{O}_X \to \mathcal{A}$ defines an isomorphism between $\mathcal{O}_X$ and the center of $\mathcal{A}$

Proof. Suppose that (a) is true. Let $\mathcal{Z}$ be the sheaf of subalgebras of $\mathcal{A}$ defined by the center of $\mathcal{A}$. Let $\{U_i\}$ be an affine open cover of X, with $U_i = \mathrm{Spec}(S_i)$, and set $W_i = \mathcal{W}(U_i)$. Over each U_i, we have $A_i := \mathcal{A}(U_i) = \mathrm{End}_{S_i}(W_i)$ and $Z_i := \mathcal{Z}(U_i) = Z(\mathrm{End}_{S_i}(W_i))$. The affine schemes $\mathrm{Spec}(Z_i)$ glue together to form a scheme $\widetilde{X}$ with $\mathcal{O}_{\widetilde{X}} = \mathcal{Z}$. The natural map $\mathcal{O}_X \to \mathcal{A}$ factors through $\mathcal{Z} \subseteq \mathcal{A}$, and induces a morphism of schemes $f : \widetilde{X} \to X$. Each W_i is a Z_i-module, and the W_i's glue together to form an $\mathcal{O}_{\widetilde{X}}$-module $\widetilde{W}$. By definition, $f_*\widetilde{\mathcal{W}}(U_i) = W_i$, and these glue to an isomorphism $W \cong f_*\widetilde{W}$. Thus by (a), $\mathcal{O}_X \to f_*\mathcal{O}_{\widetilde{X}} = \mathcal{Z}$ is an isomorphism. This proves (b).

Conversely, suppose that (b) is true. Let $f : \widetilde{X} \to X$ be a morphism of schemes, and suppose that there is a $\mathcal{O}_X$-module isomorphism $\varphi : W \cong f_*\widetilde{\mathcal{W}}$. Over each U_i, φ defines an isomorphism $W_i \cong \widetilde{W}_i := f_*\widetilde{\mathcal{W}}(U_i)$. Setting $\widetilde{S}_i = f_*\mathcal{O}_{\widetilde{X}}(U_i)$, we have an injection

$$\mathrm{End}_{\widetilde{S}_i}(\widetilde{W}_i) \to \mathrm{End}_{S_i}(\widetilde{W}_i) \cong \mathrm{End}_{S_i}(W_i).$$

Since $\widetilde{W}$ is torsion-free, the natural map $\widetilde{S}_i \to \mathrm{End}_{\widetilde{S}_i}(\widetilde{W}_i)$ defines an injection $S_i \to \mathrm{End}_{S_i}(W_i)$, which sends $\widetilde{S}_i$ into the center of $\mathrm{End}_{S_i}(W_i)$, which by assumption is S_i. This defines an injective S_i-algebra homomorphism $\widetilde{S}_i \to S_i$, and this must be an isomorphism. □

Definition 3.2.2.4. We define a **rank N Krichever quintuple** to be a quintuple of data $(X, \mathcal{W}, \infty, t, \varphi)$ where X is a projective curve, $\mathcal{W}$ is a coherent, rank N torsion-free sheaf on X, ∞ a smooth point of X, t a local uniformizer of X in a neighborhood of t, and φ a local trivialization of W (ie $\mathcal{W}_\infty \cong V\mathcal{O}_{X,\infty}$) in a neighborhood of ∞. A quintuple is called **maximal** if X and W satisfy either of the equivalent conditions in the previous lemma.

If (W, A_W) is rank 1, then W is torsion-free of rank $N = \dim_k(V)$ over S_W, so that $\mathcal{W}$ is torsion-free of rank N on X_W. The element $t \in \mathrm{Rees}(S_W)$ defines a smooth point $\infty \in X_W$ along with a uniformizer in a neighborhood of ∞, and the inclusion $W \subseteq \mathbb{V}$ defines a trivialization φ of $\mathcal{W}$ in a neighborhood of ∞. Thus we have a correspondence

$$\{\text{rank 1 Schur pairs } (W, A_W)\} \longleftrightarrow \{\text{maximal rank } N \text{ Krichever quintuples } (X, \mathcal{W}, \infty, t, \varphi)\}.$$

Proposition 3.2.2.5 (Krichever Correspondence)**.** *The Krichever correspondence defines a bijection between rank* 1 *Schur pairs and maximal rank N Krichever quintuples* $(X, \mathcal{W}, \infty, t, \varphi)$, *where*

$$X = \mathrm{Proj}(\mathrm{Rees}(S_W)), \quad \mathcal{W} = \widetilde{\mathrm{Rees}(W)}, \quad \textit{and} \quad W = \mathcal{W}(X\backslash\{\infty\}),$$

∞ *is the maximal ideal corresponding to* $t \in \mathrm{Rees}(S_W)$, t *is a uniformizer of the local ring*

$$\mathcal{O}_{X,\infty} = \left\{ \frac{a}{b} t^m : a, b \in S_W,\ m \geq \deg(a) - \deg(b) \geq 0 \right\},$$

and φ *is the trivialization of* $\mathcal{W}$ *defined by*

$$\mathcal{W}_\infty = \left\{ \frac{a}{b} t^m : a \in W, b \in S_W,\ m \geq \deg(a) - \deg(b) \geq 0 \right\} = V\mathcal{O}_{X,\infty}.$$

Proof. Suppose that (W, A_W) is a robust, rank 1 Schur pair, and define the Krichever quintuple $(X, \mathcal{W}, \infty, t, \varphi)$ as above. Since (W, A_W) is rank 1, the natural map $\mathcal{O}_X \to \mathcal{A} =$

$\mathcal{H}om(\mathcal{W},\mathcal{W})$ is an isomorphism onto the center $\mathcal{Z}$ of $\mathcal{A}$, and our Krichever quintuple is maximal. Thus the map from robust, rank 1 Schur pairs to maximal Krichever quintuples makes sense. The value of W is recovered as the image of $\mathcal{W}(X\backslash\{\infty\})$ under $VF(\mathcal{O}_{X,\infty}) \to \mathbb{V}$, so this map is injective.

Conversely, given a maximal Krichever quintuple $(X,\mathcal{W},\infty,t,\varphi)$ we define a point $W \in \mathrm{Gr}(R)$ by setting W equal to the image of $\mathcal{W}(X\backslash\{\infty\})$ under

$$\mathcal{W}(X\backslash\{\infty\}) \to F(O_{X,\infty}) \otimes_{\mathcal{O}_{X,\infty}} \mathcal{W}_\infty \xrightarrow{\varphi} VF(\mathcal{O}_{X,\infty}) \to VF(\widehat{\mathcal{O}_{X,\infty}}) = VF((t)) = \mathbb{V}.$$

Then since

$$A_W = \{a \in R((t)) : Wa \subseteq W\} = \{\varphi \in \mathrm{End}_L(\mathbb{V}) : \varphi(W) \subseteq W\},$$

maximality tells us that $Z_W = \mathcal{O}_X(X\backslash\{\infty\})$. Using this, one may check that the quintuple associated to (W, A_W) is exactly $(X,\mathcal{W},\infty,t,\varphi)$, up to isomorphism. Thus the correspondence is bijective. $\square$

The natural next question is how one may deduce various algebro-geometric invariants of the quintuple from the Schur pair (W, A_W). It turns out that many of these invariants have useful descriptions in terms of the Schur pair, such as the Jacobian of X and the cohomology of $\mathcal{W}$.

Proposition 3.2.2.6. *Let (W, A_W) be a robust, rank 1 Schur pair and $(X,\mathcal{W},\infty,t,\varphi)$ its associated Krichever quintuple. Let g be the geometric genus of X. Then*

$$\dim H^i(X,\mathcal{W}) = \begin{cases} \dim\ker(\pi_+|_{z^{-1}W}) & i=0 \\ \dim\mathrm{coker}(\pi_+|_{z^{-1}W}) & i=1 \\ 0 & i>1 \end{cases}$$

In particular if $\mathcal{W}$ has index μ, then the Euler characteristic of $\mathcal{W}$ is $\mu+1$.

Proof. The sheaf $\mathcal{W}$ is coherent, so the vanishing of $H^i(X,W)$ for $i>1$ is immediate and we may calculate its cohomology using Čech cohomology. We cover X by $U \cup V$, where $U =$

$X\backslash\{\infty\}$ and V is some affine neighborhood of ∞. Then we can write $H^0(X,\mathcal{W}) \cong \ker(d_V)$, and $H^1(X,\mathcal{W}) \cong \operatorname{coker}(d_V)$, for

$$d_V : W \oplus \mathcal{W}(V) \to \mathcal{W}(U \cap V),$$

where we are using the fact that $W = \mathcal{W}(U)$. Taking the limit over all open neighborhoods V of ∞, we get $H^0(X,\mathcal{W}) \cong \ker(d)$, and $H^1(X,\mathcal{W}) \cong \operatorname{coker}(d)$, for

$$d : W \oplus \mathcal{W}_\infty \to \mathcal{W}_\infty \otimes_{\mathcal{O}_{X,\infty}} F(\mathcal{O}_{X,\infty}).$$

Tensoring with the completion $\widehat{\mathcal{O}_{X,\infty}} = \mathbb{C}[[t]]$ of $\mathcal{O}_{X,\infty}$, this says $H^0(X,\mathcal{W}) \cong \ker(\widehat{d})$, and $H^1(X,\mathcal{W}) \cong \operatorname{coker}(\widehat{d})$, for

$$\widehat{d} : W \oplus V[[t]] \to V((t)).$$

Thus

$$H^0(X,\mathcal{W}) \cong W \cap V[[t]] \cong W \cap z\mathbb{V}_- \cong z^{-1}W \cap \mathbb{V}_- \cong \ker(\pi_+|_{z^{-1}W}),$$

and

$$H^1(X,\mathcal{W}) \cong V((t))/(V[[t]] + W) \cong \mathbb{V}/(z\mathbb{V}_- + W) \cong \mathbb{V}/(\mathbb{V}_- + z^{-1}W) \cong \operatorname{coker}(\pi_+|_{z^{-1}W}).$$

□

For a line bundle $\mathcal{L}$ on X, two trivializations of $\mathcal{L}$ in a neighborhood of ∞ differ by mulitiplication by some element of $\mathcal{O}_{X,\infty}$. Via Krichever's correspondence, these different trivializations will lead to different values $L_1, L_2 \in \mathrm{Gr}(k[[x]])$, but they will differ only by multiplication by an element of $k[[x]]$.

Definition 3.2.2.7. We say that $W_1, W_2 \in \mathrm{Gr}(R)$ are **homothety equivalent** if there exists an invertible $f(z) \in K((z^{-1}))$ with $W_1 f(z) = W_2$.

Homothety defines an equivalence relation on $\mathrm{Gr}(R)$, by which we may easily define the Picard group of X. Let $\mathrm{Gr}(R)_X$ denote the set of all points in $\mathrm{Gr}(R)$ whose associated scheme under Krichever correspondence is X.

Proposition 3.2.2.8. *Let (W, A_W) be a robust, rank 1 Schur pair and $(X, \mathcal{W}, \infty, t, \varphi)$ its associated Krichever quintuple. Then*

$$\mathrm{Pic}(X) = \mathrm{Gr}(k[[x]])_X / \sim$$

$$\mathrm{Pic}_0(X) = \mathrm{Gr}(k[[x]]; -p_a)_X / \sim$$

where p_a is the arithmentic genus of X and $\sim$ represents homothety equivalence.

Proof. Since $(X, \mathcal{W}, \infty, t, \varphi)$, Krichever correspondence sends a line bundle $\mathcal{L}$ on X to a unique point $L \in \mathrm{Gr}(k[[x]])_X$ modulo $\sim$. By Riemann-Roch:

$$h^1(X, \mathcal{L}) - h^0(X, \mathcal{L}) = \deg(L) - p_a + 1$$

and therefore $L \in \mathrm{Gr}(k[[x]])_X$ corresponds to a point $\mathcal{L} \in \mathrm{Pic}_0(X)$ if and only if the index of $\mathcal{L}$ is $-p_a$. □

The tensor product of two line bundles $\mathcal{L}_1, \mathcal{L}_2$ corresponds to the product of the associated values L_1, L_2. Since degree is multiplicative, Riemann-Roch implies

$$\mu(L_1 L_2) = \mu(L_1) + \mu(L_2) + p_a.$$

We can also describe the theta divisor of the Jacobian as

$$\Theta(X) = \{L \in \mathrm{Gr}(k[[x]]; -p_a)_X : \dim\ker(\pi_+|_{z^{-1}L}) > 0\} / \sim .$$

3.2.3 Genus 1 Examples

We pause here to consider several examples of the Krichever correspondence in action. In our first examples, we will focus on the case when the scheme X of the Krichever quintuple has arithmetic genus $p_a = 1$. Also, to avoid unnecessary complications we will assume that the characteristic of k is not 2 or 3.

Let X_{ns} denote the nonsingular locus of X. For any Weil divisor $D = \sum_{p \in X} d_p p$ supported on X_{ns}, we let $\mathcal{L}(D)$ be the line bundle defined by

$$\Gamma(U, \mathcal{L}(D)) = \left\{ f \in F(X) : \sum_{p \in U \cap X_{ns}} (\nu_p(f) + d_p) \geq 0 \right\}.$$

Fix a smooth point ∞ of X. Then since the arithmetic genus of X is 1, the bundle $\mathcal{L}(3\infty)$ is very ample. The Riemann-Roch theorem tells us $h^0(\mathcal{L}(3\infty)) = 3$. Since $h^0(\mathcal{L}(3\infty) \otimes \mathcal{L}(-p)) = h^0(\mathcal{L}(3\infty - p)) = 2$ for all p, the bundle $\mathcal{L}(3\infty)$ is base point free, and since $h^0(\mathcal{L}3\infty) \otimes \mathcal{L}(-p-q)) = h^0(\mathcal{L}(3\infty - p - q))$ for all p, q we see that $\mathcal{L}(3\infty)$ is very ample. Choose a basis of global sections $s_0, s_2, s_3 \in \Gamma(X, \mathcal{L}(3\infty))$. Then

$$X \to \mathbb{P}^2_k : \ p \mapsto [s_0(p) : s_2(p) : s_3(p)]$$

defines an embedding of X into projective space.

The arithmetic genus of X implies that it has no principal divisor of the form $p - \infty$, since the existence of such a divisor is equivalent to an isomorphism to $\mathbb{P}^1$. Using this, one may argue that the nonconstant elements of $H^0(X, \mathcal{L}(\infty))$ must have principal divisors of the form

$$2a - 2\infty \quad \text{or} \quad a + b + c - 3\infty$$

for some distinct $a, b, c \in X$. Consequently without loss of generality we may choose $s_0(p) = 1$ and $s_2(p), s_3(p)$ such that $(s_2) = 2a - 2\infty$ and $(s_3) = a + b + c - 3\infty$ for some distinct points $a, b, c \in X$. Then $s_j(p)$ has a pole of order j at ∞ for each j, and the function $t = s_2(p)/s_3(p)$ has a zero of order 1 at ∞ and must therefore define a uniformizer of the local ring $\mathcal{O}_{X,\infty}$. Hence we may write $\mathcal{O}_{X,\infty} = \mathbb{C}[t](u)$, for $u = s_2(p)t^2 = s_3(p)t^3$. Let $u(t) = \sum_{n=0}^{\infty} u_n t^n$ be the image of u in $\widehat{\mathcal{O}_{X,\infty}} = k[[t]]$. Since the characteristic is different from 2, we can find $v(t) = \sum_{n=0}^{\infty} v_n t^n \in k[[t]]$ with $v(t)^2 = u(t)$. Then under the change of variables, $t \mapsto tv(t)$, we get $s_2 = t^{-2}$ and $s_3 = t^{-3}v(t)$.

Now given a line bundle $\mathcal{L}$ in $\mathrm{Pic}_0(X)$ on X, along with a trivialization in a neighborhood of ∞, Krichever's correspondence gives us a subspace $L \subseteq k((z^{-1}))$ which is preserved under multiplication by z^2 and $z^3 v(z^{-1})$ for $z = t^{-1}$. Since the arithmetic genus is 1, we also know that the index of L should be 1. Therefore up to homothety, this says $L = k[z^2, z^3 v(z^{-1})]$ or else

$$L = k[z^2]\{z, z^2 + w(L; z^{-1})\}$$

for some $w(L; z^{-1}) = \sum_{n=0}^{\infty} w_n(L)z^{-n}$ whose value depends on L. Since the arithmetic genus is 1, the curve X has the wonderful property that the linear equivalence classes of degree 0 line bundles on X/k are in bijection with the k-rational points on the nonsingular locus X_{ns} of X.

Proposition 3.2.3.1 ([41]). *Let X be a projective curve of arithmetic genus 1, and let X_{ns} be the complement of the singular points of X. Given a smooth point $\infty \in X$, the map $X_{ns} \to \mathrm{Pic}_0(X)$ defined by $p \mapsto \mathcal{L}(p - \infty)$ defines a bijection between X_{ns} and $\mathrm{Pic}_0(X)$.*

This in particular endows X_{ns} with the structure of algebraic group, where $p + \widetilde{p}$ is the point in X_{ns} satisfying

$$\mathcal{L}(p + \widetilde{p} - \infty) = \mathcal{L}(p - \infty) \otimes \mathcal{L}(\widetilde{p} - \infty).$$

Writing $w(p; z^{-1})$ in place of $w(\mathcal{L}(p - \infty), z^{-1})$ and $w_n(p)$ in place of $w_n(\mathcal{L}(p - \infty))$, we can now ask the question of whether the functions $w_n(p)$ are nice relative to X. We claim that they are nice, and that they are in fact rational functions on X.

Proposition 3.2.3.2. *The functions $w_n(p) : X \to k$ defined as above are rational functions on X.*

Proof. The restrictions of $s_2(p)$ and $s_3(p)$ to $X \backslash \infty$ are in $\Gamma(X \backslash \{\infty\}, \mathcal{O})$. Moreover, every non-constant element of $\Gamma(X \backslash \{\infty\}, \mathcal{O})$ must have a pole at ∞ because $h^0(X, \mathcal{O}) = 1$. Consequently, we actually have an equality $\Gamma(X \backslash \{\infty\}, \mathcal{O}) = k[s_2, s_3]$. Therefore for any $q \in X$, we may write

$$H^0(X \backslash \{\infty\}, \mathcal{L}(\infty - q)) = \{f(s_2, s_3) \in k[s_2, s_3] : f(s_2(q), s_3(q)) = 0\}.$$

Furthermore, multiplication by t defines an isomorphism $\phi_q : \mathcal{L}(\infty - q)_\infty \to \mathcal{O}_{X,\infty}$. Using this, Krichever's correspondence sends $(X, \mathcal{L}(\infty - p), \infty, t, \phi_p)$ to

$$\{z^{-1} f(z^2, z^3 v(z^{-1})) \in k[s_2, s_3] : f(s_2(p), s_3(p)) = 0\}.$$

Up to homothety, this is

$$\begin{aligned} L_{-p} &:= k[z^2]\left\{z, \frac{s_3(p)z - z^4 v(z^{-1})}{s_2(p) - z^2}\right\} \\ &= k[z^2]\left\{z, z^2 + \left(\frac{v_2}{v_0} + s_2(p)\right) + \left(\frac{v_3}{v_0} + \frac{v_1}{v_0}s_2(p) - \frac{1}{v_0}s_3(p)\right)z^{-1} + \dots\right\}. \end{aligned}$$

In particular, by reading of the various coefficients in the series representation we see that

$$\begin{aligned} w_0(p) &= \frac{v_2}{v_0} + s_2(-p) \\ w_1(p) &= \frac{v_3}{v_0} + \frac{v_1}{v_0}s_2(-p) - \frac{1}{v_0}s_3(-p) \end{aligned}$$

and more generally every element $w_j(p)$ is a product of global sections of $\mathcal{L}(3\infty)$, composed with the inversion operation on X. In particular they are rational. $\square$

Example 3.2.3.3. Let k be a field, and consider the cuspidal cubic curve

$$X = \mathrm{Proj}(k[u, v, w]/(v^3 - w^2 u)),$$

Then X is rational with rational morphism $f : \mathbb{P}^1_k \to X$ defined by

$$[x_0 : x_1] \mapsto [x_0^3 : x_0 x_1^2 : x_1^3].$$

The scheme X has one singular point $[1 : 0 : 0]$. The group operation on $X_{ns} = X\backslash\{[1 : 0 : 0]\}$ is given by

$$[1 : a^{-2} : a^{-3}] + [1 : b^{-2} : b^{-3}] = [1 : (a + b)^{-2} : (a + b)^{-3}]$$

The point at infinity $\infty = [0 : 0 : 1]$ has uniformizer $t = v/w$. The complement $X\backslash\{\infty\}$ is affine, and isomorphic to $\mathrm{Spec}(k[z^2, z^3])$ for $z = t^{-1} = w/v$. The line bundles on X satisfy

$$\mathcal{L}_c(X\backslash\{\infty\}) = (cz - 1)k \oplus z^2 k[z],$$

for $c \neq 0$, where here $\mathcal{L}_c = \mathcal{L}([1 : c^{-2} : c^{-3}] - \infty)$. Up to homothety, Krichever's correspondence sends $\mathcal{L}_c$ to

$$L_c := k[z^2]\left\{z, z^2 + \sum_{n=0}^{\infty} \frac{1}{c^{n+2}} z^{-n}\right\}.$$

In particular, in this case $w_0([1 : c^{-2} : c^{-3}]) = c^{-2}$ and $w_1([1 : c^{-2} : c^{-3}]) = c^{-3}$, which are rational.

Example 3.2.3.4. Let $g_2, g_3 \in \mathbb{C}$, and consider the elliptic curve

$$E = \text{Proj}(\mathbb{C}[u,v,w]/(v^2w - 4u^3 - g_2uw^2 - g_3w^3)).$$

In this case $\text{Pic}_0(X)$ is in bijection with the points on E, so that in particular line bundles on E correspond to pairs (p, n) for p a $\mathbb{C}$-point of E and $n \in \mathbb{Z}$. Let $\wp(z)$ be the Weierstrass elliptic function associated to E, with periods ω_1, ω_2. Then there is a holomorphic covering map $\mathbb{C} \to E$ induced by

$$z \mapsto [1 : \wp(z), \wp'(z)], \quad z \notin \mathbb{Z}\{\omega_1, \omega_2\}.$$

The functions $w_1(p), w_2(p)$ pull back under this covering map to functions in the span of $1, \wp(z)$ and $\wp'(z)$.

3.3 Differential Operators and the Sato Grassmannian

To relate the Sato Grassmannian to differential operators, we will assume that R has a smooth point and that K is characteristic zero. In this case, each monic, normalized differential operator gives rise to a Schur pair (W, A), and thereby a geometric data $(X, W, \infty, t, \varphi)$ via Krichever's correspondence. In this way, differential operators themselves actually encode algebro-geometric data.

3.3.1 Pseudo-differential Operators and Gr(R)

Here we show how a smooth point I of R induces a map $\mathfrak{P}(R) \to \text{Gr}(R)$.

Definition 3.3.1.1. We define a **smooth point of** R to be a two-sided ideal I of R with $R/I = K$ and $\bigcap_n I^n = 0$ such that the induced right K-module homomorphisms

$$I^{n+1}/I^{n+2} \xrightarrow{\partial} I^n/I^{n+1}$$

are isomorphisms for all $n \geq 0$ (with $I^0 := R$).

Note that each smooth point I induces a natural differential ring homomorphism $R \to K[[x]]$ defined by

$$r \mapsto \sum_{n=0}^{\infty} r_n x^n, \text{ where } r_n \in K \text{ with } n! r_n \equiv \partial^n \cdot r \mod I.$$

Furthermore, if $\partial^n \cdot r = 0 \mod I$ for all I then $r \in I^n$ for all n. Therefore the assumption that $\bigcap_n I^n = 0$ tells us that $R \to K[[x]]$ is injective.

For our arguments below, assume that R has a smooth point I. Two particular examples to keep in mind are $R = \mathbb{C}[x]$ with $K = k = \mathbb{C}$, and $R = M_N(Holo(Y))$ for some complex Riemann surface Y, with $K = M_N(\mathbb{C})$ and $k = \mathbb{C}$.

For any wave operator $\mathfrak{v} \in \mathfrak{P}(R)$, we define an element of the Sato grassmannian as follows. Consider the $R, \mathfrak{P}(R)$-bimodule $\mathfrak{P}(R)/I\mathfrak{P}(R)$. Tensoring with V, on the left we obtain a right $\mathfrak{P}(R)$-module $V \otimes_R (\mathfrak{P}(R)/I\mathfrak{P}(R))$, which in turn has a canonical k-linear isomorphism with $\mathbb{V}$:

$$\mathbb{V} \xrightarrow{\cong} V \otimes_R (\mathfrak{P}(R)/I\mathfrak{P}(R)) : \quad \sum_{n=-\infty}^{\ell} v_n z^n \mapsto \sum_{n=-\infty}^{\ell} v_n \otimes_R \partial^n.$$

In this way, we endow $\mathbb{V}$ with the structure of a right $\mathfrak{P}(R)$-module. Note that $\mathfrak{P}(R)$-module structure of $\mathbb{V}$ depends on the choice of ideal I. As such, we will use $\mathbb{V}_I$ to denote $\mathbb{V}$ with the module structure induced by I. Then if we set $W_{\mathfrak{v},I} := (\mathbb{V}_I)_+ \cdot \mathfrak{v}$, we obtain an element of the big cell of the index-0 grassmannian. We summarize this in the next proposition.

Proposition 3.3.1.2. *Let $\mathfrak{v} \in \mathfrak{P}(R)$ and $W = W_{\mathfrak{v},I}$. Then W is an element of* $\mathrm{Gr}(R;0)_+$.

Proof. We shall write $\mathbb{V}_+$ to mean $(\mathbb{V}_I)_+$. Then if $f(z) \in \mathbb{V}_+$, the degree of $f(z)$ and $f(z) \cdot \mathfrak{v}$ agree. Consequently W has an element of degree n with leading coefficient v for all $n \geq 0$ and $v \in V$, and no elements of negative degree. Thus $W \in \mathrm{Gr}(R;0)_+$. □

Now if $\mathfrak{d} \in \mathfrak{D}(R)$ is a monic, normalized differential operator of order ℓ, we can choose a wave operator $\mathfrak{v}$ such that $\mathfrak{v}^{-1}\mathfrak{d}\mathfrak{v} = \partial^\ell$. Assuming that the Schur pair (W, A_W) (for $W = W_{\mathfrak{v},I}$) is rank 1, the Krichever correspondence gives us a quintuple $(X, \mathcal{W}, \infty, t, \varphi)$. In this way, we see that differential operators actually encode geometric information. We will show later that the endomorphism ring of $\mathcal{W}$ over X encodes the centralizer $C(\mathfrak{d})$ of $\mathfrak{d}$, and that $\mathcal{W}$ represents the vector bundle induced by the common eigenfunctions of the operators in $C(\mathfrak{d})$.

Conversely, given a quintuple of algebro-geometric data $(X, \mathcal{W}, \infty, t, \varphi)$ one might be inclined to try to construct a differential operator $\mathfrak{d}$ to which this information corresponds. By Krichever's correspondence, we have a Schur pair (W, A_W) automatically. However, the hitch is that W may not be of the form $W_{\mathfrak{v},I}$ for some $\mathfrak{v} \in \mathfrak{P}(R)$. Therefore an interesting and important question is which elements of $\mathrm{Gr}(R;0)_+$ are of the form $W_{\mathfrak{v},I}$.

Consider the differential monomorphism $R \to K[[x]]$ induced by I. This induces monomorphisms $\mathfrak{D}(R) \to \mathfrak{D}(K[[x]])$ and $\mathfrak{P}(R) \to \mathfrak{P}(K[[x]])$. Furthermore $\mathrm{Gr}(R;0)_+$ and $\mathrm{Gr}(K[[x]],0)_+$ are identical, and if $\mathfrak{v} \in \mathfrak{P}(R)$ then $W_{\mathfrak{v},I} \in \mathrm{Gr}(R;0)_+$ and $W_{\mathfrak{v},(x)} \in \mathrm{Gr}(K[[x]];0)_+$ agree. The next proposition shows that every $W \in \mathrm{Gr}(K[[x]];0)_+$ is of the form $W_{\mathfrak{v},(x)}$ for some wave operator $\mathfrak{v} \in \mathfrak{P}(K[[x]])$.

Proposition 3.3.1.3 (Formal Local Lifting)**.** *Let $W \in \mathrm{Gr}(K[[x]];0)_+$. Then there exists a wave operator $\mathfrak{v} \in \mathfrak{P}(K[[x]])$ with $W = W_{\mathfrak{v},(x)}$.*

Proof. Recall that by the Artin-Wedderburn theorem, there exists a division ring $k \subseteq D \subseteq K$ such that the natural map $K \to \mathrm{End}_D(V)$ is an isomorphism. Let $v_1, \ldots, v_d$ be a D-basis for V. Since $W \in \mathrm{Gr}(R;0)_+$ for each integer $n \geq 0$ and $1 \leq j \leq d$ there exists a unique $w_{j,n} \in W$ of the form

$$w_{j,n} = v_j \partial^n + \sum_{m=1}^{\infty} w_{j,n,m} \partial^{-m}.$$

Furthermore since $\mathrm{End}_D(V) = K$, for all $n \geq 1$ we may choose $a_{n,0} \in K$ such that $v_j a_{n,0} = w_{j,0,n}$. Then For each $m \geq 1$, choose $a_{m0} \in K$ such that $v_j a_{m0} = w_{j,0,m}$ for all $1 \leq j \leq d$. More generally, we can define $a_{t,\ell}$ for $t \geq 0$ and $\ell > 0$ by

$$v_j a_{t,\ell} = -v_j \sum_{m=0}^{\ell-1} \binom{\ell}{m} a_{\ell-m+t,m} + w_{j,\ell,t} + \sum_{s=1-\ell}^{\min(0,m-\ell)} w_{j,s,t} \binom{\ell}{m} a_{\ell-m+s,m}.$$

Then setting $a_n = \sum_{m=0}^{\infty} a_{n,m} x^m$ and $\mathfrak{v} = \sum_{n=0}^{\infty} a_n \partial^n$ we have $W = W_{\mathfrak{v},(x)}$. □

The question still remains as to whether/when $\mathfrak{v} \in \mathfrak{P}(K[[x]])$ lifts to an element $\mathfrak{v} \in \mathfrak{P}(R)$. This is difficult to answer in general. Lifting to $\mathfrak{P}(R)$ requires that the coefficient functions $a_0, a_1, \cdots \in K[[x]]$ actually have a preimage in R. Intuitively, one can think of this as a sort of convergence result for power series. As such, the standard route to supplying conditions for lifting $\mathfrak{v}$ to $\mathfrak{P}(R)$ is to start to include some analysis. This is exactly what we do in the next chapter.

3.3.2 Centralizers of Differential Operators

In the case that a Schur pair (W, A) does satisfy $W = W_{I,\mathfrak{v}}$ for some $\mathfrak{v} \in \mathfrak{P}(R)$, then the algebra A corresponds to a subalgebra of $\mathfrak{D}(R)$. To prove this, we will first establish a lemma. For simplicity, take $A_{I,\mathfrak{v}} := A_{W_{I,\mathfrak{v}}}$.

Lemma 3.3.2.1. *A pseudo-differential operator $\mathfrak{d} \in \mathfrak{P}(R)$ belongs to $\mathfrak{D}(R)$ if and only if*

$$(\mathbb{V}_I)_+ \cdot \mathfrak{d} \subseteq (\mathbb{V}_I)_+.$$

Proof. Since $\mathfrak{D}(R)$ and $\mathfrak{P}(R)$ inject into $\mathfrak{D}(K[[x]])$ and $\mathfrak{P}(K[[x]])$ and the induced action on $\mathbb{V}$ is the same, we will write $\mathbb{V}$ in place of $\mathbb{V}_I$ or $\mathbb{V}_{(x)}$. We first want to prove that if $\mathfrak{d} \in \mathfrak{D}(R)$, then $\mathbb{V}_+ \cdot \mathfrak{d} \subseteq \mathbb{V}_+$. Since the action of $\mathfrak{d}$ on $\mathbb{V}$ as an operator over R or $K[[x]]$ is the same, we may consider it as an element of $\mathfrak{P}(K[[x]])$. Also since $z^k \cdot \partial^n = z^{n+k}$, it is clear that $\mathbb{V}_+ \cdot \partial^n \subseteq \mathbb{V}_-$ for all integers $n \geq 0$. Thus it suffices to show that for any $a \in R$ we have $vz^k \cdot a \in \mathbb{V}_+$ for all $k \geq 0$ and $v \in (\mathbb{C}^N)^T$. Consider the image of a in $K[[x]]$, ie. $a = \sum_{j=0}^{\infty} a_j x^j$. For any integer $k \in \mathbb{Z}$, we calculate

$$\partial^k x = x\partial^k + k\partial^{k-1} \equiv k\partial^{k-1} \mod x\mathfrak{P}(K[[x]]).$$

Therefore $vz^k \cdot x = kvz^{k-1}$, and it follows that for $k \geq 0$:

$$z^k \cdot x^j = \begin{cases} \frac{k!}{(k-j)!} z^{k-j}, & 0 \leq j \leq k \\ 0, & j > k \end{cases}$$

Thus we find

$$z^k \cdot a = z^k \cdot \sum_{j=0}^{\infty} a_j x^j = \sum_{j=0}^{k} a_j \frac{k!}{(k-j)!} z^{k-j}.$$

Consequently $\mathbb{V}_+ \cdot a \subseteq \mathbb{V}_+$ and this proves the first direction.

Conversely, suppose that $\mathfrak{d} = \sum_{n=-\infty}^{\ell} a_n \partial^n$ satisfies $\mathbb{V}_- \cdot \mathfrak{d} \subseteq \mathbb{V}_- \cdot \mathfrak{d}$. We calculate

$$\mathrm{Ad}_{\partial}^m(\mathfrak{d}) = \sum_{n=-\infty}^{\ell} a_n^{(m)} \partial^n.$$

Moreover, since $\mathfrak{d}$ and ∂ both preserve $\mathbb{V}_-$, so too does $\mathrm{Ad}_{\partial}^m(\mathfrak{d})$. Thus for all $v \in (\mathbb{C}^{\oplus N})^T$ we have

$$v \cdot \mathrm{Ad}_{\partial}^m(\mathfrak{d}) = v \sum_{n=-\infty}^{\ell} \overline{a_n^{(m)}} z^n \in \mathbb{V}_-$$

for all m. This implies that $\overline{a_n^{(m)}} = 0$ for all $m \geq 0$ and $n < 0$, and therefore that a_n is identically zero for all $n < 0$ by the injectivity of $R \to K[[x]]$. Hence $\mathfrak{d}$ is a differential operator, and this proves the converse. □

Proposition 3.3.2.2. *Let $\mathfrak{v}$ be a wave operator, $W = W_{I,\mathfrak{v}}$, and $A = A_W = A_{I,\mathfrak{v}}$. Then conjugation by $\mathfrak{v}$ defines an isomorphism between A and the algebra*

$$\{\mathfrak{d} \in \mathfrak{D}(R) : \mathfrak{v}^{-1}\mathfrak{d}\mathfrak{v} \in K((\partial^{-1}))\}.$$

In particular, if $\mathfrak{v}$ is a wave operator taking $\mathfrak{d}$ to ∂^ℓ with ℓ different from the characteristic of R, then conjugation by $\mathfrak{v}$ defines an isomorphism between the centralizer $C(\mathfrak{d})$ of $\mathfrak{d}$ in $\mathfrak{D}(R)$ and A.

Proof. Suppose that $\mathfrak{d} \in \mathfrak{D}(R)$ with $\mathfrak{v}^{-1}\mathfrak{d}\mathfrak{v} = f(\partial^{-1}) \in K((\partial^{-1}))$. Then since $\mathfrak{d}$ is a differential operator,

$$Wf(z^{-1}) = W \cdot f(\partial^{-1}) = \mathbb{V}_+ \cdot \mathfrak{v} f(\partial^{-1}) = \mathbb{V}_+ \cdot \mathfrak{d}\mathfrak{v} \subseteq \mathbb{V}_+ \cdot \mathfrak{v} = W.$$

Hence $f(z^{-1}) \in A_W$.

Conversely, suppose that $f(z^{-1}) \in A_W$. Then for $\mathfrak{d} = \mathfrak{v}\mathfrak{f}(\partial^{-1})\mathfrak{v}^{-1}$, the same calculation shows that $\mathbb{V}_+ \cdot \mathfrak{d} \subseteq \mathbb{V}_+$, so that $\mathfrak{d} \in \mathfrak{D}(R)$. This proves our proposition. □

3.3.3 Rational Lifting

Let $\mathfrak{d}$ be a differential operator of order ℓ relatively prime to the characteristic of R, and let $\mathfrak{v}$ be a wave operator satisfying $\mathfrak{v}^{-1}\mathfrak{d}\mathfrak{v} = \partial^\ell$. As above, we've seen that a pseudo-differential operator $\mathfrak{v}$ determines a Schur pair (W, A_W) with A_W isomorphic to the centralizer $C(\mathfrak{d})$ of $\mathfrak{d}$ in $\mathfrak{D}(R)$. Next, if (W, A_W) is rank 1, then Krichever's correspondence gives us a quintuple $(X, \mathcal{W}, \infty, t, \varphi)$ corresponding to (W, A), and describing in geometric terms the structure of $C(\mathfrak{d})$.

A natural question is whether given a quintuple $(X, \mathcal{W}, \infty, t, \varphi)$, we can construct a differential operator $\mathfrak{d}$ whose centralizer the quintuple describes. We can already go part of the way, in that we can construct a Schur pair (W, A_W). Unfortunately, it is not necessarily true that $W = \mathfrak{W}_{I,\mathfrak{v}}$ for some smooth point I of R and some pseudo-differential operator $\mathfrak{v} \in \mathfrak{P}(R)$. If it were, then we could easily construct $\mathfrak{d}$ via conjugation by $\mathfrak{v}$. As we mentioned above, showing that $\mathfrak{v}$ exists typically involves some stronger assumptions about the structure of R and some analysis.

Even without analysis, we can squeeze by at least one special case of lifting a $\mathfrak{v} \in \mathfrak{P}(K[[x]])$ to $\mathfrak{v} \in \mathfrak{P}(R)$ when $W = W_{\mathfrak{v},(x)}$ is "rational".

Definition 3.3.3.1. A point $W \in \mathrm{Gr}(R)$ is called **rational** if there exists polynomials $p(z), q(z) \in k[z]$ and $r(z) \in K((z^{-1}))^\times$ such that $Wp(z) \subseteq W$ and $Wr(z) \subseteq \mathbb{V}_+ q(z)^{-1}$.

The use of rational here is justified by the next proposition.

Proposition 3.3.3.2. *Let (W, A_W) be a robust Schur pair corresponding to a quintuple $(X, \mathcal{W}, \infty, t, \varphi)$ under Krichever's correspondence. Then W is rational if and only if X is a rational curve.*

Proof. Suppose that W is rational. By replacing W with $Wr(z)$, without loss of generality we may assume that there exists $q(z) \in k[z]$ such that $W \subseteq \mathbb{V}_+ q(z)^{-1}$. Then $W \subseteq V(z)$ and since the map $\pi_+|_W$ has finite cokernel, we also know $Wk(z) = V(z)$. Therefore if $a \in A_W$ we must have

$$V(z)a = Wk(z)a = Wak(z) \subseteq Wk(z) = V(z),$$

and it follows that $a \in K(z)$. Hence $S_W \subseteq k(z)$, so that $F_W \subseteq k(z)$. The rationality of X then follows by Lüroth's theorem.

Conversely, if X is rational then $\mathcal{O}_{X,p} = k[z^{-1}]$ and therefore S_W and W embed into $k(z)$ and $VF(\mathcal{O}_{X,p}) = V(z)$, respectively. In particular, we may choose $r(z)$ such that $Wr(z) \subseteq V(z)$. Since S_W is finitely generated, we can choose $b(z)$ such that $b(z)S_W \subseteq \mathbb{C}[z]$. Then since $Wr(z)$ is finitely generated over S_W, we can choose $q(z)$ such that $Wr(z)q(z) \subseteq V[z] = \mathbb{V}_+$. Choosing a nontrivial element $p(z) \in S_W$, we have $Wp(z) \subseteq W$, so this proves W is rational. □

Proposition 3.3.3.3. *Let R be a differentially closed ring. Assume that $W \in \mathrm{Gr}(R;0)$ is rational. Then there exists $\mathfrak{v} \in \mathfrak{P}(R)$ such that $W = W_{\mathfrak{v},I}$.*

Proof. Without loss of generality, we may take p and q to be monic. Choose $\mathfrak{v} \in K[[x]]$ with $W = W_{\mathfrak{v},(x)} = (\mathbb{V}_I)_+ \cdot \mathfrak{v}$. Then we have that

$$(\mathbb{V}_I)_+ \cdot \mathfrak{v}q(\partial) \subseteq (\mathbb{V}_I)_+$$

and therefore $\mathfrak{v}q(\partial) = \mathfrak{d} \in \mathfrak{D}(K[[x]])$ for some monic differential operator $\mathfrak{d}$. Consequently, $\mathfrak{v} = \mathfrak{d}q(\partial)^{-1}$. Next since $Wp(z) \subseteq W$, we also have that $\mathfrak{v}p(\partial)\mathfrak{v}^{-1} = \mathfrak{b} \in \mathfrak{D}(K[[x]])$. From this, we calculate $\mathfrak{d}p(\partial)\mathfrak{d}^{-1} = \mathfrak{b}$ so that $\mathfrak{d}p(\partial) = \mathfrak{b}\mathfrak{d}$. Conseqently, $p(\partial) \cdot \ker_{K[[x]]}(\mathfrak{d}) \subseteq \ker_{K[[x]]}(\mathfrak{d})$. Since $\ker_{K[[x]]}(\mathfrak{d})$ is finite dimensional, there exists $\widetilde{p}(\partial) \in \mathbb{C}[\partial]$ such that $\widetilde{p}(\partial) \cdot \ker_{K[[x]]}(\mathfrak{d}) = 0$. This implies that $\ker_{K[[x]]}(\mathfrak{d}) \subseteq \ker_{K[[x]]}(\widetilde{p}(\partial))$. Now using the fact that R is differentially closed, we actually have $\ker_{K[[x]]}(\widetilde{p}(\partial)) = \ker_R(\widetilde{p}(\partial)) \subseteq R$. Hence $\ker_{K[[x]]}(\mathfrak{d}) \subseteq R$, and it follows that $\mathfrak{d} \in \mathfrak{D}(R)$. □

Corollary 3.3.3.3.1. *Let S be a k-algebra whose fraction field F is isomorphic to a subfield of $k(x)$, and let W be a torsion-free S-module of rank $N = \dim_k(V)$ with $S \cong Z(\mathrm{End}_S(W))$. Then there exists $\mathfrak{d} \in \mathfrak{D}(R)$ whose centralizer is isomorphic to $\mathrm{End}_S(W)$.*

Proof. This follows form Krichever's correspondence and rational lifting. □

Chapter 4

THE TAU FUNCTION

4.1 The Hilbert Space Grassmannian and Loops

Krichever's correspondence relates sufficiently nice Schur pairs (W, A) to quintuples of algebro-geometric data $(X, \mathcal{W}, \infty, t, \varphi)$, where here: X is a projective curve; $\mathcal{W}$ is a torsion-free sheaf on X of rank N; ∞ is a smooth point of X; t is a uniformizer of X at ∞; and φ is a local trivialization of $\mathcal{W}$ in a neighborhood of ∞. However, it partially fails to relate differential operators to geometric data, since we cannot necessarily construct an operator from a specified quintuple of geometric data. The barrier we must overcome is whether we can represent the induced element W of the Sato grassmannian as $W_{\mathfrak{v},I}$ for a pseudo-differential operator $\mathfrak{v} \in \mathfrak{P}(R)$.

In the case that R is the ring of $N \times N$ matrix-valued holomorphic functions on a simply connected open subset of $\mathbb{C}$ (whereby $k = \mathbb{C}$, $K = M_N(\mathbb{C})$, and $V = (\mathbb{C}^{\oplus N})^T$), it turns out that we can always find a $\mathfrak{v}$ with $W_{\mathfrak{v},I} = W$ for W arising from geometric data. To do this, we need to introduce some analysis in the form of a Hilbert space version $\overline{\mathrm{Gr}}(R)$ of Sato's grassmannian $\mathrm{Gr}(R)$. This extended analytic structure allows us to consider the action of loops (eg. continuous functions $S^1 \to \mathbb{C}^\times$) on $\overline{\mathrm{Gr}}(R)$. Throughout this section, we will let $R = M_N(Holo(U))$ for some open subset U of the complex plane.

4.1.1 The Hilbert Space Grassmannian

Given a point $W \in \mathrm{Gr}(R)$, we can ask whether W is contained in the Hilbert space $\overline{\mathbb{V}} = L^2(S^1, V)$ of square-integrable V-valued functions on S^1 with inner product

$$\langle v(z), w(z)\rangle = \int_{S^1} v(z)w(z)^* dz,$$

where here * denotes the conjugate transpose in $V = (\mathbb{C}^{\oplus N})^T$.

Definition 4.1.1.1. We define the **Hilbert space grassmannian** to be

$$\overline{\mathrm{Gr}}(R) := \{H \subseteq \overline{\mathbb{V}} : H \text{ closed, } \pi_-|_H \text{ is compact, and } \pi_+|_H \text{ is Fredholm}\},$$

where here $\pi_\pm$ is the projection map

$$\pi_\pm : \overline{\mathbb{V}} \to \overline{\mathbb{V}}/\overline{\mathbb{V}}_\mp = \overline{\mathbb{V}}_\pm$$

induced by the orthogonal decomposition $\overline{\mathbb{V}} = \overline{\mathbb{V}}_+ \oplus \overline{\mathbb{V}}_-$ for

$$\overline{\mathbb{V}}_+ = \left\{ \sum_{n=0}^{\infty} v_n z^n \in \overline{\mathbb{V}} : v_n \in V \right\} \quad \text{and} \quad \overline{\mathbb{V}}_- = \left\{ \sum_{n=-\infty}^{-1} v_n z^n \in \overline{\mathbb{V}} : v_n \in V \right\}.$$

Here by π_+ being Fredholm, we mean that the kernel and cokernel are both finite dimensional. We define the index-μ Hilbert space grassmannian and the big cell of the index-0 Hilbert space grassmannian to mirror the same constructions for the Sato grassmannian.

The Hilbert space grassmannian can be viewed as a proper subset of the Sato grassmannian, since there is an injective map

$$\overline{\mathrm{Gr}}(R;\mu) \to \mathrm{Gr}(R;\mu) : H \mapsto \pi_+^{-1}(\mathbb{V}_+).$$

Note that this map restricts to the respective big cells of the index-0 grassmannians.

Lemma 4.1.1.2. *Let H be a closed subspace of $\overline{\mathbb{V}}$. Then $H \in \mathrm{Gr}(R;\mu)$ if and only if there exist continuous linear maps $\iota_\pm : \overline{\mathbb{V}}_+ \to \overline{\mathbb{V}}_\pm$ with ι_- compact and ι_+ Fredholm of index μ so that the map $\iota_+ \oplus \iota_- : \overline{\mathbb{V}}_+ \to \overline{\mathbb{V}}$ is injective with image H.*

Proof. Suppose that $\iota_\pm$ exist for some H. Then $\mathrm{img}(\pi_+|_H) = \mathrm{img}(\iota_+)$, and $\ker(\pi_+|_H) = \iota_-(\ker(\iota_+))$. Since $\iota_+ \oplus \iota_i$ is injective, the dimension of $\iota_-(\ker(\iota_+))$ must be the same as the dimension of $\ker(\iota_+)$. It follows that $\pi_+|_H$ is Fredholm of index μ. Furthermore, we see that

$$\pi_-|_H \circ (\iota_+ \oplus \iota_-) = \iota_-,$$

Since the image H of $\iota_+ \oplus \iota_-$ is closed, we can invert it on its image. Thus

$$\pi_-|_H = \iota_- \circ (\iota_+ \oplus \iota_-)^{-1},$$

and since the right hand side is the composition of a bounded linear operator and a compact linear operator, it must be compact. Thus $\pi_-|_H$ is compact, and this proves $H \in \mathrm{Gr}(R;\mu)$.

Conversely, suppose $H \in \mathrm{Gr}(R;\mu)$. Since $\ker(\pi_+|_H)$ is finite dimensional, we may choose z^n such that $\pi_+|_{z^nH}$ is injective. Set $\widetilde{H} = \pi_+(z^nH)$. Then $\widetilde{H}$ is a subspace of $\overline{\mathbb{V}}_+$ of codimension $n-\mu$ and there exists a Hilbert space isomorphism $f : \overline{\mathbb{V}}_+ \to \widetilde{H}$. For all $\psi \in \overline{\mathbb{V}}_+$ define

$$\iota_\pm(\psi) = \pi_\pm(z^{-n}(\pi_+|_{z^nH})^{-1}(f(\psi))).$$

Then $\iota_\pm$ satisfy the desired properties.

□

Definition 4.1.1.3. We will call a point of the Sato grassmannian $W \in \mathrm{Gr}(R;\mu)$ **analytic** if $W \subseteq \overline{\mathbb{V}}$ with $\overline{W} \in \overline{\mathrm{Gr}}(R;\mu)$. We denote the collection of analytic points of $\mathrm{Gr}(R;\mu)$ by $\mathrm{Gr}(R;\mu)^{an}$.

Remark 4.1.1.4. If W is analytic, then $\pi_+|_{\overline{W}}^{-1}(\mathbb{V}_+) = W$, so the value of W may be recovered from the value of $\overline{W}$.

Remark 4.1.1.5. It is possible for $W \subseteq \overline{\mathbb{V}}$, but for W to not be analytic. For example

$$W = \mathrm{span}_{\mathbb{C}}\{z^n + z^{-n} | n \in \mathbb{Z}\} \subseteq \overline{\mathbb{V}},$$

however $\pi_\pm|_{\overline{W}}(z^n + z^{-n}) = z^{\pm n}$ for all $n > 0$. In particular, $\pi_-|_{\overline{W}}$ is not compact.

Lemma 4.1.1.6. *Let $W \in \mathrm{Gr}(R)$, and suppose that $W \subseteq \overline{\mathbb{V}}$. Then W is analytic if and only if $\pi_-|_{\overline{W}}$ is compact.*

Proof. By considering the basis expansion of W in terms of various degree, one readily finds that $\ker(\pi_+|_W) = \ker(\pi_+|_{\overline{W}})$ and that $\mathrm{coker}(\pi_+|_W) = \mathrm{coker}(\pi_+|_{\overline{W}})$. Therefore $\pi_+|_W$ is Fredholm, with index equal to the index of W. Thus if $\pi_-|_{\overline{W}}$ is compact, we automatically get W is analytic. □

For every quintuple $(X, \mathcal{W}, \infty, t, \varphi)$ we can construct a point H in the Hilbert space Grassmannian. The main idea is to show that the associated point in the Sato Grassmannian is analytic (and then take its closure). To do so, note that t may be viewed in an analytic neighborhood of ∞ on X as a univalent function mapping ∞ to 0. Consequently, there exists analytic open neighborhoods $U_\infty, U'_\infty, U''_\infty$ of $\infty \in X_W$, and $V_\infty, V'_\infty, V''_\infty$ of $\infty \in \mathbb{C}^*$ with

$$\overline{U}'_\infty \subset U_\infty \text{ and } \overline{U}_\infty \subseteq U''_\infty,$$

$$\overline{V}'_\infty \subset V_\infty \text{ and } \overline{V}_\infty \subseteq V''_\infty,$$

and $V_\infty = \mathbb{C}^* \backslash \overline{\mathbb{D}}$, along with a diffeomorphism $\phi : U''_\infty \to V''_\infty$, with $\phi(U_\infty) = V_\infty$, $\phi(U'_\infty) = V'_\infty$, and $\phi(\infty) = \infty$ with $t = 1/\phi$ (see Figure 4.1).

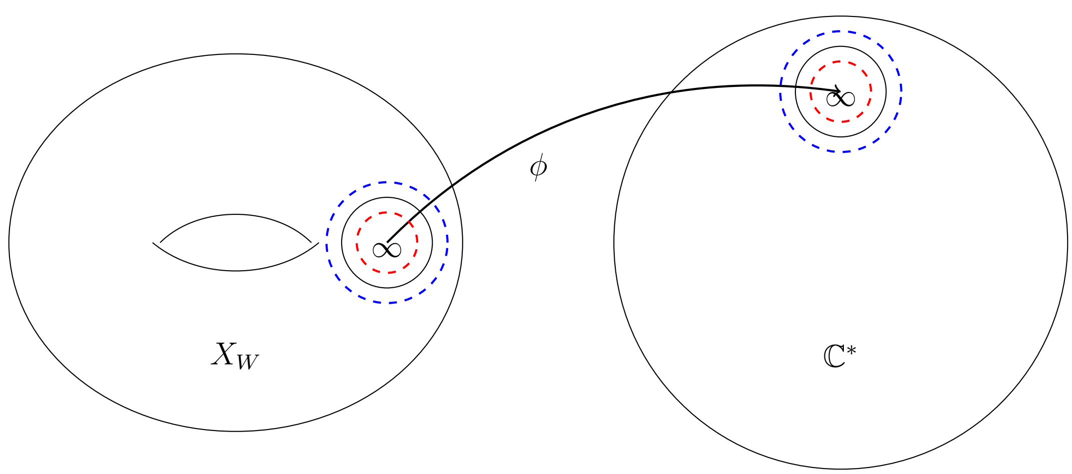

Figure 4.1: A neighborhood of ∞ in X_W is mapped to a neighborhood of ∞ in the Riemann sphere. The red dashed lines correspond to U'_∞ and V'_∞, and the blue dashed lines represent U''_∞ and V''_∞, while the black solid line represents the neighborhoods U_∞ and V_∞.

Note that the complement $X \backslash \{\infty\}$ is affine, so that the $\mathcal{O}_X$-module $\mathcal{W}$ is represented by a module W. Let $U_0 = X \backslash \overline{U}_\infty$, $U'_0 = X \backslash \overline{U}''_\infty$, and $U''_0 = X \backslash \overline{U}'_\infty$, so that $U'_0 \subseteq U_0 \subseteq U''_0$. The open sets U_0 and $\widetilde{U}_\infty$ cover X, with intersection an open set isomorphic to $\widetilde{V}_\infty \cap \mathbb{D}$, open neighborhood of S^1 in $\mathbb{C}$ under ϕ. Therefore each $w \in W$ defines (via restriction to U_0'', precomposition with ϕ^{-1}, and restriction to S^1) an element in $\overline{\mathbb{V}} = L^2(S^1, V)$ which when composed with ϕ^{-1} extends to a holomorphic map $U_0'' \to V$. Note that the value of an element $w \in W$ is uniquely determined by its values on S^1, and therefore this allows us to identify W with a $\mathbb{C}$-linear subspace of $\overline{\mathbb{V}}$. This is exactly the point in $W \in \mathrm{Gr}(R)$ corresponding to $(X, \mathcal{W}, \infty, t, \varphi)$. Using this description, we will show that W is analytic.

Proposition 4.1.1.7. *The Krichever correspondence associates each geometric quintuple* $(X, \mathcal{W}, \infty, t, \varphi)$ *to a Schur pair* (W, A_W) *with* W *analytic.*

Proof. It is clear from the above description that each $w \in W$ is the restriction to S^1 of a V-valued function holomorphic in a neighborhood of S^1. Hence $W \subseteq \overline{\mathbb{V}}$. Moreover, we know that $W \in \mathrm{Gr}(R)$. Therefore to prove that W is analytic, it suffices to show that the projection map $\pi_-|_W$ is compact.

To do so, for $\epsilon > 0$ and $\lambda \in \mathbb{C}$ with $|\lambda^{\pm 1}| < 1 + \epsilon$, consider the annulus $A_\epsilon = \{1 - \epsilon \leq |z| \leq 1 + \epsilon\}$, along with the map

$$T_\lambda : VHolo(A_\epsilon) \to VHolo(A_{\epsilon/\lambda}), \quad f(z) \mapsto f(\lambda z).$$

Then there exists an $\epsilon > 0$ such that $W \subseteq VHolo(A_{\epsilon/\lambda})$ and therefore T_λ restricts to a map on W. Note that $T_\lambda \circ \pi_- \circ T_{1/\lambda} = \pi_-$ and therefore we may write

$$\pi_-|_W = T_{1/\lambda} \circ (\pi_- \circ T_\lambda|_W).$$

Choosing $|\lambda| < 1$, $T_{1/\lambda}$ is a compact operator on $\overline{\mathbb{V}}_-$. Hence $\pi_-|_W$ is a composition of compact operator with a bounded operator and therefore compact. $\square$

4.1.2 Loops

Definition 4.1.2.1. We define the **loop group** to be the group Γ of all continuous maps $g : S^1 \to \mathbb{C}^\times$. Elements of Γ are called **loops**.

The loop group Γ has a distinguished subgroup Γ_+ consisting of real analytic functions $g : S^1 \to \mathbb{C}^\times$ extending to holomorphic functions on the closed unit disk $\overline{\mathbb{D}}$ with $g(0) = 1$. The following proposition shows that Γ_+ has a group action on $\overline{\mathrm{Gr}}(R)$ via multiplication.

Proposition 4.1.2.2. *Let $H \in \overline{\mathrm{Gr}}(R;\mu)$ and $g \in \Gamma_+$. Then $Hg \in \overline{\mathrm{Gr}}(R;\mu)$.*

Proof. Consider the map $T : \overline{\mathbb{V}} \to \overline{\mathbb{V}}$ defined by $T(f(z)) = f(z)g$. Since $\overline{\mathbb{V}}_+ g \subseteq \overline{\mathbb{V}}_+$, this decomposes with respect to $\overline{\mathbb{V}} = \overline{\mathbb{V}}_+ \oplus \overline{\mathbb{V}}_-$ as

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_4 \end{pmatrix}$$

where $T_1 : \overline{\mathbb{V}}_+ \to \overline{\mathbb{V}}_+$, $T_2 : \overline{\mathbb{V}}_- \to \overline{\mathbb{V}}_+$, and $T_4 : \overline{\mathbb{V}}_- \to \overline{\mathbb{V}}_-$. One may verify that T_1 is Fredholm of index 0, T_2 is an operator of trace class (since g is differentiable), and T_3 is compact.

Now choose $\iota_\pm : \overline{\mathbb{V}}_+ \to \overline{\mathbb{V}}_\pm$ with $W = \mathrm{img}(\iota_+ \oplus \iota_-)$. Then $Wg = \mathrm{img}(\iota'_+ \oplus \iota'_-)$ with $\iota'_+ = T_1\iota_+ + T_2\iota_-$ and $\iota'_- = T_4\iota_-$. Now ι'_- is compact since it is a composition of compact operators. Furthermore, $T_1\iota_+$ is a composition of two Fredholm operators of index μ and 0, and is therefore Fredholm of index μ. Therefore ι'_+ is a compact perturbation of a Fredholm operator of index μ, and is thus Fredholm of index μ. We conclude that $Wg \in \overline{\mathrm{Gr}}(R;\mu)$. □

Note that each $g(z) \in \Gamma_+$ defines a holomorphic function on $\overline{\mathbb{D}}$ which avoids 0. Therefore by considering the series expansion of $\log g(z)$ in $\mathbb{D}$, we may write

$$g(z) := \exp\left(\sum_{n=0}^{\infty} t_n z^n\right)$$

for some $\vec{t} = (t_0, t_1, \dots) \in \ell^2(\mathbb{N})$. Notationally, we will denote this element by $g(\vec{t}; z)$ or $g(t_0, t_1, \dots; z)$. We will also write $g(t_0, t_1, \dots, t_n; z)$ in place of $g(t_0, t_1, \dots, t_n, 0, 0, \dots; z)$, and we may occasionally write $g(\vec{t})$ in place of $g(\vec{t}; z)$ since $g(\vec{t})$ is an element of Γ_+ and therefore implicitly a function of z.

4.2 Baker-Alkheizer Function

The big theorem now is that if W is analytic, then $W = W_{\mathfrak{v},I}$ for some $\mathfrak{v} \in \mathfrak{P}(R)$. The proof of this relies on the construction of a special function $\psi_W(x,y)$, called the Baker-Alkheizer function of W. Classically, a (stationary) Baker-Alkheizer function represents a canonical choice of the family of eigenfunctions of an algebra of commuting differential operators.

Definition 4.2.0.3. Let U be a simply connected open subset of $\mathbb{C}$, $R = M_N(Holo(U))$, and $b \in U$ a fixed base point. Furthermore, let $A \subseteq \mathfrak{D}(R)$ be a subalgebra (not necessarily commutative). A **stationary Baker-Alkheizer function** for A is a function $\psi(x,y)$ defined on $U_0 \times V$ for some open neighborhoods V of ∞ and U_0 of b in $\mathbb{C}$, which satisfies the following properties

(a) $\psi(x,y)e^{-xy}$ is holomorphic in x and y, and approaches 1 as $|y| \to \infty$

(b) for each $\mathfrak{d} \in A$ there exists a holomorphic function $f(\mathfrak{d};y)$ on V satisfying

$$\mathfrak{d} \cdot \psi(x,y) = \psi(x,y)f(y).$$

Remark 4.2.0.4. Note that if $N = 1$, then for A to have a Baker-Alkheizer function it must be commutative.

Taking a power series expansion at ∞, we may write

$$\psi(x,y) = \left(1 + \sum_{n=1}^{\infty} v_n(x)z^{-n}\right) e^{xy}.$$

Then one may check $\mathfrak{v} = 1 + \sum_{n=1}^{\infty} v_n(x)\partial^{-n}$ satisfies

$$\mathfrak{v}^{-1}\mathfrak{d}\mathfrak{v} = f(\mathfrak{d};\partial), \quad \forall \mathfrak{d} \in A.$$

Example 4.2.0.5. Let U be a simply connected domain in the complex plane excluding 0, and consider the subalgebra A of $\mathfrak{D}(Holo(U))$ generated by the commuting operators

$$\delta = \partial^2 - \frac{2}{x}, \quad \eta = \partial^3 - \frac{3}{x^2}\partial + \frac{3}{x^3}.$$

Then a Baker-Alkheizer function for A is given by

$$\psi(x,y) = \left(1 - \frac{1}{xy}\right) e^{xy}.$$

Example 4.2.0.6. Let $\wp(x)$ the Weierstrass $\wp$-function for an elliptic curve E, and let U be a simply connected, open subset on which $\wp(x)$ is defined. Consider the subalgebra A of $\mathfrak{D}(Holo(U))$ generated by the commuting operators

$$\delta = \partial^2 - 2\wp(x), \quad \eta = \partial^3 - 3\wp(x)\partial - \frac{3}{2}\wp'(x).$$

Then a stationary Baker-Alkheizer function for A is given by

$$\psi(x,y) = \frac{\theta(x-1/y)}{\theta(x)} e^{xy} = \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \frac{\theta^{(n)}(x)}{\theta(x)} y^{-n}\right) e^{xy},$$

where θ is the Jacobi θ-function of E, which satisfies $\wp = -\ln(\theta)'' + c$ for some constant c.

Example 4.2.0.7. Let $U = \mathbb{C}$, and consider the subalgebra $A = \mathbb{C}[\mathfrak{d}] \subseteq \mathfrak{D}(U)$ for $\mathfrak{d} = \partial^2 - \frac{1}{x^2}$. Then A has no Baker-Alkheizer function. One reason is because $1/x^2$ decreases rapidly as $x \to \infty$, and so we can study the structure of its eigenfunctions by means of scattering theory. The fact that the potential $1/x^2$ has a nonzero reflection coefficient in its scattering data implies that its eigenfunctions (times e^{-xy}) are not analytic in a neighborhood of ∞. (In contrast to the potential $2/x^2$ above!)

Example 4.2.0.8. Let U be a simply connected domain in the complex plane excluding two values a and b, and consider the subalgebra A of $\mathfrak{D}(Holo(U))$ of operators commuting with

$$\delta = \partial^2 - \begin{pmatrix} (x-a)^{-1} & (x-a)^{-1}(x-b)^{-1} \\ 0 & (x-b)^{-1} \end{pmatrix}.$$

Note that A is *not* commutative. Then a Baker-Alkheizer function for A is given by

$$\psi(x,y) = \begin{pmatrix} 1 - y^{-1}(x-a)^{-1} & y^{-1}(x-a)^{-1}(x-b)^{-1} \\ 0 & 1 - y^{-1}(x-b)^{-1} \end{pmatrix} e^{xy}.$$

4.2.1 Basic Definitions and Facts

It's entirely unclear if and when a stationary Baker-Alkheizer function exists. We will prove that when the algebra A is rank 1, then a Baker-Alkheizer function exists. To do so, we introduce a generalized notion of the Baker-Alkheizer function, defined by means of the action of the loop group Γ_+ on $\overline{\mathrm{Gr}}(R;0)$.

Definition 4.2.1.1. Let $H \in \overline{\mathrm{Gr}}(R;0)_+$, and define $\Gamma_H = \{g \in \Gamma_+ : Hg^{-1} \in \mathrm{Gr}(R;0)_+\}$. The **Baker-Alkheizer function** for H is a function $\psi_H : \Gamma_H \to L^2(S^1, K)$ satisfying the following properties for all $g \in \Gamma_H$

(a) $V\psi_H(g) \subseteq H$

(b) $\psi_H(g)g^{-1} = 1 + \sum_{n=1}^{\infty} w_n(g)z^{-n}$ for some $w_n(g) \in K$

We call the functions $w_n(g) : G \to K$ the **Baker-Alkheizer coefficient functions**.

For notational convenience, we will write $\psi_H(\vec{t})$ in place of $\psi_H(g(\vec{t}))$. Since all of the above functions are defined in some neighborhood of S^1, we can actually think of ψ_H as a function of z also, defined for all $z \in S^1$. To emphasize this, we may write $\psi_H(g;z)$ or $\psi_H(\vec{t}, z)$ in place of $\psi_H(g)$ or $\psi_H(\vec{t})$.

Of particular interest are the values $\psi_H(x;z) = \psi_H(g(x);z)$ for $g(x) = e^{xz} \in \Gamma_+$. In this case, the Baker-Alkheizer function looks like

$$\psi_H(x;z) = \left(1 + \sum_{n=1}^{\infty} w_n(x)z^{-n}\right) e^{xz},$$

which has the same "form" as a stationary Baker-Alkheizer function for an algebra of differential operators. We will show that when H originates from the Schur pair of an algebra A of differential operators, that this *is* a stationary Baker-Alkheizer function for A. In this way, $\psi_H(x;z)$ generalizes the classical notion of the Baker-Alkheizer function.

Proposition 4.2.1.2. *Let $R = M_N(Holo(U))$ for some simply connected, open subset of U. Fix a base point $b \in U$ and let $I = (x-b)R$. Suppose $W \in \mathrm{Gr}(R;0)$ is analytic. Then the*

Baker-Alkheizer coefficient functions $w_n(x) := w_n(g(x))$ *are holomorphic in a neighborhood of* 0. *The element* W *is of the form* $W = W_{\mathfrak{v},I}$ *for some pseudo-differential operator* $\mathfrak{v} \in \mathfrak{P}(R)$ *if and only if the functions* $w_n(x+b)$ *extend to holomorphic functions on* U. *In this case, the pseudo-differential operator*

$$\mathfrak{v} := 1 + \sum_{n=1}^{\infty} w_n(x+b)\partial^n \in \mathfrak{P}(R)$$

satisfies $W = W_{\mathfrak{v},I}$. *Conjugation by* $\mathfrak{v}$ *relates* A_W *to an algebra of pseudo-differential operators* $A \subseteq \mathfrak{D}(R)$ *for which* $\psi_{\overline{W}}(x;z)$ *is a stationary Baker-Alkheizer function.*

Proof. Without loss of generality, we may assume $b = 0$. We will prove later that the Baker-Alkheizer coefficient functions $w_n(\vec{t}) := w_n(g(\vec{t}))$ are holomorphic in each variable locally in a neighborhood of $\vec{0}$ by relating $\psi_{\overline{W}}$ to the τ function of $\overline{W}$, and then using properties of the τ function. We will also be able to use the τ function to show that W and $W_{\mathfrak{v},I}$ agree. For the last part, recall from our discussion above that $A := \mathfrak{v}A_W\mathfrak{v}^{-1}$ is an algebra of differential operators. Furthermore, if $f(z) \in A_W$ and $\mathfrak{d} = \mathfrak{v}\mathfrak{f}(\partial)\mathfrak{v}^{-1}$, we calculate $e^{-xy}f(\partial)e^{xy} = f(\partial+y)$ and therefore

$$\mathfrak{v}e^{xy}f(\partial+y) = \mathfrak{v}f(\partial)e^{xy} = \mathfrak{d}\mathfrak{v}e^{xy}.$$

This shows that $\mathfrak{v}e^{xy}f(\partial+y) = \mathfrak{d}\mathfrak{v}e^{xy}$ as pseudo-differential operators with coefficients analytic in a neighborhood of 0. Comparing coefficients, we find

$$\mathfrak{d} \cdot \psi_{\overline{W}}(x;y) = \psi_{\overline{W}}(x;y)f(y).$$

□

The values of $t_1, \ldots, t_n$ where $\psi_H(t_1, \ldots, t_n; z)$ is not defined turn out to be poles with a certain geometric significance. Recall that the Picard group $\mathrm{Pic}(X)$ of isomorphism classes of holomorphic line bundles on X is isomorphic to the cohomology group $H^1(X, \mathcal{O}_X^*)$ via the map $\mathcal{L} \mapsto [\varphi_{ij}]$, where $\varphi_{ij} : V_i \cap V_j \to \mathbb{C}^*$ are transition maps for $\mathcal{L}$ on intersections of members of an analytic open cover $\{V_i\}_{i=1}^n$ over which $\mathcal{L}$ trivializes. The exponential sequence

$$0 \to 2\pi i\mathbb{Z} \to \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \to 0$$

induces an isomorphism $H^1(X, \mathcal{O}_X) \to \mathrm{Pic}_0(X)$ defined by sending $[f_{ij}] \in H^1(X)$ to $[\exp(f_{ij})]$.

As a special case of this, suppose $g(z) \in \Gamma$ extends to a holomorphic function in an open annulus A containing S^1. Then we can choose an open cover $\{V_0, V_\infty\}$ of X with the uniformizer t at ∞ defined on V_∞ and with $z(V_0 \cap V_\infty) \subseteq A$, for $z = 1/t$. It follows that the function $g(z)$ is holomorphic function on $V_0 \cap V_\infty$, and therefore any holomorphic function g on the image of z restricted to $V_0 \cap V_\infty$ defines an element $[g(z)]$ on $H^1(X, \mathcal{O}_X)$, and thereby an element $[\exp g(z)]$ of $\mathrm{Pic}_0(X)$. In particular we have a map sending each smooth loop g to a line bundle L_g on X. This map is surjective.

Now suppose that $H = \overline{W}$ for some analytic $W \in \mathrm{Gr}(R)$ corresponding to a Krichever quintuple $(X, \mathcal{W}, \infty, t, \varphi)$. One may verify $H \exp(g(z))$ corresponds under Krichever's correspondence to $\mathcal{W} \otimes \mathcal{L}_g$, along with the trivialization at ∞ obtained by taking the tensor product of the trivializations. In particular if $\mathcal{W} = \mathcal{L}[p_a]$ for some $\mathcal{L} \in \mathrm{Pic}_0(X)$ (with p_a the arithmetic genus of X) then $He^{g(z)} \notin \overline{\mathrm{Gr}}(R; 0)_+$ exactly when $\mathcal{L} \otimes \mathcal{L}_g \in \Theta$, the theta divisor of X. Thus we may view $\psi_H(g; z)e^{-xz}$ as a function on $\mathrm{Pic}_0(X)$, which is undefined exactly on a translation of the theta divisor.

4.3 Fredholm Determinants

Throughout this section, let $\mathcal{H}$ be a separable Hilbert space.

4.3.1 Basic Definition

Fredholm determinants began their life as a key ingredient in Fredholm's theory of integral equations. Specifically, Fredholm proved that the integral equation of the second kind

$$u(x) + z \int_a^b K(x, y)u(y)dy = f(x) \tag{4.1}$$

has a unique solution if and only if Fredholm's determinant

$$d(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{[a,b]^n} \det(K(y_i, y_j))_{i,j=1}^n dy_1 \dots dy_n$$

is nonzero [4]. Here the kernel $K(x,y)$ and right hand side $f(x)$ are assumed to be continuous on $[a,b]^2$ and $[a,b]$, respectively. Fredholm's determinant $d(z)$ should be viewed as the determinant of the associated operator $I + zT$ on $L^2([a,b])$, where I is the identity operator and T is the convolution operator defined by the kernel $K(x,y)$. This was broadly generalized by Grothendieck to the definition we present here [34].

The idea is to generalize the formula for the characteristic polynomial $p_A(x)$ of an $n \times n$ matrix A in terms of the traces of its exterior powers

$$p_A(z) = \det(zI - A) = \sum_{k=0}^{n} z^{n-k}(-1)^k \mathrm{Tr}\Lambda^k(A),$$

Equivalently, this says that

$$\det(I + zA) = \sum_{k=0}^{n} z^k \mathrm{Tr}\Lambda^k(A),$$

and this is the formula that Grothendieck's definition generalizes. In order to do so, we need to work with operators on $\mathcal{H}$ for which it makes sense to take the trace. This motivates the definition of operators of trace class.

Definition 4.3.1.1. Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for $\mathcal{H}$. A linear operator $T : \mathcal{H} \to \mathcal{H}$ is said to be **trace class** if

$$\sum_{k=1}^{\infty} \langle |T| e_k, e_k \rangle < \infty.$$

If T is in trace class, then we define the **trace** of T to be the limit of the absolutely convergent sum

$$\mathrm{Tr}(T) = \sum_{k=1}^{\infty} \langle T e_k, e_k \rangle$$

Example 4.3.1.2. Let $K(x,y)$ be a continuous function on the square $[a,b]^2$. Then the integral operator T on $L^2([a,b])$ defined by

$$T : f(x) \mapsto \int_a^b K(x,y) f(y) dy$$

is trace class.

For each n, let $\Lambda^n(T)$ be the induced map on the exterior power $\Lambda^n\mathcal{H}$.

Lemma 4.3.1.3. *Suppose T is trace class. Then $\Lambda^n(T)$ is also trace class, and furthermore*

$$\mathrm{Tr}|\Lambda^n(T)| \leq \frac{(\mathrm{Tr}|T|)^n}{n!}.$$

Proof. This follows by direct calculation on the usual orthonormal basis for $\Lambda^n\mathcal{H}$. □

Definition 4.3.1.4. A linear operator $F : \mathcal{H} \to \mathcal{H}$ is said to be **Fredholm** if $F = I + T$, where I is the identity on $\mathcal{H}$ and T is trace class. In this case, we define the **Fredholm determinant** of F to be the absolutely convergent series

$$\det(F) = \sum_{n=0}^{\infty} \mathrm{Tr}\Lambda^n(T).$$

4.3.2 Properties of Fredholm Determinants

Lemma 4.3.2.1. *The Fredholm determinant is multiplicative, ie. for any two Fredholm operators $F_1, F_2 : \mathcal{H} \to \mathcal{H}$ we have*

$$\det(F_1F_2) = \det(F_1)\det(F_2).$$

Proof. This follows from multiplicativity of determinants over finite-dimensional vector spaces, along with the fact that the Fredholm determinant may be calculated as a limit of determinants over finite-dimensional subspaces. □

Theorem 4.3.2.2. *[6] Let $V_1, V_2, \ldots$ be a sequence of subspaces of $\mathcal{H}$ with $\dim(V_j) = j$ for all j and with $\bigcup_j V_j$ dense in $\mathcal{H}$. Moreover, for each j let P_j be the associated projection map. Then for any $T : \mathcal{H} \to \mathcal{H}$ of trace class*

$$\det(I + zP_jTP_j) \to \det(I + zT)$$

locally uniformly in z.

4.4 The Sato-Segal-Wilson Tau Function

The tau function was originally introduced by the Kyoto school as a link between solutions of the KdV equation

$$u_t = u_{xxx} + 6uu_x \tag{4.2}$$

and elements of the Sato grassmannian Gr. This connection was further explained and expanded by Segal and Wilson, who considered the Hilbert space version $\overline{\text{Gr}}$ of Sato's grassmannian Gr and identified the tau function of a point $H \in \overline{\text{Gr}}$ as the Fredholm determinant of a Hilbert space endomorphism associated to H. To each point $H \in \overline{\text{Gr}}$, Segal and Wilson associate a tau function $\tau_H(\vec{t})$ with $\vec{t} = (t_1, t_2, t_3, \dots) \in \ell^2(\mathbb{N})$. Here by convention $x = t_1$, $t = t_2$, $\tau_H(x) := \tau_H(x, 0, 0, \dots)$, and $\tau_H(x,t) := \tau_H(x, t, 0, 0, \dots)$.

The connection between solutions of the KdV equation 4.2 and points $W \in \text{Gr}$ is clarified by the Lax formulation of the KdV equation

$$\mathfrak{p}'(t) = [\mathfrak{q}(t), \mathfrak{p}(t)] \tag{4.3}$$

where $\mathfrak{p}(t)$ and $\mathfrak{q}(t)$ are the 1-parameter families of differential operators

$$\mathfrak{p}(t) = \partial_x^2 + u(x,t), \text{ and } \mathfrak{q}(t) = 4\partial_x^3 + 6u(x,t)\partial_x + 3u_x(x,t). \tag{4.4}$$

Differential operators $\mathfrak{p}(t), \mathfrak{q}(t)$ satisfying Equation 4.3 are called **Lax pairs**. It is easy to check that $u(x,t)$ is a solution of the KdV equation 4.2 if and only if $\mathfrak{p}(x,t)$ and $\mathfrak{q}(x,t)$ as defined in Equation 4.4 are a Lax pair.

Let $\psi(x,t,z)$ be an eigenfunction with eigenvalue z^2 of the Schrödinger operator $\mathfrak{p}(t)$ in some suitable space of functions. Note that $\psi(x,t,z)$ is not unique, but for the KdV equation there is a natural choice. The restatement of the KdV equation in terms of a Lax pair allows us to relate $\psi(x,t,z)$ to $\psi(x,0,z)$ *in a linear fashion.* Then under certain niceness assumptions, we may recover the potential $u(x,t)$ of the Schrödinger operator $\mathfrak{p}(t)$ from $\psi(x,t,z)$. This allows us to determine a solution $u(x,t)$ of the nonlinear partial differential equation 4.2 linearly!

To the Schrödinger operator $\mathfrak{p}(0)$, we may associate a point $H \in \overline{\mathrm{Gr}}$, assuming $u(x,0)$ is reasonably nice. Then the tau function $\tau_H(x)$ tells us how to relate $\psi(x,t,z)$ to the potential $u(x,t)$. Specifically, we have the two fundamental identities giving us a (specific) family of eigenfunctions of the operator $\mathfrak{p}(t)$ for each t, and a solution $u(x,t)$ of the KdV equation 4.2

$$\psi(x,t,z) = \frac{\tau_H(x - z^{-1}, t - z^{-2}/2)}{\tau_H(x,t)} e^{xz+tz^2}, \tag{4.5}$$

$$u(x,t) = 2\ln(\tau_H(x,t))_{xx}. \tag{4.6}$$

In summary, the tau function $\tau_H(x,t)$ works as the glue connecting a solution of the KdV equation to the eigenfunctions of a Schrödinger operator.

4.4.1 Basic Definition

For any point $H \in \overline{\mathrm{Gr}}(R;0)_+$, we define $\iota_H : \overline{\mathbb{V}} \to H$ to be the unique map satisfying $\iota_H \circ \pi_+|_H = \mathrm{id}_H$ and $\pi_+ \circ \iota_H|_{\mathbb{V}_+} = \mathrm{id}_{\mathbb{V}_+}$. Then we define a linear function $R(H,g) : \mathbb{V}_+ \to \mathbb{V}_+$ by

$$R(H,g) : \mathbb{V}_+ \xrightarrow{\iota_H} H \xrightarrow{L_g^{-1}} g^{-1}H \xrightarrow{\pi_+} \mathbb{V}_+ \xrightarrow{L_g} \mathbb{V}_+. \tag{4.7}$$

Here $L_g : \mathbb{V}_+ \to \mathbb{V}_+$ denotes right multiplication: $L_g(v(z)) = v(z)g(z)$.

Lemma 4.4.1.1. *For all $H \in \overline{\mathrm{Gr}}(R;0)_+$ and $g \in \Gamma_+$, the operator $R(H,g)$ of Equation 4.7 is Fredholm.*

Proof. Let $H \in \overline{\mathrm{Gr}}(R;0)_+$. Then there exist operators $\iota_\pm : \overline{\mathbb{V}}_+ \to \overline{\mathbb{V}}_\pm$ such that ι_- is compact and $\iota_+ \oplus \iota_-$ is injective with image H. Then there exists an automorphism of T of $\overline{\mathbb{V}}_+$ such that $(\iota_+ \oplus \iota_-) \circ T = \iota_H$. In particular, $\iota_- \circ T$ is still compact, so without loss of generality we may assume $\iota_+ \oplus \iota_- = \iota_H$. In particular, writing

$$L_g = \begin{pmatrix} T_1 & T_2 \\ 0 & T_4 \end{pmatrix},$$

where $T_1 : \overline{\mathbb{V}}_+ \to \overline{\mathbb{V}}_+$, $T_2 : \overline{\mathbb{V}}_- \to \overline{\mathbb{V}}_+$, and $T_4 : \overline{\mathbb{V}}_- \to \overline{\mathbb{V}}_-$, we see that

$$R(H,g) = T_1(\mathrm{id}_{\overline{\mathbb{V}}_+} \;\; 0) \begin{pmatrix} T_1^{-1} & -T_1^{-1}T_2T_4^{-1} \\ 0 & T_4^{-1} \end{pmatrix} \begin{pmatrix} \iota_+ \\ \iota_- \end{pmatrix} = \iota_+ - T_2T_4^{-1}\iota_-.$$

By our choice of $\iota_\pm$, we have $\iota_+ = \mathrm{id}_{\overline{\mathbb{V}}}$. Therefore $R(H,g)$ differs from the identity by a compact operator, and consequently $R(H,g)$ is Fredholm of index 0. □

For $H \in \overline{\mathrm{Gr}}(R;0)_+$, define $\Gamma_H = \{g \in \Gamma_+ : Hg^{-1} \in \overline{\mathrm{Gr}}(R;0)_+\}$.

Definition 4.4.1.2. The **Sato-Segal-Wilson tau function** of $H \in \overline{\mathrm{Gr}}_+(R;0)$ is the function $\tau_H(g) : \Gamma_+ \to \mathbb{C}$ defined by $\tau_H(g) = 0$ for $g \notin \Gamma_H$ and for $g \in \Gamma_H$ by the Fredholm determinant

$$\tau_H(g) = \det(R(H,g)).$$

4.4.2 Properties

Proposition 4.4.2.1. *Suppose that $g, gh \in \Gamma_W$. Then $h \in \Gamma_{Wg^{-1}}$ and*

$$\tau_W(gh) = \tau_{Wg^{-1}}(h)\tau_W(g).$$

Proof. Note that $\pi_{Wg^{-1}}\pi_+ : \overline{\mathbb{V}} \to \overline{\mathbb{V}}$ restricts to the identity on Wg^{-1}. Using this, we calculate

$$\begin{aligned} R(W,gh) &= L_g\pi_+L_g^{-1}\pi_W \\ &= L_gL_h\pi_+L_h^{-1}(\pi_{Wg^{-1}}\pi_+)L_g^{-1}\pi_W \\ &= (L_gL_h\pi_+L_h^{-1}\pi_{Wg^{-1}}L_g^{-1})(L_g\pi_+L_g^{-1}\pi_W) \\ &= L_gR(Wg^{-1},h)L_g^{-1}R(W,g). \end{aligned}$$

The statement of the theorem then follows by multiplicativity of Fredholm determinants. □

Corollary 4.4.2.1.1. *Suppose that $g \in \Gamma_W$. Then*

$$\tau_W(g)^{-1} = \tau_{Wg^{-1}}(g^{-1}).$$

Proof. This follows from the previous proposition, along with the fact that $\tau_W(1) = 1$. $\square$

Proposition 4.4.2.2. *The function $\tau_H(\vec{t}; z)$ is holomorphic in each of its (infinitely many) variables, and in fact extends to a holomorphic function on $\{\vec{t} : g(\vec{t}) \in \Gamma_+\}$, which vanishes exactly at points $\vec{t}$ with $g(\vec{t}) \in \Gamma_+ \backslash \Gamma_H$.*

Sketch of proof. One may obtain an estimate that shows $\det(R_n(H, g(\vec{t})))$ converges locally uniformly in t to $\det(R(H, g(\vec{t}))) = \tau_H(\vec{t}; z)$. Then by applying Moreora's theorem, the limit $\tau_H(\vec{t}; z)$ is seen to be holomorphic. $\square$

4.4.3 Sato's Formula for the Baker-Alkheizer Function

In this section, we consider the classical case when $N = 1$, so that $K = k = \mathbb{C}$ and derive Sato's formula for the Baker-Alkheizer function, thereby verifying several key properties of the Baker-Alkheizer function $\psi_H(\vec{t}; z)$ of a point $H \in \overline{\mathrm{Gr}}(R; 0)_+$. To do so, we first establish a key lemma linking the Baker-Alkheizer function and the tau function.

Lemma 4.4.3.1. *Suppose that $N = 1$ so that $V = k = K = \mathbb{C}$, and let $H \in \overline{\mathrm{Gr}}(R; 0)_+$. Then the Baker-Alkheizer function is given by*

$$\psi_H(g; y) = \tau_{Hg^{-1}}(\rho(y); z) g(y),$$

where $\rho(y) = \rho(y; z) = 1 - z/y$.

Proof. Using the decomposition $\overline{\mathbb{V}} = \overline{\mathbb{V}}_+ \oplus \overline{\mathbb{V}}_-$, we may decompose $L^{-1}_{\rho(y;z)}$ and $L_{\rho(y;z)}$ as

$$L^{-1}_{\rho(y;z)} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \text{ and } L_{\rho(y;z)} = \begin{pmatrix} a^{-1} & -a^{-1}bd^{-1} \\ 0 & d^{-1} \end{pmatrix}$$

for some $a : \overline{\mathbb{V}}_+ \to \overline{\mathbb{V}}_+$, $b : \overline{\mathbb{V}}_- \to \overline{\mathbb{V}}_+$, and $d : \overline{\mathbb{V}}_- \to \overline{\mathbb{V}}_-$, where we have used the fact that $\rho(y; z)^{-1}\overline{\mathbb{V}}_+ \subseteq \overline{\mathbb{V}}_+$.

Let $\widetilde{\widetilde{H}} = \widetilde{H} g^{-1}$. For each $n \geq 0$, let $v(z; n) \in z^{-1}\mathbb{C}[[z^{-1}]]$ be the unique element of the form $\iota_{\widetilde{H}}(z^n) = z^n + v(z; n)$. Then

$$R(\widetilde{H}, \rho(y; z))(z^n) = z^n + a^{-1}b(v(z; n))).$$

Now for $n \geq 0$ we calculate

$$\rho(y;z)^{-1}(z^{-n}) = \sum_{j=0}^{\infty} \frac{z^{j-n}}{y^j},$$

and therefore

$$b(z^{-n}) = y^{-n} \sum_{j=n}^{\infty} \frac{z^{j-n}}{y^{j-n}} = \rho(y;z)^{-1} y^{-n}.$$

Thus $a^{-1}b(z^{-n}) = \rho(y;z)b(z^{-n}) = y^{-n}$. It follows that $a^{-1}b(v(z;n)) = v(y;n)$, and therefore

$$P_0 R(\widetilde{H}, \rho(y;z))P_0 = 1 + v(y,0)$$

and also

$$(\mathrm{id} - P_0)R(\widetilde{H}, \rho(y;z))P_0 = 0.$$

Therefore $\tau_{\widetilde{H}}(\rho(y;z)) = 1 + v(y;0)$. Since $v(y;0) = v(y)$, this shows that the function

$$f(g,y) := \tau_{\widetilde{H}g^{-1}}(\rho(y))g(y) = (1 + v(y))g(y)$$

satisfies $f(g,z) \in H$ for all $g \in \Gamma_H$ and $f(g,z)g(z)^{-1} = 1 + \sum_{j=1}^{\infty} w_n z^{-n}$ for $\sum_{j=1}^{\infty} w_n z^{-n} = v(y;0)$. Thus $f(g,z)$ is the Baker-Alkheizer function for H. □

Theorem 4.4.3.2 (Sato's Formula)**.** *Suppose that $N = 1$ so that $V = k = K = \mathbb{C}$. Then the Baker-Alkheizer function of H is given by*

$$\psi_H(\vec{t}, z) = \frac{\tau_H(\vec{t} - [z^{-1}])}{\tau_H(\vec{t})}, \tag{4.8}$$

where here $[z] = (t, t^2/2, t^3/3, \ldots)$.

Proof. Using the fact that

$$\tau_H(g\rho(y)) = \tau_{Hg^{-1}}(\rho(y))\tau_H(g),$$

the previous proposition tells us

$$\psi_H(g) = \tau_H(g\rho(y))\tau_H(g)^{-1}.$$

Now since

$$\rho(y) = 1 - \frac{z}{y} = \exp\ln\left(1 - \frac{z}{y}\right) = \exp\left(-\sum_{n=1}^{\infty}\frac{z^n}{ny^n}\right) = g(-[y^{-1}]),$$

in terms of series, for $g = g(\vec{t})$ we can write $g\rho(y) = g(\vec{t} - [y^{-1}])$. Using this, the statement of the theorem follows immediately. $\square$

4.5 A Matrix-Valued Tau Function

For an $N \times N$-matrix Schrödinger operator $\mathfrak{d} = \partial^2 I + u(x)$, the general solution of the eigenvalue problem

$$\mathfrak{d}\psi(x,z) = z^2\psi(x,z)$$

will involve N linearly independent vector-valued solutions, or equivalently an $N \times N$ fundamental matrix of solutions. Indeed the Baker-Alkheizer function $\psi_W(\vec{t}, z)$ of the associated point $W \in \overline{\mathrm{Gr}}_+^N(0)$ will be matrix-valued. Thus Sato's formula 4.8 for the Baker-Alkheizer function will not work out as stated. Here we develop a matrix-valued version where an analog of Sato's formula works directly. Specifically, we will define an h-twisted matrix tau function $\mathcal{T}^h(g)$ whose determinant agrees with the definition of the Sato-Segal-Wilson tau function, and which satisfies the following matrix version of Sato's formula

$$\psi_W(\vec{t}, z) = \mathcal{T}^{g(\vec{t})}(g(\vec{t} - [z^{-1}]))\mathcal{T}^{g(\vec{t})}(g(\vec{t}))^{-1}.$$

Our definition relies on the notion of a quasi-determinant – a tool which unifies many different notions of determinants over noncommutative rings. It is worth noting that the idea of using quasi-determinants in the context of differential equations is not new. In particular, quasi-determinants are related to Darboux transformations [26], and have been related to exact solutions of the noncommutative Sine-Gordon equation [66], and the non-commutative Bäcklund transformation in the context of the noncommutative Yang-Mills equations [30]. However, to the best of the author's knowledge, quasi-determinants have not before been used to define a tau function.

4.5.1 Quasideterminants

We define quasi-determinants, following the popular exposition [26].

Definition 4.5.1.1. Let $A \in M_N(K)$ be an $N \times N$ matrix with entries in a ring K (not necessarily commutative) and let i, j be integers between 1 and N. If the i, j'th minor matrix $\text{minor}_{ij}(A)$ is a unit in $M_{N-1}(K)$, we define the i, j'th quasi-determinant to be the element of K defined by

$$\text{qdet}_{ij}(A; K) = a_{ij} - \vec{r}_i^T (\text{minor}_{ij}(A))^{-1} \vec{c}_j$$

where here a_{ij} is the i, j'th entry of A, A_{ij} is the i, j'th minor matrix, $\vec{r}_i$ is the i'th row vector, and $\vec{c}_j$ is the j'th column vector.

In the case that the ring K is clear, we will denote $\text{qdet}_{ij}(A; K)$ as $\text{qdet}_{ij}(A)$.

Example 4.5.1.2. Let $K = M_2(\mathbb{C})$, and consider the matrix $T \in M_2(K)$ defined by

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $A, B, C, D \in K^\times$. Then all the various quasi-determinants of A exist and are defined by

$$\text{qdet}_{11}(T) = A - BD^{-1}C \qquad \text{qdet}_{12}(T) = B - AC^{-1}D$$
$$\text{qdet}_{21}(T) = C - DB^{-1}A \qquad \text{qdet}_{22}(T) = D - CA^{-1}B$$

For notational convenience, given $A \in M_N(K)$ and $I, J \subseteq \{1, 2, \ldots, N\}$, we will use $M_{I,J}(A)$ to denote the $(N - |I|) \times (N - |J|)$ matrix formed by removing all rows with index in I and columns with index in J. In the special case that I and J are singletons, we will write $M_{i,j}(A)$ in place of $M_{\{i\},\{j\}}$.

Proposition 4.5.1.3 (Quasideterminants over Commutative Rings)**.** *If K is commutative and $A \in M_N(K)$, then the usual determinant of A is defined and*

$$\text{qdet}_{ij}(A) = (-1)^{i+j} \frac{\det(A)}{\det(M_{ij}(A))}$$

whenever $\mathrm{qdet}_{ij}(A)$ *is defined.*

Proof. Let $A = [a_{ij}]$. For $i \neq \ell$, expanding along the i'th row of $M_{\ell,j}(A)$ yeilds

$$\det(M_{\ell,j}(A)) = (-1)^i \sum_{k \neq j} a_{ik} \det(M_{\{i,\ell\},\{j,k\}}(A))(-1)^k.$$

Then by using the cofactor representation for the inverse of the matrix $M_{i,j}(A)$, we calculate

$$\begin{aligned}
\mathrm{qdet}_{ij}(A) - a_{ij} &= \sum_{\ell \neq i} \sum_{k \neq j} a_{ik} M_{ij}(A)^{-1}_{k\ell} a_{\ell j} \\
&= \frac{1}{\det M_{ij}(A)} \sum_{\ell \neq i} \sum_{k \neq j} (-1)^{\ell+k} a_{ik} \det(M_{\{i,\ell\},\{j,k\}}) a_{\ell} j \\
&= \frac{1}{\det M_{ij}(A)} (-1)^i \sum_{\ell \neq i} (-1)^{\ell} \det(M_{\ell,j}) a_{\ell j} \\
&= \frac{1}{\det M_{ij}(A)} (-1)^{i+j} [\det(A) - (-1)^{i+j} a_{ij} \det M_{ij}(A)] \\
&= \frac{(-1)^{i+j} \det(A)}{\det M_{ij}(A)} - a_{ij}.
\end{aligned}$$

This proves the proposition. □

One of the most interesting properties of quasi-determinants is that the quasi-determinant of a quasi-determinant is a quasi-determinant. This property is called heredity.

Proposition 4.5.1.4 (Heredity [26]). *Let K be a ring and $S = M_d(K)$ and $T \in M_N(S)$. Then we can view T as both an $N \times N$ matrix over S, as well as an $Nd \times Nd$ matrix over K, and for $1 \leq i, j \leq N$ and $1 \leq a, b \leq d$, we have*

$$\mathrm{qdet}_{ab}(\mathrm{qdet}_{ij}(T; S); K) = \mathrm{qdet}_{(id+a)(jd+b)}(T; K).$$

Example 4.5.1.5. Let $K = M_N(\mathbb{C})$, and consider the matrix $T \in M_2(K)$ defined by

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $D \in K^{\times}$, and consider the quasi-determinant

$$\mathrm{qdet}_{11}(T;K) = A - BD^{-1}C.$$

Heredity combined with knowledge of the value of the quasi-determinant over a commutative ring tells us that

$$\mathrm{qdet}_{ij}(\mathrm{qdet}_{11}(T;K);\mathbb{C}) = \mathrm{qdet}_{ij}(T;\mathbb{C}) = (-1)^{i+j}\frac{\det(T)}{\det(M_{ij}(T))}.$$

Therefore by Cramer's rule

$$\frac{1}{\mathrm{qdet}_{ij}(\mathrm{qdet}_{11}(T;K);\mathbb{C})} = T^{-1}_{ji}$$

where the expression on the right denotes the ji'th index of the inverse of T. On the other hand,

$$\mathrm{qdet}_{ij}(\mathrm{qdet}_{11}(T;K);\mathbb{C}) = (-1)^{i+j}\frac{\det(\mathrm{qdet}_{11}(T;K))}{\det(M_{ij}(\mathrm{qdet}_{11}(T;K)))}$$

so that

$$\frac{1}{\mathrm{qdet}_{ij}(\mathrm{qdet}_{11}(T;K);\mathbb{C})} = \mathrm{qdet}_{11}(T;K)^{-1}_{ji}.$$

This shows that $\mathrm{qdet}_{11}(T;K)^{-1}$ is equal to the upper 2×2 block of T^{-1}. Assuming that the remaining four quasi-determinants also exist, this gives a "block formula" for the inverse of a 2×2 block matrix.

$$T^{-1} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & (C - DB^{-1}A)^{-1} \\ (B - AC^{-1}D)^{-1} & (D - CA^{-1}B)^{-1} \end{pmatrix}.$$

Quasideterminants satisfy many interesting relations, including the following.

Theorem 4.5.1.6 (Generalized homological relations[28][26][27])**.** *Let $I, J \subseteq \{1, 2, \ldots, n\}$ with $|I| = |J| - 1 = r$, and let $A \in M_N(K)$. Recall that for any $j \in J$, $M_{I,J\setminus\{j\}}(A)$ represents the $(N-r)\times(N-r)$ matrix formed by deleting the rows indexed by I and the columns indexed by J. Define an $(N\times r)\times(r+1)$ matrix X by setting*

$$X_{p,j} := \mathrm{qdet}_{pj}(M_{I,J\setminus\{j\}}(A)), \quad j \in J, p \notin I,$$

and let Q be the $(r+1)\times N$ matrix of quasi-determinants

$$Q_{j,\ell} := \frac{1}{\mathrm{qdet}_{\ell,j}(A)}, \quad j \in J.$$

Then

$$(XQ)_{p,\ell} = \delta_{p,\ell}.$$

Lemma 4.5.1.7. *Let A be an $n\times n$ matrix with coefficients in a commutative field, and let $I, J \subseteq \{1,2,\dots,n\}$ with $|I| = |J|$, and assume that $\det(A)$ and $\det(M_{I,J}(A)) \neq 0$. Then*

$$\det(A) = \det(M_{I,J}(A))\det(M_{J',I'}(A^{-1}))^{-1}$$

Proof. First, let $[k] := \{1,2,\dots,k\}$, and set $P = M_{[k]',[k]'}(A)$, $S = M_{[k],[k]}(A)$. Then there exist matrices $Q, R, \widetilde{P}, \widetilde{Q}, \widetilde{R}, \widetilde{S}$ such that

$$A = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} \widetilde{P} & \widetilde{Q} \\ \widetilde{R} & \widetilde{S} \end{pmatrix}.$$

Note that $\widetilde{P} = M_{[k]',[k]'}(A^{-1})$. Assume P is invertible. Then since S is also invertible, we have

$$A = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} I & QS^{-1} \\ RP^{-1} & I \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & S \end{pmatrix},$$

and therefore

$$A^{-1} = \begin{pmatrix} (P - QS^{-1}R)^{-1} & 0 \\ 0 & (S - R^{-1}P^{-1}Q)^{-1} \end{pmatrix} \begin{pmatrix} I & -QS^{-1} \\ -RP^{-1} & I \end{pmatrix}$$

In particular, this shows that $\widetilde{P} = (P - QS^{-1}R)^{-1}$. Since inversion is smooth, we can actually drop the assumption that P is smooth by taking an appropriate limit. Furthermore, we calculate

$$\begin{pmatrix} P & Q \\ R & S \end{pmatrix} \begin{pmatrix} I & 0 \\ -S^{-1}R & I \end{pmatrix} = \begin{pmatrix} P - QS^{-1}R & Q \\ 0 & S \end{pmatrix},$$

and taking determinants tells us

$$\det(A) = \det(P - QS^{-1}R)\det(S) = \det(\widetilde{P})^{-1}\det(S).$$

This proves our theorem in the case $I = J = \{1, 2, \ldots, k\}$.

More generally, let α, β be any two permutations of $\{1, 2, \ldots, n\}$ of order 2, and let T_α, T_β be the associated permutation matrices. Then replacing A with $T_\alpha A T_\beta$, we get

$$\det(T_\alpha A T_\beta) = \det(M_{[k],[k]}(T_\alpha A T_\beta))\det(M_{[k]',[k]'}(T_\beta A^{-1} T_\alpha))^{-1}$$

which simplifies to

$$\det(A) = \det(M_{\alpha([k]),\beta([k])}(A))\det(M_{\beta([k])',\alpha([k])'}(A^{-1}))^{-1}$$

Since transpositions generate all permutations, our formula follows immediately from repeated application of this calculation. □

Corollary 4.5.1.7.1. *Let $A \in M_n(K)$ for $K = M_N(\mathbb{C})$, and for $i, j \in \{1, 2, \ldots, n\}$. Then*

$$\det(A) = \det(\text{qdet}_{ij}(A; K))\det(M_{i,j}(A))$$

where here by $M_{i,j}(A)$ we mean the exclusion of the i'th row and j'th column viewing A as an element of K.

Proof. Set $u = N(i-1), v = N(j-1)$. Let $B := \text{qdet}_{j,i}(A) \in M_N(\mathbb{C})$ and let Q be the matrix whose entries are

$$Q_{p,q} = \frac{1}{\text{qdet}_{q,p}(B; \mathbb{C})}.$$

By heredity of quasi-determinants and our cofactor formula for quasi-determinants over commutative rings , we know

$$\begin{aligned} Q_{p,q} &= \frac{1}{\text{qdet}_{v+q,u+p}(A; \mathbb{C})} \\ &= (-1)^{u+v+p+q}\frac{\det(M_{v+q,u+p}(A))}{\det(A)} = (A^{-1})_{u+p,v+q}. \end{aligned}$$

Furthermore, by the cofactor formula over commutative rings, we know

$$Q_{p,q} = \frac{(-1)^{p+q}\det(M_{q,p}(B))}{\det(B)} = (B^{-1})_{p,q}$$

Thus we see that the entry of i,j'th entry of A^{-1} (viewed as an element of $M_n(K)$) is B^{-1}. Combining this with the result of the previous lemma, the statement of the corollary follows immediately. □

4.5.2 The Twisted Tau Function

For any $n \geq 0$, let $\overline{\mathbb{V}}_n$ be the subspace of $\overline{\mathbb{V}}$ consisting of polynomials of degree at most $n-1$, with coefficients in $\overline{\mathbb{V}}$. Let $P_n : \overline{\mathbb{V}} \to \overline{\mathbb{V}}_n$ be the truncation map, which in particular is a projection map, and let $P_n^{\perp} : \overline{\mathbb{V}} \to \overline{\mathbb{V}}_n^{\perp}$ represent the projection onto the orthogonal space $\overline{\mathbb{V}}_n^{\perp} := z^n\overline{\mathbb{V}}$, and let $V_n = \overline{\mathbb{V}}_n \cap \overline{\mathbb{V}}_{n-1}^{\perp} \cong V$ with $V_0 := V$.

Given any linear operator $R : \overline{\mathbb{V}}_+ \to \overline{\mathbb{V}}_+$, the restriction $R^{(n)}$ of R to $\overline{\mathbb{V}}_n^{\perp}$ decomposes as

$$R^{(n)} = \begin{pmatrix} R_{11}^{(n)} & R_{12}^{(n)} \\ R_{21}^{(n)} & R_{22}^{(n)} \end{pmatrix}$$

where $R_{11}^{(n)} : V_n^{\rightarrow} V_n$, $R_{12}^{(n)} : \overline{\mathbb{V}}_n^{\perp} \to V_n$, $R_{21}^{(n)} : V_n \to \overline{\mathbb{V}}_n^{\perp}$ and $R_{22}^{(n)} : \overline{\mathbb{V}}_n^{\perp} \to \overline{\mathbb{V}}_n^{\perp}$.

Definition 4.5.2.1. Let $R : \overline{\mathbb{V}}_+ \to \overline{\mathbb{V}}_+$. Assuming $R_{22}^{(n)}$ is invertible, we define the **quasi-determinant**

$$\operatorname{qdet}(R^{(n)}) := R_{11}^{(n)} - R_{12}^{(n)}(R_{22}^{(n)})^{-1}R_{21}^{(n)}.$$

Furthermore, we define the **matrix-valued determinant** to be the infinite product

$$\operatorname{mdet}(R^{(n)}) = \prod_{j=n}^{\infty} \operatorname{qdet}(R^{(j)}) := \operatorname{qdet}(R^{(n)})\operatorname{qdet}(R^{(n+1)})\operatorname{qdet}(R^{(n+2)})\ldots,$$

assuming the each of the terms in the product on the right hand side is defined and that the limit exists. In the special case $n = 0$, we will write $\operatorname{qdet}(R)$ instead of $\operatorname{qdet}(R^{(n)})$.

Remark 4.5.2.2. Proving the convergence of the above limit in generality is outside the scope of this document. Suffice it to say that in the cases we consider there will exist an n_0 such that $\mathrm{qdet}(R^{(n)}) = I$ for all $n \geq n_0$, so that the limit will exist trivially.

Definition 4.5.2.3. Let $H \in \overline{\mathrm{Gr}}(0)_+$, $g \in \Gamma_H$, and $h \in \Gamma_+$. We define the **matrix tau function** of H to be

$$\mathcal{T}_H(g) := \mathrm{mdet}(R(H,g)) = \mathrm{qdet}(R(H,g))\mathrm{mdet}(R(H,g)^{(1)}).$$

whenever the matrix determinant on the right hand side is defined. More generally, we define the h-**twisted matrix tau function** of H to be

$$\mathcal{T}_H^h(g) := \mathrm{mdet}(R^h(H,g)) = \mathrm{qdet}(R^h(H,g))\mathrm{mdet}(M(R^h(H,g))^{(1)}),$$

where here $R^h(H,g) = L_h^{-1}R(H,g)L_h$.

For any $y \in \mathbb{C}$ with $|y| > 1$, let $\rho_y = 1 - z/y$.

Lemma 4.5.2.4. *Let $H \in \overline{\mathrm{Gr}}(R,0)_+$ be analytic, and choose $y \in \mathbb{C}$ with $|y| > 1$. Then*

$$\mathcal{T}_W(\rho(y)) = 1 + w(y)$$

for $w(z) \in z^{-1}K[[z^{-1}]]$ the unique element satisfying $\pi_+|_H^{-1}(v) = v(1 + w(z))$ for all $v \in V$.

Proof. Using the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, we may decompose $L_{\rho(y,z)}^{-1}$ and $L_{\rho(y,z)}$ as

$$L_{\rho(y,z)}^{-1} = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \text{ and } L_{\rho(y,z)} = \begin{pmatrix} a^{-1} & -a^{-1}bd^{-1} \\ 0 & d^{-1} \end{pmatrix}$$

for some $a : \mathcal{H}_+ \to \mathcal{H}_+$, $b : \mathcal{H}_- \to \mathcal{H}_+$, and $d : \mathcal{H}_- \to \mathcal{H}_-$, where we have used the fact that $\rho(y,z)^{-1}\mathcal{H}_+ \subseteq \mathcal{H}_+$.

For each $n \geq 0$, let $w(z;n) \in K[[z^{-1}]]$ be the unique element satisfying

$$v(z^n + w(z;n)) = \pi_+|_H^{-1}(vz^n) = \iota_H(vz^n)$$

for all $v \in V$. Then for all $v \in V$ we have

$$R(H, \rho(y,z))(vz^n) = vz^n + a^{-1}b(v\vec{w}(z;n))).$$

Now for $n \geq 0$ we calculate

$$\rho(y,z)^{-1}(vz^{-n}) = v\sum_{j=0}^{\infty}\frac{z^{j-n}}{y^j},$$

and therefore

$$b(vz^{-n}) = vy^{-n}\sum_{j=n}^{\infty}\frac{z^{j-n}}{y^{j-n}} = \rho(y,z)^{-1}vy^{-n}.$$

Thus $a^{-1}b(vz^{-n}) = \rho(y,z)b(vz^{-n}) = vy^{-n}$. It follows that $a^{-1}b(vw(z;n)) = vw(y;n)$, and consequently $R(H, \rho(y,z))$ decomposes with respect to $\overline{\mathbb{V}}_0 \oplus \overline{\mathbb{V}}_0^{\perp}$ as

$$R(H, \rho(y,z)) = \begin{pmatrix} I + w(y;0) & * \\ 0 & \mathrm{id}_{\overline{\mathbb{V}}_0^{\perp}} \end{pmatrix},$$

from which we calculate $\mathcal{T}_W(\rho(y,z)) = I + w(y;0)$. Since $w(y;0) = w(y)$, this completes the proof. □

Lemma 4.5.2.5. *Let $H \in \overline{\mathrm{Gr}}(0)_+$. Then the Baker-Alkheizer function of H is given by*

$$\psi_H(g,y) = \mathcal{T}_{Hg^{-1}}(\rho_y).$$

Proof. From our previous calculation, we have $\mathcal{T}_{Hg^{-1}}(\rho_y) = 1 + w(g,y)$ for $w(g,z)$ the unique element of $z^{-1}K[[z^{-1}]]$ satisfying $v(1 + w(g,z)) = \iota_{Hg^{-1}}(v)$. This shows that $f(g,y) := \mathcal{T}_{Hg^{-1}}(\rho_y)g(y)$ satisfies $f(g,z) = v(1 + w(g,z))g(z) \in H$ satisfies the definition of the Baker-Alkheizer function for H. □

Theorem 4.5.2.6. *Let $H \in \overline{\mathrm{Gr}}(0)_+$. Then the Baker-Alkheizer function of H is given by*

$$\psi_H(g,y) = \mathcal{T}_H^g(g\rho_y)\mathcal{T}_H^g(g)^{-1}g(y).$$

Proof. We have that

$$R(H, g\rho_y) = L_g R(Hg^{-1}, \rho_y) L_g^{-1} R(H, g),$$

from which we find

$$L_g^{-1} R(H, g\rho_y) L_g = R(Hg^{-1}, \rho_y) L_g^{-1} R(H, g) L_g,$$

or equivalently

$$R^g(H, g\rho_y) = R(Hg^{-1}, \rho_y) R^g(H, g).$$

Now we have to be careful, because the multiplicative properties that we saw with tau functions don't extend directly to matrix tau functions. However, for any $h \in \Gamma_+$ we may decompose $R^g(H, h)$ with respect to $\overline{\mathbb{V}}_0 \oplus \overline{\mathbb{V}}_0^\perp$ as

$$R^g(H, h) := \begin{pmatrix} a(h) & b(h) \\ c(h) & d(h) \end{pmatrix}.$$

In particular, this gives

$$\mathcal{T}_H^g(h) = (a(h) - b(h)d(h)^{-1}c(h))\mathrm{mdet}(d(h)).$$

Furthermore, for some $e : \overline{\mathbb{V}}_0^\perp \to \overline{\mathbb{V}}_0$ and $\kappa = \mathcal{T}_{Hg^{-1}}(\rho_y) \in K$ we have

$$R(Hg^{-1}, \rho_y) = \begin{pmatrix} \kappa & e \\ 0 & \mathrm{id}_{\overline{\mathbb{V}}_0^\perp} \end{pmatrix}.$$

From this, we calculate

$$R^g(H, g\rho_y) = \begin{pmatrix} \kappa & e \\ 0 & \mathrm{id}_{\overline{\mathbb{V}}_0^\perp} \end{pmatrix} \begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix} = \begin{pmatrix} \kappa a(g) + ec(g) & \kappa b(g) + ed(g) \\ c(g) & d(g) \end{pmatrix},$$

and therefore

$$\begin{aligned} \mathcal{T}_H^g(g\rho_y) &= (\kappa a(g) + ec(g) - (\kappa b(g) + ed(g))d(g)^{-1}c(g))\mathrm{mdet}(d(g)) \\ &= \kappa(a(g) - b(g)d(g)^{-1}c(g))\mathrm{mdet}(d(g)) \\ &= \mathcal{T}_{Hg^{-1}}(\rho_y)\mathcal{T}_H^g(g). \end{aligned}$$

Consequently,

$$\mathcal{T}_H^g(g\rho_y)\mathcal{T}_H^g(g)^{-1} = \mathcal{T}_{Hg^{-1}}(\rho_y),$$

and from this the statement of the theorem follows immediately. □

Remark 4.5.2.7. Note that the calcuation above actually shows

$$\psi_H(g, y) = \mathrm{qdet}(R^g(H, g\rho_y))\mathrm{qdet}(R^g(H, g))^{-1}g(y),$$

and so we don't actually need the infinite product described in the definition of the matrix tau function to exist. We only need the first term in the product! However, we use the infinite product in our definition of the matrix tau function, as it feels more natural than just taking the first term.

Corollary 4.5.2.7.1. *Fix a normalized, monic matrix differential operator* $\mathfrak{d}$ *of order* ℓ*. Let* $\mathfrak{v}$ *be a pseudo-differential operator satisfying* $\mathfrak{v}^{-1}\mathfrak{d}\mathfrak{v} = \partial_x^\ell$*, and let* $H \in \mathrm{Gr}$ *be the associated point in the Sato Grassmannian. Suppose that* H *is analytic (ie.* $\overline{H} \in \overline{\mathrm{Gr}}$*). Then a stationary Baker-Alkheizer function* $\psi_H(x, y)$ *of* H *is given by*

$$\psi_H(x, y) = e^{xy}\mathcal{T}_H^{g(x)}(x - 1/y, -y^{-2}/2, -y^{-3}/3, \dots)\mathcal{T}_H^{g(x)}(x)^{-1}.$$

Proof. This follows immediately from the previous result and the observation that $g(\vec{t})\rho_y = g(\vec{t} - [y^{-1}])$, with the usual notation $[z] = (z, z^2/2, z^3/3, \dots)$. □

Theorem 4.5.2.8. *The matrix tau function satisfies the property*

$$\det(\mathcal{T}_H^h(g)) = \det(R(H, g)) = \tau_H(g).$$

In particular, when $N = 1$ *the matrix tau function agrees with the Sato-Segal-Wilson tau function on its domain of definition.*

Proof. For any integer $n \geq 0$, let $\overline{\mathbb{V}}_n$ be the subspace of $\overline{\mathbb{V}}_+$ consisting of polynomials of degree at most n with coefficients in K. Let $P_n : \overline{\mathbb{V}}_+ \to \overline{\mathbb{V}}_n$ be the projection map, and set $R_n^h(H, g) = \mathrm{id} + P_n R^h(H, g) P_n$. Then for all n,

$$\det(R_n^h(H, g)) = \det(\mathrm{qdet}_{11}(R_n^h(H, g)))\det(M_{1,1}(R_n^h(H, g)))$$

and therefore taking a limit gives

$$\begin{aligned}\det(R^h(H,g)) &= \det(\mathrm{qdet}(R^h(H,g)))\det(M_{1,1}(R^h(H,g))) \\ &= \det(\mathrm{qdet}(R^h(H,g)))\det(R^h(H,g)^{(1)})\end{aligned}$$

Using the same argument, one may show

$$\det(R^h(H,g)^{(n)}) = \det(\mathrm{qdet}(R^h(H,g)^{(n)}))\det(R^h(H,g)^{(n+1)}).$$

and therefore

$$\det(R^h(H,g)) = \prod_{j=0}^{n}\det(\mathrm{qdet}(R^h(H,g)^{(n)}))\det(R^h(H,g)^{(n+1)}).$$

Since $R^h(H,g)^{(n)} \to \mathrm{id}$, we obtain

$$\det(R^h(H,g)) = \prod_{j=0}^{\infty}\det(\mathrm{qdet}(R^h(H,g)^{(n)})) = \det(\mathrm{mdet}(R^h(H,g))) = \det(\mathcal{T}_H^h(g)).$$

Furthermore, since $R^h(H,g)$ differs from $R(H,g)$ by conjugation by L_h, $\det(R^h(H,g)) = \det(R(H,g))$. This completes the proof. □

4.6 Examples

4.6.1 A first example

Consider the point $\overline{W} \in \overline{\mathrm{Gr}}$ determined by the analytic $W \in \mathrm{Gr}$ defined by

$$W = Vp(z)(z-r)^{-1} \oplus Vz\mathbb{C}[z],$$

where $V = (\mathbb{C}^N)^T$, $p(z) = zI + b$ for some fixed matrix $b \in M_N(\mathbb{C})$ and some $r \in \mathbb{C}$ with $|r| < 1$. We calculate

$$\begin{aligned}\frac{p(z)}{z-r} &= p'(r) + \frac{p(r)}{z-r} \\ &= I + p(r)z^{-1}(1 + rz^{-1} + r^2z^{-2} + \dots).\end{aligned}$$

Therefore for all $v \in V$, we have $\iota_W(vz^k) = vz^k$ for $k > 0$ and

$$\iota_W(v) = v\left(I + \frac{p(r)}{z-r}\right).$$

We calcualte $R^h(W,g)(vz^j) = vz^j$ for all $j > 0$ and therefore writing

$$R^h(W,g)(v) = vR^h_{00}(W,g) + zvR^h_{10}(W,g) + \mathcal{O}(z^2),$$

it follows that relative to the canonical basis for $\overline{\mathbb{V}}_+$

$$R^h(W,g) = \begin{pmatrix} R^h_{00}(W,g) & 0I & 0I & 0I & \dots \\ * & I & 0I & 0I & \dots \\ * & 0I & I & 0I & \dots \\ * & 0I & 0I & I & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and therefore $\mathcal{T}^h_W(g) = R^h_{00}(W,g)$. We calculate

$$R(W,g)(v) = v\left(I + p(r)\frac{1 - g(z)g(r)^{-1}}{z-r}\right)$$

and therefore for $r \neq 0$

$$\mathcal{T}^h_W(g) = R^h_{00}(W,g) = R_{00}(W,g) = I - p(r)\frac{g(r) - g(0)}{rg(r)},$$

and for $r = 0$

$$\mathcal{T}^h_W(g) = R^h_{00}(W,g) = R_{00}(W,g) = I - p(0)\frac{g'(0)}{g(0)}.$$

This also gives us the Baker-Alkheizer function for $r \neq 0$ as

$$\psi_W(g;y) = \left(-b\frac{g(r)}{g(0)} + \frac{p(r)}{(1 - r/y)}\right)\left(-b\frac{g(r)}{g(0)} + p(r))\right)^{-1} g(y)$$

and for $r = 0$ as

$$\psi_W(g;y) = \left(I - b\frac{g'(0) - g(0)/y}{g(0)}\right)\left(I - b\frac{g'(0)}{g(0)}\right)^{-1} g(y).$$

In particular, we obtain stationary Baker-Alkheizer functions for $r \neq 0$

$$\psi_W(x,y) = \left(I + \frac{1}{y-r}\frac{rp(r)}{p(r)-be^{xr}}\right)e^{xy}$$

and for $r = 0$

$$\psi_W(x,y) = \left(I + \frac{1}{y}\frac{b}{I-bx}\right)e^{xy}.$$

For example, if $r = 0$ and $b = \begin{pmatrix} 1/b_1 & 1/b_1b_2 \\ 0 & 1/b_2 \end{pmatrix}$ then we have a stationary Baker-Alkheizer function

$$\psi_W(x,y) = \left[I - \frac{1}{y}\begin{pmatrix} (x-b_1)^{-1} & -(x-b_1)^{-1}(x-b_2)^{-1} \\ 0 & (x-b_2)^{-1} \end{pmatrix}\right]e^{xy}.$$

4.6.2 A second example

Consider the point $\overline{W} \in \overline{\mathrm{Gr}}$ determined by the analytic $W \in \mathrm{Gr}$ defined by

$$W = V \oplus Vp(z)(z-r)^{-1} \oplus Vz^2\mathbb{C}[z],$$

where $V = (\mathbb{C}^N)^T$, $p(z) = z^2I + az + b$ for some fixed matrices $a, b \in M_N(\mathbb{C})$ and some $r \in \mathbb{C}$ with $|r| < 1$. We calculate

$$\begin{aligned}\frac{p(z)}{z-r} &= \frac{1}{2}p''(r)(z-r) + p'(r) + \frac{p(r)}{z-r} \\ &= (z-r)I + p'(r) + p(r)z^{-1}(1 + rz^{-1} + r^2z^{-2} + \dots).\end{aligned}$$

Hence for all $v \in V$, we have $\iota_W(vz^k) = vz^k$ for $k \geq 0$ with $k \neq 1$, and

$$\iota_W(vz) = vz + v\frac{p(r)}{z-r}.$$

Since $h(z)vz^k \in \overline{W}$ for $k \geq 2$, we have that $\iota_W(hvz^k) = hvz^k$ and therefore

$$R^h(W,g)(vz^k) = h^{-1}g\pi(g^{-1}\iota_W(hvz^k)) = h^{-1}g\pi(g^{-1}hvz^k) = h^{-1}g(g^{-1}hvz^k) = \vec{v}z^k.$$

Writing

$$R^h(W,g)(v) = vR^h_{00}(W,g) + zvR^h_{10}(W,g) + \mathcal{O}(z^2),$$

$$R^h(W,g)(vz) = vR^h_{01}(W,g) + zvR^h_{11}(W,g) + \mathcal{O}(z^2),$$

for some matrices $R^h_{ij}(W,g) \in M_N(\mathbb{C})$, this tells us that in terms of the canonical ordered basis for $\overline{\mathbb{V}}_+$

$$R^h(W,g) = \begin{pmatrix} R^h_{00}(W,g) & R^h_{01}(W,g) & 0I & 0I & \dots \\ R^h_{10}(W,g) & R^h_{11}(W,g) & 0I & 0I & \dots \\ * & * & I & 0I & \dots \\ * & * & * & I & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

from which we see

$$\mathrm{qdet}(R^h(W,g)) = R^h_{00}(W,g) - R^h_{01}(W,g)R^h_{11}(W,g)^{-1}R^h_{10}(W,g)$$

and therefore

$$\mathcal{T}^h_W(g) = (R^h_{00}(W,g) - R^h_{01}(W,g)R^h_{11}(W,g)^{-1}R^h_{10}(W,g))R^h_{11}(W,g)).$$

Furthermore,

$$\begin{pmatrix} R^h_{00}(W,g) & R^h_{01}(W,g) \\ R^h_{10}(W,g) & R^h_{11}(W,g) \end{pmatrix} = \begin{pmatrix} I & 0I \\ h_1 I & I \end{pmatrix} \begin{pmatrix} R_{00}(W,g) & R_{01}(W,g) \\ R_{10}(W,g) & R_{11}(W,g) \end{pmatrix} \begin{pmatrix} I & 0I \\ -h_1 I & I \end{pmatrix},$$

for $h_1 := h'(0)/h(0)$ and $R_{ij}(W,g) := R^1_{ij}(W,g)$. We calculate

$$R(W,g)(v) = v,$$

for all $v \in V$, and therefore $R_{00} = I$ and $R_{10} = 0I$. Moreover,

$$\begin{aligned} R(W,g)(vz) &= g(z)\pi\left(g(z)^{-1}\left(vz + v\frac{p(r)}{z-r}\right)\right) \\ &= g(z)\pi\left(vg(z)^{-1}z + vp(r)\frac{g(z)^{-1}-g(r)^{-1}}{z-r} + vp(r)\frac{g(r)^{-1}}{z-r}\right) \\ &= v\left(zI + p(r)\frac{1-g(z)g(r)^{-1}}{z-r}\right), \end{aligned}$$

So that

$$\begin{aligned} R_{01}(W,g) &= p(r)\frac{g(0)g(r)^{-1}-1}{r} \\ R_{11}(W,g) &= I + \frac{p(r)}{g(r)}\frac{rg'(0)+g(0)-g(r)}{r^2} \end{aligned}$$

Hence

$$\mathcal{T}_W^h(g) = I + \frac{p(r)}{g(r)}\frac{rg'(0)+g(0)-g(r)}{r^2}$$

and a Baker-Alkheizer function

$$\psi_W(g) = \left(I + \frac{p(r)}{g(r)(1-r/y)}\frac{rg'(0)+(g(0)-g(r))(1-r/y)}{r^2}\right)\left(I + \frac{p(r)}{g(r)}\frac{rg'(0)+g(0)-g(r)}{r^2}\right)^{-1}$$

Note in particular that if r is a root of $p(z)$, then $\mathcal{T}_W^h(g) = I$, which makes sense since in this case $\overline{W} = \overline{\mathbb{V}}_+$. This also gives us the Baker-Alkheizer function

Chapter 5

BISPECTRAL DIFFERENTIAL OPERATOR ALGEBRAS

In this section, we explore the structure of bispectral algebras of differential operators with matrix-valued coefficients. We prove several structure results for bispectral differential operator algebras of rank 1.

For this section fix two simply connected, open subsets $U, V \subseteq \mathbb{C}$ and differential rings $R = M_N(Holo(U)), \widetilde{R} = M_N(Holo(V))$, with ring of constants $K = \widetilde{K} = M_N(\mathbb{C})$. Also fix a bispectral setup (A, B, M) for $A = \mathfrak{D}(R)$, $B = \mathfrak{D}(\widetilde{R})^{op}$ and M the A, B-bimodule $M_N(Holo(U \times V))$. For simplicity, we will use ∂_x to refer to the derivative of R and ∂_y to refer to the derivative of $\widetilde{R}$. We will also fix smooth points I and $\widetilde{I}$ of R and $\widetilde{R}$, respectively. For any $r \in R$, we will write $r(x)$ to denote the image of r in $K[[x]]$ under the map $R \to K[[x]]$ induced by I. Similarly, for $\widetilde{r} \in \widetilde{R}$, we will write $\widetilde{r}(y)$ to denote the image of $\widetilde{r}$ in $\widetilde{K}[[y]]$.

5.1 Rank 1 *Bispectral Operator Algebras*

5.1.1 The Ad-Condition

Most structure theorems for algebras of bispectral differential operators with commutative coefficients rely on the fact that bispectral operators satisfy a certain Ad-condition, attributed by Grünbaum to Duistermaat [32]. In particular if $(\mathfrak{d}, \mathfrak{b}, \psi)$ is a bispectral triple with

$$\mathfrak{d} \cdot \psi = \psi g \quad \text{and} \quad \psi \cdot \mathfrak{b} = f\psi,$$

then

$$(\mathfrak{d}f - f\mathfrak{d})\psi = \mathfrak{d}f\psi - f\mathfrak{d}\psi = \psi(g\mathfrak{d} - \mathfrak{d}g),$$

and more generally for all $m \geq 1$

$$\text{Ad}_{\mathfrak{d}}^m(f)\psi = \psi \text{Ad}_g(\mathfrak{b}), \quad \text{and} \quad \text{Ad}_f^m(\mathfrak{d})\psi = \psi \text{Ad}_{\mathfrak{b}}(g).$$

For differential operators, the order of $\mathrm{Ad}_f^m(\mathfrak{d})$ is at most $d-m$ for d the order of $\mathfrak{d}$. Therefore $\mathrm{Ad}_f^{d+1}(\mathfrak{d}) = 0$, and (assuming that ψ has trivial left and right annihilators) this implies $\mathrm{Ad}_{\mathfrak{b}}^{d+1}(g) = 0$. This latter equation strongly resticts the coefficients of $\mathfrak{b}$. It is this Ad-condition which is used in several papers to derive and classify bispectral triples [42][70][15].

In the noncommutative case, we can establish a similar Ad-condition. However, there is a bit of a hitch coming from the fact that $\mathrm{Ad}_f^m(\mathfrak{d})$ could still potentially have the same order as $\mathfrak{d}$, due to interactions between the noncommutative coefficients. To get around this, we will restrict our attention to bispectral algebras which include bispectral triples which are in some sense regular. Note that this isn't really that dire a restriction for rank-1 bispectral differential operator algebras, since these *must* include such a triple. With this in mind, we provide the following definition.

Definition 5.1.1.1. Let $(\mathfrak{d}, \mathfrak{b}, \psi)$ be a bispectral triple for the bispectral context (A, B, M). We call $(\mathfrak{d}, \mathfrak{b}, \psi)$ **normalized** if $\mathfrak{d}$ and $\mathfrak{b}$ are both monic with submaximal coefficient 0. We call $(\mathfrak{d}, \mathfrak{b}, \psi)$ **regular** if there exist nonconstant functions $f \in Z(R)$ and $g \in Z(\widetilde{R})$ such that

$$\mathfrak{d} \cdot \psi = \psi g \quad \text{and} \quad \psi \cdot \mathfrak{b} = f\psi.$$

Lemma 5.1.1.2. *Let $\mathfrak{d} \in \mathfrak{D}(R)$ be a differential operator of order d. Then for any $f \in Z(R)$, the order of $\mathrm{Ad}_f^m(\mathfrak{d})$ is at most $d - m$. In particular*

$$\mathrm{Ad}_f^{d+1}(\mathfrak{d}) = 0.$$

Proof. Let $\mathfrak{q} \in \mathfrak{D}(R)$ be an operator of order ℓ. Then we may write

$$\mathfrak{q} = q_\ell \partial_x^\ell + \mathfrak{q}_1,$$

for some $q_\ell \in R$ and $\mathfrak{q}_1 \in \mathfrak{D}(R)$. Using this, we calculate

$$\begin{aligned}
\mathrm{Ad}_f(\mathfrak{q}) &= f q_\ell \partial_x^\ell - q_\ell \partial_x^\ell f + f\mathfrak{q}_1 - \mathfrak{q}_1 f \\
&= f q_\ell \partial_x^\ell - q_\ell f \partial_x^\ell - \sum_{n=1}^{\ell} \binom{\ell}{n} f^{(n)} \partial_x^{\ell-n} + f\mathfrak{q}_1 - \mathfrak{q}_1 f \\
&= -\sum_{n=1}^{\ell} \binom{\ell}{n} f^{(n)} \partial_x^{\ell-n} + f\mathfrak{q}_1 - \mathfrak{q}_1 f,
\end{aligned}$$

From this we see $\mathrm{Ad}_f(\mathfrak{q})$ has order at most $\ell - 1$. Then since $\mathrm{Ad}_f^m(\mathfrak{d}) = \mathrm{Ad}_f(\mathrm{Ad}_f^{m-1}(\mathfrak{d}))$, the statement of the lemma follows from a simple inductive argument. □

Lemma 5.1.1.3 (Ad-condition)**.** *Let $(\mathfrak{d}, \mathfrak{b}, \psi)$ be a normalized, regular bispectral triple. Then if $g = \Lambda_L(\mathfrak{p})$ for some operator $\mathfrak{p} \in \mathrm{Bis}_L(\psi)$ of degree ℓ and $f = \Lambda_R(\mathfrak{q})$ for some operator $q \in \mathrm{Bis}_R(\psi)$ of degree m we have*

$$\mathrm{Ad}_{\mathfrak{d}}^{m+1}(f) = 0, \quad \textit{and} \quad \mathrm{Ad}_{\mathfrak{b}}^{\ell+1}(g) = 0.$$

Proof. Let $f_0 \in Z(R)$ an $g_0 \in Z(\widetilde{R})$ with $\Lambda_L(\mathfrak{d}) = g_0$ and $\Lambda_R(\mathfrak{b}) = f_0$. By bispectrality and the previous lemma, we calculate

$$0 = \mathrm{Ad}_{f_0}^{\ell+1}(\mathfrak{p})\psi = \psi \mathrm{Ad}_{\mathfrak{b}}^{\ell+1}(g).$$

Since ψ has trivial right annihilator, this implies $\mathrm{Ad}_{\mathfrak{b}}^{\ell+1}(g) = 0$. The fact that $\mathrm{Ad}_{\mathfrak{d}}^{m+1}(f) = 0$ is proved similarly. □

Lemma 5.1.1.4. *Let $(\mathfrak{d}, \mathfrak{b}, \psi)$ be a normalized, regular bispectral triple. Then if $g = \Lambda_L(\mathfrak{p})$ for some operator $\mathfrak{p} \in \mathrm{Bis}_L(\psi)$ of degree ℓ and $f = \Lambda_R(\mathfrak{q})$ for some operator $q \in \mathrm{Bis}_R(\psi)$ of degree m we have Then $f(x) \in K[x]$ is a polynomial of degree m and $g(y) \in K[y]$ is a polymomial of degree ℓ. In other words, the eigenvalue homomorphisms send differential operators to polynomials with degree at most the order of the operators.*

Proof. Choose wave operators $\mathfrak{v} \in \mathfrak{P}(R)$ and $\mathfrak{w} \in \mathfrak{P}(\widetilde{R})^{op}$ satisfying $\mathfrak{v}^{-1}\mathfrak{d}\mathfrak{v} = \partial_x^\ell$ and $\mathfrak{w}\mathfrak{b}\mathfrak{w}^{-1} = \partial_y^m$. Then by the previous lemma $\mathrm{Ad}_{\mathfrak{d}}^{m+1}(f) = 0$, and conjugating by $\mathfrak{v}$ gives

$$0 = \mathrm{Ad}_{\partial_x^\ell}^{m+1}(\mathfrak{v}^{-1} f \mathfrak{v}).$$

Note that $\mathfrak{v}^{-1} f \mathfrak{v}$ is a pseudo-differential operator of order 0. Choose $r_0(\partial^{-1}), r_1(\partial^{-1}), \dots \in K[[\partial^{-1}]]$ such that

$$\mathfrak{v}^{-1} f \mathfrak{v} = \sum_{n=0}^{\infty} x^n r_n(\partial^{-1}) \in \mathfrak{P}(K[[x]]).$$

Let $r_n(\partial^{-1}) = \sum_{j=0}^{\infty} r_{nj}\partial^{-j}$. Note that by comparing the coefficents of order 0, $f(x) = \sum_{n=0}^{\infty} x^n r_0$.

We calculate

$$\mathrm{Ad}^j_{\partial_x^\ell}(\mathfrak{v}^{-1} f \mathfrak{v}) = \sum_{n=0}^{\infty} \mathrm{Ad}^j_{\partial_x^\ell}(x^n) r_n(\partial^{-1}).$$

Since $\mathrm{Ad}^j_{\partial_x^\ell}(x^n)$ is a non-zero divisor in $\mathfrak{D}(R)$ for $n \geq j$, the condition that $\mathrm{Ad}^{m+1}_{\partial_x^\ell}(\mathfrak{v}^{-1} f \mathfrak{v}) = 0$ then implies that $r_n(\partial^{-1}) = 0$ for all $n > m$. Consequently $f(x)$ is a polynomial of degree at most m. The proof for $g(y)$ is similar. □

Lemma 5.1.1.5. *Let $(\mathfrak{d}, \mathfrak{b}, \psi)$ be a normalized, regular bispectral triple in the bispectral context (A, B, M), and let $\mathfrak{v} \in \mathfrak{P}(R)$ and $\mathfrak{w} \in \mathfrak{P}(\widetilde{R})^{op}$ be a wave operators satisfying*

$$\mathfrak{v}^{-1}\mathfrak{d}\mathfrak{v} = g_0(\partial_x), \quad \mathfrak{w}\mathfrak{b}\mathfrak{w}^{-1} = f_0(\partial_y),$$

for some polynomials f_0, g_0. Then the coefficients of $\mathfrak{v}$ are rational functions of x, and the coefficients of $\mathfrak{w}$ are rational functions of y.

Proof. Let $\mathfrak{v} = I + \sum_{n=1}^{\infty} v_n(x)\partial_x^{-n}$. Then the Ad-condition tells us that $\mathrm{Ad}^{\ell+1}_{\mathfrak{d}}(f(x)) = 0$. Conjugating by $\mathfrak{v}$, this says $\mathrm{Ad}^{\ell+1}_{g_0(\partial_x)}(\mathfrak{v}^{-1} f(x) \mathfrak{v}) = 0$. Writing

$$\mathfrak{v}^{-1} f(x) \mathfrak{v} = \sum_{m=0}^{\infty} w_m(x)\partial_x^{-m},$$

for some $w_0(x), w_1(x), \cdots \in M_N(\mathcal{H}(U))$, the vanishing adjoint implies that $w_m(x)$ is a polynomial of degree at most ℓ for all $m \geq 0$ (by the argument of the previous lemma).

Multiplying both sides by $\mathfrak{v}$ and expanding, we then obtain

$$f(x) + \sum_{n=1}^{\infty} f(x)v_n(x)\partial_x^{-n} = w_0(x) + \sum_{m=1}^{\infty} w_m(x)\partial_x^{-m} + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} v_n(x)\partial_x^{-n} w_m(x)\partial_x^{-m}.$$

Using Leibniz rule, we calculate

$$\partial_x^{-n} w_m(x) = \sum_{j=0}^{\infty} \binom{-n}{j} w_m^{(j)} \partial_x^{-n-j} = \sum_{j=0}^{\ell} \binom{-n}{j} w_m^{(j)} \partial_x^{-n-j},$$

where we have used the fact that $w_m(x)$ has degree at most ℓ. Inserting this in the previous sum expression, we find

$$f(x) + \sum_{n=1}^{\infty} f(x)v_n(x)\partial_x^{-n} = w_0(x) + \sum_{m=1}^{\infty} w_m(x)\partial_x^{-m} + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{j=0}^{\ell} \binom{-n}{j} v_n(x) w_m^{(j)} \partial_x^{-n-m-j}.$$

Comparing coefficients of similar powers of ∂_x on both sides, we see that $f(x) = w_0(x)$, $v_1(x) = 0$ and that v_{k-1} is defined in terms of $v_0, \ldots, v_{k-2}$ for $k > 2$ by

$$(k-1)v_{k-1}(x)w_0'(x) = \sum_{n=1}^{k-2}\sum_{m=0}^{k-n}\binom{-n}{k-n-m}v_n(x)w_m^{(k-n-m)}(x).$$

This recursion relation, along with the fact that $w_0'(x) = f'(x)$ is a nonzero polynomial implies that $v_k(x)$ is rational for all k. The same proof works for $\mathfrak{w}$ mutatis mutandis. $\square$

5.1.2 Rank 1 Bispectral Algebras

Consider a bispectral triple $(\mathfrak{d}, \mathfrak{b}, \psi)$. From the previous lemmas, we can say *some* things about the structure of the bispectral algebras $\text{Bis}_L(\psi)$ and $\text{Bis}_R(\psi)$. For example, the map sending a value in $\text{Bis}_L(\psi)$ to its function of eigenvalues embeds $\text{Bis}_L(\psi)$ into the polynomial ring $K[y]$ sending $\mathfrak{d}$ to an element in $k[y]$. Similarly, we obtain an embedding $\text{Bis}_R(\psi)$ into the polynomial ring $K[x]$ sending $\mathfrak{b}$ to an element in $k[x]$. It follows from this that $\text{Bis}_L(\psi)$ is contained in the centralizer $C(\mathfrak{d})$ of $\mathfrak{d}$ in A, and similarly $\text{Bis}_R(\psi)$ is contained in the $C(\mathfrak{b})$ of $\mathfrak{b} \in B$.

Definition 5.1.2.1. Let $\psi \in M$. We call the left bispectral algebra $\text{Bis}_L(\psi)$ **rank** r if there exists a normalized, regular bispectral triple $(\mathfrak{d}, \mathfrak{b}, \psi)$ and a wave operator $\mathfrak{v} \in \mathfrak{P}(R)$ such that $\mathfrak{v}^{-1}\mathfrak{d}\mathfrak{v} \in k((\partial^{-1}))$ and the Schur pair $(W_{\mathfrak{v},I}, A_{\mathfrak{v},I})$ is rank r. We call $\text{Bis}_L(\psi)$ **full of rank** 1 if there exists an integer $n_0 > 0$ and such that for all $n \geq n$ and all $c \in K$ the algebra $\text{Bis}_L(\psi)$ contains an operator of order n with leading coefficient c. Similar definitions are taken for $\text{Bis}_R(\psi)$ similarly.

Remark 5.1.2.2. Note that $A_{\mathfrak{v},I}$ is isomorphic to $C(\mathfrak{d})$, so the choice of wave operator $\mathfrak{v}$ is unimportant. Also, since $\mathfrak{d}$ commutes with every element of $\text{Bis}_L(\psi)$, each element of $\text{Bis}_L(\psi)$ must have constant leading term. The same holds for $\text{Bis}_R(\psi)$.

Proposition 5.1.2.3. *If* $\text{Bis}_L(\psi)$ *is full of rank* 1*, then* $\text{Bis}_L(\psi)$ *is also rank* 1*.*

In the case that $\mathrm{Bis}_L(\psi)$ (or $\mathrm{Bis}_R(\psi)$) is rank 1, it also has a (stationary) Baker-Alkheizer function ψ_{BA}. We can then prove that ψ and ψ_{BA} are related by multiplication by a function of a single variable, and by exploiting this we can classify such bispectral triples and provide very specific structure theorems for the associated bispectral algebras.

Lemma 5.1.2.4. *Suppose $\psi \in M$, and that the algebra $\mathrm{Bis}_L(\psi)$ is full of rank 1. Then the eigenvalue homomorphism $\Lambda_L : \mathrm{Bis}_L(\psi) \to K[y]$ sends the center of $\mathrm{Bis}_L(\psi)$ to $k[y]$ and sends operators to polynomials whose degree is equal to the order of the operator. A similar statement holds for Λ_R when $\mathrm{Bis}_R(\psi)$ is full of rank 1.*

Proof. Let $N = \dim_k(K)$. First of all, since Λ_L is an anti-homomorphism it must send the center of $\mathrm{Bis}_L(\psi)$ to the center of the image of Λ_L. Let $\mathrm{Bis}_L(\psi)_n$ be the subset of $\mathrm{Bis}_L(\psi)$ consisting of operators of order at most n. By the assumption that $\mathrm{Bis}_L(\psi)$ is full of rank 1, there exists an n_0 such that for $n \geq n_0$ and $c \in K$ there exists an operator in $\mathrm{Bis}_L(\psi)$ of order n with leading coefficient c. Consequently for $n \geq n_0$, $\dim_k(\mathrm{Bis}_L(\psi)_{n+1}/\mathrm{Bis}_L(\psi)_n) \geq N^2$, and we may choose $r > 0$ such that $\dim_k(\mathrm{Bis}_L(\psi)_n) \geq N^2 n - r$.

The eigenvalue homomorphism Λ_L is injective, and sends $\mathrm{Bis}_L(\psi)_n$ injectively into the k-linear space $\mathcal{P}_n$ of polynomials of degree at most n. Now suppose that $\mathrm{Bis}_L(\psi)$ contains an element $\mathfrak{p}$ such that $g(y) = \Lambda(\mathfrak{p})$ has degree m, and is less than the order ℓ of $\mathfrak{p}$. Without loss of generality, we may take $\mathfrak{p}$ to have a leading coefficient which is not a zero divisor, so that $\mathfrak{p}\mathfrak{q}$ has order ℓ more than the order of q for all $q \in \mathfrak{D}(R)$. Consequently, $\mathfrak{p}\mathrm{Bis}_L(\psi)_n$ and $\mathrm{Bis}_L(\psi)_{m+n}$ are disjoint, and both are mapped to $\mathcal{P}_{m+n}$ under Λ_L. Consequently

$$N^2 n - r + N^2(m+n) - r \leq \dim_k(\mathrm{Bis}_L(\psi)_n) + \dim_k(\mathrm{Bis}_L(\psi)_{m+n}) \leq N^2(n+m+1),$$

which tells us that

$$N^2 n \leq 2r + N^2.$$

For n large enough this is a contradiction and thus Λ_L must send operators of whatever order to polynomials of equal degree.

This also implies that for $n \geq n_0$, the image of Λ_L contains polynomials of degree n with every possible coefficient $c \in K$. As a conseqence, the center of the image of Λ_L is contained in $k[x]$. This completes the proof. □

Lemma 5.1.2.5. *Suppose $\psi \in M$, and that the algebra $\mathrm{Bis}_L(\psi)$ is full of rank 1, and choose $\mathfrak{d}$ of positive order in the center of $\mathrm{Bis}_L(\psi)$. Let $\mathfrak{v} \in \mathfrak{P}(R)$ be a wave operator satisfying $\mathfrak{v}^{-1}\mathfrak{d}\mathfrak{v} = g(\partial_x) \in k[\partial_x]$, and let $W = W_{\mathfrak{v},I}$ be the point in $\mathrm{Gr}(R;0)_+$ corresponding to $\mathfrak{v}$. Then W is rational.*

Proof. By definition $Wg(z) \subseteq W$. Therefore to prove that W is rational, it suffices to show that there exists $q(z) \in k[z]$ such that $Wq(z) \subseteq \mathbb{V}_+$. Note that $\mathrm{Bis}_L(\psi)$ is contained in the centralizer of $\mathfrak{d}$, and so $\mathfrak{v}^{-1}\mathrm{Bis}_L(\psi)\mathfrak{v} \subseteq A_W \subseteq K((z^{-1}))$. Since $\mathrm{Bis}_L(\psi)$ is rank 1 and conjugation by ν preserves the leading coefficient, this means there exists an integer n_0 such that A_W contains an element of degree n with leading coefficient c for all $c \in K$ and all $n \geq n_0$. Now the center $Z(\mathrm{Bis}_L(\psi))$ of $\mathrm{Bis}_L(\psi)$ is embedded in $k[z]$ by the eigenvalue homomorphism. Hence by Luröth's theorem $\mathrm{Spec}(Z(\mathrm{Bis}_L(\psi)))$. Note also that the fact $\mathrm{Bis}_L(\psi)$ is full of rank 1 implies that $\nu^{-1}Z(\mathrm{Bis}_L(\psi))\nu$ is contained in $k((z^{-1}))$. Degree arguments tell us that the cokernel of $\nu^{-1}Z(\mathrm{Bis}_L(\psi))\nu \subseteq S_W$ is finite-dimensional over k. Consequently, S_W is rational and it follows that W is rational. □

5.1.3 First Structure Theorem

Our first theorem regarding the structure of bispectral differential operator algebras says that if $(\mathfrak{d}, \mathfrak{b}, \psi)$ is a normalized, regular bispectral triple, then $\mathrm{Bis}_L(\psi)$ and $\mathrm{Bis}_R(\psi)$ are full of rank 1 if and only if the function ψ is equal to an exponential function times a rational function. In this case, ψ is closely related to a stationary Baker-Alkheizer function for $\mathfrak{d}$ and $\mathfrak{b}$.

Theorem 5.1.3.1. *Let $(\mathfrak{d}, \mathfrak{b}, \psi)$ be a normalized, regular bispectral triple. Then $\mathrm{Bis}_L(\psi)$ and $\mathrm{Bis}_R(\psi)$ are full of rank 1 if and only if $\psi(x,y) = e^{p(x)q(y)}h(x,y)$ for $p(x), q(y)$ degree 1 polynomials, and some $h(x,y) \in M_N(\mathbb{C}(x,y))$ with $0 \neq \det h(x,y) \in \mathbb{C}(x)\mathbb{C}(y)\mathbb{C}[x,y]$.*

Proof. Let $g = \Lambda_L(\mathfrak{d}) = \sum_{n=0}^{\ell} g_n y^n$, $f = \Lambda_L(\mathfrak{d}) = \sum_{n=0}^{m} f_n x^n$. Consider new coordinates $\widetilde{x} = a_1 x + a_0$ and $\widetilde{y} = b_1 y + b_0$, for $a_1 = f_m^{1/(m-\ell)}$, $ma_0 = -f_{m-1}/f_m$, $b_1 = g_\ell^{1/(\ell-m)}$, $b_0 = -g_{m-1}/g_m$. Set $\widetilde{d} = a_1^{-\ell}\mathfrak{d} \in \mathfrak{D}(K[[\widetilde{x}]])$, $\widetilde{b} = b_1^{-m}\mathfrak{b} \in \mathfrak{D}(K[[\widetilde{x}]])$, $\widetilde{f} = f((\widetilde{x} - a_0)/a_1)$, $\widetilde{g} = g((\widetilde{y} - b_0)/b_1)$, and $\widetilde{\psi} = \psi((\widetilde{x} - a_0)/a_1, (\widetilde{y} - b_0)/b_1)$. Then $(\widetilde{\mathfrak{d}}, \widetilde{\mathfrak{b}}, \widetilde{\psi})$ is a normalized regular bispectral triple with $\widetilde{f}_m = 1, \widetilde{f}_{m-1} = 0$, $\widetilde{g}_\ell = 1$, and $\widetilde{g}_{\ell-1} = 0$. Thus without loss of generality, we may assume that $f_m = 1, f_{m-1} = 0, g_\ell = 1$, and $g_{\ell-1} = 0$.

Then we may choose wave operators $\mathfrak{v} \in \mathfrak{P}(R)$ and $\mathfrak{w} \in \mathfrak{P}(\widetilde{R})$ such that $\mathfrak{v}^{-1}\mathfrak{d}\mathfrak{v} = g(\partial_x)$ and $\mathfrak{w}\mathfrak{b}\mathfrak{w}^{-1} = f(\partial_y)$. Then by assumption, $W_L = W_{\mathfrak{v},I}$ is rank 1, and so there exists a Baker-Alkheizer function $\psi_{\overline{W}_L}(x, y)$. The algebra S_{W_L} is isomorphic to a subalgebra of $k[y]$ via the eigenvalue homomorphism. Hence it is unirational and thus rational. It follows that $\psi_{\overline{W}_L}(x, y)$ is defined for all but finitely many values of y and

$$\psi_{\overline{W}_L}(x, y) = e^{xy} \sum_{n=0}^{t} a_n(x) y^n,$$

for some $a_0, \ldots, a_n \in R$. Furthermore, we know

$$\psi_{\overline{W}_L}(x, y) = e^{xy} \left(1 + \sum_{n=1}^{\infty} v_n(x) y^{-n}\right) \quad \text{for} \quad \mathfrak{v} = 1 + \sum_{n=1}^{\infty} v_n(x) \partial_x^{-n}.$$

By a previous lemma, the $v_n(x)$ are all rational functions of x, and consequently the $a_n(x)$ above are all rational functions of x. By the same kind of argument, we can obtain a stationary Baker-Alkheizer function for $\mathrm{Bis}_R(\psi)$

$$\psi_{\overline{W}_R}(x, y) = e^{xy} \sum_{n=0}^{s} x^n b_n(y),$$

where again the $b_n(y)$ are rational functions of y.

Comparing on the domain of $\psi(x, y)$, we see that for each y both $\psi(x, y)$ and $\psi_{\overline{W}_L}(x, y)$ are joint eigenfunctions of $\mathrm{Bis}_L(\psi)$. Since $\mathrm{Bis}_L(\psi)$ is full of rank 1, this implies that for each y there exists a constant $c_R(y)$ such that $\psi(x, y) = \psi_{\overline{W}_L}(x, y) c_R(y)$. Similarly, there exist $c_L(x)$ such that $\psi(x, y) = c_L(x) \psi_{\overline{W}_R}(x, y)$. Therefore

$$\psi_{\overline{W}_L}(x, y) c_R(y) = c_L(x) \psi_{\overline{W}_R}(x, y),$$

from which it follows that both $c_R(y)$ and $c_L(x)$ are rational. This proves one direction.

To prove the converse, suppose that $\psi(x,y) = e^{xy}h(x,y)$ for some $h(x,y) \in M_N(\mathbb{C}(x,y))$ with $\det(h(x,y)) \in \mathbb{C}(x)\mathbb{C}(y)\mathbb{C}[x,y] \neq \{0\}$. Then there exists $p(x) \in \mathbb{C}[x]$ and $q(y) \in \mathbb{C}[y]$ such that $p(x)h(x,y)q(y) \in M_N(\mathbb{C}[x,y])$. Choose

$$\mathfrak{v}_L := \sum_{i=0}^{m} a_i(x)\partial_x^i, \quad \text{for} \quad \sum_{i=0}^{m} a_i(x)y^i = h(x,y)q(y),$$

$$\mathfrak{v}_R := \sum_{i=0}^{n} \partial_y^i b_i(y), \quad \text{for} \quad \sum_{i=0}^{m} x^i b_i(y) = p(x)h(x,y).$$

Then $\psi(x,y)q(y) = \mathfrak{v}_L \cdot e^{xy}I$ and therefore $\mathfrak{d}\mathfrak{v}_L \cdot e^{xy}I = \mathfrak{v}_L e^{xy} I g(y) = \mathfrak{v}_L g(\partial_x) \cdot e^{xy}$. Therefore $(\mathfrak{d}\mathfrak{v}_L - \mathfrak{v}_L g(\partial_x)) \cdot e^{xy} = 0$ for all y, and consequently $\mathfrak{d}\mathfrak{v}_L = \mathfrak{v}_L g(\partial_x)$. It follows that $g(\partial_x) \cdot \ker(\mathfrak{v}_L) \subseteq \ker(\mathfrak{v}_L)$, and therefore that for some polynomial $G(g) \in \mathbb{C}[g]$, $G(g(\partial_x)) \cdot \ker(\mathfrak{v}_L) = 0$. Hence $\ker(\mathfrak{v}_L) \subseteq \ker(G(g(\partial_x)))$, and consequently $G(g(\partial_x)) = \mathfrak{r}\mathfrak{v}_L$ for some $\mathfrak{r} \in M_N(\mathfrak{D}(U))$. Therefore for all $k(\partial_x) \in M_N(\mathbb{C}[\partial_x])$, we have that $\mathfrak{v}_L k(\partial_x) G(g(\partial_x))\mathfrak{v}_L^{-1} = \mathfrak{v}_L k(\partial_x)\mathfrak{r}$ is a matrix differential operator and

$$\mathfrak{v}_L k(\partial_x)G(g(\partial_x))\mathfrak{v}_L^{-1} \cdot \psi(x,y) = \mathfrak{v}_L k(\partial_x)G(g(\partial_x))v_L^{-1}v_L \cdot e^{xy}q(y)^{-1} = \psi(x,y)k(y)G(g(y)).$$

Consequently, $\text{Bis}_L(\psi)$ contains $\mathfrak{v}_L G(g(\partial_x))M_N(\mathbb{C}[\partial_x])\mathfrak{v}_L^{-1}$, and it follows that $\text{Bis}_L(\psi)$ is full of rank 1. The same argument using $\mathfrak{v}_R$ shows that $\text{Bis}_R(\psi)$ is also full of rank 1. This proves the converse. □

5.1.4 Second Structure Theorem

For our second structure theorem, we show that we can relate the condition of inclusion of an element $\mathfrak{d} \in \mathfrak{D}(R)$ in $\text{Bis}_L(\psi)$ (and similarly the inclusion of $\mathfrak{b} \in \mathfrak{D}(\widetilde{R})^{op})$ in $\text{Bis}_R(\psi)$) to finitely many linear conditions. To do so, for each $k \in K$ we will consider the evaluation homomorphisms

$$\langle \delta^m(x-a), \cdot \rangle : M \to B$$

defined for each integer $m \geq 0$ and $a \in k$ by

$$\langle \delta^m(x-a), \varphi(x,y) \rangle = \partial_x^m \cdot \varphi(x,y)|_{x=a},$$

along with

$$\langle \cdot, \delta^n(y-b)\rangle : M \to A$$

defined for each integer $n \geq 0$ and $b \in k$ by

$$\langle \varphi(x,y), \delta^n(y-b)\rangle = \varphi(x,y) \cdot \partial_y^n|_{y=b}.$$

Using these, we consider the vector spaces

$$\Delta_R = \left\{ \langle \chi(x), \cdot \rangle = \sum_{j=0}^{m} \delta^{r_j}(x-a_j) : 0 \leq m \in \mathbb{Z}, \text{ and } a_j \in k,\ 0 \leq r_j \in \mathbb{Z} \text{ for } 1 \leq j \leq m \right\},$$

$$\Delta_L = \left\{ \langle \cdot, \chi(y) \rangle = \sum_{j=0}^{n} \delta^{s_j}(x-b_j) : 0 \leq n \in \mathbb{Z}, \text{ and } b_j \in k,\ 0 \leq s_j \in \mathbb{Z} \text{ for } 1 \leq j \leq n \right\},$$

and for any $\varphi(x,y)$ the vector space

$$\Sigma_R(\varphi) = \left\{ \langle \chi(x), \cdot \rangle \in \Delta_R : \langle \chi(x), \varphi(x,y) \rangle = 0 \right\},$$

$$\Sigma_L(\varphi) = \left\{ \langle \cdot, \chi(y) \rangle \in \Delta_L : \langle \varphi(x,y), \chi(y) \rangle = 0 \right\}.$$

Lemma 5.1.4.1. *Let $(\mathfrak{d}, \mathfrak{b}, \psi)$ be a normalized, regular matrix bispectral triple with $\psi(x,y) = e^{xy}h(x,y)$ for some $h(x,y) \in K(x,y)$ with $h(x,y) = p(x)^{-1}g(x,y)q(y)^{-1}$ for some $p(x) \in k[x], q(y) \in k[y], g(x,y) \in K[x,y]$. Set $\varphi(x,y) = e^{xy}g(x,y)$ and*

$$\mathfrak{v}_L := \sum_{i=0}^{m} a_i(x)\partial_x^i, \quad \textit{for} \quad \sum_{i=0}^{m} a_i(x)y^i = h(x,y)q(y),$$

$$\mathfrak{v}_R := \sum_{i=0}^{n} \partial_y^i b_i(y), \quad \textit{for} \quad \sum_{i=0}^{m} x^i b_i(y) = p(x)h(x,y).$$

Then

$$\Sigma_L(\varphi) \to \ker(\mathfrak{v}_L), \ \textit{defined by } \chi(y) \mapsto \langle e^{xy}, \chi(y)\rangle$$

$$\Sigma_R(\varphi) \to \ker(\mathfrak{v}_R), \ \textit{defined by } \chi(x) \mapsto \langle \chi(x), e^{xy}\rangle$$

are $\mathbb{C}$-vector space isomorphisms.

Proof. We will prove that the map $\Sigma_L(\varphi) \to \ker(\mathfrak{v}_L)$ is well-defined and an isomorphism. The proof for $\Sigma_R(\varphi) \to \ker(\mathfrak{v}_R)$ is similar. Note that $\mathfrak{v}_L \cdot e^{xy} = \psi(x,y)q(y)$ and $e^{xy} \cdot \mathfrak{v}_R = p(x)\psi(x,y)$. Suppose that $\chi(y) \in \Sigma_L(\varphi)$. Then

$$0 = \langle \varphi(x,y), \chi(y)\rangle = p(x)\langle \psi(x,y)q(y), \chi(y)\rangle = p(x)\langle \mathfrak{v}_L \cdot e^{xy}, \chi(y)\rangle = p(x)\mathfrak{v}_L \cdot \langle e^{xy}, \chi(y)\rangle,$$

and therefore $\langle e^{xy}, \chi(y)\rangle \in \ker(\mathfrak{v}_L)$. One may check that the induced map $\Sigma_L(\varphi) \to \ker(\mathfrak{v}_L))$ is an injective vector space homomorphism.

Next, suppose that $k(x) \in \ker(\mathfrak{v}_L)$. Since $\mathfrak{d}\mathfrak{v}_L = \mathfrak{v}_L g(\partial_x)$, it follows that $g(\partial_x) \cdot \ker(\mathfrak{v}_L) \subseteq \ker(\mathfrak{v}_L)$. Since $\ker(\mathfrak{v}_L)$ is finite dimensional, it follows that there exists $G(g) \in \mathbb{C}[g]$ satisfying $G(g(\partial_x)) \cdot \ker(\mathfrak{v}_L) = 0$. This means that $\ker(\mathfrak{v}_L) \subseteq \ker(G(g(\partial_x)))$, and therefore $k(x) \in \ker(G(g(\partial_x))$. It follows that there exist matrices $\{k_{ij}\} \subseteq M_N(\mathbb{C})$ and complex numbers $\{c_j\} \subseteq \mathbb{C}$ such that $k(x) = \sum_{j=1}^n \sum_{i=0}^{d_j} k_{ij} x^i e^{c_j x}$. Hence $k(x) = \langle e^{xy}, \chi(y)\rangle$ for $\chi(y) = \sum_{j=1}^n \sum_{i=0}^{d_j} k_{ij} \delta^{(i)}(y - c_j)$. Moreover, we see that

$$0 = f(x)\mathfrak{v}_L \cdot k(x) = f(x)\mathfrak{v}_L \langle e^{xy}, \chi(y)\rangle = \langle \psi(x,y), \chi(y)\rangle.$$

Thus $\chi(y) \in \Sigma_L(\varphi)$, and we conclude that $\Sigma_L(\varphi) \to \ker(\mathfrak{v}_L)$ is surjective. Since the map is also injective, this proves that it is an isomorphism. □

Theorem 5.1.4.2. *Let $(\mathfrak{d}, \mathfrak{b}, \psi)$ be a normalized, regular matrix bispectral triple defined on $U \times V \subseteq \mathbb{C} \times \mathbb{C}$ with $\psi(x,y) = e^{xy}h(x,y)$ for some $h(x,y) \in M_N(\mathbb{C}(x,y))$ with $h(x,y) = p(x)^{-1}g(x,y)q(y)^{-1}$ for some $p(x) \in \mathbb{C}[x], q(y) \in \mathbb{C}[y], g(x,y) \in M_N(\mathbb{C}[x,y])$. Set $\varphi(x,y) = e^{xy}g(x,y)$ and*

$$\mathfrak{v}_L := \sum_{i=0}^m a_i(x)\partial_x^i, \quad \textit{for} \quad \sum_{i=0}^m a_i(x)y^i = h(x,y)q(y),$$

$$\mathfrak{v}_R := \sum_{i=0}^n \partial_y^i b_i(y), \quad \textit{for} \quad \sum_{i=0}^m x^i b_i(y) = p(x)h(x,y).$$

Then

$$\mathrm{Bis}_L(\psi) = \mathfrak{v}_L\{f(\partial_x) : f(y) \cdot \Sigma_L(\varphi) \subseteq \Sigma_L(\varphi)\}\mathfrak{v}_L^{-1},$$

and also

$$\mathrm{Bis}_R(\psi) = \mathfrak{v}_R^{-1}\{g(\partial_y) : \Sigma_R(\varphi) \cdot g(x) \subseteq \Sigma_R(\varphi)\}\mathfrak{v}_R.$$

Proof. Suppose that $k(y) \cdot \Sigma_L(\varphi) \subseteq \Sigma_L(\varphi)$. Then for all $\chi \in \Sigma_L(\varphi)$, we calculate

$$0 = \langle \varphi(x,y), k(y)\chi(y) \rangle = \mathfrak{v}_L \cdot \langle e^{xy}k(y), \chi(y) \rangle = \mathfrak{v}_L \cdot k(\partial_x) \cdot \langle e^{xy}, \chi(y) \rangle.$$

Therefore by the previous lemma, $k(\partial_x) \cdot \ker(\mathfrak{v}_L) \subseteq \ker(\mathfrak{v}_L)$. Therefore $\mathfrak{v}_L k(\partial_x) \mathfrak{v}_L^{-1}$ is a differential operator and

$$\mathfrak{v}_L k(\partial_x)\mathfrak{v}_L^{-1} \cdot \psi(x,y) = \mathfrak{v}_L k(\partial_x)\mathfrak{v}_L^{-1} v_L \cdot e^{xy} q(y)^{-1} = \psi(x,y)k(y).$$

Hence $\mathfrak{v}_L k(\partial_x)\mathfrak{v}_L^{-1} \in \mathrm{Bis}_L(\psi)$.

Conversely, suppose that $\mathfrak{t} \in \mathrm{Bis}_L(\psi)$. Then $\mathfrak{t} \cdot \psi(x,y) = \psi(x,y)t(y)$ for some $t(y) \in M_N(\mathbb{C}[y])$. It follows that

$$\mathfrak{t}\mathfrak{v}_L \cdot e^{xy} = \mathfrak{v}_L e^{xy} t(y) = \mathfrak{v}_L t(\partial_x) e^{xy},$$

and therefore $(\mathfrak{t}\mathfrak{v}_L - \mathfrak{v}_L t(\partial_x)) \cdot e^{xy} = 0$. This implies that $\mathfrak{t}\mathfrak{v}_L - \mathfrak{v}_L t(\partial_x) = 0$, and therefore $\mathfrak{v}_L t(\partial_x)\mathfrak{v}_L^{-1} = \mathfrak{t}$ and $t(\partial_x) \cdot \ker(\mathfrak{v}_L) \subseteq \ker(\mathfrak{v}_L)$. This implies $t(y) \cdot \Sigma_L(\varphi) \subseteq \Sigma_L(\varphi)$ by the same calculation as above. Hence

$$\mathrm{Bis}_L(\psi) = \mathfrak{v}_L \{k(\partial_x) : k(y) \cdot \Sigma_L(\varphi) \subseteq \Sigma_L(\varphi)\} \mathfrak{v}_L^{-1}.$$

□

5.1.5 Third Structure Theorem

Lemma 5.1.5.1. *Consider $(\mathfrak{d}, \mathfrak{b}, \psi)$ a normalized, regular matrix bispectral triple defined on $U \times V \subseteq \mathbb{C} \times \mathbb{C}$. Let $\Sigma_L(\varphi)$ and $\mathfrak{v}_L$ be defined as in Theorem 5.1.4.2, and let W_L be defined as in Theorem 5.1.5.2. Let $\mathfrak{v}$ be a wave operator for $\mathfrak{d}$ satisfying $\mathfrak{v}^{-1}\mathfrak{d}\mathfrak{v} = g(\partial_x)I$ for $g(y) \in \mathbb{C}[y]$ satisfying $\mathfrak{d} \cdot \psi(x,y) = \psi(x,y)g(y)$, and let (W, A) be the Schur pair for $\mathfrak{v}$ using base point $b \in V$. Then there exists $v(\partial_x) \in M_N(\mathbb{C}((\partial_x^{-1})))$ satisfying $\mathfrak{v}v(\partial_x) = \mathfrak{v}_L$ and furthermore there is an isomorphism $W \to W_L$ defined by*

$$w(z) \mapsto w(y)v(y).$$

Proof. First of all, since $\mathfrak{v}_L^{-1}\mathfrak{d}\mathfrak{v}_L = g(\partial_x)$ and $\mathfrak{v}^{-1}\mathfrak{d}\mathfrak{v} = g(\partial_x)$, we see $\mathfrak{v}_L^{-1}\mathfrak{v}g(\partial_x)\mathfrak{v}^{-1}\mathfrak{v}_L = g(\partial_x)$. Therefore $\mathfrak{v}^{-1}\mathfrak{v}_L$ commutes with $g(\partial_x)$. The centralizer of $g(\partial_x)$ is $M_N(\mathbb{C}((\partial_x^{-1})))$ and therefore there exists $v(\partial_x)$ satisfying $\mathfrak{v}v(\partial_x) = \mathfrak{v}_L$. Note that it also follows $v(\partial_x)$ has nonzero determinant.

Fix a base point $b \in \mathbf{V}$, and suppose that $w(z) \in W$. Then there exists $\widetilde{w}(z) \in \mathbf{V}_+$ satisfying $w(z) = \widetilde{w}(z) \cdot \mathfrak{v}$. It follows that $w(z)v(z) = \widetilde{w}(z) \cdot \mathfrak{v}v(\partial_x) = \widetilde{w}(z) \cdot \mathfrak{v}_L \in \mathbf{V}_+$ and furthermore

$$\widetilde{w}(z) \cdot \mathfrak{v}_L = e^{-by}(\widetilde{w}(\partial_x)\mathfrak{v}_L \cdot e^{xy})|_{x=b}.$$

Therefore we calculate for all $\chi \in \Sigma_L(\varphi)$ and $k(y) \in M_N(\mathbb{C}[y])$

$$\begin{aligned}
\langle w(y)v(y), e^{by}\chi(y)\rangle &= \langle e^{-by}(\widetilde{w}(\partial_x)\mathfrak{v}_L \cdot e^{xy})|_{x=b}, e^{by}\chi(y)\rangle \\
&= \left(\widetilde{w}(\partial_x) \cdot \langle \mathfrak{v}_L \cdot e^{xy}, \chi(y)\rangle\right)|_{x=b} \\
&= \left(\widetilde{w}(\partial_x) \cdot \langle \varphi(x,y), \chi(y)\rangle\right)|_{x=b} = 0.
\end{aligned}$$

It follows that $w(y)v(y) \in W_L$, and thus the map $W \to W_L$ defined by $w(z) \mapsto w(y)v(y)$ is well defined. Since $v(y)$ has nonzero determinant, this map is also injective.

To prove surjectivity, suppose that $t(y) \in W_L$ satisfies $\langle t(y), e^{by}\chi(y)\rangle = 0$ for all $\chi \in \Sigma_L(\varphi)$ and integers $n \geq 0$. For each $f(x) \in \ker(\mathfrak{v}_L)$, there exist $\chi \in \Sigma_L(\varphi)$ satisfying $f(x) = \langle e^{xy}, \chi(y)\rangle$, and therefore $t(\partial_x) \cdot f(x)|_{x=b} = \langle t(y), e^{by}\chi(y)\rangle = 0$. Thus we have $t(z) \cdot \mathfrak{v}_L^{-1} \in V$. Taking $\widetilde{w}(z) = t(z) \cdot \mathfrak{v}_L^{-1}$ and $w(z) = \widetilde{w}(z) \cdot \mathfrak{v}$, we have $w(z)v(z) = \widetilde{w}(z) \cdot \mathfrak{v}_L = t(z)$, and hence $w(y)v(y) = t(y)$. This proves surjectivity and completes the proof of the lemma. ☐

Theorem 5.1.5.2. *Let $(\mathfrak{d}, \mathfrak{b}, \psi)$ satisfy the assumptions of the previous theorem, and let $\Sigma_L(\varphi), \Sigma_R(\varphi)$ be defined as in the previous theorem. Fix base points $b \in V$ and $a \in U$ and define*

$$W_L := \{w(y) \in (\mathbb{C}[y]^{\oplus N})^T : \langle w(y), e^{by}\chi(y)\rangle = 0, \quad \forall \chi(y) \in \Sigma_L(\varphi)\},$$

$$W_R := \{w(x) \in \mathbb{C}[x]^{\oplus N} : \langle \chi(x)e^{ax}, w(x)\rangle = 0, \quad \forall \chi(x) \in \Sigma_R(\varphi)\},$$

$$A_L := \{g(y) \in M_N(\mathbb{C}(y)) : W_L g(y) \subseteq W_L\},$$

$$A_R := \{f(x) \in M_N(\mathbb{C}(x)) : f(x)W_R \subseteq W_R\}.$$

The spectrum of the center $Z(A_L)$ of A_L is a rational curve. Furthermore W_L is a torsion-free module over $Z(A_L)$ and is generically free of rank N. Similarly, the spectrum of the center $Z(A_R)$ of A_R is a rational curve, and W_R is a torsion-free module over $Z(A_R)$ of A_R, which is generically free of rank N. Moreover,

$$\mathrm{Bis}_L(\psi) \cong A_L = \mathrm{End}_{Z(A_L)}(W_L), \quad \textit{and} \;\; \mathrm{Bis}_R(\psi) \cong A_R = \mathrm{End}_{Z(A_R)}(W_R).$$

Proof of Theorem 5.1.5.2. Let (W, A) be the Schur pair for the wave operator ν taking $\mathfrak{d}$ to $g(y)$ for $g(y) \in \mathbb{C}[y]$, as in the proof of the previous lemma. The previous lemma tells us that for some $v(y) \in M_N(\mathbb{C}((y^{-1})))$ we have

$$\begin{aligned} A &= \{k(z) \in M_N(\mathbb{C}((z^{-1}))) : Wk(z) \subseteq W\} \\ &= \{k(y) \in M_N(\mathbb{C}((y^{-1}))) : W_L v(y)^{-1} k(y) \subseteq W_L v(y)^{-1}\} \\ &= \{k(y) \in M_N(\mathbb{C}((y^{-1}))) : W_L v(y)^{-1} k(y) v(y) \subseteq W_L\} \\ &= v(y)\{k(y) \in M_N(\mathbb{C}((y^{-1}))) : W_L k(y) \subseteq W_L\} v(y)^{-1} \end{aligned}$$

Then since W_L contains $r(y)(\mathbb{C}[y]^{\oplus N})^T$ for some nonzero $r(y) \in \mathbb{C}[y]$, we see that if $k(y) \in M_N(\mathbb{C}((y^{-1})))$ satisfies $W_L k(y) \subseteq W_L$ then $k(y) \in M_N(\mathbb{C}(y))$. Therefore $A = v(y)A_L v(y)^{-1}$. Since $\mathfrak{v}^{-1}\mathrm{Bis}_L(\psi)\mathfrak{v} = A$ (itentifying z with ∂_x), it follows that $A_L \cong \mathrm{Bis}_L(\psi)$. Next, the fact that $Z(A_L) \subseteq \mathbb{C}[y]$ and $W_L \subseteq (\mathbb{C}[y]^{\oplus N})^T$ implies that W_L is torsion free over $Z(A_L)$. Furthermore, since W is generically free of rank over $Z(A_L)$, and the center of $Z(A_L)$ is $\mathbb{C}(y)$, we have that $\mathrm{End}_{Z(A)}(W_L) = \{f(y) \in M_N(\mathbb{C}(y)) : W_L f(y) \subseteq W_L\} = A$. The proof for $A_R, \mathrm{Bis}_R(\psi)$ and W_R follows similarly. □

5.2 Examples

5.2.1 A First Example

Let $N = 1$, $U = \mathbb{D}(1,1) = V$, and consider the bispectral triple $(\mathfrak{d}, \mathfrak{b}, \psi)$ defined by

$$\mathfrak{d} = \partial_x^2 - \frac{2}{x^2}, \quad \mathfrak{b} = \partial_y^2 - \frac{2}{y^2}, \quad \text{and} \quad \psi(x,y) = e^{xy}h(x,y) \text{ for } h(x,y) = \left(1 - \frac{1}{xy}\right).$$

Then choosing $p(x) = x$ and $q(y) = y$, we see that $h(x,y)p(x)q(y) \in M_N(\mathbb{C}[x,y])$. Setting $\varphi(x,y) = p(x)\psi(x,y)q(y)$,

$$\mathfrak{v}_L := x\partial_x - 1, \text{ and } \mathfrak{v}_R := \partial_y y - 1,$$

we have that $\mathfrak{v}_L \cdot e^{xy} = \varphi(x,y)$ and $e^{xy} \cdot \mathfrak{v}_R = \varphi(x,y)$. Furthermore, we calculate $\ker(\mathfrak{v}_L) = \mathbb{C}x$, and therefore $\Sigma_L(\varphi) = \mathbb{C}\delta'(y)$. From this, one easily calculates

$$A_L = W_L = \mathbb{C} + y^2\mathbb{C}[y].$$

Similarly $\Sigma_R(\varphi) = \mathbb{C}\delta'(x)$ and $A_R = W_R = \mathbb{C} + x^2\mathbb{C}[x]$. Thus by Theorem (5.1.5.2)

$$\mathrm{Bis}_L(\psi) = \mathfrak{v}_L(\mathbb{C} + \partial_x^2\mathbb{C}[\partial_x])\mathfrak{v}_L^{-1},$$

and also

$$\mathrm{Bis}_R(\psi) = \mathfrak{v}_R^{-1}(\mathbb{C} + \partial_y^2\mathbb{C}[\partial_y])\mathfrak{v}_R.$$

5.2.2 A Second Example

Let $N = 2$, $U = \mathbb{D}(0,1)$, $V = \mathbb{D}(1,1)$, $a, b \in \mathbb{C}\backslash U$ with $a \neq b$, and consider the bispectral triple $(\mathfrak{d}, \mathfrak{b}, \psi)$ defined by

$$\mathfrak{d} = \partial_x^2 I - \begin{pmatrix} \frac{1}{x-a} & \frac{1}{(x-a)(x-b)} \\ 0 & \frac{1}{x-b} \end{pmatrix}, \quad \mathfrak{b} = \partial_y^2 I - \partial_y \frac{2}{y} I - \begin{pmatrix} a^2 + \frac{2a}{y} & 0 \\ 0 & b^2 - \frac{2b}{y} \end{pmatrix},$$

and

$$\psi(x,y) = e^{xy}h(x,y) \text{ for } h(x,y) = \begin{pmatrix} y - \frac{1}{x-a} & \frac{1}{(x-a)(x-b)} \\ 0 & y - \frac{1}{x-b} \end{pmatrix}.$$

Take $p(x) = (x-a)(x-b)$ and $q(y) = 1$. Then $p(x)h(x,y)q(y) \in M_N(\mathbb{C}[x,y])$. Setting $\varphi(x,y) = p(x)\psi(x,y)q(y)$, we find $\mathfrak{v}_L \cdot e^{xy}I = \psi(x,y)$ and $e^{xy}I \cdot \mathfrak{v}_R$ for

$$\mathfrak{v}_L = \begin{pmatrix} (x-a)(x-b)\partial_x - (x-b) & 1 \\ 0 & (x-a)(x-b)\partial_x - (x-a) \end{pmatrix},$$

and

$$\mathfrak{v}_R = \begin{pmatrix} \partial_y^2 y - \partial_y((a+b)y+1) + aby + b & 1 \\ 0 & \partial_y^2 y - \partial_y((a+b)y+1) + aby + a \end{pmatrix}.$$

The kernel of $\mathfrak{v}_L$ is given by

$$\ker(\mathfrak{v}_L) = \begin{pmatrix} x-a & 1 \\ 0 & x-b \end{pmatrix} M_2(\mathbb{C}),$$

and therefore

$$\Sigma_L(\varphi) = \begin{pmatrix} \delta'(y) - a\delta(y) & \delta(y) \\ 0 & \delta'(y) - b\delta(y) \end{pmatrix} M_2(\mathbb{C}).$$

Thus

$$W_L = \mathbb{C}\begin{pmatrix} (aby+b) & 1 \end{pmatrix} \oplus \begin{pmatrix} y^2\mathbb{C}[y] & (by+1)\mathbb{C} \oplus y^2\mathbb{C}[y] \end{pmatrix},$$

and

$$A_L = \mathbb{C}I \oplus y^2 M_2(\mathbb{C}[y]).$$

The kernel of $\mathfrak{v}_R$ is given by

$$\ker(\mathfrak{v}_R) = M_2(\mathbb{C}) \begin{pmatrix} (b-a)y+1 & y \\ 0 & 1 \end{pmatrix} e^{ay} \oplus M_2(\mathbb{C}) \begin{pmatrix} b-a & 1 \\ 0 & (a-b)y+1 \end{pmatrix} e^{by},$$

and therefore $\Sigma_R(\varphi) = M_2(\mathbb{C})\chi_a(x) \oplus M_2(\mathbb{C})\chi_b(x)$ for

$$\chi_a(x) = \begin{pmatrix} (b-a)\delta'(x-a) + \delta(x-a) & \delta'(x-a) \\ 0 & \delta(x-a) \end{pmatrix}$$

$$\chi_b(x) = \begin{pmatrix} (b-a)\delta(x-b) & \delta(x-b) \\ 0 & (a-b)\delta'(x-b) + \delta(x-b) \end{pmatrix}.$$

It follows that $W_R = \mathbb{C}w_1 \oplus \mathbb{C}w_2 \oplus \mathbb{C}w_3 \oplus \mathbb{C}w_4 \oplus (x-a)^2(x-b)^2\mathbb{C}[x]^{\oplus 2}$ for

$$w_1 = \begin{pmatrix} -1 \\ x-a \end{pmatrix}, \quad w_2 = \begin{pmatrix} (x-a)^2(x-b) \\ 0 \end{pmatrix}$$

$$w_3 = \begin{pmatrix} x-b \\ 0 \end{pmatrix}, \quad w_4 = \begin{pmatrix} (x-a)(x-b) \\ (x-a)(x-b)^2 \end{pmatrix}.$$

In fact, one may check $W_R = u(x)\widetilde{W}_R$ for

$$u(x) = \begin{pmatrix} -1 & x-b \\ \\ x-a & 0 \end{pmatrix}, \quad \widetilde{W}_R = \begin{pmatrix} \mathbb{C} \oplus (x-b)^2\mathbb{C}[x] \\ \mathbb{C} \oplus (x-a)^2\mathbb{C}[x] \end{pmatrix},$$

and therefore $A_R = u(x)\widetilde{A}_R u(x)^{-1}$ for

$$\begin{aligned} \widetilde{A}_R &= \{f(x) \in M_2(\mathbb{C}(x)) : f(x)\widetilde{W}_R \subseteq \widetilde{W}_R\} \\ &= \begin{pmatrix} \mathbb{C} \oplus (x-b)^2\mathbb{C}[x] & (x-b)^2\mathbb{C}[x] \\ \\ (x-a)^2\mathbb{C}[x] & \mathbb{C} \oplus (x-a)^2\mathbb{C}[x] \end{pmatrix}. \end{aligned}$$

From this, we can obtain all of the elements of $\mathrm{Bis}_R(\psi)$ via conjugation.

Chapter 6

THE ALGEBRA OF DIFFERENTIAL OPERATORS FOR A WEIGHT MATRIX

Note that much of this section is taken verbatim from a preprint written by the author which has been submitted for publication [9].

6.1 Introduction

A weight function is a nonzero measureable function $w : \mathbb{R} \to [0, \infty)$ satisfying the condition that the moments $\int w(x)x^n dx$ are all finite. A weight function gives rise to an inner product on the space of polynomials in x, defined by

$$\langle f(x), g(x) \rangle_w := \int f(x)w(x)\overline{g(x)}dx.$$

By Gram-Schmidt, we may obtain a sequence of pairwise orthogonal polynomials $\{p(x, n)\}$ such that $p(x, n)$ has degree n for each integer $n \geq 0$. Such a sequence is called a sequence of orthogonal polynomials for w, or if w is implied, simply a sequence of orthogonal polynomials.

Bochner's problem, introduced in the paper [5], is to determine for which weights w the associated polynomials are a family of eigenfunctions of some second order differential operator. Lucky for us, Bochner provides a solution to his problem in the very same paper. Up to affine changes of coordinates, the only orthogonal polynomials satisfying this property are the classical families of Hermite, Laguerre, and Jacobi.

The question Bochner addresses generalizes naturally to operators of higher order. Consider a weight function w and a sequence of orthogonal polynomials $\{p(x, n)\}$. The set $D(w)$ of all differential operators for which the sequence is a family of eigenfuntions forms an algebra. Moreover, this algebra is independent of the choice of sequence of orthogonal polynomials for w, since after all the value of $p(x, n)$ is unique up to a scalar multiple for each n. In terms of $D(w)$, Bochner's result tells us for which w the algebra $D(w)$ contains an operator of order two. However, the classification of weights w for which the algebra $D(w)$ contains an operator of higher order seems to be very hard. The classification for operators of order 4 was done by Krall [47], but for higher order is still open.

Bochner's problem and the notion of orthogonal polynomials also extends naturally to matrix-valued polynomials. A weight matrix is a matrix-valued function $w(x) : \mathbb{R} \to M_N(\mathbb{C})$ which is sufficiently nice (explained below) so as to induce a nondegenerate matrix-valued inner product on the space of matrix-valued polynomials defined by

$$\langle f(x), g(x) \rangle_w := \int f(x)w(x)g(x)^* dx.$$

Here $g(x)^*$ refers to the Hermitian-conjugate of $g(x)$. Again by a process identical to Gram-Schmidt, but with matrix values, we may obtain a sequence of polynomials $\{p(x,n)\}$ such that $p(x,n)$ is degree n with *nonsingular* leading coefficient for all integers $n \geq 0$ and such that $\langle p(x,n), p(x,m) \rangle_w = 0I$ for $m \neq n$. A sequence of matrix-valued polynomials satisfying these properties is called a sequence of matrix-valued orthogonal polynomials for w.

To extend Bochner's problem to matrix orthogonal polynomials, one should consider the algebra $M_N(\Omega)$ of matrix-valued differential operators, eg. operators of the form

$$\mathfrak{d} = a_0(x) + \partial a_1(x) + \cdots + \partial^\ell a_\ell(x),$$

where the $a_i(x) \in M_N(\mathbb{C}[[x]])$ and $\mathfrak{d}$ acts on $M_N(\mathbb{C}[[x]])$ by

$$f(x) \cdot \mathfrak{d} := f(x)a_0(x) + f'(x)a_1(x) + \cdots + f^{(\ell)}(x)a_\ell(x).$$

In particular, our differential operators act on the right. This is required because of the noncommutativity of the coefficients: the differential operators must act on the right in order to be compatible with the matrix-valued inner product [17]. A function $f(x)$ is called an eigenfunction of $\mathfrak{d}$ if there exists a matrix $\lambda \in M_N(\mathbb{C})$ such that $f(x) \cdot \mathfrak{d} = \lambda f(x)$. With this notion of matrix differential operators, we may state the matrix version of Bochner's problem.

Problem 6.1.0.1 (Bochner's Problem for Matrix Differential Operators)**.** Let $w(x)$ be a weight matrix, and let $\{p(x,n)\}$ be a sequence of orthogonal matrix polynomials for $w(x)$. When does there exists a matrix differential operator of order two for which $p(x,n)$ is an eigenfunction for every integer $n \geq 0$?

Bochner's problem for matrix differential operators is considered in numerous papers, including many of the papers mentioned in the references below. For a helpful survey, see [19]. Unlike the scalar case, general classification results for Bochner's problem remain elusive, even for 2×2 matrices. Many papers therefore have focused instead on providing new examples of Bochner pairs $(w, \mathfrak{d})$, eg. a weight matrix w and a second order operator $\mathfrak{d} \in D(w)$.

More recent papers have explored the structure of the algebra $D(w)$ of all matrix differential operators for which the $p(x,n)$ are eigenfunctions. In particular, [68] [11] [40] provide examples of generators and relations for $D(w)$ for various values of w. Even more examples are [20][39][55][72]. These examples demonstrate that the structure of $D(w)$ can be nuanced and interesting, unlike in the scalar case. However, despite an ever increasing list of examples, more general results regarding the structure of $D(w)$ remain a mystery. In particular, current methods of finding generators and relations for $D(w)$ are often ad hoc, or based on computational evidence, and can involve extended calculation.

The purpose of the current paper is to demonstrate how under sufficiently nice conditions one may use Darboux transformations to create a new Bochner pair $(\widetilde{w}, \widetilde{\mathfrak{d}})$ from a known Bochner pair $(w, \mathfrak{d})$. Moreover, we show that when $(\widetilde{w}, \widetilde{\mathfrak{d}})$ arises from $(w, \mathfrak{d})$ by a Darboux transformation then the algebras $D(\widetilde{w})$ and $D(w)$ are closely related, so that knowledge of the structure of $D(w)$ leads to knowledge of the structure of $D(\widetilde{w})$. As a result, this paper leads both to new families of examples of Bochner pairs not currently in the literature, as well as very efficient derivations of the structure of the algebra $D(\widetilde{w})$ of matrix differential operators for several examples of weights $\widetilde{w}$ already featured in the literature.

A Darboux transformation of a differential operator $\mathfrak{d}$ is a new differential operator $\widetilde{\mathfrak{d}} = \mathfrak{y}\mathfrak{v}$

obtained by means of a factorization of the original operator $\mathfrak{D} = \mathfrak{v}\mathfrak{y}$. For a short survey of Darboux transformations, see [57], and for a great read about the role of Darboux transformations in the context of orthogonal polynomials satisfying differential equations, see [33]. Further background on the Darboux transformations relevant to this paper can be found in [35][22]. Finally, for a very recent article exploring Darboux transformations in a noncommutative context, applied in particular to matrix differential operators, see [25]. This notion works equally as well for matrix differential operators as it does for ordinary differential operators. However, given an arbitrary factorization of $\mathfrak{D}$, there's no reason to expect that the transformed operator $\widetilde{\mathfrak{D}}$ will be in $D(\widetilde{w})$ for some new weight matrix $\widetilde{w}$. In order to guarantee this, we must be more methodical about our choice of the factorization of $\mathfrak{D}$.

The method presented in this paper relies on the application of two different adjoints $*$ and $\dagger$ on the algebra of differential operators. The first adjoint $*$ is the standard notion of the adjoint, which extends the Hermitian-conjugate on matrix-valued functions and satisfies

$$(\partial^m)^* = (-1)^m \partial^m.$$

The second $\dagger$ is the formal w-adjoint of $\mathfrak{D}$, and is defined in terms of the first by

$$\mathfrak{D}^\dagger := w(x)\mathfrak{D}^* w^{-1}(x),$$

in some neighborhood of $x = 0$. The requisite properties of w are discussed further in the section on adjoints later in the paper. For clarity, we provide a formula for the formal w-adjoint of a first order differential operator in the next example.

Example 6.1.0.2. Consider the first-order matrix differential operator $\mathfrak{D} = a_0(x) + \partial a_1(x)$. Then the standard adjoint of $\mathfrak{D}$ is given by

$$\mathfrak{D}^* = a_0(x)^* - a_1'(x)^* - \partial a_1(x)^*,$$

and the formal w-adjoint of $\mathfrak{D}$ is given by

$$\mathfrak{D}^\dagger = w(x)[a_0(x)^* - a_1'(x)^* - w(x)^{-1}w'(x)a_1(x)^*]w(x)^{-1} - \partial w(x)a_1(x)^* w(x)^{-1}.$$

Using the two adjoint definitions we have introduced, we are now able to state our Main Theorem, which provides a way of obtaining a Darboux transformation of a Bochner pair $(w, \mathfrak{d})$ to a new Bochner pair $(\widetilde{w}, \widetilde{\mathfrak{d}})$.

Theorem 6.1.0.3 (Main Theorem)**.** *Let $(w, \mathfrak{d})$ be a Bochner pair supported on (x_0, x_1) with $\lim_{x\to x_i}(|x|^n + 1)w(x) = 0$ for $i = 0, 1$ and $n \geq 0$ and*

$$\mathfrak{d} = a_0(x) + \partial a_1(x) + \partial^2 a_2(x) I,$$

with $a_0(x), a_1(x)$ matrix-valued functions satisfying $a_1(x)^\dagger = a_1(x)$, with $a_2(x)$ a scalar-valued function which is nonzero on (x_0, x_1), and with $\mathfrak{d}$ degree-preserving. Suppose that there exist matrix polynomials $v_1(x), v_0(x)$ satisfying the following properties:

- $\deg(v_i(x)) = i$, $\det(v_i(x)) \neq 0$ *on* (x_0, x_1) *for* $i = 0, 1$ *and* $(v_0 v_1^{-1})^\dagger = v_0 v_1^{-1}$
- $\mathrm{Tr}((v_1(x)^{-1} a_2(x)(v_1(x)^\dagger)^{-1})) \in L^2(\mathrm{Tr}(w(x))dx)$
- $(v_0(x)v_1(x)^{-1})^2 a_2(x) + (v_0(x)v_1(x)^{-1})' a_2(x) + (v_0(x)v_1(x)^{-1})a_1(x) + a_0(x) = 0$

Then the following is true

(a) there exists a smooth, matrix-valued function f on (x_0, x_1) satisfying $v_1 f (v_1 f)^\dagger = a_2 I$

(b) $\widetilde{w} = f w f^*$ is a weight matrix

(c) $\mathfrak{v} := \partial v_1(x) - v_0(x)$ and $\mathfrak{y} := -f(x) f^\dagger(x) \mathfrak{v}^\dagger$ are degree-preserving and $\mathfrak{d} = \mathfrak{v}\mathfrak{y}$

(d) $\widetilde{p}(x, n) := p(x, n) \cdot \mathfrak{v}$ is a sequence of orthogonal matrix polynomials for $\widetilde{w}$

(e) $(\widetilde{w}, \widetilde{\mathfrak{d}})$ is a Bochner pair for $\widetilde{\mathfrak{d}} := \mathfrak{y}\mathfrak{v}$

(f)

$$D(\widetilde{w}) \supseteq (\mathfrak{v}^{-1} D(w) \mathfrak{v}) \cap M_N(\Omega) = \{\mathfrak{v}^{-1} \eta \mathfrak{v} : \eta \in D(w),\ \ker(\mathfrak{v}) \cdot \eta \subseteq \ker(\mathfrak{v})\}.$$

The Main Theorem is not as general as possible. The idea is to decompose $\mathfrak{d}$ as $\mathfrak{d} = -\mathfrak{v}ff^\dagger\mathfrak{v}^\dagger$ such that $\mathfrak{v}$ is degree-preserving and w-adjointable. The Main Theorem simply provides a specific context where this holds. Even more generally, we could find $\mathfrak{v}$ such that $\mathfrak{d}\mathfrak{v} = \mathfrak{v}\widetilde{\mathfrak{d}}$ for some differential operator $\widetilde{\mathfrak{d}}$, eg. a Darboux conjugation rather than a Darboux transformation. It is also important to note that the function f in the Main Theorem is not unique – and in fact there are many such functions up to a choice of smooth, unitary matrix-valued function. To find such an f, we split $w(x)$ as $u(x)u(x)^*$, which we may do since $w(x)$ is Hermitian. Then $u^{-1}v_1^{-1}a_2(v_1^\dagger)^{-1}u$ is also Hermitian, and therefore may be split as hh^* for some matrix-valued function h. Then taking $f = uhu^{-1}$ completes the construction.

Motivated by our Main Theorem, we make the following definition

Definition 6.1.0.4. Let $(w, \mathfrak{d})$ be a Bochner pair, and let $\{p(x, n)\}$ be a sequence of orthogonal matrix polynomials for w. We call $(\widetilde{w}, \widetilde{\mathfrak{d}})$ a **Darboux transformation** of $(w, \mathfrak{d})$ if there exist matrix differential operators $\mathfrak{y}$ and $\mathfrak{v}$ such that $\mathfrak{d} = \mathfrak{v}\mathfrak{y}$, $\widetilde{\mathfrak{d}} = \mathfrak{y}\mathfrak{v}$, and $\widetilde{p}(x, n) := p(x, n) \cdot \mathfrak{v}$ defines a sequence of orthogonal matrix polynomials for $\widetilde{w}$. We also say that $\mathfrak{v}$ is a Darboux transformation from the Bochner pair $(w, \mathfrak{d})$ to the Bochner pair $(\widetilde{w}, \widetilde{\mathfrak{d}})$. We define the **associated subalgebra** $D(\widetilde{w}, \mathfrak{v}, w)$ of $D(\widetilde{w})$ by

$$D(\widetilde{w}, \mathfrak{v}, w) := (\mathfrak{v}^{-1}D(w)\mathfrak{v}) \cap \mathfrak{D}(M_N(\mathbb{C}[x]))^{op} = \{\mathfrak{v}^{-1}\mathfrak{d}\mathfrak{v} : \mathfrak{d} \in D(w),\ \ker(\mathfrak{v}) \cdot \mathfrak{d} \subseteq \ker(\mathfrak{v})\}.$$

In general, $D(\widetilde{w}, \mathfrak{v}, w)$ will be a proper subalgebra of $D(\widetilde{w})$. However, we should expect the subalgebra to contain *most* of $D(\widetilde{w})$. In particular, we have the following proposition, the proof of which is found in 6.3.2.

Proposition 6.1.0.5. *Let $\mathfrak{v}$ be a Darboux transformation from the Bochner pair $(w, \mathfrak{d})$ to the Bochner pair $(\widetilde{w}, \widetilde{\mathfrak{d}})$. Suppose $\widetilde{\mathfrak{d}} = \partial^2\widetilde{a}_2 + \partial\widetilde{a}_1 + \widetilde{a}_0$ with $\det(\widetilde{a}_2) \neq 0$ and that there exists a positive integer ℓ such that*

$$\dim_{\mathbb{C}}(D(\widetilde{w}, \mathfrak{v}, w)_i / D(\widetilde{w}, \mathfrak{v}, w)_{i-1}) = \dim_{\mathbb{C}}(D(\widetilde{w}, \mathfrak{v}, w)_{i+2} / D(\widetilde{w}, \mathfrak{v}, w)_{i+1}),\ \forall i \geq \ell.$$

where here $D(\widetilde{w}, \mathfrak{v}, w)_i$ represents the linear subspace of $D(\widetilde{w}, \mathfrak{v}, w)$ of operators of order *at most i. Then $D(\widetilde{w})$ is a finitely generated algebra over $D(\widetilde{w}, \mathfrak{v}, w)$, and is generated by elements of order less than ℓ.*

In practice, the previous proposition will allow us to determine the structure of $D(\widetilde{w})$ from the structure of $D(w)$.

6.2 Background

6.2.1 Classical Orthogonal Polynomials and Bochner's Problem

This section is intended to provide a reader with a quick recap of the basic points in the theory of orthogonal polynomials. We begin with the definition of a weight function. For simplicity, we dodge technical measure-theoretic details by working with smooth weights.

Definition 6.2.1.1. Let $-\infty \leq x_0 < x_1 \leq \infty$. A **weight function** supported on (x_0, x_1) is a nonnegative function $r : \mathbb{R} \to (0, \infty)$ which is 0 off of (x_0, x_1) and positive and smooth on (x_0, x_1), satisfying the condition that the moments $\int_{\mathbb{R}} x^n r(x) dx < \infty$ for all integers $n \geq 0$. The interval (x_0, x_1) on which r is nonzero is called the **support of** r.

Given a weight function $r(x)$, we define an inner product on $\mathbb{C}[x]$ by

$$\langle p(x), q(x) \rangle_r := \int_{\mathbb{R}} p(x) r(x) \overline{q(x)} dx.$$

By the process of Gram-Schmidt elimination, we can construct a sequence of pairwise-orthogonal polynomials $p(x, 0), p(x, 1), p(x, 2), \ldots$ with $p(x, n)$ a polynomial of degree n. If we impose the additional constraint that each $p(x, i)$ is monic, then this sequence of polynomials is unique.

Definition 6.2.1.2. Let $r(x)$ be a weight function. A sequence of polynomials $\{p(x, n)\}_{n=0}^{\infty}$ is called a **sequence of orthogonal polynomials** for r if $p(x, n)$ has degree n for all integers $n \geq 0$ and $\langle p(x, n), p(x, m) \rangle_r = 0$ for all $m, n \geq 0$ with $n \neq m$.

The sequence of monic orthogonal polynomials for a weight function $r(x)$ can in practice be calculated recursively.

Proposition 6.2.1.3 (Three-Term Recurrence Relation). *Let $r(x)$ be a weight function. Then for all integers $n \geq 0$, there exist constants $s_n, t_n \in \mathbb{C}$ such that*

$$xp(x,n) = p(x,n+1) + s_n p(x,n) + t_n p(x,n-1). \tag{6.1}$$

The values of s_n and t_n may be defined explicitly in terms of the moments of $r(x)$.

Sequences of orthogonal polynomials that are simultaneously eigenfunctions of a second-order differential operator arise naturally in Sturm-Liouville theory. Three families of weights whose corresponding orthogonal polynomials satisfy a second-order differential operator were known classically. These weights are listed in the table in Figure 6.1. The sequences of orthogonal polynomials for these weights are referred to as the classical orthogonal polynomials.

Family	weight
Hermite	e^{-x^2}
Laguerre	$x^b e^{-x} 1_{(0,\infty)},\ b > -1$
Jacobi	$(1-x)^a(1+x)^b 1_{(-1,1)},\ a, b > -1$

Figure 6.1: The Classical Orthogonal Polynomial Weights

The importance and applicability of the classical orthogonal polynomials lead Bochner to ask and answer the following question:

Problem 6.2.1.4 (Bochner's Problem). For which weight functions $r(x)$ do there exists second-order differential operators $\mathfrak{d} \in \mathfrak{D}(\mathbb{C}[x])$ for which the sequence of monic orthogonal polynomials $p(x,n)$ are simultaneously eigenfunctions?

With this question in mind, we make the following definition.

Definition 6.2.1.5. The pair of data (r, ϵ) with r a weight function and ϵ a differential operator is called a **Bochner pair** if the sequence of monic orthogonal polynomials for r are all eigenfunctions for ϵ.

In other words, (r, ϵ) is a Bochner pair if and only if there exists a sequence of complex numbers $\{\lambda_n\}_{n=0}^{\infty}$ such that $p(x, n) \cdot \epsilon = \lambda_n p(x, n)$ for all $n \geq 0$. The next theorem characterizes solutions to Bochner's problem. See [69].

Theorem 6.2.1.6. *Suppose that (r, ϵ) is a Bochner pair, with r supported on (x_0, x_1) for $-\infty \leq x_0 < x_1 \leq \infty$. Without loss of generality, we may assume that $\epsilon = \partial^2 q(x) + \partial(l(x) + q'(x))$ for some polynomials $l(x), q(x)$ of degree 1 and 2, respectively. Then the following are true:*

(i) the weight function $r(x)$ satisfies the Pearson equation

$$q(x)r'(x)r^{-1}(x) = l(x), \quad x \in (x_0, x_1) \tag{6.2}$$

(ii) the polynomials $q(x)$ and $l(x)$ satisfy the boundary condition

$$\lim_{x \to a+} q(x)r(x) = \lim_{x \to b-} l(x)r(x) = 0.$$

Conversely, any weight function $r(x)$ supported on an interval (x_0, x_1) satisfying (i) and (ii) for some polynomials $l(x)$ and $q(x)$ of degree 1 and 2, respectively gives a solution (r, ϵ) to Bochner's problem.

In the case that (r, ϵ) is a Bochner pair, we can obtain the sequence of monic orthogonal polynomials for r in a far sleeker way than the recurrence relation of Equation (6.1) [69].

Theorem 6.2.1.7. *Let $r(x)$ be a weight function satisfying the assumptions of Theorem 6.2.1.6, and let $q(x)$ and $l(x)$ be as in the statement of the theorem. Then the sequence of monic orthogonal polynomials $p(x, n)$ may be obtained by the Rodrigues formula*

$$p(x, n) = c_n (q(x)^n r(x))^{(n)} r^{-1}(x) \tag{6.3}$$

for some constants c_n. Moreover, the generating function

$$F(x,t) = \sum_{n=0}^{\infty} \frac{p(x,n)}{n! c_n} t^n$$

is analytic in $(x_0, x_1) \times I$ for some open interval I of 0 and is given by

$$F(x,t) = \frac{r(\lambda)}{r(x)} (1 - tq'(\lambda))^{-1}, \tag{6.4}$$

where $\lambda = \lambda(x,t)$ satisfies

$$\lambda - x - tq(\lambda) = 0.$$

The answer to Bochner's question is that the classical orthogonal polynomials are the only solutions to Bochner's problem. This is the content of the next theorem.

Theorem 6.2.1.8 (Bochner[5])**.** *Up to an affine change of coordinates, the only weight functions $r(x)$ which are solutions of Bochner's problem are the classical weight functions.*

We can extend Bochner's problem by asking what other differential operators $\mathfrak{d}$ have the monic orthogonal polynomials of r as common eigenfunctions. Let $\{p(x,n)\}$ be the sequence of monic orthogonal polynomials for some weight function r. The set

$$D(r) = \{\mathfrak{d} \in \mathfrak{D}(\mathbb{C}[x]) : p(x,n) \text{ is an eigenfunction of } \mathfrak{d} \text{ for all } n\}$$

forms a subalgebra of the algebra of differential operators. For (r, ϵ) a Bochner pair, it turns out that the answer is very straightforward.

Theorem 6.2.1.9 (Miranian[51])**.** *If (r, ϵ) is a Bochner pair, then*

$$D(r) = C_{\mathfrak{D}(\mathbb{C}[x])}(\epsilon) = \mathbb{C}[\epsilon],$$

where here $C_{\mathfrak{D}(\mathbb{C}[x])}(\epsilon)$ represents the centralizer of ϵ in the Weyl algebra $\mathfrak{D}(\mathbb{C}[x])$.

Remark 6.2.1.10. We point out that the centralizer $C_{\mathfrak{D}(\mathbb{C}[[x]])}(\epsilon)$ of ϵ in $\mathfrak{D}(\mathbb{C}[[x]])$ may properly contain $C_{\mathfrak{D}(\mathbb{C}[x])}(\epsilon)$. For example, the operator $\epsilon = \partial^2(1-x^2) - \partial x$ commutes with the first-order operator $\partial(1-x^2)^{1/2}$, but the coefficients of the latter operator are not polynomial.

The previous theorem shows that a second-order differential operator ϵ forming part of the Bochner pair (r, ϵ) generates the algebra $D(r)$, and is unique up to a scalar multiple. Values of ϵ for various weight functions are listed in the table in Figure 6.2.

Weight	2nd-order operator
Hermite	$\partial^2 - 2\partial x$
Laguerre	$\partial^2 x + \partial(b + 1 - x)$
Jacobi	$\partial^2(1 - x^2) + \partial(b - a - (a + b + 2)x)$

Figure 6.2: Classical Solutions to Bochner's Problem

6.2.2 Matrix Orthogonal Polynomials and Bochner's Problem

We next review the basic theory of orthogonal matrix polynomials and Bochner's problem.

Definition 6.2.2.1. A **weight matrix** w supported on an interval (x_0, x_1) is defined to be a function $w : \mathbb{R} \to M_N(\mathbb{C})$ satisfying the condition that w vanishes outside of (x_0, x_1), that everywhere in (x_0, x_1) the matrix w is entrywise-smooth, Hermitian, and positive-definite, and finally the condition that w has finite moments:

$$\int x^m w(x) dx < \infty, \quad \forall m \geq 0.$$

The interval (x_0, x_1) where the matrix is nonsingular is called the **support of** w.

Note that we work only with "smooth" weight matrices in order to avoid more delicate analytic considerations. For a survey on the analytic theory of matrix orthogonal polynomials, see [12]. Every weight matrix w defines a matrix-valued inner product $\langle \cdot, \cdot \rangle_w$ on the set $M_N(\mathbb{C}[x])$ of all $N \times N$ complex matrix polynomials by

$$\langle p, q \rangle_w := \int_{\mathbb{R}} p(x) w(x) q(x)^* dx, \quad \forall p, q \in M_N(\mathbb{C}[x]).$$

Though it will not play an important role in our paper, the matrix-valued inner product above begets a traditional (scalar-valued) inner product by taking trace $\mathrm{Tr}(\langle p, q\rangle_w)$.

A Gram-Schmidt type argument with our matrix-valued inner product shows that for each integer $n \geq 0$ there exists a polynomial p_n of degree n with nonsingular leading coefficient, uniquely defined up to its leading coefficient, such that $\langle p_n, p_m\rangle_w = 0I$ for all $m \neq n$. We observe that orthogonality with respect to the matrix-valued inner product is a stronger condition than orthogonality with respect to $\mathrm{Tr}(\langle p, q, \rangle_w)$.

Definition 6.2.2.2. We call a sequence of matrix polynomials $\{p(x, n)\}$ a **sequence of orthogonal matrix polynomials (OMP) for** w if $\deg(p(x, n)) = n$ for all integers $n \geq 0$, with non-singular leading coefficient, and

$$\langle p_n, p_m\rangle_w = 0I, \quad \forall m \neq n.$$

A OMP $\{p(x, n)\}$ for w will be called **monic** if it satisfies the additional constraint that $p(x, n)$ has leading coefficient equal to the identity matrix I for each fixed n. Once again, there exists a unique sequence of monic OMP $\{p(x, n)\}$ for any fixed w.

Generalizing Favard's theorem, Durán and López-Rodriguez proved that a sequence of polynomials $\{p(x, n)\}$ being the sequence of monic orthogonal matrix polynomials for some weight matrix w is equivalent to the sequence $\{p(x, n)\}$ satisfying a three-term recurrence relation under certain regularity conditions.

Theorem 6.2.2.3 (Durán and López-Rodriguez [21])**.** *Suppose that $\{p(x, n)\}$ is a sequence of monic orthogonal matrix polynomials for a weight matrix w. Then there exist sequences of complex-valued matrices $\{s_n\}$ and $\{t_n\}$ such that*

$$xp(x, n) = p(x, n + 1) + s_n p(x, n) + t_n p(x, n - 1), \ \forall n \geq 1 \tag{6.5}$$

Conversely, given reasonably nice sequences of matrices $\{s_n\}, \{t_n\}$, there exists a weight matrix w for which the sequence of polynomials $\{p(x, n)\}$ defined by Equation (6.5) *is a sequence of monic orthogonal matrix polynomials.*

Matrix polynomials $p(x)$ have a natural action by matrix differential operators, ie. differential operators whose coefficients are matrix polynomials. This action is on the right in order to make it compatible with the matrix-valued inner product defined by w and the corresponding three-term recursion relation with coefficients on the left. In particular, for polynomials $a_0(x), a_1(x), a_2(x) \in M_N(\mathbb{C}[x])$ the second-order matrix differential operator $\epsilon = a_0(x) + \partial a_1(x) + \partial^2 a_2(x)$ acts on a matrix polynomial $p(x) \in M_N(\mathbb{C}[x])$ by

$$p(x) \cdot \epsilon = p(x)a_0(x) + p'(x)a_1(x) + p''(x)a_2(x).$$

With this in mind, Bochner's problem was reposed by Durán [17] as the following

Problem 6.2.2.4 (Bochner's Problem for Matrix Differential Operators)**.** Determine all weight matrices $w(x)$ such that there exists a second-order matrix differential operator $\mathfrak{d} \in \mathfrak{D}(M_N(\mathbb{C}[x]))^{op}$ for which the associated sequence of monic orthogonal matrix polynomials are simultaneously eigenfunctions.

For reasons that will be discussed below, when such an $\mathfrak{d}$ exists it may be taken to be w-symmetric. This motivates the following definition.

Definition 6.2.2.5. We define a (matrix) **Bochner pair** to be a pair $(w, \mathfrak{d})$ with w a weight matrix and $\mathfrak{d} \in \mathfrak{D}(M_N(\mathbb{C}[x]))^{op}$ a w-symmetric second-order differential operator for which the sequence of monic OMP for w are simultaneously eigenfunctions.

Thus we are interested in the question of what Bochner pairs exist. More generally, we are interested in the question of calculating the algebra $D(w)$ of all matrix differential operators having the monic OMP of w as simultaneous eigenfunctions.

Definition 6.2.2.6. Let $w(x)$ be a weight matrix, and let $p(x, n)$ be the associated sequence of monic orthogonal matrix polynomials. Then the **algebra of matrix differential operators associated to** w, denoted $D(w)$, is the set of all $\mathfrak{d} \in \mathfrak{D}(M_N(\mathbb{C}[x]))^{op}$ such that $p(x, n)$ is an eigenfunction of $\mathfrak{d}$ for all n.

The behavior in the scalar case ($N = 1$) leads us to ask questions such as whether the solutions to Bochner's problem for matrix differential operators must satisfy a form of Pearson's equation or a Rodrigues-type recurrence relation. Durán and Grünbaum [18] provide a partial result in this direction in the following theorem, restated in terms of formal adjoints.

Theorem 6.2.2.7 (Durán-Grünbaum). *Let w be a weight matrix. Then $D(w)$ contains a second-order differential operator if and only if there exists*

$$\mathfrak{d} = \partial^2 a_2 + \partial a_1 + a_0,$$

with $a_i \in M_N(\mathbb{C}[x])$ and $\deg(a_i) \leq i$ for all i satisfying

(a) $a_2 w = w a_2^*$

(b) $w \cdot \mathfrak{d}^* = a_0 w$ for $\mathfrak{d}^*$ the formal adjoint of $\mathfrak{d}$

(c) $a_2 w$ and $(a_2 w)' - a_1 w$ vanish as x approaches the endpoints of the support of w

In this case, $(w, \mathfrak{d})$ is a solution to Bochner's problem for matrix differential operators.

Note that the first two conditions translate to the statement that $\mathfrak{d}$ is formally w-symmetric, while the third condition implies that the formal w-adjoint of $\mathfrak{d}$ is an adjoint (see Def. 6.2.3.3). In other words, the content of the above theorem is that a weight matrix w has a second-order differential operator in $D(w)$ if and only if there exists a w-symmetric second-order differential operator. The above conditions imply that w satisfies the non-commutative Pearson equation

$$2(a_2 w)' = a_1 w + w a_1^*. \tag{6.6}$$

6.2.3 Adjoints of Differential Operators

The study of adjoints of differential operators can be considered a subset of the study of adjoints of unbounded linear operators on certain well-chosen Hilbert spaces. For this reason, we briefly recall the definition of an unbounded linear operator and its adjoint. For a great reference on adjoints of unbounded linear operators, as well as adjoints of differential operators, see [16].

Definition 6.2.3.1. Let $\mathcal{H}$ be a Hilbert space. An **unbounded linear operator** on $\mathcal{H}$ is a linear function $T : \mathcal{D}(T) \to V$, where $\mathcal{D}(T)$ is a dense subspace of $\mathcal{H}$, called the **domain of** T. The adjoint of T is an unbounded linear operator T^* defined on the set

$$\mathcal{D}(T^*) := \{y \in \mathcal{H} : x \to \langle Tx, y\rangle \text{ is continuous}\}$$

and satisfying $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in D(T), y \in D(T^*)$.

Note that the fact that $D(T)$ is dense is important for the existence of T^*, which then exists as a consequence of the Hahn-Banach theorem.

Differential operators make great examples of unbounded linear operators. An interesting question is when for given differential operator the adjoint is also a differential operator.

Example 6.2.3.2. For example, consider the first-order differential operator $\mathfrak{d} = \partial a_1(x) + a_0(x)$ with $a_0(x), a_1(x) \in \mathbb{C}[x]$. Given a weight function $r(x)$ supported on (x_0, x_1), we may view this as an unbounded linear operator on the inner product space $L^2(r(x)dx)$, induced by the action of $\mathfrak{d}$ on $\mathbb{C}[x]$.

Using integration by parts, one finds that for polynomials $p(x), q(x)$:

$$\begin{aligned}\langle p(x)\cdot\mathfrak{d}, q(x)\rangle_r &= \int p'(x)a_1(x)r(x)q(x)^* + p(x)a_0(x)r(x)q(x)^*\\ &= p(x_1)a_1(x_1)r(x_1)q(x_1) - p(x_0)a_0(x_0)r(x_0)q(x_0)\\ &+ \langle p(x), q(x)(-(a_1(x)^*r(x)\partial r(x)^{-1} + a_0(x)^*)\rangle\end{aligned}$$

Thus if $a_1(x_0)r(x_0) = a_1(x_1)r(x_1) = 0$, we see that the adjoint of $\mathfrak{d}$ is the differential operator $\mathfrak{d}^\dagger = -a_1(x)^*r(x)\partial r(x)^{-1} + a_0^*(x)$. In particular, if $r(x)$ is constant on its support, and $a_1(x_i) = 0$ for $i = 0, 1$, then the adjoint $\mathfrak{d}^\dagger$ of $\mathfrak{d}$ is $\mathfrak{d}^\dagger = -a_1(x)^*\partial + a_0^*(x)$.

A similar observation applies to matrix orthogonal polynomials, and this motivates the definition of a $*$-operation on the algebra of matrix differential operators.

The algebra $\mathfrak{D}(M_N(\mathbb{C}[[x]]))^{op}$ of matrix differential operators with coefficients in the power series ring $\mathbb{C}[[x]]$ is equipped with a canonical $*$-operation, extending the usual $*$-operation on $M_N(\mathbb{C}[[x]])$ defined by Hermitian conjugates and satisfying

$$(xI)^* = xI, \quad (\partial I)^* = -\partial I.$$

Definition 6.2.3.3. Let w be a weight matrix supported on an interval (x_0, x_1) containing 0 and let $\mathfrak{d} \in \mathfrak{D}(M_N(\mathbb{C}[[x]]))^{op}$. A **formal adjoint of $\mathfrak{d}$ with respect to** w, or **formal w-adjoint of $\mathfrak{d}$** is the unique differential operator $\mathfrak{d}^\dagger \in \mathfrak{D}(M_N(\mathbb{C}[[x]]))^{op}$ defined on (x_0, x_1) by

$$\mathfrak{d}^\dagger := w\mathfrak{d}^* w^{-1}.$$

(Note that this definition relies on the assumption that w is Hermitian.) In particular, the $*$-operation is exactly the formal w-adjoint for $w(x) = 1_{(x_0,x_1)}(x)I$. The assumption that 0 is in (x_0, x_1) may always be achieved by means of an affine change of coordinates, and is necessary for w to have a local representation as a power series based at 0. Moreover, the assumption that w is positive-definite also implies that w^{-1} has a power series expansion at 0. Consequently, the formal w-adjoint is indeed an element of $\mathfrak{D}(M_N(\mathbb{C}[[x]]))^{op}$. As a notational point, we will use $*$ to denote the canonical adjoint on $\mathfrak{D}(M_N(\mathbb{C}[[x]]))^{op}$ and will use $\dagger$ to denote the formal adjoint with respect to a particular weight matrix w (the value of w will be implied from the context).

Definition 6.2.3.4. Let w be a weight matrix, and $\mathfrak{d} \in \mathfrak{D}(M_N(\mathbb{C}[[x]]))^{op}$ a differential operator. Then $\mathfrak{d}$ is called **formally w-symmetric** if $\mathfrak{d}^\dagger = \mathfrak{d}$.

We next define the adjoint of a differential operator. To do so, we must define the Hilbert space upon which it acts.

Definition 6.2.3.5. Let w be a weight matrix. We define the **Hilbert space of** w to be

$$\mathcal{H}(w) := M_N(L^2(\mathrm{Tr}(w)dx)).$$

We call a matrix differential operator $\mathfrak{d} \in \mathfrak{D}(M_N(\mathbb{C}[[x]]))^{op}$ **amenable** if $p(x) \cdot \mathfrak{d} \in \mathcal{H}(w)$ for all matrix-valued polynomials $p(x) \in M_N(\mathbb{C}[x])$.

Definition 6.2.3.6. Let w be a weight matrix and let $\mathfrak{d} \in \mathfrak{D}(M_N(\mathbb{C}[[x]]))^{op}$ be an amenable matrix differential operator. An amenable matrix differential operator $\widetilde{\mathfrak{d}} \in \mathfrak{D}(M_N(\mathbb{C}[x]))^{op}$ satisfying the identity

$$\langle p \cdot \mathfrak{d}, q \rangle_w = \langle p, q \cdot \widetilde{\mathfrak{d}} \rangle_w, \ \forall p, q \in M_N(\mathbb{C}[x]),$$

is called an **adjoint of $\mathfrak{d}$ with respect to** w, or w**-adjoint of $\mathfrak{d}$**. If $\mathfrak{d} = \widetilde{\mathfrak{d}}$, then $\mathfrak{d}$ is called w**-symmetric**.

Every differential operator has a formal w-adjoint, but not necessarily a w-adjoint. Put another way, even though each differential operator $\mathfrak{d}$ will have an adjoint as an unbounded operator on $\mathcal{H}(w)$, this linear operator adjoint need not be a differential operator.

Proposition 6.2.3.7. *Let w be a weight matrix and let $\mathfrak{d} \in \mathfrak{D}(M_N(\mathbb{C}[[x]]))^{op}$ be an amenable matrix differential operator. Then an adjoint of $\mathfrak{d}$ with respect to w, if one exists, is equal to the formal w-adjoint $\mathfrak{d}^\dagger$.*

Example 6.2.3.8. Consider the weight function $r(x) = e^{-x}1_{(0,\infty)}(x)$ supported on the interval $(0, \infty)$. The formal r-adjoint of the differential operator ∂ is

$$\partial^\dagger = e^{-x}\partial^* e^x = e^{-x}(-\partial)e^x = -\partial + 1.$$

However,

$$\langle 1 \cdot \partial, 1 \rangle_r = \langle 0, 1 \rangle_r = 0$$

and

$$\langle 1, 1 \cdot \partial^\dagger \rangle_r = \langle 1, 1 \rangle_r = 1.$$

Therefore $\partial^\dagger$ is not an r-adjoint for ∂ and thus ∂ has no r-adjoint.

As another example, the formal adjoint of the operator ∂x is

$$(\partial x)^\dagger = e^{-x}(\partial x)^* e^x = e^{-x}(-x\partial)e^x = -\partial x + x - 1.$$

Moreover, integration by parts shows us that

$$\langle p'(x)x, q(x)\rangle_r = \langle p(x), -q'(x)x + q(x)(x-1)\rangle_r$$

for all polynomials $p(x), q(x) \in \mathbb{C}[x]$. Therefore $-\partial x + x - 1$ is in fact an r-adjoint of ∂x.

Expanding on these examples, we have the following lemma

Lemma 6.2.3.9. *Let $\mathfrak{v} = \partial f_1(x) + f_0(x)$ be a matrix differential operator for $f_1, f_0 \in M_N(\mathbb{C}[x])$. If $f_1(x_i)w(x_i) = 0$ for $i = 0, 1$ then $\mathfrak{v}$ has a w-adjoint.*

Proof. For any polynomials $p(x), q(x) \in M_N(\mathbb{C}[x])$, integration by parts tells us that

$$\langle p(x) \cdot \mathfrak{v}, q(x)\rangle_w = p(x)f_1(x)w(x)q^*(x)|_{x_0}^{x_1} + \langle p(x), q(x) \cdot \mathfrak{v}^\dagger\rangle.$$

Thus if $f_1(x_i)w(x_i) = 0$ for $i = 0, 1$ the formal w-adjoint $\mathfrak{v}^\dagger$ of $\mathfrak{v}$ is the w-adjoint. □

The application of adjoints of differential operators to our situation is provided by the following theorem.

Theorem 6.2.3.10 (Grünbaum-Tirao [40])**.** *Let w be a weight matrix, and let $\mathfrak{d} \in D(w)$. Then a w-adjoint of $\mathfrak{d}$ exists and is in $D(w)$. The operator $\mathfrak{d} \mapsto \mathfrak{d}^\dagger$ is an involution on $D(w)$ giving $D(w)$ the structure of a $*$-algebra.*

As a consequence, we have the following corollary originally proved in [40].

Corollary 6.2.3.10.1. *Let w be a weight matrix. Then $D(w)$ contains a differential operator of order m if and only if $D(w)$ contains a w-symmetric differential operator of order m.*

Proof. If $D(w)$ contains a second-order matrix differential operator ω, then by the previous theorem $D(w)$ contains $\omega + \omega^\dagger$ and $i(\omega - \omega^\dagger)$. Both operators are clearly symmetric, and by definition of the adjoint of order at most m. Since

$$\omega = \frac{1}{2}(\omega + \omega^\dagger) - \frac{1}{2}i(i(\omega - \omega^\dagger))$$

and ω is of order m, at least one of the two symmetric operators must be order m. □

6.3 *Proof of the Main Theorem*

6.3.1 Degree-Preserving Differential Operators

Definition 6.3.1.1. We call a matrix differential operator $\mathfrak{d} \in \mathfrak{D}(M_N(\mathbb{C}[x]))^{op}$ **degree-filtration preserving** if for all polynomials $q(x) \in M_N(\mathbb{C}[x])$, the degree of $q(x) \cdot \mathfrak{d}$ is no greater than the degree of $q(x)$. We call $\mathfrak{d}$ **degree-preserving** if the degree of $q(x) \cdot \mathfrak{d}$ is equal to the degree of $q(x)$ for all $q(x) \in M_N(\mathbb{C}[x])$. In particular, degree-preserving differential operators necessarily act injectively on the algebra of matrix polynomials.

We use $(\mathfrak{D}(M_N(\mathbb{C}[x]))^{op})^F$ to denote the subalgebra of $\mathfrak{D}(M_N(\mathbb{C}[x]))^{op}$ of degree-filtration preserving matrix differential operators. Due to its distinguished role in the following, we fix the notation $s = \partial x$.

Lemma 6.3.1.2. *The subalgebra* $(\mathfrak{D}(M_N(\mathbb{C}[x]))^{op})^F$ *is equal to the* $M_N(\mathbb{C})$*-subalgebra of* $\mathfrak{D}(M_N(\mathbb{C}[x]))^{op}$ *generated by* s *and* ∂

Proof. Note that both s and ∂ are degree-filtration preserving, and therefore the $M_N(\mathbb{C})$-subalgebra of $\mathfrak{D}(M_N(\mathbb{C}[x]))^{op}$ that they generate is contained in $(\mathfrak{D}(M_N(\mathbb{C}[x]))^{op})^F$. Thus to prove our lemma, it suffices to show the opposite containment. Suppose that $\mathfrak{d}$ is degree-filtration preserving, of order n. Since $\mathfrak{d} \in \mathfrak{D}(M_N(\mathbb{C}[x]))^{op}$, we know that

$$\mathfrak{d} = \sum_{i=0}^{n} \partial^i a_i(x)$$

for some $a_i(x) \in M_N(\mathbb{C}[x])$.

We claim that $\deg(a_i(x)) \leq i$ for all $0 \leq i \leq n$. To see this, suppose otherwise. Then let j be the smallest nonnegative integer satisfying $\deg a_j(x) > j$. Then

$$x^j I \cdot \mathfrak{d} = \sum_{i=0}^{j} \frac{j!}{(j-i)!} x^{j-i} a_i(x)$$

is a polynomial of degree greater than j, contradicting the assumption that $\mathfrak{d}$ is degree-filtration preserving. This proves our claim.

Next note that for all integers $j \geq 1$,

$$\partial^j x^j = s(s-1)(s-2)\dots(s-j+1).$$

Therefore if $a(x) \in M_N(x)$ is of degree $\leq j$, then $\partial^j a_j(x)$ is in the $M_N(\mathbb{C})$-subalgebra of $\mathfrak{D}(M_N(\mathbb{C}[x]))^{op}$ generated by ∂ and s. From this it follows that $\mathfrak{d}$ is in the subalgebra of $\mathfrak{D}(M_N(\mathbb{C}[x]))^{op}$ generated by s and ∂. This proves our lemma. □

Remark 6.3.1.3. The expression $\partial^j x^j = s(s-1)(s-2)\dots(s-j+1)$ found in the proof above is exactly the reason why the substitution $x = e^t$ may be used to re-express an Euler-Cauchy equation as a linear ordinary differential equation with constant coefficients.

Proposition 6.3.1.4. *The natural map*

$$\bigoplus_{n=0}^{\infty} \partial^n M_N(\mathbb{C}[s]) \to (\mathfrak{D}(M_N(\mathbb{C}[x]))^{op})^F$$

is an isomorphism of $M_N(\mathbb{C}[s])$-modules. In particular, every $\mathfrak{d} \in (\mathfrak{D}(M_N(\mathbb{C}[x]))^{op})^F$ has a natural expression of the form

$$\mathfrak{d} = \sum_{i=0}^{n} \partial^i a_i(s)$$

for some integer $n \geq 0$ and some matrices $a_i(s) \in M_N(\mathbb{C}[s])$.

Proof. Note that $s\partial^j = \partial^j(s+j)$, and therefore $a(s)\partial^j = \partial^j a(s+j)$ for all $a(s) \in M_N(\mathbb{C}[s])$. Using this relation, the above follows. □

Proposition 6.3.1.5. *Let w be a weight matrix, and let $p(x,n)$ be a sequence of monic orthogonal matrix polynomials for w. If $\mathfrak{d} \in D(w)$, then $\mathfrak{d}$ is degree-filtration preserving. Consequently $\mathfrak{d} = \sum_{i=0}^{n} \partial^i a_i(s)$, for some $a_i(s) \in M_N(\mathbb{C}[s])$, in which case*

$$p(x,n) \cdot \mathfrak{d} = a_0(n)p(x,n), \ \forall n \in \mathbb{Z}_+.$$

Proof. Suppose that $\mathfrak{d} \in D(w)$. Then for all integers $i \geq 0$, there exists $\lambda_i \in M_N(\mathbb{C})$ such that $p(x,i) \cdot \mathfrak{d} = \lambda_i p(x,i)$. Furthermore, since the $p(x,i)$ form a basis for $M_N(\mathbb{C}[x])$, given a

$q(x) \in M_N(\mathbb{C}[x])$ of degree n, we may write $q(x) = \sum_{i=0}^n c_i p(x,i)$ for some $c_0, \dots c_n \in M_N(\mathbb{C})$. Then

$$q(x) \cdot \mathfrak{D} = \sum_{i=0}^n c_i \lambda_i p(x,i),$$

which has degree at most n. Since $q(x)$ was arbitrary, this shows that $\mathfrak{D}$ is degree-filtration preserving. Therefore by the previous proposition, $\mathfrak{D} = \sum_{i=0}^r \partial^i a_i(s)$ for some $a_i(s) \in M_N(\mathbb{C}[s])$.

We know that $p(x,n) = Ix^n +$ (lower degree terms), and therefore

$$p(x,n) \cdot \mathfrak{D} = \sum_{i=0}^r p(x,n) \cdot \partial^i a_i(s) = a_0(n)x^n + \text{(lower degree terms)}.$$

Therefore since $p(x,n) \cdot \mathfrak{D} = \lambda_n p(x,n) = \lambda_n x^n +$(lower degree terms), we have that $\lambda_n = a_0(n)$ for all integers $n \geq 0$. □

The previous proposition in particular shows that if $p(x,n)$ is a sequence of monic orthogonal matrix polynomials for a weight matrix w and if $\mathfrak{D} \in D(w)$, then the sequence $\{\lambda_n\}_{n=0}^\infty \subseteq M_N(\mathbb{C})$ satisfying $\lambda_n p(x,n) = p(x,n) \cdot \mathfrak{D}$ will be a polynomial in n. Thus we may define a map $\Lambda : D(w) \to M_N(\mathbb{C}[n])$ satisfying the property

$$\Lambda(\mathfrak{D})(n)p(x,n) = p(x,n) \cdot \mathfrak{D}.$$

We denote the image of the map Λ in $M_N(\mathbb{C}[n])$ by $E(w)$.

Definition 6.3.1.6. We call the subalgebra $E(w)$ the **algebra of eigenvalue sequences associated to the weight matrix** w. We call the map Λ the **eigenvalue isomorphism**.

The eigenvalue homomorphism defines an injection of $D(w)$ into $M_N(\mathbb{C}[n])$. This was shown in [40], but is reproved here in our notation.

Proposition 6.3.1.7. *The map Λ is injective, and in particular defines an isomorphism of $D(w)$ onto $E(w)$.*

Proof. Suppose that $\Lambda(\mathfrak{d}) = \Lambda(\mathfrak{d}') = \lambda(n) \in M_N(\mathbb{C}[n])$ for some $\mathfrak{d}, \mathfrak{d}' \in D(w)$. Then this in particular implies that for all n,

$$p(x,n) \cdot (\mathfrak{d} - \mathfrak{d}') = (\lambda(n) - \lambda(n))p(x,n) = 0.$$

It follows that the kernel of $\mathfrak{d} - \mathfrak{d}'$ contains $M_N(\mathbb{C})\{p_0, p_1, \dots\} = M_N(\mathbb{C}[x])$. Therefore $\mathfrak{d} - \mathfrak{d}'$ must be identically zero. □

6.3.2 The Proof

Before proving the Main Theorem, we introduce one more lemma to help with some moment estimates.

Lemma 6.3.2.1. *Suppose that* $a, b \in M_N(\mathbb{C})$ *are positive-semidefinite. Then*

$$\mathrm{Tr}(ab) \leq \mathrm{Tr}(a)\mathrm{Tr}(b).$$

Proof. By Cauchy-Schwartz,

$$\mathrm{Tr}(ab) \leq (\mathrm{Tr}(a^2)\mathrm{Tr}(b^2))^{1/2}.$$

Furthermore, since a is positive-semidefinite,

$$\mathrm{Tr}(a^2) \leq \mathrm{Tr}(a)^2$$

and similarly for b. From this the statement of the lemma follows immediately. □

We now have everything in place for the proof of the Main Theorem (Theorem (6.1.0.3)).

Proof of Main Theorem.

(a) Note that since $\det(v_1(x)) \in \mathbb{C}\backslash\{0\}$, $v_1(x)^{-1}$ is well-defined. Moreover since $w(x)$ is smooth and positive-definite on its support (x_0, x_1), we may factor $w(x) = u(x)u(x)^*$ for some smooth function $u(x)$ of full rank on (x_0, x_1). Next, since $\mathfrak{d}$ is w-symmetric, the leading coefficient a_2 must be w-symmetric also. Therefore $v_1^{-1}a_2(v_1^{-1})^\dagger$ is also w-symmetric, and consequently $u^{-1}v_1^{-1}a_2(v_1^{-1})^\dagger u$ is smooth and Hermitian on (x_0, x_1). Thus we may factor it as

$$u^{-1}v_1^{-1}a_2(v_1^{-1})^\dagger u = hh^*$$

for some smooth function h on (x_0, x_1). Taking $f = uhu^{-1}$, we have that

$$ff^\dagger = uhu^{-1}w(uhu^{-1})^*w^{-1} = uhh^*u^{-1} = v_1^{-1}a_2(v_1^{-1})^\dagger$$

and therefore f satisfies the properties stated in (a)

(b) From the definition of f, we have that $|\det(f)|^2 = |\det(h)|^2 = |\det(v)|^{-2}\det(a_2) \neq 0$ on (x_0, x_1). Therefore $f(x)$ has full rank for all $x \in (x_0, x_1)$, and it follows that $\widetilde{w}(x) = f(x)w(x)f(x)^*$ is positive-definite for all $x \in (x_0, x_1)$. Moreover, $\widetilde{w}(x)$ is smooth on (x_0, x_1), since it is the product of three smooth matrix-valued functions. Therefore to prove that $\widetilde{w}(x)$ is a weight matrix, all that is left to show is that $\widetilde{w}(x)$ has finite moments. Equivalently, we must show

$$\int_{x_0}^{x_1} |x|^n \mathrm{Tr}(\widetilde{w})dx < \infty.$$

To show this, first note that

$$\mathrm{Tr}(\widetilde{w}) = \mathrm{Tr}(fuu^*f^*) = \mathrm{Tr}(uhh^*u^*) = \mathrm{Tr}(hh^*u^*u).$$

By Lemma (6.3.2.1)

$$\begin{aligned}\mathrm{Tr}(hh^*u^*u) &\leq \mathrm{Tr}(hh^*)\mathrm{Tr}(u^*u) = \mathrm{Tr}(u^{-1}v_1^{-1}a_2(v_1^{-1})^\dagger u)\mathrm{Tr}(w)\\ &= \mathrm{Tr}((v_1^{-1})^\dagger a_2 v_1^{-1})\mathrm{Tr}(w)\end{aligned}$$

and therefore

$$\int_{x_0}^{x_1} |x|^n \mathrm{Tr}(\widetilde{w})dx < \int_{x_0}^{x_1} |x|^n \mathrm{Tr}(v_1^{-1}(x)a_2(x)(v_1^{-1}(x))^\dagger)\mathrm{Tr}(w(x))dx < \infty$$

by Hölder's inequality and the fact that $w(x)$ has finite moments. Therefore $\widetilde{w}$ is a weight matrix.

(c) We calculate

$$\begin{aligned}\mathfrak{y} &= -v_1^{-1}a_2(v_1^{-1})^\dagger \mathfrak{v}^\dagger \\ &= -v_1^{-1}a_2(\partial - v_0v_1^{-1})^\dagger \\ &= -v_1^{-1}a_2(-\partial - w'w^{-1} - (v_0v_1^{-1})^\dagger) \\ &= \partial v_1^{-1}a_2 + v_1^{-1}a_2w'w^{-1} + v_1^{-1}a_2(v_0v_1^{-1})^\dagger + (v_1^{-1}a_2)'\end{aligned}$$

Furthermore, the noncommutative Pearson equation tells us that

$$(v_1^{-1}a_2)' = (v_1^{-1})'a_2 + v_1^{-1}\frac{1}{2}(a_1 + a_1^\dagger) - v_1^{-1}a_2w'w^{-1}.$$

Substituting this in and simplifying, we find

$$\mathfrak{y} = \partial v_1^{-1}a_2 - v_0^{-1}a_0 + v_0^{-1}[(v_0v_1^{-1})a_2(v_0v_1^{-1})^\dagger - (v_0v_1^{-1})^2a_2]$$

Then using the fact that $(v_0v_1^{-1})^\dagger = v_0v_1^{-1}$ the last summand cancels out, leaving

$$\mathfrak{y} = \partial v_1^{-1}a_2 - v_0^{-1}a_0$$

Using this, we calculate

$$\begin{aligned}\mathfrak{v}\mathfrak{y} &= (\partial v_1 - v_0)(\partial v_1^{-1}a_2 - v_0^{-1}a_0) \\ &= \partial^2 a_2 + \partial[(v_1)'v_1^{-1}a_2 - v_0v_1^{-1}a_2 - v_1v_0^{-1}a_0] + a_0 \\ &= \partial^2 a_2 + \partial[-(v_0v_1^{-1})^{-1}(v_0v_1^{-1})'a_2 - v_0v_1^{-1}a_2 - v_1v_0^{-1}a_0] + a_0 \\ &= \partial^2 a_2 + \partial - (v_0v_1^{-1})^{-1}[(v_0v_1^{-1})'a_2 + (v_0v_1^{-1})^2a_2 + a_0] + a_0 \\ &= \partial^2 a_2 + \partial a_1 + a_0 = \mathfrak{d}.\end{aligned}$$

Next note that $\mathfrak{v}$ is degree-filtration preserving. If $\mathfrak{v}$ is not degree preserving, then there exists a nonzero polynomial $p(x)$ satisfying $p(x)\cdot\mathfrak{v} = 0$. Then

$$\deg(p) = \deg(p\cdot\mathfrak{d}) = \deg(p\cdot\mathfrak{v}\cdot\mathfrak{v}^{-1}\mathfrak{d}) = \deg(0) = -\infty.$$

This is a contradiction, and therefore $\mathfrak{v}$ is degree-preserving. Since $\mathfrak{d} = \mathfrak{v}\mathfrak{y}$ and $\mathfrak{v}$ are degree-preserving, it follows that $\mathfrak{y}$ is also degree-preserving.

(d) The assumption that $\lim_{x\to x_i}(|x|^n+1)w(x) = 0$ implies that $\mathfrak{v}$ is w-adjointable by Lemma (6.2.3.9). Also, we note

$$\begin{aligned}\langle p(x,m)\cdot\mathfrak{v}f, p(x,n)\cdot\mathfrak{v}f\rangle_w &= \int_{x_0}^{x_1} p(x,n)\cdot\mathfrak{v}f(x)w(x)(p(x,n)\cdot\mathfrak{v}f)^*dx \\ &= \int_{x_0}^{x_1} p(x,n)\cdot\mathfrak{v}w(x)(w(x)^{-1}f(x)w(x))(p(x,n)\cdot\mathfrak{v}f)^*dx \\ &= \int_{x_0}^{x_1} p(x,n)\cdot\mathfrak{v}w(x)(f^\dagger)^*(p(x,n)\cdot\mathfrak{v}f)^*dx \\ &= \int_{x_0}^{x_1} p(x,n)\cdot\mathfrak{v}w(x)(p(x,n)\cdot\mathfrak{v}ff^\dagger)^*dx \\ &= \langle p(x,m)\cdot\mathfrak{v}, p(x,n)\cdot\mathfrak{v}ff^\dagger\rangle_w\end{aligned}$$

Using this, we calculate

$$\begin{aligned}\langle\widetilde{p}(x,m),\widetilde{p}(x,n)\rangle_{\widetilde{w}} &= \langle p(x,m)\cdot\mathfrak{v}, p(x,n)\cdot\mathfrak{v}\rangle_{\widetilde{w}} = \langle p(x,m)\cdot\mathfrak{v}f, p(x,n)\cdot\mathfrak{v}f\rangle_w \\ &= \langle p(x,m)\cdot\mathfrak{v}, p(x,n)\cdot\mathfrak{v}ff^\dagger\rangle_w = \langle p(x,m), p(x,n)\cdot\mathfrak{v}ff^\dagger\mathfrak{v}^\dagger\rangle_w \\ &= \langle p(x,m), p(x,n)\cdot(\mathfrak{v}\mathfrak{y})\rangle_w = \langle p(x,m), p(x,n)\cdot\mathfrak{d}\rangle_w\end{aligned}$$

Since $\mathfrak{d}$ is degree-filtration preserving, the above formula implies that the polynomials $\widetilde{p}(x,n)$ will be orthogonal with respect to $\widetilde{w}$.

(e) We know that $p(x,n)\cdot\mathfrak{d} = \lambda_n p(x,n)$ for some $\lambda_n \in M_N(\mathbb{C})$ for all $n \geq 0$. We calculate

$$\widetilde{p}(x,n)\cdot\widetilde{\mathfrak{d}} = p(x,n)\cdot(\mathfrak{v}\mathfrak{y}\mathfrak{v}) = p(x,n)\cdot(\mathfrak{d}\mathfrak{v}) = \lambda_n p(x,n)\cdot\mathfrak{v} = \lambda_n\widetilde{p}(x,n).$$

Since $\widetilde{p}(x,n)$ is a sequence of orthogonal matrix polynomials for $\widetilde{w}$, this shows that $\widetilde{\mathfrak{d}} \in D(\widetilde{w})$. To complete the proof, we must show that $\widetilde{\mathfrak{d}}$ is $\widetilde{w}$-symmetric. Since $\widetilde{\mathfrak{d}} \in D(\widetilde{w})$, we know that $\widetilde{\mathfrak{d}}$ is $\widetilde{w}$-adjointable, so it suffices to prove that $\widetilde{\mathfrak{d}}$ is formally $\widetilde{w}$-symmetric, ie. that

$$\widetilde{w}(\widetilde{\mathfrak{d}})^*(\widetilde{w})^{-1} = \widetilde{\mathfrak{d}}.$$

We calculate

$$\begin{aligned}\widetilde{w}(\widetilde{\mathfrak{d}})^*(\widetilde{w})^{-1} &= fwf^*(ff^\dagger\mathfrak{v}^\dagger\mathfrak{v})^*(f^*)^{-1}w^{-1}f^{-1}\\ &= fwf^*\mathfrak{v}^*(\mathfrak{v}^\dagger)^*(f^\dagger)^*f^*(f^*)^{-1}w^{-1}f^{-1}\\ &= fwf^*\mathfrak{v}^*(\mathfrak{v}^\dagger)^*(f^\dagger)^*w^{-1}f^{-1} = fw(\mathfrak{v}f)^*(\mathfrak{v}f)^{\dagger *}w^{-1}f^{-1}\\ &= fw((\mathfrak{v}f)^\dagger(\mathfrak{v}f))^*w^{-1}f^{-1} = f((\mathfrak{v}f)^\dagger(\mathfrak{v}f))^\dagger f^{-1}\\ &= f(\mathfrak{v}f)^\dagger(\mathfrak{v}f)f^{-1} = ff^\dagger\mathfrak{v}^\dagger\mathfrak{v} = \widetilde{\mathfrak{d}}.\end{aligned}$$

This proves (f).

(f) Suppose that $\widetilde{\mathfrak{d}}' \in (\mathfrak{v}^{-1}D(w)\mathfrak{v}) \cap M_N(\Omega)$. Then there exists $\mathfrak{d}' \in D(w)$ satisfying $\mathfrak{d}'\mathfrak{v} = \mathfrak{v}\widetilde{\mathfrak{d}}'$. Moreover, there exists a sequence $\lambda_0', \lambda_1', \dots \in M_N(\mathbb{C})$ satisfying $p(x,n)\cdot\mathfrak{d}' = \lambda_n' p(x,n)$ for all $n \geq 0$. Therefore we have that

$$\widetilde{p}(x,n)\cdot\widetilde{\mathfrak{d}}' = p(x,n)\cdot\mathfrak{v}\widetilde{\mathfrak{d}}' = p(x,n)\cdot(\mathfrak{d}'\mathfrak{v}) = \lambda_n' p(x,n)\cdot\mathfrak{v} = \lambda_n'\widetilde{p}(x,n).$$

This shows that $\mathfrak{d}' \in D(\widetilde{w})$, proving that $(\mathfrak{v}^{-1}D(w)\mathfrak{v}) \cap M_N(\Omega) \subseteq D(\widetilde{w})$. The fact that $(\mathfrak{v}^{-1}D(w)\mathfrak{v}) \cap M_N(\Omega) = \{\mathfrak{v}^{-1}\mathfrak{d}\mathfrak{v} : \mathfrak{d} \in D(w),\ \ker(\mathfrak{v})\cdot\mathfrak{d} \subseteq \ker(\mathfrak{v})\}$ follows from the results of Chapter 2.

□

We next prove Proposition 6.1.0.5. Before doing so, we establish the following lemma.

Lemma 6.3.2.2. *Let $\mathfrak{v}$ be a Darboux transformation from a Bochner pair $(w,\mathfrak{d})$ to a Bochner pair $(\widetilde{w},\widetilde{\mathfrak{d}})$. Then $D(\widetilde{w})\widetilde{\mathfrak{d}} \subseteq D(\widetilde{w},\mathfrak{v},w)$.*

Proof. Suppose that $\widetilde{\eta} \in D(\widetilde{w})$. Then there exists $\mathfrak{y} \in \mathfrak{D}(M_N(\mathbb{C}[[x]]))^{op}$ satisfying $\mathfrak{d} = \mathfrak{v}\mathfrak{y}$ and $\widetilde{\mathfrak{d}} = \mathfrak{y}\mathfrak{v}$ and therefore

$$\eta := \mathfrak{v}(\widetilde{\eta}\widetilde{\mathfrak{d}})\mathfrak{v}^{-1} = \mathfrak{v}\widetilde{\eta}\mathfrak{y} \in \mathfrak{D}(M_N(\mathbb{C}[[x]]))^{op}.$$

Let $\{p(x,n)\}$ be a sequence of orthogonal matrix polynomials for w. Then there exists a sequence $\lambda_0, \lambda_1, \dots \in M_N(\mathbb{C})$ satisfying $p(x,n)\cdot\mathfrak{d} = \lambda_n p(x,n)$ for all $n \geq 0$. Also $\widetilde{p}(x,n) :=$

$p(x,n)\cdot\mathfrak{v}$ defines a sequence of orthogonal matrix polynomials for $\widetilde{w}$, and therefore there exists a sequence $\widetilde{\lambda}_0,\widetilde{\lambda}_1,\dots\in M_N(\mathbb{C})$ satisfying $\widetilde{\lambda}_n\widetilde{p}(x,n)=\widetilde{p}(x,n)\cdot\widetilde{\eta}$. We calculate

$$\begin{aligned} p(x,n)\cdot\eta &= p(x,n)\cdot(\mathfrak{v}\widetilde{\eta}\mathfrak{y}) = \widetilde{p}(x,n)\cdot(\widetilde{\eta}\mathfrak{y}) \\ &= \widetilde{\lambda}_n\widetilde{p}(x,n)\cdot\mathfrak{y} = \widetilde{\lambda}_n p(x,n)\cdot(\mathfrak{v}\mathfrak{y}) \\ &= \widetilde{\lambda}_n p(x,n)\cdot\mathfrak{d} = \widetilde{\lambda}_n\lambda_n p(x,n). \end{aligned}$$

Therefore $\eta\in D(w)$. Furthermore $\mathfrak{v}^{-1}\eta\mathfrak{v}=\widetilde{\eta}\widetilde{\mathfrak{d}}$, and it follows that $\widetilde{\eta}\widetilde{\mathfrak{d}}\in D(\widetilde{w},\mathfrak{v},w)$. Since $\widetilde{\eta}\in D(\widetilde{w})$ was arbitrary, this proves our Lemma. □

Proof of Proposition 6.1.0.5. Suppose that $\widetilde{\mathfrak{d}}$ and ℓ satisfy the assumptions of the statement of the proposition. Then for all $\widetilde{\eta}\in\mathfrak{D}(M_N(\mathbb{C}[x]))^{op}$, the order of $\widetilde{\eta}\widetilde{\mathfrak{d}}$ is the order of $\widetilde{\eta}$ plus two.

By the previous lemma, $D(\widetilde{w})\widetilde{\mathfrak{d}}\subseteq D(\widetilde{w},\mathfrak{v},w)$. Therefore multiplication by $\widetilde{\mathfrak{d}}$ defines a $\mathbb{C}$-linear monomorphism $D(\widetilde{w})_i\to D(\widetilde{w},\mathfrak{v},w)_{i+2}$. This in turn restricts to a monomorphism

$$D(\widetilde{w})_i/D(\widetilde{w})_{i-1}\to D(\widetilde{w},\mathfrak{v},w)_{i+2}/D(\widetilde{w},\mathfrak{v},w)_{i+1}.$$

Furthermore, the inclusion $D(\widetilde{w},\mathfrak{v},w)\subseteq D(\widetilde{w})$ induces an injection

$$D(\widetilde{w},\mathfrak{v},w)_i/D(\widetilde{w},\mathfrak{v},w)_{i-1}\to D(\widetilde{w})_i/D(\widetilde{w})_{i-1}$$

and therefore

$$\dim_{\mathbb{C}}\left(\frac{D(\widetilde{w},\mathfrak{v},w)_i}{D(\widetilde{w},\mathfrak{v},w)_{i-1}}\right)\le\dim_{\mathbb{C}}\left(\frac{D(\widetilde{w})_i}{D(\widetilde{w})_{i-1}}\right)\le\dim_{\mathbb{C}}\left(\frac{D(\widetilde{w},\mathfrak{v},w)_{i+2}}{D(\widetilde{w},\mathfrak{v},w)_{i+1}}\right).$$

Thus for $i\ge\ell$ all of the above dimensions are equal, and therefore

$$D(\widetilde{w},\mathfrak{v},w)_i/D(\widetilde{w},\mathfrak{v},w)_{i-1}\xrightarrow{\cong}D(\widetilde{w})_i/D(\widetilde{w})_{i-1},\ \forall i\ge\ell.$$

Hence $D(\widetilde{w})$ is generated over $D(\widetilde{w},\mathfrak{v},w)$ by elements of order $<\ell$. □

6.4 Explicit Examples

6.4.1 Darboux Transformations of Classical Bochner Pairs

In this section we provide explicit examples of the Main Theorem in action. The method of finding examples is straightforward: we consider a specific Bochner pair $(w, \mathfrak{d})$ satisfying the assumptions of the Main Theorem, and then attempt to find a degree 1 matrix polynomial $v_1(x)$ and a constant matrix v_0 satisfying the differential equation presented in the Main Theorem. Once we find such a pair $v_0, v_1(x)$, we are left only with checking that the remaining assumptions of the Main Theorem are also satisfied. One helpful thing to point out in our search is that if $v_0, v_1(x)$ satisfy the required assumptions, then $\tau(x) := a_2(x)v_0v_1(x)^{-1}$ will be a Hermitian polynomial of degree one

$$\tau(x) = \tau_1 x + \tau_0$$

satisfying the differential equation

$$\tau(x)^2 + \tau(x)'a_2(x) - \tau(x)a_2'(x) + \tau(x)a_1(x) + a_0a_2(x) = 0.$$

Writing $a_2(x) = a_{20} + a_{21}x + a_{22}x^2$ and $a_1(x) = a_{11}x + a_{10}$ for some $a_{20}, a_{21}, a_{22} \in \mathbb{C}$ and $a_{10}, a_{11} \in M_N(\mathbb{C})$ and comparing coefficients of various degrees, we obtain a system of three equations for the two unknown matrices τ_1, τ_2:

$$\begin{aligned} \tau_1^2 - \tau_1 a_{22} + \tau_1 a_{11} + a_0 a_{22} &= 0 \\ \tau_1\tau_0 + \tau_0\tau_1 - 2\tau_0 a_{22} + \tau_1 a_{10} + \tau_0 a_{11} + a_0 a_{21} &= 0 \\ \tau_0^2 + \tau_1 a_{20} - \tau_0 a_{21} + \tau_0 a_{10} + a_0 a_{20} &= 0 \end{aligned} \tag{6.7}$$

Remark 6.4.1.1. If $a_0 + \tau_1$ is nonsingular, then the original equation for τ implies

$$\tau^{-1} = -a_2(x)^{-1}(\tau(x) - a_2'(x) + a_1(x))(a_0 + \tau_1)^{-1}.$$

In particular, τ^{-1} has an inverse which is equal to matrix polynomial divided by $a_2(x)$. Take $v_0(x) = -(a_0+\tau_1)c$ and $v_1(x) = (\tau(x)-a_2'(x)+a_1(x))c$ for some invertible matrix $c \in M_N(\mathbb{C})$.

Since $a_2(x)$ must be non-vanishing on the support (x_0, x_1) of $w(x)$, we have $\det(v_i(x)) \neq 0$ on (x_0, x_1). Moreover since $\tau(x)^{-1} = a_2(x)^{-1}v_1(x)v_0^{-1}$, we also have $\tau(x) = v_0v_1(x)^{-1}a_2(x)$, and consequently $v_0v_1^{-1}(x)$ is symmetric and satisfies the differential equation in the Main Theorem. Therefore v_0 and $v_1(x)$ satisfy the assumptions of the Main Theorem when both $a_0 + \tau_1$ is nonsingular and

$$\mathrm{Tr}((v_1(x))^{-1}a_2(x)(v_1(x)^\dagger)^{-1}) = a_2(x)^{-1}\mathrm{Tr}((v_0^{-1})^\dagger v_0^{-1}\tau(x)\tau(x)^\dagger)$$

is in $L^2(\mathrm{Tr}(w(x))dx)$.

Remark 6.4.1.2. Note that if $\mathfrak{d} = \partial^2 a_2(x) + \partial a_1(x) + a_0$ forms part of a Bochner pair, so does $t\mathfrak{d} = \partial^2(ta_2(x)) + \partial(ta_1(x)) + (ta_0)$ for any real number t. However, because the above equations are nonlinear in τ, replacing $\mathfrak{d}$ with $t\mathfrak{d}$ actually results in a nontrivial change Equation 6.7. In practice, this leads to even more weight matrices arising from Darboux transformations of Bochner pairs.

Our examples in this section will focus on Darboux transformations $(\widetilde{w}, \widetilde{\mathfrak{d}})$ of Bochner pairs $(w, \mathfrak{d})$ where $\mathfrak{d} = \epsilon I - B$ and $w(x) = r(x)I$ for some Hermitian matrix $B \in M_N(\mathbb{C})$ and some choice of classical Bochner pair $(r(x), \epsilon)$. In this case, the weight matrix will be given by

$$\widetilde{w}(x) = r(x)a_2(x)v(x)^{-1}(v(x)^{-1})^* = r(x)a_2(x)^{-1}a_0^{-1}\tau(x)^2(a_0^{-1})^*$$

and our calculation of the algebra $D(\widetilde{w})$ is aided by the following proposition.

Proposition 6.4.1.3. *Let (r, ϵ) be a classical Bochner pair, and consider the Bochner pair $(w, \mathfrak{d})$ where $w = rI$ and $\mathfrak{d} = \epsilon I - B$ for some Hermitian matrix B. Suppose $\mathfrak{v} = \partial v_1 - v_0$ is a Darboux transformation $(\mathfrak{d}, w)$ to $(\widetilde{\mathfrak{d}}, \widetilde{w})$ with* $\det(v_1)$ *not zero at* 0. *Then*

$$D(\widetilde{w}) = D(\widetilde{w}, \mathfrak{v}, w) = \mathfrak{v}^{-1}\{f(\epsilon) \in M_N(\mathbb{C}[\epsilon]) : f(B) \in K\}\mathfrak{v},$$

where here K is the $\mathbb{C}$-linear span of all $\widetilde{w}$-symmetric constant functions and $f(B)$ is the left evaluation of f on B (ie. if $f(x) = a_0 + a_1x + a_2x^2 + \dots$, then $f(B) := a_0 + Ba_1 + B^2a_2 + \dots$).

To prove Proposition 6.4.1.3, we first establish the following lemma.

Lemma 6.4.1.4. *Suppose that $f(z) \in M_N(\mathbb{C}(z))$ is Hermitian (ie. $f(z)^* = f(z)$) and algebraic over $\mathbb{C}[z]$. Then $f(z) \in M_N(\mathbb{C}[z])$.*

Proof. Let $f(z)$ be as in the statement of the lemma, and suppose that $h(z)$ is not in $M_N(\mathbb{C}[z])$. Let $f_{ij}(z) \in \mathbb{C}(z)$ be the i,j'th entry of $f(z)$ for all $1 \leq i,j \leq N$. Then at least one $f_{ij}(z)$ must have a pole at some complex number c. Fix such a value of c and choose $d \geq 0$ so that $-(d+1)$ is the minimum of the degrees of all the $f_{ij}(z)$ at c. Then by the choice of c, we can write

$$f(z) = \frac{A}{(z-c)^{d+1}} + \frac{h(z)}{(z-c)^d}$$

for some integer $d \geq 0$, nonzero $A \in M_N(\mathbb{C})$ and $h(z) \in M_N(\mathbb{C}(z))$ with the entries of h having non-negative degree at c. Furthermore, since $f(z)$ is algebraic over $\mathbb{C}[z]$, we may choose a monic polynomial $Q(t) = t^m + \sum_{j=0}^{m-1} q_j(z)t^j$ with coefficients $q_0(z), \ldots, q_{m-1}(z) \in \mathbb{C}[z]$ satisfying $Q(f(z)) = 0$. However, inserting $f(z)$ into Q, we find

$$Q(f(z)) = \frac{A^m}{(z-c)^{md+m}} + \frac{g(z)}{(z-c)^{md+m-1}}$$

for some $g(z) \in M_N(\mathbb{C}(z))$ with the entries of $g(z)$ having non-negative degree at c. Now since $f(z)$ is $*$-symmetric, so too is A. In particular, since A is nonzero this means that A cannot be nilpotent. Thus $A^m \neq 0$. However, this contradicts the fact that $Q(f(z)) = 0$. Thus our original assumption that $f(z)$ is not in $M_N(\mathbb{C}(z))$ must have been false. This completes the proof of the lemma. $\square$

Proof of Proposition 6.4.1.3. Throughout the proof, $f(B)$ will always refer to the left evaluation of f on B for all $f \in M_N(\mathbb{C}[x])$. Let $K' = \mathfrak{v}^{-1}M_N(\mathbb{C})\mathfrak{v} \cap \mathfrak{D}(M_N(\mathbb{C}[[x]]))^{op}$. Note that if $D(\widetilde{w}) = D(\widetilde{w}, \mathfrak{v}, w)$, then $K = K'$ because the $\widetilde{w}$-symmetric constants are exactly the $\widetilde{w}$-symmetric constants in $D(\widetilde{w})$.

We claim that

$$\mathfrak{v}D(\widetilde{w}, \mathfrak{v}, w)\mathfrak{v}^{-1} = \{f(\epsilon) \in M_N(\mathbb{C}[\epsilon]) : f(B) \in K'\}.$$

To see this, consider $\mathfrak{v}$ as stated in the proposition. Then there exists a unit $\psi \in M_N(\mathbb{C}[[x]])$ with $\ker(\mathfrak{v}) = M_N(\mathbb{C})\psi$. Moreover by the choice of w and Miranian's theorem, $D(w) = M_N(\mathbb{C}[\epsilon])$. Thus

$$\mathfrak{v}D(\widetilde{w}, \mathfrak{v}, w)\mathfrak{v}^{-1} = \{f(\epsilon) \in M_N(\mathbb{C}[\epsilon]) : \exists\lambda \in M_N(\mathbb{C}) \text{ such that } \psi f(\epsilon) = \lambda\psi\}.$$

Now by definition, there exists a differential operator $\mathfrak{y}$ such that $\mathfrak{d} = \mathfrak{v}\mathfrak{y}$, and consequently $\psi\mathfrak{d} = \psi\mathfrak{v}\mathfrak{y} = 0\mathfrak{y} = 0$. Since $\mathfrak{d} = \epsilon I - B$, it follows that $\psi \cdot \epsilon = \psi B$, and more generally $\psi \cdot f(\epsilon) = \psi f(B)$ for all $f(x) \in M_N(\mathbb{C}[x])$. Thus we can write

$$\mathfrak{v}D(\widetilde{w}, \mathfrak{v}, w)\mathfrak{v}^{-1} = \{f(\epsilon) \in M_N(\mathbb{C}[\epsilon]) : \exists\lambda \in M_N(\mathbb{C}) \text{ such that } \psi f(B) = \lambda\psi\}.$$

However, $\psi f(B) = \lambda\psi$ if and only if $\ker(\mathfrak{v})f(B) \subseteq \ker(\mathfrak{v})$, which by definition means $f(B) \in K'$. This proves our claim.

Now let $p(z) \in \mathbb{C}[z]$ be the minimal polynomial of B. Then $\psi \cdot p(\epsilon) = \psi p(B) = 0$. Consequently $\ker(\mathfrak{v}) \cdot p(\epsilon) = 0$ and $\mathfrak{v}^{-1}p(\epsilon) \in \mathfrak{D}(M_N(\mathbb{C}[[x]]))^{op}$. It follows that $\mathfrak{v}^{-1}p(\epsilon)\mathfrak{v} \in D(\widetilde{w}, \mathfrak{v}, w)$. In fact, the same argument shows that $\mathfrak{v}^{-1}M_N(\mathbb{C}[\epsilon])p(\epsilon)\mathfrak{v} \subseteq D(\widetilde{w}, \mathfrak{v}, w)$. Set $\widetilde{\eta} = \mathfrak{v}^{-1}p(\epsilon)\mathfrak{v}$.

Now take a $\widetilde{w}$-symmetric $\widetilde{\mathfrak{d}} \in D(\widetilde{w})$ and consider the pseudo-differential operator $\beta = \mathfrak{v}\widetilde{\mathfrak{d}}\mathfrak{v}^{-1}$. Then $\beta p(\epsilon) = \mathfrak{v}\widetilde{\mathfrak{d}}\widetilde{\eta}\mathfrak{v}^{-1} = \mathfrak{v}\widetilde{\mathfrak{d}}(\mathfrak{v}^{-1}p(\epsilon))$ is a differential operator. By construction, $\beta p(\epsilon) \in D(w)$. Hence there exists $f(z) \in M_N(\mathbb{C}[z])$ such that $\beta p(\epsilon) = f(\epsilon)$. In particular, this shows that $\beta = f(\epsilon)/p(\epsilon)$. Furthermore, since $\widetilde{\mathfrak{d}}$ is $\widetilde{w}$-symmetric, β must be formally w-symmetric. Hence in particular $f(\epsilon)$ is formally w-symmetric, and it follows that $f(z)$ is Hermitian. The same argument applied to $\widetilde{\mathfrak{d}}^2$ tells us that there exists $g(z) \in M_N(\mathbb{C}[z])$ such that $\beta^2 = g(\epsilon)/f(\epsilon)$. Consequently $g(\epsilon) = f(\epsilon)^2/p(\epsilon)$. Since $g(z)$ is algebraic over $\mathbb{C}[z]$ and $f(z)^2/p(z)$ is Hermitian, the previous lemma then tells us $f(z)^2/p(z) \in M_N(\mathbb{C}[z])$. Moreover, since B is Hermitian its minimal polynomial $p(z)$ is made up of a product of distinct linear factors. Let $\lambda_1, \dots, \lambda_m$ be the associated roots of p. Then since $f(z)^2/p(z) \in M_N(\mathbb{C}[z])$, $f(\lambda_i)^2 = 0$ for all i. Therefore since $f(z)$ is Hermitian, we must actually have $f(\lambda_i) = 0$ for all i. Since $f(z)$ has polynomial entries, it follows that the entries of $f(z)$ are all divisible by $p(z)$. Hence $f(z)/p(z) \in M_N(\mathbb{C}[z])$. Now since $g(\epsilon)/p(\epsilon) \in M_N(\mathbb{C}[\epsilon])$, this proves that β is a differential operator in $D(w)$ and thus $\widetilde{\mathfrak{d}} \in D(\widetilde{w}, \mathfrak{v}, w)$. Since $\widetilde{\mathfrak{d}}$ was an arbitrary $\widetilde{w}$-symmetric element of $D(\widetilde{w})$, it immediately follows that $D(\widetilde{w}) = D(\widetilde{w}, \mathfrak{v}, w)$ and this completes the proof. □

Equation 6.7 is our main tool for finding examples of Darboux transformations of classical Bochner pairs (r, ϵ), and Proposition 6.4.1.3 will be our primary method of determining their associated differential operator algebras. Many of the results in the example section below were double-checked using python code with the Sympy symbolic computation library [67].

Remark 6.4.1.5. In the examples we look at below we will only observe weight matrices $\widetilde{w}$ such that $D(\widetilde{w})$ consists of operators of even order. We should emphasize that there do exist examples of weight matrices $\widetilde{w}$ whose algebras $D(\widetilde{w})$ contain operators of odd order, far different from the classical case [20]. We can also obtain weight matrices with this property (and specifically the weight in [20]) by considering Darboux transformations of weights of the form

$$w(x) = \begin{pmatrix} x^b e^{-x} & 0 \\ 0 & x^{b+1} e^{-x} \end{pmatrix}.$$

In this way, Darboux transformations of direct sums of classical weights are an even wider source of examples of new weight matrices. Moreover, whenever we can relate a weight $\widetilde{w}$ to a nicer weight w, the calculation of the algebra $D(\widetilde{w})$ is facilitated by the Main Theorem.

6.4.2 A Family of Examples of Hermite Type

Consider a Bochner pair of the form $(e^{-x^2}, \mathfrak{d})$ for $\mathfrak{d} = \partial^2 I - \partial 2xI - B$ for some Hermitian matrix $B \in M_N(\mathbb{C})$. Any such matrix B will still give a Bochner pair, so we will leave it as flexible for now. In this case, $a_{22} = 0, a_{21} = 0, a_{20} = 1$, $a_{11} = -2$, and $a_{10} = 0$. Thus Equation 6.7 gives

$$\tau_1^2 - 2\tau_1 = 0$$

$$\tau_1\tau_0 + \tau_0\tau_1 - 2\tau_0 = 0$$

$$B = \tau_0^2 + \tau_1.$$

Since B can be taken arbitrarily, the third equation can be ignored and we can restrict our attention to the first two equations. Also, since $\tau(x)$ is Hermitian, τ_1 and τ_0 are Hermitian. In particular, τ_1 is diagonalizable and the first equation is equivalent to the statement that all of τ_1 are all either 0 or 2. The second equation is then equivalent to the statement that τ_0 sends the eigenspace of eigenvalue 2 to the eigenspace of eigenvalue 0, and the eigenspace of eigenvalue 0 to the eigenspace of eigenvalue 2. Thus up to conjugation by a unitary matrix in $M_N(\mathbb{C})$ we can write τ in the block matrix form

$$\tau(x) = \begin{pmatrix} 2I_m & 0 \\ 0 & 0I_{N-m} \end{pmatrix} x + \begin{pmatrix} 0I_m & S \\ S^* & 0I_{N-m} \end{pmatrix}$$

for some integer $0 \leq m \leq N$ and some $m \times (N-m)$ matrix S. Note that for $\det(\tau)$ to not be zero, we must have $m = N$ or $m = N/2$ and S non-singular.

For the remainder of our examples of Hermite type, we will use $(w, \mathfrak{d})$ to refer to the Bochner pair with $w = e^{-x^2}I$ and $\mathfrak{d} = \epsilon - B$ for $\epsilon = \partial^2 - \partial 2xI$. Let $N = 2m$ be even and take $S \in M_m(\mathbb{C})$ nonsingular and $c \in M_N(\mathbb{C})$ nonsingular.

Proposition 6.4.2.1. *Let τ be defined as above and set*

$$v_1(x) = (\tau(x) - a_2'(x) + a_1(x))c = \left[\begin{pmatrix} 0 & 0 \\ 0 & -2I \end{pmatrix} x + \begin{pmatrix} 0 & S \\ S^* & 0 \end{pmatrix}\right] c$$

$$v_0 = -(a_0 + \tau_1)c = \begin{pmatrix} SS^* & 0 \\ 0 & S^*S \end{pmatrix} c$$

Then v_0 and $v_1(x)$ satisfy the assumptions of the Main Theorem for the Bochner pair $(w, \mathfrak{d})$

with

$$B = \tau_0^2 + \tau_1 = \begin{pmatrix} SS^* + 2I & 0 \\ 0 & S^*S \end{pmatrix}.$$

Proof. Since S is nonsingular, the matrix

$$a_0 + \tau_1 = -\begin{pmatrix} SS^* & 0 \\ 0 & S^*S \end{pmatrix}$$

is nonsingular. Thus by Remark 6.4.1.1 it suffices to show that the function

$$a_2(x)^{-1}\mathrm{Tr}((v_0^{-1})^\dagger v_0^{-1}\tau(x)\tau(x)^\dagger)$$

is in $L^2(\mathrm{Tr}(w(x)))$. Since $a_2(x) = 1$ and $\dagger = *$ here, this function is actually a polynomial. Therefore this condition follows immediately. □

Consequently, $\mathfrak{v} = \partial v_1(x) - v_0$ defines a Darboux transformation from the Bochner pair $(w, \mathfrak{d})$ to the Bochner pair $(\widetilde{w}, \widetilde{\mathfrak{d}})$ where

$$\widetilde{w} = e^{-x^2} v_0^{-1} \begin{pmatrix} 4x^2I + SS^* & 2xS \\ 2xS^* & S^*S \end{pmatrix} (v_0^{-1})^*,$$

$$\widetilde{\mathfrak{d}} = c^{-1}\left(\partial^2 I + \partial \begin{pmatrix} -2xI & 4(S^*)^{-1} \\ 0 & -2xI \end{pmatrix} - B\right) c.$$

Proposition 6.4.2.2. *The monic orthogonal matrix polynomials $\widetilde{p}(x,n)$ for the weight matrix $\widetilde{w}$ defined above are given by the generating function formula*

$$\sum_{n=0}^\infty \frac{t^n}{n!} c^{-1}\begin{pmatrix} SS^* & 0 \\ 0 & S^*S + 2nI \end{pmatrix} \widetilde{ch}_n(x) = \exp(xt - t^2/4)c^{-1}\begin{pmatrix} SS^* & -St \\ -S^*t & 2tx + S^*S \end{pmatrix} c.$$

Proof. The (monic) Hermite polynomials $h_n(x)I$ define a sequence of monic orthogonal polynomials for $e^{-x^2}I$, and satisfy the generating function formula

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} h_n(x)I = \exp(xt - t^2/4)I.$$

By the Main Theorem, $h_n(x)I \cdot \mathfrak{v}$ defines a sequence of orthogonal matrix polynomials for the weight matrix $\widetilde{w}$. Since the leading coefficient of $h_n(x)I$ is I, it's easy to see that the leading coefficient of $h_n(x)I \cdot \mathfrak{v}$ is

$$c_n = \begin{pmatrix} -SS^* & 0 \\ 0 & -S^*S - 2nI \end{pmatrix} c.$$

Thus $\widetilde{p}(x, n) := c_n^{-1} h_n I \cdot \mathfrak{v}$ is the monic sequence of orthogonal matrix polynomials for $\widetilde{w}$. Multiplying both sides of the generating function formula for the Hermite polynomials by $\mathfrak{v}$ and using this expression for $\widetilde{p}(x, n)$ then results in the generating function formula in the statement of the theorem. □

For $N = 2$, $S = s \in \mathbb{C}\backslash\{0\}$, and c chosen so that $v_0 = I$ this weight matrix appears first in [18] and later in [11], where explicit generators and relations of the associated algebra $D(\widetilde{w})$ are listed without proof. This set of generators and relations is verified in [68] in a 30-page tour-de-force. The center of $D(\widetilde{w})$ is also determined explicitly, though misidentified as being an elliptic curve, rather than a singular cubic plane curve. In the following, we demonstrate the utility of the main theorem by rederiving the structure of $D(\widetilde{w})$ and the center of $D(\widetilde{w})$ succinctly. Better yet, we show that $D(\widetilde{w})$ is naturally identified with a certain subalgebra of $N \times N$ matrices over a polynomial ring, as shown in Equation (6.8). Proposition 6.4.1.3 then gives us a succinct The Main Theorem combined with Proposition (6.1.0.5) gives us a means to calculate the structure of the algebra $D(\widetilde{w})$ associated with the weight matrix $\widetilde{w}$.

Proposition 6.4.2.3. *The algebra $D(\widetilde{w})$ is given by*

$$D(\widetilde{w}) = \left\{ \mathfrak{v}^{-1}v_0 \begin{pmatrix} f_1(\epsilon) & f_2(\epsilon) \\ f_3(\epsilon) & f_4(\epsilon) \end{pmatrix} v_0^{-1}\mathfrak{v} : f_i(\epsilon) \in M_{N/2}(\mathbb{C}), \substack{f_2(SS^*+2I)=0,\ f_3(S^*S)=0, \\ S^*f_1(SS^*+2I)(S^*)^{-1}=f_4(S^*S), \\ [f_1(SS^*+2I), SS^*]=0} \right\}. \tag{6.8}$$

In the expressions above $f_i(X)$ of $X \in M_{N/2}(\mathbb{C})$ always refers to the left evaluation of the matrix polynomial f_i on the matrix X.

Proof. Let $a \in M_N(\mathbb{C})$ be a constant matrix. Then from our equation for $\widetilde{w}$, we calculate the $\widetilde{w}$-adjoint of a to be

$$\widetilde{w}(x)a^*\widetilde{w}(x)^{-1} = v_0^{-1} \begin{pmatrix} 4x^2I + SS^* & 2xS \\ 2xS^* & S^*S \end{pmatrix} (v_0 a v_0^{-1})^* \begin{pmatrix} 4x^2I + SS^* & 2xS \\ 2xS^* & S^*S \end{pmatrix}^{-1} v_0$$

Thus if a is $\widetilde{w}(x)$-symmetric, then

$$v_0 a v_0^{-1} \begin{pmatrix} 4x^2I + SS^* & 2xS \\ 2xS^* & S^*S \end{pmatrix} = \begin{pmatrix} 4x^2I + SS^* & 2xS \\ 2xS^* & S^*S \end{pmatrix} (v_0 a v_0^{-1})^*.$$

Writing $v_0 a v_0^{-1} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$, this is equivalent to

$$[A_1, SS^*] = 0, \quad A_1^* = A_1, \quad A_2 = 0, \quad A_3 = 0, \quad A_4 S^* = S^* A_1.$$

We conclude that the set K of constant elements in $D(\widetilde{w})$ is

$$K = \left\{ v_0^{-1} \begin{pmatrix} A_1 & 0 \\ 0 & S^*A_1(S^*)^{-1} \end{pmatrix} v_0 : A_1 \in M_{N/2}(\mathbb{C}) \right\}.$$

Let $g(x) \in M_N(\mathbb{C}[x])$ with $f(x) = v_0^{-1} g(x) v_0 = \begin{pmatrix} f_1(x) & f_2(x) \\ f_3(x) & f_4(x) \end{pmatrix}$. Then from the form of

B we calculate the left evaluation of f on B to be

$$f(B) = \begin{pmatrix} f_1(SS^* + 2I) & f_2(SS^* + 2I) \\ f_3(S^*S) & f_4(S^*S) \end{pmatrix},$$

and from our choice of $f(x)$, we have $g(B) \in K$ if and only if

$$[f_1(SS^* + 2I), SS^*] = 0,\ f_2(SS^* + 2I) = 0,\ f_3(S^*S) = 0,\ S^* f_1(SS^* + 2I)(S^*)^{-1} = f_4(S^*S).$$

Combining this with Proposition 6.4.1.3, the statement of the proposition follows immediately. □

One works out from the description of $D(\widetilde{w})$ in the previous proposition that the center $Z(\widetilde{w})$ of $D(\widetilde{w})$ is

$$Z(\widetilde{w}) = \mathfrak{v}^{-1}\{f(\epsilon)I : f(\epsilon) \in \mathbb{C}[\epsilon],\ f(SS^* + 2I) = f(SS^*)\}\mathfrak{v}$$

The affine curve $X = \text{Spec}(Z(\widetilde{w}))$ associated to $Z(\widetilde{w})$ is rational, because $Z(\widetilde{w})$ is isomorphic to a subalgebra of $\mathbb{C}[\epsilon]$. Moreover, this inclusion induces a map of affine varieties $\mathbb{A}^1_{\mathbb{C}} \to X$. Let $\lambda_1, \ldots, \lambda_{N/2}$ be the (not necessarily distinct) eigenvalues of SS^*. For each i let p_i and q_i be the images of λ_i and $\lambda_i + 2$ under the map $\mathbb{A}^1_{\mathbb{C}} \to X$. Then $\mathfrak{v}^{-1}f(\epsilon)\mathfrak{v} \in I$ if and only if $f(\lambda_i) = f(\lambda_i + 2)$ for all i. This means that the rational functions on X may be identified with the rational functions on $\mathbb{A}^1_{\mathbb{C}}$ which identify λ_i and $\lambda_i + 2$. Consequently $p_i = q_i$ for all i, and X can be though of as a quotient of $\mathbb{A}^1_k$, where λ_i is glued transversely to λ_{i+1} for each i. In particular X is a rational singular curve with singularities corresponding to the eigenvalues of SS^*, and when $N = 2$ this is a nodal cubic curve.

6.4.3 A Family of Examples of Laguerre Type

Consider a Bochner pair of the form $(x^b e^{-x} 1_{(0,\infty)}, \mathfrak{d})$ for $\mathfrak{d} = \partial^2 xI + \partial((b+1) - x)I - B$ for some Hermitian matrix $B \in M_N(\mathbb{C})$. Any such B will give a Bochner pair, so we may use B

as an additional variable. Then $a_{22} = 0, a_{21} = 1, a_{20} = 0, a_{11} = -1$, and $a_{10} = (b+1)$. Using this, Equation 6.7 becomes

$$\tau_1^2 - \tau_1 = 0$$

$$\tau_0^2 + b\tau_0 = 0$$

$$B = \tau_1\tau_0 + \tau_0\tau_1 + (b+1)\tau_1 - \tau_0$$

Again, we can choose B to satisfy the third equation, and restrict our attention to only the first two equations. Since τ is Hermitian, so too are τ_0 and τ_1. The first equation says that the eigenvalues of τ_1 need to be 0 or 1. The second equation says that the eigenvalues of τ_0 need to be 0 or $-b$. Thus up to conjugation by a unitary matrix in $M_N(\mathbb{C})$, we can write τ in the block matrix form

$$\tau(x) = \begin{pmatrix} I_m & 0 \\ 0 & 0I_{N-m} \end{pmatrix} x + u \begin{pmatrix} 0I_\ell & 0 \\ 0 & -bI_{N-\ell} \end{pmatrix} u^*$$

for some integers $0 \leq \ell, m \leq N$ and some unitary matrix $u \in M_N(\mathbb{C})$. Note that if $b = 0$ then for $a_0 + \tau_1$ to be invertible we would have to have $m = N$, and the associated value of τ is uninteresting. If $b \neq 0$, then we must take $\ell = m$ for $a_0 + \tau_1$ to be invertible. Then writing

$$u = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix},$$

we find $a_0 + \tau_1$ is nonzero if $1 - U_2U_2^*$ and $U_4U_4^*$ are both nonsingular.

For the remainder of our examples of Laguerre type, we will use $(w, \mathfrak{d})$ to refer to the Bochner pair with $w = x^b e^{-x} 1_{(0,\infty)} I$, we will take $b > 1$, and set $\mathfrak{d} = \epsilon - B$ for $\epsilon = \partial^2 x + \partial(b+1-x)$. Let $\ell = m$ and take $c \in M_N(\mathbb{C})$ nonsingular and u orthogonal as above with $1 - U_2U_2^*$ and $U_4U_4^*$ nonsingular.

Proposition 6.4.3.1. *Let τ be defined as above and set*

$$\begin{aligned} v_1(x) &= (\tau(x) - a_2'(x) + a_1(x))c \\ &= \left[\begin{pmatrix} 0 & 0 \\ 0 & -xI \end{pmatrix} + bI - u \begin{pmatrix} bI & 0 \\ 0 & 0 \end{pmatrix} u^*\right] c \\ v_0 &= -(a_0 + \tau_1)c \\ &= b \begin{pmatrix} I - U_2U_2^* & 0 \\ 0 & U_4U_4^* \end{pmatrix} c \end{aligned}$$

Then v_0 and $v_1(x)$ satisfy the assumptions of the Main Theorem for the Bochner pair $(w, \mathfrak{d})$ with

$$B = \tau_0\tau_1 + \tau_1\tau_0 + (b+1)\tau_1 - \tau_0 = \begin{pmatrix} (b+1)I - bU_2U_2^* & 0 \\ 0 & bU_4U_4^* \end{pmatrix}.$$

Proof. By assumption, $a_0 + \tau_1$ is nonsingular. Thus by Remark 6.4.1.1 it suffices to show that the function

$$a_2(x)^{-1}\mathrm{Tr}((v_0^{-1})^\dagger v_0^{-1}\tau(x)\tau(x)^\dagger)$$

is in $L^2(\mathrm{Tr}(w(x)))$. Since $a_2(x) = x$ and $\dagger = *$ here, the above function is equal to $p(x)/x$ for some polynomial $p(x)$. Since $b > 1$, $p(x)/x$ is in $L^2(\mathrm{Tr}(w(x)))$ and this completes the proof. □

Consequently, $\mathfrak{v} = \partial v_1(x) - v_0$ defines a Darboux transformation from the Bochner pair $(w, \mathfrak{d})$ to the Bochner pair $(\widetilde{w}, \widetilde{\mathfrak{d}})$ where

$$\widetilde{w} = x^{b-1}e^{-x}1_{(0,\infty)}v_0^{-1} \begin{pmatrix} x^2I + (-2bx + b^2)U_2U_2^* & (-bx + b^2)U_2U_4^* \\ (-bx + b^2)U_4U_2^* & b^2U_4U_4^* \end{pmatrix} (v_0^{-1})^*.$$

$$\widetilde{\mathfrak{d}} = c^{-1}\left(\partial^2 xI + \partial \begin{pmatrix} (b - x + 2)I & -2(1 - U_2U_2^*)^{-1}U_2U_4^* \\ 0 & (b - x)I \end{pmatrix} - B\right) c.$$

Note that U_2, U_4 are coming from the unitary operator u, so $U_2^*U_2 + U_4^*U_4 = I$.

Proposition 6.4.3.2. *The monic orthogonal matrix polynomials $\widetilde{p}(x,n)$ for the weight matrix $\widetilde{w}$ defined above are given by the generating function formula*

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} c^{-1} c_n c t^n \widetilde{p}(x,n) = \frac{1}{(1-t)^{b+1}} e^{tx/(t-1)} c^{-1} \left(\frac{t}{t-1} v_1(x) - v_0 \right) c,$$

where here

$$c_n = \left[\begin{pmatrix} 0 & 0 \\ 0 & -n \end{pmatrix} + b \begin{pmatrix} 1 - U_2U_2^* & 0 \\ 0 & U_4U_4^* \end{pmatrix} \right].$$

Proof. A generating function formula for the monic Laguerre polynomials $p(x,n)$ is given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^n p(x,n) = \frac{1}{(1-t)^{b+1}} e^{tx/(t-1)}.$$

Moreover $p(x,n) \cdot \mathfrak{v}$ is a sequence of orthogonal polynomials for $\widetilde{w}$, with leading coefficient $c_n c$, so $\widetilde{p}(x,n) = c^{-1} c_n^{-1} p(x,n) \cdot \mathfrak{v}$. Multiplying both sides by $\mathfrak{v}$ and using this formula for $\widetilde{p}(x,n)$, we obtain the formula in the statement of the proposition. □

Just like in the Hermite example, we can use Proposition 6.4.1.3 to determine the structure of the algebra $D(\widetilde{w})$. In this case, the expression for the algebra is a bit more complicated.

Proposition 6.4.3.3. *The algebra $D(\widetilde{w})$ is given by*

$$D(\widetilde{w}) = \left\{ \mathfrak{v}^{-1} v_0 \begin{pmatrix} f_1(\epsilon) & f_2(\epsilon) \\ f_3(\epsilon) & f_4(\epsilon) \end{pmatrix} v_0^{-1} \mathfrak{v} : \begin{matrix} f_2((b+1)I - bU_2U_2^*) = 0, \quad [f_1((b+1)I - bU_2U_2^*), U_2U_2^*] = 0 \\ f_3(bU_4U_4^*) = 0, \quad f_4(bU_4U_4^*)U_4U_2^* = U_4U_2^* f_1((b+1)I - bU_2U_2^*) \end{matrix} \right\}, \tag{6.9}$$

where in the above $f_i(X)$ refers to the left evaluation of the matrix polynomial f_i on X and f_1, f_2, f_3, f_4 have $\ell \times \ell$, $\ell \times (N-\ell)$, $(N-\ell) \times \ell$, and $(N-\ell) \times (N-\ell)$ matrix coefficients, respectively.

Proof. Let $a \in M_N(\mathbb{C})$ be a constant matrix. From our equation for $\widetilde{w}$, we calculate

$$\widetilde{w}(x)a^*\widetilde{w}(x)^{-1} = v_0^{-1}\tau(x)^2(v_0av_0^{-1})^*\tau(x)^{-2}v_0.$$

Therefore if a is $\widetilde{w}$-symmetric, then $v_0av_0^{-1}\tau(x)^2 = \tau(x)^2(v_0av_0^{-1})^*$. Writing $v_0av_0^{-1} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$, this is equivalent to

$$A_1^* = A_1, \quad [A_1, U_2U_2^*] = 0, \quad A_4(U_4U_2^*) = (U_4U_2^*)A_1, \quad A_2 = 0, \quad A_3 = 0.$$

Using this, we conclude that the set K of constant elements in $D(\widetilde{w})$ is

$$K = \left\{ v_0^{-1} \begin{pmatrix} A_1 & 0 \\ 0 & A_4 \end{pmatrix} v_0 : {\scriptstyle A_1 \in M_\ell(\mathbb{C}),\ A_4 \in M_{N-\ell}(\mathbb{C}), \atop \scriptstyle [A_1, U_2U_2^*]=0,\ A_4(U_4U_2^*)=(U_4U_2^*)A_1.} \right\}.$$

Let $g(x) \in M_N(\mathbb{C}[x])$ with $f(x) = v_0^{-1}g(x)v_0 = \begin{pmatrix} f_1(x) & f_2(x) \\ f_3(x) & f_4(x) \end{pmatrix}$. Then from the form of B we calculate the left evaluation of f on B to be

$$f(B) = \begin{pmatrix} f_1((b+1)I - bU_2U_2^*) & f_2((b+1)I - bU_2U_2^*) \\ f_3(bU_4U_4^*) & f_4(bU_4U_4^*) \end{pmatrix},$$

and from our choice of $f(x)$, we have $g(B) \in K$ if and only if

$$f_2((b+1)I - bU_2U_2^*) = 0, \quad f_3(bU_4U_4^*) = 0, \quad [f_1((b+1)I - bU_2U_2^*), U_2U_2^*] = 0,$$

$$f_4(bU_4U_4^*)U_4U_2^* = U_4U_2^*f_1((b+1)I - bU_2U_2^*).$$

Combining this with Proposition 6.4.1.3, the statement of the proposition follows immediately. $\square$

Note specifically in the case $N = 2$, $U_2U_2^* = p$, and $U_4U_4^* = 1 - p$ we have

$$D(\widetilde{w}) = \left\{ \mathfrak{v}^{-1}v_0 \begin{pmatrix} f_1(\epsilon) & f_2(\epsilon) \\ f_3(\epsilon) & f_4(\epsilon) \end{pmatrix} v_0^{-1}\mathfrak{v} : {\scriptstyle f_2((b+1)I-bp)=0, f_3(b-bp)=0, \atop \scriptstyle f_4(b-bp)=f_1((b+1)I-bp)} \right\},$$

This algebra is isomorphic to the algebra derived in the Hermite case for $N = 2$. However, for larger values of N, this is not the case. In particular in the Hermite case, we were required to assume N is even, but here we can have interesting examples of algebras for odd N.

6.4.4 A Family of Examples of Jacobi Type

Consider a Bochner pair of the form $((1-x^2)^{r/2}, \mathfrak{d})$ for $\mathfrak{d} = \partial^2(1-x^2)I - \partial x(r+2)I - B$ for some Hermitian matrix $B \in M_N(\mathbb{C})$. Again, any such matrix B will give a Bochner pair, so we may use B as an additional variable. In this case $a_{22} = -1$, $a_{21} = 0$, $a_{20} = 1$, $a_{11} = -(r+2)$, and $a_{10} = 0$. Therefore Equation 6.7 becomes

$$B = \tau_0^2 + \tau_1$$

$$\tau_1^2 + \tau_0^2 - r\tau_1 = 0$$

$$\tau_1\tau_0 + \tau_0\tau_1 - r\tau_0 = 0$$

These last two equations imply

$$(\tau_1 \pm \tau_0)^2 = r(\tau_1 \pm \tau_0)$$

Since $\tau(x)$ is Hermitian, so too are τ_0, τ_1 and thus also $\tau_0 \pm \tau_1$. The previous equation tells us that $\tau_1 \pm \tau_0$ has eigenvalues 0 and r only. Therefore up to conjugation by a unitary matrix in $M_N(\mathbb{C})$, we can write τ in the block matrix form

$$\tau(x) = \frac{1}{2}\left[\begin{pmatrix} rI_m & 0 \\ 0 & 0I_{N-m} \end{pmatrix} + u\begin{pmatrix} 0I_\ell & 0 \\ 0 & rI_{N-\ell} \end{pmatrix}u^*\right]x$$
$$+\frac{1}{2}\left[\begin{pmatrix} rI_m & 0 \\ 0 & 0I_{N-m} \end{pmatrix} - u\begin{pmatrix} 0I_\ell & 0 \\ 0 & rI_{N-\ell} \end{pmatrix}u^*\right]$$

for some integers $0 \leq \ell, m \leq N$ and some unitary matrix $u \in M_N(\mathbb{C})$. The condition that $a_0 + \tau_1$ is nonsingular is equivalent here to the condition that τ_0 is nonsingular. This is exactly the case when $\ell = m$ and both $U_4 U_4^*$ and $1 - U_2 U_2^*$ are nonsingular, for

$$u = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}.$$

The above expression for $\tau(x)$ is a little too general to be convenient to work with. For this reason, for the remainder of our examples of Jacobi type (unless specified otherwise) we will consider the case when N is even, $\ell = m = N/2$, and $\tau(x)$ is of the form

$$\tau(x) = \begin{pmatrix} S^2 r & 0 \\ 0 & T^2 r \end{pmatrix} x + \begin{pmatrix} 0 & -STr \\ -TSr & 0 \end{pmatrix}$$

for some nonsingular Hermitian $S, T \in M_{N/2}(\mathbb{C})$ with $S^2 + T^2 = I$. Note that this function τ satisfies Equation 6.7 in this case, and must therefore be some unitary conjugate of our previous expression of τ for a suitably chosen value of u. We will use $(w, \mathfrak{d})$ to refer to the Bochner pair with $w = (1 - x^2)^{r/2} 1_{(-1,1)} I$, we will take $r > 2$, and we will set $\mathfrak{d} = \epsilon - B$ for $\epsilon = \partial^2(1 - x^2) - \partial x(r + 2)$. We also fix a unit $c \in M_N(\mathbb{C})$.

Proposition 6.4.4.1. *Let τ be defined as above and set*

$$\begin{aligned} v_1(x) &= (\tau(x) - a_2'(x) + a_1(x))c \\ &= ((\tau_1 - rI)x + \tau_0)c \\ &= -\left[\begin{pmatrix} T^2 & 0 \\ 0 & S^2 \end{pmatrix} x + \begin{pmatrix} 0 & ST \\ TS & 0 \end{pmatrix}\right] rc \\ v_0 &= -(a_0 + \tau_1)c \\ &= r^2 \begin{pmatrix} S^2T^2 & 0 \\ 0 & S^2T^2 \end{pmatrix} c \end{aligned}$$

Then v_0 and $v_1(x)$ satisfy the assumptions of the Main Theorem for the Bochner pair $(w, \mathfrak{d})$ with $B = \tau_0^2 + \tau_1$ ie

$$B = \begin{pmatrix} (r+r^2)S^2 - r^2S^4 & 0 \\ 0 & (r+r^2)T^2 - r^2T^4 \end{pmatrix} = \begin{pmatrix} rS^2 + r^2S^2T^2 & 0 \\ 0 & rT^2 + r^2S^2T^2 \end{pmatrix}.$$

Proof. By assumption, $a_0 + \tau_1 = -\tau_0^2$ is nonsingular. Thus by Remark 6.4.1.1 it suffices to show that the function

$$a_2(x)^{-1}\mathrm{Tr}((v_0^{-1})^\dagger v_0^{-1}\tau(x)\tau(x)^\dagger)$$

is in $L^2(\mathrm{Tr}(w(x)))$. Since $a_2(x) = 1 - x^2$ and $\dagger = *$ here, the above function is equal to $p(x)/(1-x^2)$ for some polynomial $p(x)$. Since $r > 2$, $p(x)/(1-x^2)$ is in $L^2(\mathrm{Tr}(w(x)))$ and this completes the proof. □

Consequently $\mathfrak{v} = \partial v_1(x) - v_0$ defines a Darboux transformation from the Bochner pair $(w, \mathfrak{d})$ to the Bochner pair $(\widetilde{w}, \widetilde{\mathfrak{d}})$ where

$$\widetilde{w} = (1-x^2)^{r/2-1}1_{(-1,1)}v_0^{-1}r^2 \begin{pmatrix} S^4(x^2-1) + S^2 & -xST \\ -xTS & T^4(x^2-1) + T^2 \end{pmatrix} (v_0^{-1})^*.$$

$$\widetilde{\mathfrak{d}} = c^{-1}\left(\partial^2(1-x^2)I + \partial \begin{pmatrix} -x(r+2) & -2ST^{-1} \\ -2TS^{-1} & -x(r+2) \end{pmatrix} - B\right)c$$

The expression for the operator $\widetilde{\mathfrak{d}}$ is consistent in the 2×2 case with the operator found in [72], for an appropriately chosen value of c. If we instead used the general form of $\tau(x)$, the previous Proposition is also seen to hold true (though the expression for B is then also more complicated). In that case, we'd get the more complicated weight matrix

$$\widetilde{w} = (1-x^2)^{r/2-1}1_{(-1,1)}v_0^{-1}\tau(x)^2(v_0^{-1})^*, \quad \text{for}$$

$$\tau(x)^2 = \frac{r^2}{4}\begin{pmatrix} 4U_2U_2^*x^2 + (1-U_2U_2^*)(x^2+2x+1) & 2U_2U_4^*(x^2-x) \\ 2U_4U_2^*(x^2-x) & U_4U_4^*(x^2-2x+1) \end{pmatrix}.$$

Note that here U_2, U_4 are coming from the unitary operator u, so $U_2^*U_2 + U_4^*U_4 = I$. We can also calculate a generating function for a sequence of orthogonal matrix polynomials for $\widetilde{w}$.

Proposition 6.4.4.2. *A sequence of (non-monic) orthogonal matrix polynomials $\widetilde{p}(x,n)$ for the weight matrix $\widetilde{w}$ defined above are given by the generating function formula*

$$\sum_{n=0}^{\infty} \widetilde{p}_n(x)t^n = \psi_x(x,t)v_1(x) - \psi(x,t)v_0(x),$$

where $\psi(x,t) = 2^{r-1}\phi(x,t)^{-1}((1+\phi(x,t))^2 - t^2)^{(1-r)/2}$ and $\phi(x,t) = (1-2xt+t^2)^{1/2}$.

Proof. The Gegenbauer polynomials $h_n(x)I$ define a sequence of monic orthogonal polynomials for $w(x)$, and satisfy the generating function formula

$$\sum_{n=0}^{\infty} h_n(x)t^n I = \psi(x,t)I.$$

By the Main Theorem, $h_n(x)I \cdot \mathfrak{v}$ defines a sequence of orthogonal matrix polynomials for the weight matrix $\widetilde{w}$. Multiplying both sides of the generating function formula for the Gegenbauer polynomials by $\mathfrak{v}$ leads to the desired generating function formula. □

Going back to the smaller family of examples of Jacobi type, for an appropriate choice of c, we can make $v_0 = r\begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix}$, in which case the weight matrix w is equal to

$$\widetilde{w} = (1-x^2)^{r/2-1}1_{(-1,1)}\begin{pmatrix} S^2x^2 + T^2 & -Ix \\ -Ix & T^2x^2 + S^2 \end{pmatrix}$$

For $N = 2$, $S^2 = p/r$, and $T^2 = 1 - p/r$, this weight matrix appears in [72], where explicit generators and relations of the associated algebra $D(\widetilde{w})$ are computed, though the effort involved is again substantial. In this example, we verify the calculation of the algebra $D(\widetilde{w})$ calculated in the paper, using the framework established above. As with Tirao's example (eg. the example of Section 6.4.2 with $N = 2$), the machinery gives the algebra $D(\widetilde{w})$ with significantly less wrangling. In fact, we will easily calculate a generalization of the algebra in question by using Proposition 6.4.1.3.

Proposition 6.4.4.3. *Assume that S^2 and T^2 have no eigenvalues in common. Then the algebra $D(\widetilde{w})$ is given by*

$$D(\widetilde{w}) = \left\{ \mathfrak{v}^{-1} v_0 \begin{pmatrix} f_1(\epsilon) & f_2(\epsilon) \\ f_3(\epsilon) & f_4(\epsilon) \end{pmatrix} v_0^{-1} \mathfrak{v} : \substack{f_2(rS^2+r^2S^2T^2)=0,\ f_3(rT^2+r^2S^2T^2)=0, \\ [f_1(rS^2+r^2S^2T^2),S^2]=0, \\ f_1(rS^2+r^2S^2T^2)ST=STf_4(rT^2+r^2S^2T^2)} \right\}. \tag{6.10}$$

Proof. Let $a \in M_N(\mathbb{C})$ be a constant matrix. From our equation for $\widetilde{w}$, we calculate

$$\widetilde{w}(x) a^* \widetilde{w}(x)^{-1} = v_0^{-1} \tau(x)^2 (v_0 a v_0^{-1})^* \tau(x)^{-2} v_0.$$

Therefore if a is $\widetilde{w}$-symmetric then $\widetilde{a} = v_0 a v_0^{-1}$ satisfies

$$\widetilde{a} \begin{pmatrix} S^4(x^2-1)+S^2 & -xST \\ -xTS & T^4(x^2-1)+T^2 \end{pmatrix} = \begin{pmatrix} S^4(x^2-1)+S^2 & -xST \\ -xTS & T^4(x^2-1)+T^2 \end{pmatrix} \widetilde{a}^*.$$

Writing $\widetilde{a} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$, comparing coefficients in various degrees, and simplifying

$$[A_1, S^2] = 0,\ A_1^* = A_1,\ [A_4, T^2] = 0,\ A_4 = A_4^*,\ A_1 ST = ST A_4,$$

$$A_2 = A_3^*,\ T^2 A_3 = A_3 S^2.$$

Furthermore, if T^2 and S^2 have no common eigenvalues then $T^2 A_3 = A_3 S^2$ implies $A_3 = 0$. We conclude that the set K of constant elements in $D(\widetilde{w})$ is

$$K = \left\{ v_0^{-1} \begin{pmatrix} A_1 & 0 \\ 0 & (ST)^{-1} A_1 ST \end{pmatrix} v_0 : A_1 \in M_{N/2}(\mathbb{C}),\ [A_1, S^2] = 0 \right\}.$$

Let $g(x) \in M_N(\mathbb{C}[x])$ with $f(x) = v_0^{-1} g(x) v_0 = \begin{pmatrix} f_1(x) & f_2(x) \\ f_3(x) & f_4(x) \end{pmatrix}$. Then from the form of B we calculate the left evaluation of f on B to be

$$f(B) = \begin{pmatrix} f_1(rS^2 + r^2 S^2 T^2) & f_2(rS^2 + r^2 S^2 T^2) \\ f_3(rT^2 + r^2 S^2 T^2) & f_4(rT^2 + r^2 S^2 T^2) \end{pmatrix}$$

and from our choice of $f(x)$, we have $g(B) \in K$ if and only if

$$f_2(rS^2 + r^2S^2T^2) = 0,\ \ f_3(rT^2 + r^2S^2T^2) = 0,$$

$$[f_1(rS^2 + r^2S^2T^2), S^2] = 0,\ \ f_1(rS^2 + r^2S^2T^2)ST = STf_4(rT^2 + r^2S^2T^2).$$

Combining this with Proposition 6.4.1.3, the statement of the proposition follows immediately. $\square$

For $N = 2$, $S^2 = p/r$, and $T^2 = 1 - p/r$, the algebra we obtain is

$$D(\widetilde{w}) = \left\{ \mathfrak{v}^{-1} v_0 \begin{pmatrix} f_1(\epsilon) & f_2(\epsilon) \\ f_3(\epsilon) & f_4(\epsilon) \end{pmatrix} v_0^{-1} \mathfrak{v} : {\scriptstyle f_2(p+p(r-p))=0,\ f_3(r-p+p(r-p))=0, \atop f_1(p+p(r-p))=f_4(r-p+p(r-p))} \right\}.$$

It is evident from the presentation of the algebra $D(\widetilde{w})$ given that $D(\widetilde{w})$ is in fact isomorphic to the algebra found in our Hermite example when $N = 2$. In particular the center is a nodal cubic. However, as $p \to r/2$ the center degenerates to a cuspidal cubic, so the algebra $D(\widetilde{w})$ is not isomorphic to the algebra found for the Hermite polynomials in Proposition 6.4.2.3 for every value of p.

6.5 General Structure Results

For the final section of the paper, we will introduce some results regarding the algebraic structure of $D(w)$ under fairly general assumptions. To prove the desired results, we will use the fact that $D(w)$ is closed under an anti-involution † and that $D(w)$ may be embedded into a matrix algebra and is therefore a PI-ring. This latter fact will allow us to apply the results of [59] and [58].

Before proving our main result, we establish a very helpful lemma.

Lemma 6.5.0.4. *The algebra $D(w)$ is a semiprime PI-algebra. The nonzero, w-symmetric elements of $D(w)$ are not nilpotent.*

Proof. Let $\{p(x,n)\}$ be a sequence of orthogonal polynomials for w. We claim that if $\mathfrak{d} \in D(w)$ is a nonzero element, then $\mathfrak{d}\mathfrak{d}^\dagger$ is also nonzero. To see this, suppose that $\mathfrak{d} \neq 0$. Then there exists an integer $n \geq 0$ such that $p(x,n) \cdot \mathfrak{d} \neq 0$. It follows that

$$0 \neq \langle p_n \cdot \mathfrak{d}, p_n \cdot \mathfrak{d} \rangle_w = \langle p_n \cdot \mathfrak{d}\mathfrak{d}^\dagger, p_n \rangle.$$

Hence we have that $p_n \cdot \mathfrak{d}\mathfrak{d}^\dagger$ is nonzero, and therefore $\mathfrak{d}\mathfrak{d}^\dagger \neq 0$.

Next suppose that $\eta \in D(w)$ is a nonzero, nilpotent, w-symmetric element. Then there exists a least integer $m > 0$ satisfying $\eta^m = 0$. Clearly $m > 1$. If m is even, then we may write

$$0 = \eta^m = \eta^{m/2}\eta^{m/2} = \eta^{m/2}(\eta^{m/2})^\dagger.$$

By the result of the previous paragraph, this means $\eta^{m/2} = 0$, and since $0 < m/2 < m$ this contradicts the minimality of m. Therefore m must be odd. However then $\eta^{m+1} = 0$ and therefore by the same argument $\eta^{(m+1)/2} = 0$. Since $m > 1$ we have that $0 < (m+1)/2 < m$ so this again contradicts the minimality of m. Since m is neither even nor odd, this is a contradiction. We conclude that nonzero, w-symmetric elements of $D(w)$ cannot be nilpotent.

Lastly, suppose that $\mathcal{I}$ is nontrivial nilpotent two-sided ideal of $D(w)$. Since $\mathcal{I}$ is nontrivial, we may choose $\mathfrak{d} \in \mathcal{I}$ with $\mathfrak{d} \neq 0$. Therefore $\mathfrak{d}\mathfrak{d}^\dagger \in I$ is also nonzero. However, since I is nilpotent $\mathfrak{d}\mathfrak{d}^\dagger$ should also be nilpotent, contradicting the result of the previous paragraph. Hence $D(w)$ has no nontrivial, nilpotent, two-sided ideals. This shows that $D(w)$ is semiprime. Lastly, the fact that the eigenvalue homomorphism embeds $D(w)$ in $M_N(\mathbb{C}[n])$, combined with the fact that a matrix algebra is a PI-algebra shows that $D(w)$ is a PI-algebra. $\square$

We are now ready to state and prove our general result on the structure of $D(w)$.

Theorem 6.5.0.5. *Suppose that $D(w)$ contains a differential operator of positive order with nonsingular leading coefficient. Then the algebra $D(w)$ is finitely generated as a module over its center $Z(w)$ and $Z(w)$ is a reduced algebra of Krull dimension* 1.

Proof. The Krull dimension of $D(w)$ is bounded by the GK-dimension of $D(w)$. Consider the image $E(w)$ of $D(w)$ under the eigenvalue homomorphism $\Sigma : D(w) \cong E(w) \subseteq M_N(\mathbb{C}[n])$. The Krull dimension of $E(w)$ is bounded by the GK-dimension of $E(w)$ as a graded vector space, graded by degree in the variable n. Since $E(w)$ is a graded subalgebra of $M_N(\mathbb{C}[n])$, the GK-dimension of $E(w)$ is further bounded by the GK-dimension of $M_N(\mathbb{C}[n])$. However, this latter algebra is Morita equivalent to $\mathbb{C}[n]$ and therefore has the same GK-dimension as $\mathbb{C}[n]$. Since the GK-dimension and Krull dimension of a commutative ring agree, this means that the GK-dimension of $M_N(\mathbb{C}[n])$ is one. Therefore the Krull dimension of $D(w)$ is at most one. Since $D(w)$ contains a differential operator of order at least one, it must contain a w-symmetric differential operator $\mathfrak{d}$ of positive order d with nonsingular leading coefficient. Then $\mathfrak{d}^n$ has order nd for each integer $n > 0$, and it follows that $\mathfrak{d}$ is transcendental over $\mathbb{C}$. Hence the Krull dimension of $D(w)$ is at least 1. We conclude that the Krull dimension of $D(w)$ is exactly 1. Combining this with the fact that $D(w)$ is a semi-prime PI-algebra, the main result of [59] tells us that $D(w)$ is finitely generated as a module over its center. Since $D(w)$ has Krull dimension 1, it follows that $Z(w)$ also has Krull dimension 1.

To show that $Z(w)$ is reduced, suppose that $\mathfrak{d} \in Z(w)$. Then $\mathfrak{d}^\dagger \in Z(w)$. To see this, suppose that $\eta \in Z(w)$. Then $\eta^\dagger \mathfrak{d} = \mathfrak{d}\eta^\dagger$ conjugating everything we find $\mathfrak{d}^\dagger\eta = \eta\mathfrak{d}^\dagger$. Since $\eta \in D(w)$ was arbitrary, this shows that $\eta \in Z(w)$. It follows that $\mathfrak{d}\mathfrak{d}^\dagger \in Z(w)$, and by our previous lemma $\mathfrak{d}\mathfrak{d}^\dagger$ is not nilpotent. Since $(\mathfrak{d}\mathfrak{d}^\dagger)^m = \mathfrak{d}^m(\mathfrak{d}^\dagger)^m$, it follows that $\mathfrak{d}^m$ is also not nilpotent. Since $\mathfrak{d} \in Z(w)$ was arbitrary, this shows that $Z(w)$ is reduced. □

In the specific case that $N = 2$, we can say even more about the center $Z(w)$ of $D(w)$, namely that it is rational.

Theorem 6.5.0.6. *Suppose that $D(w)$ contains a differential operator of positive order with nonsingular leading coefficient. If $D(w)$ is noncommutative and $N = 2$, then the center $Z(w)$ of $D(w)$ has a spectrum isomorphic to a rational curve.*

Proof. The previous theorem tells us that the spectrum of $Z(w)$ is a reduced (affine) curve. To prove that $Z(w)$ is rational, Lüroth's Theorem tells us that it suffices to find a ring monomorphism $Z(w) \hookrightarrow \mathbb{C}[n]$. We will do so by using the eigenvalue homomorphism $\Sigma : D(w) \cong E(w) \subseteq M_N(\mathbb{C}[n])$. Since $D(w) \neq Z(w)$ we may choose operators $\mathfrak{d}_1, \mathfrak{d}_2 \in D(w)$ such that $\mathfrak{d}_1\mathfrak{d}_2 \neq \mathfrak{d}_2\mathfrak{d}_1$. Moreover we may choose $\mathfrak{d}_1$ and $\mathfrak{d}_2$ to be w-symmetric. Then $\theta := i(\mathfrak{d}_1\mathfrak{d}_2 - \mathfrak{d}_2\mathfrak{d}_1)$ is a nonzero w-symmetric differential operator and hence cannot be nilpotent. Setting $d_i(n) := \Lambda(\mathfrak{d}_i)$ and $t(n) := \Lambda(\theta)$, we see that $t(n) = i(d_1(n)d_2(n) - d_2(n)d_1(n))$ is a trace-free, non-nilpotent matrix in $M_N(\mathbb{C}[n])$. Since $t(n)$ is trace-free, $t(n)^2 = \det(t(n))$. Thus since $t(n)$ is not nilpotent, $\det(t(n)))$ is not identically zero. Thus for all but finitely many values of n $d_1(n)d_2(n) - d_2(n)d_1(n)$ is a nonsingular matrix. Therefore $d_1(n), d_2(n)$ generates the full matrix ring $M_2(\mathbb{C})$ for all but finitely many values of n by [1]. It follows that anything in $\Lambda(Z(w))$ must commute with all of $M_2(\mathbb{C})$, and therefore that $\Lambda(Z(w)) \subseteq \mathbb{C}[n]I$. Hence $Z(w)$ has rational spectrum. $\square$

6.6 Classification of Weight Matrices

In this final section we will supply a partial classification of solutions to Bochner's problem. In particular, we will show that if the algebra $D(w)$ associated to the weight matrix w is large, then w is determined by a Darboux conjugation of a direct sum of classical weights. Throughout this section, we will use $\mathfrak{R}$ to denote the algebra $\mathfrak{D}(\mathbb{C}(x))^{op}$ of differential operators with rational coefficients acting on the right.

The main idea behind our classification is to try to find simulateous eigenvectors $\vec{\mathfrak{v}} \in \mathfrak{R}^N$ of the center $Z(w)$ of $D(w)$. For an arbitrary element $\mathfrak{d} \in M_N(\mathfrak{R})$, there may exist no eigenvectors in $\mathfrak{R}^N$. However, it turns out that we can exploit the algebraic structure of $D(w)$ to find them in our case.

If we can find enough eigenvectors, then we can diagonalize $Z(w)$. If we do this carefully enough, then the diagonalized $Z(w)$ will be symmetric operators of a new weight matrix which is a direct sum of classical weights.

6.6.1 Constructing Eigenvectors of $Z(w)$

In this section we will focus on constructing eigenvectors of $Z(w)$ which are sufficiently nice for our purposes. To do so, we will assume that $D(w)$ is big in the following sense.

Definition 6.6.1.1. We call the algebra $D(w)$ **full** if $E(w) \subseteq M_N(\mathbb{C}[n])$ is generically isomorphic to an $N \times N$ matrix algebra over its center, by which we mean:

(1) $E(w)$ is finitely generated over its center $Z(w)$ and $\mathbb{C}I \subsetneq Z(w) \subseteq \mathbb{C}[n]I$

(2) the localization of $E(w)$ at the generic point of $Z(w)$ is equal to $M_N(F(Z(w)))$

Remark 6.6.1.2. By Luroth's theorem, if $Z(w) \subseteq \mathbb{C}[n]I$ then the fraction field $F(Z(w)) = \mathbb{C}(p(n))I$ for some rational function $p(n) \in \mathbb{C}(n)$.

The involution $\dagger$ of $D(w)$ induces an involution of $E(w)$. If $D(w)$ is full, then we can be very specific about the structure of the involution.

Lemma 6.6.1.3. *Suppose that $D(w)$ is full. Then there exists $h(n) \in M_N(\mathbb{C}(n))$ with determinant 1, satisfying $h(n)^* = h(n)$ and*

$$\Lambda(\mathfrak{d}^\dagger) = h(n)\Lambda(\mathfrak{d})^* h(n)^{-1}.$$

Proof. The involution $\dagger$ of $D(w)$ defines an involution of $E(w)$, which we will also denote by $\dagger$. Note that this involution must preserve the center Z of $E(w)$. By our assumption that $E(w)$ is full, we have $Z \subseteq \mathbb{C}[n]I$ with the fraction field $F(Z)$ of Z equal to $\mathbb{C}(q(n))I$ for some $q(n) \in \mathbb{C}[n]$. The involution $\dagger$ extends to the quotient $F(Z)E(w) = M_N(\mathbb{C}(q(n)))$, and the composition of involutions $*$ and $\dagger$ thereby defines an automorphism $\varphi : M_N(\mathbb{C}(q(n))) \to M_N(\mathbb{C}(q(n)))$ with $\varphi^2 = \text{id}$. Then since $\mathbb{C}(q(n))I$ is preserved by φ, the restriction of φ to $\mathbb{C}(q(n))I$ defines an automorphism σ of $\mathbb{C}(q(n))$.

We claim that σ is the identity. To see this, note that for any $f(n) \in E(w)$, there exists $\mathfrak{d} \in D(w)$ with $p(x,n) \cdot \mathfrak{d} = f(n)p(x,n)$. Using this, we see that for any integer m

$$f(m)^\dagger = \|p(x,n)\|_w^2 f(n)^* \|p(x,n)\|_w^{-2},$$

and therefore for $f(n) \in Z$ we have $f(n)^\dagger = f(n)^*$, so that $\sigma = \mathrm{id}$.

Next by applying the Skolem-Noether theorem, we have

$$\varphi(f(n)) := (f(n)^*)^\dagger = h(n)f(n)h(n)^{-1}.$$

for some unit $h(n) \in M_N(\mathbb{C}(q(n)))$. Without loss of generality, we may take $\det(h(n)) = 1$. Thus

$$f(n)^\dagger = h(n)f(n)^*h(n)^{-1},$$

and since $\dagger$ is an involution we must have $h(n)^* = h(n)$. □

Remark 6.6.1.4. The previous lemma is also interesting in the sense that it partially characterizes $E(w)$. Since $E(w)$ must be closed under the involution $\dagger$, the algebra $E(w)$ is contained in the set of all $f(n) \in M_N(\mathbb{C}[n])$ such that $h(n)f(n)h(n)^{-1} \in M_N(\mathbb{C}[n])$. Since $h(n)$ can have rational coefficients, this can be a nontrivial condition.

Remark 6.6.1.5. If $p(x,n)$ are the monic orthogonal polynomials of w, then the above calculation also shows that $\|p(x,n)\|_w^2 = h(n)a(n)$ for $a(n) = \det(\|p(x,n)\|_w^2)^{1/N}\det(h(n))^{-1/N}$. This in turn has consequences for the three-term recursion relation associated to w.

Lemma 6.6.1.6. *Suppose that $D(w)$ is full. Then there exist w-symmetric matrix differential operators $\mathfrak{v}_1, \mathfrak{v}_2, \dots, \mathfrak{v}_N \in D(w)$ such that $\mathfrak{v}_i\mathfrak{v}_j = 0$ for $i \neq j$ and such that $\mathfrak{v}_1 + \cdots + \mathfrak{v}_N$ is not a zero divisor.*

Proof. Let $h(n)$ be the matrix from the previous lemma. Then $h(n)$ has an LDL decomposition $h(n) = u(n)d(n)u(n)^*$ where $u(n), d(n) \in M_N(\mathbb{C}(n))$. For each i, let e_i be the diagonal matrix with a 1 in the i'th diagonal entry and 0's elsewhere. Then for each i there exists $f_i(n) \in \mathbb{R}[n]$ such that $v_i(n) := u(n)f_i(n)e_iu(n)^{-1}$ is in $E(w)$. Let $\mathfrak{v}_i$ be the associated element of $D(w)$. Then by virtue of its construction $v_i(n)^\dagger = v_i(n)$, and therefore $\mathfrak{v}_i^\dagger = \mathfrak{v}_i$. Moreover, since $v_i(n)v_j(n) = 0$ for $i \neq j$ we also know $\mathfrak{v}_i\mathfrak{v}_j = 0$ for $i \neq j$. Lastly, we calculate

$$v_1(n) + \cdots + v_N(n) = u(n)\mathrm{diag}(f_1(n), \dots, f_N(n))u(n)^{-1},$$

from which we see that $\mathfrak{v}_1 + \cdots + \mathfrak{v}_n$ is an eventually degree-preserving differential operator, and thus not a zero divisor. $\square$

Lemma 6.6.1.7. *Suppose that $D(w)$ is full. Let $\mathfrak{v}_1, \ldots, \mathfrak{v}_N$ be defined as in the previous lemma. For all i, set $M_i = \{\vec{\mathfrak{w}} \in \mathfrak{R}^N : \mathfrak{v}_i \vec{\mathfrak{w}} = 0\}$. Then for all i, there exists $\vec{\mathfrak{v}_i} \in \mathfrak{D}(\mathbb{C}[x])^N$ such that $\bigcap_{j \neq i} M_j = \vec{\mathfrak{v}_i}\mathfrak{R}$.*

Proof. Let $\mathfrak{K}$ be the smallest skew subfield of $\mathfrak{P}(\mathbb{C}(x))$ containing $\mathfrak{R}$ and consider the free $\mathfrak{K}$-module $\mathfrak{K}^N$. Then since $\mathfrak{v}_1 + \cdots + \mathfrak{v}_n$ is a nonzero divisor in $\mathfrak{D}(\mathbb{C}[x])$, it is invertible as an element of $M_N(\mathfrak{K})$. Consider the (right) submodule $\mathfrak{V}_j \subseteq \mathfrak{K}^N$ defined by

$$\mathfrak{V}_j = \{\vec{\mathfrak{w}} \in \mathfrak{K}^N : \mathfrak{v}_j \vec{\mathfrak{w}} = \vec{0}\}.$$

Then if $I, J \subseteq \{1, 2, \ldots, N\}$ with $I \subsetneq J$, then $\bigcap_{i \in I} \mathfrak{V}_i$ properly contains $\bigcap_{j \in J} \mathfrak{V}_j$ because the former contains the column vectors of $\mathfrak{v}_j$ for $j \in J \backslash I$, while the latter does not. Thus counting dimension over $\mathfrak{K}$, we see that $\bigcap_{j \neq i} V_j$ is one dimensional over $\mathfrak{K}$. In particular, we may write $\bigcap_{j \neq i} V_j = \vec{\mathfrak{w}}_i \mathfrak{K}$ for some $\vec{\mathfrak{w}}_i$. Then

$$\bigcap_{j \neq i} M_j = \vec{\mathfrak{w}}_i \mathfrak{K} \cap \mathfrak{R}^N = \{\vec{\mathfrak{w}}_i \mathfrak{a} : \mathfrak{a} \in I\}$$

where I is the subset of $\mathfrak{K}$ consisting of elements $\mathfrak{a}$ such that $\vec{\mathfrak{w}}_i \mathfrak{a} \in \mathfrak{R}^N$. Note that I is nonempty, since the intersection $\bigcap_{j \neq i} M_j$ is nonempty. Futhermore, I has the structure of a right $\mathfrak{R}$-module. Let $\mathfrak{a}$ be an nonzero element in I of smallest order. Then one may verify that $I = \mathfrak{a}\mathfrak{R}$, so that $\bigcap_{j \neq i} M_j = \vec{\mathfrak{w}}_i \mathfrak{a} \mathfrak{R}$. Note that we may also choose $\mathfrak{a}$ so that $\vec{\mathfrak{w}}_i \mathfrak{a}$ has polynomial coefficients. Taking $\vec{\mathfrak{v}_i} = \vec{\mathfrak{w}}_i \mathfrak{a}$ completes the proof. $\square$

Lemma 6.6.1.8. *Let $\mathfrak{v}_1, \ldots, \mathfrak{v}_N$ and $\vec{\mathfrak{v}_1}, \ldots, \vec{\mathfrak{v}_N}$ be defined as in the previous lemma. For each $\vec{\mathfrak{v}_i}$, let $\mathfrak{v}_{ij} \in \mathfrak{R}$ denote the entries of $\vec{\mathfrak{v}_i}$. Then for each i, the left ideal I_i generated by $\mathfrak{v}_{i1}, \ldots, \mathfrak{v}_{iN}$ is equal to $\mathfrak{R}$.*

Proof. Since $\mathfrak{R}$ is a left PID, there exists $\mathfrak{e}_i \in \mathfrak{R}$ such that $I_i = \mathfrak{R}\mathfrak{e}_i$. This implies that there exist $\mathfrak{u}_{ij}$ such that $\mathfrak{v}_{ij} = \mathfrak{u}_{ij}\mathfrak{e}_i$. Consequently $\vec{\mathfrak{v}_i} = \vec{\mathfrak{u}_i}\mathfrak{e}_i$ for $\vec{\mathfrak{u}_i}$ the vector whose entries are the $\mathfrak{u}_{ij}$'s. This implies that $\mathfrak{v}_j\vec{\mathfrak{u}_i}\mathfrak{e}_i = 0$ for all $i \neq j$, and since $\mathfrak{R}$ is a domain, we have $\mathfrak{v}_j\vec{\mathfrak{u}_i} = 0$. Thus by definition of the $\vec{\mathfrak{v}_i}'s$, we must have $\vec{\mathfrak{u}_i} = \vec{\mathfrak{v}_i}\mathfrak{b}_i$ for some $\mathfrak{b}_i \in \mathfrak{R}$. Hence $\vec{\mathfrak{u}_i} = \vec{\mathfrak{u}_i}\mathfrak{e}_i\mathfrak{b}_i$, from which it follows that $\mathfrak{e}_i\mathfrak{b}_i = 1$. This implies that $\mathfrak{e}_i$ is a unit, and therefore that $I_i = \mathfrak{R}$. □

Lemma 6.6.1.9. *Suppose that $D(w)$ is full, and let $\mathfrak{v}_1, \ldots, \mathfrak{v}_N \in D(w)$ and $\vec{\mathfrak{v}_1}, \ldots, \vec{\mathfrak{v}_N}$ be as in the previous lemma. Then for all i, there exists $\mathfrak{r}_i \in \mathfrak{R}$ such that $\mathfrak{r}_i = \mathfrak{r}_i^*$ and*

$$\mathfrak{v}_i = \vec{\mathfrak{v}_i}\mathfrak{r}_i\vec{\mathfrak{v}_i}^* w(x)^{-1}.$$

Proof. Choose a decomposition $w(x) = u(x)u(x)^*$. Then $u(x)^{-1}\mathfrak{v}_i u(x)$ is formally $*$-symmetric. Moreover, by the definition of $\vec{\mathfrak{v}_i}$ we may write

$$u(x)^{-1}\mathfrak{v}_i u(x) = u(x)^{-1}\vec{\mathfrak{v}_i}\vec{\mathfrak{a}_i}^*.$$

for some $\vec{\mathfrak{a}_i} \in \mathfrak{R}^N$. Then by formal $*$-symmetry

$$u(x)^{-1}\vec{\mathfrak{v}_i}\vec{\mathfrak{a}_i}^* = \vec{\mathfrak{a}_i}\vec{\mathfrak{v}_i}^*(u(x)^*)^{-1}.$$

Consequently $u(x)\vec{\mathfrak{a}_i} \in \bigcap_{j\neq i} M_j$ so that $\vec{\mathfrak{a}_i} = u(x)^{-1}\mathfrak{v}_i\mathfrak{r}_i$ for some $\mathfrak{r}_i \in \mathfrak{R}$. Thus we have

$$u(x)^{-1}\mathfrak{v}_i u(x) = u(x)^{-1}\vec{\mathfrak{v}_i}\mathfrak{r}_i\vec{\mathfrak{v}_i}^*(u(x)^*)^{-1},$$

where by $*$-symmetry $\mathfrak{r} = \mathfrak{r}^*$. From this, our desired equation for $\mathfrak{v}_i$ follows immediately. □

Definition 6.6.1.10. Let $Z(w)$ denote the center of $D(w)$. A vector $\vec{\mathfrak{u}} \in \mathfrak{R}^N$ will be called a **central eigenoperator** of $D(w)$ if for all $\mathfrak{d} \in Z(w)$ there exists $\mathfrak{b} \in \mathfrak{R}$ such that $\mathfrak{d}\vec{\mathfrak{u}} = \vec{\mathfrak{u}}\mathfrak{b}$. A collection of central eigenoperators $\vec{\mathfrak{u}_1}, \ldots, \vec{\mathfrak{u}_m}$ of $D(w)$ will be called **orthogonal central eigenoperators** if $\vec{\mathfrak{u}_i}^* w(x)^{-1}\vec{\mathfrak{u}_j} = 0$ for all $i \neq j$.

Lemma 6.6.1.11. *Suppose that $D(w)$ is full. Let $\vec{\mathfrak{v}_i}$ for $i = 1, \ldots, N$ be as in the previous lemma. Choose a central element $\mathfrak{d} \in D(w)$. Then for all i, there exists $\mathfrak{d}_i \in \mathfrak{R}$ such that $\mathfrak{d}\vec{\mathfrak{v}_i} = \vec{\mathfrak{v}_i}\mathfrak{d}_i$. In particular, the vectors $\vec{\mathfrak{v}_i}$ form a collection of orthogonal central eigenoperators.*

Proof. Since $\mathfrak{d}$ and $\mathfrak{v}_i$ commute, we know

$$\mathfrak{d}\vec{\mathfrak{v}_i}\mathfrak{r}_i\vec{\mathfrak{v}_i}^* w(x)^{-1} = \vec{\mathfrak{v}_i}\mathfrak{r}_i\vec{\mathfrak{v}_i}^* w(x)^{-1}\mathfrak{d}.$$

Using this, we see

$$\vec{\mathfrak{v}_j}^*\mathfrak{d}\vec{\mathfrak{v}_i}\mathfrak{r}_i\vec{\mathfrak{v}_i}^* w(x)^{-1} = 0,$$

and from this it follows that $\vec{\mathfrak{v}_j}^*\mathfrak{d}\vec{\mathfrak{v}_i} = 0$. Consequently $\mathfrak{d}\vec{\mathfrak{v}_i} = \vec{\mathfrak{v}_i}\mathfrak{d}_i$ for some $\mathfrak{d}_i \in \mathfrak{R}$. □

6.6.2 The Classification Theorem

The eigenoperators $\vec{\mathfrak{v}_i}$ of the previous lemma will allow us to diagonalize the center $Z(w)$ of $D(w)$.

Theorem 6.6.2.1. *Suppose that $D(w)$ is full and contains a central, symmetric second order element $\mathfrak{d}$ with real coefficients such that $D(w)$ is equal to the centralizer of $\mathfrak{d}$ in $M_N(\mathfrak{D}(\mathbb{C}[x]))$. Let $\vec{\mathfrak{v}_1},\dots,\vec{\mathfrak{v}_N}$ be defined as above, and assume moreover that*

(a) for each j, the span of $(\mathbb{C}[x]^N)^* \cdot \vec{\mathfrak{v}_j}$ is $\mathbb{C}[x]$

(b) the matrix $\mathfrak{v} = [\vec{\mathfrak{v}_1}\vec{\mathfrak{v}_2}\dots\vec{\mathfrak{v}_N}]$ is w-adjointable

Then the Bochner pair $(w,\mathfrak{d})$ is a Darboux conjugation of a Bochner pair obtained from a classical sum of weights.

Proof. Let $\vec{\mathfrak{v}_i}$ for $i = 1,\dots,N$ be as in the previous lemma. Since $\mathfrak{d}$ is real, we may without loss of generality take each of the $\vec{\mathfrak{v}_i}$'s to also be real. Then by the previous lemma, there exist $\mathfrak{d}_1,\dots,\mathfrak{d}_n \in \mathfrak{R}$ such that $\mathfrak{d}\vec{\mathfrak{v}_i} = \vec{\mathfrak{v}_i}\mathfrak{d}_i$. Moreover, by comparing order we see that each of the $\mathfrak{d}_i$ is second order for each i. By comparing coefficients, we also see that $\mathfrak{d}_i$ has real coefficients for each i.

Since $\mathfrak{d}_i$ is real and second-order, it will have a factorization of the form

$$\mathfrak{d}_i = f_{i0}(x)\partial f_{i1}(x)\partial f_{i2}(x)$$

for some real functions $f_0(x), f_1(x), f_2(x)$. Thus in particular by setting $r_i(x) = f_{i0}(x)/f_{i2}(x)$, we have

$$\mathfrak{d}_i = r_i(x)\mathfrak{d}_i^* r_i(x)^{-1}.$$

Define $\mathfrak{v} = [\vec{\mathfrak{v}_1}\ \vec{\mathfrak{v}_2}\ \dots\ \vec{\mathfrak{v}_N}]$ and note

$$\mathfrak{d}\mathfrak{v} = \mathfrak{v}\mathrm{diag}(\mathfrak{d}_1, \dots, \mathfrak{d}_N),$$

and also

$$\mathfrak{v}^\dagger\mathfrak{d} = \mathrm{diag}(\mathfrak{d}_1, \dots, \mathfrak{d}_N)^\dagger\mathfrak{v}^\dagger$$

so that for

$$ff^\dagger = \mathrm{diag}(r_1, \dots, r_N)w^{-1},$$

we have

$$\mathfrak{d}(\mathfrak{v}ff^\dagger\mathfrak{v}^\dagger) = (\mathfrak{v}ff^\dagger\mathfrak{v}^\dagger)\mathfrak{d}.$$

Thus by the assumption that $C(\mathfrak{d}) = D(w)$ we have $\mathfrak{v}ff^\dagger\mathfrak{v}^\dagger \in D(w)$. Moreover, we calculate

$$\mathfrak{v}ff^\dagger\mathfrak{v}^\dagger = \sum_i \vec{\mathfrak{v}_i} r_i(x)\vec{\mathfrak{v}_i}^* w^{-1},$$

so by the choice of the $\mathfrak{v}_i$'s, we have $\mathfrak{v}ff^\dagger\mathfrak{v}^\dagger$ is not a zero divisor in $M_N(\mathfrak{R})$. Hence the associated value in $E(w)$ is not a zero divisor in $M_N(\mathbb{C}[n])$.

Since $D(w)$ is full and $\mathfrak{d}$ is in the center of $D(w)$, we must have $\Lambda(\mathfrak{d}) = \lambda(n)I$ for some $\lambda(n) \in \mathbb{C}[n]$. Each $\mathfrak{d}_j$ is formally $r_j(x)$-symmetric. Furthermore, we calculate

$$p(x,n)\cdot\vec{\mathfrak{v}_j}\mathfrak{d}_j = p(x,n)\cdot\mathfrak{d}\vec{\mathfrak{v}_j} = \lambda(n)p(x,n)\cdot\vec{\mathfrak{v}_j}.$$

Set $q_{ij}(x,n) = (p(x,n)\cdot\vec{\mathfrak{v}_j})_i$. We calculate $q_{ij}(x,n)\cdot\mathfrak{d}_j = \lambda(n)\mathfrak{d}_j$. In particular $q_{ij}(x,n)$ is an eigenpolynomial for all i,j,n. By assumption (a) in the lemma, the $q_{ij}(x,n)$ span $\mathbb{C}[x]$ for each fixed j, so this implies that $\mathfrak{d}_j$ is degree-preserving for all j.

Next, using assumption (b) we calculate

$$\begin{aligned}\lambda(n)\|p(x,n)\|_W^2\delta_{m,n} &= \langle p(x,n)\mathfrak{v}ff^\dagger\mathfrak{v}^\dagger, p(x,m)\rangle_w \\ &= \langle p(x,n)\mathfrak{v}f, p(x,m)\mathfrak{v}f\rangle_w \\ &= \langle p(x,n)\mathfrak{v}, p(x,m)\mathfrak{v}\rangle_{\widetilde{w}}.\end{aligned}$$

for

$$\widetilde{w} = fwf^* = ff^\dagger w = \operatorname{diag}(r_1, \dots, r_n).$$

Thus for each j, the $q_{ij}(x,n)$ are orthogonal polynomials for r_j which are eigenfucntions of $\mathfrak{d}_j$. Since they span $\mathbb{C}[x]$, this implies that $(r_j, \mathfrak{d}_j)$ is a classical solution to Bochner's problem and in particular that r_j is a classical weight. This shows that $(w, \mathfrak{d})$ comes form a Darboux conjugation of a direct sum of classical weights. $\square$

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