

©Copyright 2017
Brandon Robinson

Supersymmetric Localization and
Probe Branes in the AdS/CFT correspondence

Brandon Robinson

A dissertation
submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy

University of Washington

2017

Reading Committee:

Andreas Karch, Chair

Stephen Sharpe

Jason Detwiler

Program Authorized to Offer Degree:
Department of Physics

University of Washington

Abstract

Supersymmetric Localization and
Probe Branes in the AdS/CFT correspondence

Brandon Robinson

Chair of the Supervisory Committee:
Professor Andreas Karch
Department of Physics

In this thesis, a precise, rigorous test of probe brane holography will be constructed. Since its discovery, the AdS/CFT correspondence has provided a window into the strongly coupled dynamics of supersymmetric gauge theories. The ability to include degrees of freedom that provide analogs for the physics of heavy quarks via the probe brane paradigm has further expanded the utility of the duality. The deformation away from a strictly conformal theory by the addition of flavor degrees of freedom induces a Landau pole outside of the 't Hooft limit where $N_c \rightarrow \infty$ and $\frac{N_f}{N_c} \ll 1$, which invites questions about the utility of the probe brane paradigm. Following from the recent application of equivariant localization to massive supersymmetric gauge theories on curved backgrounds, a precise question can be formulated to compare, e.g., the free energy of a supersymmetric probe brane embedding and that of the localized dual field theory. This thesis will apply those concepts to the D3/D7 probe brane system dual to $N_f \mathcal{N} = 2$ fundamental hypermultiplets on an S^4 and the D3/D5 probe brane system dual to $N_f \mathcal{N} = 2$ fundamental hypermultiplets living on a co-dimension one defect— an equatorial $S^3 \subset S^4$. In that framework, exact matching to the localization results are found.

TABLE OF CONTENTS

	Page
List of Figures	iii
List of Tables	iv
Preface	1
Chapter 1: Introduction	4
1.1 The AdS/CFT Correspondence	8
1.2 A Novel Application To Black Hole Physics	20
Chapter 2: A Tour of Further Background Material	30
2.1 Probe Brane Holography	30
2.2 Exact Results in Gauge Theories on Curved Backgrounds	48
Chapter 3: A Precision Test of Probe Brane Holography	63
3.1 Introduction	63
3.2 Supersymmetric D7 branes in AdS ₄ -sliced AdS ₅ ×S ⁵	65
3.3 Topologically distinct classes of embeddings	83
3.4 Flavored $\mathcal{N} = 4$ SYM on (two copies of) AdS ₄	96
3.5 Supersymmetric D3/D7 for massive flavors on S ⁴	110
3.6 Supersymmetric Holographic Observables	116
3.7 Localization with Quenched Flavors	122
3.8 Discussion	126
Chapter 4: Supersymmetric D3/D5 and Localization	131
4.1 Introduction	131
4.2 Geometry and embedding ansatz	133
4.3 Linearized κ -symmetry	137

4.4	Non-linear κ -symmetry	146
4.5	General Linearized Solutions	158
4.6	Spherical Embedding	165
4.7	Holographic one-point functions and match to localization	168
4.8	Discussion	178
4.9	Appendix A: Holographic renormalization with irrelevant deformations	178
4.10	Appendix B: Guide to the Shorthand Notation	184
	Bibliography	186

LIST OF FIGURES

Figure Number	Page
3.1	84
3.2	88
3.3	89
3.4	90
3.5	92
3.6	94
3.7	95
3.8	96
3.9	101
3.10	121

LIST OF TABLES

Table Number	Page
1.1 D3-brane Embedding	11
1.2 $\mathcal{N} = 4$ $SU(N_c)$ field content	13
1.3 Holographic Dictionary for bulk fields of spin s	18
2.1 D7- and D5-branes in D3 background	32
2.2 R-charges and conformal dimensions (Δ) of defect spectrum	40
2.3 Linearized spectrum of D5-brane open string modes	41
2.4 Dictionary entries for lowest ℓ modes	42
2.5 One-loop determinants for $\mathcal{N} = 2$ multiplets on an S^4	61

ACKNOWLEDGMENTS

I would like to thank first my advisor, Andreas Karch, for the opportunity to learn under his guidance. It is an experience that has forever shaped my approach- not only to scientific research- but to the broader aspects of philosophical inquiry. I would also like to acknowledge my faculty mentor, Larry Yaffe, for likewise providing an exemplar of how to ask questions and demand precision in the scientific process. No less an impact has been made by a frequent collaborator, Christoph Uhlemann, without whose conversation, insight, and perseverance much of the research in this thesis would not have been possible. Further, I would like to thank the faculty of the Particle Theory Group for providing an intellectually engaging and productive atmosphere in which to work.

Additionally, I would like to thank those that took part in building a stable, nurturing, and successful research environment, and those providing an opportunity to engage interesting social and work related conversation: Catherine Provost, James Sloan, Jon Craig, Aaron Feinberg, Colin Lamont, John Lombard, David Sommer, John Fuini, and Michael Wagman among many others.

Finally, I would like to thank the people in my personal sphere that have provided the love and support that brought me here in the first place: my partner, Kate Thorman and my parents, Lisa and Richard. Without their encouragement, I could not have begun to conceive that I could begin from the hills of rural Pennsylvania, wend my way to this point my academic career, and strike out further still.

DEDICATION

To my parents.

Preface

The paper that was the basis of the example presented in §1.2 was published: Andreas Karch and Brandon Robinson, *Holographic Black Hole Chemistry*, JHEP 1512 (2015) 073. My contribution to the work contained therein was to provide support in the calculations of the bulk thermodynamic quantities and field theory relations. Andreas Karch conceived of the project through discussions at the Strings, Black Holes and Quantum Information Workshop at the Tohoku Forum for Creativity in Sendai, Japan, and provided the physical interpretation of the results, insight into the nuances of the holographic dictionary, and the main parts of the field theory calculation of the Universal Smarr Formula including derivation and implications. The duties of writing, establishing notational conventions, and editing were shared. The text that appears is a condensed version the original paper.

The content of Ch.3 is based on two papers that have been published: (1) Andreas Karch, Brandon Robinson, and Christoph F. Uhlemann, *Supersymmetric D3/D7 for holographic flavors on curved space*, JHEP 1511 (2015) 112 (2) Andreas Karch, Brandon Robinson, and Christoph F. Uhlemann, *Precision Test of Gauge/Gravity Duality with Flavor*, Phys.Rev.Lett. 115 (2015) no.26, 261601.

1. The original concept of the project had been attempted by Andreas and former students, and so he provided the impetus for its undertaking. Andreas also provided support on calculations in the early days of the kappa symmetry analysis, the writing of the section containing and the work done on analysis of the results for massive supersymmetric flavors on two copies of AdS and their subtle boundary conditions, and large portions of the physical interpretation. Christoph provided much needed insight into the issues plaguing the search for projectors when the project had stalled, large section of the writing and notation, and the preparation of figures. My responsibilities were driving the initial phases of the project through lengthy, detailed calculations, compiling a prodigious amount of notes that became the foundation of the paper, the conversion of the AdS results into usable information in the spherical embedding, and

finding all of the ways that kappa symmetry projectors shouldn't be constructed.

2. This project was the desired end goal of the work began in (1), and was thus the brainchild of Andreas. My contribution to this project was to develop from the work done in [1] as workable form of the localized matrix model that would allow for the comparison to the supersymmetric probe brane embedding with spherical slicing. That included building and evaluating the quenched flavor free energy as a function of the mass deformation. Further it was primarily my duty to prepare the work for publication in and meeting the exacting standard of PRL. Andreas provided interpretation of the phase transition, the historical context, and insight into the scheme dependences appearing on the gravitational side and their role in the matrix model. Christoph provided calculations in support of understanding the scheme dependences, the bulk quantities, and the phase transition as well as the help in the shared burden of writing and editing for publication.

Lastly, Ch. 4 is based on work that is being prepared for submission to JHEP in collaboration with Christoph F. Uhlemann. My duties in the project were its initial conception, establishment of the overall structure of the project, initial calculations, the calculation of the free energies, holographic one-point functions with irrelevant deformations, the non-linear continuation to spherical embedding, decoupling analysis, and the construction, analysis, and interpretation of the matrix model resulting from supersymmetric localization found to be dual to the spherical embedding with massive flavors on a spherical defect. Christoph again was instrumental in overcoming the subtle issues that seem to plague kappa symmetry calculations and provided the intensely laborious calculations that determined the gauge field in the non-linear kappa symmetry analysis and the full non-linear kappa symmetry equations for the AdS sliced embedding with generically massive flavors on AdS defects. Christoph also provided the general perturbative framework and solutions in the AdS slicing provided the bulk of the typesetting of calculations including the notational conventions used. The text that appears in the introduction and throughout the body of the chapter differs from

the current, and likely final, state of the to-be-published work in order to fit the overall tone of the thesis. The last figures in this chapter figs. (4.1(a)) and (4.1(b)) will likely not appear in the final publication but were kept to make a broader point about the simplicity of the matrix model compared to the overwhelming complexity of the probe brane calculation.

Chapter 1

INTRODUCTION

*“Thus the unfacts, did we possess them,
are too precisely few to warrant our certitude...”*

-James Joyce, *Finnegan’s Wake*

Throughout the extraordinary arc of scientific progress in describing the physical universe beginning at a nascent stage in the latter part of the 19th century through the accelerating developments of a golden age of discovery spanning the 20th century, we have created ever more successful predictive models. These models straddle scales of the microscopic description of fundamental particle physics to the immensity of supra-galactic structures and much of what lies between. This mastery has not left us bereft of questions as our most-used tools to describe the physics of our universe still lack a grounding that allow for the solution to oft-cited problems such as an elegant unification of the known forces and a suitable predictive theory that would describe the origin of the universe and its dynamics all the way down to its smallest scales. To be clear, we have a coarse set of candidates for some of these standing issues, e.g. string theory as necessarily containing quantum gravity, but as of writing this thesis, there is still live, active debate about the precise nature of our proposals and the correct path forward.

The successes of these models and their theoretical headwaters — as they are revealed during one’s graduate career — are taught in two parallel tracks: microscopic physical models are generically constructed by the methods of quantum field theories, while cosmological scale physics has at its heart the geometric description of gravity furnished by General Relativity. The mystery of how to unify those two descriptions into a true and fundamental theory of

quantum gravity is pointed to in some circles to as the most concern outstanding problem and one that has proved most resilient to our attempts at solution. Not accounting for taste, it is true that this is an unsolved puzzle but one that may continue to elude our collective grasp for the foreseeable future despite the dedicated efforts of many researchers and the championing of features of their favorite models. However, there are somewhat more quotidian aspects of the realm of quantum gravity that, while not generating as heated a public debate, are essential in understanding how best to build models that can move forward from our current paradigmatic orientation.

One of these aspects that is intimately tied to questions related to the work presented in this thesis is how best we can understand quantum — and for our purposes supersymmetric — field theories defined on curved background geometries, which allows for interesting and novel phenomena once considered carefully [2]. This is not a novel question [3, 4], but it is one that curiously has not been given quarter in even the most up-to-date textbooks designed for graduate curricula in theoretical high energy and particle physics (e.g. [5]). Even with the knowledge of the non-zero, yet small, curvature of the observable universe available to us for the past two decades (i.e. [6]) most of the pedagogical narrative of quantum field theory strictly takes place with regard to flat backgrounds. While locally we can describe any collider experiment conducted on Earth as well approximated by taking place in an exactly flat background, it is still just that: an approximation.

Further, we have known for the past half century that the Standard Model in its current conception — even beyond its definition on a flat Minkowski background and lack of gravitational sector — does not include the most general possible set of symmetries of the S-matrix describing scattering processes, and thus could be simply a pinhole view of a wide vista of physics accessible in the vast region between our current experimental probes and any putative fundamental scale, e.g. the Planck scale. The extensions of the base (Poincaré) symmetries of the S-matrix by a \mathbb{Z}_2 grading known as “supersymmetry” [7] has enjoyed nearly a half century of concerted study both in the context of applications to model building (see [8] for a nice overview) and as a theoretical framework in itself ([9] is a beautiful

example) and will play a central role in this thesis. The focus of the following text will simply require understanding a small portion of the endlessly fascinating mathematics that underlie how supersymmetric theories are normally discussed in modern applications in order to fully tell the story the following work.

What will be of primary concern in the bulk of the text will be how the above concepts work in concert in the context of a striking relationship that advertises itself to be a window into the worlds of both quantum gravity and strongly coupled, often supersymmetric, field theories known as the Anti-de Sitter space/Conformal Field Theory (AdS/CFT) correspondence [10, 11, 12, 13, 14]. A working definition of this correspondence, or duality, will be presented in more detail in § 1.1, but it can be best summarized in the following schematic:

- Strong correspondence demands equality of the partition function of a full, non-perturbative definition of a string theory, which includes a graviton in its spectrum, with the full quantum partition function of a field theory in one fewer dimensions sans gravitational sector.
- Weak correspondence implies a relation between the low energy, on-shell supergravity action, obtained from a perturbative string theory, including bulk fields, and the generating functional for correlation functions of field theory operators for which the asymptotic values of the bulk fields act as sources.

This does not do the beauty of the correspondence true justice, but it will serve as a baseline for further elaboration. A small refinement of this hierarchy will be discussed as well.

The motivating question that one can keep in mind and that played a role in the author’s interest in this project is “how does one establish confidence in a concept when it is necessary to utilize it in the process of its own verification?”. As empirically minded agents, when handed a tool and informed of its designed use, we verify through experiment, *gedanken* or otherwise, repeatedly across a range of initial conditions to establish that in those successful contexts we can proceed with application of the tool in related, novel settings. It is in this

way when we find agreement that we can add it to the collection of evidentiary support in arguing that the duality in question is a meaningful relation that exists.

That is precisely the frame from which the work contained in this thesis should be viewed. Our tool in this case is the probe brane paradigm in the AdS/CFT correspondence — e.g. [15]. The tribal knowledge about its function is that it can reliably be used to include heavy degrees of freedom to the boundary field theory transforming in the fundamental representation of the Lie algebra of the gauge group $SU(N_c)$, i.e. quark analog content, in such a way that the objection concerning conical deficits associated with brane insertions can be ignored with confidence in the large brane tension limit. The context of verification is through utilization of supersymmetric localization in the field theory to reduce the infinite dimensional path integral to a deformed Gaussian unitary matrix model [1] evaluated in the limit of quenched flavors and κ -symmetry for theory of the brane’s worldvolume to project out extraneous fermionic degrees of freedom establishing a $\frac{1}{2}$ - or $\frac{1}{4}$ -BPS embedding [16]. After suitable fixing of scheme dependent counterterms arising in holographic renormalization, a comparison of derivatives of the supersymmetric holographic one-point functions and the matrix model free energy will be shown to match exactly [17, 18]. All of these concepts will be explicated in further sections.

In what follows, we will begin with a review of the basic concepts of the AdS/CFT correspondence including its best current tests and an example of its utilization. Following this discussion, a review of more specialized, technical background material including the introduction of the probe brane paradigm and supersymmetric localization will be given in Ch. 2. This will lead into the main results that comprise the bulk of this thesis where a precision test of probe brane holography via supersymmetric localization is constructed in Ch. 3. Lastly, a presentation of an extension of this line of reasoning into a less well understood system including supersymmetric mass deformations constrained to a codimension one defect surface and its impact on localization calculations holographically realized by the D3/D5 probe system will be laid out in Ch. 4 followed by a discussion of future directions in this plastic research program.

1.1 The AdS/CFT Correspondence

Since the central theme of the paper centers around the AdS/CFT correspondence and its uses in some novel settings, we should at the introductory level lay out the core principles of the duality and argue for why it is an intriguing tool. This section is meant by no means to be a comprehensive review of all of the knowledge required to construct the current status of, nor the many subtleties that can arise in, the correspondence. Rather, the goal of this section is to prime the discussion of the features of correspondence that will be necessary to understand the novel work that is to be presented in later chapters. To that end, let us first establish the central elements at work in the AdS/CFT correspondence, which will be referred to by AdS/CFT for brevity.

1.1.1 A String Story

The starting point for most discussions of AdS/CFT is the work in the seminal papers [10, 11, 12]. While these papers, as well as [14], should be read by any student or researcher interested in thinking about AdS/CFT, the benefit of studying such a popular topic two decades after its discovery is that there are a number of useful pedagogical treatments from which one can more systematically learn the subject. As such, much of this section will be drawn from resources dedicated to providing an depth analysis of, AdS/CFT and its string theoretic underpinnings [19, 20, 21]. Most of the notation will follow [19].

Although there are many possible incarnations of AdS/CFT to be used as a basis for discussion, for clarity the focus of this section will be on the following statement:

An equivalent description (read exact equality of partition functions) of $d = 4$ $SU(N_c)$ $\mathcal{N} = 4$ Super Yang-Mills theory (SYM) with coupling g_{YM} can be formulated as a type IIB string theory with string length determined by the Regge slope parameter $l_s = \sqrt{\alpha'}$ and string coupling g_s on $AdS_5 \times S^5$ with bulk AdS length scale L and N_c units of RR 5-form $F_5 (= dC_4)$ flux through the S^5 where bulk and field theory variables are mapped by $g_{YM}^2 = 2\pi g_s$ and

$$\frac{L^4}{l_s^4} = 2N_c g_{YM}^2.$$

As briefly mentioned in the opening of this thesis, the above statement reflects what can be regarded as the strongest expression of the correspondence where g_s is finite and the AdS geometry is strongly curved while the gauge group and coupling can take any value. With neither a full non-perturbative definition of string theory nor a usable analytic description of $SU(N_c)$ gauge theories at arbitrary rank and coupling, there is no way to fully probe this statement. It could thus be softened to strong correspondence if we treat a string theory as weakly coupled, $g_s \ll 1$, which corresponds to the large N_c limit of the gauge theory. The statement would still read as an equivalence between a string theory and a supersymmetric gauge theory — albeit a slightly more tractable one. However, for the purposes of this review, and thesis as a whole, we will explicitly consider only the weak form of the correspondence. In this case, the string scale is parametrically small compared to the background curvature $\alpha' \ll L^2$ and $g_s \ll 1$. This relates then the low energy (type IIB supergravity) description of the bulk physics with the large N_c , large λ gauge theory.

Open and Closed Strings: A Duet

To flesh out the weak form of the correspondence, we need to understand the objects are described by a perturbative type IIB string theory and how they descend to supergravity fields. First due to supersymmetric contributions to anomaly cancellation constraints, the critical dimension for the target space of the supersymmetric non-linear sigma model living on the worldsheet of a type IIB superstring is $d = 10$. Thus, all of the discussion in the following sections will have embeddings of string theory objects into flat $\mathbb{R}^{(9,1)}$. As an aside, a consequence of this is that the Killing spinors that generate local supersymmetry transformations in $d = 10$ are generically 32 component spinors (see the always helpful App. B of [20]).

Apart from strings, there exist dynamical extended objects called D-branes that span directions along which the endpoint of an open string can move freely but is fixed from the

point of view of the transverse directions. That is, we associate D-branes with objects on which an open string can “end”. In a type IIB string theory, the only such D-branes that exist are ones with even dimensional worldvolume, e.g. a Dp-brane has a $(p+1)$ -dimensional worldvolume. A consequence of embedding such an object in the supergravity background is that at least half of the supersymmetries will be broken. A further discussion of this point will be made in § 2.1.2, but for clarity we will consider these $\frac{1}{2}$ -BPS branes at the moment.

In order to have a meaningful perturbative, low energy description of the open string sector, we need to study a limit in which the string coupling is small $g_s N_c \ll 1$ and the massive spectrum of the open string can be effectively ignored. This is the so-called infinite tension limit where the energy scales in question are parametrically small compared to the cost of exciting anything but a massless string mode $E \ll 1/\sqrt{2\pi\alpha'}$. In addition to the role that D-branes have in the open string sector, they are in essence solitons in the supergravity description. The fluctuations of the worldvolume are naturally described in the perturbative string theory as part of the closed string spectrum. This sector contains information about how the insertion of the branes gravitationally backreact on the $\mathbb{R}^{(9,1)}$. The low energy description of this sector necessitates that the brane insertion does not significantly warp the background geometry, which implies a separation of bulk and string length scales such that $\frac{L^4}{l_s^4} \gg 1$. However, this is the opposite limit of the weak coupling string limit as $\frac{L^4}{l_s^4} \propto g_s N_c \gg 1$. Thus, questions that can be asked in the weakly coupled supergravity regime of the closed string sector will necessarily be mapped to questions in the strong coupling regime of the open string sector. This so-called *S-duality* feature is part of what makes testing and formalizing AdS/CFT difficult.

Treating the above story more explicitly, consider the low energy effective action for N_c coincident D3-branes embedded in $\mathbb{R}^{(9,1)}$ spanning the x^0, \dots, x^3 directions. This can be encoded in a useful mnemonic where the spanned directions are indicated by “•” and transverse directions ‘o’:

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D3	•	•	•	•	○	○	○	○	○	○

Table 1.1: D3-brane Embedding

In the respective limits that allow the isolation of simple effective descriptions of the string sectors, the excitations can be grouped into fairly simple supermultiplets. For the closed string sector, we use a $d = 10$ $\mathcal{N} = 1$ gravity supermultiplet, and for the open string sector we have a $d = 4$ $\mathcal{N} = 4$ SYM gauge multiplet. The discrepancy in dimensions and \mathcal{N} is owing to the fact that the string endpoints live on the D3-brane worldvolume and a supersymmetric gauge theory in $d = 4$ with 16 supercharges has four Dirac spinors worth of supersymmetry generators, i.e. $\mathcal{N} = 4$.

Without systematically treating the construction of the bosonic part of the low energy closed string sector of D3-brane fluctuations, it can be shown to take the form

$$S_{closed} = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{-G} e^{-2\Phi} (R_G + 4\partial_a \Phi \partial^a \Phi) + \mathcal{O}(\alpha'), \quad (1.1)$$

where $2\kappa^2$ encodes the $d = 10$ gravitational constant, G is the determinant of the $d = 10$ metric, R_G is the Ricci scalar for that metric, and Φ is the dilaton. κ is related to string parameters by $2\kappa^2 = (2\pi)^7 \alpha'^4 g_s^2$. The role of the dilaton is to set the string coupling constant through its vacuum expectation value $g_s = \langle e^\Phi \rangle$. Missing from this expression is the contribution from the anti-symmetric two-form Kalb-Ramond potential B_{ab} , which enters the supergravity Lagrangian through its 3-form field strength as $-\frac{1}{12} H_{abc} H^{abc}$. For the remainder of the text, only brane embeddings with $B_{ab} = 0$ will be considered, which is consistent with type IIB supergravity equations of motion.

The bosonic part of the low energy, effective action of the open string sector describing the embedding of a general Dp-brane is governed by the Dirac-Born-Infeld (DBI) action

$$S_{DBI} = -T_p \int_{\Sigma_{p+1}} d^{p+1}\xi e^{-\Phi} \sqrt{-\det(P[G]_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})}, \quad (1.2)$$

where T_p is the tension of the brane spanning the worldvolume indicated by Σ_{p+1} with coordinates ξ^μ . In addition, $P[G]_{\mu\nu}$ denotes the pullback of the ten-dimensional metric G_{ab} onto the worldvolume, and $F_{\mu\nu}$ is a worldvolume gauge field. This will be the starting point for the discussion of probe brane holography as well.

In addition to the DBI action, there are non-trivial contributions coupling the worldvolume field strength to the type IIB RR $2i$ -form potentials, C_{2i} , in the Wess-Zumino (WZ) action:

$$S_{WZ} = \frac{T_P}{g_s} \int_{\Sigma_{p+1}} \sum_i P[C_{2i}] \wedge e^{\mathcal{F}}, \quad (1.3)$$

where $\mathcal{F} = 2\pi\alpha'F$. Expanding the exponential produces enough \wedge -products to span Σ_{p+1} for any given C_{2i} . This term will play a crucial role the analysis of supersymmetric probe brane analysis.

What has been neglected so far is the interaction between open and closed string sectors, which outside of taking limits discussed above must be dealt with in kind. These interaction terms are naturally derived from the DBI action. To see them, it suffices to study the linearized fluctuations of the DBI action around the flat $G_{ab} = \eta_{ab} + \kappa h_{ab}$ and dilaton-free background $e^{-\Phi} \sim 1 - \kappa\Phi$. The factor of κ is necessary for canonically normalized graviton and dilaton kinetic terms, such that up to $\mathcal{O}(\alpha')$ terms

$$S_{DBI} \simeq -T_3 \int_{\Sigma_4} d^4\xi \text{Tr} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \eta^{\mu\nu} \partial_\mu \phi^I \partial_\nu \phi^I - \sum_{I,J} [\phi^I, \phi^J]^2 + \frac{\kappa}{4} \Phi F_{\mu\nu} F^{\mu\nu} \right). \quad (1.4)$$

The six adjoint valued worldvolume scalars $\phi^I = \phi^{I,a} T_a$, $I = 1, \dots, 6$ are produced in the pullback of the spacetime directions transverse to the brane embedding (see Table. 1.1). Note that the interaction term $\kappa\Phi F^2$ vanishes in the limit $\alpha' \rightarrow 0$, and so to decouple the open and closed string modes, this is the limit that we should take. This is not the whole story, though, as the worldvolume supersymmetry implies a fermionic sector as well. Since the fermionic sector is derivable from the bosonic theory by applying supersymmetry transformations, it will be omitted from the discussion.

The take away message is that the worldvolume action in the low energy limit as described in eq. (1.4) along with the completion by the fermionic sector is exactly that of a single gauge multiplet $\mathcal{N} = 4$ SYM theory. The field content organized $\mathcal{N} = 2$ language for use later is:

$\mathcal{N} = 2$ Multiplet	Field content
Vector	$\{A_\mu, \lambda_L, \lambda_R, \phi^5, \phi^6\}$
Hyper	$\{\psi_L, \psi_R, \phi^1, \dots, \phi^4\}$

Table 1.2: $\mathcal{N} = 4$ $SU(N_c)$ field content

Here the four Weyl fermions have been split into two pairs of left and right handed spinors λ, ψ , the four dimensional gauge field is A_μ and the six real scalars have been grouped such that in this language they realize an $SO(4)_R \times SO(2)_R \subset SO(6)_R$ R-symmetry, which will be explicitly manifested in the probe brane analysis in the following chapter. As noted, since all of the fields belong to an $\mathcal{N} = 4$ gauge multiplet they all must transform in the adjoint representation of the corresponding gauge group.

A point of clarification is warranted: the adjoint representation in this context refers to a set of non-dynamical factors that are associated with the labeling of open string endpoints attached to a brane called *Chan-Paton Factors*. For a single open string attached to a single D-brane, the Chan-Paton factors amount to a charge under a $U(1)$. Generalizing to a string attached to N_c parallel, coincident D-branes, the endpoints are charged in the (anti-)fundamental representation under $U(N_c) \cong SU(N_c) \times U(1)$. The $U(1)$ factor describes the collective motion of the D3-branes. If both endpoints are attached to the stack of branes, then the state formed is in the $\square \otimes \bar{\square}$ of $SU(N_c)$, which is the adjoint rep. Note that the $U(N_c)$ can be Higgsed by separating some of the N_c coincident branes in transverse directions where the scalar in the direction of transverse displacement gets a vev proportional to the distance of separation from the stack, R . So, the *decoupling limit* is implemented not only by $\alpha' \rightarrow 0$ but also keeping quantities like R/α' fixed in order to understand the low energy $\mathcal{N} = 4$ SYM in its Higgs phase.

What has been constructed so far shows the decoupling limit of the open string and interaction sectors producing pure $\mathcal{N} = 4$ $SU(N_c)$ SYM theory in $d = 4$. The closed string sector/supergravity description still must be accounted for, which necessitates taking the opposite ($g_s N_c \gg 1$) limit. The specific supergravity solution for embedding the $\frac{1}{2}$ -BPS D3-branes into $\mathbb{R}^{(9,1)}$ is given by

$$\begin{aligned}
 ds^2 &= \left(1 + \frac{L^4}{r^4}\right)^{-\frac{1}{2}} \eta_{\mu\nu} dX^\mu dX^\nu + \left(1 + \frac{L^4}{r^4}\right)^{\frac{1}{2}} \delta_{AB} dx^A dx^B \\
 C_4 &= \frac{L^4}{L^4 + r^4} dx^0 \wedge \dots \wedge dx^3 + \dots
 \end{aligned}
 \tag{1.5}$$

where, $\mu, \nu \in \{0, \dots, 3\}$ and $A, B \in \{4, \dots, 9\}$. By the above ansatz the dilaton has a constant profile in the bulk radial coordinate $r = \sqrt{\sum_A X_A^2}$ and the Kalb-Ramond field has been set to zero. Note that there is no unique way to fix C_4 , but the F_5 flux is fixed to be proportional to the sum of the volume forms on the AdS_5 and S^5 spaces. The above form for C_4 is a particular gauge-fixed form for the RR potential that gives the correct F_5 ; the so-called Freund-Rubin ansatz. Further, quantization of the F_5 flux in terms of the number of D3-branes in the stack, N_c , is used to fix the AdS length scale as advertised earlier $L^4 = 2g_{YM}^2 N_c \alpha'^2$. This relationship between the AdS length and N_c , among other field theory parameters, will be exploited to great effect in §1.2.

Since the solution above corresponds to embedding N_c coincident, dynamical, extended objects in a previously flat supergravity vacuum, one would rightly surmise that the result of taking $N_c \gg 1$ is that there is a strong deformation of the background curvature near the locus, in the transverse sense, of the brane stack insertion. In this case, there are two distinct regions: (i) $\frac{r}{L} \gg 1$, corresponding to the asymptotically flat region far from the branes that still looks like $\mathbb{R}^{(9,1)}$ (ii) $\frac{r}{L} \ll 1$, describing the geometry deep in the warped (“throat” or “near horizon”) region close to the branes. As is common with charged black holes, expanding around the horizon at $r = 0$ one finds that the geometry looks locally like a product $\text{AdS}_p \times S^q$, where $p + q = d$. Indeed after expanding eq. (1.5) and writing the $d = 6$

transverse space as $\mathbb{R}^+ \times S^5$ with coordinates $\{r = \frac{L^2}{z^2}, \theta_i\}$, the metric reads

$$ds^2 = \frac{L^2}{z^2}(dz^2 + \eta_{\mu\nu}dx^\mu dx^\nu) + L^2 d\Omega_5^2. \quad (1.6)$$

By $d\Omega_p^2$ we denote the line element on the unit S^p . We recognize the non-spherical part of the line element as being the Poincarè patch of AdS_5 .

We now have the first piece in the canonical story of AdS/CFT: the emergence of $\text{AdS}_5 \times S^5$ from the insertion of a large number of coincident D3-branes. Further, we see another piece of the puzzle: low energy closed string modes in the throat region — even in the $g_s N_c \gg 1$ regime where small excitations have parametrically large energy — are extremely redshifted as seen by an observer in the asymptotic region. This results a decoupling of the string states seeing $\text{AdS}_5 \times S^5$ and those seeing $\mathbb{R}^{(9,1)}$. Thus, we can understand the low energy effective behavior of fluctuations around this D3-brane embedding in terms of classical (super-)gravity on $\text{AdS}_5 \times S^5$.

Finally, we that the decoupling limit explored in the open string sector is exactly the same as zooming in on the throat geometry. This can be seen by starting with $L^4 = 2g_{YM}^2 N_c \alpha'^2$ and dividing by r^4 , then exploring the $\frac{r}{L} \ll 1$ regime

$$\frac{r^4}{L^4} = \left(2g_{YM}^2 N_c \frac{\alpha'^4}{r^4}\right)^{-1} \alpha'^2 \ll 1. \quad (1.7)$$

Since the decoupling limit holds powers of α'/r fixed, the term in (\dots) is constant while $\alpha' \rightarrow 0$. The open string decoupling limit corresponds exactly to the limit where the closed string modes decouple from the asymptotic geometry! We now have that the low energy, decoupling limit in the open string sector corresponds to strongly coupled $\mathcal{N} = 4$ $SU(N_c)$ SYM theory in $d = 4$ and in the closed string sector is just a theory of fluctuations around $\text{AdS}_5 \times S^5$, i.e. weakly coupled classical gravity. We have thus seen the weak form of the correspondence emerge quite naturally. Moreover, we can see that the isometry group on the gravity side is $SO(4, 2) \times SO(6)$, while the bosonic subgroup of the superconformal symmetry in the field theory is $SO(4, 2) \times SO(6)_R \subset PSU(2, 2|4)$. Since dual theories are supposed to describe the same underlying physics, it is necessary for the global symmetries to match, which is the case in this example of AdS/CFT

Holographic Dictionary

The above discussion has established the existence of a limiting procedure that gives equivalent descriptions of the physics of a particular low energy effective type IIB string theory in terms of weak classical gravity and the dynamics of a large N_c superconformal field theory. However, it should be made more precise with an accounting for how fields in the gravitational theory map onto observables in the dual CFT. This map is often called the holographic dictionary and is essential for the construction of precise questions on either side of the duality. This section will contain a brief overview of the arguments and a preview of how they will be applied in later sections.

The isometry group in the gravitational description is a subgroup of the dimensionally reduced $\mathcal{N} = 1$ supergravity symmetry group. In addition, the resulting metric has a simple product structure. Together, this means that we can organize fields according to representations of the $d = 5$ Lorentz group and $SO(6)$ rotation group of the S^5 by

$$\phi^I(z, x^\mu, \theta^i) = \sum_{\ell=0}^{\infty} \tilde{\phi}_\ell^I(z, x^\mu) Y_\ell(\theta^i), \quad (1.8)$$

where $\tilde{\phi}$ carries the appropriate Lorentz indices. The S^5 components organized according to the familiar basis of scalar spherical harmonics on the unit S^p , $Y_\ell(\theta^i)$, that satisfy the eigenvalue equation $\square_{S^p} Y_{\ell m} = -\ell(\ell + p - 1)Y_\ell$. The set of spinor and vector spherical harmonics on the S^5 are needed for the other fields in the gauge multiplet, which are uglier, but the spirit of the decomposition is the same.

To get a better sense of where this leads, consider linearized fluctuations of eq. (1.1) around eq. (1.6), $\tilde{g}_{MN} = g_{MN} + \delta g_{MN}$ and $\tilde{F}_{(5)} = F_{(5)} + \delta F_{(5)}$. Fixing the gauge for the background RR four-form potential by the Freund-Rubin ansatz [22, 23], we can satisfy the condition that the field strength is component-wise proportional to the volume forms: $LF_{(5)} \sim 4(\text{vol}(\text{AdS}_4) + \text{vol}(S^5))$. The metric fluctuations of the AdS_5 and S^5 yield generically coupled PDEs, but changing basis and using the eigenvalue equation for the scalar spherical harmonics to study decoupled modes $\sigma^{(\ell)}(z, x^\mu)$ yields the equations of motion for a free $d = 5$

scalar of mass $m(\ell)^2 L^2 = \ell(\ell - 4)$:

$$(\square_{AdS_5} - \frac{1}{L^2} \ell(\ell - 4)) \sigma^{(\ell)}(z, x^\mu) = 0. \quad (1.9)$$

This generalizes to arbitrary AdS_d where the spectrum for scalar (and massive spin-2) fields is $m^2 L^2 = \ell(\ell - d + 1)$.

To establish a dictionary to correspond with the soon-to-be analyzed field theory, we require knowledge of the solutions to the above equations of motion and their asymptotic behavior as they approach the conformal boundary of AdS_5 at $z = 0$. Using the form of the AdS_5 metric in Poincaré coordinates and that the Laplacian is given by $\square \equiv \frac{1}{\sqrt{g}} \partial^a (\sqrt{g} \partial_a)$, the equations of motion become

$$(z^2 \partial_z^2 - 3z \partial_z - \ell(\ell - 4) + z^2 \partial^\mu \partial_\mu) \sigma^{(\ell)} = 0. \quad (1.10)$$

We can consider $\sigma^{(\ell)}$ expanded in plane waves on the $d = 4$ slices of constant z and set $p_\mu = 0$ to extract the radial profile. The resulting differential equation in z is recognizable as the confluent hypergeometric equation, whose near boundary behavior falls into two towers

$$\sigma^{(\ell)} \sim z^{2-\sqrt{4+\ell(\ell-4)}} (\sigma_{0,-}^{(\ell)} + \mathcal{O}(z^2)) + z^{2+\sqrt{4+\ell(\ell-4)}} (\sigma_{0,+}^{(\ell)} + \mathcal{O}(z^2)). \quad (1.11)$$

Here we can make the identification of the scaling dimension of the field near the boundary, $\Delta_{\pm}^{(\ell)} = 2 \pm \sqrt{4 + \ell(\ell - 4)}$. Analyzing the finiteness of the action evaluated on the particular solutions above, the Δ_- tower consists of non-normalizable modes, and the Δ_+ tower contains normalizable modes. In the holographic description of the field theory, the non-normalizable modes correspond to the insertion of sources for operators of dimension Δ_- , and the Δ_+ is mapped to the responses to the sources.

How does this play out explicitly in the field theory? In four dimensional $\mathcal{N} = 4$ $SU(N_c)$ SYM, as is the standard procedure in discussing observables in a QFT, we classify our gauge invariant operators according to their representations that they transform under the global symmetries. Here we will only consider representations of the bosonic subalgebra $\mathfrak{so}(4, 2) \oplus \mathfrak{so}(6)_R$ of the superconformal algebra; the fermionic operators can be studied

easily following supersymmetry transformations. First, the field content of the theory is a single $\mathcal{N} = 4$ gauge multiplet containing one gauge field A_μ , six real scalars ϕ^I , and four Weyl fermions ψ_A — all transforming in the adjoint representation of $SU(N_c)$. The gauge invariant operators are color singlets, and the $\frac{1}{2}$ -BPS operators among those are formed by taking traces over products of the fields. Specifically, the chiral primary operators transforming in the totally symmetric rank Δ tensor rep is given by

$$\mathcal{O}_\Delta = \text{sTr} \left[\phi^{I_1} \dots \phi^{I_\Delta} \right], \quad (1.12)$$

where the scalars in the gauge multiplet transform in the $\mathbf{6}$ of $\mathfrak{so}(6)_R$, and sTr is the symmetrized trace over gauge indices. As is befitting a BPS operator, the R-charge is equal to the scaling dimension. Comparing to the spectrum of linearized fluctuations in the gravity side, the angular momentum, ℓ , on the internal S^5 of the scalar field above can now be read off as the R-charge, Δ , of the operator \mathcal{O}_Δ .

Were we to consider fields in the bulk gravity theory with spin $s \geq 0$, we could follow the same logic as above — solving the equations of motion, reading off the asymptotic expansion, and identifying the field theory operator according to its R-charge and spin — and arrive at the following enumeration for the masses and scaling dimension

Bulk Field	Dictionary Entry
$s = 0$, Massive $s = 2$	$m^2 L^2 = \Delta(\Delta - 4)$
$s = \frac{2k+1}{2}$ ($k = 0, 1$)	$m^2 L^2 = (\Delta - 2)^2$
q-form	$m^2 L^2 = (\Delta - q)(\Delta + q - 4)$
Massless $s = 2$	$m^2 L^2 = 0, \Delta = 4$
Rank q symmetric, traceless tensor	$m^2 L^2 = (\Delta + s - 2)(\Delta - s - 2)$

Table 1.3: Holographic Dictionary for bulk fields of spin s

The above table is specific to AdS_5 but can be easily generalized. Further, the massless spin-2 dictionary only contains $\Delta = 4$ and not $\Delta = 0$ due to unitarity constraints.

This identification is far from establishing a full holographic dictionary between fields in the gravitational theory and operators in the field theory, but it is the essential part that will be used repeatedly to great effect. However, what the table above represents is a statement about the asymptotic behavior of bulk fields giving a deformation of the field theory action by a source term for a scalar operator: $S_{src} \equiv \int d^x \sigma_{0,-}^{(\ell)} \mathcal{O}_\ell$. This can be encoded in usefully in a statement about the partition functions of the two descriptions

$$\begin{aligned} e^{-S_{grav}[\sigma]} \Big|_0 &= e^{-W[\sigma_{0,-}]} = \langle e^{S_{src}} \rangle, \\ \Rightarrow \quad \langle \mathcal{O}_{i_1}(x_1) \cdots \rangle &= - \frac{\delta^n W[\sigma_{0,-}]}{\delta \sigma_{0,-}^{i_1} \cdots} \Big|_{\sigma_{0,-}^{i_k}=0}. \end{aligned} \tag{1.13}$$

That is, the on-shell gravitational action is equated with the generating functional for the n -point connected correlation functions of operator insertions \mathcal{O}_{i_k} sourced by the non-normalizable part of the asymptotic behavior of the field σ^{i_k} . The precise holographic treatment of one (and higher) point functions requires the technology of *holographic renormalization* which will be discussed in detail in Ch. 2.

Before moving on to an application of AdS/CFT technology in black hole thermodynamics, there is a small caveat to this identification of source and response that will feature prominently in later chapters. If a bulk field in AdS_5 sources an operator whose conformal dimension lies in the window $1 \leq \Delta \leq 2$ then there is an ambiguity in which term in the asymptotic expansion leads to a source and which to a vev. The reason is that in this window, where the mass of the field lies between $m_{BF}^2 \leq m^2 \leq m_{BF}^2 + 1$ — where $m_{BF}^2 = -\frac{(d-1)^2}{4}$ is the lower bound on the mass for a field in AdS_d such that there is no induced instability through backreaction called the *Breitenlohner-Freedman bound* (BF) — there are two equivalent ways to impose boundary conditions at $z = 0$ consistent with supergravity. Choosing one or the other is perfectly allowed and leads to changing what field data corresponds to the source and vev called *alternative quantization*.

1.2 A Novel Application To Black Hole Physics

To round out this discussion of AdS/CFT, let us consider an example of how the above correspondence can be utilized to resolve questions that involve black hole physics in curved spacetimes. To that end, we will explore a how AdS/CFT is used to inform interpretations of black hole thermodynamics. The full discussion of the example considered below is contained in [24] co-authored with Andreas Karch with which there is text overlap.

1.2.1 Holographic Black Hole Chemistry

In the first example, let us explore the thermodynamics of black holes in AdS spacetimes as introduced in the previous subsections. This subject has been long studied, but recent proposals have aimed to cast the precise interpretation of the cosmological constant in a new light [25, 26]. To start, through AdS/CFT and the standard interpretation of black hole thermodynamics, the thermal properties of AdS black holes can be reinterpreted as those of a conformal field theory at finite temperature [11] such that the grand canonical free energy Ω in the field theory is expressible in terms of the on-shell action S_{os} of the Euclidean bulk solution as

$$TS_{os} = \Omega(\mu, T) = U - TS - \mu Q, \quad (1.14)$$

Here U , T , S , μ and Q are used to denote black hole thermodynamic quantities; specifically energy, temperature, entropy, chemical potential and charge of the black hole. For black branes with a homogeneous planar horizon the extensive quantities Ω , U , S and Q are spatially independent, which makes the densities defined by

$$\Omega = V\omega, \quad U = Vu, \quad S = Vs, \quad Q = Vq, \quad (1.15)$$

more natural variables. As a consequence the pressure, $p = -d\Omega/dV|_{T,\mu} = -\omega$, is simply the opposite of the grand canonical free energy density. For spherical horizons, the spatial volume of the dual field is finite and completely determined by the radius R of the sphere

on which the field theory lives. In this case, the free energy depends on R explicitly via dimensionless ratios like TR and μR in addition to the overall prefactor of $V = s_n R^n$, where n is the number of spatial dimensions of the field theory and s_n is the volume of the unit n -sphere implying the pressure takes the form

$$p = -\frac{dR}{dV} \frac{d\Omega}{dR} = -\frac{Rd_R\Omega}{nV} = -\omega - \frac{Rd_R\omega}{n} \equiv -\omega + p_R/n. \quad (1.16)$$

The same analysis holds for \mathbb{H}^n horizons as well. The point is that the pressure follows directly from Ω , and so what is needed is just the functional form for the grand canonical partition function.

In the alluded to recent reinterpretation [25, 26], which has become known as “black hole chemistry” [27], the pressure for black hole thermodynamics, p_b , is identified with cosmological constant, Λ such that the usual first law of black hole mechanics expressed in terms of a black hole’s mass, M , and area A , takes a form more in line with the standard thermodynamical notion of the first law

$$dM = \frac{\kappa}{8\pi} dA - p_b V_b, \quad (1.17)$$

where the ‘thermodynamic volume’ of the black hole is the corresponding conjugate variable, $V_b \equiv \partial M / \partial p_b|_A$ [28, 29, 30, 31, 32]. The subscript b is used for “bulk” in order to distinguish between the separate notions of black hole quantities and those in the dual field theory.

However as noted, the pressure of the dual theory is fixed by thermodynamic relations once the on-shell action has been identified as the grand canonical free energy. Thus, it is not a quantity that should nor can be defined separately. However, the inhomogeneity in the bulk metric owing to its radial dependence implies the identification of $\Lambda = p_b$ and V_b as an effective bulk (thermodynamic) volume of the black hole. Note that the notion of thermodynamic volume can differ from the geometric volume that would be calculated from simply integrating the volume form from the origin to the outer horizon of the black hole [32]. The upshot is that in black hole chemistry the story is not just a simple tidying of

thermodynamic concepts but also gives rise to a remarkably simple (Smarr) relation [33]:

$$(d - 3)M = (d - 2)TS - 2P_b V_b + (d - 3)\mu Q, \quad (1.18)$$

where $d = n + 2$ is the number of spacetime dimensions of the gravitational theory.

While this new interpretation implies some unexpectedly rich thermodynamics phase behavior such as reentrant phase transitions, the holographic treatment of these new relations shows a striking origin. Consider that if we were to treat Λ as a thermodynamic parameter, then as it is generally interpreted in the holographic language we would be treating the rank of the dual field theory gauge group, i.e. the number of colors, N_c as a thermodynamic variable [34, 35, 36, 37, 38]. This is unusual from the perspective of standard statistical mechanics as the question literally being asked in calculating variations of $\log \Omega$ with respect to N_c is how the free energy changes when we change the theory. While an unconventional question, it can be addressed by defining the color susceptibility χ_{N^2} as

$$\chi_{N^2} = \left. \frac{\partial \Omega}{\partial N^2} \right|_{\lambda, \mu, T, R}, \quad (1.19)$$

where as usual in holography, we are working at fixed 't Hooft coupling, λ , which will be omitted in further expressions.

In the limit of a large number of colors, N_c only appears in an overall prefactor in the grand canonical free energy:

$$\Omega(N_c, \mu, T, R) = N_c^2 \Omega_0(\mu, T, R) \quad \Rightarrow \quad N_c^2 \chi_{N_c^2} = \Omega. \quad (1.20)$$

The simple relation obeyed by the color susceptibility is argued to be the holographic origin of eq. (1.18) and is insensitive to the underlying details of the quantum fluid; only relying on the large N_c limit. The only difference among large N_c theories that can arise is in the exponent of N_c . For example M2 brane theories famously have $N_c^{3/2}$ scaling, while M5 branes have N_c^3 scaling, but the schematic form $N_c^\bullet \chi_{N_c^\bullet} = \Omega$ holds. For the purpose of clarity, we consider theories dual to type IIB branes where the exponent is 2.

Given the universal Smarr relation (1.20), many derived identities can be formulated for a particular system using the equation of state relating $p_R = -R\partial_R\omega$, $s = -\partial_T\omega$, and

$q = -\partial_\mu \omega$, which implicitly appear on the right hand side via $\Omega = U - TS - \mu Q$. Of particular interest are conformal field theories, whose equation of state is fixed by considering the behavior of the thermodynamic quantities under an infinitesimal scale transformation (under which all energies are rescaled by $\lambda = 1 + d\lambda$ and all length by $\lambda^{-1} = 1 - d\lambda$):

$$dU = U d\lambda, \quad dS = 0, \quad dQ = 0, \quad dV = -nV d\lambda. \quad (1.21)$$

The fact that a scale transformation takes one physical configuration into another, and so has to represent a set of variations consistent with the first law of thermodynamics

$$dU = TdS - pdV + \mu dQ \quad \Rightarrow \quad U = npV, \quad (1.22)$$

or using the definition of the pressure from (1.16), the equation of state above can be rewritten

$$(n+1)u = nTs + n\mu Q - R\partial_R \omega. \quad (1.23)$$

At this point, we need to identify the correct holographic dictionary to facilitate the link between the (1.20) and eq. (1.18). As noted in the previous section, the relation the bulk gravitational coupling and the dual field theory is, for some theory dependent constant α ,

$$\alpha \frac{L^{d-2}}{16\pi G_N} = N_c^2, \quad (1.24)$$

which in conjunction with the definition of the cosmological constant $\Lambda = (d-1)(d-2)/(2L^2)$ builds the bridge. Note that while the details of the exact holographic system in question is necessary to specify α , it would only appear in $p_b V_b$ where it drops out. Using these two expressions, the scaling behavior of Λ then implies that

$$-2\Lambda \partial_\Lambda = L \partial_L = (d-2)N_c^2 \partial_{N_c^2} + R \partial_R. \quad (1.25)$$

In deriving eq. (1.25), one should note that variations in the bulk length scale, L , not only change N_c according to eq. (1.24), but also change the length scale in the boundary geometry $R = L$ through the defining function, i.e. the boundary metric

$$ds^2 = -dt^2 + L^2 d\Sigma_k^2. \quad (1.26)$$

Lastly if one considers an AdS Einstein-Maxwell black hole solution of charge Q , i.e.

$$S = \frac{1}{16\pi G_N} \int d^d x \sqrt{-g} (R - 2\Lambda - F_b^2), \quad (1.27)$$

then a careful treatment of the holographic system reveals more factors of AdS length in

$$A_b = LA, \quad \mu_b = L\mu, \quad Q_b = L^{-1}Q. \quad (1.28)$$

Here the gauge field A has canonical mass dimension 2, and μ is the associated chemical potential.

Now, we have the necessary information to derive eq. (1.18) holographically from (1.20). From the definition of p_b and V_b we have

$$-2p_b V_b = -2\Lambda \partial_\Lambda M|_{S, Q_b} = (d-2)N^2 \partial_{N^2} U|_{S, Q} + R \partial_R U|_{S, Q} + Q \partial_Q U|_{S, L}, \quad (1.29)$$

where we used the expression of L derivatives in terms of boundary quantities from eq. (1.25) as well as the relation between derivatives at fixed Q versus fixed Q_b

$$\partial_L f(L, Q(Q_b, L))|_{Q_b} = \partial_L f|_Q + \partial_Q f|_L \partial_L Q|_{Q_b} = \partial_L f|_Q + \frac{Q}{L} \partial_Q f|_L. \quad (1.30)$$

Evaluating each of the three terms individually, note that since U is the Legendre transform of Ω and using (1.20), the first term in (1.29) becomes

$$N^2 \partial_{N^2} U|_{S, Q} = N^2 \partial_{N^2} \Omega|_{\mu, T} = \Omega. \quad (1.31)$$

The derivative in the third term simply evaluates to μQ . For the second term in (1.29), we use the definition of pressure from (1.16) and use (1.22) with the fact that U and Ω are related by Legendre transforms¹ to arrive at,

$$R \partial_R U|_{S, Q} = R \partial_R \Omega|_{\mu, T} = -npV = -U. \quad (1.33)$$

¹Explicitly, take

$$\begin{aligned} \partial_R U|_{S, Q} &= \partial_R (\Omega(\mu(S, Q, R), T(S, Q, R), R) + \mu Q + ST)|_{S, Q} \\ &= \partial_\mu \Omega|_{T, R} \partial_R \mu|_{S, Q} + \partial_T \Omega|_{\mu, R} \partial_R T|_{S, Q} + \partial_R \Omega|_{\mu, T} + Q \partial_R \mu|_{S, Q} + S \partial_R T|_{S, Q} = \partial_R \Omega|_{\mu, T} \end{aligned} \quad (1.32)$$

where we used $\partial_\mu \Omega|_{T, R} = -Q$ and $\partial_T \Omega|_{\mu, R} = -S$.

Putting the three terms back together,

$$-2p_b V_b = (d-2)\Omega + \mu Q - U = (d-3)U - (d-2)TS - (d-3)\mu Q, \quad (1.34)$$

which is, in fact, exactly the standard Smarr relation eq. (1.18). Note that, as expected, both the universal formula (1.20) as well as the conformal equation of state (1.22) were needed to derive this result.

The task, now that the boundary analysis only requires one to know scaling relations that define dual equation of state to input into the universal Smarr formula, is to check against the appropriate combinations of bulk quantities calculated using standard techniques in General Relativity. Considering the AdS-Reissner Nordstrom black hole, whose metric function and bulk gauge field are

$$h(r) = \frac{r^2}{L^2} + k - \frac{m}{r^{d-3}} + \frac{q^2}{r^{2d-6}}, \quad A_b = \left(-\frac{1}{c} \frac{q}{r^{d-3}} + \mu_b \right) dt, \quad c = \sqrt{\frac{2(d-3)}{d-2}}, \quad (1.35)$$

the intensive bulk thermodynamic quantities are

$$T = \frac{(d-1)r_+^2 + (d-3)L^2(k - c^2\mu_b^2)}{4\pi L^2 r_+}, \quad \mu = \frac{\mu_b}{L} = \frac{q}{cLr_+^{d-3}}, \quad R = L \quad (1.36)$$

and extensive

$$\begin{aligned} Q &= LQ_b = \sqrt{2(d-2)(d-3)} \frac{Ls_{d-2}q}{8\pi G_N}, \\ U &= M = \frac{(d-2)s_{d-2}m}{16\pi G_N}, \\ S &= \frac{A}{4G_N} = \frac{s_{d-2}r_+^{d-2}}{4G_N}. \end{aligned} \quad (1.37)$$

Here r_+ denotes the largest real positive root of the metric function such that

$$m = kr_+^{d-3} + \frac{r_+^{d-1}}{L^2} + \frac{q^2}{r_+^{d-3}}. \quad (1.38)$$

The equation of state is determined by comparing $TS + \mu_b Q_b$ to the ADM mass M , which reads

$$(d-1)M = (d-2)(TS + \mu_b Q_b) + 2k \frac{s_{d-2}}{16\pi G} r_+^{d-3}. \quad (1.39)$$

The left hand side and the first two terms on the right hand side match (1.23) with $n = d - 2$, but identifying the last term as $p_R = -R\partial_R\omega$ is more subtle. The boundary pressure, p_R , is found by first calculating the thermodynamic potential

$$\Omega = E - TS - \mu Q = -\frac{s_{d-2}}{16\pi G_N} \left(\frac{r_+^{d-1}}{L^2} + \frac{q^2}{r_+^{d-3}} - kr_+^{d-3} \right), \quad (1.40)$$

and then taking derivatives with respect to L at fixed L^{d-2}/G_N , μ and T . All that is then needed is to convert all of the variables (r_+, q) in (1.39) to quantities that are held fixed. Using $G_N \sim L^{d-2}N^{-2}$, $r_+ \sim L^2T$ and $q \sim \mu Lr_+^{d-3}$. With these replacements the first and second term in the density $\omega = \Omega/(s_{d-2}R^{d-2})$ scale as L^0 , whereas the last term proportional to k scales as L^{-2} . Consequently

$$R\partial_R\omega|_{N,\mu,T} = L\partial_L\omega|_{L^{d-2}/G_N,\mu,T} = -2\frac{kr_+^{d-3}}{16\pi G_N L^{d-2}}. \quad (1.41)$$

This is exactly what is needed to confirm that (1.23) indeed holds for these black holes.

A non-trivial check on this construction is to consider geometries dual to large N_c gauge theories with hyperscaling violation. Top-down examples of such theories are maximally supersymmetric gauge theories in $n + 1$ dimensions dual to black Dn branes, which are described by [13]

$$ds^2 = H^{-\frac{1}{2}} \left(-h(r)dt^2 + dx_n^2 \right) + H^{\frac{1}{2}} \left(\frac{dr^2}{h(r)} + r^2 d\Omega_{8-n}^2 \right), \quad (1.42)$$

$$e^\Phi = H^{\frac{3-n}{4}}, \quad C_{01\dots n} = H^{-1},$$

where $H = 1 + \left(\frac{L}{r}\right)^{7-n}$ and $h(r) = 1 - \left(\frac{r_\pm}{r}\right)^{7-n}$. The temperature can be found by calculating the surface gravity $\kappa = 2\pi T$ or demanding that the analytic continuation of the time direction be free of conical singularities such that

$$T = \frac{7-n}{4\pi L} \left(\frac{r_+}{L} \right)^{\frac{5-n}{2}}, \quad (1.43)$$

which demonstrates the scaling behavior of the horizon radius $r_+ \sim T^{\frac{2}{5-n}} L^{\frac{7-n}{5-n}}$ as seen above for the $n = 3$ case. In $n + 2$ dimensional Einstein frame with $L = 1$, the metric takes the form

$$ds_{n+2}^2 = r^{2(8-n)} H^{\frac{1}{n}} \left(-h(r)dt^2 + dx_n^2 + \frac{dr^2}{h(r)} \right). \quad (1.44)$$

From this, we can read off the scaling for the field theory entropy, energy, and charge (if present) densities [39]

$$[s] = n - \theta, \quad [u] = n + 1 - \theta, \quad [q] = n - \theta, \quad \theta = -\frac{(n-3)^2}{5-n}. \quad (1.45)$$

Again using these scaling relations in a first law calculation, we find that the equation of state is given by

$$(n+1-\theta)u = (n-\theta)(Ts + \mu q) = (n+1-\theta)np. \quad (1.46)$$

These identities are indeed obeyed by the thermodynamic quantities derived in [13]; see e.g. the expressions presented in [40]. For $n=3$ or $\theta=0$, we recover the equation of state found previously as expected for non-hyperscaling violating gauge theories. ω still scales as N_c^2 as appropriate for a large N_c gauge theory and so the ‘universal’ holographic Smarr relation, (1.20) still holds for any n .

Finite N corrections

In the large N_c limit, to leading order all extensive thermodynamics quantities in a gauge theory simply scale with an overall prefactor of N_c^2 . Note that this simple fact also has far-reaching consequences when thinking about the phase diagram of the theory. For the free energy

$$\Omega(N_c, \mu, T) = N_c^2 \Omega_0(\mu, T) \quad (1.47)$$

to have any non-trivial phase transitions, these discontinuities must all arise from the behavior of Ω_0 , which implies that no non-trivial phase transitions can possibly occur as a function of N_c . Thus, treating Λ as a new thermodynamical parameter may give the phase diagram an extra dimension, but when carefully accounting for the L dependence in N_c , μ and T , the black hole chemistry phase diagram is just a trivial extension of the phase diagram living in the μ - T plane. Indeed, the black hole chemistry literature finds [41] that the standard first order Hawking phase transition simply extends into a line of first order phase transitions, while the Reissner-Nordstrom black hole retains its Van der Waals behavior that was already

found in the traditional holographic analysis of the same system in [42]. All of this is trivially due to the N_c scaling behavior and the fact that varying with respect to Λ mixes up the trivial variation of N_c with a variation of the volume the field theory lives on.

The situation changes dramatically when we go beyond the leading large N_c limit. On the bulk side, this corresponds to including higher curvature terms. In fact, studies of black hole chemistry involving higher curvature couplings [25, 43, 44] find an array of exotic behaviors such as reentrant phase transitions and isolated critical points with unusual exponents. For general values of N_c , we expect N_c to be a genuinely new variable on which Ω non-trivially depends. In this case, our universal Smarr formula (1.20) no longer holds as written. Despite this fact [25, 43, 44] still find generalized Smarr relations to be valid even when higher curvature couplings are included as long as one treats the coefficients of the extra curvature terms as independent couplings which can be varied as well. This appears to be an artifact of including only a finite number of curvature terms. Treating the first, subleading $1/N_c$ terms, such that

$$\Omega = N_c^2 \Omega_0 + a N_c^0 \Omega_1, \quad (1.48)$$

we still have a relation of the form:

$$N_c^2 \partial_{N_c}^2 \Omega + a \partial_a \Omega = \Omega. \quad (1.49)$$

Continuing the expansion on in this manner, (1.49) will be spoiled by further $1/N_c$ corrections unless one includes a new thermodynamic variable for every new term in the series. In any consistent theory of quantum gravity, such as string theory, it is believed that one has an infinite tower of higher curvature corrections, and so no useful relation of this sort will hold in general.

Maybe the simplest example where one has an interesting approximation with only two terms is the case of flavor branes [15]. In this case the expansion reads

$$\Omega = N_c^2 \Omega_0 + a N_c^1 \Omega_1. \quad (1.50)$$

These two terms are singled out by being large in the large N_c limit and so are dominated by a semi-classical saddle. Here we have denoted $a \sim N_f$ as the number of flavors. So we can derive a generalized holographic Smarr relation in the field theory by independently varying the number of colors and flavors in the field theory. In the bulk, this corresponds to varying G_N as well as the tension of the brane.

Chapter 2

A TOUR OF FURTHER BACKGROUND MATERIAL

2.1 *Probe Brane Holography*

In this chapter, we will build off of the preceding section wherein it was argued that the classical gravitational dynamics in the near horizon geometry, $\text{AdS}_5 \times \text{S}^5$, of N_c coincident D3 branes in the large N_c limit is captured by the physics of the strong coupling — to be precise ‘t Hooft — limit of $SU(N_c)$ $\mathcal{N} = 4$ super Yang-Mills theory on a background identical to the conformal boundary of the AdS_5 . The obvious utility of exploiting the well studied and easily accessible physics of classical gravity on a background with negative scalar curvature as a stand in for a strongly coupled quantum field theory, even if that theory is extremely restricted by the maximal superconformal algebra, has a glaring drawback apart from the those noted at above; every field in the gauge multiplet is in the adjoint representation of $SU(N_c)$; it is a theory of pure glue.

To remedy this and include in the correspondence objects that could provide analogs for quark-like matter in order to bring it in closer phenomenological contact with the physics of the world that we have thus far uncovered, we require a mechanism that allows for the introduction of fields transforming the fundamental representation of $SU(N_c)$. Given that $d = 4$ $\mathcal{N} = 4$ supersymmetry constrains the dynamical content of a Yang-Mills theory to reside entirely within the gauge multiplet, our new tool should be one that adds, in the case of preserving the maximal amount of supersymmetry, $\mathcal{N} = 2$ hypermultiplets on the field theory side. This is accomplished on the gravity side of the duality by so-called “probe brane” embeddings. One can see that by the inclusion of the probe Dp-branes is an enlarged open string sector such that in addition to the $3 - 3$ modes — strings with both endpoints on the D3-branes — we have $p - p$, $p - 3$, and $3 - p$ modes. The $p - p$ modes are the mesons

in the worldvolume theory of the newly added D p -brane, while the $p - 3$ and $3 - p$ modes are (anti)-fundamental degrees of freedom.

As an advertisement, there is a wealth of literature exploring the utility of probe branes in a host of holographic applications. Since the probe limit acts to greatly simplify any use, the simplifications provided make them a very versatile tool. To wit, they are used, as mentioned above, to add the quark content to attempts to build holographic QCD-like theories [45, 46]. Even in the context of applications of holography to condensed matter systems, probe branes provide one of the simplest holographic realizations of compressible and conducting matter [47]. More examples abound, but we should work to define the paradigm first.

2.1.1 Probe Brane Schematics

As we have seen in Ch. 1, certain non-perturbative extended objects in a string theory exist such that open string end points can be fixed to the directions spanned by the object. Indeed, that open string sector on the worldvolume of these D-branes through the Chan-Paton factors carried by the endpoints are what gave rise in the low energy limit to the gauge theory dual to the gravitational theory supplied by the closed string sector. What would be desirable is to have a separate set of D-branes on which the string endpoints could be affixed such that in the worldvolume theory of the original collection of D-branes, we would have a new set of degrees of freedom that carry color charge under the $SU(N_c)$ gauge group, i.e. are isolated objects in the fundamental representation. That is in the colloquial language of particle physics, we want to add ‘quarks’ into our spectrum that previously only contained ‘mesons’. To be clear, what will be described in the following sections is not an attempt at describing AdS/QCD but rather a toy model for strongly coupled physics including quark-like matter. Specifically in the end, we will want to carry out this construction that explicitly is aimed at preserving as many supersymmetries as possible, which makes the connection to QCD or any chiral theory tenuous if one starts out with $\mathcal{N} = 4$ SYM theory (see [48]). However, exploration of those models, such as Hanany-Witten type constructions [49], is outside of the scope of this thesis. As a brief programming note, we will work for the rest of this section

and much of the following chapters with the AdS length $L_{AdS} = 1$ unless otherwise noted for clarity.

While there are a number of possible ways to proceed with this construction, the focus will remain on type IIB string theory and in particular adding branes to a background of a large number of coincident D3-branes. The probe brane paradigm that we will be considering in this text has branes wrapping particular maximal cycles in the S^5 directions at the boundary [15]. In the ten dimensional language, the branes span the following directions

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9
D3	•	•	•	•	○	○	○	○	○	○
D7	•	•	•	•	•	•	•	•	○	○
D5	•	•	•	○	•	•	•	○	○	○

Table 2.1: D7- and D5-branes in D3 background

That is the D7 fills all of the directions of the background N_c D3 branes, while the D5 sits at a fixed x_3 and provides a co-dimension 1 defect on the D3 worldvolume. This is not the complete story for D7 and D5 probe branes as there are other embeddings where the branes break supersymmetry but are stabilized by additional fluxes (e.g. [50]) or describe supersymmetric flavors higher co-dimension defects (e.g. [51, 52]).

Adding new branes to the supergravity background must break some supersymmetries. After all, adding D3-branes into $\mathbb{R}^{(9,1)}$ broke 16 of the original 32 supersymmetries, and so the best we can hope for is that the probe branes being added are $\frac{1}{2}$ -BPS with respect to the D3 background. A thorough accounting of which supersymmetries are preserved will be done in §2.1.2, but for now we can do a quick accounting by examining the number of directions wrapped by the additional branes transverse to the original D3 worldvolume.

To understand this accounting note that the original 32 supersymmetry generators for $d = 10$ IIB supergravity can be written as Majorana-Weyl spinors of equal chirality such that the supercharges $\{Q_\alpha, \tilde{Q}\}$ transform in the **16** of $SO(9, 1)$. These are broken to the set

in $\mathcal{Q}_\alpha \equiv \{Q_\alpha + ((\prod_{m_\perp} (\Gamma^{m_\perp} \Gamma_{11}) \tilde{Q})_\alpha)\}$, where m_\perp span the directions transverse to the D3-branes and $\Gamma_{11} \equiv \Gamma_{0\dots 9}$. Other branes would break a different set due to the span of transverse directions being changed so that supercharges of the form $\mathcal{Q}'_\alpha \equiv \{Q_\alpha + ((\prod_{m'_\perp} (\Gamma^{m'_\perp} \Gamma_{11}) \tilde{Q})_\alpha)\}$. Thus, the remaining supersymmetries are those in $\mathcal{Q} \cap \mathcal{Q}'$. This is given then by the set of supercharges invariant under $\prod_{m_\perp, m'_\perp} (\Gamma^{m_\perp} \Gamma_{11})^{-1} (\Gamma^{m'_\perp} \Gamma_{11})$, which owing to the action of any $\Gamma^{m_\perp} \Gamma_{11}$ on a spinor as a reflection in the n_\perp -plane amounts to reflections in the planes spanned exclusively by either set of branes, e.g. those seen by $3 - p$ or $p - 3$ strings with mixed Neumann-Dirichlet (ND or DN) boundary conditions.

Adding new branes to the background D3s would certainly deform the open and closed string sectors if done haphazardly, potentially ruining the nice picture constructed in the previous chapter. However, organizing the hierarchy of limits such that (i) $1 \ll N_f \ll N_c$, which implies (ii) $1 \ll T_{brane} \ll G_N^{-1}$, we can construct a reasonable picture of the new degrees of freedom. The (ii) limit effectively decouples the closed string sector of the probe brane fluctuations such that there is no deformation of the background $\text{AdS}_5 \times S^5$ geometry as the low energy limit of the D3-branes. The (i) limit ensures that we are in a suitable decoupling limit with respect to the worldvolume (open string) theory of the probe branes such that we can talk about purely low energy, non-stringy physics being added to the background $\mathcal{N} = 4$ $SU(N_c)$ SYM theory to which the new degrees of freedom is thus minimally coupled. If we then employ the probe branes embeddings that wrap minimal cycles, we find exactly that $\#_{ND} = 4$ which corresponds to $\frac{1}{4}$ -BPS degrees of freedom coming from the probe sector. Note that this is a generic statement, but the special case where the probe branes are not separated from the D3 stack in any of the transverse directions can preserve conformal symmetry as well leaving a total of 16 superconformal charges unbroken.

As a general statement, let us consider the symmetries that will manifest in the probe brane paradigm. In the case of a single probe Dp-brane wrapping $\text{AdS}_n \times S^m \subset \text{AdS}_5 \times S^5$ ($n + m = p + 1$) in the manner of Table. 2.1, there will be an overall $SO(n - 1, 2) \times SO(m)$ preserved. From the ten dimensional perspective, there is an additional $SO(6 - (m + 1))$ isometry in the transverse direction to the original stack of D3-branes that acts to rotate

the embedding functions of the worldvolume theory. This will generically be broken when we consider massive embeddings where the probes are separated by a distance l_{probe} away from the D3-branes. Moreover, if we consider a large number of coincident flavor branes N_f then, just as in the previous discussion of the D3-branes, we have an emergent $U(N_f) \cong SU(N_f) \times U(1)_B$ flavor symmetry in the holographic description. The fundamental matter carries $U(1)_B$ charge ± 1 .

All of the above discussion has left a very critical point out: the deformation of $\mathcal{N} = 4$ SYM theory by adding flavors leads to the theory no-longer being conformal. This can be seen at the level of the 1-loop β -function, where for a generic $SU(N_c)$ gauge theory in $d = 4$ is given by

$$\beta = -\frac{g^3}{16\pi^2} \left(\frac{11}{3}C_{adj} - \frac{2}{3}n_f C_{R_f} - \frac{1}{6}n_s C(R_s) \right), \quad (2.1)$$

where n_f is the number of Weyl fermions transforming in the R_f representation of $SU(N_c)$, n_s is the number of real scalars in the R_s representation, and C_X is the quadratic Casimir for the X representation — see e.g. [53]. For $\mathcal{N} = 4$ SYM theory, every field is in the adjoint, and with $n_f = 4$ and $n_s = 6$, the right hand side vanishes. However by deforming the theory with N_f additional fundamental degrees of freedom, it is clear that $\beta \propto +\frac{N_f}{N_c}$. Thus at low energies, the theory develops a Landau pole; except for the case in which we are particularly interested, the ‘quenched’ flavor limit where $\frac{N_f}{N_c} \ll 1$. To leading order, then, there is no Landau pole. It does leave one asking, though, whether this paradigm can truly be trusted, and establishing that trust will comprise the next two chapters.

D3/D7

The canonical starting point for a pedagogical introduction to supersymmetric probe branes is the D3/D7 system. While most of the previous introductory material followed [19, 20] including notation, for ease in application to the later chapters the notation in this discussion will be more closely aligned with [15, 54]. Referring to Table. 2.1, what will be considered is the D7-brane wrapping all of the AdS direction and a $S^3 \subset S^5$ such that no additional fluxes

are required. For this and the D3/D5 setup, we will consider the background AdS_5 metric with $L_{\text{AdS}} = 1$ in Fefferman-Graham coordinates (see e.g. [55])

$$ds^2 = g_{IJ} dx^I dx^J = \frac{dr^2}{r^2} + \frac{1}{r^2} \tilde{g}_{ij} dx^i dx^j, \quad (2.2)$$

where $I, J = 0, \dots, 4$ run over all AdS_5 directions and $i, j = 0, \dots, 3$ span the radial slices. The conformal boundary lies at $r = 0$. For the case at hand, it is easiest to parameterize the S^5 geometry such that

$$d\Omega_5^2 = d\theta^2 + \sin^2 \theta d\psi^2 + \cos^2 \theta d\Omega_3^2. \quad (2.3)$$

The worldvolume metric found by computing the pullback of eq. (2.2). For simplicity and expediency, we will consider only the radial profile of the scalar fields found by taking the pullback, e.g. $\theta = \theta(r)$, such that

$$ds_{\text{WV}}^2 = \frac{1}{r^2} ((1 + r^2 \theta'^2) dr^2 + \tilde{g}_{ij} dx^i dx^j) + \cos^2 \theta(z) d\Omega_3^2. \quad (2.4)$$

Note that form of eq. (2.4) is a particular gauge choice, “static gauge”. That is, we have identified the worldvolume coordinates $\{z, x_i, \phi_i\}$ (where ϕ span the S^3) with those in the background geometry. While this is convenient, it is far from necessary, and as will be seen in Ch. 4, there are benefits to making other choices. Further in this construction, our interest is in probe brane embeddings that sit at a fixed distance away from the D3-branes, and so the pullback of the transverse directions is ignored.

As we saw for the D3-branes in the previous chapter, the low energy description of the worldvolume theory is given by the DBI action (with no worldvolume gauge fields the WZ term vanishes), which allowing for generic coordinate dependence, takes the form

$$S_{D7} = -T_7 \int_{\Sigma_8} d^8 \xi \sqrt{-\det P[g]_{\mu\nu}} = -T_7 \int_{\Sigma_8} d^8 \xi \sqrt{-|g| \sqrt{g_{S^3}} \cos^3 \theta \sqrt{1 + \partial^I \theta \partial_J \theta}}, \quad (2.5)$$

where $\int d^3 \alpha \sqrt{g_{S^3}}$ is the volume of the unit S^3 parameterized by α . The general treatment should include the angular profile of θ on the S^3 , but for the purposes of this illustration, we can just consider θ to have a constant profile on the internal space.

At this point with the above simplifications implemented, the equations of motion can be read off

$$\square_{AdS}\theta + 3 \tan \theta - \frac{\partial^I \theta \partial_I (\partial^J \theta \partial_J \theta)}{2(1 + \partial^J \theta \partial_J \theta)} = 0, \quad (2.6)$$

where $\square_{AdS} = \frac{1}{\sqrt{g}} \partial^I (\sqrt{g} \partial_I)$. In the simplest possible set up where, the Fefferman-Graham form of the metric reduces to the Poincare patch $\tilde{g} = \eta$ and $\theta = \theta(r)$, then the solution to eq. (2.6) is given by

$$\theta(r) = \arcsin(mr). \quad (2.7)$$

This solution has an interesting feature in that at a finite value of the radial coordinate $r_c = \frac{1}{m}$, $\theta(r_c) = \frac{\pi}{2}$. From the form of eq. (2.4) at $\theta = \frac{\pi}{2}$ the internal S^3 that is supporting the D7-brane has moved to a pole of the S^5 and degenerates to point. Thus from the perspective of the classical background geometry, the brane has ended. In actuality, what this demonstrates is that the D7-branes are sitting at a finite distance away from the original stack of D3-branes — in these coordinates located at $r = \infty$ — in a transverse direction. As is apparent, the D7-branes coincide in all directions with the D3-branes in the limit where $m \rightarrow 0$.

Note that the behavior near the conformal boundary is given by

$$\theta \simeq mr + \frac{m^3}{6} r^3 + \mathcal{O}(r^4), \quad (2.8)$$

which means that $\Delta_- = 1$. Appealing back to the previous chapter's discussion of the mass of the scalar mode with scaling dimension $\Delta = 1$, $m_\theta = -3 > m_{BF}$, and so there is no instability associated with this mode. Thus, in the field theory θ corresponding to $M\mathcal{O}_\theta$ sources a dimension 3 operator insertion — easily identifiable as a fermion bilinear $\bar{\psi}\psi$. The leading coefficient m is then related to the mass. This is entirely consistent with the story from the perspective of the branes where there is a minimal distance that a 3 – 7 string can reach, and owing to the tension in the fundamental string, $T_f = \frac{\sqrt{\lambda}}{2\pi}$, the field theory mass is given by $M = mT$.

In order to extract further field theory information, e.g. the free energy of the configuration, from the above solution, we will need a way to deal with the various near boundary divergences that appear in eq. (2.5). This is accomplished by the process of *holographic renormalization* (see the lecture notes [56]) where a UV cutoff is imposed as a radial slice is introduced away from the boundary $r = \delta \ll 1$, covariant counterterms are added to the action evaluated on the radial cutoff slice, and the cutoff is removed $\delta \rightarrow 0$ yielding at finite result. For application to the probe brane paradigm, we need to account not only for the divergent volume contributions inherent in the AdS geometry but also any divergence associated with worldvolume fields — conveniently a list of appropriate counterterms exists [54]:

$$\begin{aligned} L_1 &= -\frac{1}{4}\sqrt{h}, & L_2 &= -\frac{1}{48}\sqrt{h}R_\delta, & L_3 &= -\frac{1}{32}\sqrt{h}(R_{\delta,ij}R_\delta^{ij} - \frac{1}{3}R_h^2) \log \delta, \\ L_4 &= \frac{1}{2}\sqrt{h}\theta_\delta^2, & L_5 &= -\frac{1}{2}\sqrt{h}(\theta_\delta \square_W \theta_\delta) \log \delta, & L_f &= \alpha\sqrt{h}\theta_\delta^4, \end{aligned} \quad (2.9)$$

where h is the determinant of the induced metric on the cutoff slice, R_δ and R_δ^{ij} are the Ricci curvatures computed with respect to h_{ab} , $\square_W = \square_h + \frac{1}{6}R_\delta$ is the Weyl covariant Laplacian on the cutoff slice, and L_f is an available finite counterterm.

The role of the finite counterterm is non-trivial. Its coefficient α is fixed by demanding that the on-shell gravitational action — dual to the free energy in the field theory — vanishes. Why should that be the case? If in the field theory the free energy of the vacuum $F \neq 0$, then supersymmetry is manifestly broken. After a simple calculation, the result is that $\alpha = -\frac{5}{12}$. This should be kept in mind when in the next chapter we will have more scheme dependent counterterms to contend with, and it will be field theory data in $\frac{dF}{dM}$ that will prove useful to fix the coefficients.

Speaking of $\frac{dF}{dM}$, it is important to note how this is computed in the holographically renormalized probe brane analysis. Recall that the leading near boundary behavior provides the source for the operator, in this case $\theta_{0,-} = m$, the field theory mass is given by $M = m\frac{\sqrt{\lambda}}{2\pi}$, and that the on-shell action provides the generating functional for connected correlation functions in the boundary theory. With this, it is clear that with the identification of the

renormalized on-shell action, S_{ren}^{os} , being dual to the free energy, that up to factors converting bulk gravity data to field theory data

$$\frac{dF}{dM} \sim \langle \mathcal{O}_\theta \rangle \equiv \frac{1}{\sqrt{h}} \frac{\delta S_{ren}^{os}}{\delta \theta_{0,-}}. \quad (2.10)$$

As can be easily seen, if enough derivatives are taken with respect to the field theory mass, the comparison to bulk data is insensitive to scheme dependence. In the case that will be considered in the next chapter, we will see that $\frac{d^3 F}{dM^3}$ is a scheme independent quantity, and so work is done to find what the correct holographic correlator is needed for comparison.

One point that should be made is that while the analysis so far has given a sufficient amount of information to understand the work in later chapters, it is not an exhaustive summary of the spectrum of $\mathcal{N} = 2$ operators sourced within the D3/D7 system. Indeed, the examination of the “slipping mode” has left out fields with higher ℓ quantum number on the internal S^3 , worldvolume gauge field, and other 3 – 7 open string states that can be realized in this way. Moreso, what will remain unaddressed in the rest of the work involving supersymmetric probe brane configurations are the 7 – 7 open string states in the spectrum — the $\mathcal{N} = 2$ mesons corresponding to probe brane fluctuations. It remains an open, interesting, and addressable question in the framework.

D3/D5

Moving on from the D3/D7 system, our focus should turn to the other probe brane construction that features prominently in this thesis: the D3/D5 system. This system realizes a supersymmetric dual field theory co-dimension one defect on which the added flavor degrees of freedom are localized. Moreover, this system is perturbatively conformal — in the case where the D5s are not separated from the D3 stack — to all orders. In addition, while this dual theory from the perspective of $d = 4$ is $\frac{1}{4}$ -BPS, the 8 supercharges that are preserved constitute $\mathcal{N} = 4$ supersymmetry on the $d = 3$ defect. As we will now see, the holographic realization of a defect theory is far richer than in the D3/D7 system.

One novelty of this probe brane system, in comparison the geometry of the D3/D7 world-volume, is that apart from needing only to wrap an internal S^2 for support the brane fills a bulk AdS_4 in the AdS_5 as in Table. 2.1. So when finding the worldvolume theory through the pullback of the metric, there will be an additional scalar field that controls the radial profile of the co-dimension one bulk surface. Importantly, the D5 embedding breaks translational symmetry in the transverse bulk AdS direction, and so in the field theory preserves only $SO(3,2)$, i.e. the conformal group on the defect, if the embedding gives rise to massless defect hypermultiplets.

The ansatz for the AdS metric in this section differs from eq. (2.4) by truncating the Fefferman-Graham expansion to the Poincarè patch

$$ds^2 = \frac{1}{z^2}(dz^2 + \eta_{ij}dx^i dx^j), \quad (2.11)$$

where the defect sits at $x^3 = 0$, and for the S^5 parameterize the S^2 being wrapped by the D5-brane by $\{\beta_1, \beta_2\}$ such that

$$d\Omega_5^2 = d\theta^2 + \sin^2 \theta d\Omega_\alpha^2 + \cos^2 \theta d\Omega_\beta^2. \quad (2.12)$$

Here we will, and for convenience study the only the θ, x_3 part of the worldvolume spectrum. The worldvolume metric is then, allowing for arbitrary dependence on S^2 directions

$$ds_{D5}^2 = \frac{1}{z^2}(dx_1^2 + dx_2^2 - dt^2) + (g_{ab} + \frac{1}{z^2}\partial_a x_3 \partial_a x_3 + \partial_a \theta \partial_a \theta) dx^a dx^b, \quad (2.13)$$

where $\{\mathbf{a}, \mathbf{b}\}$ runs over $\{z, \beta_1, \beta_2\}$.

Apart from preserving only a reduced conformal group, there is a more dramatic consequence of this configuration as seen in the R-symmetry. As can be seen in the form of eq. (2.12), the preserved R-symmetry is $SO(3) \times SO(3) \cong SU(2)_H \times SU(2)_V \subset SO(6)_R$. A bit of clarification of regarding the notation, in coupling the flavor fields to the ‘‘ambient’’ $\mathcal{N} = 4$ gauge multiplet, we need to study the symmetries of the dimensional reduction from the $d = 4$ theory onto the defect, which splits into a $\mathcal{N} = 4$ vector multiplet and $\mathcal{N} = 4$ hypermultiplet. The $SU(2)_H$ acts to rotate the scalars in the hypermultiplet, X_H^I , and likewise

$SU(2)_V$ acts to rotate the scalars of the vector multiplet, X_V^A . Recall that in the dimensional reduction, the component of the ambient gauge field transverse to the defect (A_3) becomes an adjoint scalar mode on the defect. The total accounting for charge assignments in this dimensionally reduced theory is in Table. 2.2, where the adjoint fermions are labeled λ_a , fundamental scalars and fermions are q_m and ψ_p respectively. This handy table can be found in [57].

Fields	Spin	$SU(2)_V$	$SU(2)_H$	Δ
X_H	0	0	1	1
X_V	0	1	0	1
A_3	0	0	0	1
A_i	1	0	0	1
λ_a	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
q_m	0	0	$\frac{1}{2}$	$\frac{1}{2}$
ψ_p	$\frac{1}{2}$	$\frac{1}{2}$	0	1

Table 2.2: R-charges and conformal dimensions (Δ) of defect spectrum

Starting from [58], the authors of [57] found the supersymmetric linearized D5-brane embedding in the flat radial slicing of AdS_5 . What distinguishes this work from the D7-brane construction [15] was that a worldvolume gauge field was a necessary part of the construction. This will serve as a useful check when constructing the embedding where the radial slices have constant scalar curvature. In that case, we are instructed as well to have a non-trivial worldvolume gauge field by the field theory construction [59], and in the case of an infinitesimally massive embedding, we should recover the following story to leading order.

Starting from the embedding action with $P[G]_{\mu\nu} \equiv$ given in eq. (2.13), $F = \frac{1}{2}F_{ab}dx^a \wedge dx^b$

and ignoring $2\pi\alpha'$ for notational ease

$$\begin{aligned} \frac{S_{D5}}{t_5} &= - \int d^6 x i \sqrt{-\det [P[G]_{\mu\nu} + F_{\mu\nu}]} + \int_{\Sigma_6} \mathcal{C}_4 \wedge F \\ &= - \int d^6 \xi \frac{1}{z^3} \sin \beta_1 \sqrt{-\det [(P[G] + F)_{ab}]} + \int_{\Sigma_6} \frac{1}{2} F_{ab} \mathcal{C}_4 \wedge dx^a \wedge dx^b. \end{aligned} \quad (2.14)$$

Noting that on the Freund-Rubin ansatz $\mathcal{C}_4 \sim z^{-4}$. Expanding eq. (2.14) to find the quadratic action for the linearized embedding and deriving the equations of motion, one finds that expanding the geometric modes — θ, x_3 — on a basis of scalar S^2 spherical harmonics, e.g. $\theta = \sum_\ell \theta_\ell Y_\ell(\vec{\beta})$ and the gauge field on a basis of vector spherical harmonics, given in terms of co-exact one forms $A = \sum_\ell A_\ell (\star_{S^2} d) Y_\ell(\beta)$

$$\begin{aligned} (\square_{AdS} + 2 - \ell(\ell + 1))\theta_\ell &= 0, \\ (\square_{AdS} - 4 - \ell(\ell + 1))\tilde{x}_{3\ell} - 4\ell(\ell + 1)A_\ell &= 0, \\ (\square_{AdS} - \ell(\ell + 1))A_\ell - 4\tilde{x}_{3\ell} &= 0, \end{aligned} \quad (2.15)$$

where $\tilde{x}_{3\ell} = \frac{x_{3\ell}}{z}$. The combined geometric and gauge sector can be decoupled by studying the free scalar modes $\sigma_\ell = \frac{(\ell+1)A_\ell - \tilde{x}_{3\ell}}{2\ell+1}$ and $\chi_\ell = \frac{\ell A_\ell + \tilde{x}_{3\ell}}{2\ell+1}$ whence

$$\begin{aligned} (\square_{AdS} - \ell(\ell - 3))\sigma_\ell &= 0, \\ (\square_{AdS} - (\ell + 1)(\ell + 4))\chi_\ell &= 0. \end{aligned} \quad (2.16)$$

The internal S^2 quantum number ℓ corresponds to the $SU(2)_H$ R-charge of the BPS field theory operator insertions.

Now, we have a system of free scalars to which we could apply the machinery of holographic renormalization should we so choose. At this point however, all that we should note is the resulting spectrum and the field theory interpretation. To wit, we can read off the masses and conformal dimensions of these decoupled scalars — and the slipping mode — fairly easily in Table. 2.3

Mode	$m^2(\ell)$	Δ_+	Δ_-
θ_ℓ	$-2 - \ell(\ell + 1)$	$\ell + 2$	$1 - \ell$
σ_ℓ	$\ell(\ell - 3)$	$\ell + 4$	$-\ell - 1$
χ_ℓ	$-(\ell + 1)(\ell + 4)$	$\frac{1}{2}(3 + 2\ell - 3)$	$\frac{1}{2}(3 - 2\ell - 3)$

Table 2.3: Linearized spectrum of D5-brane open string modes

First, note that based on unitarity considerations $\ell \geq 1$ for χ_ℓ . Second, for all ℓ , the asymptotic behavior of σ_ℓ provides the source for irrelevant operators in the boundary theory. While we would normally disregard this in AdS/CFT either on the grounds that as deformation in the UV it would spoil the asymptotically AdS geometry necessary for the dictionary to be constructed or from unitarity considerations. The latter objection will be cast aside when considering D3/D5 with spherical boundary geometry, and the former will be addressed in Ch. 4.

The question then becomes of what to make of the field theory operators sourced by these modes. Again turning to [57], the dual operators through a superspace boundary formalism (see [60]) in the field theory were identified. While the exact calculation is too involved for the current discussion, the pertinent dictionary entries for lowest dimension states, which will appear again in later chapters, is contained in Table.2.4. Note that in Table.2.4 A indexes the scalars in the defect $\mathcal{N} = 4$ vector multiplet and I runs over the scalars in the adjoint $\mathcal{N} = 4$ hypermultiplet.

Mode	Min $[\Delta_+]$	Min $[\Delta]$ \mathcal{O} dual
θ_ℓ	2	$\bar{\psi}_p \sigma_{pq}^A \psi_q + 2\bar{q}^m X_V^A q^m$
χ_ℓ	$\frac{1}{2}(3 + 2\ell - 3)$	$\bar{q}^m \sigma_{mn}^I q^n$

Table 2.4: Dictionary entries for lowest ℓ modes

If we were to attempt to construct dictionary entries for higher ℓ modes such that the operators sourced are single trace, the correct quantum numbers are captured by symmetric traceless combinations, $\bar{q}^m \sigma_{mn}^{(I_1} q^n \mathcal{O}^{I_2} \dots \mathcal{O}^{I_\ell)}$ with \mathcal{O}^I either $\bar{q}^m \sigma_{mn}^I q^n$ or X_H^I . The authors of [57] on the basis of T-dual brane pictures that X_H^I is the correct choice.

2.1.2 κ -symmetry

The above constructions, even their massive incarnations, have in their simplicity of considering flat radial slicing of the bulk AdS geometry the afforded benefit of preserving many supersymmetries. However, it will be necessary for us to consider the case where the conformal boundary has non-trivial but constant scalar curvature. This splits our consideration into two distinct cases: massless and massive embeddings. The massless embedding does not introduce new scales into the theory and thus enjoy the full accommodation of preserved conformal symmetry; these types of embeddings are equivalent to the flat space construction modulo conformal rescalings. The massive embeddings require more work insofar as the new scale induced by the addition of massive hypermultiplets acts to break conformal symmetry and thus are sensitive to whether the theory is defined on a sphere versus flat space through the coupling to background scalar curvature. To that end, we will now review the process to ensure that the addition of branes preserves the maximum amount of supersymmetry possible.

It has long been known that the worldvolume theory of a D-brane embedded in a background solution to $d = 10$ type IIA/B supergravity contains an asymmetric number of bosonic and fermionic degrees of freedom necessitating an additional fermionic symmetry that projects out the excess number of fermionic modes (see e.g. [61]). That additional fermionic worldvolume symmetry goes by the name ‘ κ -symmetry’. For simplicity and ease of application in later sections, we will focus on the construction of supersymmetric branes in type IIB backgrounds, but it should be noted that the type IIA follows the same basic approach with parsable distinctions.

The starting point is the same as in the general discussion of probe branes: the action describing the embedding consisting of the DBI action and Wess-Zumino term

$$S_{Dp} = -T_p \int d^{p+1}\xi \sqrt{-\det [P[G]_{ij} + \mathcal{F}_{ij}]} + T_p \int_{\Sigma_{p+1}} \mathcal{C} \wedge e^{\mathcal{F}}, \quad (2.17)$$

where ξ^i are the $p + 1$ -dimensional worldvolume coordinates indexed by $\{i, j\}$, $P[G]_{ij}$ is the pullback of the $d = 10$ spacetime metric onto the worldvolume, \mathcal{C} is the tower of R-R

potentials which in type IIB backgrounds is given by a sum over even forms $\mathcal{C} = \sum_{k=0}^5 C_{2k}$, and $\mathcal{F} = F - B$ is the worldvolume 2-form built out of the usual Yang-Mills gauge field strength $F = dA + \dots$ and the 2-form potential B . For our purposes, we will not explicitly require the B -field but will be kept in the following discussion for completeness.

Treating the worldvolume theory described in the above action as non-linear sigma model with the $d = 10$ supergravity background as target space, we count the degrees of freedom in the bosonic action above for any p coming from the embedding functions and gauge fields to total 8 . Studying type IIB backgrounds means that the Killing spinor generating the supersymmetry transformations is in the self-conjugate **16** irrep of a prescribed chirality (e.g. Majorana-Weyl representation) of $SO(9, 1)$. Thus pulled back onto the world volume will give rise to twice as many fermionic degrees of freedom as bosonic. Note that once we consider a continuation to Euclidean signature we no longer have the Majorana condition, which will cause a serious headache if not treated properly, but those issues will be sorted in subsequent sections. However, the thrust here is that we must have that the κ -symmetry constraints provide a suitable projector to eliminate half of the fermionic degrees of freedom if we hope to preserve any amount of supersymmetry.

The projection condition is derived from a constraint equation following κ -symmetry variations that we will use is schematically of the form

$$(\mathbb{1} - \Gamma_\kappa)\epsilon = 0. \tag{2.18}$$

By ϵ we will denote the spacetime Killing spinor generating supersymmetry transformations. The subtlety that arises is that the projector that kills off extra fermionic degrees of freedom must necessarily be built out of worldvolume quantities, while it is acting on a spinor that lives in the ambient spacetime. Thus, we will see that there is a lot of information contained eq. (2.18) that depends on the specific application. Furthermore, in order to solve this equation to arrive a projection condition that is valid over the whole worldvolume, we need to act on constant spinors.

Due to the fact that we must deal with spinors in the worldvolume theory as derived by

reductions from the $d = 10$ supergravity, we must understand how to translate between the two tangent bundles. Let us denote the $d = 10$ spacetime and frame indices by $\{M, N\}$ and $\{A, B\}$ respectively and the worldvolume frame indices by $\{a, b\}$. In the $d = 10$ language, the coordinates on the tangent bundle are given by the vielbein e_M^A where $G_{MN} = e_M^A e_M^B \eta_{AB}$. It would seem natural then to describe the worldvolume spinors using a ‘ $(p+1)$ -bein’ similarly, but that is not quite the case. Since we are describing the embedding of the Dp -brane by pulling the spacetime geometry onto a $(p+1)$ -dimensional worldvolume, we must likewise consider the worldvolume frame as the pullback of the ‘10-bein’. It may seem like a small, semantic point of distinction, but in order to properly construct the κ -symmetry projector and solve the constraints, we must carefully treat the spinor structures and track their associated dependences on the worldvolume coordinates and embedding functions.

To that end, let us define the following objects

$$J_p^{(n)} = (-1)^n \sigma^{n+\frac{p-3}{2}} i\sigma_2 \otimes \Gamma_{(0)} \quad (2.19a)$$

$$\Gamma_{(0)} = \frac{1}{(p+1)!} \epsilon^{i_1 \dots i_{p+1}} \gamma_{i_1 \dots i_{p+1}} \quad (2.19b)$$

where $\gamma_{ij\dots k} = \gamma_{[i} \gamma_j \dots \gamma_{k]}$ and brackets denote unit weight antisymmetrization. The fact the tensor product $i\sigma_2 \otimes \Gamma_{(0)}$ arises in the above definition is that the type IIB Killing spinor that we are using to express the generator of supersymmetry transformations are Majorana-Weyl of definite chirality. Note that in the type IIA case $J_p^{(n)} = (\Gamma_{11})^{n+\frac{p-3}{2}} \Gamma_{(0)}$. The κ -symmetry projector can then be built using the following object [16, 62]

$$\Gamma_\kappa = \frac{1}{L_{DBI}} \sum_{n=0}^{\infty} \frac{1}{2^n n!} \gamma^{i_1 j_1 \dots i_n j_n} \mathcal{F}_{i_1 j_1} \dots \mathcal{F}_{i_n j_n} J_p^{(n)} \quad (2.20)$$

where for brevity $L_{DBI} = \sqrt{-\det(P[G]_{ij} + \mathcal{F}_{ij})}$. While the formal definition of Γ_κ contains an infinite sum over contractions of the worldvolume 2-form \mathcal{F} and worldvolume Clifford algebra structures $\gamma^{ij\dots}$, it should be clear that for any given brane embedding the sum truncates at a finite order. Even turning on an arbitrary number of worldvolume gauge fields leaves a finite number of contractions that can be made between $\mathcal{F}_{jk} = (\sum_i F^{(i)} + B)_{jk}$; assuming of course that the brane itself spans a finite number of directions.

In order for this to be an appropriate object with which we can build the κ -symmetry projector, it must be idempotent. This can easily be seen to be the case given that the worldvolume frame is given by $e^{\underline{i}} = E^i_M (\partial_j X^M) dx^j$, where E^i is the spacetime 10-bein, and $\gamma_i = e^M_i \Gamma_M$. The total antisymmetrization in the Clifford algebra structures in Γ_κ ensures that the numerator will produce a determinant to cancel that in L^2_{DBI} .

To elucidate the utility of the κ -symmetry framework, let us consider a simple example: the embedding of a D7 probe brane in $\text{AdS}_5 \times \text{S}^5$ with flat geometry on the radial slices as in eq. (2.11). As described above since the massless D7 embedding is a spacetime filling probe brane in that it wraps all of the AdS_5 directions, the geometric parameterization of the embedding in static gauge is done with a lone scalar field that characterizes the radial profile of the zenith position of the $\text{S}^3 \subset \text{S}^5$ that is also part of the D7 worldvolume. If we do not turn on any worldvolume gauge fields, then the calculation is quite simple in that the series eq. (2.20) terminates at $n = 0$, which implies that $\Gamma_\kappa = \frac{1}{\sqrt{g}} \Gamma_{(0)}$. The κ -symmetry constraint $\Gamma_\kappa \varepsilon = \varepsilon$ now requires an explicit expression for the Killing spinors of $\text{AdS}_5 \times \text{S}^5$.

Since this is a crucial step in any κ -symmetry calculation, let us consider the construction of the Killing spinors of $\text{AdS}_5 \times \text{S}^5$ as an aside. The work done in [63] is the basis for the analysis of Killing spinors in this and following sections. The starting point for finding an explicit form for the Killing spinors in any vacuum solution to supergravity is vanishing of the variation of the gravitino in the gravity multiplet. This is obviously dependent upon the specific solution being studied.

The end result is that the Killing spinor is required to satisfy the following constraints — see e.g. [23, 63] —

$$D_\mu \varepsilon = \frac{i}{2} \Gamma_{AdS} \Gamma_\mu \varepsilon \quad (2.21a)$$

$$D_I \varepsilon = \frac{i}{2} \Gamma_{S^5} \Gamma_I \varepsilon \quad (2.21b)$$

where μ, ν label the AdS_5 directions and I, J the S^5 directions, and generically the covariant derivative $D_a \varepsilon = \partial_a \varepsilon - \frac{i}{4} \omega_a^{\underline{bc}} [\Gamma_{\underline{b}}, \Gamma_{\underline{c}}]$. The underline index notation specifically refer to Lorentz indices and will be utilized in later chapters. In the more feature rich background

geometries considered in the following sections and in any future work, our starting point will be solving eq. (2.21) anew. Since we are not considering any brane embeddings into backgrounds apart from $\text{AdS}_5 \times \text{S}^5$, the schematic form of the solution to eq. (2.21) can be written $\varepsilon = R_{AdS}[\Gamma_{AdS}]R_S[\Gamma_S]\varepsilon_0$ where ε_0 is a constant spinor. The matrices R_{AdS} and R_S depend only on and contain even numbers of AdS_5 and S^5 gamma matrices respectively, i.e. $[R_{AdS}, R_S] = 0$. The explicit forms of the R -matrices are generally quite cumbersome and will be calculated in detail in later chapters for specific cases, but for a

Returning to the κ -symmetry analysis with the form of the Killing spinor above, the constraint is written in terms of

$$R_S^{-1}R_{AdS}^{-1}\frac{1}{\sqrt{g}}\Gamma_{(0)}R_{AdS}R_S\varepsilon_0 = \varepsilon_0 \quad (2.22)$$

From this point, to derive a condition designed to project out the correct fermionic modes to produce a BPS embedding we need to find a condition that has a coordinate independent Clifford algebra structure acting on a constant spinor. As has been mentioned, it is only in this way that one can make an invariant statement about the embedding to ensure that the resulting worldvolume theory preserves supersymmetries globally.

The mildly tedious task in this simple exercise is in tracking which Clifford algebra structures and coordinate dependences are pulled down from the exponentials and where sign flips occur when massaging the constraint into the form of $\tilde{\Gamma}\varepsilon_0 = \varepsilon_0$. However, from the analysis above the massless embedding is known to wrap a trivial cycle in the internal space with no radial profile, i.e. it remains equatorial for all z , and so we can, without loss of generality set the values of the transverse direction to the worldvolume S^3 on the internal space to $\theta = \frac{\pi}{2}$, $\psi = 0$. This results in the constant projection condition for massless BPS D7 embeddings in a background of D3 branes

$$-\Gamma_{AdS}\Gamma^\psi\varepsilon_0 = \varepsilon_0 \quad \Rightarrow \quad \Gamma^\theta\Gamma_{S^3}\varepsilon_0 = \varepsilon_0, \quad (2.23)$$

where in the last equality it was used that the type IIB spinor must have definite chirality according to $\Gamma_{11}\varepsilon = \varepsilon$ and standard Clifford algebra relations.

While the κ -symmetry projectors are easy to identify in this instance, that will not be the case when we want to consider less contrived examples. In fact, the bulk of the work in constructing the precision test of the probe brane paradigm is in first constructing the appropriate set of additional projection conditions that are necessary for the massive embeddings with curved radial slices that commute with Γ_κ , and from that point one can derive the coupled non-linear differential equations that result from imposing eq. (2.18). Even in the relatively mild relaxation of assumptions to the case where D7-branes wrap and AdS_5 with constant, non-zero scalar curvature on the radial slices with non-trivial worldvolume gauge field on the internal S^3 , the process of determining the κ -symmetry equations for the gauge field and slipping mode is exceedingly complicated. However, before presenting that calculation in full detail we still need to account for the necessary foundational part of the field theory analysis that provided the motivation attempting such a difficult construction.

2.2 Exact Results in Gauge Theories on Curved Backgrounds

It is far beyond the scope of this thesis to provide a thorough accounting of the vast subject of equivariant localization — or even the more focussed of its application to supersymmetric field theories. However, we should pause the discussion of branes to appreciate the beauty of a technique that provides 1-loop exact statements about a theory among a subset of questions that can be posed in a restrictive cohomological construction. It is in localization calculations that we will find results in the massively flavored field theory that will be at this point in the narrative putatively dual to brane embeddings obeying κ -symmetry conditions. Thus in the spirit of meticulous examination of our toolset, we will explore the foundations of equivariant localization of path integrals in the context of supersymmetric gauge theories on curved backgrounds.

The main point of this chapter is to establish the language necessary to complete the justificatory framework for the test of probe brane holography. The take-away for those readers uninterested in the mathematical grounding of the field theory calculation is that supersymmetric localization hinges on studying a type of cohomological construction very

much akin to that carried out in Witten-type topological field theories [64]. That is, one begins with a supersymmetric gauge theory, singles out a supercharge \mathcal{Q}_{SUSY} , defines the BRST supercharge \mathcal{Q}_{BRST} , and studies the total $\mathcal{Q} = \mathcal{Q}_{SUSY} + \mathcal{Q}_{BRST}$ cohomology of the theory. A deformation of the theory of the type $t\mathcal{Q}_{SUSY}V$ is added to the theory, and as long as the supercharge is nilpotent, V as a functional of the field content of the undeformed theory has suitable behavior, and no anomalies are generated by the path integral measure, the partition function is insensitive to the exact value of the coupling, t , with which the deformation is added. Taking $t \rightarrow \infty$, one can perform a saddle point analysis and discover that the theory reduces to locus on the manifold of field configurations where the deformation vanishes. Calculating the Euler class of the normal bundle of this locus gives the functional determinant for the 1-loop fluctuations of the locus configuration. As a result of the topological nature of this process of deformation, the semi-classical (1-loop) result is exact. The question that will remain until a theory and background geometry are specified is what the form of this semi-classical result is and whether it affords us any recourse by way of techniques that apply to ordinary, possible matrix valued, sums or integrals.

The organizational theme of this chapter begins with the mathematical underpinnings of equivariant cohomology. This naturally leads to the subsequent discussion of the Atiyah-Bott(-Berline-Vergne) localization formulae [65, 66] and the Duistermaat-Heckman theorem [67]. Reemerging into the slightly less abstract atmosphere, we will follow the application of these theorems to supersymmetric gauge theories on compact manifolds culminating in the exploration of localization in massively deformed $\mathcal{N} = 4$ SYM theory on an S^4 . A brief discussion will follow concerning similar calculations done on an S^3 for use in Ch. 4.

2.2.1 Atiyah-Bott Localization Formula

Without presenting a full mathematical treatise on equivariant cohomology (see [68]), we should at least acquaint ourselves with the language used to establish one-loop exactness of certain deformed path integrals. The notation in this section will follow that of [69] and which follows lectures given by the author at the Geometry of String and Fields summer

school program at the Galileo Galilei Institute in 2013 — the recording of which is freely available on the internet. We will take as the starting point the standard definition of the de Rahm differential generating the complex among spaces of differential forms on a manifold $\Omega^\bullet(\mathcal{M})$

$$\dots \xrightarrow{d} \Omega^p(\mathcal{M}) \xrightarrow{d} \Omega^{p+1}(\mathcal{M}) \xrightarrow{d} \dots \quad (2.24)$$

The de Rahm cohomology group is then defined as $H^\bullet(\mathcal{M}) = \ker d / \text{Im } d$. That is, the p^{th} cohomology group is the set of d -closed p -forms modulo d -exact p -forms on \mathcal{M} .

To define equivariant cohomology, we begin with a manifold \mathcal{M} with a defined G -action ($G \curvearrowright \mathcal{M}$) by a compact Lie group, G , . We then define a basis element of the Lie algebra \mathfrak{g} of G as T_a . Owing to the G -action on \mathcal{M} , we can then import the action of the Lie algebra generator through a vector field v_a on \mathcal{M} . The equivariant differential, d_G , is then defined using a coordinate ϵ^a on \mathfrak{g} (e.g. for $h \in \mathfrak{g}$, $h = \epsilon^a T_a$) by

$$d_G = d - \epsilon^a \iota_{v_a}, \quad (2.25)$$

where d is the de Rahm differential and ι_{v_a} is the inclusion map whose action is given by contraction with v_a . This may, at first blush, seem like a strange way to define a differential operator whose cohomology we wish to study as it is obvious that on the space $\tilde{\Omega}^\bullet(\mathcal{M}, \mathfrak{g}) = \Omega^\bullet(\mathcal{M}) \otimes C[\mathfrak{g}]$, where $C[\mathfrak{g}]$ is the space of polynomials of \mathfrak{g} -valued functions, that

$$d_G^2 = -\epsilon^a (d\iota_{v_a} + \iota_{v_a}d) \equiv -\epsilon^a L_{v_a} \neq 0, \quad (2.26)$$

where L_{v_a} is the Lie derivative on \mathcal{M} in the direction of the vector field v_a . Now, if we restrict ourselves to the G -invariant subspace of $\tilde{\Omega}^\bullet(\mathcal{M}, \mathfrak{g})$, then it is clear that $\epsilon^a L_{v_a} = 0$, and so d_G is nilpotent on this subspace and is thus a good starting point to study equivariant cohomology.

At this point we are well situated to proceed, but we will need to assemble just a few more notions before moving on to the localization formulae in earnest. Consider the vector bundle $(E, \mathcal{M}, \sigma, \pi)$ where E is the total space, \mathcal{M} is the base manifold, section σ , and projection

$\pi : E \rightarrow \mathcal{M}$. Through the projection there is a natural definition of the pullback π^* acting on the cohomology $\pi^* : H^\bullet(\mathcal{M}) \rightarrow H^\bullet(E)$, and with a bit of work using the Poincaré duality one can take the similar natural definition of the pushforward π_* acting on the homologies $\pi_* : H_\bullet(E) \rightarrow H_\bullet(\mathcal{M})$ and extend, under suitable conditions, to the cohomologies as well. The reason that we would want to use this language is that there is then a relationship between integration on the fibres and the pushforward of the projection as

$$\int_E a = \pi_* a, \quad a \in H^\bullet(E). \quad (2.27)$$

If we then restrict to the cohomology groups that have compact support on the fibres $H_c^\bullet(E)$, then we are equipped to define an essential part of the localization formulae: the equivariant Euler class. Consider the Thom isomorphism induced by the projection π

$$H_c^\bullet(E) \xrightleftharpoons[\pi^*]{\pi_*} H^\bullet(\mathcal{M}). \quad (2.28)$$

The definition of the equivariant Euler class $e_G(E)$ is given by the pullback of the pushforward of the identity $\pi^* \pi_* \mathbf{1} = e_G(E)$. We are now in a position to state the Atiyah-Bott localization formula [65] despite the broad strokes with which the background has been painted. Let a be a p -form in $H_c^\bullet(E)$ associated to the G -bundle $(E, \mathcal{M}', \sigma, \pi)$ where \mathcal{M}' is taken to be the set of G -fixed points in E ,

$$\int_E a = \int_{\mathcal{M}'} \frac{\pi^* a}{e_G(\mathcal{N}_{\mathcal{M}'})}, \quad (2.29)$$

where $e_G(\mathcal{N}_{\mathcal{M}'})$ is the equivariant Euler class of the normal bundle to \mathcal{M}' ($\mathcal{N}_{\mathcal{M}'}$) in E . This follows from the definition of integration over the fibres by action of the pushforward and that $\pi_* e_G(E)^{-1} \pi^* = \mathbf{1}$.

An important manifestation of equivariant localization is provided by the work of Duistermaat and Heckman [67], which preceded and provided the impetus for the work of Atiyah and Bott. One begins first with a symplectic manifold, i.e. a manifold \mathcal{M} with a non-vanishing symplectic two-form ω , together with a Hamiltonian k -torus action, T^k with moment map $f : \mathcal{M} \rightarrow \mathbb{R}^k$ (e.g. $\mathfrak{t} = \mathfrak{so}(2k)$), in the context that we will be concerned this will be the circle

($U(1)$) action on the manifold. Given a non-Abelian G -manifold the appropriate G -action is furnished by the maximal torus (e.g. Cartan subgroup) of G . The Duistermaat-Heckman fixed point theorem then states that there is an exact representation of the integral of ω as a weighted sum over the fixed points f .

Consider the case of the Hamiltonian circle action ($k = 1$) on a $2n$ -dimensional manifold \mathcal{M} with a discrete set of fixed points $x_i \in \mathcal{M}$. The moment map for a Hamiltonian torus action then obeys $df_a + \iota_{v_a}\omega = 0$ where as above v_a is the action generated by a basis element (Lie generator) of \mathfrak{t} . This then implies that the object $\omega_G = \omega + \epsilon^a \iota_{v_a}$ with ϵ^a as above is d_G -closed by virtue of the Hamiltonian action constraint on df_a . Any function that can be represented as a polynomial in ω_G can then be shown to be d_G -closed. Thus, returning to the specification of $k = 1$, writing $\epsilon^a f_a = -itf$, as should be an obviously valid form from the Abelian case, it is clear that the application of the Atiyah-Bott formula to e^{ω_G} then yields

$$\int_{\mathcal{M}} e^{-itf} \frac{\omega^l}{l!} = \sum_{x_i} \frac{e^{-itf(x_i)}}{(it)^l e(x_i)} \quad (2.30)$$

where $e(x_i)$ is the equivariant Euler class of the normal bundle at x_i as described above. For applications of the Duistermaat-Heckman theorem in physical contexts see [70, 71].

Supersymmetric Localization

At this point it may not seem clear that the above construction has much to do with supersymmetric gauge theories. As will be explained below, once we can establish an operator that behaves like an equivariant exterior derivatives exists quite naturally descending from the supersymmetric algebra itself, then employing a version of the Atiyah-Bott localization formula will have profound results for the theory. However before reading off the equivariant differential, there are a few points that must be clarified, and even after all of the preliminary concerns are addressed, there will be subtleties that can obscure the process.

Our goal is to translate the somewhat abstract components in eq. (2.29) and eq. (2.30) to the language of supersymmetric gauge theories. A natural candidate for the oscillatory integral in eq. (2.30) that appears in any typical field theory to which we can apply a

saddle point approximation and implement a localization procedure is that most central of actors in this thesis: the partition function. As discussed in the introduction, the partition function \mathcal{Z} is represented in quantum field theories by the functional path integration over field configurations whose measure is weighted by the oscillatory — in the case of Lorentzian signature — factor e^{iS} . Schematically, we then want the lefthand side of either eq. (2.29) or eq. (2.30) to be represented by

$$\mathcal{Z}[J] = \frac{1}{\mathcal{Z}[0]} \int [\mathcal{D}X] e^{iS[X, J]} \quad (2.31)$$

where X is a placeholder for any field content of which the action S is a functional, J represents any possible sources, $[\mathcal{D}X]$ is the formally infinite product of ordered ordinary integral measures necessary for path integration, and $\mathcal{Z}[0]^{-1}$ is a typical normalization such that $\langle 0|0 \rangle = 1$.

Specifying further, we can consider a theory that possesses a local symmetry encoded in a compact Lie group G , i.e. a ‘gauge’ theory. The path integrand is now no longer just a manifold of field configurations, F but contains many copies of equivalent configurations according to the local transformation laws furnished by G -action on F . This of course assuming that the G -action is smooth in that it contains no fixed points. The definition of the properly normalized path integral must be over the G -invariant submanifold, i.e. the quotient space F/G

$$\mathcal{Z}[J] = \frac{1}{\mathcal{Z}[0] \text{vol } G} \int_{F/G} [\mathcal{D}X] e^{iS[X, J]}, \quad (2.32)$$

where $\text{vol } G$ is the volume of the group manifold of G . We should notice here is that we have a connection beginning to form between gauge theories and the abstract spaces in which Atiyah-Bott was formulated beyond the superficial: we have a Lie group, G , acting freely on, at worst, submanifolds of F over which we are integrating. In the cases of gauge symmetries generated by bosonic charges, G acting freely on all of F poses no problems. However in the cases of supersymmetry, this is not as straightforward.

In a typical field theory, one starts with identifying desired aspects of the feature-set encoded in the partition function and isolating characteristics such as the global and gauge

symmetries that one would like to exhibit. The typical symmetries that are built into a given quantum field theory are drawn from the Cartan classification of compact, (semi-)simple Lie groups, which notable exceptions being two dimensional conformal field theories whose conformal symmetry group is infinite dimensional and gravity whose gauge symmetries are diffeomorphisms. What is special about the extension into the realm of supersymmetric field theories where the Lie algebra is \mathbb{Z}_2 graded is that the additional set of symmetry generators \mathcal{Q} , i.e. the conserved supercharges associated to the supersymmetry transformations, have unusual statistics: they are Grassmann valued fermionic charges. It is that Grassmann valuing that provides the magic that allows one to use objects like \mathcal{Q} as the equivariant differential central to localization in the previous section.

The aspects of localization are then distilled in supersymmetric gauge theories in the following way. Given a supercharge, \mathcal{Q} that is nilpotent up to gauge transformations, we can form the localized theory by utilizing the assumed non-anomalous behavior of the path integral measure and that the action is \mathcal{Q} -closed. These two features imply that the path integral is insensitive to addition of \mathcal{Q} -exact terms

$$\mathcal{Z}[t] \sim \int_{F/G} [\mathcal{D}X] e^{iS[X]+t\mathcal{Q}V} \quad \Rightarrow \quad \partial_t \mathcal{Z}[t] \sim \langle \mathcal{Q}(\dots) \rangle = 0. \quad (2.33)$$

Since the theory is independent of the parameter with which we encode the \mathcal{Q} -exact deformation of the action, taking the $t \rightarrow \infty$ limit will then reduce to a saddle point approximation to the original path integral. The beauty of this method is that one can then establish that once the critical submanifold in the quotient space F/G has been found, the 1-loop fluctuations about the locus (i.e. the semi-classical analysis if one treats $t = \hbar^{-1}$) are the only non-vanishing correction that must be calculated. That is to say, that \mathcal{Z} has a 1-loop exact expression.

There are caveats to this optimistic statement, though. First, the assumption is that $\mathcal{Q}V$ has a positive definite bosonic part. In practice, one typically wants, then, to express the bosonic part of the localizing term as a sum of squares of fields such that on the saddle point each square must independently vanish. While that may not always be possible, it should

be the case that any sufficiently ‘good’ supersymmetric gauge theory has a positive bosonic part. The second caveat is that the fermionic part decays sufficiently fast in the localizing limit. Because the fermionic part of the localizing term cannot be strictly bounded from below by zero, the requirement of that it vanishes in the localizing limit is to ensure that there is no runaway behavior and that the critical submanifold is smooth and isolated.

To see localization working in a supersymmetric case, consider the extremely simple example in where the base space is just a point: $\mathcal{M} = pt$ [72]. A simple example of a supersymmetric field theory in $d = 0$ formulated in terms of real bosonic variable ϕ and real fermionic variables ψ_a can be written as

$$\mathcal{Z} \sim \int [\mathcal{D}\phi][\mathcal{D}\psi_a] e^{-\frac{1}{2}f'(\phi)^2 + \frac{1}{2}\epsilon^{ab}\psi_a\psi_b f''(\phi)} \quad (2.34)$$

where f is an arbitrary, smooth function of ϕ with isolated zeros of f' at ϕ_i and $\epsilon^{12} = -\epsilon^{21} = 1$ such that $\epsilon^{ab}\psi_a\psi_b = 2\psi_1\psi_2$. Generically away from the zeros of f' , one can use the supersymmetry algebra infinitesimally generated by ϵ^a ,

$$\delta_\epsilon \phi = \epsilon^a \psi_a, \quad \delta_\epsilon \psi_a = \epsilon^{ab} \epsilon_b f', \quad (2.35)$$

to exchange one of the ψ_a/f' with $\epsilon^{ab}\epsilon_b$. Obviously at the critical points this redefinition is singular and invalid. However, it is indeed these critical points that give non-zero contributions to \mathcal{Z} . One can see easily that by affecting some simple shifts, Grassmann integration, and disregarding total derivatives, that away from ϕ_i , $\mathcal{Z} = 0$ exactly! Seriously treating the behavior around the critical points, expanding f to second order in $\phi - \phi_i$, the partition function reduces to

$$\begin{aligned} \mathcal{Z} &\sim \mathcal{Z}_0 + \sum_i \int [\mathcal{D}\phi][\mathcal{D}\psi_a] e^{-\frac{1}{2}f''(\phi_i)^2(\phi-\phi_i)^2 + f''(\phi_i)\psi_1\psi_2} \\ &\sim \sum_i \left. \frac{f''}{|f''|} \right|_{\phi=\phi_i}, \end{aligned} \quad (2.36)$$

where \mathcal{Z}_0 is the vanishing contribution away from the critical points. What all of this implies is that $\mathcal{Z} = -1, 0, 1$ depending on whether f is an odd function $\phi(0)$, or f is an

even function giving ± 1 depending on the boundary conditions, i.e. sign of $f(\pm\infty)$. While this is an incredibly simple, n e contrived, example to illustrate the idea of localization in a supersymmetric field theory, in the generalization to complex fields one uncovers $d = 0$ Landau-Ginzburg theory. The connection between complex supersymmetric field theories and Landau-Ginzburg theory in higher dimensions would take us too far afield to elucidate, but the upshot is that this connection plays a central role in physical interpretations and proofs of yet another fascinating duality: mirror symmetry [9]. While further less trivial examples could be given, in order to move the narrative along, let us jump straight to supersymmetric gauge theories in $d = 4$.

2.2.2 Application to Gauge Theories on S^4

One of the triumphs of the supersymmetric localization paradigm, and indeed the result that spurred the questions that lead to this thesis, is the example of the mass deformed $\mathcal{N} = 4$ super-Yang-Mills theory (SYM) on an S^4 [1]. Since its publication, there have been many other applications of the techniques to gauge theories with a varying amount of supersymmetry, in various dimensions [73, 74], and even relaxing the condition that the space be compact [75, 76, 77]. Since the test of probe brane holography that will be presented in the subsequent chapter is motivated by [1], the focus of this section will be on understanding the necessary components to localize such a non-trivial theory. The case of mass deformations of $\mathcal{N} = 4$ SYM on an S^3 will be briefly discussed as it is relevant for applying localization techniques in the field theory dual to massive D3/D5 embeddings described in the previous sections of this chapter.

Recalling the field content of $\mathcal{N} = 4$ SYM as discussed in § 1.1, we can better organize the $\mathcal{N} = 4$ gauge multiplet in $\mathcal{N} = 2$ language to facilitate the discussion of localization more easily. That is, in $d = 4$ the $\mathcal{N} = 4$ gauge multiplet decomposes into an $\mathcal{N} = 2$ gauge multiplet and $\mathcal{N} = 2$ adjoint valued hypermultiplet (see Table. 1.2). The advantage of this is that if we want to study, say, $\mathcal{N} = 2^*$ SYM theory, we would want to leave the gauge field massless while giving fields living in the adjoint hypermultiplet a mass. The accounting

is thus much easier in this $\mathcal{N} = 2$ language. For the purposes of performing the same type of deformation with the addition of massive $\mathcal{N} = 2$ hypermultiplets in the fundamental representation to $\mathcal{N} = 4$ SYM theory, we will see the advantage of this treatment as well.

While there was a great deal of effort to correctly identify the terms necessary to preserve supersymmetry on a curved background in [1], later work by Seiberg and Festuccia in [59] used a particularly elegant method that finds the proper ‘compensating’ term on backgrounds with constant curvature. The core idea in [59, 78] is that in studying a supersymmetric theory on flat space, coupling to a background gravity multiplet, and then taking $M_{pl} \rightarrow \infty$ to decouple the gravitational sector, the resulting ‘rigid’ theory manifestly preserves the appropriate number of supersymmetries while living on a curved background. During this process, one eschews standard practices by keeping auxiliary terms in the background, off-shell supergravity multiplet around, i.e. not integrating them out, and not performing a Weyl rescaling to give the graviton canonical kinetic term and instead rescale the auxiliary fields to have mass dimension one. The reason for not putting the kinematic part of the supergravity sector into the canonical form is that in determining supersymmetric vacua, the condition that the variation of the gravitino, χ_μ^α ($\bar{\chi}_{\mu\dot{\alpha}}$), vanishes, which generates Killing spinor equations as discussed in §2.1.2, is then independent of the fields living in other multiplets.

To be explicit, denote the complex auxiliary scalar M and real auxiliary vector β_μ . The complex auxiliary scalar is what will parameterize a mass term deformation of the theory as compared to the flat background analysis. Then for Killing spinor ϵ , the constraint emerging from the variation of the gravitino becomes [59, 78]

$$\begin{aligned}\delta\chi_\mu^\alpha &= -2\nabla_\mu\epsilon^\alpha + \frac{i}{3}(M(\varepsilon\sigma_\mu\bar{\chi})^\alpha + 2\beta_\mu\chi^\alpha + 2\beta^\nu(\chi\sigma_{\nu\mu})^\alpha) = 0, \\ \delta\bar{\chi}_{\mu\dot{\alpha}} &= -2\nabla_\mu\bar{\epsilon}_{\dot{\alpha}} - \frac{i}{3}(\bar{M}(\chi\sigma_\mu)_{\dot{\alpha}} + 2\beta_\mu\bar{\chi}_{\dot{\alpha}} + 2\beta^\nu(\bar{\chi}\bar{\sigma}_{\nu\mu})_{\dot{\alpha}}) = 0.\end{aligned}\tag{2.37}$$

Here $\varepsilon^{\alpha\dot{\alpha}}$ is used to raise and lower spinor indices, $\sigma^\mu = \sigma_{\alpha\dot{\alpha}}^\mu$, $\bar{\sigma}_\mu^{\dot{\alpha}\alpha}$, and $\sigma_{\mu\nu} = \frac{i}{4}[\sigma_\mu, \sigma_\nu]_\alpha^\delta$ is the Lorentz generator. Just as in the analysis of κ -symmetry, we need to specify a background on which we are placing this rigidifying supergravity theory in order to solve these constraints.

Two examples that will be pertinent will be in the case when the background is AdS₄ and S⁴ without going through the full calculation, what solve the constraints in these two cases are

$$\begin{aligned} \text{AdS}_4 : \quad M = \bar{M} = -\frac{3}{L}, \quad \beta_\mu = 0 \\ \text{S}^4 : \quad M = \bar{M} = -\frac{3i}{L}, \quad \beta_\mu = 0 \end{aligned} \tag{2.38}$$

Note that the convention for these terms is that the scalar curvature of AdS₄ $\mathcal{R}_A = +\frac{12}{L^2}$ and the S⁴ $\mathcal{R}_S = -\frac{12}{L^2}$, which is opposite of standard conventions in, say general relativity textbooks. Further, on the S⁴ we see that the theory has a complex mass deformation of the theory where \bar{M} is explicitly not the conjugate of M , and so, the Lagrangian for the theory on a sphere is not unitary. For a case with non-vanishing auxiliary vector, consider S³ × S¹_τ where along the S¹_τ, $\beta_\tau = -\frac{3i}{L}$ and all other auxiliary field components vanishing. This would be an case interesting to consider for holographic studies of supersymmetric indices with regards to the probe brane analysis in the following chapters.

To connect this back to terms that will enter into the Lagrangian, specifying to S⁴ gives a superpotential mass deformation of the form (see e.g. [79] or [17] where this is succinctly summarized)

$$V_{comp} = \frac{iM}{R}(Q\tilde{Q} + \text{c.c.}), \tag{2.39}$$

where $W = MQ\tilde{Q}$ is the superpotential, and Q (\tilde{Q}) are commonly, and frustratingly, used to denote both the hypermultiplet and the scalars contained therein. This is in addition to the conformal coupling of the scalars to the background curvature and the F-term generated mass

$$V_{conf} + V_F = \frac{2}{R^2}(|Q|^2 + |\tilde{Q}|^2) + |M|^2(|Q|^2 + |\tilde{Q}|^2). \tag{2.40}$$

The point is that the compensating term necessary to have a supersymmetric theory on the S⁴ is an *imaginary mass term*. Moreover, the two terms that make up V_{comp} are not conjugate to one another (both have + sign). While this might be concerning at first blush, it has

long been known that while there exist unitary superconformal theories on S^4 (or dS_4 in the Lorentzian case) there are no unitary, reflection positive purely supersymmetric theories on these spaces. That is, because those backgrounds are conformally related to flat space, the superconformal representations are equivalent to those on \mathbb{R}^4 (or $\mathbb{R}^{3,1}$). However, if we want a massive supersymmetric theory on the sphere, we need to give up unitarity.

The construction of the action for an adjoint mass deformation of $\mathcal{N} = 4$ SYM on an S^4 that still preserves supersymmetry is, in the original calculation, a cumbersome task. However, with the knowledge of the appropriate compensating term from the rigid supergravity limit and a bit of algebra and perseverance, one can construct the action for $\mathcal{N} = 2^*$ SYM. The detailed calculation will not be undertaken here, but rather quote form of the action that was given in [1] in the compact, if slightly obtuse formulation:

$$S_{\mathcal{N}=2^*} = -\frac{1}{g^2} \int_{S^4} d^4x \sqrt{g_{S^4}} \text{Tr} \left[\frac{1}{4} \hat{F}^2 - \hat{\Psi} \mathcal{D}_{10} \hat{\Psi} + 2\hat{\Phi}^2 - \frac{1}{4} (R^T M)_{ij} \hat{\Phi}^i \hat{\Phi}^j - \hat{K}^2 \right]. \quad (2.41)$$

Here the ‘ $\hat{}$ ’ indicates that these are all fields constructed with respect to the original $d = 10$ $\mathcal{N} = 1$ language, \mathcal{D}_{10} is the gauge covariant derivative with respect to \hat{A} contracted with the $d = 10$ Clifford algebra Γ^M , R is a rank 4 anti-self-dual matrix with unusual normalization $R_{ij} R^{ij} = 4$, M_{ij} is a generator of $SU(2)_R^R \subset SO(6)_R$ transformations, the $(R_{pi} M_{jp}) \hat{\Phi}^i \hat{\Phi}^j$ term in the action is necessary to ensure the action remains invariant under the correct supergroup ($O\text{Sp}(2|4)$), and the K_i are auxiliary scalars added for off-shell supersymmetry (à la Berkovits [80]) necessary to identify a proper localizing supercharge \mathcal{Q} . To unpack all of the ten dimensional fields and translate them into the field content of the four dimensional theory in Table. 1.2 is straightforward but unnecessary for the current review.

With the action in hand, we can proceed with the story of localization by introducing the localizing term widely used in the literature (e.g. [1, 73])

$$V = \text{Tr} \psi \overline{\mathcal{Q}\psi}, \quad (2.42)$$

where the trace is over indices originating in the Lie algebra, or to keep with the notation akin to [1] $V = (\hat{\Psi}, \mathcal{Q}\hat{\Psi})$. In any case, it can be shown that the bosonic part of the localizing

term can be expressed as a sum of squares of bosonic fields which satisfies the positive definiteness condition — i.e. $\mathcal{Q}V|_B = \text{Tr}(\mathcal{Q}\psi\overline{\mathcal{Q}\psi})$. What remains is to find the saddle points of the $\mathcal{Q}V$ term and determine the non-vanishing field configurations that parameterize the locus.

Without dwelling too much on the details (see [1, 81, 82]), the locus is parameterized by a single non-zero, constant adjoint valued scalar, denoted a . In particular, the locus is the point in field space at which

$$\begin{aligned} A_\mu, \hat{\Phi}^5, \dots, \hat{\Phi}^9, K_1, \dots, K_4 &= 0 \\ \tilde{\Phi}_0 = a, \quad K_m &= -w_m a \end{aligned} \tag{2.43}$$

where $m = 1, \dots, 4$ and the w_m satisfying $w_m w^m = 1$ — on an S^4 with radius L $w_m w^m = L^{-2}$ — are related to the Killing spinors generating supersymmetry transformations and the auxiliary spinors associated with the off-shell formalism.

That form of the locus in eq. (2.43) is the same for pure $\mathcal{N} = 4$ SYM theory and any of the $\mathcal{N} = 2$ mass deformations of the theory. The classical contribution to the partition function, after localization, is then expressed not as an infinite dimensional path integral over field configurations but an ordinary Gaussian matrix model. Again, this is the power of equivariant localization. The leading order saddle point analysis, evaluating eq. (2.41) on eq. (2.43) gives a partition function of the form

$$\mathcal{Z}|_{saddle} \simeq \int da e^{-\frac{8\pi^2}{\lambda} N_c a^2}, \tag{2.44}$$

where $\lambda = g_{YM}^2 N_c$ is the 't Hooft coupling.

Now that the locus has been identified, what remains to be calculated is the one-loop determinant encoding the fluctuations in field space around the locus. This is no easy task, but to start one goes about determining the BRST charge \mathcal{Q}_{BRST} in order to gauge fix the G -action on the field space \mathcal{M} ; the combined fermionic charge $\mathcal{Q}' := \mathcal{Q} + \mathcal{Q}_{BRST}$ acts exactly as the equivariant differential needed to apply the Atiyah-Bott localization formula. Albeit, the abstract manifold of field space \mathcal{M} and the infinite dimensional gauge group

G are non-trivial extensions of the summary discussion in the preceding sections. One expands the action about the locus, identifies the quadratic action for the fluctuations, and picks a favorite method of computing the resulting functional determinant for the kinetic operators therein. For example, one can straightforwardly compute eigenvalues, take care of any zero modes, and regulate the resulting product as in [73], or taking the approach in [1] compute it using the Atiyah-Singer index theorem suitably generalized for differential operators that are elliptic on a space orthogonal to some orbit. Either way, one arrives at the following mnemonic for the contributions of the various multiplets to the one loop determinant displayed in Table. 2.5

$\mathcal{N} = 2$ multiplet	One-loop determinant
Vector	$\prod_{i < j} H(a_{[ij]})^2$
Massive Adjoint Hyper	$\prod_{i < j} \frac{1}{H(a_{[ij]} + M_{adj})H(a_{[ij]} - M_{adj})}$
Massive Fundamental Hyper	$\prod_{i,f} \frac{1}{\sqrt{H(a_i - M_f)H(a_i + M_f)}}$

Table 2.5: One-loop determinants for $\mathcal{N} = 2$ multiplets on an S^4

The notation being used is that $H(x) = G(1 + ix)G(1 - ix)$, $G(x)$ is the Barnes-G function, a_i label the weights of the fundamental rep of $\mathfrak{g} = \mathfrak{su}(N_c)$, $a_{[ij]} = a_i - a_j$ label the roots (adjoint rep weights), M_f labels possibly distinct masses for the flavor degrees of freedom, and M_{adj} is the mass of the adjoint hypermultiplet. Thus, the final result for, say, an addition of N_f $\mathcal{N} = 2$ hypermultiplets — with $SU(N_f) \times U(1)_B$ flavor symmetry in the fundamental representation of $\mathfrak{su}(N_c)$ all with mass M to pure $\mathcal{N} = 4$ SYM theory on an S^4 after localization is — after diagonalizing into the Cartan subalgebra of the gauge group and picking up a factor of the Vandermonde determinant from the Jacobian of the transformation

$$\mathcal{Z} = \int \left(\prod_i da_i \right) \frac{\prod_{i < j} a_{[ij]}^2}{\prod_i H^{\frac{N_f}{2}}(a_i + M) H^{\frac{N_f}{2}}(a_i - M)} e^{-\frac{8\pi}{\lambda} N_c \sum_i a_i^2}. \quad (2.45)$$

Note that the one-loop determinants for pure $\mathcal{N} = 4$ SYM theory on an S^4 cancel leaving

behind only the Gaussian unitary matrix model. This particular example may well be useful to keep in mind in Ch. 3.

With the above mnemonic for the contributions to the matrix model corresponding to mass deformed supersymmetric theories on an S^4 , we will be able to build the field theory dual to an appropriate D3/D7 probe brane set up with boundary S^4 geometry. However, in Ch. 4 we will need new ingredients in order to formulate the matrix model dual to a D3/D5 with the new flavor degrees of freedom living on an $S^3 \subset S^4$. The work done to understand supersymmetric localization on an S^3 was done in [73]. The only results that we will need are that the localizing supercharge \mathcal{Q}' is the same as in [1] (see e.g. the discussion in [83]), and that the entry for the one-loop determinant for the fundamental hypermultiplet to be added to Table. 2.5

$$(d = 3) \text{ Fundamental Hyper : } \quad \prod_i \frac{1}{\cosh(\pi(a_i + M))}.$$

With that, we have all of the necessary ingredients to perform the analysis in the field theory once suitable κ -symmetric D3/D7 and D3/D5 embeddings have been found. Again, we could very well consider an example of pure $\mathcal{N} = 4$ SYM theory on an S^4 deformed by N_f equally massive $\mathcal{N} = 2$ hypermultiplets with $U(N_f)$ flavor symmetry group in the fundamental representation of $\mathfrak{su}(N_c)$ confined to live a co-dimension one defect, say an equatorial $S^3 \subset S^4$ coupled minimally to the dimensional reduction of the $\mathcal{N} = 2$ vector multiplet part $\mathcal{N} = 4$ gauge multiplet defined in the ambient space, such that the result of localization calculation is

$$\mathcal{Z} = \int \left(\prod_i da_i \right) \frac{\prod_{i < j} a_{[ij]}^2}{\prod_i \cosh^{N_f}(\pi(a_i + M))} e^{-\frac{8\pi}{\lambda} N_c \sum_i a_i^2}. \quad (2.46)$$

Again, this may be a result that the reader may want to keep in mind for use in Ch. 4.

Chapter 3

A PRECISION TEST OF PROBE BRANE HOLOGRAPHY

This chapter is based on work done in collaboration with Andreas Karch and Christoph Uhlemann contained in two papers with which there is text overlap. The first constructs supersymmetric probe brane embeddings with curved boundary geometry [17]. The second explicitly compares the case of a spherical boundary geometry with the exact free energy (or derivatives thereof) computed using supersymmetric localization [18].

3.1 Introduction

As noted in Ch. 1, the non-trivial nature of the dynamics of a simple free field theory in curved space makes it interesting and worthwhile to study strongly-coupled QFTs in curved spacetimes. Spacetimes of constant curvature are certainly the natural starting point for departure from Minkowski space, and we will primarily focus on AdS_4 .

Efforts to understand holographic investigations of QFT on (A)dS₄ have been going on since the early days of AdS/CFT [84, 58, 85] and continue to provide a fertile ground of inquiry, see e.g. [86, 87]. Pertinent for the following analysis, the choice of $\mathcal{N} = 4$ SYM is a natural starting point for holographic investigations, and detailed studies on AdS_4 and dS_4 were initiated in [88, 89]. Taken in isolation, $\mathcal{N} = 4$ SYM also is a rather special theory having its superconformal symmetry and all fields in the adjoint representation. One aspect of that is adding fundamental matter, which can be done holographically by adding D7-branes [15]. The resulting theory is $\mathcal{N} = 2$ supersymmetric and has a non-trivial UV fixed point in the quenched approximation, where the rank of the gauge group is large compared to the flavor group. For massless quenched flavors, the superconformal generators remain unbroken and the AdS_4 discussion could just as well be carried out on flat space.

To move into more non-trivial embeddings, holographic studies of $\mathcal{N} = 4$ SYM coupled to massive flavor hypermultiplets on AdS_4 can be found in [90, 91], but the numerical construction of such embeddings manifestly broke all of the supersymmetries. To that end, the following sections will show how to add massive $\mathcal{N} = 2$ flavors and thereby explicitly break conformal symmetry in AdS_4 while preserving 8 real supercharges. We will also employ the quenched or probe approximation throughout, such that the D7-branes on the holographic side can be described by a classical action, i.e. DBI and WZ actions, and backreaction effects are small. Note that the desire to include supersymmetry makes AdS the preferred curved space to look at in Lorentzian signature, as the formulation of supersymmetric QFTs on dS faces additional issues with unitarity unless the theory is conformal [92].

In the brane construction preserving supersymmetry for flavored $\mathcal{N} = 4$ SYM on AdS_4 necessitates solving κ -symmetry conditions [16, 93, 94] §2.1.2. While on the field theory side, formulating supersymmetric field theories in curved space needs some care. Non-minimal curvature couplings may be needed, and can be understood systematically by taking the rigid limit of supergravity [1, 59] (see §2.1.2) The extra terms needed on the field theory side to preserve supersymmetry suggest that varying the slipping mode alone will not be enough to get massive supersymmetric embeddings, and so we will also have to include worldvolume flux.

The following sections begin with systematic analysis of the constraint imposed by κ -symmetry on the embeddings and extract necessary conditions for the slipping mode and worldvolume flux in §3.2. From the resulting conditions we will be able to extract analytic supersymmetric D7-brane embeddings in a nice closed form, which are given in §3.2.5, with the conventions laid out in §3.2.1. The solutions we find allow to realize a surprisingly rich set of supersymmetric embeddings, which we categorize into short, long and connected embeddings. We study those in more detail in §3.3. In §3.4 we focus on implications for flavored $\mathcal{N} = 4$ SYM. We carry out the process of holographic renormalization, compute the chiral and scalar condensates, and attempt an interpretation of the various embeddings found in §3.3 from the QFT perspective. In §3.5 we similarly construct supersymmetric D7-

brane embeddings into S^4 -sliced and dS_4 -sliced $AdS_5 \times S^5$, so we end up with a comprehensive catalog of D7-brane embeddings to holographically describe massive $\mathcal{N} = 2$ supersymmetric flavors on spaces of constant curvature. Following the construction of the spherical embedding, the holographic construction of supersymmetric observables will be carried out where a particular combination of correlation functions will be computed, § 3.6 in order to compared to a localization calculation in § 3.7. To end, a discussion of the results, their interpretation, and questions left unanswered is contained in § 3.8

3.2 Supersymmetric D7 branes in AdS_4 -sliced $AdS_5 \times S^5$

In this section we evaluate the constraint imposed by κ symmetry to find supersymmetric D7-brane embeddings into AdS_4 -sliced $AdS_5 \times S^5$. The κ -symmetry constraint for embeddings with non-trivial fluxes has a fairly non-trivial Clifford-algebra structure, and the explicit expressions for $AdS_5 \times S^5$ Killing spinors are themselves not exactly simple. That makes it challenging to extract the set of necessary equations for the embedding and flux from it, and this task will occupy most of the next section. On the other hand, the non-trivial Clifford-algebra structure will allow us to separate the equations for flux and embedding. Once the κ -symmetry analysis is done, the pay-off is remarkable. Instead of heaving to solve the square-root non-linear coupled differential equations resulting from variation of the DBI action with Wess-Zumino term, we will be able to explicitly solve for the worldvolume gauge field in terms of the slipping mode. The remaining equation then is a non-linear but reasonably simple differential equation for the slipping mode alone. As we verified explicitly to validate our derivation, these simple equations indeed imply the full non-linear DBI equations of motion. We set up the background, establish conventions and motivate our choices for the embedding ansatz and worldvolume flux in Sec. 3.2.1. Generalities on κ -symmetry are set up in Sec. 3.2.2, and small-mass embeddings are discussed in Sec. 3.2.3. The fully massive embeddings are in Sec. 3.2.4. To find the solutions, we take a systematic approach to the κ -symmetry analysis in the next subsections. Readers interested mainly in the results can directly proceed from Sec. 3.2.1 to the the embeddings given in Sec. 3.2.5.

3.2.1 Geometry and embedding ansatz

Our starting point will be Lorentzian signature and the AdS_4 slices in Poincaré coordinates. For the global structure, it does make a difference whether we choose global AdS_4 or the Poincaré patch as slices, and the explicit expressions for the metric, Killing spinors etc. are also different. However, the field equations and the κ -symmetry constraint are local conditions, and our final solutions will thus be valid for both choices.

We choose coordinates such that the $\text{AdS}_5 \times \text{S}^5$ background geometry has a metric

$$g_{\text{AdS}_5} = d\rho^2 + \cosh^2 \rho [dr^2 + e^{2r}(-dt^2 + d\vec{x}^2)] , \quad g_{\text{S}^5} = d\theta^2 + \cos^2 \theta d\psi^2 + \sin^2 \theta d\Omega_3^2 , \quad (3.1)$$

where $d\Omega_3^2 = d\chi_1^2 + \sin^2 \chi_1 (d\chi_2^2 + \sin^2 \chi_2 d\chi_3^2)$. We use the $\text{AdS}_5 \times \text{S}^5$ Killing spinor equation in the conventions of [23]

$$D_\mu \epsilon = \frac{i}{2} \Gamma_{\text{AdS}} \Gamma_\mu \epsilon , \quad \mu = 0 \dots 4 , \quad D_\mu \epsilon = \frac{i}{2} \Gamma_{\text{S}^5} \Gamma_\mu \epsilon , \quad \mu = 5 \dots 9 , \quad (3.2)$$

and we have $\Gamma_{\text{AdS}} := \Gamma^{01234} = -\Gamma_{01234}$ along with $\Gamma_{\text{S}^5} := \Gamma^{56789}$. Generally, we follow the usual convention and denote coordinate indices by Greek letters from the middle of the alphabet and local Lorentz indices by latin letters from the beginning of the alphabet. We will use an underline to distinguish Lorentz indices from coordinate indices whenever explicit values appear. The ten-dimensional chirality matrix is $\Gamma_{11} = \Gamma_{\text{AdS}} \Gamma_{\text{S}^5}$.

For the κ -symmetry analysis we will need the explicit expressions for the Killing spinors solving eq. (4.8). They can be constructed from a constant chiral spinor ϵ_0 with $\Gamma_{11} \epsilon_0 = \epsilon_0$ as

$$\epsilon = R_{\text{S}^5} \times R_{\text{AdS}} \times \epsilon_0 . \quad (3.3)$$

The matrices R_{AdS} , R_{S^5} denote products of exponentials of even numbers of Γ -matrices with indices in AdS_5 and S^5 , respectively. For the S^5 part we find¹

$$R_{\text{S}^5} = e^{\frac{\theta}{2} i \Gamma^{\underline{\psi}} \Gamma_{\underline{\chi}}} e^{\frac{\psi}{2} i \Gamma_{\underline{\chi}} \Gamma^{\underline{\theta}}} e^{\frac{1}{2} \chi_1 \Gamma^{\underline{\theta} \chi_1}} e^{\frac{1}{2} \chi_2 \Gamma^{\underline{\chi}_1 \chi_2}} e^{\frac{1}{2} \chi_3 \Gamma^{\underline{\chi}_2 \chi_3}} , \quad (3.4)$$

¹For $\psi = 0$ our eq. (3.4) agrees with the S^4 Killing spinors constructed in [63]. But this is different from (86) of [23] by factors of i in the S^3 part.

where we have defined $\Gamma_{\vec{\chi}} := \Gamma^{\chi_1} \Gamma^{\chi_2} \Gamma^{\chi_3}$. The exponent in all the exponentials is the product of a real function f and a matrix A which squares to -1 . We will also encounter the product of a real function and a matrix B which squares to $+1$ in the exponential. The explicit expansions are

$$e^{fA} = \cos f \cdot \mathbf{1} + \sin f \cdot A , \quad e^{fB} = \cosh f \cdot \mathbf{1} + \sinh f \cdot B . \quad (3.5)$$

The corresponding R -matrix for AdS₄-sliced AdS₅ can be constructed easily starting from the AdS Killing spinors given in [95, 63]. With the projectors $P_{r\pm} = \frac{1}{2}(\mathbf{1} \pm i\Gamma_{\underline{r}}\Gamma_{\text{AdS}})$, the AdS₅ piece reads

$$R_{\text{AdS}} = e^{\frac{r}{2}i\Gamma_{\underline{r}}\Gamma_{\text{AdS}}} R_{\text{AdS}_4} , \quad R_{\text{AdS}_4} = e^{\frac{r}{2}i\Gamma_{\underline{r}}\Gamma_{\text{AdS}}} + ie^{r/2}x^\mu \Gamma_{\underline{x}\mu} \Gamma_{\text{AdS}} P_{r-} . \quad (3.6)$$

For the D7 branes we explicitly spell out the DBI action and WZ term to fix conventions. For the κ -symmetry analysis we will not actually need it, but as a consistency check we want to verify that our final solutions solve the equations of motion derived from it. We take

$$S_{\text{D7}} = -T_7 \int_{\Sigma_8} d^8\xi \sqrt{-\det(g + 2\pi\alpha'F)} + 2(2\pi\alpha')^2 T_7 \int_{\Sigma_8} C_4 \wedge F \wedge F , \quad (3.7)$$

with g denoting the pullback of the background metric and the pullback on the four-form gauge field C_4 is understood. We absorb $2\pi\alpha'$ by a rescaling of the gauge field, so it is implicit from now on. To fix conventions on the five-form field strength we use [96]: to get $R_{\mu\nu} = 4L^{-2}g_{\mu\nu}$, we need $F_5 = L^{-1}(1 + \star) \text{vol}(\text{AdS}_5)$. So we take

$$C_4 = L^{-1}\zeta(\rho) \text{vol}(\text{AdS}_4) + \dots , \quad \zeta'(\rho) = \cosh^4 \rho . \quad (3.8)$$

The dots in the expression for C_4 denote the part producing the volume form on S⁵ in F_5 , which will not be relevant in what follows. As usual, C_4 is determined by F_5 only up to gauge transformations, and we in particular have an undetermined constant in ζ , which will not play any role in the following.

Embedding ansatz

We will be looking for D7-brane embeddings to holographically describe $\mathcal{N} = 4$ SYM coupled to massive $\mathcal{N} = 2$ flavors on AdS_4 . So we in particular want to preserve the AdS_4 isometries. The ansatz for the embedding will be such that the D7-branes wrap entire $\text{AdS}_4 \times \text{S}^3$ slices in $\text{AdS}_5 \times \text{S}^5$, starting at the conformal boundary and reaching into the bulk possibly only up to a finite value of the radial coordinate ρ . The S^3 is parametrized as usual by the “slipping mode” θ as function of the radial coordinate ρ only. We choose static gauge such that the entire embedding is characterized by θ .

To gain some intuition for these embeddings, we recall the Poincaré AdS analysis of [15]. From that work we already know the $\theta \equiv \pi/2$ embedding, which is a solution regardless of the choice of coordinates on AdS_5 . So we certainly expect to find that again, also with our ansatz. This particular D3/D7 configuration preserves half of the background supersymmetries, corresponding to the breaking from $\mathcal{N} = 4$ to $\mathcal{N} = 2$ superconformal symmetry in the boundary theory.² For Poincaré AdS_5 with radial coordinate z , turning on a non-trivial slipping mode $\theta = \arcsin mz$ breaks additional, but not all supersymmetries. The configuration is still 1/4 BPS [15], corresponding to the breaking of $\mathcal{N} = 2$ superconformal symmetry to just $\mathcal{N} = 2$ supersymmetry in the boundary theory on Minkowski space.

Our embedding ansatz, on the other hand, is chosen such that it preserves AdS_4 isometries, and the slipping mode depends non-trivially on a different radial coordinate. These are, therefore, geometrically different embeddings. As we will see explicitly below, supersymmetric embeddings can not be found in that case by just turning on a non-trivial slipping mode. From the field-theory analyses in [1, 59], we know that in addition to the mass term for the flavor hypermultiplets we will have to add another purely scalar mass term to preserve some supersymmetry on curved backgrounds. This term holographically corresponds to a certain mode of the worldvolume gauge field on the $\text{S}^3 \subset \text{S}^5$, an $\ell = 1, -$ mode in the language of [97]. Including such worldvolume flux breaks the $\text{SO}(4)$ isometries of the S^3 to $\text{SU}(2) \times \text{U}(1)$.

²The preserved conformal symmetry is a feature of the quenched approximation with $N_f/N_c \ll 1$ only.

The same indeed applies to the extra scalar mass term on the field theory side: it breaks the R-symmetry from SU(2) to U(1). The SU(2) acting on the $\mathcal{N} = 2$ adjoint hypermultiplet coming from the $\mathcal{N} = 4$ vector multiplet is not altered by the flavor mass term (see e.g. [98]). The bottom line for our analysis is that we should not expect to get away with a non-trivial slipping mode only.

For the analysis below we will not use the details of these arguments as input. Our ansatz is a non-trivial slipping mode $\theta(\rho)$ and a worldvolume gauge field $A = f(\rho)\omega$, where ω is a generic one-form on S^3 . This ansatz can be motivated just by the desire to preserve the AdS₄ isometries.³ Whether the supersymmetric embeddings we will find reflect the field-theory analysis will then be a nice consistency check, rather than input. As we will see, the κ -symmetry constraint is enough to determine ω completely, and the result is indeed consistent with the field-theory analysis.

3.2.2 κ -symmetry generalities

The κ -symmetry condition projecting on those Killing spinors which are preserved by a given brane embedding were derived in [16, 93, 94]. We take the conventions of [16]. The pullback of the ten-dimensional vielbein E^a to the D7 worldvolume is denoted by $e^a = E^a_\mu(\partial_i X^\mu)dx^i$, and the Clifford algebra generators pulled back to the worldvolume are denoted by $\gamma_i = e^a_i \Gamma_a$. We follow [16] and define $X^i_j := g^{ik} F_{kj}$. Their κ -symmetry requirement then is $(1 - \Gamma_\kappa)\epsilon = 0$, where

$$\Gamma_\kappa = \frac{1}{\sqrt{\det(1 + X)}} \sum_{n=0}^{\infty} \frac{1}{2^n n!} \gamma^{j_1 k_1 \dots j_n k_n} X_{j_1 k_1} \dots X_{j_n k_n} J_{(p)}^{(n)} , \quad (3.9a)$$

$$J_{(p)}^{(n)} = (-1)^n (\sigma_3)^{n+(p-3)/2} i\sigma_2 \otimes \Gamma_{(0)} , \quad (3.9b)$$

$$\Gamma_{(0)} = \frac{1}{(p+1)! \sqrt{-\det g}} \varepsilon^{i_1 \dots i_{p+1}} \gamma_{i_1 \dots i_{p+1}} . \quad (3.9c)$$

³A generalization which we will not study here is to also allow for non-trivial ρ -dependence in ψ .

For embeddings characterized by a non-trivial slipping mode as described above, the induced metric on the D7-branes reads

$$g = (1 + \theta'^2)d\rho^2 + \cosh^2\rho ds_{\text{AdS}_4}^2 + \sin^2\theta d\Omega_3^2 . \quad (3.10)$$

The pullback of the ten-bein to the D7 worldvolume is given by

$$e^a = E^a , \quad a = \underline{0} \dots \underline{7} , \quad e^{\underline{8}} = \theta' d\rho , \quad e^{\underline{9}} = 0 . \quad (3.11)$$

The κ -symmetry condition eq. (3.9) for type IIB supergravity is formulated for a pair of Majorana-Weyl spinors. We will find it easier to change to complex notation, such that we can deal with a single Weyl Killing spinor, without the Majorana condition.

Complex notation

Eq. (3.9) is formulated for a pair of Majorana-Weyl Killing spinors (ϵ_1, ϵ_2) . It is the index labeling the two spinors on which the Pauli matrices act. To switch to complex notation we define a single Weyl spinor $\epsilon = \epsilon_1 + i\epsilon_2$, containing the two Majorana-Weyl spinors. With the Pauli matrices

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad (3.12)$$

we then find that $i\sigma_2(\epsilon_1, \epsilon_2)$ translates to $-i\epsilon$ and $\sigma_3(\epsilon_1, \epsilon_2)$ to $C\epsilon^*$. With these replacements the action of $\sigma_{2/3}$ commutes with multiplication by Γ -matrices, as it should: the Γ -matrices in eq. (3.9) should be understood as $\mathbb{1}_2 \otimes \Gamma$. We thus find

$$J_{(7)}^{(n)} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \rightarrow -i\Gamma_{(0)}\epsilon \quad \text{for } n \text{ even} , \quad J_{(7)}^{(n)} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \rightarrow -iC(\Gamma_{(0)}\epsilon)^* \quad \text{for } n \text{ odd} . \quad (3.13)$$

Note that Γ_κ contains an involution and does not act as a \mathbb{C} -linear operator. To fix conventions, we choose the matrix B_1 defined in the appendix of [20], and set $C = B_1$. C then is the product of four Hermitian Γ s that square to $\mathbb{1}$, so we immediately get $C^\dagger = C$ and $C^2 = \mathbb{1}$. Furthermore, we have

$$C\Gamma^\mu = (\Gamma^\mu)^*C , \quad C^*C = \mathbb{1} . \quad (3.14)$$

With eq. (3.13) it is straightforward now to switch to complex notation in eq. (3.9).

Projection condition for our embedding ansatz

We not set up the κ -symmetry condition in complex notation for our specific ansatz for embedding and worldvolume flux. As explained above, for our analysis we do not make an a priori restriction on the S^3 gauge field to be turned on. So, for the worldvolume gauge field we set $A = f(\rho)\omega$, with ω a generic one-form on the S^3 . The field strength is $F = df \wedge \omega + fd\omega$, and we find the components $F_{\rho\alpha} = f'\omega_\alpha$ and $F_{\alpha\beta} = \partial_\alpha\omega_\beta - \partial_\beta\omega_\alpha$. We only have 4 non-vanishing components of F , which means that the sum in eq. (3.9a) terminates at $n = 2$. We thus get

$$\Gamma_{\kappa\epsilon} = \frac{-i}{\sqrt{\det(1+X)}} \left[\left(\mathbf{1} + \frac{1}{8}\gamma^{ijkl}F_{ij}F_{kl} \right) \Gamma_{(0)\epsilon} + \frac{1}{2}\gamma^{ij}F_{ij}C(\Gamma_{(0)\epsilon})^\star \right]. \quad (3.15)$$

The pullback of the vielbein to the D7 branes has been given in eq. (3.11) above, and we find

$$\Gamma_{(0)} = \frac{-1}{\sqrt{1+\theta'^2}}\hat{\Gamma}, \quad \hat{\Gamma} = \left[\mathbf{1} + \theta'\Gamma_{\underline{\theta}}\Gamma_{\underline{\rho}} \right] \Gamma_{\text{AdS}}\Gamma_{\bar{\chi}}. \quad (3.16)$$

The equations eq. (3.15) and eq. (3.16) will be our starting point for the analyses in the next subsections.

3.2.3 Infinitesimally massive embeddings

Our construction of supersymmetric embeddings will proceed in two steps. We would first want to know what exactly the preserved supersymmetries are and what the general form of the S^3 gauge field is. These questions can be answered from a linearized analysis, which we carry out in this section. With that information in hand, the full non-linear analysis will be easier to carry out, and we come to that in the next section.

So, for now want to solve the κ -symmetry condition in a small-mass expansion, i.e. with θ and F different from $\pi/2$ and zero only infinitesimally. The leading order is straightforward: we know that $\theta \equiv \pi/2$ with $F = 0$ is the solution from the flat slicing. But we will need the

specific part of supersymmetry this embedding preserves for later. We expand $\theta = \frac{\pi}{2} + \delta\theta + \dots$ and analogously for f . We use f without explicit δ , though, as it is zero for the massless embedding and there should be no confusion. The κ -symmetry condition $\Gamma_\kappa \epsilon = \epsilon$ can then be expanded up to linear order in δm . The leading-order equation reads

$$\Gamma_\kappa^{(0)} \epsilon^{(0)} = \epsilon^{(0)} , \quad \Gamma_\kappa^{(0)} = i\Gamma_{\text{AdS}} \Gamma_{\vec{\chi}} , \quad \epsilon^{(0)} = \epsilon|_{\theta=\pi/2} , \quad (3.17)$$

where we use the superscript to indicate the order in the expansion in δm . For the next-to-leading order we need to take into account that not only the projector changes, but also the location where the Killing spinor is evaluated – the κ -symmetry condition is evaluated on the D7s. This way we get

$$\Gamma_\kappa^{(0)} \epsilon^{(1)} + \Gamma_\kappa^{(1)} \epsilon^{(0)} = \epsilon^{(1)} , \quad \epsilon^{(1)} = \frac{i}{2} \delta\theta \Gamma^\psi \Gamma_{\vec{\chi}} \epsilon^{(0)} . \quad (3.18)$$

To find out which supersymmetries can be preserved, if any, we need to see under which circumstances the projection conditions eqs. (3.17) and (3.18) can be satisfied. To work this out, we note that we can only impose *constant* projection conditions on the constant spinor ϵ_0 that was used to construct the Killing spinors in eq. (3.3): any projector with non-trivial position dependence would only allow for trivial solutions when imposed on a constant spinor. For the massless embedding we can straightforwardly find that projector on ϵ_0 , by acting on the projection condition in eq. (3.17) with inverse R -matrices. This gives

$$\epsilon_0 = R_{\text{AdS}}^{-1} R_{\text{S}^5}^{-1} \Gamma_\kappa^{(0)} R_{\text{S}^5} R_{\text{AdS}} \epsilon_0 = -\Gamma_{\text{AdS}} \Gamma^\psi \epsilon_0 . \quad (3.19)$$

We have used the fact that Γ_{AdS} commutes with all the Γ -matrices in the AdS_5 part, and also with R_{S^5} . The last equality holds only when the left hand side is evaluated at $\theta = \pi/2$. Using that $\Gamma_{11} \epsilon_0 = \epsilon_0$, this can be written as a projector involving S^5 Γ -matrices only

$$P_0 \epsilon_0 = \epsilon_0 , \quad P_0 = \frac{1}{2} (\mathbf{1} + \Gamma^\theta \Gamma_{\vec{\chi}}) . \quad (3.20)$$

This is the desired projection condition on the constant spinor: those $\text{AdS}_5 \times \text{S}^5$ Killing spinors constructed from eq. (3.4) with ϵ_0 satisfying eq. (3.20) generate supersymmetries that are preserved by the D3/D7 configuration. We are left with half the supersymmetries of the $\text{AdS}_5 \times \text{S}^5$ background.

Projection condition at next-to-leading order

For the small-mass embeddings we expect that additional supersymmetries will be broken, namely those corresponding to the special conformal supersymmetries in the boundary theory. We can use the massless condition, eq. (3.17), to simplify the projection condition eq. (3.18) before evaluating it. With $\{\Gamma_\kappa^{(0)}, \Gamma_\psi \Gamma_{\bar{\chi}}\} = 0$ and $\Gamma_\kappa^{(0)} \epsilon^{(0)} = \epsilon^{(0)}$, we immediately see that $\Gamma_\kappa^{(0)} \epsilon^{(1)} = -\epsilon^{(1)}$. The next-to-leading-order condition given in eq. (3.18) therefore simply becomes

$$\Gamma_\kappa^{(1)} \epsilon^{(0)} = 2\epsilon^{(1)} \quad (3.21)$$

The determinants entering Γ_κ in eq. (3.15) contribute only at quadratic order, so we get

$$\Gamma_\kappa^{(1)} = \theta' \Gamma_{\underline{\theta}} \Gamma_{\underline{\rho}} \Gamma_\kappa^{(0)} - \frac{1}{2} \gamma^{ij} F_{ij} C(\Gamma_\kappa^{(0)} \cdot)^* . \quad (3.22)$$

We use that in eq. (3.21) and multiply both sides by $\Gamma_\psi \Gamma_{\bar{\chi}}$. With $\Gamma_\kappa^{(0)} \epsilon^{(0)} = \epsilon^{(0)}$ and $\Gamma_{S^5} \epsilon^{(0)} = -\Gamma_{\text{AdS}} \epsilon^{(0)}$, we find the explicit projection condition

$$\left[\delta\theta' \Gamma_{\underline{\rho}} \Gamma_{\text{AdS}} - i\delta\theta \mathbf{1} \right] \epsilon^{(0)} = \frac{1}{2} \Gamma_\psi \Gamma_{\bar{\chi}} \gamma^{ij} F_{ij} C \epsilon^{(0)*} . \quad (3.23)$$

The left hand side has no Γ -structures on S^5 , except for those implicit in the Killing spinor.

We turn to evaluating the right hand side further, and note that

$$\frac{1}{2} \gamma^{ij} F_{ij} = f' \omega_\alpha \gamma^{\rho\alpha} + f \partial_\alpha \omega_\beta \gamma^{\alpha\beta} . \quad (3.24)$$

For the perturbative analysis, the pullback to the D7 brane for the γ -matrices is to be evaluated with the zeroth-order embedding, i.e. for the massless $\theta \equiv \pi/2$ one. Then eq. (3.23) becomes

$$\left[\delta\theta' \Gamma_{\underline{\rho}} \Gamma_{\text{AdS}} - i\delta\theta \mathbf{1} \right] \epsilon^{(0)} = \Gamma_\psi \Gamma_{\bar{\chi}} [f' \omega_\alpha \Gamma^{\underline{\rho}\alpha} + f \partial_\alpha \omega_\beta \Gamma^{\chi\alpha\chi\beta}] C \epsilon^{(0)*} . \quad (3.25)$$

Note that some of the S^3 Γ -matrices on the right hand side come without underline, and include non-trivial dependence on the S^3 coordinates through the vielbein.

Next-to-leading order solutions: projector and S^3 harmonic

We now come to evaluating eq. (3.25) more explicitly, starting with the complex conjugation on $\epsilon^{(0)}$. Commuting R_{AdS} and R_{S^5} through C acts as just complex conjugation on the coefficients in eq. (3.4) and eq. (3.6). We define R -matrices with a tilde such that $\tilde{R}_{\text{AdS}}C = CR_{\text{AdS}}^*$ and analogously for \tilde{R}_{S^5} . Acting on eq. (3.25) with $R_{S^5}^{-1}$, we then get

$$\left[\delta\theta' \Gamma_{\underline{\rho}} \Gamma_{\text{AdS}} - i\delta\theta \mathbf{1} \right] R_{\text{AdS}} \epsilon_0 = R_{S^5}^{-1} \Gamma_{\underline{\psi}} \Gamma_{\underline{\chi}} [f' \omega_{\alpha} \Gamma^{\underline{\rho}} \Gamma^{\chi\alpha} + f \partial_{\alpha} \omega_{\beta} \Gamma^{\chi\alpha\chi\beta}] \tilde{R}_{S^5} \tilde{R}_{\text{AdS}} C \epsilon_0^* , \quad (3.26)$$

The noteworthy feature of this equation is that the left hand side has no more dependence on S^3 directions. To have a chance at all to satisfy this equation, we therefore have to ensure that S^3 dependence drops out on the right hand side as well. Since f and f' are expected to be independent as functions of ρ , this has to happen for each of the two terms individually. We start with the first one, proportional to f' , and solve for an ω s.t. the dependence on S^3 coordinates implicit in the $\Gamma^{\chi\alpha}$ matrices drops out. The Clifford-algebra structure on S^5 is dictated by the terms we get from evaluating $R_{S^5}^{-1} \Gamma_{\underline{\psi}} \Gamma_{\underline{\chi}} \Gamma^{\chi\alpha} \tilde{R}_{S^5}$. We just want to solve for the coefficients to be constants. That is, with three constants c_i we solve for

$$\omega_{\alpha} R_{S^5}^{-1} \Gamma_{\underline{\psi}} \Gamma_{\underline{\chi}} \Gamma^{\chi\alpha} \tilde{R}_{S^5} P_0 = c_i \Gamma^{\chi_i} \Gamma_{\underline{\psi}} \Gamma^{\theta} P_0 . \quad (3.27)$$

We only need this equation to hold when acting on $\epsilon^{(0)}$, i.e. only when projected on P_0 . This fixes ω . Using eq. (3.20), we can find a solution for arbitrary c_i . The generic solution satisfies $\star_{S^3} d\omega = -(\ell + 1)\omega$ with $\ell = 1$. The S^3 one-form ω thus is precisely the $\ell = 1, -$ mode we had speculated to find in Sec. 3.2.1 when we set up the ansatz, and the bulk analysis indeed reproduces the field-theory results. Holographically speaking, this is a result solely about matching symmetries and may not be overly surprising, but it is a nice consistency check anyway. The solutions parametrized by c_i are equivalent for our purposes, and we choose a simple one with $c_1 = 1, c_2 = c_3 = 0$. This yields

$$\omega = -\cos \chi_2 d\chi_1 + \sin \chi_1 \cos \chi_1 \sin \chi_2 d\chi_2 + \sin^2 \chi_1 \sin^2 \chi_2 d\chi_3 . \quad (3.28)$$

The second term on the right hand side of eq. (3.26) can then easily be evaluated using that $(\partial_\alpha \omega_\beta) \gamma^{\alpha\beta} = -2 \csc \theta \omega_\alpha \Gamma_{\bar{\chi}} \gamma^\alpha$, for ω as given in eq. (3.28). The κ -symmetry condition becomes

$$\left[\delta\theta' \Gamma_{\underline{\rho}} \Gamma_{\text{AdS}} - i\delta\theta \mathbf{1} \right] R_{\text{AdS}} \epsilon_0 = \left[f' \Gamma_{\underline{\rho}} \Gamma_{\text{AdS}} + 2if \mathbf{1} \right] \Gamma^\theta \Gamma^{\chi_1} \tilde{R}_{\text{AdS}} C \epsilon_0^* . \quad (3.29)$$

There are no more S^5 Γ -matrices on the left hand side, so we need to get rid of those on the right hand side, too. We expected to find at most one fourth of the background supersymmetries preserved, and now indeed see that we can not get away without demanding an additional projection condition on ϵ_0 . We will demand that

$$\tilde{\Gamma} C \epsilon_0^* = \lambda \epsilon_0 , \quad \tilde{\Gamma} = \Gamma_{\underline{\rho}} \Gamma_{\text{AdS}} \Gamma^{\chi_1} \Gamma^\theta , \quad (3.30)$$

where $\lambda^* \lambda = 1$. We can achieve that by setting $\epsilon_0 = \eta + \lambda^* \tilde{\Gamma} C \eta^*$, noting that $\tilde{\Gamma}^2 = \mathbf{1}$ and $C \tilde{\Gamma}^* = \tilde{\Gamma} C$. We also see that, due to $[\Gamma_{\text{AdS}} \Gamma^\psi, \tilde{\Gamma}] = 0$, ϵ_0 satisfies eq. (3.19) if η does. So the two conditions are compatible. Note that in Majorana-Weyl notation eq. (4.55) relates the two spinors to each other, rather than acting as projection condition on each one of them individually. This is different from the flat slicing. From eq. (3.29) we then get with eq. (4.55)

$$\left[\delta\theta' \Gamma_{\underline{\rho}} \Gamma_{\text{AdS}} - i\delta\theta \mathbf{1} \right] R_{\text{AdS}} \epsilon_0 = \lambda \left[2if \Gamma_{\text{AdS}} - f' \Gamma_{\underline{\rho}} \right] \tilde{R}_{\text{AdS}} \Gamma_{\underline{\rho}} \epsilon_0 . \quad (3.31)$$

There is no dependence on the S^5 Γ -matrices anymore. As a final step we just act with R_{AdS}^{-1} on both sides. To evaluate the result we use a relation between \tilde{R}_{AdS} and R_{AdS} , and define a short hand $\tilde{\Gamma}_{\rho A}$ as follows

$$\Gamma_{\underline{\rho}} \tilde{R}_{\text{AdS}} \Gamma_{\underline{\rho}} = e^{-\rho} i \Gamma_{\underline{\rho}} \Gamma_{\text{AdS}} R_{\text{AdS}} , \quad \tilde{\Gamma}_{\rho A} := R_{\text{AdS}}^{-1} \Gamma_{\underline{\rho}} \Gamma_{\text{AdS}} R_{\text{AdS}} . \quad (3.32)$$

We then find that acting with R_{AdS}^{-1} on eq. (3.31) yields

$$[\delta\theta' - i\lambda (2f \cosh \rho + f' \sinh \rho)] \tilde{\Gamma}_{\rho A} \epsilon_0 = (i\delta\theta - \lambda f' \cosh \rho - 2f \sinh \rho) \epsilon_0 . \quad (3.33)$$

These are independent Γ -structures on the left and on the right hand side, so the coefficients have to vanish separately.

The main results for this section are the massive projector eq. (4.55) and the one-form on the S^3 given in eq. (3.28). They will be the input for the full analysis with finite masses in the next section. To validate our results so far, we still want to verify that the κ -symmetry condition eq. (3.33) for small masses can indeed be satisfied with the linearized solutions for θ and f . The solutions to the linearized equations of motion resulting from eq. (3.7) with eq. (3.28) (or simply eq. (3.56) below) read

$$f = \mu \operatorname{sech}^2 \rho (1 - \rho \tanh \rho) , \quad \theta = m \operatorname{sech} \rho (\rho \operatorname{sech}^2 \rho + \tanh \rho) , \quad (3.34)$$

We find that eq. (3.33) is indeed satisfied exactly if $i\lambda\mu = m$. To get a real gauge field, λ should be chosen imaginary, which is compatible with consistency of eq. (4.55).

Before coming to the fully massive embeddings, we want to better understand the projector eq. (4.55) we found above. The massive embedding is expected to break what acts on the conformal boundary of AdS_5 as special conformal supersymmetries, leaving only the usual supersymmetries intact. Now, what exactly the usual supersymmetries are depends on the boundary geometry. To explain this point better, we view the $\mathcal{N} = 4$ SYM theory on the boundary as naturally being in a (fixed) background of $\mathcal{N} = 4$ conformal supergravity. How the conformal supergravity multiplet and its transformations arise from the AdS supergravity fields has been studied in detail for $\mathcal{N} = 1, 2$ subsectors in [99, 100]. The Q- and S-supersymmetry transformations of the gravitino in the $\mathcal{N} = 4$ conformal supergravity multiplet schematically take the form

$$\delta_Q \psi_\mu = D_\mu \epsilon_Q + \dots , \quad \delta_S \psi_\mu = i\gamma_\mu \epsilon_S + \dots , \quad (3.35)$$

where the dots denote the contribution from other fields in the multiplet. Holographically, these transformations arise as follows: for a local bulk supersymmetry transformation parametrized by a bulk spinor ϵ , the two classes of transformations arise from the two chiral components of ϵ w.r.t. the operator we called $\Gamma_\rho \Gamma_{\text{AdS}}$ above [99, 100]. A quick way to make our point is to compare this to the transformation for four-dimensional (non-conformal) Poincaré and AdS supergravities. They take the form $\delta\psi_\mu = D_\mu \epsilon$ for Poincaré

and $\delta\psi_\mu = D_\mu\varepsilon - i\gamma_\mu\varepsilon$ for AdS supergravities. If we now break conformal symmetry on Minkowski space, we expect to preserve those conformal supergravity transformations which correspond to the former, for AdS₄ those corresponding to the latter. From eq. (3.35) we see that the Poincaré supergravity transformations arise purely as Q-supersymmetries. So holographically we expect a simple chirality projection on the bulk Killing spinor, of the form $\Gamma_\rho\Gamma_{\text{AdS}}\varepsilon = \varepsilon$, to give the supersymmetries preserved by a massive D7 embedding, and this is indeed the case. For an AdS₄ background, on the other hand, the transformations arise as a particular combination of Q- and S-supersymmetries of the background $\mathcal{N} = 4$ conformal supergravity multiplet. That means we need both chiral components of the bulk spinor, with specific relations between them. This is indeed reflected in our projector eq. (4.55).⁴

3.2.4 Finite mass embeddings

We now turn to the full non-linear κ -symmetry condition, i.e. with full slipping mode and gauge-field dependence. From the linearized analysis we will take the precise form of ω given in eq. (3.28), and the projection conditions on the constant spinors ϵ_0 , eq. (3.20), and eq. (4.55). We will assume that ψ is constant, and then set $\psi = 0$ w.o.l.g. whenever explicit expressions are given.

We have two overall factors in the definition of $\Gamma^{(0)}$ and Γ_κ , and we pull those out by defining

$$h(\rho) := \sqrt{1 + \theta'^2} \sqrt{\det(1 + X)} = \sqrt{1 + 4f^2 \csc^4 \theta} \sqrt{1 + \theta'^2 + f'^2 \csc^2 \theta} . \quad (3.36)$$

For the explicit evaluation we used eq. (3.28). Note that there is no dependence on the S^3

⁴The global fermionic symmetries of $\mathcal{N} = 4$ SYM actually arise from the conformal supergravity transformations as those combinations of Q- and S-supersymmetries which leave the background invariant. A more careful discussion should thus be phrased in terms of the resulting (conformal) Killing spinor equations along similar lines. For a nice discussion of (conformal) Killing spinor equations on curved space we refer to [101].

coordinates in h . We can then write the κ -symmetry condition eq. (3.15) as

$$\left(\mathbf{1} + \frac{1}{8} \gamma^{ijkl} F_{ij} F_{kl} \right) \hat{\Gamma} \epsilon + \frac{1}{2} \gamma^{ij} F_{ij} \hat{\Gamma} C \epsilon^* = -i h \epsilon , \quad (3.37)$$

where we have used $C \hat{\Gamma}^* = \hat{\Gamma} C$ since θ is supposed to be real. This compact enough expression will be our starting point, and we now evaluate the bits and pieces more explicitly. With the expression for ω in eq. (3.28), the F^2 -term evaluates to

$$\frac{1}{8} \gamma^{ijkl} F_{ij} F_{kl} \hat{\Gamma} = \frac{1}{2} \gamma_\rho \gamma_{\chi_1 \chi_2 \chi_3} \hat{\Gamma} \epsilon_{ijk} F^{\rho \chi_i} F^{\chi_j \chi_k} = -2 f' f \csc^3 \theta \Gamma_\rho \Gamma_{\text{AdS}} . \quad (3.38)$$

For the last equality we used $\gamma^\rho \Gamma_{\bar{\chi}} \hat{\Gamma} = \Gamma_\rho \Gamma_{\text{AdS}}$. To evaluate $C \epsilon^*$ in eq. (3.37), we recall the definition of \tilde{R}_{AdS} by $C R_{\text{AdS}} = \tilde{R}_{\text{AdS}} C$ and analogously for R_{S^5} (see above eq. (3.26)), and use eq. (4.55). With eq. (3.38) we then find

$$\hat{\Gamma} \epsilon + \frac{\lambda}{2} \gamma^{ij} F_{ij} \hat{\Gamma} \tilde{R}_{\text{S}^5} \tilde{R}_{\text{AdS}} \tilde{\Gamma} \epsilon_0 = 2 f' f \csc^3 \theta \Gamma_\rho \Gamma_{\text{AdS}} \epsilon - i h \epsilon . \quad (3.39)$$

There are no more S^5 Γ -matrices except for those implicit in ϵ due to R_{S^5} on the right hand side, and also no explicit dependence on the S^3 coordinates. So the remaining task is to find out whether we can dispose of all the non-trivial S^3 dependence and S^5 Γ -matrices on the left hand side with just the projectors we already have derived in Sec. 3.2.3 – the amount of preserved supersymmetry and the form of the Killing spinors is not expected to change when going from infinitesimally small to finite masses.

S³ dependence

To evaluate the left hand side of eq. (3.39) further, we have to work out the term linear in F . With the specific form of ω given in eq. (3.28) we find $(\partial_\alpha \omega_\beta) \gamma^{\alpha\beta} = -2 \csc \theta \omega_\alpha \Gamma_{\bar{\chi}} \gamma^\alpha$, and from eq. (3.24) we then get

$$\frac{1}{2} \gamma^{ij} F_{ij} \hat{\Gamma} = [f' \gamma^\rho - 2 f \csc \theta \Gamma_{\bar{\chi}}] \omega_\alpha \gamma^\alpha \hat{\Gamma} = - \left[f' \Gamma_\rho \Gamma_{\text{AdS}} + 2 f \csc \theta \hat{\Gamma} \right] \Gamma_{\bar{\chi}} \omega_\alpha \gamma^\alpha . \quad (3.40)$$

For the last equality we have used $[\Gamma_{\bar{\chi}} \Gamma_{\underline{\chi}_i}, \hat{\Gamma}] = 0$ and $\gamma^\rho \gamma^\alpha \hat{\Gamma} = -\Gamma_\rho \Gamma_{\text{AdS}} \Gamma_{\bar{\chi}} \gamma^\alpha$. With eq. (3.28) we easily find the generalization of eq. (3.27) to generic θ , and this allows us to

eliminate all explicit S^3 dependence. We have

$$\omega_\alpha \gamma^\alpha \tilde{R}_{S^5} P_0 = -\csc \theta \Gamma^\psi \Gamma_{\tilde{\chi}} R_{S^5} \Gamma^\theta \Gamma^\psi \Gamma^{\chi_1} P_0 . \quad (3.41)$$

Since P_0 commutes with R_{S^5} and $\tilde{\Gamma}$, we can pull it out of ϵ_0 in eq. (3.39) and use it when applying eq. (3.41). When acting on ϵ_0 as in eq. (3.39), we thus find

$$\frac{1}{2} \gamma^{ij} F_{ij} \hat{\Gamma} \tilde{R}_{S^5} = \csc \theta \left[f' \Gamma_{\underline{\rho}} \Gamma_{\text{AdS}} + 2f \csc \theta \hat{\Gamma} \right] \Gamma^\psi R_{S^5} \Gamma^\theta \Gamma^\psi \Gamma^{\chi_1} . \quad (3.42)$$

As desired, the right hand side does not depend on the S^3 coordinates anymore. Using the explicit expression for $\tilde{\Gamma}$ and the massless projector eq. (3.19), we find $\Gamma^\theta \Gamma^\psi \Gamma^{\chi_1} \tilde{R}_{\text{AdS}} \tilde{\Gamma} \epsilon_0 = \tilde{R}_{\text{AdS}} \Gamma_{\underline{\rho}} \epsilon_0$. So we get

$$\frac{1}{2} \gamma^{ij} F_{ij} \hat{\Gamma} \tilde{R}_{S^5} \tilde{R}_{\text{AdS}} \tilde{\Gamma} \epsilon_0 = \csc \theta \left[f' \Gamma_{\underline{\rho}} \Gamma_{\text{AdS}} + 2f \csc \theta \hat{\Gamma} \right] \Gamma^\psi R_{S^5} \tilde{R}_{\text{AdS}} \Gamma_{\underline{\rho}} \epsilon_0 \quad (3.43)$$

$$= \csc \theta \left[f' \Gamma_{\underline{\rho}} \Gamma_{\text{AdS}} \Gamma^\psi - 2f \csc \theta (\Gamma^\theta - \theta' \Gamma_{\underline{\rho}}) \right] R_{S^5} \tilde{R}_{\text{AdS}} \Gamma_{\underline{\rho}} \epsilon_0 . \quad (3.44)$$

For the second equality we have used $\hat{\Gamma} \Gamma^\psi = (\Gamma^\theta - \theta' \Gamma_{\underline{\rho}}) \Gamma_{11}$. The second term in round brackets does not have any S^5 Γ -matrices, and can go to the right hand side of eq. (3.39).

So combining eq. (3.39) with eq. (3.44), we find

$$\begin{aligned} \text{LHS} &:= \hat{\Gamma} \epsilon + \lambda \csc \theta \left[f' \Gamma_{\underline{\rho}} \Gamma_{\text{AdS}} \Gamma^\psi - 2f \csc \theta \Gamma^\theta \right] R_{S^5} \tilde{R}_{\text{AdS}} \Gamma_{\underline{\rho}} \epsilon_0 \\ &= -i h \epsilon + 2f' f \csc^3 \theta \Gamma_{\underline{\rho}} \Gamma_{\text{AdS}} \epsilon - 2\lambda f \theta' \csc^2 \theta \Gamma_{\underline{\rho}} R_{S^5} \tilde{R}_{\text{AdS}} \Gamma_{\underline{\rho}} \epsilon_0 =: \text{RHS} . \end{aligned} \quad (3.45)$$

Nicely enough, the left hand side is linear in the gauge field and its derivative – it appears non-linearly only on the right hand side.

Identities for S^5 Γ -matrices

With eq. (3.45) we are still not in a position to extract necessary conditions for non-trivial solutions to $\Gamma_\kappa \epsilon = \epsilon$ to exist, due to the coordinate dependence implicit in ϵ through R_{AdS} and R_{S^5} . So, we want to evaluate LHS in eq. (3.45) further, and that needs a bit of work. To eliminate the implicit coordinate dependence we need to act with $R_{S^5}^{-1} R_{\text{AdS}}^{-1}$ on

the left hand side of eq. (3.45). We use $\tilde{\Gamma}_{\rho A}$ defined in eq. (3.32), and define the operator $\mathcal{R}[\Gamma] := R_{S^5}^{-1} \Gamma R_{S^5}$. Noting that $\hat{\Gamma}\epsilon = \left[\Gamma^\psi \Gamma^\theta + \theta' \Gamma_\theta \Gamma_{\tilde{\chi}} \Gamma_{\underline{\rho}} \Gamma_{\text{AdS}} \right] \epsilon$, we then find

$$\begin{aligned} R_{S^5}^{-1} R_{\text{AdS}}^{-1} \hat{\Gamma}\epsilon &= \mathcal{R}[\Gamma^\psi \Gamma^\theta] \epsilon_0 + \theta' \mathcal{R}[\Gamma_\theta \Gamma_{\tilde{\chi}}] \tilde{\Gamma}_{\rho A} \epsilon_0 \\ &= \mathcal{R}[\Gamma_\theta \Gamma_{\tilde{\chi}}] \left[i \cot \theta \cdot \mathbf{1} + \theta' \tilde{\Gamma}_{\rho A} \right] \epsilon_0 - i \csc \theta \epsilon_0 . \end{aligned} \quad (3.46)$$

For the second line we used $\mathcal{R}[\Gamma^\psi \Gamma^\theta] = i \cot \theta \mathcal{R}[\Gamma_\theta \Gamma_{\tilde{\chi}}] - i \csc \theta \Gamma_\theta \Gamma_{\tilde{\chi}}$. This will allow us to evaluate the first term on the left hand side in eq. (3.45). For the second term we use the relation for \tilde{R}_{AdS} of eq. (3.32), to find

$$R_{S^5}^{-1} R_{\text{AdS}}^{-1} \Gamma_{\underline{\rho}} \Gamma_{\text{AdS}} \Gamma^\psi R_{S^5} \tilde{R}_{\text{AdS}} \Gamma_{\underline{\rho}} \epsilon_0 = \mathcal{R}[\Gamma_\theta \Gamma_{\tilde{\chi}}] \left[\cosh \rho \cdot \mathbf{1} - i \sinh \rho \tilde{\Gamma}_{\rho A} \right] \epsilon_0 . \quad (3.47)$$

This will allow us to evaluate the first term in the brackets of the second term on the left hand side in eq. (3.45). For the last term we note that

$$\Gamma_\theta R_{S^5} \tilde{R}_{\text{AdS}} \Gamma_{\underline{\rho}} \epsilon_0 = \Gamma^\psi \Gamma_{\tilde{\chi}} R_{S^5} \left[\cosh \rho \Gamma_{\underline{\rho}} \Gamma_{\text{AdS}} + i \sinh \rho \cdot \mathbf{1} \right] R_{\text{AdS}} \epsilon_0 . \quad (3.48)$$

So we find

$$R_{S^5}^{-1} R_{\text{AdS}}^{-1} \Gamma_\theta R_{S^5} \tilde{R}_{\text{AdS}} \Gamma_{\underline{\rho}} \epsilon_0 = \mathcal{R}[\Gamma^\psi \Gamma_{\tilde{\chi}}] \left[\cosh \rho \tilde{\Gamma}_{\rho A} + i \sinh \rho \cdot \mathbf{1} \right] \epsilon_0 . \quad (3.49)$$

We will now use $\mathcal{R}[\Gamma^\psi \Gamma_{\tilde{\chi}}] = i \csc \theta \mathcal{R}[\Gamma_\theta \Gamma_{\tilde{\chi}}] \Gamma_\theta \Gamma_{\tilde{\chi}} - i \cot \theta \cdot \mathbf{1}$, which gives us

$$R_{S^5}^{-1} R_{\text{AdS}}^{-1} \Gamma_\theta R_{S^5} \tilde{R}_{\text{AdS}} \Gamma_{\underline{\rho}} \epsilon_0 = [\csc \theta \mathcal{R}[\Gamma_\theta \Gamma_{\tilde{\chi}}] - \cot \theta \mathbf{1}] \times \left[i \cosh \rho \tilde{\Gamma}_{\rho A} - \sinh \rho \cdot \mathbf{1} \right] \epsilon_0 . \quad (3.50)$$

We now have the tools needed to identify the parts with non-trivial dependence on the internal space on the left hand side of eq. (3.45).

Solving the κ -symmetry condition

With eqs. (3.46), (3.47) and (3.50) derived above, we are now in a position to evaluate the κ -symmetry condition eq. (3.45). For the left hand side we find

$$\begin{aligned} R_{\text{AdS}}^{-1} R_{S^5}^{-1} \text{LHS} &= (i \cot \theta + \lambda \csc \theta f' \cosh \rho + 2f\lambda \csc^3 \theta \sinh \rho) \mathcal{R}[\Gamma_\theta \Gamma_{\tilde{\chi}}] \epsilon_0 \\ &\quad + (\theta' - i\lambda \csc \theta f' \sinh \rho - 2if\lambda \csc^3 \theta \cosh \rho) \mathcal{R}[\Gamma_\theta \Gamma_{\tilde{\chi}}] \tilde{\Gamma}_{\rho A} \epsilon_0 \\ &\quad - i \csc \theta \epsilon_0 + 2f\lambda \csc^2 \theta \cot \theta \left(i \cosh \rho \tilde{\Gamma}_{\rho A} - \sinh \rho \cdot \mathbf{1} \right) \epsilon_0 , \end{aligned} \quad (3.51)$$

where $\tilde{\Gamma}_{\rho A}$ was defined in eq. (3.32). Note that the RHS in eq. (3.45) has no dependence on the S^3 -directions, but $\mathcal{R}[\Gamma^\theta \Gamma_{\tilde{\chi}}]$ in eq. (3.51) does. So the coefficients of the two terms involving $\mathcal{R}[\Gamma^\theta \Gamma_{\tilde{\chi}}]$ in eq. (3.51) have to vanish. Moreover, they have to vanish separately, since they multiply different AdS₅ Γ -matrix structures. So we get

$$i \cot \theta + \lambda \csc \theta f' \cosh \rho + 2f\lambda \csc^3 \theta \sinh \rho = 0 , \quad (3.52a)$$

$$\theta' - i\lambda \csc \theta f' \sinh \rho - 2if\lambda \csc^3 \theta \cosh \rho = 0 . \quad (3.52b)$$

The non-trivial Clifford-algebra structure of the κ -symmetry condition has thus given us two independent 1st-order differential equations. Moreover, since f and f' only appear linearly, we can actually solve eq. (3.52) for f and f' . This yields

$$f = \frac{i}{2\lambda} \sin^3 \theta (\sinh \rho \cot \theta - \theta' \cosh \rho) , \quad f' = \frac{i}{\lambda} (\theta' \sin \theta \sinh \rho - \cosh \rho \cos \theta) . \quad (3.53)$$

Note that the expression for f' does not contain second-order derivatives of θ , which we would get if we just took the expression for f and differentiate. Comparing the expressions for f and f' , we can thus derive a second-order ODE for θ alone, with no more gauge fields. It reads

$$\theta'' + 3\theta'^2 \cot \theta + 4 \tanh \rho \theta' - \cot \theta (1 + 2 \csc^2 \theta) = 0 . \quad (3.54)$$

With the solutions for f and f' s.t. the first two lines of eq. (3.51) vanish, the κ -symmetry condition eq. (3.45) simplifies quite a bit. Collecting the remaining terms according to their Γ -matrix structure gives

$$\begin{aligned} 0 = & [ih - i \csc \theta + 2f\lambda \csc^2 \theta (\theta' \cosh \rho - \cot \theta \sinh \rho)] \epsilon_0 \\ & - 2f \csc^2 \theta [f' \csc \theta + i\lambda (\theta' \sinh \rho - \cot \theta \cosh \rho)] \tilde{\Gamma}_{\rho A} \epsilon_0 . \end{aligned} \quad (3.55)$$

With the solution for f' in terms of θ given in eq. (3.53), we see that the term in square brackets in the second line vanishes exactly if λ is purely imaginary, s.t. $\lambda^{-1} = -\lambda$. So we are left with the first line only. This once again vanishes when plugging in the explicit expressions of eq. (3.36) and eq. (3.53), and using imaginary λ . So, any solution for the

slipping mode satisfying eq. (3.54) which is accompanied by the gauge field eq. (3.53) gives a supersymmetric D7-brane embedding into AdS₄-sliced AdS₅. These equations are our first main result.

As a consistency check, one wants to verify that each such combination of slipping mode satisfying eq. (3.54) with gauge field eq. (3.53) indeed satisfies the highly non-linear and coupled equations of motion resulting from the D7-brane action. To derive those, we first express eq. (3.7) explicitly in terms of θ and f . That is, we use $A = f\omega$ with ω given in eq. (3.28), but not any of the other κ -symmetry relations. Also, for ω we only use that our ω satisfies $\star_{S^3}d\omega = -2\omega$, i.e. that we found an $\ell = 1, -$ mode in the language of [97]. The combination of DBI action and WZ term then becomes

$$S_{D7} = -T_7 V_{S^3} \int d^5\xi \sqrt{g_{\text{AdS}_4}} \left[\zeta' \sqrt{\sin^4\theta + 4f^2} \sqrt{f'^2 + (1 + \theta'^2) \sin^2\theta} + 8\zeta f' f \right], \quad (3.56)$$

where $\zeta' = \cosh^4\rho$ as defined in eq. (4.5), and we have integrated over the S^3 . Working out the resulting equations of motion gives two coupled second-order non-linear equations. In the equation for the slipping mode one can at least dispose of the square root, by a suitable rescaling of the equation. But for the gauge field even that is not possible, due to the WZ term. The resulting equations are bulky, and we will not spell them out explicitly. Finding an analytic solution to these equations right away certainly seems hopeless. But we do find that using eq. (3.53) to replace f , along with replacing θ'' using eq. (3.54) actually solves both of the equations of motion resulting from eq. (3.56).

3.2.5 Solutions

We now have a decoupled equation for the slipping mode alone in eq. (3.54), and an immediate solution for the accompanying f in eq. (3.53). So it does not seem impossible to find an explicit solution for the embedding in closed form. To simplify eq. (3.54), we reparametrize the slipping mode as $\cos\theta(\rho) = 2 \cos\left(\frac{1}{3} \cos^{-1} \tau(\rho)\right)$, which turns it into a simple linear equation for τ , namely

$$\tau'' + 4 \tanh \rho \tau' + 3\tau = 0. \quad (3.57)$$

This can be solved in closed form, and as a result we get three two-parameter families of solutions for θ , corresponding to the choice of branch for the \cos^{-1} . Restricting \cos^{-1} to the principle branch where it takes values in $[0, \pi]$, we can write them as

$$\theta = \cos^{-1} \left(2 \cos \frac{2\pi k + \cos^{-1} \tau}{3} \right), \quad \tau = \frac{6(m\rho - c) + 3m \sinh(2\rho)}{4 \cosh^3 \rho}, \quad (3.58)$$

with $k \in \{0, 1, 2\}$. Only $k = 2$ gives real θ , though: To get real θ , we need $|\cos \frac{2\pi k + \cos^{-1} \tau}{3}| \leq \frac{1}{2}$. This translates to $\cos^{-1} \tau \in [\pi, 2\pi] + (3n - 2k)\pi$. Since we have chosen the branch with $\cos^{-1} \tau \in [0, \pi]$ in eq. (3.58), this only happens for $k = 2$. For $\rho \rightarrow \infty$ we then have $\tau \rightarrow 0$ and $\theta \rightarrow \frac{\pi}{2}$, so the branes wrap an equatorial S^3 in the S^5 . As ρ is decreased, τ increases and the branes potentially cap off – we need $|\tau| \leq 1$ to have real θ . The remaining constant c may then be fixed from regularity constraints, and we will look at this in more detail below. So these are finally the supersymmetric embeddings we were looking for: the slipping mode θ given in eq. (3.58) with $k = 2$, accompanied by the gauge field $A = f\omega$, with f given eq. (3.53) and ω in eq. (3.28). The naming of the constants is anticipating our results for the one-point functions in eq. (3.86) below: m will be the flavor mass in the boundary theory and c will appear in the chiral condensate.

3.3 Topologically distinct classes of embeddings

In the previous section we have obtained the general solution to the κ -symmetry condition, giving the two-parameter family of embeddings in eq. (3.58) with the accompanying gauge field eq. (3.53). In this section we will study the parameter space (m, c) , and whether and where the branes cap off depending on these parameters. A crucial part in that discussion will be demanding regularity of the configurations, e.g. that the worldvolume of the branes does not have a conical singularity and a similar condition for the worldvolume gauge field.

To cover either of global or Poincaré AdS_5 with AdS_4 slices, we need two coordinate patches with the corresponding choice of global or Poincaré AdS_4 slices, as illustrated in Fig. 3.10. They can be realized by just letting ρ run through the entire \mathbb{R} . The figure illustrates global AdS , but we do not need to commit to one choice at this point. For the

massless embeddings we know from Poincaré AdS, where $\theta \equiv \pi/2$ is the known supersymmetric solution, the D7 branes wrap all of the AdS₅ part. For the massive case, again from Poincaré-AdS₅ intuition, we naively expect this to be different. However, that discussion will turn out to be more nuanced for AdS₄-sliced AdS₅.

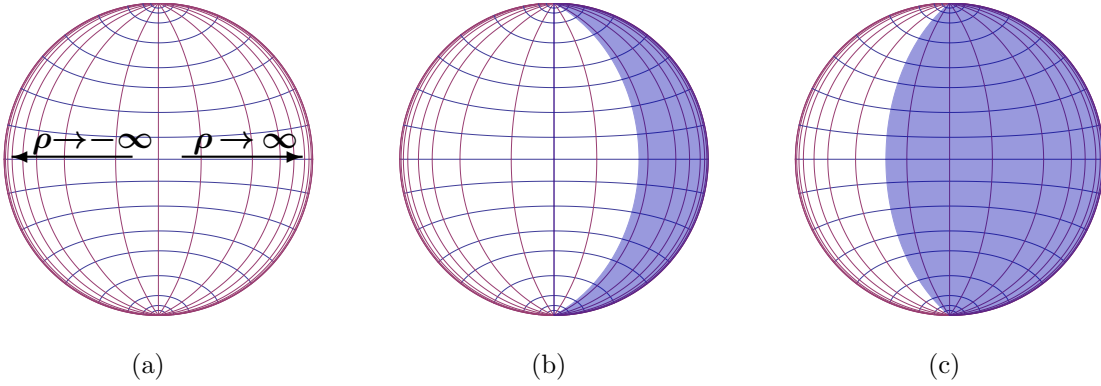


Figure 3.1: The left hand side illustrates how global AdS₅ is sliced by AdS₄ slices. The vertical lines are slices of constant ρ , the horizontal ones correspond to constant AdS₄ radial coordinate. Connected embeddings cover the entire AdS₅ and stretch out to the conformal boundary in both patches. The figure in the middle illustrates what we call a short embedding, where the branes, wrapping the shaded region, stretch out to the conformal boundary in one patch and then cap off at a finite value of the radial coordinate in the same patch. On the right hand side is what we call a long embedding, where the branes stretch out to one half of the conformal boundary, cover the entire patch and cap off in the other one.

The two options for the branes to cap off are the (arbitrarily assigned) north and south poles of the S⁵, which we take as $\theta = 0$ or $\tau = -1$ and $\theta = \pi$ or $\tau = 1$, respectively. With eq. (3.58), the condition for the branes to cap off at the north/south pole at $\rho = \rho_*$ then becomes

$$c = m\rho_* + \frac{1}{2}m \sinh(2\rho_*) \pm \frac{2}{3} \cosh^3 \rho_* =: c_{n/s}(\rho_*). \quad (3.59)$$

Due to the non-linear nature of the equation, it is a bit tricky to get closed-form expressions e.g. for ρ_* . A fruitful way to look at this equation is as follows: for the branes with a given m to cap off at the north/south pole at $\rho = \rho_*$, we have to fix $c = c_{n/s}(\rho_*)$.

There is no a priori relation between the masses we choose for the two patches, so we start the discussion from one patch, say the one with $\rho \geq 0$. For $\rho \rightarrow \infty$ we have $\theta \rightarrow \pi/2$, and what happens as we move into the bulk depends on whether and what sort of solutions ρ_* there are to eq. (3.59). Depending on m and c , we can distinguish 3 scenarios:

- (i) There is a $\rho_* \geq 0$ such that either $c = c_n(\rho_*)$ or $c = c_s(\rho_*)$. In that case, the branes cap off in the patch in which they started, and we call this a short embedding.
- (ii) There is no $\rho_* \geq 0$ as above, but there is a $\rho_* < 0$ such that $c = c_n(\rho_*)$ or $c = c_s(\rho_*)$. In that case, the branes cover the entire $\rho > 0$ patch and part of the $\rho < 0$ patch. We call this a long embedding.
- (iii) We have $c \neq c_n(\rho_*)$ and $c \neq c_s(\rho_*)$ for all $\rho_* \in \mathbb{R}$. In that case the branes never reach either of the poles and do not cap off at all. They cover all of AdS_5 , connecting both parts of the conformal boundary. We call these connected embeddings.

For the cases where the branes do cap off, we want to ensure that they do so in a regular way, and this will impose an additional constraint. For the connected embeddings we get a relation between the flavor masses on the two copies of AdS_4 . For the disconnected embeddings we can potentially look at arbitrary combinations of long and short embeddings in each of the patches. The embeddings are illustrated in Fig. 3.10, and we will study them in more detail below.

3.3.1 Cap-off points

If a cap-off point exists at all, we want the branes to cap off smoothly. The relevant piece of the induced metric to check for a conical singularity is $g = (1 + \theta'^2)d\rho^2 + \sin^2 \theta d\Omega_3^2 + \dots$. Expanding around $\rho = \rho_*$ with eq. (3.58) and eq. (3.59) gives

$$\sin \theta \Big|_{c=c_{n/s}^*(\rho_*)} = \alpha_{n/s}(\rho - \rho_*)^{1/4} + \mathcal{O}(\sqrt{\rho - \rho_*}) , \quad \alpha_{n/s} = \sqrt[4]{8 \operatorname{sech} \rho_* (\sinh \rho_* \pm m)} . \quad (3.60)$$

The induced metric with that scaling is smooth without a conical singularity. For the gauge field and the on-shell action the story is a bit more interesting.

So we examine the regularity of the gauge field $A = f\omega$. We fix $\chi_1 = \chi_2 = \pi/2$, s.t. we look at a plane around $\rho = \rho_*$ and the pullback of the gauge field to the plane is $A = fd\chi_1$. To have this smooth at the origin, $\rho = \rho_*$, we need f to vanish there. For small $\sin\theta$, we see from eq. (3.53) that

$$f = -\frac{i}{2\lambda} \cosh\rho \theta' \sin^3\theta + \mathcal{O}(\sin^2\theta) . \quad (3.61)$$

From the expansion eq. (3.60), we then find the condition for f to vanish at $\rho = \rho_*$ when the branes cap off at the north/south pole

$$\theta' \sin^3\theta \Big|_{\rho=\rho_*} = \pm \frac{1}{4} \alpha_{\text{n/s}}^4 \stackrel{!}{=} 0 \quad \iff \quad \rho_* = \rho_{\text{n/s}} , \quad \rho_{\text{n/s}} = \mp \sinh^{-1}m . \quad (3.62)$$

We thus find that for any given m there are in principle two options for the branes to cap off smoothly. For positive m they can potentially cap off smoothly at the south pole in the $\rho > 0$ patch and at the north pole in the $\rho < 0$ patch, and for negative m the other way around. With $\alpha_{\text{n/s}}$ fixed like that, the slipping mode shows the usual square root behavior as it approaches the north/south pole.

The constraint eq. (3.62) can also be obtained by looking at the on-shell action. We know from the κ -symmetry discussion that the combination in square brackets in the first line of eq. (3.55) vanishes. By slightly reorganizing the terms, this gives us

$$h = \csc\theta (1 - 4\lambda^2 f^2 \csc^4\theta) . \quad (3.63)$$

This allows us to get rid of the square root in the on-shell DBI Lagrangian, which will also be useful for the discussions below. The DBI Lagrangian of eq. (3.7) expressed in terms of h reads

$$L_{\text{DBI}} = -T_7 \cosh^4\rho \sin^3\theta \cdot h \cdot \sqrt{g_{\text{AdS}_4}} \sqrt{g_{\text{S}^3}} , \quad (3.64)$$

where $\sqrt{g_{\text{AdS}_4}}$ and $\sqrt{g_{\text{S}^3}}$ are the standard volume elements on AdS_4 and S^3 of unit curvature radius, respectively. For the full D7-brane action eq. (3.7), we then find

$$S_{\text{D7}} = -T_7 \int d^7\xi d\rho \sqrt{g_{\text{AdS}_4}} \sqrt{g_{\text{S}^3}} [h \cosh^4\rho \sin^3\theta + 8\zeta f'f] , \quad (3.65)$$

To have the first term in square brackets not blow up at $\rho = \rho_*$, we once again need $f(\rho_*) = 0$, leading to the constraint in eq. (3.62). So once again, the branes only have two options to cap off smoothly, and we will look at the embeddings in more detail now.

3.3.2 Short embeddings

The first option we want to discuss are short embeddings, where the branes cap off in the same patch in which they reach out to the conformal boundary. Our expectation would be that these always exist. Say we look at the $\rho > 0$ patch. The discussion for the other patch is analogous, and we will stick with that one. For positive/negative m , we just take $\rho_{s/n}$ from eq. (3.62) and fix

$$c = c_s(\rho_s) \quad \text{for } m > 0, \quad c = c_n(\rho_n) \quad \text{for } m < 0, \quad (3.66)$$

with $c_{n/s}$ defined in eq. (3.59). This gives a smooth cap-off point at $\rho = \rho_{s/n} \geq 0$ – in the same patch where we assumed the D7 branes to extend to the conformal boundary. For the other patch the choices are simply reversed.

There is a slight subtlety with that, though, which gives us some insight into the curves $c_{s/n}(\rho_*)$. For the embeddings to really be smooth, there must be no other cap-off points between the conformal boundary and our smooth cap-off point, i.e. no cap-off points for $\rho > \rho_{s/n}$. This is indeed not the case, which can be seen in a nice way from eq. (3.59) as follows. For any given (m, c) , the solutions ρ_* to eq. (3.59) are cap-off points. So we want to look at the curves $c_{s/n}(\rho_*)$ as functions of ρ_* . The specific values $\rho_{n/s}$ happen to also be the only extrema of the curves $c_{n/s}$, respectively. That is, c_n has a minimum at ρ_n and c_s has a maximum at ρ_s . In the patch where $c_{n/s}$ take their minimum/maximum, respectively, $c_{s/n}$ is always strictly smaller/greater than $c_{n/s}$. This ensures that there are no cap-off points in between. See Fig. 3.2(a) for an illustration. In the (m, c) -plane, these embeddings correspond to the thick solid lines in Fig. 3.2(b). We will complete filling in the details of Fig. 3.2 in the next subsections. What we can tell already is what happens when we depart from the choice eq. (3.66) for large masses. From Fig. 3.2(a), we see that for large enough masses there will

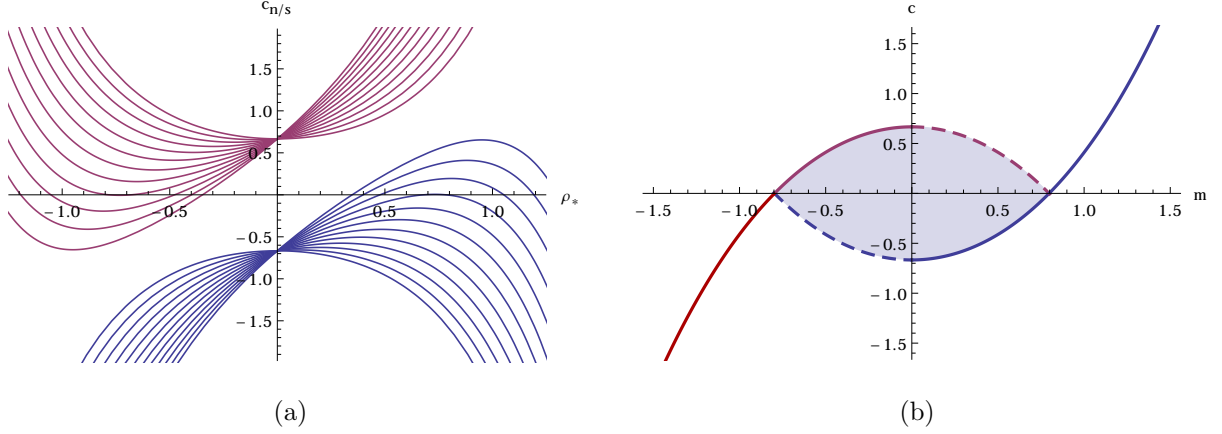


Figure 3.2: The plot on the left hand side shows c_n and c_s , as defined in eq. (3.59), as the family of red upper and blue lower curves, respectively. The symmetric curves are $m = 0$ for both, and as m is increased the curves tilt. For $m = 0$ and $|c| \leq \frac{2}{3}$, there is no solution ρ_* to eq. (3.59), and the branes do not cap off. As m is increased, the maximum value taken on the lower curve increases and the minimum taken on the upper curve decreases. So the window for c to get continuous embeddings shrinks. The plot on the right hand side shows the smooth brane embeddings in the (m, c) plane. For large m these are given by the thick blue curve only, which corresponds to the disconnected embeddings. For lower m the blue-shaded region is possible and corresponds to connected embeddings. On the dashed lines the embedding covers all of one patch and caps off smoothly in the other one. The $\theta \equiv \frac{\pi}{2}$ embedding corresponds to $c = m = 0$. \mathbb{Z}_2 -symmetric connected embeddings correspond to the axes, i.e. $c = 0$ or $m = 0$ in this plot.

be solutions ρ_* to eq. (3.59) for any real c (since $\text{Im } c_n \cup \text{Im } c_s = \mathbb{R}$). However, that cap off will not be regular in the sense discussed above around eq. (3.62). The discussion for small masses is more interesting, and we will come to that below.

So, what we found in this section are indeed the short embeddings illustrated in Fig. 3.1(b). As we will see below, these are the only ones which exist for any choice of m . With these embeddings, the masses in the two patches can be chosen completely independently. Some sample configurations are shown in Fig. 3.3.

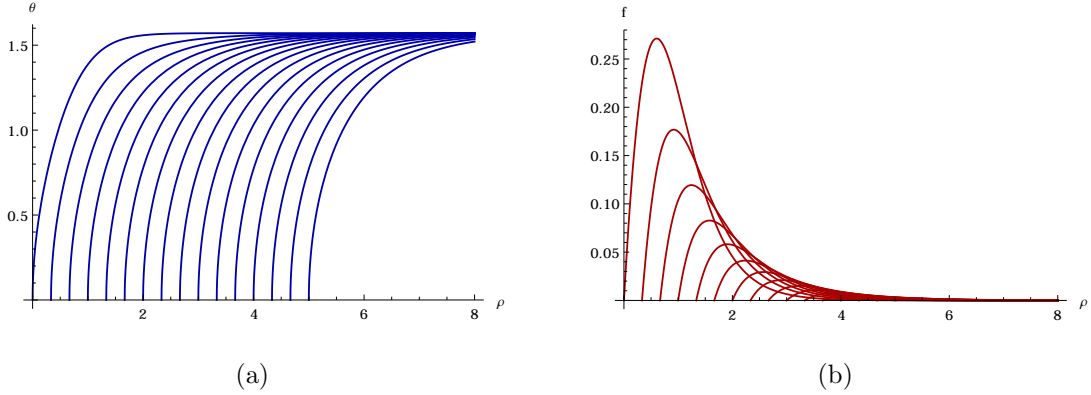


Figure 3.3: Slipping mode on the left hand side and the accompanying gauge field on the right hand side for short embeddings with masses m in $[0, 5]$, as function of the radial coordinate ρ . The smaller the mass, the deeper the branes extend into the bulk. For $m = 0$ we find an embedding which caps off at $\rho = 0$, which is the boundary to the other patch. So we already see that, with $\theta \equiv \pi/2$, there are at least two massless embeddings.

3.3.3 Long embeddings

We have seen in the previous section that there is at least one smooth D7-brane embedding for any choice of m . In this section we will start to look at less generic configurations, which turn out to be possible for small enough masses only. The plots in Fig. 3.2(a) already indicate that small m is special, and we will look at this in more detail now.

We go back to eq. (3.59) and Fig. 3.2(a). For small enough mass, the maximum of the lower curve in Fig. 3.2(a), which is c_s , is strictly smaller than the minimum of the upper curve, which is c_n . We denote the critical value of the mass, below which this happens, by m_ℓ . It can be characterized as the mass for which the maximum of c_s is equal to the minimum of c_n , which translates to

$$c_s(\rho_s) = c_n(\rho_n) \quad \Longleftrightarrow \quad \sqrt{m_\ell^2 + 1} (m_\ell^2 - 2) + 3m_\ell \sinh^{-1} m_\ell = 0, \quad (3.67)$$

or $m_\ell \approx 0.7968$. For $m < m_\ell$, we can make the opposite choice for c as compared to eq. (3.66) and still get a smooth cap-off point. As discussed above, if we were to reverse the

choice for c for larger mass, the branes would hit a non-smooth cap-off point before reaching the other patch. But for $m < m_\ell$ we can fix

$$c = c_n(\rho_n) \quad \text{for } m > 0, \quad c = c_s(\rho_s) \quad \text{for } m < 0. \quad (3.68)$$

There is no cap-off point in the patch in which we start, so the branes wrap all of it. In the second patch they do not stretch out to the conformal boundary, but rather cap off smoothly at $\rho_{n/s}$ for positive/negative m . This is what we call a long embedding, as illustrated in Fig. 3.1(c). In Fig. 3.2(b) those correspond to the dashed thick lines. A sample plot for $m = \frac{1}{2}$ can be found in Fig. 3.4. For $c_s(\rho_s) < c < c_n(\rho_n)$ we get connected embeddings, which we will discuss in the next section. For the long embeddings the maximal mass m_ℓ translates to a maximal depth up to which the branes can extend into the second patch.

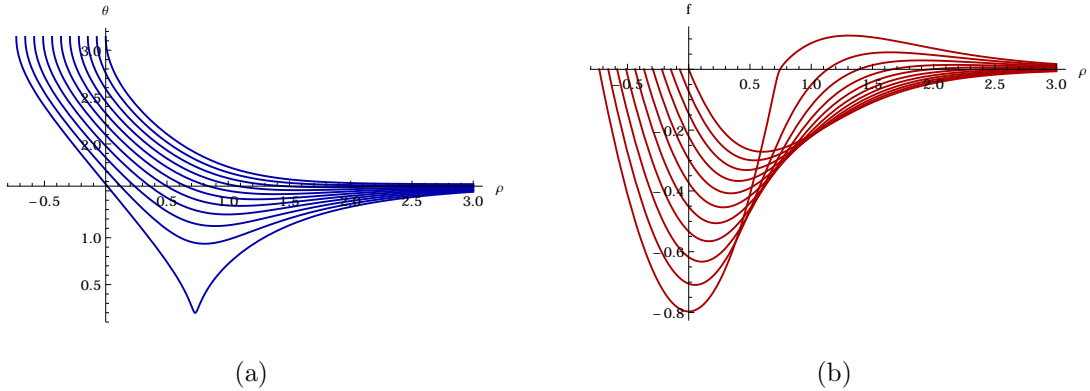


Figure 3.4: The left hand side shows the slipping mode for long embeddings, from top to bottom corresponding to the mass m increasing from 0 to m_ℓ , as function of the radial coordinate ρ . The right hand side shows the accompanying gauge field. As compared to the short embeddings with the same mass, the long ones cap off on the other pole. We see that there is a maximal depth up to which the long embeddings can extend into the second patch. The sharp feature developing for the bottom curve on the left hand side turns into a cap-off for a short embedding as we approach the critical mass m_ℓ , given in eq. (3.67). This way the plot connects to Fig. 3.1(b). In the (m, c) plane of Fig. 3.2(b), this corresponds to following one of the thick solid lines coming from large $|m|$, and then at m_ℓ switching to the dashed line instead of further following the solid one.

Thinking of holographic applications, this offers interesting possibilities for adding flavors in both patches. In addition to the short-short embeddings discussed above, which can be realized for arbitrary combinations of masses, we now get the option to combine a short embedding in one patch with a long one in the other. Moreover, we could combine two long embeddings, which would realize partly overlapping stacks of D7-branes from the AdS₅ perspective. Whether the branes actually intersect would depend on the chosen mass in each of the patches: For masses of the same sign, they can avoid each other – they cap off at different poles on the S⁵ and do not actually intersect. But for masses of opposite sign they would have to intersect. See also Fig. 3.5.

3.3.4 Connected embeddings

The last class of embeddings we want to discuss are the connected ones, which cover all of AdS₅, including both parts of the conformal boundary. In contrast to Poincaré AdS₅, where finite-mass embeddings always cap off, such embeddings exist for non-zero masses. The critical value is the same m_ℓ given in eq. (3.67) for the long embeddings. As discussed in the section above, for $m < m_\ell$ the images of c_n and c_s do not cover the entire \mathbb{R} , which means there are choices of c for which there is no ρ_* to satisfy eq. (3.59), and thus no cap-off points. These are given by

$$c_s(\rho_s) < c < c_n(\rho_n) , \quad (3.69)$$

where $c_{n/s}$ were defined in eq. (3.59) and $\rho_{n/s}$ in eq. (3.62). With no cap-off points there are no regularity constraints either, and these are accepted as legitimate embeddings right away. Due to the very fact that the embeddings are connected, we immediately get a relation between the masses in the two patches: they have the same modulus but opposite signs.

One of the open questions is how the massive embeddings for AdS₄-sliced AdS₅ connect to the massless $\theta \equiv \pi/2$ embedding with $f \equiv 0$. This embedding, which we know and appreciate from Poincaré-AdS₅, clearly is a solution regardless of the chosen coordinates on AdS₅. So far, we have only seen massless embeddings capping off at either of the poles (see

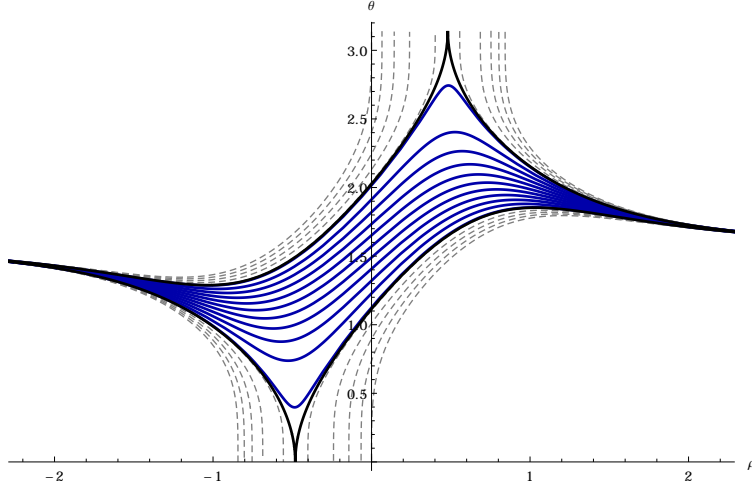


Figure 3.5: Connected, long and short embeddings with $m = \frac{1}{2}$ in the $\rho > 0$ patch. The shown configurations correspond to a vertical section through the phase diagram shown in Fig. 3.2(b). The thick blue lines are a family of embeddings with values of c satisfying eq. (3.69). The solid black lines show the disconnected limiting embeddings: the upper/lower ones correspond to positive/negative c saturating the inequalities in eq. (3.69). The gray dashed lines correspond to embeddings with irregular cap off. For the limiting case where the branes cap off, we get to choose among short/short, long/short, short/long and long/long embeddings. We see how the branes do not intersect for this choice of masses, even for the long/long embedding.

Fig. 3.3 and 3.4). The $\theta \equiv \pi/2$ embedding is at the origin in Fig. 3.2(b), and the blue-shaded region around it are the connected embeddings. The embeddings corresponding to a vertical section through Fig. 3.2(b) are shown in Fig. 3.5. For the figure we chose $m = \frac{1}{2}$ in the $\rho > 0$ patch and $m = -\frac{1}{2}$ in the $\rho < 0$ patch, such that the connected embeddings exist. As m is decreased, the embeddings become more and more symmetric around $\rho = 0$, and we eventually find the $\theta \equiv \pi/2$ embedding. This will be seen more explicitly in the next section.

For each chosen m , one can ask which embedding among those with different c has the minimal action. The on-shell action as given in eq. (3.65) is divergent, as common for this quantity on asymptotically-AdS spaces. But the divergences are given purely in terms of m , and do not depend on c . So a simple background subtraction will be sufficient for now. We

will return to performing the holographic renormalization with all the bells and whistles in Sec. 3.4. For now we simply define the finite quantity

$$\delta S_{D7}(m, c) = S_{D7}(m, c) - S_{D7}(m, 0) . \quad (3.70)$$

Strictly speaking, δS_{D7} is still divergent due to the infinite volume of the AdS_4 slices. But this is a simple overall factor and we indeed just look at the “action density”, with the volume of AdS_4 divided out. Using $\lambda^2 = -1$, the explicit expression for h in eq. (3.63) and integration by parts, we can further simplify the action eq. (3.65) to

$$S_{D7} = -T_7 \int d^7 \xi \sqrt{g_{\text{AdS}_4}} \sqrt{g_{S^3}} \left[[4\zeta f^2] + \int d\rho \cosh^4 \rho (\sin^2 \theta + 4f^2 \cot^2 \theta) \right] . \quad (3.71)$$

To evaluate this further we introduce $\sigma = \cos \theta$, such that $\sigma = 2 \cos \frac{4\pi + \cos^{-1} \tau}{3}$. We can then isolate the volume divergence, which is there regardless of the value for m and c , and write

$$S_{D7} = -T_7 \int d^7 \xi \sqrt{g_{\text{AdS}_4}} \sqrt{g_{S^3}} \left[[\zeta(1 + 4f^2)] - \int d\rho \zeta' \sigma^2 \left(1 - (1 - \sigma^2) (\sigma \cosh \rho)^2 \right) \right] . \quad (3.72)$$

The prime in the last term denotes a derivative w.r.t. ρ . Prepared like that, δS_{D7} as defined in eq. (3.70) is easily evaluated numerically, and we find the plot shown in Fig. 3.6.

3.3.5 \mathbb{Z}_2 -symmetric configurations

In the last section for this part we look at a special class of configurations with an extra \mathbb{Z}_2 symmetry relating the two patches. The slipping mode may be chosen either even or odd under the \mathbb{Z}_2 transformation exchanging the two patches, and we can see from eq. (3.53) that the gauge field f consequently will have the opposite parity, i.e. odd and even, respectively.

For the disconnected embeddings, the extra symmetry simply tells us how to combine the embeddings for the two patches. It narrows the choices down to either short/short or long/long, and depending on the parity the long/long embeddings will be intersecting or not. So let us come to the connected embeddings. For the $k = 2$ solutions in eq. (3.58), we find

$$\theta(\rho) = \theta(-\rho) \Big|_{m \rightarrow -m} = \pi - \theta(-\rho) \Big|_{c \rightarrow -c} , \quad f(\rho) = f(-\rho) \Big|_{c \rightarrow -c} = -f(-\rho) \Big|_{m \rightarrow -m} . \quad (3.73)$$

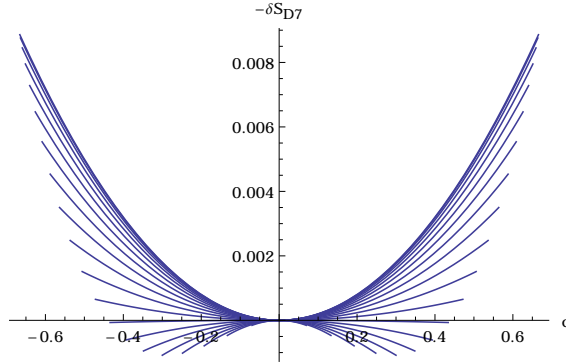


Figure 3.6: The plot shows $-\delta S_{D7}$ defined in eq. (3.70), as function of the parameter c controlling the chiral condensate. This corresponds to the free energy with the $c = 0$ value subtracted off. The quantity δS_{D7} then is independent of the chosen renormalization scheme. From top to bottom the curves correspond to increasing $|m|$. For $m = 0$, corresponding to the top curve, the $\theta \equiv \pi/2$ embedding with $c = 0$ is the one with lowest energy. As $|m|$ is increased, this changes, seemingly through a 1st-order phase transition, and the “marginal” embeddings with maximal allowed c become the ones with minimal free energy.

We can immediately tell that for connected solutions, where smoothness forces us to choose the same solution in both patches, imposing even \mathbb{Z}_2 parity corresponds to $m = 0$ and imposing odd \mathbb{Z}_2 parity to $c = 0$. From the connected \mathbb{Z}_2 -even configurations we therefore get an entire family of massless solutions. They correspond to the vertical axis in Fig. 3.2(b). The \mathbb{Z}_2 -odd solutions with $c = 0$ loosely correspond to vanishing chiral condensate in $\mathcal{N} = 4$ SYM, but that statement depends on the chosen renormalization scheme, as we will see below.

With these results in hand, we can now understand how the connected embeddings can be connected to the short or long disconnected embeddings discussed above. Say we assume \mathbb{Z}_2 symmetry for a start, which for the connected embeddings confines us to the axes in Fig. 3.2(b). That still leaves various possible trajectories through the (m, c) diagram.

If we pick an even slipping mode, we are restricted to the vertical axis for connected embeddings. Starting out from large mass and the short embeddings, one could follow the thick lines in Fig. 3.2(b) all the way to $m = 0$, where the cap-off point approaches $\rho_\star = 0$.

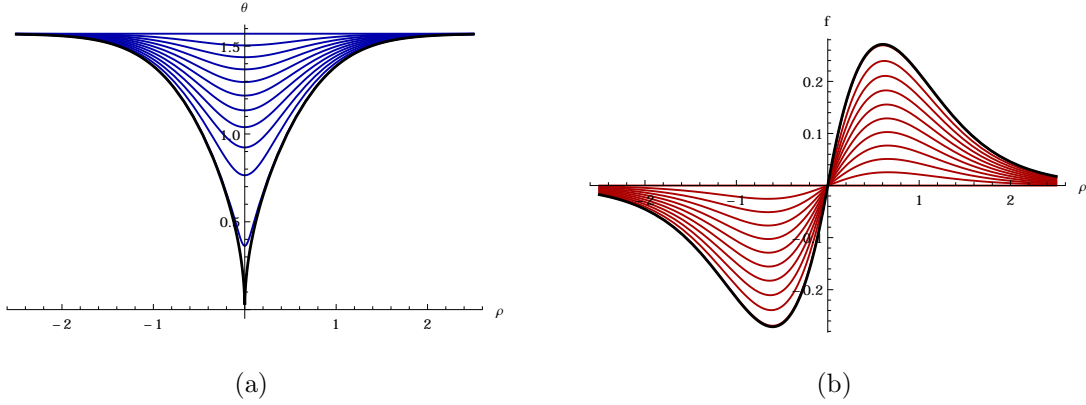


Figure 3.7: Embeddings with \mathbb{Z}_2 -even slipping mode on the left hand side and the accompanying \mathbb{Z}_2 -odd gauge field on the right hand side, as function of the radial coordinate ρ . The $\theta \equiv \frac{\pi}{2}$ solution corresponds to the flat slipping mode and $f \equiv 0$. The lower the curve on the left hand side, the larger c . On the right hand side larger c corresponds to a larger peak. We have restricted to $c > 0$ for the plot, the $c < 0$ embeddings are obtained by a reflection on the equator of the S^5 for the embedding and a sign flip on the gauge field. Note how the solutions interpolate between $\theta \equiv \frac{\pi}{2}$ and the massless solution capping off at $\rho_* = 0$ discussed in Sec. 3.3.2, which is shown as thick black line in both plots.

Another option would be to change to the dashed line at $m = m_\ell$, corresponding to a long embedding. In either case, once we hit the vertical axis in Fig. 3.2(b), this corresponds to the massless embedding with maximal $|c|$. From there one can then go along the vertical axis to the $\theta \equiv \pi/2$ embedding at the origin. This interpolation is shown in Fig. 3.7(a).

If we pick an odd slipping mode, we are restricted to the horizontal axis in Fig. 3.2(b) for connected embeddings. Coming in again from large mass and a short embedding, the thick line eventually hits $c = 0$ as the mass is decreased. From there the branes can immediately go over to a connected embedding, which corresponds to going over from the thick black line in Fig. 3.8(a) to the blue ones. These connect to the $\theta \equiv \pi/2$ embedding along the horizontal axis in Fig. 3.2(b). Another option would be to make the transition to a long/long embedding.

If we decide to not impose parity at all, the transition to the $\theta \equiv \pi/2$ embedding does not have to proceed along one of the axes. The transition from connected to disconnected

embeddings may then happen at any value of m , small enough to allow for connected embeddings. Fig. 3.5 shows an example for the transition, corresponding to following a vertical line in Fig. 3.2(b) at $m = \frac{1}{2}$.

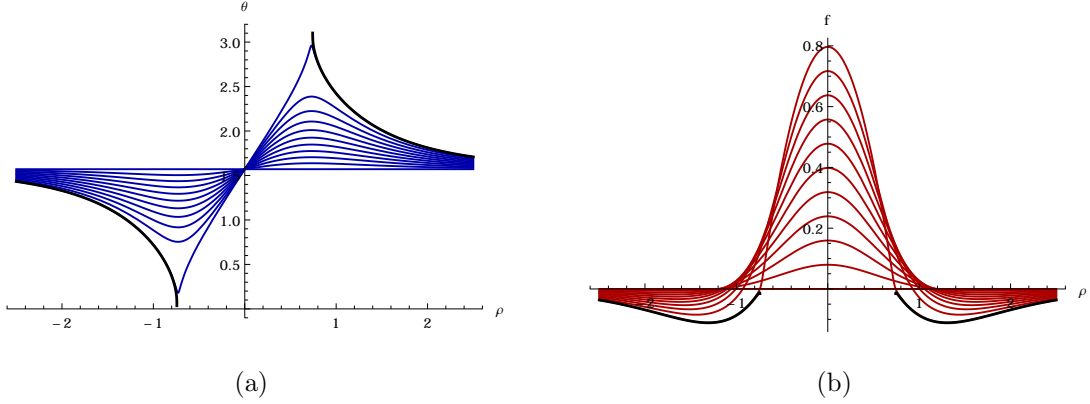


Figure 3.8: Embeddings with \mathbb{Z}_2 -odd slipping mode on the left hand side and the accompanying \mathbb{Z}_2 -even gauge field on the right hand side, as function of the radial coordinate ρ . $m = 0$ corresponds to $\theta \equiv \frac{\pi}{2}$ and the further the embedding departs from that, the larger is m . The $\theta \equiv \frac{\pi}{2}$ solution has $f = 0$, and the higher the peak of the curves on the right hand side, the larger is m . Note how these curves interpolate between the $\theta \equiv \frac{\pi}{2}$ solution and the massive embedding capping off at $\rho_\star = \sinh^{-1} m_\ell$, shown as thick black line. The critical mass m_ℓ was defined in eq. (3.67), and corresponds to $\rho_\star \approx 0.73$.

3.4 Flavored $\mathcal{N} = 4$ SYM on (two copies of) AdS_4

In the previous section we have studied the various classes of brane embeddings for their own sake, to get a catalog of allowed embeddings and also of how they may be combined when we combine the two coordinate patches to cover AdS_5 . In this section we will turn more towards AdS/CFT and take first steps to understanding what all this tells us about $\mathcal{N} = 4$ SYM on two copies of AdS_4 . One can get down to one copy of AdS_4 by taking a \mathbb{Z}_2 quotient, identifying the two patches. But we do not make this restriction to \mathbb{Z}_2 -symmetric configurations here.

3.4.1 Supersymmetric field theories in curved space

While it is well understood how to construct supersymmetric Lagrangians on Minkowski space, for example using superfields, the study of supersymmetric gauge theories on curved spaces takes a little bit more care. For a generic supersymmetric field theory, supersymmetry is completely broken by the connection terms in the covariant derivatives when naively formulating the theory on a curved background. One simple class of supersymmetric curved space theories is provided by superconformal field theories on spacetimes that are conformal to flat space. That is, once we have constructed a superconformal field theory on flat space, such as $\mathcal{N} = 4$ SYM without any flavors or with massless quenched flavors, we can simply obtain the same theory on any curved space with metric

$$ds^2 = \Omega(t, \vec{x}) (-dt^2 + d\vec{x}^2). \quad (3.74)$$

For the theory to be conformal, all fields have to have the conformal coupling to the background curvature (see e.g. [102] for a review). While this is of course without any consequence in the original Minkowski space theory, the corresponding mass-like terms have to be included after the conformal transformation.

One interesting choice for the conformal factor Ω is $\Omega = z^{-2}$ where z is one of the spatial directions. In this case the resulting metric is locally AdS_4 in Poincare coordinates. In fact, the resulting metric is two copies of AdS_4 , since both negative and positive z give rise to their own Poincare patch metric. These two AdS spaces are linked with each other along their common boundary at $z = 0$ via boundary conditions. To be more specific, we can derive the appropriate boundary conditions from the conformal transformation. In Minkowski space, $z = 0$ is not a special place. All fields as well as their z -derivatives, which we denote by primes, have to be continuous at this codimension one locus. Denoting the fields at positive (negative) values of z with a subscript of R (L), the boundary conditions for a massless scalar field X read

$$X_L(z = 0) = X_R(z = 0), \quad X'_L(z = 0) = X'_R(z = 0) \quad (3.75)$$

The generalization to fermions and vector fields is straightforward. Under a conformal transformation, the left and right hand sides of these conditions change in the same way, and so these boundary conditions have to be kept in place when studying the same field theory on the conformally related two copies of AdS. These “transparent” boundary conditions were discussed as very natural from the point of view of holography in [84] and many subsequent works. They preserve the full supersymmetry of the field theory, as is obvious from their flat space origin. From the point of view of the field theory on AdS they are unusual. For physics to be well defined on one copy of AdS₄, we need one boundary condition on (say) X_L and X'_L alone. Typical examples are the standard Dirichlet or Neumann boundary conditions. For two separate copies of AdS₄ we do need two sets of boundary conditions in total, but what is unusual is that the transparent boundary conditions relate L and R fields to each other. A different set of boundary conditions would for example be given by $X_L(z=0) = X_R(z=0) = 0$ with no restrictions on the derivatives. These double Dirichlet boundary conditions are the standard boundary conditions typically used for field theories on AdS₄. With these boundary conditions the two copies are entirely decoupled. Generic boundary conditions will break all supersymmetries, but it is well known how to impose boundary conditions on $\mathcal{N} = 4$ SYM on a single AdS₄ space in a way that preserves half of the supersymmetries. These boundary conditions follow from the analysis in [103] and correspond to the field theory living on D3 branes ending on stacks of NS5 or D5 branes. The detailed choice of boundary conditions dramatically changes the dynamics of field theory on AdS₄, as comprehensively discussed in [89]. While [103, 89] completely classified the boundary conditions preserving at least half of the supersymmetries one can impose on a single copy of AdS₄, more general supersymmetry preserving boundary conditions are possible on two copies. We already saw one example, the transparent boundary conditions above, which preserve the full supersymmetry. It is easy to write down boundary conditions that interpolate between transparent and two copies of Dirichlet, even though we have not yet attempted a complete classification of supersymmetric boundary conditions of $\mathcal{N} = 4$ on two copies AdS₄.

While conformal theories are the simplest supersymmetric field theories to formulate on curved space, one can also formulate non-conformal supersymmetric field theories, e.g. with masses for at least some of the fields, on curved spaces. The non-invariance of the connection terms can be compensated by adding additional terms to the Lagrangians. One classic example of such compensating terms is provided by the procedure of “topological twisting” [64]. One can think of topological twisting as adding a background expectation value to the background R-charge gauge field. If the background gauge field is chosen to be equal to the spin connection⁵, some supercharges can be made to transform as scalars under parallel transport in the gravitational and R-charge background. The background expectation value of the gauge fields adds extra terms to the Lagrangian proportional to the R-current. These serve as the compensating terms that restore supersymmetry. This twisting procedure puts no constraints on the geometry of the background space. In principle it will be straightforward to implement this procedure also in the dual bulk prescription. The R-charge gauge field is now dynamical. The topological twisting implies that its leading (radially independent) piece at the boundary no longer is taken to vanish but is set equal to the spin connection. It would in fact be very interesting to construct the corresponding supergravity solutions.

While topological twisting restores some supersymmetry for generic curved backgrounds, there are simpler compensating terms that one can add for very special backgrounds. In the simple case of the 4-sphere, it was shown in [1] that for an $\mathcal{N} = 2$ supersymmetric gauge theory with massive hypermultiplets a simple scalar mass can act as a compensating term. Denoting the two complex scalars in a hypermultiplet by Q and \tilde{Q} , which in a common abuse of notation we also use for the whole chiral multiplet they are part of, one recalls that the

⁵More precisely, for $\mathcal{N} = 4$ SYM one picks [104] a $SU(2)$ subgroup of the $SU(4)_R$ symmetry and sets the corresponding gauge field equal to the $SU(2)_r$ part of the spin connection, which transforms in the $(1, 3) \oplus (3, 1)$ representation of the $\text{Spin}(4)=SU(2)_l \times SU(2)_r$ Euclidean Lorentz group. We will continue to loosely refer to this procedure as setting the gauge field equal to the spin connection.

superpotential term

$$W = MQ\tilde{Q} \quad (3.76)$$

gives rise to a fermion mass m as well as an F-term scalar mass term in the potential,

$$V_m = |M|^2 \left(|Q|^2 + |\tilde{Q}|^2 \right) . \quad (3.77)$$

This theory as it stands is not supersymmetric, but can be made supersymmetric by adding a particular dimension 2 operator to the Lagrangian. The full scalar mass term then reads $V = V_m + V_c^{S^4}$ with

$$V_c^{S^4} = i \frac{M}{R} \left(Q\tilde{Q} + \text{cc} \right) , \quad (3.78)$$

where R is the curvature radius of the sphere, which we from now on set to 1. Since the compensating mass term is purely imaginary, the resulting action is not real. We construct supersymmetric D7-brane embeddings to holographically realize this supersymmetric combination of mass terms on S^4 in § 3.5. The compensating terms have been understood systematically in [59]. The perhaps most natural way to construct supersymmetric field theories in curved space is to couple a supersymmetric gauge theory to a background supergravity multiplet. One then obtains a supersymmetric field theory for every supersymmetric configuration of the background supergravity. To have a non-trivial curved-space configuration preserving supersymmetry, the supergravity background has not just the metric turned on but also additional dynamical or auxiliary fields. The expectation values of these extra fields appear as the desired compensating terms in the field theory Lagrangian. Following this logic, the simple compensating term (3.79) for the S^4 can easily be generalized to AdS_4 , but this time with a real coefficient,

$$V_c^{\text{AdS}_4} = \frac{M}{R} \left(Q\tilde{Q} + \text{cc} \right) . \quad (3.79)$$

Including the superpotential mass term, the conformal curvature coupling, as well as the supersymmetry restoring compensating term, the full scalar mass matrix for a field theory

in AdS_{d+1} with 8 supercharges reads

$$M_{Q\bar{Q}} = \begin{pmatrix} -\frac{1}{4}(d^2 - 1) + M^2 & M \\ M & -\frac{1}{4}(d^2 - 1) + M^2 \end{pmatrix}. \quad (3.80)$$

For this work we are of course most interested in the AdS_4 case, that is $d = 3$. The eigenvalues of this mass matrix are given by

$$M_{\pm} = \frac{1}{4}(1 - d^2 + 4M^2 \pm 4M). \quad (3.81)$$

The full spectrum is symmetric under $M \rightarrow -M$ with the two branches being exchanged. The minimal eigenvalue ever reached is $M_{\min} = -d^2/4$, which for every d is reached at $M = \pm 1/2$. This is exactly the BF bound in d -dimensions. So, reassuringly, the supersymmetric theory never becomes unstable. The full scalar mass spectrum is depicted in figure 3.9.

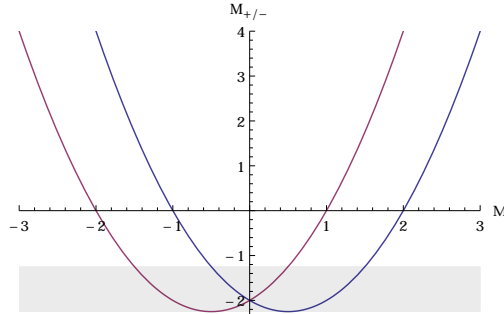


Figure 3.9: Eigenvalues of the scalar mass matrix as a function of superpotential mass M for flavored $\mathcal{N} = 4$ SYM on AdS_4 . Alternative quantization is possible for eigenvalues in the gray-shaded region.

While the compensating term restores supersymmetry, it breaks some of the global symmetries. Let us discuss this in detail for the case of $\mathcal{N} = 4$ SYM with N_f flavors. The massless theory has a global $\text{SU}(2)_{\Phi} \times \text{SU}(2)_R \times \text{U}(1)_R \times \text{U}(N_f)$ symmetry. Holographically, the first 3 manifest themselves by the preserved $\text{SO}(4) \times \text{U}(1)$ isometry of the D7 brane embedding, whereas the $\text{U}(N_f)$ global symmetry corresponds to the worldvolume gauge field. All flavor fields are invariant under $\text{SU}(2)_{\Phi}$, which acts only on the adjoint hypermultiplet

that is part of the $\mathcal{N} = 4$ Lagrangian written in $\mathcal{N} = 2$ language. So this symmetry will not play a role in the discussion below. Under $SU(2)_R$ the fundamental scalars transform as a doublet, but the fundamental fermions are neutral. A superpotential mass breaks the $U(1)_R$. In the holographic dual $U(1)_R$ corresponds to shifts in ψ and are manifestly broken by the D7 brane localized at a fixed value of ψ . The superpotential mass term, however, preserves the full $SO(4)$ symmetry since $|Q|^2 + |\tilde{Q}|^2$ is the $SU(2)$ invariant “length” of the doublet. Not so for the compensating term, which explicitly breaks $SU(2)_R$ down to a $U(1)$ subgroup [59]. This pattern mirrors exactly the symmetry breaking patterns we found in our supersymmetric probe branes.

The structure of supersymmetry being restored by a compensating term is somewhat reminiscent of supersymmetric Janus solutions, that is field theories with varying coupling constants. Since under a supersymmetry variation the Lagrangian typically transforms into a total derivative, position-dependent coupling constants generically break supersymmetry. It was found in [105, 106] that supersymmetry can, once again, be restored by compensating terms. In fact, this discussion is related to field theories on AdS by a conformal transformation and we will find this picture useful in some of our analysis, so we briefly introduce it. We start with a massive theory on AdS. In the presence of a mass term, the conformal transformation from AdS to flat space is not a symmetry, but is explicitly broken by the mass term. If, however, the mass is the only source of conformal symmetry breaking (as is the case in $\mathcal{N} = 2^*$ or flavored $\mathcal{N} = 4$ SYM), we can restore the conformal symmetry by treating the mass M as a spurion. That is, by letting it transform explicitly. This way a field theory with constant M on AdS can be mapped into a field theory on flat space with a position-dependent superpotential mass $M \sim 1/z$. Correspondingly the fermion mass goes as $1/z$ whereas the superpotential induced scalar mass as $1/z^2$. Such a position-dependent $M(z)Q\tilde{Q}$ superpotential is precisely the framework discussed in [105]. It was shown that supersymmetry can be restored by a compensating term proportional to $M'(z)Q\tilde{Q}$. This $1/z^2$ compensating term is exactly the conformally transformed AdS compensating term from (3.79). So supersymmetric field theories on AdS are indeed conformally related to Janus like

configuration with $1/z$ mass terms.

As compared to the compensating terms of the topologically twisted $\mathcal{N} = 4$ theory, the implementation of the compensating terms of [1, 59, 105] is easier to accomplish holographically, since this time we only need to turn on a single extra scalar field. For $\mathcal{N} = 2^*$ this was done in [79], for super Janus in [107, 108] and for flavored $\mathcal{N} = 4$ SYM in this work.

3.4.2 Holographic renormalization

The most puzzling aspect of our supersymmetric probe configurations was the fact that for a given leading term, that is for a given mass M in the field theory, we found families of solutions that differed in the subleading term. Holographic intuition suggests that this should correspond to different allowed vacuum expectation values for a given mass. In order to confirm that this is indeed the case, let us map out which field theory operators get non-vanishing expectation values before we attempt an interpretation. It should also be noted that similar ambiguities were also found in numerical studies of non-supersymmetric flavors on AdS in [91].

The D7-brane action eq. (3.7) is divergent as usual, and we have to carry out its holographic renormalization before extracting information about the flavor sector of the dual theory [109, 110]. The counterterms for the slipping mode θ have been given in [54], so we only need to construct those for the gauge field. In many regards, f can be seen as a scalar at the BF bound, and it is tempting to just take those counterterms from [110]. There are some subtleties, however, as we will discuss momentarily. The counterterms for the slipping mode as given in [54] are

$$L_{\text{ct},\theta} = -\frac{1}{4}\left[1 - \frac{1}{12}R\right] + \frac{1}{2}\tilde{\theta}^2 - \log \epsilon \left[\frac{1}{32}(R_{ij}R^{ij} - \frac{1}{3}R^2) + \frac{1}{2}\tilde{\theta}\square_W\tilde{\theta} \right] \\ + \alpha_1\tilde{\theta}^4 + \alpha_2\tilde{\theta}\square_W\tilde{\theta} + \frac{\alpha_3}{32}(R_{ij}R^{ij} - \frac{1}{3}R^2) . \quad (3.82)$$

Note that g_ϵ denotes the metric on the cut-off surface induced from the *background* metric (as opposed to the worldvolume metric), and R denotes the curvature of g_ϵ . \square_W is the Weyl-covariant Laplacian, $\square_W = \square + \frac{1}{6}R$. We have also defined $\tilde{\theta} := \theta - \frac{\pi}{2}$, since our θ

is shifted by $\frac{\pi}{2}$ compared to the coordinates used in [54]. In the curvature conventions of [54], the AdS₄ slices have $R_{ij} = 3g_{\epsilon ij}$. The coefficients of the finite counterterms α_i are not determined by the renormalization, and reflect a scheme dependence. For α_1 , demanding the free energy to vanish for Poincaré AdS fixes [54]

$$\alpha_1 = -\frac{5}{12}. \quad (3.83)$$

From the gravity side the holographic counterterms are supposed to be universal, in the sense that they should be fixed once and for all, regardless of the background. That means that the same argument for a supersymmetry-preserving scheme in Poincaré AdS also fixes α_1 for us – it is still the same theory, just evaluated on a different background. That leaves the scheme dependence coming from α_2 and α_3 . They can not be fixed from flat-space supersymmetry considerations, since the counterterms just vanish for flat boundaries.

We now come back to the extra terms for the S³ gauge field. There are two possible ways to look at it. The first one would be the approach taken in [54] to deal with the slipping mode. For fixed ω as given in eq. (3.28), the gauge field $A = f\omega$ from the AdS perspective reduces to the radial profile f , which can be treated as a scalar field. The other one would be to take the 6 + 1-dimensional cut-off surface introduced by the bulk radial cut-off on the worldvolume of the D7-branes for what it is. We would then only allow covariant and (sufficiently) gauge-invariant counterterms like $\sqrt{g_\epsilon}F(A_\star)^2$, $A_\star \wedge F(A_\star) \wedge C_4$ etc., where the star denotes pullback to the cut-off surface, and determine their coefficients. Since we already have ω and at the end of the day the two approaches are equivalent, we follow the first one.

The radial profile of the S³ gauge field f is almost like a scalar at the BF bound. After integrating the WZ term in eq. (3.7) by parts, the bulk action takes exactly the same form. But precisely due to this integration by parts, the action picks up an extra boundary term as compared to a scalar at the BF bound. Such boundary terms can not be ignored on AdS, and here the extra boundary term cancels a \log^2 divergence usually expected for scalars at the BF bound. The counterterms are therefore slightly different from those given e.g. in

[110, 54, 100], and we find

$$L_{\text{ct},f} = -\lambda^2 f^2 \left(\frac{1}{2 \log \epsilon} + \frac{\alpha_4}{(\log \epsilon)^2} \right) . \quad (3.84)$$

This introduces an additional scheme-dependent counterterm with coefficient α_4 . For the κ -symmetric embeddings, where θ and f are related by eq. (3.53), these finite counterterms are related as well. So if one stays within this family of supersymmetric embeddings, α_4 could be absorbed in a redefinition of α_2 . But we will not do that. For the renormalized D7-brane action corresponding to eq. (3.7) we then have

$$S_{\text{D7,ren}} = S_{\text{D7}} - T_7 \int_{\rho = -\log(\epsilon/2)} d^7 \xi \sqrt{g_\epsilon} [L_{\text{ct},\theta} + L_{\text{ct},f}] . \quad (3.85)$$

To get to Fefferman-Graham coordinates where the boundary metric is AdS_4 with unit curvature radius, we have set $\rho = -\log(z/2)$. The cut-off surface is then given by $\rho = -\log(\epsilon/2)$. For the covariant counterterms it does not matter how we parametrize the cut-off surface, but for those involving explicit logarithms a change in the parametrization results in a change of the finite counterterms. The log-terms in eq. (3.82), eq. (3.84) are chosen such that the coefficients of the finite counterterms agree with the usual Fefferman-Graham gauge conventions.

3.4.3 One-point functions

With the holographic renormalization carried out we can now compute the renormalized one-point functions for the chiral and scalar condensates in $\mathcal{N} = 4$ SYM. To get the near-boundary expansion for the embedding eq. (3.58) and gauge field eq. (3.53), we change to FG gauge. As explained above, this amounts to setting $\rho = -\log(\epsilon/2)$. Expanding then in small z yields

$$\theta = \frac{\pi}{2} + mz - mz^3 \log(z) + z^3 \left[\frac{m}{4} (2m^2 + 4 \log 2 - 3) - c \right] + \mathcal{O}(z^5 \log z) , \quad (3.86a)$$

$$f = \frac{m}{i\lambda} z^2 \log z + \frac{3c - m^3 + 3m(1 - \log 2)}{3i\lambda} z^2 + \mathcal{O}(z^4 \log z) . \quad (3.86b)$$

The $\mathcal{O}(z)$ term of the slipping mode as usual sets the mass, while the $\mathcal{O}(z^3)$ term loosely corresponds to the chiral condensate. The leading term of the gauge field expansion sets the extra scalar mass. The $\mathcal{O}(z^2)$ term encodes the corresponding scalar condensate.

To actually get the one-point functions, we have to compute the variation of the action eq. (3.7) and evaluate it on shell. Going on shell and varying does not necessarily commute, so we will not use any of the κ -symmetry relations to simplify the action. The starting point will be eq. (3.65), where h was computed in eq. (3.36). We can, however, use relations like eq. (3.63) after the variation. That gives

$$\delta_\theta S_{D7} = -T_7 V_{S^3} \int d^5 \xi \partial_\rho [\sqrt{g_{\text{AdS}_4}} \zeta' \theta' \sin^4 \theta \delta \theta] + \text{EOM} , \quad (3.87a)$$

$$\delta_f S_{D7} = -T_7 V_{S^3} \int d^5 \xi \partial_\rho [\sqrt{g_{\text{AdS}_4}} (\zeta' \sin^2 \theta f' + 8\zeta f) \delta f] + \text{EOM} , \quad (3.87b)$$

where EOM denotes contributions which vanish when evaluated on shell. Combining that with the variation of the counterterms yields

$$\langle \mathcal{O}_\theta \rangle = -\frac{1}{\sqrt{-g_{\text{AdS}_4}}} \frac{\delta}{\delta \theta^{(0)}} S_{D7, \text{ren}} = T_7 V_{S^3} \left[2c + (1 + 4\alpha_1) m^3 + \frac{5 + 8\alpha_2 - 4 \log 2}{2} m \right] , \quad (3.88a)$$

$$\langle \mathcal{O}_f \rangle = -\frac{1}{\sqrt{-g_{\text{AdS}_4}}} \frac{\delta}{\delta f^{(0)}} S_{D7, \text{ren}} = T_7 V_{S^3} i \lambda \left[c - \frac{m}{3} (m^2 - 3 + \log(8) - 6\alpha_4) \right] , \quad (3.88b)$$

where $\theta^{(0)}$ and $f^{(0)}$ denote the leading coefficients in the near-boundary expansion, i.e. $\theta = \pi/2 + \theta^{(0)} z + \dots$ and $f = z^2 \log z f^{(0)} + \dots$. The fact that the one-point functions are finite validates the holographic counterterms. As expected, the scheme-dependent terms only contribute pieces proportional to m , and for $m = 0$ there is no scheme dependence at all. The term with coefficient α_3 does not contribute at all to the one-point functions. The precise value of c depends on the embedding we choose. We give the explicit values for $m \geq 0$ in the $\rho > 0$ patch. For the short and long embeddings, we find from eq. (3.66) and eq. (3.68)

$$c_{\text{short}} = \frac{m^2 - 2}{3} \sqrt{m^2 + 1} + m \sinh^{-1}(m) , \quad c_{\text{long}} = -c_{\text{short}} . \quad (3.89)$$

Recall that the long embeddings are possible only for $m < m_\ell$, with m_ℓ given in eq. (3.67). For the connected embeddings, which also only exist in this mass range, c is free to vary within the ranges given by eq. (3.69), that is between c_{short} and c_{long} .

3.4.4 Interpretation

We found that, for a range of masses, a one-parameter family of solutions exists with a different values of the chiral and scalar condensates for one and the same mass. Here we attempt to interpret what these solutions could potentially mean in the field theory. Our interpretation will be somewhat similar to the one offered in [91] for the case of non-supersymmetric flavors on AdS_4 . It seems that the family of massless solutions should be easiest to understand. Recall that there is one solution among this family that is singled out: the $m = c = 0$ connected solution. This is the only massless solution in which chiral symmetry is not spontaneously broken. This solution can be conformally mapped to massless flavors on flat space, as described at the beginning of this section. Correspondingly, this particular solution should correspond to flavored $\mathcal{N} = 4$ SYM with transparent boundary conditions.

Under the same conformal transformations our other solutions also map into massless flavors on all of flat space, but with a position dependent chiral condensate that falls off as $1/z^3$ as a function of distance to the plane at $z = 0$. Our basic suggestion is that these other embeddings should correspond to supersymmetric flavors in the field theory with different boundary conditions imposed on the flavor fields at $z = 0$. Only the standard transparent boundary conditions will yield a vanishing chiral condensate. This case is easiest to make for the other extreme case: the disconnected embedding with $m = 0$ and $c = -2/3$. In this case, we can decide to only study flavors in one of the two asymptotic AdS spaces (or, alternatively, one one half of Minkowski space). The disconnected embedding at positive ρ is perfectly smooth and well behaved without the second disconnected embedding at negative ρ . Since we only added flavors in the $z > 0$ half of Minkowski space, this embedding can not correspond to a field theory with transparent boundary conditions. In the notation introduced above eq. (3.75), only the R fields exist, there are no L fields we could relate

them to at $z = 0$. In this case one has to impose boundary conditions at $z = 0$ on the R flavor fields alone, presumably either Dirichlet or Neumann conditions. Either of these choices is expected to give rise to a position dependent condensate, just as we observed in our brane embedding. This happens already in the free field theory, as can be seen by employing the standard method of images, as was e.g. done in this context in [105]. For simplicity, let us consider the case of a scalar field. The propagator of a scalar field on half space is given by

$$G(x, y) = \frac{1}{4\pi} \left(\frac{1}{(x - y)^2} \pm \frac{1}{(x - Ry)^2} \right) \quad (3.90)$$

where $(Ry)_\mu = (t_y, x_y, y_y, -z_y)$ for $y_\mu = (t_y, x_y, y_y, z_y)$ and upper/lower sign corresponds to Neumann/Dirichlet boundary conditions. To calculate the expectation value $\langle X^2(x) \rangle$ we simply need to evaluate $G(x, x)$. The first term gives rise to a divergent contribution that needs to be subtracted in order to properly define the composite operator X^2 . In $\mathcal{N} = 4$ SYM, the corresponding divergence was even shown to cancel between contributions from different fields in [105]. We get a non-vanishing expectation value from the mirror charge term that goes as $1/z^2$, as appropriate for a dimension 2 operator. For a fermion a similar calculation gives $1/z^3$, which are the expectation values as we found here. So in principle, at least for this special configuration, the behavior we found holographically makes qualitative sense. It would be nice to see whether non-renormalization theorems could allow a quantitative comparison between free and strongly coupled field theories, as was done in the context of Janus solutions in [105].

For all the other embeddings, that is the connected embeddings with $m = 0$, $c \neq 0$ and the disconnected embeddings with $m = 0$ in both halves of spacetime, we have no strong argument for what the field theory boundary conditions should be. But we suspect that they interpolate between the transparent boundary conditions at $m = c = 0$ and Dirichlet (or Neumann) boundary conditions for the maximal c . It is easy to write down such interpolating boundary conditions that give a $1/z^3$ condensate with growing coefficient, but the choice is not unique. Potentially, a careful study of supersymmetry together with our results for the expectation values could help pin this down. But at least our results for the massless

embeddings appear to be consistent with this interpretation.

For massive embeddings there is no such simple argument. Mapping to flat space gives position-dependent masses that diverge at $z = 0$, and so any discussion of boundary conditions is more involved. But it is tempting to relate the presence of connected and long embeddings for small mass to boundary conditions in a similar way. In the window $m_{\text{BF}}^2 \leq m^2 < m_{\text{BF}}^2 + 1$ we can do standard and alternative quantization for scalar fields on AdS_4 . That similarly allows for a family of boundary conditions, and hence presumably a family of expectation values. But which mass is it that approaches $m_{\text{BF}}^2 + 1$ when we dial m from zero to its critical value given in eq. (3.67)? Note that for a given leading coefficient m in our expansion of the fluctuating field θ in the bulk, the field theory mass M is actually given by [97] $M = \sqrt{\lambda}/(2\pi)m$. Since this M is thus much larger than 1 except for infinitesimally small values of m , we find that both mass eigenvalues in (3.81) are large and positive for any finite m . So none of the fundamental fields is even close to the window in which two different boundary conditions are allowed. However, while the fundamental fields have masses of order $\sqrt{\lambda}$ in all our embeddings, the gauge invariant meson fields have actually order one masses [97]. This makes the mesons a natural candidate for a field that obeys different boundary conditions in our different embeddings for one and the same mass. We can indeed see directly from the geometry that the meson spectrum is strongly affected by the difference of embeddings. Corresponding to the different classes of brane embeddings, we get different classes of mesons: those built from pairs of L quarks and R quarks separately (again in the notation introduced above eq. (3.75)), and those with mixed content. Denoting the mesons by their quark content, we see that for connected embeddings both LL, RR as well as LR and RL mesons are light, as they all correspond to fluctuations of the brane. For disconnected embeddings, only LL and RR mesons can be light, whereas LR and RL mesons correspond to semi-classical strings stretched between the two disconnected branes, and hence to order $\sqrt{\lambda}$ masses. A more quantitative discussion of this suggestion requires an analysis of the meson spectrum as encoded in the spectrum of linearized fluctuations around our embeddings, and is beyond the scope of the present work.

3.5 Supersymmetric D3/D7 for massive flavors on S^4

We now go through the derivation of κ -symmetric D7 embeddings into $\text{AdS}_5 \times S^5$, where AdS_5 is Euclidean and in global coordinates. The result will allow us to holographically describe $\mathcal{N} = 4$ SYM coupled to (massive) flavor hypermultiplets on S^4 . So we take $g = g_{\text{AdS}_5} + g_{S^5}$, with the S^5 metric given in eq. (3.1). For the S^4 part of AdS_5 we use conformally flat coordinates to simplify the explicit computations, but note that the resulting embeddings are independent of the chosen S^4 coordinates. The metric then takes the form

$$g_{\text{AdS}_5} = dR^2 + \sinh^2 R d\Omega_4^2, \quad d\Omega_4^2 = W^{-2} d\vec{x}^2, \quad W = 1 + \vec{x}^2. \quad (3.91)$$

The Killing-spinor equation for Euclidean AdS differs by a factor of i from the Lorentzian one, and there is a sign convention to be fixed. For the analytic continuation to be discussed momentarily, we have

$$D_\mu \epsilon = \frac{1}{2} \Gamma_{\text{AdS}} \Gamma_\mu \epsilon, \quad \mu = 0 \dots 4. \quad (3.92)$$

We have denoted by Γ (as opposed to Γ) the Euclidean Clifford-algebra generators. The conventions for the Euclidean Clifford algebra will be laid out in more detail along with the analytic continuation below. The Killing spinor equation for the S^5 part stays the same, and is given in eq. (4.8). The $\text{AdS}_5 \times S^5$ Killing spinors are again of the form eq. (3.3), i.e. $\epsilon = R_{\text{AdS}} R_{S^5} \epsilon_0$, with R_{S^5} given in eq. (3.4) and

$$R_{\text{AdS}} = W^{-1/2} e^{\frac{1}{2} \rho \Gamma_\rho \Gamma_{\text{AdS}}} \left[\mathbf{1} + x_i \Gamma_\rho \Gamma^{x_i} \right]. \quad (3.93)$$

For the embedding and gauge field we choose the same ansatz as before and motivated in Sec. 3.2.1. That is, we take a non-trivial slipping mode as function of the radial coordinate ρ and a worldvolume gauge field $A = f\omega$. Note that although we have used the same name for the radial coordinate, ρ , it does not have the same geometric meaning as in the AdS_4 -sliced case, and consequently the embeddings are geometrically different. The ten-bein pulled back to the worldvolume is again given by eq. (3.11), and the induced metric reads

$$g = (1 + \theta'^2) d\rho^2 + \sinh^2 \rho d\Omega_4^2 + \sin^2 \theta d\Omega_3^2. \quad (3.94)$$

3.5.1 Analytic continuation

For the S^3 mode ω appearing in the gauge field we again take eq. (3.28). Evaluating the κ -symmetry projection conditions eq. (3.9) then proceeds almost in the same way as for the AdS_4 slicing in Lorentzian signature all the way up to eq. (3.37). To make this more precise, we will have to set up the analytic continuation to Euclidean signature. The metric quantities as we set them up are taken care of and are already in Euclidean signature. The more subtle step is to implement the analytic continuation on the Clifford algebra and spinors. The Killing spinors we have given in eq. (3.93) also assume a Euclidean-signature Clifford algebra already. But we have to implement the continuation in the κ -symmetry condition eq. (3.9). To reflect the change in signature we set

$$\Gamma_{\underline{x}_0} \rightarrow \Gamma_{\underline{x}_0} = i\Gamma_{\underline{x}_0}, \quad \Gamma^{\underline{x}_0} \rightarrow \Gamma^{\underline{x}_0} = -i\Gamma^{\underline{x}_0}. \quad (3.95)$$

With this continuation, we note that $C\Gamma_{\underline{x}_0}^* = -\Gamma_{\underline{x}_0}C$. The S^5 part is not affected by the analytic continuation, and for the AdS part we define Γ_{AdS} using the same expressions as before below eq. (4.8). That is, with indices up but Γ s replaced by Γ s. This gives $\Gamma_{\text{AdS}} = -i\Gamma_{\text{AdS}}$ (lowering the indices, however, does not produce a sign for Γ_{AdS}). This is the matrix showing up in eq. (3.93). For the chirality projector we define $\Gamma_{11} = -i\Gamma_{\text{AdS}}\Gamma_{S^5} = \Gamma_{11}$.

We can now take a closer look at the matrix \tilde{R}_{AdS} , defined by $CR_{\text{AdS}} = \tilde{R}_{\text{AdS}}C$ as before. We follow [111] (extending earlier work in [112]), in including a time reflection in the complex structure on Euclidean space to be compatible with analytic continuation, i.e. $x_0^* = -x_0$. The charge conjugation matrix is kept as the Lorentzian one. Noting that now $C\Gamma_{\text{AdS}} = -\Gamma_{\text{AdS}}C$, we then find

$$\tilde{R}_{\text{AdS}} = e^{\rho\Gamma_{\text{AdS}}\Gamma_{\underline{t}}R_{\text{AdS}}}. \quad (3.96)$$

Comparing to eq. (3.32), we note that this relation is different from the one we found for the AdS_4 slicing in Lorentzian signature. That means we will also have to change the projection condition eq. (4.55) to solve the κ -symmetry constraint.

3.5.2 κ -symmetry

Although we never had to explicitly unpack the AdS₄-slice Γ -matrices in Sec. 3.2, the analytic continuation still has implications. We start with the κ -symmetry condition as spelled out in eq. (3.9). The lowercase γ 's are now defined as $\gamma_i = e_i^a \Gamma_a$, i.e. with the Euclidean Γ -matrices, and we have

$$\Gamma_{(0)} = \frac{i}{(p+1)! \sqrt{\det g}} \epsilon^{i_1 \dots i_{p+1}} \gamma_{i_1 \dots i_{p+1}} . \quad (3.97)$$

The extra i is due to the change in sign for the metric determinant. Since our embedding ansatz is formally the same as for the AdS₄ slicing, we get to an analog of eq. (3.15) in just the same way, and find

$$\Gamma_\kappa \epsilon = \frac{-i}{\sqrt{\det(1+X)}} \left[\left(\mathbf{1} + \frac{1}{8} \gamma^{ijkl} F_{ij} F_{kl} \right) \Gamma_{(0)} \epsilon + \frac{1}{2} \gamma^{ij} F_{ij} C (\Gamma_{(0)} \epsilon)^\star \right] . \quad (3.98)$$

Since F does not have timelike components, its continuation is trivial. The matrices appearing in eq. (3.98) are now

$$\Gamma_{(0)} = \frac{i}{\sqrt{1+\theta'^2}} \hat{\Gamma} , \quad \hat{\Gamma} = \left[\mathbf{1} + \theta' \Gamma_{\underline{\theta}} \Gamma_{\underline{\rho}} \right] \Gamma_{\text{AdS}} \Gamma_{\underline{X}} . \quad (3.99)$$

As compared to the Lorentzian case, there is now no relative sign between eq. (3.99) and eq. (3.97), since Γ_{AdS} is equal to the product of all AdS₅ Γ -matrices with indices down. In proceeding to the analog of eq. (3.37), we have to be a bit careful. The function h again evaluates to eq. (3.36). Since now $C(i\Gamma_{\text{AdS}})^\star = i\Gamma_{\text{AdS}}C$, we get the same sign in the term with the charge conjugation matrix as before in eq. (3.37), and find

$$\left(\mathbf{1} + \frac{1}{8} \gamma^{ijkl} F_{ij} F_{kl} \right) \hat{\Gamma} \epsilon + \frac{1}{2} \gamma^{ij} F_{ij} \hat{\Gamma} C \epsilon^\star = h \epsilon . \quad (3.100)$$

Since we kept the S³ mode eq. (3.28), we again find eq. (3.38) for the F^2 -term. As pointed out above eq. (3.96), we define R -matrices with a tilde analogously to Sec. 3.2. Since R_{S^5} does not contain any AdS₅ Γ -matrices, it is not affected by the Wick rotation and this procedure results in the same matrix as in Sec. 3.2. Instead of the projection condition

eq. (4.55), we now use

$$\tilde{\Gamma} C \epsilon_0^* = \lambda \epsilon_0 , \quad \tilde{\Gamma} = \Gamma^{\underline{\chi}1} \Gamma^\theta . \quad (3.101)$$

We then find that eq. (3.100) evaluates to

$$\hat{\Gamma} \epsilon + \frac{\lambda}{2} \gamma^{ij} F_{ij} \hat{\Gamma} \tilde{R}_{S^5} \tilde{R}_{\text{AdS}} \tilde{\Gamma} \epsilon_0 = 2f' f \csc^3 \theta \Gamma_\rho \Gamma_{\text{AdS}} \epsilon + h \epsilon . \quad (3.102)$$

We have to first evaluate the $\gamma \cdot F$ term. With the definitions above we find that on the spinor subspace singled out by P_0 (which stays the same as in Lorentzian signature) and $(\mathbf{1} + \Gamma_{11})$,

$$\frac{1}{2} \gamma^{ij} F_{ij} \hat{\Gamma} \tilde{R}_{S^5} \Gamma^\theta \Gamma^{\underline{\chi}1} = \csc \theta \left[i f' \Gamma_\rho \Gamma^\psi - 2f \csc \theta \left(\Gamma^\theta - \theta' \Gamma_\rho \right) \Gamma_{\text{AdS}} \right] R_{S^5} . \quad (3.103)$$

With that in hand we can evaluate eq. (3.102), for which we find

$$\begin{aligned} \text{LHS} &:= \hat{\Gamma} \epsilon - \lambda \csc \theta \left[i f' \Gamma_\rho \Gamma^\psi - 2f \csc \theta \Gamma_{\text{AdS}} \right] R_{S^5} \tilde{R}_{\text{AdS}} \epsilon_0 \\ &= 2\lambda f \theta' \csc^2 \theta R_{S^5} \Gamma_\rho \Gamma_{\text{AdS}} \tilde{R}_{\text{AdS}} \epsilon_0 + 2f' f \csc^3 \theta \Gamma_\rho \Gamma_{\text{AdS}} \epsilon + h \epsilon =: \text{RHS} . \end{aligned} \quad (3.104)$$

The left hand side once again is linear in f and its derivative. The right hand side does not have any more S^5 Γ -matrices. To proceed further we need the analogues of eqs. (3.46), (3.47) and (3.50). With $\tilde{\Gamma}_{\rho A} = R_{\text{AdS}}^{-1} \Gamma_\rho \Gamma_{\text{AdS}} R_{\text{AdS}}$, we find

$$R_{S^5}^{-1} R_{\text{AdS}}^{-1} \hat{\Gamma} \epsilon = \mathcal{R}[\Gamma^\theta \Gamma_{\underline{\chi}}] \left[\theta' \tilde{\Gamma}_{\rho A} - \cot \theta \cdot \mathbf{1} \right] \epsilon_0 + \csc \theta \epsilon_0 , \quad (3.105)$$

$$R_{S^5}^{-1} R_{\text{AdS}}^{-1} \Gamma_\rho \Gamma^\psi R_{S^5} \tilde{R}_{\text{AdS}} \epsilon_0 = i \mathcal{R}[\Gamma^\theta \Gamma_{\underline{\chi}}] \left[\sinh \rho \cdot \mathbf{1} - \cosh \rho \tilde{\Gamma}_{\rho A} \right] \epsilon_0 , \quad (3.106)$$

$$R_{S^5}^{-1} R_{\text{AdS}}^{-1} \Gamma^\theta \Gamma_{\text{AdS}} R_{S^5} \tilde{R}_{\text{AdS}} \epsilon_0 = \left[\csc \theta \mathcal{R}[\Gamma_\rho \Gamma_{\underline{\chi}}] - \cot \theta \mathbf{1} \right] \left[\cosh \rho \cdot \mathbf{1} - \sinh \rho \tilde{\Gamma}_{\rho A} \right] \epsilon_0 . \quad (3.107)$$

We now come back to the left hand side of eq. (3.104). Since the right hand side does not involve S^5 Γ -matrices, any terms involving those will have to vanish on the left hand side.

We find

$$\begin{aligned} R_{S^5}^{-1} R_{\text{AdS}}^{-1} \text{LHS} &= (\theta' - \lambda f' \csc \theta \cosh \rho - 2\lambda f \csc^3 \theta \sinh \rho) \mathcal{R}[\Gamma^\theta \Gamma_{\underline{\chi}}] \tilde{\Gamma}_{\rho A} \\ &\quad - (\cot \theta - \lambda f' \csc \theta \sinh \rho - 2\lambda f \csc^3 \theta \cosh \rho) R[\Gamma^\theta \Gamma_{\underline{\chi}}] \\ &\quad + \csc \theta \epsilon_0 - 2\lambda f \csc^2 \theta \cot \theta \left(\cosh \rho \cdot \mathbf{1} - \sinh \rho \tilde{\Gamma}_{\rho A} \right) \epsilon_0 . \end{aligned} \quad (3.108)$$

Just like in the AdS₄-sliced case, the round brackets of the upper two lines on the right hand have to vanish separately, since they multiply linearly independent Γ -matrix structures and nothing on the right hand side of eq. (3.104) can cancel them. Since f and f' appear linearly, we can solve for them and find

$$f = \frac{1}{2\lambda} \sin^3 \theta (\cot \theta \cosh \rho - \theta' \sinh \rho) , \quad f' = \frac{1}{\lambda} (\theta' \sin \theta \cosh \rho - \cos \theta \sinh \rho) . \quad (3.109)$$

Since both of these are functions of θ and θ' only, we can derive a second-order ODE for θ , which reads

$$\theta'' + 3\theta'^2 \cot \theta + 4\theta' \coth \rho - \cot \theta (1 + 2 \csc^2 \theta) = 0 . \quad (3.110)$$

These will once again be our main results. With eq. (3.109), we can simplify eq. (3.104) – after applying $R_{S^5}^{-1} R_{\text{AdS}}^{-1}$ – to find the remaining condition

$$\begin{aligned} & [\csc \theta - h + 2\lambda f \csc^2 \theta (\theta' \sinh \rho - \cot \theta \cosh \rho)] \epsilon_0 \\ & + 2f \csc^2 \theta [f' \csc \theta + \lambda (\theta' \cosh \rho - \cot \theta \sinh \rho)] \tilde{T}_{\rho A} \epsilon_0 = 0 . \end{aligned} \quad (3.111)$$

The terms in brackets multiply independent Γ -matrix structures and have to vanish separately. Thanks to eq. (3.109) they do indeed vanish separately if $\lambda^2 = -1$. To complete the analysis, we once again checked that eqs. (3.109) and (3.110) together imply that the equations of motion for f and θ resulting from the DBI action are satisfied.

The solutions to eq. (3.110) again come in 3 branches, and the one which gives real slipping mode is

$$\theta = \cos^{-1} \left(2 \cos \frac{2\pi k + \cos^{-1} \tau}{3} \right) , \quad \tau = \frac{3m \sinh(2\rho) - 6c - 6m\rho}{4 \sinh^3 \rho} , \quad (3.112)$$

with $k = 2$. The function τ is related to that in eq. (3.58) by a simple analytic continuation $\rho \rightarrow \rho + i\pi/2$ along with a redefinition of the parameters m and c . The same applies for the accompanying gauge field eq. (3.109), which now is imaginary. That latter feature had to be expected from the discussion of the field-theory side in Sec. 3.4.1, and specifically the results of [1, 59] and [79]. We note that even though the final embeddings could have been obtained

by a simple analytic continuation from the AdS solutions, the evaluations of the κ -symmetry constraint differs from the AdS case by more than that, due to the changed Killing spinors and projectors. While this simple continuation could be anticipated, the subtleties that arise in implementing κ -symmetry in Euclidean signature for type IIB Killing spinors can non-trivially effect the resulting BPS equations, and so it will always behoove one to carefully verify this in considering novel embeddings.

Before ending, we note that the above AdS₄ sliced solutions, just as for the continuation to S⁴, can be Wick rotated or otherwise affected with coordinate transformations to fill out solutions describing embeddings with any other constant curvature (maximally symmetric) radial slice geometry. For example take D7-brane embeddings into dS₄-sliced AdS₅ with metric

$$g_{\text{AdS}_5} = d\rho^2 + \sinh^2 \rho g_{\text{dS}_4} . \quad (3.113)$$

It can be readily seen that the S⁴ sliced solutions work equally well for the dS₄ slicing. The reason is simple: Once the S³-mode ω is fixed, the field equations are only sensitive to radial dependences and warp factors, and not to the metric on the slices. Since these parts are the same for global Euclidean and dS₄-sliced Lorentzian AdS, the solutions found here work on dS₄-sliced AdS₅ just as well. While we have not gone through the κ -symmetry analysis for that case, we expect the solutions to trivially be supersymmetric.

As far as geometries where AdS₅ is sliced by other spaces of constant curvature are concerned, that only leaves hyperbolic space \mathbb{H}_4 (or Euclidean AdS₄) to consider. However since \mathbb{H}_4 is related to the original AdS₄ slice geometry, the solutions describing the D7-brane embedding can be obtained by simply Wick-rotating $t \rightarrow it$ in (eq. (3.1)). By the same arguments as applied in the S⁴ slicing, the embeddings found in Sec. 3.2.5 are still solutions with that analytic continuation. We thus have a comprehensive catalog of supersymmetric D7-brane embeddings to describe $\mathcal{N} = 4$ SYM with massive flavors on four dimensional spacetimes of constant curvature.

3.6 Supersymmetric Holographic Observables

In 2.2.2 where the construction $\mathcal{N} = 2^*$, i.e. adjoint mass deformed $\mathcal{N} = 4$ SYM theory, on S^4 was reviewed, it was seen that by a particular off-shell construction, a subset of supersymmetries were preserved that facilitated the use of supersymmetric localization to reduce the partition function—defined via the infinite-dimensional path integral—to a Gaussian matrix model deformed by explicable one-loop factors. The beauty of the exact statements readily available in a localized theory has led to a large body of work dedicated to the studying the import in the context of AdS/CFT through the solving the matrix model mapping out the RG flow of $\mathcal{N} = 2^*$ down to $\mathcal{N} = 2$ SYM theory and the subsequent construction of the, numerical, bulk gravitational dual [79, 113, 114]. Building on the construction of supersymmetric probe brane embeddings dual to $\mathcal{N} = 4$ SYM coupled to massive fundamental matter on S^4 in the previous section, we are now in a position to perform a precise check of this theory using localization, with both sides of the correspondence under complete analytic control.

Even for a gauge theory on a compact manifold, there can be non-trivial phase structure owed to large N_c [115]. Including fundamental matter raises interesting puzzles on the field theory side, where localization calculations with fundamental matter hint at a complicated and sometimes poorly understood phase structure [116]. In the theory we are studying we have complete control over the localization calculation and can identify a single well-characterized phase transition as a function of mass.

3.6.1 Further details of S^4 solutions

Starting from the discussion in § 3.5 regarding the continuation of the AdS_4 slicing solutions to S^4 , it would help if the solutions were cast in a form more amenable to holographic

analysis. Thus, changing to Fefferman-Graham gauge the line element reads

$$ds^2 = \frac{dz^2}{z^2} + \frac{(1 - \frac{z^2}{4})^2}{z^2} d\Omega_4^2 + d\Omega_5^2, \quad (3.114a)$$

$$d\Omega_5^2 = d\theta^2 + \sin^2 \theta d\Omega_3^2 + \cos^2 \theta d\psi^2. \quad (3.114b)$$

The action describing the embedding of the D7-brane into this background takes the form

$$S_{D7} = -T_7 \int d^8x \sqrt{\det [g + 2\pi\alpha' F]} + 2(2\pi\alpha')^2 \int_{\Sigma_8} C_4 \wedge F \wedge F. \quad (3.115)$$

From this point, we can follow the same construction as in §3.5.

After switching to Fefferman-Graham gauge, the solutions to the BPS equations afforded by κ -symmetry take the form

$$\cos \theta(z) = 2 \cos \left(\frac{2\pi k + \cos^{-1} \tau(z)}{3} \right), \quad (3.116a)$$

$$\tau(z) = \frac{96z^3(c - m \log \frac{z}{2}) + 6mz(z^4 - 16)}{(z^2 - 4)^3}, \quad (3.116b)$$

$$f(z) = -i \sin^3 \theta \frac{z(z^2 - 4)\theta' - (z^2 + 4) \cot \theta}{8z}. \quad (3.116c)$$

Again, we will use $k=2$ such that θ is real at the boundary. The parameter m is identified, up to a factor of the tension of a fundamental string, with the mass of the flavor fields as $M = m\sqrt{\lambda}/2\pi$ [97]. The normalization factor $\mu \equiv \sqrt{\lambda}/(2\pi)$ will be crucial in the field theory analysis: for any m which is not infinitesimally small, we will deal with heavy flavors in the field theory. That is, our quarks have mass of order $\sqrt{\lambda}$ in units of the S^4 radius. c appears in the condensate $\langle \bar{\psi}\psi \rangle$ and controls its non-analytic behavior. The relation eqgauge-field-from-slipping-mode in particular links the near-boundary expansions of f and θ , which should be expected given that, on the field theory side, the coefficient of the compensating term is fixed by the superpotential mass [1].

The D7 brane embeddings come in two distinct classes: the branes can either smoothly cap off at a $z_* \in (0, 2)$, or they can extend all the way to the center of AdS at $z = 2$. The D7-brane geometry is a cone with an $S^3 \times S^4$ base, where the S^3 lives in the internal space and the S^4 is the radial slice in AdS. For the first type of embeddings, the S^3 shrinks at

the tip of the cone, whereas for the second it is the S^4 that shrinks. These two types of embeddings are connected by a critical embedding where the brane caps off at $z_* = 2$. In that case the spheres simultaneously collapse at the tip. The condition that must be satisfied for a brane to cap off at a $z_* \in (0, 2)$ is $\theta(z_*) \in \{0, \pi\}$, which determines c as

$$c = \frac{96mz_*^3 \log \frac{z_*}{2} - 6mz_*(z_*^4 - 16) \pm (z_*^2 - 4)^3}{96z_*^3}. \quad (3.117a)$$

The gauge field configuration at $z = z_*$ is singular unless $f(z_*) = 0$, which fixes the cap off point in terms of the mass as

$$z_* = 2(m - \sqrt{m^2 - 1}). \quad (3.117b)$$

Note that these capped embeddings only exist for $m > 1$. For $m < 1$, including the case of massless flavors, we instead find embeddings that fill all of AdS. A smooth embedding in that case requires $\theta'(z = 2) = 0$, which translates to $c = 0$. These two topologically distinct families merge, as we will see, in a continuous phase transition at $m = 1$. For a plot of the corresponding slipping modes see Fig. 1(a).

3.6.2 One-Point Functions and Free Energy

With the embedding in hand, the computation of CFT one-point functions follows the standard AdS/CFT prescription. Making contact with CFT data requires computing a finite on-shell action employing holographic renormalization [109, 110, 54]. The on-shell action diverges near the conformal boundary. It can be regulated by introducing a radial cut-off $z \geq \delta$, and renormalized by then introducing a set of covariant counterterms at $z = \delta$. The chiral (scalar) condensates, \mathcal{O}_θ (\mathcal{O}_f), are then computed by varying the on-shell action w.r.t. the boundary values of θ (f).

We start with the D7-branes described by the DBI action with Wess-Zumino term eq.

(3.115) The covariant counterterms needed for the slipping mode at $z = \delta$ are [54]

$$L_1 = -\frac{1}{4} \left[1 - \frac{R}{12} + \frac{\log \delta}{8} \left(R_{ij} R^{ij} - \frac{R^2}{3} \right) \right] , \quad (3.118)$$

$$L_2 = \frac{1}{2} \left(-\tilde{\theta} \square_W \tilde{\theta} \log \delta + \tilde{\theta}^2 \right) , \quad (3.119)$$

$$L_{\text{fin}} = \alpha_1 \tilde{\theta}^4 + \alpha_2 \tilde{\theta} \square_W \tilde{\theta} + \frac{\alpha_3}{32} \left(R_{ij} R^{ij} - \frac{R^2}{3} \right) , \quad (3.120)$$

where $\square_W = \square + \frac{R}{6}$ is the Weyl covariant Laplacian and $\tilde{\theta} = \theta - \pi/2$. In addition to these, we need the following counterterms associated with the gauge field

$$L_3 = \frac{f^2}{2 \log \delta} \left(1 + \frac{2\alpha_4}{\log \delta} \right) . \quad (3.121)$$

The renormalized action, $S_{D7,\text{ren}} = S_{D7} - S_{\text{ct}}$, obtained by supplementing S_{D7} in eq. (3.115) with these counterterms, is now finite as $\delta \rightarrow 0$. The chiral condensate is given by

$$\mu \langle \mathcal{O}_\theta \rangle = -\frac{1}{\sqrt{g_{S^4}}} \frac{\delta S_{D7,\text{ren}}}{\delta \theta^{(0)}} . \quad (3.122)$$

The factor of μ on the left hand side accounts for the fact that the coefficient of the $\mathcal{O}(z)$ term in the slipping mode, $\theta^{(0)} = m$, is related to the actual source of the fermion bilinear, M , by a factor of μ . The computation of the scalar condensate proceeds analogously.

We now spell out the variations of eq. (3.115). After the variation is carried out, we can use identities following from the κ -symmetry analysis in the previous sections to simplify the result, which yields

$$\frac{\delta_\theta S_{D7}}{T_0 V_4} = \int_{z_*}^{\delta} dz \partial_z \left(z^2 \xi' \sin^4 \theta \theta' \delta \theta \right) , \quad (3.123)$$

$$\frac{\delta_f S_{D7}}{T_0 V_4} = \int_{z_*}^{\delta} dz \partial_z \left[(z^2 \xi' \sin^2 \theta f' + 8 \xi f) \delta f \right] , \quad (3.124)$$

where $\xi' = z^{-5} (1 - z^2/4)^4$. The variation of the counterterms is straightforward and reads

$$\begin{aligned} \delta_\theta L_{\text{ct}} &= T_0 \delta \theta \left(\theta + \frac{\theta}{6} R \log \delta + 4\alpha_1 \theta^3 - \frac{\theta}{3} \alpha_2 R \right) , \\ \delta_f L_{\text{ct}} &= T_0 \delta f \left(\frac{f}{\log \delta} + \frac{2\alpha_4 f}{(\log \delta)^2} \right) . \end{aligned} \quad (3.125)$$

With these results in hand, we can compute the one-point functions. The last ingredient is the asymptotic expansion of the solutions eq. (3.116), which reads

$$\tilde{\theta} \simeq mz - \left(c - \frac{m}{2}(m^2 + \frac{3}{2})\right)z^3 + mz^3 \log \frac{z}{2} + \dots \quad (3.126)$$

$$f \simeq imz^2 \log \frac{z}{2} - \frac{i}{3} (3c - m(m^2 + 3)) z^2 + \dots \quad (3.127)$$

We then find the condensates

$$\begin{aligned} \langle \mathcal{O}_\theta \rangle &= \frac{T_0}{\mu} \left[2c - \frac{m}{2}(5 + 8\alpha_2 - 4 \log 2) + m^3(1 + 4\alpha_1) \right], \\ \langle \mathcal{O}_f \rangle &= \frac{iT_0}{\mu} \left[\frac{m^3}{3} - c + m(1 + 2\alpha_4 - \log 2) \right]. \end{aligned} \quad (3.128)$$

We can vary the asymptotic values of θ and f independently, and thus calculate the chiral condensate \mathcal{O}_θ and the scalar condensate \mathcal{O}_f individually. However, if we insist on staying within the family of supersymmetric embeddings, the variations are related, and we only get a particular linear combination $\mathcal{O}_s \equiv \mathcal{O}_\theta + i\mathcal{O}_f$. This is the only expectation value we will have access to in the localization calculation. Note though that in $\langle \mathcal{O}_s \rangle = \langle \mathcal{O}_\theta \rangle + i\langle \mathcal{O}_f \rangle$, only a linear combination of α_2 and α_4 appears, and so in what follows $\beta \equiv 2\alpha_2 + \alpha_4 - \frac{3}{2} \log 2$ will be used. Combining the above condensates appropriately and identifying the field theory mass $M = m\mu$, we find

$$\frac{\mu}{T_0} \langle \mathcal{O}_s \rangle = 3c + \frac{2m^3}{3}(1 + 6\alpha_1) - \frac{m}{2}(7 + 4\beta), \quad (3.129)$$

where $T_0 = T_7 V_{S^3}$ and α_1, β are scheme-dependent coefficients of finite counterterms. That is, they are ambiguities in the renormalization procedure. Demanding the renormalization scheme to preserve supersymmetry on flat space/Poincaré AdS fixes $\alpha_1 = -\frac{5}{12}$ [54]. To translate T_0 in eq. (3.129) to field theory quantities, we use (see e.g. the table in [117])

$$T_0 V_4 / N_c^2 = \lambda \zeta / 6\pi^2 = 2\mu^2 \zeta / 3, \quad (3.130)$$

where V_4 denotes the volume of the unit S^4 . This in particular results in a free energy proportional to λ at strong coupling, which has long been recognized as a puzzling feature

of the probe brane analysis, and the localization calculation will have to reproduce that. Note that $V_4\langle\mathcal{O}_s\rangle = dF/dM$, so eq. (3.129) with eq. (3.117), eq. (3.130)) can be readily compared to the field theory side. For a plot see Fig. 1(b).

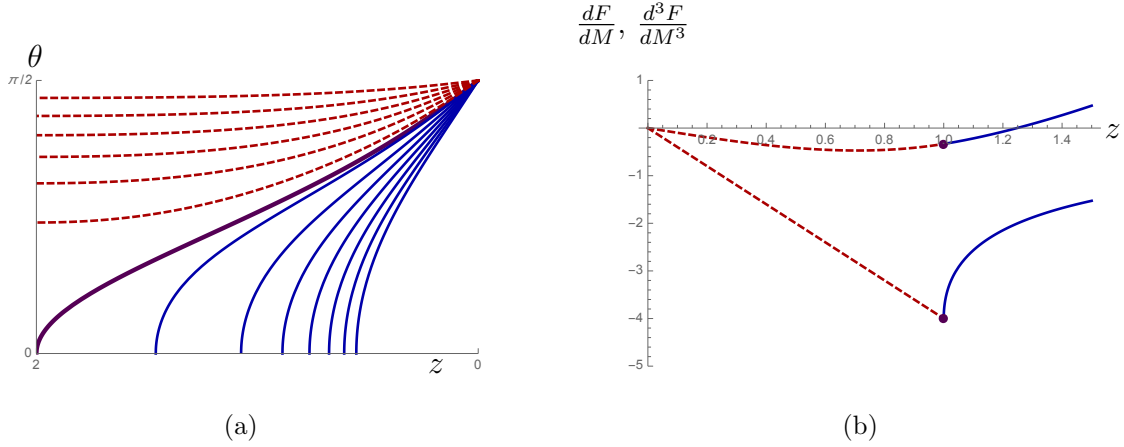


Figure 3.10: On the left hand side the slipping mode is shown for $m \in \{0, 0.025, \dots, 2\}$. The dashed red curves ending on the vertical axis are embeddings where the S^4 collapses at $z=2$. The solid blue curves ending on the horizontal axis are embeddings where the brane caps off at a $z_* \in (0, 2)$ with the internal S^3 collapsing. The critical embedding with $m = 1$, where the S^3 and S^4 collapse concurrently, is shown as the thick purple curve ending at the origin. The upper and lower curves on the right hand side show $dF/dM = V_4\langle\mathcal{O}_s\rangle$ and d^3F/dM^3 , respectively. The linear term in dF/dM is scheme dependent, and we subtracted off $2m \log \frac{z}{2}$ for the plot. Since the matrix model and holographic calculations match analytically, we only plot the curves once. The color/line coding reflects the kind of embedding from which the results are obtained holographically. In the matrix model calculation, the red dashed parts are obtained from (3.147) and the blue solid parts from (3.142). They meet at the purple dots, which correspond to the hyper moving on to the eigenvalue distribution.

On the matrix model side, analyses of massive large- N_c $\mathcal{N} = 2$ gauge theories on S^4 have seen peculiar, infinite in multitude, phase transitions as one takes the decompactification limit at strong coupling, when more and more resonances are excited on the eigenvalue distribution [116, 118]. In our setup on the probe brane side, we see exactly one, topology changing, transition between the phases where we have spacetime filling branes for $m < 1$ and branes that cap off smoothly for $m > 1$. The (quantum) critical point occurs exactly at $m = 1$, where the wrapped $S^3 \subset S^5$ collapses concurrently with the S^4 at the origin. To determine the critical exponent we expand eq. (3.129) around the critical embedding. For

$m = 1 + \epsilon$ with $\epsilon \ll 1$, we find

$$\frac{V_4}{\zeta \mu N_c^2} \langle \mathcal{O}_s \rangle \simeq -\frac{27 + 12\beta}{9} - \frac{13 + 4\beta}{3} \epsilon - 2\epsilon^2 + \frac{16\sqrt{2}}{15} \epsilon^{5/2} + \mathcal{O}(\epsilon^3). \quad (3.131)$$

The striking feature of this expansion is that we have full analytical control over extracting the critical exponents for this manifestly quantum phase transition. It should be mentioned that these are distinct from the study of non-SUSY flavors in [90]. The difference can be traced back to the imaginary gauge field which gives non-trivial cancellations in the action that modify the general scaling analysis of [90].

3.7 Localization with Quenched Flavors

Before we derive dF/dM on the field theory side, we review where the components in the matrix model originate. The localization calculation [1] begins by identifying a Grassmann scalar symmetry, Q , that is nilpotent up to gauge transformations. This procedure requires closure of the supersymmetry algebra off shell, which needs an appropriate set of auxiliary fields. After adding a Q -exact term $\delta_V = tQV$ to the Lagrangian, to which the partition function is insensitive, one can take the limit $t \rightarrow \infty$ and study the saddles of the path integral where δ_V vanishes. The partition function reduces to an integral over the locus in field space where $\delta_V = 0$.

In the study of $\mathcal{N} = 2$ gauge theories on S^4 , the locus ends up being where all of the fields vanish, except for one constant adjoint-valued scalar. Computing the 1-loop fluctuations about the locus exactly determines the partition function, up to instanton corrections. The latter are exponentially suppressed in the large N_c limit [119], and so we ignore them here. For $\mathcal{N} = 4$ SYM, the 1-loop determinants evaluate to unity, and one ends up solving a simple unitary Gaussian matrix model

$$Z = \int da^{N_c-1} \prod_{i < j} a_{[ij]}^2 e^{S_0}, \quad S_0 = -\frac{8\pi^2}{\lambda} N_c \sum_i a_i^2, \quad (3.132)$$

where $a_{[ij]} = a_i - a_j$ labels the roots of $\mathfrak{su}(N_c)$ and a_i labels the weights. The Vandermonde determinant factor $\prod_{i < j} a_{[ij]}^2$ comes from gauge fixing into the Cartan subalgebra. It provides a repulsive logarithmic interaction term for the eigenvalues.

For any $\mathcal{N} = 2$ gauge theory on S^4 with massive hypermultiplets, the 1-loop fluctuations can be encoded in the following mnemonic: for each vector multiplet, adjoint hyper, and fundamental hyper we acquire a 1-loop factor

$$\begin{aligned}
\text{Vector} &: \prod_{i < j} H^2(a_{[ij]}), \\
\text{Adjoint} &: \prod_{i < j} \frac{1}{H(a_{[ij]} + M_A)H(a_{[ij]} - M_A)}, \\
\text{Fundamental} &: \prod_{i, f} \frac{1}{\sqrt{H(a_i + M_f)H(a_i - M_f)}},
\end{aligned} \tag{3.133}$$

where $H(x) = G(1 + ix)G(1 - ix)$ and $G(x)$ is the Barnes G-function, M_A is the adjoint hyper mass, and M_f are independent flavor masses indexed by f .⁶ For notational ease, we will denote $H(x \pm M) \equiv H_{\pm}(x)$ and $x_{\pm} = x \pm M$. The result for $\mathcal{N} = 4$ SYM coupled to massive $\mathcal{N} = 2$ flavors is a matrix model given by

$$Z = \int d^{N_c-1} a \frac{\prod_{i < j} a_{[ij]}^2}{\prod_i \sqrt{H_+^{N_f}(a_i)H_-^{N_f}(a_i)}} e^{S_0}. \tag{3.134}$$

Rearranging such that all factors appear in the exponent, we get an integrand e^S with

$$S = S_0 - N_c \sum_i \frac{\zeta}{2} \log(H_+ H_-) + \sum_{i < j} \log a_{[ij]}^2. \tag{3.135}$$

The quenched approximation then becomes a matter of evaluating the partition function with $1 \ll N_f \ll N_c$. Note that the sums \sum_i and $\sum_{i < j}$ are $\mathcal{O}(N_c)$ and $\mathcal{O}(N_c^2)$, respectively. The calculation of the free energy can be organized according to an expansion in ζ by using $F \approx -S|_{\text{saddles}}$ and

$$\frac{S}{N_c^2} = \tilde{S}_0|_{\rho_0} + \zeta(S_1|_{\rho_0} + \delta\tilde{S}_0|_{\rho_0}) + \mathcal{O}(\zeta^2), \tag{3.136}$$

where $N_c^2 \tilde{S}_0 = S_0 + \sum_{i < j} \log a_{[ij]}^2$, and S_1 is the contribution of the flavors. Here we have denoted the solution to the Gaussian matrix model corresponding to pure $\mathcal{N} = 4$ SYM in

⁶This counting scheme for fundamental hypers differs from [116, 118, 120], where the authors count ‘fundamental’ and ‘anti-fundamental’ hypers with $\prod_i H^{-1}(a_i \pm M)$, respectively, suggestive of $\mathcal{N} = 1$ chiral and anti-chiral multiplets.

the continuum limit as ρ_0 , which is the Wigner semicircle distribution

$$\rho_0(x) = \frac{2}{\pi\mu^2} \sqrt{\mu^2 - x^2}, \quad (3.137)$$

where $\mu = \sqrt{\lambda}/2\pi$ is the maximum eigenvalue. Note that, since ρ_0 extremizes \tilde{S}_0 , $\delta\tilde{S}_0|_{\rho_0} = 0$. Thus, our analysis only requires the knowledge of $S_1|_{\rho_0}$.

Since we want to compare to AdS/CFT, in addition to $\zeta \ll 1$ we need to be working in the strong coupling, $\lambda \gg 1$, limit. The masses of our flavors in the field theory are set by the string tension and is of order $\sqrt{\lambda}$, and the typical eigenvalue contributing to our integral is also of order $\mu \sim \sqrt{\lambda}$. So the arguments of H are large, validating the use of the asymptotic expansion of the log derivatives

$$H'(x_{\pm})/H(x_{\pm}) = -x_{\pm} \log x_{\pm}^2 + 2x_{\pm} + \mathcal{O}(x_{\pm}^{-1}). \quad (3.138)$$

When $|M| < \mu$ and the hypers lie on the eigenvalue distribution, using the semicircle distribution and large-argument expansion is only justified outside of a region of width $1/\sqrt{\lambda}$ around $x = M$, where the hypers are parametrically light. But the contribution of that region is negligible at large λ . Consequently, the flavor contribution is

$$F' = \frac{\zeta N_c^2}{2} \int_{-\mu}^{\mu} dx \rho_0(x) [4M - x_+ \log x_+^2 + x_- \log x_-^2], \quad (3.139)$$

where $F' = dF/dM$. Integrating explicitly in the regime where $\mu < M$, we find

$$F' = \frac{\zeta N_c^2}{3\mu^2} \left[2\sqrt{M^2 - \mu^2}(M^2 + 2\mu^2) - 2M^3 + 3M\mu^2 \left(1 - 2 \log \frac{M + \sqrt{M^2 - \mu^2}}{2} \right) \right]. \quad (3.140)$$

In the matrix model, phase transitions can occur when some of the hypers become light, as demonstrated e.g. in $\mathcal{N} = 2^*$ in [118]. That is, as we take the hyper mass on to the eigenvalue distribution, $M \leq \mu$, there can be resonances driving the hypers effectively massless. Zooming in on the potential phase transition point, $M = m\mu$, with $m = 1 + \epsilon$ and expanding for $\epsilon \ll 1$, we find

$$\frac{dF/dm}{N_c^2 \mu^2 \zeta} = \frac{1}{3} - \log \frac{\mu^2}{4} - \epsilon \left(1 + \log \frac{\mu^2}{4} \right) - 2\epsilon^2 + \frac{16\sqrt{2}}{15} \epsilon^{5/2} + \mathcal{O}(\epsilon^3). \quad (3.141)$$

This expansion reproduces the non-analytic behavior and matches exactly the coefficients of all the $\epsilon^{(2n+1)/2}$ terms that were seen on the gravitational side in (3.131), which is strong evidence that the topology changing phase transition is captured by the transition associated with bringing hypers on to the distribution.

If we set the remaining scheme dependent counterterm to $\beta = -\frac{5}{2} + \frac{3}{2} \log \frac{\mu}{2}$, we can achieve an exact match of the holographic result eq. (3.129) with eq. (3.117), eq. (3.130) to eq. (3.140) for $\mu < M$:

$$\frac{V_4 \langle \mathcal{O}_s \rangle}{\zeta \mu N_c^2} = \frac{2}{3} \sqrt{m^2 - 1} (m^2 + 2) - \frac{2m^3}{3} + m \left[1 + 2 \log \frac{2(m - \sqrt{m^2 - 1})}{\mu} \right] = \frac{F'}{\zeta \mu N_c^2}. \quad (3.142)$$

Since we have a good holographic description also of $m < 1$ embeddings, we should be able to perform the same calculation of dF/dM also in the matrix model for $M < \mu$. In this case, the mass of the hypermultiplets sits directly on the Wigner distribution. As was seen in, e.g. [116], were it not for the strong coupling and probe limits being taken in this calculation, the non-trivial backreaction on the semicircular distribution can be problematic. However as was argued previously, any correction to the Wigner distribution is suppressed outside of a parametrically small window of width $1/\sqrt{\lambda}$.

Even though we are examining the regime in which $M < \mu$, the strong coupling limit still facilitate a large argument expansion the integrand of dF/dM eq. (3.139) which becomes

$$\frac{2F'}{\zeta N_c^2} = \int_{-\mu}^{\mu} dx \rho_0(x) [4M - x_+ \log x_+^2 + x_- \log x_-^2]. \quad (3.143)$$

The first term in the integrand is just a constant multiplying the Wigner distribution and is fixed by its normalization. The remaining part of the integrand can be split up into the $x_{\pm} \log x_{\pm}^2$ parts. Rewriting our variables in the usual way, $M = m\mu$ and $x = (a \mp m)\mu$, we find

$$\frac{2F'}{\zeta N_c^2} = 4m\mu - I(m) + I(-m), \quad (3.144)$$

where we have introduced the convenient short hand

$$I(m) = \frac{2\mu}{\pi} \int_{m-1}^{m+1} da a \log(a^2 \mu^2) \sqrt{1 - (a - m)^2}. \quad (3.145)$$

The remaining integrals to take can easily be calculated analytically, which results in

$$\frac{1}{\zeta N_c^2} \frac{dF}{dM} = \frac{m\mu}{3} (3 - 2m^2 - 6 \log \frac{\mu}{2}) . \quad (3.146)$$

Thus, we indeed find again perfect agreement in this regime:

$$V_4 \langle \mathcal{O}_s \rangle = m\mu \zeta N_c^2 \left(1 - \frac{2}{3} m^2 - 2 \log \frac{\mu}{2} \right) = F' . \quad (3.147)$$

3.8 Discussion

The main results of this work begins with the construction of a class of supersymmetric D7-brane embeddings into $\text{AdS}_5 \times \text{S}^5$, which allows to holographically describe $\mathcal{N} = 4$ SYM coupled to massive flavor hypermultiplets on spaces of constant curvature. We studied AdS_4 as a natural starting point– from the point of view of supersymmetry– and continued to S^4 and dS_4 afterwards. Preserving supersymmetry in the generalization from flat to curved space needs additional care for non-conformal theories, and in particular requires extra compensating terms to make mass terms supersymmetric [59]. This needs to be taken into account in the construction of holographic duals as well. For $\mathcal{N} = 2^*$ on S^4 the holographic construction was carried out in [79], and for flavored $\mathcal{N} = 4$ SYM on constant-curvature spaces in this work. We focused on AdS_4 for the main part, and found embeddings in a simple analytic form, for which the flavor mass term preserves $\mathcal{N} = 2$ supersymmetry.

In Sec. 3.2 we systematically went through the κ -symmetry discussion, resulting in the embeddings given in Sec. 3.2.5. The method can be generalized straightforwardly to obtain analogous embeddings for other geometries. In § 3.5 we complete the list, to geometries where AdS_5 is sliced by spacetimes of constant curvature. This in particular includes (global) Euclidean AdS_5 , where the dual theory is defined on S^4 . That latter geometry naturally suggests itself to conducting a test for probe-brane holography using localization, and we address that in a separate paper. Another example for future applications is the computation of the superconformal index for flavored $\mathcal{N} = 4$ SYM, for which one needs the analogous embeddings into $\text{S}^1 \times \text{S}^3$ -sliced AdS_5 .

In this work we have focused on AdS_4 -sliced AdS_5 , which is a preferred choice among the constant-curvature spaces in Lorentzian signature. The corresponding supergroups have unitary representations and in contrast to dS it is relatively straightforward to realize supersymmetric QFTs on that background. Holographically, we naturally get two copies of AdS_4 as boundary geometries. Each one is the boundary in one of the two coordinate patches needed to cover AdS_5 with AdS_4 slices (as illustrated in Fig. 3.1(a)). In Sec. 3.3 we discussed in detail the families of regular massive D7-brane embeddings, and how they can be combined for the two patches. That revealed a surprisingly rich set of options for small masses. For generic large masses, the D7-branes cap off in the coordinate patch where they extend to the conformal boundary, much like they do on Poincaré AdS. This feature of those “short” embeddings reflects that the massive flavors do not affect the deep IR of the QFT, as they are simply gapped out. On Poincaré AdS this is the generic behavior, regardless of the value of the mass [15]. For the AdS_4 slices, on the other hand, we found that for small masses there are also “long” brane embeddings, which cover all of the patch in which they extend to the conformal boundary. They extend into the second patch and cap off at a finite value of the radial coordinate there. See Fig. 3.10 for an illustration. In addition to that, we found families of connected embeddings, which cover both copies of AdS_4 on the conformal boundary, again also for small but finite values of the flavor mass. Similar non-supersymmetric embeddings had been found numerically in [91] before. We found that the connected embeddings become available below the same critical mass as the long embeddings. They then come in one-parameter families for each fixed mass, corresponding to different values for the chiral and scalar condensates. Generically, that family of connected embeddings interpolates between long and short embeddings, as shown in Fig. 3.5. For the particular case of massless flavors, we also find a one-parameter family of connected embeddings, which includes the one conformally related to massless flavors on flat space as one special case. A kind of phase diagram summarizing the possible embeddings can be found in Fig. 3.2(b).

The embeddings stretching all through at least one of the coordinate patches, which are available for low enough values of the mass, suggest that those flavors can indeed affect

also the deep IR regime of the dual QFT, despite their non-zero mass. A natural candidate feature of QFT on AdS₄, that is suggestive of this behavior, is the possibility of having stable negative-mass scalars [121, 122]. We turned to a more detailed discussion focussing on the QFT side in Sec. 3.4, where we carried out the holographic renormalization and computed the one-point functions. For the massless embeddings we found that the one which is familiar already from Poincaré AdS is the only one where chiral symmetry is not spontaneously broken. All others come with non-vanishing (constant) chiral and scalar condensates. We also gave a simple toy model to explain the one-parameter family of massless embeddings, based on the possible choices of boundary conditions at the conformal boundary for each of the AdS₄ spaces. For the massless embeddings we could employ a conformal map to two halves of Minkowski space, which maps the constant condensates on AdS₄ to position-dependent condensates on Minkowski space. For a family of free theories with boundary conditions interpolating between transparent and either Dirichlet or Neumann boundary conditions, we found exactly the position-dependent scalar condensates that came out of our holographic calculation. This suggests that going through the different embeddings available at zero mass might be a holographic analog of the transitions studied in [123].

For massive embeddings we suspect a qualitatively similar reason behind the families of embeddings. The option for standard and alternative quantization (imposing the analog of Dirichlet and Neumann b.c. on AdS) for scalars on AdS has been studied already in [121, 122] as well. If the reason for the existence of those curious embeddings were rooted in the availability of extra boundary conditions, this would indeed also explain why they are possible only for small masses: the choice of boundary condition is only available for $m_{\text{BF}}^2 \leq m^2 < m_{\text{BF}}^2 + 1$. As explained in Sec. 3.4, however, none of the masses of the fundamental fields is in that that window – at least not in the regime where the bulk theory can be described by weakly-coupled supergravity. This suggests the mesons as natural candidates, and it would be interesting to carry out the fluctuation analysis to determine the meson spectrum. The mesons analysis promises an interesting structure also due to the variety of possibilities to combine quarks from each of the two AdS₄ geometries. Naively, one would expect that

mesons built out of quarks which are localized in the same AdS_4 geometry should be light, and mesons built out of one quark from each AdS_4 part should be very heavy. However, the existence of the connected embeddings suggests that this is only true for large masses: for small masses and connected embeddings also the mixed mesons simply correspond to infinitesimal fluctuations of the D7-branes, as opposed to macroscopic strings stretching between different branes. So, calculating the meson spectrum with our analytic embeddings should not only be feasible, but indeed contribute an interesting piece of information to the story.

The second part of the calculation concerned the phase structure of $\mathcal{N} = 4$ SYM coupled to massive $\mathcal{N} = 2$ flavor hypermultiplets on S^4 , using holography and direct QFT computations independently. The crucial ingredients to allow for localization of the path integral on the field theory side are the preservation of some supersymmetry and the formulation on a compact space. Holographically, this translates to the supersymmetry of the D7-brane embeddings we derived from the continuation of the AdS sliced embedding. We found one continuous phase transition at the same value of the flavor mass in both calculations, and analytically matching dF/dM in both phases. The remaining constant, which enters when this relation is integrated to get the free energies, is scheme dependent. So matching the free energies themselves then merely amounts to choosing compatible renormalization schemes.

Our results give strong support to the validity of the probe brane constructions used so frequently in AdS/CFT. The theories we studied are non-conformal, and the quantities we compared are not special, in the sense that they are not extrapolated from weak to strong coupling using non-renormalization theorems. Moreover, the theory described by the D3/D7 setup has a non-trivial UV fixed point only in the quenched approximation, which frequently means that extra care is needed when establishing the validity of holographic results. The fact that we found such nicely matching results therefore truly provides a non-trivial test of the dualities.

In the next chapter, another of the standard type IIB probe brane systems, D3/D5, [58], will be considered as a bulk dual candidate for a boundary theory employing supersymmetric

localization on S^4 with $\mathcal{N} = 2$ (from the $d = 4$ perspective) hypermultiplets living on an S^3 [57]. However, there are many more theories where supersymmetric localization has been carried out in a context that has immediate import into probe brane systems such as computing supersymmetric indices [124, 125, 126], higher co-dimension defect theories [127], or using κ -symmetry with type IIA probe branes to study $d = 3$ supersymmetric localization where some interesting generalizations have recently been construction [128, 77, 75, 76]. The crucial ingredient on the holographic side in either case will be to find the corresponding supersymmetric embeddings. It would also be desirable to further investigate why other massive $\mathcal{N} = 2$ theories see curiously rich phase structures.

Chapter 4

SUPERSYMMETRIC D3/D5 AND LOCALIZATION

This chapter is based on collaborative work— to appear— with Christoph Uhlemann, whom provided insight into calculations and interpretations as well as work in finding the final form of the non-linear kappa symmetry equations and the general linear solutions.

4.1 Introduction

In the previous chapter, we studied the construction of supersymmetric D7-brane embeddings into $\text{AdS}_5 \times \text{S}^5$ backgrounds where the AdS_5 space had non-trivial constant curvature spaces \mathcal{M}_4 on each radial slice submanifold. Of particular interest was the case where $\mathcal{M}_4 = \text{S}^4$ where the results allowed for the formulation of a precision test of AdS/CFT where the boundary $\mathcal{N} = 4$ SYM theory was deformed by massive flavor fields in a $\mathcal{N} = 2$ hypermultiplet. Such a boundary theory, by virtue of a particular off-shell construction introducing an superpotential mass term for the hypermultiplet, admitted analysis by supersymmetric localization such that the statements compared on either side of the duality were one-loop exact. The success of the calculation as seen in the analytic matching of derivatives of the free energy with respect to field theory mass in the boundary and the holographic one-point function for a combination of worldvolume fields gives the impetus to ask if the question that motivated the κ -symmetry analysis could be turned around to use probe branes to give information about localized theories and perhaps shed light on new quantities that can be calculated in such reduced descriptions.

In this chapter, that question will be addressed by taking the lessons learned in the D7-brane embedding to construct supersymmetric D5-brane embeddings into constant curvature sliced AdS_5 . The D5 probe brane system has long been— and continues to be [129, 130, 131]—

studied as a paradigmatic system in which to discuss AdS/dCFT with flavors (see 2.1.1 reference). In short, the primary difference between this and the supersymmetric D7-brane embedding that was constructed in the previous chapter is that the branes are now wrapping an AdS_4 submanifold in the AdS_5 geometry— and to be clear an $S^2 \subset S^5$ as well. Thus, the maximal subgroup of the original group of superconformal symmetries, at least its bosonic part, is $SO(3, 2) \times SU(2)_H \times SU(2)_V \times U(1)$. From the boundary theory this perspective localizes (geometrically speaking) the hypermultiplet to a co-dimension one defect coupled to the pullback of the ambient $\mathcal{N} = 2$ vector multiplet onto the defect hypersurface. This is a relatively simple setup in the field theory, but the extra degree of freedom describing fluctuations of the brane transverse to its defect location has rather drastic import into the κ symmetry analysis and construction of the BPS equations for the worldvolume fields. Even in static gauge, which we keep for the Lorentzian AdS sliced portion of the calculation, the generically first order, coupled, non-linear ODEs that characterized the supersymmetric D7-brane embedding become equally complicated PDEs, which greatly reduce the chance of analytic solubility in the general case.

Moreover, the story of localization of $\mathcal{N} = 4$ SYM theory on an S^4 in the presence of defect operators is much less well understood in most instances. There have been some examples where explicit calculations have been done beyond line operator insertions [127], but theories with domain walls and those defined on manifolds with boundaries are still poorly understood and are being actively investigated [83, 75]. That leaves the supersymmetric D5-brane constructed below without the same level of guidance that was present in the previous chapter for what to expect to comprise the set of supersymmetric observables calculable in the holographic framework. Indeed, at the end of this chapter, there will be indications that there may be a whole tower of non-trivial observables revealed holographically calculable in supersymmetric localization that were previously unexamined.

To systematically construct the holographic story, we will first establish the conventions and pertinent worldvolume field content for the D5-brane embedding in Lorentzian AdS_4 sliced AdS_5 and note the distinctions to be made in the κ -symmetry analysis as compared

to how it functions for the D7-brane. Much of the overall strategy is imported directly from the previous chapter, but the analysis has strikingly different complications. In the subsequent sections, the full non-linear κ symmetry equations will be calculated by first solving constraints for the worldvolume gauge field in terms of the scalars describing the embedding and reducing the set of eight BPS equations down to two. The general linearized solutions to reduced BPS equations will be calculated with an infinite family of (largely irrelevant) supersymmetric deformations revealed. The mass deformation beyond linear order will be found, and all of these solutions will be continued to S^4 sliced AdS_5 embeddings with the defect hypermultiplet living on an equatorial S^3 in the boundary theory. A framework for the matrix model after supersymmetric localization is established and checked against the holographic computation much as in the D7 case, where we find exact matching between the D5 analog of $\langle \mathcal{O}_{SUSY} \rangle$ and $\frac{dF}{dM}$ to cubic order in small mass.

4.2 Geometry and embedding ansatz

In this section, we will follow the same line of reasoning as in the previous chapter by considering D5-brane embeddings in a Lorentzian AdS sliced bulk geometry. This will avoid any subtleties associated with Euclidean signature. As with the D7-brane embedding, we begin with $\text{AdS}_5 \times S^5$ in the following coordinate system

$$g_{\text{AdS}_5 \times S^5} = d\rho^2 + \cosh^2 \rho g_{\text{AdS}_4} + d\theta^2 + \cos^2 \theta g_{S^2} + \sin^2 \theta g_{\tilde{S}^2} . \quad (4.1)$$

The particular form for the radial slices will be that of AdS_4 with AdS_3 surfaces at constant values of the AdS_4 radial coordinate r , i.e.

$$g_{\text{AdS}_4} = dr^2 + \cosh^2 r g_{\text{AdS}_3} , \quad g_{\text{AdS}_3} = dx^2 + e^{2x}(-dt^2 + dy^2) . \quad (4.2)$$

Further, the explicit form for the coordinates on the S^2 and \tilde{S}^2 will be chosen respectively as such that

$$g_{S^2} = d\beta_1^2 + \sin^2 \beta_1 d\beta_2^2 , \quad g_{\tilde{S}^2} = d\alpha_1^2 + \sin^2 \alpha_1 d\alpha_2^2 . \quad (4.3)$$

To describe the embedding of the D5-branes in the the background eq. (4.1), we will use the, by now in this thesis painfully, familiar DBI action– using our noted convention for the string length $2\pi\alpha' = 1$ ¹

$$S_{\text{D5}} = -T_5 \int_{\Sigma_6} d^6\xi \sqrt{-\det(g + F)} + T_5 \int_{\Sigma_6} C_4 \wedge F , \quad (4.4)$$

where g is the pullback of the background spacetime metric on to the worldvolume Σ_6 . For the 4-form gauge field evaluated using the Freund-Rubin ansatz²

$$C_4 = L^{-1}\zeta(\rho) \text{vol}(\text{AdS}_4) + \dots , \quad \zeta'(\rho) = 4 \cosh^4 \rho . \quad (4.5)$$

Further, the gauge fixing of coordinate reparameterizations into static gauge gives us that $(r, x, t, y, \vec{\beta})$ span the worldvolume where the embedding functions– neè slipping and bending modes– take the form

$$\theta = \theta(r, \vec{\beta}) , \quad \rho = \rho(r, \vec{\beta}) . \quad (4.6)$$

To round out the listing of the ansätze for the embedding, we will take heed from the previous D7-brane analysis once again– and from [59, 78]– and posit that a worldvolume gauge field is necessary and is given by

$$A = A_1(r, \vec{\beta})d\beta_1 + A_2(r, \vec{\beta})d\beta_2 . \quad (4.7)$$

4.2.1 Killing spinors

As we proceed turning the crank on the machinery introduced in the previous chapter, we will need to specify again the background Killing spinors on which the κ -symmetry projector will act to ensure worldvolume supersymmetry. To that end, the Killing spinor equation for

¹The sign of the WZ term is that of e.g. [93, 16, 97] and [17]. It's different from the choice in [57].

²This is the normalization for gauge field and the WZ term of [57, 97], where $dC_4 = 4L^{-1} \text{vol}(\text{AdS}_5)$. Note that we used a different normalization in [17], which is why we had 2 instead of 1/2 as coefficient of the WZ term.

the above parameterization of $\text{AdS}_5 \times \text{S}^5$ in the conventions of [23]– alternatively one could use [63]– read

$$D_\mu \epsilon = \frac{i}{2} \Gamma_{\text{AdS}} \Gamma_\mu \epsilon, \quad \mu = 0 \dots 4, \quad D_\mu \epsilon = \frac{i}{2} \Gamma_{\text{S}^5} \Gamma_\mu \epsilon, \quad \mu = 5 \dots 9, \quad (4.8)$$

Owing to the fact that the trivial product structure of the background geometry allows one to solve the Killing spinor equations on the submanifolds independently, the solutions to eq. (4.8) are schematically found to be

$$\epsilon = R_{\text{AdS}} R_{\text{S}^5} \epsilon_0. \quad (4.9)$$

Explicitly, the solution for R_{AdS} is expressed with the AdS_3 projector $P_{x\pm} = \frac{1}{2}(\mathbb{1} \pm i\Gamma_{\underline{x}}\Gamma_{\text{AdS}})$, is

$$R_{\text{AdS}} = e^{\frac{i\rho}{2}\Gamma_{\underline{\rho}}\Gamma_{\text{AdS}}} e^{\frac{i\tau}{2}\Gamma_{\underline{\tau}}\Gamma_{\text{AdS}}} R_{\text{AdS}_3}, \quad R_{\text{AdS}_3} = e^{\frac{i\mathbf{x}}{2}\Gamma_{\underline{\mathbf{x}}}\Gamma_{\text{AdS}}} + ie^{\frac{\mathbf{x}}{2}} \left(t\Gamma_{\underline{t}} + y\Gamma_{\underline{y}} \right) \Gamma_{\text{AdS}} P_{x-}. \quad (4.10)$$

Likewise, the solution for R_{S^5} is explicitly given with the convenient definitions of $\Gamma_{\underline{\alpha}} = \Gamma_{\underline{\alpha}_1} \Gamma_{\underline{\alpha}_2}$ and analogously for $\Gamma_{\underline{\beta}}$, by

$$R_{\text{S}^5} = e^{\frac{i\theta}{2}\Gamma_{\underline{\theta}}\Gamma_{\text{S}^5}} R_{\tilde{\text{S}}^2} R_{\text{S}^2}, \quad R_{\tilde{\text{S}}^2} = e^{\frac{\alpha_1}{2}\Gamma_{\underline{\theta}}\Gamma_{\underline{\alpha}_1}} e^{\frac{\alpha_2}{2}\Gamma_{\underline{\alpha}}}, \quad R_{\text{S}^2} = e^{\frac{i\beta_1}{2}\Gamma_{\underline{\beta}_1}\Gamma_{\text{S}^5}} e^{\frac{\beta_2}{2}\Gamma_{\underline{\beta}}}. \quad (4.11)$$

With the solutions for the Killing spinors in hand, we can now proceed with the calculation of the other primary players in the κ -symmetry analysis. Note that for later convenience, we define $\tilde{R}_{\text{AdS}} = C(R_{\text{AdS}})^*C$ and analogously for the other R -matrices, which then allows for the identification of useful relations:

$$\tilde{R}_{\text{AdS}} = e^{-i\rho\Gamma_{\underline{\rho}}\Gamma_{\text{AdS}}}\Gamma_{\underline{\rho}}R_{\text{AdS}}\Gamma_{\underline{\rho}}, \quad \tilde{R}_{\text{S}^5} = -e^{-i\theta\Gamma_{\underline{\theta}}\Gamma_{\text{S}^5}}\Gamma_{\underline{\beta}}R_{\text{S}^5}\Gamma_{\underline{\beta}}. \quad (4.12)$$

4.2.2 κ -symmetry generalities

Now, we may proceed with the evaluation of Γ_κ as defined in eq. (2.20) adapted specifically for a D5-brane embedding

$$\Gamma_\kappa = \frac{1}{\sqrt{\det(1+X)}} \left(J_{(5)}^{(0)} + \frac{1}{2}\gamma^{jk}F_{jk}J_{(5)}^{(1)} \right), \quad J_{(5)}^{(n)} = (-1)^n(\sigma_3)^{n+1}i\sigma_2 \otimes \Gamma_{(0)}, \quad (4.13)$$

Since F only has non-trivial components in the $r, \vec{\beta}$ directions, the sum in Γ_κ terminates after the linear term instead of the quadratic term as was seen in the previous chapter. As in Ch. 3, it is more convenient to express action of J on the type IIB Killing spinor by switching to complex notation

$$J_{(5)}^{(0)} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} = iC (\Gamma_{(0)}\epsilon)^* , \quad J_{(5)}^{(1)} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} = i\Gamma_{(0)}\epsilon . \quad (4.14)$$

Once this change in notation is affected, the κ -symmetry condition can succinctly be expressed as

$$iC (\Gamma_{(0)}\epsilon)^* + \frac{i}{2} \gamma^{ij} F_{ij} \Gamma_{(0)}\epsilon = \sqrt{\det(1+X)}\epsilon , \quad \Gamma_{(0)} = \frac{1}{6! \sqrt{-\det g}} \varepsilon^{i_1 \dots i_6} \gamma_{i_1 \dots i_6} , \quad (4.15)$$

where $\gamma_i = e_i^a \Gamma_a$ and $e^a = E_\mu^a (\partial_i X^\mu) dx^i$ is the pullback of the ten-dimensional vielbein E^a to the D5 worldvolume.

To evaluate $\Gamma_{(0)}$ we need the exact form of the pullback of the background Lorentz frame, which has the easy to peel off components:

$$e^{\alpha_1} = e^{\alpha_2} = 0 , \quad e^a = E^a , \quad a = \underline{\beta}_1, \underline{\beta}_2, \underline{t}, \underline{y}, \underline{x}, \underline{r} . \quad (4.16)$$

For the less trivial components, it is convenient—especially when we get deeper into the calculations—to introduce the indices $\mathbf{m}, \mathbf{n} = \{r, \beta_1, \beta_2\}$, such that the remaining parts of the pullback read

$$e^\theta = d\theta = (\partial_{\mathbf{m}}\theta) d\xi^{\mathbf{m}} , \quad e^\rho = d\rho = (\partial_{\mathbf{m}}\rho) d\xi^{\mathbf{m}} . \quad (4.17)$$

Having found the necessary ingredients, i.e. the pullback of the vielbein, we can now explicitly write the universal part of the κ -symmetry projector as

$$\Gamma_{(0)} = \frac{1}{\sqrt{-\det g}} \gamma_{r\beta_1\beta_2} = \frac{\cosh^3 \rho \cosh^3 r \sqrt{-\det g_{\text{AdS}_3}}}{\sqrt{-\det g}} \hat{\Gamma} . \quad (4.18)$$

Here we have denoted the metric on the unit AdS_3 as g_{AdS_3} , and $\hat{\Gamma}$ is given by

$$\hat{\Gamma} = \Gamma_{\text{AdS}_3} \gamma_{r\beta_1\beta_2} , \quad \Gamma_{\text{AdS}_3} = \Gamma^{\underline{xy}} = -\Gamma_{\underline{xy}} . \quad (4.19)$$

The explicit expressions for the leftover (worldvolume) γ -matrices are determined with the help of the pullback of the vielbein such that

$$\gamma_m = \Gamma_m + (\partial_m \rho) \Gamma_\rho + (\partial_m \theta) \Gamma_\theta . \quad (4.20)$$

Note that the (background) Γ -matrices involve the diagonal $\text{AdS}_5 \times S^5$ vielbein.

Pulling together all of the above calculations and definitions, the κ -symmetry condition then becomes

$$iC(\hat{\Gamma}\epsilon)^* + \frac{i}{2}\gamma^{ij}F_{ij}\hat{\Gamma}\epsilon = h\epsilon , \quad h = \frac{\sqrt{-\det(g+F)}}{\cosh^3 \rho \cosh^3 r \sqrt{-\det g_{\text{AdS}_3}}} . \quad (4.21)$$

These last two equations, together with the Killing spinors given previously, are all of the pieces that need to be evaluated. We can simplify the above expressions a bit further. First, there are no explicit factors of i in $\hat{\Gamma}$, so we can use $C^2 = \mathbb{1}$ and $C(\Gamma^\mu)^*C = \Gamma^\mu$ to get

$$i\hat{\Gamma}C\epsilon^* + \frac{i}{2}\gamma^{ij}F_{ij}\hat{\Gamma}\epsilon = h\epsilon . \quad (4.22)$$

Second, evaluating h a little more explicitly yields $h = \sqrt{\det(g_{mn} + F_{mn})}$, where

$$g_{mn} = (g_{\text{AdS}_5 \times S^5})_{mn} + (\partial_m \rho)(\partial_n \rho) + (\partial_m \theta)(\partial_n \theta) . \quad (4.23)$$

It is the focussed analysis of eq. (4.22) that will consume the bulk of the remaining work. As we will see, the departure from the simple construction in Ch. 3 introduces an unforeseen set of complexities that make the full non-linear analysis challenging.

4.3 Linearized κ -symmetry

As a basic step in constructing and solving the equations that follow from eq. (4.22), we first need to figure out what symmetries are preserved by the massless embedding with $\rho = A = \theta = 0$. This will afford us the most basic projector that is consistent with $\frac{1}{2}$ -BPS embeddings that additionally preserve the superconformal symmetries. To facilitate further, non-linear, analysis, we will denote the order at which terms are constructed in a perturbative expansion in the bulk ‘mass’ parameter by a superscript in parenthesis, i.e.

$$\hat{\Gamma}^{(0)} = -h^{(0)}\Gamma_{\underline{\rho}}\Gamma_{\text{AdS}}\Gamma_{\underline{\beta}} , \quad h^{(0)} = \sin \beta_1 . \quad (4.24)$$

Note that $\Gamma_{\text{AdS}} = \Gamma^{01234} = -\Gamma_{01234}$. For the massless embedding the κ -symmetry condition (4.22) becomes

$$i\hat{\Gamma}^{(0)}C\epsilon^{(0)\star} = h^{(0)}\epsilon^{(0)} \quad \Leftrightarrow \quad -i\Gamma_{\underline{\rho}}\Gamma_{\text{AdS}}\Gamma_{\underline{\beta}}C\epsilon^{(0)\star} = \epsilon^{(0)} . \quad (4.25)$$

Denoting $C(R_{\text{AdS}})^{\star} = \tilde{R}_{\text{AdS}}C$ and analogously for R_{S^5} , the resulting projection condition on the constant spinor ϵ_0 in $\epsilon = R_{\text{AdS}}R_{\text{S}^5}\epsilon_0$ is

$$-iR_{\text{AdS}}^{-1}R_{\text{S}^5}^{-1}\Gamma_{\underline{\rho}}\Gamma_{\text{AdS}}\Gamma_{\underline{\beta}}\tilde{R}_{\text{AdS}}\tilde{R}_{\text{S}^5}C\epsilon_0^{\star} = \epsilon_0 . \quad (4.26)$$

With (4.12) and $\rho = \theta = 0$ corresponding to the exactly massless embedding, the resulting projection condition acting on the constant spinor is

$$-i\Gamma_{\underline{\rho}}\Gamma_{\text{AdS}}\Gamma_{\underline{\beta}}C\epsilon_0^{\star} = \epsilon_0 . \quad (4.27)$$

This gives us a projection condition that must be obeyed by all D5 embeddings in order to preserve half of the supersymmetries. So, we can keep this in our catalog for use in the κ -symmetry equations at linear order in mass.

4.3.1 Linearized massive embeddings

Moving on from the massless embedding, we now need to find the projection condition that will remove the superconformal symmetry generators while still preserving half of the supersymmetries. That is, we need a linear order projector that commutes with the one found in eq. (4.27). To that end, we solve eq. (4.22) at linear order. As a starting point, we will use an ansatz for the slipping mode $\theta = \theta(r)$ that will be relaxed when studying the non-linear embeddings. With this in mind, eq. (4.22) at linear order reads

$$i\hat{\Gamma}^{(1)}C\epsilon^{(0)\star} + i\hat{\Gamma}^{(0)}C\epsilon^{(1)\star} + \frac{i}{2}\gamma^{(0)ij}F_{ij}^{(1)}\hat{\Gamma}^{(0)}\epsilon^{(0)} = h^{(0)}\epsilon^{(1)} . \quad (4.28)$$

On the right hand side we have used that corrections to h are at least quadratic, so $h^{(1)} = 0$. One subtlety that should not be overlooked is that the Killing spinor also contains information

about the order to which we are working in this perturbative analysis. As such, the linear order Killing spinor take the form:

$$\epsilon^{(1)} = \frac{i}{2} \left(\rho \Gamma_{\underline{\rho}} \Gamma_{\text{AdS}} + \theta \Gamma_{\underline{\theta}} \Gamma_{S^5} \right) \epsilon^{(0)} =: \delta R \epsilon^{(0)} . \quad (4.29)$$

Applying the charge conjugation operator yields $C(\delta R)^* C = -\delta R$ as one can see from the explicit factors of i . We then find

$$i \left(\hat{\Gamma}^{(1)} - \hat{\Gamma}^{(0)} \delta R \right) C \epsilon^{(0)*} + \frac{i}{2} \gamma^{(0)ij} F_{ij}^{(1)} \hat{\Gamma}^{(0)} \epsilon^{(0)} = h^{(0)} \delta R \epsilon^{(0)} . \quad (4.30)$$

Using $(\hat{\Gamma}^{(0)})^2 = h^{(0)2} \mathbf{1}$, the massless projection condition eq. (4.27) can be written as $C \epsilon^{(0)*} = -i \hat{\Gamma}^{(0)} \epsilon^{(0)} / h^{(0)}$. So we can eliminate $C \epsilon^{(0)*}$ such that all of the Clifford algebra structures act on unconjugated Killing spinors, which leads to

$$\frac{1}{h^{(0)}} \left(\hat{\Gamma}^{(1)} - \hat{\Gamma}^{(0)} \delta R \right) \hat{\Gamma}^{(0)} \epsilon^{(0)} + \frac{i}{2} \gamma^{(0)ij} F_{ij}^{(1)} \hat{\Gamma}^{(0)} \epsilon^{(0)} = h^{(0)} \delta R \epsilon^{(0)} . \quad (4.31)$$

While it is not entirely obvious, a short calculation gives a useful relation: $\hat{\Gamma}^{(0)} \delta R \hat{\Gamma}^{(0)} = h^{(0)2} \delta R$. Applying this to the most recent expression for the linear order κ -symmetry condition yields

$$\frac{1}{h^{(0)}} \hat{\Gamma}^{(1)} \hat{\Gamma}^{(0)} \epsilon^{(0)} + \frac{i}{2} \gamma^{(0)ij} F_{ij}^{(1)} \hat{\Gamma}^{(0)} \epsilon^{(0)} = 2h^{(0)} \delta R \epsilon^{(0)} . \quad (4.32)$$

For the computation of the term involving the field strength, it is useful to note that $\gamma^{(0)ij} F_{ij} = \Gamma^{ij} F_{ij}$, since for $\rho = \theta = 0$ the induced metric g just coincides with the corresponding part of $g_{\text{AdS}_5 \times S^5}$ for the components of interest. At this point for explicit calculation, we will need to employ an ansatz for the gauge potential. One that accords with intuition gleaned from the previous chapter and the form used for the slipping mode ansatz is

$$A_i = f(r) \omega_i(\vec{\beta}) , \quad (4.33)$$

such that $A = f\omega$ with ω a one-form on S^2 . To line up with [57] we would hope that the κ -symmetry analysis forces ω to be co-exact on the S^2 , but rather than enforce that from the outset, we will keep ω general for the time being.

For this subsection we use the shorthand notation $\omega_{\beta_{1/2}} = \omega_{1/2}$, $\partial_{\beta_{1/2}} = \partial_{1/2}$ and a prime to denote ∂_r . Setting all of the $^{(0)}$ to appropriate values– i.e. using $\rho = \theta = 0$ – we then find

$$\frac{1}{2}\gamma^{(0)ij}F_{ij}^{(1)} = f'\Gamma_r \left(\omega_1\Gamma_{\underline{\beta}_1} + \omega_2 \csc \beta_1\Gamma_{\underline{\beta}_2} \right) + f(\star d\omega)\Gamma_{\underline{\beta}}, \quad (4.34)$$

where $\star d\omega = \csc \beta_1(\partial_1\omega_2 - \partial_2\omega_1)$. For the combination which appears in (4.32) this yields

$$\frac{i}{2}\gamma^{(0)ij}F_{ij}^{(1)}\hat{\Gamma}^{(0)} = i \left[f'\Gamma_r \left(\omega_2\Gamma_{\underline{\beta}_1} - \sin \beta_1\omega_1\Gamma_{\underline{\beta}_2} \right) + f \sin \beta_1(\star d\omega) \mathbf{1} \right] \Gamma_{\underline{\rho}}\Gamma_{\text{AdS}} \quad (4.35)$$

The last piece we have to work out is $\hat{\Gamma}^{(1)}$ with $\hat{\Gamma}$ given in (4.19). The combination appearing in (4.32) becomes

$$\frac{1}{h^{(0)}}\hat{\Gamma}^{(1)}\hat{\Gamma}^{(0)} = \sin \beta_1 \left(\theta'\Gamma_{\underline{\theta}} + \rho'\Gamma_{\underline{\rho}} \right) \Gamma_r + \sin \beta_1(\partial_1\rho)\Gamma_{\underline{\rho}}\Gamma_{\underline{\beta}_1} + (\partial_2\rho)\Gamma_{\underline{\rho}}\Gamma_{\underline{\beta}_2}. \quad (4.36)$$

Assembling all of the above expressions for the components of (4.32) and dividing by $\sin \beta_1$ yields

$$0 = \left[\theta'\Gamma_{\underline{\theta}}\Gamma_r + \rho'\Gamma_{\underline{\rho}}\Gamma_r + (\partial_1\rho)\Gamma_{\underline{\rho}}\Gamma_{\underline{\beta}_1} + \csc \beta_1(\partial_2\rho)\Gamma_{\underline{\rho}}\Gamma_{\underline{\beta}_2} - i\rho\Gamma_{\underline{\rho}}\Gamma_{\text{AdS}} - i\theta\Gamma_{\underline{\theta}}\Gamma_{S^5} \right. \\ \left. - if'(\omega_1\Gamma_{\underline{\beta}_2} - \omega_2 \csc \beta_1\Gamma_{\underline{\beta}_1})\Gamma_{\underline{\rho}}\Gamma_r\Gamma_{\text{AdS}} + if(\star d\omega)\Gamma_{\underline{\rho}}\Gamma_{\text{AdS}} \right] \epsilon^{(0)}. \quad (4.37)$$

Multiplying by $\Gamma_{\underline{\theta}}\Gamma_{\underline{\rho}}$ and using the type IIB chirality condition, which can be expressed $\Gamma_{11}\epsilon_0 = \epsilon_0 \Leftrightarrow \Gamma_{S^5}\epsilon = -\Gamma_{\text{AdS}}\epsilon$, yields

$$\left[\theta'\Gamma_{\underline{\rho}}\Gamma_r + i\theta\Gamma_{\underline{\rho}}\Gamma_{\text{AdS}} \right] \epsilon^{(0)} = \left[i(\rho - f \star d\omega)\Gamma_{\text{AdS}} - (\partial_1\rho)\Gamma_{\underline{\beta}_1} - \csc \beta_1(\partial_2\rho)\Gamma_{\underline{\beta}_2} \right] \Gamma_{\underline{\theta}}\epsilon^{(0)} \\ + \left[if'(\omega_2 \csc \beta_1\Gamma_{\underline{\beta}_1} - \omega_1\Gamma_{\underline{\beta}_2})\Gamma_r\Gamma_{\text{AdS}} - \rho'\Gamma_r \right] \Gamma_{\underline{\theta}}\epsilon^{(0)}. \quad (4.38)$$

The term on the left hand side has no non-trivial dependence on the S^5 coordinates. That is, after hitting the equation with $R_{S^5}^{-1}$ all dependence drops out. The first term on the right hand side only has dependence on the AdS_5 directions through ρ and f , no non-trivial AdS_5 Γ -matrix structures. That means it is independent of the AdS_3 directions x, t, y after hitting the equation with R_{AdS}^{-1} . The interesting term is the second bracketed structure on the right hand side contained in the second line. It has non-trivial dependence on the S^5 and the AdS_3

directions, due to the appearance of Γ_r .³ That means it either has to vanish all by itself, or at least one of the non-trivial dependences on the S^5 and AdS_3 directions has to drop out for it to cancel with one of the other terms. But there's no way to cancel the non-trivial AdS_3 dependence, so it must be the dependence on the S^5 directions which cancels. This allows it to combine with the term on the left hand side, which also has non-trivial dependence on the AdS_3 directions. In fact, all of the Γ -matrix structures in $R_{\text{AdS}}^{-1}\Gamma_r R_{\text{AdS}}$ have non-trivial dependence on the AdS_3 directions. So in the first term on the right hand side, which does not have such dependence, the S^5 dependence has to cancel as well!

Solving for S^2 dependences

From the fact that the S^5 dependence in the first term on the rhs of (4.38) has to cancel, we now see that either f and ρ have the same dependence on r up to overall constants, or $\star d\omega = \text{const}$. After some thought, we will choose the former and express the ‘bending mode’ in terms of the gauge field’s radial part as $\rho = f\psi$. Note that the relative normalization of ω and ψ now matters – they can not both be chosen normalized to unity without loss of generality. Using also $\Gamma_{11}\epsilon^{(0)} = \epsilon^{(0)}$ to convert Γ_{AdS} to Γ_{S^5} ,⁴ we then find

$$\begin{aligned} \left[\theta' \Gamma_{\underline{\rho}} \Gamma_r + i \theta \Gamma_{\underline{\rho}} \Gamma_{\text{AdS}} \right] \epsilon^{(0)} &= f \left[i(\psi - \star d\omega) \Gamma_{S^5} - (\partial_1 \psi) \Gamma_{\underline{\beta}_1} - \csc \beta_1 (\partial_2 \psi) \Gamma_{\underline{\beta}_2} \right] \Gamma_{\underline{\theta}} \epsilon^{(0)} \\ &+ f' \left[i\omega_2 \csc \beta_1 \Gamma_{\underline{\beta}_1} - i\omega_1 \Gamma_{\underline{\beta}_2} + \psi \Gamma_{S^5} \right] \Gamma_{\underline{\theta}} \Gamma_r \Gamma_{\text{AdS}} \epsilon^{(0)}. \end{aligned} \quad (4.39)$$

To start the analysis of the S^5 dependences, consider the second term on the right hand side, in which all S^5 dependence has to drop out after hitting it with $R_{S^5}^{-1}$, as argued above. The relevant feature is that this term is algebraic in ω and ψ . To make the structure more apparent, rewrite (4.39) as

$$\left[\theta' \Gamma_{\underline{\rho}} \Gamma_r + i \theta \Gamma_{\underline{\rho}} \Gamma_{\text{AdS}} \right] \epsilon^{(0)} = f Y \epsilon^{(0)} + f' Z \Gamma_r \Gamma_{\text{AdS}} \epsilon^{(0)}, \quad (4.40)$$

³At $\rho = 0$, $R_{\text{AdS}}^{-1}\Gamma_r R_{\text{AdS}}$ has four independent Γ -matrix structures whose coefficients depend on (x, t, y) in such a way that the dependence can not be cancelled by imposing a projector on ϵ_0 . See (4.53) below.

⁴Had we left it at Γ_{AdS} , the S^5 dependence would have to cancel in those terms separately.

where Y and Z are independent of r , and are given by

$$Y = \left[i(\psi - \star d\omega)\Gamma_{S^5} - (\partial_1\psi)\Gamma_{\underline{\beta}_1} - \csc\beta_1(\partial_2\psi)\Gamma_{\underline{\beta}_2} \right] \Gamma_{\underline{\theta}} , \quad (4.41)$$

$$Z = \left[i\omega_2 \csc\beta_1 \Gamma_{\underline{\beta}_1} - i\omega_1 \Gamma_{\underline{\beta}_2} + \psi \Gamma_{S^5} \right] \Gamma_{\underline{\theta}} . \quad (4.42)$$

So the task is to find ψ, ω such that $R_{S^5}^{-1}ZR_{S^5}$ is independent of S^5 directions. We can see right away that $R_{S^5}^{-1}YR_{S^5}$ and $R_{S^5}^{-1}ZR_{S^5}$ will involve the same Clifford algebra structures. We can now utilize the freedom to generically set $\alpha_1 = 0$, and evaluating the expression at $\theta = 0$, we find

$$R_{S^5}^{-1}\Gamma_{\underline{\beta}_1}\Gamma_{\underline{\theta}}R_{S^5} = \cos\beta_1(\cos\beta_2\Gamma_{\underline{\beta}_1} + \sin\beta_2\Gamma_{\underline{\beta}_2})\Gamma_{\underline{\theta}} - \sin\beta_1\Gamma_{S^5}\Gamma_{\underline{\theta}} , \quad (4.43a)$$

$$R_{S^5}^{-1}\Gamma_{\underline{\beta}_2}\Gamma_{\underline{\theta}}R_{S^5} = (\cos\beta_2\Gamma_{\underline{\beta}_2} - \sin\beta_2\Gamma_{\underline{\beta}_1})\Gamma_{\underline{\theta}} , \quad (4.43b)$$

$$R_{S^5}^{-1}\Gamma_{S^5}\Gamma_{\underline{\theta}}R_{S^5} = \cos\beta_1\Gamma_{S^5}\Gamma_{\underline{\theta}} - i\sin\beta_1(\cos\beta_2\Gamma_{\underline{\beta}_1} + \sin\beta_2\Gamma_{\underline{\beta}_2})\Gamma_{\underline{\theta}} . \quad (4.43c)$$

We see three independent Clifford algebra structures, and to get rid of S^5 dependence we have to solve

$$R_{S^5}^{-1}ZR_{S^5} = (ic_1\Gamma_{\underline{\beta}_1} + ic_2\Gamma_{\underline{\beta}_2} + c_3\Gamma_{S^5})\Gamma_{\underline{\theta}} , \quad (4.44)$$

with generically complex constants c_1, c_2, c_3 . This is a system of three linear equations for ω_i and ψ , and can be solved. Demanding $R_{S^5}^{-1}YR_{S^5}$ to be independent of S^5 directions as well adds another three equations, overconstraining the system. If the solution we get from eq. (4.44) automatically satisfies the additional equations from $R_{S^5}^{-1}YR_{S^5}$, then we will not have to search for additional projection conditions to achieve an S^5 independent set of constraints. Explicitly calculating eq. (4.44) is

$$\begin{pmatrix} -\sin\beta_1\cos\beta_2 & \sin\beta_2 & \cot\beta_1\cos\beta_2 \\ -\sin\beta_1\sin\beta_2 & -\cos\beta_2 & \cot\beta_1\sin\beta_2 \\ \cos\beta_1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \psi \\ \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} . \quad (4.45)$$

For all c_i real we get real solutions, which we shall assume from here onwards. The general solution to eq. (4.45) can be found to be

$$\psi = c_3 \cos \beta_1 - \sin \beta_1 (c_1 \cos \beta_2 + c_2 \sin \beta_2) , \quad (4.46a)$$

$$\omega_1 = c_1 \sin \beta_2 - c_2 \cos \beta_2 , \quad (4.46b)$$

$$\omega_2 = \sin \beta_1 (\cos \beta_1 (c_1 \cos \beta_2 + c_2 \sin \beta_2) + c_3 \sin \beta_1) . \quad (4.46c)$$

Reassuringly, the above solution satisfies $\square_{S^2} \psi = -2\psi$, so ψ is an $\ell = 1$ mode. Furthermore, $\omega_i = \sqrt{g_{S^2}} \varepsilon_{ij} g_{S^2}^{jk} \partial_k \psi$ with $\varepsilon_{12} = 1$, and we also have $\psi = \frac{1}{2} \star d\omega$. Thus, ω is exactly a co-exact one form– i.e. S^2 vector spherical harmonic– as desired. For $c_1 = c_2 = 0$ there is no β_2 dependence and translations in β_2 parameterize the preserved $U(1)$. Finally, using that general solution for (ψ, ω_i) to (4.45) produces

$$R_{S^5}^{-1} Y R_{S^5} = -i R_{S^5}^{-1} Z R_{S^5} . \quad (4.47)$$

So the S^5 dependence drops out in the first term on the right hand side of (4.40) as well. Hence we do not need a projector here.

Solving for the radial profiles

After hitting eq. (4.40) with $R_{S^5}^{-1}$ and evaluating using eq. (4.46), the κ -symmetry condition becomes

$$\left[\theta' \Gamma_{\underline{\rho}} \Gamma_{\underline{r}} + i \theta \Gamma_{\underline{\rho}} \Gamma_{\text{AdS}} \right] R_{\text{AdS}}^{(0)} \epsilon_0 = \left[f \mathbf{1} + i f' \Gamma_{\underline{r}} \Gamma_{\text{AdS}} \right] R_{\text{AdS}}^{(0)} \Gamma_p \epsilon_0 , \quad (4.48)$$

where we have defined

$$\Gamma_p = (c_1 \Gamma_{\underline{\beta}_1} + c_2 \Gamma_{\underline{\beta}_2} - i c_3 \Gamma_{S^5}) \Gamma_{\underline{\theta}} . \quad (4.49)$$

Note that $\Gamma_p^2 = -|c|^2 \mathbf{1}$ with $|c|^2 = c_1^2 + c_2^2 + c_3^2$, and Γ_p commutes with AdS_5 Γ -matrices. So it can be used straightforwardly for constructing projectors.

To solve (4.48) systematically we turn to the dependence on the AdS_3 directions (x, t, y) . We introduce an operator \mathcal{R}_A and for later convenience also \mathcal{R}_S defined by

$$\mathcal{R}_A[\Gamma] = R_{\text{AdS}}^{-1} \Gamma R_{\text{AdS}} , \quad \mathcal{R}_S[\Gamma] = R_{S^5}^{-1} \Gamma R_{S^5} . \quad (4.50)$$

At $\rho = 0$, we then note the useful identity

$$\mathcal{R}_A^{(0)}[\Gamma_{\underline{\rho}}\Gamma_{\underline{r}}] = i \tanh r \mathcal{R}_A^{(0)}[\Gamma_{\underline{\rho}}\Gamma_{\text{AdS}}] + \text{sech } r \Gamma_{\underline{\rho}}\Gamma_{\underline{r}} . \quad (4.51)$$

Hitting (4.48) with $(R_{\text{AdS}}^{(0)})^{-1}$, we then find

$$i(\theta' \tanh r + \theta) \mathcal{R}_A^{(0)}[\Gamma_{\underline{\rho}}\Gamma_{\text{AdS}}]\epsilon_0 + \theta' \text{sech } r \Gamma_{\underline{\rho}}\Gamma_{\underline{r}} = f \Gamma_p \epsilon_0 + i f' \mathcal{R}_A^{(0)}[\Gamma_{\underline{r}}\Gamma_{\text{AdS}}]\Gamma_p \epsilon_0 . \quad (4.52)$$

The parts without \mathcal{R}_A already suggest a form of the projector to impose, but to really see which one we need, we need the (x, t, y) dependent terms. The explicit identities we'll use for that are

$$\mathcal{R}_A^{(0)}[\Gamma_{\underline{r}}\Gamma_{\text{AdS}}] = i e^x (y \Gamma_{\underline{y}} + t \Gamma_{\underline{t}}) \Gamma_{\underline{r}} - i (e^x (y^2 - t^2 - 1) + e^{-x}) \Gamma_{\underline{x}} P_{x+} \Gamma_{\underline{r}} + e^x \Gamma_{\underline{r}} \Gamma_{\text{AdS}} , \quad (4.53)$$

$$\mathcal{R}_A^{(0)}[\Gamma_{\underline{\rho}}\Gamma_{\text{AdS}}]\Gamma_{\underline{\rho}}\Gamma_{\underline{r}} = \cosh r \mathcal{R}_A^{(0)}[\Gamma_{\underline{r}}\Gamma_{\text{AdS}}] + i \sinh r \mathbb{1} . \quad (4.54)$$

The first one shows that the AdS_3 dependence can not be canceled within $\mathcal{R}_A^{(0)}[\Gamma_{\underline{\rho}}\Gamma_{\text{AdS}}]$ and $R_A^{(0)}[\Gamma_{\underline{r}}\Gamma_{\text{AdS}}]$ separately, since there are too many independent Clifford algebra structures with non-trivial dependence. The only way to do it would be $\theta' \tanh r + \theta = 0$ and $f' = 0$, so that the terms vanish separately altogether in (4.52). The second identity tells us precisely which projector we have to impose on ϵ_0 to have a chance to cancel the non-trivial AdS_3 dependence between the two terms. Using it in (4.52) gives

$$i \mathcal{R}_A^{(0)}[\Gamma_{\underline{\rho}}\Gamma_{\text{AdS}}](\theta' \tanh r + \theta - f' \text{sech } r \Gamma_{\underline{\rho}}\Gamma_{\underline{r}}\Gamma_p)\epsilon_0 = (f + f' \tanh r) \Gamma_p \epsilon_0 - \theta' \text{sech } r \Gamma_{\underline{\rho}}\Gamma_{\underline{r}}\epsilon_0 .$$

The projector we need to impose for canceling the left hand side is

$$\frac{1}{|c|} \Gamma_{\underline{\rho}}\Gamma_{\underline{r}}\Gamma_p \epsilon_0 = \lambda \epsilon_0 , \quad \lambda = \pm 1 , \quad (4.55)$$

and this should be the only projector that does it. The equation for κ -symmetry becomes

$$i \mathcal{R}_A^{(0)}[\Gamma_{\underline{\rho}}\Gamma_{\text{AdS}}](\theta' \tanh r + \theta - \lambda |c| f' \text{sech } r) \epsilon_0 = -(\theta' \text{sech } r + \lambda |c| (f + f' \tanh r)) \Gamma_{\underline{\rho}}\Gamma_{\underline{r}} \epsilon_0 .$$

Now the terms in brackets on each side have to vanish separately and we get our two 1st-order equations

$$\theta' \tanh r + \theta - \lambda |c| f' \text{sech } r = 0 , \quad \theta' \text{sech } r + \lambda |c| (f + f' \tanh r) = 0 . \quad (4.56)$$

This yields an algebraic equation for f in terms of θ , and a second-order equation for θ alone, namely

$$f = -\frac{\theta \sinh r + \theta' \cosh r}{\lambda|c|}, \quad \theta'' + 3\theta' \tanh r + 2\theta = 0. \quad (4.57)$$

Checking vs. EOM

To check that the (local) solution for supersymmetric embeddings we derived in the previous section satisfies the EOM, we evaluate the action (4.4) a bit more explicitly. We find

$$S_{D5} = -T_5 \int d^6 \xi \cosh^3 \rho \cosh^3 r e^{2x} h + T_5 \int d^6 \xi \cosh^3 r e^{2x} \zeta(\rho) F_{\beta_1 \beta_2}, \quad (4.58)$$

where h was spelled out explicitly around (4.23). To find the linearized equations of motion we expand that action to quadratic order in the fluctuations. Writing $S_{D5} = -T_5 \int d^6 \xi L_{\text{DBI}} + T_5 \int d^6 \xi L_{\text{WZ}}$, this yields

$$L_{\text{DBI}} = \frac{1}{2} \sqrt{-g_{\text{AdS}_3 \times \text{S}^2}} \left[\frac{A_2'^2 + (\partial_2 \rho)^2 + F_{\beta_1 \beta_2}^2}{\sin^2 \beta_1} + A_1'^2 + (\partial_1 \rho)^2 + \rho'^2 + 4\rho^2 + \theta'^2 - 2\theta^2 \right], \quad (4.59a)$$

$$L_{\text{WZ}} = 4\sqrt{-g_{\text{AdS}_3}} \rho F_{\beta_1 \beta_2}. \quad (4.59b)$$

where $\sqrt{-g_{\text{AdS}_3}} = \cosh^3 r e^{2x}$ and $\sqrt{-g_{\text{AdS}_3 \times \text{S}^2}} = \sqrt{-g_{\text{AdS}_3}} \sin \beta_1$. The WZ term produces a total derivative at linear order in the fluctuations, which we have dropped.

From that quadratic action we can now derive the linearized equations of motion. That yields an ODE for θ and PDEs for ρ , A_1 and A_2 . From (4.57) we derive that f satisfies the second-order equation

$$f'' + 3f' \tanh r + 2f = 0. \quad (4.60)$$

Note that (4.57) has more information than that: with $\lambda|c|$ and the two parameters in the general solution for θ , (4.57) has a 3-parameter family of solutions. Solving the second-order equations for θ and f separately, on the other hand, produces a 4-parameter family of solutions. Anyway, writing $\rho = k(r)\psi(\vec{\beta})$, using $k = f$ and (4.46) together with the second-order equations for f , θ , solves all the equations of motion derived from (4.59).

4.4 Non-linear κ -symmetry

In this section, we will set up the general non-linear κ -symmetry equations. While the calculation of the final equation describing the κ -symmetric embedding in eq. (4.128) is lengthy and the final result compact, we will note that there are serious hurdles concerning their general solubility that have yet to be overcome. Nevertheless, we will lay out the general framework for their analysis beyond leading order in the perturbative framework started above. In the end, we will find that there are interesting features that one would not have naively anticipated from the perspective of the supersymmetric D3/D7 system that warrant report and further investigation.

Our starting point will be to return to the original form of the κ -symmetry condition, eq. (4.22), and approach the calculation one piece at a time. Handling the first term, we would like to see that by use of the previous projection conditions that we can get rid of the conjugate spinor. With eq. (4.12) and a bit of algebra, we find that the conjugate spinor can generically be expressed as

$$C\epsilon^\star = -e^{-i\rho\Gamma_\rho\Gamma_{\text{AdS}}} e^{-i\theta\Gamma_\theta\Gamma_{S^5}} \Gamma_\rho\Gamma_{\underline{\beta}} R_{\text{AdS}} R_{S^5} \Gamma_\rho\Gamma_{\underline{\beta}} C(\epsilon_0)^\star . \quad (4.61)$$

At this point, we are in a position to employ the massless projection condition eq. (4.27), which results in

$$C\epsilon^\star = ie^{-i\rho\Gamma_\rho\Gamma_{\text{AdS}}} e^{-i\theta\Gamma_\theta\Gamma_{S^5}} \Gamma_\rho\Gamma_{\underline{\beta}} \Gamma_{\text{AdS}} \epsilon . \quad (4.62)$$

Utilizing this expression, eq. (4.22) becomes

$$-\hat{\Gamma} e^{-i\rho\Gamma_\rho\Gamma_{\text{AdS}}} e^{-i\theta\Gamma_\theta\Gamma_{S^5}} \Gamma_\rho\Gamma_{\underline{\beta}} \Gamma_{\text{AdS}} \epsilon + \frac{i}{2} \gamma^{ij} F_{ij} \hat{\Gamma} \epsilon = h\epsilon . \quad (4.63)$$

No further immediate manipulation will be done on the first term, nor the right hand side. For the gauge field term we'll use

$$\frac{1}{2} \gamma^{ij} F_{ij} = \gamma_{r\beta_i} F^{r\beta_i} + \gamma_{\beta_1\beta_2} F^{\beta_1\beta_2} . \quad (4.64)$$

4.4.1 Isolating AdS and S dependencies

Informed by the linearized analysis, we will start by partitioning eq. (4.63) into the AdS₅ and S⁵ Clifford algebra and coordinate dependences. This will better facilitate finding necessary, independent vanishing conditions.

To evaluate $\hat{\Gamma}$ explicitly, we start from eq. (4.19) and eq. (4.20). To better organize the partitioning, let us use the decomposition along the lines of particular recurrent AdS₅ gamma matrix structures such that

$$\gamma_{r\beta_1\beta_2} = \mathcal{A}^{\mathbb{1}} + \mathcal{A}^\rho \Gamma_\rho + \mathcal{A}^r \Gamma_r + A^{\rho r} \Gamma_{\rho r} , \quad (4.65)$$

where, with $\tau^m = \varepsilon^{mnr}(d\rho)_n(d\theta)_r$, $\varepsilon^{r\beta_1\beta_2} = 1$, and the following convenient shorthands are

$$\mathcal{A}^{\mathbb{1}} = (d\theta)_r \Gamma_\theta \Gamma_{\vec{\beta}} , \quad \mathcal{A}^\rho = (d\rho)_r \Gamma_{\vec{\beta}} - \tau^{\beta_i} \Gamma_{\beta_i} \Gamma_\theta , \quad (4.66a)$$

$$\mathcal{A}^r = \Gamma_{\vec{\beta}} - \varepsilon^{r\mu\nu} (d\theta)_\mu \Gamma_\nu \Gamma_\theta , \quad \mathcal{A}^{\rho r} = \varepsilon^{r\mu\nu} (d\rho)_\mu \Gamma_\nu - \tau^r \Gamma_\theta . \quad (4.66b)$$

Using the decomposition and shorthands as well as the type IIB chirality condition, $\Gamma_{11}\epsilon = \epsilon$, we find

$$-\hat{\Gamma} e^{-i\rho \Gamma_{\underline{r}} \Gamma_{\text{AdS}}} e^{-i\theta \Gamma_{\underline{\theta}} \Gamma_{\text{S}^5}} \Gamma_{\underline{\rho}} \Gamma_{\underline{\beta}} \Gamma_{\text{AdS}} \epsilon = \left[\mathcal{M}^{\mathbb{1}} + \mathcal{M}^\rho \Gamma_\rho + \mathcal{M}^r \Gamma_r + \mathcal{M}^{\rho r} \Gamma_{\rho r} \right] e^{-i\theta \Gamma_{\underline{\theta}} \Gamma_{\text{S}^5}} \Gamma_{\underline{\beta}} \epsilon , \quad (4.67)$$

where we used yet another set of shorthands for common structures:

$$\mathcal{M}^{\mathbb{1}} = -\cosh^2 \rho (\mathcal{A}^r + i \tanh \rho A^{\rho r} \Gamma_{\text{S}^5}) , \quad \mathcal{M}^\rho = -\cosh^2 \rho (A^{\rho r} - i \tanh \rho \mathcal{A}^r \Gamma_{\text{S}^5}) , \quad (4.68a)$$

$$\mathcal{M}^r = -\mathcal{A}^{\mathbb{1}} - i \tanh \rho \mathcal{A}^\rho \Gamma_{\text{S}^5} , \quad \mathcal{M}^{\rho r} = -\mathcal{A}^\rho + i \tanh \rho \mathcal{A}^{\mathbb{1}} \Gamma_{\text{S}^5} . \quad (4.68b)$$

This will unfortunately be necessary in order to express the manipulations in any sort of readable way. All of these types of expressions will be collected in a set of tables in the appendix in §4.10 for reference purposes.

We will proceed with the partitioning of eq. (4.63) by analyzing the term involving the field strength. Let us expand and collect all of the relevant AdS gamma matrix structures as above:

$$\frac{1}{2} \gamma^{ij} F_{ij} = \mathcal{B}^{\mathbb{1}} + \mathcal{B}^\rho \Gamma_\rho + \mathcal{B}^r \Gamma_r + B^{\rho r} \Gamma_{\rho r} , \quad (4.69)$$

where this new set of shorthand notation is given by

$$\mathcal{B}^{\mathbb{1}} = (d\theta)_r F^{r\beta_i} \Gamma_\theta \Gamma_{\beta_i} + F^{\beta_1\beta_2} (\Gamma_{\bar{\beta}} + \Gamma_\theta \varepsilon^{r\text{mn}} (d\theta)_m \Gamma_n) , \quad \mathcal{B}^r = -F^{r\beta_i} (\Gamma_{\beta_i} + (d\theta)_{\beta_i} \Gamma_\theta) , \quad (4.70a)$$

$$\mathcal{B}^\rho = F^{\text{mn}} (d\theta)_m (d\rho)_n \Gamma_\theta + F^{\beta_i\text{m}} (d\rho)_m \Gamma_{\beta_i} , \quad \mathcal{B}^{\rho r} = -F^{r\beta_i} (d\rho)_{\beta_i} \mathbb{1} . \quad (4.70b)$$

Inserting this schematic form into the relevant combination appearing in eq. (4.63) yields

$$\frac{i}{2} \gamma^{ij} F_{ij} \hat{\Gamma} \epsilon = -i [\mathcal{E}^{\mathbb{1}} + \mathcal{E}^\rho \Gamma_\rho + \mathcal{E}^r \Gamma_r + \mathcal{E}^{\rho e} \Gamma_{\rho r}] \Gamma_{S^5} \epsilon , \quad (4.71)$$

where yet another set of shorthands has been used to shield the reader from an acute onset of eyestrain and malaise. Now that the reader has had a short respite, the translation into previous quantities is given by

$$\mathcal{E}^{\mathbb{1}} = -\cosh \rho (\mathcal{B}^{\rho r} \mathcal{A}^{\mathbb{1}} + \mathcal{B}^{\mathbb{1}} \mathcal{A}^{\rho r} + \mathcal{B}^\rho \mathcal{A}^r - \mathcal{B}^r \mathcal{A}^\rho) , \quad (4.72a)$$

$$\mathcal{E}^\rho = -\cosh \rho (\mathcal{B}^r \mathcal{A}^{\mathbb{1}} - \mathcal{B}^{\mathbb{1}} \mathcal{A}^r + \mathcal{B}^\rho \mathcal{A}^{\rho r} + \mathcal{B}^{\rho r} \mathcal{A}^\rho) , \quad (4.72b)$$

$$\mathcal{E}^r = -\text{sech} \rho (-\mathcal{B}^\rho \mathcal{A}^{\mathbb{1}} + \mathcal{B}^{\mathbb{1}} \mathcal{A}^\rho) - \cosh \rho (\mathcal{B}^r \mathcal{A}^{\rho r} + \mathcal{B}^{\rho r} \mathcal{A}^r) , \quad (4.72c)$$

$$\mathcal{E}^{\rho r} = \text{sech} \rho (\mathcal{B}^{\mathbb{1}} \mathcal{A}^{\mathbb{1}} + \mathcal{B}^\rho \mathcal{A}^\rho) + \cosh \rho (\mathcal{B}^r \mathcal{A}^r - \mathcal{B}^{\rho r} \mathcal{A}^{\rho r}) . \quad (4.72d)$$

We can then assemble the full κ -symmetry condition, which becomes

$$[\mathcal{N}^{\mathbb{1}} + \mathcal{N}^\rho \Gamma_\rho + \mathcal{N}^r \Gamma_r + \mathcal{N}^{\rho r} \Gamma_{\rho r}] \epsilon = h \epsilon , \quad (4.73)$$

where

$$\mathcal{N}^X = \mathcal{M}^X e^{-i\theta \Gamma_{\underline{\theta}} \Gamma_{S^5} \Gamma_{\bar{\beta}}} - i \mathcal{E}^X \Gamma_{S^5} , \quad X \in \{\mathbb{1}, \rho, r, \rho r\} . \quad (4.74)$$

If only this were the end of the condensation of notation, we would breath a bit easier. However, the task before us now is to open up the Killing spinors and drag through R_{AdS}^{-1} and $R_{S^5}^{-1}$ which blow up the short hands necessitating more condensation.

Using the massive projector

To that end, we would like to use the other projector discovered in the linearized analysis above. Further manipulation begins with eq. (4.73), unpacking the Killing spinor to extract

the R_{AdS} and R_{S^5} matrices, and applying $R_{\text{AdS}}^{-1}R_{S^5}^{-1}$. Using the operator expression eq. (4.50), the result is

$$\left(\mathcal{R}_S[\mathcal{N}^1] + \mathcal{R}_S[\mathcal{N}^\rho]\mathcal{R}_A[\Gamma_\rho] + \mathcal{R}_S[\mathcal{N}^r]\mathcal{R}_A[\Gamma_r] + \mathcal{R}_S[\mathcal{N}^{\rho r}]\mathcal{R}_A[\Gamma_{\rho r}]\right)\epsilon_0 = h\epsilon_0. \quad (4.75)$$

While this does not immediately present itself as a tractable question, we have the following identities at our disposal:

$$\mathcal{R}_A[\Gamma_{\underline{\rho}}] = \cosh r \mathcal{K} \Gamma_{\underline{\rho}} + i \sinh r \Gamma_{\underline{\rho r}} \Gamma_{\text{AdS}}, \quad (4.76a)$$

$$\mathcal{R}_A[\Gamma_{\underline{r}}] = \mathcal{K} \left(\cosh \rho \Gamma_{\underline{r}} - \sinh \rho \sinh r \Gamma_{\underline{\rho}} \right) - i \sinh \rho \cosh r \Gamma_{\underline{\rho r}} \Gamma_{\text{AdS}}, \quad (4.76b)$$

$$\mathcal{R}_A[\Gamma_{\underline{\rho r}}] = -i\mathcal{K} \left(\sinh \rho \Gamma_{\underline{r}} - \cosh \rho \sinh r \Gamma_{\underline{\rho}} \right) \Gamma_{\text{AdS}} + \cosh \rho \cosh r \Gamma_{\underline{\rho r}}. \quad (4.76c)$$

The newest entry into our expanding collection of shorthand notations is the matrix \mathcal{K} , which handily encodes the entire dependence on the AdS_3 directions and is given by

$$\mathcal{K} = [e^{-x} + e^x(y^2 - t^2)]P_{x+} + e^x P_{x-} - ie^x(t\Gamma_{\underline{t}} + y\Gamma_{\underline{y}})\Gamma_{\text{AdS}}, \quad (4.77)$$

where $P_{x\pm}$ were defined above eq. (4.10). We are now at the starting point to apply the massive projector given that the above structures act on ϵ_0 , which satisfies $\Gamma_{11}\epsilon_0 = \epsilon_0$ and eq. (4.55). To aid in the computation, we have $\Gamma_{\text{AdS}}\epsilon_0 = -\Gamma_{S^5}\epsilon_0$ and $\Gamma_{\underline{\rho r}}\epsilon_0 = -\hat{\Gamma}_p\epsilon_0$, with $\hat{\Gamma}_p = \frac{\lambda}{|c|}\Gamma_p$ such that $\hat{\Gamma}_p^2 = -\mathbb{1}$. Exploiting all of these resources the above relations become

$$\mathcal{R}_A[\Gamma_{\underline{\rho}}]\epsilon_0 = \cosh r \mathcal{K} \Gamma_{\underline{\rho}}\epsilon_0 + i \sinh r \hat{\Gamma}_p \Gamma_{S^5}\epsilon_0, \quad (4.78a)$$

$$\mathcal{R}_A[\Gamma_{\underline{r}}]\epsilon_0 = -\left(\cosh \rho \hat{\Gamma}_p + \sinh \rho \sinh r \mathbb{1}\right)\mathcal{K} \Gamma_{\underline{\rho}}\epsilon_0 - i \sinh \rho \cosh r \hat{\Gamma}_p \Gamma_{S^5}\epsilon_0, \quad (4.78b)$$

$$\mathcal{R}_A[\Gamma_{\underline{\rho r}}]\epsilon_0 = i\left(\sinh \rho \hat{\Gamma}_p + \cosh \rho \sinh r \mathbb{1}\right)\Gamma_{S^5}\mathcal{K} \Gamma_{\underline{\rho}}\epsilon_0 - \cosh \rho \cosh r \hat{\Gamma}_p \epsilon_0. \quad (4.78c)$$

We also used that $\hat{\Gamma}_p$ has an even number of S^5 Γ -matrices and commutes with AdS Γ -matrices, and that \mathcal{K} commutes with S^5 Γ -matrices.

With these identities in hand, bringing to bear on eq. (4.75) we see that

$$\mathcal{Q}_{\mathcal{K}} \mathcal{K} \Gamma_{\underline{\rho}}\epsilon_0 + \mathcal{Q}_{\mathbb{1}}\epsilon_0 = h\epsilon_0, \quad (4.79)$$

where

$$\begin{aligned} \mathcal{Q}_{\mathcal{K}} &= \cosh r \mathcal{R}_S[\mathcal{N}^\rho] - \cosh^2 \rho \mathcal{R}_S[\mathcal{N}^r] (\hat{\Gamma}_p + \tanh \rho \sinh r \mathbf{1}) \\ &\quad + i \cosh^2 \rho \mathcal{R}_S[\mathcal{N}^{\rho r}] (\tanh \rho \hat{\Gamma}_p + \sinh r \mathbf{1}) \Gamma_{S^5} , \end{aligned} \quad (4.80)$$

$$\begin{aligned} \mathcal{Q}_{\mathbf{1}} &= \mathcal{R}_S[\mathcal{N}^{\mathbf{1}}] + i \sinh r \mathcal{R}_S[\mathcal{N}^\rho] \hat{\Gamma}_p \Gamma_{S^5} - i \sinh \rho \cosh \rho \cosh r \mathcal{R}_S[\mathcal{N}^r] \hat{\Gamma}_p \Gamma_{S^5} \\ &\quad - \cosh^2 \rho \cosh r \mathcal{R}_S[\mathcal{N}^{\rho r}] \hat{\Gamma}_p . \end{aligned} \quad (4.81)$$

We have used up all the projectors we have to reduce the AdS Clifford algebra structures to just one, and we can hence now formulate the conditions for κ -symmetry as two operator equations which have to be satisfied simultaneously. Namely,

$$\mathcal{Q}_{\mathcal{K}} = 0 , \quad \mathcal{Q}_{\mathbf{1}} - h \mathbf{1} = 0 . \quad (4.82)$$

4.4.2 The BPS equations

We want to solve $\mathcal{Q}_{\mathcal{K}} = 0$ and $\mathcal{Q}_{\mathbf{1}} - h \mathbf{1} = 0$, or equivalently $\tilde{\mathcal{Q}}_{\mathcal{K}} = 0$ with $\tilde{\mathcal{Q}}_{\mathcal{K}} = R_{S^5} \mathcal{Q}_{\mathcal{K}} R_{S^5}^{-1}$ and analogously for the other one, which is easier. For the projector we shall from now on set $c_1 = c_2 = 0$ and $c_3 = 1$ in (4.55). There is no explicit dependence on β_2 in the metric and with this choice of projector the Clifford algebra manipulations do not introduce dependence either. So the equations can be solved with only trivial dependence on β_2 .

An equation we can exploit right away is

$$\text{tr} \left(\tilde{\mathcal{Q}}_{\mathcal{K}} \Gamma_{\underline{\theta\beta_1\beta_2}} \right) = 0 \iff F^{r\beta_1} \left[g_{rr} \cos^2 \theta + (\tau^{\beta_2})^2 + \cosh^2 \rho ((\partial_{\beta_1} \rho)^2 + (\partial_{\beta_1} \theta)^2) \right] = 0 . \quad (4.83)$$

In the bracket is a sum of squares and the first term is non-zero. Moreover, the combination is precisely $F_{r\beta_1}$, so we conclude

$$F^{r\beta_1} = F_{r\beta_1} = 0 . \quad (4.84)$$

With nothing depending on β_2 , this means A_{β_1} is a constant and can be set to zero. This stands as the first signpost towards the resulting differential equations that have thus far passed. It is a highly non-trivial result to eliminate a component of the field strength tensor. The rest of the constraints are, unfortunately, a bit more opaque.

The remaining equations resulting from $\mathcal{Q}_{\mathcal{K}} = 0$ are then

$$\sqrt{g_{\mathcal{S}^2}}(\sin \theta \hat{Z}_3 + \lambda \tanh \rho F_{r\beta_2}) - \sinh r X_1 - \lambda \cos \beta_1 X_3 = 0, \quad (4.85)$$

$$\begin{aligned} & \lambda \sin \beta_1 \cos \theta \partial_r \theta - \tau^{\beta_2} \sin \theta \sinh r + \lambda \sinh \rho X_1 \\ & + \cosh \rho (F_{r\beta_2} \sinh r \csc \beta_1 - \lambda X_3 \tan \theta + \cosh r Z_1) = 0, \end{aligned} \quad (4.86)$$

$$\lambda \sinh \rho (F_{r\beta_2} \cot \beta_1 + X_1 \tan \theta) + \sinh r \cos \theta \tau^{\beta_2} + \cosh \rho (\lambda X_3 - \cosh r Z_2) = 0, \quad (4.87)$$

$$\sqrt{g_{\mathcal{S}^2}}(\cos \theta \hat{Z}_3 - \lambda \tan \theta \tanh \rho F_{r\beta_2}) + \tanh \rho (\lambda \cos \beta_1 X_1 - X_3 \sinh r) + \cosh r F_{\beta_1\beta_2} = 0. \quad (4.88)$$

By now, we know the drill: we have defined the shorthand notation

$$X_1 = \varepsilon^{\beta_2 mn} \partial_m \theta F_{n\beta_2}, \quad X_3 = -\varepsilon^{\beta_2 mn} \partial_m \rho F_{n\beta_2}, \quad Z_1 = -\partial_{\beta_1} (\sinh \rho \cos \theta), \quad (4.89)$$

$$Z_2 = \partial_{\beta_1} (\sinh \rho \sin \theta), \quad \hat{Z}_3 = \operatorname{sech} \rho (\lambda \cos \beta_1 \partial_r \theta + \sinh r \partial_r \rho) - \sinh \rho \cosh r. \quad (4.90)$$

The equations resulting from $\mathcal{Q}_1 - h\mathbb{1} = 0$ are formidable and given by

$$\begin{aligned} & -h \csc \beta_1 + \cos \theta \cosh^2 \rho (\lambda \tanh \rho \sinh r (\sin \beta_1 \partial_{\beta_1} \theta - \cos \beta_1 \sin \theta \cos \theta) + \cos^2 \theta) \\ & + \lambda \cosh r (\sin \theta \cos \theta (\cos \beta_1 \cos \theta \partial_r \rho + F_{r\beta_2} \cosh \rho) - X_1 \cot \beta_1 \cosh \rho - \tau^{\beta_2} \sin \beta_1 \cos \theta) = 0, \end{aligned} \quad (4.91)$$

$$\begin{aligned} & \lambda \cosh r (\cosh \rho (F_{r\beta_2} \cot \beta_1 + X_1 \tan \theta) - \sin \beta_1 \cos \theta \partial_r \rho + X_3 \sinh \rho - \tau^{\beta_2} \cos \beta_1 \sin \theta) \\ & - \cosh \rho (Z_5 + \lambda \sinh r (F_{\beta_1\beta_2} - \cos \beta_1 Z_1 - \sinh \rho \sin \beta_1 \cos \theta)) = 0, \end{aligned} \quad (4.92)$$

$$\begin{aligned} & Z_4 - \lambda \sinh r (\tan \theta F_{\beta_1\beta_2} + \cos \beta_1 Z_2) \\ & + \lambda \cosh r (X_3 \tan \theta \tanh \rho - X_1 + \tau^{\beta_2} \operatorname{sech} \rho \cos \beta_1 \cos \theta) = 0, \end{aligned} \quad (4.93)$$

$$\begin{aligned} & \lambda \cosh r (\cos \beta_1 (\sqrt{g_{\mathcal{S}^2}} \cos \theta \operatorname{sech} \rho \partial_r \rho - X_3 \tanh \rho) + F_{r\beta_2} \sqrt{g_{\mathcal{S}^2}}) - \cosh \rho \sin \theta \sqrt{g_{\mathcal{S}^2}} \\ & - \lambda \sinh r \sin^2 \beta_1 \cos \theta \cosh \rho \partial_{\beta_1} \rho - \lambda \cos \beta_1 \sinh r (\sqrt{g_{\mathcal{S}^2}} \cos \theta \sinh \rho - F_{\beta_1\beta_2}) = 0, \end{aligned} \quad (4.94)$$

where

$$Z_4 = \partial_{\beta_1} (\cosh \rho \cos \theta), \quad Z_5 = \partial_{\beta_1} (\cosh \rho \sin \theta). \quad (4.95)$$

There are not enough repeated structures in the above expressions to add to our continuously growing list of shorthands.

Let's do a quick counting here. We have 8 equations for two functions ρ, θ and the two field strength components $F_{r\beta_2}, F_{\beta_1\beta_2}$. So the system is overconstrained. In addition, F has to satisfy its Bianchi identity. For any potential solution that we find, we have a robust system of checks on its validity. This will serve us well.

Coordinate transformations

Given how difficult it is to even parse the above conditions from $\mathcal{Q}_1 - h\mathbb{1} = 0$ and $\mathcal{Q}_K = 0$, it would do us well to ponder for a moment if there are any coordinate systems that would be more convenient for analysis. To that end, we will set

$$r = 2 \tanh^{-1} \tan \frac{z}{2}, \quad \beta_1 = \cos^{-1} x. \quad (4.96)$$

The new coordinates range over $z \in (-\frac{\pi}{2}, \frac{\pi}{2})$ — in order to cover both patches of global AdS— and $x \in [0, 1]$. Our field strength components pick up expected factors from this change of coordinates

$$\partial_r \rightarrow \cos z \partial_z, \quad \partial_{\beta_1} \rightarrow -\sqrt{1-x^2} \partial_x. \quad (4.97)$$

We now recast eqs. (4.85)-(4.88) and (4.91)-(4.94) in the coordinates eq. (4.96). As shorthands we introduce

$$F_z := F_{z\beta_2} = \frac{\partial r}{\partial z} F_{r\beta_2}, \quad F_x := F_{x\beta_2} = \frac{\partial \beta_1}{\partial x} F_{\beta_1\beta_2}. \quad (4.98)$$

Denoting a new set of Fraktur indices $\mathfrak{a}, \mathfrak{b} = z, x$ and $\epsilon^{zx} = 1$, the quantities quadratic in derivatives are expressed

$$B_1 = \epsilon^{ab} \partial_a \theta F_b, \quad B_2 = \epsilon^{ab} \partial_a \rho F_b, \quad B_3 = \epsilon^{ab} \partial_a \rho \partial_b \theta. \quad (4.99)$$

Much the same can be said for those linear in derivatives

$$A_1 = \partial_x(\sinh \rho \cos \theta) , \quad A_2 = \partial_x(\sinh \rho \sin \theta) , \quad (4.100)$$

$$A_4 = \partial_x(\cosh \rho \cos \theta) , \quad A_5 = \partial_x(\cosh \rho \sin \theta) , \quad (4.101)$$

$$A_3 = \lambda x \partial_z \theta + \tan z \partial_z \rho , \quad \hat{A}_3 = \operatorname{sech} \rho A_3 - \sec^2 z \sinh \rho . \quad (4.102)$$

In these new coordinates, the $\mathcal{Q}_{\mathcal{K}} = 0$ equations become

$$\cos^2 \theta (\sin \theta \hat{A}_3 + \lambda \tanh \rho F_z) + \tan z B_1 - \lambda x B_2 = 0 , \quad (4.103)$$

$$\lambda \partial_z \sin \theta + \sin \theta \tan z B_3 - \lambda \sinh \rho B_1 + \cosh \rho \left[\frac{F_z \tan z}{1 - x^2} - \lambda \tan \theta B_2 - \sec^2 z A_1 \right] = 0 , \quad (4.104)$$

$$\lambda \sinh \rho \left[\frac{x F_z}{1 - x^2} - \tan \theta B_1 \right] - \tan z \cos \theta B_3 + \cosh \rho (\lambda B_2 + \sec^2 z A_2) = 0 , \quad (4.105)$$

$$\cos^2 \theta (\cos \theta \hat{A}_3 - \lambda \tan \theta \tanh \rho F_z) - \tanh \rho (\lambda x B_1 + \tan z B_2) - \sec^2 z F_x = 0 . \quad (4.106)$$

Similarly the, still quite formidable, $\mathcal{Q}_1 - h\mathbf{1} = 0$ equations become

$$-\frac{h}{\sqrt{1 - x^2}} + \cos \theta \cosh^2 \rho \left[\cos^2 \theta - \lambda \tanh \rho \tan z \left((1 - x^2) \partial_x \theta + \frac{x}{2} \sin 2\theta \right) \right] \quad (4.107)$$

$$+ \frac{\lambda}{2} \sin 2\theta (x \cos \theta \partial_z \rho + \cosh \rho F_z) + \lambda x \cosh \rho B_1 + \lambda (1 - x^2) \cos \theta B_3 = 0 ,$$

$$\cosh \rho \left[\frac{x F_z}{1 - x^2} - \tan \theta B_1 + \tan z (F_x + x A_1 + \sinh \rho \cos \theta) + \lambda A_5 \right] \quad (4.108)$$

$$- \cos \theta \partial_z \rho + \sinh \rho B_2 + x \sin \theta B_3 = 0 ,$$

$$\lambda \tan z (\tan \theta F_x + x A_2) - A_4 + \lambda (\tan \theta \tanh \rho B_2 + B_1 - x \operatorname{sech} \rho \cos \theta B_3) = 0 , \quad (4.109)$$

$$\lambda x \cos^3 \theta (\operatorname{sech} \rho \partial_z \rho - \tan z \sinh \rho) - \lambda x \tanh \rho B_2 \quad (4.110)$$

$$+ \cos^2 \theta (\lambda F_z - \sin \theta \cosh \rho) + \lambda \tan z ((1 - x^2) \cos \theta \partial_x \sinh \rho - x F_x) = 0 .$$

While this may not seem that dramatic improvement, once we begin to analyze the above equations, we will see that obtaining solutions for the gauge field and reducing the redundancy in the system will be well served with the switch.

4.4.3 Solving for the gauge potential

The equations are still not pretty, eq. (4.107) has a square root hiding in h and all the other ones are quadratic in derivatives. The appeal of the BPS equations following from κ -symmetry was that we were supposed to be left with first order differential equations. To find a set of first order differential equations we will need to a bit more work. First eqs. (4.103)-(4.105) can be regarded as a set of linear equations for (B_1, B_2, B_3) and we can solve for them. This yields

$$\sinh \rho B_1 = \cos \theta \left[\lambda \sec^2 z \cosh^2 \rho \partial_x \rho + \cos^2 \theta \partial_z \theta + \frac{F_z}{1-x^2} C_+ \right], \quad (4.111)$$

$$\begin{aligned} x \sec^2 \theta B_2 &= \coth \rho \tan z \sec^2 z (A_1 + A_2 \tan \theta) + \lambda \hat{A}_3 \sin \theta + \lambda \operatorname{csch} \rho \tan z \partial_z \sin \theta \\ &+ F_z \tanh \rho + \frac{\lambda F_z}{1-x^2} \sec \theta \operatorname{csch} \rho \tan z C_+, \end{aligned} \quad (4.112)$$

$$\begin{aligned} x \sec \theta B_3 &= \cosh \rho \left(x \csc z \sec z (A_2 - \tan \theta A_1) + \hat{A}_3 \sin \theta \cot z \right. \\ &\left. + \lambda \coth \rho \sec^2 z (A_1 + A_2 \tan \theta) \right) - \lambda \operatorname{csch} \rho \cot z C_- \partial_z \theta \\ &+ \frac{\lambda F_z}{1-x^2} (\sinh \rho \cot z + \cosh \rho \coth \rho \tan z), \end{aligned} \quad (4.113)$$

where

$$C_{\pm} = x \sinh \rho \sin \theta \pm \lambda \tan z \cos \theta \cosh \rho. \quad (4.114)$$

While we could just as easily choose another set of equations to derive the solutions for the B_i 's, will not use eq. (4.107) to avoid introducing square roots when it is not explicitly necessary. Using the solution for (B_1, B_2, B_3) thus leaves us with four equations which are linear in derivative terms, namely eq. (4.106) and (4.108)-(4.110).

We solve eq. (4.106) and eq. (4.108) with B_i inserted for $\partial_x \rho$ and $\partial_x \theta$, and use the result in eq. (4.109). After some rewriting the resulting equation becomes

$$\partial_z (x \sinh \rho \cos \theta - \lambda \tan z \cosh \rho \sin \theta) + F_z = 0. \quad (4.115)$$

Eq. (4.110) becomes the same equation. Eq. (4.115) can be integrated straightforwardly for

the gauge potential, and we find

$$A_{\beta_2} = \lambda \tan z \cosh \rho \sin \theta - x \sinh \rho \cos \theta + \mathcal{A}(x) . \quad (4.116)$$

With this solution the Bianchi identity is automatically satisfied and we “only” have to solve the κ -symmetry equations. This is the second victory that we can claim in this analysis. We have determined, explicitly, yet another component of the field strength and determined up to an x -dependent factor the gauge potential. The remaining task is to solve for the remaining x dependence in A_{β_2} .

To bring the equations resulting after solving for B_i into a nice form, we first solve eq. (4.106) and eq. (4.107) for $(\partial_z \rho, \partial_x \rho)$, and use the result in the two remaining equations. They become equivalent then and both read

$$\begin{aligned} (x^2 - 1 + \sec^2 z) \partial_x \theta - \frac{F_z \operatorname{sech} \rho}{1 - x^2} (\sin \theta \tanh \rho \tan z - \lambda x \cos \theta) \\ - \lambda F_x \sec \theta \operatorname{sech} \rho \tan z - \cos \theta (x \sin \theta + \lambda \cos \theta \tanh \rho \tan z) = 0 . \end{aligned} \quad (4.117)$$

Repeating the same procedure just now eliminating one of $(\partial_z \theta, \partial_x \theta)$ and (F_z, F_x) yields

$$\begin{aligned} (x^2 - 1 + \sec^2 z) \partial_x \rho + \lambda \frac{F_z \operatorname{sech} \rho}{1 - x^2} (x \sin \theta \tanh \rho + \lambda \cos \theta \tan z) \\ + x \sec \theta \operatorname{sech} \rho F_x + \cos \theta (x \cos \theta \tanh \rho - \lambda \sin \theta \tan z) = 0 , \end{aligned} \quad (4.118)$$

$$\begin{aligned} - \cos^2 \theta (\tan z \partial_z \theta - \lambda x \partial_z \rho) - \frac{1}{2} \sin 2\theta \tanh \rho (\tan z \partial_z \rho + \lambda x \partial_z \theta) \\ - \lambda (1 - x^2) (\tan z \partial_x \rho + \lambda x \partial_x \theta) - \frac{1}{2} \sin 2\theta (\sec^2 z + x^2 - 1) = 0 . \end{aligned} \quad (4.119)$$

Once again, the pairs of equations left over turn out to be equivalent in both cases. So of the four equations, only three are independent.

Solving eq. (4.117) for F_z and using the result along with eq. (4.116) in eq. (4.118) then fixes

$$\mathcal{A}'(x) = 0 . \quad (4.120)$$

So $\mathcal{A}(x)$ is a constant and can be set to zero by a gauge transformation. This completely determines the worldvolume gauge field consistent with κ -symmetry. Victory #3 can be

claimed: We have now used and solved two out of the three equations linear in derivatives. The remaining one is eq. (4.119), which, using eq. (4.116), can be written as (it can be obtained more straightforwardly by solving eq. (4.117) for F_x and using the result in eq. (4.118))

$$(1 - x^2) \cosh \rho (\lambda x \partial_x \theta + \tan z \partial_x \rho - \lambda \sin \theta \cos \theta) + \cos \theta F_z = 0 . \quad (4.121)$$

4.4.4 The remaining equations

Upon using the solution for the gauge potential eq. (4.116) and the terms first order in differentials eq. (4.121), the $\mathcal{Q}_{\mathcal{K}} = 0$ eqs. (4.103)-(4.106) and the last three of the $\mathcal{Q}_{\mathbb{1}} - h\mathbb{1} = 0$ eqs. (4.108)-(4.110) all become equivalent.⁵ This leaves only the first of the $\mathcal{Q}_{\mathbb{1}} - h\mathbb{1} = 0$ equations, eq. (4.107); we must now reckon with the square root lurking there. We are thus left with three equations for two functions ρ, θ , which we repeat for convenience. Of eqs. (4.103)-(4.106) and eqs. (4.108)-(4.110) we pick eq. (4.106), so we are left with

$$(1 - x^2) \cosh \rho (\lambda x \partial_x \theta + \tan z \partial_x \rho - \lambda \sin \theta \cos \theta) + \cos \theta F_z = 0 , \quad (4.122)$$

$$\cos^2 \theta (\cos \theta \hat{A}_3 - \lambda \tan \theta \tanh \rho F_z) - \tanh \rho (\lambda x B_1 + \tan z B_2) - \sec^2 z F_x = 0 , \quad (4.123)$$

$$\begin{aligned} -\frac{h}{\sqrt{1-x^2}} + \cos \theta \cosh^2 \rho \left[\cos^2 \theta - \lambda \tanh \rho \tan z \left((1-x^2) \partial_x \theta + \frac{x}{2} \sin 2\theta \right) \right] \\ + \frac{\lambda}{2} \sin 2\theta (x \cos \theta \partial_z \rho + \cosh \rho F_z) + \lambda x \cosh \rho B_1 + \lambda (1-x^2) \cos \theta B_3 = 0 . \end{aligned} \quad (4.124)$$

We made it from eight equations for three functions to 3 equations for two functions, which imply all the others and the Bianchi identity. Using the shorthands B_i , we can write h as

$$\begin{aligned} \frac{h^2}{1-x^2} = \cos^4 \theta \cosh^2 \rho + \cos^2 z [B_1^2 + B_2^2 + (1-x^2) \cos^2 \theta B_3^2 + ((\partial_z \theta)^2 + (\partial_z \rho)^2) \cos^4 \theta] \\ + (1-x^2) [(\partial_x \theta)^2 + (\partial_x \rho)^2] \cos^2 \theta \cosh^2 \rho + F_x^2 \cosh^2 \rho + \frac{F_z^2 \cos^2 \theta \cos^2 z}{1-x^2} \end{aligned} \quad (4.125)$$

This function could have been an order 6 polynomial in derivative terms (it is the determinant of the 3x3 matrix g_{mn} , the components of which are quadratic in derivative terms) but it

⁵A way to see this is to solve eq. (4.121) for $\partial_x \theta$ and replace it everywhere, after using (4.116).

only is of order 4, since the dependence on β_2 is trivial. It can be reduced to order 2 using (4.111)-(4.113).

With the last equation in a convenient enough form, we can now show that (4.122) and (4.123) together imply (4.124). The argument proceeds using cumbersome but otherwise straightforward algebraic manipulations: We isolate the term involving h in (4.124) and square the equation afterwards, to get rid of the square root. This results in

$$\frac{h^2}{1-x^2} = \left[\cos \theta \cosh^2 \rho \left[\cos^2 \theta - \lambda \tanh \rho \tan z \left((1-x^2) \partial_x \theta + \frac{x}{2} \sin 2\theta \right) \right] + \frac{\lambda}{2} \sin 2\theta (x \cos \theta \partial_z \rho + \cosh \rho F_z) + \lambda x \cosh \rho B_1 + \lambda (1-x^2) \cos \theta B_3 \right]^2 . \quad (4.126)$$

With (4.125) we see that this equation is quadratic in the B_i , and thus quartic in derivative terms. However, using (4.111), (4.112) and (4.111) reduces it to an equation which is only quadratic in derivative terms. To further process it, we solve (4.122) for $\partial_x \theta$ and eliminate it in (4.123) and in (4.126). Eq. (4.123) is quadratic in derivative terms. If we tried to use (4.111)-(4.113) on it, we would end up with an equation that is equivalent to (4.122), which is useless. So we have to take (4.123) as it is. We solve it for $(\partial_z \rho)(\partial_z \theta)$ and use the result to eliminate that particular quadratic derivative term in (4.126). After this step (4.126) collapses to zero.

The remaining equations are therefore only (4.122) and (4.123). These two equations for slipping and bending mode, together with the solution for the gauge field (4.116), (4.120), are *equivalent* to the entire set of equations (4.85)-(4.88) and (4.91)-(4.94). To reduce the number of short hands introduced especially in (4.123), we rewrite them as follows. With

$$G_a = \lambda \sinh \rho \sin^2 \theta \partial_a (x \cot \theta) - \partial_a (\tan z \cosh \rho) , \quad (4.127)$$

they become

$$\begin{aligned} -(1-x^2) \cosh \rho G_x + \sinh \rho \cos \theta F_z &= 0 , \\ \sinh \rho (G_z F_x - G_x F_z) - \cos^3 \theta G_z - \sec^2 z \cosh^3 \rho \cos^3 \theta &= 0 . \end{aligned} \quad (4.128)$$

4.5 General Linearized Solutions

4.5.1 General linearized solution

In this section, we confront the non-linear κ -symmetry equations with a return to the perturbative analysis. More than simply serving as a check on the construction– the previously found solutions had better work– we will find the most generic linearized solutions possible. we start from eq. (4.128). Expanding to linear order in the fields gives

$$\partial_z(\lambda \tan z \theta - x\rho) + (1 - x^2)\partial_x(\lambda x\theta + \tan z \rho) - 2\lambda(1 - x^2)\theta = 0 , \quad (4.129a)$$

$$\partial_z(\lambda x\theta + \tan z \rho) - \sec^2 z \partial_x(\lambda \tan z \theta - x\rho) - 2\sec^2 z \rho = 0 . \quad (4.129b)$$

We collect the terms in the round brackets on which the derivatives act, and define

$$\theta = \lambda(\sin z \cos z \phi + x\zeta) , \quad \rho = \tan z \zeta - x \cos^2 z \phi . \quad (4.130)$$

eqs. (4.129) then become

$$\cos^2 z \partial_z \phi + (1 - x^2)\partial_x \zeta = 0 , \quad (4.131a)$$

$$\partial_z \zeta - \partial_x \phi = 0 . \quad (4.131b)$$

Hitting the first equation with ∂_x and then using the second leaves us with an equation for ζ alone,

$$\cos^2 z \partial_z^2 \zeta + \partial_x(1 - x^2)\partial_x \zeta = 0 . \quad (4.132)$$

This equations can be solved by separation of variables. We write $\zeta = p_\ell(x)\zeta_\ell(z)$ and conveniently introduce a constant ℓ such that the above equation implies

$$\partial_x(1 - x^2)\partial_x p = -\ell(\ell + 1)p , \quad \cos^2 z \partial_z^2 \zeta_\ell = \ell(\ell + 1)\zeta_\ell . \quad (4.133)$$

We recognize the x -dependent part as the Legendre equation. The requirement of regularity at $x = 1$ and $x = 0$, corresponding to $\beta_1 = 0$ and $\beta_1 = \pi$, respectively, forces us to choose

the Legendre functions of the first kind, P_ℓ , with ℓ an integer.⁶ Since $P_\ell = P_{-\ell-1}$, we can restrict to non-negative ℓ w.o.l.g. The result should not come as a surprise, as the P_ℓ are the polynomials appearing in the spherical harmonics $Y_{\ell,0}$. Summing up, we have

$$\zeta = \sum_{\ell=0}^{\infty} P_\ell(x)\zeta_\ell(z) , \quad \cos^2 z \partial_z^2 \zeta_\ell = \ell(\ell+1)\zeta_\ell . \quad (4.134)$$

Solving for ζ_ℓ yields

$$\zeta_\ell = c_{\ell,1}f_\ell + c_{\ell,2}f_{-\ell-1} , \quad f_\ell = (\cos z)^{\ell+1} {}_2F_1\left(\frac{\ell+1}{2}, \frac{\ell+1}{2}, \ell + \frac{3}{2}, \cos^2 z\right) . \quad (4.135)$$

Note that $\ell(\ell+1)$ is invariant under $\ell \rightarrow -\ell-1$, but f_ℓ is not.

We can now solve for ϕ as follows. Acting on the second equation of (4.131) with $\partial_x(1-x^2)$ leaves us with

$$\partial_x(1-x^2)\partial_z\zeta - \partial_x(1-x^2)\partial_x\phi = 0 . \quad (4.136)$$

Expanding ϕ in Legendre polynomials as $\phi = \sum_\ell \phi_\ell(z)P_\ell(x)$ and using (4.135) leaves us with

$$\sum_{\ell=0}^{\infty} \partial_x [(1-x^2)P_\ell(x)] \zeta'_\ell + \sum_{\ell=0}^{\infty} \ell(\ell+1)P_\ell(x)\phi_\ell(z) = 0 . \quad (4.137)$$

Using the recursion formulas for Legendre polynomials and Bonnet's recursion formula ($\ell+1)P_{\ell+1}(x) = (2\ell+1)xP_\ell(x) - \ell P_{\ell-1}$, we have

$$\partial_x [(1-x^2)P_\ell(x)] = \frac{\ell(\ell-1)}{2\ell+1}P_{\ell-1}(x) - \frac{(\ell+1)(\ell+2)}{2\ell+1}P_{\ell+1}(x) . \quad (4.138)$$

Using this in (4.137) and rearranging the sums such that they start at the same ℓ , we find

$$0 = \sum_{\ell=0}^{\infty} \ell(\ell+1)P_\ell(x) \left[\frac{\zeta'_{\ell+1}}{2\ell+3} - \frac{\zeta'_{\ell-1}}{2\ell-1} + \phi_\ell \right] . \quad (4.139)$$

Note the overall factor of $\ell(\ell+1)$, which means that ϕ_0 is undetermined. We thus find

$$\phi_\ell = \frac{\zeta'_{\ell-1}}{2\ell-1} - \frac{\zeta'_{\ell+1}}{2\ell+3} , \quad \ell \geq 1 . \quad (4.140)$$

⁶Note that regularity of θ and ρ implies regularity of ζ , ϕ for $z \in (0, \frac{\pi}{2})$.

Feeding that solution back into (4.131) gives no constraints from the second equation, and the first one collapses to

$$2\zeta_1 + 3 \cos^2 z \phi'_0 = 0 . \quad (4.141)$$

Using the second equation in (4.134), this is equivalent to $\zeta_1'' + 3\phi'_0 = 0$. We thus find $\phi_0 = -\frac{1}{3}\zeta_1' + d$ with an arbitrary constant d . However, ζ_{-1}' is indeed also just a constant (which flips sign with the sign of z , though, unless we make $c_{-1,2}$ flip as well), so we can equivalently write $\phi_0 = -\frac{1}{3}\zeta_1' - \zeta_{-1}'$. Of the two constants $c_{-1,1}$ and $c_{-1,2}$ only $c_{-1,2}$ matters, $c_{-1,1}$ drops out altogether. Summing up, we therefore have

$$\phi = \sum_{\ell=0}^{\infty} P_{\ell}(x)\phi_{\ell}(z) , \quad \phi_{\ell} = \frac{\zeta'_{\ell-1}}{2\ell-1} - \frac{\zeta'_{\ell+1}}{2\ell+3} , \quad \ell = 0, 1, \dots . \quad (4.142)$$

To translate back to θ and ρ we use (4.130). This yields

$$\theta = \sum_{\ell=0}^{\infty} P_{\ell}(x)\theta_{\ell}(z) \quad \theta_{\ell} = \lambda \left[\sin z \cos z \phi_{\ell} + \frac{\ell \zeta_{\ell-1}}{2\ell-1} + \frac{(\ell+1)\zeta_{\ell+1}}{2\ell+3} \right] , \quad (4.143)$$

$$\rho = \sum_{\ell=0}^{\infty} P_{\ell}(x)\rho_{\ell}(z) \quad \rho_{\ell} = \tan z \zeta_{\ell} - \cos^2 z \left[\frac{(\ell+1)\phi_{\ell+1}}{2\ell+3} + \frac{\ell\phi_{\ell-1}}{2\ell-1} \right] . \quad (4.144)$$

Finally, the solution for the gauge field (4.116) becomes

$$A_{\beta_2} = \lambda \tan z \theta - x \rho = -(1-x^2) \cos^2 z \phi + \phi . \quad (4.145)$$

With only A_{β_2} non-vanishing, we automatically have $d_{S^2}^{\dagger} A = 0$, or $\nabla^i A_i = 0$ with i running over the S^2 indices corresponding to (β_1, β_2) only. That was the gauge fixing used in [57]. Simply substituting the expressions for ρ and θ would yield an expansion in scalar spherical harmonics, which would be straightforward and possible but make regularity obscure. The more natural expansion is in terms of 1-form spherical harmonics. To make the normalization convention explicit, we take

$$Y_{\ell,0} = \star_{S^2} dP_{\ell} = (1-x^2) \frac{d}{dx} P_{\ell}(x) . \quad (4.146)$$

The first term on the right hand side of (4.145) almost is in the correct form already and can be rewritten using $(2n + 1)P_n = \frac{d}{dx}(P_{n+1} - P_{n-1})$. The second term can be rearranged starting from (4.142) using $(n + 1)P_{n+1} = (2n + 1)xP_n - nP_{n-1}$. We find

$$A_{\beta_2} = \sum_{\ell=1}^{\infty} A_{\ell}(z)(1 - x^2) \frac{d}{dx} P_{\ell}(x) - \zeta'_{-1} + \zeta'_0 P_1 , \quad (4.147)$$

$$A_{\ell} = \cos^2 z \left[\frac{\phi_{\ell+1}}{2\ell + 3} - \frac{\phi_{\ell-1}}{2\ell - 1} \right] - \frac{\zeta'_{\ell}}{\ell(\ell + 1)} .$$

Regularity of the gauge field on S^2 requires $A_{\beta_2} = 0$ at both poles, corresponding to $x = \pm 1$ (otherwise the limit to the pole is ill defined) and the gauge field not smooth. All terms in the sum manifestly implement this by the $(1 - x^2)$ factor. The remaining terms are both non-zero, which means their combination has to vanish at both of $x = \pm 1$. This fixes the extra terms to zero and we are left with

$$A_{\beta_2} = \sum_{\ell=1}^{\infty} A_{\ell}(z)(1 - x^2) \frac{d}{dx} P_{\ell}(x) , \quad A_{\ell} = \cos^2 z \left[\frac{\phi_{\ell+1}}{2\ell + 3} - \frac{\phi_{\ell-1}}{2\ell - 1} \right] - \frac{\zeta'_{\ell}}{\ell(\ell + 1)} . \quad (4.148)$$

All functions appearing as radial profiles are hypergeometric functions with argument $\cos^2 z$. They have the usual regular singular points for $\cos^2 z = 0$ and $\cos^2 z = 1$ and are regular for $\cos^2 z \in (0, 1)$. For us $\cos^2 z = 0$ corresponds to the conformal boundaries at $r \rightarrow \pm\infty$, while $\cos^2 z = 1$ corresponds to $r = 0$ where we need regular solutions. Indeed, for any choice of the constants the solutions for ρ and θ are regular at $z = 0$. This is reminiscent of the AdS_4 situation for D3/D7 [17], where, due to the absence of an additional regularity condition from the central slice (at least for small masses) there was a one-parameter family of supersymmetric states for a given choice of mass. We linked this to the availability of a variety of supersymmetric boundary conditions.

To analyze the near-boundary behavior of the solutions we switch to Fefferman-Graham

coordinates $r = -\log(u/2)$ or $z = 2 \tan^{-1}(\frac{2}{u}) - \frac{\pi}{2}$. We then find

$$\theta_\ell = \lambda u^{1-\ell}(d_{\ell,2} + \dots) + \lambda u^{2+\ell}(d_{\ell,1} + \dots) , \quad (4.149)$$

$$\frac{\rho_\ell - \ell \mathcal{A}_\ell}{2\ell + 1} = \left(\frac{d_{\ell+1,2}}{\ell + 1} + \dots \right) u^{-\ell-1} - \left(\frac{\ell + 3}{(2\ell + 3)(2\ell + 5)} d_{\ell+1,1} + \dots \right) u^{4+\ell} , \quad (4.150)$$

$$\frac{\rho_\ell + (\ell + 1) \mathcal{A}_\ell}{2\ell + 1} = \left(\frac{2 - \ell}{(2\ell - 3)(2\ell - 1)} d_{\ell-1,2} + \dots \right) u^{3-\ell} + \left(\frac{d_{\ell-1,1}}{\ell} + \dots \right) u^\ell . \quad (4.151)$$

The dots denote a series in positive even powers of u (plus possible log terms) and we have redefined the coefficients as

$$d_{\ell,1} = \frac{\ell(\ell + 1)}{4\ell^2 - 1} c_{\ell-1,1} + c_{\ell+1,1} , \quad d_{\ell,2} = c_{\ell-1,2} + \frac{\ell(\ell + 1)c_{\ell+1,2}}{(2\ell + 1)(2\ell + 3)} . \quad (4.152)$$

We see that the slipping mode corresponds to an operator with $\Delta_+ = 2 + \ell$ and $\Delta_- = 1 - \ell$, as discussed in [57] [around their (3.28)]. The bending mode mixes with the gauge field, and together they source two operators. One of them has $\Delta_+ = \ell + 4$ and $\Delta_- = -1 - \ell$ [see (3.53) of [57]]. The other one has $\Delta_+ = \ell$ and $\Delta_- = 3 - \ell$ for $\ell \geq 2$ and $\Delta_+ \leftrightarrow \Delta_-$ for $\ell \leq 1$ [see (3.55) of [57]]. In standard quantization the $d_{\ell,2}$ are the sources and the $d_{\ell,1}$ parametrize the expectation values. For the low-lying operators the roles are exchanged if alternative quantization is chosen.

What we see is that each deformation can be turned on individually (and preserves supersymmetry). Moreover, as discussed above, there are no further regularity conditions from the interior of the bulk AdS, which means the $d_{\ell,1}$ and $d_{\ell,2}$ can be chosen independently. In particular, we can set all sources to zero and still dial the expectation values of the various operators. The expectation values of all three operators will be linked, since they are all proportional to the $d_{\ell,1}$, but we nevertheless find a large “moduli space” of supersymmetric states. An interesting question is whether the deformations stay independent at the non-linear level. We will study this for the particular case of a mass deformation below and see that turning on one deformation triggers all other ones at the non-linear level.

4.5.2 Mass deformation to cubic order

We studied the general linear solution above and found a discrete series of supersymmetric deformations. In this section we specialize to a mass deformation and study higher orders in the perturbative expansion in the mass. As detailed in the previous subsection, at the linearized level we can dial mass and condensate separately. To keep track of the order in the expansion we therefore explicitly introduce a small parameter κ – not to be confused with the symmetry – and expand

$$\theta = \sum_{n=0}^{\infty} \kappa^n \theta^{(n)}, \quad \rho = \sum_{n=0}^{\infty} \kappa^n \rho^{(n)}. \quad (4.153)$$

The linearized solution to the final form of the kappa-symmetry equations eq. (4.128), are actually a special case

$$\begin{aligned} \theta^{(1)} &= P_0(x) \cos z (m \sin z + c \cos z), \\ \rho^{(1)} &= \lambda P_1(x) \cos z (c \sin z - m \cos z). \end{aligned}$$

Continuing the expansion to second order, we find that the linearized solutions do not provide source terms, and so the quadratic equations are exactly the same as the linearized one. It is entirely consistent to set $\theta^{(2)} = \rho^{(2)} = 0$, which we will do for the remainder.

The same cannot be said for the expansion to cubic order in κ . At this point in the perturbative analysis, the linear terms provide sources and it is no longer consistent to work with $\ell = 0$ and $\ell = 1$ modes for θ and ρ respectively. The linear solutions thus provide sources for higher dimension operators insofar as the cubic order solutions can be expressed in terms of modes as

$$\theta^{(3)} = P_0(x)\theta_0^{(3)}(z) + P_2(x)\theta_2^{(3)}(z), \quad \rho^{(3)} = P_1(x)\rho_1^{(3)}(z) + P_3(x)\rho_3^{(3)}(z) \quad (4.154)$$

The explicit form of the modes is bulky and not very illuminating, but the main features can be summarized quickly: The linearized solution appears as source for the cubic-order solutions, and in particular for $\theta_2^{(3)}$ and $\rho_3^{(3)}$. This forces these modes to be non-zero – for no

combination of m , c does the source vanish. Moreover, the equations for $\theta_2^{(3)}$, $\rho_3^{(3)}$ decouple from $\theta_0^{(3)}$, $\rho_1^{(3)}$. The general solution for $\theta_2^{(3)}$, $\rho_3^{(3)}$ involve solutions to both the particular and homogenous solution. The homogenous solution are informed in the same way as the previous section– the mass and condensate parameter can be dialed independently: So we can always choose a combination of particular and homogeneous solution such that $\theta_2^{(3)}$, $\rho_3^{(3)}$ are non-zero only at orders that are further subleading compared to the source and the vev terms.

Since we have the freedom to dial the source and vev terms independently– controlled by m and c respectively– we can study the much simpler expression for the $c = 0$ solutions, which are given by

$$\theta_2^{(3)} = -\frac{5\lambda}{3}(\rho_3^{(3)'} + 3\rho_3^{(3)}) \tan z, \quad (4.155a)$$

$$\rho_3^{(3)} = \frac{c_1 z}{2} + c_2 + \frac{c_1}{4} \sin(2z) - \lambda \frac{m^3}{480} (15 \cos(2z) + 6 \cos(4z) + \cos(6z)), \quad (4.155b)$$

$$\rho_1^{(3)} = -\lambda \theta_0^{(3)'} - \lambda \tan z (\theta_0^{(3)} - \theta_2^{(3)}) - \rho_3^{(3)} - \frac{m^3}{3} \lambda \cos(4z) \cos^2 z, \quad (4.155c)$$

$$\theta_0^{(3)} = \cos z \left[\left(c_3 - \frac{m^3}{6} \right) \sin z + c_4 \cos z - \frac{m^3}{12} \sin(3z) - \frac{m^3}{24} \sin(5z) \right]. \quad (4.155d)$$

By changing to a suitable Fefferman-Graham-like coordinate $z = 2 \tan^{-1} \left(\frac{2}{u} \right) - \frac{\pi}{2}$, we can read off the near-boundary expansions

$$\theta_2^{(3)} \simeq \frac{\lambda(\pi c_1 + 4c_2) + \frac{m^3}{12}}{16} \left(-\frac{20}{u} + 5u \right) - \frac{\lambda c_1}{3} u^4 - \frac{m^3 u^5}{3} + \mathcal{O}(u^6), \quad (4.156a)$$

$$\rho_3^{(3)} \simeq \left(\frac{\pi c_1}{4} + c_2 + \frac{m^3 \lambda}{48} \right) - \frac{c_1 u^3}{3} + \frac{3c_1 u^5}{20} - \frac{\lambda m^3 u^6}{15} + \mathcal{O}(u^7), \quad (4.156b)$$

$$\rho_1^{(3)} \simeq \frac{(\pi c_1 + 4c_2) + \frac{\lambda m^3}{12}}{16} \left(-\frac{20}{u} + 6 \right) + \lambda c_4 u + \mathcal{O}(u^2), \quad (4.156c)$$

$$\theta_0^{(3)} \simeq \left(c_3 - \frac{m^3}{8} \right) u + c_4 u^2 + \left(\frac{25m^3}{96} - \frac{3c_3}{4} \right) u^3 + \mathcal{O}(u^4). \quad (4.156d)$$

We can choose appropriate values of the integration constants such that the leading parts of $\theta_2^{(3)}$, $\rho_3^{(3)}$ vanish:

$$c_1 = 0, \quad c_2 = -\frac{\lambda m^3}{48}. \quad (4.157)$$

This, by virtue of the relation between θ , ρ , and A_{β_2} will set to zero the sources for the dual operators in the field theory once a proper decoupling, evaluation of the DBI action on-shell, and holographic renormalization takes place. This, however, is not the ultimate aim of this project, which will be attended to in the next section.

4.6 Spherical Embedding

While purely for supersymmetry considerations, Lorentzian AdS slicing is most convenient, the field theory tools available to make direct comparisons to the holographic calculations require spherical boundary geometry [1, 73, 74]. Firstly, the final kappa symmetry equations eq. (4.128) do not depend on the chosen coordinates for the defect AdS₃ slices. So, we can analytically continue

$$r \rightarrow \hat{r} = r + \frac{i\pi}{2}, \quad g_{AdS_3} \rightarrow -g_{S^3}. \quad (4.158)$$

Affecting this change of coordinates, the AdS part of the metric becomes

$$ds^2 = d\rho^2 + \cosh^2 \rho (d\hat{r}^2 + \sinh^2 \hat{r} g_{S^3}). \quad (4.159)$$

This metric describes Euclidean AdS₅, or rather \mathbb{H}^5 , sliced by \mathbb{H}^4 . While studying these Euclidean AdS embeddings would be interesting– and is left to future work– it is the immediate goal to obtain Euclidean AdS₅ sliced by S^4 . This will need us to affect one more coordinate transformation such that the metric takes the form

$$ds^2 = dR^2 + \sinh^2 R (d\chi^2 + \sin^2 \chi g_{S^3}). \quad (4.160)$$

This metric can be obtained from the hyperbolic slicing by the identification

$$\cosh R = \cosh \rho \cosh \hat{r}, \quad \sin \chi = \frac{\sinh \rho}{\sqrt{\cosh^2 \rho \cosh^2 \hat{r} - 1}}. \quad (4.161)$$

Keep in mind that R and χ , through their dependence on ρ , are now implicitly defined functions of both \hat{r}, β_1 . The embedding preserving the superconformal symmetries $\rho = 0$ is now where at $\chi = 0$ which describes an equatorial $S^3 \subset S^4$.

4.6.1 Continuing perturbative solutions

Now that we have successfully identified the continuation of the background AdS geometry to Euclidean signature and S^4 slicing, we can use eq. (4.161) to translate the perturbative solutions found in the previous sections out to cubic order. To that end, we employ the framework introduced for the AdS slicing to expand R and χ first to linear order in κ :

$$R = \hat{r} + \mathcal{O}(\kappa^2), \quad \chi = \kappa \rho^{(1)} \operatorname{csch} \hat{r}. \quad (4.162)$$

While this is a nice form, it will not be so simple at cubic order because of the implicit definitions. However, we can use that to linear order $\chi = \kappa \chi^{(1)}(R)$. Note that at higher orders we will need to work to cast everything in terms of the spherically sliced radial coordinate R . The linearized solutions in spherical slicing take the form

$$\theta^{(1)} = -\operatorname{csch}^2 R (c + im \cosh R), \quad (4.163a)$$

$$\chi^{(1)} = \lambda \operatorname{csch}^3 R (m - ic \cosh R) P_1(\cos \beta_1), \quad (4.163b)$$

$$A_{\beta_2}^{(1)} = \lambda \operatorname{csch}^2 R (m - ic \cosh R) \sin^2 \beta_1. \quad (4.163c)$$

The gauge field is still determined by θ and χ just as in the AdS slicing story, and note that $\sin^2 \beta_1 = -(\star_{S^2} dP_1(\cos \beta_1))$.

Moving on to the continuation of the cubic order solutions to spherical slicing, it can be seen to be subtler than the linearized analysis. The reason is that the original spacetime radial coordinate, \hat{r} , is implicitly defined in terms of the worldvolume radial coordinate, R , as

$$\hat{r}(R) = \cosh^{-1} \left(\cosh R \operatorname{sech} \left(\sum_i \kappa^i \rho_i(\hat{r}(R)) \right) \right). \quad (4.164)$$

Treating the expansion first in κ to determine $\hat{r}(R) = \sum_i r_i(\hat{r}(R)) \kappa^i$, we then expand again in κ such that to $\mathcal{O}(\kappa^4)$

$$\begin{aligned} \hat{r}(R) \simeq & R - \frac{1}{2} \coth R \rho_1^2(R) \kappa^2 \\ & - \frac{\kappa^4 \rho_1(R) \coth R}{24} (24(\rho_3(R) - (1 + \coth^2 R) \rho_1^3(R) - 4\partial_R(\coth R \rho_1(R))). \end{aligned} \quad (4.165)$$

The translation from the \mathbb{H}^4 from of the embedding functions to spherical slicing to $\mathcal{O}(\kappa^3)$ then takes the form

$$\theta(\hat{r}(R)) \simeq \theta_1(R)\kappa + (\theta_3(R) + r_2(R)\theta_1'(R))\kappa^3 + \dots \quad (4.166a)$$

$$\chi(\hat{r}(R)) \simeq \frac{\rho_1(R)}{\sinh R} \kappa + \frac{\kappa^3}{6 \sinh R} (\rho_1^3(R) - \partial_R(\rho_1^3(R) \coth R) + 6\rho_3(R)) \quad (4.166b)$$

$$\begin{aligned} A_{\beta_2}(\hat{r}(R)) &\simeq (\lambda \sinh R \theta_1(R) - \cos \beta_1 \rho_1(R))\kappa \\ &+ \frac{\kappa^3}{6} \left[i\lambda \cosh R (6\theta_3(R) - \theta_1(R)^3 - 3 \coth R \rho_1(R)^2 \theta_1'(R)) \right. \\ &\left. + \cos \beta_1 (3\theta_1(R)^2 \rho_1(R) - \rho_1(R)^3 + 3 \coth R \rho_1(R)^2 \rho_1'(R) - 6\rho_3(R)) \right] \end{aligned} \quad (4.166c)$$

where the $X_i(R)$ are the forms of the embedding functions as used in previous sections but with $\hat{r}_0(R) = R$ as the argument.

At this point, one could easily check that the above continued solutions satisfy the equations of motion out to $\mathcal{O}(\kappa^6)$. The explicit form of the Lagrangian and the associated equations of motion is too cumbersome to spell out and are in some small way unilluminating. There is one subtlety that arises, and that is in the action

$$S_{D5} = T_5 \int d^6 \xi \sqrt{\det g + F} + T_5 \int_{\Sigma_6} C_4 \wedge F, \quad (4.167)$$

care must be taken in order to properly treat the WZ term. Specifically, on the Freund-Rubin ansatz

$$C_4 = \zeta(\chi) \sinh^4 R dR \wedge \text{vol}(S^4) + \dots, \quad \Rightarrow \quad \zeta'(\chi) = -4 \cos^3 \chi. \quad (4.168)$$

The $-$ sign in ζ' accounts for the ordering in $\text{vol}(AdS_5) \sim dR \wedge d\chi \wedge \dots$

Regularity conditions

Just as in the D3/D7 analysis, once we go into spherical slicing, we need to concern ourselves with regularity in the IR. That is, as the probe branes cap-off or otherwise move away from the D3-brane stack, we need to ensure that the embedding is smooth near the origin on the worldvolume. There are issues that conspire to make regularity subtle: primarily, the

complex embeddings make the concept of ‘regularity’ unclear. However, we can understand that smoothness of the embedding near the origin from the perspective that the Lagrangian density should be integrable– i.e. the action not singular at the origin– near $R = 0$. In the linearized theory the near origin behavior of the Lagrangian density is

$$\mathcal{L}^{(1)} \equiv \frac{L_{DBI} + L_{WZ}}{2\pi V_3 \sin \beta_1} = -\frac{\kappa^2 \cos^2 \beta_1 (c + im)^2}{R^3} + \frac{3\kappa^2 (3 \cos 2\beta_1 + 1)(c + im)^2}{4R} + \mathcal{O}(R). \quad (4.169)$$

Thus, integrability of the linearized Lagrangian density implies that

$$c = -im \quad (4.170)$$

Regularity including cubic order deformations can be established by expanding the embedding Lagrangian to $\mathcal{O}(\kappa^6)$ to capture the quadratic fluctuations for linear and cubic modes and looking at the behavior near $R = 0$. What we find is that a convenient choice for the cubic order integration constants $c_1 = i\kappa^3 \lambda/5$ and $c_3 = (k + ic_4)\kappa^3$ can be made, which after integrating over the S^2 leaves only a $\mathcal{O}(R^{-3})$ piece to eliminate

$$\mathcal{L}^{(3)} \simeq \frac{(16775 - 75936k + 80640k^2)\kappa^6}{120960R^3} + \mathcal{O}(R) \quad \Rightarrow \quad k_{\pm} = \frac{791 \pm 18\sqrt{119}}{1680}. \quad (4.171)$$

While not as aesthetically pleasing as the linearized regularity condition, k_{\pm} does the job in ensuring integrability of \mathcal{L} near the origin.

4.7 Holographic one-point functions and match to localization

Now that we have completed the continuation of the linear and cubic order solutions to spherical slicing and established regularity conditions, we can begin calculations that can be compared to analogous quantities in the field theory found using supersymmetric localization of massive fundamental $\mathcal{N} = 2$ hypermultiplets on an S^3 [73] coupled to the gauge multiplet of $\mathcal{N} = 4$ SYM theory on an S^4 [1], which will be addressed in the following section. To achieve that end, we can follow the calculation done in [17, 18] and compute the one point function for a particular combination of slipping mode and free scalar mode dual to the source of the so-called compensating mass term.

4.7.1 Decoupling

With both the continuation to spherical slicing and a generic treatment of holographic renormalization and calculation of one point functions completed, we can orient ourselves in trying to compute quantities in the holographic context that can be related to calculable observables in the supersymmetric localization of a defect theory including a mass deformation. With the continuation to functions describing regular embeddings in spherical slicing to $\mathcal{O}(\kappa^3)$, we can better facilitate discussions of holographic one-point functions by identifying the analog of the decoupled scalar modes, $\tilde{\tau}$ and $\tilde{\sigma}$, used in App. 4.9. Starting at linear order, $X = X_1\kappa + \mathcal{O}(\kappa^3)$, the quadratic Lagrangian density expressed in spherical slicing, after integrating over the S^3 directions and over β_2 ,

$$L_{D5} = \left(-\frac{8 \sinh R}{\sin \beta_1} \chi_1 \partial_{\beta_1} A_{\beta_2,1} + \frac{(\partial_{\beta_1} A_{\beta_2,1})^2 + (\partial_R A_{\beta_2,1})^2}{\sin^2 \beta_1} + (\partial_{\beta_1} \theta_1)^2 \right. \\ \left. + (\partial_R \theta_1)^2 + \sinh^2 R ((\partial_{\beta_1} \chi_1)^2 + (\partial_R \chi_1)^2) - 2(\theta_1)^2 - 3(\chi_1)^2 \right), \quad (4.172)$$

where the quadratic action is

$$\frac{S_{D5}}{2\pi V_{S^3} T_5} = \frac{1}{2} \int dR d\beta_1 \sin \beta_1 \sinh^3 R L_{D5}. \quad (4.173)$$

From the quadratic action, we can easily derive the equations of motion. The real work that needs to be done is in the coupled χ_1 and $A_{\beta_2,1}$ equations:

$$\chi : 0 = 4 \frac{\partial_{\beta_1} A_{\beta_2,1}}{\sinh R \sin \beta_1} + \frac{3\chi_1}{\sinh^2 R} + \square_{S^2} \chi_1 + \frac{1}{\sinh^5 R} \partial_R (\sinh^5 R \partial_R \chi_1), \quad (4.174a)$$

$$A_{\beta_2} : 0 = 4 \sinh R \partial_{\beta_1} \chi_1 - \frac{1}{\sin \beta_1 \sinh^3 R} \partial_R (\sinh^3 R \partial_R A_{\beta_2,1}) - \partial_{\beta_1} \left(\frac{1}{\sin \beta_1} \partial_{\beta_1} A_{\beta_2,1} \right), \quad (4.174b)$$

where $\square_{S^2} = \frac{1}{\sin \beta_1} \partial_{\beta_1} (\sin \beta_1 \partial_{\beta_1})$. The equations of motion can be made more palatable if we redefine $\chi_1 = \cos \beta_1 \operatorname{csch} R C_1^{(1)}(R)$ and $A_{\beta_2,1} = \sin^2 \beta_1 a_1^{(1)}(R)$, while hitting eq. (4.174b) with $\csc \beta_1 \partial_{\beta_1}$. The free scalars can then easily be identified as $\varpi_1^{(1)} = a_1^{(1)} - C_1^{(1)}$ and $\varsigma = 2a_1^{(1)} + C_1^{(1)}$, which results in

$$(\square_A - 10)\varpi_1^{(1)} = 0, \quad (\square_A + 2)\varsigma_1^{(1)} = 0. \quad (4.175)$$

Here we have denoted $\square_A = \frac{1}{\sinh^3 R} \partial_R (\sinh^3 R \partial_R)$. This matches the linearized spectrum for the coupled sector analysis found in [57] for $\ell = 1$.

Repeating the procedure for the $\mathcal{O}(\kappa^3)$ fields has one subtlety in that the linear order solutions now provide source terms for the cubic modes. However, the overall form of the quadratic action is the same, and so we can apply the same prescription as above: finding the equations of motion, rescaling $\chi_3 = \text{csch } R \tilde{\chi}_3$, hitting the $A_{2,3}$ equation with $\text{csc } \beta_1 \partial_{\beta_1}$, and expanding in S^2 modes. For the cubic order solution, we know that we have $\ell = 0, 2$ modes turned on for the slipping mode and $\ell = 1, 3$ modes for the bending mode and gauge field such that $\tilde{\chi}_3 = C_3^{(1)} P_1 + C_3^{(3)} P_3$, $\theta_3 = \theta_3^{(0)} P_0 + \theta_3^{(2)} P_2$, and $A_{2,3} = a_3^{(1)} (\star d)_{S^2} P_1 + a_3^{(3)} (\star d)_{S^2} P_3$. After a bit of algebra, we arrive at

$$\theta_3 : \quad ((\square_A + 2)\theta_3^{(0)}) + P_2((\square_A - 4)\theta_3^{(2)}) = f_\theta , \quad (4.176a)$$

$$\tilde{\chi}_3 : \quad P_1(\square_A C_3^{(1)} - 2(3C_3^{(1)} - 4a_3^{(1)})) + P_3(\square_A C_3^{(3)} - 16(C_3^{(3)} - 3a_3^{(3)})) = f_\chi , \quad (4.176b)$$

$$A_2 : \quad P_1(\square_A a_3^{(1)} - 2(a_3^{(1)} - 2\chi_{31})) + 6P_3(\square_A a_3^{(3)} - 4(3a_3^{(3)} - C_3^{(3)})) = f_A , \quad (4.176c)$$

where the source terms $f_X = f_X(X_1, R, \beta_1)$. The specific forms of the f are not particularly important at this point, and so will only be treated by breaking up into sources for individual modes, $f_X = \sum_\ell f_X^{(\ell)} P_\ell$. Turning attention on the coupled modes, we can treat the P_1 and P_3 independently. The definitions of the free scalars follow easily as $\varpi_3^{(1)} = a_3^{(1)} - C_3^{(1)}$, $\varsigma_3^{(1)} = 2a_3^{(1)} + C_3^{(1)}$ for $\ell = 1$ modes and $\varpi = 3a_3^{(3)} - C_3^{(3)}$, $\varsigma_3^{(3)} = 4a_3^{(3)} + C_3^{(3)}$ for $\ell = 3$ modes.

The equations of motion then take a simple form

$$P_1 : \quad (\square_A - 10)\varpi_3^{(2)} = f_\varpi^{(1)}, \quad (\square_A + 2)\varsigma_3^{(1)} = f_\varsigma^{(1)}, \quad (4.177a)$$

$$P_3 : \quad (\square_A - 28)\varpi_3^{(3)} = f_\varpi^{(3)}, \quad \square_A \varsigma_3^{(3)} = f_\varsigma^{(3)}, \quad (4.177b)$$

where $f_\varpi^{(1)} = f_A^{(1)} - f_\chi^{(1)}$ and so on. Again the mass terms in the $\ell = 3$ equations reassuringly match exactly [57]. With those definitions in hand, we can construct the necessary near boundary data setting $R = -\log u/2$ and expanding around $u = \epsilon \ll 1$. However, at this point it is convenient to combine the $\mathcal{O}(\kappa)$ and $\mathcal{O}(\kappa^3)$ contributions organized according to

sector and mode, e.g. $\varsigma = \varsigma^{(1)}(u, \kappa)P_1 + \varsigma^{(3)}(u, \kappa)P_3$. Doing so allows organization into non-linear corrections to mass deformations and, separately, sources for new higher dimension operator insertions in the field theory. This leaves us with expansions that then read, with the usual rescaling by mode to ensure canonical kinetic term in the quadratic action,

$$\theta^{(0)} \simeq -im(\kappa, c_i)\epsilon - \mathbf{c}(\kappa, c_i)\epsilon^2 + \mathcal{O}(\epsilon^3), \quad (4.178a)$$

$$\theta^{(2)} \simeq -\frac{i\mathbf{m}_3(\kappa, c_i)}{\epsilon} - im_3(\kappa, c_i)\epsilon + \frac{i\kappa^3}{15}\epsilon^3 + \mathcal{O}(\epsilon^4), \quad (4.178b)$$

$$\varsigma^{(1)} \simeq -i\lambda\mathbf{c}(\kappa, c_i)\epsilon + \lambda m(\kappa, c_i)\epsilon^2 + \mathcal{O}(\epsilon^3), \quad (4.178c)$$

$$\varsigma^{(3)} \simeq \frac{\lambda\mathbf{m}_3(\kappa, c_i)}{3} - \frac{\lambda\kappa^3}{9}\epsilon^3 + \mathcal{O}(\epsilon^4), \quad (4.178d)$$

$$\varpi^{(1)} \simeq -\frac{15\lambda\mathbf{m}_3(\kappa, c_i)}{14\epsilon^2} - \frac{28\lambda\mathbf{m}_3(\kappa, c_i)}{9} - \frac{224\lambda\mathbf{m}_3(\kappa, c_i)}{15}\epsilon^2 + \frac{2\lambda\kappa^3}{5}\epsilon^3 + \mathcal{O}(\epsilon^4), \quad (4.178e)$$

$$\varpi^{(3)} \simeq \frac{\kappa^3\lambda}{105}\epsilon^5 + \mathcal{O}(\epsilon^6), \quad (4.178f)$$

where

$$m(\kappa, c_i) = \kappa - \frac{\kappa^3}{12}(7 - 12(k_{\pm} + ic_4)), \quad (4.179a)$$

$$\mathbf{c}(\kappa, c_i) = -i(\kappa + \kappa^3(\frac{1}{6} + ic_4)), \quad (4.179b)$$

$$\mathbf{m}_3(\kappa, c_i) = \kappa^3\left(\frac{13}{480} + \lambda c_2\right). \quad (4.179c)$$

4.7.2 Holographic one-point functions

The Lagrangian for the combined fluctuations can be found working from eq. (4.175) and eq. (4.177), remembering to include the slipping mode in the process, as $\mathcal{L}_3 = \mathcal{L}_\theta + \mathcal{L}_\varpi + \mathcal{L}_\varsigma$ where, e.g.

$$\mathcal{L}_\varsigma = \frac{\sqrt{g}}{2} \left((\varsigma^{(1)'})^2 - 2(\varsigma^{(1)})^2 + 2\varsigma^{(1)}f_\varsigma^{(1)} + (\varsigma^{(3)'})^2 + 2\varsigma^{(3)}f_\varsigma^{(3)} \right). \quad (4.180)$$

Here $' \equiv u\partial_u$, and $\sqrt{g} = \frac{(1-\frac{u}{2})^3}{u^3} \sin\beta_1\sqrt{g_{S^3}}$. Switching to $R = -\log u/2$, evaluating on-shell using the equations of motion for each ℓ , and integrating over the S^3 and S^2 directions gives

results in, e.g., for the ς contribution to the on-shell action

$$\frac{S_\varsigma^{os}}{4\pi V_{S^3} T_5} = \int_0^2 du \left(\frac{F(u)}{2u} (\varsigma^{(1)} f_3^{(1)} + \varsigma^{(3)} f_3^{(3)}) \right) + \frac{1}{2} \left[(uF(u)\varsigma_3^{(1)} \partial_u \varsigma^{(1)}) + \left[(uF(u)\varsigma^{(3)} \partial_u \varsigma^{(3)}) \right]_\epsilon^2 \right], \quad (4.181)$$

where $F(u) = \frac{(1-\frac{u^2}{4})^3}{u^3}$. The overall form of the on-shell action is the same θ_3 and ϖ_3 .

Focussing attention on the purely massive deformation, i.e. setting $\mathbf{m}_3 = 0$, we can turn the crank on the holographic renormalization mechanism outlined in App. 4.9, writing schematically the on-shell action for any given mode as

$$\frac{S_{X,os}^{(\ell)}}{4\pi V_{S^3} T_5} = \left(\int_0^2 du \frac{F(u)}{2u} X^{(\ell)} f_X^{(\ell)} + \frac{1}{2} [uF(u)X^{(\ell)} \partial_u X^{(\ell)}] \right) \quad (4.182)$$

We can first calculate the total on-shell action in this sector as a function of κ and c_i , and then calculate $\langle \mathcal{O}_{SUSY}(\kappa, c_i) \rangle := \langle \mathcal{O}_{\theta^{(0)}} + i\lambda \mathcal{O}_{\varsigma^{(1)}} \rangle$. While they have been neglected in the overall discussion thus far, it can be shown that the residual ‘source’ Lagrangian for the θ and ς modes has at worst $\mathcal{O}(\epsilon)$ contributions in the near boundary region and are thus inconsequential. So, we can focus our attention on the total derivatives. Since the asymptotic expansion of $\theta^{(0)}$ and $\varsigma^{(1)}$ have the same structure as the linearized analysis, the same counterterms will be needed. That is, the counterterm Lagrangians can be built from

$$\mathcal{L}_{CT,\theta}^{(0)} = \sum_{k=0} a_{\theta,k}^{(0)} \frac{(1-\frac{\epsilon^2}{4})^3}{\epsilon^3} (\theta^{(0)})^2 R_\epsilon^k, \quad \mathcal{L}_{CT,\varsigma}^{(1)} = \sum_{k=0} a_{\varsigma,k}^{(1)} \frac{(1-\frac{\epsilon^2}{4})^3}{\epsilon^3} (\varsigma^{(1)})^2 R_\epsilon^k, \quad (4.183)$$

where R_ϵ is the Ricci scalar of the radial cutoff slice. Treating both the on-shell and counterterm actions by expanding $\theta^{(0)} = \theta_0^{(0)} \epsilon + \theta_1^{(0)} \epsilon^2 + \dots$ and $\varsigma^{(1)} = \varsigma_0^{(1)} \epsilon + \varsigma_1^{(1)} \epsilon^2 + \dots$, it can be seen that indeed $a_{\theta,0}^{(0)} = -1$, $a_{\varsigma,0}^{(1)} = -1$, and all others vanish. Thus the holographically renormalized on-shell action can be written in the supersymmetric sector is constructed from

$$\frac{S_{D5,\theta}^{(0)} + S_{CT,\theta}^{(0)}}{4\pi V_{S^3} T_5} = -\frac{1}{2} \theta_0^{(0)} \theta_1^{(0)} + \mathcal{O}(\epsilon), \quad \frac{S_{D5,\varsigma}^{(1)} + S_{CT,\varsigma}^{(1)}}{4\pi V_{S^3} T_5} = -\frac{1}{2} \varsigma_0^{(1)} \varsigma_1^{(1)} + \mathcal{O}(\epsilon), \quad (4.184)$$

where $\theta_{0,1}^{(0)}$ etc. can be read off from eq. (4.178) and are related by $\varsigma_1^{(1)} = i\lambda \theta_0^{(0)}$ and $\varsigma_0^{(1)} = i\lambda \theta_1^{(0)}$.

Moving on to calculating the one point functions to compare to $\frac{dF}{dM}$, we should recall the discussion surrounding regularity in the linearized theory, where it was argued that we should redefine $\kappa \rightarrow i\kappa$ to accord with field theory knowledge. Why we would want to do this, more precisely than making the slipping mode real, from the perspective of the field theory is in following the construction of the supersymmetric compensating term on the spherical background geometry in [59], the mass for the fermions $MQ\tilde{Q}$ was real while the superpotential (compensating) mass term was imaginary, $iM(Q^2 + \tilde{Q}^2)$. Since the same localizing supercharge is used on the S^4 and S^3 in the calculations of [1] and [73] respectively (see [83]), we should maintain that the slipping mode gives a real mass for the dual fermions in the hypermultiplet and the coupled sector giving an imaginary mass for the appropriate scalars. How the redefinition, $\kappa \rightarrow i\kappa$, works at cubic order is to say that $m \rightarrow \tilde{m}$ and $\mathbf{c} \rightarrow \tilde{\mathbf{c}}$ such that

$$\tilde{m} = i\left(\kappa + \frac{\kappa^3}{12}(7 - 12(k + ic_4))\right), \quad \tilde{\mathbf{c}} = -\kappa + \frac{\kappa^3}{6}(1 + 6ic_4). \quad (4.185)$$

In addition, owing to the relation $\varsigma_1^{(1)} = i\lambda\theta_0^{(0)}$ and $\varsigma_0^{(1)} = i\lambda\theta_1^{(0)}$, it is more natural to affect the variation with respect to $\varsigma_1^{(1)}$, which is allowed since $\varsigma^{(1)}$ lies in the window for alternative quantization. Further, note that the appearance of ic_4 can be removed by making the redefinition $\kappa \rightarrow \tilde{\kappa} = \kappa(1 - ic_4\kappa^2)$. We can see that c_4 if kept around would need to be purely imaginary for the free energy to be real. Moreover, while this shift would generate a terms at $\mathcal{O}(\kappa^5)$, those are outside of our current working order. As such, the redefined m and \mathbf{c} take the form

$$\hat{m} = i\left(\tilde{\kappa} + \tilde{\kappa}^3\left(\frac{7}{12} - k\right)\right), \quad \hat{\mathbf{c}} = -\tilde{\kappa} + \frac{\tilde{\kappa}^3}{6}. \quad (4.186)$$

Now, we can calculate the appropriate variations to import into $\langle \mathcal{O}_{SUSY} \rangle$, which are

$$\frac{1}{4\pi V_{S^3} T_5} \frac{\delta(S_{D_5}^{os} + S_{CT})}{\delta\theta_0^{(0)}} = -\frac{1}{2}\hat{\mathbf{c}} + \mathcal{O}(\epsilon), \quad \frac{1}{4\pi V_{S^3} T_5} \frac{\delta(S_{D_5}^{os} + S_{CT})}{\delta\varsigma_1^{(1)}} = \frac{i\lambda}{2}\hat{\mathbf{c}} + \mathcal{O}(\epsilon). \quad (4.187)$$

We can then construct the supersymmetric one point function including the non-linear de-

formation by

$$\langle \mathcal{O}_{SUSY} \rangle = -\frac{1}{V_{S^3}} \left(\frac{\delta(S_{D5}^{os} + S_{CT})}{\delta\theta_0^{(0)}} + \frac{i}{\lambda} \frac{\delta(S_{D5}^{os} + S_{CT})}{\delta\varsigma_1^{(0)}} \right), \quad (4.188)$$

which results in

$$\frac{V_{S^3}}{N_c^2} \langle \mathcal{O}_{SUSY} \rangle = -2\zeta\tilde{\kappa} + \frac{\zeta\tilde{\kappa}^3}{3}. \quad (4.189)$$

Completing the story, we need the map to field theory language through $4\pi T_5 = N_f N_c \frac{\sqrt{\lambda}}{2\pi^3}$, $V_{S^3} = 2\pi^2$, and $\mu = \frac{\sqrt{\lambda}}{2\pi}$.

$$\frac{V_{S^3}}{N_c^2} \langle \mathcal{O}_{SUSY} \rangle \equiv \frac{dF}{dM} = -\frac{2\zeta M}{\mu} + \frac{\zeta M^3}{3\mu^3} + \dots, \quad (4.190)$$

where we have identified $M = \tilde{\kappa}\mu$ as before. If we compare to the expression calculated first without shifting the factor of ic_4 into the definition of the linear order mass parameter and affecting the shift at the end, then we would end up with the same expression up to terms subleading in $1/\mu$, which are further subleading our strong coupling expansion.

The takeaway from this analysis is that in the supersymmetric sector the holographic one-point function for the purely massive deformation will always end up with the form, following the definitions and sign conventions above,

$$\langle \mathcal{O}_{SUSY} \rangle = 2\zeta\mathfrak{c}(\tilde{\kappa}). \quad (4.191)$$

This is remarkably simple result that could be, in principle, checked to any order in κ , but having established the pattern at the linearized and cubic orders, it stands to reason that further non-linear analysis will bear this out.

4.7.3 Comparison to localization

Here is where the spherical kappa symmetry rubber meets the asphalt of the supersymmetric localization highway: How do we compare the calculation of the one point functions above with anything that could be calculated using localization? As in the case of the dual to

supersymmetric probe D3/D7 [17, 18], the idea is that the free energy of localized $\mathcal{N} = 4$ SYM theory on S^4 coupled to N_f $\mathcal{N} = 2$ hypermultiplets that live on an S^3 defect can be expanded

$$F \simeq S_0|_{\rho_W} + \zeta S_1|_{\rho_W} + \mathcal{O}(\zeta^2) \tag{4.192}$$

where $\frac{N_f}{N_c} \equiv \zeta \ll 1$, $\rho_W(x) = \frac{2}{\pi\mu^2} \sqrt{\mu^2 - x^2}$ is the Wigner semi-circular eigenvalue distribution, S_0 contains the purely S^4 contribution from $\mathcal{N} = 4$ SYM theory, and S_1 encodes the one-loop determinant due to the flavor fields on the S^3 defect. The justification for this is seen as studying perturbatively massive supersymmetric probe D3/D5 embeddings meaning that we are in the regimes where $1 \ll N_f \ll N_c$, $1 \ll \lambda$, and $\frac{N_f}{N_c} \ll 1$. The large N_c limit facilitates the continuum analysis of the saddle point equations found in the $\lambda \gg 1$ regime whose solution (ρ_W) is corrected only in a parametrically narrow region of width $1/\sqrt{\lambda}$ which is extinguished by the strong coupling limit. Hence, the expansion in $\zeta \equiv \frac{N_f}{N_c}$, the evaluation of S_1 on the Wigner distribution, and the absence of and $\delta S_0|_{\rho_W}$ contribution to the $\mathcal{O}(\zeta)$ term.

Since the flavor fields are localized on an S^3 defect and only contribute to the matrix model through their one-loop fluctuations about the locus of the gauge theory in the ambient S^4 , we can use the results of [73, 74] folded in to the mnemonic of one-loop contributions

$$d = 4 : \begin{cases} \text{Vector :} & \prod_{i < j} H^2(a_{[ij]}) , \\ \text{Adj. Hyper :} & \prod_{i < j} \frac{1}{H(a_{[ij]} + M_{adj})H(a_{[ij]} - M_{adj})} , \\ \text{Fund. Hyper :} & \prod_{i, f} \frac{1}{\sqrt{H^{N_f}(a_i + M_f)H^{N_f}(a_i - M_f)}} , \end{cases} \tag{4.193a}$$

$$d = 3 : \quad \text{Fund. Hyper :} \quad \prod_{i, f} \frac{1}{\cosh^{N_f}(\pi(a_i M_f) L_3)} , \tag{4.193b}$$

where $H(x) = G(1 + ix)G(1 - ix)$, $G(x)$ the Barnes' G-function, a is the non-vanishing adjoint valued scalar field parameterizing the $\mathcal{N} = 4$ SYM theory locus, a_i label the weights after diagonalizing into the Cartan, $a_{[ij]} = a_i - a_j$ are the roots, f runs over the flavor sectors with masses M_f , and L_3 is the length scale on the $d = 3$ defect. Note that at the boundary,

the S^3 is equatorial, which means $L_3 = L_4$ which can be normalized to unity. However, for clarity we will keep L_3 around during the calculations and set it to unity when convenient. Since in the ambient S^4 we only have pure $\mathcal{N} = 4$ SYM theory, the contributions from the $d = 4$ vector and massless adjoint hyper multiplets cancel one another leaving behind just the classical Gaussian matrix model by the defect hypers:

$$\mathcal{Z} = \int \left(\prod_i da_i \right) \frac{\prod_{i < j} (a_{[ij]})^2}{\prod_i \cosh^{N_f}(\pi(a_i + M)L_3)} e^{-\frac{8\pi}{\lambda} \sum_i a_i^2}, \quad (4.194)$$

where the factor of $a_{[ij]}^2$ appears due the Vandermonde determinant resulting from diagonalizing the Cartan of the gauge group. Compiling the above information in order to compare to the one point function calculated holographically, we need first to calculate

$$S_1 = -\frac{1}{2} \int_{-\mu}^{\mu} dx \rho_W(x) \log \cosh(\pi(x + M)L_3), \quad (4.195)$$

or rather $\frac{dS_1}{dM}$. Taking advantage of the various limits afforded to us as discussed above, we first affect the small M limit and take the derivative with respect to M . We then exploit the fact that we are, by virtue of comparing to a supergravity calculation, in the $\lambda \rightarrow \infty$ limit. This means that we can take the integration bounds to be $\pm\infty$, and expand the integrand in powers of $\frac{1}{\mu}$. The result is that at linear order in M , we end up with the integral

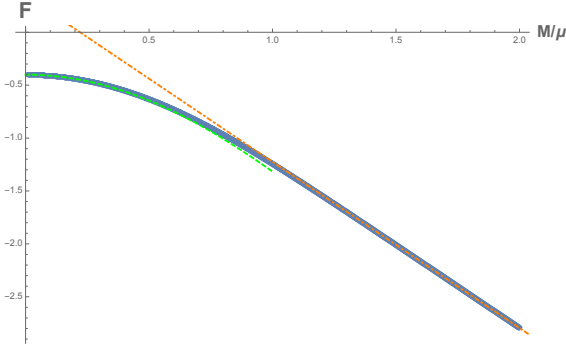
$$M \int_{-\infty}^{\infty} dx \frac{\operatorname{sech}^2(\pi x)}{\pi} \left(1 - \frac{x^2}{2\mu} - \frac{x^4}{8\mu^3} - \dots \right) = \frac{2M}{\mu} - \frac{M}{12\mu^3} - \dots, \quad (4.196)$$

where in the large argument expansion $\frac{M}{\mu} \ll 1$, we are only interested in the leading powers of μ^{-1} at each order in M . Carefully tracking leading contributions, we find

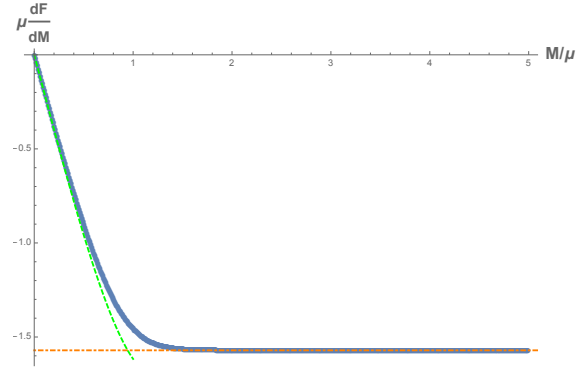
$$\frac{dF}{dM} = -\frac{2\zeta M}{\mu} + \frac{\zeta M^3}{3\mu^3} + \frac{\zeta M^5}{20\mu^5}. \quad (4.197)$$

Comparing to eq. (4.190), we can see that the coefficients of M and M^3 in eq. (4.197) match exactly.

It is interesting to note that, while we do not have full holographic access to the matrix model, even in the probe limit detailed above, the form of the defect flavor contribution to



(a) Numerical integration of eq. (4.195) in thick blue with $\zeta = 1$. Orange dot-dashed corresponds to large M : $F \simeq \log \sqrt{2} - \pi \frac{M}{2\mu}$. Green dashed corresponds to fit against small M : $F \simeq -4 - \frac{M^2}{\mu^2} - \frac{M^4}{12\mu^4}$



(b) Numerical integration of $\frac{dS_1}{dM}$ in thick blue with $\zeta = 1$. Orange dot-dashed corresponds to large M behavior: $\mu \frac{dF}{dM} \simeq -\frac{\pi}{2}$. Green dashed corresponds to fit against small M : $\mu \frac{dF}{dM} \simeq -2 \frac{M}{\mu} + \frac{M^3}{3\mu^3} + \frac{M^5}{20\mu^5}$

the free energy is readily integrable in the large mass regime $M > \mu$. That is for sufficiently large M , the $\mathcal{O}(\zeta)$ contribution to the free energy becomes

$$S_1|_{\rho_W} \simeq -\frac{1}{2}(\pi M - \log 2) \int_{-\mu}^{\mu} dx \rho_W(x) = -\frac{1}{2}(\pi M - \log 2), \quad (4.198)$$

where the last equality follows from the normalization of the Wigner distribution. This implies that $\mu \frac{dF}{dM}|_{M \gg \mu} \simeq -\frac{\pi}{2}$. Indeed, numerical investigations have born out this behavior as in figs. 4.1(a) and 4.1(b).

Even though this data does not give an explicit clue for how the full non-analytic kappa symmetry solutions can be solved, it does give an extremely simple data point to compare for any holographic calculation dual to parametrically large masses. Although, to be clear that this result would be suspect in the weak coupling limit owing to any back reaction on the Wigner semicircle for values of M near the width of the distribution. In the case of large N , $SU(N) \mathcal{N} = 2^*$ on an S^4 , this regime of $M \gtrsim \mu$ leads to resonance-like cusps forming on the semicircle [118, 116].

4.8 Discussion

In the preceding work, we have seen that, while the explicit non-linear construction of a κ -symmetric embedding of a D5-brane into $\text{AdS}_5 \times S^5$ with radial slices of the AdS_5 possessing constant, non-zero curvature is not as straightforward as the D3/D7 system explored in [17, 18], it displays a feature rich perturbative analysis. That is, we have seen for the case where the slices have AdS geometry, there exist an infinite family of supersymmetric vacua parameterized by the tower of modes on the internal space even at the linearized level. Further, the spherically sliced system exhibits an exact matching of $\langle \mathcal{O}_{SUSY} \rangle$ computed via holographic renormalization to $\frac{dF}{dM}$ computed in the matrix model via supersymmetric localization out to the first non-linear order in the mass deformation.

Accompanying this supersymmetric mass deformation are sources for higher dimension operators with non-trivial vevs. An open question remains if these can be seen in a more careful localization analysis. A more fundamental question remains in this framework: whether a full analytic solution to the coupled non-linear partial differential equations resulting from kappa symmetry can be found. Standard approaches and intuition gleaned from the perturbative analysis has suggested that a separable solution is not possible, and so more refined analysis of the system is needed. Further given that the large mass regime of the matrix model is so simple, it gives hope that there may be a way to find an infinitely massive embedding and a function to interpolate between the two regimes.

4.9 Appendix A: Holographic renormalization with irrelevant deformations

This appendix follows the method that was constructed in [132] for treating holographic renormalization in the presence of irrelevant sources. This is a delicate question to tackle since an irrelevant deformation strongly deforms the UV of the theory, and the dual picture to this would be a spacetime that is no longer asymptotically AdS. However since we are treating both the probe limit and a perturbative framework in which such sources only enter at non-linear orders, we can proceed carefully to understand how these deformations can

be understood holographically. The hope would be to uncover potentially new sources for operators that naively do not appear in the spectrum of known localization results.

To begin, let us treat first the most general linearized solutions in the Lorentzian AdS slicing to gain an understanding of how holographic renormalization works for the most general deformations. We start at the form of the quadratic Lagrangian density for the linearized theory written using the change in coordinates $r = -\log \frac{u}{2}$:

$$L_{D5} = \left(-4\rho \partial_x A_{\beta_2} - \frac{(\partial_x A_{\beta_2})^2}{2} - \frac{u^2 (\partial_u A_{\beta_2})^2}{2(1-x^2)} + \frac{1}{2}(2\theta^2 - 4\rho^2 - (1-x^2)((\partial_x \theta)^2 + (\partial_x \rho)^2) - u^2((\partial_u \theta)^2 + (\partial_u \rho)^2)) \right), \quad (4.199)$$

where the quadratic action is

$$\frac{S_{D5}}{2\pi V_{S^3} T_5} = \frac{1}{2} \int dudx \frac{(1 + \frac{u^2}{4})^3}{u^4} L_{D5}. \quad (4.200)$$

In order to facilitate taking holographic one point functions into field theory statements, we should abandon using ρ and A as our functions describing our embedding and switch to framing calculations in terms of the free scalar fields σ and τ . These free scalars are defined in a mode expansion $\sigma = \sum_\ell \sigma^{(\ell)} P_\ell(x)$ and $\tau = \sum_\ell \tau^{(\ell)} P_\ell(x)$ by taking $\theta = \sum_\ell \theta^{(\ell)} P_\ell(x)$, $\rho = \sum_\ell \rho^{(\ell)} P_\ell(x)$, and $A_{\beta_2} = \sum_\ell \mathcal{A}^{(\ell)} (\star d)_{S^2} P_\ell(x)$ and defining

$$\tau^{(\ell)} \equiv \frac{\rho^{(\ell)} - \ell \mathcal{A}^{(\ell)}}{2\ell + 1}, \quad \sigma^{(\ell)} \equiv \frac{\rho^{(\ell)} + (\ell + 1) \mathcal{A}^{(\ell)}}{2\ell + 1}. \quad (4.201)$$

The near boundary, $u = \epsilon \ll 1$, expansions for the general ℓ modes can then be read off as

$$\tau^{(\ell)} = \epsilon^{-\ell-1} \left(\frac{d_{\ell+1,2}}{\ell+1} + \mathcal{O}(\epsilon^2) \right) - \epsilon^{\ell+4} \left(\frac{\ell+3}{(2\ell+3)(2\ell+5)} d_{\ell+1,1} + \mathcal{O}(\epsilon^2) \right), \quad (4.202a)$$

$$\sigma^{(\ell)} = \epsilon^{3-\ell} \left(\frac{2-\ell}{(2\ell-3)(2\ell-1)} d_{\ell-1,2} + \mathcal{O}(\epsilon^2) \right) + \epsilon^\ell \left(\frac{d_{\ell-1,1}}{\ell} + \mathcal{O}(\epsilon^2) \right), \quad (4.202b)$$

$$\lambda \theta^{(\ell)} = \epsilon^{1-\ell} (d_{\ell,2} + \mathcal{O}(\epsilon^2)) + \epsilon^{\ell+2} (d_{\ell,1} + \mathcal{O}(\epsilon^2)), \quad (4.202c)$$

where the $d_{\ell,i}$ are related to the original constants $c_{\ell,i}$ by

$$d_{\ell,1} = \frac{\ell(\ell+1)}{4\ell^2-1} c_{\ell-1,1} + c_{\ell+1,1}, \quad d_{\ell,2} = c_{\ell-1,2} + \frac{\ell(\ell+1)}{(2\ell+1)(2\ell+3)} c_{\ell+1,2}. \quad (4.203)$$

In order to make the following more readable, we will treat the expansions schematically. For example, the slipping mode expansion is regarded as

$$\lambda\theta^{(\ell)} = \epsilon^{1-\ell}(\theta_0^{(\ell)} + \dots) + \epsilon^{\ell+2}(\theta_1^{(\ell)} + \dots). \quad (4.204)$$

After making the change to using σ and τ and rescaling

$$\tau^{(\ell)} \rightarrow \frac{\tau^{(\ell)}}{\sqrt{(\ell+1)}} \equiv \tilde{\tau}_\ell, \quad \sigma^{(\ell)} \rightarrow \frac{\sigma^{(\ell)}}{\sqrt{\ell}} \equiv \tilde{\sigma}_\ell, \quad (4.205)$$

in order to have canonically normalized kinetic terms, eq. (4.199) takes the form

$$\frac{S_{D5}}{4\pi V_3 T_5} = -\frac{1}{2} \sum_\ell \int du f(u) \left(L_\ell(\theta) + L_\ell(\tilde{\tau}) + L_\ell(\tilde{\sigma}) \right), \quad (4.206)$$

where

$$L_\ell(X) = u^2(\partial_u X^{(\ell)})^2 - \Delta_+^X(\ell)\Delta_-^X(\ell)(X^{(\ell)})^2, \quad f(u) = \frac{(1 + \frac{u^2}{4})^3}{u^3}, \quad (4.207)$$

and the $\Delta_\pm^X(\ell)$ can be read off from the above expansions. The equations of motion can easily be computed in this schematic form as

$$\frac{1}{f(u)} \partial_u(u^2 f(u) \partial_u X^{(\ell)}) + \Delta_+^X \Delta_-^X X^{(\ell)} = 0, \quad (4.208)$$

which we would expect for a massive scalar on AdS. Evaluating the on-shell action is then a simple matter of determining the boundary term

$$\frac{S_{D5}^{os}}{4\pi V_3 T_5} = -\frac{1}{2} \left[\sum_\ell u^2 f(u) (\theta^{(\ell)} \partial_u \theta^{(\ell)} + \tilde{\tau}_\ell \partial_u \tilde{\tau}_\ell + \tilde{\sigma}_\ell \partial_u \tilde{\sigma}_\ell) \right]_\epsilon, \quad (4.209)$$

where $|_\epsilon$ denotes evaluating at the UV cutoff slice $u = \epsilon \ll 1$.

In order to complete the calculation of field theory quantities from the general linearized solutions, we need to understand how holographic renormalization would proceed in the presence of irrelevant sources. To that end, lets start with eq. (4.209) before expanding in S^2 modes and integrating over the S^3 . That is,

$$\frac{S_{D5}^{os}}{T_5} = -\frac{1}{2} \int_\partial d^5 y \sqrt{-g_3 g_{S^2}} u f(u) (\theta \partial_u \theta + \tilde{\tau} \partial_u \tilde{\tau} + \tilde{\sigma} \partial_u \tilde{\sigma}), \quad (4.210)$$

where $\int_{\partial} d^5 y$ is evaluated on the radial cutoff slice $u = \epsilon$. The variations of the on-shell action then take the form

$$\frac{\delta_X S_{D5}}{T_5} = - \int_{\partial} d^5 y \sqrt{-g_3} f(u) u^2 \partial_u X \delta X ,$$

where X is any of the free scalar fields. To make the calculation of the counterterms explicit, let us focus on the slipping mode θ , and note that all of the techniques can be applied equally well to $\tilde{\tau}$ and $\tilde{\sigma}$. First, we can organize terms more easily by making a schematic expansion such that the infinite tower of possible modes is truncated, e.g. for the slipping mode

$$\theta \sim \epsilon^{\Delta_{\theta-}} (\theta_{0-} + \theta_{2-} \epsilon^2 + \dots) + \epsilon^{\Delta_{\theta+}} (\theta_{0+} + \theta_{2+} \epsilon^2 + \dots) . \quad (4.211)$$

This treatment is general and the limit that arbitrarily large ℓ s are included can be taken at the end. Extracting the θ part of the on-shell action in eq. (4.209) and insert eq. (4.211) such that

$$\begin{aligned} \left. \frac{S_{D5}^{os}}{T_5} \right|_{\theta} = & -\frac{1}{2} \int_{\partial} d^5 y \sqrt{-g_3} \left(\frac{1}{\epsilon^2} + \frac{3}{4} + \dots \right) \left(\Delta_{\theta-} \epsilon^{2\Delta_{\theta-}-1} (\theta_{0-}^2 + \dots) \right. \\ & \left. + \epsilon^2 (2\theta_{0-} \theta_{0+} + \dots) + \epsilon^{2\Delta_{\theta+}-1} (\Delta_{\theta+} \theta_{0+}^2 + \dots) \right), \end{aligned} \quad (4.212)$$

where $\Delta_+ + \Delta_- = 3$ was used. Following [132], we now want to invert the asymptotic expansion $\theta_{0-} = \epsilon^{-\Delta_{\theta-}} \theta + \dots$ and insert into the action above. We then identify the leading divergent power of ϵ , and a counterterm that appropriately cancels this off out of powers of θ , covariant derivatives thereof and the volume form on the cutoff slice. For example, above the leading term is $-\frac{\Delta_{\theta-}}{2} \epsilon^{2\Delta_{\theta-}-3} \theta_{0-}$, which suggests an appropriate counterterm would be

$$\frac{S_{ct,1\theta}}{T_5} = \frac{\Delta_{\theta-}}{2} \int_{\partial} d^5 y \sqrt{-g_3} \frac{(1 + \frac{\epsilon^2}{4})^3}{\epsilon^3} \theta_{\epsilon}^2, \quad (4.213)$$

which we would naively have guessed. We can then construct higher order counterterms through establishing a recursion relation to link subleading coefficients in the asymptotic expansion to the leading coefficient in a given tower $\theta_{0\pm}$. Starting with the equations of motion, expanding in modes, and looking at the near boundary behavior gives relations like

$$\theta_{2k\pm} = -h(\Delta_{\theta\pm} + 2k) \square_0 \theta_{2(k-1)\pm}, \quad (4.214)$$

where h is dependent on k and Δ_{\pm} and $\square_0 = \delta^{ij}\partial_i\partial_j$ is the rescaled covariant Laplacian on the cutoff slice ($\square_{\epsilon} = \epsilon^2\square_0$). Explicitly, we can find a recursion relation for the coefficients of the form

$$\theta_{2k\pm} = \frac{3(-)^k\Delta_{\theta\pm}\theta_{0\pm}}{2^{2k-1}(-2 + \Delta_{\theta\pm}(1 + \Delta_{\theta+} + \Delta_{\theta-}))} \prod_{n=2}^k \frac{2(n-1)(2n+1) + \Delta_{\theta\pm}(4n-2 + \Delta_{\theta+} + \Delta_{\theta-})}{2n(2n-3) + \Delta_{\theta\pm}(4n-3 + \Delta_{\theta+} + \Delta_{\theta-})}. \quad (4.215)$$

While not particularly pleasant to look at, the relation is expressible as $\theta_{2k\pm} = H_{\pm}(2k)\theta_{0\pm}$ and so could be used to construct the counterterms as

$$\int_{\partial} d^5y \sqrt{-g_3} \frac{(1 + \frac{\epsilon^2}{4})^3}{\epsilon^3} \left(-H_{-}(\Delta_{\theta-} + 2)\theta\square_{\epsilon}\theta + \sum_k c_k \theta\square_{\epsilon}^k\theta \right), \quad (4.216)$$

where $1 < k < \frac{\Delta_{\theta+} - \Delta_{\theta-}}{2}$ and c_k are some constant coefficients determined recursively, built with $\prod_{i=2}^k H(\Delta_{\theta-} + 2i)$ and other $\mathcal{O}(1)$ factors.

Moving out the the general treatment and coming back to our original task, we can combine the θ part of eq. (4.206) with eq. (4.213) and using eq. (4.215), we find that

$$\begin{aligned} \frac{S_{D_5}^{os}|_{\theta} + S_{ct,1\theta}}{4\pi V_3 T_5} &= -\frac{1}{2} \left(\frac{1}{\epsilon^3} + \frac{3}{4\epsilon} + \dots \right) \left(\epsilon^{2\Delta_{\theta-}+2} (2H_{-}(2)\theta_{0-}^2 + \dots) \right. \\ &\quad \left. + \epsilon^3 ((\Delta_{\theta+} - \Delta_{\theta-})\theta_{0-}\theta_{0+} + \dots) + \epsilon^{2\Delta_{\theta+}}(\dots) \right), \end{aligned} \quad (4.217)$$

Iterating the procedure further, adding counterterms of the form $a_p R_{\epsilon}^p \theta_{\epsilon}^2$, where R_{ϵ} is the Ricci scalar on the cutoff slice pulled back to the worldvolume, we find that after expanding in S^2 the holographically renormalized on-shell action receives a contribution from the slipping mode of the form

$$\frac{S_{D_5}^{os}|_{\theta} + S_{ct,\theta}}{4\pi V_3 T_5} \Big|_{\epsilon \rightarrow 0} = -\frac{1}{2} \sum_{\ell} (\Delta_{\theta+} - \Delta_{\theta-}) \theta_{0-}^{(\ell)} \theta_{0+}^{(\ell)} + \mathcal{O}(\epsilon). \quad (4.218)$$

Affecting the same calculation for $\sigma^{(\ell)}$ and $\tau^{(\ell)}$, we find that the complete renormalized on-shell action is

$$\frac{S_{D_5}^{os} + S_{ct}}{4\pi V_3 T_5} \Big|_{\epsilon \rightarrow 0} = -\frac{1}{2} \sum_{\ell} \left((\Delta_{\theta+} - \Delta_{\theta-}) \theta_{0-}^{(\ell)} \theta_{0+}^{(\ell)} + (\Delta_{\tau+} - \Delta_{\tau-}) \tilde{\tau}_{0-}^{(\ell)} \tilde{\tau}_{0+}^{(\ell)} + (\Delta_{\sigma+} - \Delta_{\sigma-}) \tilde{\sigma}_{0-}^{(\ell)} \tilde{\sigma}_{0+}^{(\ell)} \right). \quad (4.219)$$

From here, we can easily construct the one point functions for any of the available fields by taking the appropriate variation of eq. (4.219). Staying with the idea that we are treating this as generally as possible, the variations take the form

$$\frac{\delta_X(S_{D_5}^{os} + S_{ct})}{4\pi V_3 T_5} = -\frac{1}{2} \left(\frac{(1 + \frac{\epsilon^2}{4})^3}{\epsilon^3} \right) \sum_{\ell} \Pi_{X^{(\ell)}} \delta X^{(\ell)} \quad (4.220)$$

where, again borrowing the notation from [132], $\Pi_{X^{(\ell)}}$ is the renormalized conjugate momentum to the mode $X^{(\ell)}$

$$\Pi_{X^{(\ell)}} = \epsilon \partial_u X^{(\ell)}|_{\epsilon} - (\Delta_{X^-} - \frac{R_{\epsilon}}{3} H_-(2)\epsilon^2 + \dots) X^{(\ell)}|_{\epsilon}. \quad (4.221)$$

The one point function is then calculated by taking the variation with respect to the source, $X_{0-}^{(\ell)}$, setting $X_{0-}^{(\ell)} \rightarrow 0$, taking $\epsilon \rightarrow 0$, mapping to field theory language by

$$V_{S^3} \langle \mathcal{O}_X \rangle = -\frac{\delta S_{D_5,ren}^{os}}{\delta X^{(0)}}, \quad (4.222)$$

and defining $4\pi T_5 = N_f N_c \frac{\sqrt{\lambda}}{2\pi^3}$, $V_{S^3} = 2\pi^2$, and $\mu = \frac{\sqrt{\lambda}}{2\pi}$. This translation of probe brane language into something more natural in the field theory is compiled in a helpful table in [117]. The one point functions then take the form

$$\frac{\langle \mathcal{O}_{\theta_{\epsilon}} \rangle}{N_f N_c \mu} = (\Delta_{\theta^-} - \Delta_{\theta^+}) \theta_{0+}^{(\ell)}, \quad \frac{\langle \mathcal{O}_{\tau_{\ell}} \rangle}{N_f N_c \mu} = (\Delta_{\tau^-} - \Delta_{\tau^+}) \tilde{\tau}_{0+}^{(\ell)}, \quad \frac{\langle \mathcal{O}_{\sigma_{\ell}} \rangle}{N_f N_c \mu} = (\Delta_{\sigma^-} - \Delta_{\sigma^+}) \tilde{\sigma}_{0+}^{(\ell)}. \quad (4.223)$$

A point to worry about here is whether or not there is a symmetry reason to exclude terms like the finite $\sqrt{g_{\epsilon}} X^3|_{\epsilon}$ -type counterterms as analyzed in [133]. From the analysis of the cubic order mass deformations, this could be problematic since we are explicitly keeping $\mathcal{O}(\kappa^3)$ and thus the contribution from say $\sqrt{g_{\epsilon}} (\zeta_1^{(1)})_{\epsilon}^3$ could be a pertinent quantity to track. The reason this wasn't considered in [54] is that the bending mode was not systematically treated and the quadratic action for the slipping mode has a \mathbb{Z}_2 symmetry $\theta \rightarrow -\theta$, and so such counterterms were disregarded so as to not explicitly break the symmetry. While there is no *a priori* reason to forbid addition counterterms that break this \mathbb{Z}_2 symmetry, the analysis done to this point does not bear out any indication that they play a role.

4.10 Appendix B: Guide to the Shorthand Notation

Here we collect the shorthand notation that was necessary for the computation of non-linear κ -symmetry together to more easily follow how various structures were repackaged during the calculation. The initial redefinitions needed to isolate the AdS₅ and S⁵ Clifford algebra structures from one another in $\gamma_{r\beta_1\beta_2}$:

$$\mathcal{A}^{\mathbb{1}} = (d\theta)_r \Gamma_\theta \Gamma_{\underline{\beta}} , \quad \mathcal{A}^\rho = (d\rho)_r \Gamma_{\underline{\beta}} - \tau^{\beta_i} \Gamma_{\beta_i} \Gamma_\theta , \quad (4.224a)$$

$$\mathcal{A}^r = \Gamma_{\underline{\beta}} - \varepsilon^{r\text{mn}} (d\theta)_m \Gamma_n \Gamma_\theta , \quad \mathcal{A}^{\rho r} = \varepsilon^{r\text{mn}} (d\rho)_m \Gamma_n - \tau^r \Gamma_\theta , \quad (4.224b)$$

$$\mathcal{M}^{\mathbb{1}} = -\cosh^2 \rho (\mathcal{A}^r + i \tanh \rho \mathcal{A}^{\rho r} \Gamma_{S^5}) , \quad \mathcal{M}^\rho = -\cosh^2 \rho (\mathcal{A}^{\rho r} - i \tanh \rho \mathcal{A}^r \Gamma_{S^5}) , \quad (4.224c)$$

$$\mathcal{M}^r = -\mathcal{A}^{\mathbb{1}} - i \tanh \rho \mathcal{A}^\rho \Gamma_{S^5} , \quad \mathcal{M}^{\rho r} = -\mathcal{A}^\rho + i \tanh \rho \mathcal{A}^{\mathbb{1}} \Gamma_{S^5} . \quad (4.224d)$$

Relations that follow from these are used in evaluating the partitioning of Clifford algebra structures in the F_{ij} part of the κ -symmetry projector which necessitated the following redefinitions

$$\mathcal{B}^{\mathbb{1}} = (d\theta)_r F^{r\beta_i} \Gamma_\theta \Gamma_{\beta_i} + F^{\beta_1\beta_2} (\Gamma_{\underline{\beta}} + \Gamma_\theta \varepsilon^{r\text{mn}} (d\theta)_m \Gamma_n) , \quad (4.225a)$$

$$\mathcal{B}^r = -F^{r\beta_i} (\Gamma_{\beta_i} + (d\theta)_{\beta_i} \Gamma_\theta) , \quad (4.225b)$$

$$\mathcal{B}^\rho = F^{\text{mn}} (d\theta)_m (d\rho)_n \Gamma_\theta + F^{\beta_i\text{m}} (d\rho)_m \Gamma_{\beta_i} , \quad (4.225c)$$

$$\mathcal{B}^{\rho r} = -F^{r\beta_i} (d\rho)_{\beta_i} \mathbb{1} , \quad (4.225d)$$

$$\mathcal{E}^{\mathbb{1}} = -\cosh \rho (\mathcal{B}^{\rho r} \mathcal{A}^{\mathbb{1}} + \mathcal{B}^{\mathbb{1}} \mathcal{A}^{\rho r} + \mathcal{B}^\rho \mathcal{A}^r - \mathcal{B}^r \mathcal{A}^\rho) , \quad (4.225e)$$

$$\mathcal{E}^\rho = -\cosh \rho (\mathcal{B}^r \mathcal{A}^{\mathbb{1}} - \mathcal{B}^{\mathbb{1}} \mathcal{A}^r + \mathcal{B}^\rho \mathcal{A}^{\rho r} + \mathcal{B}^{\rho r} \mathcal{A}^\rho) , \quad (4.225f)$$

$$\mathcal{E}^r = -\text{sech} \rho (-\mathcal{B}^\rho \mathcal{A}^{\mathbb{1}} + \mathcal{B}^{\mathbb{1}} \mathcal{A}^\rho) - \cosh \rho (\mathcal{B}^r \mathcal{A}^{\rho r} + \mathcal{B}^{\rho r} \mathcal{A}^r) , \quad (4.225g)$$

$$\mathcal{E}^{\rho r} = \text{sech} \rho (\mathcal{B}^{\mathbb{1}} \mathcal{A}^{\mathbb{1}} + \mathcal{B}^\rho \mathcal{A}^\rho) + \cosh \rho (\mathcal{B}^r \mathcal{A}^r - \mathcal{B}^{\rho r} \mathcal{A}^{\rho r}) , \quad (4.225h)$$

$$\mathcal{N}^X = \mathcal{M}^X e^{-i\theta \Gamma_{\underline{\theta}} \Gamma_{S^5}} \Gamma_{\underline{\beta}} - i \mathcal{E}^X \Gamma_{S^5} , \quad X \in \{\mathbb{1}, \rho, r, \rho r\} . \quad (4.225i)$$

After a bit of algebra, we needed to define, using the notation $\mathcal{R}_\bullet[X] = R_\bullet^{-1}XR_\bullet$,

$$\begin{aligned} \mathcal{Q}_K &= \cosh r \mathcal{R}_S[\mathcal{N}^\rho] - \cosh^2 \rho \mathcal{R}_S[\mathcal{N}^r] (\hat{\Gamma}_p + \tanh \rho \sinh r \mathbb{1}) \\ &\quad + i \cosh^2 \rho \mathcal{R}_S[\mathcal{N}^{\rho r}] (\tanh \rho \hat{\Gamma}_p + \sinh r \mathbb{1}) \Gamma_{S^5} , \end{aligned} \quad (4.226)$$

$$\begin{aligned} \mathcal{Q}_1 &= \mathcal{R}_S[\mathcal{N}^1] + i \sinh r \mathcal{R}_S[\mathcal{N}^\rho] \hat{\Gamma}_p \Gamma_{S^5} - i \sinh \rho \cosh \rho \cosh r \mathcal{R}_S[\mathcal{N}^r] \hat{\Gamma}_p \Gamma_{S^5} \\ &\quad - \cosh^2 \rho \cosh r \mathcal{R}_S[\mathcal{N}^{\rho r}] \hat{\Gamma}_p . \end{aligned} \quad (4.227)$$

Lastly, we have the final piece of shorthand worth noting: the final form of the BPS equations

$$G_a = \lambda \sinh \rho \sin^2 \theta \partial_a (x \cot \theta) - \partial_a (\tan z \cosh \rho) , \quad (4.228)$$

$$\begin{aligned} -(1 - x^2) \cosh \rho G_x + \sinh \rho \cos \theta F_z &= 0 , \\ \sinh \rho (G_z F_x - G_x F_z) - \cos^3 \theta G_z - \sec^2 z \cosh^3 \rho \cos^3 \theta &= 0 . \end{aligned} \quad (4.229)$$

BIBLIOGRAPHY

- [1] Vasily Pestun. Localization of gauge theory on a four-sphere and supersymmetric Wilson loops. *Commun. Math. Phys.*, 313:71–129, 2012.
- [2] W. G. Unruh. Notes on black-hole evaporation. *Phys. Rev. D*, 14:870–892, Aug 1976.
- [3] Leonard Parker. Quantized fields and particle creation in expanding universes. i. *Phys. Rev.*, 183:1057–1068, Jul 1969.
- [4] Stephen A. Fulling. Nonuniqueness of canonical field quantization in riemannian space-time. *Phys. Rev. D*, 7:2850–2862, May 1973.
- [5] Matthew D. Schwartz. *Quantum Field Theory and the Standard Model*. Cambridge University Press, 2014.
- [6] S. Perlmutter et al. Measurements of Omega and Lambda from 42 high redshift supernovae. *Astrophys. J.*, 517:565–586, 1999.
- [7] J. Wess and B. Zumino. Supergauge transformations in four dimensions. *Nuclear Physics B*, 70(1):39 – 50, 1974.
- [8] Mary K. Gaillard and Bruno Zumino. Supersymmetry and Superstring Phenomenology. *Eur. Phys. J.*, C59:213–221, 2009.
- [9] Kentaro Hori and Cumrun Vafa. Mirror symmetry. 2000.
- [10] Juan Martin Maldacena. The Large N limit of superconformal field theories and supergravity. *Int. J. Theor. Phys.*, 38:1113–1133, 1999. [Adv. Theor. Math. Phys.2,231(1998)].
- [11] Edward Witten. Anti-de Sitter space, thermal phase transition, and confinement in gauge theories. *Adv. Theor. Math. Phys.*, 2:505–532, 1998.
- [12] S. S. Gubser, Igor R. Klebanov, and Alexander M. Polyakov. Gauge theory correlators from noncritical string theory. *Phys. Lett.*, B428:105–114, 1998.
- [13] Nissan Itzhaki, Juan Martin Maldacena, Jacob Sonnenschein, and Shimon Yankielowicz. Supergravity and the large N limit of theories with sixteen supercharges. *Phys. Rev.*, D58:046004, 1998.
- [14] Ofer Aharony, Steven S. Gubser, Juan Martin Maldacena, Hirosi Ooguri, and Yaron Oz. Large N field theories, string theory and gravity. *Phys. Rept.*, 323:183–386, 2000.
- [15] Andreas Karch and Emanuel Katz. Adding flavor to AdS / CFT. *JHEP*, 06:043, 2002.
- [16] E. Bergshoeff and P. K. Townsend. Super D-branes. *Nucl. Phys.*, B490:145–162, 1997.

- [17] Andreas Karch, Brandon Robinson, and Christoph F. Uhlemann. Supersymmetric D3/D7 for holographic flavors on curved space. *JHEP*, 11:112, 2015.
- [18] Andreas Karch, Brandon Robinson, and Christoph F. Uhlemann. Precision Test of Gauge-Gravity Duality with Flavor. *Phys. Rev. Lett.*, 115(26):261601, 2015.
- [19] Martin Ammon and Johanna Erdmenger. *Gauge/gravity duality*. Cambridge Univ. Pr., Cambridge, UK, 2015.
- [20] J. Polchinski. String theory. Vol. 2: Superstring theory and beyond. 1998.
- [21] Elias Kiritsis. *String theory in a nutshell*. 2007.
- [22] Peter G. O. Freund and Mark A. Rubin. Dynamics of Dimensional Reduction. *Phys. Lett.*, B97:233–235, 1980.
- [23] Marcus T. Grisaru, Robert C. Myers, and Oyvind Taffjord. SUSY and goliath. *JHEP*, 08:040, 2000.
- [24] Andreas Karch and Brandon Robinson. Holographic Black Hole Chemistry. *JHEP*, 12:073, 2015.
- [25] David Kastor, Sourya Ray, and Jennie Traschen. Smarr Formula and an Extended First Law for Lovelock Gravity. *Class. Quant. Grav.*, 27:235014, 2010.
- [26] Brian P. Dolan. Where is the PdV term in the first law of black hole thermodynamics? 2012.
- [27] David Kubiznak and Robert B. Mann. Black Hole Chemistry. 2014.
- [28] Marco M. Caldarelli, Guido Cognola, and Dietmar Klemm. Thermodynamics of Kerr-Newman-AdS black holes and conformal field theories. *Class. Quant. Grav.*, 17:399–420, 2000.
- [29] Jolien D. E. Creighton and Robert B. Mann. Quasilocal thermodynamics of dilaton gravity coupled to gauge fields. *Phys. Rev.*, D52:4569–4587, 1995.
- [30] G. W. Gibbons, M. J. Perry, and C. N. Pope. The First law of thermodynamics for Kerr-anti-de Sitter black holes. *Class. Quant. Grav.*, 22:1503–1526, 2005.
- [31] David Kastor, Sourya Ray, and Jennie Traschen. Enthalpy and the Mechanics of AdS Black Holes. *Class. Quant. Grav.*, 26:195011, 2009.
- [32] M. Cvetič, G. W. Gibbons, D. Kubiznak, and C. N. Pope. Black Hole Enthalpy and an Entropy Inequality for the Thermodynamic Volume. *Phys. Rev.*, D84:024037, 2011.
- [33] Larry Smarr. Mass formula for Kerr black holes. *Phys. Rev. Lett.*, 30:71–73, 1973. [Erratum: *Phys. Rev. Lett.* 30,521(1973)].
- [34] David Kastor, Sourya Ray, and Jennie Traschen. Chemical Potential in the First Law for Holographic Entanglement Entropy. *JHEP*, 11:120, 2014.
- [35] Clifford V. Johnson. Holographic Heat Engines. *Class. Quant. Grav.*, 31:205002, 2014.

- [36] Jia-Lin Zhang, Rong-Gen Cai, and Hongwei Yu. Phase transition and thermodynamical geometry for Schwarzschild AdS black hole in AdS₅ S⁵ spacetime. *JHEP*, 02:143, 2015.
- [37] Jia-Lin Zhang, Rong-Gen Cai, and Hongwei Yu. Phase transition and thermodynamical geometry of Reissner-Nordström-AdS black holes in extended phase space. *Phys. Rev.*, D91(4):044028, 2015.
- [38] Elena Caceres, Phuc H. Nguyen, and Juan F. Pedraza. Holographic entanglement entropy and the extended phase structure of STU black holes. 2015.
- [39] Xi Dong, Sarah Harrison, Shamit Kachru, Gonzalo Torroba, and Huajia Wang. Aspects of holography for theories with hyperscaling violation. *JHEP*, 06:041, 2012.
- [40] Andreas Karch, Manuela Kulaxizi, and Andrei Parnachev. Notes on Properties of Holographic Matter. *JHEP*, 11:017, 2009.
- [41] David Kubiznak and Robert B. Mann. P-V criticality of charged AdS black holes. *JHEP*, 07:033, 2012.
- [42] Andrew Chamblin, Roberto Emparan, Clifford V. Johnson, and Robert C. Myers. Charged AdS black holes and catastrophic holography. *Phys. Rev.*, D60:064018, 1999.
- [43] Antonia M. Frassino, David Kubiznak, Robert B. Mann, and Fil Simovic. Multiple Reentrant Phase Transitions and Triple Points in Lovelock Thermodynamics. *JHEP*, 09:080, 2014.
- [44] Brian P. Dolan, Anna Kostouki, David Kubiznak, and Robert B. Mann. Isolated critical point from Lovelock gravity. *Class. Quant. Grav.*, 31(24):242001, 2014.
- [45] J. Babington, J. Erdmenger, Nick J. Evans, Z. Guralnik, and I. Kirsch. Chiral symmetry breaking and pions in nonsupersymmetric gauge / gravity duals. *Phys. Rev.*, D69:066007, 2004.
- [46] Tadakatsu Sakai and Shigeki Sugimoto. Low energy hadron physics in holographic QCD. *Prog. Theor. Phys.*, 113:843–882, 2005.
- [47] Andreas Karch and Andy O’Bannon. Metallic AdS/CFT. *JHEP*, 09:024, 2007.
- [48] Andreas Karch, Dieter Lust, and Douglas J. Smith. Equivalence of geometric engineering and Hanany-Witten via fractional branes. *Nucl. Phys.*, B533:348–372, 1998.
- [49] Amihay Hanany and Edward Witten. Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics. *Nucl. Phys.*, B492:152–190, 1997.
- [50] Robert C. Myers and Matthias C. Wapler. Transport Properties of Holographic Defects. *JHEP*, 12:115, 2008.
- [51] Shamit Kachru, Andreas Karch, and Sho Yaida. Holographic Lattices, Dimers, and Glasses. *Phys. Rev.*, D81:026007, 2010.
- [52] Shamit Kachru, Andreas Karch, and Sho Yaida. Adventures in Holographic Dimer Models. *New J. Phys.*, 13:035004, 2011.

- [53] Michael E. Peskin and Daniel V. Schroeder. *An Introduction to quantum field theory*. 1995.
- [54] Andreas Karch, Andy O'Bannon, and Kostas Skenderis. Holographic renormalization of probe D-branes in AdS/CFT. *JHEP*, 04:015, 2006.
- [55] Charles Fefferman and C. Robin Graham. The ambient metric. 2007.
- [56] Kostas Skenderis. Lecture notes on holographic renormalization. *Class. Quant. Grav.*, 19:5849–5876, 2002.
- [57] Oliver DeWolfe, Daniel Z. Freedman, and Hiroshi Ooguri. Holography and defect conformal field theories. *Phys. Rev.*, D66:025009, 2002.
- [58] Andreas Karch and Lisa Randall. Open and closed string interpretation of SUSY CFT's on branes with boundaries. *JHEP*, 06:063, 2001.
- [59] Guido Festuccia and Nathan Seiberg. Rigid Supersymmetric Theories in Curved Superspace. *JHEP*, 06:114, 2011.
- [60] Johanna Erdmenger, Zachary Guralnik, and Ingo Kirsch. Four-dimensional superconformal theories with interacting boundaries or defects. *Phys. Rev.*, D66:025020, 2002.
- [61] E. Bergshoeff, E. Sezgin, and P. K. Townsend. Supermembranes and Eleven-Dimensional Supergravity. *Phys. Lett.*, B189:75–78, 1987.
- [62] E. Bergshoeff, R. Kallosh, T. Ortin, and G. Papadopoulos. Kappa symmetry, supersymmetry and intersecting branes. *Nucl. Phys.*, B502:149–169, 1997.
- [63] Hong Lu, C. N. Pope, and J. Rahmfeld. A Construction of Killing spinors on S^{2n} . *J. Math. Phys.*, 40:4518–4526, 1999.
- [64] Edward Witten. Topological Quantum Field Theory. *Commun. Math. Phys.*, 117:353, 1988.
- [65] M. F. Atiyah and R. Bott. The Moment map and equivariant cohomology. *Topology*, 23:1–28, 1984.
- [66] N. Berline and M. Vergne. Classes caractéristiques équivariantes. Formule de localisation en cohomologie équivariante. *C. R. Acad. Sci. Paris Sér. I Math*, 295:539–541, 1982.
- [67] J. J. Duistermaat and G. J. Heckman. On the Variation in the cohomology of the symplectic form of the reduced phase space. *Invent. Math.*, 69:259–268, 1982.
- [68] Richard J. Szabo. Equivariant localization of path integrals. 1996.
- [69] Vasily Pestun. Review of localization in geometry. 2016.
- [70] Douglas Stanford and Edward Witten. Fermionic Localization of the Schwarzian Theory. 2017.
- [71] Edward Witten. Two-dimensional gauge theories revisited. *J. Geom. Phys.*, 9:303–368, 1992.

- [72] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow. *Mirror symmetry*, volume 1 of *Clay mathematics monographs*. AMS, Providence, USA, 2003.
- [73] Anton Kapustin, Brian Willett, and Itamar Yaakov. Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter. *JHEP*, 03:089, 2010.
- [74] Anton Kapustin, Brian Willett, and Itamar Yaakov. Nonperturbative Tests of Three-Dimensional Dualities. *JHEP*, 10:013, 2010.
- [75] Edi Gava, K. S. Narain, M. Nouman Muteeb, and V. I. Giraldo-Rivera. $N = 2$ gauge theories on the hemisphere HS^4 . *Nucl. Phys.*, B920:256–297, 2017.
- [76] Justin R. David, Edi Gava, Rajesh Kumar Gupta, and Kumar Narain. Localization on $AdS_2 \times S^1$. *JHEP*, 03:050, 2017.
- [77] Benjamin Assel, Dario Martelli, Sameer Murthy, and Daisuke Yokoyama. Localization of supersymmetric field theories on non-compact hyperbolic three-manifolds. *JHEP*, 03:095, 2017.
- [78] Thomas T. Dumitrescu, Guido Festuccia, and Nathan Seiberg. Exploring Curved Superspace. *JHEP*, 08:141, 2012.
- [79] Nikolay Bobev, Henriette Elvang, Daniel Z. Freedman, and Silviu S. Pufu. Holography for $N = 2^*$ on S^4 . *JHEP*, 07:001, 2014.
- [80] Nathan Berkovits. A Ten-dimensional superYang-Mills action with off-shell supersymmetry. *Phys. Lett.*, B318:104–106, 1993.
- [81] Vasily Pestun et al. Localization techniques in quantum field theories. 2016.
- [82] Jörg Teschner, editor. *New Dualities of Supersymmetric Gauge Theories*. Mathematical Physics Studies. Springer, Cham, Switzerland, 2016.
- [83] Nadav Drukker, Davide Gaiotto, and Jaume Gomis. The Virtue of Defects in 4D Gauge Theories and 2D CFTs. *JHEP*, 06:025, 2011.
- [84] Andreas Karch and Lisa Randall. Locally localized gravity. *JHEP*, 05:008, 2001. [,140(2000)].
- [85] Alex Buchel. Gauge / gravity correspondence in accelerating universe. *Phys. Rev.*, D65:125015, 2002.
- [86] Donald Marolf, Mukund Rangamani, and Toby Wiseman. Holographic thermal field theory on curved spacetimes. *Class. Quant. Grav.*, 31:063001, 2014.
- [87] Mukund Rangamani, Moshe Rozali, and Mark Van Raamsdonk. Cosmological Particle Production at Strong Coupling. *JHEP*, 09:213, 2015.
- [88] Donald Marolf, Mukund Rangamani, and Mark Van Raamsdonk. Holographic models of de Sitter QFTs. *Class. Quant. Grav.*, 28:105015, 2011.
- [89] Ofer Aharony, Donald Marolf, and Mukund Rangamani. Conformal field theories in anti-de Sitter space. *JHEP*, 02:041, 2011.

- [90] Andreas Karch, Andy O’Bannon, and Laurence G. Yaffe. Critical Exponents from AdS/CFT with Flavor. *JHEP*, 09:042, 2009.
- [91] Adam B. Clark, Nathan Crossette, George M. Newman, and Andrea Rommal. AdS-Sliced Flavor Branes and Adding Flavor to the Janus Solution. *Phys. Rev.*, D89(2):026014, 2014.
- [92] Tarek Anous, Daniel Z. Freedman, and Alexander Maloney. de Sitter Supersymmetry Revisited. *JHEP*, 07:119, 2014.
- [93] Martin Cederwall, Alexander von Gussich, Bengt E. W. Nilsson, and Anders Westerberg. The Dirichlet super three-brane in ten-dimensional type IIB supergravity. *Nucl. Phys.*, B490:163–178, 1997.
- [94] Martin Cederwall, Alexander von Gussich, Bengt E. W. Nilsson, Per Sundell, and Anders Westerberg. The Dirichlet super p-branes in ten-dimensional type IIA and IIB supergravity. *Nucl. Phys.*, B490:179–201, 1997.
- [95] Hong Lu, C. N. Pope, and P. K. Townsend. Domain walls from anti-de Sitter space-time. *Phys. Lett.*, B391:39–46, 1997.
- [96] H. J. Kim, L. J. Romans, and P. van Nieuwenhuizen. The Mass Spectrum of Chiral N=2 D=10 Supergravity on S^5 . *Phys. Rev.*, D32:389, 1985.
- [97] Martin Kruczenski, David Mateos, Robert C. Myers, and David J. Winters. Meson spectroscopy in AdS / CFT with flavor. *JHEP*, 07:049, 2003.
- [98] Sungho Hong, Sukjin Yoon, and Matthew J. Strassler. Quarkonium from the fifth-dimension. *JHEP*, 04:046, 2004.
- [99] Vijay Balasubramanian, Eric G. Gimon, Djordje Minic, and Joachim Rahmfeld. Four-dimensional conformal supergravity from AdS space. *Phys. Rev.*, D63:104009, 2001.
- [100] Thorsten Ohl and Christoph F. Uhlemann. The Boundary Multiplet of N=4 SU(2) \times U(1) Gauged Supergravity on Asymptotically-AdS₅. *JHEP*, 06:086, 2011.
- [101] Claudius Klare, Alessandro Tomasiello, and Alberto Zaffaroni. Supersymmetry on Curved Spaces and Holography. *JHEP*, 08:061, 2012.
- [102] E. S. Fradkin and Arkady A. Tseytlin. CONFORMAL SUPERGRAVITY. *Phys. Rept.*, 119:233–362, 1985.
- [103] Davide Gaiotto and Edward Witten. Supersymmetric Boundary Conditions in N=4 Super Yang-Mills Theory. *J. Statist. Phys.*, 135:789–855, 2009.
- [104] Cumrun Vafa and Edward Witten. A Strong coupling test of S duality. *Nucl. Phys.*, B431:3–77, 1994.
- [105] A. B. Clark, D. Z. Freedman, A. Karch, and M. Schnabl. Dual of the Janus solution: An interface conformal field theory. *Phys. Rev.*, D71:066003, 2005.
- [106] Eric D’Hoker, John Estes, and Michael Gutperle. Interface Yang-Mills, supersymmetry, and Janus. *Nucl. Phys.*, B753:16–41, 2006.

- [107] A. Clark and A. Karch. Super Janus. *JHEP*, 10:094, 2005.
- [108] Eric D'Hoker, John Estes, and Michael Gutperle. Ten-dimensional supersymmetric Janus solutions. *Nucl. Phys.*, B757:79–116, 2006.
- [109] Sebastian de Haro, Sergey N. Solodukhin, and Kostas Skenderis. Holographic reconstruction of space-time and renormalization in the AdS / CFT correspondence. *Commun. Math. Phys.*, 217:595–622, 2001.
- [110] Massimo Bianchi, Daniel Z. Freedman, and Kostas Skenderis. Holographic renormalization. *Nucl. Phys.*, B631:159–194, 2002.
- [111] C. Wetterich. Spinors in euclidean field theory, complex structures and discrete symmetries. *Nucl. Phys.*, B852:174–234, 2011.
- [112] H. Nicolai. A Possible constructive approach to (SUPER ϕ^{**3}) in four-dimensions. 1. Euclidean formulation of the model. *Nucl. Phys.*, B140:294–300, 1978.
- [113] Alex Buchel, Jorge G. Russo, and Konstantin Zarembo. Rigorous Test of Non-conformal Holography: Wilson Loops in $N=2^*$ Theory. *JHEP*, 03:062, 2013.
- [114] Alex Buchel. Localization and holography in $N=2$ gauge theories. *JHEP*, 08:004, 2013.
- [115] David J. Gross and Edward Witten. Possible third-order phase transition in the large- n lattice gauge theory. *Phys. Rev. D*, 21:446–453, Jan 1980.
- [116] J. G. Russo and K. Zarembo. Massive $N=2$ Gauge Theories at Large N . *JHEP*, 11:130, 2013.
- [117] Han-Chih Chang and Andreas Karch. Entanglement Entropy for Probe Branes. *JHEP*, 01:180, 2014.
- [118] Jorge G. Russo and Konstantin Zarembo. Evidence for Large- N Phase Transitions in $N=2^*$ Theory. *JHEP*, 04:065, 2013.
- [119] Nikita Nekrasov and Andrei Okounkov. Seiberg-Witten theory and random partitions. *Prog. Math.*, 244:525–596, 2006.
- [120] Xinyi Chen-Lin, James Gordon, and Konstantin Zarembo. $\mathcal{N} = 2^*$ super-Yang-Mills theory at strong coupling. *JHEP*, 11:057, 2014.
- [121] Peter Breitenlohner and Daniel Z. Freedman. Positive Energy in anti-De Sitter Backgrounds and Gauged Extended Supergravity. *Phys. Lett.*, B115:197, 1982.
- [122] Peter Breitenlohner and Daniel Z. Freedman. Stability in Gauged Extended Supergravity. *Annals Phys.*, 144:249, 1982.
- [123] Alfred D. Shapere, Frank Wilczek, and Zhaoxi Xiong. Models of Topology Change. 2012.
- [124] Satoshi Nawata. Localization of $N=4$ Superconformal Field Theory on $S^{**1} \times S^{**3}$ and Index. *JHEP*, 11:144, 2011.

- [125] Benjamin Assel, Davide Cassani, and Dario Martelli. Localization on Hopf surfaces. *JHEP*, 08:123, 2014.
- [126] Leonardo Rastelli and Shlomo S. Razamat. The supersymmetric index in four dimensions. 2016.
- [127] Joel Lamy-Poirier. Localization of a supersymmetric gauge theory in the presence of a surface defect. 2014.
- [128] Cyril Closset, Heeyeon Kim, and Brian Willett. Supersymmetric partition functions and the three-dimensional A-twist. 2017.
- [129] Isak Buhl-Mortensen, Marius de Leeuw, Asger C. Ipsen, Charlotte Kristjansen, and Matthias Wilhelm. A Quantum Check of AdS/dCFT. *JHEP*, 01:098, 2017.
- [130] Isak Buhl-Mortensen, Marius de Leeuw, Asger C. Ipsen, Charlotte Kristjansen, and Matthias Wilhelm. One-loop one-point functions in gauge-gravity dualities with defects. *Phys. Rev. Lett.*, 117(23):231603, 2016.
- [131] Marius de Leeuw, Charlotte Kristjansen, and Konstantin Zarembo. One-point Functions in Defect CFT and Integrability. *JHEP*, 08:098, 2015.
- [132] Balt C. van Rees. Holographic renormalization for irrelevant operators and multi-trace counterterms. *JHEP*, 08:093, 2011.
- [133] Daniel Z. Freedman, Krzysztof Pilch, Silviu S. Pufu, and Nicholas P. Warner. Boundary Terms and Three-Point Functions: An AdS/CFT Puzzle Resolved. 2016.