K-rigidity in Modular Representation Theory

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Abstract

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The theory of K-rigidity aims to study representation rings (a.k.a 'K-rings') up to a 'deformation of the tensor product', in a certain sense. Calculating representation rings for a group of tame or finite representation type over some field is a difficult problem, and if a group is of wild representation type over some field, it is a quantifiably untenable problem. For every nonabelian 2-group of tame representation type over a field of characteristic 2, calculating the ring of representations remains an open problem. By studying which calculations hold 'up to a deformation of the tensor product', one gains deeper insight into the role of certain homological tools in said calculations.

In this thesis, a relationship between representation type and representation ring calculations is established for certain families of finite group schemes. Those of finite representation type are shown to have 'total K-rigidity', meaning that upon any deformation of their tensor product, the ring of representations is unchanged. Those of tame representation type are shown to have the more delicate property of 'noble K-rigidity', meaning that they obey a geometric criterion for which tensor products of representations are unchanged in their isomorphism class, upon deformation. Those of wild representation type are shown to be 'not noble K-rigid', so that even under nice geometric conditions, one can still produce representations whose tensor product changes isomorphism class upon deformation.

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To bringers of light and joy to an otherwise dark world:

To my family. To the survivors. To the defenders.

INTRODUCTION

In this thesis, I have developed a theory termed 'K-rigidity'. The letter 'K' here is in reference to K-theory, a topic which originates in algebraic geometry and topology, and, briefly, is a tool for studying vector bundles up to isomorphism. The 'rigidity' of 'K-rigidity' is with respect to a 'deformation of a tensor product', in an appropriate sense, to be made precise.

Modular representation theory is the theory of matrices over fields of positive characteristic which solve certain polynomial equations. Most uses for modular representation theory have been purely algebraic, towards algebraic number theory and finite group theory. But in recent years, modular representation theory has progressively become more connected with homotopy theory, using new geometric methods to understand both topics in unison [Bal05].

The theory of K-rigidity is primarily concerned with a categorical analogue of the question 'which ring structures can exist on a given abelian group?'. The analogy goes as follows: an abelian group is replaced with an abelian category, where the group operation on objects is a 'direct sum'. Indeed, for an abelian category $\mathcal{A}$ with some mild smallness assumptions, one can form $K_0(\mathcal{A})$, an abelian group, such that the group is generated by the building blocks, or atoms, of $\mathcal{A}$, known as (isomorphism classes of) 'indecomposable objects'. A ring structure on a fixed abelian group is replaced with a tensor category structure on a fixed abelian category $\mathcal{A}$, in which the tensor product of objects plays the role of the ring multiplication. Indeed, if $\mathcal{A}$ is given the structure of a tensor category, then our abelian group $K_0(\mathcal{A})$, from before, becomes a ring. By design, the ring structure of $K_0(\mathcal{A})$ can be completely understood by tensoring each pair of atoms together. Now, which tensor category structures can even exist on a given abelian category? An even richer question arises beyond the one raised by analogy, that is: how can we compare the induced ring structures of different tensor structures, and what does this look like atom-by-atom, so to speak?

A warning, which ought to be obligatory for any author introducing tensor categories by way of a similar analogy, is precisely where K-rigidity comes in. That is, sometimes two different tensor category structures on an underlying abelian category, denoted as $(\mathcal{A}, \otimes), (\mathcal{A}, \otimes')$ may have *identical* induced ring structures $(K_0(\mathcal{A}), \otimes), (K_0(\mathcal{A}), \otimes')$. The theory of K-rigidity is concerned, more broadly, with certain subgroups of $K_0(\mathcal{A})$ which are guaranteed closed under both of $\otimes, \otimes'$. Determining which of these subgroups $\mathcal{A}$ have identical ring structure as subrings of $(K_0(\mathcal{A}), \otimes), (K_0(\mathcal{A}), \otimes')$, respectively, is what is studied with K-rigidity.

In Chapter 1, I review the representation theory of group schemes over a commutative ring. In Chapter 2, I review the representation theory of a split reductive group, and I demonstrate how some homological properties of constant SL_2-representations lead to an original proof of a well-known result: that the representation ring of modular representations of the group $\mathbb{Z}/p$ is 'K-rigid' with respect to any deformation. To do this, I use a certain $\mathbb{A}^1$-family of finite group schemes contained in SL_2.

In Chapter 3, I investigate what I interpret as one instance of the metamathematical duality between complexity and rigidity. To elaborate, 'rigidity' phenomena are generally defined with respect to some theory of deformations. On the other hand 'complexity' is generally defined over some class of mathematical problems, and is usually related to the dimension of some moduli space associated to a given problem. Then, within a class of problems admitting some deformation theory, as complexity increases, one should find that rigidity phenomena become increasingly sporadic. Dually, problems demonstrating some regular form of rigidity should only exist when they are of low complexity.

In representation theory, a given algebra has a moduli problem classifying its indecomposable modules. The dimension of the moduli space of indecomposable modules for a given finite algebra is either 0, 1, or ∞, depending on if the algebra is respectively of 'finite', 'tame', or 'wild' representation type, indicating complexity. In Chapter 3, I defined 'noble K-rigidity' as a kind of regular form of rigidity that some algebras are shown to demonstrate. I conjecture that a cocommutative finite Hopf algebra (meeting some technical cohomological hypothesis) is of wild representation type if and only if it is *not* 'noble K-rigid'. I show evidence of my conjecture by devising some theory for producing obstructions to K-rigidity that apply to Hopf algebras of wild representation type. Conversely, I make novel computations of certain rings of representations for Hopf algebras of finite and tame representation type group in order to show that they *are* 'noble K-rigid'.

General conventions and notations

To be followed throughout this document.

1. We will work within the standard model for set theory ZFC.

2. Category theory follows definitions as in [ML98]

3. Commutative algebra follows definitions as in [AM69].

4. Modules over a ring are always assumed to be left modules unless they are called right modules. Similarly, group actions are assumed to be left group actions unless they are called right actions.

5. Rings are assumed to be associative and with identity, and homomorphisms of rings are assumed to be additive, multiplicative, and preserve the identity.

6. If k is a field:

   F1. An extension L/k is a field L with an embedding of rings $k \hookrightarrow L$.

   F2. We denote by $\overline{k}$ the algebraic closure of k.

   F3. The characteristic of k is the nonnegative integer $p \geq 0$ such that the principal ideal $(p) \subset \mathbb{Z}$ is the kernel of the unique homomorphism of rings $\mathbb{Z} \to k$.

7. If R is a commutative ring with identity:

   R1. We say R is of characteristic $p \geq 0$ if there is a field k of characteristic p and a homomorphism of rings $k \to R$.

   R2. For any R-module M, there is a dual module denoted $M^\vee$, defined by $M^\vee = \mathrm{Hom}(M, R)$.

   R3. If M, N are R-modules, we denote $M \otimes_R N$ the tensor product over R, usually suppressing the subscript by denoting it $M \otimes N$ when the context is clear.

R4. For i a positive integer and M an R-module, we denote by $T^i(M)$ the i-fold tensor power of M. We denote by $S^i(M)$ and by $\Lambda^i(M)$ the quotients of $T^i(M)$, given by the i^{th} symmetric and exterior powers, respectively. We declare $T^0(M) = S^0(M) = \Lambda^0(M) = R$ for any module M, as well.

R5. An algebra over R is assumed to be associative and with identity, except when it is called a 'Lie algebra', or a 'restricted Lie algebra' (in case R is a field of characteristic $p > 0$). Associative algebras with identity will be named usually using uppercase Latin alphabet letters (e.g. $A, B, C, \dots$), while Lie algebras (and restricted Lie algebras) will be named exclusively using lowercase Fraktur letters (e.g. $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \dots$, and often $\mathfrak{g}$) to emphasize the distinction.

R6. A graded algebra over R is an algebra A^* with a direct sum decomposition $A^* = \bigoplus_{i\in\mathbb{Z}} A^i$, such that $ab \in A^{i+j}$ whenever $a \in A^i$, $b \in A^j$.

R7. A graded algebra $A^* = \bigoplus_{i\in\mathbb{Z}} A^i$ over R is called a graded-commutative algebra over R if $ab = (-1)^{ij}ba$ whenever $a \in A^i$, $b \in A^j$. **Caution:** A commutative graded algebra is in general not graded-commutative, and vice versa.

R8. For any R-module M, we let

$$T^*(M) = \bigoplus_{i\geq 0} T^i(M), \quad S^*(M) = \bigoplus_{i\geq 0} S^i(M), \quad \Lambda^*(M) = \bigoplus_{i\geq 0} \Lambda^i(M).$$

Then $T^*(M)$ (called the tensor algebra) has the structure of a graded algebra, $S^*(M)$ (called the symmetric algebra) is a commutative graded algebra, and $\Lambda^*(M)$ (called the exterior algebra) is a graded-commutative algebra.

R9. If A^* is a graded-commutative algebra, then there is a commutative graded subalgebra $A^\bullet \subset A^*$ consisting of

$$A^\bullet = \begin{cases} A^* & \text{if } 1 = -1 \text{ in } R, \\ \bigoplus_{i\in\mathbb{Z}} A^{2i} & \text{otherwise} \end{cases}$$

R10. An affine scheme over R is a covariant representable functor from commutative R-algebras to sets.

S1. If X is an affine scheme over R and A is a commutative algebra over R, then $X(A)$ is called the set of A-points of X.

S2. If A is a commutative algebra over R, the affine scheme $\mathrm{Hom}(A, ?)$ is denoted by $\operatorname{Spec} A$.

S3. Natural maps between affine schemes are called morphisms, and for affine schemes X, Y, the set of all morphisms $X \to Y$ is denoted by $\mathrm{Mor}(X, Y)$.

S4. Let $A = R[x]$. The affine scheme $\operatorname{Spec} A$ is called the affine line, and is denoted by $\mathbb{A}^1_R$, or if the base ring R is clear, by $\mathbb{A}^1$. **Note:** The affine line is naturally isomorphic to the forgetful functor from commutative algebras over R to sets and is therefore endowed with the structure of a commutative R-algebra object in the category of affine schemes over R.

S5. If X is an affine scheme over R, then the R-algebra $\mathrm{Mor}(X, \mathbb{A}^1)$ is called the coordinate algebra of X and is denoted $\mathcal{O}(X)$. **Note:** It is a direct consequence of the Yoneda lemma that the commutative algebra $\mathcal{O}(X)$ is a representing object for the functor X.

R11. An affine scheme X over R is called finite (resp. flat) if $\mathcal{O}(X)$ is finitely generated (resp. flat) as a module over R. Moreover, we follow the convention from algebraic geometry that a finite affine scheme is simply called a finite scheme.

R12. More generally, schemes over R are defined as locally affine functors (over R), as in Jantzen [Jan03]. It is a great convenience, however, to model the category of schemes (over R) in the classical geometric fashion instead, as locally ringed spaces $(X, \mathcal{O}_X)$ which are locally affine, equipped with a structure morphism $X \to \operatorname{Spec} R$, as found in Hartshorne [Har77].

AG1. Affine schemes (over R) are a full subcategory of the category of schemes (over R). The set of morphisms $X \to Y$ for schemes X, Y over R is also denoted $\mathrm{Mor}(X, Y)$, and by fullness there is no ambiguity in notation.

AG2. If X is a scheme over R (in any sense), the 'functor of points' defined by $\mathrm{Mor}(\operatorname{Spec}(?), X)$ from commutative R-algebras to sets, is a locally affine functor, and is canonically isomorphic to X when modeled as a functor (by

[EGAI], the category of schemes is a full subcategory of functors from commutative R-algebras to sets, so this is the Yoneda lemma).

AG3. If X is a scheme over R (in any sense), we denote $|X|$ the 'underlying topological space' of X. When modeled as a locally ringed space, then the terminology 'underlying' is appropriate. When modeled as a functor, the terminology 'associated topological space' is better suited, but Grothendieck [EGAI] has allowed us to make no distinction between models, so we will always say 'underlying'.

AG4. If A is a commutative R-algebra, then the underlying space $|\operatorname{Spec} A|$ is the set of prime ideals of the ring A, with the Zariski topology.

AG5. If $A^* = \bigoplus_{i\in\mathbb{Z}} A^i$ is a graded-commutative R-algebra, we denote by $\operatorname{Proj} A$ the scheme $\operatorname{Proj} A^\bullet$ as defined in [Har77] for commutative graded algebras. The underlying topological space $|\operatorname{Proj} A|$ consists of *homogeneous* prime ideals of $A^\bullet$ (or equivalently, of A^*) which don't contain all of $A^+ = \bigoplus_{i>0} A^i$. We also denote by $\operatorname{Spec} A$ the affine scheme $\operatorname{Spec} A^\bullet$, as is defined for commutative algebras.

AG6. For ideals (resp. homogeneous ideals) $I \subset A$ of a commutative R-algebra (resp. commutative graded algebra) A, we denote by $\mathscr{V}(I)$ to be the closed subscheme of $X = \operatorname{Spec} A$ (resp. $X = \operatorname{Proj} A$) associated to I. As a scheme, $\mathscr{V}(I) = \operatorname{Spec} A/I$ (resp. $\operatorname{Proj} A/I$), and the inclusion $\mathscr{V}(I) \hookrightarrow X$ is determined by the canonical projection of coordinate algebras $A \to A/I$ (resp. homogeneous coordinate algebras). The underlying space of $\mathscr{V}(I)$ is canonically identified as a closed subspace

$$|\mathscr{V}(I)| = \{\mathfrak{p} \in |X| : I \subset \mathfrak{p}\}$$

R13. If A is an R-algebra, then a homomorphism of algebras $A \to R$ is called an augmentation of A (or on A) and the pairing $(A, A \to R)$ is called an augmented algebra (or augmented algebra structure on the R-module A).

R14. If A is an R-algebra, then the left ideal generated by a subset $S \subset A$ or elements

$x_i, \ldots$ is denoted as a ket $|S\rangle$ or $|x_i, \ldots\rangle$ and the right ideal is denoted as a bra $\langle S|$ or $\langle x_i, \ldots|$. The two-sided ideal will be denoted as (S) or $(x_i, \ldots)$.

Chapter 1
BACKGROUND

Here we will review the essential topics for K-rigidity in modular representation theory. We will review properties of cohomology of modules for augmented algebras and Hopf algebras. As a special case, we will see properties of the cohomology of finite groups. We will define group schemes, (geometrically) reductive groups, and restricted Lie algebras, and review their representation theory.

Conventions

Used throughout our background chapter.

- We fix R to be a commutative ring. 'Schemes' generally will mean 'R-schemes', i.e. 'schemes over R', and schemes taken over any other base will be specified as necessary.

- We denote the category of sets by $\mathcal{S}et$ and the category of groups by $\mathcal{G}r$. The faithful functor taking any group to its underlying set is denoted by For:$\mathcal{G}r \to \mathcal{S}et$ (the 'forgetful functor').

- We denote the category of R-modules by $\mathcal{QC}oh_R$, and the category of commutative R-algebras by $\mathcal{C}_R$.

- If $\mathcal{Y}$ is a category, then a $\mathcal{Y}$-functor is a functor from some category into the category $\mathcal{Y}$. If $\mathcal{Y}$ is a category with a named type of object, like $\mathcal{S}et$ or $\mathcal{G}r$, we may also call a $\mathcal{Y}$-functor by e.g. a set functor, or group functor, respectively.

1.1 Hopf algebras

We'll primarily follow the conventions of [Wat79], [Jan03], for Hopf algebras over R. We'll review modules and comodules over a Hopf algebra, as well as some general operations for (co)modules over a Hopf algebra.

1.1.1 Definitions and examples

Recall that, for any algebras A, B, over R, an algebra $A \otimes_R B$ is always defined with multiplication of simple tensors defined as

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb'.$$

Definition 1.1.1.1. Let V be an R-module. Then a coalgebra structure on V consists of R-linear maps

$$\Delta: V \to V \otimes V, \quad \varepsilon: V \to R$$

such that Δ is a coassociative comultiplication, i.e.

$$(\Delta \otimes V) \circ \Delta = (V \otimes \Delta) \circ \Delta: V \to V \otimes V \otimes V$$

and such that ε is a counital with respect to Δ, i.e.

$$(V \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes V) \circ \Delta = \mathrm{id}$$

A homomorphism $f: V \to V'$ of coalgebras $(V, \Delta, \varepsilon), (V', \Delta', \varepsilon')$ is an R-linear map such that

$$\varepsilon' \circ f = \varepsilon, \quad (f \otimes f) \circ \Delta = \Delta' \circ f.$$

Definition 1.1.1.2. If V is a coalgebra over R, and R is regarded as a trivial coalgebra over itself, then a homomorphism $u: R \to V$ is called a coaugmentation or coaugmentation structure. The data of a coalgebra with a coaugmentation structure is called a coaugmented coalgebra.

Definition 1.1.1.3. Let A be an R-algebra. A bialgebra structure on A is a coalgebra structure on the R-module A, such that Δ and ε are homomorphisms of algebras.

Definition 1.1.1.4. A bialgebra structure on an algebra A is called a Hopf algebra structure if there exists an 'antipode' $S\colon A \to A$, i.e. an R-linear map satisfying

$$m \circ (A \otimes S) \circ \Delta = m \circ (S \otimes A) \circ \Delta = u \circ \varepsilon,$$

where $m\colon A \otimes A \to A$ is the multiplication and $u\colon R \to A$ is the unit structure morphism. It follows that an antipode S is an antiautomorphism of A, and S is unique if it exists for some bialgebra structure.

Example 1.1.1.5. Let G be any group, and let RG be the free R-module generated by the set G. Then RG is given a ring structure by extending R-linearly from the multiplication of elements in G. In fact, RG is a Hopf algebra over R, with structure maps Δ, ε, S defined by extending R-linearly from

$$\Delta\colon g \mapsto g \otimes g, \quad \varepsilon\colon g \mapsto 1, \quad S\colon g \mapsto g^{-1},$$

defined for elements $g \in G$.

Example 1.1.1.6. The singular cohomology $H^*(X;R)$ defines a (graded) algebra over R under the cup product whenever X is a topological space. If G is a Lie group, the singular cohomology $H^*(G;R)$ is a (graded) Hopf algebra over R, with e.g. comultiplication defined by contravariance

$$H^*(G;R) \xrightarrow{H^*(m;R)} H^*(G \times G;R) \xrightarrow{\sim} H^*(G;R) \otimes H^*(G;R).$$

Here, $m\colon G \times G \to G$ is the smooth group multiplication, and the second arrow is the isomorphism from the Künneth theorem. The cohomology of Lie groups and topological groups was historically among the original motivations for studying Hopf algebras.

Example 1.1.1.7. Not every algebra admits a Hopf algebra structure. Let k be a field and let $A = \mathrm{End}_k(k^n)$ be the endomorphism ring. For $n = 1$, A is the trivial algebra, which admits a trivial Hopf algebra structure. For $n > 1$, the algebra A is Morita equivalent to k, and one can argue then that there is no augmentation $A \to k$, since the unique simple A-module is given by the canonical action on k^n. The noncommutative Weyl algebra $k\langle x,y\rangle/(xy-yx-1)$ is an example of a basic algebra, of infinite dimension, that does not admit any Hopf algebra structure.

Example 1.1.1.8. A given algebra may have nonisomorphic Hopf algebra structures on it. For instance, if k is a field of characteristic $p > 0$, and $\eta \in k$ is any element, then the algebra $A = k[t]/t^p$ over k has a Hopf algebra structure with comultiplication given by

$$\Delta^\eta : t \mapsto t \otimes 1 + 1 \otimes t + \eta t \otimes t,$$

with $\varepsilon(t) = 0$, and antipode $S^\eta : A \to A$ uniquely determined by any Δ^η. The comultiplications Δ^0, Δ^1 will always define nonisomorphic Hopf algebra structures on A, and if k is algebraically closed, then any Hopf algebra structure on A is isomorphic to Δ^0 or Δ^1 by Theoreom A.2.0.5, attributed to Oort [Oor66].

Some additional notes:

- We'll often say an algebra is a bialgebra (resp. Hopf algebra) when it has a bialgebra structure (resp. Hopf algebra structure).

- If $R \to R'$ is any commutative algebra over R, the base change $A_{R'} = A \otimes_R R'$ is a Hopf algebra over R' whenever A is a Hopf algebra.

- The R-linear dual of any coalgebra is an algebra, and the R-linear dual of any algebra is a coalgebra. If A is a bialgebra over R, then we take $A^\vee$ to mean the dual bialgebra, with (co)algebra structure determined by dualizing the (co)coalgebra structure of A. If A is a Hopf algebra, its antipode dualizes to also make $A^\vee$ a Hopf algebra.

1.1.2 Operations on modules

Let A be an algebra over a commutative ring R. Now A-modules have an underlying R-module.

Since R is commutative, we have operations on the category $\mathcal{QC}oh_R$ of R-modules like tensoring, and dualizing, but generally speaking, performing these operations on the underlying R-module of an A-module will not result in an A-module in any natural way. But we do have natural way for these operations on A-modules to result in another A-module when A is given a Hopf algebra structure over R.

We will first define comodules over a coalgebra. Then we will review operations on modules over a Hopf algebra, and we'll note that for all of these operations and properties that we review, there is a corresponding notion for comodules over the Hopf algebra as well.

Definition 1.1.2.1. Let V be a coalgebra over R with comultiplication Δ, and let M be any module over R. A left V-coaction (or left V-comodule structure) on M is a map

$$\Delta_M: M \to V \otimes M$$

such that $(V \otimes \Delta_M) \circ \Delta_M = (\Delta \otimes M) \circ \Delta_M$ as R-linear maps $M \to V \otimes V \otimes M$.

A homomorphism $f: M \to M'$ of comodules over V is an R-linear map such that $\Delta_{M'} \circ f = (V \otimes f) \circ \Delta_M$ as maps $M \to V \otimes M'$.

Definition 1.1.2.2. Let A be a Hopf algebra over R, with the usual convention Δ, ε, S for structure maps.

1. The counit $\varepsilon: A \to R$ defines R to be a module over A.

2. If M is a (left) module over R, then the R-linear dual $M^\vee$ has a canonical A-module structure, by defining $xf: M \to R$ by $(af)(x) = f(S(a)x)$ for any R-linear $f: M \to R$.

3. If M, N are modules over R, then the R-module $M \otimes N$ is canonically a $A \otimes A$-module. The pullback along $\Delta: A \to A \otimes A$ then defines a canonical A-module structure on $M \otimes N$. The tensor product then endows the category of A-modules with the structure of a monoidal category, with monoidal unit R.

4. If M, N are modules over A, then the R-module $\mathrm{Hom}_R(M, N)$ is canonically an (A, A)-

bimodule, (hence a left $A \otimes A^{\mathrm{op}}$-module) by defining $a_1 f a_2 \colon M \to N$ by

$$(a_1 f a_2)(x) = a_1 f(a_2 x)$$

for any R-linear $f \colon M \to N$. The pullback along $(A \otimes S) \circ \Delta \colon A \to A \otimes A^{\mathrm{op}}$ then defines a canonical A-module structure on $\mathrm{Hom}_R(M, N)$.

5. If M is any A-module, there is a R-submodule M^A, given by

$$M^A = \{x \in M \mid am = \varepsilon(a)m\},$$

called the invariants of M. Then there is a canonical isomorphism

$$M^A \xrightarrow{\sim} \mathrm{Hom}_A(R, M)$$

of R-modules, taking x to $f_x \colon c \mapsto cx$, for $x \in M^A$, $c \in R$.

6. If M is an A-module which is projective and finitely generated as a module over R, then the canonical isomorphism

$$N \otimes M^\vee \xrightarrow{\sim} \mathrm{Hom}_R(M, N)$$

is A-linear given the canonical A-module structures on the source and target. In this case there is a tensor-hom adjunction giving us a natural isomorphism

$$\mathrm{Hom}_A(Q \otimes M, N) \cong \mathrm{Hom}_A(Q, \mathrm{Hom}_R(M, N))$$

which is natural in the arguments Q, N. In particular $(N \otimes M^\vee)^A \cong \mathrm{Hom}_A(M, N)$.

Proposition 1.1.2.3. *Let A be a Hopf algebra and let P be a projective A-module, and let Q be an injective A-module. If M is any A-module which is projective and finitely generated as a module over R, then $P \otimes M$ is projective over A and $\mathrm{Hom}_R(M, Q)$ is injective over A.*

Proof. The functors $\mathrm{Hom}_A(P \otimes M, ?)$ and $\mathrm{Hom}_A(?, \mathrm{Hom}_R(M, Q))$ are exact by the tensor-hom adjunction, given that tensoring over R with $M, M^\vee$ is exact, since M is projective and finitely generated over R. □

Example 1.1.2.4. Consider the algebra $A = k[t]/t^p$, for a field k of characteristic $p > 0$, and the comultiplication Δ^η from before. The indecomposable modules over A are all of the form $J_i = k[t]/t^i$, for $0 \leq i \leq p$. The tensor product of A-modules over R depends, a priori, on the choice of $\eta \in k$ defining Δ^η. Since A is a local algebra, any projective module is free of some rank, i.e. isomorphic to some J_p^r. Therefore, independent of choice of η, we know that $J_p \otimes J_i$ is projective and hence isomorphic to J_p^i. For a detailed account of how to compute any $J_i \otimes J_j$, see 2.3.2. Perhaps surprisingly, the isomorphism class of any $J_i \otimes J_j$ is always independent of choice of η.

Example 1.1.2.5. Consider the algebra $A = k[t_1, t_2]/(t_1^2, t_2^2)$, for a field k of characteristic $p = 2$. Let $\eta \in k$, and define $\Delta^\eta \colon A \to A \otimes A$ by

$$\Delta^\eta(t_1) = t_1 \otimes 1 + 1 \otimes t_1, \quad \Delta^\eta(t_2) = t_2 \otimes 1 + 1 \otimes t_2 + \eta t_1 \otimes t_1.$$

Then for any η, the Δ^η defines a Hopf algebra structure (with counit sending each t_i to 0). Consider $V = A/t_2$, an A-module of dimension two. Consider the A-module $V \otimes V$, of dimension four. The isomorphism class of $V \otimes V$ now depends on η. To see this, notice

$$t_2 \cdot ((a + bt_1) \otimes (c + dt_1)) = \eta(at_1 \otimes ct_1),$$

so $V \otimes V$ is annihilated by t_2, if and only if $\eta = 0$.

1.2 Representations of group schemes

1.2.1 Group functors and group schemes

Definition 1.2.1.1. A group scheme (over R) is a group object in the category of schemes (over R). A flat, resp. affine, finite, group scheme over R is a group scheme over R which is flat, resp. affine, finite, as a scheme over R.

By [EGAI], a group scheme is equivalent to a group functor

$$G: \mathcal{C}_R \to \mathcal{G}r$$

such that the composition $\mathcal{C}_R \xrightarrow{G} \mathcal{G}r \xrightarrow{\mathrm{For}} \mathcal{S}et$ is representable by a scheme.

In this fashion, we say an abelian group scheme is a group scheme that is an abelian group functor. Abelian group schemes may also be called commutative group schemes.

Definition 1.2.1.2. Let V be a module over R. The group functor V_a is defined by

$$V_a(A) = A \otimes V,$$

using the underlying additive group structure of the resultant A-module.

Definition 1.2.1.3. Let V be a module over R. The group functor $\mathrm{GL}(V)$ is defined by

$$\mathrm{GL}(V)(A) = \mathrm{Aut}_{\mathcal{QC}oh_A}(A \otimes V).$$

Lemma 1.2.1.4. *The assignment $V \mapsto V_a$ is a functor from $\mathcal{QC}oh_R$ to the functor category $\mathcal{F}un(\mathcal{C}_R, \mathcal{S}et)$.*

Proof. The induced maps and subsequent functor property of the assignment all follow from $? \otimes ?$ being a functor $\mathcal{QC}oh_R \times \mathcal{QC}oh_R \to \mathcal{QC}oh_R$. □

Proposition 1.2.1.5. *If V is projective and finitely generated over R, then V_a and $\mathrm{GL}(V)$ are affine group schemes.*

Proof. We find some algebras to serve as representing objects for the two functors of interest.

By the lemma, each homomorphism of R-modules $V \to V'$ induces a natural map in $\mathrm{Mor}(V_a, V'_a)$. Notice that R_a is merely the affine line $\mathbb{A}^1$, (here we are viewing R as a module over itself). Then the functor V_a in general is viewed as merely an 'affine-schemified' version of V itself, so that elements of $V^\vee = \mathrm{Hom}(V, R)$ are realizable as elements of the coordinate algebra $\mathcal{O}(V_a) = \mathrm{Mor}(V_a, \mathbb{A}^1)$.

Indeed, universal properties of the symmetric algebra give us a natural isomorphism $\operatorname{Spec} S^*(V^\vee) \cong V_a$ whenever V is projective and finitely generated (and hence reflexive, in the sense that $(V^\vee)^\vee \cong V$). On points, we have:

$$\begin{aligned}\operatorname{Hom}_{\mathcal{C}_R}(S^*(V^\vee), A) &\cong \operatorname{Hom}_{\mathcal{QC}oh_R}(V^\vee, A)\\ &\cong \operatorname{Hom}_{\mathcal{QC}oh_R}(R, A \otimes V)\\ &\cong A \otimes V.\end{aligned}$$

Verifying naturality being straightforward, this shows the functor V_a is representable by the algebra $S^*(V^\vee) \cong \mathcal{O}(V_a)$, in which $V^\vee$ is a submodule.

There are two common approaches to the second claim, on GL(V). The first approach involves the construction of a determinant, and has the advantage of realizing the subfunctor inclusion $\iota\colon \mathrm{GL(V)} \hookrightarrow (V \otimes V^\vee)_a$ as an open immersion of schemes. This is the approach found in e.g. [Jan03].

The second approach is the one we will take, which is to realize GL(V) as a closed subscheme of $(V \otimes V^\vee)_a \times (V \otimes V^\vee)_a$. First, for R-modules V_1, V_2, V_3, we see a well-defined composition of homomorphisms, which behaves nicely under tensoring with any commutative R-algebra. If they each V_i is projective and finitely generated, this gives us a morphism of schemes (not of group schemes)

$$\mathrm{comp}\colon ((V_3 \otimes V_2^\vee) \times (V_2 \otimes V_1^\vee))_a \to (V_3 \otimes V_1^\vee)_a.$$

Let X be the geometric fiber as in the cartesian square of affine schemes within the diagram below:

$$\begin{tikzcd} \mathrm{GL}(V) \arrow[r, dotted] \arrow[rr, bend left, "(\iota,\ \iota\circ\mathrm{inv}_{\mathrm{GL}(V)})"] \arrow[dr] & X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (V \otimes V^\vee)_a \times (V \otimes V^\vee)_a \arrow[d, "\mathrm{comp}"] \\ & \operatorname{Spec} R \arrow[r, "\mathrm{id}_V"'] & (V \otimes V^\vee)_a \end{tikzcd}.$$

The arrow id_V is the image under the Yoneda embedding of the identity element of $(V \otimes V^\vee)_a(R) = V \otimes V^\vee$. The map $\mathrm{inv}_G\colon G \to G$ is the natural inverse, part of the structure of any group functor G. The functor category has pullbacks so X exists, and assuming V is projective and finitely generated, we get X is also an affine scheme. Since all the non-dotted arrows in the diagram commute, there exists a dotted arrow by the cartesian property in the functor category. The dotted arrow is easily shown to be an isomorphism of functors, so $\mathrm{GL}(V)$ is representable. □

Example 1.2.1.6.

1. The affine scheme $\operatorname{Spec} R$ is also an affine group scheme, that we'll refer to as the trivial group scheme $\underline{1}$.

2. If $V = R^n$ is the free module of finite rank $n \geq 0$, then $\mathrm{GL}(V)$ is called GL_n, and the points of $\mathrm{GL}_n(A)$ are regarded as $n \times n$ invertible matrices with coefficients in A.

3. In the case of $n = 1$, GL_1 is also called $\mathbb{G}_m$, the general multiplicative group. We have that $\mathcal{O}(\mathbb{G}_m) = R[x, \frac{1}{x}]$, and that $\mathbb{G}_m$ is an abelian group scheme.

4. For any $n \geq 1$, the abelian group functor μ_n is defined on points by $\mu_n(A) = \{a \in A : a^n = 1\}$, a group under multiplication. Then μ_n is a subgroup functor of $\mathbb{G}_m$, and has coordinate algebra $\mathcal{O}(\mathbb{G}_m) = R[x]/(x^n - 1)$.

5. If $V = R$, the free module of rank 1, then V_a is called $\mathbb{G}_a$, the general additive group, an abelian group scheme. The underlying scheme of $\mathbb{G}_a$ is canonically isomorphic to the affine line $\mathbb{A}^1$. Indeed, in our general convention S4., we note how $\mathbb{A}^1$ has a canonical structure of a ring functor, and we see that the underlying abelian group functor of the ring functor is canonically isomorphic to the group scheme $\mathbb{G}_a$.

6. If k is an algebraically closed field, then a group scheme over k is called an algebraic group if it is a variety over k, i.e. if it is integral, and of finite type over k (any group scheme is separated over its base). A group scheme over k is called an abelian variety

if it is a projective algebraic group, i.e. a projective variety over k. Abelian varieties are abelian group schemes. Elliptic curves (projective curves of genus 1) are examples of abelian varieties.

Proposition 1.2.1.7. *Let G be any group scheme with structure maps*

$$m: G \times G \to G, \quad e: \operatorname{Spec} R \to G, \quad \operatorname{inv}: G \to G.$$

Then the induced maps between tensor powers of the algebra $\mathcal{O}(G)$ of regular sections define a Hopf algebra structure on $\mathcal{O}(G)$ over R. Conversely, any Hopf algebra structure on a commutative algebra A over R induces a group scheme structure on the affine scheme Spec A. *In this way, affine group schemes are anti-equivalent to commutative Hopf algebras over R, via the usual anti-equivalence betweeen affine schemes and commutative algebras.*

1.2.2 Representations of group functors and schemes

Definition 1.2.2.1. Let $G: \mathcal{C}_R \to \mathcal{G}r$ be a group functor. A representation of G (over R) is an R-module V, together with a natural map of group functors

$$\rho: G \to \mathrm{GL}(V).$$

A representation of a group scheme is a representation of its group functor of points.

Thus, for each commutative algebra A, we have an A-linear action of the group $G(A)$ on $V_a(A)$, and this uniquely determines a representation. Often, in notation, the map ρ is omitted, and the object V alone is referred to as the representation, with $\rho_A(g)(v)$ denoted by $g \cdot v$, for $g \in G(A), v \in V_a(A)$, given A in $\mathcal{C}_R$. For a given R-module V, any such ρ giving an action will be referred to as a G-representation structure on the underlying R-module V.

Definition 1.2.2.2. A homomorphism of representations V, V' is an R-linear homomorphism $f: V \to V'$, such that the induced map $f_a: V_a \to V'_a$ is G-equivariant, i.e., that

$$g \cdot f_{a,A}(v) = f_{a,A}(g^{-1} \cdot v)$$

for each $g \in G(A), v \in V_a(A)$, given any A in $\mathcal{C}_R$. The set of homomorphisms $V \to V'$ is denoted $\mathrm{Hom}_G(V, V')$, and is an R-submodule of $\mathrm{Hom}_R(V, V')$.

Definition 1.2.2.3. Suppose that R is a noetherian ring, and that G is an affine group scheme over R. A representation V of G is called a lattice over G if the underlying R-module V is projective and finitely generated.

If G is an affine group scheme and R is noetherian, and if V is an R-module which is projective and finitely generated, then a G-representation structure on V will also be called a G-lattice structure on V.

1.2.2.4. [Jan03, I, §2.8] Let G be an affine group scheme over R, and let V be any R-module. Then to each G-representation structure on V, we associate a canonical homomorphism of R-modules

$$V \to \mathcal{O}(G) \otimes V$$

as follows: Take $u \in G(\mathcal{O}(G))$ to be the 'universal' group element, which is associated to the identity in $\mathrm{Mor}(G, G)$. Given a G-representation structure on V, we have a group action of $G(\mathcal{O}(G))$ on $V_a(\mathcal{O}(G)) = \mathcal{O}(G) \otimes V$. Then the associated $V \to \mathcal{O}(G) \otimes V$ is defined to take $v \mapsto u \cdot (1 \otimes v)$, and is R-linear by the linearity of the action of G on V.

Further, the group action property shows that the associated map is an $\mathcal{O}(G)$-comodule structure on V. Similarly, any $\mathcal{O}(G)$-comodule structure $\Delta : V \to \mathcal{O}(G) \otimes V$ defines a natural map

$$G \times V_a \to V_a,$$

on points as

$$(g, 1 \otimes v) \mapsto f_{1,A}(g) \otimes v_2$$

for each $g \in G(A)$, and each $v \in V$ with Sweedler notation $\Delta(v) = f_1 \otimes v_2$, and then extending A-linearly, given any A in $\mathcal{C}_R$. The action of any $g \in G(A)$ is then A-linear by assumption, and further this defines a natural group action of $G(A)$, by the comodule property of Δ. Therefore there is an associated G-representation structure on V. One verifies that these associated structures form a bijective correspondence between G-representation structures and $\mathcal{O}(G)$-comodule structures on V.

Definition 1.2.2.5. Let G be an affine group scheme. The category of G-representations has operations (c.f. 1.1.2.2) such as taking duals, tensor products, and invariants, as defined by the corresponding operations on comodules over the Hopf algebra $\mathcal{O}(G)$.

For instance, if R is noetherian and V is a G-lattice, then there is a dual $V^\vee$, such that there is a natural isomorphism

$$\mathrm{Hom}_G(Q \otimes V, M) \cong \mathrm{Hom}_G(Q, M \otimes V^\vee).$$

The invariants of V, for any G-representation V, will be denoted V^G and is, per definition, the subspace

$$\{v \in V \mid g(1 \otimes v) = 1 \otimes v \in V_a(A) \quad \forall g \in G(A), A \in \mathcal{C}_R\},$$

canonically identified with $\mathrm{Hom}_G(R, V)$, where R is the trivial representation. Then if V is again a G-lattice, and M is any G-representation, we have canonical isomorphisms

$$M \otimes V^\vee \cong \mathrm{Hom}_R(V, M), \quad (M \otimes V^\vee)^G \cong \mathrm{Hom}_G(V, M).$$

Example 1.2.2.6. Let G be a group scheme over R.

1. If V is any R-module, the trivial homomorphism $G \to \mathrm{GL}(V)$ (factoring through $\underline{1}$) is called the trivial representation structure on V.

2. If $V = R$, then G-representation structures on V are called characters of G, and form a set denoted $\chi(G)$. Since $\mathrm{GL}(V) = \mathbb{G}_m$ is an abelian group scheme, the set $\chi(G)$ is given the structure an abelian group, for any G. The group $\chi(G)$ coincides with the grouplike elements of the Hopf algebra $\mathcal{O}_G(G)$ (a.k.a $\mathcal{O}(G)$ when G is affine). When context is clear, R will often denote the trivial character, which is otherwise a G-lattice structure on R.

3. If $G \xrightarrow{\phi} G'$ is any homomorphism of group schemes and V is a G' representation, by

some map $G' \to \mathrm{GL}(V)$, then the composition

$$G \xrightarrow{\phi} G' \to \mathrm{GL}(V)$$

is called the pullback of V along ϕ and is denoted $\phi^*(V)$.

4. If G_1, G_2 are group schemes over R, and V_i is a G_i-representation for $i = 1, 2$, then $V_1 \otimes V_2$ is naturally a $G_1 \times G_2$-representation. If $G_1 = G_2 = G$, we say the canonical G-representation structure on $V_1 \otimes V_2$ is the pullback $\Delta^*(V_1 \otimes V_2)$ along the diagonal $\Delta\colon G \to G \times G$. Then the tensor product $\otimes$ defines a symmetric monoidal structure on the category of G-representations and on the category of finite G-representations, for any group scheme G. Further, the faithful functor from (finite, all) G-representations to R-modules is given a natural structure of a symmetric monoidal functor. The tensor product as defined in this way agrees with the tensor product of $\mathcal{O}(G)$-comodules defined whenever G is affine, as before.

5. If G is affine and acts on an affine scheme X on the left, we get a left comodule structure $\mathcal{O}(X) \to \mathcal{O}(G) \otimes \mathcal{O}(X)$ on $\mathcal{O}(X)$, and hence a G-representation structure. This is called the regular representation given the action.
The action of $g \in G(A)$ is defined on $\mathcal{O}(X)_a(A) = A \otimes \mathcal{O}(X) = \mathcal{O}(X_A)$ by taking a natural map $f\colon X_A \to \mathbb{A}^1_A$ over A to $g \cdot f\colon X_A \to \mathbb{A}^1_A$, defined on A'-points by

$$g \cdot f(x) = f(g_{A'}^{-1} x),$$

whenever A' is a commutative A-algebra, with $x \in X_A(A')$, and with $g_{A'} \in G(A')$ the image of $g \in G(A)$ under $G(A \to A')$.

6. If G, X are as above, and $\mathcal{M}$ is any quasicoherent sheaf on X, we can identify $M \otimes \mathcal{O}(G) = \mathrm{Mor}(G, M_a)$, where $M = \mathcal{M}(X)$ is the module of global sections. By giving M the trivial representation, we define a representation structure formed by the tensor product $M \otimes \mathcal{O}(G)$ of representations.

7. If G is affine, then 'the left regular representation of G' will refer to the regular representation given the left action of G on itself by multiplication.

1.2.3 Restriction and induction

Suppose G is an affine group scheme and H is a subgroup scheme of G, with canonical inclusion $\iota: H \hookrightarrow G$.

1.2.3.1. If V is any G-module, then we define $V\downarrow_H^G = \iota^*(V)$, called the restriction of V to H.

1.2.3.2. Consider the action of H on G by left multiplication, and let V by any H-representation. We define a G-representation $V\Uparrow_H^G$, called the induced representation of V to G, as the G-subrepresentation of $V \otimes \mathcal{O}(G) = \mathrm{Mor}(G, V_a)$, given by

$$\{f: G \to V_a \mid f_A(gh) = h^{-1} f_A(g), \quad g \in G(A), h \in H(A)\}$$

Theorem 1.2.3.3. (Frobenius Reciprocity) *The functors*

$$\langle (?)\downarrow_H^G,\ (?)\Uparrow_H^G \rangle$$

form an adjoint pair. That is, there is a natural isomorphism

$$\mathrm{Hom}_H(V\downarrow_H^G, W) \cong \mathrm{Hom}_G(V, W\Uparrow_H^G)$$

for G-representations V, and H-representations W.

1.3 Frobenius kernels

The stacks project [STACKS, Section 0CC6, 'Frobenii'] will be our primary reference for the surrounding algebraic geometry necessary for this topic.

For smooth connected affine group schemes over a field of positive characteristic, Frobenius kernels give us a way of approximating the smooth group with finite connected group schemes, which are also known as 'infinitesimal group schemes'.

1.3.1 Frobenii

Suppose that S is a scheme over $\mathbb{F}_p$. Then there is a morphism of schemes

$$F_S\colon S \to S,$$

which is the identity of topological spaces, but with $F_S^\sharp\colon \sigma \mapsto \sigma^p$ for any local section $\sigma \in \mathcal{O}_S(U)$, for open $U \subset S$. Then if $X \to S$ is any morphism over $\mathbb{F}_p$, we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d] & X \arrow[d] \\ S \arrow[r, "F_S"'] & S \end{tikzcd}$$

of schemes over $\mathbb{F}_p$. For any scheme X over S, (with a structure morphism $X \to S$), a scheme $X^{(p)}$ over S is defined by base changing along F_S, so that we have a Cartesian square

$$\begin{tikzcd} X^{(p)} \arrow[r] \arrow[d] & X \arrow[d] \\ S \arrow[r, "F_S"'] & S \end{tikzcd}$$

of schemes over $\mathbb{F}_p$.

Definition 1.3.1.1. The two diagrams and the Cartesian property give us a unique morphism $F_{X/S}\colon X \to X^{(p)}$ of schemes over S such that the diagram commutes

$$\begin{tikzcd} X \arrow[dr, dashed, "F_{X/S}"] \arrow[drr, bend left, "F_X"] \arrow[ddr, bend right] & & \\ & X^{(p)} \arrow[r] \arrow[d] & X \arrow[d] \\ & S \arrow[r, "F_S"'] & S \end{tikzcd} .$$

The morphism $F_{X/S}\colon X \to X^{(p)}$ of schemes over S is called the relative Frobenius (over S).

Definition 1.3.1.2. For each $r > 1$, the iterated absolute Frobenius is defined recursively by $F_S^r = F_S \circ F_S^{r-1}\colon S \to S$. Then for any scheme $X \to S$ over S we have an iterated base change $X^{(p)^r}$ along F_S^r for each $r \geq 1$, and an iterated relative Frobenius $F_{X/S}^r\colon X \to X^{(p)^r}$.

Lemma 1.3.1.3. *The iterated relative Frobenius is natural, in the sense that whenever $f\colon X \to Y$ is a morphism of schemes over S, we have a commutative diagram*

$$\begin{tikzcd} X \arrow[r, "F^r_{X/S}"] \arrow[d, "f"'] & X^{(p)^r} \arrow[d, "f^{(p)^r}"] \\ Y \arrow[r, "F^r_{Y/S}"'] & Y^{(p)^r} \end{tikzcd}.$$

Proposition 1.3.1.4. *The iterated relative Frobenius $F_{X/S}^r\colon X \to X^{(p)^r}$ over S is the composite*

$$X \xrightarrow{F_{X/S}} X^{(p)} \to \cdots \to X^{(p)^i} \xrightarrow{F_{X^{(p)^i}/S}} X^{(p)^{i+1}} \to \cdots \to X^{(p)^{r-1}} \xrightarrow{F_{X^{(p)^{r-1}}/S}} X^{(p)^r}$$

Proof. Follows by way of induction on r, directly applying Lemma 1.3.1.3, noting that $(X^{(p)^i})^{(p)} = X^{(p)^{i+1}}$. $\square$

1.3.2 Group schemes

Now we return to schemes over R, i.e. schemes over $S = \operatorname{Spec} R$. We assume that R is of characteristic p, so that S is a scheme over $\mathbb{F}_p$.

Proposition 1.3.2.1. *If G is any group scheme over S, then so is $G^{(p)}$, and the relative Frobenius $F_{G/S}\colon G \to G^{(p)^r}$ is a group homomorphism.*

Proof. Any base change of a group scheme is again a group scheme over the new base. The rest follows from Lemma 1.3.1.3, applied to the morphism $m\colon G \times G \to G$ of schemes over S defining multiplication for G. $\square$

Definition 1.3.2.2. If G is any group scheme over S, we let G_r or $G_{(r)}$ denote the 'r^{th} Frobenius kernel', defined by the kernel of the homomorphism $F_{G/S}\colon G \to G^{(p)^r}$.

Proposition 1.3.2.3. *Suppose k is a field of characteristic $p > 0$, and G is a group scheme over k.*

1. *Each Frobenius kernel G_r is connected for $r \geq 1$.*

2. *If G is locally of finite type over k, then G_r is finite over k.*

3. *If G is smooth, locally of finite type over k, and of dimension ℓ, then the algebra $\mathcal{O}(G_r)$ is a $p^{r\ell}$-dimensional vector space over k.*

4. *If G is smooth, locally of finite type over k, and of dimension ℓ, and if either G is affine or k is a perfect field, then the algebra $\mathcal{O}(G_r)$ is isomorphic to*

$$k[y_1, \dots, y_\ell]/(y_i^{p^r})_{i=1,\dots,\ell}.$$

Proof. By [STACKS, Lemma 0CCB], and Proposition 1.3.1.4, the relative Frobenius

$$F^r_{G/k} \colon G \to G^{(p)^r}$$

is a universal homeomorphism. Therefore, since G_r is defined as a kernel, i.e. a fiber

$$\begin{tikzcd} G_r \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & G \arrow[d, "F^r_{G/k}"] \\ \operatorname{Spec} k \arrow[r, "e"] & G^{(p)^r} \end{tikzcd},$$

we get a (universal) homeomorphism $G_r \to \operatorname{Spec} k$. In particular G_r is topologically a point, and hence connected, proving claim 1.

Now suppose that G is locally of finite type, and let $U = \operatorname{Spec} A$ be an affine open neighborhood of the identity, with A a finitely generated k-algebra. By Lemma 1.3.1.3, we have a Cartesian square

$$\begin{tikzcd} G_r \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, "F^r_{U/k}"] \\ \operatorname{Spec} k \arrow[r, "e"] & U^{(p)^r} \end{tikzcd}.$$

Since A is finitely generated algebra over k, so is the base change $A^{(p)^r} = \mathcal{O}(U^{(p)^r})$, so let $x_1, \dots, x_\ell \in A$ be a generating set. Let $\varepsilon \colon A^{(p)^r} \to k$ be the associated ring homomorphism to

$e\colon \operatorname{Spec} k \to U^{(p)^r}$. Then the cocartesian square in the category of commutative k-algebras lets us conclude that

$$\mathcal{O}(G_r) \cong A/((x_i)^{p^r} - \varepsilon(x_i))_{i=1,\dots\ell},$$

which is a vector space of dimension $\le p^{r\ell}$, proving claim 2.

If, in addition, we assume that G is smooth and of dimension ℓ, then the generators $x_1, \dots, x_\ell \in A$ can be chosen to also be a basis for the tangent space of G at the identity. It follows that

$$\mathcal{O}(G_r) \cong k[x_1, \dots, x_\ell]/((x_i)^{p^r} - \varepsilon(x_i))_{i=1,\dots\ell},$$

proving claim 3.

If, in addition, k is perfect, then a simple coordinate change $y_i = x_i - a_i$, such that $a_i^{p^r} = \varepsilon(x_i)$, proves claim 4 in this case. If instead, we have in addition that G is affine, then G admits a faithful representation over k. Then it suffices to show claim 4 in the case of GL_n, with $\ell = n^2$, from which it follows that $\varepsilon(x_i) \in k$ belongs to a perfect subfield of k so that the same coordinate change is possible. Indeed, for GL_n, the generators x_i can be chosen such that each $\varepsilon(x_i)$ is either 0 or 1. This concludes the proof of claim 4. □

1.3.3 Infinitesimal group schemes

Here we'll discuss an important class of finite group schemes arising from Frobenius kernels.

Definition 1.3.3.1. Let k be a field. A group scheme G over k is called infinitesimal if it is finite and connected, as a scheme over k.

Proposition 1.3.3.2. *Let k be a field of characteristic 0, and let G be an infinitesimal group scheme over k. Then G is the trivial group scheme.*

Proof. By [Wat79], a finite group scheme over k is étale, since k is of characteristic 0. Any scheme which is connected, and étale over a field k is necessarily isomorphic to $\operatorname{Spec} k'$ for some finite separable field extension k'/k. Since group schemes over k always have a k-point, we conclude $G \cong \operatorname{Spec} k$ as schemes, so G is the trivial group scheme. □

Now assume that k is of positive characteristic $p > 0$. By Proposition 1.3.2.3, we know that G_r is then infinitesimal whenever G is a group scheme which is locally of finite type over k.

Proposition 1.3.3.3. *Let G be infinitesimal over k. Then there exists $r \geq 0$ such that $G_r = G$.*

Proof. A scheme is finite and connected over k only if its coordinate ring is a local ring which is finite dimensional over k. Then the Jacobson radical of $\mathcal{O}(G)$ is also the augmentation ideal, and is nilpotent, hence some iterated Frobenius $\mathcal{O}(G^{(p)^r}) \to \mathcal{O}(G)$ annihilates the Jacobson radical of $\mathcal{O}(G^{(p)^r})$. We conclude that the iterated Frobenius $F^r_{G/S}: G \to G^{(p)^r}$ is equal to the trivial map of group schemes, so its kernel G_r is all of G. □

Definition 1.3.3.4. Let G be infinitesimal over k. The height of G is defined to be the smallest r such that $G_r = G$.

Proposition 1.3.3.5. *Let G be a finite abelian group scheme over k. Then G is infinitesimal if and only if the Cartier dual $G^\#$ is unipotent.*

Proof. A *finite* scheme over k is connected if and only if its coordinate algebra is a local ring. Therefore, G being finite and abelian, we have that G is connected if and only if G is infinitesimal, if and only if $\mathcal{O}(G) = \mathcal{O}(G^\#)^\vee$ is local, if and only if $G^\#$ has a unique simple representation (representations of any finite group scheme H correspond to modules over $\mathcal{O}(H)^\vee$). □

Example 1.3.3.6.

1. Frobenius kernels of algebraic groups are infinitesimal, as shown in Proposition 1.3.2.3.

2. Take $\Gamma = \mathbb{Z}/p^r$. Then the finite group scheme $\underline{\Gamma}$ over k is abelian and unipotent, but not connected for $r > 0$. We then have that the Cartier dual $G = \underline{\Gamma}^\#$ is infinitesimal. Indeed, the Cartier dual of any constant group scheme $\underline{\mathbb{Z}/n}$ over any field is isomorphic to the group scheme μ_n, and over k, we then have $G \cong \mu_{p^r} = \mathbb{G}_{m(r)}$ is infinitesimal of height r.

3. Let $\mathbb{W}_n$ denote the unipotent smooth algebraic group of 'length n Witt vectors' over k [DG70]. Then any Frobenius kernel $\mathbb{W}_{n(r)}$ is abelian, unipotent, and infinitesimal of height r. The Cartier dual $\mathbb{W}_{n(r)}$ is then also abelian, unipotent, and infinitesimal of some height. The theory of Dieudonné modules lets us actually conclude that

$$\mathbb{W}^{\#}_{n(r)} \cong \mathbb{W}_{r(n)},$$

from the relative Frobenius on $\mathbb{W}_n$ being 'dual' in some sense to a shifting operation on Witt vectors, otherwise known as 'Verschiebung' (German for 'shift'), defined $V_{\mathcal{A}}: \mathcal{A}^{(p)} \to \mathcal{A}$ for general abelian algebraic groups $\mathcal{A}$ over k. When $n = 1$, we have $\mathbb{W}_1 = \mathbb{G}_a$, and $\mathbb{W}_{1(r)} = \mathbb{G}_{a(r)}$ is often denoted α_{p^r}.

1.3.4 Restricted Lie algebras

Now we review Lie algebras, and show how 'restricted Lie algebras', due to N. Jacobson [Jac41], also known as 'Lie p-algebras', form a category, and the category of finite-dimensional restricted Lie algebras over a field are equivalent to infinitesimal group schemes of height 1.

1.3.4.1. Let A be any coalgebra over a commutative ring R, with a 'coaugmentation' $u: R \to A$, with R the trivial coalgebra over R. We denote

$$P(A, u) = \{x \in A \mid \Delta(x) = x \otimes u(1) + u(1) \otimes x\}.$$

The elements in $P(A)$ are called primitive (respective of the coaugmentation). Immediately, $P(A)$ is an R-submodule of A. If $m, m': A \otimes A \to A$ are two homomorphisms of coalgebras, such that u is unital with respect to m, m' as bilinear products, then we have

$$m(x \otimes y) - m'(x \otimes y) \in P(A)$$

whenever $x, y \in P(A)$. The common use for this occurs when A is a bialgebra with given associative multiplication m, and taking m' to be the opposite multiplication $m'(x \otimes y) =$

$m(y \otimes x)$. Then $m(x \otimes y) - m'(x \otimes y)$ is usually denoted as a bracket operation

$$[x, y] = xy - yx.$$

In this case, we get $[x, x] = 0$ for any $x \in A$, and using that m is associative, we get the the Jacobi identity

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$$

for any $x, y, z \in A$, defining both A and its submodule $P(A)$ to be a Lie algebra over R. If A is a bialgebra and R is of characteristic p, then it's also true that x^p is primitive whenever x is primitive.

If A is any associative algebra over a commutative ring R, then $\mathfrak{Lie}(A)$ denotes the Lie algebra over R with underlying module A and bracket defined by $[x, y] = xy - yx$.

1.3.4.2. Let $\mathfrak{g}$ be a Lie algebra over a commutative ring R. Recall a representation of $\mathfrak{g}$ is given by an R-module V with the structure of a homomorphism $\mathfrak{g} \to \mathfrak{Lie}(\mathrm{End}_R(V))$. The 'adjoint representation' is defined to have underlying R-module $\mathfrak{g}$, with

$$\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{Lie}(\mathrm{End}_R(\mathfrak{g}))$$

defined by $\mathrm{ad}(x)(y) = [x, y]$.

Definition 1.3.4.3. Let $\mathfrak{g}$ be a Lie algebra over a commutative ring R of prime characteristic $p > 0$. A restriction on $\mathfrak{g}$ is a skew-linear map $[p]\colon \mathfrak{g} \to \mathfrak{g}$, given on elements using the notation $x \mapsto x^{[p]}$, satisfying the following axioms:

1. (Skew-linear) $(tx)^{[p]} = t^p x^{[p]}, \quad \forall\ t \in R, x \in \mathfrak{g}$,

2. $\mathrm{ad}(x^{[p]}) = (\mathrm{ad}\, x)^p \quad \forall\ x \in \mathfrak{g}$,

3. $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} \frac{\sigma_i(x,y)}{i}$, where each $\sigma_i(x, y) \in \mathfrak{g}$ is given by the coefficient of τ^{i-1} in the formal expression $(\mathrm{ad}(\tau x + y))^{p-1}(x)$, for any $x, y \in \mathfrak{g}$.

A Lie algebra with a restriction is called a restricted Lie algebra, or a Lie p-algebra. Homomorphisms $\varphi\colon \mathfrak{g} \to \mathfrak{g}'$ of restricted Lie algebras are defined to be homomorphisms of Lie algebras satisfying $\varphi(x^{[p]}) = \varphi(x)^{[p]}$ for each $x \in \mathfrak{g}$.

1.3.4.4. The Poincaré–Birkhoff–Witt (PBW) theorem gives us, for any Lie algebra $\mathfrak{g}$ over a field k, a filtered k-algebra $U(\mathfrak{g})$, called the universal enveloping algebra of $\mathfrak{g}$, with associated graded $\operatorname{gr} U(\mathfrak{g}) \cong S^*(\mathfrak{g})$. The elements of $U(\mathfrak{g})$ are taken to be formal R-linear combinations of ordered words in an ordered basis for in $\mathfrak{g}$. In fact, $U(?)$ defines a functor from Lie algebras over k, to associative algebras over k, which is left-adjoint to $\mathfrak{Lie}(?)$. Therefore, given a vector space V, any representation structure $\mathfrak{g} \to \operatorname{End}_k(V)$ defines also a module structure $U(\mathfrak{g}) \to \operatorname{End}_k(V)$ on V, and vice-versa, so that representations of $\mathfrak{g}$ are equivalent to $U(\mathfrak{g})$-modules.

1.3.4.5. Given any associative algebra A over a field k of characteristic $p > 0$, we define $\mathfrak{Lie}^{[p]}(A)$ to be the restricted Lie algebra, with underlying Lie algebra $\mathfrak{Lie}(A)$, and with restriction defined by $x^{[p]} = x^p$. In [Jac41], the structures on $\mathfrak{Lie}^{[p]}(A)$ are shown to define a restricted Lie algebra, and for each restricted Lie algebra $\mathfrak{g}$, a filtered k-algebra $u(\mathfrak{g}) = U(\mathfrak{g})/\langle x^p - x^{[p]}\rangle_{x\in\mathfrak{g}}$ is defined such that $u(?)$ defines a functor, from restricted Lie algebras to associative algebras, which is left-adjoint to $\mathfrak{Lie}^{[p]}(?)$. The filtration on $u(\mathfrak{g})$ is such that the associated graded $\operatorname{gr} u(\mathfrak{g})$ is isomorphic to the homogeneous quotient $S^*(\mathfrak{g})/(x^p)_{x\in\mathfrak{g}}$ of the symmetric algebra.

1.3.4.6. Representations of a restricted Lie algebra $\mathfrak{g}$ over k are defined to be vector spaces V, together with a homomorphism $\mathfrak{g} \to \mathfrak{Lie}^{[p]}(\operatorname{End}_k(V))$. By the adjunction, representations of $\mathfrak{g}$ are modules over $u(\mathfrak{g})$. Representations of a restricted Lie algebra are sometimes called 'restricted representations' so as to distinguish them from representations of the underlying (unrestricted) Lie algebra.

1.3.4.7. Let V_1, V_2 be representations of a Lie algebra $\mathfrak{g}$ over a field k. Then $V_1 \otimes V_2$ is a representation via the Leibniz rule

$$x(v_1 \otimes v_2) = xv_1 \otimes v_2 + v_1 \otimes xv_2.$$

A classical consequence of the PBW theorem shows how $U(\mathfrak{g})$ is actually a Hopf algebra over k, with comultiplication $U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ defined by the map $\mathfrak{g} \mapsto \mathfrak{Lie}(U(\mathfrak{g}) \otimes U(\mathfrak{g}))$ given by $x \mapsto x \otimes 1 + 1 \otimes x$. The tensor product of $\mathfrak{g}$-representations then agrees with the tensor product of modules over the Hopf algebra $U(\mathfrak{g})$.

If k if of characteristic 0, then $P(U(\mathfrak{g})) = \mathfrak{g}$ as Lie algebras, and if A is any Hopf algebra with a monomorphism $\mathfrak{g} \hookrightarrow P(A) \subset \mathfrak{Lie}(A)$ of Lie algebras, the induced map $U(\mathfrak{g}) \to A$ is a monomorphism and a homomorphism of Hopf algebras. By noting that any Hopf algebra map takes primitive elements to primitive elements, we conclude that Lie algebras over a field are categorically equivalent to 'primitively generated Hopf algebras', i.e. those Hopf algebras which are generated by their subspace of primitive elements in characteristic 0.

1.3.4.8. If $\mathfrak{g}$ is a restricted Lie algebra, then the ideal $I = \langle x^p - x^{[p]} \rangle \subset U(\mathfrak{g})$ is a Hopf ideal, i.e., we have $\Delta(I) \subset I \otimes U(\mathfrak{g}) + U(\mathfrak{g}) \otimes I$, and $\varepsilon(I) = 0$, and $S(I) = I$. We conclude that $u(\mathfrak{g})$ is a Hopf algebra, and $\mathfrak{g} = P(u(\mathfrak{g}))$, as a restricted Lie subalgebra of $\mathfrak{Lie}^{[p]}(u(\mathfrak{g}))$. By the PBW theorem, if $\mathfrak{g}$ is of dimension n over k, then $u(\mathfrak{g})$ is of dimension p^n over k. We see similarly to above that finite dimensional restricted Lie algebras are equivalent to primitively generated finite dimensional Hopf algebras.

By construction, the Hopf algebra $u(\mathfrak{g})$ is 'cocommutative', so that $u(\mathfrak{g})^\vee$ is a commutative Hopf algebra. By the PBW filtration, the Hopf algebra $u(\mathfrak{g})$ has a unique simple sub-comodule, given by the linear span of the unit. This shows that the dual Hopf algebra $u(\mathfrak{g})^\vee$ is commutative, local, and finite dimension over k so $G = \operatorname{Spec} u(\mathfrak{g})^\vee$ defines an infinitesimal group scheme. The height of G must be one, and any infinitesimal group scheme G of height one must have $P(\mathcal{O}(G)^\vee)$ generate the dual Hopf algebra $\mathcal{O}(G)^\vee$. In this way there is an equivalence of categories between finite dimensional restricted Lie algebras over k and infinitesimal group schemes of height one over k.

1.4 Cohomology of finite flat group schemes

Throughout this section we assume that R is a commutative noetherian ring. Recall a finite group scheme G over R is an affine group scheme for which the coordinate algebra $\mathcal{O}(G)$ is finitely generated as a module over R. When G is finite and flat as a scheme, the

G-representations coincide with modules over the group algebra $RG = \mathcal{O}(G)^\vee$.

1.4.1 Graded rings and modules of cohomology

If G is a finite group scheme over R, then both abelian categories, of G-representations, and of finitely generated RG-modules have enough projectives and enough injectives. Thus, if V_1, V_2 are two G-representations, we have that the cohomology modules $\mathrm{Ext^i(V_1, V_2)}$ can be computed using either a projective resolution of V_1, or an injective resolution of V_2.

We will first work in a more general framework, which is cohomology rings and modules for augmented algebras.

1.4.1.1. Let A be an algebra over R and let M, M', M'' be modules over A. There is a product given by splicing of Yoneda extensions

$$\mathrm{Ext}^i(M', M'') \otimes_R \mathrm{Ext}^j(M, M') \to \mathrm{Ext}^{i+j}(M, M'').$$

This product is associative in the sense that A-modules form a category enriched over the symmetric monoidal category of graded R-modules, using 'total Ext'

$$\mathrm{Ext}^*(M, N) = \bigoplus_{i \geq 0} \mathrm{Ext}^i(M, N)$$

as morphisms. See e.g. [Eis95, Appendix A3].

1.4.1.2. By the above, we have each R-module $\mathrm{Ext}^*(M, M)$ of 'total endomorphisms' is a graded R-algebra, and each $\mathrm{Ext}^*(M_1, M_2)$ is a graded bimodule, with a 'natural left action' of $\mathrm{Ext}^*(M_2, M_2)$ and a 'natural right action' of $\mathrm{Ext}^*(M_1, M_1)$.

If R is given an A-module structure by an augmentation $A \to R$, and M is any module over A, then we denote $H^i(A, M) = \mathrm{Ext}^i(R, M)$, and similarly $H^*(A, M) = \mathrm{Ext}^*(R, M)$. So any $H^*(A, M)$ has a 'natural graded module structure' over $H^*(A, R)$.

1.4.1.3. If G is a finite flat group scheme over R, then representations of G coincide with modules over RG.

The hom and cohomology sets of representations will be denoted

$$\mathrm{Hom}_G(?,?), \quad \mathrm{Ext}^i_G(?,?)$$

if notation would otherwise not distinguish the sets from those of their underlying R-module.

The trivial lattice R is also a module defined by an augmentation $RG \to R$. In this case we denote

$$H^i(G,V) = H^i(RG,V) = \mathrm{Ext}^i_G(R,V),$$

for representations V, and similarly $H^*(G,V)$. Hence each $H^*(G,V)$ is a right graded module over $H^*(G,R)$.

Theorem 1.4.1.4. *Let $\varepsilon\colon A \to R$ be an augmentation of an R-algebra A which is projective over R. If ε is the counit of some Hopf algebra structure on A, then the graded algebra $H^*(A,R)$ over R is graded-commutative (general conventions R7.).*

Proof. If R is a field, this is well known (see e.g. [SA04]). Over a general commutative ring, the classical argument still holds, under the hypothesis that A is projective over R. That is, the Hopf algebra structure in this case gives the category of modules a monoidal product. If A is projective over R, this allows another product of extensions, as graded endomorphisms of the monoidal unit, to be defined. There is a law of exchange between the two products, such that a Hilton–Eckmann argument applies and shows that the two operations coincide and are graded-commutative. □

Corollary 1.4.1.5. *Let G be a finite flat group scheme over a commutative ring R. Then the ring $H^*(G,R)$ is graded-commutative.*

Proof. Since G is finite and flat, we let $A = RG$ and apply the theorem.

In fact, whenever G is affine and flat, this is still true, except that $H^*(G,R)$ is not something that we've defined, since representations are not modules over an algebra a priori, rather they are comodules over a Hopf algebra which is projective as a module over R. □

Example 1.4.1.6. Let $R = k$ be a field of characteristic p, and let $G = \underline{\mathbb{Z}/p}$. Then the group algebra kG is isomorphic to $k[x]/(x^p - 1) \cong k[t]/t^p$, with $t = x - 1$. If we let k denote the trivial G-representation, a periodic minimal resolution of k gives us that each $\operatorname{Ext}^i(k,k)$ is a one-dimensional vector space over k. Further, there exists a nonzero cohomology $\zeta \in \operatorname{Ext}^2(k,k)$ which is not nilpotent. If we take ξ to be a nonzero cohomology in degree 1, then $\xi\zeta^n$ is nonzero in degree $2n+1$, so that if $p \neq 2$, then

$$\operatorname{Ext}^*(k,k) \cong \frac{k\langle \xi, \zeta\rangle}{\xi^2, \xi\zeta + \zeta\xi}$$

as graded algebras. If $p = 2$, then we can show ξ^n is a nonzero scalar multiple of ζ and therefore

$$H^*(G,k) = \operatorname{Ext}^*(k,k) \cong k[\xi].$$

By the Künneth theorem, we have by induction on r that if $p \neq 2$, then

$$H^*(G^r,k) \cong \Lambda^*(\xi_1,\ldots,\xi_r) \otimes S^*(\zeta_1,\ldots,\zeta_r) = \frac{k\langle \xi_1,\ldots,\xi_r,\zeta_1,\ldots,\zeta_r\rangle}{(\xi_i\xi_j + \xi_j\xi_j, \xi_i\zeta_j + \zeta_j\xi_i)_{1\le i\le j\le r}},$$

and if $p = 2$, then

$$H^*(G^r,k) \cong k[\xi_1,\ldots,\xi_r].$$

1.4.2 Finite generation of cohomology

Let R be a noetherian commutative ring. Let A be an algebra over R which is projective as a module over R. Let $A \to R$ be an augmentation, and we'll suppose that the cohomology algebra $H^*(A,R)$ is graded commutative. Here we'll explore two properties on cohomology of A-modules that will allow for sheaf theoretic methods to help understand the categories of modules for the possibly noncommutative A.

Definition 1.4.2.1.

FG1 A has FG1 if the algebra $H^*(A,R)$ is noetherian.

FG2 A has FG2 if $H^*(A,V)$ is a noetherian right module over $H^*(A,R)$ for each A-module V which is finitely generated and flat as an R-module.

1.4.2.2. We assumed that $H^*(A,R)$ was a graded-commutative algebra over R, so the algebra A has FG1 if and only if $H^*(A,R)$ is a finitely generated algebra over R, and A has FG2 if each $H^*(A,V)$ is a finitely generated module over $H^*(A,R)$.

A recent theorem of van der Kallen:

Theorem 1.4.2.3. [Kal23] *If A is finite and flat over R and if $A \to R$ is the counit of a cocommutative Hopf algebra structure, then A has FG1 and FG2.*

Results on FG1 and FG2 go back to [Gol59], [Ven59], [Eve61], when R is a field and A is a group algebra $R\underline{G}$ of a finite group G (and its constant group scheme $\underline{G}$). Group algebras for arbitrary finite group schemes over a field were shown to have FG1 and FG2 first by Friedlander and Suslin [FS97].

1.4.3 Homogeneous primes and witnesses

So far we've seen that $H^* = H^*(G,R)$ is a finitely generated algebra over R, meaning we have a commutative graded subalgebra $H^\bullet$ (see general convention R9.) which is also a finitely generated algebra. Alas, finding the generators and relations for $H^\bullet$, to the extent of describing the geometry of the projective scheme $\operatorname{Proj} H$, or of the affine cone $\operatorname{Spec} H^\bullet$, is a very difficult problem in general. So we'll review some techniques for understanding the geometry directly from structural and representation-theoretic aspects of the finite flat group scheme G.

Definition 1.4.3.1. Let G be a finite flat group scheme over R, and let R' be a commutative R-algebra. Let $\mathfrak{p} \in |\operatorname{Proj} H(G,R)|$ be some homogeneous prime. A prewitness for $\mathfrak{p}$, over a commutative R-algebra R', consists of an augmented algebra $A \to R'$ over R', and a flat map

$$\alpha: A \to R'G_{R'}$$

of augmented R'-algebras such that the induced map

$$H^*(\alpha): H^*(G_{R'},R') \to \operatorname{Ext}^*_A(R',R')$$

has $\sqrt{\ker(H^*(\alpha))} = \mathfrak{p}$.

As we will see, in practice, we'll choose A with a particularly nice cohomology ring $\operatorname{Ext}^*_A(R', R')$, such that certain kinds of maps α have that $\sqrt{\ker(H^*(\alpha))}$ is automatically some prime in $|\operatorname{Proj} H(G, R)|$

Definition 1.4.3.2. Let G, R', $\mathfrak{p}$, $A \to R'$ as above. A prewitness $\alpha\colon A \to R'G_{R'}$ for $\mathfrak{p}$ over R' is called a witness (for $\mathfrak{p}$, over R') if, in addition, we have, for every G-lattice V over R, that $\alpha^*(V_{R'})$ is not projective over A if and only if

$$\ker(V \otimes ?) \subset \mathfrak{p} = \sqrt{\ker(H^*(\alpha))},$$

where $V \otimes ?$ is the map on extensions

$$V \otimes ?\colon \operatorname{Ext}_G(R, R) \to \operatorname{Ext}_G(V, V).$$

1.4.3.3. The witness condition on a prewitness α for $\mathfrak{p}$ has a geometric interpretation: when M is a finitely generated graded module over a commutative graded algebra S, the set

$$\begin{aligned} |\mathscr{V}(\operatorname{Ann}_S M)| &= \{\mathfrak{p} \in |\operatorname{Proj} S| : \operatorname{Ann}_S M \subset \mathfrak{p}\} \\ &= \{\mathfrak{p} \in |\operatorname{Proj} S| : M_{\mathfrak{p}} \neq 0\} \end{aligned}$$

is called the support of M over S. In the next section 1.4.4, we will review the relationship between the set $\{\mathfrak{p} : \ker(V \otimes ?) \subset \mathfrak{p}\}$ with the support of $H^*(G, V)$ as a right module over $H^*(G, R)$.

Example 1.4.3.4. Suppose $R = k$ is a field of characteristic $p > 0$, and let $G = \underline{(\mathbb{Z}/p)^r}$ be the constant group scheme over k. If $g_1, \ldots, g_r$ are the order p generators for $(\mathbb{Z}/p)^r$, then the group algebra kG is isomorphic to

$$\frac{k[g_1, \ldots, g_r]}{(g_i^p - 1)_{1 \le i \le j}} \cong \frac{k[t_1, \ldots, t_r]}{(t_i^p)_{1 \le i \le j}},$$

with $t_i = g_i - 1$. We gave an explicit calculation of the graded commutative ring $H^*(G, k)$ in Example 1.4.1.6 by reducing to the case of $r = 1$ and applying the Künneth theorem. The commutative graded algebra $H^\bullet(G, k)$, modulo nilpotents, is isomorphic to a polynomial algebra generated freely by elements $\zeta_1, \dots, \zeta_r$ in degree 2 when $p \neq 2$, and if $p = 2$, the algebra is freely generated by $\xi_1, \dots, \xi_1$ in degree 1. In any case, we have coordinates for the topological space

$$X = |\operatorname{Proj} H(G,k)| = |\operatorname{Proj} H(G,k)_{\mathrm{red}}| \cong |\mathbb{P}_k^{r-1}|.$$

Carlson's cyclic shifted subgroups [Car83] give examples of prewitnesses for the closed points of X, over the base field k. A cyclic shifted subgroup is by definition a subalgebra of kG generated by $t = a_1t_1 + \cdots + a_rt_r$ for some choice of elements $a_1, \dots, a_r \in k$, not all zero. In the case that k is algebraically closed, Carlson showed directly that the inclusion

$$k\langle t\rangle \cong k[t]/t^p \hookrightarrow kG$$

is a prewitness over k for the closed point $[a_1 : \cdots : a_r] \in X$ (using the coordinate system for X implied by the homeomorphism with $\mathbb{P}_k^{r-1}$ above). In this way, every closed point in X has a prewitness over k. It follows that, even if k is not algebraically closed, every closed point in X has a prewitness over an algebraic extension of k, and every point in X has a prewitness over some transcendental extension of k.

Carlson conjectured, in part, that the prewitnesses given by his cyclic shifted subgroups were indeed witnesses, having proven the 'only if' direction of Definition 1.4.3.2. Carlson's conjecture was proven by Avrunin and Scott [AS82].

Now we'll review a type of "representation theoretic" witness introduced by Friedlander and Pevtsova for finite group schemes over a field, which serves as the motivation for our definition of prewitness and witnesses in the first place. In [FPev05], 'p-points' were first defined, over the base field, with one goal of giving a description of the homogeneous variety associated to the graded cohomology algebra, but without direct reference to cohomology. In the erratum [FPev06], it was pointed out that not every p-point is a prewitness for some point in the variety, and 'abelian p-points' were defined as p-points with an extra condition that is sufficient to conclude the prewitness property, for some point in the variety. We will fastforward into [FPev07], where so-called 'π-points' are defined, over field extensions of the base, which give us witnesses for the non-closed points in the scheme.

Recall, over a field of characteristic 0, that finite group schemes are semisimple and have no positive degree cohomologies of their representations. Therefore there are no relevant homogeneous primes of the cohomology ring, so it's vacuously true that every prime has a witness.

Definition 1.4.3.5. [FPev07] Let $R = k$ be a field of characteristic $p > 0$, and let L/k be an extension of fields, and let G be a finite group scheme over k. A π-point over L is a flat map of algebras

$$\alpha\colon L[t]/t^p \to LG_L$$

which factors through the inclusion $LU \hookrightarrow LG_L$ of group algebras for some abelian unipotent subgroup scheme U of G_L.

The following restates a result of [FPev07], in the language of witnesses:

Proposition 1.4.3.6. *Let $R = k$ be a field of characteristic $p > 0$, and let L/k be an extension of fields, and let G be a finite group scheme over k. Every π-point over some field extension of k has that $\mathfrak{p} = \sqrt{\ker(H^*(\alpha))}$ is a prime ideal in $|\operatorname{Proj} H(G,k)|$, hence every π-point is a prewitness for some prime $\mathfrak{p}$. Furthermore, for every prime $\mathfrak{p} \in |\operatorname{Proj} H(G,k)|$, there exists a field extension L/k and a π-point over L which is a prewitness for $\mathfrak{p}$.*

Moreover, the witness property is also proven by [FPev07]:

Proposition 1.4.3.7. *Let $R = k$ be a field of characteristic $p > 0$, and let L/k be an extension of fields, and let G be a finite group scheme over k. If a π-point α is a prewitness for $\mathfrak{p}$ over L, then α is a witness for $\mathfrak{p}$ over L.*

Finally, we the restate the main result of [FPev07] in our language:

Theorem 1.4.3.8. *Let $\mathfrak{p} \in |\operatorname{Proj} H(G,k)|$, for a finite group scheme G over a field k of characteristic $p > 0$. Then there exists a field extension L/k and a π-point α over L such that α is a witness for $\mathfrak{p}$.*

Example 1.4.3.9. Carlson's cyclic shifted subgroups [Car83] are π-points (over k, i.e. p-points c.f. [FPev05]). If $R = k$ is a field of characteristic $p > 0$, and $G = (\mathbb{Z}/p)^r$, with generators as in Example 1.4.3.4, then G is a unipotent abelian group scheme. Therefore any flat map $\alpha\colon k[t]/t^p \to kG$ is a π-point over k, and the map taking t to some $a_1t_1+\cdots+a_rt_r \in kG$, with $a_1, \ldots, a_r \in k$ not all zero (i.e. the inclusion of a cyclic shifted subgroup) is always flat.

Let $J \subset kG$ be the Jacobson radical and let $z \in J^2$. Consider the map $\beta\colon k[t]/t^p \to kG$ defined by

$$t \mapsto a_1t_1 + \cdots + a_rt_r + z.$$

Then β is also a flat map, hence a π-point over k, and hence a witness for some $\mathfrak{p} \in |\operatorname{Proj} H(G,k)|$. We claim that α is also a witness for $\mathfrak{p}$. This follows from [FPev05, Proposition 2.2], which gives us, for any G-representation M over k, that $\alpha^*(M)$ is projective if and only if $\beta^*(M)$ is projective. Then the definition of witness, and Proposition 1.4.3.7, show that α and β must then witness the same prime. A converse can also be argued, that two π-points

$$\alpha, \beta\colon k[t]/t^p \to kG$$

over k witness the same prime only if there is some nonzero scalar $\gamma \in k$ such that $\alpha(t) - \gamma\beta(t) \in J^2$. Further, every π-point over k is in the form β for some $a_1, \ldots, a_r \in k$ not all 0, and some $z \in J^2$.

1.4.4 Support spaces over a field

Let G be a finite group scheme over $R = k$, a field of characteristic $p > 0$, let V be a G-lattice over k, i.e., a representation which is a finite dimensional vector space. We define two closed subspaces of $|\operatorname{Proj} H(G,k)|$ associated to V, and we review the argument using local cohomology due to Benson, Iyengar, and Krause [BIK08], showing that both spaces are the same.

Definition 1.4.4.1. A support datum for rep_G consists of a topological space X and an assignment σ of closed subspace $\sigma(W) \subset X$ to each lattice W such that:

Σ1. $\sigma(R) = X$.

Σ2. $\sigma(\Omega W) = \sigma(W)$ for any lattice W.

Σ3. $\sigma(W) = \varnothing$ if and only if W is a projective kG-module.

Σ4. If $0 \to W_1 \to W_2 \to W_3 \to 0$ is a short exact sequence of G-lattices over k, then $\sigma(W_2) \subset \sigma(W_1) \cup \sigma(W_3)$.

Σ5. If W_1, W_2 are lattices then $\sigma(W_1 \oplus W_2) = \sigma(W_1) \cup \sigma(W_2)$.

Definition 1.4.4.2. For $s \in S$ be a simple G-represention over k, and V any G-lattice V over k, denote $H^*(s, V) = \operatorname{Ext}^*_G(s, V)$. Then for any V, $H^*(s, V)$ is naturally a graded right module over $H^*(s, s)$.

Definition 1.4.4.3. Let V be a G-lattice over k. Recall how $\operatorname{Ext}^*(V, V)$ is an algebra over $H^*(G, k)$ via

$$V \otimes ?\colon H^*(G, k) = \operatorname{Ext}^*(k, k) \to \operatorname{Ext}^*(V, V),$$

and recall the notion of support for modules over a commutative graded ring (discussed in 1.4.3.3).

Let S denote a set of G-lattices such that any simple G-lattice is isomorphic to an element of S.

1. The simple support of V, denoted $\mathscr{X}_G(V)$, is the union over $s \in S$ of supports of the natural graded right-modules $H^*(s, V)$ over the commutative graded algebra $H^\bullet(G, k)$ (via $s \otimes ?$).

2. The big support of V, denoted $\operatorname{supp}_G(V)$, is $|\mathscr{V}(I)|$, where I is the homogeneous ideal given by the kernel of

$$H^\bullet(G, k) = \operatorname{Ext}^\bullet(k, k) \xrightarrow{? \otimes V} \operatorname{Ext}^\bullet(V, V).$$

1.4.4.4. The main result of [FPev07], stated as Theorem 1.4.3.8, allows the spaces $\operatorname{supp}_G(V)$ to be reconstructed from π-points, using the witness property. Two π-points α_1, α_2, say over field extensions $L_1, L_2/k$ respectively, are said to be equivalent if, for any G-lattice V over k, we have $\alpha_1^*(V_{L_1})$ is projective if and only if $\alpha_2^*(V_{L_2})$ is projective. The witness property for π-points says that α_1 and α_2 are equivalent if and only if they are witnesses for the same prime in $|\operatorname{Proj} H(G,k)|$. By Theorem 1.4.3.8, we have that each support space $\operatorname{supp}_G(V)$ is reconstructed (as a set) from the set of equivalence classes of π-points α, over some L, such that $\alpha^*(V_L)$ is not projective. Further, there is a "representation theoretic" Zariski topology on the set of equivalence classes of π-points, which reconstructs the support spaces $\operatorname{supp}_G(V)$ as topological spaces.

1.5 Reductive groups

We fix R to be an integral domain for all of Section 1.5. We will review some representation theory for reductive group schemes over R as found in [Jan03]. Reductive groups over an arbitrary base scheme were first defined by Demazure in [SGA3III], generalizing classical results of Chevalley. Over a field of characteristic 0, they are 'linearly reductive', meaning that all finite representations are a direct sum of simple representations. Over other fields, this fails to hold. Our primary reference for the structure theory and representation theory of reductive groups over integral domains is Jantzen [Jan03].

1.5.1 Unipotent group schemes

We take a recursive definition from [DG70]: unipotent group schemes over any commutative ring R are a class of affine group schemes over R that is closed under isomorphisms. Unipotent group schemes are built out of subgroup schemes of $\mathbb{G}_a$ in the sense that any subgroup scheme of $\mathbb{G}_a$ is declared unipotent, and any affine group scheme admitting a filtration with unipotent subquotients is also unipotent, and the class of unipotent group schemes is minimal in affine group schemes, subject to these properties. Over a field k, a group scheme is unipotent if and only if any simple representation is isomorphic to the trivial representation structure on k, and any base change of a unipotent group scheme is unipotent.

Proposition 1.5.1.1. *If $p > 0$ is prime and G is a p-group and R is an algebra over the prime field $\mathbb{F}_p$, then the constant group scheme $\underline{G}$ over R is unipotent.*

Proof. The constant group scheme over R is a base change of the constant group scheme over $\mathbb{F}_p$. Therefore it suffices to prove this over the prime field. By the class equation, any nonzero finite representation $G \to \mathrm{GL}_n(\mathbb{F}_p)$ of a p-group G over the finite field $\mathbb{F}_p$ has at least p fixed points so the constant group scheme is unipotent. □

Example 1.5.1.2. Suppose R is a commutative algebra over $\mathbb{F}_p$, and let $\eta \in R$. The closed subfunctor

$$U^\eta(A) = \{x \in \mathbb{G}_a(A) \mid x^p = \eta x\}$$

is a subgroup scheme over R, and U^η is then a finite unipotent group scheme over R. If $\eta = 0$, U^η is the Frobenius kernel $\mathbb{G}_{a(1)}$ (1.3). If $\eta = 1$ then U^η is isomorphic to the constant group scheme $\underline{\mathbb{Z}/p}$. This example is revisited in Section 2.3.1.

1.5.2 Definition and examples of reductive groups

Definition 1.5.2.1. Let G a be an affine group scheme over an algebraically closed field k, and assume G is smooth and connected. The unipotent radical of G, denoted $\mathrm{rad}^u(G)$, is the largest unipotent smooth connected normal subgroup scheme of G.

Note: The unipotent radical exists and is unique, for any G over k defined in 1.5.2.1.

Definition 1.5.2.2. An affine group scheme G over R is reductive if the geometric fiber $\overline{\kappa(\mathfrak{p})} \times G$ has trivial unipotent radical for each $\mathfrak{p} \in \operatorname{Spec} R$.

Example 1.5.2.3.

1. Let k be an algebraically closed field and V a finite dimensional vector space over k. Then the group scheme $\mathrm{GL}(V)$ over k is reductive, because any normal solvable subgroup is central and hence diagonal. It follows that if R is any commutative ring and M any finitely generated R-module, that the group scheme $\mathrm{GL}(M)$ over R is a reductive group.

2. Suppose the canonical $\mathrm{GL}(V)$-representation V is of dimension n. Then the exterior power $\Lambda^n V$ is one-dimensional, and there is a natural map $\mathrm{GL}(V) \to \mathrm{GL}(\Lambda^n V) \cong \mathbb{G}_m$, making $\Lambda^n V$ a representation called the 'determinant representation' of $\mathrm{GL}(V)$. Define

$\mathrm{SL}(V)$ to be the kernel of the determinant $\mathrm{GL}(V) \to \mathrm{GL}(\Lambda^n V)$. Since $\mathrm{SL}(V)$ is a normal subgroup of $\mathrm{GL}(V)$, again its normal solvable subgroups are diagonal. Then the group scheme $\mathrm{SL}(V)$ over k is reductive. We conclude that if M is any finitely generated projective R-module, that $\mathrm{SL}(M)$ is reductive. For $n \geq 1$, we also write $\mathrm{SL}_n = \mathrm{SL}(R^n)$, a reductive group.

1.5.3 Structure theory: weights and root data

When G is a (split) reductive group over $\mathbb{Z}$, we use T to denote a choice of split maximal torus subgroup of G. We fix an isomorphism $T \xrightarrow{\sim} \mathbb{G}_m^\ell$ for some ℓ, and hence an isomorphism $\chi(T) \xrightarrow{\sim} \mathbb{Z}^\ell$, with canonical basis of elements ν_i. We then also have an isomorphism from the Grothendieck ring $K_0(T) \cong \mathbb{Z}[\chi(T)]$ of T-representations to the ring $\mathbb{Z}[\nu_1, \nu_1^{-1}, \dots, \nu_\ell, \nu_\ell^{-1}]$ of Laurent polynomials.

Definition 1.5.3.1. If V is a G-representation, the weight character, denoted $\operatorname{ch} V$, is the image in $\mathbb{Z}[\nu_1, \nu_1^{-1}, \dots, \nu_\ell, \nu_\ell^{-1}]$ of the restriction $V \downarrow_T^G$, as a T-representation in $K_0(T)$. Then ch defines a generalized character.

Example 1.5.3.2. Take G to be SL_2 from 1.5.2.3. The maximal torus T consists of diagonal matrices of determinant 1. Taking a 2×2 matrix to its upper-left entry is a well-defined isomorphism $T \xrightarrow{\sim} \mathbb{G}_m$, and a character ν in $\chi(T)$. Let $V = \mathbb{Z}^2$ be the canonical SL_2-representation (given by the embedding $\mathrm{SL}_2 \hookrightarrow \mathrm{GL}(V)$), and let $\nabla(n) = S^n(V)$ denote the symmetric power for each integer $n \geq 0$. Then the weight character $\operatorname{ch} \nabla(n)$ is also called a 'Weyl character', and is the Laurent polynomial given by

$$\sum_{i=0}^{n} \nu^{2i-n} = \frac{\nu^{n+1} - \nu^{-n-1}}{\nu - \nu^{-1}} = \nu^n + \nu^{n-2} + \cdots + \nu^{2-n} + \nu^{-n}.$$

Recall the support of a Laurent polynomial is the finite set of Laurent monomials having nonzero coefficient.

Definition 1.5.3.3. Let V be any G-representation. A weight of V is a character $\chi(T)$ with corresponding Laurent monomial in the support of $\operatorname{ch} V$. If $\lambda \in \chi(T)$ is any character, then we let V_λ denote the 'weight space' of V, i.e. the maximal T-subrepresentation of $V \downarrow_T^G$

which is isomorphic to some direct sum copies of λ. Then λ is a weight of V if and only if V_λ is not 0.

Definition 1.5.3.4. A root for G is a nonzero weight in $\chi(T)$ (i.e. a nontrivial character) of the adjoint representation $\mathfrak{g}$. The set of roots will be denoted Φ.

Example 1.5.3.5. The adjoint representation $\mathfrak{sl}_2$ of SL_2 is the space of trace-zero 2×2 matrices given the action of conjugation. We can directly check now, under our previous notation:

$$\operatorname{ch} \mathfrak{sl}_2 = \nu^2 + 1 + \nu^{-2},$$

so $\Phi = \{\nu^2, \nu^{-2}\}$, keeping multiplicative notation for $\chi(T)$.

1.5.3.6. If $\alpha \in \Phi$ is any root of G, the exponential $\mathfrak{g} \to G$ restricted to the weight space $\mathfrak{g}_\alpha$ gives us a root homomorphism

$$x_\alpha \colon \mathbb{G}_a \to G,$$

with the property on A-points

$$t x_\alpha(a) t^{-1} = x_\alpha(\alpha(t) a)$$

for any $t \in T(A)$ and $a \in \mathbb{G}_a(A)$. The image of x_α in G is called U_α, the 'root subgroup'.

1.5.3.7. We may define an inner product on Euclidean space $\chi(T) \otimes_{\mathbb{Z}} \mathbb{R}$, such that Φ is a root system. We won't explicitly use reductive groups not of type $\mathbf{A}_n$ in this thesis, so we will omit the details of this (consult [Jan03]). We'll merely note some indispensable related notions for general reductive groups.

Using additive convention for $\chi(T)$, we may write $\Phi = \Phi^+ \sqcup \Phi^-$, a disjoint union, for some subset $\Phi^+ \subset \Phi$ with $\Phi^- = -\Phi^+$. The subgroups U_α for $\alpha \in \Phi^+$ together generate a smooth, connected unipotent subgroup we call $U^+ < G$, and similarly U^-. Now we denote

$$B^+ = TU^+ = T \ltimes U^+, \quad B^- = TU^- = T \ltimes U^-,$$

the 'positive' and 'negative' Borel subgroups of G, respectively. Indeed, B^+, B^- are maximal solvable subgroup schemes of G. We also have that $U^{\pm} = \mathrm{rad}^u(B^{\pm})$, when taken over an algebraically closed field, so that the unipotent radical is defined.

Using the additive convention for $\chi(T)$, let

$$\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha \in \chi(T)\otimes_{\mathbb{Z}}\mathbb{Q}.$$

This will be important later. To increase the likelihood that the reader remembers this notation when it is needed, we'll note that ρ actually belongs to $\chi(T) \subset \chi(T)\otimes_{\mathbb{Z}}\mathbb{Q}$ whenever the center $Z(G)$ is a finite group scheme (G is then said to be semi-simple, see [Jan03]).

Example 1.5.3.8.

1. Let $G = \mathrm{GL}_2$. A maximal torus T is given by the 2×2 invertible diagonal matrices, and is isomorphic to $\mathbb{G}_m^2$. Let ν_1 be the projection of a diagonal matrix to its upper left entry, and ν_2 the projection to the lower right entry. Then the adjoint representation $\mathfrak{gl}_2$ is all 2×2 matrices, and has weight character

$$\mathrm{ch}\,\mathfrak{gl}_2 = \nu_1\nu_2^{-1} + 2 + \nu_1^{-1}\nu_2.$$

   We take $\Phi^+ = \{\nu_1\nu_2^{-1}\}$ and $\Phi^- = \{\nu_1^{-1}\nu_2\}$. Then U^+ consists of upper triangular matrices with ones along the diagonal, and similarly U^- is lower triangular. We also have B^+, consisting of all upper triangular matrices, and B^- all lower triangular matrices. The character ρ is given by $\nu_1^{\frac{1}{2}}\nu_2^{-\frac{1}{2}}$ (using multiplicative convention). It is easily verified that $Z(G)\cong\mathbb{G}_m$, which is not a finite group scheme, so G is not a semi-simple reductive group, and indeed $\rho\notin\chi(T)$.

2. Let $G = \mathrm{SL}_2$, a subgroup of SL_2 from the example above. From our work before, we have $\Phi = \{\nu^2, \nu^{-2}\}$, so we take $\Phi^+ = \{\nu^2\}$ and $\Phi^- = \{\nu^{-2}\}$. The subgroups $T, U^{\pm}, B^{\pm}, Z(G)$ of SL_2 are all intersections of SL_2 with the relevant subgroups of GL_2. In particular, $Z(G)\cong\mu_2$ (1.2.1.6). Indeed, $\rho = \nu\in\chi(T)$, and SL_2 is a semi-simple reductive group.

1.5.3.9. [Jan03, II, §1.5] The Weyl group scheme of G, denoted W_G, is the subquotient $N_G(T)/T$ of G. Then W_G is isomorphic to (the constant group scheme associated to) a Coxeter group associated with the root data, which we won't need in general. Instead, we will denote $W = W_G(\mathbb{Z})$, a finite group. Then W acts on Φ by acting on the root subgroups of G by conjugation. That is, for $w \in W$, $\alpha \in \Phi$, the action $w\alpha$ is defined such that

$$\dot{w} U_\alpha \dot{w}^{-1} = U_{w(\alpha)},$$

where $\dot{w} \in N_G(T)(\mathbb{Z})$ is a representative for the element

$$w \in W_G(\mathbb{Z}) = (N_G(T)/T)(\mathbb{Z}) \cong N_G(T)(\mathbb{Z})/T(\mathbb{Z}).$$

There is a unique $w_0 \in W$ such that $w_0(\Phi^+) = \Phi^-$.

The action of W on Φ agrees with a natural action of W on characters $T \to \mathbb{G}_m$ in $\chi(T)$, where $w \in W_G(\mathbb{Z})$ takes f to the map given on A-points

$$wf : t \mapsto f(\dot{w} t \dot{w}^{-1})$$

for $t \in T(A)$, for any commutative ring A. Since this additive action on $\chi(T)$ restricts to Φ, the action in terms of root subgroups is additive. This gives us certain commutator relations on the root subgroups corresponding to addition on the roots.

Example 1.5.3.10. It is a classical result that if $G = \mathrm{SL}_n$ or GL_n, the Weyl group W is isomorphic the symmetric group S_n. If $n = 2$, then the nontrivial permutation w_0 has a representative given by

$$\dot{w}_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in N_G(T)(\mathbb{Z}).$$

We have another representative given by

$$\ddot{w}_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in N_G(T)(\mathbb{Z}),$$

and in case $G = \mathrm{GL}_2$, we have two more representatives. Indeed, taking $\alpha \in \Phi^+$ (by previous conventions, we take $\alpha = \nu$ or $\alpha = \nu_1\nu_2^{-1}$), we have

$$\dot{w}_0 x_\alpha(a) \dot{w}_0^{-1} = \ddot{w}_0 x_\alpha(a) \ddot{w}_0^{-1} = x_{-\alpha}(-a),$$

given the root homomorphisms

$$x_\alpha \colon a \mapsto \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \qquad x_{-\alpha} \colon a \mapsto \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}.$$

Hence we have $\dot{w}_0 U_\alpha \dot{w}_0^{-1} = \ddot{w}_0 U_\alpha \ddot{w}_0^{-1} = U_{-\alpha}$.

1.5.4 Related combinatorial identities

We present two combinatorial identities that hold for Weyl characters for SL_2. These identities are purely in terms of Laurent polynomials and won't tell us much about the representation theory of SL_2 over, say, $\mathbb{Z}$. But once we change base to a field, we will see in Chapter 2 how these identities give us information about some representations called 'tilting modules', which depend on characteristic.

1.5.4.1. For each $n \geq 0$, denote the Weyl character as

$$[n] = \mathrm{ch}\, \nabla(n) = \sum_{i=0}^{n} \nu^{2i-n} \in \mathbb{Z}[\nu, \nu^{-1}].$$

Then we have the 'Weyl identity' for any $n \geq m \geq 0$:

$$\begin{aligned} [n][m] &= \sum_{i=0}^{m} [n+m-2i] \\ &= [n+m] + [n+m-2] + \cdots + [n-m]. \end{aligned}$$

1.5.4.2. Fix an integer $p \geq 1$ (in practice, p is prime), and suppose that

$$0 \leq m \leq n \leq p-1, \quad \text{and} \quad n+m \geq p-1.$$

From the Weyl identity, we also get the 'p-truncated Weyl identity'

$$[n][m] = [p-1][n+m+1-p] + \sum_{i=0}^{p-n-2} [n-m+2i].$$

Here, if $n = p-1$, the big sum is empty, interpreted as 0, and the identity holds in a trivial way.

Now we give a visual representation of the Weyl identity and the truncated Weyl identity as follows. We'll represent the Weyl character $[n]$ by a string of length n, i.e. a connected undirected graph with $n+1$ vertices, with no cycles, and with each vertex having degree ≤ 2.

1.5.4.3. The ordinary Weyl identity for $[n][m]$ can be seen by drawing a diagram of strings starting with a $(n+1)\times(m+1)$ rectangular lattice of vertices. The first string of length $n+m$ can be drawn along the upper-right boundary of the rectangle, so that the remaining vertices form a $n\times m$ rectangular lattice. Then repeat recursively by drawing strings along the upper-right boundary of the remaining rectangle until there are m distinct strings (assuming still that $m \leq n$) that cover the rectangular lattice. Notice, in this process, that the string drawn has a length which decreases by 2 after each recursive step. The resulting strings covering the rectangular lattice therefore correspond to the characters $[n+m-2i]$ appearing in the Weyl identity for $[n][m]$.

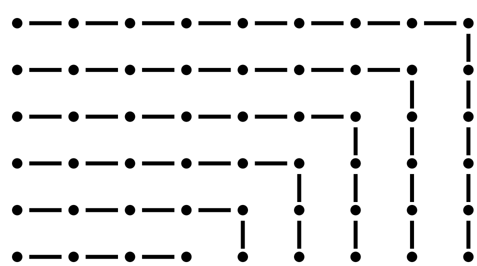


Figure 1.1: Weyl identity for [8][5]

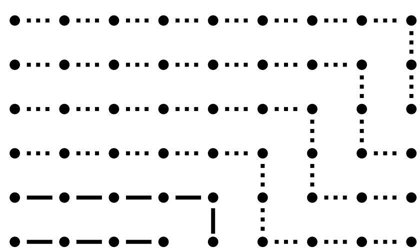
1.5.4.4. Our visual representation of the p-truncated Weyl identity will also be a covering of a $(n+1)\times(m+1)$ rectangular lattice by strings. For reasons that will become more clear when we look at some modular representation theory, we want all of the strings to be of length $\leq p-1$. For now it suffices to say that we are simply visualizing how to make the correct 'cutoff' in the non-truncated Weyl identity, so that we recover the index of $i = p-n-2$ in the formula for the p-truncated identity given above. We will assume that $0 \leq m \leq n \leq p-2$ to avoid the degenerate case where the truncation is trivial.

Strings of length $p-1$ will be dotted, and strings of length $< p-1$ will be solid black as usual. We begin by drawing a string along the upper-right boundary of the rectangular lattice, starting at the upper-left corner and drawing until a string of length $p-1$ is drawn. Make it dotted! Now the remaining vertices in the lattice won't form a rectangle unless $n+m = p-1$. Repeat recursively, starting at the upper-left corner of the remaining lattice, and drawing a string of length $p-1$ (dotted) along the upper-right boundary of the non-rectangular shape, until the remaining lattice *is* rectangular. Then the remaining rectangular lattice is $(n'+1)\times(m'+1)$, for $n' \geq m'$ such that the non-truncated Weyl identity gives

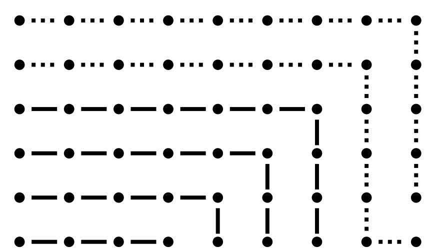
$$[n'][m'] = \sum_{i=0}^{p-n-2} [2(p-2)-(n+m)-2i] = \sum_{i=0}^{p-n-2} [n-m+2i].$$

Indeed, one finds $n' = p-m-2$ and $m' = p-n-2$. On the remaining $(n'+1)\times(m'+1)$ rectangular lattice, cover with solid strings as according to the algorithm in the non-truncated case. The resulting covering now has a 'dotted block' and a 'solid block'. The dotted block consists of $n+m+2-p$ many strings of length $p-1$, and represents the character $[p-1][n+m+1-p]$, which we will learn how to interpret using characteristic p representations in Chapter 2.

Warning: The dotted block of our visual p-truncated Weyl identity does *not* tell us that $[p-1][n+m+1-p] = (n+m+2-p)[p-1]$. Indeed, this is simply not an identity that holds in the ring of Laurent polynomials. Our algorithm should not give us more than a heuristic for reproducing the index $i = p-n-2$ appearing in the formula. After all, the p-truncated Weyl identity is just regrouping the terms of the ordinary Weyl identity in a way that depends on p.



For [8][5], with $p = 11$



For [8][5], with $p = 13$

Figure 1.2: p-Truncated Weyl identities

Chapter 2

TILTING MODULES

Conventions

Used throughout this chapter.

- We let k be a field.

- If G is an affine group scheme over a noetherian ring R, then full subcategory G-representations given by G-lattices over R will be denoted rep_G.

2.1 *For reductive groups*

In this section we let $G_{\mathbb{Z}}$ be a reductive group scheme over $\mathbb{Z}$ (1.5.2.2), we denote by $G_{R'}$ the base change of $G_{\mathbb{Z}}$ to a commutative ring R', and we set $G = G_k$. We fix $T < G$ a maximal torus, with roots $\Phi = \Phi^+ \sqcup \Phi^-$, and $U^\pm < G$ unipotent subgroup schemes of G with $B^\pm = TU^\pm \cong T \ltimes U^\pm$ a Borel subgroup scheme of G, and any root subgroup $U_\alpha < G$ for $\alpha \in \Phi^\pm$ is contained in $U^\pm$.

2.1.1 Induced modules and Weyl modules

Let $\pi\colon B^\pm \to T$ be the projection. Given any $\lambda \in \chi(T)$, we get the pullback $\pi^*(\lambda) \in \chi(B^\pm)$ as a B-representation.

2.1.1.1. We'll define a G-representation by

$$\begin{aligned}\Delta(\lambda) &= (\pi^*(\lambda))\Uparrow_{B^-}^{G}\\ &= \{\ f \in \mathcal{O}(G) \mid f_A(gb) = \lambda^{-1}(b) f_A(g), \quad g \in G(A), b \in B^-(A)\}.\end{aligned}$$

The $\lambda \in \chi(T)$ such that $\Delta(\lambda) \neq 0$ are called dominant weights, and coincide with with weights meeting a root-theoretic criterion depending on the inner product we have refrained from defining. The set of all dominant weights is denoted $\chi(T)_+$. The representation $\Delta(\lambda)$ will often be called the 'induced module of highest weight λ'.

2.1.1.2. Given $\lambda \in \chi(T)_+$, a dual notion, the 'Weyl module of highest weight λ' is also defined, by

$$\nabla(\lambda) = \Delta(-w_0\lambda)^\vee$$

One checks that the weight spaces $(\Delta(\lambda))_\lambda$ and $(\nabla(\lambda))_\lambda$ are both nonzero, and that for a root-theoretic ordering on characters, we have $(\Delta(\lambda))_\mu \neq 0$ implies that $\mu \leq \lambda$, and similarly for $\nabla(\lambda)$. This is what is meant by 'highest weight', and we'll continue to use this terminology for other representations, but we will explicitly describe the ordering only when, rather than defining the root-theoretic ordering.

Proposition 2.1.1.3. [Jan03] *Let $\lambda \in \chi(T)_+$. Then $\Delta(\lambda)$ and $\nabla(\lambda)$ are finite dimensional G-representions.*

2.1.2 Definition and general theory

Definition 2.1.2.1. [Jan03, II, §4.16] Let V be a G-representation. A chain of subrepresentations

$$0 = V_0 \subset V_1 \subset V_2 \subset \dots$$

is called a good filtration if $V = \bigcup_{i \geq 0} V_i$ and each V_i/V_{i-1} is isomorphic to a representation in the form $\Delta(\lambda_i)$ with $\lambda_i \in \chi(T)_+$.

Definition 2.1.2.2. [Jan03, II, §4.19] Let V be a G-representation. A chain of subrepresentations

$$0 = V_0 \subset V_1 \subset V_2 \subset \dots$$

is called a Weyl filtration if $V = \bigcup_{i \geq 0} V_i$ and each V_i/V_{i-1} is isomorphic to a representation in the form $\nabla(\lambda_i)$ with $\lambda_i \in \chi(T)_+$.

Definition 2.1.2.3. [Jan03] A G-representation V over R is called a tilting module if V has both a good filtration and a Weyl filtration.

Proposition 2.1.2.4. [Jan03] *For each $\lambda \in \chi(T)_+$, there exists an indecomposable tilting module $T(\lambda)$ of highest weight λ. If M is any tilting module of highest weight λ, then $T(\lambda)$ is isomorphic to some direct summand of M.*

Corollary 2.1.2.5. [Jan03, E.6] *If M is any tilting module, then there is a unique finite subset $X_M \subset \chi(T)_+$ and unique multiplicities $c_\lambda \in \mathbb{Z}_{>0}$ for $\lambda \in X_M$ such that there is a direct sum decomposition*

$$M = \sum_{\lambda \in X_M} c_\lambda T(\lambda).$$

It's also worth mentioning that the weight spaces

$$(\Delta(\lambda))_\lambda, (\nabla(\lambda))_\lambda, (T(\lambda))_\lambda$$

are always one-dimensional, for dominant weights $\lambda \in \chi(T)_+$.

Proposition 2.1.2.6. [Jan03, E.7] *The tensor product of tilting modules is again a tilting module.*

Proposition 2.1.2.7. [Jan03] *Let $\lambda \in \chi(T)_+$ be any dominant weight. Then the following are equivalent:*

1. *$\Delta(\lambda)$ is a simple G-representation.*
2. *$\Delta(\lambda) \cong \nabla(\lambda)$*
3. *$\Delta(\lambda)$ is a tilting module over G.*
4. *$\Delta(\lambda) \cong T(\lambda)$.*

2.2 SL_2

Let $G = \mathrm{SL}_2$ over k. Explicitly, G is defined by the functor of points

$$G(A) = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathrm{GL}_2(A) : a_{11}a_{22} - a_{12}a_{21} = 1 \right\}.$$

We also fix subgroup schemes $U, B, T < G$ by

$$U^- = \mathscr{V}(a_{11} - 1, a_{12}, a_{22} - 1), \quad B^- = \mathscr{V}(a_{12}), \quad T = \mathscr{V}(a_{12}, a_{21})$$

so that there is a split exact sequence of group schemes

$$0 \to U^- \to B^- \to T \to 0,$$

and isomorphisms $\mathcal{O}(T) \xrightarrow{\sim} k[t, \frac{1}{t}]$, and $\mathcal{O}(U) \xrightarrow{\sim} k[x]$, given respectively by $a_{11} \mapsto t$, and $a_{12} \mapsto x$. The group G is reductive, with $T \cong \mathbb{G}_m$ a maximal torus and with $U^+ = U$, and $B^+ = B$, and the induced isomorphism $\mathbb{G}_a \to U$ is canonically a root homomorphism for the positive root

$$2\colon a_{11} \mapsto a_{11}^2,$$

defined on points $T(A) \to \mathbb{G}_m(A)$.

2.2.1 The induced modules $\Delta(i)$

We can define a character

$$\lambda = a_{11}\colon T \to \mathbb{G}_m$$

and identify $\lambda = 1 \in \chi(T) = \mathbb{Z}$, with the usual ordering, so that $i = i\lambda = a_{11}^i$.

Recall the embedding $\mathrm{SL}_2 \hookrightarrow \mathrm{GL}_2$ defines a canonical representation structure on $V = k^2$. There is an isomorphism $S^i(V) \cong \Delta(i)$ for each symmetric power $S^i(V)$, for $0 \leq i \leq p-1$, or any $i \geq 0$ if $p = 0$ and also $\Delta(i) = 0$ whenever $i < 0$. This can be argued by identifying G/B^- with the projective G-variety by the restricted canonical action of GL_2 on $\mathbb{P}^1 = \mathbb{P}(V) = \operatorname{Proj} S^*(V^\vee)$. Then each induced module $\Delta(i)$ is a natural G-representation structure on the global sections of some equivariant structure on the line bundle $\mathcal{O}(i)$ over $\mathbb{P}^1 = \operatorname{Proj} S^*(V^\vee)$. Finally, by construction of a representative for w_0, we get $\nabla(i) = \Delta(i)^\vee$, so

$$\nabla(i) \cong S^i(V^\vee)^\vee \cong S^i(V)$$

whenever $i \geq 0$ and is 0 otherwise. What remains to show follows from the proposition:

Proposition 2.2.1.1. *Let $p \geq 0$ be the characteristic of k.*

1. *If $p = 0$, then $\Delta(i)$ is a simple G_k-representation of dimension $i + 1$ for each $i \geq 0$.*

2. *If $p > 0$, then $\Delta(i)$ is a simple G_k-representation of dimension $i+1$ for each $0 \leq i \leq p-1$.*

Proof. If $x < y$ is the canonical ordered basis for $V \cong \Delta(1)$, then an ordered basis for $S^i(V) \cong \Delta(i)$ is given by

$$x^i < x^{i-1}y < \ldots xy^{i-1} < y.$$

Consider the unipotent subgroups $U^+, U^- < G_k$ given respectively by upper and lower triangular matrices with ones along the diagonal. Then, as long as $p = 0$ or $i < p$,

$$0 \subset \langle x^i \rangle \subset \langle x^i, x^{i-1}y \rangle \subset \cdots \subset S^i(V) \downarrow^{G_k}_{U^+},$$

$$0 \subset \langle y^i \rangle \subset \langle xy^{i-1}, y^i \rangle \subset \cdots \subset S^i(V) \downarrow^{G_k}_{U^-}$$

are the *unique* composition series for the respective restricted modules of $S^i(V)$. Now take W to be any G_k-subrepresentation of $S^i(V)$, so that the restrictions of W to U^+, U^- are subrepresentations of the restrictions of $S^i(V)$. By uniqueness, the restrictions of W must appear as one of the terms in both of the composition series above. But the only vector spaces the two composition series have in common are 0 and $S^i(V)$. $\square$

We apply 2.1.2.7, and immediately we have

Corollary 2.2.1.2. *Let k be a field of characteristic $p \geq 0$, and consider the base change G_k.*

1. *If $p = 0$ then $\Delta(i) \cong \nabla(i) \cong T(i)$, the indecomposable tilting module of highest weight i, for each $i \geq 0$.*

2. *If $p > 0$, then $\Delta(i) \cong \nabla(i) \cong T(i)$ for each $0 \leq i \leq p - 1$.*

2.2.2 The Weyl identities revisited

We can now derive explicit identities using tilting modules of highest weight $< p$ (or of any highest weight, if $p = 0$) using the Laurent polynomial identities of 1.5.4 and the theory in 2.1.2.

2.2.2.1. By the uniqueness of the direct sum decomposition 2.1.2.5, tilting modules are uniquely identified by their weight character in $K_0(T) = \mathbb{Z}[\lambda, \lambda^{-1}]$. This is argued by induction on the highest weight.

Since tilting modules are closed under tensor product 2.1.2.6, we can use this to derive tensor-identities among tilting modules by looking only at identities of Laurent polynomials. In other words, if $T_1, \dots, T_n$ are any tilting modules and $f(x_1, \dots, x_n), g(x_1, \dots, x_n)$ are polynomials of with positive integer coefficients, such that

$$f(\operatorname{ch} T_1, \dots, \operatorname{ch} T_n) = g(\operatorname{ch} T_1, \dots, \operatorname{ch} T_n)$$

as Laurent polynomials in $K_0(T)$, then there exists an isomorphism

$$f(T_1, \dots, T_n) \cong g(T_1, \dots, T_n)$$

where a (positive coefficient) polynomial of representations is defined taking sum and product to be direct sum and tensor product of G-representations.

By Corollary 2.2.1.2 and Example 1.5.3.2, the weight character $\operatorname{ch} T(i)$ is the Weyl character $\operatorname{ch} \nabla(i)$ whenever $p = 0$ or $0 \le i < p - 1$. We conclude with the Weyl identity and p-truncated Weyl identity for SL_2 tilting modules.

Theorem 2.2.2.2. *Let $0 \le m \le n$ be integers.*

1. *If $p = 0$, then*

$$\begin{aligned} T(n) \otimes T(m) &\cong \sum_{i=0}^{m} T(n + m - 2i) \\ &\cong T(n + m) \oplus T(n + m - 2) \oplus \cdots \oplus T(n - m). \end{aligned}$$

2. If $p > 0$, and $n \leq p-1$, and $n+m \geq p-1$, then

$$T(n) \otimes T(m) \cong T(p-1) \otimes T(n+m+1-p) \oplus \sum_{i=0}^{p-n-2} T(n-m+2i).$$

If $n \leq p-1$ and $n+m \leq p-1$, then the usual Weyl identity holds

$$T(n) \otimes T(m) \cong \sum_{i=0}^{m} T(n+m-2i).$$

In general, $T(p-1) \otimes T(n+m+1-p)$ is not indecomposable (if it is, then it is isomorphic to $T(n+m)$). More thorough calculations can determine each weight character $\operatorname{ch} T(\lambda)$ for $\lambda \geq p$ and derive the decomposition of any $T(\lambda) \otimes T(\mu)$ into indecomposable representations as a result, using the same technique. We have left the identities as stated above because our only direct use for them is in the next section, where we restrict to some finite subgroup schemes of SL_2 in positive charactersitic.

2.3 Application: Jordan blocks and rigidity

For this section, we let k be an algebraically closed field of characteristic $p > 0$, and we let $R = k[\eta]$. Whenever X is an object over R (i.e. an algebra, coalgebra, scheme, module over an algebra, comodule over a coalgebra, structure morphisms thereof, homomorphisms of objects, etc.), for each $\eta_0 \in k$, we will denote by X_{η_0} the specialization of X to the point $(\eta - \eta_0) \in \operatorname{Spec} R$ (i.e. the base change to the residue field $R/(\eta - \eta_0)$).

2.3.1 Finite subgroup schemes of $\mathbb{G}_a$

Definition 2.3.1.1. Define a subgroup scheme $U^\eta < \mathbb{G}_a$ over R by the functor of points

$$U^\eta(A) = \{a \in \mathbb{G}_a(A) : a^p = \eta a\}.$$

2.3.1.2. We have assumed k is algebraically closed and therefore there are two isomorphism classes of specializations of U^η, as group schemes over k: We have U^η_0 is the first Frobenius kernel $\mathbb{G}_{a(1)} < \mathbb{G}_a$ over k, while any other specialization $U^\eta_{\neq 0}$ is isomorphic to the constant group scheme $\underline{\mathbb{Z}/p}$ over k.

2.3.1.3. We'll fix the coordinate algebra $\mathcal{O}(U^\eta)$ as $R[d]/(d^p - \eta d)$, a Hopf algebra over R such that d is primitive. This coordinate algebra is free over R, on a basis of monomials d^i, for $i = 0, \dots, p-1$. We'll let x denote the dual element to d with respect to the basis of monomials. A straightforward calculation shows that

$$RU^\eta = \mathcal{O}(U^\eta)^\vee \cong R[x]/x^p,$$

a cocommutative Hopf algebra, with comultiplication defined by

$$\Delta(x) = x \otimes 1 + 1 \otimes x + \eta x \otimes x.$$

2.3.1.4. As a finite group scheme, U^η-lattices over R coincide with modules over $RU^\eta = R[x]/x^p$ which are free of finite rank as R-modules (R is a principal ideal domain). The isomorphism classes of U^η-lattices over R then coincide with isomorphism classes of R-matrices with a p^{th} power of 0. We define a set of such lattices called Jordan blocks: for $i = 0, \dots, p-1$, let L^i denote the free R-module of rank $i+1$, equipped with U^η-lattice structure determined by x acting as the $(i+1) \times (i+1)$ upper nilpotent Jordan matrix

$$\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}.$$

Note: each L^i is an indecomposable representation of U^η, and is isomorphic to the symmetric power $S^i(L^1)$

2.3.1.5. As we discussed, each specialization of U^η is isomorphic to either $\mathbb{G}_{a(1)}$ or $\underline{\mathbb{Z}/p}$ over k. Either way the specialized group algebra $kU^\eta_{\eta_0}$ can be identified as the algebra $k[x]/x^p$. For this reason, representations of any specialization $U^\eta_{\eta_0}$ of U^η coincide with modules over an algebra which is independent of η_0. Moreover, $A = k[x]/x^p$ being a quotient of a principal ideal domain, we can see that every indecomposable A-module is isomorphic to some $A/(x^{i+1})$, which is in turn the specialization $L^i_{\eta_0}$ of our Jordan blocks from 2.3.1.4, for $0 \le i \le p-1$, again, independent of η_0. Therefore any finite dimensional representation of $\underline{\mathbb{Z}/p}$ or of $\mathbb{G}_{a(1)}$ over k is a specialization of a direct sum of the U^η-lattices L_i over R.

2.3.2 A rigidity theorem

The remainder of Section 2.3 will be towards the theorem:

Theorem 2.3.2.1. *There is an exact equivalence of abelian categories*

$$\Psi\colon rep_{\mathbb{G}_{a(1)}} \to rep_{\underline{\mathbb{Z}/p}},$$

and an isomorphism

$$\Theta^{n,m}\colon \Psi(L^n_0) \otimes \Psi(L^m_0) \to \Psi(L^i_0 \otimes L^j_0)$$

of functors $rep_{\mathbb{G}_{a(1)}} \times rep_{\mathbb{G}_{a(1)}} \to rep_{\underline{\mathbb{Z}/p}}$, for each n, m.

2.3.2.2. Construction of Ψ: We identify group schemes over k with specializations as follows: $\mathbb{G}_{a(1)} = U^\eta_0$ and $\underline{\mathbb{Z}/p} = U^\eta_1$. Then we define Ψ as the functor given by the composition

$$rep_{\mathbb{G}_{a(1)}} \to rep_{U^\eta} \to rep_{\underline{\mathbb{Z}/p}}.$$

The first functor $rep_{\mathbb{G}_{a(1)}} \to rep_{U^\eta}$ is the base change from modules over $k\mathbb{G}_{a(1)} = k[d]/d^p$ to modules over $R[d]/d^p$ which are free over R. The second functor $rep_{U^\eta} \to rep_{\underline{\mathbb{Z}/p}}$ being specialization of modules to the point $(\eta - 1) \in \operatorname{Spec} R$.

By construction, Ψ is exact, and can be checked on indecomposable representations L^i_0 to be fully faithful and essentially surjective, with $\Psi(L^i_0) = L^i_1$ for each $i = 0, \ldots, p-1$.

2.3.2.3. Let $G = \mathrm{SL}_2$ over k. By our work from 2.2.1, we have that each $T(i)$ is isomorphic to the symmetric power $S^i(V)$, for V the canonical G-representation structure on k^2. We identify the subgroup scheme $U^\eta < \mathbb{G}_a$ over R with a corresponding subgroup of $\mathbb{G}_a \cong U^+_R < G_R$. Then, for instance, we recover the group schemes $\mathbb{G}_{a(1)}$, $\underline{\mathbb{Z}/p}$, as subgroups of $U^+ < G$.

To be precise, we define a faithful, exact, symmetric monoidal functor

$$\mathrm{res}^{\eta}\colon rep_G \xrightarrow{?\otimes_k R} rep_{G_R} \xrightarrow{(?)\downarrow^{G_R}_{U^\eta}} rep_{U^\eta},$$

which we'll call restriction in abuse of terminology; now for each $\eta_0 \in k$, an honest restriction functor is defined by the composite

$$\mathrm{res}_{\eta_0}\colon rep\ G \xrightarrow{\mathrm{res}^\eta} rep\ U^\eta \xrightarrow{(?)_{\eta_0}} rep\ U^\eta_{\eta_0}.$$

We conclude that each U^η-lattice L^i is $\mathrm{res}^\eta(T(i))$, and each Jordan block, in the form $L^i_{\eta_0}$, is the restriction $\mathrm{res}_{\eta_0} T(i)$.

Lemma 2.3.2.4. *Let $0 \le m \le n \le p-1$. If $n+m \ge p-1$, then*

$$L^n \otimes L^m \cong (n+m+2-p)L^{p-1} \oplus \sum_{i=0}^{p-n-2} L^{n-m+2i}.$$

If $n+m \le p-1$, then

$$L^n \otimes L^m \cong \sum_{i=0}^{m} L^{n+m-2i}.$$

Proof. In the first case, by applying our functor res^η to Theorem 2.2.2.2, we get

$$L^n \otimes L^m \cong L^{p-1} \otimes L^{n+m+1-p} \oplus \sum_{i=0}^{p-n-2} L^{n-m+2i}.$$

Now notice that L^{p-1} is the unique indecomposable U^η-lattice which is a projective RU^η-module (it is isomorphic to the regular representation of the finite unipotent group scheme U^η). It follows that $L^{p-1}\otimes L \cong (\mathrm{rk}\, L)L^{p-1}$ for any U^η-lattice L of finite rank. Since $\mathrm{rk}\, L^i = i+1$ for $0 \le i \le p-1$, we're done.

The second case follows similarly, this time from directly applying res^η to Theorem 2.2.2.2, with no additional considerations. □

2.3.2.5. Construction of Θ: We start by defining a map

$$\Theta^{n,m}\colon \Psi(L^n_0) \otimes \Psi(L^m_0) \to \Psi(L^n_0 \otimes L^m_0)$$

for $0 \le m \le n \le p-1$, noting that by definition we have

$$L_0^i = \operatorname{res}_0 T(i)$$

for each i.

For such n, m, set

$$E^\eta(n,m) = (n+m+2-p)\operatorname{res}^\eta T(p-1) \oplus \sum_{i=0}^{p-n-2} \operatorname{res}^\eta T(n-m+2i)$$

if $n+m \ge p-1$, and

$$E^\eta(n,m) = \sum_{i=0}^{m} \operatorname{res}^\eta T(n+m-2i)$$

for $n+m \le p-1$. By Lemma 2.3.2.4, there is an isomorphism

$$\theta^{n,m}\colon \operatorname{res}^\eta T(n) \otimes \operatorname{res}^\eta T(m) \to E^\eta(n,m)$$

for each $0 \le m \le n \le p-1$.

Then for each such n, m, define

$$\Theta^{n,m}\colon \Psi(\operatorname{res}_0 T(n)) \otimes \Psi(\operatorname{res}_0 T(m)) \to \Psi(\operatorname{res}_0(T(n) \otimes T(m)))$$

by the composition

$$\Theta^{n,m} = \Psi(\theta_0^{n,m})^{-1} \circ \theta_1^{n,m},$$

where $\theta_{\eta_0}^{n,m}$ is the specialization of $\theta^{n,m}$, defined

$$\operatorname{res}_{\eta_0} T(n) \otimes \operatorname{res}_{\eta_0} T(m) \to E_{\eta_0}(n,m),$$

and $E_{\eta_0}(n,m) = (E^\eta(n,m))_{\eta_0}$, so that e.g.

$$E_{\eta_0}(n,m) = \sum_{i=0}^{m} \operatorname{res}_{\eta_0} T(n+m-2i)$$

when $n+m \le p-1$.

Since each $\theta^{n,m}$ is an isomorphism over U^η, we have $\theta^{n,m}_{\eta 0}$ is an isomorphism over $U^\eta_{\eta 0}$. In particular, both of $\Psi(\theta^{n,m}_0)^{-1}$, and $\theta^{n,m}_1$ are isomorphisms of U^η_1-representations. The composition $\Theta^{n,m}$ is well defined since $\mathrm{res}_1 = \Psi \circ \mathrm{res}_0$, as functors over the category of tilting modules $T(i)$ with all k-linear maps, for $0 \leq i \leq p-1$. Then $E_1(n,m) = \Psi E_0(n,m)$. Then each $\Theta^{n,m}$ is an isomorphism of U^η_1-represention for each n, m. We extend Θ to include pairs $\mathrm{res}_0 T(n), \mathrm{res}_0 T(m)$ for $0 \leq n \leq m$ as well, by defining $\Theta^{n,m} = b \circ \Theta^{m,n} \circ b$, with b the symmetric braiding.

2.3.2.6. Next chapter, especially Section 3.3, will give us the language to state the results of this section more succinctly. The finite group scheme U^η is an example of an $\mathbb{A}^1$-deformation of either of its specializations $\underline{\mathbb{Z}/p}$, $\mathbb{G}_{a(1)}$. In particular, U^η is a generic twist of $\underline{\mathbb{Z}/p}$ over $\mathbb{A}^1$, since it is isomorphic to the constant group scheme $\mathbb{A}^1$ away from $\eta = 0$. The main result of this section is then one of 'geometric K-rigidity'.

Chapter 3

K-RIGIDITY

Conventions

Used throughout this chapter.

1. Exact categories are as defined by Quillen [Qui73]. An exact monoidal category is an exact category equipped with a biexact monoidal functor.

2. If G is a finite group scheme over a field k.

   (a) We let $K_0(G)$ denote the Grothendieck ring of the exact monoidal category of finite kG-modules. That is, $K_0(G)$ is a quotient of the free abelian group generated by classes $[X]$ for finite kG-modules X, modulo relations in the form $[X'] + [X''] - [X]$ whenever

   $$0 \to X' \to X \to X'' \to 0$$

   is any exact sequence of kG-modules. Then $K_0(G)$ has a well-defined ring multiplication given by the tensor product.

   (b) We let $\mathfrak{R}(G)$ denote the representation ring, or 'Green ring' of G. That is, $\mathfrak{R}(G)$ is a quotient of the free abelian group generated by classes $[X]$ for finite kG-modules X, modulo relations in the form $[X'] + [X''] - [X]$ whenever

   $$0 \to X' \to X \to X'' \to 0$$

   is a *split* exact sequence of kG-modules. Then $\mathfrak{R}(G)$ has a well-defined ring multiplication given by the tensor product.

3.1 Preliminaries

Let G be a finite group scheme over a field k. Then the underlying additive category of kG-modules has two exact structures of note: one given by taking all exact sequences to be conflations, and one given by taking only the split exact sequences of kG-modules to be conflations. The latter has a Grothendieck group of $\mathfrak{R}(G)$, the representation ring of G, and the latter has a Grothendieck group of $K_0(G)$, following our conventions above. When possible, we will refrain from calling $\mathfrak{R}(G)$ a Grothendieck group, instead of a 'representation ring', to avoid confusion, despite Quillen's work being sufficiently general to view both $K_0(G)$ and $\mathfrak{R}(G)$ as algebraic K-theory.

3.1.1 A tale of two Grothendieck rings

As noted in our conventions, the tensor product of G-modules endows both $\mathfrak{R}(G)$ and $K_0(G)$ with the structure of a ring.

Proposition 3.1.1.1. *The underlying abelian group of $\mathfrak{R}(G)$ is generated freely by isomorphism classes of indecomposable finite kG-modules.*

Proof. This is an immediate consequence of the Krull–Schmidt theorem: for any finite module V, there is a direct sum decomposition

$$V \cong V_1 \oplus \cdots \oplus V_n,$$

for indecomposable V_i. Further, up to isomorphism and permutation of indecomposable direct sum factors, this decomposition is unique. Hence there is a unique way to write any class in $\mathfrak{R}(G)$ as an integer linear combination of isomorphism classes of indecomposable modules. □

Proposition 3.1.1.2. *The underlying abelian group of $K_0(G)$ is generated freely by isomorphism classes of simple finite kG-modules.*

Proof. This is a consequence of the Jordan–Hölder theorem: for any finite module V, there is a filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

called a composition series, with each 'composition factor' V_i/V_{i-1} a simple module. Further, up to isomorphism and permutation, the composition factors are unique. It is straightforward to see that for any exact sequence of finite kG-modules

$$0 \to V' \to V \to V'' \to 0$$

that the composition factors of V are a disjoint union of the composition factors of V' and of V''.

By inducting on the length of a finite module, we see then that each class in $K_0(G)$ is written uniquely as an integer linear combination of isomorphism classes of simple modules.

□

Proposition 3.1.1.3. *Suppose the simple kG-modules are given by $s_1, \ldots s_n$. For each finite kG-module V, let $c_i(V)$ denote the number of copies of s_i that appear as a composition factor for V. Then the map taking each V to $\sum_{i=1}^n c_i(V)[s_i]$ defines a surjective homomorphism*

$$\mathfrak{R}(G) \to K_0(G)$$

of rings.

Proof. First, kG is an artinian ring, since G is a finite group scheme over k, and hence there are indeed finitely many kG-modules. The Jordan–Hölder theorem tells us that the coefficients $c_i(V)$ are well-defined nonnegative integers for finite kG-modules V. Further, we have, by induction on the length of a module, for each finite kG-module V, an identity

$$[V] = \sum_{i=1}^n c_i(V)[s_i]$$

in the Grothendieck ring $K_0(G)$.

Therefore, our map $\mathfrak{R}(G) \to K_0(G)$ is identical to that which takes each generator class $[V] \in \mathfrak{R}(G)$ to $[V]$ as a class in $K_0(G)$. This is a well-defined homomorphism of rings, using a more general fact of exact monoidal categories: suppose $\mathcal{C}$ is an essentially small additive category, with a class of conflations $\mathcal{E}$ making $\mathcal{C}$ an exact category, and with a $\mathcal{E}$-biexact monoidal functor $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. Suppose $\mathcal{E}' \subset \mathcal{E}$ is a subclass of conflations which also makes $\mathcal{C}$ into an exact category. Then the Grothendieck groups $K_0(\mathcal{C}, \mathcal{E})$ and $K_0(\mathcal{C}, \mathcal{E}')$ both carry a ring structure given by $\otimes$ (since $\otimes$ is also $\mathcal{E}'$-biexact), and there is a canonical quotient homomorphism of rings

$$K_0(\mathcal{C}, \mathcal{E}') \to K_0(\mathcal{C}, \mathcal{E})$$

induced by taking any object X of $\mathcal{C}$ to its class $[X]$ in $K_0(\mathcal{C}, \mathcal{E})$. To prove our claim, we then only note that we can take $\mathcal{C}$ to be the category of finite kG-modules and $\mathcal{E}, \mathcal{E}'$ to be the classes of exact sequences, and of split exact sequences, respectively, with $\otimes$ the tensor product. □

3.1.1.4. By 3.1.1.2, the ring $K_0(G)$ is always finitely generated. By Proposition 3.1.1.1, $\mathfrak{R}(G)$ is finitely generated as an abelian group if and only if G is of 'finite representation type', i.e. there only finitely many indecomposable kG-modules, up to isomorphism. If G is of 'wild representation type', then it is necessarily the case that $\mathfrak{R}(G)$ is not finitely generated as a ring. We will see examples of finite group schemes G of 'tame representation type' where $\mathfrak{R}(G)$ is also not finitely generated as a ring. It's not straightforward to show that $\mathfrak{R}(G)$ being a finitely generated ring implies that G is of finite representation type.

Definition 3.1.1.5. A generalized character for G is a commutative ring C and a homomorphism $\mathfrak{R}(G) \to C$.

Example 3.1.1.6.

1. The homomorphism $\mathfrak{R}(G) \to K_0(G)$ of Proposition 3.1.1.3 is a generalized character. We call this the Jordan–Hölder character. Group schemes for which the Jordan–Hölder character is an isomorphism are said to be *linearly reductive*.

2. If $k = \mathbb{C}$, then finite group schemes are constant group schemes for some finite group. If n is the number of conjugacy classes of G, we identify $\mathbb{C}^n$ with the topological ring of class functions. Then classical character theory gives a (generalized) character

$\mathfrak{R}(G) \to \mathbb{C}^n$, taking each representation V to the class function taking an element of g to the trace of its action on V. The character $\mathfrak{R}(G) \to \mathbb{C}^n$ is moreover an embedding. This shows how the ring of representations is determined purely by character theory in this case.

3. Let k be algebraically closed of characteristic $p > 0$, and G a constant group scheme over k for some finite group. In this case, [BN41] constructed an embedding $K_0(G) \hookrightarrow \mathbb{C}^n$, where n is the number of elements of G, up to conjugation, which are of order coprime to p. Then the composition $\mathfrak{R}(G) \to K_0(G) \hookrightarrow \mathbb{C}^n$ is often called the Brauer character, and its image (which is isomorphic to $K_0(G)$) is sometimes called the ring of Brauer characters.

4. With k, G, p, n as above, if we suppose further that p is coprime to the order of G, by Maschke's theorem kG is semisimple and it follows that the Jordan–Hölder character $\mathfrak{R}(G) \to K_0(G)$ is an isomorphism (hence G is linearly reductive as a constant group scheme over k). In this case the Brauer character $\mathfrak{R}(G) \to \mathbb{C}^n$ is an embedding and in this way the ring of representations is again determined purely by character theory.

5. We always have a generalized character $\mathfrak{R}(G) \to \mathbb{Z}$ taking any representation to its dimension over k. The dimension character always factors through the Jordan–Hölder character. Suppose that U is a unipotent group scheme over k. Then the dimension character descends to an isomorphism $K_0(U) \xrightarrow{\sim} \mathbb{Z}$, and in this way the Jordan–Hölder character is equivalent to the dimension character.

6. Generalized characters into the ring $\mathbb{C}$ are called 'species' by Benson in [Ben24]. The Examples 2. and 3. of classical and of Brauer characters, give rise to species for each projection from the product $\mathbb{C}^n$. The semisimple Examples 2. and 4. are small examples of the commutative Banach algebras and representation rings studied in [Ben24].

3.1.2 Deformations of the tensor product

Here we will define deformations of the tensor product for representations of finite group schemes, noting some elementary considerations.

Definition 3.1.2.1. Let G be a finite group scheme over a field k. A *weak deformation* of G is a finite group scheme G' over k, together with an isomorphism of underlying augmented algebras $kG' \xrightarrow{\sim} kG$.

Thus, if G is a finite group scheme over k and $\varphi\colon kG' \to kG$ is a weak deformation, there is a canonical equivalence of categories $rep_G \to rep_{G'}$ given by pulling back any kG-module along φ to become a kG'-module. Further, this is an equivalence of 'pointed categories', since the pullback of the trivial G-representation is the trivial G'-representation, given that φ is a homomorphism of *augmented* algebras.

3.1.2.2. The theory of Tannakian categories affords us the perspective that (weak) deformations of finite group schemes are '(weak) deformations of the tensor product'. Specifically, if $A \to k$ is a finite augmented algebra over k, we consider the category A-*mod* of finite left A-modules as a k-linear pointed abelian category, with a faithful exact functor

$$\mathrm{For}\colon A\text{-}mod \to \mathrm{vec}_k,$$

the 'forgetful functor' to the category of vector spaces vec_k, taking any module to its underlying vector space over k. A 'Tannakian formalism' on A-*mod*, by For, consists of a rigid tensor category structure on A-*mod*, such that the trivial module k is the monoidal unit, and which is compatible with For in the sense that, given the canonical symmetric monoidal category structure on vec_k, there are additional structure maps that make For into a symmetric monoidal functor.

'Tannaka duality' then gives us that the Tannakian formalisms on A-*mod*, by For, all correspond to one which canonically arises from a cocommutative Hopf algebra structure on A, such that $A \to R$ is the counit. See [EGNO15].

In this way, a weak deformation $kG' \to kG$, is essentially the same thing as a change of symmetric monoidal structure on the category of modules over the underlying algebra kG, which is isomorphic to kG'.

Definition 3.1.2.3. Let G be a finite group scheme over a field k, and let $\varphi\colon kG' \to kG$ be a weak deformation.

I. Any weak deformation φ will also be 'type I'.

II. $\varphi\colon kG' \to kG$ is 'type II' if the pullback φ^* from kG'-modules to kG-modules induces an isomorphism $\chi(G) \xrightarrow{\sim} \chi(G')$ of character groups.

III. $\varphi\colon kG' \to kG$ is 'type III' if the pullback φ^* induces an isomorphism of Grothendieck groups $K_0(G) \to K_0(G')$.

IV. $\varphi\colon kG' \to kG$ is 'type IV' if, for each simple G-representation s, there exists a natural isomorphism

$$\varphi^*(s\otimes ?) \xrightarrow{\sim} \varphi^*(s) \otimes \varphi^*(?)$$

of functors from kG-modules to kG'-modules.

A *deformation* (or *strong deformation*, for emphasis) of G is a type IV weak deformation.

Lemma 3.1.2.4. *Let G be a finite group scheme over a field k.*

1. *Any deformation is type III.*

2. *Any type III weak deformation is type II.*

3. *If G is abelian and k is algebraically closed, then any type II weak deformation is a deformation.*

4. *If G is unipotent, then any weak deformation is a deformation.*

Or, diagrammatically:

$$\begin{tikzcd}
\mathrm{IV} \arrow[r, Rightarrow, "1."] & \mathrm{III} \arrow[r, Rightarrow, "2."] & \mathrm{II} \arrow[r, Rightarrow] \arrow[ll, Rightarrow, bend right, "3.\ \textit{abelian},\ k=\overline{k}"'] & \mathrm{I} \arrow[lll, Rightarrow, bend left, "4.\ \textit{unipotent}"]
\end{tikzcd}$$

Proof. Claim 1. follows from Proposition 3.1.1.2. Claim 2. follows from the fact that $\chi(G)$ is the group of (nonnegative) units of $K_0(G)$, for any finite group scheme G. Claim 4. holds because group schemes are unipotent if and only if they have only one simple module, up to isomorphism, and our definition of weak deformations having that φ is augmented, so the trivial module is a monoidal unit being pulled back to a monoidal unit.

Claim 3. requires some work: if G is abelian and k is algebraically closed, then any simple kG-module is a character. This can be seen from the fact that any maximal left ideal $\mathfrak{m} \subset kG$ is two-sided, so the quotient $kG/\mathfrak{m}$ is an algebraic field extension of k, and is hence 1-dimensional over k, being algebraically closed.

Now if $\varphi: kG' \to kG$ is a weak deformation, we have that G' is also abelian. Dualizing, we have the weak deformation gives an isomorphism between the underlying schemes for the Cartier duals of G, G'. Call the underlying scheme X, so that, as sets, the character groups $\chi(G)$, $\chi(G')$ correspond naturally with the connected components of the topological space $|X|$. Since φ is an isomorphism of augmented algebras, we can identify a point $e \in |X|$, thought of as the identity for either the Cartier dual of G or of G'.

If $|X|_e$ is the connected component of $e \in |X|$, then $|X|$ is equivalent to the product $|X|_e \times \chi(G)$ as a $\chi(G)$-space, where $|X|_e$ is given the trivial action and $\chi(G)$ is given the action by left multiplication. Similarly, $|X|$ is equivalent to $|X|_e \times \chi(G')$ as a $\chi(G')$-space.

If φ is type II, then our isomorphism $\varphi^*: \chi(G) \to \chi(G')$ tells us that the actions on $|X|$ are identical. Finally, we note that tensoring a G-representation with a character $g \in \chi(G)$ corresponds with the pushforward of some quasicoherent sheaf along the action $g: X \to X$. Hence we can make our canonical identifications giving the desired natural isomorphism. □

Notation 3.1.2.5. Often, we will use a symbol $\mathcal{G}$ to denote a groupoid of finite group scheme weak deformations.

Objects in $\mathcal{G}$ will be denoted by decorating the symbol G with one or many diacritics on the symbol G. For instance, we may have G^{ϕ} or G_{ϕ} for an arbitrary symbol ϕ, or $\hat{G}, \widetilde{G}, G', G'', \underline{G}$, as different finite group schemes which are objects in the groupoid $\mathcal{G}$.

The morphisms from G_1 to G_2 in the groupoid $\mathcal{G}$ will be denoted with squiggly arrows $G_1 \rightsquigarrow G_2$ and are by definition weak deformations $kG_1 \to kG_2$. The set of weak deformations

$G_1 \rightsquigarrow G_2$ is denoted by $\mathrm{Def}_{\mathrm{I}}(G_1, G_2)$. We say such categories G are 'deformation groupoids'.

3.1.2.6. We will usually construct our deformation groupoids G with a choice of base point $\widetilde{G}$, and such that G is connected.

Now for $f^{\phi}: G^{\phi} \rightsquigarrow \widetilde{G}$, $\hat{f}: \hat{G} \rightsquigarrow \widetilde{G}$, etc. we will denote 'relative Hopf algebra structure' by applying the relevant diacritic to the symbol Δ, to form

$$\Delta^{\phi}, \quad \hat{\Delta}, \quad \text{etc.} : k\widetilde{G} \to k\widetilde{G} \otimes k\widetilde{G}.$$

Our definition is then, for instance, $\Delta^{\phi} = (f^{\phi} \otimes f^{\phi}) \circ \Delta \circ (f^{\phi})^{-1}$ etc., where $\Delta: kG^{\phi} \to kG^{\phi} \otimes kG^{\phi}$ is abused notation to always be defined as the canonical.

We also denote the 'relative tensor product'

$$?\otimes^{\phi}?, \quad ?\hat{\otimes}?, \quad \text{etc.} : k\widetilde{G}\text{-mod} \times k\widetilde{G}\text{-mod} \to k\widetilde{G}\text{-mod},$$

defined by $?\hat{\otimes}? = (\hat{f}^{-1})^* \circ (\hat{f}^*(?) \otimes \hat{f}^*(?))$, etc.

Proposition 3.1.2.7. *Suppose $\varphi: G_{\phi} \rightsquigarrow \widetilde{G}$ is a (weak) deformation. Then $?\otimes_{\phi}?$ is the tensor product of modules over the noncanonical Hopf algebra structure $(k\widetilde{G}, \Delta_{\phi})$ on the underlying augmented algebra $k\widetilde{G}$.*

Proof. As an exercise, the reader should prove a version of this for arbitrary bialgebras, rather than limit themselves to finite dimensional cocommutative Hopf algebras.

We only note that in the case of finite cocommutative Hopf algebras this proposition is also stating that the pair $\Delta_{\phi}, \otimes_{\phi}$ is Tannaka dual. □

3.1.2.8. The set of type II, III, IV weak deformations $G_1 \rightsquigarrow G_2$ will be denoted

$$\mathrm{Def}_{\mathrm{II}}(G_1, G_2), \quad \mathrm{Def}_{\mathrm{III}}(G_1, G_2), \quad \mathrm{Def}_{\mathrm{IV}}(G_1, G_2),$$

respectively. It's straightforward to show that each type of deformation in 3.1.2.3 is closed under composition and hence we have 'deformation categories' as faithful subcategories of G, denoted Def_n for $n = \mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}$, and $G = \mathrm{Def}_{\mathrm{I}}$

By Lemma 3.1.2.4, we have canonical faithful embeddings of categories

$$\mathrm{Def}_{\mathrm{IV}} \subset \mathrm{Def}_{\mathrm{III}} \subset \mathrm{Def}_{\mathrm{II}} \subset \mathrm{Def}_{\mathrm{I}},$$

with $\mathrm{Def}_{\mathrm{I}} = \mathrm{Def}_{\mathrm{IV}}$ when restricted to unipotent group schemes as objects, and with $\mathrm{Def}_{\mathrm{II}} = \mathrm{Def}_{\mathrm{IV}}$ when restricted to abelian group schemes as objects.

3.1.3 Strong deformations preserve support

We will demonstrate a deep connection between deformations and support theory (1.4.4), as well as a connection to the theory of witnesses (1.4.3).

3.1.3.1. If $\varphi: G' \rightsquigarrow \widetilde{G}$ is a weak deformation of a finite group scheme $\widetilde{G}$ over k, then the pullback of modules gives an isomorphism of graded k-modules

$$\mathrm{Ext}^*_{\widetilde{G}}(V_1, V_2) \to \mathrm{Ext}^*_{G'}(\varphi^*(V_1), \varphi^*(V_2))$$

for any $\widetilde{G}$-representations V_1, V_2. These isomorphisms are compatible with yoneda splicing. Therefore we have an isomorphism

$$H^*(\varphi): H^*(\widetilde{G}, k) \to H^*(G', k)$$

of graded rings.

Further, we have, for each $\widetilde{G}$-representation V, that the graded right module structure $H^*(G', \varphi^* V)$ over $H^*(G', k)$ is isomorphic to the base change $H^*(\widetilde{G}, V) \otimes_{H^*(\widetilde{G},k)} H^*(G', k)$ along the homomorphism $H^*(\varphi)$.

Regarding $\widetilde{G}$ as a base point for a connected deformation groupoid G, we'll let $\mathscr{X}_G = \mathscr{X}_{\widetilde{G}}(k) = |\operatorname{Proj} H(\widetilde{G}, k)|$ and $\mathscr{X}_G(V) = \mathscr{X}_{\widetilde{G}}(V) \subset \mathscr{X}_G$ as well. So whenever $\varphi: G' \to \widetilde{G}$ is a weak deformation, we get a homeomorphism $\varphi_!: \mathscr{X}_{G'}(k) \to \mathscr{X}_G$ from an isomorphism of schemes, induced from the isomorphism $H^*(\varphi)$.

Theorem 3.1.3.2. *Let $\widetilde{G}$ be a finite group scheme over a field k and let $\varphi: G' \rightsquigarrow \widetilde{G}$ be a* *deformation. Then for any finite $k\widetilde{G}$-module V, we have*

$$\varphi_!(\mathscr{X}_{G'}(\varphi^*(V))) = \mathscr{X}_G(V)$$

as subspaces of $\mathscr{X}_G$.

Proof. Recall how the simple supports $\mathscr{X}_G(V)$ are defined by a union of supports of the right modules $H^*(s,V)$, as s ranges over simple $k\widetilde{G}$ modules. It is then sufficient to establish an equality

$$\varphi_!(\operatorname{supp} H^*(\varphi^*(s), \varphi^*(V))) = \operatorname{supp} H^*(s,V))$$

for each s. The structure of $H^*(\varphi^*(s),\varphi^*(V))$ is pulling back its natural right module structure over $H^*(\varphi^*(s),\varphi^*(s))$ via the tensor product $\varphi^*(s)\otimes ?\colon H^*(G',k) \to H^*(\varphi^*(s),\varphi(s))$.

Since deformations are assumed to be strong, we have a natural isomorphism

$$\varphi^*(s)\otimes ? \cong \varphi^*(s \otimes (\varphi^{-1})^*(?)),$$

and hence an equality when regarded as maps of cohomology modules.

Let $\mathfrak{p} \in \operatorname{supp} H^*(\varphi^*(s),\varphi^*(V))$. By definition of support, we get for each cohomology $\xi \in H^*(\widetilde{G},k)$ such that $\varphi^*(s\otimes\xi)$ annihilates $H^*(\varphi^*(s),\varphi^*(V))$, that $\varphi^*(\xi) \in \mathfrak{p}$. We want that $\varphi_!(\mathfrak{p})$ contains the annihilator of $H^*(s,V)$. Any cohomology $\xi \in H^*(\widetilde{G},k)$ annihilates $H^*(s,V)$ if and only if $s\otimes\xi$ annihilates $H^*(s,V)$, by the natural action. By naturality of the type IV condition for strong deformations, we know $s\otimes\xi$ annihilates $H^*(s,V)$ if and only if $\varphi^*(s\otimes\xi)$ annilitates $H^*(\varphi^*(s),\varphi^*(V))$. It follows that any cohomology ξ annihilating $H^*(s,V)$ has that $\varphi^*(\xi) \in \mathfrak{p}$, and hence $\xi \in \varphi_!(\mathfrak{p})$. We have demonstrated containment $\varphi_!(\operatorname{supp} H^*(\varphi^*(s),\varphi^*(V))) \subset \operatorname{supp} H^*(s,V))$, and the other containment follows by a similar argument. $\square$

One consequence of the theorem tells us that the support of a tensor product of arbitrary finite modules, not just those which are completely reducible, is actually well defined up to a deformation of the tensor product.

Corollary 3.1.3.3. *Let $\widetilde{G}$ be a finite group scheme over a field k, and let $G' \rightsquigarrow \widetilde{G}$ be a deformation. Let V_1, V_2 be any finite modules over $k\widetilde{G}$. Then*

$$\mathscr{X}_G(V_1 \widetilde{\otimes} V_2) = \mathscr{X}_G(V_1 \otimes' V_2)$$

as subspaces of $\mathscr{X}_G$.

Proof. For any weak deformation $\psi: \hat{G} \rightsquigarrow \widetilde{G}$, we'll define G_ψ to be Spec A, where A is the dual of the noncanonical Hopf algebra structure $(k\widetilde{G}, \hat{\Delta})$. Then a weak deformation $\varpi: G_\psi \rightsquigarrow \widetilde{G}$ exists as the identity $kG_\psi = k\widetilde{G}$, and various induced maps $\varpi^*, H^*(\varpi), \varpi_!$ are all identity too. We also have a weak deformation which which we'll abuse notation and call $\psi: \hat{G} \rightsquigarrow G_\psi$. Since $\psi: k\hat{G} \to kG_\psi$ is an isomorphism of Hopf algebras, it is type IV.

Each $\mathscr{X}_{G_\psi}(V)$ is now an honest subspace of $\mathscr{X}_G$, and we have

$$\mathscr{X}_{G_\psi}(V) = \psi_!(\mathscr{X}_{\hat{G}}(\psi^*(V)))$$

by the theorem. Therefore, if the original $\psi: \hat{G} \rightsquigarrow \widetilde{G}$, is type IV, we have

$$\mathscr{X}_{G_\varphi}(V) = \mathscr{X}_G(V)$$

for each finite representation V.

By the tensor product property of support (1.4.4.1, proved in [FPev07]), we have

$$\mathscr{X}_H(V_1 \otimes V_2) = \mathscr{X}_H(V_1) \cap \mathscr{X}_H(V_2)$$

for any finite group scheme H and any H-representations V_1, V_2. Since our V_1, V_2 are simultaneously representations of $\widetilde{G}, G_\varphi$, we have

$$\begin{aligned}
\mathscr{X}_G(V_1 \otimes' V_2) &= \mathscr{X}_{G_\varphi}(V_1 \otimes V_2) \\
&= \mathscr{X}_{G_\varphi}(V_1) \cap \mathscr{X}_{G_\varphi}(V_2) \\
&= \mathscr{X}_G(V_1) \cap X_G(V_2) = \mathscr{X}_G(V_1 \widetilde{\otimes} V_2).
\end{aligned}$$

□

We can also now prove a critical property of witnesses: for a prewitness, as a map into a group algebra, the witness property 1.4.3.2 only depends on the cocommutative Hopf algebra structure of a group algebra 'up to deformation', so-to-speak. Thus, for a unipotent group scheme, by Lemma 3.1.2.4, the witness property is completely independent of the cocommutative Hopf algebra structure.

Corollary 3.1.3.4. *Let $\widetilde{G}$ be a finite group scheme over a field k and let $G' \rightsquigarrow \widetilde{G}$ be a weak deformation. Let R be a commutative k-algebra and suppose $\alpha\colon A \to RG'_R$ is a prewitness for a prime $\mathfrak{p} \in \mathscr{X}_{G'}(k)$ over R. Then*

1. *The composition*
$$\varphi_R \circ \alpha\colon A \to RG'_R \to R\widetilde{G}_R$$
is a prewitness for $\varphi_!(\mathfrak{p})$, and

2. *If, in addition, φ is a* deformation*, and α is a* witness *for $\mathfrak{p}$, then the composition $\varphi_R \circ \alpha$ is a witness for $\varphi_!(\mathfrak{p})$.*

Proof. Of claim 1.: The composition $\varphi_R \circ \alpha$ is flat. Now, we find the kernel of the induced map $H^*(\varphi_R \circ \alpha) = H^*(\alpha) \circ H^*(\varphi_R)$. Since φ_R is an isomorphism of augmented algebras, $H^*(\varphi_R)$ is an isomorphism of graded algebras. So indeed, the kernel is the inverse image, under $H^*(\varphi_R)$, of $H^*(\alpha)$, and the radical is therefore $\varphi_!(\mathfrak{p})$, given that α is a prewitness for $\mathfrak{p}$.

Of claim 2.: consider a $k\widetilde{G}$-module V. Since α is a witness for $\mathfrak{p}$, we have we have $\alpha^*((\varphi^*(V))_R)$ is projective if and only if $\mathfrak{p} \in \mathscr{X}_{G'}(\varphi^*(V))$. By the theorem, this holds if and only if $\varphi_!(\mathfrak{p}) \in \mathscr{X}_G(V) = \operatorname{supp}_{\widetilde{G}}(V)$. It follows that the composition $\varphi_R \circ \alpha$ is a witness for $\varphi_!(\mathfrak{p})$, as
$$\alpha^*((\varphi^*(V))_R) = \alpha^*(\varphi_R^*(V_R)) = (\varphi_R \circ \alpha)^*(V_R).$$

□

3.2 Rigidity properties

3.2.1 Noble primes of cohomology

Definition 3.2.1.1. Let G be a finite group scheme over a field k. A closed point $\mathfrak{p} \in \mathscr{X}_G$ is called 'noble' if there exists a field extension L/k and a subgroup scheme $H < G_L$ such that the inclusion $LH \hookrightarrow LG_L$ is a witness for $\mathfrak{p}$ over L.

Example 3.2.1.2. Nobility is not preserved upon deformation. The cardinality of the set of noble points is an invariant for isomorphism classes of finite group schemes that distinguishes the isomorphism classes of all G admitting a deformation $G \rightsquigarrow \mathbb{G}_{a(1)}^2$, over an algebraically closed field. Here, since $\mathbb{G}_{a(1)}$ is unipotent, weak deformations are deformations. By a theorem of X. Wang [Wan13], included as A.2.0.4, the group schemes are each isomorphic to one of

$$\mathbb{G}_{a(2)},\, \mathbb{G}_{a(1)} \times \underline{\mathbb{Z}/p},\, \underline{(\mathbb{Z}/p)^2},\, \mathbb{G}_{a(1)}^2,$$

and we will show in 3.4.3.2 that they have one, two, $p-1$, or infinitely many noble points.

Example 3.2.1.3. Let k be an algebraically closed field of characteristic $p > 0$, and let $\widetilde{G}$ be an infinitesimal group scheme of height one, i.e. $k\widetilde{G}$ is generated by primitive elements. If L/k is a field extension, and $x \in L\widetilde{G}_L$ is any primitive element such that $x^p = 0$, then $L\langle x\rangle \cong L[x]/x^p$ is a Hopf subalgebra of $L\widetilde{G}_L$, and the inclusion $\alpha_x : k\langle x\rangle \hookrightarrow k\widetilde{G}$ is a π-point over L. Therefore α_x is a witness for some point $\mathfrak{p} \in X_{\widetilde{G}}$ over L, and $\mathfrak{p}$ is then noble, by construction. The results of Friedlander and Parshall [FPar86] and of Suslin, Friedlander, and Bendel [SFB97] show that any $\mathfrak{p} \in |\operatorname{Proj} H(\widetilde{G}, k)|$ is noble in this way.

Example 3.2.1.4. Suppose Γ is a finite group. If $p \nmid |\Gamma|$, then it is vacuous that every point in $|\operatorname{Proj} H(\Gamma, k)| = \varnothing$ is noble. Supposing that Γ has a subgroup isomorphic to $(\mathbb{Z}/p)^r$, $r > 0$, Quillen's stratification [Qui71] shows that the dimension of $|\operatorname{Proj} H(\Gamma, k)|$ is at least $r-1$. We conclude, again from Quillen's stratification, that if $p \mid |\Gamma|$, then $|\operatorname{Proj} H(\Gamma, k)|$ has at least one noble point. Further, every closed point in $|\operatorname{Proj} H(\Gamma, k)|$ is noble if and only if $r = 1$ is the largest r such that Γ has a subgroup isomorphic to $(\mathbb{Z}/p)^r$. As one consequence, every closed point in $|\operatorname{Proj} H(\Gamma, k)|$ is noble if and only if every point in $|\operatorname{Proj} H(S, k)|$ is noble for a Sylow p-subgroup $S < \Gamma$. Examples of p-groups Γ with this property include cyclic p-groups, and if $p = 2$, the generalized Quaternion groups of order 2^n, for $n \geq 3$.

Conjecture 3.2.1.5. *Let G be a finite group scheme over an algebraically closed field k of characteristic $p > 0$. Then every closed point in $\mathscr{X}_G$ is noble if and only if the largest value $r + s$ such that G has subgroup isomorphic to $\mathbb{G}_{a(r)} \times (\underline{\mathbb{Z}/p})^s$ is 0 or 1.*

The machinery of π-points shows one direction of the conjecture above: any point $\mathfrak{p} \in \mathscr{X}_G$ is witnessed by a π-point over k, say $\alpha: k[t]/t^p \to kG$, which factors through the inclusion $kE \hookrightarrow kG$ of group algerbas for some subgroup scheme $E < G$, with $E \cong \mathbb{G}_{a(r)} \times (\underline{\mathbb{Z}/p})^s$. If $r + s = 1$ (i.e. if one of r, s is 1 and the other is 0), then the inclusion $kE \hookrightarrow kG$ is itself a π-point witnessing $\mathfrak{p}$, making $\mathfrak{p}$ noble. Hence if the largest value of $r + s$ is 1, then every point is noble. If the largest value of $r + s$ is 0, then actually $\mathscr{X}_G$ is the empty set and it vacuously holds that every point is noble. The converse is mysterious.

3.2.2 Noble K-rigidity

We define K-rigidity of a closed point $\mathfrak{q} \in \mathscr{X}_{G'}$, with respect to an arrow $G' \rightsquigarrow \widetilde{G}$ in a maximal connected deformation groupoid G over a finite group scheme $\widetilde{G}$ (3.1.2.5). We assume that every point in $\mathscr{X}_{\widetilde{G}}$ is noble for our base point $\widetilde{G}$, and note that this property is not preserved under arbitrary deformation. Following our convention of diacritics, we denote

$$\mathscr{X}' = \mathscr{X}_{G'} = |\operatorname{Proj} H(G', k)|$$

for each finite group scheme G' with deformation $f': G' \rightsquigarrow \widetilde{G}$, and similarly $\widetilde{\mathscr{X}}, \mathscr{X}'', \hat{\mathscr{X}}$, etc. We'll also take $\mathfrak{R}' = \mathfrak{R}(G')$, $\widetilde{\mathfrak{R}} = \mathfrak{R}(\widetilde{G})$, etc. for rings of representations for objects in the groupoid G.

For a given closed point $\mathfrak{p} \in \mathscr{X}_{\widetilde{G}}$, the tensor product and direct sum properties of support (1.4.4.1, proved in [FPev07]) show how the subcategory of representations with support contained in $\{\mathfrak{p}\}$ is a 'tensor ideal'. It follows that, if $\widetilde{\mathfrak{I}}_{\mathfrak{p}}$ is the subset of $\widetilde{\mathfrak{R}}$ consisting of classes of objects with support contained in $\{\mathfrak{p}\}$, then $\widetilde{\mathfrak{I}}_{\mathfrak{p}} \subset \widetilde{\mathfrak{R}}$ is an ideal, and hence we have a subring

$$\widetilde{\mathfrak{R}}_{\mathfrak{p}} = \mathbb{Z} + \widetilde{\mathfrak{I}}_{\mathfrak{p}} \subset \widetilde{\mathfrak{R}}.$$

If $f: kG' \to k\widetilde{G}$ is a weak deformation, we have an isomorphism of abelian groups given by the pullback

$$f^*: \widetilde{\mathfrak{R}} \to \mathfrak{R}'$$

on classes $[V] \mapsto [f^*V]$. By Theorem 3.1.3.2, if $f: G' \rightsquigarrow \widetilde{G}$ is a strong deformation, then there are induced isomorphisms given by restriction of f^*, as in the diagram

$$\begin{tikzcd}
\widetilde{\mathfrak{I}}_{f_!\mathfrak{q}} \arrow[r, hook] \arrow[d, "f^*"] & \widetilde{\mathfrak{R}}_{f_!\mathfrak{q}} \arrow[r, hook] \arrow[d, "f^*"] & \widetilde{\mathfrak{R}} \arrow[d, "f^*"] \\
\mathfrak{I}'_{\mathfrak{q}} \arrow[r, hook] & \mathfrak{R}'_{\mathfrak{q}} \arrow[r, hook] & \mathfrak{R}'
\end{tikzcd}$$

of abelian groups, for any $\mathfrak{q} \in \mathscr{X}'$.

Strong deformations don't preserve tensor product of representations in general. So a priori, a pullback f^* is not a homomorphism of rings $\widetilde{\mathfrak{R}} \to \mathfrak{R}'$, even when restricted $\widetilde{\mathfrak{R}}_{f_!\mathfrak{q}} \to \mathfrak{R}'_{\mathfrak{q}}$.

Definition 3.2.2.1. Let $\widetilde{G}$ be a finite group scheme over k, and let $f: G' \rightsquigarrow \widetilde{G}$ be any deformation of $\widetilde{G}$. We say a closed point $\mathfrak{q} \in \mathscr{X}'$ is K-rigid with respect to f if the restricted pullback

$$f^*: \widetilde{\mathfrak{R}}_{f_!\mathfrak{q}} \to \mathfrak{R}'_{\mathfrak{q}}$$

is an isomorphism of rings.

Some elementary considerations before defining 'noble K-rigidity':

Lemma 3.2.2.2. *If $f: G' \to \widetilde{G}$ is an isomorphism of groups, taken as a deformation, then each closed point $\mathfrak{q} \in \mathscr{X}'$ is K-rigid with respect to f.*

Proof. The unrestricted pullback $f^*: \widetilde{\mathfrak{R}} \to \mathfrak{R}'$ is already an isomorphism of rings in this case, and hence restricts to isomorphisms of subrings. □

Lemma 3.2.2.3. *Suppose $f\colon G' \rightsquigarrow \widetilde{G}$ and $g\colon G'' \rightsquigarrow G'$ be deformations, and consider a closed point $\mathfrak{q} \in \mathscr{X}''$. Then if $\mathfrak{q}$ is K-rigid with respect to g, and if $g_!\mathfrak{q}$ is K-rigid with respect to f, then $\mathfrak{q}$ is K-rigid with respect to the composition $f \circ g$.*

Proof. Our constructions $\mathfrak{R}, \mathfrak{I}_\mathfrak{p}, \mathfrak{R}_\mathfrak{p}$ define contravariant functors from the groupoid G to abelian groups using the pullback for induced maps, with composition as

$$(f \circ g)^* = g^* \circ f^*\colon \widetilde{\mathfrak{R}} \to \mathfrak{R}' \to \mathfrak{R}'',$$

and restriction as

$$(f \circ g)^*\colon \widetilde{\mathfrak{R}}_{(f \circ g)!\mathfrak{q}} = \widetilde{\mathfrak{R}}_{f_! g_! \mathfrak{q}} \to \mathfrak{R}'_{g_!\mathfrak{q}} \to \mathfrak{R}''_{\mathfrak{q}}.$$

The claim then follows from the composition of ring homomorphisms being a ring homomorphism □

Definition 3.2.2.4. Let $\widetilde{G}$ be a finite group scheme over k, such that every closed point in $\widetilde{\mathscr{X}}$ is noble. We say that $\widetilde{G}$ is noble K-rigid if, for every deformation $f\colon G' \rightsquigarrow \widetilde{G}$, we have every noble point $\mathfrak{q} \in \mathscr{X}'$ is K-rigid with respect to f.

Lemma 3.2.2.5. *Let $\widetilde{U}$ be a unipotent finite group scheme over k, such that every closed point in $\widetilde{\mathscr{X}}$ is noble. Let $\widetilde{\mu}$ be any semisimple finite group scheme over k. Then $\widetilde{U}$ is noble K-rigid if and only if $\widetilde{\mu} \times \widetilde{U}$ is noble K-rigid.*

Proof. First, we have that $H^*(\widetilde{\mu}, k) = k$ as a graded algebra, since $\widetilde{\mu}$ is semisimple. By the Künneth theorem, we have an isomorphism

$$H^*(\widetilde{\mu} \times \widetilde{U}, k) \xrightarrow{\sim} H^*(\widetilde{\mu}, k) \otimes H^*(\widetilde{U}, k) = H^*(\widetilde{U}, k)$$

of graded algebras over k. It follows that every point $\mathfrak{q} \in |\operatorname{Proj} H(\widetilde{\mu} \times \widetilde{U}, k)|$ is noble, as for any point $\mathfrak{q}$, we have a corresponding point $\mathfrak{p} \in |\operatorname{Proj} H(\widetilde{U}, k)|$, which is assumed noble. Then if $H < \widetilde{U}$ is any subgroup scheme witnessing $\mathfrak{p}$, we have $1 \times H < \widetilde{\mu} \times \widetilde{U}$ is a witness for $\mathfrak{q}$, so $\mathfrak{q}$ is noble.

Now let $\widetilde{G} = \widetilde{\mu} \times \widetilde{G}$, and let $f\colon G' \rightsquigarrow \widetilde{G}$ be any (strong) deformation. We can argue that G'

decomposes as $\mu' \times U'$, for some semisimple μ' and unipotent U', such that f decomposes as a product of deformations $\mu' \rightsquigarrow \widetilde{\mu}$ and $U' \rightsquigarrow \widetilde{U}$.

If V is a $\widetilde{\mu}$-representation and W is a $\widetilde{U}$-representation, then the canonical $\widetilde{G}$-representation structure on $V \otimes W$ will be denoted (V, W). Then every indecomposable $\widetilde{G}$-representation is in the form (s, V) for an indecomposable $\widetilde{U}$-representation V and a simple $\widetilde{\mu}$-representation s. The tensor product of $\widetilde{G}$-representations given by (W_1, V_1) and (W_2, V_2) is $(W_1 \otimes W_2, V_1 \otimes V_2)$.

Consider the symmetric tensor category of $\widetilde{G}$-representations with its fiber functor to vector spaces $\mathscr{V}ec_k$. Pulling back the projection $\pi_1 : \mu \times \widetilde{G} \twoheadrightarrow \mu$ gives a symmetric tensor equivalence over $\mathscr{V}ec_k$, from μ-representations, to the full subcategory of semisimple $\mu \times \widetilde{G}$-representations (those in the form (V, k) for V some μ-representation). Since the deformation f is an isomorphism of augmented algebras, the composition $f^* \circ \pi_1^*$ gives an exact equivalence over $\mathscr{V}ec_k$ from $\widetilde{\mu}$-representations to the full subcategory of semisimple G'-representations. Since f is type IV (3.1.2.3), we can conclude that the full subcategory of semisimple G'-representations is closed under tensor product, and is therefore endowed with the structure of a symmetric tensor category fibered over $\mathscr{V}ec_k$. If we let μ' be the group scheme of symmetric monoidal automorphisms of this fiber functor (restricted to semisimple representations), then by Tannakian duality, we have a homomorphism of group schemes $G' \to \mu'$, and a deformation $\mu' \rightsquigarrow \widetilde{\mu}$, such that the diagram commutes:

$$\begin{tikzcd} G' \arrow[r] \arrow[d, rightsquigarrow] & \mu' \arrow[d, rightsquigarrow] \\ \widetilde{G} \arrow[r, two heads, "\pi_1"] & \widetilde{\mu} \end{tikzcd} .$$

With some extension theory for Hopf algebras, we can conclude, by examining the group algebras, that $G' \to \mu'$ is epic, split, and with some split unipotent kernel U' admitting a deformation $U' \rightsquigarrow \widetilde{U}$. Hence, the deformation $G' \rightsquigarrow \widetilde{G}$ decomposes as claimed.

Now we show the main claim of the Lemma, that $\widetilde{G}$ is noble K-rigid if and only if $\widetilde{U}$ is noble K-rigid. Suppose that $\widetilde{U}$ is noble K-rigid, and consider the deformation $f : G' \rightsquigarrow \widetilde{G}$, with $f_1 : \mu' \rightsquigarrow \widetilde{\mu}$ and $f_2 : U' \rightsquigarrow \widetilde{U}$ such that $f = f_1 \times f_2$. Note that $f^*(s, V) = (f_1^* s, f_2^* V)$, for any $\widetilde{\mu}$-representation s and $\widetilde{G}$-representation V. Let us identify $\mathscr{X}' = \mathscr{X}_{U'} = \mathscr{X}_{G'}$, and $\widetilde{\mathscr{X}} = \mathscr{X}_{\widetilde{U}} = \mathscr{X}_{\widetilde{G}}$ from the prior discussion. Let $\mathfrak{q} \in \mathscr{X}'$ be a noble point (as discussed, noble relative to G' is the same as noble relative to U'). Similarly, under this identification, $f_! \mathfrak{q} \in \widetilde{\mathscr{X}}$ also corresponds with $(f_2)_! \mathfrak{q} \in \mathscr{X}_{\widetilde{U}}$. Let (s_1, V_1) and (s_2, V_2) be indecomposable $\widetilde{G}$-representations, both of support contained in $\{f_! \mathfrak{q}\} \subset \mathscr{X}'$. Then

$$\begin{aligned} f^*((s_1, V_1) \otimes (s_2, V_2)) &\cong f^*(s_1 \otimes s_2, V_1 \otimes V_2) \\ &\cong (f_1^*(s_1 \otimes s_2), f_2(V_1 \otimes V_2)). \end{aligned}$$

Under the identification, the support of the V_i is the point $\{(f_2)_! \mathfrak{q}\}$ for the noble point $\mathfrak{q} \in \mathscr{X}_{U'}$. By assumption, we have then that $\mathfrak{q}$ is K-rigid with respect to f_2. We also have $f_1^*(s_1 \otimes s_2) \cong f_1^*(s_1) \otimes f_1^*(s_2)$, since s_1, s_2 are simple and f_1 is a strong deformation. Then

$$f^*((s_1, V_1) \otimes (s_2, V_2)) \cong f^*(s_1, V_1) \otimes f^*(s_2, V_2),$$

and we've shown that this holds for arbitrary deformation, arbitrary noble point $\mathfrak{q}$, and arbitrary indecomposable (s_i, V_i) of the given support. Then $\widetilde{G}$ is noble K-rigid by Proposition 3.1.1.1.

Suppose $\widetilde{U}$ is not noble K-rigid. Then there exists a deformation $f_2 \colon U' \rightsquigarrow \widetilde{U}$, with a noble point $\mathfrak{q} \in \mathscr{X}'$, and $\widetilde{U}$-representations V_1, V_2 with support $\{(f_2)_! \mathfrak{q}\}$, with

$$f_2^*(V_1 \otimes V_2) \ncong f_2^*(V_1) \otimes f_2^*(V_2).$$

Now let $G' = \widetilde{\mu} \times U'$, so that $f = \mathrm{id}_{\widetilde{\mu}} \times f_2 \colon G' \rightsquigarrow \widetilde{G}$ defines a deformation. Then $\mathfrak{q}$, as a point in $\mathscr{X}_{G'}$, is also noble, $(k, V_1), (k, V_2)$ $\widetilde{G}$-representations with support contained in $\{f_! \mathfrak{q}\}$. But immediately, we have

$$\begin{aligned} f^*((k, V_1) \otimes (k, V_2)) = (k, f_2(V_1 \otimes V_2)) &\ncong (k, f_2(V_1) \otimes f_2(V_2)) \\ &= f^*(k, V_1) \otimes f^*(k, V_2). \end{aligned}$$

Hence, $\widetilde{G}$ is not noble K-rigid. □

Our deformation groupoids give us a notational advantage of diacritics, but in practice, we often need to show noble K-rigidity using explicit computations, and so we need to leverage that our deformation groupoids are essentially small. This is accomplished with the following:

Lemma 3.2.2.6. *Let $\widetilde{G}$ be a finite group scheme over k, such that every closed point in $\widetilde{\mathscr{X}}$ is noble.*

1. *If $\widetilde{G}$ is noble K-rigid, then every closed point in $\widetilde{\mathscr{X}}$ is K-rigid with respect to any autodeformation $\widetilde{G} \rightsquigarrow \widetilde{G}$ of $\widetilde{G}$.*

2. *If every closed point in $\widetilde{\mathscr{X}}$ is K-rigid with respect to any autodeformation $\widetilde{G} \rightsquigarrow \widetilde{G}$, then $\widetilde{G}$ is noble K-rigid if and only if there exists a set I, indexing a set of 'distinguished deformations' $\{f_i : G_i \rightsquigarrow \widetilde{G}\}_{i \in I}$, such that:*

   (a) Any finite group scheme G' admitting a deformation $G' \rightsquigarrow \widetilde{G}$ is isomorphic to G_i for some $i \in I$, and

   (b) For each $i \in I$, every noble point in $\mathscr{X}_i$ is K-rigid with respect to f_i.

Proof. The first claim is immediate. For the second claim, we make use of our associated 'noncanonical comultiplication' $\Delta' : k\widetilde{G} \to k\widetilde{G} \otimes k\widetilde{G}$, given a deformation $G' \rightsquigarrow \widetilde{G}$. Given any augmented algebra A, the comultiplications $A \to A \otimes A$ defining a Hopf algebra structure on A form a set. Given weak deformations $G' \rightsquigarrow \widetilde{G}$ and $G'' \rightsquigarrow \widetilde{G}$, we have $\Delta' = \Delta''$ as maps $k\widetilde{G} \to k\widetilde{G} \otimes k\widetilde{G}$ if and only if the composition $(f'')^{-1} \circ f' : G' \rightsquigarrow G''$ is an isomorphism of group schemes. Therefore, the isomorphism classes of finite group schemes G' admitting a (strong) deformation $G' \rightsquigarrow \widetilde{G}$ form a set. Call this set I, and for each $i \in I$, we use the axiom of choice to pick an object G_i in the class i, and a deformation $f_i : G_i \rightsquigarrow \widetilde{G}$. By construction, the set $\{f_i : G_i \rightsquigarrow \widetilde{G}\}_{i \in I}$ satisfies the first property (a). If $\widetilde{\mathscr{X}}$ is noble K-rigid, then by definition the second property (b) is also satisfied. This shows the 'only if' direction of the second claim.

Suppose that every closed point in $\widetilde{\mathscr{X}}$ is K-rigid with respect to any autodeformation $\widetilde{G} \rightsquigarrow \widetilde{G}$, and that there exists I, with a set $\{f_i\}_{i \in I}$ of deformations such that properties (a) and (b) hold. Let $f' : G' \rightsquigarrow \widetilde{G}$ be any deformation. Then take $i \in I$ such that that there exists an isomorphism of groups $\varphi\colon G' \xrightarrow{\sim} G_i$. Then there exists a unique autodeformation $\widetilde{G} \rightsquigarrow \widetilde{G}$ making the diagram commute

$$\begin{tikzcd} G' \arrow[r, "\varphi"] \arrow[dr, rightsquigarrow, "f'"'] & G_i \arrow[r, rightsquigarrow, "f_i"] & \widetilde{G} \arrow[dl, rightsquigarrow] \\ & \widetilde{G} & \end{tikzcd},$$

thus factoring the deformation f' into a composition of a group isomorphism, a distinguished deformation f_i, and an autodeformation $\widetilde{G}$. Now let $\mathfrak{q} \in \mathscr{X}'$ be a noble point. Then $\mathfrak{q}$ is K-rigid with respect to the group isomorphism φ, and $\varphi_!\mathfrak{q}$ is noble. Then by assumption $\varphi_!\mathfrak{q} \in \mathscr{X}_i$ is K-rigid with respect to the distinguished deformation f_i, and $(f_i \circ \varphi)_!\mathfrak{q} \in \widetilde{\mathscr{X}}$ is K-rigid with respect to any autodeformation. By Lemma 3.2.2.3, we have $\mathfrak{q}$ is K-rigid with respect to f', and so we have shown that $\widetilde{G}$ is noble K-rigid. □

3.2.3 The trichotomy theorem, and a conjecture

The representation type of an algebra A over a field indicates the difficulty of classifying finite dimensional left A-modules up to isomorphism. The study of 'tame' and 'wild' matrix problems is classical, but the modern definition for the representation type of an algebra is attributed to Freislich and Donovan [FD72], and proven to form a strict trichotomy for all finite dimensional algebras by Drozd [Dro80]. We'll use [Erd90] as a more complete reference. The main result of this thesis is Theorem 3.2.3.3 below, which identifies a class of algebras for which representation type is evidently determined by K-rigidity.

Recall the Krull–Schmidt theorem, which lets us conclude that a classification of finite modules in general follows from a classification of indecomposable modules. Let's review the definitions of representation type as found in [Erd90], and Drozd's seminal trichotomy theorem.

Definition 3.2.3.1. [Erd90] Let k be a field and let A be an algebra over k.

I. The algebra A is said to be of 'finite representation type' if there are finitely many indecomposable A-modules up to isomorphism. If A is not of finite representation type then A is also said to be of 'infinite representation type'.

II. If A is of infinite representation type, we say that A is of 'tame representation type' if in addition, for each integer $d > 0$, there are some finitely many $(A, k[T])$-bimodules M_i which are free as right $k[T]$-modules, such that all but finitely many isomorphism classes of indecomposable A-modules of dimension d given by $M_i \otimes_{k[T]} N$ for some simple $k[T]$-module N.

III. If A is of infinite representation type, we say that A is of 'wild representation type' if there exists a finitely generated $(A, k\langle X, Y\rangle)$-bimodule B, which is free as a right $k\langle x, y\rangle$-module, and such that the functor $B \otimes_{k\langle X,Y\rangle} ?$ from left $k\langle X, Y\rangle$-modules to left A-modules preserves indecomposability and reflects isomorphisms (is essentially injective).

Theorem 3.2.3.2. [Dro80] (see also [Erd90]) *Let A be a finite dimensional algebra over an algebraically closed field k, and assume A is of infinite representation type. Then A is of tame representation type if and only if A is not of wild representation type.*

Now, we state the main result of this thesis: let k be an algebraically closed field of characteristic $p > 0$.

Theorem 3.2.3.3. *Let $\widetilde{G}$ be an abelian infinitesimal group scheme of height one over k. Then:*

I. Suppose $\widetilde{G}$ is of finite representation type, and let $E < \widetilde{G}$ be a maximal unipotent subgroup scheme. If $p = 2$ or if the dimension of kE is p, then $\widetilde{G}$ is noble K-rigid.

II. If $\widetilde{G}$ is of tame representation type, then $\widetilde{G}$ is noble K-rigid.

III. If $\widetilde{G}$ is of wild representation type, then $\widetilde{G}$ is not noble K-rigid.

Proof. This will be proven in many steps in the next section of this chapter. For reference, part I. is proven as Corollary 3.4.2.3. Part II. is proven as Corollary 3.4.3.4. Part III. is proven as Corollary 3.4.4.4. $\square$

Conjecture 3.2.3.4. *Let $\widetilde{G}$ be a finite group scheme over k, such that every closed point in $\widetilde{\mathscr{X}}$ is noble. Then $\widetilde{G}$ is noble K-rigid if and only if $\widetilde{G}$ is not of wild representation type.*

Roughly speaking, we can see that 'finite', 'tame', and 'wild' representation types are stated in order of increasing complexity of the representation theory of a given algebra. We may also observe how, given a finite group scheme $\widetilde{G}$, K-rigidity with respect to some non-trivial deformations $G' \rightsquigarrow \widetilde{G}$ should function as an abstract upper bound on the complexity of the representation theory $\widetilde{G}$. In these terms, we reinterpret the conjecture to say how 'noble K-rigidity' serves as an equivalent upper bound to the classical notion of 'tameness'.

3.2.4 Obstruction theory

As we will see, noble K-rigidity is atypical for group schemes, a priori. To show a given group scheme $\widetilde{G}$ is *not* noble K-rigid, ultimately, one needs to show the existence of a deformation $f: G' \rightsquigarrow \widetilde{G}$, a noble point $\mathfrak{q} \in \mathscr{X}'$, and a pair of indecomposable $\widetilde{G}$-representations V, W such that $\widetilde{\mathscr{X}}(V) = \widetilde{\mathscr{X}}(W) = \{f_! \mathfrak{q}\}$, and such that $f^*(V \otimes W)$ and $f^*(V) \otimes f^*(W)$ are not isomorphic as G'-representations. The data of such $f, \mathfrak{q}, V, W$ together is what we'll call a *noble obstruction*. Intuitively, when $\widetilde{G}$ is of wild representation type, noble obstructions should be in ample supply: after choosing a nontrivial deformation f, one should be able to pick $\mathfrak{q}$ arbitrarily, and find that there is such a complex array of representations with support $\{f_! \mathfrak{q}\}$ that a random choice of indecomposable V, W is surely a noble obstruction. Indeed, it seems likely that any wild representation type group scheme $\widetilde{G}$ (such that every point in $\widetilde{\mathscr{X}}$ is noble) is not noble K-rigid. The problem we're presented with is a general method of producing and actually verifying obstructions, and there seems to be no easy solution to this problem.

Definition 3.2.4.1. Let $\widetilde{G}$ be a finite group scheme over a field of characteristic p. Let $P \in \mathbb{Z}[X_1, \ldots, X_m]$ be some integer polynomial. A *P-obstruction* is a tuple $\varphi, \mathfrak{q}, V_1, \ldots, V_m$, for an autodeformation $\varphi \in \mathrm{Def}_{\mathrm{IV}}(\widetilde{G}, \widetilde{G})$, a closed point $\mathfrak{p} \in \widetilde{\mathscr{X}}$, and representations V_i, such that $\widetilde{\mathscr{X}}(V_i) = \{\varphi_! \mathfrak{q}\}$ for each $i = 1, \ldots, m$, and such that:

1. The polynomial P is contained in the kernel of the homomorphism $\mathbb{Z}[X_1, \ldots, X_m] \to \widetilde{\mathfrak{R}}$ of rings determined by $X_i \mapsto [V_i]$.

2. The polynomial P is not contained in the kernel of the homomorphism of rings $\mathbb{Z}[X_1,\ldots,X_m] \to \widetilde{\mathfrak{R}}$ defined by $X \mapsto [\varphi^* V_i]$.

Lemma 3.2.4.2. *Let $\widetilde{G}$ be a finite group scheme over k, such that every closed point in $\widetilde{\mathscr{X}}$ is noble, and suppose that there exists a P-obstruction, for some $P \in \mathbb{Z}[X_1,\ldots,X_m]$. Then $\widetilde{G}$ is not noble K-rigid.*

Proof. Suppose that $\widetilde{G}$ is noble K-rigid and that $\varphi, \mathfrak{q}, V_1, \ldots, V_m$ is a P-obstruction. Then each $[V_i]$ belongs to the subring $\widetilde{\mathfrak{R}}_{\varphi_! \mathfrak{q}}$, and hence the homomorphism taking $X_i \mapsto [V_i]$ is well defined as $\mathbb{Z}[X_1,\ldots,X_m] \to \widetilde{\mathfrak{R}}_{\varphi_! \mathfrak{q}}$. We've assumed that $\mathfrak{q}$ and $\varphi_! \mathfrak{q}$ are both noble in $\widetilde{\mathscr{X}}$, hence noble K-rigidity implies that $\varphi^* : \widetilde{\mathfrak{R}}_{\varphi_! \mathfrak{q}} \to \widetilde{\mathfrak{R}}_{\mathfrak{q}}$ is an isomorphism of rings. Then the composition

$$\mathbb{Z}[X_1,\ldots,X_m] \to \widetilde{\mathfrak{R}}_{\varphi_! \mathfrak{q}} \xrightarrow{\varphi^*} \widetilde{\mathfrak{R}}_{\mathfrak{q}} \hookrightarrow \widetilde{\mathfrak{R}}$$

is a homomorphism of rings. But the composition is precisely the map defined by $X_i \mapsto [\varphi^* V_i]$. From here, we see that conditions 1 and 2 of Definition 3.2.4.1 contradict one another. $\square$

The existence of a *noble obstruction* from such a P-obstruction is implicit from the Krull–Schmidt theorem. In practice we'll look for P-obstructions for $P = X_1^2 - \sum_{i=1}^m n_i X_i, \quad n_i \geq 0$, and for indecomposable V_i, so that $\varphi, \mathfrak{p}, V_1, V_1$ is a noble obstruction.

Definition 3.2.4.3. Let $\widetilde{G}$ be a finite group scheme over a field of characteristic p. Let $P \in \mathbb{Z}[X_1,\ldots,X_m]$ be some integer polynomial, and let $\rho : \widetilde{\mathfrak{R}} \to \mathfrak{C}$ be a generalized character, for $\mathfrak{C}$ a commutative ring. A (ρ, P)*-obstruction* is a tuple $\varphi, \mathfrak{q}, V_1, \ldots, V_m$, for $\varphi \in \mathrm{Def}_{\mathrm{IV}}$, a closed point $\mathfrak{q} \in \widetilde{\mathscr{X}}$, and representations V_i, such that $\widetilde{\mathscr{X}}(V_i) = \{\varphi_! \mathfrak{q}\}$ for each $i = 1, \ldots, m$, and such that:

1. The polynomial P is contained in the kernel of the homomorphism $\mathbb{Z}[X_1,\ldots,X_m] \to \widetilde{\mathfrak{R}}$ of rings defined by $X_i \mapsto [V_i]$.

2. The polynomial P is not contained in the kernel of $\mathbb{Z}[X_1,\ldots,X_m] \to \mathfrak{C}$ defined by $X_i \mapsto \rho([\varphi^* V_i])$.

Any (ρ, P)-obstruction is a P-obstruction, and any P-obstruction is a (ρ, P)-obstruction for some suitable ρ (clearly taking $\mathfrak{C} = \widetilde{\mathfrak{R}}$ and $\rho = \mathrm{id}$ works, but there are more interesting generalized characters we'll work with).

Now we'll go over some techniques for producing obstructions in the special case of infinitesimal group schemes $\widetilde{G}$ of height 1 (restricted Lie algebras). Recall our notation of Jordan blocks J_i, the unique indecomposable $\mathbb{W}_{n(1)}$-module of dimension i, for $1 \leq i \leq p^n$.

Lemma 3.2.4.4. *Let $\widetilde{G}$ be an infinitesimal group scheme of height one over k, and $C < \widetilde{G}$ a subgroup scheme. Assume that $C \cong \mathbb{W}_{n(1)}$ for some n and that C is central in $\widetilde{G}$ (i.e. that the subalgebra kC is central in $k\widetilde{G}$). Let $V = J_1 \uparrow_C^{\widetilde{G}}$ be the induced module. If $\varphi \in \mathrm{Def}_{\mathrm{IV}}(\widetilde{G}, \widetilde{G})$ is an autodeformation with the property that $\varphi^*(V) \downarrow_C^{\widetilde{G}}$ is not isomorphic to mJ_{p^ℓ} for any m, ℓ, then $\varphi, \mathfrak{q}, V$ is a (ρ, P)-obstruction for $P = X^2 - p^{-n}|\widetilde{G}|X \in \mathbb{Z}[X]$, with $\mathfrak{q} \in \widetilde{\mathscr{X}}$ such that $\varphi_! \mathfrak{q}$ is the point witnessed by $kC \hookrightarrow k\widetilde{G}$ over k, and $\rho : \mathfrak{R}(\widetilde{G}) \to \mathfrak{R}(C)$ the restriction character.*

Proof. First, $\widetilde{\mathscr{X}}(V) = \{\mathfrak{p}\}$, where $\mathfrak{p}$ is the point witnessed by $kC \hookrightarrow k\widetilde{G}$. Since C is a subgroup of $\widetilde{G}$, the inclusion $kC \hookrightarrow k\widetilde{G}$ is flat. Further, the inclusion is a witness to some $\mathfrak{p} \in \widetilde{\mathscr{X}}$ because $C \cong \mathbb{W}_{n(1)}$, so $kC \cong k[t]/t^{p^n}$, so the composition $k\mathbb{G}_{a(1)} \hookrightarrow kC \hookrightarrow k\widetilde{G}$ defines a π-point. Since C is central, we can view t as a central element of the group algebra $k\widetilde{G}$ and get that $V \cong k\widetilde{G}/(t)$, as $k\widetilde{G}$-modules. Then $V \downarrow_C^{\widetilde{G}} \cong p^{-n}|\widetilde{G}|J_1$, which is not projective, and hence $\mathfrak{p} \in \widetilde{\mathscr{X}}(V)$. No other point in the restricted nullcone, independent from the powers of t, can be in the support of V by the PBW theorem, hence $\widetilde{\mathscr{X}}(V) = \{\mathfrak{p}\}$ as claimed.

Since $V \downarrow_C^{\widetilde{G}} \cong p^{-n}|\widetilde{G}|J_1$, we have by Frobenius property that

$$V \otimes V \cong p^{-n}|\widetilde{G}|V.$$

Therefore $P = X^2 - p^{-n}|\widetilde{G}|X$ is in the kernel of the homomorphism of rings $\mathbb{Z}[X] \to \widetilde{\mathfrak{R}}$ defined by $X \mapsto [V]$.

By assumption, $\varphi^*(V) \downarrow_C^{\widetilde{G}}$ is not isomorphic to mJ_{p^ℓ} for any m, ℓ. Some explicit work with the ring of representations $\mathfrak{R}(\mathbb{W}_{n(1)})$ shows that only those representations of the form $V' = mJ_{p^\ell}$ can satisfy $V' \otimes V' \cong m'V$ for some m. In particular, $P = X^2 - p^{-n}|\widetilde{G}|X$ is not in the kernel of the homomorphism of rings $\mathbb{Z}[X] \to \mathfrak{R}(C)$ defined by $X \mapsto \rho([\varphi^* V])$. Now let $\mathfrak{q} = \varphi_!^{-1}\mathfrak{p}$. It follows that $\varphi, \mathfrak{q}, V$ is then a (ρ, P)-obstruction, for $P = X^2 - p^{-n}|\widetilde{G}|X \in \mathbb{Z}[X]$ and $\mathfrak{q}$ such that $\varphi_! \mathfrak{q}$ is the point witnessed by $kC \hookrightarrow k\widetilde{G}$ over k. □

3.3 Geometric K-rigidity

Let k be an algebraically closed field. We have given definitions for group schemes, lattices, etc. over a commutative ring in our background section. We should note that with very little work we can extend these definitions to an arbitrary base scheme, and with no work at all these definitions apply to arbitrary *affine* base schemes.

3.3.1 Geometric deformations

Let k be an algebra and let X be an affine k-scheme of finite type. We choose a k-point $\mathfrak{p} \in X(k)$, and regard X as based at $\mathfrak{p}$. For objects (e.g. schemes, morphisms, representations of group schemes, etc.) F over X, and k-points $\mathfrak{q} \in X(k)$, we denote by $F_\mathfrak{q}$ the base change of F along $\mathfrak{q}$. If X is an integral k-scheme with generic point ξ, then we denote by $F_{\bar{\xi}}$ for the base change along $\operatorname{Spec}\overline{\kappa(\xi)} \to X$, i.e. $F_{\bar{\xi}}$ denotes the geometric generic fiber.

Definition 3.3.1.1. Let $\widetilde{G}$ be an affine group scheme over k. A *weak X-deformation* of $\widetilde{G}$, denoted $f\colon G_X \rightsquigarrow \widetilde{G}_X$, is an affine group scheme G over X, together with an isomorphism $f^\sharp\colon \mathcal{O}(X) \otimes_k \mathcal{O}(\widetilde{G}) \xrightarrow{\sim} \mathcal{O}(G)$ of coaugmented coalgebras over $\mathcal{O}(X)$, such that the specialization

$$f_\mathfrak{p}^\sharp\colon \kappa(\mathfrak{p}) \otimes_{\mathcal{O}(X)} \mathcal{O}(X) \otimes_k \mathcal{O}(\widetilde{G}) \xrightarrow{\sim} \kappa(\mathfrak{p}) \otimes_{\mathcal{O}(X)} \mathcal{O}(G)$$

is also an isomorphism of commutative algebras.

3.3.1.2. We've assumed k is algebraically closed and X is of finite type over k. Supposing $\widetilde{G}$ is finite over k, we conclude that, for any weak X-deformation $G_X \rightsquigarrow \widetilde{G}$, and any point $\mathfrak{q} \in X(k)$, we have a weak deformation $f_\mathfrak{q}\colon G_\mathfrak{q} \rightsquigarrow \widetilde{G}$ given by the linear dual

$$f_\mathfrak{q} = (f_\mathfrak{q}^\sharp)^\vee\colon kG_\mathfrak{q} \to k\widetilde{G},$$

an isomorphism of augmented algebras, using

$$kG_{\mathfrak{q}} = (\kappa(\mathfrak{q}) \otimes_{\mathcal{O}(X)} \mathcal{O}(G))^{\vee},$$

and

$$k\widetilde{G} = (\kappa(\mathfrak{q}) \otimes_{\mathcal{O}(X)} \mathcal{O}(\widetilde{G}_X))^{\vee} = \kappa(\mathfrak{q}) \otimes_{\mathcal{O}(X)} \mathcal{O}(X)$$

for any $\mathfrak{q} \in X(k)$.

We'll say a weak X-deformation $f\colon G_X \rightsquigarrow \widetilde{G}$ is type I, II, III, IV (or a strong deformation, or a deformation), if $f_{\mathfrak{q}}\colon G_{\mathfrak{q}} \rightsquigarrow \widetilde{G}$ is type I, II, III, IV (or a strong deformation, or a deformation), for each $\mathfrak{q} \in X(k)$, respectively (Definition 3.1.2.3).

Example 3.3.1.3. The group scheme U^{η} of Section 2.3.2 is an example of a group scheme over $\mathbb{A}^1$, admitting a strong $\mathbb{A}^1$-deformation $U^{\eta} \rightsquigarrow \mathbb{G}_{a(1)}$ of the Frobenius kernel $\mathbb{G}_{a(1)}$. Indeed, since $\mathbb{G}_{a(1)}$ is unipotent, any weak deformation of $\mathbb{G}_{a(1)}$ is strong, and the isomorphism

$$f^{\sharp}\colon \mathcal{O}(X) \otimes_k \mathcal{O}(\mathbb{G}_{a(1)}) \to \mathcal{O}(U^{\eta})$$

of coaugmented coalgebras is given as

$$k[\eta] \otimes \frac{k[t]}{t^p} \to \frac{k[\eta][x]}{x^p - \eta x},$$

defined $k[\eta]$-linearly on the basis by $f^{\sharp}(t^i) = x^i$, for $0 \le i < p$, where $\mathcal{O}(\mathbb{A}^1) = k[\eta]$. Per our convention, we need $\mathbb{A}^1$ to be based at the k-point $\mathfrak{p}$ given by $\eta \mapsto 0$, in order for the specialization $f^{\sharp}_{\mathfrak{p}}$ to define an isomorphism of algebras.

Similarly, if $\mathbb{A}^1$ is based at $\eta \mapsto 1$, then U^{η} also admits an $\mathbb{A}^1$-deformation $U^{\eta} \rightsquigarrow \underline{\mathbb{Z}/p}$, given as

$$k[\eta] \otimes \frac{k[t]}{t^p - t} \to \frac{k[\eta][x]}{x^p - \eta x}$$

also defined $k[\eta]$-linearly by $t^i \mapsto x^i$. By design, given any $\mathfrak{q} \in X(k)$, any X-deformation $G_X \rightsquigarrow \widetilde{G}_X$ based at $\mathfrak{p}$ also admits an X-deformation $G_X \rightsquigarrow G_{\mathfrak{q}}$ based at $\mathfrak{q}$.

3.3.2 Integral deformations

Definition 3.3.2.1. If $f\colon G_X \rightsquigarrow \widetilde{G}_X$ is a weak X-deformation of $\widetilde{G}$, and V is some $\widetilde{G}$-representation, say with $\mathcal{O}(\widetilde{G})$-comodule structure $\Delta\colon V \to \mathcal{O}(\widetilde{G}) \otimes_k V$, then the *lift* of V to G, denoted f^*V, is the $\mathcal{O}(G)$-comodule structure on the base change $\mathcal{O}(X) \otimes_k V$, given by pushing forward the base-changed $\mathcal{O}(\widetilde{G}_X)$-comodule structure

$$\begin{aligned}\mathcal{O}(X) \otimes_k \Delta\colon \mathcal{O}(X) \otimes_k V &\to \mathcal{O}(X) \otimes_k \mathcal{O}(\widetilde{G}) \otimes_k V\\ &= \mathcal{O}(\widetilde{G}_X) \otimes_{\mathcal{O}(X)} \mathcal{O}(X) \otimes_k V\end{aligned}$$

along the coalgebra map

$$f^\sharp\colon \mathcal{O}(\widetilde{G}_X) = \mathcal{O}(X) \otimes_k \mathcal{O}(\widetilde{G}) \to \mathcal{O}(G).$$

3.3.2.2. Notice, that if $f\colon G_X \rightsquigarrow \widetilde{G}$ is any weak X-deformation of a finite group scheme $\widetilde{G}$ over k, and $\mathfrak{q} \in X(k)$ is any k-point, then there is a natural isomorphism $(f^*?)_{\mathfrak{q}} \xrightarrow{\sim} f^*_{\mathfrak{q}}?$ of functors from $rep_{\widetilde{G}}$ to $rep_{G_{\mathfrak{q}}}$, canonically defined, between specializing the lift f^* and pulling back the specialized weak deformation $f_{\mathfrak{q}}\colon G_{\mathfrak{q}} \rightsquigarrow \widetilde{G}$.

Definition 3.3.2.3. Let X be an integral affine k-scheme of finite type with some base point, and with generic point $\xi \in |X|$. Let $\widetilde{G}$ be a finite group scheme over k, and $f\colon G_X \rightsquigarrow \widetilde{G}_X$ a weak X-deformation.

1. We say f is a *generic twist* (over X) if there exists an isomorphism $\varphi\colon G_{\overline{\xi}} \xrightarrow{\sim} \widetilde{G}_{X,\overline{\xi}}$ of group schemes over $\overline{\kappa(\xi)}$.

2. Suppose f is a generic twist, and let $\mathcal{C}$ be some subcategory (not necessarily full) of $\widetilde{G}$-representations. We say $\mathcal{C}$ is *stable* with respect to f, φ, if there exists a natural isomorphism

$$\Xi\colon f^*_{\overline{\xi}}(?_{\overline{\xi}}) \xrightarrow{\sim} \varphi^*(?_{\overline{\xi}})$$

of functors from $\mathcal{C}$ to $rep_{G_{\overline{\xi}}}$

Example 3.3.2.4. If $f\colon G_X \rightsquigarrow \widetilde{G}_X$ is a weak X-deformation such that the base change $f_{\overline{\xi}}\colon G_{\overline{\xi}} \rightsquigarrow \widetilde{G}_{X,\overline{\xi}}$ is an isomorphism of group schemes, then f is a generic twist and any full subcategory of $rep_{\widetilde{G}}$ (including $rep_{\widetilde{G}}$ itself) is stable with respect to f, φ, taking $\varphi = f$.

Example 3.3.2.5. Take the $G = U^\eta$ and $\widetilde{G} = \underline{\mathbb{Z}/p}$ with the $\mathbb{A}^1$-deformation $f\colon U^\eta \rightsquigarrow \underline{\mathbb{Z}/p}$, based at $\eta \mapsto 1$, from before. We claim f is a generic twist: let $K = \overline{\kappa(\xi)} = \overline{k(\eta)}$, and so $\mathcal{O}(\widetilde{G}_{X,\overline{\xi}}) = \frac{K[t]}{t^p - t}$, while $\mathcal{O}(G_{\overline{\xi}}) = \frac{K[x]}{x^p - \eta x}$. If we choose $\alpha \in K$ such that $\alpha^{p-1} = \eta^{-1}$, then an isomorphism $\varphi\colon G_{\overline{\xi}} \xrightarrow{\sim} \widetilde{G}_{X,\overline{\xi}}$ of group schemes is given by the map of coordinate algebras

$$\varphi^\sharp\colon \frac{K[t]}{t^p - t} \to \frac{K[x]}{x^p - \eta x}$$

defined by $\varphi^\sharp(t) \mapsto \alpha x$.

Example 3.3.2.6. Using $G = U^\eta$ over $\mathbb{A}^1$ and $\widetilde{G} = \underline{\mathbb{Z}/p}$ with generic twist f and isomorphism φ as above, we can construct an interesting subcategory $\mathcal{C}$ which is stable with respect to f, φ, and which is maximal in a sense. The objects of $\mathcal{C}$ are defined to be the symmetric powers $L^i = S^i(V)$ of the two-dimensional $\mathcal{O}(\widetilde{G})$-comodule $V = \langle v_0, v_1 \rangle$ with coaction

$$v_0 \mapsto 1 \otimes v_0, \quad v_1 \mapsto 1 \otimes v_1 + t \otimes v_0.$$

For each i, j, the set of homomorphisms $\mathcal{C}(L^i, L^j) \subset \mathrm{Hom}_{\widetilde{G}}(L^i, L^j)$ is defined to be 0 if $i < j$ and the set of those homomorphisms uniquely determined by $v_1^i \mapsto \beta v_0^j$ if $i \geq j$. One checks that these are well defined, and make $\mathcal{C}$ a subcategory.

The pullback φ^* is by definition the pushforward of comodule structures, and similarly the lift f^* and its base change $f^*_{\overline{\xi}}$. Now we'll define a map

$$\Xi(L^i)\colon f^*_{\overline{\xi}}(L^i_{\overline{\xi}}) \xrightarrow{\sim} \varphi^*(L^i_{\overline{\xi}})$$

for each $L^i = S^i(V)$.

First define $\Xi(V)\colon f^*_{\overline{\xi}}(V_{\overline{\xi}}) \to \varphi^*(V_{\overline{\xi}})$ on the K-basis v_0, v_1 by

$$v_0 \mapsto \alpha v_0, \quad v_1 \mapsto v_1.$$

Then $\Xi(V)$ is a K-linear isomorphism, and one checks that it is a coalgebra homomorphism and hence an isomorphism of $G_{\overline{\xi}}$-representations. Now, for each symmetric power $L^i = S^i(V)$, note that the underlying space is unchanged under pullbacks, so there is a well-defined linear isomorphism

$$S^i(\Xi(V))\colon f_{\overline{\xi}}^*(L_{\overline{\xi}}^i) \to \varphi^*(L_{\overline{\xi}}^i),$$

and since $\Xi(V)$ is $\widetilde{G}$-equivariant, so is $S^i(\Xi(V))$. So let's define $\Xi(L^i) = S^i(\Xi(L^i))$. Now for each i, j, we claim that $\mathcal{C}(L^i, L^j) \subset \operatorname{Hom}_{\widetilde{G}}(L^i, L^j)$ is the set of all homomorphisms ϖ such the square

$$\begin{tikzcd} f_{\overline{\xi}}^*(L_{\overline{\xi}}^i) \arrow[r, "\Xi(L^i)"] \arrow[d, "f_{\overline{\xi}}^*(\varpi_{\overline{\xi}})"'] & \varphi^*(L_{\overline{\xi}}^i) \arrow[d, "\varphi^*(\varpi_{\overline{\xi}})"] \\ f_{\overline{\xi}}^*(L_{\overline{\xi}}^j) \arrow[r, "\Xi(L^j)"] & \varphi^*(L_{\overline{\xi}}^j) \end{tikzcd}$$

commutes.

For each L^i, a homomorphism $L^i \to L$ of $\widetilde{G}$-representations is uniquely determined by the image of $v_1^i \in L^i$, and for $L = L^j$, v_1^i can be mapped freely to any linear combination of monomials $v_0^n v_1^m$, for $n + m = j$ and $m \leq i$. Of these homomorphisms $L^i \to L^j$, only those which map v_1^i to a scalar multiple of v_1^j make the diagram commute. Hence, the maximal set of homomorphisms is, indeed, given by

$$\mathcal{C}(L^i, L^j) = \begin{cases} 0 & i < j, \\ \langle v_1^i \mapsto v_1^j \rangle & i \geq j. \end{cases}$$

Lemma 3.3.2.7. *Let $\widetilde{G}$ be a finite group scheme over k such that $k\widetilde{G} = k[t]/t^{p^n}$* (c.f. A.2.0.5)*, with $J_i = k\widetilde{G}/t^i$ the $\widetilde{G}$-representation, for each $1 \leq i \leq p^n$. Let X be an integral affine k-scheme of finite type, based at some $\mathfrak{p} \in X(k)$, and let $f\colon G_X \rightsquigarrow \widetilde{G}_X$ be a generic twist. Fix some i, j and for any $\mathfrak{q} \in |X|$, let $b(\mathfrak{q})$ denote the number of indecomposable direct summands of $f_{\mathfrak{q}}^*(J_i) \otimes f_{\mathfrak{q}}^*(J_j)$, as a representation of $G_{\mathfrak{q}}$. Then $b(\mathfrak{q}) \geq b(\mathfrak{p})$ for all $\mathfrak{q}$.*

Proof. Take $R = \mathcal{O}(X)$, an integral domain. Since $f\colon G_X \rightsquigarrow \widetilde{G}_X$ is an X-deformation, we may write the group algebra RG as $R[x]/x^{p^n}$, with the deformation $f\colon G_X \rightsquigarrow \widetilde{G}_X$ defined to take $x \mapsto t$.

By its definition, the function $b(\mathfrak{q})$ is the dimension of the kernel of the map $\overline{x}\colon f_{\mathfrak{q}}^*(J_i) \otimes f_{\mathfrak{q}}^*(J_j) \to f_{\mathfrak{q}}^*(J_i) \otimes f_{\mathfrak{q}}^*(J_j)$. We'll relate this to a dimension function for a coherent sheaf over X as follows. Let $\mathbf{J}$ be the tautological X-scheme associated to the trivial vector bundle $J = f^*(J_i) \otimes f^*(J_j)$ over X. That is, we let

$$\mathbf{J} = \mathbf{Spec}\ S^*(J^\vee)/S^2(J^\vee).$$

Then $\mathbf{J}$ is a finite scheme over X, and the map $\overline{x}\colon J \to J$ induces a morphism of X-schemes $\overline{x}\colon \mathbf{J} \to \mathbf{J}$. The zero map $0 \to J$ also induces a 'zero section' of X-schemes $X \to \mathbf{J}$, using that $R = S^*(0^\vee)/S^2(0^\vee)$. Now define the X-scheme $\mathbf{F}$ by the fiber product as in the diagram below:

$$\begin{tikzcd} \mathbf{F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbf{J} \arrow[d, "\overline{x}"] \\ X \arrow[r, "0"] & \mathbf{J} \end{tikzcd}.$$

Define a sheaf $\mathcal{F}$ over X by the sheaf of sections of the X-scheme $\mathbf{F}$. Then for any $\mathfrak{q} \in |X|$, we have

$$b(\mathfrak{q}) = \dim_{\kappa(\mathfrak{q})} \mathcal{F}_x \otimes \kappa(\mathfrak{q}) - 1.$$

We have $\mathcal{F}$ is a coherent sheaf over X, and that X is noetherian by the Hilbert basis theorem. Hence $b(\mathfrak{q})$ is upper semi-continuous by Nakayama's lemma [Har77, Ch. II, 5.8]. □

3.3.2.8. Within the lemma above, let $b^\ell(\mathfrak{q})$ denote the number of indecomposable direct summands of $f_{\mathfrak{q}}^*(J_i) \otimes f_{\mathfrak{q}}^*(J_j)$ having dimension $\geq \ell$. Then, taking $B^m(\mathfrak{q}) = \sum_{\ell=1}^m b^\ell(\mathfrak{q})$, we have $B^m(\mathfrak{q}) \geq B^m(\mathfrak{p})$ for any m and for any $\mathfrak{q} \in |X|$. This also follows by upper semi-continuity, this time of a dimension function for $\mathbf{F}^m$, the same scheme theoretic kernel of $\overline{x}^m\colon \mathbf{J} \to \mathbf{J}$.

3.4 Abelian restricted Lie algebras

Let k be a field of characteristic p. The abelian restricted Lie algebras over k are equivalent to abelian infinitesimal group schemes of height one. A skew-polynomial ring Ω is defined so that those of height one correspond to Ω-modules. Over a perfect field, Ω-modules are classified using a structure theorem for modules over the skew-PID Ω, due to N. Jacobson [Jac43], and over an algebraically closed field, all indecomposable modules over Ω are classified explicitly, also in [Jac43].

We conclude, when k is algebraically closed, an explicit classification of all abelian infinitesimal group schemes and their representation type. Those of finite and tame representation type are then shown to be noble K-rigid. Those of wild representation type shown to not be noble K-rigid.

3.4.1 The structure theorem

Let k be a field of characteristic $p > 0$. Let Ω denote the skew-polynomial ring on a single variable F, subject to $F\lambda = \lambda^p F$ for $\lambda \in k$. Then any restriction $[p]\colon \mathfrak{g} \to \mathfrak{g}$ on an abelian Lie algebra $\mathfrak{g}$ is equivalent to a left action of F on $\mathfrak{g}$, endowing the vector space $\mathfrak{g}$ with the structure of a left Ω-module. Therefore Ω-modules are also equivalent to abelian infinitesimal group schemes of height one. Jacobson [Jac43] showed that if k is a perfect field, then Ω is a skew-PID, and consequently that indecomposable Ω-modules are cyclic. Further, if k is algebraically closed, a complete description of cyclic modules are given: any cyclic left Ω-module is isomorphic to either $\Omega/\Omega(F-1)$ or $\Omega/\Omega(F^s)$ for some integer $s \geq 0$.

We summarize with the following theorems of abelian infinitesimal group schemes. Assume that k is an algebraically closed field of characteristic p below:

Theorem 3.4.1.1. *Let $\widetilde{G}$ be an abelian infinitesimal group scheme over k of height one. Then $\widetilde{G}$ has a direct sum decomposition as*

$$\widetilde{G} \cong \mu_p^r \oplus \sum_{i \geq 1} \mathbb{W}_{i(1)}^{s_i}$$

for some $r \geq 0$ and finitely many nonzeero $s_i \geq 0$.

For any $\widetilde{G}$ as in the theorem, the unipotent component of $\widetilde{G}$ (dual to the connected component of $\widetilde{G}^{\#}$) is the sum $\sum_{i \geq 1} \mathbb{W}_{i(1)}^{s_i}$, and the group algebra of the unipotent component coincides also with the principal block of the group algebra $k\widetilde{G}$. Therefore the representation type of $\widetilde{G}$ (finite, tame, wild) coincides with that of its unipotent component (see e.g. Erdmann [Erd90]). The representation type of any unipotent abelian infinitesimal group scheme of height one can be determined by noting that $k\mathbb{W}_{n(1)} \cong k[t]/t^{p^n}$, so any such unipotent $\widetilde{U}$ admits a deformation $\underline{\Gamma} \rightsquigarrow \widetilde{U}$ for some abelian p-group Γ.

Since deformations preserve representation type (indeed, even weak deformations), we deduce the following theorem as a straightforward consequence of the structure theorem, and from knowledge of the representation type of finite groups over positive characteristic fields [BD77].

Theorem 3.4.1.2. *Let $\widetilde{G}$ be an abelian infinitesimal group scheme over k of height one, with unipotent component $\widetilde{U}$, and let p^n be the dimension of the group algebra $k\widetilde{U}$.*

I. *If $\widetilde{U} \cong \mathbb{W}_{n(1)}$, then $\widetilde{G}$ is of finite representation type.*

II. *If $\widetilde{U}$ is not isomorphic to $\mathbb{W}_{n(1)}$, and $p^n = 4$ (i.e. $p = 2$ and $\widetilde{U} = \mathbb{G}^2_{a(1)}$), then $\widetilde{G}$ is of tame representation type.*

III. *In any other case, $\widetilde{G}$ is of wild representation type.*

Corollary 3.4.1.3. *Let $\widetilde{G}$ be an abelian infinitesimal group scheme over k of height one and of wild representation type. Suppose that $\widetilde{G}$ has no nontrivial wild direct summands (for any decomposition $\widetilde{G} \cong G_1 \oplus G_2$, if G_1 is of wild representation type then $G_2 = 0$).*

I. *If $p = 2$ then either $\widetilde{G} \cong \mathbb{G}_{a(1)} \oplus \mathbb{G}_{a(1)} \oplus \mathbb{G}_{a(1)}$, or $\widetilde{G} \cong \mathbb{W}_{n(1)} \oplus \mathbb{W}_{m(1)}$, for some $n+m \geq 3$ and $n, m \geq 1$.*

II. *If $p > 2$, then $\widetilde{G} \cong \mathbb{W}_{n(1)} \oplus \mathbb{W}_{m(1)}$ for $n, m \geq 1$.*

3.4.2 Finite representation type

Recall that $|\operatorname{Proj} H(\mathbb{W}_{n(1)}, k)|$ is given by a singleton $\{\mathfrak{p}\}$, so for any deformation $f\colon G' \rightsquigarrow \mathbb{W}_{n(1)}$, we have $|\operatorname{Proj} H(G', k)|$ is given by a singleton $\{\mathfrak{q}\}$, with $f_!\mathfrak{q} = \mathfrak{p}$. Supposing either $p = 2$ or $n = 1$, we can first prove the K-rigidity of each $\mathfrak{q}$, for an explicit class of deformations

$G' \rightsquigarrow \mathbb{W}_{n(1)}$, c.f. Theorem A.2.0.5. Recall that, since $\mathbb{W}_{n(1)}$ is unipotent, we have by Lemma 3.1.2.4 that any weak deformation $G_i \rightsquigarrow \mathbb{W}_{n(1)}$ is a deformation.

Definition 3.4.2.1. For each $0 \le i \le n$ we let G_i be the group scheme given by the kernel of the map $(\mathfrak{F} - \mathfrak{V}^i): \mathbb{W}_n \to \mathbb{W}_n$. We define a deformation $G_i \rightsquigarrow \mathbb{W}_{n(1)}$ as follows:

First assume $i > 0$, so that G_i is also the kernel of $(\mathfrak{F} - \mathfrak{V}^i): \mathbb{W}_{n(n)} \to \mathbb{W}_{n(n)}$. The coordinate algebra for $\mathbb{W}_{n(n)}$ is given by

$$\mathcal{O}(\mathbb{W}_{n(n)}) = \frac{k[x_0, \dots, x_{n-1}]}{x_0^{p^n}, \dots, x_{n-1}^{p^n}},$$

and we have that $\mathbb{W}_{n(n)}^{\#} \cong \mathbb{W}_{n(n)}$, so the algebra above can also be taken as the group algebra $k\mathbb{W}_{n(n)}$. Let x_{-1} denote 0 in $k\mathbb{W}_{n(n)}$. Then the maps $\mathfrak{F}, \mathfrak{V}^i$ are given at the level of group algebras $k\mathbb{W}_{n(n)} \to k\mathbb{W}_{n(n)}$ by

$$\mathfrak{F}: x_i \mapsto x_{i-1}$$
$$\mathfrak{V}^i: x_i \mapsto x_i^{p^i}.$$

The group algebra kG_i is then the Hopf subalgebra of $k\mathbb{W}_{n(n)}$ equalizing the maps $\mathfrak{F}$ and $\mathfrak{V}^i$. One can show that kG_i is generated by the element

$$t_i = \sum_{j=0}^{\infty} x_j^{p^{ij}} = \sum_{j=0}^{\lfloor \frac{n}{i} \rfloor} x_j^{p^{ij}},$$

such that $t_i \mapsto t \in k[t]/t^{p^n}$ defines an isomorphism of algebras $kG_i \to k\mathbb{W}_{n(1)}$, i.e. a deformation $G_i \rightsquigarrow \mathbb{W}_{n(1)}$. Let $f_i: G_i \rightsquigarrow \mathbb{W}_{n(1)}$ denote this deformation, for any n and for any $0 < i \le n$.

If $i = 0$, notice G_0 is the constant group scheme $\underline{\mathbb{Z}/p^n}$, so we have $kG_0 = k[g]/(g^{p^n} - 1)$. Then let $f_0: G_0 \rightsquigarrow \mathbb{W}_{n(1)}$ be defined by $g \mapsto t + 1$.

The following is the main result for this section:

Theorem 3.4.2.2. *Let k be a field of characteristic p, and let $\widetilde{G} = \mathbb{W}_{n(1)}$ be the group scheme over k. Suppose that $p = 2$, or that $n = 1$. Then, for $f_i: G_i \rightsquigarrow \widetilde{G}$ the deformation* *defined above, we have* $\mathfrak{q}$ *is* K*-rigid, where* $|\operatorname{Proj} H(G_i, k)| = \{\mathfrak{q}\}$*, for any* $0 \le i \le n$*.*

This theorem is proven by way of induction on $n \ge 1$, when $p = 2$. It also holds trivially for $n = 0$. Before proving this, we show how the corollary below follows from this theorem, proving the 'finite representation type' case of Theorem 3.2.3.3

Corollary 3.4.2.3. *Let* k *be an algebraically closed field of characteristic* $p > 0$ *and let* $\widetilde{G}$ *be an abelian infinitesimal group scheme of finite representation type and of height one over* k*. If* $p = 2$ *or the dimension of* kE *is* p*, for a maximal unipotent subgroup scheme* $E < \widetilde{G}$*, then* $\widetilde{G}$ *is noble* K*-rigid.*

Proof. From the Theorem 3.4.1.2, we have $\widetilde{G} \cong \mu_p^\ell \oplus \mathbb{W}_{n(1)}$ for some ℓ, n. In this case $E \cong \mathbb{W}_{n(1)}$, so we have assumed either that $p = 2$ or that $n = 1$. Since μ_p^ℓ is semisimple, we have $\widetilde{G}$ is noble K-rigid if and only if $\mathbb{W}_{n(1)}$ is noble K-rigid, by Lemma 3.2.2.5. We have that Gouthier's theorem A.2.0.5 classifies deformations $G' \rightsquigarrow \mathbb{W}_{n(1)}$ up to isomorphism of the group scheme G'. In particular, if we take $I = \{0, \ldots, n\}$, we get that our set $\{f_i : G_i \to \mathbb{W}_{n(1)}\}$ is a set of distinguished deformations satisfying property (a) of Lemma 3.2.2.6, and Theorem 3.4.2.2 says that the set also satisfies property (b), given our hypothesis on p or n. Therefore it suffices to show that any closed point in $\mathscr{X}_{\mathbb{W}_{n(1)}}$ is K-rigid with respect to any autodeformation.

Notice, there are, up to isomorphism, exactly p^n indecomposable representations of $\mathbb{W}_{n(1)}$, with only one indecomposable module of dimension i, for each $1 \le i \le p^n$. By Proposition 3.1.1.1, the pullback $\varphi^* : \mathfrak{R}(\mathbb{W}_{n(1)}) \to \mathfrak{R}(\mathbb{W}_{n(1)})$ of any autodeformation $\varphi : \mathbb{W}_{n(1)} \rightsquigarrow \mathbb{W}_{n(1)}$ must then be the identity on $\mathfrak{R}(\mathbb{W}_{n(1)})$, since the pullback of any weak deformation is always compatible with the dimension character. □

3.4.2.4. The case of $n = 1$: Supposing that $n = 1$, we have only one nontrivial deformation, namely $G_0 \cong \mathbb{Z}/p \rightsquigarrow \mathbb{W}_{n(1)} \cong \mathbb{G}_{a(1)}$, for which we must check K-rigidity. But K-rigidity is just a restatement of our Theorem 2.3.2.1 from the previous chapter, so there's nothing more to prove.

3.4.2.5. The case of $p = 2$: Notice, for arbitrary n and p, we have a projection $\mathbb{W}_{n(1)} \to \mathbb{W}_{(n-1)(1)}$. By the same machinery as demonstrated in A.2.0.6, these projections are compatible with all of our deformations f_i, in the sense that there exists a projection of G_i to a finite group scheme G_i' (equal to some G_j, by an abuse of notation, since these group schemes are not labeled correctly to indicate how they depend on n) admitting a deformation $G_i' \rightsquigarrow \mathbb{W}_{(n-1)(1)}$ (by the same abuse of notation, this deformation is one of our f_j) and a commutative diagram

$$\begin{tikzcd} G_i \arrow[r, rightsquigarrow] \arrow[d, two heads] & \mathbb{W}_{n(1)} \arrow[d, two heads] \\ G_i' \arrow[r, rightsquigarrow] & \mathbb{W}_{(n-1)(1)} \end{tikzcd}.$$

Now, for any $0 \le m \le p^{n-1}$, the unique indecomposable $\mathbb{W}_{n(1)}$-representation J_m of dimension m over k is actually the pullback of the same J_m along $\mathbb{W}_{(n-1)(1)}$. It is easily checked that the indecomposable $\mathbb{W}_{n(1)}$-representation J_{p^n-m} is (stably) isomorphic to the syzygy module ΩJ_m for any $1 \le m \le p^n$.

Therefore, in the case of $p = 2$, every indecomposable $\mathbb{W}_{n(1)}$-representation is either a pullback of an indecomposable module along the projection, or a syzygy thereof. K-rigidity then follows by induction on n, using $n = 1$ as a base case.

3.4.3 Tame representation type implies noble K-rigid

First we define three nontrivial deformations $G' \rightsquigarrow \widetilde{G} = \mathbb{G}_{a(1)}^2$, and an explicit description of their noble closed points, c.f. Theorem A.2.0.4. Notice, since $\widetilde{G}$ is unipotent, any weak deformation is a deformation, by Lemma 3.1.2.4.

3.4.3.1. We'll fix coordinates for the group algebra as

$$k\widetilde{G} = k[t_1, t_2]/(t_1^p, t_2^p) = \mathcal{O}(\widetilde{G}),$$

with t_1, t_2 primitive elements, defining a self-dual Hopf algebra. Consider the group schemes $G_1 = \mathbb{G}_{a(2)}$, $G_2 = \mathbb{G}_{a(1)} \oplus \mathbb{Z}/p$, and $G_3 = (\mathbb{Z}/p)^2$. A deformation

$$f_1 : G_1 \rightsquigarrow \widetilde{G}$$

can be defined from dualizing a coalgebra isomorphism

$$\mathcal{O}(\mathbb{G}_{a(1)}^2) = k[t_1, t_2]/(t_1^p, t_2^p) \to k[t]/t^{p^2} = \mathcal{O}(\mathbb{G}_{a(2)})$$

defined on monomials by

$$t_1^i t_2^j \mapsto t^{i+pj}.$$

One can show this is indeed a coalgebra isomorphism using that the coordinates t_1, t_2, t, t^p are all primitive in their respective Hopf algebras.

We also get two deformations

$$f_2\colon G_2 \rightsquigarrow \widetilde{G}, \quad f_3\colon G_3 \rightsquigarrow \widetilde{G},$$

using the deformation $\mathbb{Z}/p \rightsquigarrow \mathbb{G}_{a(1)}$ discussed in the previous section, and taking products.

3.4.3.2. Every point of $\widetilde{\mathscr{X}}$ is noble, given that $\widetilde{G}$ is infinitesimal of height one. The variety $\widetilde{\mathscr{X}}$ is identified with $\mathbb{P}^1$, such that the closed point given by homogeneous coordinates $[a : b] \in \mathbb{P}^1$ is prewitnessed by

$$k[t]/t^p \to k[t_1, t_2]/(t_1^p, t_2^p),$$

defined as $t \mapsto at_1 + bt_2$. Taking t to be primitive makes the map an inclusion of group algebras, and a π-point over k, and hence a witness for the closed point $[a : b]$. Below, we describe the noble points of each spectrum $\mathscr{X}_1, \mathscr{X}_2, \mathscr{X}_3$, by calculating their image in $\widetilde{\mathscr{X}}$ under the homeomorphism $f_i!$, for each deformation $f_i\colon G_i \rightsquigarrow \widetilde{G}$.

1. The subgroup schemes of $G_1 = \mathbb{G}_{a(2)}$ are the trivial group scheme, the canonically defined subgroup scheme $\mathbb{G}_{a(1)}$, and G_1 itself. Of these, the only group algebra inclusion which is a prewitness for a closed point in $\mathscr{X}_1$ is $k\mathbb{G}_{a(1)} \hookrightarrow kG_1$. Indeed, the inclusion is a π-point over k, and hence witnesses some closed $\mathfrak{q} \in \mathscr{X}_1$. Then $\mathfrak{q}$ is the only noble point of $\mathscr{X}_1$, and, given the deformation $f_1\colon G_1 \rightsquigarrow \widetilde{G}$, we have that $f_1!(\mathfrak{q}) = [1 : 0] \in \mathbb{P}^1 = \widetilde{\mathscr{X}}$. To see this, notice that the noncanonical comultiplication $\Delta_1\colon k\widetilde{G} \to k\widetilde{G} \otimes k\widetilde{G}$ (from 3.1.2.6) defines the element $t_1 \in k\widetilde{G}$ to be primitive, and the

inclusion $k\mathbb{G}_{a(1)} \hookrightarrow kG_1$ of group algebras is essentially the inclusion of the subalgebra generated by t_1 into $k\widetilde{G}$.

2. Any subgroup scheme H of G_2, being abelian, is the product of a connected group H^o with a constant group scheme Γ. Further, the inclusion $H \hookrightarrow G_2 = \mathbb{G}_{a(1)} \times \underline{\mathbb{Z}/p}$ factors as the direct product of inclusions $H^o \hookrightarrow \mathbb{G}_{a(1)}$ and $\Gamma \hookrightarrow \underline{\mathbb{Z}/p}$. We conclude that there are four subgroup schemes of G_2, only two of which, given by $\mathbb{G}_{a(1)} \times \underline{1}$ and $\underline{1} \times \underline{\mathbb{Z}/p}$, have an inclusion that is a prewitnesses for a closed point in $\mathscr{X}_2$. Further, the two inclusions are π-points and hence witnesses. Given the deformation $f_2\colon G_2 \rightsquigarrow \widetilde{G}$, the image under $f_2!$ of the points witnessed by these inclusions are $[1:0]$ and $[0:1]$ in $\mathbb{P}^1 = \widetilde{\mathscr{X}}$. This can, again, be seen easily by calculating the noncanonical comultiplication $\Delta_2\colon k\widetilde{G} \to k\widetilde{G} \otimes k\widetilde{G}$.

3. Viewing the group $\Gamma = (\mathbb{Z}/p)^2$ as a two-dimensional vector space over $\mathbb{F}_p$, we see that the subgroups of Γ are all $\mathbb{F}_p$-subspaces, and so the proper and nontrivial subgroups are all one-dimensional. Hence there are $p+1$ subgroup schemes of the constant group scheme $G_3 = \underline{\Gamma}$, which are proper, and nontrivial. The only inclusions of subgroup schemes of $\underline{\Gamma}$ which are prewitnesses for a closed point in $\mathscr{X}_3$ are proper and nontrivial, hence a π-point, and hence a witness. Given the deformation $f_3\colon G_3 \rightsquigarrow \widetilde{G}$, the images, under the homeomorphism $f_3!$, of the primes witnessed by the nontrivial proper subgroups, are $[1:i] \in \mathbb{P}^1$, for $0 \le i \le p-1$, and $[0:1]$. This can, again, be seen easily by calculating the noncanonical comultiplication $\Delta_3\colon k\widetilde{G} \to k\widetilde{G} \otimes k\widetilde{G}$.

The main result of this section is the following:

Theorem 3.4.3.3. *Let k be an algebraically closed field of characteristic $p = 2$, and let $\widetilde{G} = \mathbb{G}_{a(1)}^2$ be the group scheme over k. Then for the deformations $f_i\colon G_i \rightsquigarrow \widetilde{G}$ defined above, we have that every noble point $\mathfrak{q} \in \mathscr{X}_i$ is K-rigid with respect to f_i, for $i = 1, 2, 3$.*

This theorem will be proven using direct computations, inspired by the original work of Bašev [Baš61] and Conlon [Con65] on the modular ring of representations for the Klein four group. Before proving this, we show how the corollary below follows from this theorem, proving the 'tame representation type' case of Theorem 3.2.3.3.

Corollary 3.4.3.4. *Let k be an algebraically closed field of characteristic $p > 0$ and let $\widetilde{G}$ be an abelian infinitesimal group scheme of tame representation type, and of height one, over k. Then $\widetilde{G}$ is noble K-rigid.*

Proof. By Theorem 3.4.1.2, we have $p = 2$ and $\widetilde{G} \cong \mu_p^\ell \oplus \mathbb{G}_{a(1)}^2$ for some ℓ. Since μ_p^ℓ is semisimple, we have $\widetilde{G}$ is noble K-rigid if and only if $\mathbb{G}_{a(1)}^2$ is noble K-rigid, by Lemma 3.2.2.5. The classification of X. Wang A.2.0.4 tells us that, for any deformation $f'\colon G' \rightsquigarrow \mathbb{G}_{a(1)}^2$, there is an isomorphism of group schemes $\varphi\colon G' \xrightarrow{\sim} G''$ for some $G'' \in \{\widetilde{G}, G_1, G_2, G_3\}$. So, taking $\{\widetilde{f} = \mathrm{id}_{\widetilde{G}}, f_1, f_2, f_3\}$ to be our class of 'distinguished deformations', the hypothesis (a) of Lemma 3.2.2.6 is met. By Theorem 3.4.3.3, the hypothesis (b) of Lemma 3.2.2.6 is met. Using direct computations of the autodeformation group of $\widetilde{G}$, and an explicit classification of indecomposable $\widetilde{G}$-representations, Lemma 3.4.3.8 below gives us that every closed point in $\widetilde{\mathscr{X}}$ is K-rigid with respect to any autodeformation $\widetilde{G} \rightsquigarrow \widetilde{G}$. Hence, we apply Lemma 3.2.2.6 and conclude that $\widetilde{G}$ is noble K-rigid. $\square$

If $p = 2$, then the indecomposable $\mathbb{G}_{a(1)}^2$-representations are classified by Conlon's work [Con65] on the modular representations of the Klein 4-group. With the assumption that k is algebraically closed, the classification goes back to Bašev [Baš61], and, ultimately, Kronecker's work on quadratic and bilinear forms [Kron]. If k is algebraically closed and of characteristic $p = 2$, then for each $\mathfrak{p} \in \widetilde{\mathscr{X}}$, we can explicitly describe all of the indecomposable $\widetilde{G}$-representations V, over k, with $\widetilde{\mathscr{X}}(V) = \{\mathfrak{p}\}$. The classification of all indecomposable $\widetilde{G}$-representations over k follows from the classification of modular representations of $\mathbb{Z}/2\times\mathbb{Z}/2$, due to Bašev [Baš61]. We summarize what we need from this classification in the proposition below.

Proposition 3.4.3.5. *Let $p = 2$ and $\widetilde{G} = \mathbb{G}_{a(2)}$. Let M denote a vector space of dimension $2n$, with k-linear decomposition into* lower and upper *blocks $M = M_\ell \oplus M_u$, with M_ℓ, M_u each of dimension n. We let $V_{2n} = M$ denote the A-module defined by the following matrix representations of the actions of $t_1, t_2 \in A$*

$$T_1 : \begin{pmatrix} 0 & I_n \\ 0 & 0 \end{pmatrix}, \quad T_2 : \begin{pmatrix} 0 & \mathfrak{N}_n \\ 0 & 0 \end{pmatrix},$$

where I_n is the diagonal ones matrix and $\mathfrak{N}_n$ is an upper triangular nilpotent Jordan block of rank $n-1$.

Then we have that

I. *The representation V_{2n} is indecomposable,*

II. *The support $\widetilde{\mathscr{X}}(V_{2n})$ is $\{[0:1]\} \subset \mathbb{P}^1$, and*

III. *Any finite indecomposable $\mathbb{G}_{a(1)}$-representation V with support $\widetilde{\mathscr{X}}(V) = \{[0:1]\}$ is of even dimension $2n$ and is isomorphic to V_{2n}, for some n.*

3.4.3.6. Consider the group of autodeformations $\mathrm{Def}(\widetilde{G}, \widetilde{G})$, i.e. of (augmented) automorphisms of the local artinian algebra $A = k[t_1, t_2]/(t_1^2, t_2^2)$. Each can restrict to a k-linear automorphism in $\mathrm{GL}(J)$ of the Jacobson radical $J \subset A$ and induces a well-defined automorphism on J/J^2. This defines a homomorphism $\mathrm{Def}(\widetilde{G}, \widetilde{G}) \to \mathrm{GL}(J/J^2)$, and this is a split epimorphism. Indeed, using the basis t_1, t_2 for J/J^2, there is an exact sequence of algebraic groups

$$\mathbb{G}_a^2 \to \mathrm{Aut}(A) \to \mathrm{GL}_2$$

using the basis t_1, t_2 for J/J^2, and for $\alpha, \beta \in k$, the element $(\alpha, \beta) \in \mathbb{G}_{a(1)}^2(k)$ defines the homomorphism taking

$$t_1 \mapsto t_1 + \alpha t_1 t_2, \quad t_2 \mapsto t_2 + \beta t_1 t_2.$$

The canonical action of GL_2 on $\mathbb{G}_{a(1)}^2 \cong \langle t_1, t_2 \rangle_a$ defines the splitting.

3.4.3.7. Let GL_2 act on $\mathbb{P}^1$ by canonical isomorphism $\mathrm{GL}_2/B^- \xrightarrow{\sim} \mathbb{P}^1$ (§1.5). Then the action of $\mathrm{Def}(\widetilde{G}, \widetilde{G})$ on $\widetilde{\mathscr{X}}$ is identical to that of the pullback along $\mathrm{Def}(\widetilde{G}, \widetilde{G}) \twoheadrightarrow \mathrm{GL}_2$ of the GL_2-action on $\widetilde{\mathscr{X}} \cong \mathbb{P}^1$.

Then, for each $\varphi \in \mathrm{Def}(\widetilde{G}, \widetilde{G})$, there is a $\widetilde{G}$-representation $V_{2n}^{\varphi} = \varphi^* V_{2n}$. By Proposition 3.4.3.5, we have V_{2n}^{φ} is indecomposable, with support $\widetilde{\mathscr{X}}(V_{2n}^{\varphi}) = \{\varphi_![0:1]\} \subset \mathbb{P}^1$, and any finite indecomposable $\mathbb{G}_{a(1)}$-representation V with suppport $\widetilde{\mathscr{X}}(V) = \{\varphi_![0:1]\}$ is of even dimension $2m$ and is isomorphic to V_{2n}^{φ}, for some m.

We define $V_{2n}([1:0]) = V_{2n}^{\psi}$ for $\psi \in \mathrm{Aut}(A)$ the involution permuting t_1 and t_2, and for any $a \in k$, we define $V_{2n}([a:1]) = V_{2n}^{\phi}$ for ϕ the automorphism fixing t_1 and taking $t_2 \mapsto at_1 + t_2$. Then $\psi![0:1] = [1:0]$ and $\phi![0:1] = [a:1]$, so we have now defined a representation $V_{2n}(\mathfrak{p})$ which is, up to isomorphism, the unique indecomposable representation of dimension $2n$ and support $\{\mathfrak{p}\}$, for each closed point $\mathfrak{p} = [a:b] \in \mathbb{P}^1$. Lastly, we define P to be the free A-module of rank 1, so that P is the unique indecomposable projective $\widetilde{G}$-representation (A being a local ring). Then any indecomposable direct summand of a representation V with $\widetilde{\mathscr{X}}(V) = \{\mathfrak{p}\}$ is isomorphic to P, or $V_{2n}(\mathfrak{p})$ for some n.

Lemma 3.4.3.8. *Let k be an algebraically closed field of characteristic $p = 2$, and let $\widetilde{G} = \mathbb{G}_{a(1)}^2$. Then $\widetilde{G}$ is K-rigid with respect to any autodeformation $\varphi\colon \widetilde{G} \rightsquigarrow \widetilde{G}$.*

Proof. The split surjection $\mathrm{Def}_{\mathrm{IV}}(\widetilde{G}, \widetilde{G}) \twoheadrightarrow \mathrm{GL}(J/J^2)$ defined before gives a decomposition

$$\mathrm{Def}_{\mathrm{IV}}(\widetilde{G}, \widetilde{G}) = \mathrm{GL}_2(k) \cdot \mathbb{G}_a^2(k),$$

where the subgroup $\mathrm{GL}_2(k)$, given by the image of the splitting map, is identified also as the subgroup $\mathrm{Aut}(\widetilde{G}) < \mathrm{Def}_{\mathrm{IV}}(\widetilde{G}, \widetilde{G})$ of group scheme automorphisms. Each element $\varphi \in \mathrm{GL}_2(k)$, being a group automorphism, has that any point in $\widetilde{\mathscr{X}}$ is K-rigid with respect to φ. Then it is sufficient to show that an arbitrary $\mathfrak{p} \in \widetilde{\mathscr{X}}$ is K-rigid with respect to those autodeformations in the subgroup $\mathbb{G}_a^2(k)$.

We've already seen in 3.4.3.7 that the pullback φ^* along any $\varphi \in \mathbb{G}_a^2(k)$ is actually the identity on any indecomposable representation of singleton support. Therefore the induced map $\varphi^*\colon \widetilde{\mathfrak{R}}_{\mathfrak{p}} \to \widetilde{\mathfrak{R}}_{\mathfrak{p}}$ is the identity, and hence an isomorphism of rings, so $\mathfrak{p}$ is K-rigid with respect to $\varphi\colon k\widetilde{G} \rightsquigarrow k\widetilde{G}$. $\square$

The following lemmas are adapted from [Baš61]. Bašev's lemma, discussed in the appendix, is a purely ring-theoretic statement that helps determine the structure of the ring $\widetilde{\mathfrak{R}}_{\mathfrak{p}}$, and $\mathfrak{R}'_{\mathfrak{q}}$ for any deformation $G' \rightsquigarrow \widetilde{G}$. Bašev's calculations of $\mathfrak{R}'_{\mathfrak{p}}$ (for $G' = \mathbb{Z}/2 \times \mathbb{Z}/2$) were in error for non-noble $\mathfrak{p}$, and I suspect this means that Bašev might have been mistaken in how to apply his lemma, which was itself in need of additional hypotheses. Chapter B of the appendix revises and expands Bašev's lemma, so that Bašev's original argument applies and gets us the true calculations of the ring $\widetilde{\mathfrak{R}}_{\mathfrak{p}}$.

Lemma 3.4.3.9. *Let $n \leq m$, and $\mathfrak{p} \in \mathbb{P}^1$ be any closed point. Let $f\colon G' \rightsquigarrow \widetilde{G}$ be any deformation. Then there exists an isomorphism of G'-representations*

$$f^*V_{2n}(\mathfrak{p}) \otimes f^*V_{2m}(\mathfrak{p}) \cong f^*V \oplus (mn-n)f^*P,$$

where $V = \sum c_{n,m,\ell}(\mathfrak{p})V_{2\ell}(\mathfrak{p})$, for some Clebsch–Gordon coefficients $c_{n,m,\ell}(\mathfrak{p})$ relative to G', satisfying $\sum c_{n,m,\ell}(\mathfrak{p})2\ell = 4n$. Equivalently, we may also say $c_{n,m,P}(\mathfrak{p}) = mn - n$.

Lemma 3.4.3.10. *Let $n \leq m$, and $\mathfrak{p} \in \mathbb{P}^1$ be any closed point. Let $f\colon G' \rightsquigarrow \widetilde{G}$ be any deformation. Let $V = \sum c_{n,m,\ell}(\mathfrak{p})V_{2\ell}(\mathfrak{p})$ such that*

$$f^*V_{2n}(\mathfrak{p}) \otimes f^*V_{2m}(\mathfrak{p}) \cong f^*V \oplus (mn-n)f^*P,$$

as in the previous lemma. Then

I. $c_{n,m,\ell}(\mathfrak{p}) = 2\delta_{n,\ell}$ whenever there is strict inequality $n < m$, and

II. For any $\mathfrak{p}$, there exists a subset $N(\mathfrak{p}) \subset \mathbb{N}$, such that no two consecutive numbers are elements of $N(\mathfrak{p})$, and such that

$$c_{n,n,\ell}(\mathfrak{p}) = \begin{cases} 2\delta_{n,\ell} & n \notin N(\mathfrak{p}), \\ \delta_{n-1,\ell} + \delta_{n+1,\ell} & n \in N(\mathfrak{p}). \end{cases}$$

By these lemmas, we have that a given noble point $\mathfrak{q} \in \mathscr{X}'$ is K-rigid with respect to each distinguished deformation $f\colon G' \rightsquigarrow \widetilde{G}$, if and only if the subset $N(f_!\mathfrak{q}) \subset \mathbb{N}$ is the same as that which is defined by the trivial deformation. For this, we show, for each distinguished $f\colon G$ and noble $\mathfrak{q} \in \mathscr{X}'$, that $N(f_!\mathfrak{q})$ is actually empty, i.e. that

$$f^*V_{2n}(f_!\mathfrak{q}) \otimes f^*V_{2n}(f_i!\mathfrak{q}) \cong 2f^*V_{2n}(f_!\mathfrak{q}) \oplus (n^2-n)P. \tag{3.4.3.11}$$

for each n, so that the ring structure $\mathfrak{R}'_{\mathfrak{q}}$ agrees with a straightforward generalization of the original unproven formulas of [Baš61] and [Con65], which apply directly only to the case of the distinguished deformation $f = f_3 \colon (\underline{\mathbb{Z}/2})^2 \rightsquigarrow \widetilde{G}$.

Definition 3.4.3.12. Recall our definition for the $\widetilde{G}$-representation V_{2n} by upper and lower blocks 3.4.3.5. For $n, m \in \mathbb{N}$ we define the *canonical extension*

$$0 \to V_{2n} \to V_{2(n+m)} \to V_{2m} \to 0$$

of V_{2m} by V_{2n}, with *canonical mono* $\mu_{n,m} \colon V_{2n} \hookrightarrow V_{2(n+m)}$ defined by the induced map from including ordered bases for respective upper and lower blocks (3.4.3.5), and *canonical epi* $\epsilon_{n,m} \colon V_{2(n+m)} \twoheadrightarrow V_{2m}$ by the quotient defined on the ordered bases for upper and lower blocks. It is clear the canonical monos and epis are A-linear and define an extension of A-modules.

If $V = \sum_\ell c_\ell V_{2\ell}$ is a finite module with no projective direct summand and $\nu \colon V_{2n}(\mathfrak{p}) \hookrightarrow V$ is a monomorphism, we also say that ν is canonical if there exists some $c_\ell > 0$, for $\ell > n$ and a direct summand inclusion $\iota \colon V_{2\ell} \hookrightarrow V$ such that ν is isomorphic to $\iota \circ \mu_{n,\ell-n}$.

For ψ, ϕ the automorphisms of 3.4.3.7, the image under ψ^*, ϕ^* of any canonical extension (resp. mono, epi) is also called the canonical extension (resp. mono, epi) of $V_{2m}([a:b])$ by $V_{2n}([a:b])$.

3.4.3.13. Notice, the lower block of $V_{2\ell}$, as defined in Theorem 3.4.3.5, agrees as a vector space with the socle submodule $\mathrm{Soc}^1(V_{2\ell})$, while the upper block agrees canonically with the vector space $\mathrm{Soc}^2(V_{2\ell})/\mathrm{Soc}^1(V_{2\ell})$. We extend the notion of lower (resp. upper) block to any $V = \sum_\ell c_\ell V_{2\ell}$ to be the sum of lower (resp. upper) blocks of each direct summand. Thus, the fact about the socle series remains true in general for such V. Therefore we know the lower block of V_{2n} always maps into the lower block of V for any such V. Further, if $V_{2n} \to V$ is a monomorphism, we have that each nonzero vector in the upper block of V_{2n} maps to a vector in V with nonzero upper block component in V (with respect to the basis of upper and lower blocks of V).

Let H denote the subgroup of $\widetilde{G}$ witnessing the point $[0:1]$. Then the restriction $V_{2\ell} \downarrow_H^{\widetilde{G}}$ is isomorphic to $2J_1 \oplus (n-1)J_2$, where a canonical basis for this H-representation agrees with total basis for upper and lower blocks given in Theorem 3.4.3.5. Each copy of J_2

intersects nontrivially with both the lower and upper blocks, while one copy of J_1 is in the upper block and the other is in the lower. The subspace J_1 which is in the upper block we call the 'near point' and the subspace J_1 which is in the lower block we call the 'far point' of $V_{2\ell}$.

Notice, the near point of any $V_{2\ell}$ is annihilated by s_2, and the upper block intersected with the submodule of $V_{2\ell}$ annihilated by s_1 is precisely the near point. Hence, if $\nu\colon V_{2n} \to V$ is a monomorphism, the image of the near point of V_{2n} is in the sum of near points of the direct summands of V. In this case, $\nu \downarrow_H^{\widetilde{G}}$, restricted to the near point of $V_{2n} \downarrow_H^{\widetilde{G}}$, is a split monomorphism.

Now we give an original proof of the Bašev–Conlon formula 3.4.3.11, using nobility in an essential way.

Theorem 3.4.3.14. *Let $f\colon G' \rightsquigarrow \widetilde{G}$ be a distinguished deformation in the set $\{\mathrm{id}, f_1, f_2, f_3\}$ and $c_{n,m,\ell}(\mathfrak{p})$ the Clebsch–Gordon coefficients relative to f and $\mathfrak{q} \in \mathscr{X}'$ be a noble point, with $\mathfrak{p} = f_!\mathfrak{q} \in \mathbb{P}^1$. Then $c_{n,n,\ell}(\mathfrak{p}) = 2\delta_{n,\ell}$ for each n, ℓ, i.e.*

$$f^*V_{2n}(\mathfrak{p}) \otimes f^*V_{2n}(\mathfrak{p}) \cong 2f^*V_{2n}(\mathfrak{p}) \oplus (n^2 - n)f^*P$$

as G'-representations.

Proof. Since $\mathfrak{q}$ is noble, let $C < G'$ be a subgroup scheme with inclusion of group algebras witnessing $\mathfrak{q} \in \mathscr{X}'$. Then C is among the subgroup schemes discussed in 3.4.3.2, and in particular $kC \cong k[t]/t^2$. We will then refer to C-representations as modules over kC, which are direct sums of the Jordan blocks J_i, $1 \le J_i \le p$.

Then for $n = 1$, we have

$$(f^*V_2(\mathfrak{p}) \otimes f^*V_2(\mathfrak{p})) \downarrow_C^{G'} \cong f^*V_2(\mathfrak{p}) \downarrow_C^{G'} \otimes f^*V_2(\mathfrak{p}) \downarrow_C^{G'},$$

which by direct computation gives $(f^*V_2(\mathfrak{p}) \otimes f^*V_2(\mathfrak{p})) \downarrow_C^{G'} \cong 4J_1$. Also by direct computation, we have $f^*V_4(\mathfrak{p}) \downarrow_C^{G'} \cong J_2 \oplus 2J_1$, we conclude that

$$f^*V_2(\mathfrak{p}) \otimes V_2(\mathfrak{p}) \cong 2f^*V_2(\mathfrak{p})$$

as G'-representations, as 3.4.3.5 shows how this is the unique kG'-module of dimension 4 up to isomorphism with the required restriction to C and a support of $\{\mathfrak{q}\} \subset \mathscr{X}'$. We find also that

$$f^*V_4(\mathfrak{p}) \otimes f^*V_4(\mathfrak{p}) \cong 2f^*V_4(\mathfrak{p}) \oplus 2f^*P,$$

but we have not found an easy reason for this. Instead this follows from direct computations for the 16-dimensional module in each case of $G' = \widetilde{G}, G_1, G_2, G_3$ and each noble $\mathfrak{q}$. By twisting by sufficiently many group scheme automorphisms, there are only five cases: one for each $\widetilde{G}, G_1, G_3$, and two for G_2.

For our fixed $\mathfrak{p} = f_!\mathfrak{q}$, we abuse notation and denote $V_{2n} = f^*V_{2n}(\mathfrak{p})$ to be the G'-representation for each n (overwriting our notation for the $\widetilde{G}$-representations in the initial construction 3.4.3.5).

Consider now the monomorphism

$$V_{2(n-1)} \otimes V_{2(n-1)} \xrightarrow{\mu = \mu_{n-1,1} \otimes \mu_{n-1,1}} V_{2n} \otimes V_{2n}.$$

Now, for each non-projective direct sum inclusion $\iota: V_{2\ell} \hookrightarrow V_{2(n-1)} \otimes V_{2(n-1)}$, we want to determine the structure of the composition $\mu \circ \iota: V_{2\ell} \to V$ in terms of canonical monos.

Let $A = kG'$, and B the subalgebra kC. Then B is generated by an element t, with $t^2 = 0$, such that t acts on each C-representation $V_{2n} \downarrow^A_B$ by the matrix T_2 of 3.4.3.5. We proceed by treating B as a Hopf subalgebra of A, and by casework that either t is primitive or $t + 1$ is grouplike. Then the restriction property for the tensor product gives us

$$V_{2(n-1)} \downarrow^A_B \otimes V_{2(n-1)} \downarrow^A_B \xrightarrow{\mu\downarrow^A_B = \mu_{n-1,1}\downarrow^A_B \otimes \mu_{n-1,1}\downarrow^A_B} V_{2n} \downarrow^A_B \otimes V_{2n} \downarrow^A_B .$$

Notice, identifying $V_{2\ell} \downarrow^A_B \cong 2J_1 \oplus (\ell - 1)J_2$ for each ℓ, with either copy of J_1 being a 'near point' or a 'far point' (c.f. 3.4.3.13), we see any finite A-module M of support $\{\mathfrak{q}\}$ restricts to a B-module of the form $2iJ_1 \oplus jJ_2$ for some i, j, and we may identify i with the number of non-projective indecomposable direct summands. Notice it follows immediately that each $V_{2\ell} \otimes V_{2\ell}$ has 2 non-projective indecomposable direct summands, which we could also know from Lemma 3.4.3.10.

By Definition 3.4.3.12 we examine the restricted canonical mono

$$\mu_{n-1,n}\downarrow_B^A: 2J_1 \oplus (n-2)J_2 \to 2J_1 \oplus (n-1)J_2,$$

and see that it takes the far point of $V_{2(n-1)}$ to the socle of a copy of J_2, and when restricted to any other direct summand of $V_{2(n-1)}\downarrow_B^A$ is split. By direct computation the monomorphism

$$\mu_{n-1,n}\downarrow_B^A \otimes \mu_{n-1,n}\downarrow_B^A: 4J_1 \oplus 2(n^2-2n)J_2 \to 4J_1 \oplus 2(n^2-1)J_2$$

sends three copies of J_1 into some copy of J_2, and every other direct summand splits. We conclude that the cokernel of μ restricts to a B-module isomorphic to $6J_1 \oplus (4n-5)J_2$.

Consider the cochain complex given as

$$Y: 0 \to V_{2n} \xrightarrow{\epsilon_{n-1,1}} V_2 \to 0$$

with V_{2n} regarded as degree 0 and V_2 regarded as degree 1. By the canonical extension, we know $H^0(Y) = V_{2(n-1)}$ and $H^1(Y) = 0$. The tensor product of complexes gives

$$Y \otimes Y: 0 \to V_{2n} \otimes V_{2n} \xrightarrow{\partial^0} V_{2n} \otimes V_2 \oplus V_2 \otimes V_{2n} \xrightarrow{\partial^1} V_2 \otimes V_2 \to 0,$$

where $\partial^0 = 1_{V_{2n}} \otimes \epsilon_{n-1,1} + \epsilon_{n-1,1} \otimes 1_{V_{2n}}$ and $\partial^1 = \epsilon_{n-1,1} \otimes 1_{V_2} + 1_{V_2} \otimes \epsilon_{n-1,1}$. By the Künneth theorem we have $H^0(Y \otimes Y) = V_{2(n-1)} \otimes V_{2(n-1)}$ and $Y \otimes Y$ is exact in every other degree. Indeed, we have an exact sequence

$$0 \to V_{2(n-1)} \otimes V_{2(n-1)} \xrightarrow{\mu} V_{2n} \otimes V_{2n} \xrightarrow{\partial^0} V_{2n} \otimes V_2 \oplus V_2 \otimes V_{2n} \xrightarrow{\partial^1} V_2 \otimes V_2 \to 0.$$

Let M denote the cokernel of μ. Then M is also the image of ∂^0 and hence the kernel of ∂^1. We calculated $M\downarrow_B^A \cong 6J_1 \oplus (4n-5)J_2$ already, hence M has precisely three indecomposable direct summands which are not projective. By Lemma 3.4.3.10, we know $V_{2n} \otimes V_2 \oplus V_2 \otimes V_{2n} \cong 4V_2 \oplus 2(n-1)P$, and we already know $V_2 \otimes V_2 \cong 2V_2$. Now we have a short exact sequence

$$0 \to M \to 4V_2 \oplus 2(n-1)P \to 2V_2 \to 0,$$

and we can conclude $M \cong 2V_2 \oplus V_4 \oplus (2n-3)P$. Starting with the short exact sequence

$$0 \to V_{2(n-1)} \otimes V_{2(n-1)} \to V_{2n} \otimes V_{2n} \to M \to 0,$$

we delete the projective component of the left hand side with rank $n^2 - 3n + 2$ and of the right hand side with rank $2n - 3$, and what remains is a short exact sequence

$$0 \to V' \to V \oplus P \to 2V_2 \oplus V_4 \to 0 \tag{3.4.3.15}$$

where V' is the sum of non-projective summands of $V_{2(n-1)} \otimes V_{2(n-1)}$ and V is the sum of non-projective summands of $V_{2n} \otimes V_{2n}$.

The last thing we need before we proceed by contradiction is to repeat essentially the same considerations as before, now for the canonical extension of V_4 by $V_{2(n-1)}$: we denote $\mu' = \mu_{n-1,2} \otimes \mu_{n-1,2}$, and we have

$$\mu' \downarrow_B^A = \mu_{n-1,2} \downarrow_B^A \otimes \mu_{n-1,2} \downarrow_B^A .$$

We compute directly using $V_{2\ell} \downarrow_B^A \cong 2J_1 \oplus (\ell - 1)J_2$ and the structure of $\mu_{\ell,2} \downarrow_B^A$ that the cokernel M' of μ' restricts to a B-module isomorphic to $6J_1 \oplus (8n-3)J_2$. We know then that M' has precisely three indecomposable direct summands which are not projective. Notice,

$$\mu' = (\mu_{n,1} \otimes \mu_{n,1}) \circ (\mu_{n-1,1} \circ \mu_{n-1,1}),$$

since by Definition 3.4.3.12 we know $\mu_{n-1,2} = \mu_{n,1} \circ \mu_{n-1,1}$. The argument above which computed the cokernel of μ, with $n+1$ in place of n, gives us another exact sequence

$$0 \to V \to V'' \oplus P \to 2V_2 \oplus V_4 \to 0 \tag{3.4.3.16}$$

where V'' is the sum of non-projective summands of $V_{2(n+1)} \otimes V_{2(n+1)}$. Since $V' \to V \oplus P$ is a direct summand of $\mu_{n-1,1} \otimes \mu_{n-1,1}$ and $V \to V'' \oplus P$ is a direct summand of $\mu_{n,1} \otimes \mu_{n,1}$, we have the composition

$$V' \to V \oplus P \to V'' \oplus 2P$$

is a direct summand of μ'. It follows that the cokernel of the composition above is a direct summand of M'.

Suppose for contradiction that for some $n > 1$, we have $V \ncong 2V_{2n}$. By Lemma 3.4.3.10 it follows that $V \cong V_{2(n-1)} \oplus V_{2(n+1)}$, that $V' \cong 2V_{2(n-1)}$, and that $V'' \cong 2V_{2(n+1)}$. With this assumption on V' and V'', there exists a unique extension in the forms 3.4.3.15, 3.4.3.16 up to isomorphism. Thus we have determined up to isomorphism the maps $V' \to V \oplus P$ and $V \to V'' \oplus P$. One checks that the cokernel of the composition $V' \to V'' \oplus 2P$ is isomorphic to $2V_6 \oplus 2V_2$. This is then a direct summand of M' which was shown to have only three indecomposable direct summands which are not projective, a contradiction. □

3.4.4 Obstructions for wild representation type

Here we show the existence of noble obstructions for any abelian infinitesimal group schemes of height one and of wild representation type. Through direct computations, we will show the existence of P-obstructions by giving ad-hoc autodeformations. Throughout, k is a field of characteristic $p > 0$.

Proposition 3.4.4.1. *Suppose $p = 2$, and let $\widetilde{G}$ be a finite group scheme of the form $\widetilde{G} = G_1 \oplus G_2$, for $G_1 = \mathbb{G}_{a(1)} \oplus \mathbb{G}_{a(1)} \oplus \mathbb{G}_{a(1)}$, and for some finite group scheme G_2, so that $k\widetilde{G} \cong \frac{k[x,y,z]}{x^2,y^2,z^2} \otimes kG_2$. Let $\varphi{:}\widetilde{G} \rightsquigarrow \widetilde{G}$ be the autodeformation $f \oplus \mathrm{id}_2$, for $\mathrm{id}_2{:}G_2 \to G_2$ the identity and f defined by*

$$f(x) = x + yz, \quad f(y) = y, \quad f(z) = z.$$

Then $\varphi^*(V) \downarrow^{\widetilde{G}}_C$ is not isomorphic to mJ_{p^ℓ} for any m, ℓ, where $V = J_1 \uparrow^{\widetilde{G}}_C$ is the induced module, and C is the subgroup scheme of $\widetilde{G}$ with subalgebra kC generated by x.

Proof. First, we assume that G_2 is trivial, so that $\widetilde{G} = G_1 = \mathbb{G}_{a(1)} \oplus \mathbb{G}_{a(1)} \oplus \mathbb{G}_{a(1)}$. Then $V = J_1 \uparrow^{\widetilde{G}}_C$ is isomorphic as a $k\widetilde{G}$-module to $k\widetilde{G}/x$, with an ordered basis $yz, z, y, 1$. Therefore

$\varphi^*(V) = f^*(V)$ is a $k\widetilde{G}$-module with x-action given by the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We conclude $\varphi^*(V)\downarrow_C^{\widetilde{G}}$ is isomorphic to $2J_1 \oplus J_2$.

Now, supposing G_2 is some nontrivial infinitesimal group scheme of order n, we have that the induced module V has the structure $k\widetilde{G}/x = (kG_1/x)\otimes kG_2$ as a $k\widetilde{G} = kG_1 \otimes kG_2$-module. Then $\varphi^*(V) = f^*(kG_1/x) \otimes kG_2$, and

$$\begin{aligned} \varphi^*(V)\downarrow_C^{\widetilde{G}} &= \varphi^*(V)\downarrow_{G_1}^{\widetilde{G}}\downarrow_C^{G_1} \\ &= f^*(kG_1/x)\downarrow_C^{G_1} \otimes kG_2\downarrow_C^{\widetilde{G}} \\ &= |G_2|(2J_1 \oplus J_2). \end{aligned}$$

The last line follows since C acts on kG_2 trivially. □

Corollary 3.4.4.2. *Suppose $p = 2$, and let $\widetilde{G}$ be an infinitesimal group scheme of the form $\widetilde{G} = G_1 \oplus G_2$, for $G_1 = \mathbb{G}_{a(1)} \oplus \mathbb{G}_{a(1)} \oplus \mathbb{G}_{a(1)}$, and for some finite group scheme G_2, so that $k\widetilde{G} \cong \frac{k[x,y,z]}{x^2,y^2,z^2} \otimes kG_2$. If $\widetilde{G}$ is of height one (i.e., if G_2 is also of height one), then $\widetilde{G}$ is not noble K-rigid.*

Proof. We apply Lemma 3.2.4.4 and conclude that the proposition above gives a (ρ, P)-obstruction for a restriction character ρ and a polynomial $P \in \mathbb{Z}[X]$. By Lemma 3.2.4.2, $\widetilde{G}$ is not noble K-rigid. □

The proof of the following theorem, concerning ad-hoc autodeformations and expanding the proposition above, can be derived from mutatis mutandis on the proposition above, using the relevant matrix computations in place of the four-by-four matrix given. We leave this as an exercise for the reader.

Theorem 3.4.4.3. *Suppose $\widetilde{G}$ is a finite group scheme over k, of the form $\widetilde{G}_1 \oplus G_2$ for $\widetilde{G}$ abelian, infinitesimal of height one, and of wild representation type with no nontrivial wild direct summands, and G_2 some finite group scheme, so that $k\widetilde{G} = kG_1 \otimes kG_2$. We define $\varphi: \widetilde{G} \rightsquigarrow \widetilde{G}$ by $f \oplus \mathrm{id}_2$, for $\mathrm{id}_2: G_2 \to G_2$ the identity, and f defined in cases as follows:*

I. *For $p = 2$:*

- *If $G_1 = \mathbb{G}_{a(1)} \oplus \mathbb{G}_{a(1)} \oplus \mathbb{G}_{a(1)}$, say $kG_1 = k[x,y,z]/(x^p, y^p, z^p)$, then define f by*

$$f(x) = x + yz, \quad f(y) = y, \quad f(z) = z.$$

- *If $G_1 = \mathbb{W}_{n(1)} \oplus \mathbb{W}_{m(2)}$ for $m > n \geq 1$, say $kG_1 = k[x,y]/(x^{p^n}, y^{p^m})$, then define f by*

$$f(x) = x + y^2, \quad f(y) = y.$$

- *If $G_1 = \mathbb{W}_{n(1)} \oplus \mathbb{W}_{n(1)}$ for $n \geq 2$, say $kG_1 = k[x,y]/(x^{p^n}, y^{p^n})$, then define f by*

$$f(x) = x + y^{p^{n-1}-1}, \quad f(y) = y.$$

II. *For $p > 2$:*

- *If $G_1 = \mathbb{W}_{n(1)} \oplus \mathbb{W}_{m(1)}$ for $n, m \geq 1$, say $kG_1 = k[x,y]/(x^{p^n}, y^{p^m})$, then define f by*

$$f(x) = x + y^2, \quad f(y) = y.$$

In any case, $\varphi^*(V) \downarrow^{\widetilde{G}}_C$ is not isomorphic to mJ_{p^ℓ} for any m, ℓ, where $V = J_1 \uparrow^{\widetilde{G}}_C$ is the induced module, and C is the subgroup scheme of $\widetilde{G}$ with subalgebra generated by x.

Corollary 3.4.4.4. *Let k be an algebraically closed field of characteristic $p > 0$ and let $\widetilde{G}$ be an abelian infinitesimal group scheme of wild representation type and of height one over k. Then $\widetilde{G}$ is not noble K-rigid.*

Proof. By the Structure Theorem and its Corollary 3.4.1.3, we know the casework of the Theorem above exhausts all abelian infinitesimal group schemes of height one and of wild representation type. Then in each case, we apply Lemma 3.2.4.4 and conclude that the theorem above gives a (ρ, P)-obstruction for a restriction character ρ and a polynomial $P \in \mathbb{Z}[X]$. By Lemma 3.2.4.2, $\widetilde{G}$ is not noble K-rigid. □

3.5 Further considerations and open problems

Let's restate two conjectures presented earlier.

Conjecture 3.2.1.5. *Let G be a finite group scheme over an algebraically closed field k of characteristic $p > 0$. Then every closed point in $\mathscr{X}_G$ is noble if and only if the largest value $r + s$ such that G has subgroup isomorphic to $\mathbb{G}_{a(r)} \times (\underline{\mathbb{Z}/p})^s$ is 0 or 1.*

Conjecture 3.2.3.4. *Let $\widetilde{G}$ be a finite group scheme over k, such that every closed point in $\widetilde{\mathscr{X}}$ is noble. Then $\widetilde{G}$ is noble K-rigid if and only if $\widetilde{G}$ is not of wild representation type.*

Let's call a finite group scheme $\widetilde{G}$ *cohomologically short* if every point in $\widetilde{\mathscr{X}}$ is noble. Conjecture 3.2.1.5 suggests there is a more computational way of determining which finite group schemes are cohomologically short. Any class of cohomologically short group schemes constitutes a sub-conjecture of 3.2.3.4. We provide some commentary on these sub-conjectures for various classes of cohomologically short group schemes, with known representation type.

3.5.1 The Heisenberg group

Consider the linear algebraic group $Y_n < \mathrm{GL}_{n+2}$ of dimension $2n+1$, given on points by

$$Y_n(A) = \left\{ \left. \begin{pmatrix} 1 & x_1 & \cdots & x_n & z \\ & 1 & & & y_1 \\ & & \ddots & & \vdots \\ & & & 1 & y_n \\ & & & & 1 \end{pmatrix} \right| x_1, \ldots, x_n, z, y_1, \ldots, y_n \in A \right\}.$$

The group $Y_n(k)$ is usually called the Heisenberg group. For any n, and for k any field of characteristic $p > 0$, the Frobenius kernel $Y_{n(1)}$ is infinitesimal of height one, and is therefore cohomologically short. The finite group scheme $Y_{n(1)}$ is of wild representation type for any $n > 1$, and if p is odd, $Y_{n(1)}$ is of wild representation type for any $n > 0$. When $p = 2$ and $n = 1$, we have that $Y_{1(1)}$ is of tame representation type.

We have the following result, as a first example of noncommutative unipotent group schemes in the theory of K-rigidity :

Theorem 3.5.1.1. *If $\widetilde{G} = Y_{n(1)}$ is of wild representation type, then $\widetilde{G}$ is not noble K-rigid.*

This theorem follows from a lengthy computation of a certain polynomial obstructions (c.f. 3.2.4.1), coming from induced modules along central subgroups.

In the tame case, all known deformations $G' \rightsquigarrow Y_{1(1)}$, when $p = 2$, happen to project to a deformation $H' \rightsquigarrow \mathbb{G}_{a(1)}^2$, forming a commutative diagram

$$\begin{tikzcd} G' \arrow[r, rightsquigarrow] \arrow[d, twoheadrightarrow] & Y_{1(1)} \arrow[d, twoheadrightarrow] \\ H' \arrow[r, rightsquigarrow] & \mathbb{G}_{a(1)}^2 \end{tikzcd} .$$

The spectrum $\widetilde{\mathscr{X}} = \operatorname{Proj} H(Y_{1(1)}, k)$ is isomorphic to two copies of $\mathbb{P}^1$ glued at a point, forming a normal crossing singularity. Some preliminary computations suggest that the $Y_{1(1)}$-representations, supported only at a nonsingular point, are simply pullbacks of such $\mathbb{G}_{a(1)}^2$-representations, as classified in 3.4.3.5. Supposing this is true, K-rigidity of nonsingular points with respect to known deformations of $Y_{1(1)}$ follows from the noble K-rigidity of $\mathbb{G}_{a(1)}^2$, shown in Theorem 3.2.3.3.

What remains to show noble K-rigidity of $Y_{1(1)}$ is a more careful examination of the representations supported at both the singular and the nonsingular points, and a rigorous proof that all deformations project to a deformation of $\mathbb{G}_{a(1)}^2$ as claimed. On the latter point, all such G' admitting a deformation $G' \rightsquigarrow Y_{1(1)}$ are known implicitly via the work of X. Wang et al. [Wan15], [NWW15], classifying all connected Hopf algebras of dimension p^3. Since $Y_{1(1)}$ is unipotent, the coordinate algebra $\mathcal{O}(G')$ is necessarily connected, but picking out which connected Hopf algebras are isomorphic, as a coalgebra, to $\mathcal{O}(Y_{1(1)})$, is a difficult task, given that the classification is cohomological in nature. Ten distinct isomorphism classes of group schemes G' are known, including the trivial deformation $Y_{1(1)}$, and the constant group scheme $\underline{D_8}$. Calculating the total ring of representations for D_8 in characteristic 2 is an open problem [Ben24, § 5.15], and further progress on the noble K-rigidity of $Y_{1(1)}$ may prove useful towards this boundary of knowledge in modular representation theory.

3.5.2 Generalized Quaternions

As discussed in 3.2.1.4, over a field k of characteristic $p > 0$, a constant group scheme $\underline{\Gamma}$ for a finite group Γ is cohomologically short if and only if for any r, such that Γ has a subgroup isomorphic to $(\mathbb{Z}/p)^r$, we have $r \leq 1$. The only cohomologically short constant group schemes of tame representation type, for a p-group Γ, occur when $p = 2$ and Γ is a 'generalized quaternion group', i.e. one having presentation

$$Q_{2^{k+1}} = \langle x, y \mid y^2 = x^{2^{k-1}}, x^{2^k} = 1, yxy^{-1} = x^{-1} \rangle.$$

Letting $\widetilde{G} = \underline{Q_{2^{k+1}}}$, no deformations $G' \rightsquigarrow \widetilde{G}$ are known to exist, except for those with $G' \cong \widetilde{G}$. Notice that, for Z the center of $Q_{2^{k+1}}$, we have that $Q_{2^{k+1}}/Z \cong (\mathbb{Z}/2)^2$. The autodeformations $\widetilde{G} \rightsquigarrow \widetilde{G}$ map the center of $k\widetilde{G}$ into itself, and therefore descend to an autodeformation $(\underline{\mathbb{Z}/2})^2 \rightsquigarrow (\underline{\mathbb{Z}/2})^2$. Moreover, these autodeformations, modulo the center, appear to all be group automorphisms of $(\mathbb{Z}/2)^2$. Therefore, assuming some simple conjectures on deformations of $\widetilde{G}$, the subring of $\widetilde{G}$-representations given by pulling back modules along the projection $\widetilde{G} \twoheadrightarrow \widetilde{G}/Z$ is K-rigid with respect to any deformation, in an appropriate sense. Unfortunately, even if these conjectures on the autodeformations are correct, this is a far cry from noble K-rigidity. An explicit classification of the indecomposable modular $Q_{2^{k+1}}$-representations is an open problem (for $k > 0$) [Ben24, §5.15], despite these groups being known to be of tame representation type. Hence, very little is known about the ring of representations, even before we try deforming the tensor product.

3.5.3 Classical reductive groups

If G is a classical reductive group (e.g. GL_n, or semisimple), then Tannaka duality may be sfficient for determining that all possible deformations $G' \rightsquigarrow G_{(1)}$ of the Frobenius kernel are group automorphisms. This is in turn sufficient to know that $G_{(1)}$, being cohomologically short, is noble K-rigid. Indeed, $G_{(1)}$ is of tame representation type.

To illustrate this idea further, consider the case of characteristic 0. Then reductive groups are linearly reductive, but any deformation of a linearly reductive group is again linearly reductive. Hence, connected deformations $G' \rightsquigarrow G$ (defined as coalgebra isomorphisms $\mathcal{O}(G) \to \mathcal{O}(G')$ meeting a similar Type IV. condition for simple comodules, and for a connected group G), when G is a reductive group, all have that G' is also reductive. But then comparing the simple modules between various reductive groups, one should find that when G is classical, there is only one possible G' with the same representation theory, making $G' \rightsquigarrow G$ an isomorphism of group schemes.

Knowing then that G is *geometrically reductive* over a positive characteristic field, and that $G_{(1)}$-representations are a restriction of G-representations, may yield a similar result by Tannaka duality.

3.5.4 Abelian restricted Lie algbras

Let's return to the main result of this thesis:

Theorem 3.2.3.3. *Let $\widetilde{G}$ be an abelian infinitesimal group scheme of height one over k. Then:*

I. Suppose $\widetilde{G}$ is of finite representation type, and let $E < \widetilde{G}$ be a maximal unipotent subgroup scheme. If $p = 2$ or if the dimension of kE is p, then $\widetilde{G}$ is noble K-rigid.

II. If $\widetilde{G}$ is of tame representation type, then $\widetilde{G}$ is noble K-rigid.

III. If $\widetilde{G}$ is of wild representation type, then $\widetilde{G}$ is not noble K-rigid.

Part I., on finite representation type, includes an awkward hypothesis. According to Conejecture 3.2.3.4, this hypothesis is unnecessary. Much of the theory of geometric K-rigidity (3.3) was developed in the hopes that it would yield noble K-rigidity for $\mathbb{W}_{n(1)}$, for any n and any p. Unfortunately this remains an open problem from what I can surmise. What is known is that, unlike when $n = 1$, there is no suitable geometric deformation

$G_X \rightsquigarrow \widetilde{G}_X$, such that G embeds into a reductive group over X, whose tilting modules restrict to sufficiently many $\widetilde{G}$-representations. When $n = 1$, all the stars align and we can use the tilting modules for SL_2, but the techniques of Section 2.3 can not be adapted for higher n.

Appendix A
DIEUDONNÉ ALGEBRA

Let $p > 0$ be a prime. A scheme $\mathbb{W}$ of Witt vectors, over $\mathbb{Z}$, is defined in e.g. [DG70] as a ring functor structure on $\prod_{i=0}^{\infty} \mathbb{A}^1$. Some key features of $\mathbb{W}$ include:

1. $\mathbb{W}(\mathbb{F}_p)$ is isomorphic to the ring of p-adic integers $\mathbb{Z}_p$.

2. $\mathbb{W}(k)$ is a complete DVR of characteristic 0, with residue field k, whenever k is a perfect field of characteristic p.

3. There is for each $n \geq 1$ a ring functor of length n Witt vectors $\mathbb{W}_n$, isomorphic as a scheme to $\mathbb{A}^n$, with $\mathbb{W}_n(k) \cong \mathbb{W}(k)/\mathfrak{m}^n$, $\mathfrak{m} \subset \mathbb{W}(k)$ being the maximal ideal of the DVR, for k a perfect field of characteristic p. There is a canonical projection of ring functors $\mathbb{W} \to \mathbb{W}_n$ taking a Witt vector $x = (x_0, x_1, \dots)$ to $(x_0, x_1, \dots, x_{n-1})$.

4. The 'coordinate Frobenius' $\mathfrak{F}\colon (x_0, x_1, \dots) \mapsto (x_0^p, x_1^p, \dots)$ is a ring homomorphism of R-points $\mathbb{W}(R) \to \mathbb{W}(R)$ for any commutative ring R of characteristic p.

5. The 'coordinate shift' $\mathfrak{V}\colon (x_0, x_1, \dots) \to (0, x_0, \dots)$ is a homomorphism of underlying abelian groups $\mathbb{W}(R) \to \mathbb{W}(R)$ when R is a commutative ring of characteristic p.

6. If R is a ring of characteristic p, and $x = (x_0, x_1, x_2, \dots) \in \mathbb{W}(R)$ is an arbitrary vector, then the element $px \in \mathbb{W}(R)$ (that is, the p-fold sum of x with itself) is given by the vector
$$(0, x_0^p, x_1^p, \dots).$$
In this way, we have $\mathfrak{F} \circ \mathfrak{V} = \mathfrak{V} \circ \mathfrak{F} = p$.

7. If R is a ring of characteristic p, then for any $x \in R$, the element $x^\tau \in \mathbb{W}(R)$ is defined by the vector
$$x^\tau = (x, 0, 0, \dots).$$
Then for any $y = (y_0, y_1, y_2, \dots) \in \mathbb{W}(R)$, we have the identity
$$x^\tau \cdot y = (xy_0, x^p y_1, x^{p^2} y_2, \dots),$$
and in particular the identity $x^\tau y^\tau = (xy)^\tau$ for any $x, y \in R$.

The Dieudonné algebra $\mathbb{D}$ is a noncommutative algebra over $\mathbb{W}(k)$ on the generators F, V, subject to the relations that

$$\begin{cases} Fx = \mathfrak{F}(x)F, \quad xV = V\mathfrak{F}(x), \quad x \in \mathbb{W}(k), \\ FV = VF = p \in \mathfrak{W}(k). \end{cases}$$

A Dieudonné module is a left $\mathbb{D}$-module, and is equivalent to the data of some $\mathbb{W}(k)$-module M, together with some skew-endomorphisms F_M, V_M of M satisfying similar relations. This chapter will review some of [DG70], including the theory of Cartier and Gabriel, which shows how unipotent abelian group schemes are anti-equivalent to a type of $\mathbb{D}$-module.

A.1 Unipotent abelian group schemes

Recall our definition of unipotent group schemes 1.5.1, as that which is built out of subgroup schemes of $\mathbb{G}_a$. Let k be a perfect field of characteristic $p > 0$. We denote by $\mathcal{U}$ the category of unipotent abelian group schemes over k, an abelian category. We will take $\mathbb{W}_n$ to denote the base change of length n Witt vectors to the field k, so that $\mathbb{W}_n$ is an object in $\mathcal{U}$.

The following theorem was first developed by Cartier and Gabriel and communicated to Oort, where it was first stated in [Oor66]. The book cited, of Demazure and Gabriel, contains a detailed proof. A $\mathbb{D}$-module M is said to be effaceable if M is V-torsion, i.e., if for any $m \in M$, there exists some n with $V^n m = 0$.

Theorem A.1.0.1. [DG70, V, §1 4.3] *There is an additive exact antiequivalence of abelian* *categories between the category $\mathcal{U}$ of unipotent abelian group schemes over k and the category of effaceable $\mathbb{D}$-modules. A given U in $\mathcal{U}$ corresponds to the $\mathbb{D}$-module given by the direct limit*

$$M(U) = \varinjlim_n \mathrm{Hom}_{\mathcal{U}}(U, \mathbb{W}_n).$$

A given U in $\mathcal{U}$ is of finite type over k if and only if the underlying $\mathbb{W}(k)$-module of $M(U)$ is finitely generated, and U is a finite group scheme if and only if $M(U)$ is finite length over $\mathbb{D}$ (or equivalently, over $\mathbb{W}(k)$).

A.1.0.2. More detail on the $\mathbb{D}$-module structure of $\mathrm{Hom}_{\mathcal{U}}(U, \mathbb{W}_n)$ for $n \to \infty$ is required to understand the $\mathbb{D}$-module structure of $M(U)$, but we omit this. For our purposes, we only need to understand how to compute the inverse of the functor $M(?)$ in some cases, translating from some finitely presented $\mathbb{D}$-modules back to group schemes in $\mathcal{U}$. To do this, we note the following:

- If M is any $\mathbb{W}(k)$-module, we denote by $M^{(p)^r}$ the pullback of M along the ring endomorphism $\mathfrak{F}^r : \mathbb{W}(k) \to \mathbb{W}(k)$, for any r. If M is a $\mathbb{D}$-module, then a $\mathbb{D}$-module structure is defined on $M^{(p)^r}$ by the usual action of $F, V \in \mathbb{D}$ on M. One checks that all the necessary relations hold. Then, for any $\mathbb{D}$-module M, the elements $F^r, V^r \in \mathbb{D}$ define skew-linear homomorphisms of $\mathbb{D}$-modules

  $$F^r : M^{(p)^r} \to M, \quad V : M \to M^{(p)^r}.$$

  If the $\mathbb{D}$-module M is effaceable, corresponding to some U, then $M^{(p)^r}$ is effaceable for each r, and corresponds to $U^{(p)^r}$. The skew-linear homomorphism F^r then corresponds to the iterated relative Frobenius

  $$F_{U/k} : U \to U^{(p)^r},$$

  while V^r corresponds to the iterated 'Verschiebung'

  $$V : U^{(p)^r} \to U.$$

- $M(\mathbb{W}_n) = \mathrm{End}_{\mathcal{U}}(\mathbb{W}_n) \cong \mathbb{D}/|V^n\rangle$ for any $n \geq 0$.

- If U is an object in $\mathcal{U}$, then the Frobenius kernel U_r has corresponding $\mathbb{D}$-module $M(U_r) \cong M(U)/|F^r\rangle M(U)$, for any $r \geq 0$. Here, we use notation IM, for I a left ideal and M a left module, to mean the left submodule generated by elements in the form xm for $x \in I, m \in M$.

- Suppose that U is a unipotent abelian group scheme of finite type over $\mathbb{F}_{p^r}$ for some $r \geq 1$, so that the base change U_k belongs to $\mathcal{U}$, and so that we have a canonical isomorphism $U_k^{(p)^r} \cong U_k$. Then the iterated relative Frobenius $F^r_{U_k/k}$ is an endomorphism in a canonical way. The equalizer U^F, as in the diagram,

$$U^{F^r} \to U_k \underset{\mathrm{id}}{\overset{F^r}{\rightrightarrows}} U_k$$

is then a group scheme isomorphic to the constant group scheme associated to the finite group of points $U(\mathbb{F}_{p^r})$. By exactness of the functor $M(?)$, we have $M(U^F) = M(U_k)/|F^r - 1\rangle M(U_k)$.

- Consider a homomorphism $\mathbb{D}^n \to \mathbb{D}^m$ of $\mathbb{D}$-modules is given by a matrix of integer polynomials of F and V. If the cokernel M is finite length, then M is isomorphic to the cokernel of some $M(\mathbb{W}^n_{r(r)}) \to M(\mathbb{W}^m_{r(r)})$ which is, again, essentially a matrix of integer polynomials of F and V. Hence, a corresponding U in $\mathcal{U}$, for which $M = M(\mathcal{U})$, can be reconstructed as an explicit kernel of a map $\mathbb{W}^m_{r(r)} \to \mathbb{W}^n_{r(r)}$ which can be written easily in terms of the coordinate functions $\mathfrak{F}$ and $\mathfrak{V}$ for Witt vectors.

A.2 Developments in classifying finite dimensional Hopf algebras

A recent result of Gouthier [Gou26] has acheived a classification of all Hopf algebra structures on the underlying algebra $k[t]/t^{p^n}$ for any n, for any algebraically closed field k of characteristic $p > 0$. Gouthier's explicit description of all such structures is quite concise if stated in terms of their corresponding $\mathbb{D}$-modules. This motivates us to review a history of classifying cocommutative Hopf algebra structures on an underlying finite local algebra in positive characteristic, and present relevant classifications of deformations used in Chapter 3 of this thesis, in the language of Dieudonné algebra.

A.2.0.1. Tate and Oort [TO70] classify group schemes of ‘prime order’ over an arbitrary scheme. These are by definition the commutative Hopf algebra structures on a prime-rank vector bundle over a given scheme. They restate a well-known proof that over an algebraically closed field, the schemes $\underline{\mathbb{Z}/p}, \mu_p, \mathbb{G}_{a(1)}$ are the only group schemes of prime order. Here, μ_p is only distinct from $\underline{\mathbb{Z}/p}$ in positive characteristic, and the Frobenius kernel $\mathbb{G}_{a(1)}$ only exists in positive characteristic. This result also appears in Oort’s earlier book [Oor66]. Part of this classification can be seen as the case of $n = 1$ for Gouthier’s classification.

A.2.0.2. Notice, over an algebraically closed field of characteristic 0, we have now that any Hopf algebra of prime dimension, which is commutative, or is cocommutative, must then be isomorphic to the group algebra for $\mathbb{Z}/p$. In fact, Y. Zhu [Zhu94] showed that any Hopf algebra of prime dimension over an algebraically closed field of characteristic 0 is commutative, i.e. isomorphic to the group algebra.

A.2.0.3. X. Wang’s thesis work [Wan13], [Wan15], together with Ng, L. Wang, and a X. Wang [NWW15], completely classifies connected Hopf algebras of dimension p^2 and p^3 over an algebraically closed field of characteristic p. Finite dimensional coalgebras are connected if and only if their dual is a local algebra. Thus finite group schemes G over an algebraically closed field are unipotent if and only if the coordinate algebra $\mathcal{O}(G)$ is a connected Hopf algebra. The case of $n = 2$ in Gouthier’s classification is included in [Wan13] as an explicit classification of Hopf algebra structures on the local algebra $k[t]/t^{p^2}$, as a corollary of classifying the connected Hopf algebras of dimension p^2. The case of $n = 3$ in Gouthier’s classification is implicit from what is found in [NWW15], which includes an explicit description of all connected Hopf algebras of dimension p^3 with a one-dimensional primitive subspace. Those which are commutative and cocommutative are precisely group algebras for finite group schemes $\widetilde{G}$ admitting a deformation $\mathbb{G}_{a(3)} \rightsquigarrow \widetilde{G}$, and hence $\widetilde{G}^{\#} \rightsquigarrow \mathbb{W}_{3(1)}$.

Now we present the following classification of certain deformations, in terms of their corresponding $\mathbb{D}$-module:

Theorem A.2.0.4. [Wan13] *Let k be an algebraically closed field of characteristic $p > 0$, and let $\widetilde{G} = \mathbb{G}_{a(1)}^2$. There are then four isomorphism classes of finite group schemes G' admitting a deformation $G' \rightsquigarrow \widetilde{G}$ (see 3.1.2), and the $\mathbb{D}$-module $M(G')$ is isomorphic to one of:*

$$\begin{gathered}\mathbb{D}/|F^2, V\rangle, \qquad (\mathbb{D}/|F, V\rangle) \oplus (\mathbb{D}/|F-1, V\rangle), \\ (\mathbb{D}/|F-1, V\rangle)^2, \quad (\mathbb{D}/|F, V\rangle)^2.\end{gathered}$$

Theorem A.2.0.5. [Oor66; Wan13; NWW15; Gou26] *Let k be an algebraically closed field of characteristic $p > 0$, and let $\widetilde{G} = \mathbb{W}_{n(1)}$ for some $n \geq 0$. Then there are $n+1$ isomorphism classes of finite group schemes G' admitting a deformation $G' \rightsquigarrow \widetilde{G}$, and the $\mathbb{D}$-module $M(G')$ is isomorphic to one of the form*

$$M_{n,i} = \mathbb{D}/|F - V^i, V^n\rangle$$

for some $0 \leq i \leq n$. Here, $M_{n,0} \cong M(\underline{\mathbb{Z}/p^n})$ and $M_{n,n} \cong M(\mathbb{W}_{n(1)})$.

A.2.0.6. For $i > n$, we may also define $M_{n,i} = \mathbb{D}/|F - V^i, V^n\rangle$, but this simply leaves us with $M_{n,i} = M_{n,n} = \mathbb{D}/|F, V^n\rangle \cong M(\mathbb{W}_{n(1)})$. Observe now that whenever $m \leq n$, we have a canonical projection $M_{n,i} \twoheadrightarrow M_{m,i}$ of $\mathbb{D}$-modules, for any $i \geq 0$, corresponding to some embedding of finite group schemes.

These projections are compatible with deformations in the following sense: First, define deformations $M \rightsquigarrow M'$ for finite length $\mathbb{D}$-modules M, M' as deformations $G' \rightsquigarrow G$ for G, G' the finite unipotent abelian group schemes corresponding to M, M' respectively.

Then for every $i \geq 0$ and every deformation $M_{n,n} \rightsquigarrow M_{n,i}$, and for any $m \leq n$, there exists a deformation $M_{m,n} = M_{m,m} \rightsquigarrow M_{m,i}$ making the diagram commute

$$\begin{tikzcd} M_{n,n} \arrow[r, two heads] \arrow[d, rightsquigarrow] & M_{m,m} \arrow[d, rightsquigarrow] \\ M_{n,i} \arrow[r, two heads] & M_{m,i} \end{tikzcd}.$$

Essentially, this all reduces to the fact that over a field of characteristic p, any cocommutative Hopf algebra structure on $k[t]/t^{p^n}$ will have that the subalgebra generated by any t^{p^j} is actually a Hopf subalgebra.

A.3 Examples of generic twists and stability

Gouthier's theorem A.2.0.5 classifies deformations $G' \rightsquigarrow \mathbb{W}_{n(1)}$ up to isomorphism. In Section 3.3 I introduce the idea of geometric deformations and generic twists of finite group schemes. In this section, I'll present some background computations used related to geometric K-rigidity, which I've found to be ultimately large enough to derail the flow of Chapter 3, while also being similar enough to some of Gouthier's original computations [Gou26].

Proposition A.3.0.1. *Let k be an algebraically closed field of characteristic $p > 0$, and let $\widetilde{G}$ be the affine group scheme over k defined by the equalizer diagram over k*

$$\widetilde{G} \to \mathbb{W}_n \underset{V^i}{\overset{F}{\rightrightarrows}} \mathbb{W}_n.$$

Then there exists a finite commutative group scheme G over $\mathbb{A}^1 = \operatorname{Spec} k[\eta]$, with base point $\eta \mapsto 1$, and with the following properties:

1. *The specialization of G to k along $\eta \mapsto 0$ is $\mathbb{W}_{n(1)}$*

2. *There exists a generic twist $f\colon G_{\mathbb{A}^1} \rightsquigarrow \widetilde{G}_{\mathbb{A}^1}$.*

Proof. Let $R = k[\eta] = \mathcal{O}(\mathbb{A}^1)$. The case of $i = 0$ is relatively straightforward: we define the finite group scheme G over R by defining a finite rank cocommutative Hopf algebra over R, serving as the group algebra. Let

$$RG = \frac{R[x]}{x^{p^n}}$$

with comultiplication defined by $x \mapsto x \otimes 1 + 1 \otimes x + \eta x \otimes x$. Immediately, we have Property 1., since specializing along $\eta \mapsto 0$ gives $k\mathbb{W}_{n(1)}$ as a Hopf algebra. We have, since $\widetilde{G}$ is by definition the Frobenius fixed points of $\mathbb{W}_n$, that $\widetilde{G}$ is isomorphic to the constant group scheme associated to the finite group $\mathbb{W}_n(\mathbb{F}_p) \cong \mathbb{Z}/p^n$. Then $R\widetilde{G}_{\mathbb{A}^1}$ is isomorphic to the algebra

$$R\widetilde{G}_{\mathbb{A}^1} \cong \frac{R[t]}{t^{p^n}}$$

with comultiplication $t \mapsto t\otimes 1 + 1\otimes t + t\otimes t$. Taking $t \mapsto x$ then defines an algebra isomorphism $RG \to R\widetilde{G}_{\mathbb{A}^1}$, and by dualizing we get a coalgebra isomorphism $f^\sharp: \mathcal{O}(\widetilde{G}_{\mathbb{A}^1}) \to \mathcal{O}(G)$. If we specialize along $\eta \mapsto 1$, we see that $f^\sharp$ is an isomorphism of Hopf algebras, and hence we have a (weak) deformation $f: G_{\mathbb{A}^1} \rightsquigarrow \widetilde{G}_{\mathbb{A}^1}$ based at the point $\eta \mapsto 1$. If K denotes an algebraic closure of the quotient field $k(\mathbb{A}^1) = k(\eta)$, the map $t \mapsto \eta x$ defines an isomorphism of Hopf algebras $KG_K \xrightarrow{\sim} K\widetilde{G}_K$, and hence we have an isomorphism $\varphi: G_K \xrightarrow{\sim} \widetilde{G}_K$ of finite group schemes over K (taking K to be algebraically closed is not necessary for this to be an isomorphism, it's only necessary that η be invertible). Therefore f is a generic twist, since there exists some isomorphism φ of group schemes over K.

If $i > 0$, then consider the finite group scheme $\mathscr{G}$ given by the coequalizer diagram over $\mathbb{A}^1$

$$\mathbb{W}_{n(n)} \underset{\eta^\tau \circ F^i}{\overset{V}{\rightrightarrows}} \mathbb{W}_{n(n)} \to \mathscr{G},$$

where $\eta^\tau: \mathbb{W}_n \to \mathbb{W}_n$ multiplies by the element η^τ in the ring $\mathbb{W}(R)$. Then we take a finite group scheme G over $\mathbb{A}^1$ defined to be the Cartier dual $\mathscr{G}^\#$. The Group algebra RG is therefore isomorphic to $\mathcal{O}(\mathscr{G})$ as a Hopf algebra. We can specialize along $\eta \mapsto 0$ and see that there is an equalizer diagram

$$\operatorname{Spec} R/\eta \times_R G \to \mathbb{W}_{n(n)} \underset{0}{\overset{F}{\rightrightarrows}} \mathbb{W}_{n(n)},$$

and similarly along $\eta \mapsto 1$ to see there is an equalizer diagram

$$\operatorname{Spec} R/(\eta - 1) \times_R G \to \mathbb{W}_{n(n)} \underset{V^i}{\overset{F}{\rightrightarrows}} \mathbb{W}_{n(n)}.$$

Then the specialization of G along $\eta \mapsto 0$ is indeed the Frobenius Kernel $\mathbb{W}_{n(1)}$ as claimed for Property 1., and the specialization of G along $\eta \mapsto 0$ is isomorphic to $\widetilde{G}$, as needed to form a generic twist $G_{\mathbb{A}^1} \rightsquigarrow \widetilde{G}_{\mathbb{A}^1}$. We can calculate the group algebra RG, realized by a certain Hopf subalgebra of $\mathcal{O}(\mathbb{W}_{n(n)})$ forming the relevant equalizer. We take coordinates

$$\mathcal{O}(\mathbb{W}_{n(n)}) = \frac{R[x_0, \dots, x_{n-1}]}{x_0^{p^n}, \dots, x_{n-1}^{p^n}},$$

and we let $x \in \mathcal{O}(\mathbb{W}_{n(n)})$ be the coordinate function

$$x = \sum_{m=0}^{\infty} \beta_m x_m^{p^{mi}} = \sum_{m=0}^{\lfloor \frac{n}{i} \rfloor} \beta_m x_m^{p^{mi}},$$

for $\beta_0 = 1$ and β_m satisfying $\beta_{m+1} = \eta^{p^{(i+1)m}} \beta_m$ for each $m \geq 0$. One checks then that $V^\sharp(x) = (\eta^\tau \circ F^i)^\sharp(x)$, that x generates the R-subalgebra equalizing the relevant diagram, and that the subalgebra is $\frac{R[x]}{x^{p^n}}$. The group algebra for $\widetilde{G}$ is $k\widetilde{G} = \frac{k[t]}{t^{p^n}}$, as calculated in 3.4.2.1 (using essentially the same method). Then taking $x \mapsto t$ defines an isomorphism of algebras $RG \to R\widetilde{G}_R$ and hence an $\mathbb{A}^1$-deformation $f: G_{\mathbb{A}^1} \rightsquigarrow \widetilde{G}_{\mathbb{A}^1}$. Using the construction from 3.4.2.1, it's easily verified that specializing to $\eta \mapsto 1$ makes f an isomorphism of group schemes.

Now notice that the Cartier dual $\widetilde{G}^\#$ is also the coequalizer as in the diagram

$$\mathbb{W}_{n(n)} \underset{F^i}{\overset{V}{\rightrightarrows}} \mathbb{W}_{n(n)} \to \widetilde{G}^\#,$$

and this remains true after passing to the algebraic closure K of $k(\mathbb{A}^1) = k(\eta)$. An isomorphism of group schemes $\varphi: G_K \xrightarrow{\sim} \widetilde{G}_K$ can be given by the dual of an isomorphism $\varphi^\#: \widetilde{G}_K^\# \xrightarrow{\sim} \mathscr{G}_K$. First, take $\alpha \in K$ such that $\alpha^{1-p^{i+1}} = \eta \in K$. Then, taking $\pi: \mathbb{W}_{n(n)} \to \mathscr{G}$ to be the canonical map, define $\hat{\varphi}: \mathbb{W}_{n(n)} \to \mathscr{G}$ by the composition

$$\hat{\varphi}: \mathbb{W}_{n(n)} \xrightarrow{\alpha^\tau} \mathbb{W}_{n(n)} \xrightarrow{\pi} \mathscr{G}_K.$$

Now, since $\alpha \in K$ is invertible (and so is $\alpha^\tau \in \mathbb{W}(K)$), the diagram

$$\mathbb{W}_{n(n)} \underset{F^i}{\overset{V}{\rightrightarrows}} \mathbb{W}_{n(n)} \xrightarrow{\hat{\varphi}} \mathscr{G}_K$$

commutes (is equalized) if and only if the diagram

$$\mathbb{W}_{n(n)} \xrightarrow{(\alpha^{-p})^\tau} \mathbb{W}_{n(n)} \underset{F^i}{\overset{V}{\rightrightarrows}} \mathbb{W}_{n(n)} \xrightarrow{\hat{\varphi}} \mathscr{G}_K$$

commutes. But this diagram is equivalent to

$$\mathbb{W}_{n(n)} \underset{(\alpha^{1-p^{i+1}})F^i}{\overset{V}{\rightrightarrows}} \mathbb{W}_{n(n)} \xrightarrow{\pi} \mathscr{G},$$

and we've assumed that $\alpha^{1-p^{i+1}} = \eta \in K$, so we know there exists a unique $\widetilde{G}_K^\# \to \mathscr{G}_K$ making the square

$$\begin{tikzcd}
\mathbb{W}_{n(n)} \arrow[r, "\alpha^\tau"] \arrow[d] & \mathbb{W}_{n(n)} \arrow[d] \\
\widetilde{G}_K^\# \arrow[r, dashed] & \mathscr{G}_K
\end{tikzcd}$$

commute, where both vertical arrows are canonical projections from $\mathbb{W}_{n(n)}$ to the respective target. Similarly, there is a unique $\mathscr{G}_K \to \widetilde{G}_K^\#$ making the square

$$\begin{tikzcd}
\mathbb{W}_{n(n)} \arrow[r, "(\alpha^{-1})^\tau"] \arrow[d] & \mathbb{W}_{n(n)} \arrow[d] \\
\mathscr{G}_K \arrow[r, dashed] & \widetilde{G}_K^\#
\end{tikzcd}$$

commute. It should be clear that the two maps between $\widetilde{G}_K^\#$ and $\mathscr{G}_K$ are mutually inverse, and hence there exists an isomorphism $\varphi : G_K \xrightarrow{\sim} \widetilde{G}_K$ of group schemes over K. This concludes the proof that our $\mathbb{A}^1$-deformation $f : G \rightsquigarrow \widetilde{G}_{\mathbb{A}^1}$ is a generic twist. $\square$

Proposition A.3.0.2. *Take $k, p, \widetilde{G}$, and G over $\mathbb{A}^1$ from Proposition A.3.0.1, and let $f : G_{\mathbb{A}^1} \rightsquigarrow \widetilde{G}_{\mathbb{A}^1}$ be the generic twist, with $\varphi : G_{\overline{\xi}} \to \widetilde{G}_{\overline{\xi}}$ the isomorphism of group schemes constructed in the proof. Taking $k\widetilde{G} = k[t]/t^{p^n}$ by construction, let $J_i = k\widetilde{G}/t^i$ be the unique* *indecomposable $\widetilde{G}$-representation of dimension i, and let $\pi\colon J_m \to J_\ell$ be the canonical projection whenever $m \geq \ell$. Then, define the subset $\mathcal{C}(J_m, J_\ell) \subset \operatorname{Hom}_{\widetilde{G}}(J_n, J_\ell)$ by*

$$\mathcal{C}(J_n, J_\ell) = \begin{cases} 0 & n < \ell \\ \langle \pi \rangle & n \geq \ell \end{cases},$$

we have that $\mathcal{C}$ is a subcategory of $\mathrm{rep}_{\widetilde{G}}$, consisting of objects J_m for $1 \leq m \leq p^n$ and morphisms $\mathcal{C}(J_m, J_\ell)$, which is stable with respect to f, φ (Definition 3.3.2.3).

Proof. We'll assume that $i > 0$, noting that the case of $i = 0$ is more straightforward. Let K be the algebraic closure of $k(\mathbb{A}^1) = k(\eta)$, with $\alpha \in K$ satisfying $\alpha^{1-p^{i+1}} = \eta \in K$. The group algebras $K\widetilde{G}_K$ and KG_K are both given by subalgebras of $\mathcal{O}(\mathbb{W}_{n(n)})_K = \frac{K[x_0,\ldots,x_{n-1}]}{x_0^{p^n},\ldots,x_{n-1}^{p^{n-1}}}$, describing the coordinate algebras of their Cartier dual. Recall, for the elements $x, t, \in \mathcal{O}(\mathbb{W}_{n(n)})_K$ given by

$$x = \sum_{m=0}^{\infty} \beta_m x_m^{p^{mi}}, \qquad t = \sum_{m=0}^{\infty} x_m^{p^{mi}},$$

that KG_K is generated by x and $K\widetilde{G}_K$ is generated by t. The base change $f_K\colon G_K \rightsquigarrow \widetilde{G}_K$ of the $\mathbb{A}^1$ deformation f is the isomorphism of algebras $KG_K \to K\widetilde{G}$ taking $x \mapsto t$. The isomorphism of group schemes $\varphi\colon G_K \xrightarrow{\sim} \widetilde{G}_K$ is given by the unique homomorphism of Hopf algebras $KG_K \to K\widetilde{G}_K$ making the diagram

$$\begin{tikzcd} KG_K \arrow[r, hook] \arrow[d, dotted, "\varphi"'] & \mathcal{O}(\mathbb{W}_{n(n)})_K \arrow[d, "(\alpha^\tau)^\sharp"] \\ K\widetilde{G}_K \arrow[r, hook] & \mathcal{O}(\mathbb{W}_{n(n)})_K \end{tikzcd}$$

commute, where the horizontal arrows are inclusions and $(\alpha^\tau)^\sharp$ is the map of coordinate algebras taking $x_m \mapsto \alpha^{p^m} x_m$ for each x_m. Then we get

$$\begin{aligned}(\alpha^\tau)^\sharp(x) &= \sum_{m=0}^{\infty} \beta_m (\alpha^{p^m} x_m)^{p^{mi}} \\ &= \sum_{m=0}^{\infty} \beta_m \alpha^{p^{m(i+1)}} x_m^{p^{mi}} \\ &= \alpha \sum_{m=0}^{\infty} x_m^{p^{mi}} = \alpha t,\end{aligned}$$

and therefore $\varphi(x) = \alpha t$. Now to show that $\mathcal{C}$ is stable with respect to f, φ, notice that $f_K^*((J_i)_K)$ and $\varphi^*((J_i)_K)$ have the same underlying vector space of $(J_i)_K = K[t]/t^i$, with both pullbacks are determined by the data of an action of the element $x \in KG_K$. The action of x on $f_K^*((J_i)_K)$ is multiplication by t, while the action of x on $\varphi^*((J_i)_K)$ is multiplication by αt. Define $\Xi(J_i)\colon f_K^*((J_i)_K) \to \varphi^*((J_i)_K)$ by the linear map taking $t^i \mapsto \alpha^i t^i$ for the basis elements $t^i \in J_i$. It's easily verified now that Ξ defines a natural map of functors from $\mathcal{C}$ to rep_{G_K}. $\square$

Appendix B
BAŠEV'S LEMMA

Следующая лемма, которую мы приведем без доказательства, понадобится в дальнейшем.

Лемма 6. *Пусть* $\mathfrak{A}$ — *ассоциативное, коммутативное кольцо с образующими* x_s $(s = 1, 2, \ldots; x_0 = 0)$, *на котором определен целочисленный линейный функционал* f *такой, что* $f(x_s) = s$, $f(x_s \cdot x_t) = 2\min(s, t)$. *Тогда* $x_m \cdot x_n = 2x_{\min(m,n)}$, *если* $m \neq n$.

Figure B.1: Bašev's lemma [Baš61, Lemma 6]

Here we prove some laborious work on commutative rings, inspired by the unproven (and, tragically, in need of stronger hypotheses) lemma of Bašev [Baš61].

Lemma B.0.0.1. *Let R be a (nonunital) associative, commutative ring with a $\mathbb{Z}$-linear basis $\{x_s \mid s = 1, 2, \ldots\}$; $x_0 = 0$. Assume that each product $x_m x_n$ is a nonnegative integer combination of the x_s for $s > 0$, and further that $x_1 x_2 = 2x_1$. If the $\mathbb{Z}$-linear functional $f : R \to \mathbb{Z}$ defined by $f(x_s) = s$ satisfies $f(x_s x_t) = 2\min(s,t)$, then $x_m x_n = 2x_{\min(m,n)}$ whenever $m \neq n$.*

Proof. We assumed $x_0 = 0$ and therefore $x_0 x_n = 2x_0$ for each $n > 0$. Now we strongly induct on m: suppose whenever $s \leq m$ that $x_s x_n = 2x_s$ for each $n > s$, and let $n > m+1$. Write $x_n x_{m+1} = \sum_s c_s x_s$ for $c_s \geq 0$, so that $2(m+1) = \sum_s c_s s$. Then

$$4x_m = x_n x_{m+1} x_m = \sum_s c_s x_s x_m = \sum_{s<m} c_s 2x_s + c_m x_m^2 + \sum_{s>m} c_s 2x_m.$$

Therefore $c_s = 0$ for $s < m$. We also get $4m = \sum_{s \geq m} c_s 2m$, and in case $m = 0$, we get $2 = \sum_{s \geq m} c_s s$ so there are at most two values $s = s_1, s_2 \geq m$ such that $c_s > 0$. So we have $2(m+1) = c_{s_1} s_1 + c_{s_2} s_2$ and also $2m = (c_{s_1} + c_{s_1})m$ for $s_1, s_2 \geq m$. Assume $s_1 < s_2$ without loss of generality.

Consider the case $m = 0$. Then $2 = c_{s_1}s_1 + c_{s_2}s_2$. We are trying to show $c_{s_2} = 0$ and $s_1 = 1, c_{s_1} = 2$, so suppose otherwise that $c_1 = 0$ and $c_2 = 1$, i.e. $x_n x_1 = x_2$. Then $x_n x_1^2 = x_2 x_1 = 2x_1$ by hypothesis. Either $x_1^2 = 2x_1$ or x_2 by the same considerations as above, but in the former case $2x_1 = x_n x_1^2 = 2x_n x_1 = 2x_2$, a contradiction, so $x_1^2 = x_2$. But $x_n x_1^2 = x_2 x_1 = 2x_1 = x_n x_2$, and since we assumed $n > 1$, $2f(x_1) = f(x_n x_2) = 4$, a contradiction either way.

Now assume $m > 0$ so we have identities $2 = c_{s_1} + c_{s_2}$ and $2(m+1) = c_{s_1}s_1 + c_{s_2}s_2$. We want that one of the c_{s_i} is 2 and the other is zero, so that $s_i = m+1$ and we conclude $x_n x_{m+1} = 2x_{m+1}$. The former identity implies if we assume otherwise, that $c_{s_1} = c_{s_2} = 1$ and so by the latter $2(m+1) = s_1 + s_2$, an even number. Then $s_2 \geq s_1 + 2 \geq m+2$, and we have $s_1 = m, s_2 = m+2$. It now follows that $x_n x_{m+1} = x_m + x_{m+2}$, hence $x_m^2 + 2x_m = x_n x_{m+1} x_m = 4x_m$, and $x_m^2 = 2x_m$.

Taking $n' = m+2$, we may repeat the arguments thus far and know that one of the following holds

$$\begin{cases} x_{m+2}x_{m+1} = 2x_{m+1} & (1) \\ x_{m+2}x_{m+1} = x_{m+2} + x_m. & (2) \end{cases}$$

Let $x_{m+1}^2 = \sum_t d_t x_t$ for $d_t \geq 0$ so that $2(m+1) = \sum_t d_t t$.

Supposing (1) we get $x_n x_{m+1}^2 = (x_m + x_{m+2})x_{m+1} = 2x_m + 2x_{m+1}$ and also

$$\begin{aligned} x_n x_{m+1}^2 &= \sum_{t \leq m} d_t 2x_t + d_{m+1}x_{m+1}x_n + \sum_{t > m+1} d_t x_t x_n \\ &= \sum_{t \leq m} d_t 2x_t + d_{m+1}(x_m + x_{m+2}) + \sum_{t > m+1} d_t x_t x_n. \end{aligned}$$

Therefore $d_t = 0$ for $t < m$, $d_m \leq 1$, and $d_{m+1} = 0$, so we get

$$2x_m + 2x_{m+1} = 2d_m x_m + \sum_{t > m+1} d_t x_t x_n.$$

At least one coefficient d_t is nonzero for $t > m+1$, with $x_t x_n = \nu_1 x_m + \nu_2 x_{m+1}$ for nonzero ν_2, and both $\nu_i \leq 2$. Then $2\min(t,n) = \nu_1 m + \nu_2(m+1) > 2(m+1)$. It follows that $\nu_1 > 0$, and indeed, $\nu_1 = \nu_2 = 2$, $d_t = 1$, and so $2m+1 = \min(t,n)$ and $x_t x_n = 2x_m + 2x_{m+1}$, with $d_s = 0$

for each $s \neq t$. Then $x_m x_t x_n = 4x_m = 4x_m + 4x_m$, contradicting $m > 0$.

Supposing (2) similarly gets

$$3x_m + x_{m+1} = 2d_m x_m + \sum_{t>m+1} d_t x_t x_n. \qquad (*)$$

There is exactly one coefficient $d_t = 1$, $t > m+1$, with $x_t x_n = \mu x_m + x_{m+1}$ for $\mu \leq 3$. Then

$$4x_m = x_m x_t x_n = 2\mu x_m + 2x_m$$

and we have $\mu = 1$. We know $d_m < 2$ and so there is some $x_{t'} x_n = \sigma x_m$ for $t' > m+1$ for nonzero $\sigma > 0$. It follows that $\sigma = 2$ for any such t', but this is impossible, the x_m coefficient is odd in $(*)$.

We conclude the inductive step, that $x_n x_{m+1} = 2x_{m+1}$ for $n > m+1$. □

Proposition B.0.0.2. *Let S be any subset of the natural numbers such that no two consecutive numbers are elements of S. Let $R = \langle x_i \rangle_{i=1}^{\infty}$ be the free abelian group, denoting $x_0 = 0$, with multiplication defined by*

$$x_s x_t = \begin{cases} 2x_{\min(s,t)} & s \neq t \\ 2x_s & s = t \notin S \\ x_{s-1} + x_{s+1} & s = t \in S, \end{cases}$$

extended linearly. Then R is an associative, commutative ring admitting a functional $f : R \to \mathbb{Z}$ as in Lemma B.0.0.1.

Proof. It is immediate that the product is commutative. Associativity may be checked on the generators x_i, x_j, x_k for $i \leq j \leq k$ in cases as follows:

$$x_i(x_jx_k) = (x_ix_j)x_k = \begin{cases} 4x_i & i < j < k, \\ 2x_{i-1} + 2x_{i+1} & i = j < k, \quad i \in S, \\ 4x_i & i = j < k, \quad i \notin S, \\ 4x_i & i < j = k, \\ 2x_{i-1} + 2x_i & i = j = k \in S, \\ 4x_i & i = j = k \notin S. \end{cases}$$

□

Proposition B.0.0.3. *Let R be a ring with functional $f : R \to \mathbb{Z}$ as in Lemma B.0.0.1. Then there is some subset S of the positive integers such that R is the ring defined in Proposition B.0.0.2.*

Proof. We proceed by induction: Since $f(x_1^2) = 2$, we have either $x_1^2 = 2x_1$ or $x_1^2 = x_2 = x_0 + x_2$. Let $m \in \mathbb{Z}_{>0}$ and suppose for each $n \leq m$ that either $x_n^2 = 2x_n$ or $x_n^2 = x_{n-1} + x_{n+1}$.

Write $x_{m+1}^2 = \sum_s c_s x_s$ for $c_s \geq 0$, so that $2(m+1) = \sum_s c_s s$. Then

$$4x_m = x_m x_{m+1}^2 = \sum_s c_s x_s x_m = \sum_{s<m} c_s 2x_s + c_m x_m^2 + \sum_{s>m} c_s 2x_m,$$

so $c_s = 0$ for $s < m$. Supposing $c_m > 0$, we have $x_m^2 = 2x_m$ and then $c_m \leq 2$. If $c_m = 1$, we have $2(m+1) = m + \sum_{s>m} c_s s$ and therefore $m + 2 = \sum_{s>m} c_s s$. We conclude $c_s = 0$ for $s \neq m+2, m$, and so $x_{m+1}^2 = x_m + x_{m+2}$. If $c_m = 2$ we have $2(m+1) = 2m + \sum_{s>m} c_s s$ and therefore $2 = \sum_{s>m} c_s s$. If $m = 1$, this implies $c_s = 0$ for $s > 2$ and $c_2 = 1$, so that $x_2^2 = 2x_1 + x_2$. Here we assumed $x_1^2 = 2x_1$ so that

$$x_1(x_2^2) = 2x_1^2 + x_1x_2 = 6x_1.$$

This violates associativity since $(x_1x_2)x_2 = 2x_1x_2 = 4x_1$.

Assume otherwise that $c_m = 0$. Now we have $\sum_{s>m} c_s = 2$. Since $2(m+1) = \sum_{s>m} c_s s$, we must have $c_{m+1} > 0$, and indeed $c_{m+1} = 2$. For otherwise, if $c_{m+1} = 0$ the sum is taken over $s > m+1$, so $2(m+1) > \sum_{s>m} c_s(m+1)$, and similarly if $c_{m+1} = 1$. We conclude that $x_{m+1}^2 = 2x_{m+1}$.

Now by induction we have shown $x_s^2 = x_{s-1} + x_{s+1}$ or $x_s^2 = 2x_s$ for all $s \in \mathbb{Z}_{>0}$. Let $S = \{s \in \mathbb{Z}_{>0} \mid x_s^2 = x_{s-1} + x_{s+1}\}$. To finish the proposition, we show that no two consecutive numbers $n, n+1$ can be elements of S. Supposing so, we have $x_n(x_{n+1}^2) = x_n(x_n + x_{n+2}) = x_{n-1} + x_{n+1} + 2x_n$, while $(x_n x_{n+1})x_{n+1} = 2x_n x_{n+1} = 4x_n$. This contradicts associativity. $\square$

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