The Koszul dual to n-Lie, n-Com algebras, and Young tableaux

Cody Tipton

A dissertation
submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy

University of Washington

2024

Reading Committee:

James Zhang, Chair

John Palmieri

Julia Pevtsova

Program Authorized to Offer Degree:

Department of Mathematics

University of Washington

Abstract

The Koszul dual to n-Lie, n-Com algebras, and Young tableaux

Cody Tipton

Chair of the Supervisory Committee:
James Zhang
Department of Mathematics

We study the operad $n\text{-}Lie_d$, whose algebras are n-Lie algebras, which was first introduced in Nambu mechanics to extend Hamiltonian mechanics to more than one Hamiltonian. We find the Koszul dual of $n\text{-}Lie_{-d+n-2}$ to be the operad $n\text{-}Com_d$, whose relations come from the Specht module $S^{(n,n-1)}$. We combine the operads n-Lie and m-Com to construct the operad $(m,n) - Poiss$, where the rewriting rule that relates them is through a generalized Leibniz rule.

We generalize the above Koszul duality to different types of generalization of *Lie* and *Com* which arise from the eigenspaces of the general Kneser graphs $\mathcal{O}_{n,s}$, where the operads $n\text{-}Lie$ ($n\text{-}Com$) and Lie^d_n (Com^d_n) appear on different sides of the spectrum based on the parameter s.

With the introduction of the new class of n-Com algebras through the Koszul duality, we take our first step in exploring these new types of algebras. Specifically, we start the work of trying to classify finite-dimensional simple 3-Com algebras C using the Peirce decomposition through semisimple idempotents e through $\chi_e = m_3(e, e, -)$ to obtain important structural properties. In particular, it decomposes the 3-Com algebra $C = \bigoplus C_e(t)$ for eigenvalues t for χ_e in which $C_e(1)$ is a commutative unital associative k-algebra acting on the other components, which is almost associative.

Furthermore, we briefly construct an analog of the Killing form for 3-Com algebras, de-

noted as κ, and define non-degeneracy when the form is non-degenerate and fully degenerate when $\kappa = 0$. We use this to show that every non-degenerate 3-Com algebra is a direct product of non-degenerate simple 3-Com algebras. However, one very interesting aspect of this is that not every finite-dimensional 3-Com algebra is non-degenerate, as our main example is fully degenerate.

Towards some classifications of simple 3-Com algebras, if e is primitive semisimple idempotent with exactly two eigenvalues, and C is simple, then its eigenvalues consist of $\{1, -1\}$ which help give a classification in dimension 2 and 3.

Finally, we construct combinatorial objects, called Young n-trees, which are just rooted trees with a local Young tableaux structure at each edge following what is happening in the n-Com operad. In particular, we use these to give an upper bound to the arities of the dimension for the operad $n\text{-}Com_d$, which gives a description for it.

TABLE OF CONTENTS

Page

List of Figures . . . iv

Chapter 1: Background . . . 10
- 1.1 Rooted Trees . . . 11
- 1.2 Catalan Numbers . . . 15
- 1.3 Young Tableaux and Representation Theory of Σ_n . . . 17
  - 1.3.1 Permutation Module . . . 17
  - 1.3.2 Dominance Ordering . . . 20
  - 1.3.3 Specht Modules . . . 20
  - 1.3.4 Basis for S^λ and Garnir Relations . . . 22
- 1.4 Operads . . . 24
  - 1.4.1 $\mathcal{O}$-Algebras . . . 28
  - 1.4.2 Free Operad . . . 29

Chapter 2: (n, m)-Quadratic Operads and Koszul Duality . . . 32
- 2.1 (n, m)-Quadratic Operads and Koszul Duality . . . 33
  - 2.1.1 Description of $F(E)^{(2)}$ . . . 33
  - 2.1.2 Koszul Dual of (n, m)-Quadratic Operads . . . 36
  - 2.1.3 Symmetric and Skew-symmetric (n, m)-Quadratic Operads . . . 39

Chapter 3: The Generalizations of *Lie* . . . 43
- 3.1 The Operad Lie_n^d . . . 44
  - 3.1.1 General Definition of Lie_n Algebras of Degree d . . . 44
  - 3.1.2 Quadratic Representation for Lie_n^d . . . 45
- 3.2 The Operad n-Lie_d . . . 45
  - 3.2.1 General Definition of n-*Lie* Algebras of Degree d . . . 46
  - 3.2.2 Quadratic Representation for n-Lie_d . . . 52

Chapter 4: The Generalizations of Com 53
4.1 The Operad Com_n^d 54
4.2 The Operad n-Com_d 54
4.2.1 General Definition of n-Com Algebras of Degree d 54
4.2.2 Associative Algebra induced by n-Com algebra 61
4.2.3 3-Com algebras of dimension m 69
4.2.4 Module n-Com algebras 74
4.2.5 Quadratic Representation for n-Com_d 79

Chapter 5: Eigen-Algebras and Simple 3-Com algebras 83
5.1 Eigen-Algebras 85
5.2 Simples n-Com algebras and Ideals 88
5.2.1 Bilinear form on 3-Com algebra 98
5.3 Invariant Properties of non-zero idempotents 104

Chapter 6: The Generalizations of $Pois$ 117
6.1 Definitions of the Algebras 118
6.2 (n, m)-Module Poisson algebras 122
6.3 (m, n)-Potential Algebras 124
6.4 The Corresponding Operads 135

Chapter 7: Odd Operads 137
7.1 Generalized Odd Graphs 138
7.1.1 Spectrum of $\mathcal{O}_n$ 140
7.2 Proof of lemma 7.1.1.2 145
7.2.1 Balanced Matrix Sequences 146
7.2.2 Properties of the vertices in $\mathcal{O}_n$ 148
7.2.3 The Adjacency Matrix of $\mathcal{O}_n$ 152
7.3 Odd Operads 157

Chapter 8: Young Trees 163
8.1 Young n-Trees 164
8.1.1 General Definitions of Young n-Trees 164
8.1.2 Young (r, n)-trees 168

8.1.3 Properties of Young (r, n)-trees and Standard Edges 171
8.1.4 Symmetric Group actions 174
8.1.5 Various Poset Structures 178
8.1.6 Monomial Order on Shuffle Young (r, n)-trees 182
8.2 Specht Operad 184
8.2.1 Polytrees 184
8.2.2 Properties of $V_{r,n}^d$ 187

Bibliography 189

LIST OF FIGURES

Figure Number		Page
1	Koszul Dual	7
1.1	Example of rooted tree	12
1.2	Grafting of Cor_3 and Cor_2 at e, where e is the 3rd input edge of Cor_3.	15
2.1	Tree with two vertices	34
5.1	Examples of ideals appearing in a decomposition	107
7.1	The graph $\mathcal{O}_2$	138
7.2	The Peterson Graph	139
7.3	Graph K_{10}	139
7.4	The parameter s	162
8.1	Example of a Young 7-Tree	166
8.2	Example of representation of Young 7-tree	167
8.3	Why Shuffle Young n-trees	168
8.4	Example of Permuting the planar order	170
8.5	Example of Co-standard Young $(3,3)$-tree	172
8.6	Description for all standard shuffle Young (r,n)-trees	173
8.7	Example of Young $(4,3)$-tree T	175
8.8	Ungrafting the subtrees	176
8.9	The Young Tree $T \circ_e \sigma$	176
8.10	Example of Ordering of Young (r,n)-trees	179
8.11	Young $(3,3)$-tree of degree d	185

ACKNOWLEDGMENTS

I express my deepest gratitude to my doctoral committee members for their invaluable guidance, support, and encouragement throughout this journey.

First and foremost, I am indebted to James Zhang for his guidance as an advisor during my time at the University of Washington, insightful feedback on my work, seminar talks, and paper writing.

I would like to express my sincere gratitude to Zihao Qi for his invaluable support, encouragement, and feedback on my papers and research. He has been very encouraging when the pressure of completing work on challenging problems and extremely helpful in their knowledge of operads and Grobner basis theory.

I would like to thank my colleague and especially my friend Albert Artiles for their constructive criticism, encouragement, and dedication to fostering intellectual curiosity throughout my work. They have always been there when the times were rough and when the times were full of joy. I do not think I could have done what I have done today without their support and making me laugh when I needed it.

I extend my sincere appreciation to Medha Sagar, who came into my life in the latter half of my PhD journey, which has supported me emotionally and given me distractions away from my mathematical pursuits in cooking together, board games, movies, and many more.

Finally, I would like to thank my family members for their unwavering support back from Texas, whose support has been a source of strength in pursuing the goal I decided to take on years ago.

This world would not have been possible without the support and contributions of every individual mentioned above, my colleagues, and friends. Thank you all for being a part of

this incredible journey.

DEDICATION

To my Friends and Family

Introduction

There are various generalizations of the operads Com and Lie to n-arity operation by either extending the defining relations for them, changing the perspective of generalization, or weakening the relation in some way. For the operad Lie, if we let ν be its generator, then its defining relation is essentially the sum

$$\nu \circ_1 \nu + (\nu \circ_1 \nu)^{(1\ 2\ 3)} + (\nu \circ_1 \nu)^{(1\ 3\ 2)} = 0$$

using the even representatives of the right cosets of $\overline{\Sigma}_1 \times \Sigma_2$ in Σ_3, (where $\overline{\Sigma}_1$ is the subgroup of the permutation group Σ_3 which fixes 1 and 2). One can generalize this for any n-arity operation by extending this sum to all the even representatives of the right cosets of $\overline{\Sigma}_{n-1} \times \Sigma_n$ in Σ_{2n-1}. This gives us the Lie^d_n-algebras, which are graded vector spaces L with a n-arity skew-symmetric bracket $[-, \ldots, -] : L^{\otimes n} \to L$ of degree d such that

$$\sum_{\sigma \in Sh(n,n-1)} \xi(\sigma, x_1, \ldots, x_n) Sgn(\sigma)[[x_{\sigma(1)}, \ldots, x_{\sigma(n)}], x_{\sigma(n+1)}, \ldots, x_{\sigma(2n-1)}] = 0$$

for every $x_1, \ldots, x_{2n-1} \in L$ and where $\xi(\sigma, x_1, \ldots, x_n)$ is the Koszul sign rule from the permutation. When $d = 2-n$ then these are essentially L_∞-algebras with only a non-zero n-arity operation, which was first discovered by Jim Stasheff in [26] and used in the celebrated formality conjecture by Maxim Kontsevich in [13] with implications for the deformation theory of Poisson manifolds.

On the other side of the spectrum, we can think of the Lie algebra relation as a derivation and generalize the relation accordingly. This gives us the n-Lie algebras of degree d, or simply as n-Lie_d-algebras, which are graded vector spaces L with an n-arity skew-symmetric bracket $[-, \ldots, -] : L^{\otimes n} \to L$ of degree d such that for any $n-1$ elements, $[-, x_1, \ldots, x_{n-1}]$ is a derivation on L. The most natural example of a n-Lie algebra of degree 0 is taking the polynomial ring $A = k[x_1, \ldots, x_n]$ and defining

$$[f_1, \ldots, f_n] = \mathrm{Jac}(f_1, \ldots, f_n),$$

where the right-hand side is the determinant of the Jacobian matrix of the n polynomials $f_1, \ldots, f_n \in A$. Nambu first introduced the concept of 3-Lie bracket in [20] to extend the principles of Hamiltonian mechanics beyond a single Hamiltonian. In the algebraic setting, the general concept of n-Lie algebras was first introduced by Filippov in [8], who studied the representation theory and constructed simple n-Lie algebras of dimension $n+1$ in characteristic not equal to 2. Later, Kasymov in [12] started the pioneering work of studying the nilpotency and solvability of n-Lie algebras and proved generalizations of Engel's theorem and Cartan's solvability criterion.

On the other side, since the operad Com is Koszul dual to Lie, we should expect to be able to construct generalizations of commutative associative algebras which are Koszul dual to the various generalizations of the operad Lie. A straightforward generalization to Com is the Com_n^d-algebras, which have symmetric n-arity operations m_n of degree d, that is totally associative:

$$m_n \circ_i m_n = m_n \circ_j m_n$$

for all $1 \leq i, j \leq n$. These are the most natural and easier to work with as total associativity is a very strong property to generalize the usual notations from commutative associative algebras. Since both Com_n^d and Lie_n^d are very similar to their classical counterparts, the Koszul duality of Lie_n^d and Com_n^{-d+n-2} holds due to essentially the same argument, with some shifts in degrees.

Regarding the Koszul dual of n-Lie_d, its relations are a lot more complicated as they come from the Specht module $S^{(n,n-1)}$. In particular, a n-Com_d-algebra is a graded k-module C equipped with a symmetric n-arity operation $m_n : C^{\otimes n} \to C$ of degree d, satisfying the $(n, n-1)$-polytabloid relation

$$\sum_{\sigma \in C_n} Sgn(\sigma)(m_n \circ_1 m_n)^\sigma = 0,$$

where C_n is the group generated by $(1\ n+1), \ldots, (n-1\ 2n-1)$. Both of these operads are connected through the eigenspaces of a sequence of graphs $\{\mathcal{O}_n\}$, called the Odd graphs, which are a certain subset of the Kneser graphs, whose eigenvalues $(-1)^{n+1}$ have multiplicity C_n, the nth Catalan number. Using these connections, we can prove the following theorem.

Theorem 0.0.0.1. *For all $n \geq 2$ and $d \in \mathbb{Z}$, the operads n-Lie_d and n-Com_{-d+n-2} are Koszul dual.*

This gives a proof that $(n\text{-}Lie_d)(2n-1) \cong \uparrow^{2d} S^{(n,n-1)^t}$, where $(n, n-1)^t$ is the transpose partition.

Since the n-Com algebras are a new type of algebra, we set out to explore these types of algebra by constructing lots of examples and exploring some invariants connected to them. In particular, we construct examples of n-Com algebras of degree 0 by deforming augmented modules over a commutative k-algebra, using a derivation on a commutative k-algebra, and finding finite-dimensional examples derived from a certain collection of matrices with some relations. For explicit examples, let $M_n(k)$ be the k-algebra of $n \times n$ matrices over k, and let $Tr : M_n(k) \to k$ be the trace map. Then we can construct a 3-Com algebra structure on $M_n(k)$ with

$$\mu^{Tr}(A, B, C) = Tr(A)Tr(B)C + Tr(A)Tr(C)B + Tr(B)Tr(C)A.$$

Alternatively, if A is a commutative k-algebra equipped with a derivation $D : A \to A$, we can define a n-Com algebra structure on A by defining

$$m_n^D(a_1, \dots, a_n) = D(a_1 \cdots a_n)$$

for $n > 2$.

To gain insight into the structure of finite dimensional n-Com algebra structure, we use the theory of Peirce decompositions on our spaces and study primitive semisimple idempotents to find examples and, in particular, give some criteria of when they are not simple. To do this, we take a primitive semisimple idempotent e in a finite-dimensional n-Com algebra C and look at the eigenvalues of the endomorphism $\chi_e = m_n(e, \dots, e, -)$, which gives us the decomposition $C = \bigoplus C_e(\alpha)$, where the sum runs over distinct eigenvalues of the endomorphism χ_e, i.e., distinct elements in the Peirce spectrum $\sigma(e)$ (the eigenvalues of χ_e including multiplicity).

In the case $n = 3$, we are able to obtain important structure properties of finite dimensional 3-Com algebras and some small classification results using these tools. In particular, the subspace $C_e(1)$ has an unital commutative associative k-algebra structure with multiplication

$$\theta_e(x, y) := m_3(e, x, y)$$

and each of the $C_e(\gamma)$ for $\gamma \in \sigma(e)$ have a $C_e(1)$-action that is almost associative. Furthermore, the multiplication is graded in the sense that $\theta_e(x, y) \in C_e(t + s - 1)$ if $x \in C_e(t)$ and $y \in C_e(s)$. Using these properties, one can determine what an arbitrary finite-dimensional 3-Com algebra structure and multiplication are if it has a semisimple idempotent and you know its spectrum $\sigma(e)$.

In the case where the only eigenvalue in $\sigma(e)$ is 1, we have $C = C_e(1)$ and this shows that the 3-Com algebra structure with multiplication m_3 actually comes from the commutative associative algebra structure on $C_e(1)$ through

$$m_3(x, y, z) = \theta_e(\theta_e(x, y), z).$$

Even more, C is simple if and only if $C_e(1)$ is a simple commutative associative k-algebra with a unit, i.e., a field. So we know when a 3-Com algebra comes from a commutative k-algebra by looking if it has only one distinct eigenvalue for a semisimple idempotent.

In the case where the semisimple primitive idempotent e has exactly 2 distinct eigenvalues, the grading structure induces strong conditions on what the eigenvalues can be in this situation.

Theorem 0.0.0.2. *If C is a m-dimensional simple 3-Com algebra with $m \geq 2$ equipped with a semisimple primitive non-zero idempotent e with exactly two Peirce numbers, then $\sigma(e) = \{1, -1, -1, \ldots, -1\}$, where the algebraic(geometric) multiplicity of -1 is $m - 1$.*

In particular, this shows that every simple 2-dimensional 3-Com algebra with a primitive semisimple idempotent is isomorphic to a 3-Com algebra SC_2 with basis e, f and multiplication

$$m_3(e,e,e) = e \qquad m_3(e,e,f) = -f$$
$$m_3(e,f,f,) = 0 \qquad m_3(f,f,f) = e.$$

Hence, we have used these two cases to classify all simple 3-Com algebras of dimension less than or equal to 2 equipped with a semisimple idempotent. We can go further into dimension 3 and show there does not exist any simple 3-dimensional 3-Com algebras.

Theorem 0.0.0.3. *There does not exist a simple* 3*-dimensional* 3*-Com algebra equipped with a semisimple primitive idempotent.*

In analog to Lie algebras, we are also able to define a symmetric associative bilinear form on any finite-dimensional 3-Com algebra as follows. We define $\chi_{x,y} = m_3(x,y,-)$, which is bilinear and symmetric in both x and y and let $\kappa(x,y) = Tr(\chi_{x,y})$ with satisfies the associative identity:

$$\kappa(m_3(x,y,z),w) = \kappa(x, m_3(y,z,w)).$$

This enables us to define fully degenerate and non-degenerate 3-Com algebras as follows.

Definition 0.0.0.1. *A finite-dimensional* 3*-Com algebra C is non-degenerate if and only if κ is non-degenerate. We say C is fully degenerate if and only if $\kappa = 0$.*

This enables us to show that any non-degenerate 3-Com algebras are a direct product of non-degenerate simple 3-Com algebras. Furthermore, if e is a semisimple idempotent of a non-degenerate 3-Com algebra C, then $C_e(1)$ is a commutative Frobenius algebra by restricting the bilinear form κ onto $C_e(1)$.

This gives us two types of simple 3-Com algebras, ones that are fully degenerate and the others that are non-degenerate. So far, the non-degenerate simple 3-Com algebras we know are fields and more closely resemble what happens for commutative associative k-algebras.

On the other hand, the fully degenerate ones are new and do not resemble anything that happens for commutative associative algebras. One can think of this as follows: we have gone up by an arity in our multiplication which enables non-traditional properties to be obtained, like simple 3-Com algebras which are not fields.

For the classical operads Com and Lie, we can combine them together to obtain the Poisson operad $Pois \cong Com \circ Lie \cong Com \wedge_\gamma Lie$ with respect to some rewriting rule γ coming from the Leibniz rule. We can apply a similar construction to the generalizations of Com and Lie above to construct various generalizations of the Poisson operad. To do this, we extend the Leibniz rule in the following way. Suppose P is a vector space with a symmetric operation $\mu : P^{\otimes m} \to P$ and a skew-symmetric operation $\nu : P^{\otimes n} \to P$. Therefore, the generalized Leibniz rule is defined as

$$\nu(\mu(f_1, \dots, f_m), g_1, \dots, g_{n-1}) = \sum_{i=1}^{m} (-1)^{\varepsilon_i} \mu(f_1, \dots, , f_{i-1}, \nu(f_i, g_1, \dots, g_{n-1}), f_{i+1}, \dots, f_m)$$

for some appropriate Koszul signs ε_i. These considerations lead to the following two definitions of algebraic structures, combining the operads mentioned above.

Definition 0.0.0.2. *A $Pois_{m,n}^{d_m,d_n}$-algebra is a graded vector space P with a m-arity symmetric operation $\mu : P^{\otimes m} \to P$ of degree d_m and a n-arity skew-symmetric operation $\nu : P^{\otimes n} \to P$ of degree d_n such that*

- *(P, μ) is a $Com_m^{d_m}$-algebra*
- *(P, ν) is a $Lie_n^{d_n}$-algebra*
- *and μ and ν satisfies the generalized Leibniz rule.*

Definition 0.0.0.3. *A $(m, n) - Pois_{d_m,d_n}$-algebra is a graded vector space P with a m-arity symmetric operation $\mu : P^{\otimes m} \to P$ of degree d_m and a n-arity skew-symmetric operation $\nu : P^{\otimes n} \to P$ of degree d_n such that*

- *(P, μ) is a $m - Com_{d_m}$-algebra*

- *(P, ν) is a n-Lie_{d_n}-algebra*

- *and μ and ν satisfies the generalized Leibniz rule.*

The $(2, n)$-Poisson algebras are classically called n-Lie Poisson algebras and have been studied in Physics through Nambu mechanics and Nambu-Poisson manifolds, see [28]. The natural example is the n-Lie algebra $A = k[x_1, \ldots, x_n]$ with $\{f_1, \ldots, f_n\} = \mathrm{Jac}(f_1, \ldots, f_n)$ as defined above satisfies the generalized Leibniz rule with the ordinary multiplication in A.

As we explained before, the connection between n-Lie_d and n-Com_{-d+n-2} is through the eigenvalues of certain graphs we mentioned above; in fact, this happens for all of the generalizations of *Lie* and *Com*. There exists a class of graphs $\mathcal{O}_{n,s}$ which can connect both sides of the generalizations through the eigenspaces in a very natural process. The graph $\mathcal{O}_{n,s}$ has vertices consisting of ordered sequences $(a_1, \ldots, a_{n-1})$ of elements $a_1, \ldots, a_{n-1} \in \{1, \ldots, 2n-1\}$ with an edge between $(a_1, \ldots, a_{n-1})$ and $(b_1, \ldots, b_{n-1})$ if and only if $|(a_1, \ldots, a_{n-1}) \cap (b_1, \ldots, b_{n-1})| < s$. These include famous graphs like the triangle, the Peterson graph, K_{10}, etc. Using these graphs we can create a vast collection of operads $Odd^d_{n,s}(\lambda)$ and $AOdd^d_{n,s}(\lambda)$ where λ is an eigenvalue of the graph $\mathcal{O}_{n,s}$ such that $Odd^d_{n,s}(\lambda)$ is Koszul dual to $AOdd^{-d+n-2}_{n,s}(\lambda)$, see figure 1. These operads give us all of the different generalizations of

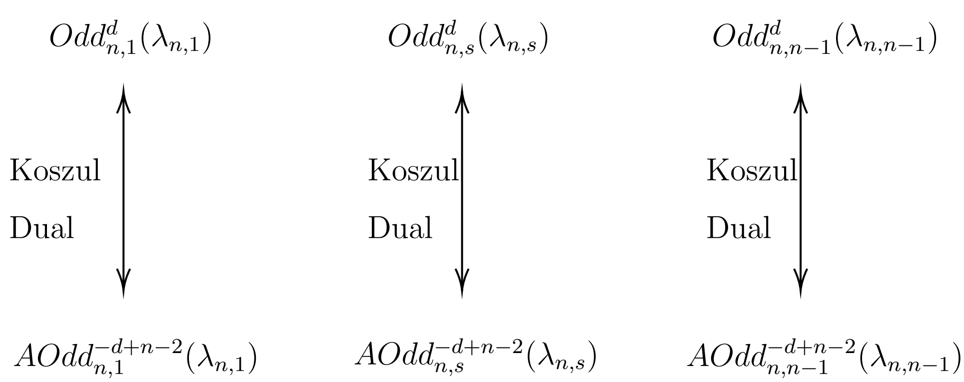


Figure 1: Koszul Dual

the *Com* and *Lie* operads, including the operads we mentioned above.

lemma 0.0.0.4. *We have the following isomorphism of operads*

$$n\text{-}Com_d \cong Odd^d_{n,1}((-1)^{n+1})$$
$$n\text{-}Lie_d \cong AOdd^d_{n,1}((-1)^{n+1})$$
$$Com^d_n \cong Odd^d_{n,n-1}(-1)$$
$$Lie^d_n \cong AOdd^d_{n,n-1}(-1)$$

for all $n \geq 2$ *and* $d \in \mathbb{Z}$.

This shows that $n\text{-}Com_d$, $n\text{-}Lie_d$, Lie^d_n, and Com^d_n exist on different sides of the spectrum for generalizations of the operads Com and Lie for $n \geq 3$, where at $n = 2$ they coincide. So with $Odd^d_{n,s}(\lambda)$, we should think of s as some parameter that deforms the different generalizations to either side of the spectrum, as in figure 1. As for the generalizations of Com, when $s = n - 1$ it is associative, and as we make s smaller towards 1, it becomes more non-associative until we get to $n\text{-}Com_d$ which has a very non-associative relation.

For the structure of this paper, chapter 1 is the background on the necessary information about rooted trees, representation of the symmetric groups, Catalan numbers, and operad theory. Chapter 2 defines (n, m)-quadratic data and Koszul duality with this information which will be used later to define our operads and prove Koszul duality of $n\text{-}Com_d$ and $n\text{-}Lie_{-d+n-2}$. Chapters 3 and 4 study the generalizations of the operads Lie and Com, respectively, and give examples for each of these operads. In chapter 5 we explore simple 3-Com algebras by defining Peirce decomposition for these types of algebras and exploring the consequences, and show that we can classify the simple 3-Com algebras of dimension 2. For chapter 6, we combine the generalizations of Lie and Com together to construct the (m, n)-Poisson algebras and its operad. We explore different examples and show that we can construct a full class of them through strong n-Lie Poisson algebras. Chapter 7 is the exploration of the Generalized Odd graphs and using them to construct the odd operads, which give pairs of Koszul dual operads, which include the operads we discussed in chapters

3 and 4. Finally, for chapter 8, we construct Young (r,n)-trees, which give us the elements in the operad n-Com_d and hence help us describe the space and a bound for the dimensions of the arities.

Notation and Conventions

Let $n \geq 2$ and denote by Σ_n the symmetric group on n letters. If $m \geq n$, we will consider Σ_n as a subgroup of Σ_m, whose elements fix the last $m-n$ elements of $\{1,\ldots,m\}$, and we denote by $\overline{\Sigma}_n$ the subgroup of Σ_m whose elements fix the first $m-n$ elements of $\{1,\ldots,m\}$. Denote by $Sh(n,m)$ the set of (n,m)-shuffles in Σ_{n+m}.

Let k be a field and denote by $GVect_k$ the category of graded k-modules, and by $Vect_k$ for the category of vector spaces over k. For the whole paper, we will assume that the characteristic of the field k is zero unless otherwise stated. It is not needed for most of the definitions, but for applying the irreducible representations of Σ_n and our results, we need to have non-positive characteristics.

Let V be a graded k-module and define the right action of Σ_n on $V^{\otimes n}$ as follows: for $v_1,\ldots,v_n \in V$, define

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = \xi(\sigma, v_1,\ldots,v_n) v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

where $\xi(\sigma, v_1,\ldots,v_n)$ is the Koszul sign rule from permutating the elements in the tensor product by σ^{-1}. Furthermore, if we have a n-arity function $f : V^{\otimes n} \to V$ of degree d, we define

$$f^{\sigma}(v_1 \otimes \cdots \otimes v_n) = f(\sigma \cdot v_1 \otimes \cdots \otimes v_n) = \xi(\sigma, v_1,\ldots,v_n) f(v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}).$$

Chapter 1

BACKGROUND

1.1 Rooted Trees

There are various ways to define the free operad in the setting of vector spaces; the route we will take is through the combinatorial description using rooted trees. There is a natural relationship between identifying operations in an operad with its corresponding rooted tree, which gives an intuitive description of the composition of the free operad in terms of grafting. Furthermore, we will need the language of rooted trees to help describe the operads $n\text{-}Com_d$ in section 8.1.1 using Young n-trees. We will follow [29] for the presentation and definitions for our rooted trees.

Definition 1.1.0.1. *Let $m, n \geq 0$. A directed (m, n)-graph is a quadruple $G = (V, E, in_G, out_G)$ consisting of*

- *a directed graph (V, E), where V are the abstract vertices and E are the ordered edges, and*
- *disjoint subsets in_G and out_G of V such that the following conditions hold.*
  - *$|in_G| = m$ and $|out_G| = n$.*
  - *Each $v \in in_G$, we have $|in(v)| = 0$ and $|out(v)| = 1$, where $in(v)$ is the set of input edges of v and $out(v)$ is the set of outgoing edges of v.*

For any directed (m, n)-graph G, define V_G^{in} to be the set of elements in V that are not in $in_G \cup out_G$. We call elements of in_G the inputs of G, the elements of out_G the outputs of G, and all elements of V_G^{in} the vertices of G or the internal vertices of G, while the vertices in $in_G \cup out_G$ are the external vertices.

Definition 1.1.0.2. *A rooted m-tree T is a connected, acyclic, directed $(m, 1)$-graph (V, E, in_T, out_T) such that $|out(v)| = 1$ for every $v \in Vt_T$.*

For an m-rooted tree T, we call the unique outgoing edge of T the root edge and denote it by e_{r_T}, and denote by r_t, the unique initial vertex v of the root edge, called the root vertex, provided that v is not in in_T, otherwise it does not exist. An edge from an external vertex to an internal vertex is called a leaf, and denote the set of leaves at an internal vertex v by $leaves_v$. Edges between internal vertices are called internal edges, and we will denote the subset of E_T consisting of internal edges as E_T^{in}.

Example 1.1.0.1. *For an example, let T be the rooted planar tree as in figure 1.1 which has*

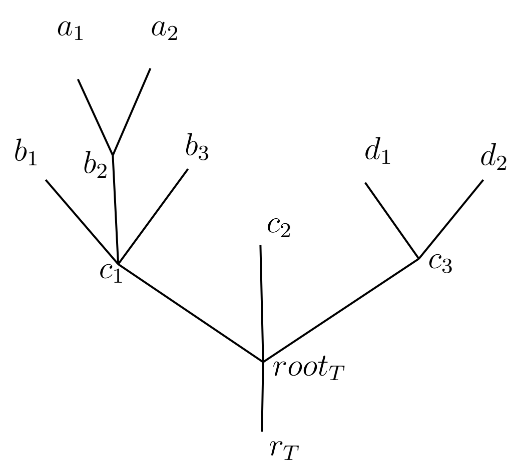


Figure 1.1: Example of rooted tree

the properties

$$V_T^{in} = \{b_2, c_1, c_3, root_T\}$$
$$E_T^{in} = \{(b_2, c_1), (c_1, root_T), (c_3, root_T)\},$$

and

$$in_T(c_1) = \{b_1, b_2, b_3\}$$
$$int_T(b_2) = \{a_1, a_2\}$$
$$in_T(c_3) = \{d_1, d_2\}$$
$$int_T(root_T) = \{c_1, c_2, c_3\}.$$

A **planar structure** on a rooted m-tree is a collection of maps $\Psi_v : [|in_T(v)] \to int_T(v)$

for $v \in V_T^{in}$, where $[n] = \{1, \ldots, n\}$. A **vertex-input labeling** on a rooted m-tree T by a set S is a collection of maps $\lambda_v : in_T(v) \to S$ for all $v \in V_T^{in}$. An **input labeling** on a rooted m-tree T by a set S is injective map $\lambda : in_T \to S$, and if $S = [m]$, then we call T a rooted m-labeled tree, which in this case we will, without loss of generality, assume $in_T = [m]$.

Definition 1.1.0.3. *Let T and S be rooted m-trees.*

- *An isomorphism between T and S is a pair of bijections $f_V : V_T \to V_S$ and $f_E : E_T \to E_S$ such that if $(u, v) \in E_T$ if and only if $(f_V(u), f_V(v)) \in E_S$. We let $Tree_m$ be the set of isomorphism classes of rooted m-trees.*

- *If T and S have planar structures $\{\Psi_v\}_{v \in V_T^{in}}$ and $\{\Psi'_u\}_{u \in V_S^{in}}$ respectively, then we say the isomorphism is an isomorphism of rooted planar trees if $f_V(\Psi_v(i)) = \Psi_{f_V(v)}(i)$ for $1 \leq i \leq |in_T(v)|$ for all $v \in V_T^{in}$. We let $PTree_m$ be the set of isomorphism classes of planar rooted m-trees.*

- *If T has vertex-input labeling $\{\lambda_v\}_{v \in V_T^{in}}$ and S has a vertex-input labeling $\{\lambda'_u\}_{u \in V_S^{in}}$ on the same set X, then the isomorphism preserves the input labeling if and only if $\lambda_v(w) = \lambda_{f_V(v)}(f_V(w))$ for all $w \in in_T(v)$ and all $v \in V_T^{in}$.*

- *If T and S have input labeling on the set $[m]$, then an isomorphism between T and S preserves the input labeling if f_V is an identity on $in_T = [m] \to in_S = [m]$. We will denote by $Tree(m)$ the set of isomorphism classes of rooted m-labeled trees.*

Suppose T is a rooted m-labeled tree and $\sigma \in \Sigma_m$, then we define $\sigma^*(T)$ to be the rooted m-labeled tree with the same vertices, but with the inputs in_T re-indexed by σ through the induced map $\sigma : int_T = [m] \to in_{\sigma^*(T)} = [m]$. This induces natural isomorphism $\sigma^* : E_T \to E_{\sigma^*(T)}$ and $\sigma^* : int_T(v) \to in_{\sigma^*(T)}(v)$ for each $v \in V_T^{in}$ by applying σ on the elements of in_T and the identity on the rest.

Next, we will briefly describe the process of grafting two trees, as this will be important for defining the composition of the free operads and the construction of the operad SpO_n^d in section 8.2. The following definition of grafting of rooted trees is taken from [29], where they go into more detail than what we will do here.

Definition 1.1.0.4. *Let T_1 be an rooted m-tree and T_2 be an rooted n-tree. Let e be a input edge of T_1 and define $S = T_1 \circ_e T_2$ to be the rooted $n+m-1$-tree with the following properties:*

- *The set of abstract vertices $V = V_{T_1} \coprod V_{T_2}$;*
- *The set of edges $E = E_{T_1} \coprod E_{T_2} \setminus (e \sim e_{r_{T_2}})$, where $e_{r_{T_2}}$ is the root edge of T_2;*
- *The root vertex $r_S = r_{T_1}$;*
- *and the inputs $in_S = (in_{T_1} \coprod in_{T_2}) \setminus \{v\}$ where v is the external vertex for the input edge e.*

In the situation where T_1 is a rooted m-labeled tree, T_2 is a rooted n-labeled tree, and e corresponds to the ith edge of T_1, then we define $in_S = [n+m-1]$ and re-index the input labels as follows: the inputs of T_1 with labels lower then i are unaffected, the original ith spot is now occupied by the inputs of T_2, so each one is bumped up by $i-1$, and the inputs of T_1 with labels greater then i are bumped up by $n-1$.

If T_1 has planar structure $\{\Psi_v\}$ be and T_2 has a planar structure $\{\Psi'_u\}$, then $S = T_1 \circ_e T_2$ has planar structure $\{\Psi''_v\}$ with $\Psi''_v = \Psi_v$ if $v \in V_{T_1}$ or $\Psi''_v = \Psi'_v$ for $v \in V_{T_2}$.

Denote by $\downarrow$ for the trivial rooted 1-labeled tree with no vertices, which is the unique tree up to 1-tree isomorphism. For $m \geq 0$, denote by Cor_m the rooted m-labeled tree with a single internal vertex v and edges (i, v) for $1 \leq i \leq m$, these trees are called the m-Corolla rooted trees and are very important since all other trees are grafting of such trees. For an example of grafting two corolla trees, see 1.1. Another essential property of Corolla trees is that they are invariant under any permutation of Σ_m: if $\sigma \in \Sigma_m$, then $\sigma^*(Cor_m) \cong Cor_m$ through an identity map.

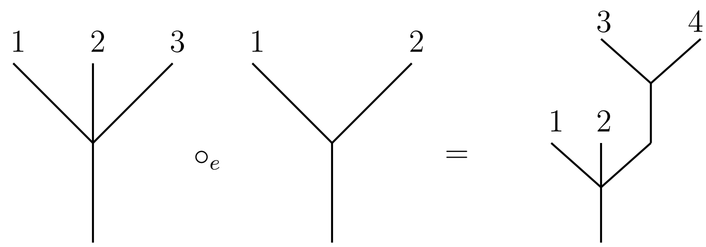


Figure 1.2: Grafting of Cor_3 and Cor_2 at e, where e is the 3rd input edge of Cor_3.

1.2 Catalan Numbers

The Catalan numbers are a sequence of numbers that occur in vast amounts of different counting problems in combinatorics to the dimensions of the Specht module S^λ for certain partitions λ of $2n-1$ which appears in the representation theory of the symmetric groups. I cannot explain all of the beautiful connections that Catalan numbers have to various counting problems, but luckily for us, there is already a very well-written book on the history and properties of Catalan numbers by Richard P. Stanley in [24]. This section will only present the basic definitions and results we need for this thesis.

One of the first instances of the Catalan numbers appeared in counting the number of different ways of triangulating a convex n-polygon. This gives a natural way to define the Catalan numbers in a recursive formula as follows.

Definition 1.2.0.1. *The Catalan numbers are the numbers C_n such that $C_0 = 1$ and $C_n = \sum_{\substack{i+j=n \\ i,j\geq 0}} C_iC_j$.*

Using the recurrence formula defined above, we can go further and find the generating function for the Catalan numbers; for more information about generating functions, see [25]. One important example of a generating function, first proved by Issac Newton, called the "generalized binomial theorem" is

$$(1+x)^a = \sum_{n\geq 0} \binom{a}{n} x^n$$

where a is any complex number or indeterminate. Note that this is just the Taylor series for the function $(1+x)^a$ based at $x = 0$, but for our purposes, we consider this as a formal equation and ignore any information about convergence.

Proposition 1.2.0.1. *Let $C(x) = \sum_{n\geq 0} C_n x^n$ be the generating function for the Catalan numbers, then*

$$C(x) = \frac{1-\sqrt{1-4x}}{2x}$$

From the generating function, we can now easily find an explicit formula for the Catalan numbers.

Theorem 1.2.0.2. *We have $C_n = \frac{1}{n+1}\begin{pmatrix} 2n \\ n \end{pmatrix}$*

One can go further and generalize the Catalan numbers to the (k)-fold convolutions of $\{C_n\}$, and this will be important for us as these will appear in the dimensions of various operads.

Definition 1.2.0.2. *The (k)-fold convolution Catalan number $C_n^{(k)}$ is defined as*

$$C_n^{(k)} = \sum_{i_1+\cdots+i_{k+1}=n} C_{i_1} \cdots C_{i_{k+1}}$$

Note that the Catalan numbers are the (0)-fold convolution $C_n^{(0)}$, and this naturally generalizes the Catalan numbers. There is an extremely nice formula for the (k)-fold convolution of the Catalan numbers, first due to Catalan in [3].

Theorem 1.2.0.3. *For every $k \geq 0$ and $n \geq 0$ we have*

$$C_n^{(k)} = \frac{k+1}{n+k+1}\begin{pmatrix} 2n+k \\ n \end{pmatrix}$$

For various different proofs and interpretation of the (k)-fold convolution of the Catalan numbers, see [21, 15, 27].

1.3 Young Tableaux and Representation Theory of Σ_n

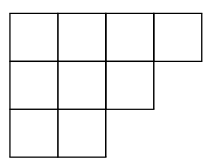
For this section, we will state the definitions and results needed about the representation theory of the symmetric groups in characteristic 0. In particular, the Specht modules for a partition λ play a big part in studying the generalizations of the operads *Com* and *Lie*. Most of the material in this section will follow [22].

1.3.1 Permutation Module

A partition of an integer n is a sequence of integers $\lambda = (\lambda_1, \ldots, \lambda_m)$ such that $\lambda_1 + \cdots + \lambda_m = n$ and we will write $\lambda \vdash n$. We also use the notation $|\lambda| = \sum_{i=1}^{m} \lambda_i$ so that a partition of n satisfies $|\lambda| = n$. We can visualize a partition using a Young diagram.

Definition 1.3.1.1. *Let* $\lambda = (\lambda_1, \ldots, \lambda_m) \vdash$. *A Young diagram of shape* λ *is an array of* n *cells having* m *left-justified rows with row* i *containing* λ_i *cells for* $1 \leq i \leq m$.

The cell in row i and column j has coordinates (i, j) as in a matrix. For example, suppose we have $\lambda = (4, 3, 2)$, then a Young diagram of this shape is of the form



.

Recall that if T is a set, then Σ_T is the set of permutations of T.

Definition 1.3.1.2. *Let* $\lambda = (\lambda_1, \ldots, \lambda_m) \vdash n$. *Then the corresponding Young subgroup of* Σ_n *is*

$$\Sigma_\lambda = \Sigma_{\{1,\ldots,\lambda_1\}} \times \Sigma_{\{\lambda_1+1,\ldots,\lambda_1+\lambda_2\}} \times \cdots \times \Sigma_{\{n-\lambda_m+1,n-\lambda_m+2,\ldots,n}$$

which is isomorphic to the group $\Sigma_{\lambda_1} \times \Sigma_{\lambda_2} \times \cdots \times \Sigma_{\lambda_m}$.

Next, we will define the various versions of Young tableaux that appear in the literature and help understand the representations of the symmetric group.

Definition 1.3.1.3. *Suppose $\lambda \vdash$. A Young tableau of shape λ is Young diagram T with the cells filled in with numbers $1, \ldots, n$ bijectively. We will also call Young tableau of shape λ a λ-tableau.*

If T is a Young tableau, we let $T_{i,j}$ stand for the entry of T in position (i, j), just as in a matrix. Note that there are $n!$ number of Young tableau for a certain shape λ. Furthermore, there is a natural left-action of the symmetric group on the set of Young tableaux of shape λ by just applying the permutation on each number in the cells, i.e., if $\sigma \in \Sigma_n$, then $\sigma T = (\sigma(T_{i,j}))$. An important subclass of Young tableaux is the standard Young tableaux, where each of the rows increases from left to right, and the columns increase from top to bottom. We will use these later to find a basis for the various irreducible representations of the symmetric groups. The following theorem is the hook length formula, which counts the number of standard Young tableaux of shape λ; see [22] for more information.

Theorem 1.3.1.1. *Let $\lambda = (\lambda_1, \ldots, \lambda_m) \vdash n$, and let T be the corresponding Young diagram. For each cell (i, j) of T, define $H_\lambda(i, j)$ to be the set of cells (a, b) such that $a = i$ and $b \geq j$ or $a \geq i$ and $b = j$. The hook length $h_\lambda(i, j) = |H_\lambda(i, j)|$. The hook length formula*

$$f^\lambda = \frac{n!}{\prod_{i,j} h_\lambda(i, j)} \tag{1.1}$$

where the product is over cells (i, j) of T, counts the number of standard Young tableaux of shape λ.

Another generalization of Young tableaux is by relaxing the condition that the set of numbers in the cells has to be a bijection.

Definition 1.3.1.4. *Let $\lambda \vdash n$. A semistandard Young tableaux of shape λ is a Young diagram with the cells filled in with the numbers $1, \ldots, n$ where the rows are weakly increasing, and the columns are strictly increasing. Note there can be repeats along the rows.*

Given two partitions $\lambda, \mu \vdash$, a semistandard Young tableau of shape λ and content μ is a semistandard Young tableaux T of shape λ such that i appears μ_i number of times in T.

Definition 1.3.1.5. *Two λ-tableaux T_1 and T_2 are row-equivalent, $T_1 \sim T_2$, if the corresponding rows of the two tableaux contain the same elements. A λ-tabloid is an equivalence class*

$$\{T\} = \{S \ : \ T \sim S\}.$$

For an example, suppose $\lambda = (2,1)$ and Young tableau

$$T = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

, then its corresponding Young tabloid is

$$\{T\} = \{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \}$$

. If $\lambda = (\lambda_1, \ldots, \lambda_m) \vdash n$, the number of λ-tabloids is just $\frac{n!}{\lambda!}$, where $\lambda! = \lambda_1! \cdots \lambda_m!$. The natural action on tableaux by the symmetric group induces an action on the Young tableaux by setting $\sigma\{T\} = \{\sigma T\}$, which is well-defined. Hence, this gives us a natural Σ_n-module as follows.

Definition 1.3.1.6. *Suppose $\lambda \vdash n$. Let M^λ be the k-module generated by all of the λ-tabloids, which has dimension $\frac{n!}{\lambda!}$. Then M^λ is called the permutation module corresponding to λ.*

Example 1.3.1.1.

- *If $\lambda = (n)$, then $M^{(n)}$ is the trivial representation of Σ_n.*

- *If $\lambda = (1^n)$, then $M^{(1^n)} \cong k[\Sigma_n]$ with the regular representation since every λ-tabloid corresponds to a unique permutation.*

- *If $\lambda = (a,b)$ for $a + b = n$ and $b < a$, then $M^{(a,b)}$ is isomorphic to the k-linear space spanned by ordered tuples $(a_1, \ldots, a_b)$ since each of the λ-tabloids are uniquely determined by the elements in the bottom row.*

1.3.2 Dominance Ordering

The collection of partitions has two important orderings that enable us to study the spaces M^λ effectively and find the irreducible submodules. We will use these orderings later in our theory of Young n-trees to find appropriate orderings for studying the operad $n\text{-}Com_d$.

Definition 1.3.2.1. *Suppose $\lambda = (\lambda_1, \dots, \lambda_m)$ and $\mu = (\mu_1, \dots, \mu_l)$ are partitions of n. Then λ dominates μ, written $\lambda \rhd \mu$, if*

$$\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$$

for all $i \geq 1$. If $i > l$, then we take $\lambda_i = 0$, and similarly for μ.

We have the fundamental lemma concerning the dominance order through the corresponding Young tableaux.

lemma 1.3.2.1 (Dominance Lemma for Partition)**.** *Let T and S be Young tableaux of shape λ and μ, respectively. If, for each index i, the elements of row i of S are all in different columns in T, then $\lambda \rhd \mu$.*

The second ordering on partitions is as follows.

Definition 1.3.2.2. *Let $\lambda = (\lambda_1, \dots, \lambda_m)$ and $\mu = (\lambda_1, \dots, \lambda_l)$ be partitions of n. Then $\lambda < \mu$ in lexicographic order if, for some index i,*

$$\lambda_j = \mu_j \qquad \textit{for } j < i \textit{ and } \lambda_i < \mu_i.$$

This gives us a total ordering on the set of partitions, which is a refinement of the dominance ordering, i.e., if $\lambda, \mu \vdash n$ with $\lambda \rhd \mu$, then $\lambda \geq \mu$.

1.3.3 Specht Modules

Here we will review the Specht modules S^λ, which give us the irreducible representations of Σ_n.

Definition 1.3.3.1. *Let T be a Young tableau of shape λ. If $R_1, \ldots, R_m$ are the rows of T and $C_1, \ldots, C_l$ are the columns of T, then we define*

$$R_T = \Sigma_{R_1} \times \cdots \times \Sigma_{R_m}$$

$$C_T = \Sigma_{C_1} \times \cdots \times \Sigma_{C_l},$$

called the row-stabilizer and column-stabilizer of T, respectively.

For each Young tableau T of shape λ, we can form a group element

$$\kappa_t = \sum_{\sigma \in C_T} Sgn(\sigma)\sigma \in k[\Sigma_n],$$

which we will use to define very important elements in M^λ.

Definition 1.3.3.2. *If T is a Young tableau of shape λ, then the associated polytabloid is*

$$e_T = \kappa_T\{T\} = \sum_{\sigma \in C_T} Sgn(\sigma)\{\sigma T\}. \tag{1.2}$$

With this, we can define our Specht modules.

Definition 1.3.3.3. *For any partition λ, define S^λ to be the submodule of M^λ spanned by the polytabloids e_T, where T is a Young tableau of shape λ.*

The spaces S^λ have the following very nice properties. For the proof of these statements, see [22].

Theorem 1.3.3.1. *Let $n \geq 0$. The S^λ for $\lambda \vdash n$ form a complete list of irreducible Σ_n-modules in characteristic* 0.

Furthermore, these irreducible modules give a decomposition of the permutation module with multiplicities counted by the Kostka numbers, see [22].

Theorem 1.3.3.2. *The permutation module decomposes as*

$$M^\lambda = \bigoplus_{\mu \rhd \lambda} K_{\mu,\lambda} S^\mu$$

where $K_{\mu,\lambda}$ are the Kostka numbers. The Kostka numbers $K_{\mu,\lambda}$ are counted by the number of semistandard Young tableaux of shape μ and content λ.

1.3.4 Basis for S^λ and Garnir Relations

In general, the set of polytabloids in S^λ do not give a linearly independent, but it turns out that the set of e_T where T is standard gives a basis using the dominance ordering to prove linear independence and the Garnir relations to show they span the space. One can extend the dominance ordering to tabloids in the following way. First, define a composition of n to be an ordered sequence of non-negative integers $(\lambda_1, \ldots, \lambda_m)$ such that $\sum_i \lambda_i = n$. Note that every partition is a composition, but not every composition is a partition. Furthermore, one can extend Young diagrams and tableaux to compositions. Let $\lambda \vdash n$ and $\{T\}$ be a λ-tabloid. For each index i such that $1 \leq i \leq n$, let $\{T^i\}$ be the tabloid formed by all elements $\leq i$ in $\{T\}$ and λ^i be the composition which is the shape of $\{T^i\}$. For an example, let

$$\{T\} = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline\end{array}$$

. Then

$$\{T^1\} = \begin{array}{|c|}\hline 0 \\ \hline 1 \\ \hline\end{array}, \quad \{T^2\} = \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline\end{array}, \quad \{T^3\} = \begin{array}{|c|c|}\cline{1-1} 2 & \multicolumn{1}{c}{} \\ \hline 1 & 3 \\ \hline\end{array}, \quad \{T^4\} = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline\end{array}$$

$$\lambda^1 = (0,1), \qquad \lambda^2 = (1,1), \qquad \lambda^3 = (1,2), \qquad \lambda^4 = (2,2)$$

Definition 1.3.4.1. *If $\{T\}$ and $\{S\}$ are two tabloids with composition sequences $\{\lambda^i\}$ and $\{\mu^i\}$, respectively. Then $\{S\}$ dominates $\{T\}$, written $\{S\} \rhd \{T\}$, if $\lambda^i \unrhd \mu^i$ for all i.*

With this, we have the following dominance lemma for tabloids.

lemma 1.3.4.1. *If $k < l$ and k appears in a lower row than l in $\{T\}$, then*

$$\{T\} \lhd (k\ l)\{T\}. \tag{1.3}$$

The dominance lemma can be used to prove a few properties about the polytabloids.

corollary 1.3.4.2. *If T is a standard Young tableau and $\{S\}$ appears in e_T, then $\{T\} \unrhd \{S\}$.*

lemma 1.3.4.3. *Let $v_1, \ldots, v_m$ be elements in M^λ. Suppose for each v_i, we choose a tabloid $\{T_i\}$ appearing in v_i such that*

- *$\{T_i\}$ is maximum in v_i, and*
- *the $\{T_i\}$ are all distinct.*

Then $v_1, \ldots, v_m$ are linearly independent.

Since the set of $\{e_T\}$ for standard Young tableaux T satisfies lemma 1.3.4.3 , then the set $\{e_T \ : \ T \text{ is standard } \lambda\text{-tabloid}\}$ is linearly independent.

On the other hand, to show that the set e_T for standard Young tableaux T spans S^λ, one has to use the straightening algorithm, or in other words, the Garnir relations to relate between different e_T.

Definition 1.3.4.2. *Let A and B be two disjoint finite sets of positive integers and choose permutations π such that*

$$\Sigma_{A\cup B} = \bigcup_{\pi} \pi(\Sigma_A \times \Sigma_B).$$

Let $G(A,B)$ be the set of π satisfying the relation above. Then, a corresponding Garnir element is

$$g_{A,B} = \sum_{\pi \in G(A,B)} Sgn(\pi)\pi.$$

These Garnir elements associated with a tableau T are used to eliminate a descent in a row of T if T has all increasing columns.

Definition 1.3.4.3. *Let T be a Young tableau with increasing columns and let A and B be subsets of the jth and $(j+1)$st columns of T, respectively. The Garnir element associated with T (and A,B) is $g_{A,B} = \sum_\pi Sgn(\pi)\pi$, where π have been chosen so that the elements of $A \cup B$ are increasing down the columns of πT.*

For T,A, and B as in the definition for Garnir element. If $|A \cup B|$ is greater than the number of elements in columns j of T, then $g_{A,B}e_Y = 0$, where e_Y is the polytabloid associated with T. Since one of the elements in $G(A, B)$ for the tableau T is the identity element, then we obtain

$$e_Y = -\sum_{\pi \in G(A,B)\setminus\{id\}} Sgn(\pi)e_{\pi Y}.$$

Using the Garnir relations, one can show that the set of e_T for standard T spans S^λ and hence gives the theorem.

Theorem 1.3.4.4. *For $\lambda \vdash n$, the set of $\{e_T : T \text{ standard } \lambda\text{-tableau}\}$ is a basis for S^λ. Furthermore, the dimension of S^λ is f^λ.*

1.4 Operads

Peter May introduced operads in [18] to study the loop spaces in algebraic topology. These objects parameterize classes of different types of algebras by essentially describing the n-ary operations and their relationships. We will be following the notation and definitions in [16] for this section.

The underlying structure for operads are **Σ-modules**, which are a family $M = (M(1), \dots, M(n), \dots)$ of right $k[\Sigma_n]$-modules for $n \geq 1$. This is equivalent to a functor $M : \Sigma^{op} \to Vect_k$, where Σ is the permutation groupoid. One should think of each $M(n)$ as holding the n-ary operations in the form of n-arity trees, which is the intuition for the free operad later discussed in section 1.4.2. A morphism $f : M \to N$ of Σ-modules is a natural transformation between their functors, i.e., a collection of right $k[\Sigma_n]$-module homomorphisms $f_n : M(n) \to N(n)$. Each Σ-module M defines a Schur functor $M : Vect_k \to Vect_k$ by defining

$$M(V) = \bigoplus_{n\geq 1} M(n) \otimes_{\Sigma_n} V^{\otimes n},$$

and we will denote elements in $M(V)$ as $(\mu; v_1, \dots, v_n)$ for $\mu \in M(n)$ and $v_1, \dots, v_n \in V$.

To help define the combinatorial free operad on a Σ-module, we will need the use of linear species, which are similar to Σ-modules but are defined on more general finite sets. Any Σ-module M can be made into a linear species $\widetilde{M}$, where for any finite set X with cardinality n, where

$$\widetilde{M}(X) = \left(\bigoplus_{f \in Bij([n],X)} M(n) \right)_{\Sigma_n}, \tag{1.4}$$

and where Bij is the category of finite sets with their morphism of bijections. Equivalence classes in $\widetilde{M}$ are represented by $(f;\mu)$, where $f \in Bij([n],X)$ and $\mu \in M(n)$, and the right Σ_n action inside of the coinvariants is defined as

$$(f;\mu)^\sigma = (f\sigma;\mu)$$

If $h : X \to Y$ is any bijection of sets with cardinality n, then we obtain a isomorphism $\widetilde{M}(h) : \widetilde{M}(Y) \to \widetilde{M}(X)$ where

$$\widetilde{M}(h)(f;\mu) = (h^{-1}f;\mu)$$

Furthermore, if $\iota : \Sigma \to Bij$ is the natural inclusion functor, then we obtain a natural isomorphism of functors $\tau : \widetilde{M}\iota \to M$ with $\tau_n([f;\mu]) = \mu^{f^{-1}}$. In other words, if $\sigma \in \Sigma_n$, then we have

$$\begin{aligned} M(\sigma)\tau_n([f;\mu]) &= M(\sigma)(\mu^{f^{-1}}) = (\mu^{f^{-1}})^\sigma = \mu^{f^{-1}\sigma} \\ &= \tau_n([\sigma^{-1}f;\mu]) = \tau_n(\widetilde{M}(\sigma)([f;\mu])). \end{aligned}$$

This implies that $\widetilde{M}([n]) \cong M(n)$ for any $n \geq 1$.

There are a few different equivalent ways to define an operad in the literature, such as the monoid in a certain monoidal category of Σ-modules or as Σ-modules with some binary partial compositions. For our purposes, we will use the latter, which is the classical definition described in [18], as it is more useful for computations with the free operad regarding rooted trees. For more information about operads and their different equivalent ways to define them in the algebraic setting, see [16].

Definition 1.4.0.1. *A **pseudo-operad** is a tuple $(\mathcal{O}, \circ, \eta)$ consisting of the following data;*

- $\mathcal{O}$ *is a* Σ*-module;*
- *for each* $n, m \geq 1$ *and* $1 \leq i \leq n$ *it is equipped with a* k*-linear map*

$$- \circ_i - : \mathcal{O}(n) \otimes \mathcal{O}(m) \to \mathcal{O}(n+m-1)$$

- *and a* k*-linear map* $\eta_{\mathcal{O}} : k \to \mathcal{O}(1)$*, which we define* $\eta_{\mathcal{O}}(1) = 1$*,*

satisfying the following axioms.

- *For* $n \geq 2$ *and* $1 \leq i < j \leq n$*, then the horizontal associativity for* $\mu \in \mathcal{O}(n)$*,* $\nu \in \mathcal{O}(m)$ *and* $\gamma \in \mathcal{O}(l)$ *is*

$$(\mu \circ_i \gamma) \circ_{j-1+l} \nu = (\mu \circ_j \nu) \circ_i \gamma.$$

- *For* $n, m \geq 1$*,* $1 \leq i \leq n$ *and* $1 \leq j \leq m$*, then the vertical associativity relation for* $\mu \in \mathcal{O}(n)$*,* $\nu \in \mathcal{O}(m)$*, and* $\gamma \in \mathcal{O}(l)$ *is*

$$\mu \circ_i (\nu \circ_j \gamma) = (\mu \circ_i \nu) \circ_{i-1+j} \gamma.$$

- *For* $n \geq 1$ *and* $1 \leq i \leq n$*, and any* $\mu \in \mathcal{O}(n)$ *we have*

$$\mu \circ_i \eta_{\mathcal{O}}(1) = \mu$$

  and

$$\eta_{\mathcal{O}}(1) \circ_1 \mu = \mu.$$

- *For* $n \geq 1$*,* $1 \leq i \leq n$*,* $\sigma \in \Sigma_n$*, and* $\tau \in \Sigma_m$*, the equivariance relation for* $\mu \in \mathcal{O}(n)$ *and* $\nu \in \mathcal{O}(m)$ *is*

$$(\mu \circ_{\sigma(i)} \nu)^{\sigma \circ_i \tau} = \mu^{\sigma} \circ_i \nu^{\tau}$$

where $\sigma \circ_i \tau \in \Sigma_{n+m-1}$ *is the permutation in*

$$(\sigma \circ_i \tau) = \sigma(id \oplus \cdots id \oplus \tau \oplus id \oplus \cdots \oplus id)$$

with $i-1$ *identity maps on the left of* τ *and* $n-i$ *identity maps on the right of* τ.

Definition 1.4.0.2. *A morphism between operads* $\mathcal{P}$ *and* $\mathcal{Q}$ *is a* Σ*-module morphism* $f : \mathcal{P} \to \mathcal{Q}$ *such that we have*

- *for* $\mu \in \mathcal{P}(m)$ *and* $\nu \in \mathcal{P}(n)$, *then*

$$f(\mu \circ_i \nu) = f(\mu) \circ_i f(\nu)$$

- *and* $f \circ \eta_{\mathcal{P}} = \eta_{\mathcal{Q}}$.

Example 1.4.0.1. *The following is a list of the traditional examples of algebraic operads in the literature.*

- *Given any vector space* V, *define* End_V *to be the* Σ*-module such that* $End_V(n) = Hom_k(V^{\otimes n}, V)$ *with the natural right* Σ_n *action induced from left action on* $V^{\otimes n}$. *For* $f \in End_V(n)$ *and* $g \in End_V(m)$, *define the partial composition*

$$(f \circ_i g)(v_1, v_2, \ldots, v_{n+m-1}) = f(v_1, \ldots, v_{i-1}, g(v_i, \ldots, v_{i+m-1}), v_{i+m}, \ldots, v_{n+m-1})$$

  and define the unit map $\eta : k \to End_V(1) = Hom_k(V, V)$ *as sending* 1 *to the identity map. This becomes an operad in a very natural way.*

- *The associative operad has* Σ*-module* Ass, *where* $Ass(n) = k[\Sigma_n]$ *for* $n \geq 1$. *Let* μ_n *be the generator of* $k[\Sigma_n]$ *as a right* $k[\Sigma_n]$*-module and we define the composition* $\mu_n \circ_i \mu_m = \mu_{n+m-1}$. *This becomes an operad in a natural way, and its algebras are exactly the non-unital associative* k*-algebras.*

- *The commutative operad has* Σ*-module* Com *with* $Com(n) = k\nu_n$, *where* ν_n *has the trivial action. Defining the partial composition as in* Ass, *this becomes an algebraic operad as well.*

- *The Lie operad has Σ-module Lie where $Lie(n)$ is the vector space of rooted planar n-trees with the inputs labeled by $\{1, \ldots, n\}$ quotient by the relations of antisymmetry and the Jacobian identity. This becomes an algebraic operad through the grafting of trees.*

1.4.1 $\mathcal{O}$-Algebras

As with any algebraic object, one would like to understand the objects it acts on. In particular, given an operad $\mathcal{O}$, the most important objects they act on are the corresponding algebras they define.

Definition 1.4.1.1. *An algebra over an operad $\mathcal{O}$, or a* ***$\mathcal{O}$-algebra*** *for short, is a vector space A with a morphism $f : \mathcal{O} \to End_A$ of operads. More specifically, A is equipped with a k-linear map $\gamma_A : \mathcal{O}(A) \to A$ such that the following axioms are satisfied*

- *For each $\mu \in \mathcal{O}(m)$, $\nu \in \mathcal{O}(n)$, and $a_1, \ldots, a_n \in A$ we have*

$$\gamma_A(\mu \circ_i \nu; a_1, \ldots, a_{n+m-1}) = \gamma_A(\mu; a_1, \ldots, \gamma_A(\nu, a_i, \ldots, a_{i+n-1}), a_{i+n}, \ldots, a_{n+m-1})$$

- *and for all $a \in A$, we have*

$$\gamma_A(\eta(1); a) = a.$$

Example 1.4.1.1.
- *For any operad $\mathcal{P}$, we can define the free $\mathcal{P}$-algebra on the vector space V to be $\mathcal{P}(V)$ with multiplication $\gamma_{\mathcal{P}(A)} : \mathcal{P}(\mathcal{P}(A)) \to \mathcal{P}(A)$ such that for $\mu \in \mathcal{P}(m)$, $\nu \in \mathcal{P}(n)$, and $a_1, \ldots, a_{n+m-1} \in V$ we have*

$$\gamma_{\mathcal{P}}(\mu; (1; a_1), \ldots, (\nu; a_i, \ldots, a_{i+n-1}), \ldots, (1; a_{n+m-1})) = (\mu \circ_i \nu; a_1, \ldots, a_{n+m-1}).$$

- *The Ass-algebras are exactly the non-unital associative algebras. In particular, if A is an Ass-algebra and we define $m_2(a_1, a_2) = \gamma_A(\mu_2; a_1, a_2)$ for all $a_1, a_2 \in A$, then we* *have the associativity relation*

$$\begin{aligned} m_2(m_2(a_1,a_2),a_3) &= \gamma_A(\mu_2 \circ_1 \mu_1; a_1,a_2,a_3) \\ &= \gamma_A(\mu_3; a_1,a_2,a_3) = \gamma_A(\mu_2 \circ_2 \mu_2; a_1,a_2,a_3) = m_2(a_1, m_2(a_2,a_3)). \end{aligned}$$

The free Ass-algebras on the vector space V is the non-unital tensor algebras $\overline{T}(V) = \bigoplus_{n\geq 1} V^{\otimes n}$.

- *The Com-algebras are exactly the non-unital commutative and associative algebras. The associativity relation is similar to how Ass gives an associativity relation and the commutative relation comes from*

$$\gamma_C(\mu_2, a_1, a_2) = \gamma_C(\mu_2^{(1\ 2)}; a_1, a_2) = \gamma_C(\mu_2; a_2, a_1).$$

The free Com-algebras on the vector space V is the non-unital symmetric algebra $\overline{S}(V)$.

1.4.2 Free Operad

To construct examples of operads, we would like to construct a free object so we can generate examples through generators and relations. One way to think of the free operad associated with a Σ-module E is as the collection of n rooted trees with the vertices labeled by the elements of E. To do this, we need to use the language of linear species as explained in 1.4 to help us define this notion. There are various ways to define the free operad associated with a Σ-module, but for our purposes, we will follow the construction outlined in [29]. For the other equivalent way to define the free operad in terms of the monoidal structure on the category of Σ-modules, see [16].

Definition 1.4.2.1. *Let M be a linear species, and suppose we have an isomorphism class $[T] \in Tree(n)$ with $m = |V_T|$. Define the M-decoration of $[T]$ as*

$$M[T] = \left(\bigoplus_{f \in Bij([m], V_T)} M(in_T(f(1))) \otimes \cdots \otimes M(in_T(f(m))) \right)_{\Sigma_m}$$

where elements are expressed as $(f;\mu_1,\ldots,\mu_m)$ *for* $f \in Bij([m],V_T)$ *and* $\mu_i \in M(in_T(f(i)))$ *and where the right action of* $\sigma \in \Sigma_m$ *on an element* $(f;\mu_1,\ldots,\mu_m)$ *is defined as*

$$(f;\mu_1,\ldots,\mu_m)^\sigma = (f\sigma;\mu_{\sigma^{-1}(1)},\ldots,\mu_{\sigma^{-1}(m)})$$

If T *is the unique tree* $\downarrow$*, then we define* $M[\downarrow] = k$*.*

Note that this definition is independent of the representative of the class $[T]$ since isomorphic trees induce identities on $M(In_T(v))$. Furthermore, if $\sigma \in \Sigma_n$, then we have an isomorphism $\sigma^* : M[\sigma^*(T)] \to M[T]$ induced by the isomorphisms $M(\sigma^*) : M(In_{\sigma^*(T)}(v)) \to M(In_T(v))$ coming from $\sigma_v^* : In_T(v) \to In_{\sigma^*(T)}(v)$ for each $v \in V_T$.

lemma 1.4.2.1. *Let* E *be a* Σ*-module and let* T_1 *be an* n*-tree with* $|V_{T_1}| = q$ *and* T_2 *be a* m*-tree with* $|V_{T_2}| = p$*. Suppose* e *is the* i*th edge of* T_1*, then we have the map*

$$\Psi : E[T_1] \otimes E[T_2] \to E[T_1 \circ_e T_2]$$
$$[f_1;\mu_1 \otimes \cdots \otimes \mu_q] \otimes [f_s;\nu_1 \otimes \cdots \otimes \nu_p] \mapsto [f_1 \times f_2;\mu_1 \otimes \cdots \otimes \mu_q \otimes \nu_1 \otimes \cdots \otimes \nu_p]$$

where

$$(f_1 \times f_2)(i) = \begin{cases} f_1(i) & \text{if } 1 \le i \le q \\ f_2(i-q) & \text{if } q+1 \le i \le q+p. \end{cases}$$

Now that we have all the main ingredients, we can define the free operad associated with a Σ-module E.

Definition 1.4.2.2. *For any* Σ*-module* E*, define* $(F(E),\circ,\eta)$ *as follows.*

- *For each* $n \ge 1$*, define*

$$F(E)(n) = \bigoplus_{[T]\in Tree(n)} E[T]$$

  with right Σ_n*-action induced by* $M(\sigma^*) : M[\sigma^*(T)] \to M[T]$ *for each* $\sigma \in \Sigma_n$*.*

- *Define the unit $\eta : k \to F(E)(1)$ as the following composition*

$$\begin{array}{ccc} k & \xrightarrow{\eta} & F(E)(1) = E[\downarrow] \\ \downarrow{=} & \nearrow_{inclusion} & \\ E[\downarrow] & & \end{array}$$

- *Define the $\circ_i$ composition as follows. Since $\otimes$ commutes with direct sums on each side, then we have the natural right $\Sigma_n \times \Sigma_m$ equivariant isomorphism*

$$\begin{aligned} (F(E))(n) \otimes (F(E)(m)) &= \left(\bigoplus_{[T_1] \in Tree(n)} E[T] \right) \otimes \left(\bigoplus_{[T_2] \in Tree(m)} E[T_2] \right) \\ &\cong \bigoplus_{([T_1],[T_2]) \in Tree(n) \times Tree(m)} E[T_1] \otimes E[T_2]. \end{aligned}$$

So it suffices to define $\circ_i$ restricted to a direct summand

$$\begin{array}{ccc} E[T_1] \otimes X[T_2] & \xrightarrow{\Psi} & X[T_1 \circ_e T_2] \\ \downarrow{\scriptstyle inclusion} & & \downarrow{\scriptstyle inclusion} \\ F(E)(n) \otimes F(E)(m) & \xrightarrow{-\circ_i-} & F(E)(n+m-1) \end{array}$$

where e is the is the ith edge of T_1 and $T_1 \circ_e T_2$ is the grafting of T_1 and T_2 along e.

There is a natural weight grading on the free operad $F(E)$ as follows: for $n \geq 0$, define a Σ-submodule $F(E)^{(n)}$ of $F(E)$ with

$$F(E)^{(n)}(m) = \bigoplus_{[T] \in Tree_n(m)} E[T]$$

where $Tree_n(m)$ is the equivalence classes of rooted m-labeled trees with n vertices.

Chapter 2

(N, M)-QUADRATIC OPERADS AND KOSZUL DUALITY

2.1 (n,m)-Quadratic Operads and Koszul Duality

Recall from [16], they define quadratic data as a pair (E,R) with E is a Σ-module and $R \subseteq F(E)^{(2)}$. For our context, we are interested in a particular type of quadratic operads, ones that are generated in arity $n,m \geq 2$. This models algebras with either one n-arity operation when $n=m$ or two operations where one is n-arity and the other is m-arity. We say a quadratic data (E,R) is an (n,m)-quadratic data if E is concentrated in arity n and m and R is a Σ-module of $F(E)^{(2)}$. When $n=m$, we say that (E,R) is a n-quadratic data.

An (n,m)-quadratic operad is a quadratic operad $\mathcal{P}(E,R) = F(E)/(R)$ for (n,m)-quadratic data (E,R). Furthermore, if $n=m$, then we call these n-quadratic operads. If $n=m=2$, these are exactly the binary quadratic operads as studied by Ginzburg and Kapranov in their study of Koszul duality in [10].

2.1.1 Description of $F(E)^{(2)}$

Here, we want to find a nice description for the weight 2 part of (n,m)-quadratic operads as this will make it easier to describe our relations and give a connection of $F(E)^{(2)}$ with various vertices of graphs.

Since our free operads are defined by rooted labeled trees, we need some description for the rooted labeled trees with exactly 2-vertices and the symmetric group action on them. Let $Tree_2(n,m)$ be the subset of $Tree_2(n+m-1)$ consisting of isomorphism classes of $[T]$ with exactly two vertices v_1 and v_2 such that $v_2 \in In_T(v_1)$ and $|in_T(v_1)| = n$ and $|in_T(v_2)| = m$, see figure 2.1. We have the following description of $Tree_2(n,m)$ in terms of the left cosets of Σ_m in Σ_{n+m-1}.

lemma 2.1.1.1. *The set* $Tree_2(n,m) \cong \Sigma_{n+m-1}/(\overline{\Sigma_{n-1}} \times \Sigma_m) \times \{[Cor_n \circ_1 Cor_m]\}$*, where* $\Sigma_{n+m-1}/(\overline{\Sigma_{n-1}} \times \Sigma_m)$ *is the collection of left cosets of* $\overline{\Sigma_{n-1}} \times \Sigma_m$*.*

Proof. Any rooted $n+m-1$-labeled tree with two vertices v_1 and v_2 such that $v_2 \in in_T(v_1)$ and $|int_T(v_1)| = n$ and $|in_T(v_2)| = m$ is isomorphic to the tree $\sigma^*(Cor_n \circ_1 Cor_m)$ for some

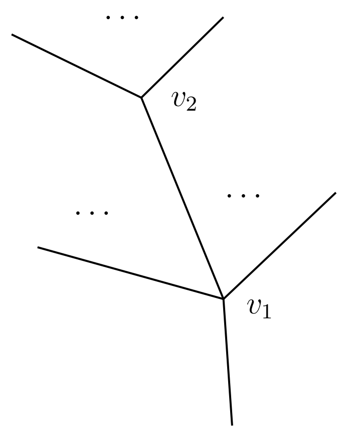


Figure 2.1: Tree with two vertices

permutation $\sigma \in \Sigma_{n+m-1}$ that preserves the order of the inputs, and where $Cor_n \circ_1 Cor_m$ has its inputs for the vertices ordered from 1 to m for the inputs into Cor_m and $m+1$ to $m+n-1$ for the inputs of Cor_n excluding the one that is connected to the root of C_m.

To do this, take our rooted m-labeled tree T and

$$in_T(v_1) = \{v_2, i_1, \ldots, i_{n-1}\}$$
$$in_T(v_2) = \{j_1, \ldots, j_m\}$$

where $i_1, \ldots, i_{n-1}, j_1, \ldots, j_m$ are distinct elements of $[n+m-1]$. We can define a permutation σ such that $\sigma(\{i_1, \ldots, i_{n-1}\} = \{m+1, \ldots, m+n-1\}$ and $\sigma(\{j_1, \ldots, j_m\}) = \{1, \ldots, m\}$ and its ordered i.e. if $i_p < i_q$ then $\sigma(i_p) < \sigma(i_q)$. Hence, we have the rooted m-labeled tree $\sigma^*(T) \cong Cor_n \circ_1 Cor_m$ and $T \cong (\sigma^{-1})^*(Cor_n \circ_1 Cor_m)$.

Next, let $\sigma_1 = 1, \sigma_2, \ldots, \sigma_l$ be the $l = \frac{(n+m-1)!}{(n-1)!m!}$ representatives for the left cosets of $\Sigma_{n-1} \times \Sigma_m$ in Σ_{n+m-1}. Take any rooted m-labeled tree T in $Tree_2(n,m)$, and by above, $T = \tau^*(Cor_n \circ_1 Cor_m)$ for some permutation $\tau \in \Sigma_{n+m-1}$. Then we have $\tau = \sigma_i \omega$ for some $1 \leq i \leq n$ and $\omega \in \Sigma_{n-1} \times \Sigma_m$. Therefore, we have

$$T = \tau^*(Cor_n \circ_1 Cor_m) = (\sigma_i \omega)^*(Cor_n \circ_1 Cor_m) = \sigma_i^*(Cor_n \circ_1 Cor_m).$$

This shows that $Tree_2(n,m) = \Sigma_{n+m-1}/(\overline{\Sigma_{n-1}} \times \Sigma_m) \times \{[Cor_n \circ_1 Cor_m]\}$. $\square$

With our description for $Tree_2(n,m)$, we can push this through and find a useful description for $F(E)^{(2)}$ in the case where E is concentrated in arities n and m.

lemma 2.1.1.2. *Let E be a Σ-module concentrated in arity n and m. Then $F(E)^{(2)}$ has the following description:*

$$F(E)^{(2)}(p) = \begin{cases} \bigoplus_{[T]\in Tree_2(p)} E[T] & \text{if } p = 2n-1, n+m-1, 2m-1 \\ 0 & \text{otherwise.} \end{cases}$$

and furthermore, we have

$$F(E)^{(2)}(n+m-1) = Ind^{\Sigma_{2n-1}}_{\Sigma_n \times \overline{\Sigma}_{m-1}}(E(m) \otimes E(n)) \oplus Ind^{\Sigma_{2n-1}}_{\Sigma_m \times \overline{\Sigma}_{n-1}}(E(n) \otimes E(m)) \qquad (2.1)$$

if $n \neq m$ and we also have

$$F(E)^{(2)}(2n-1) = Ind^{\Sigma_{2n-1}}_{\Sigma_n \times \overline{\Sigma}_{n-1}}(E(n) \otimes E(n)) \qquad (2.2)$$

$$F(E)^{(2)}(2m-1) = Ind^{\Sigma_{2n-1}}_{\Sigma_m \times \overline{\Sigma}_{m-1}}(E(m) \otimes E(m)). \qquad (2.3)$$

Proof. We will prove equation 2.1, as the other are similar. By definition, we have

$$\begin{aligned} F(E)^{(2)}(n+m-1) &= \bigoplus_{[T]\in Tree_2(n+m-1)} E[T] \\ &= \bigoplus_{[T]\in Tree_2(n,m)} E[T] \oplus \bigoplus_{[S]\in Tree_2(m,n)} E[S] \end{aligned}$$

since if a tree with two vertices did not have a vertex with n inputs or m inputs, then the E-decoration of that tree would be zero. Pick representatives $\sigma_1 = 1, \sigma_2, \ldots, \sigma_l$ for $l = \binom{n+m-1}{m}$ for the left cosets of $\Sigma_m \times \overline{\Sigma}_{n-1}$ in Σ_{n+m-1}. Then by lemma 2.1.1.1, we have

$$\begin{aligned} \bigoplus_{[T]\in Tree_2(n,m)} E[T] &= \bigoplus_{i=1}^{l} E[\sigma_i^*(Cor_n \circ_1 Cor_m)] \\ &\cong \bigoplus_{i=1}^{l} E[Cor_n \circ_1 Cor_m]\sigma_i^{-1}. \end{aligned}$$

As right $\Sigma_m \times \overline{\Sigma}_{n-1}$-modules, we have

$$E[Cor_n \circ_1 Cor_m] = E(int_T(v_1)) \otimes E(int_T(v_2))$$

where v_1 and v_2 are the vertices with $int_T(v_1) = \{v_2, m+1, \ldots, m+n-1\}$, $in_T(v_2) = [m]$, and $int_T(r_{Cor_n \circ_1 Cor_m}) = \{v_1\}$. By the cardinalities of the input sets for each of the internal vertices we obtain

$$E(in_T(v_1)) \otimes E(in_T(v_2)) \cong E(n) \otimes E(m)$$

as right $\Sigma_m \times \overline{\Sigma}_{n-1}$-modules. This finally shows

$$\bigoplus_{[T] \in Tree_2(n,m)} E[T] \cong \mathrm{Ind}_{\Sigma_m \times \overline{\Sigma}_{n-1}}(E(n) \otimes E(m)),$$

which completes the proof. □

The elements in E in $F(E)/(R)$ can be thought of as operations μ which act on formal elements $x_1, \ldots, x_n$ through $\mu(x_1, \ldots, x_n)$. Therefore, elements of $F(E)^{(2)}(n+m-1)$ can be thought of as elements $\mu \circ_\sigma \nu$, where μ acts on n elements and ν acts on m elements, who act on $n+m-1$ formal elements $x_1, \ldots, x_{n+m-1}$ where

$$\begin{aligned}(\mu \circ_\sigma \nu)(x_1, \ldots, x_{n+m-1}) &= (\mu \circ_1 \nu)^\sigma(x_1, \ldots, x_{n+m-1}) \\ &= (\mu \circ_1 \nu)(x_{\sigma(1)}, \ldots, x_{\sigma(n+m-1)}) \\ &= \mu(\nu(x_{\sigma(1)}, \ldots, x_{\sigma(m)}), x_{\sigma(m+1)}, \ldots, x_{\sigma(n+m-1)}).\end{aligned}$$

2.1.2 Koszul Dual of (n,m)-Quadratic Operads

For this subsection, we will describe how to find the Koszul dual for any (n,m)-quadratic operad using some bilinear pairing on the weighted 2 part of the free operad by following the ideas in [17], where they generalized the Koszul duality of binary operads to quadratic operads with n-arity generators.

For any Σ-module E, define $E \otimes Sgn$ to be the Σ-module with $(E \otimes Sgn)(n) = E(n) \otimes Sgn_n$, where Sgn_n is the signed representation of Σ_n. Furthermore, let E^* be the Σ-module with $E^*(n) = E(n)^*$, the k-linear dual with the induced right Σ_n-action. Combining these two definitions, define $E^\vee$ to be the Σ-module with $E^\vee(a) = \uparrow^{a-2} E^*(a) \otimes Sgn_a$, where $\uparrow^{a-2}$ denotes the suspension iterated $a-2$ times, which is naturally a Σ-module. More explicitly the action of Σ_n on the right is defined as follows: if $f \in \uparrow^{n-2} E^*(n)$, $v \in E(n)$, and $\sigma \in \Sigma_n$, then

$$f^\sigma(v) = Sgn(\sigma) f(v^{\sigma^{-1}}).$$

Hence, if E is concentrated in arity n with degree d_n and in arity m with degree d_m, then $E^\vee$ is concentrated in arity n with degree $-d_n + n - 2$ and in arity m with degree $-d_m + m - 2$.

Definition 2.1.2.1. *A pairing between two Σ-modules M and N, is a Σ-module map*

$$\langle -, - \rangle : M \otimes_H N \to Com \otimes Sgn.$$

If L is a sub Σ-module of N, then $L^\perp$ is the sub Σ-module of M with $L^\perp(n) = \{x \in M \ : \ \langle x, L(n) \rangle = 0\}$.

Explicitly, the pairing has the following relation: if $f \in M(n)$, $g \in N(n)$, and $\sigma \in \Sigma_n$, then

$$\langle f^\sigma, g^\sigma \rangle = Sgn(\sigma) \langle f, g \rangle.$$

Let (E, R) be (n, m)-quadratic data. Following the binary quadratic operad situation, we want to define a non-degenerate bilinear pairing $\langle -, - \rangle : F(E^\vee)^{(2)} \otimes_H F(E)^{(2)} \to Com \otimes Sgn$. To start, let $S_{i,j}^{i+j-1}$ be a collection of representatives of the right cosets of $\Sigma_j \times \overline{\Sigma}_{i-1}$ in Σ_{n+m-1} consisting of even permutations for $i, j \in \{n, m\}$. By definition of induced representation, we have

$$\mathrm{Ind}_{\Sigma_j \times \overline{\Sigma}_{i-1}}^{\Sigma_{i+j-1}} (E(i) \otimes E(j)) = \bigoplus_{\sigma \in S_{i,j}^{i+j-1}} E(i) \otimes E(j) \sigma \tag{2.4}$$

and denote our elements in this space as $\mu \circ_\sigma \nu$ to represent the image of the elements $(\mu \otimes \nu)\sigma$ for $\mu \in E(i)$, $\nu \in E(j)$, and $\sigma \in S_{i,j}^{i+j-1}$ for all $i, j \in \{n, m\}$. If $\tau \in \Sigma_{i+j-1}$, then

$$(\mu \circ_\sigma \nu)^\tau = \mu^y \circ_\omega \nu^h$$

for some $y \times h \in \overline{\Sigma}_{i-1} \times \Sigma_j$ and $\omega \in S_{i,j}^{i+j-1}$ such that $(y \times h)\omega = \sigma\tau$. Let $i, j, p, q \in \{n, m\}$, then for $\alpha^* \in E(p)$, $\beta^* \in E(q)$, $\mu \in E(i)$, $\nu \in E(j)$, $\sigma \in S_{p,q}^{p+q-1}$, and $\tau \in S_{i,j}^{i+j-1}$, define

$$\langle \alpha^* \circ_\sigma \beta^*, \mu \circ_\tau \nu \rangle = \begin{cases} \alpha^*(\mu)\beta^*(\nu) & \text{if } i = p,\ j = q, \text{ and } \sigma = \tau \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

This is exactly the same bilinear form defined in [17], where here we are only composing elements at the first components, which does not produce any extra signs.

lemma 2.1.2.1. *The pairing $\langle -, - \rangle : F(E^\vee) \otimes_H F(E)^{(2)} \to Com \otimes Sgn$ is a non-degenerate bilinear pairing of Σ-modules.*

Proof. It is obvious that this is non-degenerate since it is defined through the basis elements, so it suffices to show it respects the permutation actions. For $i, j \in \{n, m\}$, let $\alpha^* \in E(i)$, $\beta^* \in E(j)$, $\mu \in E(i)$, $\nu \in E(j)$, $\sigma, \omega \in S_{i,j}^{i+j-1}$, and $\tau \in \Sigma_{i+j-1}$. By definition, we have

$$\langle (\alpha^* \circ_\sigma \beta^*)^\tau, (\mu \circ_\omega \nu)^\tau) \rangle = \langle (\alpha^*)^{y_\sigma} \circ_{\sigma'} (\beta^*)^{h_\sigma}, \mu^{y_\omega} \circ_{\omega'} \nu^{h_\omega} \rangle$$

where $(y_\sigma \times h_\sigma)\sigma' = \sigma\tau$ and $(y_\omega \times h_\omega)\omega' = \omega\tau$ for some $h_\sigma, h_\omega \in \Sigma_j$, $y_\sigma, y_\omega \in \overline{\Sigma_{i-1}}$, and $\sigma', \omega' \in \Sigma_{i+j-1}$.

By definition of our pairing,

$$\langle \alpha^* \circ_{\sigma'} (\beta^*)^{h_\sigma}, \mu \circ_{\omega'} \nu^{h_\omega} \rangle = \begin{cases} Sgn(y_\sigma)Sgn(h_\sigma)\alpha^*(\mu^{y_\omega y_\sigma^{-1}})\beta^*(\nu^{h_\omega h_\sigma^{-1}}) & \text{if } \sigma' = \omega' \\ 0 & \text{otherwise} \end{cases}$$

In the case where $\sigma' = \omega'$, we have

$$(y_\omega^{-1} \times h_\omega^{-1})\omega\tau = \omega' = \sigma' = (y_\sigma^{-1} \times h_\sigma^{-1})\sigma\tau,$$

which implies $(y_\sigma \times h_\sigma)(y_\omega^{-1} \times h_\omega^{-1})\omega = \sigma$. This says that ω and σ are representatives of the same right coset, hence they are equal. Therefore, we obtain $h_\omega = h_\sigma$ and $y_\sigma = y_\omega$, and this gives

$$\begin{aligned}\langle \alpha^* \circ_{\sigma'} (\beta^*)^{h_\sigma}, \mu \circ_{\omega'} \nu^{h_\omega} \rangle &= \begin{cases} Sgn(y_\sigma)Sgn(h_\sigma)\alpha^*(\mu^{y_\omega y_\sigma^{-1}})\beta^*(\nu^{h_\omega h_\sigma^{-1}}) & \text{if } \sigma' = \omega' \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} Sgn(y_\sigma)Sgn(h_\sigma)\alpha^*(\mu)\beta^*(\nu) & \text{if } \sigma' = \omega' \\ 0 & \text{otherwise} \end{cases} \\ &= Sgn(y_\sigma)Sgn(h_\sigma)\langle \alpha^* \circ_\sigma \beta^*, \mu \circ_\omega \nu \rangle.\end{aligned}$$

Since $(y_\sigma \times h_\sigma)\sigma' = \sigma\tau$ and both σ and σ' are even permutations, then $Sgn(y_\sigma)Sgn(h_\sigma) = Sgn(y_\sigma \times h_\sigma) = Sgn(\tau)$. This completes the proof. □

As in the binary quadratic operad, we have the same description for the Koszul dual for a n-quadratic operad; see [16] and [17].

Definition 2.1.2.2. *Let $\mathcal{P} = \mathcal{P}(E, R)$ be a n-quadratic operad. Then its Koszul dual $\mathcal{P}^!$ is the n-quadratic operad*

$$\mathcal{P}^! = \mathcal{P}(E^\vee, R^\perp).$$

2.1.3 Symmetric and Skew-symmetric (n, m)-Quadratic Operads

The operads we are interested in are the ones with n-arity operations and m-arity operations that are either symmetric or skew-symmetric. In this scenario, the weight 2 part of the free operads has a particularly useful description suitable for computations. To help with the cluttering of symbols, define $\Sigma_{i-1,j} = \overline{\Sigma}_{i-1} \times \Sigma_j$ for any i, j.

Let $n, m \geq 2$ for this section. If $\sigma, \tau \in \Sigma_{n+m-1}$, then $\sigma_{n-1,m}\sigma = \Sigma_{n-1,m}\tau$ if and only if $\sigma^{-1}(i) = \tau^{-1}(h(i))$ and $\sigma^{-1}(j) = \tau^{-1}(y(j))$ for some $y \times y \in \Sigma_{n-1,m}$ and for $1 \leq i \leq m$ and $m+1 \leq j \leq n+m-1$. This shows that the distinct right cosets of $\Sigma_{n-1,m}$ are entirely based on where the inverse of the representatives send the set of numbers $m+1,\dots,n+m-1$, up to any permutation in $\overline{\Sigma}_{n-1}$.

Let $a_1,\dots,a_{n-1}$ be distinct elements in $\{1,\dots,n+m-1\}$ and define $\Sigma_{n-1,m}\{a_1,\dots,a_{n-1}\}$ to be the collection of permutations $\sigma \in \Sigma_{n+m-1}$ such that

$$\sigma^{-1}(\{m+1,\dots,2n-1\}) = \{a_1,\dots,a_{n-1}).$$

The set of $\Sigma_{n-1,m}\{a_1,\dots,a_{n-1}\}$ for distinct elements $a_1,\dots,a_{n-1} \in \{1,\dots,n+m-1\}$ is equivalent to the set of right cosets of $\Sigma_{n-1,m}$ by sending $\Sigma_{n-1,m}\sigma \mapsto \Sigma_{n-1,n}\{\sigma^{-1}(m+1),\dots,\sigma^{-1}(n+m-1)\}$. Denote by $\Lambda_{n,m}$ to be the set of finite sets $\{a_1,\dots,a_{n-1}\}$ for distinct elements $a_1,\dots,a_{n-1} \in \{1,\dots,n+m-1\}$. The natural action of Σ_{n+m-1} on the right cosets of $\Sigma_{n-1,m}$ transfers to a natural action on the collection of $\Sigma_{n-1,m}\{a_1,\dots,a_{n-1}\}$ in the following way: for $\sigma \in \Sigma_{n+m-1}$, define $\Sigma_{n-1,m}\{a_1,\dots,a_{n-1}\}\sigma = \Sigma_{n-1,m}\{\sigma^{-1}(a_1),\dots,\sigma^{-1}(a_{n-1})\}$.

If M is a right $\Sigma_{n-1,n}$-module where M either the signed representation or the trivial representation, then we choose representatives for the right cosets of $\Sigma_{n-1,m}$ to be even so that

$$\mathrm{Ind}_{\Sigma_{n-1,m}}^{\Sigma_{n+m-1}}(M) \cong \bigoplus_{\{a_1,\dots,a_{n-1}\}} M\{a_1,\dots,a_{n-1}\},$$

where $M\{a_1,\dots,a_{n-1}\}$ are isomorphism to M as graded k-modules with elements denoted as $e\{a_1,\dots,a_{n-1}\}$ for $e \in M$ and $\{a_1,\dots,a_{n-1}\} \in \Lambda_{n,m}$. The action on the right is defined as

$$e\{a_1,\dots,a_{n-1}\}\sigma = \begin{cases} e\{\sigma^{-1}(a_1),\dots,\sigma^{-1}(a_{n-1})\} & \text{if } M \text{ is the trivial representation} \\ Sgn(\sigma)e\{\sigma^{-1}(a_1),\dots,\sigma^{-1}(a_{n-1})\} & \text{if } M \text{ is the signed representation} \end{cases}$$

for $\sigma \in \Sigma_{n+m-1}$.

The above formulation gives us a way to show that the space $F(E)^{(2)}(i+j-1)$ corresponds to the vertices of certain graphs that we will explain later in section 7.3, and gives us nice tools to study these spaces. Next, we define a few free operads that will be important to us and which all of our operads in this paper will be a quotient of.

Definition 2.1.3.1. *Let $n \geq 2$ and $d \in \mathbb{Z}$.*

- *Let $E_{n,d}$ be a graded Σ-module such that $E_{n,d}(n) = \uparrow^{2d} k\nu_{n,d}$ with $\nu_{n,d}^{\sigma} = Sgn(\sigma)\nu_{n,d}$ for every $\sigma \in \Sigma_n$, and zero everywhere else. Define the free operad $ASMag_{n,d} = F(E_{n,d})$.*

- *Let $H_{n,d}$ be the graded Σ-module such that $H_{n,d}(n) = \uparrow^{2d} k\mu_{n,d}$ such that $\mu_{n,d}^{\sigma} = \mu$ for all $\sigma \in \Sigma_n$, and zero everywhere else. Define the free operad $SMag_{n,d} = F(H_{n,d})$.*

The algebras over $ASMag_{n,d}$ are exactly the graded k-modules L with a degree d n-arity bracket $[-, \ldots, -]$ with $Sgn(\sigma)\xi(\sigma, v_1, \ldots, v_n)[v_1, \ldots, v_n] = [v_{\sigma(1)}, \ldots, v_{\sigma(n)}]$ for all $v_1, \ldots, v_n \in L$ and $\sigma \in \Sigma_n$. The n-quadratic operads, which are a quotient of $ASMag_{n,d}$, will be called *Lie*-type operads. In chapter 3, we will describe an entire collection of *Lie*-type operads and their algebras, by generalizing the Jacboian identity in various ways. In particular, the operad of n-Lie algebras will be a *Lie*-Type operad and is in some ways a maximal one in the spectrum of *Lie*-type operads.

Similarly, the algebras over $SMag_{n,d}$ are exactly the graded k-modules C with a degree d n-arity operation $(-, \ldots, -)$ such that $(v_1, \ldots, v_n) = \xi(\sigma, v_1, \ldots, v_n)(v_{\sigma(1)}, \ldots, v_{\sigma(n)})$ for all $v_1, \ldots, v_n \in C$ and $\sigma \in \Sigma_n$. The n-quadratic operads which are a qoutient of $SMag_{n,d}$ will be called Com-type operads. In chapter 4, we will describe an entire collection of Com-type operads and their algebras. The relations for these operads will come from the relations in S^{λ} for various partitions λ of $2n - 1$ and these operads will be the non-trivial minimal Com-type operads.

We can also combine E_{n,d_n} and H_{m,d_m} together to obtain the free operad $FP_{(n,d_n),(m,d_m)} = F(E_{n,d_n} \oplus H_{m,d_m})$. We will use this free operad to construct various generalizations of the Poisson operads that come from combining the *Lie*-type operads and Com-type operads in chapter 6.

By lemma 2.1.1.2, the weight 2 parts of the free operads above can be described as follows:

for $n, m \geq 2$ and $d_n, d_m \in \mathbb{Z}$ we have

$$F(E_{n,d_n})^{(2)}(2n-1) = \bigoplus_{a_1<\cdots<a_{n-1}} E_{n,d_n}(n) \otimes E_{n,d_n}(n)\{a_1,\ldots,a_{n-1}\}$$
$$F(H_{m,d_m})^{(2)}(2m-1) = \bigoplus_{b_1<\cdots<b_{m-1}} H_{m,d_m}(m) \otimes H_{m,d_m}(m)\{b_1,\ldots,b_{m-1}\}$$

and $F(E_{n,d} \oplus H_{m,d_m})^{(2)}$ has the following spaces in their respective arities:

$$F(E_{n,d_n})^{(2)}(2n-1)$$
$$\bigoplus_{a_1<\cdots<a_{n-1}} E_{n,d_n}(n) \otimes H_{m,d_m}(m)\{a_1,\ldots,a_{n-1}\} \oplus \bigoplus_{b_1<\cdots<b_{m-1}} H_{m,d_m}(m) \otimes E_{n,d_n}\{b_1,\ldots,b_{m-1}\}$$
$$F(H_{m,d_m})^{(2)}(2m-1).$$

In particular, we let $\nu^d_{\{a_1,\ldots,a_{n-1}\}}$ represent the image of $\nu_{n,d} \otimes \nu_{n,d}\{a_1,\ldots,a_{n-1}\}$ in $E_{n,d_n}(n) \otimes E_{n,d}(n)\{a_1,\ldots,a_{n-1}\}$, and similarly we let $\mu^d_{\{b_1,\ldots,b_{m-1}\}}$ represent the image of the element $\mu_{m,d_m} \otimes \mu_{m,d_m}\{b_1,\ldots,b_{m-1}\}$ in $H_{m,d_m}(m) \otimes H_{m,d_m}(m)\{b_1,\ldots,b_{m-1}\}$. For the cross terms in $F(E_{n,d_n} \oplus H_{m,d_m})$, we let $\mu^{d_m}\nu^{d_n}_{\{b_1,\ldots,b_{m-1}\}}$ to represent $\mu_{m,d_m} \otimes \nu_{n,d_n}\{b_1,\ldots,b_{m-1}\}$ and we let $\nu^{d_n}\mu^{d_m}\{a_1,\ldots,a_{n-1}\}$ represent the elements $\nu_{n,d_n} \otimes \mu_{m,d_m}\{a_1,\ldots,a_{n-1}\}$.

The cross terms in $FP_(n, d_n), (m, d_m)$ are used to set up the relation between the *Lie*-type and *Com*-type operads. There are various relations one can use here to relate both of the operations, but the most natural choice is the Leibniz relation, which we will use to construct various generalizations of the Poisson operad.

Chapter 3

THE GENERALIZATIONS OF *LIE*

In this chapter, we will give a few examples of generalizations of the Operad *Lie* in various ways. In particular, we will give the quadratic representations for the operads Lie_n^d and n-Lie_d, which generalize the Jacobian identity. We will construct more examples of these types of operads in section 7.3 when we give a large class of operads that happen to be a generalization of *Lie* through the Odd graphs.

3.1 The Operad Lie_n^d

3.1.1 General Definition of Lie_n Algebras of Degree d

The relations for *Lie* are generated by the sum of all the even representatives of the right cosets of $\overline{\Sigma}_1 \times \Sigma_2$ in Σ_3, i.e the relation

$$\nu_{2,d} + \nu_{2,d}^{(1\ 2\ 3)} + \nu_{2,d}^{(1\ 3\ 2)}. \tag{3.1}$$

Therefore, one natural generalization of *Lie* with an n-arity operation is to make the relation the sum of all the even representatives of the right cosets of $\Sigma_{n-1,n}$ in Σ_{2n-1}. In terms of the algebras, these are expressed using (p, q)-shuffles as in the following definition.

Definition 3.1.1.1. *Let $n \geq 2$. A Lie n-algebra of degree d is a graded k-module L with a skew-symmetric n-arity bracket $[-, \dots, -] : L^{\otimes n} \to L$ satisfying the following properties.*

- *For every $v_1, \dots, v_n \in L$ and $\sigma \in \Sigma_n$ we have*

$$Sgn(\sigma)\xi(\sigma, v_1, \dots, v_n)[v_1, \dots, v_n] = [v_{\sigma(1)}, \dots, v_{\sigma(n)}];$$

- *and for every $v_1, \dots, v_{2n-1}$ we have*

$$\sum_{\sigma \in Sh(n,n-1)} Sgn(\sigma)\xi(\sigma, v_1, \dots, v_{2n-1})[[v_{\sigma(1)}, \dots, v_{\sigma(n)}], v_{\sigma(n+1)}, \dots, v_{\sigma(2n-1)}] = 0.$$

More compactly, if $\nu = [-, \dots, -]$ is the bracket, then we want

$$\sum_{\sigma \in Sh(n,n-1)} Sgn(\sigma)(\nu \circ_1 \nu)^{\sigma^{-1}} = 0. \tag{3.2}$$

In the case when $d = n-2$, then *Lie* n-algebras of degree $n-2$ are exactly the L_∞-algebras L with operations $\{l_m\}_{m\geq 1}$ such that $l_m = 0$ for all $m \neq n$.

The relation for n-Lie algebras of degree d can be simplified even further so that we do not have any negative signs that appear in the front, i.e., it has only even permutations acting on it. For each permutation $\sigma \in Sh(n, n-1)$, we can choose a permutation t_σ such that $t_\sigma = (\sigma(1), \sigma(2))$ if σ is odd, or $t_\sigma = id$ if σ is even. In this way, if L is a *Lie* n-algebra of degree d with skew-symmetric operation ν, then we can rewrite the relation as

$$\sum_{\sigma \in Sh(n,n-1)} (\nu \circ_1 \nu)^{t_\sigma \sigma^{-1}} = 0 \tag{3.3}$$

This shows that we can express the relation using only even permutations, which makes it possible to write the relations for the corresponding operad.

3.1.2 Quadratic Representation for Lie_n^d

Fix $n \geq 2$ and $d \in \mathbb{Z}$. Let $LR_{n,d}(2n-1)$ be the right Σ_{2n-1}-module of $F(E_{n,d})^{(2)}(2n-1)$ in degree $2d$ generated as a right $k[\Sigma_{2n-1}]$-module by

$$lr_{n,d} = \sum_{a_1 < \cdots < a_{n-1}} \nu_{\{a_1, \ldots, a_{n-1}\}}$$

and $LR_{n,d}(j) = 0$ for all $j \neq 2n-1$. It is easy to see $LR_{n,d} \cong S^{(1,\ldots,1)}$, for partition $(1, \ldots, 1)$ with $2n-1$ 1's, and hence 1-dimensional representation of Σ_{2n-1}. By equation 3.3, these relations give us exactly the relations for *Lie* n-algebras of degree d so we can define its corresponding operad.

Definition 3.1.2.1. *For every* $n \geq 2$*, and* $d \in \mathbb{Z}$*, we define* $Lie_n^d = ASMag_{n,d}/(RL_{n,d})$.

In particular, when $n = 2$ and $d = 0$, then Lie_2^0 is just the classical *Lie* operad.

3.2 The Operad n-Lie_d

3.2.1 General Definition of n-Lie Algebras of Degree d

For the *n-Lie* algebras, these are the generalizations where we think of the Jacobian identity as a derivation property, which is very different from the relations in Lie_n^d for $n > 2$.

Definition 3.2.1.1. *Let $n \geq 2$. An n-Lie algebra of degree d is a graded k-module L with a n-ary operation $l_n = [-, \ldots, -] : L^{\otimes n} \to L$ of degree d satisfying the following properties:*

- *for any n elements $v_1, \ldots, v_n \in L$, and any $\sigma \in \Sigma_n$,*

$$Sgn(\sigma)[v_1, \ldots, v_n] = \xi(\sigma, v_1, \ldots, v_n)[v_{\sigma(1)}, \ldots, v_{\sigma(n)}]$$

- *and for any $2n-1$ elements $v_1, \ldots, v_{2n-1} \in L$, the n-ary operation satisfies the following generalized Jacobi-identity:*

$$[[v_1, \ldots, v_n], v_{n+1}, \ldots, v_{2n-1}] = \sum_{i=1}^{n} (-1)^{\varepsilon_i} [v_1, \ldots, v_{i-1}, [v_i, v_{n+1}, \ldots, v_{2n-1}], v_{i+1}, \ldots, v_n]$$

where

$$\varepsilon_i = d(\sum_{j=1}^{i-1} |v_j|) + (\sum_{j=i+1}^{n} |v_j|)(\sum_{r=n+1}^{2n-1} |v_r|).$$

In this definition, if $d = 0$ and the underlying algebra is concentrated in degree 0, then we will call these just n-Lie algebras. Note in this definition, we see that $[-, x_1, \ldots, x_{n-1}]$ are derivations on the space L.

Example 3.2.1.1.

- *Let $A = k[x_1, \ldots, x_{n+1}]$ be the polynomial ring on $n+1$-variables. Then we can define a $n+1$-ary bracket on the elements $p_1, \ldots, p_{n+1} \in A$*

$$[p_1, \ldots, p_{n+1}] = Jac(p_1, \ldots, p_{n+1})$$

where the right-hand side is the determinant of the associated Jacobian matrix of the polynomials $p_1, \ldots, p_{n+1}$. This is naturally a $n+1$-Lie algebra and of great interest, specifically for $n = 3$ in Nambu mechanics.

- *More generally, suppose A is a commutative k-algebra with commuting derivations $D_1, \ldots, D_n$. Then A is an n-Lie algebra with bracket*

$$[a_1, \ldots, a_n] = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} Sgn(\sigma) D_{\sigma(1)}(a_1) \cdots D_{\sigma(n)}(a_n) = det \begin{pmatrix} D_1(a_1) & \cdots & D_1(a_n) \\ \vdots & \cdots & \vdots \\ D_n(a_1) & \cdots & D_n(a_n). \end{pmatrix}$$

- *Let $n \geq 3$ and let L be any n-Lie algebra with bracket $[-, \ldots, -]$. Pick $\Omega \in L$, which is called the potential, and define a $n-1$-ary bracket*

$$[p_1, \ldots, p_{n-1}]_\Omega = [p_1, \ldots, p_{n-1}, \Omega]$$

which gives L a $n-1$-Lie algebra structure.

- *In Filippov's study of n-Lie algebras and their algebraic properties in [8], they defined the following example which helped in the classifications for simple n-Lie algebras of finite dimensions. Let A_n be a $n+1$ dimensional vector space with basis $\{v_1, \ldots, v_{n+1}\}$. Define the n-bracket*

$$[v_1, \ldots, \widehat{v_i}, \ldots, v_{n+1}] = (-1)^{n+1+i} v_i$$

which gives A_n a n-Lie algebra structure. Filippov showed that every $n+1$-dimensional simple n-Lie algebra is isomorphic to one of the form A_n.

- *More generally, let L be a finite-dimensional vector space with basis $e_1, \ldots, e_m$ and let*

$$[e_{i_1}, \ldots, e_{i_n}] = \sum_{l=1}^{m} a_{i_1,\ldots,i_n} e_l$$

for some $a_{i_1,\ldots,i_n} \in k$ such that $Sgn(\sigma) a^l_{i_1,\ldots,i_n} = a_{i_{\sigma(1)},\ldots,i_{\sigma(n)}}$ for all $\sigma \in \Sigma_n$. For the above bracket to be a n-Lie algebra, we must have the following relations with the coefficients

$$\sum_{l=1}^{m} a^l_{i_1,\ldots,i_n} a^q_{l,j_1,\ldots,j_{n-1}} = \sum_{t=1}^{n} \sum_{r=1}^{m} a^r_{i_t,j_1,\ldots,j_{n-1}} a^q_{i_1,\ldots,i_{t-1},r,i_{t+1},\ldots,i_n}$$

for all $i_1, \ldots, i_n, j_1, \ldots, j_{n-1}, q$.

Example 3.2.1.2. *There exists an very nice example of a 3-Lie algebra structure on the general linear lie algebra* $\mathfrak{gl}_n(k)$ *of* $n \times n$ *with the commutator bracket* $[A, B]$ *from [1]. They define*

$$[A, B, C] = Tr(A)[B, C] + Tr(B)[C, A] + Tr(C)[A, B]$$

where Tr *is the normal trace, which gives* $\mathfrak{gl}_n(k)$ *a 3-Lie algebra structure.*

More generally, if L *is a* $(n-1)$*-Lie algebra with* $n-1$ *arity bracket* $[-, \dots, -]$ *and* $g : L \to k$ *is a* k*-linear map such that* $g([x_1, \dots, x_{n-1}]) = 0$*. Then there is a* n*-Lie algebra structure on* L *with bracket*

$$[x_1, \dots, x_n]^g = \sum_{i=1}^{n} (-1)^{i-1} g(x_i)[x_1, \dots, \widehat{x_i}, \dots, x_n]$$

where $\widehat{x_i}$ *means to take it out of the sequence.*

Example 3.2.1.3. *Let* A *be a symmetric Frobenius algebra, i.e. an associative unital algebra* A *with a non-degenerate symmetric bilinear form* $\langle -, - \rangle : A \times A \to k$ *such that* $\langle ab, c\rangle = \langle a, bc\rangle$ *for all* $a, b, c \in A$*. Then we have* $\mathfrak{e} = \langle 1, - \rangle = \langle -, 1 \rangle \in A^*$ *which has the property*

$$\mathfrak{e}(ab) = \langle 1, ab\rangle = \langle a, b\rangle = \langle b, a\rangle = \langle 1, ba\rangle = \mathfrak{e}(ba).$$

Therefore, $\mathfrak{e} : A \to k$ *is a linear functional which induces a 3-Lie algebra structure on* A *with*

$$[a, b, c]^{\mathfrak{e}} = \mathfrak{e}(a)[b, c] + \mathfrak{e}(b)[c, a] + \mathfrak{e}(c)[a, b].$$

Example 3.2.1.4. *A metric* n*-Lie algebra is a* n*-Lie algebra* L *equipped with a non-degenerate symmetric bilinear form* $B : L \times L \to k$ *such that*

$$B([x_1, \dots, x_{n-1}, y], z) = -B([x_1, \dots, x_{n-1}, z], y) \tag{3.4}$$

for $x_1, \dots, x_{n-1}, y, z \in L$*. These were first introduced by Figueroa-O'Farril and Papadopoulos in their study of the classification of maximally supersymmetric type IIB supergravity, see [6, 7]. Let* $n \geq 3$ *and* L *is a metric* n*-Lie algebra. We can choose a potential* $\Omega \in L$ *and* *define* $[-,\dots,-]_\Omega$ *to be a* $n-1$ *bracket which makes* L *into a metric* $(n-1)$*-Lie algebra with form* B*. Then we can define a* k*-linear map* $B_\Omega : L \to k$ *as*

$$B_\Omega(x) = B(x,\Omega)$$

which has the property

$$B_\Omega([x_1,\dots,x_n]_\Omega) = B([x_1,\dots,x_n]_\Omega,\Omega) = -B([x_1,\dots,x_{n-1},\Omega]_\Omega,x_n) = 0.$$

Therefore, by example 3.2.1.2 we have a n *bracket* $[-,\dots,-]_{B_\Omega}$ *on* L *with*

$$[x_1,\dots,x_n]_{B_\Omega} = \sum_{i=1}^{n}(-1)^{i-1}B_\Omega(x_i)[x_1,\dots,\widehat{x_i},\dots,x_n]_\Omega.$$

To help with defining the corresponding operad to *n-Lie* algebras of degree d, we need to be able to express the relations for these types of algebras using only permutations and the operations in the following way.

lemma 3.2.1.1. *Let* L *be a* n*-Lie algebra of degree* d *and let* $l_n = [-,\dots,-]$ *be the* n*-ary bracket on* L*. If* $\lambda = (1,2,\dots,2n-1)$ *is the standard* $2n-1$*-cycle and* $\omega = (1,2,\dots,n)$ *is the standard* n*-cycle in* Σ_{2n-1}*, then the defining relation for a* n*-Lie algebra of degree* d *is*

$$l_n \circ_1 l_n + (-1)^n \sum_{i=0}^{n-1}(-1)^{i(n+1)}(l_n \circ_1 l_n)^{\lambda^n\omega^i} = 0 \tag{3.5}$$

where $f \circ_i g$ *is defined in operad example 1.4.0.1.*

Proof. For ease of calculations, here are the expressions for $\lambda^n\omega^i$ for $0 \le i \le n-1$:

$$\lambda^n\omega^i = \begin{cases} \begin{pmatrix} 1 & 2 & \cdots & n-1 & n & n+1 & \cdots & 2n-1 \\ n+1 & n+2 & \cdots & 2n-1 & 1 & 2 & \cdots & n \end{pmatrix} & i=0 \\ \begin{pmatrix} 1 & 2 & \cdots & n-i-1 & n-i & n-i+1 & \cdots & n & n+1 & \cdots \\ n+i+1 & n+i+2 & \cdots & 2n-1 & 1 & n+1 & \cdots & n+i & 2 & \cdots \end{pmatrix} & 1 \le i \le n - \\ \begin{pmatrix} 1 & 2 & 3 & \cdots & n & n+1 & n+2 & \cdots & 2n-1 \\ 1 & n+1 & n+2 & \cdots & 2n-1 & 2 & 3 & \cdots & n \end{pmatrix} & i=n-1 \end{cases}$$

Let $v_1, \ldots, v_{2n-1} \in L$, then the equation 3.5 applied to these elements give us

$$(l_n \circ_1 l_n)(v_1, \ldots, v_{2n-1}) + (-1)^n \sum_{i=0}^{n-1} (-1)^{i(n+1)} (l_n \circ_1 l_n)^{\lambda^n \omega^i}(v_1, \ldots, v_{2n-1})$$
$$= l_n(l_n(v_1, \ldots, v_n), v_{n+1}, \ldots, v_{2n-1})$$
$$+ (-1)^n \xi(\lambda^n, v_1, \ldots, v_{2n-1}) l_n(l_n(v_n, \ldots, v_{2n-1}), v_1, \ldots, v_{n-1})$$
$$+ (-1)^n \sum_{i=1}^{n-2} (-1)^{i(n+1)} \xi(\lambda^n \omega^i, v_1, \ldots, v_{2n-1}) l_n(l_n(v_{n-i}, v_{n+1}, v_{n+2}, \ldots, v_{2n-1}), v_{n-i+1}, \ldots, v_n, v_1, \ldots, v_{n-i-1})$$
$$+ (-1)^n \xi(\lambda^n \omega^{n-1}, v_1, \ldots, v_{2n-1}) l_n(l_n(v_1, v_{n+1}, \ldots, v_{2n-1}), v_2, \ldots, v_n)$$

For each of these terms in the equation above, if we move the $l_n(*, \ldots, *)$ term to the right to their appropriate place, it will cancel the appropriate permuted terms in $\xi(\lambda^n \omega^i, v_1, \ldots, v_{2n-1})$ and we keep the terms

$$\left(\sum_{j=i+1}^{n} |v_j| \right) \left(\sum_{r=n+1}^{2n-1} |v_r| \right),$$

and since we are moving a degree d operation l_n as well, we add in $d\left(\sum_{j=1}^{n-i-1} |v_j|\right)$. This will give us exactly the relations for a n-Lie algebra of degree d. □

In the last lemma, if n is odd, then $\lambda^n \omega^i$ is always an even permutation and we obtain the relation

$$(l_n \circ_1 l_n) - \sum_{i=0}^{n} (l_n \circ_1 l_n)^{\lambda^n \omega^i} = 0$$

In the case when n is even, we have that $\lambda^n \omega^i$ is odd if and only if i is odd. Therefore, to make sure that we can use these equations in the construction of our operads, we need to make sure the permutations acting on our operations are even. Therefore, in the case when

n is even, we have

$$
\begin{aligned}
(l_n \circ_1 l_n) + \sum_{i=0}^{n-1}(-1)^i(l_n \circ_1 l_n)^{\lambda^n\omega^i} &= (l_n \circ_1 l_n) + \sum_{i=0}^{n-1}(-1)^i(l_n \circ_1 l_n)^{(1\ 2)^i(1\ 2)^i\lambda^n\omega^i} \\
&= (l_n \circ_1 l_n) + \sum_{i=0}^{n-1}(-1)^i(-1)^i(l_n \circ_1 l_n)^{(1\ 2)^i\lambda^n\omega^i} \\
&= (l_n \circ_1 l_n) + \sum_{i=0}^{n-1}(l_n \circ_1 l_n)^{(1\ 2)^i\lambda^n\omega^i} = 0
\end{aligned}
$$

where we used the fact that $(1\ 2)$ acting on l_n is $-l_n$. Hence, we have a representation of the relations for *n-Lie* algebras of degree d using even permutations.

The relations in lemma 3.5 can be thought of as some generalized Garnir relation as explained in [9]. This gives some explanation of why the relations for the Koszul dual of $n\text{-}Lie_d$ would be coming from Young symmetrizer relations as shown in section 7.3

We can generalize the relations in definition 3.2.1.1 slightly based on a integer $1 \leq j \leq n$. We will see later in section 7.3 that this generalization of the relation is enough to constitute all of the "maximal" n-quadratic operads with skew-symmetric generators.

Definition 3.2.1.2. *Let $n \geq 2$, $d \in \mathbb{Z}$, $1 \leq j \leq n$, $\lambda = (1\ 2\ \cdots\ 2n-1)$ be the standard $2n-1$ cycle. and $\omega = (1\ 2\ \cdots\ n)$ be the standard n-cycle in Σ_{2n-1}. A n-Lie algebra of type j and degree d is a graded k-module L with a n-arity operation $l_n = [-, \ldots, -] : L^{\otimes n} \to L$ satisfying the following properties:*

- *for any n elements $v_1, \ldots, v_n \in L$, and any $\sigma \in \Sigma_n$,*

$$Sgn(\sigma)[v_1, \ldots, v_n] = \xi(\sigma, v_1, \ldots, v_n)[v_{\sigma(1)}, \ldots, v_{\sigma(n)}];$$

- *and the n-arity operation satisfies the following generalized Jacboi-identity of type j:*

$$(-1)^{n+j}j(l_n \circ_1 l_n) - \sum_{i=0}^{n-1}(-1)^{i(n+1)}(l_n \circ_1 l_n)^{\lambda^n\omega^i} = 0.$$

For the case when $j = 1$, we get the ordinary n-*Lie* as defined in definition 3.2.1.1. Note that even if we are just multiplying the first component by an integer, this will drastically change the relations it can have. We will show later that when $j = n$, then we can reduce these relations to more simple relations that look like associativity relations.

3.2.2 Quadratic Representation for n-Lie_d

Fix $n \geq 2$, $d \in \mathbb{Z}$, $1 \leq j \leq n$, and let $\lambda = (1, 2, \ldots, 2n-1)$ be the standard $(2n-1)$-cycle and let $\omega = (1, 2, \ldots, n)$ be the standard n-cycle in Σ_{2n-1}. Let $R^j_{n,d}(2n-1)$ be the right Σ_{2n-1}-submodule of $F(E_{n,d})^{(2)}(2n-1)$ in degree $2d$ generated by

$$r^j_{n,d} = jv_{\{n+1,\ldots,2n-1\}} + (-1)^{n+j-1}v_{\{1,2,\ldots,n-1\}}(-1)^{n+j-1}\sum_{i=2}^{n-2} v_{\{n-i+1,\ldots,n-1,n,1,\ldots,n-i-1\}} + (-1)^{n+j-1}v_{\{2,\ldots,n\}},$$

and $R^j_{n,d}(i) = 0$ for all $i \neq 2n-1$. By lemma 3.2.1.1, the relation $r^j_{n,d}$ gives us exactly the relation for an n-Lie algebra of degree d and of type j. By applying every element of Σ_{2n-1} to the generator of $R^j_{n,d}$, then $R^j_{n,d}$ is spanned by the elements

$$\begin{aligned}(r^j_{n,d})^{\sigma^{-1}} := Sgn(\sigma)v_{\{\sigma(n+1),\ldots,\sigma(2n-1)\}} &+ Sgn(\sigma)(-1)^{n+j-1}v_{\{\sigma(1),\ldots,\sigma(n-1)\}} \\ &+ Sgn(\sigma)(-1)^{n+j-1}\sum_{i=2}^{n-2} v_{\{\sigma(n-i+1),\ldots,\sigma(n-1),\sigma(n),\sigma(1),\ldots,\sigma(n-i-1)\}} \\ &+ Sgn(\sigma)(-1)^{n+j-1}v_{\{\sigma(2),\ldots,\sigma(n)\}}\end{aligned}$$

for all $\sigma \in \Sigma_{2n-1}$.

Definition 3.2.2.1. *For every $n \geq 2$, $1 \leq j \leq n$ and $d \in \mathbb{Z}$, we define n-Lie^j_d = $ASMag_{n,d}/(R^j_{n,d})$.*

For $d = 0$ and $n = 2$ this is just the ordinary operad *Lie*.

Chapter 4

THE GENERALIZATIONS OF COM

4.1 The Operad Com_n^d

For the algebras over Com, its main relations are commutativity and associativity, the former being easy to generalize while the latter can have a lot of different generalizations. The most natural generalization of associativity, called total associativity in [11], for an n-arity operation μ is to assume $\mu \circ_i \mu = \mu \circ_j \mu$ for all $1 \leq i, j \leq n$. This gives us our first natural generalization of commutative associative algebras.

Definition 4.1.0.1. *Let $n \geq 2$. A Com n-algebra of degree d is a graded k-module C with a symmetric n-arity operation $\mu_n : C^{\otimes n} \to C$ such that*

$$\mu_n \circ_j \mu_n = \mu_n \circ_i \mu_n$$

for all $1 \leq i, j \leq n$.

Note that for $n = 2$, these are exactly the non-unital commutative and associative graded k-algebras with a degree d binary operation. For $n \geq 3$, one can always make a Com n-algebra of degree d by defining μ_n as an iterated application of the original binary operation.

To define the corresponding operad, let $CS_{n,d}$ be the $k[\Sigma_{2n-1}]$ subspace of $F(H_{n,d})^{(2)}(2n-1)$ generated by the relations

$$\mu_{\{n+1,\ldots,2n-1\}} - \mu_{\{\sigma(n+1),\ldots,\sigma(2n-1)\}} \tag{4.1}$$

for all $\sigma \in \Sigma_{2n-1}$, which is exactly the associative relations with symmetric operations.

Definition 4.1.0.2. *For $n \geq 2$ and $d \in \mathbb{Z}$, define $Com_n^d = SMag_{n,d}/(CS_{n,d})$.*

4.2 The Operad n-Com_d

4.2.1 General Definition of n-Com Algebras of Degree d

The notion of n-Com algebras of degree d is a generalization of commutative algebras in the direction n-arity operations that are not associative for $n \geq 3$, but associative up to some twisted terms. Let $n \geq 2$ and fix the partition $(n, n-1)$ of $2n-1$. Let T_n be the standard Young tableau on $(n, n-1)$ whose entries first increase along the rows and then increase along the columns as in

$$T_n = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & \cdots & n-1 & n \\ \hline n+1 & n+2 & \cdots & 2n-1 & \\ \cline{1-4} \end{array} \tag{4.2}$$

Furthermore, let C_n be the columns-stabilizer of T_n i.e. it is generated by the disjoint transpositions $(1\ n+1), \ldots, (n-1\ 2n-1)$.

Definition 4.2.1.1. *An n-Com algebra of degree d is a graded k-module C with a n-ary operation $m_n : C^{\otimes n} \to C$ of degree d satisfying the following properties.*

- *For any n elements $u_1, \ldots, u_n \in C$, and for $\sigma \in \Sigma_n$, we have*

$$m_n(u_1, \ldots, u_n) = \xi(\sigma, u_1, \ldots, u_n) m_n(u_{\sigma(1)}, \ldots, u_{\sigma(n)}),$$

- *and we have the following relation*

$$\Phi(m_n) = \sum_{\sigma \in C_n} Sgn(\sigma)(m_n \circ_1 m_n)^{\sigma} = 0.$$

Explicitly, for any $2n-1$ elements $u_1, \ldots, u_{2n-1} \in C$, we have the following relation:

$$\begin{aligned} &\Phi(m_n)(u_1, \ldots, u_{2n-1}) \\ &\quad = \sum_{\sigma \in C_n} Sgn(\sigma)\xi(\sigma, u_1, \ldots, u_{2n-1})((u_{\sigma^{-1}(1)}, \ldots, u_{\sigma^{-1}(n)}), u_{\sigma^{-1}(n+1)}, \ldots, u_{\sigma^{-1}(2n-1)}) = 0. \end{aligned}$$

The relation for n-Com_d is independent of the standard Young tableau of type $(n, n-1)$, as they are all permutations of each other. Furthermore, if the degree $d = 0$ and the graded k-module is concentrated in degree 0, then we just call them n-Com algebras. We denote by $Alg_{n\text{-}Com_d}$ the category of n-Com algebras of degree d, and if $d = 0$ we denote by $Alg_{n\text{-}Com}$

the category of n-Com algebras. A morphisms of n-Com algebras A and C is just a k-linear map $F : A \to C$ such that

$$m_n^C(F(a_1), \ldots, F(a_n)) = F(m_n^A(a_1, \ldots, a_n))$$

for any $a_1, \ldots, a_n \in A$.

Example 4.2.1.1. *For these examples, we will look at the relation for when m_n is of degree 0 to simplify the presentation. When m_n is of degree d, there are extra signs coming from permuting the inputs.*

- *For $n = 2$, these are just the non-unital commutative associative k-algebras, since $C_{T_{(2,1)}} = \langle (1\ 3) \rangle$ and this gives us the associative relation*

  $$m_n(m_n(u_1, u_2), u_3) - m_n(u_1, m_n(u_2, u_3)) = 0$$

  using commutativity of the multiplication.

- *For $n = 3$, if C is a 3-Com algebra, then $C_{T_{(3,2)}}$ is generated by $(1\ 4), (2\ 5)$ and this gives us the relation*

  $$\begin{aligned} m_n(m_n(u_1, u_2, u_3), u_4, u_5) - m_n(u_1, m_n(u_2, u_3, u_4), u_5) + m_n(u_1, u_2, m_n(u_3, u_4, u_5)) \\ = m_n(m_n(u_1, u_3, u_5), u_2, u_4). \end{aligned}$$

- *For $n = 4$, $C_{T_{(4,3)}}$ is generated by $(1\ 5), (2\ 6), (3\ 7)$ and this gives us the relation*

  $$\begin{aligned} & m_n(m_n(u_1, u_2, u_3, u_4), u_5, u_6, u_7) - m_n(u_1, m_n(u_2, u_3, u_4, u_5), u_6, u_7) \\ & + m_n(u_1, u_2, m_n(u_3, u_4, u_5, u_6), u_7) - m_n(u_1, u_2, u_3, m_n(u_4, u_5, u_6, u_7)) \\ & = m_n(m_n(u_1, u_3, u_4, u_6), u_2, u_5, u_7) + m_n(m_n(u_1, u_2, u_4, u_7), u_3, u_5, u_6) \\ & - m_n(m_n(u_2, u_4, u_5, u_7), u_1, u_3, u_6) - m_n(m_n(u_1, u_4, u_6, u_7), u_2, u_3, u_5). \end{aligned}$$

In the examples above, for $n \geq 3$, we have some partial associativity up to some twisted factors, which shows that these are non-trivial generalizations of commutative associative algebras.

Next, we will give a few examples of n-Com algebras of degree 0 to give some idea of where these structures can arise.

Example 4.2.1.2. *Let A be any commutative k-algebra and let D a derivations on A. We will show that we can define a n-Com algebra structure on B using this derivation. For $n \geq 3$, define*

$$m_n(f_1, \ldots, f_n) = D(f_1 \cdots f_n)$$

for $f_1, \ldots, f_n \in B$. For $a_1, \ldots, a_{2n-1} \in A$, we have

$$\begin{aligned}
&\sum_{\sigma \in C_n} Sgn(\sigma) m_n(m_n(a_{\sigma(1)}, \ldots, a_{\sigma(n-1)}, a_n), a_{\sigma(n+1)}, \ldots, a_{\sigma(2n-1)}) \\
&\quad = \sum_{\sigma \in C_n} Sgn(\sigma) D(D(a_{\sigma(1)} \cdots a_{\sigma(n-1)}, a_n) a_{\sigma(n+1)} \cdots a_{\sigma(2n-1)}) \\
&\quad = \sum_{\sigma \in C_n} \sum_{p=1}^{n-1} Sgn(\sigma) D(a_{\sigma(1)} \cdots D(a_{\sigma(p)}) \cdots a_{\sigma(n-1)} a_n a_{\sigma(n+1)} \cdots a_{\sigma(2n-1)}) \\
&\quad + \sum_{\sigma \in C_n} Sgn(\sigma) D(a_{\sigma(1)} \cdots a_{\sigma(n-1)} D(a_n) a_{\sigma(n+1)} \cdots a_{\sigma(2n-1)}).
\end{aligned}$$

By commutativity, the last sum is just $D(a_1 \cdots D(a_n) \cdots a_{2n-1})$ multiplied by a finite alternating sum of 1's, with an even amount of elements, which is zero. Let H_p be the subgroup of C_n that fixes p, which is generated by the transpositions $(1, n+1), \ldots, (p-1, n+p-1), (p+1, n+p+1), \ldots, (n-1, 2n-1)$. Using commutativity, the first sum becomes

$$\begin{aligned}
&\sum_{\sigma \in C_{T_{\lambda_n}}} \sum_{p=1}^{n-1} Sgn(\sigma) D(a_{\sigma(1)} \cdots D(a_{\sigma(p)}) \cdots a_{\sigma(n-1)} a_n a_{\sigma(n+1)} \cdots a_{\sigma(2n-1)}) \\
&\quad = \sum_{\sigma \in H_p} \sum_{p=1}^{n-1} Sgn(\sigma) D(a_1 \cdots D(a_p) \cdots a_{n-1} a_n a_{n+1} \cdots a_{2n-1}) \\
&\quad + \sum_{\sigma \in C_n \setminus H_p} \sum_{p=1}^{n-1} Sgn(\sigma) D(a_1 \cdots a_{n+p-1} D(a_{n+p}) a_{n+p+1} \cdots a_{2n-1}).
\end{aligned}$$

Since both H_p and C_n have even cardinality, then both of these sums are zero. This shows that we have

$$\sum_{\sigma\in C_n} Sgn(\sigma)(m_n \circ_1 m_n)^\sigma = 0.$$

Therefore, m_n gives B a n-Com algebra structure. Note that $m_{n-1}(f_1,\dots,f_{n-1}) = m_n(f_1,\dots,f_{n-1},1)$ if 1 is the unit in B.

For an explicit example, suppose $B = k[x_1,\dots,x_m]$ is the polynomial ring on m variables. We have m natural derivations $\frac{\partial}{\partial x_i}$ for all $1 \le i \le m$ and we can define

$$m_n^i(f_1,\dots,f_n) = \frac{\partial}{\partial x_i}(f_1\cdots f_n),$$

for each i, which gives B a n-Com algebra structure. Even more, we can define

$$m_n(f_1,\dots,f_n) = \sum_{i=1}^{m} m_n^i(f_1,\dots,f_n) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}(f_1\cdots f_n)$$

and this gives the polynomial ring B a n-Com algebra structure since the sum of derivations is a derivation.

Next, we will explore some examples of n-Com algebras on finite rank modules over a commutative k-algebra A. Let M be any A-module of finite rank and with basis elements $e_1,\dots,e_m$. Let $T : M^{\otimes n} \to M$ be any symmetric A-linear map and using the basis elements we can express T as follows:

$$T(e_{i_1},\dots,e_{i_n}) = \sum_{i=1}^{m} \lambda^j_{i_1,\dots,i_n} e_j$$

for some symmetric coefficients $\lambda^j_{i_1,\dots,i_n} \in A$ in the $i_1,\dots,i_n$. Plugging these into the defining equation for n-Com, we have

$$\begin{aligned}
&\sum_{\sigma\in C_n} Sgn(\sigma) T(T(e_{i_{\sigma(1)}},\dots e_{i_{\sigma(n-1)}}, e_{i_n}), e_{i_{\sigma(n+1)}},\dots,e_{i_{\sigma(2n-1)}}) \\
&\quad = \sum_{\sigma\in C_n}\sum_{j=1}^{m} Sgn(\sigma)\lambda^j_{i_{\sigma(1)},\dots,i_{\sigma(n-1)},i_n} T(e_j, e_{i_{\sigma(n+1)}},\dots,e_{i_{\sigma(2n-1)}}) \\
&\quad = \sum_{\sigma\in C_n}\sum_{j=1}^{m}\sum_{l=1}^{m} Sgn(\sigma)\lambda^j_{i_{\sigma(1)},\dots,i_{\sigma(n-1)},i_n}\lambda^l_{j,i_{\sigma(n+1)},\dots,i_{\sigma(2n-1)}} e_l
\end{aligned}$$

Therefore, for T to be give M a n-Com algebra structure, it suffices for us to have

$$\sum_{\sigma\in C_n}\sum_{j=1}^{m} Sgn(\sigma)\lambda^{j}_{i_{\sigma(1)},\ldots,i_{\sigma(n-1)},i_n}\lambda^{l}_{j,i_{\sigma(n+1)},\ldots,i_{\sigma(2n-1)}}=0$$

for all $1\le l\le m$.

Example 4.2.1.3. *Let $n\ge 3$, $\delta\in A$, and define $\lambda^{j}_{i_1,\ldots,i_n}=\delta$ whenever $j\in\{i_1,\ldots,i_n\}$, and zero otherwise. Then we have*

$$\begin{aligned}
&\sum_{\sigma\in C_n}\sum_{j=1}^{m} Sgn(\sigma)\lambda^{j}_{i_{\sigma(1)},\ldots,i_{\sigma(n-1)},i_n}\lambda^{l}_{j,i_{\sigma(n+1)},\ldots,i_{\sigma(2n-1)}}\\
&\qquad=\sum_{\sigma\in C_n}\sum_{r=1}^{n-1} Sgn(\sigma)\lambda^{i_{\sigma(r)}}_{i_{\sigma(1)},\ldots,i_{\sigma(n-1)},i_n}\lambda^{l}_{i_{\sigma(r)},i_{\sigma(n+1)},\ldots,i_{\sigma(2n-1)}}\\
&\qquad+\sum_{\sigma\in C_n} Sgn(\sigma)\lambda^{i_n}_{i_{\sigma(1)},\ldots,i_{\sigma(n-1)},i_n}\lambda^{l}_{i_n,i_{\sigma(n+1)},\ldots,i_{\sigma(2n-1)}}\\
&\qquad=\sum_{\sigma\in C_n}\sum_{r=1}^{n-1} Sgn(\sigma)\delta\lambda^{l}_{i_{\sigma(r)},i_{\sigma(n+1)},\ldots,i_{\sigma(2n-1)}}\\
&\qquad+\sum_{\sigma\in C_n} Sgn(\sigma)\delta\lambda^{l}_{i_n,i_{\sigma(n+1)},\ldots,i_{\sigma(2n-1)}}.
\end{aligned}$$

If $l\notin\{i_1,\ldots,i_{2n-1}\}$, then the last two sums are automatically zero by definition. Otherwise, l could be any number of the elements $i_1,\ldots,i_{2n-1}$ (there could be some redundancy in this list of indices). If $l=i_n$, then the last sum is always zero since C_n is of even order. On the other hand, suppose $l=i_{s_1}=\cdots=i_{s_t}$ for some $1\le t\le 2n-1$, then we can find a subgroup H_l consisting of the permutations that fix any one of the $s_1,\ldots,s_t$. Since $C_{T_{\lambda_n}}$ is even, then subgroup H_l has to be even as well by Lagrange's theorem. Therefore, the last sum is

$$\sum_{\sigma\in C_n} Sgn(\sigma)\delta\lambda^{l}_{i_n,i_{\sigma(n+1)},\ldots,i_{\sigma(2n-1)}}=\sum_{\sigma\in H_l} Sgn(\sigma)\delta^2=0$$

By the same argument for the other sum, when we fix r, this shows that A-module M with the A-linear map T is a n-Com algebra. Note that this can fail when $n=2$, as it is not generally associative.

Example 4.2.1.4. *Here is an example derived from geometry. Let* $X = \mathbf{R}^n$ *for* $n \geq 1$ *and let* $Vect(X)$ *be the collection of vector fields on* X*. The space* $Vect(X)$ *is a* $C^\infty(X)$*-module of finite rank with basis* $\frac{\partial}{\partial x^i}$ *for* $i = 1, \ldots, N$*. Pick a section* $\pi \in \Gamma(S^m(T^*M) \otimes TM)$ *for* $m \geq 3$ *expressed as*

$$\pi = \sum_{i_1 \leq \cdots \leq i_m} \sum_k \delta^k_{i_1,\ldots,i_m} dx^{i_1} \otimes \cdots \otimes dx^{i_m} \otimes \frac{\partial}{\partial x^k}$$

where

$$\delta^k_{i_1,\ldots,i_m} = \begin{cases} 1 & \text{if } k \in \{i_1, \ldots, i_m\} \\ 0 & \text{if otherwise} \end{cases}.$$

Note that $\delta^k_{i_1,\ldots,i_m}$ *is symmetric in the* $i_1, \ldots, i_m$ *indices. We can define a* m*-ary multiplication on* $Vect(X)$ *through* $\pi_m : Vect(X)^{\otimes_{\mathbf{R}} m} \to Vect(X)$ *defined as*

$$\pi_m(V_1, \ldots, V_m) = \pi(V_1, \ldots, V_m) = \sum_{i_1,\ldots,i_m,k} \delta^k_{i_1,\ldots,i_m} V_1(x^{i_1}) \cdots V_n(x^{i_m}) \frac{\partial}{\partial x^k}.$$

On the basis elements of $Vect(X)$*, we have*

$$\pi_m\left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_m}}\right) = \sum_k \delta^k_{j_1,\ldots,j_m} \frac{\partial}{\partial x^k}$$

This is exactly the case as in example 4.2.1.3 with $\delta = 1$ *and this gives* $Vect(X)$ *a* n*-Com algebra structure.*

Explicitly, if $X = \mathbf{R}$ *and* $m \geq 2$*, we have*

$$\pi_m(V_1, \ldots, V_m) = V_1(x) \cdots V_m(x) \frac{d}{dx}$$

and one can easily see this satisfies the relation for a n*-Com algebra for* $m \geq 2$*.*

In the next few sections we will explore more constructions and examples of n-Com algebras.

4.2.2 Associative Algebra induced by n-Com algebra

In this section, we will show that we can construct an associative algebra from some certain operators on a n-Com algebra A. As a motivation, suppose C is a Com-algebra, then it is very natural to construct $\mu_c : C \to C$ for every $c \in C$ such that $\mu_c(a) = ca$. We can combine these together to get elements of the endomorphism ring $\mu_c \in \mathrm{End}(C)$ of C, which is usually not commutative. But if we take the subcollection of μ_c, these are in the center of $\mathrm{End}(C)$ using the commutativity and associativity of C.

Let C be any n-Com algebra and we have the associative algebra $\mathrm{End}(C)$ of endomorphisms of C, and we will construct similar endomorphisms as μ_c for this case. In this case, because our algebras are not necessarily associative, these new elements will not necessarily commute.

Definition 4.2.2.1. *Let C be a n-Com algebra with multiplication m_n and let $y_1, \dots, y_{n-1} \in C$, then we define $\chi_{y_1,\dots,y_{n-1}} = m_n(y_1, \dots, y_{n-1}, -)$, which is a endomorphism of C. Let $\Gamma(C)$ be the sub associative k-algebra of End(C) generated by $\chi_{y_1,\dots,y_{n-1}}$ for $y_1, \dots, y_{n-1} \in C$.*

This defines a linear map $\chi : S^{n-1}(C) \to \mathrm{End}(C)$, where $S^{n-1}(C) = (C^{\otimes n-1})_{\Sigma_{n-1}}$ is the $n-1$th symmetric power of C.

The following lemma is just a consequence of the definition of χ and the relation we have in n-Com algebras. To make the presentation clearer, let $\widehat{C}_n$ be the subgroup of Σ_{2n-2} generated by the transpositions $(1\ n), \dots, (n-1\ 2n-2)$.

lemma 4.2.2.1. *For any $y_1, \dots, y_{n-1}, y_n, \dots, y_{2n-2} \in C$ we have*

$$\sum_{\sigma \in \widehat{C}_n} Sgn(\sigma) \chi_{y_{\sigma(1)},\dots,y_{\sigma(n-1)}} \chi_{y_{\sigma(n)},\dots,y_{\sigma(2n-2)}} = 0 \tag{4.3}$$

and for any $z_1, \dots, z_{n-2}, x, y \in C$

$$\chi_{z_1,\dots,z_{n-2},x}(y) = \chi_{z_1,\dots,z_{n-2},y}(x).$$

We can rewrite the equation 4.3

$$\sum_{\substack{\sigma\in\widehat{C}_n\\ \sigma(1)=1}} Sgn(\sigma)\chi_{y_1,y_{\sigma(2)},\dots,y_{\sigma(n-1)}}\chi_{y_n,y_{\sigma(n+1)},\dots,y_{\sigma(2n-2)}}$$

$$= \sum_{\substack{\sigma\in\widehat{C}_n\\ \sigma(1)=n}} Sgn(\sigma)\chi_{y_n,y_{\sigma(2)},\dots,y_{\sigma(n-1)}}\chi_{y_1,y_{\sigma(n+1)},\dots,y_{\sigma(2n-2)}}$$

$$= (-1)^{n-2}\sum_{\substack{\sigma\in\widehat{C}_n\\ \sigma(1)=n}} Sgn(\sigma)\chi_{y_n,y_{\sigma(n+1)},\dots,y_{\sigma(2n-2)}}\chi_{y_1,y_{\sigma(2)},\dots,y_{\sigma(n-1)}}.$$

Proof. This is just an easy consequence of the definition of n-Com algebra. □

Example 4.2.2.1. *Here we will explore the equation* **??** *for different n and see how far it is from being commutative.*

- *For $n = 2$, we have*

$$\chi_y\chi_z = \chi_z\chi_y$$

  which is exactly what we should expect since $\chi_y(b) = yb$.

- *For $n = 3$, we obtain the equations*

$$\chi_{y_1,y_2}\chi_{y_3,y_4} - \chi_{y_1,y_4}\chi_{y_3,y_2} = -(\chi_{y_3,y_4}\chi_{y_1,y_2} - \chi_{y_3,y_2}\chi_{y_1,y_4})$$

- *For $n = 4$, we have the equations*

$$\chi_{y_1,y_2,y_3}\chi_{y_4,y_5,y_6} - \chi_{y_1,y_5,y_3}\chi_{y_4,y_2,y_6} - \chi_{y_1,y_2,y_6}\chi_{y_4,y_5,y_3} + \chi_{y_1,y_5,y_6}\chi_{y_4,y_2,y_3}$$
$$= \chi_{y_4,y_5,y_6}\chi_{y_1,y_2,y_3} - \chi_{y_4,y_2,y_6}\chi_{y_1,y_5,y_3} - \chi_{y_4,y_5,y_3}\chi_{y_1,y_2,y_6} + \chi_{y_4,y_2,y_3}\chi_{y_1,y_5,y_6}$$

Now suppose C is a finite-dimensional n-Com algebra with basis $e_1, \dots, e_m$, and we can define

$$\chi_{i_1,\dots,i_n} := \chi_{e_{i_1},\dots,e_{i_n}}$$

for all $i_1, \ldots, i_n \in [m]$.

If V is a m-dimensional vector space with basis $e_1, \ldots, e_m$, then define the k-linear maps $\gamma_i^j : V \to V$ for any $i, j \in [m]$ such that

$$\gamma_i^j(e_t) = \begin{cases} e_j & \text{if } t = i \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that the γ_i^j satisfy $\gamma_i^j \gamma_r^s = \delta_i^s \gamma_r^j$ for all $i, j, r, s \in [m]$, where δ_i^s is 1 if $i = s$ and 0 otherwise. Therefore, γ_i^j in matrix form is the matrix with 0's everywhere, and 1 at position (i, j) where i is the column and j is the row, which the collection gives a basis for $M_m(k)$.

lemma 4.2.2.2. *If C is finite dimensional n-Com algebra with basis $e_1, \ldots, e_m$, then*

$$\sum_{\sigma \in \widehat{C}_n} Sgn(\sigma) \chi_{i_{\sigma(1)}, \ldots, i_{\sigma(n-1)}} \chi_{i_{\sigma(n)}, \ldots, i_{\sigma(2n-2)}} = 0$$

for any $i_1, \ldots, i_{2n-2} \in [m]$. Furthermore, for any $i_1, \ldots, i_{n-2}, i, j \in [m]$ we have

$$\chi_{i_1, \ldots, i_{n-2}, i} \gamma_i^j = \chi_{i_1, \ldots, i_{n-2}, j} \gamma_i^i.$$

Proof. The first equation is the obvious consequence of the definition of n-Com algebra. Furthermore, we have

$$\begin{aligned} \chi_{i_1, \ldots, i_{n-2}, i} \gamma_i^j(e_r) &= \begin{cases} \chi_{i_1, \ldots, i_{n-2}, i}(e_j) & \text{if } r = i \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \chi_{i_1, \ldots, i_{n-2}, j}(e_i) & \text{if } r = i \\ 0 & \text{otherwise} \end{cases} \\ &= \chi_{i_1, \ldots, i_{n-2}, j} \gamma_i^i(e_r). \end{aligned}$$

□

The last lemma shows how we can systematically figure out when a finite-dimensional vector space has a n-Com algebra structure. In particular, we will show that finite dimensional vector spaces that are left modules over a particular associative algebra give n-Com algebra structure.

Definition 4.2.2.2. *For $m \geq 1$ let ∇_m be the associative k-algebra*

$$\nabla_m = k\langle \xi_i^j \mid 1 \leq i, j \leq m \rangle / I_m$$

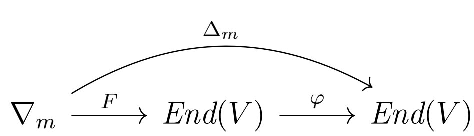
where I_m is the ideal generated by

$$\xi_i^j \xi_r^s - \delta_{i,s} \xi_r^j.$$

The associative k-algebra ∇_m is finite dimensional with basis 1 and ξ_i^j for all i, j, i.e. we have $\nabla_m = k \oplus (\bigoplus_{i,j} k\xi_i^j)$. In some cases we don't even need the unit in ∇_m, so we define $\overline{\nabla}_m$ to be the non-unital k-algebra generated by only ξ_i^j for all i, j. In particular, we have ∇_m is isomorphic to the subalgebra of $M_m(k)$ generated by the matrices with only a 1 in the (i, j)-spot and zero everywhere else, and the identity matrix.

For any m, there is a natural algebra homomorphism $\Delta_m : \nabla_m \to \text{End}(V)$ for any m-dimensional vector space V by sending $\Delta_m(\xi_i^j) = \gamma_i^j$. Next, we will show that if M is a module over ∇_m with vector space dimension m and has a non-trivial action then we can apply a change of basis to get the action through γ_i^j.

lemma 4.2.2.3. *If we have a k-algebra homomorphism $F : \nabla_m \to End(V)$ such that $F(\xi_i^j) \neq 0$ for some i, j, then there exists an k-algebra isomorphism $\varphi : End(V) \to End(V)$ such that*



commutes.

Proof. Let $h_i^j = F(\xi_i^j)$ and we will show h_i^j is a basis for End(V). We know dim(End(V)) is of dimension m^2 so we just need to show it is linearly independent. Suppose $h_s^t \neq 0$, then by the relations in ∇_m we have

$$h_s^t = h_j^t h_s^j = h_i^t h_j^i h_s^j$$

and since $h_s^t \neq 0$, then we must have $h_j^i \neq 0$ for all i, j. Furthermore, if we let $\sum_{i,j} a_{i,j} h_i^j = 0$ then we can multiply it on the right by h_t^s to get

$$0 = \sum_{i,j} a_{i,j} h_i^j h_t^s = \sum_{i,j} a_{i,j} \delta_i^s h_t^j = a_{s,j} h_t^j$$

which shows $a_{s,j} = 0$ and hence this shows linear independence. Therefore, h_i^j gives a basis for $\mathrm{End}(V)$ and gives us a way to define a k-algebra isomorphism $\varphi : \mathrm{End}(V) \to \mathrm{End}(V)$ by sending h_i^j to γ_i^j and we get the following commutative diagram

$$\nabla_m \xrightarrow{F} \mathrm{End}(V) \xrightarrow{\varphi} \mathrm{End}(V), \qquad \Delta_m : \nabla_m \to \mathrm{End}(V).$$

□

The last lemma states that as long as we have a non-trivial action of ∇_m on a m-dimensional vector space then we can go through a change of basis to get the action through γ_i^j.

Furthermore, we can define the following associative k-algebra with the correct relations needed for particular modules over it to have a n-Com algebra structure, but note that not every m-dimensional module will have a n-Com algebra structure as we need some compatibility with ∇_m.

Definition 4.2.2.3. *Let $n \geq 2$ and $m \geq 1$. Define the associative k-algebra $\Omega_{n,m}$ to be*

$$\Omega_{n,m} := k\langle x_{i_1,\dots,x_{n-1}} \ : \ i_1, \dots, i_{n-1} \in [m] / J_{n,m}$$

where $J_{n,m}$ is generated by

$$\sum_{\sigma \in \widehat{C}_n} Sgn(\sigma) x_{i_{\sigma(1)},\dots,i_{\sigma(n-1)}} x_{i_{\sigma(n)},\dots,i_{\sigma(2n-2)}} \tag{4.4}$$

for $i_1, \dots, i_{2n-2} \in [m]$ and by

$$x_{i_{\sigma(1)},\dots,i_{\sigma(n-1)}} - x_{i_1,\dots,x_{n-1}}$$

for $i_1, \dots, i_{n-1} \in [m]$ and $\tau \in \Sigma_{n-1}$.

Note that $\Omega_{n,m}$ is a locally finite connected $\mathbf{N}$-graded algebra, i.e. the 0 degree component is just k. Furthermore, $\Omega_{n,m}$ is a quadratic algebra, since we can take V be the finite dimensional vector space with basis elements $x_{i_1,\ldots,i_{n-1}}$ for $i_1,\ldots,i_{n-1} \in [m]$ with the natural Σ_{n-1} action acting on the indices and we have $\Omega_{n,m} = T(V_{\Sigma_{n-1}})/(R)$ where R is the relation in equation 4.4. We can combine both $\Omega_{n,m}$ and ∇_m to obtain the free product $\Omega_{n,m} \star \nabla_m$, which is the coproduct in the category of associative k-algebras:

$$\Omega_{n,m} \star \nabla_m = k\langle x_{i_1,\ldots,i_{n-1}}, \xi_i^j \ : \ i_1,\ldots,i_{n-1},i,j \in [m]\rangle / J_{n,m} \star I_{n,m} \tag{4.5}$$

with $J_{n,m} \star I_{n,m}$ is generated by the relations in $J_{n,m}$ and $I_{n,m}$. Note that if V is a module over $\Omega_{n,m}$ which is also m-dimensional vector space then it is a module over the free product $\Omega_{n,m} \star \nabla_m$ using the γ_i^j actions. But this is not enough give us a n-Com algebra structure as we need the compatibility condition $x_{i_1,\ldots,i_{n-2},i}\xi_i^j = x_{i_1,\ldots,i_{n-2},j}\xi_i^i$. In particular, we define $\Gamma_{n,m} = \Omega_{n,m} \star \nabla_m / W_{n,m}$ where $W_{n,m}$ is generated by

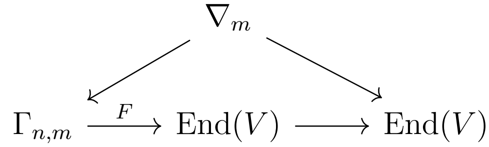
$$x_{i_1,\ldots,i_{n-2}i}\xi_i^j - x_{i_1,\ldots,x_{n-2},j}\xi_i^i \tag{4.6}$$

for all $i_1,\ldots,i_{n-2},i,j \in [m]$.

lemma 4.2.2.4. *Let V be a m-dimensional vector. The space V has a n-Com algebra structure if and only V is a left $\Gamma_{n,m}$-module with at least one of the ξ_i^j acts non-trivially on V.*

Proof. If V is a n-Com algebra structure, then by lemma 4.2.2.2 we have a k-algebra homomorphism $F : \Gamma_{n,m} \to \mathrm{End}(V)$ with $F(\xi_i^j) \neq 0$ for all i,j.

On the other hand, suppose V is a left $\Gamma_{n,m}$-module with ξ_s^t acts non-trivally on V for some s,t. Then we have a k-algebra homomorphism $F : \Gamma_{n,m} \to \mathrm{End}(V)$ with $F(\xi_s^t)$ is non-zero. Then by lemma 4.2.2.3 there is a change of basis such that we have the commutative diagram



where $F(\xi_i^j)$ gets sent to γ_i^j in $\mathrm{End}(V)$. This composition shows that we can induce a n-Com algebra structure on V. □

In particular, this gives us the following definition for modules in $\Omega_{n,m}$.

Definition 4.2.2.4. *Let $n \geq 2$ and $m \geq 1$. We say a $\Omega_{n,m}$-module V is a n-Com module algebra if and only if V is m-dimensional and we have*

$$x_{i_1,\ldots,i_{n-2}i}\gamma_i^j(v) = x_{i_1,\ldots,i_{n-2},j}\gamma_i^i(v)$$

for all $v \in V$ and $i_1, \ldots, i_{n-2}, i, j \in [m]$.

If V and W are isomorphic as n-Com module algebras, then they are isomorphic as n-Com algebras, by taking appropriate changes of basis. This change of perspective gives us a way to study finite dimensional n-Com algebras through the tools on modules over an associative k-algebra. Note that in general, $\Omega_{n,m}$ and $\Gamma_{n,m}$ are not finite dimensional algebras and, in general, not commutative for $n > 2$ and $m > 1$.

Let $m \geq 1$ and let I_i^j be the $m \times m$ matrix with a 1 in the ith column and in the jth row and zero everywhere else, which gives a basis for $M_m(k)$. If we have the matrix $A = (a_{i,j}) \in M_m(k)$ with

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,m} \end{pmatrix}$$

then AI_i^j extracts the jth column of A and puts it at the ith column, i.e. we have

$$AI_i^j = \begin{pmatrix} 0 & \cdots & a_{1,j} & \cdots & 0 \\ 0 & \cdots & a_{2,j} & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & 0 \\ 0 & \cdots & a_{n,j} & \cdots & 0 \end{pmatrix}$$

corollary 4.2.2.5. *Let $n \geq 2$ and $m \geq 1$. An m-dimensional vector space V is a n-Com algebra if and only if we can find matrices $X_{i_1,\ldots,i_{n-1}} \in M_m(k)$ for $i_1, \ldots, i_{n-1} \in [m]$ such that*

$$\sum_{\sigma \in \overline{C}_n} Sgn(\sigma) X_{i_{\sigma(1)},\ldots,i_{\sigma(n-1)}} X_{i_{\sigma(n)},\ldots,i_{\sigma(2n-2)}} = 0,$$

$$X_{i_1,\ldots,i_{n-1}} = X_{i_{\sigma(1)},\ldots,i_{\sigma(n-1)}}$$

for $\sigma \in \Sigma_{n-1}$, and

$$X_{i_1,\ldots,i_{n-2},i} I_i^j = X_{i_1,\ldots,i_{n-2},j} I_i^i$$

for all $i_1, \ldots, i_{n-2}, i, j \in [m]$.

Example 4.2.2.2.

- *For the case $n = 2$ and $m \geq 1$, we have*

$$\Omega_{2,m} = k[x_1, \ldots, x_m]$$

and

$$\Gamma_{2,m} \cong \frac{k[x_1, \ldots, x_m] \star \nabla_m}{(x_i h_i^j - x_j h_i^i)}.$$

If V is a m-dimensional $\Gamma_{2,m}$-module with ξ_i^j acts through γ_i^j, then V has a 2-Com algebra structure by defining $e_i e_j = x_i(e_j) = x_j(e_i)$, which is a non-unital commutative associative k-algebra. Hence, to find the commutative algebras of dimension m, one just needs to find a collection of m commuting matrices with the additional property $X_i I_i^j = X_j I_i^i$.

- *For any $n \geq 2$ and $m = 1$ we have*

$$\Omega_{n,1} = k[x]$$

and hence the n-Com algebras of dimension 1 are very easy to describe, i.e. we just have $m_n(e, \ldots, e) = \lambda e$ for any $\lambda \in k$.

4.2.3 3-Com algebras of dimension m

For this section, we will use the algebras $\Omega_{3,m}$ to help us find examples of 3-Com algebras of dimension m. By definition, the algebra $\Omega_{3,m}$ has generators $x_{i,j}$ for $i,j \in [m]$ with $x_{i,j} = x_{j,i}$ and relations

$$x_{i,j}x_{s,t} - x_{i,t}x_{s,j} - x_{s,j}x_{i,t} + x_{s,t}x_{i,j} = 0.$$

Note that if $i = j$ and s or t is i then this gives us trivial relations and does not give us anything new.

lemma 4.2.3.1. *For $m \geq 2$, the only non-trivial relations in $\Omega_{3,m}$ are*

$$x_{i,i}x_{s,t} - x_{i,t}x_{s,i} - x_{s,i}x_{i,t} + x_{s,t}x_{i,i} = 0$$

for $1 \leq s < t \leq m$ with $s,t \neq i$ for all $1 \leq i \leq m$ and

$$x_{i,i}x_{j,j} + x_{j,j}x_{i,i} = 2x_{i,j}^2$$

Proof. Through the relations, if $i \neq j$, then we have the relations

$$x_{i,i}x_{j,j} - x_{i,j}x_{j,i} - x_{j,i}x_{i,j} + x_{j,j}x_{i,i} = 0$$

which simplifies down to

$$x_{i,i}x_{j,j} + x_{j,j}x_{i,i} = 2x_{i,j}^2.$$

On the other hand, if $1 \leq s < t \leq m$ with $s,t \neq i$ then we have

$$x_{i,i}x_{s,t} - x_{i,t}x_{s,i} - x_{s,i}x_{i,t} + x_{s,t}x_{i,i} = 0$$

which is non-trivial since s,t are not equal to i, otherwise if one of the s or t was equal to i, say t, then we would have $x_{i,i}x_{s,i}$ and $x_{s,i}x_{i,i}$ repeat and hence gives a trivial relation. □

We will focus on the casee $m = 2$ and use this to find the simple 3-Com algebras. In this case, the algebra $\Omega_{3,2}$ is

$$\Omega_{3,2} = \frac{k\langle x, y, z\rangle}{(xz + zx - 2y^2)}$$

where in the original notation we have $x = x_{1,1}$, $y = x_{1,2} = x_{2,1}$ and $z = x_{2,2}$. We won't need it for the results in this section, but by the work in [30], the algebra $\Omega_{3,2}$ is a non-Noetherian AS-regular algebra of global dimension 2 and Gorenstien index 0.

To find examples of 3-Com algebras of dimension 2 we need to find matrices $X, Y, Z \in M_2(k)$ such that

$$XZ + ZX = 2Y^2$$

with $XI_1^2 = YI_1^1$ and $ZI_2^1 = YI_2^2$. If we have

$$X = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \qquad Z = \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} \tag{4.7}$$

then the relations $XI_1^2 = Y_1^1$ and $ZI_2^1 = Y_2^2$ tell us the the first column of Y is the second column of X and the second column of Y is the first column of Z, i.e. we have

$$Y = \begin{pmatrix} a_{1,2} & b_{1,1} \\ a_{2,2} & b_{2,1}. \end{pmatrix}$$

With this, we can find an entire family of 3-Com algebras of dimension 2 when we are over the field $\mathbf{C}$ of complex numbers.

Proposition 4.2.3.2. *If $k = \mathbf{C}$ and V is a 2-dimensional vector space over k and suppose $X, Y, Z \in M_2(k)$ such that $XZ + ZX = 2Y^2$ and $XI_1^2 = YI_1^1$ and $ZI_2^1 = YI_2^2$, then X is diagonalizable. The matrices X, Y, Z can only be one of the following cases, where $X = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.*

- *If* $\lambda_1 = \lambda_2$*, then we have*

$$X = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} \qquad Z = \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & b_{1,1} \\ \lambda_1 & b_{2,1} \end{pmatrix}$$

where $\lambda b_{1,2} = b_{1,1}b_{2,1}$ *and* $\lambda(b_{2,2} - b_{1,1}) = b_{2,1}^2$ *i.e. we have* $\lambda Z = Y^2$*.*

- *If* $\lambda_1 \neq \lambda_2$*, then we have the following cases.*

  – *If* $\lambda_1 = 0$*, then we have*

$$X = \begin{pmatrix} 0 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad Z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ \lambda_2 & 0 \end{pmatrix}.$$

  – *If* $\lambda_2 = 0$*, then we have*

$$X = \begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

  – *If* λ_1 *and* λ_2 *are both non-zero, then*

$$X = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad Z = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ \lambda_2 & 0 \end{pmatrix}$$

  or

$$X = \begin{pmatrix} -\lambda_2 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad Z = \begin{pmatrix} 0 & b_{1,2} \\ 0 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ \lambda_2 & 0 \end{pmatrix}.$$

Proof. Since we are over $\mathbf{C}$, we can find the Jordan canonical form for the matrix X by a change of basis in V, i.e. it is of the form $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ or $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Suppose for a contradiction X is of the form $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ and let Z be as in equation 4.7 so that Y becomes the matrix

$\begin{pmatrix} 1 & b_{1,1} \\ \lambda & b_{2,1} \end{pmatrix}$. Computing $XZ + ZX$ and $2Y^2$ we obtain

$$XZ + ZX = \begin{pmatrix} 2\lambda b_{1,1} + b_{2,1} & 2\lambda b_{2,1} + b_{1,1} + b_{2,2} \\ 2\lambda b_{1,2} & 2\lambda b_{2,2} + b_{1,2} \end{pmatrix}$$
$$2Y^2 = \begin{pmatrix} 2 + 2\lambda b_{1,1} & 2b_{1,1} + 2b_{1,1}b_{1,2} \\ 2\lambda + 2\lambda b_{1,2} & 2\lambda b_{1,1} + 2b_{12}^2 \end{pmatrix}.$$

Since both of these have to be equal to each other, then we must have

$$2\lambda + 2\lambda b_{1,2} = 2\lambda b_{1,2}$$
$$2\lambda b_{1,1} + b_{2,1} = 2 + 2\lambda b_{1,1}$$

and hence $\lambda = 0$ and $b_{2,1} = 2$. But this would imply

$$2 \cdot 2^2 = 2b_{1,2}^2 = b_{1,2} = 2$$

on the $(2, 2)$-position of the matrices, which is a contradiction. Therefore, X must be diagonalizable with the geometric and algebraic multiplicities are the same, i.e. $X = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ for some $\lambda_1, \lambda_2 \in k$. In this case and going through the equations we have

$$XZ + ZX = \begin{pmatrix} 2\lambda_1 b_{1,1} & (\lambda_1 + \lambda_2)b_{1,2} \\ (\lambda_1 + \lambda_2)b_{2,1} & 2\lambda_2 b_{2,2} \end{pmatrix}$$
$$2Y^2 = \begin{pmatrix} 2\lambda_2 b_{1,1} & 2b_{1,1}b_{2,1} \\ 2\lambda_2 b_{2,1} & 2\lambda_2 b_{1,1} + 2b_{2,1}^2 \end{pmatrix}.$$

If $\lambda_1 = \lambda_2 = \lambda$, then we get $ZX + XZ = 2Y^2$ if and only if $\lambda b_{1,2} = b_{1,1}b_{2,1}$ and $\lambda(b_{2,2} - b_{1,1}) = b_{2,1}^2$. On the other hand, if $\lambda_1 \neq \lambda_2$, then equating both of these equations, we obtain

$$(\lambda_1 - \lambda_2)b_{1,1} = 0$$
$$(\lambda_1 + \lambda_2)b_{2,1} - 2\lambda_2 b_{2,1} = (\lambda_1 - \lambda_2)b_{2,1} = 0$$

which implies $b_{2,1} = b_{1,1} = 0$, since λ_1 and λ_2 are distinct. This simplifies the equations to

$$(\lambda_1 + \lambda_2)b_{1,2} = 0$$
$$\lambda_2 b_{2,2} = 0$$

in this case.

This gives us a few cases for $b_{1,2}$, $b_{2,2}$ and the eigenvalues λ_1, λ_2. If $\lambda_2 = 0$ or $\lambda_1 = 0$ then we must have $b_{2,2} = b_{1,2} = 0$ and hence Z is the zero matrix. On the other hand, if both λ_1 and λ_2 are not zero, then we must have $b_{2,2} = 0$ and either $b_{1,2} = 0$ or $\lambda_1 = -\lambda_2$. This gets us all of our possible matrices X, Y, Z which satisfies the equations. □

The last lemma shows what all the possible 3-Com algebra structures are on 2-dimensional **C** vector space as shown by the next lemma, which gives the explicit 3-arity multiplications.

lemma 4.2.3.3. *If V is a 2-dimensional $\mathbf{C}$-vector space with basis e_1, e_2, with a 3-Com algebra structure m_3, then m_3 can is one of the following structures.*

(1) Let $\lambda \in k$ and $b_{1,1}, b_{1,2}, b_{2,1}, b_{2,2} \in k$ such that $\lambda b_{1,2} = b_{1,1}b_{2,1}$ and $\lambda(b_{2,2} - b_{1,1}) = b_{2,1}^2$ then

$$m_3(e_1, e_1, e_1) = \lambda e_1 \qquad m_3(e_1, e_1, e_2) = \lambda e_2$$
$$m_3(e_1, e_2, e_2) = b_{1,1}e_1 + b_{2,1}e_2 \qquad m_3(e_2, e_2, e_2) = b_{1,2}e_1 + b_{2,2}e_2.$$

(2) Let $\lambda_1, \lambda_2 \in k$ such that $\lambda_1 \neq \lambda_2$, then we have the following cases.

(a) If $\lambda_1 = 0$, then

$$m_3(e_1, e_1, e_1) = 0 \qquad m_3(e_1, e_1, e_2) = \lambda_2 e_2$$
$$m_3(e_1, e_2, e_2) = 0 \qquad m_3(e_2, e_2, e_2) = 0.$$

(b) If $\lambda_2 = 0$, then we have

$$m_3(e_1, e_1, e_1) = \lambda_1 e_1 \qquad m_3(e_1, e_1, e_2) = 0$$
$$m_3(e_1, e_2, e_2) = 0 \qquad m_3(e_2, e_2, e_2) = 0.$$

(c) If λ_1 and λ_2 are both non-zero, then we either have

$$m_3(e_1, e_1, e_1) = \lambda_1 e_1 \qquad m_3(e_1, e_1, e_2) = \lambda_2 e_2$$
$$m_3(e_1, e_2, e_2) = 0 \qquad m_3(e_2, e_2, e_2) = 0$$

or we have $\lambda_1 = -\lambda_2$ and some $b \in k$ such that

$$m_3(e_1, e_1, e_1) = -\lambda_2 e_1 \qquad m_3(e_1, e_1, e_2) = \lambda_2 e_2$$
$$m_3(e_1, e_2, e_2) = 0 \qquad m_3(e_2, e_2, e_2) = be_1.$$

Later in chapter 5, we will show from the examples we constructed above, only one of them is simple, up to isomorphism. To do this, we will develop some useful tools and invariants to determine when a 3-Com algebra is simple.

4.2.4 Module n-Com algebras

In this section we will give a construction on any module over a commutative algebra R equipped with a R-module homomorphism into R and construct a n-Com algebra. For this section, we let R be any commutative k-algebra and Mod_R be the category of R-modules.

Definition 4.2.4.1. *Let R be a commutative k-algebra. An augmented R-module is a R-module M equipped with an R-module homomorphism $f : M \to R$. Let Mod_R^{Aug} be the category consiting of augmented R-modules (M, f_M), where a morphism $F : (M, f_M) \to (N, f_N)$ is a R-module homomorphism $F : M \to N$ such that $f_N F = f_M$. Furthermore, there is a forgetful functor $Mod_R^{Aug} \to Mod_R$.*

Example 4.2.4.1. *Here, we will give examples of augmented R-modules, and we will use these later to define examples of n-Com algebras.*

- *Let $M = R[x_1, \ldots, x_n]$ be the polynomial ring over R. Let $a = (a_1, \ldots, a_n) \in R^n$ and define $ev_a : M \to R$ as $ev_a(f(x)) = f(a)$, which is an R-module homomorphism.*

- *Let F be a free module over R with generating set E. For each $x \in E$, there is a unique linear combination $x = \sum_{e \in E} a_e e$ where all but a finite number of the a_e are non-zero. Then we can define $\pi : F \to R$ as $\pi(x) = \sum_{e \in E} a_e$, which is a finite sum and unique. In particular, $F = R$, then $\pi = id_R$. For R-module homomorphisms $R \to R$, these are parameterized by elements $a \in R$ by defining $\sigma_a : R \to R$ as $\sigma_a(b) = ab$.*

- *Suppose M is a R-module and (N, f_N) is an augmented R-module, then we can make $Hom_R(M, N)$ into an augmented $Hom_R(M, R)$-module, where $Hom_R(M, R)$ is a commutative R-algebra with product $(f \cdot g)(a) = f(a)g(a)$, with the augmentation*

$$Hom_R(M, N) \xrightarrow{f_M^*} Hom_R(M, R).$$

- *If (M, f_M) and (N, f_N) are augmented R-modules, then $M \otimes_R N$ is an augmented R-module with augmentation*

$$f_M \otimes f_N : M \otimes_R N \to R \otimes_R R \cong R.$$

- *Let $GL_n(R)$ be the general linear group of $n \times n$ invertable matrices. We can induce the determinant map to a R-algebra homomorphism $det : R[GL_n(R)] \to R$.*

- *Let $M_n(R)$ be the R-algebra of $n \times n$ matrices over R. Then we have the R-module homomorphism $Tr : M_n(R) \to R$, which is just the ordinary trace map.*

- *Given a Hopf algebra $(H, m, u, \Delta, \varepsilon)$ over R, then the counit $\varepsilon : H \to R$ defines a R-module homomorphism.*

If $g : R \to S$ is a morphism of k-algebras, then $- \otimes_R S$ defines a functor from the category Mod_R^{Aug} to Mod_S^{Aug}. Furthermore, submodules of an augmented R-module are augmented in a natural way.

Next, we can use a augmented R-module M to define a n-arity multiplication on M based on the augmented map.

Definition 4.2.4.2. *Let R be a commutative k-algebra and M is an augmented R-module with augmentation $f : M \to R$. Define $\mu_n^f : M^{\otimes n} \to M$ as*

$$\mu_n^f(m_1, \ldots, m_n) = \sum_{i=1}^{n} f(m_1) \cdots \widehat{f(m_i)} \cdots f(m_n) m_i \tag{4.8}$$

for $m_1, \ldots, m_n \in M$ and where $\widehat{f(m_i)}$ means to take it out of the product. We denote by $M(f) = M$ for the space M with the product μ_n^f.

Note that the multiplication μ_n^f is not generally associative, even in the case $n = 2$. But, this is enough to satisfy the n-Com relation as shown next.

Proposition 4.2.4.1. *Let R be a commutative k-algebra and M is a augmented R-module with augmentations $f : M \to R$. The space $(M(f), \mu_n^f)$ is a n-Com algebra. Furthermore, the construction defines a functor $L_R^n : Mod_R^{Aug} \to Alg_{n\text{-}Com}$ sending (M, f) to $(M(f), \mu_n^f)$ and sending a morphism F to $L_R^n(F) = F$.*

Proof. For $m_1, \ldots, m_{2n-1} \in M$ we have

$$\begin{aligned}
&\sum_{\sigma \in C_n} Sgn(\sigma) \mu_n^f(\mu_n^f(m_{\sigma(1)}, \ldots, m_{\sigma(n-1)}, m_n), m_{\sigma(n+1)}, \ldots, m_{\sigma(2n-1)} \\
&\quad = \sum_{\sigma \in C_n} \sum_{i=1}^{n-1} f(m_{\sigma(1)}) \cdots \widehat{f(m_{\sigma(i)})} \cdots f(m_{\sigma(n-1)}) f(m_n) \mu_n^f(m_{\sigma(i)}, m_{\sigma(n+1)}, \ldots, m_{\sigma(2n-1)}) \\
&\quad + \sum_{\sigma \in C_n} sgn(\sigma) f(m_{\sigma(1)}) \cdots f(m_{\sigma(n-1)}) \mu_n^f(m_n, m_{\sigma(n+1)}, \ldots, m_{\sigma(2n-1)}) \\
&\quad = \sum_{\sigma \in C_n} \sum_{i=1}^{n-1} sgn(\sigma) f(m_{\sigma(1)}) \cdots \widehat{f(m_{\sigma(i)})} \cdots f(m_{\sigma(2n-1)}) m_{\sigma(i)} \\
&\quad + \sum_{\sigma \in C_n} \sum_{i=1}^{n-1} \sum_{j=n+1}^{2n-1} sgn(\sigma) f(m_{\sigma(1)}) \cdots \widehat{f(m_{\sigma(j)})} \cdots f(m_{\sigma(2n-1)}) m_{\sigma(j)} \\
&\quad + \sum_{\sigma \in C_n} sgn(\sigma) f(m_{\sigma(1)}) \cdots \widehat{f(m_n)} \cdots f(m_{\sigma(2n-1)}) m_n \\
&\quad + \sum_{\sigma \in C_n} \sum_{j=n+1}^{2n-1} sgn(\sigma) f(m_{\sigma(1)}) \cdots \widehat{f(m_{\sigma(j)})} \cdots f(m_{\sigma(2n-1)}) m_{\sigma(j)}.
\end{aligned}$$

In each of these sums, we can rewrite

$$\sum_{\sigma\in C_n} sgn(\sigma)f(m_{\sigma(1)})\cdots\widehat{f(m_{\sigma(i)})}\cdots f(m_{\sigma(2n-1)})$$
$$= \sum_{\substack{\sigma\in C_n\\ \sigma(i)=i}} sgn(\sigma)f(m_1)\cdots\widehat{f(m_i)}\cdots f(m_{2n-1})m_i$$
$$+ \sum_{\substack{\sigma\in C_n\\ \sigma(i)=n+i}} sgn(\sigma)f(m_1)\cdots\widehat{f(m_{n+i})}\cdots f(m_{2n-1})m_{n+i} = 0.$$

Hence, this gives M a n-Com algebra structure.

For the functorial part, suppose $F : (M, f_M) \to (N, f_N)$, then F defines a n-Com algebra morphism since for any $m_1, \ldots, m_n$ we have

$$\begin{aligned} F\mu_n^{f_M}(m_1,\ldots,m_n) &= \sum_{i=1}^{n} f_A(m_1)\cdots\widehat{f_A(m_i)}\cdots f_A(m_n)F(m_i) \\ &= \sum_{i=1}^{n} f_B(F(m_1))\cdots f_B(\widehat{F(m_i)})\cdots f_B(F(m_n))F(m_i) \\ &= \mu_n^{f_B}(F(m_1),\ldots,F(m_n)). \end{aligned}$$

□

Example 4.2.4.2. • *Let $a \in R$ and let $\sigma_a : R \to R$ be the R-module homomorphism defined in example 4.2.4.1. Then this induces a n-Com algebra structure on R by*

$$\begin{aligned} \mu^{\sigma_a}(r_1,\ldots,r_n) &= \sum_{i=1}^{n}(ar_1)\cdots\widehat{ar_i}\cdots ar_n a_i \\ &= \sum_{i=1}^{n} a^{n-1}r_1\cdots r_n \\ &= na^{n-1}r_1\cdots r_n. \end{aligned}$$

• *Let $n \geq 1$ and let $\pi : R^m \to R$ be the augmentation from example 4.2.4.1. Then R^m*

obtains the n-Com algebra structure

$$\begin{aligned}\mu^{\pi}((a_1^1,\dots,a_m^1),\dots,(a_1^n,\dots,a_m^n)) &= \sum_{i=1}^{n}(\sum_{j_1=1}^{m} a_{j_1}^1)\cdots\widehat{(\sum_{j_i=1}^{m} a_{j_i}^i)}\cdots(\sum_{j_n=1}^{m} a_{j_n}^n)(a_1^i,\dots,a_m^i)\\ &= \sum_{i=1}^{n}\sum_{(j_1,\dots,\widehat{j_i},\dots,j_n)\in[m]^{n-1}} a_{j_1}^1\cdots\widehat{a_{j_i}^i}\cdots a_{j_n}^n(a_1^i,\dots,a_m^i)\end{aligned}$$

- *If (M, f_M) and (N, f_N) are augmented R-modules and $M \otimes_R N$ has its induced augmented R-module structure as in example 4.2.4.1, then it has a n-Com algebra structure with*

$$\begin{aligned}&\mu^{f_M\otimes f_N}((a_1\otimes b_1),\dots,(a_n\otimes b_n))\\ &\quad= \sum_{i=1} f_M(a_1)f_N(b_1)\cdots\widehat{f_M(a_i)}\widehat{f_N(b_i)}\cdots f_M(a_n)f_N(b_n)a_i\otimes b_i.\end{aligned}$$

- *Let $det : R[GL_n(R)] \to R$ be the induced map from the determinant map. Then $R[GL_n(R)]$ has a n-Com algebra structure μ^{det} with*

$$\mu^{det}(A_1,\dots,A_n) = \sum_{i=1}^{n} det(A_1\cdots\widehat{A_i}\cdots A_n)A_i$$

for $A_1,\dots,A_n \in GL_n(R)$. More generally, given any group homomorphism $G \to R^{\times}$, this induces a R-algebra homomorphism $f : R[G] \to R$, and hence we can induce a μ_n^f on $R[G]$ to give it a n-Com algebra structure.

- *Let $Tr : M_n(R) \to R$ be the ordinary trace map and we have a n-Com algebra structure on $M_n(R)$ with*

$$\mu_n^{Tr}(A_1,\dots,A_n) = \sum_{i=1}^{n} Tr(A_1)\cdots\widehat{Tr(A_i)}\cdots Tr(A_n)A_i.$$

- *Let $A = R[x_1,\dots,x_m]$ be the polynomial ring over R and let $p = (p_1,\dots,p_m) \in R^m$. Then $\mathfrak{m}_p = (x_1 - p_1,\dots,x_m - p_m)$ is a maximal ideal, which is the kernel of* *the augmentation map* $ev_p : R[x_1, \dots, x_m] \to R$*. Then we can define* $\mu_n^{ev_p}$ *to give* $R[x_1, \dots, x_m]$ *a n-Com algebra structure with*

$$\mu^{ev_p}(f_1, \dots, f_m) = \sum_{i=1}^{n} f_1(p) \cdots \widehat{f_i(p)} \cdots f_n(p) f_i(x).$$

Note that if $f_1, \dots, f_n$ *are homogeneous polynomials of degree* r*, then* $\mu_n^{ev_p}(f_1, \dots, f_n)$ *is a homogeneous polynomial of degree* r *by definition. So each of the graded components of the polynomial ring are also n-Com algebras.*

Since $ker(ev_p) = \mathfrak{m}_p$*, then* $\mathfrak{m}_p^t \subseteq \mathfrak{m}_p$ *and hence we have the maps* $ev_p : A/\mathfrak{m}_p^t \to R$ *for all* $t \geq 1$*. We can then define* $\widehat{A}_p = \lim_{t\to\infty} A/\mathfrak{m}_A^t$ *which has a R-algebra homomorphism* $ev_p : \widehat{A}_p \to R$ *through the universal property of the limit. Therefore, we can also define an n-Com algebra structure on* $\widehat{A}_p$ *through*

$$\mu^{ev_p}([f_1], \dots, [f_n]) = \sum_{i=1}^{n} f_1(p) \cdots \widehat{f_i(p)} \cdots f_n(p)[f_i].$$

- *Let* M *be any R-module and* (N, f_N) *is an augmented R-module, then with the augmentation structure on* $Hom_R(M, N)$ *over the commutative R-algebra* $Hom_R(M, R)$*, we have the n-Com algebra structure*

$$\mu^{f_N^*}(h_1, \dots, h_n) = \sum_{i=1}^{n} (f_N \circ h_1) \cdots \widehat{f_N \circ h_i} \cdots (f_N \circ h_N) h_i.$$

4.2.5 Quadratic Representation for n-Com$_d$

To construct the operad associated to n-Com algebras of degree d, we will find a certain submodule of $F(H_{n,d}^\vee)(2n-1)$ that is isomorphic to $\uparrow^{2d} S^{(n,n-1)}$, the Specht module with respect to the partition $(n, n-1)$ of $2n-1$. This will give us exactly the relations we want for our n-Com algebras as defined in definition 4.2.1.1. Let $\mathfrak{L}_{n,d} = F(H_{n,d})^{(2)}(2n-1)$ and $S_{n,d}$ be a sub $k[\Sigma_{2n-1}]$-module of $\mathfrak{L}_n$ generated by

$$s_{n,d} = \sum_{\sigma \in C_{T_\lambda}} Sgn(\sigma) u^{\sigma}_{\{n+1,\dots,2n-1\}} = \sum_{\sigma \in C_{T_\lambda}} Sgn(\sigma) u_{\{\sigma^{-1}(n+1),\dots,\sigma^{-1}(2n-1)\}}$$

which is in degree $2d$.

lemma 4.2.5.1. *The relation s_n defined in above is exactly the relations for n-Com algebras.*

Proof. Let $x_1, \ldots, x_{2n-1}$ be formal graded elements in which μ acts on. Then we have

$$\begin{aligned} s_{n,d}(x_1, \ldots, x_{2n-1}) &= \sum_{\sigma \in C_{T_\lambda}} Sgn(\sigma) u^{\sigma}_{[n+1,\ldots,2n-1]}(x_1, \ldots, x_{2n-1}) \\ &= \sum_{\sigma \in C_{T_\lambda}} Sgn(\sigma) \xi(\sigma, x_1, \ldots, x_{2n-1}) u_{[n+1,\ldots,2n-1]}(x_{\sigma(1)}, \ldots, x_{\sigma(2n-1)}) \\ &= \sum_{\sigma \in C_{T_\lambda}} Sgn(\sigma) \xi(\sigma, x_1, \ldots, x_{2n-1}) \mu(\mu(x_{\sigma(1)}, \ldots, x_{\sigma(n)}), x_{\sigma(n+1)}, \ldots, x_{\sigma(2n-1)}). \end{aligned}$$

This gets us exactly the relations for n-Com algebras. □

Since $S_{n,d}$ gives us exactly the relations for a n-Com algebra of degree d, we can define the following operad to represent these algebras.

Definition 4.2.5.1. *Define the n-Com_d operad to be the quadratic operad $SMag_{n,d}/(S_{n,d})$.*

From our intuition so far about n-Com_d having relations coming from a Standard Young Tableau, we should expect a relation with the irreducible representation corresponding to the Young diagram, in fact, we will show $S_{n,d} \cong \uparrow^{2d} S^{(n,n-1)}$.

Let $n \geq 2$ and choose any Young tableau T of shape $(n, n-1)$, for each (i,j)th cell of T, let $T_{(i,j)}$ represent the number in that cell. Let $M^{(n,n-1)}$ represent the right $k[\Sigma_{2n-1}]$-module generated by all $(n, n-1)$-tabloids. Define $\Phi_{n,d} : \uparrow^{2d} M^{(n,n-1)} \to \mathfrak{L}_{n,d}$ by taking a $(n, n-1)$-tabloid $\uparrow^{2d} \{T\}$ and sending it to

$$\Phi(\uparrow^{2d} \{T\}) = u_{\{T_{(1,2)},\ldots,T_{(n-1,2)}\}},$$

which is well-defined for any representative $S \in \{T\}$ since they are equivalent to any permutation of the rows in the Young Tableau. This gives an isomorphism between $\uparrow^{2d} M^{(n,n-1)}$ and $\mathfrak{L}_{n,d}$ as right $k[\Sigma_{2n-1}]$-modules through the generators.

Next, take $\uparrow^{2d} e_{T_n} = \sum_{\sigma \in C_{T_n}} Sgn(\sigma) \uparrow^{2d} \{T_n \sigma\}$ be the shifted $(n, n-1)$-polytabloid corresponding to T_n as in the standard Young tableau in figure 4.2 and apply $\Phi_{n,d}$ on to this element to obtain

$$\begin{aligned}\Phi\left(\uparrow^{2d} e_{T_n}\right) &= \sum_{\sigma\in C_{T_n}} Sgn(\sigma)\Phi(\uparrow^{2d}\{T_n\sigma\}) \\ &= \sum_{\sigma\in C_{T_n}} Sgn(\sigma)\Phi(\uparrow^{2d}\{T_n\})^{\sigma} \\ &= s_{n,d}.\end{aligned}$$

This induces an isomorphism between $S^{(n,n-1)}$ and $S_{n,d}$ using the generators, and by the hook length formula we have $\dim S_{n,d} = \dim S^{\lambda_n} = C_n$, where C_n is the nth Catalan number.

By lemma 1.3.3.2, the space $M^{(n,n-1)}$ decomposes into irreducible representations as

$$M^{(n,n-1)} \cong S^{(n,n-1)} \oplus \bigoplus_{(n,n-1)\lhd\mu} k_{\mu,(n,n-1)} S^{\mu}.$$

The only partitions μ of $2n-1$ with $\mu \rhd (n,n-1)$ are the partitions

$$\mu_{n,k} = (n+k, n-k-1)$$

for $1 \le k \le n-1$, where if there is a 0 in the partition we ignore it. Recall from section 1.3.3.2, the Kostka numbers $K_{\mu,\lambda}$ for partitions μ and λ of n is the number of semistandard Young tableaux of shape μ and content λ. Therefore, the number of semistandard Young tableaux of shape $\mu_{n,k}$ with content $(n,n-1)$ for $0 \le k \le n-1$ is $K_{\mu_{n,k},(n,n-1)} = 1$ by a simple observation. Therefore, we have

$$M^{(n,n-1)} \cong \bigoplus_{k=0}^{n-1} S^{\mu_{n,k}}$$

which gives us an explicit decomposition of $\mathfrak{L}_{n,d}$ through Φ_n as

$$\mathfrak{L}_{n,d} = \bigoplus_{k=0}^{n-1} S^{k}_{n,d}$$

where $S^0_{n,d} = S_{n,d} \cong S^{(n,n-1)}$ and $S^k_{n,d} \cong S^{\mu_{n,k}}$ for $1 \le k \le n-1$.

The above decomposition shows that we can define even more operads through the other irreducible representations based on $S^k_{n,d}$, but in general it is unclear what these relations would be except for some special cases.

Definition 4.2.5.2. *For $n \geq 2$, $0 \leq k \leq n-1$, and $d \in \mathbb{Z}$, let n-$Com_d^{\mu_{n,k}}$ be the operad $F(H_{n,d})/(S^k_{n,d})$. Note n-$Com_d^{\mu_{n,0}} = n$-Com_d.*

For an example with $\mu_{n,n-1} = (2n-1)$, suppose we take $s^{n-1}_{n,d} = \sum_{a_1<\cdots<a_{n-1}} u_{\{a_1,\ldots,a_{n-1}\}}$ which gives us a 1-dimensional sub-representation which is equal to $S^{n-1}_{n,d}$. Therefore, we have $S^{n-1}_{n,d}$ generated by $s^{n-1}_{n,d}$ defined above, and this gives us a way to describe the algebras over the operad n-$Com_d^{(2n-1)}$. A better way to describe the relation $s^{n-1}_{n,d}$ is as follows: since we want to essentially permute through all possible distinct sets $\{a_1, \ldots, a_{n-1}\}$ such that $a_1 < \cdots < a_{n-1}$, this is the same as looking at all $\{\sigma(n+1), \ldots, \sigma(2n-1)\}$ for all $\sigma \in Sh(n, n-1)$. Since being even or an odd permutation does not produce any signs then we can rewrite the relation as

$$\sum_{\sigma\in Sh(n,n-1)} u_{\{\sigma(n+1),\ldots,\sigma(2n-1)\}},$$

which gives us a way to write the relation for the algebras.

Definition 4.2.5.3. *An n-Com algebra of type $\mu_{n,n-1} = (2n-1)$ of degree d is a graded k-module C with a degree d symmetric operation $m_n : C^{\otimes n} \to C$ such that*

$$\sum_{\sigma\in Sh(n,n-1)} \xi(\sigma, v_1, \ldots, v_n) m_n(m_n(v_{\sigma(1)}, \ldots, v_{\sigma(n)}), v_{\sigma(n+1)}, \ldots, v_{\sigma(2n-1)}) = 0$$

More generally, can we find a description for the algebras over n-$Com_d^{\mu_{n,k}}$ for any $0 \leq k \leq n-1$?

More generally, for any $\mu_{n,k}$, we can define $S^{\mu_{n,k}}_{n,d}$ to be the sub $k[\Sigma_{2n-1}]$ module of $\mathfrak{L}_n$ generated by

$$s^{\mu_{n,k}}_{n,d} = \sum_{\sigma\in C_{n,k}} Sgn(\sigma) u^{\sigma}_{\{n+1,\ldots,2n-1\}}$$

Chapter 5

EIGEN-ALGEBRAS AND SIMPLE 3-COM ALGEBRAS

To study simple finite dimensional associative k-algebra, one usually finds idempotents in the space and uses it to try to decompose the underlying algebra into smaller pieces. In particular, the classical non-commutative simple k-algebras are the matrix k-algebras, which has a basis of orthogonal idempotents, and one can show that if an algebra has such a basis, then it is isomorphic to a matrix algebra.

For the non-associative world, one can still define idempotents, but it becomes a lot harder to use the decomposition of the space as the lack of associativity makes it harder to define 'nice' ideals and subalgebras. A central tool called the Peirce decomposition is used to study non-associative algebras and extract useful information for them. These were used effectively for Jordan algebras to give a general description for their decomposition and determine when they are simple, see [5]. There, they were able to show that Jordan algebras can contain only the Peirce numbers $0, \frac{1}{2}, 1$ and the eigenspace corresponding to $\frac{1}{2}$ determines when a Jordan algebra is simple or not. Furthermore, one can study the geometry of the underlying space of idempotents as in [14], which can help determine how many idempotents a 'generic' non-associative commutative k-algebras one has.

Here, we will define simple n-Com algebras, idempotents, ideals, and the Peirce decomposition in this case. We will then focus on finite-dimensional 3-Com algebras and use the Peirce decomposition to extract some important invariants and conditions for when they are simple. The Peirce decomposition for 3-Com algebras has a particular algebraic structure that is almost commutative associative up to some 3-arity factor and contains commutative associative unital k-algebras coming from the eigenvalue 1 for the corresponding idempotent. Furthermore, we briefly talk about non-degenerate 3-Com algebras and show that we can define a symmetric bilinear form on 3-Com algebras that is associative with respect to the 3-arity multiplication; this is the analog of the Killing form on Lie algebras. Furthermore, we define a non-degenerate 3-Com algebra to be such that the bilinear form is non-degenerate and show that we can decompose such a 3-Com algebra as a direct product of simple 3-Com algebras.

5.1 Eigen-Algebras

In lots of finite dimensional structures, we would like to explore the eigenvalues coming from the endomorphisms induced by the elements in that space. This provides a tractable way to understand the underlying structure by decomposing the space into its eigenspaces. In this section, we will define a commutative structure that is associative up to some 3-arity function. Later in section 5.2, we show that eigen-algebras arise naturally from 3-Com algebras, which gives us a binary operation one can potentially use to study algebra.

Definition 5.1.0.1. *A eigen-algebra is a graded space $A = \bigoplus_{i \in k} A_i$, where we denote $\alpha(x) = i$ for homogenous elements $x \in A_i$, equipped with a symmetric product $m : A \otimes A \to A$, a symmetric 3-arity map $f : A^{\otimes 3} \to A$, and a element $e \in A_1$ satisfying the following properties.*

- *The product is graded such that for $i, j \in k$, we have $m(A_i, A_j) \subseteq A_{i+j-1}$.*
- *The element is a "eigen"-unit in the sense*

$$m(e, a) = \alpha(a)a$$

  for any homogenous element $a \in A$.

- *The product m is associative up to a affine factor of f in the following way: if $x, y, z \in A$ are homogenous elements, then*

$$m(m(x, y), z) - m(x, m(y, z)) = (\alpha(x) - \alpha(z))f(x, y, z).$$

- *Finally, the function f satisfies the following compatibility condition:*

$$m(f(x_1, x_2, x_3), x_4) - m(x_1, f(x_2, x_3, x_4)) = f(m(x_1, x_3), x_2, x_4) - f(x_1, x_2, m(x_3, x_4))$$

  for all elements $x_1, x_2, x_3, x_4 \in A$.

We denote eigen-algebras as (A, m, e, f).

From the definitions, we can see that we can define an endomorphism $\mu_e = (e, -)$ and we can see that A_i are the eigenspaces concerning the eigenvalue i; this is the reasoning for the name on e.

lemma 5.1.0.1. *If (A, m, e, f) is a eigen-algebra, then A_1 is a unital commutative associative k-algebra. Furthermore, A_i has a A_1-action, which is associative up to some factor of f:*

$$m(m(x, y)h) = m(x, m(y, h)) + (1 - \alpha(h))f(x, y, h)$$

for $x, y \in C_e(1)$ and $h \in A_i$.

Proof. First, by definition, $m(A_1, A_1) \subseteq A_1$ and hence m can be restricted to a well-defined binary operation on A_1. Furthermore, if $x, y, z \in A_1$, then we have

$$m(m(x, y), z) - m(x, m(y, z)) = (1 - 1)f(x, y, z) = 0$$

and this shows A_1 is associative. Furthermore, every element of A_1 is an eigenvector of e with eigenvalue 1, so hence it is a unit. The action of A_1 on to A_i is clear, and the associativity up to some factor comes from the relations. □

For our application towards studying simple 3-Com algebras, we want to study the ideals of the eigen-algebras and their properties when they are simple. But, because of the extra map f, there are two types of ideals that are of interest to us. An ideal of an eigen-algebra (A, m, e, f) is a subspace I such that $m(a, b) \in I$ if $b \in I$ for all $a \in A$, which is the traditional definition. An f-ideal is a subspace I such that I is an ideal, and it has $f(a, b, c) \in I$ if $a \in I$ for $b, c \in A$. Note that every f-ideal is an ideal by definition, but it does not go the other way, as we can show later. Furthermore, we can descend ideals from A to its underlying commutative algebra A_1 by just taking the intersection: if I is an ideal of A, then $I \cap A$ is an ideal of A using the grading of A.

Let S be a subset of an eigen-algebra (A, m, e, f) and define $\langle S \rangle_f$ to be the subspace spanned by the elements

$$m(s,a), f(s,a,b), f(m(s,a),b,c), \ldots, f(m(m(\ldots m(s,a_1),a_2),\ldots,a_n),b).$$

for $s,a,b,c,a_1,\ldots,a_n \in A$ for any $n \geq 1$. The subspace $\langle S\rangle$ is an f-ideal since for every $s \in S$ and $a,b,c \in A$, we have

$$m(m(s,a),b) = m(s,m(a,b)) + (\alpha(s) - \alpha(b))f(s,a,b) \in \langle S\rangle$$

and

$$m(f(s,a,b),c) = m(s,f(a,b,c)) + f(m(s,b),b,c) - f(s,a,m(b,c)) \in \langle S\rangle$$

for any $s,a,b,c \in A$, and the other cases come from a similar relation. On the other hand, let $\langle S\rangle$ be the subspace of A spanned by the elements

$$m(m(\cdots m(m(s,a_1),a_2),\ldots,a_{n-1}),a_n)$$

for $s \in S$ and $a_1,\ldots,a_n \in A$ and any $n \geq 1$. It is clear $\langle S\rangle$ is an ideal, and normally it is not a f-ideal since we won't necessarily contain $f(s,a,b)$ if $s \in I$ and s,a,b are all of the same degrees.

Definition 5.1.0.2. *We say an eigen-algebra (A,m,e,f) is simple if its multiplication m and f are not-trivial and it does not have any non-trivial ideals (neither the 0 ideal nor the whole space). We say it is f-simple if m and f are not-trivial and have no non-trivial f-ideals.*

We can see that an eigen-algebra A is simple, implying it is f-simple since every f-ideal is an ideal. Simple eigen-algebras are important to us since if there is a 3-Com algebra connected to one, as we will construct in the next section, the property that A is simple implies that the 3-Com algebra is simple. But, in the converse direction, it will only imply f-simple.

5.2 Simples n-Com algebras and Ideals

As with any algebraic structure, one would like to quotient the objects by equivalence relations to obtain new objects with the same algebraic structure. Furthermore, we would like to determine which of these algebraic objects can not be quotient into a smaller non-trivial algebra. Here, we will define ideals for n-Com algebras in an obvious way and define when a n-Com algebra is simple. In the latter half of this section, we will define the Peirce decomposition for n-Com algebra and idempotents and give a few results of these structures.

Definition 5.2.0.1. *Let C be a n-Com algebra with n-arity multiplication m_n. A n-Com ideal of C, or just an ideal, is a subspace I of C such that $m_n(c_1, \ldots, c_n) \in I$ if and only if one of the $c_i \in I$ for some i.*

For any two ideals I and J of an n-Com algebra, the intersection $I \cap J$ and the sum $I + J$ are both ideals in C. Furthermore, the usual isomorphism theorems hold that we can take the quotient C/I as an ideal to obtain a new n-Com algebra in our context. We say that an ideal of a n-Com algebra C is trivial if it is the 0 ideal or equal to all of C.

Since n-Com algebras are usually not associative for $n > 2$, a lot of the ideal constructions that use associativity are not as nice. For example, let C be a n-Com algebra and S is a subset of C, then we define $\langle S \rangle$ as the intersection of all ideals that contain S. In this case, this will have products of the form $m_n(m_n(\cdots(m_n(a, b_1, \ldots, b_{n-1}), \ldots,), \ldots), \ldots,)$ which will not necessarily simplify down to something of the form $m_n(x, y_1, \ldots, y_{n-1})$ for $x \in S$.

But, for any n-Com algebra C, we can always define the ideal $m_n(C, \ldots, C)$, consisting of finite sums of elements of the form $m_n(c_1, \ldots, c_n)$ for $c_i \in C$. But, replacing C with any other ideals does not work in general, except for some special cases. For example, suppose C is a 3-Com algebra and I,J are both ideals of C and define $m_3(I, J, C)$ be the linear span of elements of the form $m_n(a, b, x)$ for $a \in I$, $b \in J$ and $x \in C$. This becomes an ideal since

$$\begin{aligned} m_3(m_3(a, b, x), y_1, y_2) &= m_3(m_3(y_1, b, x), a, y_2) + m_3(m_3(a, y_2, x), y_1, b) \\ &\quad - m_3(m_3(y_1, y_2, x), a, b) \in m_3(I, J, C). \end{aligned}$$

If we were to increase 3 to any $n > 3$, this would not work in general as the transpositions actions will make it not possible for both I and J to be in separate spots.

Definition 5.2.0.2. *An n-Com algebra C with n-arity multiplication m_n is simple if $m_n(C, \ldots, C) \neq 0$ and C has no non-trivial ideals.*

For examples of simple n-Com algebras when $n > 2$, we can always take the 1-dimensional n-Com algebra k with multiplication $m_n(1, \ldots, 1) = \lambda$ for any $\lambda \in k^\times$ and by dimension there is no other ideals. The following example is an example of a 2-dimensional simple 3-Com algebra.

Example 5.2.0.1. *Let V be a 2-dimensional 3-Com algebra with basis elements e_1, e_2, $\lambda, b \in k^\times$ and multiplication*

$$m_3(e_1, e_1, e_1) = \lambda e_1 \qquad m_3(e_1, e_1, e_2) = -\lambda e_2$$
$$m_3(e_1, e_2, e_2) = 0 \qquad m_3(e_2, e_2, e_2) = be_1.$$

This is a 3-Com algebra by 4.2.3.3 and we will show it is simple. If I is an ideal of V that contains either e_1 or e_2, then it must contain both using the operations above and hence be equal to V. Next, suppose $ae_1 + be_2 \in I$ such that $a, b \neq 0$, then we have

$$m_3(ae_1 + be_2, e_1, e_2) = -\lambda ae_2$$

which implies $e_2 \in I$ since $\lambda a \neq 0$. Therefore, I must be all of C, which shows that V is simple.

For $n = 2$, the simple 2-Com algebras are exactly the fields over k which is implied by the fact that every a has a unique b such that $ab = 1$.

lemma 5.2.0.1. *If C is a finite-dimensional n-Com algebra with basis $e_1, \ldots, e_m$, then for each i there exists non-zero elements $a_1^i, \ldots, a_n^i \in C$ such that*

$$e_i = m_n(a_1^i, \ldots, a_n^i).$$

Proof. Since C is simple, then $m_3(C,\dots,C)$ is a non-zero ideal and hence $m_3(C,\dots,C) = C$. This implies the result. $\square$

An idempotent for commutative associative algebras is an element $c \in C$ such that $c^2 = c$, and we say it is 2-nilpotent if $c^2 = 0$. These notations can be naturally generalized to n-Com algebras as follows.

Definition 5.2.0.3. *Let C be a n-Com algebra and let $(c_1,\dots,c_{n-1})$ be a $n-1$-tuple of elements in C. We say $(c_1,\dots,c_{n-1})$ is an idempotent tuple if*

$$m_n(c_1,\dots,c_{n-1},c_i) = c_i$$

for any $i \in \{1,\dots,n-1\}$. In particular, if $c_1 = \cdots = c_{n-1} = c$, then we say c is idempotent if $m_n(c,\dots,c) = c$. We denote by $Idm(C)$ to be the set of non-zero tuples $(c_1,\dots,c_{n-1})$ that are idempotent on C, and let $Idm_1(C)$ to be the set of $x \in C$ that are idempotent, which can be thought of as a subset of $Idm(C)$ through $x \mapsto (x,\dots,x)$.

On the other hand, we say $(c_1,\dots,c_{n-1})$ is n-nilpotent if

$$m_n(c_1,\dots,c_{n-1},c_i) = 0$$

for any $i \in \{1,\dots,n-1\}$. Furthermore, if $c_1 = \cdots = c_{n-1} = c$, then we say c is n-nilpotent if $m_n(c,\dots,c) = 0$. We denote by $Nil(C)$ be the collection of elements $c \in C$ which are n-nilpotent.

If $c = (c_1,\dots,c_{n-1})$ is a $n-1$ tuple of elements in C, then as before, we define $\chi_c = \chi_{c_1,\dots,c_{n-1}} = m_n(c_1,\dots,c_{n-1},-)$, which is an endomorphism of the space C. If $c = (x,\dots,x)$ then we define $\chi_x := \chi_{x,\dots,x}$ for simplicity. If C is finite-dimensional, then we can compute the eigenvalues and the eigenspaces for each endomorphism as usual. Define $ch_c(t) = \det(\chi_c - t\mathrm{I})$ for $t \in k$ to be the characteristic polynomial of the endomorphism χ_c for any $n-1$ tuple $c = (c_1,\dots,c_{n-1})$. Furthermore, we define $\sigma(c)$ to be the set of roots of the polynomial $ch_c(t)$ counting multiplicity, which is called the **Peirce spectrum**, and we call the distinct numbers of $\sigma(c)$ the **Peirce numbers** of c, and we denote the set of them as $P_\sigma(c)$. Furthermore, we define

$$C_c(t) := \ker(\chi_c - t\mathrm{I})$$

called the **Peirce subspace**, which is just the eigenspace if t is a eigenvalue of χ_c. For $x \in C_c(t)$, we define $\alpha(x) = c$ the Peirce subspace it is contained in. We will assume that our n-Com algebra C is finite-dimensional to use all of our linear algebraic tools. Furthermore, in practice, the ones that are more useful for us are the endomorphisms $\chi_e = m_n(e, \dots, e, -)$ for a single element $e \in C$, as the others are a lot harder to calculate.

Suppose e is a non-zero idempotent in C, then e is a eigenvector of the endomorphism χ_e with eigenvalue 1, so that $1 \in \sigma(e)$. On the other hand, if f is a n-nilpotent element then f is an eigenvector of χ_f with eigenvalue 0 and hence $0 \in \sigma(f)$. If e is idempotent (f is n-nilpotent) and $C_e(1)$ is 1-dimensional ($C_f(1)$ is 1-dimensional) , then we call e **primitive** (f primitive). On the other hand, if e is a non-zero idempotent such that the only Peirce number for e is 1, then we call e **unipotent**. Let $\sigma(C)$ be the set of all Peirce numbers of C concerning only idempotent $e \in \mathrm{Idm}_1(C)$ of C, and hence we will always have $1 \in \sigma(C)$ if there exists a non-zero idempotent. We call a non-zero idempotent $e \in \mathrm{Idm}_1(C)$ **semisimple** if C is decomposable as the sum of the corresponding Peirce subspaces:

$$C = \bigoplus_{\lambda} C_e(\lambda)$$

where λ are the Peirce numbers of e. We also say a non-zero n-nilpotent f is semisimple if it decomposes the space in a similar fashion with respect to its Peirce numbers.

If C and D are both n-Com algebra that are isomorphic through $F : C \to D$, then idemopotents e of C give us idempotents $F(e)$ of D such that $\sigma(e) = \sigma(F(e))$ and therefore $\sigma(C) = \sigma(D)$. This shows that the collection of idempotents and their corresponding Peirce spectrums give an invariant for the n-Com algebras. All n-Com algebras might have the same collection of Peirce numbers with respect to an idempotent, in which case it will be easier to study the spaces. This happens for other non-associative algebras, like Jordan algebras, which can only have Peirce numbers $0, 1, \frac{1}{2}$, and they can obtain information about the space using $\frac{1}{2}$. The following examples show that we can get drastically different Peirce numbers for different n-Com algebras, but we can use this to our advantage to give a good invariant for n-Com algebras and use these to determine when they are simple.

Example 5.2.0.2. *Lets look back at the 2-dimensional simple 3-Com algebra from example 5.2.0.1. In this case, there are only two idempotent elements*

$$c_1 = \frac{1}{\lambda^{1/2}} e_1$$
$$c_2 = \frac{1}{i\sqrt{3}\sqrt{\lambda}} e_1 + \frac{4^{1/3}}{3i^{2/3}\lambda^{1/3}b^{2/3}} e_2$$

which one can show by finding all the possible $p, q \in \mathbf{C}$ such that $m_3(pe_1+qe_2, pe_1+qe_2, pe_1+qe_2) = pe_1 + qe_2$. For c_1, it has eigenvector c_1 with eigenvalue 1 and eigenvector e_2 with eigenvalue -1. On the other hand, χ_{c_2} has the matrix

$$\chi_{c_2} = \begin{pmatrix} -1/3 & \frac{4^{2/3}b^{1/3}}{3i^{2/3}\lambda^{1/3}} \\ -\frac{2\cdot 4^{1/3}\lambda^{1/3}}{i^{4/3}3b^{1/3}} & 1/3 \end{pmatrix}$$

which has eigenvalues 1 and -1. Therefore, both c_1 and c_2 are semisimple idempotents so that

$$C = C_{c_1}(1) \oplus C_{c_1}(-1) = C_{c_2}(1) \oplus C_{c_2}(-1)$$

and $\sigma(C) = \{1, -1\}$.

Example 5.2.0.3. *Here is another example of a 2-dimensional 3-Com algebra that is not simple. Let C be 2-dimensional with basis e_1, e_2 and numbers $\lambda_1, \lambda_2 \in \mathbf{C}$, and define*

$$m_3(e_1, e_1, e_1) = \lambda_1 e_1 \qquad m_3(e_1, e_1, e_2) = \lambda_2 e_2$$
$$m_3(e_1, e_2, e_2) = 0 \qquad m_3(e_2, e_2, e_2) = 0$$

which is not simple since $\langle e_2 \rangle$ does not contain e_1. Suppose $pe_1 + qe_2$ is a non-trivial idempotent element of C, then we must have

$$p^3\lambda_1 = p$$
$$3p^2 q\lambda_2 = q$$

If $\lambda_2/\lambda_1 \neq 1/3$, then C has only one non-trivial idempotent which is

$$c = \frac{1}{\lambda_1^{1/2}} e_1$$

with eigenvalues 1 and λ_2/λ_1 and it is semisimple and hence $\sigma(C) = \{1, \lambda_2/\lambda_1\}$ in this case. We can always choose $\lambda_1 = -\lambda_2$ to get $\sigma(C) = \{1, -1\}$ which shows that having -1 in $\sigma(C)$ is not enough to make it simple. But, we will see later that this is a singularity, i.e., from example 5.2.0.2 for the 3-Com algebra in this situation, if we let $b = 0$, then we get this example.

If $\lambda_2/\lambda_1 = 1/3$, then the non-trivial idempotents are

$$c_1 = \frac{1}{\lambda_1^{1/2}} e_1$$
$$c_2 = \frac{1}{\lambda_1^{1/2}} e_1 + q e_2$$

for any non zero $q \in \mathbf{C}$. Both χ_{c_1} and χ_{c_2} have eigenvalues 1 and 1/3 and hence shows $\sigma(C) = \{1, 1/3\}$ in this case.

Example 5.2.0.4. *Let (M, f) be a $m \geq 2$ dimensional augmented R-module and let $(M(f), \mu_n^f)$ be its corresponding n-Com algebra from section 4.2.4. An idempotent e of $M(f)$ is one such that*

$$\mu_n^f(e, \ldots, e) = n f(e)^{n-1} e = e$$

so that

$$f(e) = \frac{1}{n^{\frac{1}{n-1}}}.$$

Since M is finite dimensional, then we have ker(f) has dimension $m-1$ and we can find a basis $e, e_1, \ldots, e_{m-1}$ in M such that $e_1, \ldots, e_{m-1}$ is a basis for the kernel of f and $f(e) = \frac{1}{n^{\frac{1}{n-1}}}$. In this case, the matrix for χ_e is a diagonal matrix with 1 1 and $m-1$ instances of $\frac{1}{n}$. Therefore, $\sigma(e) = \{1, \frac{1}{n}, \ldots, \frac{1}{n}\}$. This is easily shown to be not simple because the ideal ker(f) is not trivial. Later, we can use a quick method to determine if it's not simple because the negative of an eigenvalue is not in $\sigma(e)$.

From all of the examples we define above, we always had a semisimple idempotent which we can use to decompose the space. But, in general, semisimple non-zero idempotents will not always exist, as follows from the last example.

Example 5.2.0.5. *Let V be the 2-dimensional 3-Com algebra with basis and multiplication*

$$m_3(e_1, e_1, e_1) = 0 \qquad m_3(e_1, e_1, e_2) = e_2$$
$$m_3(e_1, e_2, e_2) = 0 \qquad m_3(e_2, e_2, e_2) = 0.$$

suppose $ae_1 + be_2$ is an idempotent, then we would have

$$m_3(ae_1 + be_2, ae_1 + be_2, ae_1 + be_2) = 3a^2be_2 = ae_1 + be_2$$

which implies $a = 0$ and hence $b = 0$ which would show it has to be zero. Therefore, there do not exist non-zero idempotents in this 3-Com algebras, but we have two non-zero 3-nilpotents e_1 and e_2 so we have

$$C = C_{e_1}(0) \oplus C_{e_1}(1)$$
$$C = C_{e_2}(0).$$

Next, let's study some properties of the idempotents and the corresponding Peirce subspaces on 3-Com algebras.

lemma 5.2.0.2. *Let C be a finite dimensional 3-Com algebra with multiplication m_3 and suppose e is an idempotent of C.*

- *If $x_1, x_2, x_3 \in C$ such that $x_i \in C_e(t_i)$ for $1 \leq i \leq 3$, then we have the equation*

$$m_3(m_3(x_3, e, x_1), e, x_2) - m_3(m_3(x_3, e, x_2), e, x_1) = (t_1 - t_2)m_3(x_1, x_2, x_3).$$

- *For $x_1 \in C_e(t_1)$ and $x_2 \in C_e(t_2)$ we have the equation*

$$\chi_e(m_3(x_1, x_3, e)) = (t_1 + t_3 - 1)m_3(x_1, x_3, e)$$

  so that $m_3(x_1, x_3, e) \in C_e(t_1 + t_3 - 1)$.

- *Finally, for $x_1, x_2, x_3 \in C$ such that $x_i \in C_e(t_i)$ for all i we have*

$$\chi_e(m_3(x_1,x_2,x_3)) + t_1 m_3(x_1,x_2,x_3) = m_3(e,x_2,m_3(x_1,e,x_3)) + m_3(e,x_3,m_3(x_1,x_2,e))$$

where $m_3(e,x_2,m_3(x_1,e,x_3))$ and $m_3(e,x_3,m_3(x_1,x_2,e))$ are elements of $C_e(t_1+t_2+t_3-2)$.

Proof. First, by definition of 3-Com algebra we have

$$\begin{aligned} m_3(e,e,m_3(x_1,x_2,x_3)) &= m_3(x_1,e,m_3(e,x_2,x_3)) + m_3(e,x_2,m_3(x_1,e,x_3)) - m_3(x_1,x_2,m_3(e,e,x_3)) \\ &= m_3(x_1,e,m_3(e,x_2,x_3)) + m_3(e,x_2,m_3(x_1,e,x_3)) - t_3 m_3(x_1,x_2,x_3) \\ &= m_3(x_2,e,m_3(e,x_1,x_3)) + m_3(x_1,x_3,m_3(e,e,x_2)) - m_3(x_2,x_3,m_3(e,e,x_1)) \\ &+ m_3(e,x_2,m_3(x_1,e,x_3)) - t_3 m_3(x_1,x_2,x_3) \\ &= 2m_3(e,x_2,m_3(e,x_1,x_3)) + (t_2-t_1-t_3) m_3(x_1,x_2,x_3). \end{aligned}$$

Permuting the variables, we also get the equation

$$\chi_e(m_3(x_1,x_2,x_3)) = 2m_3(e,x_1,m_3(e,x_2,x_3)) + (t_1-t_2-t_3) m_3(x_1,x_2,x_3)$$

and taking the difference between these two equations gets us

$$\begin{aligned} 0 = {} & 2m_3(e,x_2,m_3(e,x_1,x_3)) + (t_2-t_1-t_3) m_3(x_1,x_2,x_3) - 2m_3(e,x_1,m_3(e,x_2,x_3)) \\ & - (t_1-t_2-t_3) m_3(x_1,x_2,x_3) \end{aligned}$$

which gives us the equation

$$m_3(e,x_2,m_3(e,x_1,x_3)) - m_3(e,x_1,m_3(e,x_2,x_3)) = (t_1-t_2) m_3(x_1,x_2,x_3)$$

since the t_3 will cancel out. This proves the first equation.

Now if we were to pick $x_2 = e$, then this equation would reduce to

$$\begin{aligned} m_3(e,e,m_3(e,x_1,x_3)) &= m_3(e,x_1,m_3(e,e,x_3)) + (t_1-1) m_3(x_1,x_2,x_3) \\ &= (t_1+t_3-1) m_3(e,x_1,x_3) \end{aligned}$$

which gives us the second equation.
Finally, for the last, this is just an application of $m_3(e, e, m_3(x_1, x_2, x_3))$ and using the relation for 3-Com algebras. □

One thing to note from the last lemma is that this enables us to define a binary operation on C as follows. Suppose e is an idempotent of C and define $\theta_e : C \otimes C \to C$ as

$$\theta_e(x_1, x_2) = m_3(e, x_1, x_2)$$

for $x_1, x_2 \in C$, which is commutative. For call elements $x \in C_e(t)$ for some $t\sigma(e)$ a homogenous element of C. We will also define a function α on homogenous elements $x \in C_e(t)$ defined as $\alpha(x) = t$ to make the equations we derive more easily readable.

Proposition 5.2.0.3. *Let C be a finite dimensional 3-Com algebra with a idempotent e. We have the following properties:*

(a) The space $C_e(1)$ is a unital commutative associative k-algebra with unit e and multiplication θ_e. In particular, if $C_e(1)$ is 1-dimensional, then $C_e(1) \cong k$.

(b) The spaces $C_e(t)$ for $t \in k$ have $C_e(1)$-actions with $\mu_e^t(x, b) = m_3(e, x, b)$ with $x \in C_e(1)$ and $b \in C_e(t)$ with the relation

$$\mu_e^t(\theta_e(x, z), b) = \mu_e^t(x, \mu_e^t(z, b)) + (1 - t)m_3(x, b, z)$$

for $x, z \in C_e(1)$ and $b \in C_e(t)$.

(c) For homogenous element $z \in C$ and elements $x, y \in C$ we have

$$\chi_e(m_3(x, y, z)) + \alpha(z)m_3(x, y, z) = \theta_e(\theta_e(x, z), y) + \theta_e(x, \theta_e(z, y)).$$

(d) If $t, s \in \sigma(e)$ such that $t + s - 1 \notin \sigma(e)$, then

$$m_3(e, x, y) = 0$$

for $x \in C_e(t)$ *and* $y \in C_e(s)$*. Even more, if* $t, s, r \in \sigma(e)$ *such that* $t+s-1, t+r-1 \notin \sigma(e)$*, then*

$$m_3(x, y, z) \in C_e(-t)$$

for $x \in C_e(t)$*,* $y \in C_e(s)$*, and* $z \in C_e(r)$*.*

(e) If $x \in C_e(t)$*,* $y \in C_e(s)$*, and* $Z \in C_e(r)$ *and* $t+s+r-2 \notin \sigma(e)$*, then*

$$m_3(x, y, z) \in C_e(-t) \cap C_e(-s) \cap C_e(-r).$$

If either of the t*,* s*, or* r *are not equal to each other then* $m_3(x, y, z) = 0$*, other wise if* $t = s = r$ *then* $m_3(x, y, z) \in C_e(-t)$*.*

(f) If e *is semisimple, then* $C = \bigoplus_{t \in k} C_e(t)$ *and* (C, θ_e, e, m_3) *is an eigen-algebra, which we denote as* $C(e)$*.*

Proof. Part (a), (b), (f) are consequences of lemma 5.2.0.2. For part (c), for homogenous element $z \in C$ and elements $x, y \in C$, we have

$$\begin{aligned}\chi_e(m_3(x, y, z)) &= m_3(m_3(x, y, z), e, e) = m_3(m_3(e, y, z), x, e) + m_3(m_3(x, e, z), e, y) \\ &\quad - m_3(m_3(e, e, z), x, y) \\ &= \theta_e(\theta_e(y, z), x) + \theta_e(\theta_e(x, z), y) - \alpha(z) m_3(x, y, z)\end{aligned}$$

which proves part (c).

For part (d), if $t, s \in \sigma(e)$ such that $t+s-1 \notin \sigma(e)$, then by lemma 5.2.0.2 we have

$$\chi_e(m_3(e, x, y)) \in C_e(t+s-1) = 0$$

for $x \in C_e(t)$ and $y \in C_e(s)$.

For part (e), this is a consequence of part (c) using the symmetry of the arguments. □

If C is a finite dimensional 3-Com algebra with semisimple idempotent e, then we can transfer between the ideals of C and $C(e)$, i.e. a subspace I of $C = C(e)$ is an ideal of C as a 3-Com algebra if and only if I is an m_3-ideal of $C(e)$ as a eigen-algebra. Furthermore, if J is an ideal of C as a 3-Com algebra, then it is also an ideal of $C(e)$ as an eigen-algebra, but this normally does not go the other way unless it is a m_3-ideal.

lemma 5.2.0.4. *Let C be a finite dimensional 3-Com algebra with semisimple idempotent e. We have the following statements:*

(1) If $C(e)$ is simple, then C is simple.

(2) The space C is simple if and only if $C(e)$ is m_3-simple.

Proof. For (1), let I be an ideal of C, then I is an ideal of $C(e)$ and since $C(e)$ is simple then we must have $I = 0$ or $I = C(e)$. Therefore, C is simple.

For (2), if $C(e)$ is m_3-simple then by the same argument as part (1), C is simple. On the other hand, suppose C is simple. If J is an m_3-ideal of $C(e)$, then J is an ideal of C and hence $J = 0$ or $J = C$ which shows that $C(e)$ is m_3-simple. □

5.2.1 Bilinear form on 3-Com algebra

For this section, we will study a natural bilinear form κ on the finite-dimensional 3-Com algebras reminiscent of Killing from Lie algebras. But, as we will see, there are some proerties that are not like the traditional scenario. For example, we have finite-dimensional simple 3-Com algebras with $\kappa = 0$, which we call fully degenerate. We think of these fully degenerate simples as something that separates 3-Com algebras from 2-Com algebras since every simple 2-Com algebra is a field. This shows the departure of 3-Com algebras from 2-Com and shows there are some interesting examples that are not trivial.

Let $x, y \in C$ and we define $\chi_{x,y} = m_3(x, y, -)$ just as before, and it is clear this is bilinear in both x and y.

lemma 5.2.1.1. *Let C be a finite-dimensional 3-Com algebra. We have the following properties with the trace $Tr(-)$.*

(a) For $x, y, z, w \in C$ we have

$$\chi_{w,y}\chi_{x,z} - \chi_{x,y}\chi_{w,z} = \chi_{w,z}\chi_{x,y} - \chi_{x,z}\chi_{w,y}.$$

Furthermore, we obtain

$$Tr(\chi_{w,y}\chi_{x,z}) = Tr(\chi_{x,y}\chi_{w,z}).$$

(b) For $x, y, z, w \in C$, we have

$$Tr(\chi_{m_3(x,y,z),w}) = Tr(\chi_{x,m_3(y,z,w)})$$

Proof. For part (a), let $x, y, z, w, a \in C$ and by definition we have

$$m_3(m_3(x,y,z),w,a) = m_3(m_3(w,y,z),x,a) + m_3(m_3(x,a,z),w,y) - m_3(m_3(w,a,z),x,y)$$
$$m_3(m_3(x,y,z),w,a) = m_3(m_3(w,y,z),x,a) + m_3(m_3(x,y,a),w,z) - m_3(m_3(w,y,a),x,z)$$

and subtracting these gets us the equation

$$m_3(m_3(x,a,z),w,y) - m_3(m_3(w,a,z),x,y) = m_3(m_3(x,y,a),w,z) - m_3(m_3(w,y,a),x,z)$$

which is true for all a. This gives us

$$\chi_{w,y}\chi_{x,z} - \chi_{x,y}\chi_{w,z} = \chi_{w,z}\chi_{x,y} - \chi_{x,z}\chi_{w,y}.$$

The second equation with the trace in part (b) is proved using the fact $Tr(ab) = Tr(ba)$ and we are in characteristic 0.

For part (b), if $x, y, z, w, a \in C$, then we have

$$m_3(m_3(x,y,z),w,a) = m_3(m_3(w,y,z),x,a) + m_3(m_3(x,a,z),w,y) - m_3(m_3(w,a,z),x,y)$$

which is true for all a and gets us the equation

$$\chi_{m_3(x,y,z),w} = \chi_{m_3(w,y,z),x} + \chi_{w,y}\chi_{x,z} - \chi_{x,y}\chi_{w,z}.$$

Applying the trace map gets us the result using part (b). □

From the above lemma, we are able to define a very special bilinear form on any 3-Com algebra that respects the 3-Com multiplication.

Definition 5.2.1.1. *Let C be a finite-dimensional 3-Com algebra and define the bilinear map $\kappa : C \otimes C \to k$ as*

$$\kappa(x, y) = Tr(\chi_{x,y})$$

which is well-defined and symmetric. We call κ the Killing form on C. Furthermore, κ respects the multiplication m_3 as follows:

$$\kappa(m_3(x, y, z), w) = \kappa(x, m_3(y, z, w))$$

for any $x, y, z, w \in C$. Let $Rad(\kappa)$ be the subspace of C defined as

$$Rad(\kappa) = \{x \in C \ : \ \kappa(x, y) = 0 \ \text{ for all } y \in C\},$$

which we call the radical of κ.

lemma 5.2.1.2. *The subspace $Rad(\kappa)$ is an ideal of C.*

Proof. For $x \in \mathrm{Rad}_C$, and $y, z \in C$, we have

$$\kappa(m_3(x, y, z), w) = \kappa(x, m_3(y, z, w) = 0$$

for all $w \in C$ and hence this shows $m_3(x, y, z) \in \mathrm{Rad}(\kappa)$. □

We define the following definition, where when κ is non-degenerate, it sort of resembles semisimplicity as in the commutative algebraic world. But, as we will see, not every simple finite-dimensional 3-Com algebra has a non-degenerate Killing form.

Definition 5.2.1.2. *Let C be a finite-dimensional 3-Com algebra. We say C is non-degenerate if κ is non-degenerate. On the other side, we say C is fully degenerate if $\kappa = 0$.*

lemma 5.2.1.3. *If C is a finite-dimensional simple 3-Com algebra, then C is either fully-degenerate or non-degenerate.*

The next example shows that we can produce a 2-dimensional 3-Com algebra that is fully degenerate and simple, showing that not every simple 3-Com algebra is non-degenerate.

Example 5.2.1.1. *Let C be the 2-dimensional 3-Com algebra as in example 5.2.0.2 with multiplications*

$$m_3(e,e,e) = e \qquad m_3(e,e,f) = -f$$
$$m_3(e,f,f) = 0 \qquad m_3(f,f,f) = e$$

and by definition we have

$$\chi_e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \chi_f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
$$\chi_{e,f} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

and we see $\kappa(e,e,) = \kappa(e,f) = \kappa(f,f) = 0$. This shows that this 3-Com algebra is fully degenerate by definition.

The fact that simple property does not imply non-degenerate is a little discouraging in using this definition, especially as it is very hard to find examples of simple 3-Com algebras in dimension 2 and 3 that have a non-zero Killing form, as we will see in the next section. But, we can always take $C = \bigoplus \mathbb{C}$ as a finite direct product of 3-Com algebras which gives us an example of a non-degenerate 3-Com algebra.

A big question I have is the following: does every non-degenerate simple finite-dimensional 3-Com algebra have to be a field? Furthermore, if we have a simple finite-dimensional 3-Com algebra with a dimension greater than or equal to 2, does it have to be fully degenerate? From the next section, the only simple 3-Com algebras with dimension less than or equal to 3 is either a field or fully degenerate.

Next, we will explore the properties we can obtain through just the definition of κ descending it to certain subspaces of a 3-Com algebra.

lemma 5.2.1.4. *Let C be a finite-dimensional 3-Com algebra. We have the following properties for the Killing form κ.*

(1) For any $x \in C$, it is a symmetric k-linear map with respect to κ :

$$\kappa(\chi_x(y), z) = \kappa(y, \chi_x(z)).$$

Furthermore, if C is non-degenerate, then eigenvectors corresponding to different eigenvalues for χ_x are orthogonal with respect to κ.

(2) If e is an idempotent of C, then κ restricts to a bilinear form on $C_e(1)$, defined as $\kappa_e = \kappa|_{C_e(1)\times C_e(1)}$, satisfying the following property:

$$\kappa_e(\theta_e(x, y), z) = \kappa_e(x, \theta_e(y, z))$$

for any $x, y, z \in C_e(1)$. Furthermore, if we let $Rad(\kappa_e)$ be the radical of κ_e, then

$$Rad(\kappa) \cap C_e(1) \subseteq Rad(\kappa_e).$$

If C is non-degenerate and e is semisimple, then we get equality and hence κ_e is non-degenerate.

(3) If I is an ideal of C, then κ restricts to the Killing form $\kappa_I = \kappa|_{I\times I}$ on I. Furthermore, we have

$$I \cap Rad(\kappa) \subseteq Rad(\kappa_I)$$

Proof. Part (a), is clear from $\chi_e = m_3(e, e, -)$ and eigenvectors corresponding to different eigenvectors are orthogonal if κ is non-degenerate is a standard linear algebra argument. For part (b), the bilinear form descends to $C_e(1)$ satisfying

$$\kappa(\theta_e(x, y), z) = \kappa(x, \theta_e(y, z))$$

since $\theta_e(-,-) = m_3(e,-,-)$.

Next, let $x \in \text{Rad}(\kappa) \cap C_e(1)$, then since $\kappa(x,y) = 0$ for all $y \in C$, then it is true for all $y \in C_e(1)$ and hence this shows $\text{Rad}(\kappa) \cap C_e(1) \subseteq \text{Rad}(\kappa_e)$. If C is non-degenerate and $1, \lambda_1, \ldots, \lambda_m$ are the $m+1$ distinct eigenvalues of χ_e, then $C_e(1), C_e(\lambda_1), \ldots, C_e(\lambda_m)$ are orthogonal with respect to κ. Furthermore, if e is semisimple, $x \in \text{Rad}(\kappa_e)$, $y \in C$, then $y = z + \sum_{i=1}^{m} f_i$ for $z \in C_e(1)$ for $f_i \in C_e(\lambda_i$. This implies

$$\kappa(x,y) = \kappa(x,z) + \sum_{i=1}^{m} \kappa(x, f_i) = 0$$

by orthogonality of eigenvectors and $x \in \text{Rad}(\kappa_e)$. This shows $\text{Rad}(\kappa_e) \subseteq C_e(1) \cap \text{Rad}(\kappa)$ and shows equality.

For part (3), let I be an ideal and κ_I is the Killing form for I. We will show $\kappa_I = \kappa|_{I \times I}$, i.e., the restriction of κ on C to I. Choose a basis for I and extend the basis to all of L, then $\chi_{x,y}$ for $x, y \in I$ has matrix of the form

$$\begin{pmatrix} \chi_{x,y}|_I & \star \\ 0 & 0 \end{pmatrix}.$$

This shows $\kappa_I = \kappa_{I \times I}$ since the trace is independent of the chosen basis. Furthermore, for $\text{Rad}(\kappa) \cap I \subseteq \text{Rad}(\kappa_I)$, this is a similar argument as part (2).

Similarly, for part (3), κ restricts to the Killing form κ_I on I and the argument $\text{Rad}(\kappa) \cap I \subseteq \text{Rad}(\kappa_I)$ is the same. □

lemma 5.2.1.5. *If C be a finite-dimensional non-degenerate 3-Com algebra and e is a semisimple non-zero idempotent, then $C_e(1)$ is a Frobenius algebra with Frobenius form κ_e.*

Next, let I be any ideal of C, and define $I^\perp$ to be the orthogonal complement of I with respect to the form κ:

$$I^\perp = \{x \in C \ : \ \kappa(x,y) = 0 \ \text{ for all } y \in I\}.$$

Then it is easy to show that $I^\perp$ is an ideal of C using the same argument as in lemma 5.2.1.2 and the fact that I is an ideal: if $x \in I^\perp$, $y, z \in C$, and $w \in I$, we have

$$\kappa(m_3(x,y,z), w) = \kappa(x, m_3(y,z,w)) = 0$$

since $m_3(y, z, w) \in I$.

lemma 5.2.1.6. *Let C be a finite-dimensional non-degenerate 3-Com algebra. If $I \neq 0$ is an ideal of C, then κ_I is non-degenerate, and hence I is also non-degenerate.*

Proof. Since I is an non-zero ideal and κ is non-degenerate, then $C = I \oplus I^{\perp}$. Let $x \in \text{Rad}(\kappa_I)$ and $y \in C$ so that $y = u + v$ for $u \in I$ and $v \in I^{\perp}$. Then we have

$$\kappa(x, y) = \kappa(x, u) + \kappa(x, v) = 0$$

for all $y \in C$. This shows $x \in \text{Rad}(\kappa) = 0$ and therefore $\text{Rad}(\kappa_I) = 0$. Hence I is non-degenerate.

□

Theorem 5.2.1.7. *Let C be a finite-dimensional 3-Com algebra. If C is non-degenerate, then*

$$C = I_1 \oplus \cdots \oplus I_m$$

for non-degenerate simple ideals $I_1, \ldots, I_m$.

Proof. We will prove this by induction on the dimension of C. If C is simple, then we are done; otherwise, suppose there exists a non-trivial ideal I, which is minimal, hence simple. Since κ is non-degenerate then $C = I \oplus I^{\perp}$ as vector spaces. Each I and $I^{\perp}$ are non-degenerate 3-Com subalgebras of lesser dimension by lemma 5.2.1.6. Since $I \cap I^{\perp}$=0, then $C = I \oplus I^{\perp}$ is a direct product of 3-Com algebras. Since $I^{\perp}$ is non-degenerate 3-Com algebras, so by induction $I^{\perp}$ is a direct sum of non-degenerate simple ideals and finishes the proof. □

5.3 Invariant Properties of non-zero idempotents

In this subsection, we will explore some of the implications we can obtain from the corresponding eigen-algebra $C(e)$ for a finite-dimensional 3-Com algebra C with non-zero idempotent e. In particular, we will give some restrictions for $\sigma(e)$ and, even more, provide restrictions for the Peirce spectrum $\sigma(C)$.

lemma 5.3.0.1. *Suppose C is a finite dimensional 3-Com algebra with a semisimple unipotent non-zero idempotent e. Then we have the following properties.*

(1) The space $C = C_e(1)$ as vector spaces.

(2) If $x, y, z \in C$, then

$$m_3(x, y, z) = \theta_e(\theta_e(x, y), z)$$

(3) The space C is simple as a 3-Com algebra if and only if $C_e(1)$ is simple as an unital commutative, associative algebra, i.e. if $C_e(1)$ is a field.

Proof. Part (1) is obvious by definition. For part (2), if $x, y, z \in C$, which are all homogenous of degree 1, then we can use part (c) in lemma 5.2.0.3 to show

$$\chi_e(m_3(x, y, z)) + m_3(x, y, z) = \theta_e(\theta_e(x, y), z) + \theta_e(\theta_e(x, z), y) = 2\theta_e(\theta_e(x, y), z).$$

Since χ_e is the identity map on $C_e(1)$, then we obtain

$$2m_3(x, y, z) = 2\theta_e(\theta_e(x, y), z)$$

which proves part (2).

For For part (3), suppose $C_e(1)$ is a field and I is an ideal of C as a 3-Com algebra. Then I is an ideal of $C_e(1)$ as a commutative k-algebra and hence $I = 0$ or $I = C_e(1) = C$. This shows that C is simple. On the other hand, suppose C is simple and let J be an ideal of $C_e(1)$ as a commutative k-algebra. Next, we will show J is an ideal in C as a 3-Com algebra. If $a \in J$ and $x, y \in C$, then we have

$$m_3(a, x, y) = \theta_e(\theta_e(a, x), y) \in J$$

which shows $m_3(a, x, y) \in J$. Therefore, J is an ideal in C as a 3-Com algebra, and since C is simple, then $J = 0$ or $J = C$. This shows $C_e(1)$ is simple as a commutative k-algebra and, therefore, a field. □

The last lemma essentially says that 3-Com algebras with unipotent semisimple non-zero idempotents come from commutative unital associative k-algebras through the multiplication θ_e, and hence do not give us any "new" 3-Com algebras.

lemma 5.3.0.2. *Let C be a finite-dimensional simple 3-Com algebra with a non-zero semisimple unipotent idempotent e.*

- *If $k = \mathbb{C}$, then $C \cong \mathbb{C}$.*
- *If $k = \mathbb{R}$, then $C \cong \mathbb{C}$ or $C \cong \mathbb{R}$.*

In essence, we have classified all of the possible finite-dimensional simple 3-Com algebras with a semisimple unipotent idempotent. Our next goal is to explore the examples with primitive idempotents and see what information we can get from them. From the conditions in 5.2.0.3, we are able to find some restrictions that must be met for finite-dimensional 3-Com algebras to be simple, which constricts what the possible Peirce numbers can be.

Theorem 5.3.0.3. *Let C be a finite dimensional 3-Com algebra with primitive idempotent e and $\gamma \in \sigma(e) \setminus \{1\}$ such that $\gamma + \beta - 1 \notin \sigma(e)$ for $\beta \in \sigma(e) \setminus \{1\}$. If C is simple, then $-\gamma \in \sigma(e)$.*

Proof. We will prove this by contrapositive. Suppose $-\gamma \notin \sigma(e)$ and we will show $C_e(\gamma)$ is a non-trivial ideal. Since $\gamma + \beta - 1 \notin \sigma(e)$ for all $\beta \in \sigma(e) \setminus \{1\}$, then we must have

$$m_3(x, y, z) = 0$$

for all $y \in C_e(s)$ and $z \in C_e(r)$ for $s, r \in \sigma(e)$ such that both r, s are not 1. For the case $r = s = 1$, then since e is primitive $C_e(1) = ke$ and therefore $m_3(C_e(\gamma), C_e(1), C_e(1)) \subseteq C_e(\gamma)$. This shows that $C_e(\gamma)$ is a non-trivial ideal in C; hence, C is not simple. □

Theorem 5.3.0.4. *Let C be a finite dimensional 3-Com algebra with a primative idempotent e and $0 \in \sigma(e)$. If $\beta - 1 \notin \sigma(e)$ for all $\beta \in \sigma(e) \setminus \{1\}$, then C is not simple.*

Proof. Since $\beta - 1 \notin \sigma(e)$ for $\beta \in \sigma(e) \setminus \{1\}$, then by theorem 5.2.0.3 we have

$$m_3(x, y, z) \in C_e(0)$$

for all $y \in C_e(t)$ and $z \in C_e(s)$. This shows $C_e(0)$ is an ideal, and it is non-trivial since $0, 1 \in \sigma(e)$. Hence C is not simple. □

One thing to note here about the preceding results is the following: if there exists an eigenvalue $\gamma \in \sigma(e)$ such that the space $C_e(\gamma)$ is 'isolated' in the decomposition, i.e. there is no way to go from $C_e(\gamma)$ to another $C_e(\beta)$ for $\beta \in \sigma(e)$ through some multiplication, then $C_e(\gamma)$ will become an ideal in C. In other words, for us to find simple 3-Com algebras, one has to make sure that every Peirce subspace can be multiplied by another distinct Peirce subspace non-trivially. More generally, if we have a collection of Peirce numbers $\gamma_1, \ldots, \gamma_n \in \sigma(e)$, such that $\bigoplus_{i=1}^n C_e(\gamma_i)$ does not equal to C such that the actions of the space C keeps them in this direct sum, then this will become an ideal as well, see figure 5.1. In particular, one

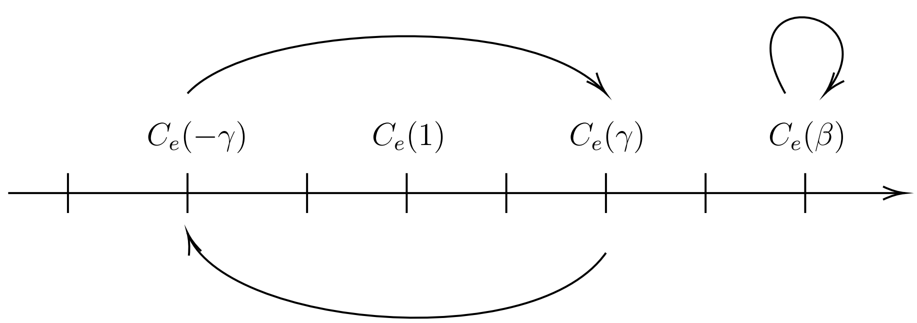


Figure 5.1: Examples of ideals appearing in a decomposition

can only have one set of Peirce numbers in the 2-dimensional case since many numbers will become isolated, except for 1 and -1 as the next result shows.

corollary 5.3.0.5. *If C is a simple m-dimensional 3-Com algebra with $m \geq 2$ with a primitive semisimple non-zero idempotent e with only 2-Peirce numbers, then we must have $\sigma(e) = \{1, -1, -1, \ldots, -1\}$, where the algebraic (geometric) multiplicity of -1 is $m - 1$.*

Proof. Since e is primitive and semisimple with only 2-eigenvalues, $\sigma(e) = \{1, \gamma\}$ for some $\gamma \in k$ that is not equal to 1. Since C is simple and $2\gamma - 1 \notin \sigma(e)$ (the only way for $2\gamma - 1 \in \sigma(e)$ is if $\gamma = 1$ which is not true in this scenario), then we must have $-\gamma \in \sigma(e)$. Therefore, we either have $-\gamma = 1$ or $-\gamma = \gamma$ which gets us $\gamma = -1$ or $\gamma = 0$. By lemma 5.3.0.4, since $0 - 1 = -1 \notin \sigma(e)$, we must have $\gamma = -1$, since C is simple. This completes the proof. □

The last lemma states that in dimension 2, if C has a semisimple idempotent e, then it must have 2-distinct eigenvalues and hence we must have $\sigma(e) = \{1, -1\}$, and furthermore $\sigma(C) = \{1, -1\}$. This gives us a hint as to what all of the simple 3-Com algebras should be, which have primitive semisimple idempotents.

Definition 5.3.0.1. *Let $b \in k$ and let $SC_2(b)$ be a 2-dimensional vector space with basis e_1, e_2. Define the symmetric 3-arity multiplication $m_3 : SC_2(b)^{\otimes 3} \to SC_2(b)$ such that*

$$m_3(e_1, e_1, e_1) = e_1 \qquad m_3(e_1, e_1, e_2) = -e_2$$

$$m_3(e_1, e_2, e_2) = 0 \qquad m_3(e_2, e_2, e_2) = be_1$$

which makes $SC_2(b)$ into a 3-Com algebra, see proposition 4.2.3.3.

It is easy to show $SC_2(b)$ has a primitive semisimple idempotent e_1 such that $\sigma(e) = \{1, -1\}$, which is simple when $b \neq 0$ and not simple when $b = 0$, by examples 5.2.0.3 and 5.2.0.2. Furthermore, we can define a 3-Com algebra isomorphism $F : SC_2(b) \to SC_2(1)$ when $b \neq 0$ with $F(e_1) = e_1$ and $F(e_2) = \frac{1}{b^{1/2}} e_2$ so that $SC_2(b) \cong SC_2(1)$ for $b \neq 0$.

corollary 5.3.0.6. *If C is a 2-dimensional simple 3-Com algebra with primitive semisimple idempotent e, then $C \cong SC_2(b)$ for $b \neq 0$. On the other hand, $SC_2(b)$ is 2-dimensional simple 3-Com algebra with a primitive semisimple idempotent when $b \neq 0$. In conclusion, every 2-dimensional simple 3-Com algebra C with a primitive semisimple idempotent is isomorphic $SC_2(1)$.*

Proof. If C is a 2-dimensional simple 3-Com algebra with a primitive semisimple idempotent e, then we must have $\sigma(e) = \{1, -1\}$ by corollary 5.3.0.5. With the decomposition $C = C_e(1) \oplus C_e(-1)$, let $f \in C_e(-1)$ be the non-zero element which gives e, f is a basis for C. By the properties of $\sigma(e) = \{1, -1\}$, we have

$$m_3(e, e, e) = e \qquad m_3(e, e, f) = -f$$
$$m_3(e, f, f) = 0 \qquad m_3(f, f, f) \in C_e(1).$$

Therefore, $m_3(f, f, f) = be$ for some $b \in k$ and since C is simple then we must have $b \neq 0$. This shows $C \cong SC_2(b)$ and proves the first part.

On the other hand, $SC_2(b)$ for $b \neq 0$ is a 2-dimensional simple 3-Com algebra with a primitive semisimple idempotent e which has $\sigma(e) = \{1, -1\}$. □

The last result gives a complete classification of all the 2-dimensional simple 3-Com algebras equipped with a primitive semisimple idempotent. Furthermore, we have also shown that there are no 2-dimensional semisimple simple 3-Com algebras since $\kappa = 0$ for $SC_2(1)$ as shown in example 5.2.1.1. In particular, combining this with lemma 5.3.0.2 we obtain the following proposition.

Proposition 5.3.0.7. *Let C be a 2-dimensional simple 3-Com algebra with a non-zero idempotent.*

- *If $k = \mathbb{C}$, then $C \cong SC_2(1)$.*
- *If $k = \mathbb{R}$, then C is either isomorphic to $\mathbb{C}$ with its induced 3-Com algebra structure through the ordinary multiplication, or $SC_2(1)$.*

Next, we will use the structure results we proved above to give examples of simple 3-Com algebras in dimensions higher than 1 and 2. Let $n \geq 2$ and $B = (b_{i,j,r})$ be a collection of elements in k symmetric in the i, j, r such that $1 \leq i, j, r \leq n - 1$ and define $SC_n(B)$ be the n-dimensional space with $SC_n(B) = C(1) \oplus C(-1)$ with $C(1)$ has basis e and $C(-1)$

has basis $f_1, \ldots, f_{n-1}$. Define the the 3-arity multiplication $m_3 : SC_n(B)^{\otimes 3} \to SC_n(B)$ as follows:

$$m_3(e,e,e) = e \qquad m_3(e,e,f_i) = -f_i$$
$$m_3(e,f_i,f_j) = 0 \qquad m_3(f_i,f_j,f_r) = b_{i,j,r}e$$

for $1 \le i,j,r \le n-1$.

lemma 5.3.0.8. *For $n \ge 3$ the space $SC_n(B)$ is a 3-Com algebra if and only if $B = 0$ and it has a primitive semisimple idempotent e Peirce numbers 1 and -1.*

Proof. The operations m_3 give us the following matrices

$$X_{0,0} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & -1 \end{pmatrix}, X_{0,i} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, X_{i,j} = \begin{pmatrix} 0 & b_{i,j,1} & \cdots & b_{i,j,n-1} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

for $1 \le i,j \le n-1$, where $X_{0,i}$ has the -1 in the ith row, and we need to show these satisfy the equations

$$X_{0,0}X_{i,i} + X_{i,i}X_{0,0} = 2X_{0,i}^2$$
$$X_{i,i}X_{j,j} + X_{j,j}X_{i,i} = 2X_{i,j}^2$$
$$X_{0,0}X_{s,t} - X_{0,s}X_{0,t} - X_{0,t}X_{0,s} + X_{s,t}X_{0,0} = 0$$
$$X_{i,i}X_{s,t} - X_{i,s}X_{i,t} - X_{i,t}X_{i,s} + X_{s,t}X_{i,i} = 0$$

for $1 \le i \le n-1$ and $0 \le s < t \le n-1$ from lemma 4.2.3.1. It is easy to see that $X_{0,0}X_{i,j} = X_{i,j}$ and $X_{i,j}X_{0,0} = -X_{i,j}$ and we have $X_{0,i}X_{0,j} = 0$ for any $1 \le i,j \le n-1$. From the last equations, we must have $b_{i,j,1}, \ldots, b_{i,j,n-1}$ must all be zero.

For $SC_n(B)$, the element e is semisimple primitive idempotent with $\sigma(e) = \{1, -1, \ldots, -1\}$ through $X_{0,0}$. This completes the proof.

□

Since the only time when $SC_n(B)$ is a 3-Com algebras, for $n \geq 3$, is when $B = 0$ and these are always non-simple since $C_e(-1)$ will be a non-trivial ideal. With this, if C is a finite-dimensional 3-Com algebra with a semisimple primitive idempotent e with Peirce numbers 1 and -1, then it must be of the form $SC_2(B)$ for some B. But, by the above lemma, we must have $B = 0$ and hence can not be simple.

lemma 5.3.0.9. *If C is a finite-dimensional simple 3-Com algebra with semisimple primitive idempotent e with Peirce numbers 1 and -1, then $C \cong SC_2(1)$ and must be 2-dimensional.*

Next, we will explore what the possible Peirce numbers are for when an idempotent has exactly 3 distinct Peirce numbers, but as we will see, there exist no examples as follows.

lemma 5.3.0.10. *There exists no finite-dimensional simple 3-Com algebra equipped with a semisimple primitive idempotent e with exactly three Peirce numbers.*

Proof. Since e has exactly 3 Peirce numbers, then $P_\sigma(e) = \{1, \gamma, \beta\}$ for distinct numbers $1, \gamma, \beta \in k$. If γ and β are such that $\gamma + \beta - 1 \notin \sigma(e)$, $2\gamma - 1, 2\beta - 1 \notin \sigma(e)$, then this would imply we must have $-\gamma, -\beta \in \sigma(e)$ for C to be simple. Therefore, we must have either of the cases:

$$-\gamma = 1 \qquad\qquad -\gamma = \gamma \qquad\qquad -\gamma = \beta.$$

If $-\gamma = 1$, then we must have $\beta = 0$ since $\beta \neq 1$, but we would have $2\beta - 1 = -1 \in \sigma(e)$ which can not happen in this case. If $-\gamma = \gamma$, then we must have $\gamma = 0$ and hence $\beta = -1$, and again, we do not have this in this case. The final case is when $-\gamma = \beta$, with the property that $\gamma - \gamma - 1 = -1 \notin \sigma(e)$, and $2\gamma - 1, -2\gamma - 1 \notin \sigma(e)$. In conclusion, we have the following case in this scenario:

$$P_\sigma(e) = \{-\gamma, 1, \gamma\}$$

where $2\gamma - 1, -2\gamma - 1 \notin \sigma(e)$. In this case, we have the following multiplications through proposition 5.2.0.3:

$$\begin{aligned}
m_3(C_e(1), C_e(\gamma), C_e(\gamma)) &= 0 \\
m_3(C_e(1), C_e(\gamma), C_e(-\gamma)) &= 0 \\
m_3(C_e(1), C_e(-\gamma), C_e(-\gamma)) &= 0 \\
m_3(C_e(\gamma), C_e(\gamma), C_e(\gamma)) &= 0 \\
m_3(C_e(\gamma), C_e(\gamma), C_e(-\gamma)) &= 0 \\
m_3(C_e(\gamma), C_e(-\gamma), C_e(-\gamma)) &= 0 \\
m_3(C_e(-\gamma), C_e(-\gamma), C_e(-\gamma)) &= 0
\end{aligned}$$

since $2\gamma - 1, -2\gamma - 1, -1 \notin \sigma(e)$. Therefore, the subspace $I = C_e(-\gamma) \oplus C_e(\gamma)$ is a non-trivial ideal. This shows that C has a non-trivial ideal and, hence, is not simple. Thus $P_\sigma(e) \neq \{-\gamma, 1, \gamma\}$ with $2\gamma - 1, -2\gamma - 1 \notin \sigma(e)$.

We have shown that in the case where $\gamma + \beta - 1, 2\gamma - 1, 2\beta - 1 \notin \sigma(e)$ does not give us any simple 3-Com algebras, so let us suppose we have at least one of the $\gamma + \beta - 1, 2\gamma - 1$, or $2\beta - 1$ to be in $\sigma(e)$. If $\gamma + \beta - 1 \in \sigma(e)$, then we either have

$$\gamma + \beta - 1 = 1 \qquad \gamma + \beta - 1 = \gamma \qquad \gamma + \beta - 1 = \beta.$$

The latter two cases can not happen since β and γ are not 1, so we must have $\gamma + \beta = 2$. This gives us the Peirce numbers $P_\sigma(e) = \{1, \gamma, 2 - \gamma\}$.

Next, we will show that the possible multiplications for the case $P_\sigma(e) = \{1, \gamma, 2 - \gamma\}$ are as follows:

$$\begin{aligned} m_3(C_e(1), C_e(1), C_e(t)) &\subseteq C_e(t) \\ m_3(C_e(1), C_e(\gamma), C_e(\gamma)) &= 0 \\ m_3(C_e(1), C_e(\gamma), C_e(2-\gamma)) &= 0 \\ m_3(C_e(1), C_e(2-\gamma), C_e(2-\gamma)) &= 0 \\ m_3(C_e(\gamma), C_e(\gamma), C_e(\gamma)) &\subseteq C_e(-\gamma) \\ m_3(C_e(\gamma), C_e(\gamma), C_e(2-\gamma)) &= 0 \\ m_3(C_e(\gamma), C_e(2-\gamma), C_e(2-\gamma)) &= 0 \\ m_3(C_e(2-\gamma), C_e(2-\gamma), C_e(2-\gamma)) &\subseteq C_e(-(2-\gamma)). \end{aligned}$$

Note that $2\gamma - 1, 3\gamma - 2, 3 - 2\gamma \notin \sigma(e)$, as these would result in $\gamma = 1$, which is not the case. Through proposition 5.2.0.3, we get the following multiplications

$$\begin{aligned} m_3(C_e(1), C_e(1), C_e(t)) &\subseteq C_e(t) \\ m_3(C_e(1), C_e(\gamma), C_e(\gamma)) &= 0 \\ m_3(C_e(1), C_e(\gamma), C_e(2-\gamma)) &\subseteq C_e(1) \\ m_3(C_e(1), C_e(2-\gamma), C_e(2-\gamma)) &= 0 \\ m_3(C_e(\gamma), C_e(\gamma), C_e(\gamma)) &\subseteq C_e(-\gamma) \end{aligned}$$

where the last equation comes from the fact that $2\gamma - 1 \notin \sigma(e)$.

For $x \in C_e(\gamma)$ and $y \in C_e(2-\gamma)$, then $m_3(e, x, y) \in C_e(1)$, so there exists $a \in k$ such that $m_e(e, x, y) = ae$. Then we have

$$m_3(m_3(x, y, e), x, y) = m_3(m_3(x, x, e), y, y) + m_3(m_3(y, y, e), x, x) - m_3(m_3(x, y, e), x, y)$$

which gives us the equation

$$2a^2e = 2m_3(m_3(x, y, e), x, y) = 0$$

through the multiplication above. This shows $a = 0$ and hence this shows $m_3(C_e(1), C_e(\gamma), C_e(2-\gamma)) = 0$.

We need to consider the other three last cases to determine where m_3 sends these elements. The equations

$$m_3(C_e(\gamma), C_e(\gamma), C_e(2-\gamma)) = 0$$
$$m_3(C_e(\gamma), C_e(2-\gamma), C_e(2-\gamma)) = 0$$

are similar arguments and we will show the first one. Let $x, y \in C_e(\gamma)$ and $a \in C_e(2-\gamma)$ and we have

$$\begin{aligned}(2-\gamma)m_3(x,y,a) &= m_3(x,y,m_3(a,e,e)) = m_3(a,y,m_3(x,e,e)) + m_3(x,e,m_3(a,y,e)) \\ &-m_3(a,e,m_3(x,y,e)) \\ &= m_3(a,y,m_3(x,e,e)) + m_3(x,e,m_3(a,y,e)) \\ &= \gamma m_3(x,y,a)\end{aligned}$$

where $m_3(x,y,e) = 0$ and $m_3(a,y,e) = 0$. This gives us the equation

$$2(1-\gamma)m_3(x,y,a) = 0$$

and since $(1-\gamma) \neq 0$ then we must have $m_3(x,y,a) = 0$. Therefore, $m_3(C_e(\gamma), C_e(\gamma), C_e(2-\gamma)) = 0$ and proves equations.

With this multiplication we just derived, if $-\gamma \notin \sigma(e)$ or $-(2-\gamma) \notin \sigma(e)$, then this would result in a non-trivial ideal $C_e(\gamma)$ or $C_e(2-\gamma)$. Therefore, we must have $-\gamma$ and $-(2-\gamma) \in \sigma(e)$ in this scenario. If $-\gamma \in \sigma(e)$, then we would have

$$-\gamma = 1 \qquad\qquad -\gamma = \gamma \qquad\qquad -\gamma = 2-\gamma$$

with the resulting cases

$$P_\sigma = \{\gamma = -1, 3, 1\}$$
$$P_\sigma = \{\gamma = 0, 2, 1\}.$$

But note that $-(2-\gamma) \notin \sigma(e)$ in both cases and hence they will have non-trivial ideals. A similar argument holds we start when we start with $-(2-\gamma) \in \sigma(e)$, and this shows that we do not get any simple 3-Com algebras in this case.

For the next case, suppose we have $2\gamma - 1 \in \sigma(e)$, then we must have either of the following cases:

$$2\gamma - 1 = 1 \qquad\qquad 2\gamma - 1 = \gamma \qquad\qquad 2\gamma - 1 = \beta.$$

The former two cases can not happen since $\gamma \neq 1$, so we must have $2\gamma - 1 = \beta$, and hence $P_\sigma(e) = \{1, \gamma, 2\gamma - 1\}$. We will show in this case, that only one spectrum is possible in which it does not give us a non-simple 3-Com algebra. Note that the numbers $3\gamma - 2, 4\gamma - 3, 5\gamma - 4, 6\gamma - 5 \notin \sigma(e)$, since if they were it would result in $\gamma = 1$, which is a contradiction. Therefore, in this case, using proposition 5.2.0.3 we have the following multiplications:

$$\begin{aligned}
m_3(C_e(1), C_e(1), C_e(t)) &\subseteq C_e(t)\\
m_3(C_e(1), C_e(\gamma), C_e(\gamma)) &\subseteq C_e(2\gamma - 1)\\
m_3(C_e(1), C_e(\gamma), C_e(2\gamma - 1)) &= 0\\
m_3(C_e(1), C_e(2\gamma - 1), C_e(2\gamma - 1)) &= 0\\
m_3(C_e(\gamma), C_e(\gamma), C_e(\gamma)) &\subseteq C_e(-\gamma)\\
m_3(C_e(\gamma), C_e(\gamma), C_e(2\gamma - 1)) &= 0\\
m_3(C_e(\gamma), C_e(2\gamma - 1), C_e(2\gamma - 1)) &= 0\\
m_3(C_e(2\gamma - 1), C_e(2\gamma - 1), C_e(2\gamma - 1)) &\subseteq C_e(-2\gamma + 1).
\end{aligned}$$

If $-2\gamma + 1, -\gamma \notin \sigma(e)$, then $C_e(\gamma) \oplus C_e(2\gamma - 1)$ would be a non-trivial ideal. Hence, we must have either $-2\gamma - 1 \in \sigma(e)$ or $-\gamma \in \sigma(e)$.

If $-2\gamma+1 \in \sigma(e)$ or $-\gamma \in \sigma(e)$, we obtain the cases:

$$P_\sigma(e) = \{-1, \gamma = 0, 1\}$$
$$P_\sigma(e) = \{-\frac{1}{3}, \gamma = \frac{1}{3}, 1, \}$$
$$P_\sigma(e) = \{-3, \gamma = -1, 1\}$$
$$P_\sigma(e) = \{0, \gamma = \frac{1}{2}, 1\},$$

where we note which element γ is equal to for reference. We go through each of the cases and show that they have to have non-trivial ideals and hence do not give simple 3-Com algebras.

- For the case $P_\sigma(e) = \{-1, \gamma = 0, 1\}$, the subspace $C_e(-1) \oplus C_e(1)$ is a non-trivial ideal using the fact that $m_3(C_e(1), C_e(1), C_e(0) = 0$ by definition and using the above multiplication.

- For the case $P_\sigma(e) = \{-\frac{1}{3}, \gamma = \frac{1}{3}, 1\}$, the subspace $C_e(-\frac{1}{3}) \oplus C_e(\frac{1}{3})$ is a non-trivial ideal.

- For the case $P_\sigma(e) = \{-3, \gamma = -1, 1\}$, the subspace $C_e(-3)$ is a non-trivial ideal.

- For $P_\sigma(e) = \{0, \gamma = \frac{1}{2}, 1\}$, the subspace $C_e(0)$ is a non-trivial ideal and hence it is not simple in this case.

This completes the proof.

□

Therefore, we have just shown there exist no examples of 3-dimensional simple 3-Com algebras with a semisimple primitive idempotent.

Chapter 6

THE GENERALIZATIONS OF *POIS*

6.1 Definitions of the Algebras

Here we will define two different classes of generalizations of Poisson algebras based on the various generalizations of *Com* and *Lie* we defined in sections 4 and 3.

Definition 6.1.0.1. *Let $n, m \geq 2$ and $d_n, d_m \in \mathbb{Z}$. A Poisson (n, m)-algebra of degree (d_n, d_m) is a graded k-module P with a degree d_n operation $m_n : P^{\otimes n} \to P$ and a degree d_m operation $l_m : P^{\otimes m} \to P$ such that*

- *(P, m_n) is a Com n-algebra of degree d_n;*
- *(P, l_m) is a Lie m-algebra of degree d_m;*
- *and for $p_1, \dots, p_{n+m-1} \in P$ we have*

$$l_m(m_n(p_1, \dots, p_n), p_{n+1}, \dots, p_{n+m-1}) = \sum_{i=1}^{n} (-1)^{\varepsilon_i} m_n(p_1, \dots, p_{i-1}, l_m(p_i, p_{n+1}, \dots, p_{n+m-1}), p_{i+1}, \dots, p_n)$$

where

$$\varepsilon_i = d_m \left(\sum_{j=1}^{i-1} |p_j| \right) + \left(\sum_{j=i+1}^{n} |p_j| \right) \left(\sum_{r=n+1}^{m} |p_r| \right).$$

Definition 6.1.0.2. *Let $n, m \geq 2$ and $d_n, d_m \in \mathbb{Z}$. An (n, m)-Poisson algebra of degree (d_n, d_m) is a graded k-module P with a degree d_n operation $m_n : P^{\otimes n} \to P$ and a degree d_m operation $l_m : P^{\otimes m} \to P$ such that*

- *(P, m_n) is an n-Com algebra of degree d_n;*
- *(P, l_m) is an n-Lie algebra of degree d_m;*
- *and for $p_1, \dots, p_{n+m-1} \in P$ we have the generalized Leibniz rule*

$$l_m(m_n(p_1, \dots, p_n), p_{n+1}, \dots, p_{n+m-1}) = \sum_{i=1}^{n} (-1)^{\varepsilon_i} m_n(p_1, \dots, p_{i-1}, l_m(p_i, p_{n+1}, \dots, p_{n+m-1}), p_{i+1}, \dots, p_n)$$

where

$$\varepsilon_i = d_m \left(\sum_{j=1}^{i-1} |p_j| \right) + \left(\sum_{j=i+1}^{n} |p_j| \right) \left(\sum_{r=n+1}^{m} |p_r| \right).$$

Example 6.1.0.1. *We have a very interesting example of a $(3, 2)$-Poisson algebra structure on the space $k[GL_n(k)]$, the group algebra over the general linear group $GL_n(k)$. By example 4.2.4.2, we have a 3-Com algebra structure with the 3-arity operation*

$$m^{det}(A, B, C) = det(AB)C + det(AC)B + det(BC)A$$

on $k[GL_n(k)]$. On the other hand, we can let $[-, -]$ be the commutator bracket on the space $k[GL_n(k)]$, which is non-zero since the group $GL_n(k)$ is non-commutative. These operations satisfy the generalized Leibniz rule as follows. Let $A_1, A_2, A_3, B \in GL_n(k)$ and we have

$$[m^{det}(A_1, A_2, A_3), B] = det(A_1A_2)[A_3, B] + det(A_1A_3)[A_2, B] + det(A_2A_3)[A_1, B].$$

On the other hand, since $det([A, B]) = 0$ we have

$$m^{det}([A_1, B], A_2, A_3) + m^{det}(A_1, [A_2, B], A_3) + m^{det}(A_1, A_2, [A_3, B]) = det(A_2A_3)[A_1, B] + det(A_1A_3)[A_2, B] + det(A_1A_2)[A_3, B]$$

which shows $k[GL_n(k)]$ with the commutator bracket and m^{det} is a $(3, 2)$-Poisson algebra.

Even more, if we let

$$[A, B, C] = Tr(A)[B, C] + Tr(B)[C, A] + Tr(C)[A, B]$$

then $k[GL_n(k)]$ has a 3-Lie algebra structure and this will also be compatible with m^{det} with the generalized Leibniz rule to make it into a $(3, 3)$-Poisson algebra structure.

In both of the generalizations of Poisson algebras, they both have the common generalized Leibniz rule and just like for the relation in n-Lie algebras, we can express it in a very compact manner using permutations useful for defining the corresponding operads. Let $\lambda_{n,m} = (1\ 2\ \cdots\ n+m-1)$ and $\omega_{n,m} = (1\ 2\ \cdots n)$ be permutations in Σ_{n+m-1}. The following lemma is a similar proof as lemma 3.2.1.1.

lemma 6.1.0.1. *Let P be either a Poisson (n,m)-algebra of degrees (d_n, d_m) or a (n,m)-Poisson algebra of degrees (d_n, d_m) with symmetric n-arity operation m_n and skew-symmetric operation l_m. Then the generalized Leibniz rule in both cases can be expressed as*

$$l_m \circ_1 m_n - \sum_{i=0}^{n-1} (m_n \circ_1 l_m)^{\lambda^m_{n,m}\omega^i_{n,m}} = 0.$$

For each $n, m \geq 2$, we have the following description for the permutations $\lambda^m_{n,m}\omega^i_{n,m}$.

- For $i = 0$ we have

$$\lambda^m_{n,m} = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n & n+1 & \cdots & n+m-1 \\ m+1 & m+2 & \cdots & n+m-1 & 1 & 2 & \cdots & m \end{pmatrix}.$$

- For $1 \leq i \leq n-2$ we have

$$\lambda^m_{n,m}\omega^i_{n,m} = \begin{pmatrix} 1 & 2 & \cdots & n+i-1 & n-i & n-i+1 & \cdots & n & n+1 & \cdots \\ m+i+1 & m+i+2 & \cdots & n+m-1 & 1 & m+1 & \cdots & m+i & 2 & \cdots \end{pmatrix}.$$

- For $i = n-1$ we have

$$\lambda^m_{n,m}\omega^{n-1}_{n,m} = \begin{pmatrix} 1 & 2 & 3 & \cdots & n & n+1 & n+2 & \cdots & n+m-1 \\ 1 & m+1 & m+2 & \cdots & n+m-1 & 2 & 3 & \cdots & m \end{pmatrix}.$$

To use these relations for the operads, we need to change the permutations slightly so that the permutations are all even. Since we are dealing with a pair of positive integers (n,m), there are four cases that we need to deal with.

lemma 6.1.0.2. *Let P be as in lemma 6.1.0.1 above. Then the Leibniz relation can be rewritten as*

$$l_m \circ_1 m_n - \sum_{i=0}^{n-1} (-1)^{m(n+i+1)+i} (m_n \circ_1 l_m)^{(1\ 2)^{m(n+i+1)+i}\lambda^m_{n,m}\omega^i_{n,m}} = 0, \tag{6.1}$$

where $\sigma_{n,m,i} := (1\ 2)^{m(n+i+1)+i}\lambda^m_{n,m}\omega^i_{n,m}$ is even permutation for all i.

Proof. We just need to explore when $\lambda^m_{n,m}\omega^i_{n,m}$ is odd or even depending on n and m and then we just put $id = (1\ 2)(1\ 2)$ on $(m_n \circ_1 l_m)$ if $\lambda^m_{n,m}\omega^i_{n,m}$ using the fact that permuting l_m by $(1\ 2)$ produces a sign.

We will go through the cases for when n, m is even or odd. Depending on the parity of n and m, this will affect the parity of $\lambda^m_{n,m}$ and $\omega^i_{n,m}$, which we can determine since these are powers of cyclic permutations.

Case 1 If n and m are odd, then $n + m - 1$ is odd which implies $\lambda_{n,m}$ and $\omega_{n,m}$ are both even permutations. Therefore, $\lambda^m_{n,m}\omega^i_{n,m}$ is even for $0 \leq i \leq n - 1$.

Case 2 If n is odd and m is even, then $n + m - 1$ is even. Therefore, $\lambda_{n,m}$ and $\omega_{n,m}$ is odd and hence $\lambda^m_{n,m}$ is even and $\omega^i_{n,m}$ is odd if and only if i is odd. Then $\lambda^m_{n,m}\omega^i_{n,m}$ is odd if and only if i is odd.

Case 3 If n is even and m is odd, then $n + m - 1$ is even. Therefore, $\lambda_{n,m}$ is a odd and $\omega_{n,m}$ is even. Since m is odd, then $\lambda^m_{n,m}$ is odd and hence $\lambda^m_{n,m}\omega^i_{n,m}$ is odd for all i.

Case 4 If n and M are both even, then $n + m - 1$ is odd which implies $\lambda_{n,m}$ is even and $\omega_{n,m}$ is odd. Therefore, $\lambda^m_{n,m}$ is even and $\omega^i_{n,m}$ is odd for all odd i. Then $\lambda^m_{n,m}\omega^i_{n,m}$ is odd if and only if i is odd.

From these cases, we have the following parity

$$Sgn(\lambda^m_{n,m}\omega^i_{n,m}) = (-1)^{m(n+i+1)+i}$$

□

The $(2, n)$-Poisson algebras of degree $(0, 0)$ are the n-Lie Poisson algebras originally explored by Nambu in [20] and more generally [28] as in the next definition.

Definition 6.1.0.3. *Let $n \geq 2$. An n-Lie Poisson algebra is a commutative associative algebra P equipped with a n-arity bracket $\{-, \ldots, -\} : P^{\otimes n} \to P$ which makes P into a* *n-Lie algebra and satisfies the following generalized Leibniz rule: for $g_1, g_2, f_1, \ldots, f_{n-1} \in P$ we have*

$$\{g_1g_2, f_1, \ldots, f_{n-1}\} = \{g_1, f_1, \ldots, f_{n-1}\}g_2 + g_1\{g_2, f_1, \ldots, f_{n-1}\}.$$

Example 6.1.0.2. *The $n+1$-Lie algebra structure on $A = k[x_1, \ldots, x_{n+1}]$ defined in example* 3.2.1.1 *with the ordinary commutative and associative multiplication gives A a $n+1$-Lie Poisson algebra structure. Even more, if $\Omega \in A$, called a potential, ten $[-, \ldots, -]_\Omega$ gives A a n-Lie Poisson algebra structure as well.*

6.2 (n, m)-Module Poisson algebras

In this section, we will construct a large class of (n, m)-Poisson algebras which naturally extend examples in **??** and 3.2.1.2 by introducing trace-like maps.

Definition 6.2.0.1. *Let $n \geq 3$, $m \geq 2$, and let M be an R-module. Suppose M is a m-Lie algebra over R with bracket $[-, \ldots, -]$ which is multi-linear over R, i.e. an m-Lie algebra over R.*

- *We say a R-linear map $f : M \to R$ is trace-like map with respect to $[-, \ldots, -]$ if*

  $$f([x_1, \ldots, x_m]) = 0$$

  for all $x_1, \ldots, x_m \in M$.

The following definition is a restatement of example 3.2.1.2 and definition 4.2.4.1 using these trace-like maps.

Definition 6.2.0.2. *Let $n \geq 3$, $m \geq 2$ and let M be an m-Lie algebra over R equipped with a trace-like map $f : M \to R$ with respect to the bracket. We define $\mu_n^f : M^{\otimes n} \to M$ as*

$$\mu_n^f(m_1, \ldots, m_n) = \sum_{i=1}^{n} f(m_1) \cdots \widehat{f(m_i)} \cdots f(m_n) m_i$$

for $y_1, \ldots, y_n \in M$*. Define* $[-,\ldots,-]_f : M^{\otimes m+1} \to M$ *as*

$$[y_1, \ldots, y_{m+1}]_f = \sum_{i=1}^{m+1} f(y_i)[y_1, \ldots, \widehat{y_i}, \ldots, y_{m+1}]$$

Proposition 6.2.0.1. *If* M *is a* m*-Lie algebra over* R *equipped with a trace-like map* $f : M \to R$ *with respect to its bracket, then* $(M, \mu_n^f, [-,\ldots,-])$ *is a* (n,m)*-Poisson algebra. In particular, if* $g : M \to R$ *is another trace-like map with respect to the bracket, then* $(M, \mu^f, [-,\ldots,-]_g)$ *is a* $(n, m+1)$*-Poisson algebra.*

Proof. We know that M is a n-Com algebra structure with μ_n^f by proposition 4.2.4.1, so we just need to show μ_n^f and $[-,\ldots,-]$ satisfy the generalized Leibniz rule. Let $x_1, \ldots, x_n, y_1, \ldots, y_m \in M$ and we have

$$\begin{aligned}
&\sum_{i=1}^{n} \mu_n^f(x_1, \ldots, [x_i, y_1, \ldots, y_m], \ldots, x_n) \\
&\quad = \sum_{i=1}^{n} \sum_{j \neq i} f(x_1) \cdots f([x_i, y_1, \ldots, y_m]) \cdots \widehat{f(x_j)} \cdots f(x_n) x_j \\
&\quad + \sum_{i=1}^{n} f(x_1) \cdots f(x_{i-1}) f(x_{i+1}) \cdots f(x_n)[x_i, y_1, \ldots, y_m] \\
&\quad = \sum_{i=1}^{n} f(x_1) \cdots f(x_{i-1}) f(x_{i+1}) \cdots f(x_n)[x_i, y_1, \ldots, y_m] \\
&\quad = [\mu_n^f(x_1, \ldots, x_n), y_1, \ldots, y_m].
\end{aligned}$$

where we used the fact $f([x_i, y_1, \ldots, y_m]) = 0$. This shows $(M, \mu_n^f, [-,\ldots,-])$ is a (n,m)-Poisson algebra.

If $g : M \to R$ is another trace-like map, then we can define $[-,\ldots,-]_g$ and since $f([-,\ldots,]) = 0$, then $f([-,\ldots,-]_g) = 0$ so that from the same argument above we have $(M, \mu_n^f, [-,\ldots,-]_g)$ is a $(n, m+1)$-Poisson algebra. □

The last theorem expresses how easy it is to find a (n,m)-Poisson structure on a m-Lie algebra given that you have some module map to a commutative k-algebra that is trivial for any bracket.

Example 6.2.0.1. *Let $M = M_n(R)$ over some commutative k-algebra R, and we have the natural trace map $Tr : M_n(R) \to R$, which is R-linear. Furthermore, if we let $[-,-]$ be the commutator bracket in $M_n(R)$, then $Tr([A,B]) = 0$ and hence we have a $(n,2)$-Poisson structure on $M_n(R)$ using μ_n^{Tr} and the commutator bracket. Even more, we can define $[-,-,-]_{Tr}$ and have a $(n,3)$-Poisson structure as well.*

Example 6.2.0.2. *Let $m \geq 3$ and let L be a metric m-Lie algebra with metric $B : L \otimes L \to k$. By example 3.2.1.4, if we pick a potential $\Omega \in L$ then we have a metric $(m-1)$-Lie algebra $L\ [-,\dots,-]_\Omega$ and we can define the trace-like map $B_\Omega : L \to k$ with $B_\Omega(x) = B(x,\Omega)$. By propostion 6.2.0.1 we have a $(n, m-1)$-Poisson structure on L with the bracket $[-,\dots,-]_\Omega$ and the n-Com algebra structure*

$$\mu_n^{B_\Omega}(x_1,\dots,x_n) = \sum_{i=1}^{n} B_\Omega(x_1)\cdots\widehat{B_\Omega(x_i)}\cdots B_\Omega(x_n)x_i.$$

Furthermore, we have the (n,m)-Poisson structure with $(L, \mu_n^{B_\Omega}, [-,\dots,-]_{B_\Omega})$.

6.3 (m,n)-Potential Algebras

For this section we will construct a vast collection of (m,n)-Poisson algebras that are induced by the n-Lie Poisson algebras $P_n(\Omega)$ by a certain subset of polynomials that satisfy a system of equations of partial derivatives and Jacobians with respect to Ω.

Let $n \geq 2$ and let P be any n-Lie Poisson algebra. We will first go through a general construction on P and give some conditions that are needed to make it into a (m,n)-Poisson algebra. Pick a finite subset Γ of P such that $|\Gamma| \geq n-1$ and for any $m \geq 3$. which we call a P-Com set, we can define the m-arity product $\mu_\Gamma^m : P^{\otimes m} \to P$ with

$$\mu_\Gamma^m(f_1,\dots,f_m) = \sum_{\gamma_1,\dots,\gamma_{n-1}\in\Gamma} \{f_1\cdots f_m, \gamma_1,\dots,\gamma_{n-1}\}. \tag{6.2}$$

By example 4.2.1.2, this gives P a m-Com algebra structure, but in general, it will not satisfy the generalized Leibniz rule.

How can we ensure that the operations $\{-,\dots,-\}$ and μ_Γ^m satisfy the generalized Leibniz rule? Lets suppose we have $f_1, \dots, f_m, g_1, \dots, g_{n-1} \in P^m(\Gamma)$ and going through the equations we obtain

$$\begin{aligned}
\{\mu_\Gamma^m(f_1, \dots, f_m), g_1, \dots, g_{n-1}\} &= \sum_{\gamma_1, \dots, \gamma_{n-1} \in \Gamma} \{\{f_1 \cdots f_m, \gamma_1, \dots, \gamma_{n-1}\}, g_1, \dots, g_{n-1}\} \\
&= \sum_{\gamma_1, \dots, \gamma_{n-1} \in \Gamma} \{\{f_1 \cdots f_m, g_1, \dots, g_{n-1}\}, \gamma_1, \dots, \gamma_{n-1}\} \\
&+ \sum_{\gamma_1, \dots, \gamma_{n-1} \in \Gamma} \sum_{i=1}^{n-1} \{f_1 \cdots f_m, \gamma_1, \dots, \gamma_{i-1}, \{\gamma_i, g_1, \dots, g_{n-1}\}, \gamma_{i+1}, \dots, \gamma_{n-1}\} \\
&= \sum_{\gamma_1, \dots, \gamma_{n-1} \in \Gamma} \sum_{j=1}^{m} \{f_1 \cdots f_{j-1}\{f_j, g_1, \dots, g_{n-1}\} f_{j+1} \cdots f_m, \gamma_1, \dots, \gamma_{n-1}\} \\
&+ \sum_{\gamma_1, \dots, \gamma_{n-1} \in \Gamma} \sum_{i=1}^{n-1} \{f_1 \cdots f_m, \gamma_1, \dots, \gamma_{i-1}, \{\gamma_i, g_1, \dots, g_{n-1}\}, \gamma_{i+1}, \dots, \gamma_{n-1}\} \\
&= \sum_{\gamma_1, \dots, \gamma_{n-1} \in \Gamma} \sum_{j=1}^{m} \mu_\Gamma^m(f_1, \dots, f_{j-1}, \{f_j, g_1, \dots, g_{n-1}\}, f_{j+1}, \dots, f_m) \\
&+ \sum_{\gamma_1, \dots, \gamma_{n-1} \in \Gamma} \sum_{i=1}^{n-1} \{f_1 \cdots f_m, \gamma_1, \dots, \gamma_{i-1}, \{\gamma_i, g_1, \dots, g_{n-1}\}, \gamma_{i+1}, \dots, \gamma_{n-1}\}.
\end{aligned}$$

So for μ_Γ^m and $\{-, \dots, -\}$ to satisfy the generalized Leibniz rule, we must have the last sum to be zero. In other words, for each $\gamma_1, \dots, \gamma_{n-1} \in \Gamma$, we define $S_{\gamma_1, \dots, \gamma_{n-1}} : (P^{\otimes n} \to P$ as

$$S_{\gamma_1, \dots, \gamma_{n-1}}(g_1, \dots, g_n) = \sum_{i=1}^{n} \{g_1, \gamma_1, \dots, \gamma_{i-1}, \{\gamma_i, g_2, \dots, g_n\}, \gamma_{i+1}, \dots, \gamma_{n-1}\}, \tag{6.3}$$

then we say P is Γ-compatible if $S_{\gamma_1, \dots, \gamma_{n-1}} = 0$ for all $\gamma_1, \dots, \gamma_{n-1} \in \Gamma$. With this definition and the calculations above, we have the following consequence.

lemma 6.3.0.1. *If P is Γ-compatible, then $(P, \{-, \dots, -\}, \mu_\Gamma^m)$ is a (m, n)-Poisson algebra.*

In particular, we can let $\mathcal{J}_\Gamma(P)$ be the n-Lie Poisson ideal of P generated by $S_{\gamma_1, \dots, \gamma_{n-1}\}(g_1, \dots, g_n)}$ for all $g_1, \dots, g_n \in P$. We can define $P(\Gamma) = P/\mathcal{J}_\Gamma(P)$, which is Γ-compatible by definition, which gives the space $(P(\Gamma), \{-, \dots, -\}, \mu_\Gamma^m)$ a (m, n)-Poisson algebra structure.

Now that we have our (m, n)-Poisson algebras, which are quotients of n-Lie Poisson algebras P through the relations $S_{\gamma_1, \dots, \gamma_{n-1}}$ for all $\gamma_1, \dots, \gamma \in \Gamma$, we need to find a way to easily describe these relations. Note that $S_{\gamma_1,\ldots,\gamma_{n-1}}$ is generally not a derivation in each of its arguments, except for the first argument as this is always a derivation. This would make it hard to find the relations coming from the n-Lie Poisson ideal, so we need to find some conditions on P so that $S_{\gamma_1,\ldots,\gamma_{n-1}}$ is a derivation in all of its arguments. This would make it possible to describe the ideal $\mathcal{J}_\Gamma(P)$ based on the generators on P. The next condition comes from determining when $S_{\gamma_1,\ldots,\gamma_{n-1}}$ is a derivation.

Definition 6.3.0.1. *Let P be a n-Lie Poisson algebra and Γ a P-Com set. We say P is Γ semi-strong if for every $f, g, h_1, \ldots, h_{n-1} \in P$ and $\gamma_1, \ldots, \gamma_{n-1} \in \Gamma$ we have*

$$T_{\gamma_1,\ldots,\gamma_{n-1}}(f, g, h_1, \ldots, h_{n-1}) = \sum_{i=1}^{n-1}(-1)^i\{\gamma_1, \ldots, \widehat{\gamma_i}, \ldots, \gamma_{n-1}, f, g\}\{\gamma_i, h_1, \ldots, h_{n-1}\} = 0.$$

The condition to be Γ semi-strong is just enough to be able to make $S_{\gamma_1,\ldots,\gamma_{n-1}}$ into a derivation in each of its arguments as follows.

lemma 6.3.0.2. *Let P be a Γ semi-strong n-Lie Poisson algebra, then for every $\gamma_1, \ldots, \gamma_{n-1} \in \Gamma$ the map $S_{\gamma_1,\ldots,\gamma_{n-1}}$ is a derivation in each of its arguments.*

Proof. Let $v_1, \ldots, v_n \in P$. It is obvious that $S_{P,\gamma_1,\ldots,\gamma_{n-1}}(v_1, \ldots, v_n)$ is a derivation at v_1 since $\{-, \ldots, -\}$ is a derivation in each of its inputs. For the $v_2, \ldots, v_n$ components and $g_j \in P$, we have

$$S_{P,\gamma_1,\dots,\gamma_{n-1}}(v_1,\dots,v_jg,\dots,v_n,\gamma_1,\dots,\gamma_{n-1}) = \sum_{i=1}^{n-1}\{v_1,\gamma_1,\dots,\gamma_{i-1},\{\gamma_i,v_2,\dots,v_jg,\dots,v_n\},\gamma_{i+1},\dots,\gamma_{n-1}\}$$

$$= \sum_{i=1}^{n-1}\{v_1,\gamma_1,\dots,\gamma_{i-1},v_j\{\gamma_i,v_2,\dots,g_j,\dots,v_n\},\gamma_{i+1},\dots,\gamma_{n-1}\}$$

$$+ \sum_{i=1}^{n-1}\{v_1,\gamma_1,\dots,\gamma_{i-1},\{\gamma_i,v_2,\dots,v_j,\dots,v_n\}g_j,\gamma_{i+1},\dots,\gamma_{n-1}\}$$

$$= \sum_{i=1}^{n-1} v_j\{v_1,\gamma_1,\dots,\gamma_{i-1},\{\gamma_i,v_2,\dots,g_j,\dots,v_n\},\gamma_{i+1},\dots,\gamma_{n-1}\}$$

$$+ \sum_{i=1}^{n-1}\{v_2,\gamma_1,\dots,\gamma_{i-1},v_j,\gamma_{i+1},\dots,\gamma_{n-1}\}\{\gamma_i,v_2,\dots,g_j,\dots,v_n\}$$

$$+ \sum_{i=1}^{n-1}\{v_1,\gamma_1,\dots,\gamma_{i-1},\{\gamma_i,v_2,\dots,v_n\},\gamma_{i+1},\dots,\gamma_{n-1}\}g_j$$

$$+ \sum_{i=1}^{n-1}\{\gamma_i,v_2,\dots,v_n\}\{v_1,\gamma_1,\dots,\gamma_{i-1},g_j,\gamma_{i+1},\dots,\gamma_{n-1}\}$$

$$= v_jS_{P,\gamma_1,\dots,\gamma_{n-1}}(v_1,v_2,\dots,g_j,\dots,v_n) + S_{P,\gamma_1,\dots,\gamma_{n-1}}(v_1,v_2,\dots,v_n)g_j$$

$$+ \sum_{i=1}^{n-1}\{v_2,\gamma_1,\dots,\gamma_{i-1},v_j,\gamma_{i+1},\dots,\gamma_{n-1}\}\{\gamma_i,v_2,\dots,g_j,\dots,v_n\}$$

$$+ \sum_{i=1}^{n-1}\{\gamma_i,v_2,\dots,v_n\}\{v_1,\gamma_1,\dots,\gamma_{i-1},g_j,\gamma_{i+1},\dots,\gamma_{n-1}\}.$$

The last two sums in the last equation are zero by the fact that P is a Γ semi-strong n-Lie Poisson algebra, and the remaining terms shows that $S_{\gamma_1,\dots,\gamma_{n-1}}$ is a derivation in v_j spot. This proves the lemma. □

To find examples of Γ semi-strong n-Lie Poisson algebras P, we can take quotients of a particular class of *strong* n-Lie Poisson algebras by a very specific n-Lie Poisson ideal. By

[4] that a **strong n-Lie Poisson algebra** is a n-Lie Poisson algebra P such that

$$\sum_{i=1}^{n}(-1)^i\{v_1,\ldots,\widehat{v_i},\ldots,v_{n+1}\}\{v_i,u_1,\ldots,u_{n-1}\}=0 \tag{6.4}$$

for all $v_1,\ldots,v_{n+1},u_1,\ldots,u_{n-1}\in P$, and we call this the strong condition. The strong condition is used to establish sufficient conditions for when the symmetric algebra $S(L)$ of a n-Lie algebra L is a n-Lie Poisson algebra, for more information see [4]. In particular, the n-Lie Poisson algebra $P_n(\Omega)$ is a strong n-Lie Poisson algebra.

lemma 6.3.0.3 ([4])**.** *For any $\Omega\in P_n$, the n-Lie Poisson algebra $P_n(\Omega)$ is a strong n-Lie Poisson algebra.*

Proof. Let $Y:(P_n(\Omega))^{\otimes 2n}\to P_n(\Omega)$ be the linear map

$$Y(v_1,\ldots,v_{n+1},u_1,\ldots,u_{n-1})=\sum_{i=1}^{n}(-1)^i\{v_1,\ldots,\widehat{v_i},\ldots,v_{n+1}\}\{v_i,u_1,\ldots,u_{n-1}\}.$$

Since Y is skew-symmetric in the variables $u_1,\ldots,u_{n-1}$ and a derivation in each of its arguments, then it suffices to look $v_1,\ldots,v_{n+1},u_1,\ldots,u_{n-1}$ are the generators $x_1,\ldots,x_{n+1}$. In particular, we have

$$\begin{aligned}Y(x_1,\ldots,x_{n+1},x_3,\ldots,x_{n+1})&=-\{x_2,\ldots,x_{n+1}\}\{x_1,x_3,\ldots,x_{n+1}\}\\&\quad+\{x_1,x_3,\ldots,x_{n+1}\}\{x_2,x_3,\ldots,x_{n-1}\}\\&=-1+1=0.\end{aligned}$$

This proves $P_n(\Omega)$ is strong. □

Suppose P is a strong n-Lie Poisson algebra and Γ is a P-Com set. We can define the functions

$$F_{\gamma_1,\ldots,\gamma_{n-1}}(f,g,h_1,\ldots,h_{n-1})=\{\gamma_1,\ldots,\gamma_{n-1},f\}\{g,h_1,\ldots,h_{n-1}\}-\{\gamma_1,\ldots,\gamma_{n-1},g\}\{f,h_1,\ldots,h_{n-1}\}$$

for every $\gamma_1,\ldots,\gamma_{n-1}\in\Gamma$ and $f,g,h_1,\ldots,h_{n-1}\in P$. With this, define $\mathcal{I}_\Gamma(P)$ be the n-Lie Poisson ideal generated by the image of $F_{\gamma_1,\ldots,\gamma_{n-1}}$ for all $\gamma_1,\ldots,\gamma_{n-1}\in\Gamma$. Since P is strong, we have

$$\sum_{i=1}^{n-1}(-1)^i\{\gamma_1,\ldots,\widehat{\gamma_i},\ldots,\gamma_{n-1},f,g\}\{\gamma_i,h_1,\ldots,h_{n-1}\} = (-1)^n F_{\gamma_1,\ldots,\gamma_{n-1}}(g,f,h_1,\ldots,h_{n-1}) \in \mathcal{I}_\Gamma$$

by definition, so that $P(\Gamma) = P/\mathcal{I}_\Gamma(P)$ is Γ semi-strong.

Now that we have our conditions which make $S_{\gamma_1,\ldots,\gamma_{n-1}}$ into a derivation, we can finally find some explicit examples with some description for the ideals.

Definition 6.3.0.2. *Let $n, m \geq 3$, $\Omega \in P_n$, and Γ is a P_n-Com set. Define $P_n(\Gamma,\Omega) = P_n(\Omega)/\mathcal{I}(P_n(\Omega))$, which is a Γ semi-strong n-Lie Poisson algebra. Furthermore, define $P_{m,n}(\Gamma,\Omega) = P_n(\Gamma,\Omega)/\mathcal{J}_\Gamma(P_n(\Gamma,\Omega))$ with the n-Lie Poisson algebra structure $\{-,\ldots,-\}_\Omega$ and the m-Com algebra structure μ_Γ^m. We call $P_{m,n}(\Gamma,\Omega)$ the (m,n)-Potential algebra with respect to Γ and Ω and this is a (m,n)-Poisson algebra by lemma 6.3.0.1.*

With all of the nice properties we have defined above, we are able to describe the ideals $\mathcal{J}_\Gamma(P_n(\Omega))$ in $P_n(\Omega)$ and $\mathcal{I}_\Gamma(P_n(\Gamma,\Omega))$ in $P_n(\Gamma,\Omega)$, which are generated by certain systems of partial differential equations. For simplicity, if $x_1,\ldots,x_{n+1}$ are $n+1$ variables, define $\mathrm{Jac}_i(f_1,\ldots,f_n)$ to be the determinant of the Jacobian matrix of $f_1,\ldots,f_n$ with respect to the variables $x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{n+1}$. Furthermore, denote by $\{f,g\}_{p,q}$ the 2-arity bracket with

$$\{f,g\}_{p,q} = \frac{\partial f}{\partial x_p}\frac{\partial g}{\partial x_q} - \frac{\partial f}{\partial x_q}\frac{\partial g}{\partial x_p}$$

for $1 \leq p,q \leq n+1$.

Theorem 6.3.0.4. *Let $m,n \geq 3$, Γ is a P_n-Com set and $\Omega \in P_n$.*

- *The ideal $\mathcal{I}_\Gamma(P_n(\Omega)$ is generated by*

$$Jac_i(\gamma_1,\ldots,\gamma_{n-1},\Omega)\frac{\partial\Omega}{\partial x_i} - Jac_j(\gamma_1,\ldots,\gamma_{n-1},\Omega)\frac{\partial\Omega}{\partial x_j}$$

  for all $1 \leq i,j \leq n+1$ and by

$$Jac_q(\gamma_1,\ldots,\gamma_{n-1},\Omega)\frac{\partial\Omega}{\partial x_p}$$

  for $q \neq p$ and for all $\gamma_1,\ldots,\gamma_{n-1} \in \Gamma$.

- *The ideal* $\mathcal{J}_\Gamma(P_n(\Gamma,\Omega))$ *is generated by*

$$\sum_{i=1}^{n} Jac_j(\gamma_1,\dots,\gamma_{i-1},[\gamma_i,\Omega]_{p,q},\gamma_{i+1},\dots,\gamma_{n-1},\Omega)$$

for all $1 \le j,p,q \le n+1$ *and* $\gamma_1,\dots,\gamma_{n-1} \in \Gamma$.

- *The space* $(P_n(\Omega),\{-,\dots,-\}_\Omega,\mu_\Gamma^m)$ *is a* (m,n)*-Poisson algebra if*

$$\frac{\partial}{\partial x_i} Jac_i(\gamma_1,\dots,\gamma_{n-1},\Omega) = Jac_i(\gamma_1,\dots,\gamma_{n-1}\frac{\partial\Omega}{\partial x_i})$$
$$Jac_i(\gamma_1,\dots,\gamma_{n-1},\Omega)\frac{\partial\Omega}{\partial x_i} = 0$$

for all $1 \le i \le n+1$ *and* $\gamma_1\dots,\gamma_{n-1} \in \Gamma$.

Proof. To find the description for the ideal $\mathcal{I}_\Gamma(P_n(\Omega))$, note that $F_{\gamma_1,\dots,\gamma_{n-1}}$ for $\gamma_1,\dots,\gamma_{n-1} \in \Gamma$ is a derivation in each of its arguments and it is skew-symmetric in the last $n-1$ arguments. It suffices to look at the generators $x_1,\dots,x_{n-1}$ and we have

$$\begin{aligned}F_{\gamma_1,\dots,\gamma_{n-1}}(x_a,x_b,x_{i_1},\dots,x_{i_{n-1}}) &= \{\gamma_1,\dots,\gamma_{n-1},x_a\}_\Omega\{x_b,x_{i_1},\dots,x_{i_{n-1}}\}_\Omega \\ &\quad - \{\gamma_1,\dots,\gamma_{n-1},x_b\}_\Omega\{x_a,x_{i_1},\dots,x_{i_{n-1}}\}_\Omega.\end{aligned}$$

such that $i_1 < \cdots < i_{n-1}$. If $a,b \in \{i_1,\dots,i_{n-1}\}$, then this sum is just zero. If exactly one of the a or b is in $\{i_1,\dots,x_{n-1}\}$, say a and j is the unique element in $\{1,\dots,n+1\}\{b,x_{i_1},\dots,i_{n-1}\}$, then we have

$$\begin{aligned}F_{\gamma_1,\dots,\gamma_{n-1}}(x_a,x_b,x_{i_1},\dots,x_{i_{n-1}}) &= \{\gamma_1,\dots,\gamma_{n-1},x_a,\Omega\}\{x_b,x_{i_1},\dots,x_{i_{n-1}},\Omega\} \\ &= (-1)^{n-1+a}\mathrm{Jac}_a(\gamma_1,\dots,\gamma_{n-1},\Omega)(-1)^b\frac{\partial\Omega}{\partial x_j}.\end{aligned}$$

In other words, we have

$$\mathrm{Jac}_i(\gamma_1,\dots,\gamma_{n-1},\Omega)\frac{\partial\Omega}{\partial x_j}$$

is a generator of $\mathcal{I}_\Gamma(P_n(\Omega))$ for $i \ne j$.

Next, suppose $a,b \in \{1,\dots,n+1\} \setminus \{i_1,\dots,i_{n-1}\}$ and $F_{\gamma_1,\dots,\gamma_{n-1}}(x_a,x_b,x_{i_1},\dots,x_{i_{n-1}})$ is of the form

$$(-1)^{n-1+a+b}\mathrm{Jac}_a(\gamma_1,\dots,\gamma_{n-1},\Omega)\frac{\partial\Omega}{\partial x_a} - (-1)^{n-1+b+a}\mathrm{Jac}_b(\gamma_1,\dots,\gamma_{n-1},\Omega)\frac{\partial\Omega}{\partial x_b}.$$

Therefore, we also have the elements

$$\mathrm{Jac}_a(\gamma_1,\dots,\gamma_{n-1},\Omega)\frac{\partial\Omega}{\partial x_i} - \mathrm{Jac}_j(\gamma_1,\dots,\gamma_{n-1},\Omega)\frac{\partial\Omega}{\partial x_j}$$

are generators for all $1 \le i, j \le n+1$.

For the ideal $\mathscr{J}_\Gamma(P_n(\Gamma,\Omega))$, the space $P_n(\Gamma,\Omega)$ is Γ semi-strong and hence the maps $S_{\gamma_1,\dots,\gamma_{n-1}}$ is a derivation in each of its arguments and skew-symmetric in the last $n-1$ inputs. For $i_1 < \cdots < i_{n-1}$, $1 \le j \le n+1$, and p, q are the unique integers in $\{1,\dots,n+1\} \setminus \{i_1,\dots,i_{n-1}\}$ we have

$$\begin{aligned}
S_{\gamma_1,\dots,\gamma_{n-1}}(x_j, x_{i_1},\dots,x_{i_{n-1}}) &= \sum_{i=1}^{n}\{x_j,\gamma_1,\dots,\gamma_{i-1},\{\gamma_i,x_{i_1},\dots,x_{i_{n-1}}\}_\Omega,\gamma_{i+1},\dots,\gamma_{n-1}\}_\Omega \\
&= \sum_{i=1}^{n}\{x_j,\gamma_1,\dots,\gamma_{i-1},\{\gamma_i,x_{i_1},\dots,x_{i_{n-1}}\},\gamma_{i+1},\dots,\gamma_{n-1}\}_\Omega \\
&= \sum_{i=1}^{n}(-1)^{n-1}\{x_j,\gamma_1,\dots,\gamma_{i-1},[\gamma_i,\Omega]_{p,q},\gamma_{i+1},\dots,\gamma_{n-1},\Omega\} \\
&= \sum_{i=1}^{n}(-1)^{n-1+j}\mathrm{Jac}_j(\gamma_1,\dots,\gamma_{i-1},[\gamma_i,\Omega]_{p,q},\gamma_{i+1},\dots,\gamma_{n-1},\Omega).
\end{aligned}$$

Therefore, $\mathscr{J}_\Gamma(P_n(\Gamma,\Omega))$ is generated by the elements

$$\sum_{i=1}^{n}\mathrm{Jac}_j(\gamma_1,\dots,\gamma_{i-1},[\gamma_i,\Omega]_{p,q},\gamma_{i+1},\dots,\gamma_{n-1},\Omega)$$

for all $1 \le j, p, q \le n+1$.

The last statement is a consequence of the fact that if the corresponding equations are zero, then the corresponding ideals are trivial, and hence we get the result. □

Now that we have our description with respect to some partial differential equations, let's find some explicit examples.

Example 6.3.0.1. *Let $n = 2$ and $P_2 = k[x, y, z]$, and pick $\gamma, \Omega \in P_2$. Then by theorem 6.3.0.4 the ideal $\mathcal{I}_\gamma(P_2(\Omega))$ is generated by*

$$
\begin{aligned}
&Jac_x(\gamma,\Omega)\frac{\partial\Omega}{\partial x} - Jac_y(\gamma,\Omega)\frac{\partial\Omega}{\partial y}, && Jac_x(\gamma,\Omega)\frac{\partial\Omega}{\partial x} - Jac_z(\gamma,\Omega)\frac{\partial\Omega}{\partial z},\\
&Jac_x(\gamma,\Omega)\frac{\partial\Omega}{\partial y}, && Jac_x(\gamma,\Omega)\frac{\partial\Omega}{\partial z},\\
&Jac_y(\gamma,\Omega)\frac{\partial\Omega}{\partial x}, && Jac_y(\gamma,\Omega)\frac{\partial\Omega}{\partial z},\\
&Jac_z(\gamma,\Omega)\frac{\partial\Omega}{\partial x}, && Jac_z(\gamma,\Omega)\frac{\partial\Omega}{\partial y}.
\end{aligned}
$$

For the ideal $\mathcal{J}_\gamma(P_2(\Gamma, \Omega)$, we have the generators

$$
\begin{array}{ccc}
Jac_x([\gamma,\Omega]_{x,y},\Omega) & Jac_x([\gamma,\Omega]_{x,z},\Omega) & Jac_x([\gamma,\Omega]_{y,z},\Omega)\\
Jac_y([\gamma,\Omega]_{x,y},\Omega) & Jac_y([\gamma,\Omega]_{x,z},\Omega) & Jac_y([\gamma,\Omega]_{y,z},\Omega)\\
Jac_z([\gamma,\Omega]_{x,y},\Omega) & Jac_z([\gamma,\Omega]_{x,z},\Omega) & Jac_z([\gamma,\Omega]_{y,z},\Omega).
\end{array}
$$

Note that there might be some redundancies and they will not be the minimal generators for the ideals. Next, we will pick appropriate γ and Ω and find the corresponding $(m, 2)$-Poisson algebras.

- *Let $\gamma = x + y + z$ and $\Omega = xyz$ and by direct calculation, we have*

$$
\begin{aligned}
Jac_x(\gamma,\Omega)\frac{\partial\Omega}{\partial x} - Jac_y(\gamma,\Omega)\frac{\partial\Omega}{\partial y} &= (xy - xz)yz - (xy - yz)xz = xy^2z - x^2yz\\
Jac_x(\gamma,\Omega)\frac{\partial\Omega}{\partial x} - Jac_y(\gamma,\Omega)\frac{\partial\Omega}{\partial y} &= xy^2z - x^2yz + xy^2 - xyz^2\\
Jac_x(\gamma,\Omega)\frac{\partial\Omega}{\partial y} &= (xy - xz)xz = x^2z(y - z)\\
Jac_x(\gamma,\Omega)\frac{\partial\Omega}{\partial z} &= (xy - xz)xy = x^2y(y - z)
\end{aligned}
$$

and similarly by symmetry and reducing the redundancies, the ideal $\mathcal{I}_\gamma(P_2(\Omega))$ is gen-

erated by

$$\begin{array}{cccc} xyz(x-y) & xyz(x-z) & x^2y(y-z), & xy^2(x-z) \\ & x^2z(y-z) & y^2z(x-z) & \\ & xz^2(x-y) & yz^2(x-y). & \end{array}$$

In $P_2(\Omega)/\mathcal{I}_\Gamma(P_2(\Omega))$ *we have*

$$\begin{aligned} Jac_x([\gamma,\Omega]_{x,y},\Omega) &= Jac_x(xz-yz,xyz) = -xyz - xz(x-y) = x^2z \\ Jac_x([\gamma,\Omega]_{x,z},\Omega) &= Jac_x(xy-yz,xyz) = xy(x-z)+xyz = x^2y \\ Jac_x([\gamma,\Omega]_{y,z},\Omega) &= Jac_x(xy-xz,xyz) = x^2y + x^2y \end{aligned}$$

and by symmetry and reducing the equations the ideal $\mathcal{J}_\gamma(P_2(\gamma,\Omega))$ *is generated by*

$$x^2z \qquad x^2y \qquad xy^2 \qquad y^2z \qquad xz^2 \qquad yz^2.$$

In conclusion, we obtain the $(m,2)$*-Poisson algebra*

$$P_{m,2}(\gamma,\Omega) = \frac{k[x,y,z]}{\langle x^2y, x^2z, xy^2, y^2z, xz^2, yz^2\rangle}$$

with the m*-Com multiplication*

$$\mu_\gamma^m(f_1,\ldots,f_m) = \frac{\partial}{\partial x}(f_1\cdots f_m)x(y-z) - \frac{\partial}{\partial y}(f_1\cdots f_m)y(x-z) + \frac{\partial}{\partial z}(f_1\cdots f_m)z(x-y)$$

This is just a finite-dimensional vector space with basis $1, x, y, z, x^2, y^2, z^2, xy, xz, yz, xyz$ *and with operations*

$$\begin{array}{lll} \{x,y\}_\Omega = xy & \{x,z\}_\Omega = xz & \{y,z\}_\Omega = yz \\ \mu_\gamma^m(x,1,\ldots,1) = x(y-z) & \mu_\gamma^m(y,1,\ldots,1) = -y(x-z) & \mu_\gamma^m(z,1,\ldots,)) = z(x-y) \end{array}$$

with μ_γ^m *on the other basis elements is zero.*

- *Let $\gamma = x^4 - y^4$ and $\Omega = x^4 - z^4$ and by direction calculation we have*

$$Jac_x(\gamma,\Omega)\frac{\partial\Omega}{\partial x} - Jac_y(\gamma,\Omega)\frac{\partial\Omega}{\partial y} = 4^3x^3y^3z^3, \quad Jac_x(\gamma,\Omega)\frac{\partial\Omega}{\partial x} - Jac_z(\gamma,\Omega)\frac{\partial\Omega}{\partial z} = 2\cdot 4^3x^3y^3z^3,$$

$$Jac_x(\gamma,\Omega)\frac{\partial\Omega}{\partial y} = 0, \qquad Jac_x(\gamma,\Omega)\frac{\partial\Omega}{\partial z} = -4^3y^3z^6,$$

$$Jac_y(\gamma,\Omega)\frac{\partial\Omega}{\partial x} = -4^3x^6z^3, \qquad Jac_y(\gamma,\Omega)\frac{\partial\Omega}{\partial z} = 4^3x^3z^6,$$

$$Jac_z(\gamma,\Omega)\frac{\partial\Omega}{\partial x} = 4^3x^6y^3, \qquad Jac_z(\gamma,\Omega)\frac{\partial\Omega}{\partial y} = 0.$$

In other words, the ideal $\mathcal{I}_\gamma(P_2(\Omega))$ is generated by $x^3y^3z^3, x^6z^3, y^3z^6, x^3z^6, x^6y^3$. For the ideal $\mathcal{J}_\gamma(P_2(\gamma,\Omega))$, computing the relations we obtain

$$Jac_x([\gamma,\Omega]_{x,y},\Omega) = -3\cdot 4^3x^3y^2z^3 \qquad Jac_x([\gamma,\Omega]_{x,z},\Omega) = 0$$

$$Jac_x([\gamma,\Omega]_{y,z},\Omega) = -3\cdot 4^3y^2z^6 \qquad Jac_y([\gamma,\Omega]_{x,y},\Omega) = -3\cdot 4^3x^2y^3z^3$$

$$Jac_y([\gamma,\Omega]_{x,z},\Omega) = 3\cdot 4^3x^2z^6 + 3\cdot 4^3x^6z^2 \qquad Jac_y([\gamma,\Omega]_{y,z},\Omega) = -3\cdot 4^3x^3y^3z^2$$

$$Jac_z([\gamma,\Omega]_{x,y},\Omega) = -3\cdot 4^3x^6y^2 \qquad Jac_z([\gamma,\Omega]_{x,z},\Omega) = 0$$

$$Jac_z([\gamma,\Omega]_{y,z},\Omega) = 0.$$

This shows that the ideal $\mathcal{J}_\gamma(P_2(\gamma,\Omega))$ is generated by

$$x^2y^3z^3, \qquad x^3y^2z^3, \qquad x^3y^3z^2, \qquad x^6y^2, \qquad x^2z^6 + x^6z^2$$

and hence

$$P_{m,2}(\gamma,\Omega) = \frac{k[x,y.z]}{\langle x^2y^3z^3, x^3y^2z^3, x^3y^3z^2, x^6y^2, x^2z^6 + x^6z^2\rangle}$$

with the m-Com multiplication

$$\mu_\gamma^m(f_1,\ldots,f_m) = \frac{\partial}{\partial x}(f_1\cdots f_m)y^3z^3 + \frac{\partial}{\partial y}(f_1\cdots f_m)x^3z^3 + \frac{\partial}{\partial z}(f_1\cdots f_m)x^3y^3.$$

In other words, $P_{m,2}(\gamma,\Omega)$ is the infinite-dimensional vector space with the operations

$$\{x,y\}_\Omega = -4z^3 \qquad \{x.z\}_\Omega = 0 \qquad \{y,z\}_\Omega = 4x^3$$

and

$$\mu_\gamma^m(x,1,\ldots,1)=y^3z^3 \qquad \mu_\gamma^m(y,1,\ldots,1)=x^3z^3 \qquad \mu_\gamma^m(z,1,\ldots,1)=x^3y^3$$

and it is not too hard to compute the others.

6.4 The Corresponding Operads

In all of these generalizations, they have one particular relation that they all have in common: the generalized Leibniz relation that relates the symmetric operation and the skew-symmetric operation. Recall from section 2.1.3, that we have the free operad $FP_{(n,d_n),(m,d_m)}$ for $n,m\geq 2$ and $d_n,d_m\in\mathbb{Z}$, which has

$$\bigoplus_{a_1<\cdots<a_{n-1}} E_{m,d_m}(m)\otimes H_{n,d_n}(n)\{a_1,\ldots,a_{m-1}\}\oplus \bigoplus_{b_1<\cdots<b_{n-1}} H_{n,d_n}(n)\otimes E_{m,d_m}\{b_1,\ldots,b_{n-1}\}$$

in the space $FP_{(n,d_n),(m,d_m)}(n+m-1)$. This is where we have the relationship between the operations μ_{n,d_n} and ν_{m,d_m} in terms of the generalized Leibniz rule. In terms of the rewriting rule, let $\gamma_{n,m}$ be the rewriting rule defined as taking

$$\gamma_{n,m}(\nu^{d_m}\mu^{d_n}_{\{n+1,\ldots,n+m-1\}})=\sum_{i=0}^{n-1}(-1)^{m(n+i+1)+i}(\mu^{d_n}\nu^{d_m}_{\{n+1,\ldots,n+m-1\}})^{\sigma_{n,m,i}}. \tag{6.5}$$

and all appropriate permutations. Thus we have $GLeib_{n,m}$ to be the Σ_{n+m-1}-submodule of $FP_{(n,d_n),(m,d_m)}(n+m-1)$ generated by $\nu^{d_m}\mu^{d_n}_{\{n+1,\ldots,2n-1\}}-\gamma_{n,m}(\nu^{d_m}\mu^{d_n}_{\{n+1,\ldots,2n-1\}})$. Then we can define the Σ-submodule $RP_{n,m}$ of $FP^{(2)}_{(n,d_n),(m,d_m)}$ with

$$RP_{n,n}(2n-1)=S_{n,d}\oplus GLeib_{n,n}\oplus R^0_{n,d}\text{otherwise} \tag{6.6}$$

if $n=m$ and

$$RP_{n,m}(2n-1)=S_{n,d}$$

$$RP_{n,m}(2m-1)=R^0_{m,d}$$

$$RP_{n,m}(n+m-1)=GLeib_{n,m}$$

when $n\neq m$.

Definition 6.4.0.1. *For* $n, m \geq 2$ *we define* $(n,m){-}Pois_{(d_n,d_m)} = FP_{(n,d_n),(m,d_m)}/(RP_{n,m}) \cong (n\text{-}Com_{d_n}) \wedge_\gamma (m - Lie_{d_m})$.

Chapter 7

ODD OPERADS

7.1 Generalized Odd Graphs

In this section, we will review the construction and properties of the generalized Kneser graphs which will be useful for constructing our operads based on these graphs.

Definition 7.1.0.1. *For $n \geq 2$, let $\Lambda(n)$ be the collection of ordered sequences $(a_1, \ldots, a_{n-1})$ of elements $a_1, \ldots, a_{n-1} \in \{1, \ldots, 2n-1\}$. For each $n \geq 2$ and $1 \leq s \leq n-1$, define $\mathcal{O}_{n,s}$ to be the graph with vertices $\Lambda(n)$ and an edge between σ and τ in $\Lambda(n)$ if and only if $|\sigma \cap \tau| < s$.*

In the literature, $\mathcal{O}_{n,s} = K(2n-1, n-1, s)$ where $K(n, r, s)$ is the generalized Kneser graphs. Furthermore, when $s = 1$ the graphs $\mathcal{O}_n$ are called the Odd graphs and are a subset of the collection of Kneser graphs whose spectrum is known in general.

Furthermore, there is a natural action Σ_{2n-1} on the set of vertices $\Lambda(n)$ by just applying $\sigma \in \Sigma_{2n-1}$ on the elements in the tuple and then reordering the tuple.

Example 7.1.0.1.

- *For $n = 2$ and $s = 1$, the graph $\mathcal{O}_{2,1} = \mathcal{O}_2$ is just the triangle in figure 7.1 with the vertices $\{1\}, \{2\}, \{3\}$ with an edge between each of them.*

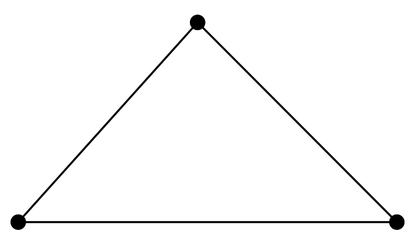


Figure 7.1: The graph $\mathcal{O}_2$

- *For $n = 3$ and $s = 1$, $\mathcal{O}_{3,1} = \mathcal{O}_3$ is the Peterson graph in figure 7.2, with vertices* $(1, 2), (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)$.

- *For $n = 3$ and $s = 2$, we obtain the graph the K_{10}, the complete graph on 10 vertices as in figure 7.3.*

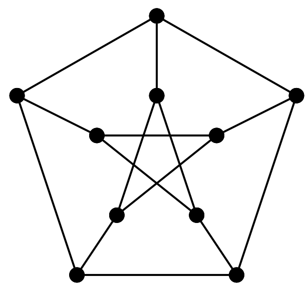


Figure 7.2: The Peterson Graph

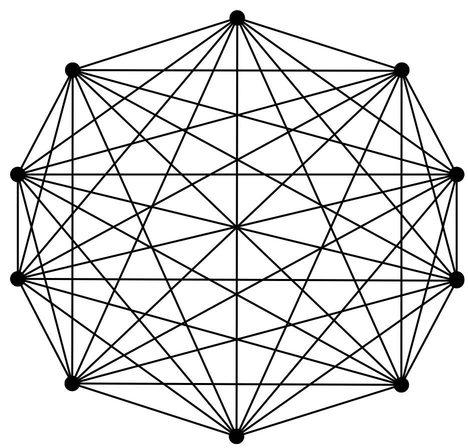


Figure 7.3: Graph K_{10}

- *In general, if $n \geq 2$ and $s = n-1$ then the graph $\mathcal{O}_{n,n-1}$ is the complete graph K_m where $m = |\Lambda(n)| = \begin{pmatrix} 2n-1 \\ n-1 \end{pmatrix}$.*

For $n \geq 2$ and $1 \leq s \leq n-1$, the adjacency matrix $B(\mathcal{O}_{n,s})$ associated with $\mathcal{O}_{n,s}$ is defined as

$$B(\mathcal{O}_{n,s})_{\tau,\omega} = \begin{cases} 1 & \text{if } |\tau \cap \omega| < s \\ 0 & \text{otherwise} \end{cases} \tag{7.1}$$

for any $\omega, \tau \in \Lambda(n)$ with respect to the ordered basis on $k\Lambda(n)$ with respect to the lexicographical ordering. The associated linear map $T_{n,s} : k\Lambda(n) \to k\Lambda(n)$ is defined as

$$T_{n,s}(\omega) = \sum_{\substack{\tau \in \Lambda(n) \\ |\tau \cap \omega| < s}} \tau. \tag{7.2}$$

. Furthermore, the maps $T_{n,s}$ are invariant under the Σ_{2n-1} actions and hence the $\mathrm{Ker}(T_{n,s} - \lambda \mathrm{I})$ and $\mathrm{Im}(T_{n,s} - \lambda \mathrm{I})$ are all left $k[\sigma_{2n-1}]$-modules for every eigenvalue λ of $T_{n,s}$.

7.1.1 Spectrum of $\mathcal{O}_n$

Recall that if we have a graph G with eigenvalues $\lambda_n \geq \cdots \geq \lambda_1$, then its spectrum is

$$Spec(G) = \begin{pmatrix} \lambda_n & \cdots & \lambda_1 \\ m_n & \cdots & m_1 \end{pmatrix} \tag{7.3}$$

where m_i is the multiplicity of λ_i. For this section, we will give a self-contained proof of the spectrum of $\mathcal{O}_n$ for all $n \geq 2$ and show that the multiplicities of the eigenvalues $(-1)^{n+1}$ are the Catalan numbers C_n.

We can compute the spectrum of O_2 and O_3 fairly easily using basic linear algebra tools.

Example 7.1.1.1.

- *For O_2, its adjacency matrix is*

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

  which it is easy to show that its spectrum is

$$Spec(O_2) = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

- *For O_3, this is the Peterson graph, and it is known in the literature to have spectrum*

$$Spec(O_3) = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 5 & 4 \end{pmatrix}$$

- *For $n \geq 2$ and $s = n-1$, the graph $\mathcal{O}_{n,n-1}$ is the complete graph K_m for $m = \binom{2n-1}{n-1}$ and hence its spectrum is*

$$\begin{pmatrix} m-1 & -1 \\ 1 & m-1 \end{pmatrix}. \tag{7.4}$$

Furthermore, the eigenspace $Ker(T_{n,n-1} + I)$ has a basis consisting of $\sigma - \tau$ for $\sigma < \tau$.

Before we compute the spectrum of O_n for $n \geq 2$, we need to state a few results about a particular proper Riordan array that is associated with the spectrum of O_n. A Riordan array is a pair $(d(t), h(t))$ of formal power series such that $d(0) \neq 0$ and $h(0) \neq 0$, as defined in [19]. This defines a infinite, lower triangular array $\{d_{n,k}\}_{n,k\in\mathbb{N}}$ such that

$$d_{n,k} = [t^n]d(t)(th(t))^k$$

where $[t^n]$ is saying to extract the coefficient in front of t^n. In other words, $d(t)(th(t))^k$ is the generating function for the kth column of this array.

Next, we will define a proper Riordan array that was first introduced by B. Shapiro in [23] in the study of a walk problem on the non-negative quadrant on the integral square lattice in two-dimensional Euclidean space, which they called the Catalan triangle.

Definition 7.1.1.1. *Let $\mathcal{E} = \{\mathcal{E}_{n,k}\}_{n,k\in\mathbb{N}}$ be the lower triangular array with*

$$\mathcal{E}_{n,k} = \mathcal{E}_{n-1,k-1} + 2\mathcal{E}_{n-1,k} + \mathcal{E}_{n-1,k+1}$$

for all $n, k \geq 0$ and define

$$\mathcal{E}_{0,0} = 1$$
$$\mathcal{E}_{n,k} = 0 \quad \textit{for } n < k.$$

One can use the explicit characterizations in [19] to show that $\mathcal{E}$ is a proper Riordan array with

$$d(t) = h(t) = \frac{c(t) - 1}{t} = \sum_{i=0}^{\infty} C_{k+1} t^i$$

where $C(t)$ is the generating function for the Catalan numbers as in section 1.2.0.1. Furthermore, it is known that $C_n^{(1)} = C_{n+1}$ for all $n \geq 1$ by theorem 1.2.0.3, where $C_n^{(k)}$ is the k-fold convolution of the Catalan numbers. Therefore, the generating function for the (1)-fold Catalan numbers is

$$d(t) = \sum_{i=0}^{\infty} C_n^{(1)} t^i$$

In other words, we have $\mathcal{E}_{n,0} = C_n^{(1)}$. Furthermore, the generating function $d_k(t)$ for the kth column of $\{\mathcal{E}_{n,k}\}$ is

$$\begin{aligned} f_k(t) &= d(t)(th(t))^k \\ &= t^k(d(t))^{k+1} \end{aligned}$$

where $d_0(t) = d(t)$. The first few lines of the lower triangular array $\mathcal{E}_{n,k}$ are

$$\begin{array}{ccccccc} 1 & & & & & & \\ 2 & 1 & & & & & \\ 5 & 4 & 1 & & & & \\ 14 & 14 & 6 & 1 & & & \\ 42 & 48 & 27 & 8 & 1 & & \\ 132 & 165 & 110 & 44 & 10 & 1 & \\ 429 & 572 & 429 & 208 & 65 & 12 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

As one may notice, the columns are exactly counted by the k-fold convolutions $C_n^{(k)}$ where k is odd, which we will now prove.

lemma 7.1.1.1. *For any $n \geq 0$ and $0 \leq k \leq n$, we have*

$$\mathcal{E}_{n,k} = C_{n-k}^{(2k+1)}.$$

Proof. It suffices to describe the generating function $d(t)^{k+1}$ and show it is the generating function for the $(2k+1)$-fold convolution of the Catalan numbers. By the Cartan multiplication, we have

$$[t^n]d(t)^{k+1} = \sum_{x_1+\cdots x_{k+1}=n} C_{x_1}^{(1)} \cdots C_{x_{k+1}}^{(1)}$$

and since each $C_{x_i}^{(1)} = \sum_{y_1^i+y_2^i=x_i} C_{y_1^i} C_{y_2^i}$ for each i, then this implies

$$\begin{aligned}\sum_{x_1+\cdots x_{k+1}=n} C_{x_1}^{(1)} \cdots C_{x_{k+1}}^{(1)} &= \sum_{\substack{y_1^i+y_2^i=x_i \ : \ 1\leq i\leq k+1 \\ x_1+\cdots+x_{k+1}=n}} C_{y_1^1} C_{y_2^1} \cdots C_{y_1^{k+1}} C_{y_2^{k+1}} \\ &= \sum_{y_1^1+y_2^1+y_1^2+\cdots+y_1^{k+1}+y_2^{k+1}=n} C_{y_1^1} C_{y_2^1} \cdots C_{y_1^{k+1}} C_{y_2^{k+1}} \\ &= C_n^{(2k+1)}.\end{aligned}$$

Therefore, we have

$$[t^n]d_k(t) = C_{n-k}^{(2k+1)}$$

and this proves $d_k(t)$ is the generating function for the $(2k+1)$-fold convolution of the Catalan numbers shifted and hence proves the lemma. □

By theorem 1.2.0.3, it is known that the (k)-fold convolution of the Catalan numbers is given by the formula

$$C_n^{(k)} = \frac{k+1}{n+k+1}\binom{2n+k}{n}.$$

Therefore, if $n \geq 0$ and $k < n$, then we have

$$\mathcal{E}_{n,k} = \frac{2k+2}{n+k+2}\binom{2n+1}{n-k}$$

On the other hand, if we let $\mu_{n,k} = (n+k, n-k-1)$ be a partition of $2n-1$ for $0 \leq k \leq n-1$, then by the hook length formula we have

$$\begin{aligned} f^{\mu_{n,k}} &= \frac{(2n-1)!}{\prod_{i=k+1}^{n-1}(i+k+2)(2k+1)!(n-k-1)!} \\ &= \frac{(2n-1)!(2k+2)}{(n+k+1)!(n-k-1)!} \\ &= \frac{2k+2}{n+k+1}\binom{2n-1}{n-k-1}. \end{aligned}$$

Therefore, we have

$$f^{\mu_{n+1,k}} = \frac{2k+2}{n+k+2}\binom{2n+1}{n-k} = \mathcal{E}_{n,k}$$

from above, which shows $\mathcal{E}_{n,k}$ is counting the number of standard Young tableaux of shape $\mu_{n+1,k}$.

Using the proper Riordan array $\mathcal{E}$, we can find the spectrum for $\mathcal{O}_n$ for all $n \geq 2$ by using a useful property about the sequence of adjacency matrices $\{B(\mathcal{O}_n)\}_{n\geq 2}$ in which we can use induction to compute the characteristic polynomials. The proof of the following lemma is postponed to section 7.2, where we will develop the required tools and lemmas needed to prove the following lemma.

lemma 7.1.1.2. *For any $n \geq 2$, O_n has eigenvalues $\lambda_1^n, \ldots, \lambda_n^n$ where*

$$\lambda_i^n = (-1)^{n+i} i.$$

If $m_{n,i}$ are the multiplicities for λ_i^n, then

$$m_{n,i} = \mathcal{E}_{n-1,i-1}.$$

We have a natural left $k[\Sigma_{2n-1}]$-module isomorphism $\varphi_n : M^{(n,n-1)} \to k\Lambda(n)$ by defining $\varphi_n(\{T\}) = (a_1, \ldots, a_{n-1})$, where $a_1 < \cdots < a_{n-1}$ is the bottom row of T. We know $M^{(n,n-1)}$ decomposes into a direct sum of $S^{\mu_{n,i}}$ for $0 \leq i \leq n-1$ by 1.3.3.2 which implies

$$\dim k\Lambda(n) = \sum_{i=0}^{n-1} f^{\mu_{n,i}}.$$

On the other hand, we have the eigenvalues for $T_{n,1}$ with each $E_i^n = \ker(T_{n,1} - \lambda_i^n \mathrm{I})$ has dimension $\mathcal{E}_{n-1,i-1} = f^{\mu_{n,i-1}}$. This shows that $k\lambda(n) = \bigoplus_{i=1}^n E_i^n$ has the decomposition into its eigenspaces. Furthermore, let $\sigma = (a_1, \ldots, a_{n-1})$ and $DS(\sigma)$ is the set of $\omega \in \Lambda(n)$ such that $\sigma \cap \omega = \emptyset$. If $g \in \Sigma_{2n-1}$ and $\omega \in DS(\sigma)$, then ω^g is in $DS(\sigma^g)$ by the fact that they permute the same numbers and they will be disjoint to each other under the action. This shows that

$$T_{n,1}(\sigma^g) = \sum_{theta \in DS(\sigma^g)} \theta = \sum_{\omega \in DS(\sigma)} \omega^g = T_{n,1}(\sigma)^g$$

and hence $T_{n,1}$ is Σ_{2n-1}-invariant. Therefore, we have E_i^n are each Σ_{2n-1}-modules which decompose the space $k\Lambda(n)$.

lemma 7.1.1.3. *The space E_i^n are irreducible and are isomorphic to $S^{\mu_{n,i-1}}$ through the isomorphism $\varphi_n : M^{(n,n-1)} \to k\Lambda(n)$.*

Proof. Through the isomorphism $\varphi_n : M^{(n,n-1)} \to k\Lambda(n)$ for each $1 \leq i, j \leq n$ this isomorphism induces Σ_{2n-1}-equivariant maps $\varphi_n^{i,j} : S^{\mu_{n,i-1}} \to E_j^n$ for $1 \leq i, j \leq n$ such that they are an isomorphism for only one of the j's through the decomposition and isomorphism. By comparing the dimensions; we must have $\varphi_n^{i,i}$ is an isomorphism, and the rest are zero. This shows the proof. □

The last lemma shows that through the isomorphism φ_n we have E_1^n is a Σ_{2n-1}-cyclic module with generator $\sum_{g \in C_n} sgn(g)(n+1, \ldots, 2n-1)^g$. These are exactly the relations in the n-Com operad as we will show later.

7.2 Proof of lemma 7.1.1.2

In this subsection, we will lay out the groundwork of definitions and technical lemmas that are used to prove lemma 7.1.1.2. First, we define balanced matrix sequences, which consist of a sequence of matrices with certain properties that make it possible to give a description of the characteristic polynomial in a recursive manner. Next, we will use the properties of

$\Lambda(n)$ to show that the sequence of matrices $\{B(\mathcal{O}_n)\}_{n\geq 1}$ is a balanced matrix sequence and hence prove lemma 7.1.1.2.

7.2.1 Balanced Matrix Sequences

Definition 7.2.1.1. *A balanced matrix sequence is a triple* $\{(B(n), C(n), D(n))\}_{n\geq 1}$ *of matrices with*

$$B(n) = \begin{pmatrix} 0 & C(n) \\ C(n)^T & J \end{pmatrix}$$

$$C(n) = \begin{pmatrix} 0 & D(n) \\ B(n-1) & J \end{pmatrix}$$

with with

$$D(n)^T D(n) + I = JB(n-1)^2 J$$

, where J *is the anti-diagonal square identity matrix. We will denote such sequences as* (B, C, D)*.*

For any such sequence, define a sequence of tuples (N_n, M_n) where $C(n)$ is a $N_n \times M_n$ matrix and all the other sizes of the matrices are derived from these two indices. For any square matrix A, denote by $P_A(x) = det(A - xI)$ the characteristic polynomial of A.

Theorem 7.2.1.1. *If* (B, C, D) *is a balanced matrix sequence, then we have the following relationship between the characteristic polynomials:*

$$P_{B(n)}(x) = -(-1)^{N_n+N_{n-1}} x^{N_n - M_n} P_{B(n-1)}(-x)^2 P_{B(n-1)}(x-1) P_{B(n-1)}(x+1)$$

Proof. This proof just uses some standard theorems from linear algebra. First, note by standard multiplication of block matrices we have

$$C(n)^T C(n) = \begin{pmatrix} B(n-1)^2 & B(n-1)J \\ JB(n-1) & D(n)^T D(n) + I \end{pmatrix}$$
$$= \begin{pmatrix} B(n-1)^2 & B(n-1)J \\ JB(n-1) & JB(n-1)^2 J \end{pmatrix}$$

By definition of characteristic polynomial

$$\begin{aligned} P_{B(n)}(\lambda) &= det(B(n) - \lambda I) \\ &= det \begin{pmatrix} -\lambda I & C(n) \\ C(n)^T & J - \lambda I \end{pmatrix} \\ &= det(-\lambda I) det(J - \lambda I + \frac{1}{\lambda} C(n)^T C(n)) \\ &= (-1)^{N_n} \lambda^{N_n - M_n} det(\lambda J - \lambda^2 I + C(n)^T C(n)), \end{aligned}$$

where we used Schur's complement for the second determinant. Row-reducing the matrix $\lambda J - \lambda^2 I + C(n)^T C(n)$, we obtain the matrix

$$\begin{pmatrix} 0 & -f(\lambda, B(n-1))J \\ JB(n-1) + \lambda J & JB(n-1)^2 J - \lambda^2 J \end{pmatrix}$$

where

$$f(x, b) = x^3 - x^2 b - x(1 + b^2) - b + b^3$$

Note that $f(x, b)$ can be decomposed as

$$f(x, b) = (b - x - 1)(b - x + 1)(b + x)$$

so that our matrix becomes

$$\begin{pmatrix} 0 & -(B(n-1) - (\lambda+1)I)(B(n-1) - (\lambda-1)I)(B(n-1) + \lambda I)J \\ JB(n-1) + \lambda J & JB(n-1)^2 J - \lambda^2 I \end{pmatrix}.$$

Computing the determinate of this gets us our result. □

7.2.2 Properties of the vertices in $\mathcal{O}_n$

Recall the definition of $\Lambda(n)$ from section 7.3, which give us the basis elements that construct our adjacency matrix $B(\mathcal{O}_n)$. For each $1 \leq i \leq n+1$, define $\Lambda_i(n)$ to be the subset of $\Lambda(n)$ consisting of elements starting with i and $\Lambda_{i+}(n)$ to be the set $\prod_{j=i+1}^{n+1} \Lambda_i(n)$. Then for each $1 \leq i \leq n$, define $\Lambda_{i,i+1}(n)$ to be the subset of $\Lambda_i(n)$ consisting of ordered lists containing both i and $i+1$, and $\Lambda_{i,i+1,+}$ to be the complement of $\Lambda_{i,i+1}(n)$ in $\Lambda_i(n)$.

Define the usual set operations on the elements of $\Lambda(n)$ by using the associated set to each sequence. Furthermore, we can put a natural lexicographic ordering on $\Lambda(n)$, which gives us a way to define a distance function between any two elements. For $\sigma, \tau \in \Lambda(n)$, define $d(\sigma, \tau)$ to be equal to one plus the number of elements between them with the ordering and 0 if they are the same element.

One particular case that will be useful for us is the following. Let $\omega = (a_1, \ldots, a_{n-1}) \in \Lambda(n)$ which is not $(n+1, \ldots, 2n-1)$, and suppose $\tau \in \Lambda(n)$ such that $\omega \leq \tau$ and $d(\omega, \tau) = 1$. In ω, there is a maximal i with $1 \leq i \leq n-1$ such that

$$\omega = (a_1, \ldots, a_i, n+1+i, n+2+i, \ldots, 2n-1)$$

with $a_i < n+i$. Then τ can be explicitly described as

$$\tau = (a_1, \ldots, a_{i-1}, a_i+1, a_i+2, \ldots, a_i+n-i).$$

For our purposes, we want to preserve the ordering and the distance at the same time. Define the signed distance function d_S with

$$d_S(\sigma, \tau) = \begin{cases} d(\sigma, \tau) & \text{if } \sigma \leq \tau \\ -d(\sigma, \tau) & \text{if } \tau \leq \sigma \end{cases}.$$

It is an easy consequence of this definition that we have the following identity for any $\sigma, \tau, \omega \in \Lambda(n)$:

$$d_S(\sigma, \tau) + d_S(\tau, \omega) = d_S(\sigma, \omega).$$

Next, we have a few technical lemmas that describe the relationship between the subsets and some properties of the signed distance function.

lemma 7.2.2.1. *We have the following properties for* $\Lambda(n)$*.*

(a) We have the following cardinalities:

$$|\Lambda(n)| = \binom{2n-1}{n-1} \qquad |\Lambda_i(n)| = \binom{2n-1-i}{n-2}$$
$$|\Lambda_{i,i+1}(n)| = \binom{2n-2-i}{n-3} \qquad |\Lambda_{i,i+1,+}(n)| = \binom{2n-2-i}{n-2}$$
$$|\Lambda_{i+}(n)| = \binom{2n-1-i}{n-1}$$

(b) We have $|\Lambda_2(n)| = |\Lambda_{2+}(n)| = \frac{1}{2}|\Lambda_{1+}(n)|$*.*

(c) For each $\sigma \in \Lambda(n)$*, the set of elements disjoint from* σ *has cardinality* n*.*

(d) For each $\sigma \in \Lambda_2(n)$*, there is a unique* $\Gamma_2(\sigma) \in \Lambda_{2+}(n)$ *such that* $\Gamma_2(\sigma) \cap \sigma = \emptyset$*.*

(e) For any $\sigma \in \Lambda_{12}(n)$*, there is a unique* $\Gamma_{12}(\sigma) \in \Lambda_{2+}(n)$ *such that* $\Gamma_{12}(\sigma) \cap \sigma = \emptyset$*.*

Proof. Part (a) and (c) uses standard counting arguments and part (b) uses binomial coefficient identities.

Part (d) and part (e) are similar proofs, so we will just prove part (d). For any $\sigma \in \Lambda_2(n)$, it is of the form

$$\sigma = (2, a_2, \ldots, a_{n-1})$$

with $a_2, \ldots, a_{n-1} \in \{3, \ldots, 2n-1\}$. The set $\{3, \ldots, 2n-1\} \setminus \{a_2, \ldots, a_{n-1}\}$ has cardinality $2n-3-n+2 = n-1$. Hence, there is only one other element that is disjoint with σ, i.e. $\Gamma_2(\sigma)$. □

Note that the functions Γ_2 and Γ_{12} are their own inverses, since if we have $\Gamma_2(\sigma) \cap \sigma = \emptyset$, then $\Gamma_2\Gamma_2(\sigma) \cap \Gamma_2(\sigma) = \emptyset$, then by uniquness we must have $\Gamma_2\Gamma_2(\sigma) = \sigma$, and a similar argument for Γ_{12}.

Next, we will see how the new functions Γ_2 and Γ_{12} interact with the signed distance function in the following lemma.

lemma 7.2.2.2. *We have the following properties.*

(a) Suppose $\omega, \tau \in \Lambda_2(n)$ such that $\omega \leq \tau$, then $\Gamma_2(\tau) \leq \Gamma_2(\omega)$.

(b) If $\omega, \tau \in \Lambda_2(n)$ such that $d_S(\tau, \omega) = 1$, then $d_S(\Gamma_2(\omega), \Gamma_2(\tau)) = 1$.

(c) For any $\omega \in \Lambda_2(n)$, we have

$$d_S((2, \dots, n), \omega) = d_S(\Gamma(\omega), (n+1, \dots, 2n-1)).$$

(d) If $\omega, \tau \in \Lambda_2(n)$, then $|\omega \cap \tau| = n-2$ if and only if $|\Gamma_2(\omega) \cap \Gamma_2(\tau)| = n-3$.

Proof. For part (a), if $\omega \leq \tau$, then there exists an i with $1 \leq i \leq n-2$ such that

$$\omega = (2, a_2, \dots, a_{i-1}, a_i, \dots, a_{n-1})$$
$$\tau = (2, a_2, \dots, a_{i-1}, b_i, \dots, b_{n-1})$$

with $a_i < b_i$. Therefore,

$$\Gamma_2(\omega) = (c_1, \dots, c_r, c_{r+1}, \dots, c_{n-1})$$
$$\Gamma_2(\tau) = (c_1, \dots, c_r, d_{r+1}, \dots, d_{n-1}),$$

where the first r elements are the same since the first $i-1$ elements are the same for τ and ω and $d_{r+1} = a_i$ since $a_{i-1} < a_i < b_i$ and c_{r+1} is any element $a_i < c_{r+1}$, and this shows $\Gamma_2(\tau) \leq \Gamma_2(\omega)$.

For part (b), suppose for a contradiction that $d_S(\Gamma_2(\omega), \Gamma_2(\tau)) \neq 1$. If $d_S(\Gamma_2(\omega), \Gamma_2(\tau)) =$

0, then $\Gamma_2(\omega) = \Gamma_2(\tau)$ and by part (a), we have $\omega = \tau$ which contradicts $d_S(\omega, \tau) = 1$. If $d_S(\Gamma_2(\Omega), \Gamma_2(\tau)) > 1$, then there exists $\sigma \in \Lambda_2(n)$ such that

$$\Gamma_2(\omega) < \sigma < \Gamma_2(\tau)$$

which implies

$$\tau < \Gamma_2(\sigma) < \omega$$

and this contradicts $d_S(\tau, \Omega) = 1$.

For part (c), we will prove this by induction. First, it is clear

$$\begin{aligned} d_S((2,\ldots,n),(2,\ldots,n)) &= d_S((n+1,\ldots,2n-1),(n+1,\ldots,2n-1)) \\ &= d_S((\Gamma_2((2,\ldots,n)),(n+1,\ldots,2n-1)). \end{aligned}$$

Next, suppose $\sigma \in \Lambda_2(n)$ such that $(2,\ldots,n) \leq \sigma$ and it is not the last element in $\Lambda_2(n)$ such that

$$d_S((2,\ldots,n),\sigma) = d_S(\Gamma_2(\sigma),(n+1,\ldots,2n-1)).$$

If $\tau \in \Lambda_2(n)$ such that $d_S(\sigma, \tau) = 1$, then we have

$$\begin{aligned} d_S((2,\ldots,n),\tau) &= d_S((2,\ldots,n),\sigma) + d_S(\sigma,\tau) \\ &= d_S(\Gamma_2(\sigma),(n+1,\ldots,2n-1)) + d_S(\Gamma_2(\tau),\Gamma_2(\sigma)) \\ &= d_S(\Gamma_2(\tau),(n+1,\ldots,2n-1)), \end{aligned}$$

which proves part (c).

For part (d), if $|\omega \cap \tau| = n-2$, then we have

$$\begin{aligned} \omega &= (2, a_1, \ldots, a_i, a_{i+1}, \ldots, a_{n-2}) \\ \tau &= (2, a_1, \ldots, a_i, b_{i+1}, a_{i+2}, \ldots, a_{n-2}) \end{aligned}$$

where $a_{i+1} \neq b_{i+1}$. We have

$$\{3,\ldots,2n-1\} \setminus \{a_1,\ldots,a_{n-2}\} \cap \{3,\ldots,2n-1\} \setminus \{a_1,\ldots,a_i,b_{i+1},a_{i+2},\ldots,a_{n-2}\}$$

has $n-3$ elements since we took out almost all the same elements except for one in both that is not common. This implies $|\Gamma_2(\omega) \cap \Gamma_2(\tau)| = n-3$. □

Similar statements hold for Γ_{12} with similar proofs.

7.2.3 The Adjacency Matrix of $\mathcal{O}_n$

With the technical information about the vertices, we can apply this to show that $B(\mathcal{O}_n)$ gives us a balanced matrix sequence.

lemma 7.2.3.1. *For any $n \geq 2$, we have*

$$B(O_n)^2_{\tau,\omega} = \begin{cases} n & \text{if } \tau = \omega \\ 1 & \text{if } |\tau \cap \omega| = n-2 \\ 0 & \text{if } 0 \leq |\tau \cap \omega| \leq n-3 \end{cases}.$$

Proof. By definition, for any $\tau, \omega \in \Lambda(n)$ we have

$$\begin{aligned}(B(O_n)^2)_{\tau,\omega} &= \sum_{\sigma \in \Lambda(n)} B(O_n)_{\tau,\sigma} B(O_n)_{\sigma,\omega} \\ &= \sum_{\substack{\sigma \in \Lambda(n) \\ \sigma \cap \tau = \sigma \cap \omega = \emptyset}} 1.\end{aligned}$$

There are only three cases we need to consider to describe our matrix:

- $\tau = \omega$,
- $|\tau \cap \omega| = n-2$,
- and $0 \leq |\tau \cap \omega| \leq n-3$.

For $\tau = \omega$, by lemma 7.2.2.1, the number of $\sigma \in \Lambda(n)$ such that $\sigma \cap \tau = \emptyset$ is n. Therefore,

$$(B(O_n)^2)_{\tau,\tau} = n.$$

When $|\tau \cap \omega| = n-2$, then the number of elements in $\{1, \ldots, 2n-1\} \setminus (\tau \cup \omega)$ is $n-1$. Hence, there are only $\binom{n-1}{n-1} = 1$ element that intersect both trivially. Therefore,

$$(B(O_n)^2)_{\tau,\omega} = 1$$

in this case.

Finally, when $0 \leq |\tau \cap \omega| \leq n-3$, we have

$$|\{1, \ldots, 2n-1\} \setminus (\tau \cup \omega)| \leq n-2.$$

Therefore, there is no other element that intersects both trivially and hence

$$(B(O_n)^2)_{\tau,\omega} = 0.$$

This gives us our result. □

Next, we will describe the adjacency matrix for O_n, using the fact that it will contain the adjacency matrix of O_{n-1} in a very particular way.

lemma 7.2.3.2. *We have the following properties for our adjacency matrices.*

(a) Let $N_n = |\Lambda_1(n)|$ and $M_n = |\Lambda_{1+}(n)|$. The adjacency matrix $B(O_n)$ satisfies the following block form

$$B(O_n) = \begin{pmatrix} 0 & C(O_n) \\ C(O_n)^T & J \end{pmatrix}$$

where J is the anti-diagonal identity-matrix of the correct size, and $C(O_n)$ is $N_n \times M_n$-submatrix for $\omega \in \Lambda_1(n)$ and $\tau \in \Lambda_+(n)$ such that

$$C(O_n)_{\omega,\tau} = B(O_n)_{\omega,\tau}.$$

(b) The matrix $C(O_n)$ satisfies the block form

$$C(O_n) = \begin{pmatrix} 0 & D(O_n) \\ B(O_{n-1}) & J \end{pmatrix}$$

for some matrix $D(O_n)$ with $D(O_n)^T D(O_n) + I = JB(O_{n-1})^2 J$, where J is the anti-diagonal matrix of the correct size.

Proof. For part (a), the top left block has rows and columns indexed by $\Lambda_1(n)$. Since every element of $\Lambda_1(n)$ has a 1 in it, then they all intersect non-trivially, hence this block is the zero matrix. The bottom right block has rows and columns indexed by $\Lambda_{1+} = \Lambda_2(n) \coprod \Lambda_{2+}(n)$. By lemma 7.2.2.1, for each $\sigma \in \Lambda_2(n)$, there is a unique $\Gamma_2(\sigma) \in \Lambda_{2+}(n)$ such that $\Gamma_2(\sigma) \cap \sigma = \emptyset$ with

$$d_S((2, \dots, n), \sigma) = d_S(\Gamma_2(\sigma), (n+1, \dots, 2n-1)).$$

This shows that the bottom right block is J from the ordering. The top right and bottom left blocks come from $B(O_n)$, which is a symmetric matrix.

For part (b), the top left block is zero since the rows are indexed by $\Lambda_{12}(n)$ and the columns are indexed by $\Lambda_2(n)$ which all have 2's in them. The bottom right block is J as a consequence of lemma 7.2.2.2. For the bottom left block of the matrix $C(O_n)$, the rows are indexed by $\Lambda_{12+}(n)$ and the columns are indexed by $\Lambda_2(n)$. We have well-defined bijections $\varphi_n : \Lambda(n-1) \to \Lambda_2(n)$ and $\psi_n : \Lambda(n-1) \to \Lambda_{12+}(n)$ defined as

$$\varphi_n((a_1, \dots, a_{n-2})) = (2, a_1 + 2, \dots, a_{n-2} + 2)$$
$$\psi_n((a_1, \dots, a_{n-2})) = (1, a_1 + 2, \dots, a_{n-2} + 2).$$

It is clear that the intersection is preserved: if $\sigma, \tau \in \Lambda(n-1)$, then $\sigma \cap \tau = \emptyset$ if and only if $\varphi_n(\sigma) \cap \psi_n(\tau) = \emptyset$. With these bijections, we have that the bottom left block is exactly $B(O_{n-1})$. For the top right block, its rows are indexed by $\Lambda_{12}(n)$ and its columns are indexed by $\Lambda_{2+}(n)$, and denote this submatrix as $D(O_n)$. By definition, for any $\tau, \omega \in \Lambda_{2+}(n)$ we have

$$\begin{aligned}(D(O_n)^T D(O_n))_{\tau,\omega} &= \sum_{\sigma \in \Lambda_{12}(n)} D(O_n)_{\sigma,\tau} D(O_n)_{\sigma,\omega} \\ &= \sum_{\substack{\sigma \in \Lambda_{12}(n) \\ \sigma \cap \tau = \sigma \cap \omega = \emptyset}} 1.\end{aligned}$$

Following the same proof as in lemma 7.2.3.1, we obtain

$$(D(O_n)^T D(O_n))_{\tau,\omega} = \begin{cases} n-2 & \text{if } \tau = \omega \\ 1 & \text{if } |\tau \cap \omega| = n-2 \\ 0 & \text{if } 0 \leq |\tau \cap \omega| \leq n-3 \end{cases}$$

Next, we will show $D(O_n)$ has the required property $D(O_n)^T D(O_n) + I = JB(O_{n-1})^2 J$. The maps φ_n and Γ_2 induce the following linear map

$$k\Lambda(n-1) \xrightarrow{\varphi_n} k\Lambda_2(n) \xrightarrow{\Gamma_2} k\Lambda_{2+}(n)$$

which has matrix representation J by lemma 7.2.2.2. By lemma 7.2.3.1 and our description for $D(O_n)^T D(O_n)$ we have the following: if $\tau = (a_1, \ldots, a_{n-1}) \in k\Lambda_{2+}(n)$, then

$$\begin{aligned}JB(\mathcal{O}_{n-1})^2 J^{-1}(\tau) &= JB(\mathcal{O}_{n-1})^2(\varphi_n^{-1}\Gamma_2(\tau)) \\ &= J((n-1)\varphi_n^{-1}\Gamma_2(\tau) + \sum_{|\varphi_n^{-1}\Gamma_2(\tau) \cap \omega| = n-3} \omega) \\ &= (n-1)\tau + \sum_{|\varphi_n^{-1}\Gamma_2(\tau) \cap \omega| = n-3} \Gamma_2(\varphi_n(\omega)) \\ &= (n-1)\tau + \sum_{|\tau \cap \Gamma_2(\varphi_n \omega))| = n-2} \Gamma_2(\varphi_n(\omega)) \\ &= (n-1)\tau + \sum_{|\tau \cap \sigma| = n-2} \sigma\end{aligned}$$

which is exactly $D(\mathcal{O}_n)^T D(\mathcal{O}_n) + I$.

, we see that $JB(O_{n-1})^2J^{-1} = JB(O_{n-1})^2J = D(O_n)^TD(O_n) + I$, which completes the proof. □

This shows that $\{(B(O_n), C(O_n), D(O_n))\}$ is a balanced matrix sequence as in the definition 7.2.1.1, which gives us a way to relate the characteristic polynomial of $B(O_n)$ with $B(O_{n-1})$ by lemma 7.2.1.1. Since $\{(B(\mathcal{O}_n), C(\mathcal{O}_n), D(\mathcal{O}_n)\}$ is a balanced matrix sequence, we have the sequence (N_n, M_n) as in definition 7.2.1.1 where

$$N_n = \binom{2n-2}{n-2}$$
$$M_n = \binom{2n-2}{n-1}$$

by lemmas 7.2.2.1 and 7.2.3.2 and hence by definition of the Catalan numbers we have

$$M_n - N_n = C_{n-1}.$$

proof of theorem 7.1.1.2 . We will prove this by induction on n. For $n = 2$, we already described the spectrum of O_2, which gives us $m_{2,1} = 2 = \mathcal{E}_{1,0}$ and $m_{2,2} = 1 = \mathcal{E}_{1,1}$.

Next, suppose $n-1 > 2$ and O_{n-1} satisfies the properties in the statement of the theorem. Since $\{(B(O_n), C(O_n), D(O_n))\}$ is a balanced matrix sequence, then by lemma 7.2.1.1, we have

$$P_{B(O_n)}(x) = -(-1)^{N_n-N_{n-1}}x^{N_n-M_n}P_{B(O_{n-1})}(-x)^2P_{B(O_{n-1})}(x-1)P_{B(O_{n-1})}(x+1)$$

where P_A is the characteristic polynomial of a matrix A, and $N_n - M_n = -C_{n-1}$. Since we know the eigenvalues and multiplicities of $B(O_{n-1})$, we have $P_{B(O_n)}(x)$ is of the form

$$\begin{aligned}\pm x^{-C_n}&(x+\lambda_1^{n-1})^{2m_{n-1,1}}\cdots(x+\lambda_{n-1}^{n-1})^{2m_{n-1,n-1}}\\ &\cdot(x-(\lambda_1^{n-1}+1))^{m_{n-1,1}}\cdots(x-(\lambda_{n-1}^{n-1}+1))^{m_{n-1,n-1}}\\ &\cdot(x-(\lambda_1^{n-1}-1))^{m_{n-1,1}}\cdots(x-(\lambda_{n-1}^{n-1}-1))^{m_{n-1,n-1}}\end{aligned}$$

In this product, we either have $(x-(\lambda_1^{n-1}-1)^{m_{n-1,1}} = x^{m_{n-1,1}}$ or $(x-(\lambda_1^{n-1}+1)^{m_{n-1,1}} = x^{m_{n-1,1}}$ depending on if n is odd or even. In either case, $m_{n-1,1} = \mathcal{E}_{n-2,0} = C_{n-1}$ and this cancels out the $x^{-C_{n-1}}$ in the front of the product. Furthermore, the eigenvalues switch signs, and we obtain $\lambda_n^n = n$ and $\lambda_i^n = -\lambda_i^{n-1}$ for $1 \leq i \leq n-1$.

From this product, the multiplicities are

$$\begin{aligned} m_{n,i} &= m_{n-1,i-1} + 2m_{n-1,i} + m_{n-1,i+1} = \\ &= \mathcal{E}_{n-2,i-2} + 2\mathcal{E}_{n-2,i-1} + \mathcal{E}_{n-2,i} \\ &= \mathcal{E}_{n-1,i-1} \end{aligned}$$

for $1 \leq i \leq n$ where $m_{i,j} = 0$ for $j > i$ and $j < 0$. This proves the result by induction. □

7.3 Odd Operads

In this section we will use the graphs $\mathcal{O}_{n,s}$ and their eigenvalues to construct various operads with symmetric operations and their natural Koszul duals with skew-symmetric operations.

For each $n \geq 2$, there is a right Σ_{2n-1}-isomorphism $\Xi_{n,d} :\uparrow^{2d} k\Lambda(n) \to F(H_{n,d})^{(2)}(2n-1)$ sending the ordered list $(a_1, \ldots, a_{n-1})$ to the element $u_{\{a_1,\ldots,a_{n-1}\}}$. On the other hand, we have a right $k[\Sigma_{2n-1}]$-isomorphism $\eta_{n,d} :\uparrow^{2d} k\Lambda(n) \otimes Sgn_n \to F(E_{n,d})^{(2)}(2n-1)$ by sending $(a_1, \ldots, a_{n-1})$ to $\nu_{\{a_1,\ldots,a_{n-1}\}}$.

If λ is an eigenvalue of $T_{n,s}$, we define $K_{n,s}^{\lambda} = \mathrm{Ker}(T_{n,s} - \lambda \mathrm{I})$ as a subspace of $k\Lambda(n)$ and we define $I_{n,s}^{\lambda} = \mathrm{Im}(T_{n,s} - \lambda \mathrm{I}) \otimes Sgn_{2n-1}$ as a subspace of $k\Lambda(n) \otimes Sgn_{2n-1}$.

Definition 7.3.0.1. *Let $n \geq 0$, $1 \leq s \leq n-1$ and λ to be an eigenvalue of $\mathcal{O}_{n,s}$. We define the two n-quadratic operads*

$$Odd_{n,s}^{d}(\lambda) = SMag_{n,d}/(\Xi_{n,d}(\uparrow^{2d} K_{n,s}^{\lambda})) \tag{7.5}$$

and

$$AOdd_{n,s}^{d}(\lambda) = ASMag_{n,d}/(\eta_{n,d}(\uparrow^{2d} I_{n,s}^{\lambda})). \tag{7.6}$$

This gives us a plethora of examples of n-quadratic operads, where $Odd^d_{n,s}(\lambda)$ has a generator which is a symmetric n-arity operation and $AOdd^d_{n,s}(\lambda)$ has a generator which is an skew-symmetric n-arity operation.

Next, we will show that the n-quadratic operads above are Koszul dual to each other up to some change in degree d. To do this, we need to extend the non-degenerate bilinear form used in the Koszul dual of quadratic operads to $k\Lambda(n)$. We define $\langle -,-\rangle : (k\Lambda(n) \otimes Sgn_{2n-1}) \otimes k\Lambda(n) \to k$ with

$$\langle \tau, \omega\rangle = \begin{cases} 1 & \text{if } \tau = \omega \\ 0 & \text{otherwise.} \end{cases}$$

This non-degenerate bilinear form has the property $\langle \tau^\sigma, \omega^\sigma\rangle = Sgn(\sigma)\langle \tau, \omega\rangle$ for any $\sigma \in \Sigma_{2n-1}$ by definition.

lemma 7.3.0.1. *For any $\tau \in k\Lambda(n) \otimes Sgn_{2n-1}$ and $\omega \in k\Lambda(n)$, we have*

$$\langle \tau, \omega\rangle = \langle \Xi_{n,d} \uparrow^{2d} \tau, \eta_{n,-d+n-2} \uparrow^{2(-d+n-2)} \omega\rangle$$

where the right-hand side is with the non-dengenerate bilinear form $\langle -,-\rangle : F(E_{n,d})^{(2)}(2n-1) \otimes F(H_{n,-d+n-2})^{(2)}(2n-1) \to k$ defined in section 2.1.3.

Proof. This is clear through the isomorphisms we have between $\uparrow^{2d} k\Lambda(n) \otimes Sgn_{2n-1}$ and $F(E_{n,d})^{(2)}(2n-1)$ and the isomorphism between $\uparrow^{2(-d+n-2)} k\Lambda(n)$ and $F(H_{n,-d+n-2})^{(2)}(2n-1)$. □

Theorem 7.3.0.2. *The operads $Odd^d_{n,s}(\lambda)$ and $AOdd^{-d+n-2}_{n,s}(\lambda)$ are Koszul dual to each other.*

Proof. Recall that we have a non-degenerate bilinear form $\langle -,-\rangle : F(E_{n,d})^{(2)}(2n-1) \otimes F(H_{n,-d+n-2})^{(2)}(2n-1) \to k$ and non-degenerate bilinear form $\langle -,-\rangle : (k\Lambda(n) \otimes Sgn_{2n-1}) \otimes k\Lambda(n) \to k$ such that for any $\tau \in k\Lambda(n) \otimes Sgn_{2n-1}$ and $\omega \in k\Lambda(n)$ we have

$$\langle \tau, \omega\rangle = \langle \Xi_{n,d} \uparrow^{2d} \tau, \eta_{n,-d+n-2} \uparrow^{2(-d+n-2)} \omega\rangle$$

by lemma 7.3.0.1.

Therefore, it suffices to show

$$(I_{n,s}^{\lambda})^{\perp} = K_{n,s}^{\lambda}$$

for any n and s. By definition, if $u = \sum_{\omega \in \Lambda(n)} \Gamma_\omega \omega \in K_{n,s}^{\lambda}$, then we have

$$\begin{aligned}
0 = (T_n - \lambda I)(u) &= \sum_{\omega \in \Lambda(n)} \Gamma_\omega (T_n(\omega) - \lambda\omega) \\
&= \sum_{\omega \in \Lambda(n)} \Gamma_\omega (\sum_{\substack{\tau \in \Lambda(n) \\ |\tau \cap \omega| < s}} \tau - \lambda\omega) \\
&= \sum_{\substack{\omega, \tau \in \Lambda(n) \\ |\tau \cap \omega| < s}} \Gamma_\omega \tau - \lambda \sum_{\omega \in \Lambda(n)} \Gamma_\omega \omega \\
&= \sum_{\substack{\omega, \tau \in \Lambda(n) \\ |\tau \cap \omega| < s}} \Gamma_\tau \omega - \lambda \sum_{\omega \in \Lambda(n)} \Gamma_\omega \omega \\
&= \sum_{\omega \in \Lambda(n)} (\sum_{\substack{\tau \in \Lambda(n) \\ |\tau \cap \omega| < s}} \Gamma_\tau - \lambda \Gamma_\omega) \omega
\end{aligned}$$

where we switched τ and ω the second the last line above. The above equation implies

$$0 = \sum_{\substack{\tau \in \Lambda(n) \\ |\tau \cap \omega| < s}} \Gamma_\tau - \lambda \Gamma_\omega$$

which shows that the coefficients in $u = \sum_{\omega \in \Lambda(n)} \Gamma_\omega \omega$ satisfies the above equation.

On the other hand, if $u = \sum_{\omega \in \Lambda(n)} \Gamma_\omega \omega \in (I_{n,s}^{\lambda})^{\perp}$, then for any x in $\Lambda(n)$, we have

$$\begin{aligned}
0 = \langle T_{n,s}(x) - \lambda x, u \rangle &= \sum_{\omega \in \Lambda(n)} \Gamma_\omega \langle T_{n,s}(x), \omega \rangle - \sum_{\omega \in \Lambda(n)} \Gamma_\omega \langle \lambda x, \omega \rangle \\
&= \sum_{\omega \in \Lambda(n)} \sum_{\substack{\tau \in \Lambda(n) \\ |\tau \cap x| < s}} \Gamma_\omega \langle \tau, \omega \rangle - \Gamma_x \lambda \\
&= \sum_{\substack{\tau \in \Lambda(n) \\ |\tau \cap x| < s}} \Gamma_\tau - \lambda \Gamma_x
\end{aligned}$$

which is exactly the same equation we got before. This shows $(I^{\lambda}_{n,s})^{\perp} = K^{\lambda}_{n,s}$.

□

Therefore, the graphs $\mathcal{O}_{n,s}$ and each of their eigenvalues gives pairs of Koszul dual operads.

Next, we will give explicit examples of these types of operads, and we will see that $Odd^d_{n,s}(\lambda)$ gives generalizations of Com and $AOdd^d_{n,s}(\lambda)$ will give generalizations of Lie.

Example 7.3.0.1. *Let $n \geq 2$, $s = n-1$, $d \in \mathbb{Z}$ and $\lambda = -1$ for the graph $\mathcal{O}_{n,n-1}$. By example 7.1.1.1, we know that $K^{-1}_{n,n-1}$ has a basis consisting of $\sigma - \tau$ for $\sigma < \tau$, which has dimension $\begin{pmatrix} 2n-1 \\ n-1 \end{pmatrix} - 1$. In its image in the operad $Odd^d_{n,n-1}(-1)$, this gives us the relations generated by*

$$\mu_{\{n+1,\ldots,2n-1\}} - \mu_{\{\sigma(n+1),\ldots,\sigma(2n-1)\}}$$

for all $\sigma \in \Sigma_{2n-1} \setminus \{id\}$, which is exactly the relations for Com^d_n. Its not hard to see $I^{-1}_{n,n-1} \cong S^{(2n-2,1)}$ by

For the Koszul dual operad $AOdd^d_{n,n-1}(-1)$, the space $I^{-1}_{n,n-1}$ is generated by

$$\sum_{\tau \in \Lambda(n)} \tau$$

which is 1 dimensional. In the corresponding operad $AOdd^d_{n,n-1}(-1)$, it has relations generated by

$$\sum_{a_1 < \cdots < a_{n-1}} \nu_{\{a_1,\ldots,a_{n-1}\}}$$

which is exactly the relations for Lie^d_n.

From theorem 7.3.0.2, we have $Odd^{-d+n-2}_{n,n-1}(-1) = Com^{-d+n-2}_n$ and $AOdd^d_{n,n-1}(-1) = Lie^d_n$ are Koszul dual to each other.

Example 7.3.0.2. *When we have $s = 1$, we know that $I^{\lambda^n_j}_{n,1}$ is generated by the elements of the form*

$$\sum_{i=1,\ldots,n} (1,\ldots,\widehat{i},\ldots,n) - (-1)^{n+j} j(n+1,\ldots,2n-1)$$

for $1 \leq j \leq n$, which through the isomorphism $\eta_{n,d}$ will give us exactly the space $R^j_{n,d}$. This shows $AOdd_{n,1}(\lambda^n_j) \cong n\text{-}Lie^j_d$.

On the other hand, we proved after lemma 7.1.1.2 that $K^{\lambda^n_j}_{n,1} \cong S^{\mu_{n,j-1}}$ for $1 \leq j \leq n$ through $\varphi_n : M^{(n,n-1)} \to k\Lambda(n)$ by lemma 7.1.1.3 and in particular this shows $Odd^d_{n,1}(\lambda^n_j) \cong n\text{-}Com^{\mu_{n,j-1}}_d$.

In conclusion, we have the following tables of dimensions for these spaces.

The last example gives us the following result giving the Koszul duality of the constructed operads we have before.

corollary 7.3.0.3. *The operads $n\text{-}Lie^j_d$ and $n\text{-}Com^{\mu_{n,j-1}}_{-d+n-2}$ are Koszul dual for all $1 \leq j \leq n$.*

Proof. Since $n\text{-}Lie^j_d = AOdd^d_{n,1}(\lambda^n_j)$ and $n\text{-}Com^{\mu_{n,j-1}}_{-d+n-2} = Odd^{-d+n-2}_{n,1}(\lambda^n_j)$, then they are Koszul dual by theorem 7.3.0.2. □

The Koszul duality we have for these operads gives us some interesting results, in particular $n\text{-}Com^{\mu_{n,j-1}}_d$ are the minimal n-quadratic operads which is a non-trivial quotient of $SMag_{n,d}$. This comes as a consequence of the fact that their relations are coming from irreducible representations of Σ_{2n-1}. In particular, if $\mathcal{P} = \mathcal{P}(H_{n,d}, Q)$ is any n-quadratic operad with generators $H_{n,d}$ and relations $Q \leq SMag^{(2)}_{n,d}(2n-1)$, then Q must contain one of the irreducible submodules $S^{\mu_{n,j}}$ for some $0 \leq j \leq n-1$ and hence we have the injective map of operads $n\text{-}Com^{\mu_{n,j}}_d \to \mathcal{P}$ using the fact that its relations are irreducible. Hence, the operads $n\text{-}Com^{\mu_{n,j}}_d$ are the minimal non-trivial Com-type operads.

On the other side, using the Koszul duality we have the following result.

Theorem 7.3.0.4. *If $\mathcal{P} = \mathcal{P}(E_{n,d}, R)$ is any n-quadratic operad, then there is an operad map into $n\text{-}Lie^j_d$ for some $1 \leq j \leq n$.*

Proof. Since $\mathcal{P}$ is a n-quadratic operad, then it has a Koszul dual $\mathcal{P}^! = \mathcal{P}(H_{n,d}, R^\perp)$. By above, there exists a injective map $n\text{-}Com^{\mu_{n,j-1}}_{-d+n-2} \to \mathcal{P}^!$ for some $1 \leq j \leq n$. Taking the Koszul duals again, we obtain the map $\mathcal{P} \to n\text{-}Lie^j_d$ and this proves the claim. □

From our work, we can think of the s in the $Odd^d_{n,s}(\lambda)$ and $AOdd^d_{n,s}(\lambda)$ deforms the defining associative and Jacobian identity, respectively for these operads, see figure 7.4. As far as we know, the defining relations for $1 < s < n-1$ are unknown, and it would be interesting to find all possible relations that come from this range of integers. Even for $s = 1$, for the other eigenvalues, it is also not known what the relations should be for $Odd^d_{n,1}(\lambda)$ for $\lambda \neq (-1)^{n+1}$.

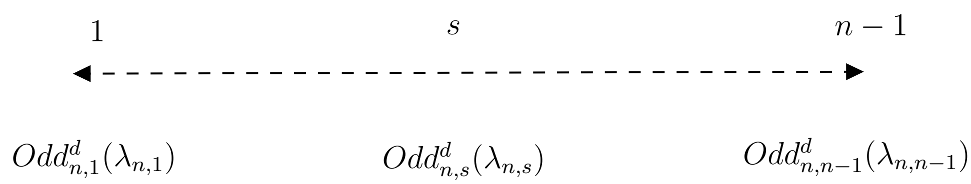


Figure 7.4: The parameter s

Some more questions one could ask: can we construct all of the possible quadratic operads with symmetric or skew-symmetric generators through combinations of the odd operads we constructed above? If so, this would show that the graphs $\mathcal{O}_{n,s}$ would classify all the possible symmetric and skew-symmetric quadratic operads that are possible.

Chapter 8

YOUNG TREES

8.1 Young n-Trees

In this section, we will introduce some combinatorial objects, called Young n-trees, that underlie the operad n-Com_d. These are essentially planar rooted trees such that at each internal edge, it is represented by a Young tableau corresponding to an input labeling at each internal vertex. This information is useful for describing the relations in the operad n-Com_d, which are derived by the Specht module $S^{(n,n-1)}$. The local and global behavior of the tree interact to create very complex relations arising in the operad n-Com_d.

For this section, recall that $Tree_n$ is the set of isomorphism classes of n-trees and we denote by $Tree_{r,n}$ to be the set of isomorphism classes of $rn-r+1$-trees with r internal vertices and each vertex has n-inputs. Furthermore, we let YT_m^r to be the set of Young tableau on any partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ of $n \leq m$ on the set $[m]$.

In this section, we will deal a lot with a non-standard Young tableau as these are the main players in the operad SpO_n^d. In particular, we say a Young tableau T of shape λ is divergent if it is not standard and co-standard if its columns are all decreasing.

8.1.1 General Definitions of Young n-Trees

The combinatorial objects Young n-trees are essentially trees with Young tableaux of shape (n, m) locally at each internal edge. There are some choices when defining these trees depending on how we want to label the internal edges. For us, we will label them using the minimal element on the initial vertex, which follows what one does in shuffle operads and Groebner basis.

Definition 8.1.1.1. *A* ***Young*** n***-tree*** *is a rooted tree T with planar structure $\{\Psi_v\}_{v\in V_T^{in}}$, an input-vertex labeling $\{\lambda_v : in_T(v) \to [n]\}_{v\in V_T^{in}}$ and a set map $\Phi_T : E_T^{in} \to YT_n^2$ satisfying the following properties.*

- *For each $v \in V_T^{in}$, we have $\lambda_v(u) = min_{1\leq i\leq n}\{\lambda_u(\Psi_u(i))\}$ for $u \in in_T(v) \cap V_T^{in}$ and we* *have the following bijection*

$$\coprod_{v \in V_T^{in}} \lambda_v : \coprod_{v \in V_T^{in}} leaves_v \to [n].$$

- *For each internal edge $e = (u, v)$ in T, with $\omega_1 = |in_T(u)|$ and $\omega_2 = |in_T(v)|$ and i is a positive integer such that $\Psi_v(i) = u$, then we define $\Phi_T(e)$ to be the Young tableau of shape $(\omega_1, \omega_2 - 1)$ defined as*

$\lambda_u(\Psi_u(1))$	$\cdots$	$\lambda_u(\Psi_u(i-1))$	$\lambda_u(\Psi_u(i))$	$\cdots$	$\lambda_v(\Psi_v(\omega_2 - 1))$	$\cdots$	$\lambda_u(\Psi_u(\omega_1))$
$\lambda_v(\Psi_v(1))$	$\ldots$	$\lambda_v(\Psi_v(i-1))$	$\lambda_v(\Psi_v(i+1))$	$\ldots$	$\lambda_u(\Psi_u(\omega_2))$		

An isomorphism of Young n-trees is an isomorphism of rooted planar trees preserving the input-vertex labeling, and such that isomorphic internal edges give the same Young tableau. We let $YTree_n$ be the set of isomorphism classes of Young n-trees.

In essence, Young n-trees partition the set $[n]$ in a tree-like way such that there are Young tableaux relationships between those sets when an internal edge connects them. We will call the Young tableau $\Phi_T(e)$ a local Young tableau at e and think of these as the "local charts" of our Young n-tree analogous to a local chart in a manifold. If $e = (u, v)$ is an internal edge of T, we define $\lambda_T : E_T^{in} \to [n]$ to be defined as $\lambda_T(e) := \lambda_v(u)$, which puts the input-vertex labeling structure onto the internal edges as well.

For our pictorial representation, we will identify external input vertices with their corresponding labeling from λ_v for ease of presentation. For example, suppose we have the Young 7-tree T whose underlying planar rooted tree is as in figure 8.1 with $\lambda_{b_2}(a_1) = 1$, $\lambda_{b_2}(a_2) = 2$, $\lambda_{c_1}(b_1) = 3$, $\lambda_{c_1}(b_3) = 4$, $\lambda_{root_T}(c_2) = 5$, $\lambda_{c_3}(d_1) = 6$, and $\lambda_{c_3}(d_2) = 7$, then we can represent this as in figure 8.2. We will also suppress the internal vertices labeling in the pictorial representation for our future figures of Young n-trees, and we will label the internal edges if we need to for clarity.

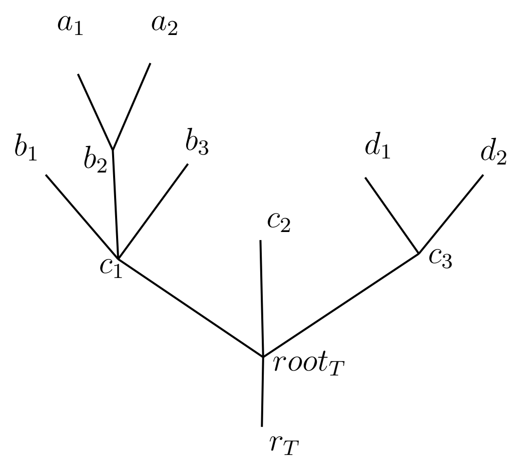


Figure 8.1: Example of a Young 7-Tree

Example 8.1.1.1. *Let T be the Young 7-tree in figure 8.2, then its local Young tableaux are*

$$\Phi_T((b_2, c_1)) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \qquad \Phi_T((c_1, root_T)) = \begin{array}{|c|c|c|}\hline 3 & 1 & 4 \\ \hline 5 & 6 & \\ \cline{1-2} \end{array}$$

$$\Phi_T((c_3, root_T)) = \begin{array}{|c|c|}\hline 6 & 7 \\ \hline 1 & 5 \\ \hline \end{array}.$$

Since Young n-trees are locally Young tableaux, we can define local properties on our trees depending on the properties of Young tableaux. If T is a Young n-tree an, e is an internal edge of T, and P is a property of a Young tableaux, then we say e is P if $\Phi_T(e)$ is P. In particular, we say that a Young n-tree is **standard** if each of its internal edges is standard. Similarly, we say a Young n-tree is **divergent (co-standard)** if each of its local tableaux is divergent (co-standard). We also define a **quasi-standard Young n-tree** as a Young n-tree with at least one standard internal edge. For Young n-tree T, we define $sta(T)$ as equal to the number of standard internal edges of T, which measures how far it is from being divergent.

Following the local nature of these objects, we can define symmetric groups on each of the edges depending on the local Young tableaux. For instance, if T is a Young n-tree

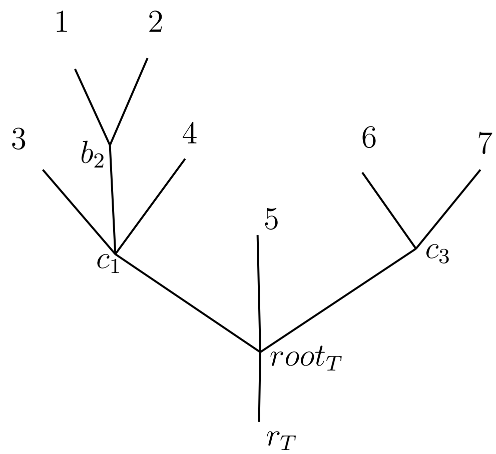


Figure 8.2: Example of representation of Young 7-tree

and $e = (u, v)$ is an internal edge of T, we define $\Sigma_T(e)$ to be the permutation group on the numbers that appear in $\Phi_T(e)$. Furthermore, define $C_T(e)$ to be the subgroup of $\Sigma_T(e)$ generated by the permutations that stabilize the columns of $\Phi_T(e)$, i.e., the columns-stabilizer group as in definition 1.3.3.1. With these local groups, we can define the global spaces

$$\Sigma_{\prod}(T) = \prod_{e\in E_T^{in}} \Sigma_T(e), \quad C_{\prod}(T) = \prod_{e\in E_T^{in}} C_T(e)$$

$$\Sigma_{\coprod}(T) = \coprod_{e\in E_T^{in}} \Sigma_T(e), \quad C_{\coprod}(T) = \coprod_{e\in E_T^{in}} C_T(e)$$

where the former with products are groups and the latter with disjoint unions are groupoids. For example, if T is the Young 7-tree as in figure 8.2, then its local permutation groups are

$$C_T((b_2, c_1)) = \langle (1\ 3), (2\ 4) \rangle$$
$$C_T((c_1, root_T)) = \langle (3\ 5), (1\ 6) \rangle$$
$$C_T((c_3, root_T)) = \langle (3\ 6), (5\ 7) \rangle.$$

As above, with the internal edges, we can look at local symmetric groups on each of the internal vertices as well. Let v be a internal vertex of T and define $R_T(v)$ to be the group $\Sigma(\lambda_v(in_T(v)))$. We can also define local properties at each of the internal vertices by saying an internal vertex is increasing if we have $\lambda_v(\Psi_v(1)) < \cdots < \lambda_v(\Psi_v(n))$. A **shuffle Young**

n**-tree** is a Young n-tree such that every internal vertex is increasing. We will denote by $\sqcup\sqcup$ YTree(n) as the set of all shuffle Young n-trees.

The condition that the internal vertices are increasing is different from the rows increasing in $\Phi_T(e)$, since we can have Young n-trees with increasing rows, but not have increasing vertex. For example, the trees in figure 8.3 have standard edges, but the bottom vertex on the left is increasing, while the one on the right is not. On the other hand, if $e = (u, v)$ is an

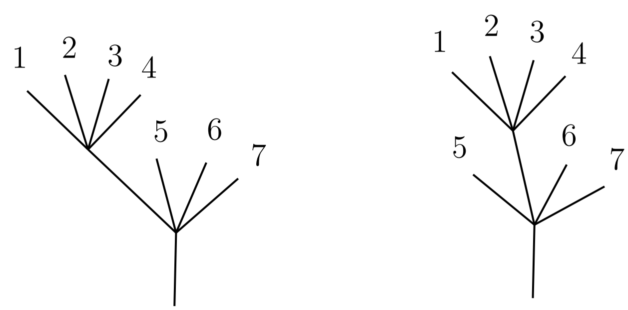


Figure 8.3: Why Shuffle Young n-trees

internal edge of T with v is increasing, then the bottom row of $\Phi_T(e)$ is increasing. Similarly, if u is increasing, then the top row of $\Phi_T(e)$ is increasing. However, note that the leftmost tree is a shuffle young tree, which shows why we would like to restrict our attention to these types of trees.

8.1.2 Young (r, n)-trees

For our context, we care about a particular class of Young m-trees where the underlying trees are n-arity trees with a specified number of internal vertices. These types of trees model the composition of r n-arity operations, making it possible to study the operad n-Com_d.

Definition 8.1.2.1. *Let $r \geq 0$ and $n \geq 2$. A Young (r, n)-tree is a Young $rn - r + 1$ tree with r internal vertices such that the local Young tableaux at each internal edge is of shape $(n, n-1)$.*

Let $YTree_{r,n}$ be the set of isomorphism classes of Young (r,n)-trees, and we let $\sqcup\!\sqcup$ $YTree_{r,n}$ to be the subset of $YTree_{r,n}$ consisting of shuffle Young (r,n)-trees.

If we have two Young trees, the usual grafting of rooted planar trees with input labeling is defined in the usual way with the induced structure, and the local Young tableaux are the same at every internal edge except for the new internal edge that appears from the grafting, which is induced by the input-vertex labeling and the planar structure. Hence, if T is a Young (r,n)-tree and S is a Young (s,n)-tree, then $T \circ_i S$ is a Young $(r+s,n)$-tree, where i is the input-vertex labeling at a leaf on T. This gives us a well-defined partial composition on these trees that we will use later to define our operads.

Just as we can define Young tabloids coming from identifying Young tableaux whose rows have the same numbers, we can do the same with Young n-trees, but for our context, we want to keep track of the degrees as these will correspond to degree d operations at each vertex. So, we need a way to keep track of these degrees to apply the Koszul sign rule when interchanging them.

Definition 8.1.2.2. *A **Young** n**-tree of degree** d is a Young n-tree T equipped with a input labeling $\{deg_v : in_T(v) \to \{0,d\}\}_{v\in V_T^{in}}$ such that*

$$deg_v(u) = \begin{cases} d + \sum_{w\in in_T(u)} deg_u(w) & \text{if } u \in V_T^{in} \\ 0 & \text{otherwise.} \end{cases}$$

We call $\{deg_v\}_{v\in V_T^{in}}$ a degree labeling on T.

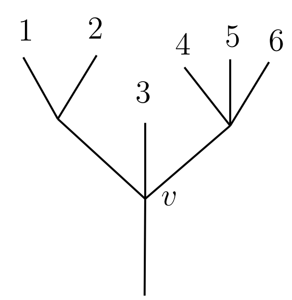
For an example, if T is the Young 7-tree of degree d with underlying rooted planar tree as in figure 8.2, then its degree labeling is

$$\deg_{c_1}(b_2) = d \qquad \deg_{root_T}(c_1) = 2d \qquad \deg_{root_T}(c_3) = d$$

and the rest are zero.

The degree labeling on a Young n-tree lets us apply the Koszul sign rule when we switch between two internal edges connected to a single internal vertex. Let $YTree^d_{r,n}$ be the set of isomorphism classes of Young (r,n)-trees of degree d and define $M^d_{r,n} = \uparrow^{rd} k[YTree^d_{r,n}]$, the k-module with basis element consisting of elements in $YTree^d_{r,n}$.

Next, we will define a symmetric group structure on each of the internal vertices that changes the planar structure underlying it. Let $T \in YTree^d_{r,n}$, v is a internal vertex, Ψ^T_v is the planar structure at v, and $\sigma \in \Sigma_n$. Define $T \wedge_v \sigma$ to be the same exact rooted tree T but with $\Psi^{T\wedge_v \sigma}_v = \Psi^T_v \sigma$. For an example, see figure 8.4, where we interchange the leftmost edge on v and the rightmost edge at v.

$T =$ 

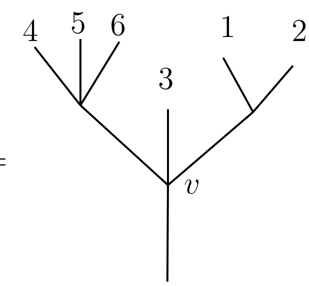
$T \wedge_v (1\ 3) =$ 

Figure 8.4: Example of Permuting the planar order

With the action $\wedge_v$, define $\mathcal{J}^d_{r,n}$ to be the subspace of $M^d_{r,n}$ generated by

$$[T \wedge_v (i\ j)] - (-1)^{\deg_v(\Psi^T_v(i))\deg_v(\Psi^T_v(j)}[T]$$

and define $RM^d_{r,n} = M^d_{r,n}/\mathcal{J}^d_{r,n}$. We denote by $\{T\}$ to represent the equivalence class for $[T]$ in $RM^d_{r,n}$, which we call symmetric Young (r,n)-trees of degree d. Let $RYTree^d_{r,n}$ be the set of all symmetric Young (r,n)-trees of degree d. The relation on $RM^d_{r,n}$ says that if we switch two internal edges on a single vertex, we have a $(-1)^{d^2} = (-1)^d$ sign appear. If d is even, then $\{T\} = \{S\}$ if and only if T and S are isomorphic as non-planar trees, preserving the input-vertex labeling and degree labeling.

In $RM^d_{r,n}$, for any element $\{T\}$, we can find $Sh(\{T\}) = \{S\}$, where S is the corresponding shuffle tree. Furthermore, there is an extra sign that comes out from this, specifically, define $Sgn(T)$ to be such that $\{T\} = Sgn(T)Sh(\{T\})$. Hence, the basis for $RM^d_{r,n}$ consist of $\{T\}$

such that T is a shuffle Young (r, n)-tree and $\dim(RM^d_{r,n})$ is equal to the number of Shuffle Young (r, n)-trees. We let $\bigsqcup RYTree^d_{r,n}$ to be the set of $\{T\}$, where T is a shuffle Young (r, n)-tree.

8.1.3 Properties of Young (r, n)-trees and Standard Edges

The following lemma shows how restrictive being standard at an internal edge is in the sense that only at most one internal edge at every vertex can be standard.

lemma 8.1.3.1. *Let T be a Young (r, n)-tree and suppose e and e' are two internal edges of T with the same terminal vertex. At most, one of the internal edges can be standard.*

Proof. Suppose for a contradiction that e and e' are both standard internal edges and let $e = (u, v)$ and $e' = (u', v)$. Suppose furthermore that $\Psi_v^{-1}(u) = i$ and $\Psi_v^{-1}(u') = j$ with $i < j$. We have that the local Young tableaux are of the form

$$\Phi_T(e) = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline a_1 & a_2 & \cdots & a_{j-1} & a_j & a_{j+1} & \cdots & a_{n-1} & a_n \\ \hline c_1 & c_2 & \cdots & c_{j-1} & b_1 & c_j & \cdots & c_{n-2} & \\ \hline \end{array}$$

$$\Phi_T(e') = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline b_1 & b_2 & \cdots & b_{i-1} & b_i & a_{i+1} & \cdots & a_{n-1} & a_n \\ \hline c_1 & c_2 & \cdots & c_{i-1} & a_1 & c_i & \cdots & c_{n-2} & \\ \hline \end{array}.$$

Since both of these are standard, then we must have

$$a_1 < a_2 < \cdots < a_{j-1} < a_j < b_1 < b_2 < \cdots < b_i < a_1$$

which is a contradiction. This proves the lemma. □

This shows that a Young (r, n)-tree that is standard at every edge must have a linear ordering on its set of internal vertices. On the other hand, we can have a Young (r, n)-tree T, which is co-standard at every edge, and its set of internal vertices does not have a linear

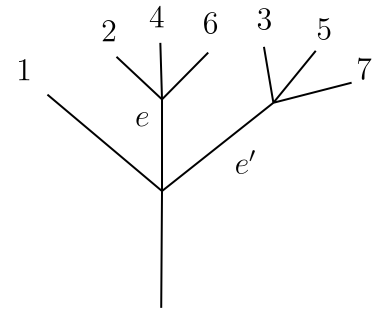


Figure 8.5: Example of Co-standard Young $(3, 3)$-tree

order. For an example, if T is the Young $(3, 3)$-tree as in figure 8.5, then both edges e and e' are co-standard since

$\Phi_T(e) =$

2	4	6
1	3	

$\Phi_T(e') =$

3	5	7
1	2	

Another property of standard edges is that the edges have to be right combs if the tree has increasing rows.

lemma 8.1.3.2. *If T is a shuffle Young (r, n)-tree and $e = (u, v)$ is a standard internal edge, then $\Psi_v(1) = u$.*

Proof. Suppose for a contradiction that $\Psi_v(i) = u$ for some $i > 1$. This implies there is a labeling $\lambda_T(\Psi_v(1)) = a$ where a is less than a labeling in the top row of $\Phi_T(e)$, which is a contradiction to e being standard. □

Combining lemmas 8.1.3.2 and 8.1.3.1, the standard edges for Shuffle trees all have to be right combs. In particular, if T is a standard shuffle Young (r, n)-tree, then they are all right combs as in figure 8.6, where every internal edge e_i is standard for $1 \leq i \leq r - 1$. Furthermore, we have the standard Young tableaux

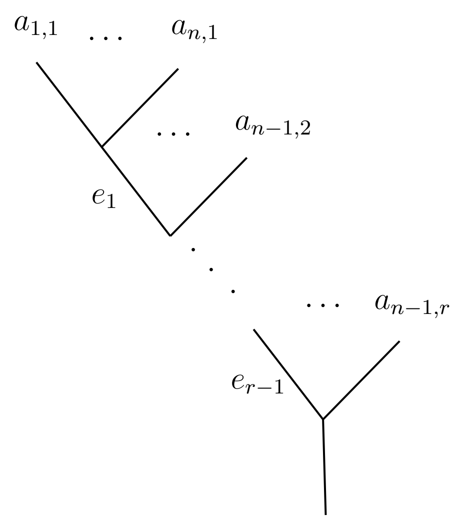


Figure 8.6: Description for all standard shuffle Young (r, n)-trees

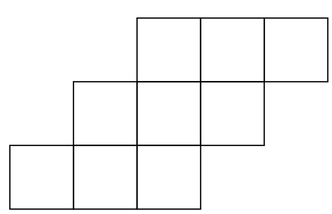
$\Phi_T(e_1) =$

$a_{1,1}$	$a_{2,1}$	$\cdots$	$a_{n-1,1}$	$a_{n,1}$
$a_{1,2}$	$a_{2,2}$	$\cdots$	$a_{n-1,2}$	

$\Phi_T(e_i) =$

$a_{1,1}$	$a_{1,i}$	$\cdots$	$a_{n-2,i}$	$a_{n-1,i}$
$a_{1,i+1}$	$a_{2,i+1}$	$\cdots$	$a_{n-1,i+1}$	

for $2 \leq i \leq r-1$. Since $a_{1,1} = 1$ by the shuffle and standard structure, then we can combine these standard Young tableaux to construct a Skew Young tableaux of shape $\lambda_{r,n}/\mu_r$, where $\lambda_{r,n} = (n+r-1, n+r-2, \ldots, n-1)$ and $\mu_r = (r-1, r-2, \ldots, 1)$. For an example, if $n = 4$ and $r = 3$, then they are of the form



In particular, this gives a bijection between the set of standard skew Young tableaux of shape $\lambda_{r,n}/\mu_r$ and standard shuffle Young (r,n)-trees. Note that if T was not standard and shuffle, the tree will not correspond to a skew Young tableau of shape $\lambda_{r,n}/\mu_r$, as many more relations can appear.

8.1.4 Symmetric Group actions

Here, we will discuss the natural symmetric group actions that occur on Young n-trees and define a particular local symmetric group action that will be useful for describing a generalization of the Specht module $S^{(n,n-1)}$ to Young Trees. First, there is a natural left action of Σ_n on the set $YTree_n$ by applying the permutation to the elements of the input labeling. This is analogous to the action of the symmetric group on the set of Young tableaux. In particular, we have a natural left action on $YTree_{r,n}$.

Continuing with our ideas of a local structure to our Young trees, we will show that we can define a local action of Σ_{2n-1} on each of the internal edges of a Young (r,n)-tree derived from the action on the local Young tableaux of shape $(n, n-1)$. This will not only permute the input labeling but also permute the subtrees whose roots are connected to any of the two vertices of the internal edge when we apply the action.

Let T be a Young (r,n)-tree, $e = (u,v)$ is an internal edge of T, and $\sigma \in \Sigma(e) \cong \Sigma_{2n-1}$. Note that we can also put an ordering on the vertices induced by the ordering $v \leq w$ if $v \in in_(w)$. We define $T \circ_e \sigma$ as follows. Let T_e to be the subtree of T with $V_{T_e}^{in} = (V_T^{in} \setminus \{w \in V_T^{in} \ : \ w \leq v\}) \coprod \{u\}$, and $int_{T_e}(w) = in_T(w)$ for all $w \in V_{T_e}^{in}$ with the same input labeling. For each internal vertex a in $in_T(u)$ and in $in_T(v) \setminus \{u\}$, define S_a to be the subtree of T with $V_{S_a}^{in} = \{w \in V_T^{in} \ : \ w \leq a\}$ and $in_{S_a}(w) = in_T(w)$ for all $w \in V_{S_a}^{in}$ with the same input labeling derived from T. Furthermore, define out_a to be the integer $\lambda_u^T(a)$ for $a \in in_T(u) \cap V_T^{in}$ or $\lambda_v^T(a)$ for $a \in (in_T(v) \setminus \{u\}) \cap V_T^{in}$. Next, we can order the elements in $(in_T(u) \coprod (in_T(v) \setminus \{u\})) \cap V_T^{in}$ with $a_1, \ldots, a_m$ where $out_{a_i} < out_{a_{i+1}}$ and define

$$T \circ_e \sigma = ((T_e)^\sigma \circ_{a_1} S_{a_1}) \cdots) \circ_{a_m} S_{a_m}$$

where the grafting does not renumber the input labeling.

Furthermore, the action of $\circ_e \sigma$ will change the subgroup $R_T(w)$ for $w \in \{u, v\}$ to $R_{T\circ_e\sigma}(w) = \sigma^{-1} R_T(w)\sigma$. In particular, if $\sigma \in C_T(e)$ and $\Phi_T(e)$ is of the form

$$\Phi_T(e) = \begin{array}{|c|c|c|c|c|c|c|c|}\hline a_1 & a_2 & \cdots & a_{i-1} & a_i & \cdots & a_{n-1} & a_n \\ \hline b_1 & b_2 & \cdots & b_{i-1} & b_{i+1} & \cdots & b_n & \\ \hline \end{array}$$

where b_i corresponds the the edge e, then

$$\Phi_{T\circ_e\sigma}(e) = \begin{array}{|c|c|c|c|c|c|c|c|}\hline \sigma^{-1}(a_1) & \sigma^{-1}(a_2) & \cdots & \sigma^{-1}(a_{i-1}) & \sigma^{-1}(a_i) & \cdots & \sigma^{-1}(a_{n-1}) & \sigma^{-1}(a_n) \\ \hline \sigma^{-1}(b_1) & \sigma^{-1}(b_2) & \cdots & \sigma^{-1}(b_{i-1}) & \sigma^{-1}(b_{i+1}) & \cdots & \sigma^{-1}(b_n) & \\ \hline \end{array}.$$

If $\tau \in R_{T\circ_e\sigma}(w)$ for $w \in \{u, v\}$, then it is clear by the action that $\sigma\tau\sigma^{-1} \in R_T(w)$ which shows $\sigma^{-1} R_T(w)\sigma = R_{T\circ_e\sigma}(w)$.

Example 8.1.4.1. *Here is a small non-trivial example. Let T be the Young $(4, 3)$-tree in figure 8.7 and we let e be the internal edge (u_2, u_4). Therefore, we have T_e and S_{u_1} are*

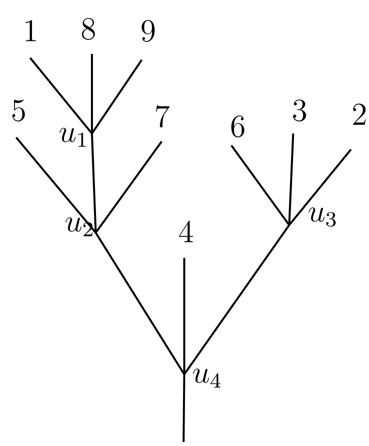


Figure 8.7: Example of Young $(4, 3)$-tree T

the rooted trees in figure 8.8. For our internal edge e, we have $\Sigma(e) = \Sigma(\{1, 4, 5, 6, 7\})$. If

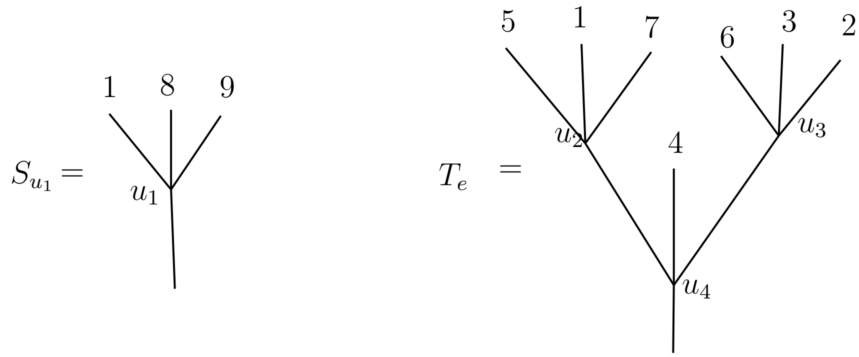


Figure 8.8: Ungrafting the subtrees

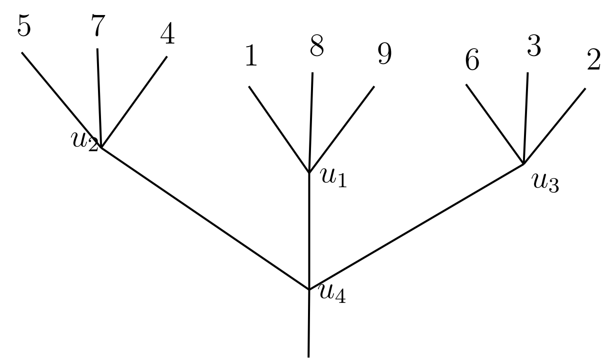


Figure 8.9: The Young Tree $T \circ_e \sigma$

$\sigma = (1\ 4\ 7)$, *then we have* $T \circ_e \sigma$ *as the tree in figure 8.9.*

Similarly, if v is an internal edge of T, we can define a similar action of $R_T(v) \cong \Sigma_n$ on T at the internal vertex v. Let $\tau \in R_T(v)$ and define $T \circ_v \tau$ to be the tree where we pluck the subtrees attached to the vertex, use the permutation τ to permute the leaves of the n leaves on v and then graft the subtrees back on the vertex v with respect to their input labeling λ_v. Furthermore, it is not hard to show that if $\tau \in R_T(w)$ for $w \in \{u, v\}$, then $C_{T \circ_v \tau}(e) = \tau^{-1} C_T(e) \tau$.

Next, we will show a relationship between the actions of $R_T(v)$ through $\circ_v$ and Σ_n through $\wedge_v$ at each internal vertex v. If we let Ψ_v^T be the planar structure of T at v, then we can define a group isomorphism $\widetilde{\Psi}_v^T : \Sigma_n \to R_T(v)$ sending a permutation $\sigma \in \Sigma_n$ to $\tau = \widetilde{\Psi}_v^T(\sigma)$

where τ is defined as

$$\tau(\lambda_v(\Psi_v^T(i))) = \lambda_v(\Psi_v^T(\sigma(i)))$$

for all $i \in \{1, \dots, n\}$. Therefore, we have the following easy consequence of the isomorphism $\widetilde{\Psi}_v^T$.

lemma 8.1.4.1. *If T is a Young (r,n)-tree with a planar structure $\{\Psi_v\}_{v \in V_T^{in}}$, v is a internal vertex of T, and $\sigma \in \Sigma_n$, then*

$$T \wedge_v \sigma = T \circ_v \widetilde{\Psi}_v^T(\sigma).$$

The following lemma is a straightforward computation locally at the internal edge, which explains what happens when you graft a new tree to a tree with a permutation action at the internal edge.

lemma 8.1.4.2. *Let $r \geq 2$. If T is a Young (r,n)-tree, e is an internal edge of T, j is the input labeling at a leaf l on T, S is a (s,n)-tree, and $\sigma \in \Sigma(e)$, then*

$$(T \circ_e \sigma) \circ_j S \cong (T \circ_j S) \circ_e \sigma',$$

where σ' is the permutation where its number changes in the same way when we reorder the leaves in the grafting of the trees.

On the other hand, if W is another Young (w,n)-tree with $w \geq 1$ and i is a input labeling for a leaf a in W, then

$$W \circ_i (T \circ_e \sigma) \cong (W \circ_i T) \circ_e \sigma''$$

where σ'' is the permutation where its number change corresponds to the same change from the grafting of the trees.

The actions of Σ_{rn-r+1} on $YTree_{r,n}$ induce actions on $YTree_{r,n}^d$ by ignoring the degree input labeling. In particular, these induce left $k[\Sigma_{rn-r+1}$ actions on $M_{r,n}^d$ and on its quotient $RM_{r,n}^d$. For $\{T\} \in RM_{r,n}^d$, e an internal edge of T, $\sigma \in \Sigma(e)$, we define $\{T\} \circ_e \sigma = \{T \circ_e$

$\sigma\}$, which is independent of the representative T with the corresponding change of the permutation σ.

With our symmetric group actions on $RM^d_{r,n}$ and the grafting of Young (r, n)-trees of degree d are well-defined, then we can define the operad of Young (r, n)-trees for all $r \geq 0$.

Definition 8.1.4.1. *Define the graded Σ-module YO^d_n with*

$$YO^d_n(rn - r + 1) = \uparrow^{rd} RM^d_{r,n}$$

for $r \geq 0$ and zero everywhere else. Then YO^d_n becomes a graded operad with grafting of Young trees as the composition, and the unique tree $[\uparrow]$ with no internal vertices is the unit element. We call YO^d_n the ***Young n-arity operad of degree*** *d.*

If we were to ignore the local Young tableaux structure on the elements in $YO^d_n(rn-r+1)$, then these are just $rn-r+1$ arity rooted trees, and it is clear that $YO^d_n \cong F(H_{n,d}) = SMag_{n,d}$ as operads.

8.1.5 Various Poset Structures

Next, we will discuss various partial orderings on specific sets of Young (r, n)-trees of degree d, which will be a crucial tool in measuring how far a symmetric Young (r, n)-tree of degree d is away from being standard or divergent. One should think of standard and divergent properties as opposite sides of the spectrum with respect to the orderings we will define.

Recall that the set of Young tabloids of shape $\omega \vdash n$ has a partial ordering $\lhd$ from section 1.3.2, which is used to prove the polytabloids corresponding to standard Young tableaux are a basis for the Specht modules. One of the defining properties of this ordering is the following: if $\{Y\}$ is a Young tabloid with decreasing columns and $\sigma \in C(Y)$, the column permutation group corresponding to Y, then $\{\sigma Y\} \lhd \{Y\}$. We will use the same ideas to generate a poset structure on $RYTree^d_{r,n}$, similar to the one for Young tabloids.

Definition 8.1.5.1. *Let $\{T\}$ and $\{S\}$ be two symmetric Young (r, n)-trees of degree d. We say $\{T\} \prec \{S\}$ if and only if there exists an internal edge e of T such that $\Phi_T(e)$ has elements*

$i < j$ such that i is in a higher row then j and we have $\{T \circ_e (i\ j)\} = \{S\}$, or $\{S\} = \{T\}$.

Note that the first condition is equivalent to $\{S\}$ having an internal edge e' such that $\Phi_S(e')$ has at least a single column that is decreasing, say $r < s$, and we have $\{T\} = \{S \circ_{e'} (r\ s)\}$.

This relation essentially takes a symmetric Young (r, n)-tree $\{T\}$ of degree d, which is standard at e, and makes it less standard by switching one of the columns to be decreasing. In essence, the bigger elements are the ones that are closer to being divergent, and the smaller elements are closer to being standard.

Example 8.1.5.1. *Here are a few examples of the relation $\prec$ applied to $YTree^d_{3,3}$. For the*

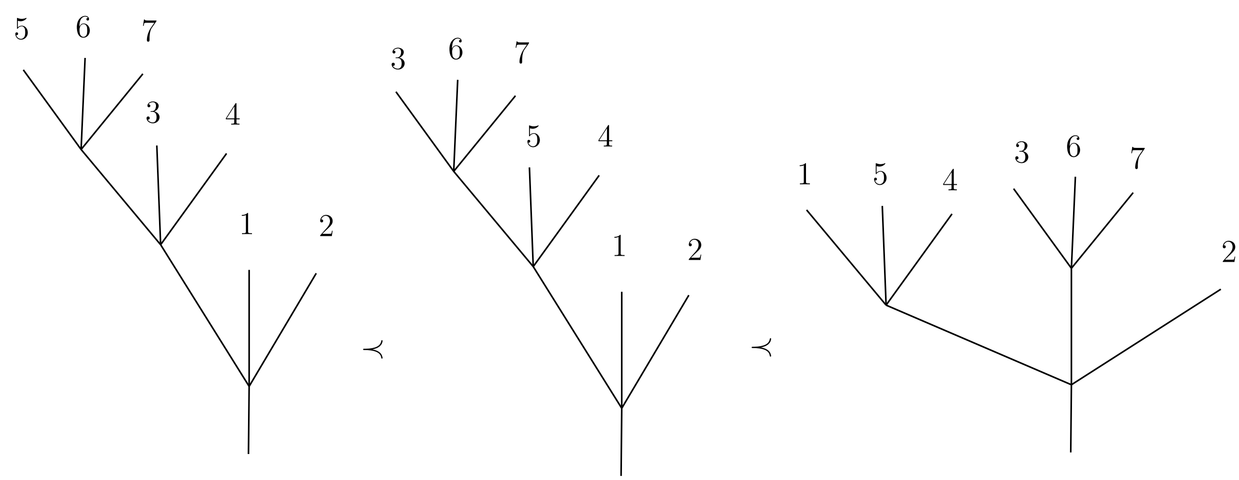


Figure 8.10: Example of Ordering of Young (r, n)-trees

first relation in 8.10, we look at 5 and 3 on the first internal edge from the top, and the second relation comes from looking between 3 and 1 on the second internal edge from the top.

lemma 8.1.5.1. *The relation $\prec$ on symmetric Young (r, n)-trees of degree d is reflexive and antisymmetric.*

Proof. The relation is automatically reflexive by definition, so we need to show it is antisymmetric. Suppose we have $\{T\} \prec \{S\}$ and $\{S\} \prec \{T\}$ with $\{T\} \neq \{S\}$. By definition, there exists an edge e_S in S and an edge e_T in T with transpositions $\sigma, \tau \in \Sigma_{2n-1}$ such that $\{S \circ_{e_S} \sigma\} = \{T\}$ and $\{T \circ_{e_T} \tau\} = \{S\}$. Without loss of generality, we may assume

$$T = S \circ_{e_S} \sigma$$
$$S = T \circ_{e_T} \tau$$

which implies

$$T = (T \circ_{e_T} \tau) \circ_{e_S} \sigma.$$

For this to happen, since τ and σ are transpositions the edges e_T and e_S must be the same edge. This implies

$$T = (T \circ_{e_T} \tau\sigma)$$

which would imply that $\tau = \sigma$ since they are transpositions. Hence, $\tau = \sigma = (i\ j)$ and $i < j$ with i is in a higher row then j at the internal edge e_T in T. But this is not true for S since $S = T \circ_{e_S} (i\ j)$ which is a contradiction. This implies $\{T\} = \{S\}$ and completes the proof. □

With this relation, we can take the transitive closure of $\prec$ to define a poset structure on $RYTree^d_{r,n}$ as in the following definition.

Definition 8.1.5.2. *We say $\{T\} \lhd \{T'\}$ if and only if there is a sequence of symmetric Young (r, n)-trees $\{S_1\}, \ldots, \{S_m\}$ of degree d such that*

$$\{T\} \prec \{S_1\} \prec \cdots \prec \{S_m\} \prec \{T'\}.$$

We call this ordering the Young ordering in $RYTree^d_{r,n}$.

The following lemma explains the statement we had before in that the smaller an element is in $RYTree^d_{r,n}$, the more standard it becomes, and vice-versa, the bigger the element, the more divergent it becomes.

lemma 8.1.5.2. *Suppose T is a shuffle Young (r, n)-tree. If $\{T\}$ is a minimal element with respect to the poset $\lhd$, then $\{T\}$ is standard. Similarly, if $\{T\}$ is a maximal element, it is co-standard.*

Proof. We will prove the case when $\{T\}$ is a minimal element and show it is standard as the other case is similar. Suppose for a contradiction $\{T\}$ is not standard, then this implies that there exists an internal edge e such that $\Phi_T(e)$ has a decreasing column, say $i < j$. We can apply $(i\ j)$ to get $\{T \circ_e (i\ j)\} < \{T\}$ which is a contradiction of the minimality of $\{T\}$. Hence, $\{T\}$ must be standard. This proves the lemma. □

Since the set $RYTree^d_{r,n}$ is finite, if $\{T\}$ is any symmetric Young (r, n)-trees, there exists a standard symmetric Young (r, n)-tree $\{S\}$ of degree d and a divergent symmetric Young (r, n)-tree $\{D\}$ of degree d such that

$$\{S\} \unlhd \{T\} \unlhd \{D\}$$

by essentially applying permutations on either increasing columns to go up or decreasing columns to go down. This shows that this relation is a measure between being standard or divergent.

On the other hand, we can define a partial ordering where we switch rows to columns for the equivalence classes. This will allow us to use the Garnir relations for Young tabloids and apply them to the relations in n-Com_d later in 8.2.2.

Let $P^d_{r,n}$ be the set of triples (T, e), where T is a Young (r, n)-tree of degree d, e is an internal edge of T. We call the tuples (T, e) **pointed Young (r, n)-tree of degree** d. An isomorphism between pointed Young (r, n)-trees of degree d is an isomorphism between the underlying Young trees such that the corresponding internal edges are mapped to each other under the isomorphism. Define $YTree^{d,\text{pointed}}_{r,n}$ to be the set of isomorphism classes of pointed Young (r, n)-trees, and we denote such a class as $[T, e]$.

Next, define $\{T, e\}_c$ to be the equivalence class of elements $[T \circ_e \sigma, e]$ for $\sigma \in C_T(e)$, and let $CYTree^{d,\text{pointed}}_{r,n}$ to be the set of these equivalence classes. Like with $RYTree^d_{r,n}$, we can put a poset structure on the set $CYTree^d_{r,n}$.

Definition 8.1.5.3. *Let $\{T, e\}_c$ and $\{S, e'\}_c$ be two elements in $CYTree_{r,n}^{d,pointed}$. We say $\{T, e\}_c \prec \{S, e'\}_c$ if and only if one of the following properties holds:*

- *either we have $\{T, e\}_c = \{S, e'\}_c$,*
- *or there exists $k < l$ in $\Phi_T^W(e)$ such that l is in a lower column then k and we have $\{S, e', W'\}_c = \{T \circ_e (k\ l), e, W\}_c$.*

In a similar proof as in lemma 8.1.5.1, the relation $\prec$ for $CYTree_{r,n}^{d,\text{pointed}}$ is reflexive and antisymmetric. Furthermore, we can take the transitive closure of the relation $\prec$ to produce a partial ordering $\lhd$ on $CYTree_{r,n}^{d,\text{pointed}}$. We have the following proof that is similar to the row version.

lemma 8.1.5.3. *Let T be a shuffle Young (r, n)-tree of degree d, e an internal edge e. If $\{T, e\}_c$ is minimal, then T is standard at e.*

8.1.6 Monomial Order on Shuffle Young (r, n)-trees

The set of shuffle n-trees are an important set of objects in the theory of Groebner basis and operads. In particular, the set of shuffle trees on a particular alphabet $\mathcal{X}$ define the free shuffle operad $\mathcal{T}_{\sqcup\sqcup}(\mathcal{X})$ as constructed in [2]. The set of shuffle Young (r, n)-trees is a subset of the set $\sqcup\!\sqcup\,\text{Tree}_{\mathcal{X}}(rn - r + 1)$ consisting of shuffle $rn - r + 1$ trees on the alphabet $\mathcal{X} = \{*\}$. The set of such trees has a natural monomial ordering that helps with studying the elements in an operad.

The next definitions are from [2], but we specify them for our context of shuffle Young n-trees.

Definition 8.1.6.1. *Let T be a shuffle Young n-tree. For each leaf l of T in the total ordering induced by the input labeling, we record the labels of internal vertices of the path from the root of T to l, forming a word consisting of $*$. The sequence of these words denoted $Path(T)$, is called the* ***path sequence*** *of the shuffle Young tree T.*

As noted in [2], the path sequence of a shuffle Young n-tree does not determine a shuffle Young n-tree uniquely. To fix this, we need to add in the permutation data.

Definition 8.1.6.2. *Let T be a shuffle Young n-tree. The **leaf permutation** of T is the permutation $\sigma(T)$ for which $\sigma(T)(j) = \lambda(l_j)$, where l_j is the jth leaf of T in the total planar order of leaves.*

*The **path-permutation data** of a shuffle Young n-tree is the pair $(Path(T), \sigma(T))$.*

lemma 8.1.6.1. *([2]) A shuffle Young n-tree is uniquely determined by its path-permutation data.*

Next, we can use this data to define a total ordering on the set of shuffle Young n-trees.

Definition 8.1.6.3. *Suppose T and S are two shuffle Young trees. The* GPATHPERMLEX *ordering is defined as follows.*

- *If T has less leaves then S, then $T \leq S$.*
- *If T and S have the same number of leaves, then we compare $Path(T)$ and $Path(S)$ word by word, comparing words in $\{*\}$ using the lexicographical ordering.*
- *If T and S have the same number of leaves and $Path(T) = Path(S)$, then we compare the permutations $\sigma(T)$ and $\sigma(S)$ using the lexicographical ordering.*

Example 8.1.6.1. *Put examples of ordering here.*

We can induce the monomial ordering we defined above to the symmetric Young (r, n)-trees as follows: if $\{T\}$ and $\{S\}$ are both symmetric Young (r, n)-trees of degree d with T and S are shuffles trees, then we define $\{T\} \leq_{\sqcup\sqcup} \{S\}$ if and only if $T \leq S$ with the monomial ordering. This can be induced to every symmetric Young tree in an obvious way, and we will call this ordering the monomial ordering on $RYTree^d_{r,n}$.

Next, we will show that the Young tableaux ordering implies the monomial ordering on $RYTree^d_{r,n}$.

lemma 8.1.6.2. *If $\{T\} \prec \{S\}$ for symmetric Young (r,n)-trees, then $\{T\} <_{\sqcup\sqcup} \{S\}$.*

8.2 Specht Operad

In this section, we will construct the operad SpO^d_n whose relations are built from the Young antisymmetrizer relation defined in the Specht module $S^{(n,n-1)}$. As it turns out, the operad $SpO^d_n \cong n\text{-}Com_d$ allows us to find bounds to the dimensions of the arities of the operads.

8.2.1 Polytrees

In this section, we generalize the permutation and Specht module of the partitions $(n, n-1)$ in terms of our Young trees.

Definition 8.2.1.1. *Let T be a Young (r,n)-tree of degree d and e an internal edge of T. Define*

$$\Omega_{T,e} = \sum_{\sigma \in C_T(e)} Sgn(\sigma)\{T \circ_e \sigma\} \in RM^d_{r,n},$$

where we extended $\circ_e$ action linearly. We call $\Omega_{T,e}$ polytrees of T at e.

Later, we will need to deal with these polytrees where the terms are shuffle trees to find our basis. Hence, we can rewrite the polytrees $\Omega_{T,e}$ as follows:

$$\Omega_{T,e} = \sum_{\sigma \in C_T(e)} Sgn(\sigma) Sgn(T \circ_e \sigma) Sh(\{T \circ_e \sigma\}).$$

The following two lemmas are used to ensure that the grafting of trees, i.e., the operadic structure in YO^d_n, is compatible with the polytrees $\Omega_{T,e}$.

lemma 8.2.1.1. *If T is a Young (r, n)-tree of degree d with $r \geq 2$, e is an internal edge of T, S is a Young (s, n)-tree with $s \geq 0$, and l is a leaf of T corresponding to the input label j, then*

$$\Omega_{T,e} \circ_j S = \Omega_{T \circ_j S, e}.$$

Additionally, if Q is a Young (q, n)-tree with $q \geq 1$, and a is a leaf of Q corresponding to an input labeling i, then

$$Q \circ_i \Omega_{T,e} = \Omega_{Q \circ_i T, e}.$$

Example 8.2.1.1. *Let T be the Young $(3, 3)$-tree as in figure 8.11, then we have*

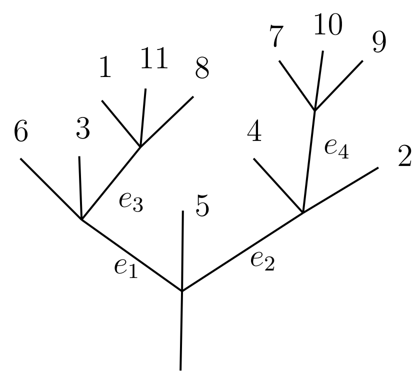


Figure 8.11: Young $(3, 3)$-tree of degree d

$$\Omega_{T,e_1} = \{T\} - \{T \circ_{e_1} (5\ 6)\} - \{T \circ_{e_1} (3\ 4)\} + \{T \circ_{e_1} (5\ 6)(3\ 4)\},$$
$$\Omega_{T,e_2} = \{T\} - \{T \circ_{e_2} (4\ 6)\} - \{T \circ_{e_2} (5\ 7)\} + \{T \circ_{e_2} (4\ 6)(5\ 7)\},$$
$$\Omega_{T,e_3} = \{T\} - \{T \circ_{e_3} (1\ 6)\} - \{T \circ_{e_3} (3\ 11)\} + \{T \circ_{e_3} (1\ 6)(3\ 11)\},$$
$$\Omega_{T,e_4} = \{T\} - \{T \circ_{e_4} (4\ 7)\} - \{T \circ_{e_4} (2\ 10)\} + \{T \circ_{e_4} (4\ 7)(2\ 10)\}.$$

Next, for $r \geq 2$, $n \geq 2$ and d even, define $V^d_{r,n}$ to be the subspace of $RM^d_{r,n}$ generated by $\Omega_{T,e}$ for all Young (r, n)-trees T of degree d and internal edges e of T. Similarly, if d is odd, we define $V^d_{r,n}$ to be generated in the same way, except we restrict ourselves to trees T that are right combs at e to help deal with the problems of negatives that appear from interchanging internal edges. For example, if T is a Young (r,n)-tree of degree d and $e=(u,v)$ is an internal edge of T that is not a right comb, then we can choose $\tau \in R_T(v)$ so that $T \circ_v \tau$ is a right comb and use $\Omega_{T\circ_v \tau, e}$. For $r<2$, we define $V^d_{1,n}=V_{0,n}=0$.

It is clear that the space $V^d_{r,n}$ are $k[\Sigma_{rn-r+1}]$-submodules of $RM^d_{r,n}$ and we can define the $k[\Sigma_{rn-r+1}]$-modules $C^d_{r,n}=RM^d_{r,n}/V^d_{r,n}$ and denote its elements as $\overline{\{T\}}$.

Example 8.2.1.2. *Here are some examples of the spaces $RM^d_{r,n}$, $V^d_{r,n}$, and $C^d_{r,n}$ for small r.*

- *When $r=0$, we have $C^d_{0,n}=RM^d_{0,n}=M^0_{r,n}\cong k[\uparrow]$, where $\uparrow$ is the unique Young $(0,n)$-tree of degree d with no internal vertices.*

- *When $r=1$, we have $C^d_{1,n}=RM^d_{1,n}\cong\uparrow^d k\{Cor_n\}$, where Cor_n is the corolla n-tree with a single internal vertex, n inputs, and with labeling ordered from left to right starting at 1 and ending at n.*

- *For $r=2$, we have $RM^d_{2,n}\cong\uparrow^{2d} M^{(n,n-1)}$, the permutation module consisting of all k-linear combinations of Young tabloids of shape $(n,n-1)$, since we have a bijection between symmetric Young $(2,n)$-trees and tabloids of shape $(n,n-1)$ through the local tabloid structure. Furthermore, $V^d_{2,n}$ is isomoprhic to $\uparrow^{2d} S^{(n,n-1)}$, which has a basis consisting of Ω_{T,e_T} for all standard Young $(2,n)$-trees T of degree d, where e_T is the unique internal edge of T.*

Next, we will use the relations $\Omega_{T,e}$ to generate a operadic ideal of YO^d_n in a natural way. Let $\mathcal{S}^d_n$ be the graded Σ-module of YO^d_n with $\mathcal{S}^d_n(rn-r+1)=\uparrow^{rd} V^d_{r,n}$ for all $r\geq 0$ and $n\geq 2$, with the rest being zero. Lemma 8.2.1.1 explicitly shows that $\mathcal{S}^d_n$ is an operadic ideal of YO^d_n and it shows that $\mathcal{S}^d_n$ is spanned by the following elements. For each Young (r,n)-tree T of degree d, with $r\geq 2$, and e an internal edge of T, there exists positive integers $a_1,\dots,a_{r-1}$, an $1\leq i\leq r-1$, an $1\leq j\leq n$, and a permutation $\sigma\in\Sigma_{rn-r+1}$ such that

$$\Omega_{T,e}=\pm(\cdots((\cdots((Cor_n \circ_{a_1} Cor_n)\circ_{a_2} Cor_n)\cdots)\circ_{a_i}\Omega_{Cor_n\circ_j Cor_n,e_j})\circ_{a_{i+1}} Cor_n)\cdots)\circ_{a_{r-1}} Cor_n)\cdots)^{\sigma}$$

where e_j is the unique internal edge of $Cor_n \circ_j Cor_n$. In conclusion, the submodule $\mathcal{S}_n^d$ is the operadic ideal generated by $V_{2,n}^d$ in arity $2n-1$.

Definition 8.2.1.2. *Define SpO_n^d to be the quotient operad $YO_n^d/\mathcal{S}_n^d$, which we call the* ***Specht n-arity Operad of degree*** *d.*

From this definition, we can see that $SpO_n^d(rn-r+1) =\uparrow^{rd} C_{r,n}^d$ for all $r \geq 0$ and $n \geq 2$. Furthermore, since $V_{2,n}^d \cong\uparrow^{2d} S_{n,d}$ in section 4.2.5.1, then the operadic ideal $\mathcal{S}_n^d \cong \mathcal{J}_{n,d}$ under the isomorphism $F(H_{n,d}) \cong YO_n^d$ and this shows n-$Com_d \cong SpO_n^d$ as graded operads.

8.2.2 Properties of $V_{r,n}^d$

The space $V_{r,n}^d$ is very reminiscent of the Specht module in the sense it is generated by a higher version of the Young antisymmetrizer relation on the internal edges of trees. However, as we can see, there are huge differences based on the interaction between the trees and the local Young tableaux structure. In particular, we will see that $V_{r,n}^d$ has a basis that does not entirely consist of $\Omega_{T,e}$ where T is standard at e.

The first property that is similar to the properties of the Specht module is the following lemma that shows when n elements are linearly independent.

lemma 8.2.2.1. *For any elements $v_1, \ldots, v_m \in RM_{r,n}^d$ such that we can choose a symmetric Young (r,n)-tree $\{T_i\}$ in v_i for all i satisfying the following:*

- *$\{T_i\}$ is minimal among the elements of v_i with respect to the monomial order or the Young order,*
- *and $\{T_i\}$ are all distinct and not negative multiples of each other.*

Then $v_1, \ldots, v_m$ are linearly independent.

Proof. Let suppose $\sum_{i=1}^{m} c_i v_i = 0$ and let $\{T_1\}$ be minimal among all of the $\{T_1\}, \ldots, \{T_m\}$ that are comparable to $\{T_1\}$. Note that $\{T_1\}$ can not appear in any v_i for all $i \neq 1$ by minimality of $\{T_1\}$ and the ones that are not comparable to $\{T_1\}$. In other words, $c_1 = 0$ by linear independence in $RM^d_{r,n}$, and we can proceed by induction to get the rest. This finishes the lemma. □

On the other hand, we can use the partial ordering $RYTree^d_{r,n}$, we can show that the set of shuffle divergent Young (r, n)-trees of degree d give a spanning set for $SpO^d_n(rn - r + 1)$ for all $r \geq 2$.

lemma 8.2.2.2. *The set of* $\overline{\{T\}}$ *for divergent shuffle Young* (r, n)*-trees* T *of degree* d *span* $C^d_{r,n}$ *for* $r \geq 1$*.*

Proof. We will prove by induction on the partial ordering on the elements $RYTree^d_{r,n}$. For maximal element $\{D\}$, it is divergent by 8.1.5.2 and is automatically in the span.

For the induction step, let $\{T\}$ be any symmetric Young (r, n)-tree of degree d in $RM^d_{r,n}$ such that all other $\{S\}$ with $\{T\} < \{S\}$, we have $\overline{\{S\}}$ is in the span. If $\{T\}$ is divergent, then we are done. Otherwise, if there exists an internal edge e that is standard, then we can use the relation from $V^d_{r,n}$ to get

$$\overline{\{T\}} = -\sum_{\sigma \in C(e)\setminus\{id\}} Sgn(\sigma)\overline{\{T \circ_e \sigma\}}$$

and from our relation we have $\{T\} < \{T \circ_e \sigma\}$, since all the columns of $\Phi_T(e)$ are increasing. By induction, $\{T \circ_e \sigma\}$ is in the span, and hence $\{T\}$ is in the span. This proves the result. □

The last lemma states that we have

$$\dim(SpO^d_n(rn - r + 1)) \leq div_{r,n},$$

where $div_{r,n}$ is the number of shuffle divergent Young (r, n)-trees, and hence we must have some subcollection of shuffle divergent trees be a basis for $SpO^d_n(rn - r + 1)$ for all $r \geq 2$. When $r = 2$, this is automatically true by what we proved in chapter 4.

BIBLIOGRAPHY

[1] Hidetoshi Awata, Miao Li, Djordje Minic, and Tamiaki Yoneya. On the quantization of nambu brackets. *Journal of High Energy Physics*, 2001(02):013–013, February 2001.

[2] M.R. Bremner and V. Dotsenko. *Algebraic Operads: An Algorithmic Companion*. CRC Press, 2016.

[3] Eugène Catalan. Sur les nombres de segner. *Rendiconti del Circolo Matematico di Palermo*, 1(1), 1887.

[4] Yuri L. Daletskii and Leon A. Takhtajan. Leibniz and lie algebra structures for nambu algebra. *Letters in Mathematical Physics*, 39(2):127–141, Jan 1997.

[5] Jacques Faraut and Adam Korányi. *Analysis on Symmetric Cones*. Oxford University Press, 12 1994.

[6] José Figueroa-O'Farrill and George Papadopoulos. Maximally supersymmetric solutions of ten- and eleven-dimensional supergravities. *Journal of High Energy Physics*, 2003(03):048–048, March 2003.

[7] José Figueroa-O'Farrill and George Papadopoulos. Plücker-type relations for orthogonal planes. *Journal of Geometry and Physics*, 49(3–4):294–331, March 2004.

[8] V.T. Filippov. n-lie algebras. *Sib Math J*, 26:879–891, 1985.

[9] Tamar Friedmann, Phil Hanlon, Richard P. Stanley, and Michelle L. Wachs. On a generalization of lie(k): A catalanke theorem. *Advances in Mathematics*, 380:107570, March 2021.

[10] Victor Ginzburg and Mikhail Kapranov. Koszul duality for operads. *Duke Mathematical Journal*, 76(1):203 – 272, 1994.

[11] Michel Goze, Nicolas Goze, and Elisabeth Remm. n-lie algebras, 2009.

[12] Sh. M. Kasymov. Theory of n-lie algebras. *Algebra and Logic*, 26(3):155–166, Jun 1987.

[13] Maxim Kontsevich. Deformation quantization of poisson manifolds. *Letters in Mathematical Physics*, 66(3):157–216, December 2003.

[14] Yakov Krasnov and Vladimir G. Tkachev. *Variety of Idempotents in Nonassociative Algebras*, page 405–436. Springer International Publishing, 2019.

[15] Peter Larcombe and Ira Gessel. A forgotten convolution type identity of catalan: Two hypergeometric proofs. *Utilitas Mathematica*, 59, 05 2001.

[16] J.-L Loday and B. Vallette. *Algebraic Operads*. Springer, Verlag Berlin Heidelberg, 2012.

[17] Martin Markl and Elisabeth Remm. (non-)koszulness of operads for $$n$$-ary algebras, galgalim and other curiosities. *Journal of Homotopy and Related Structures*, 10(4):939–969, Dec 2015.

[18] J. May. The geometry of iterated loop spaces. 271, 08 2011.

[19] Donatella Merlini, Douglas G. Rogers, Renzo Sprugnoli, and M. Cecilia Verri. On some alternative characterizations of riordan arrays. *Canadian Journal of Mathematics*, 49(2):301–320, 1997.

[20] Yoichiro Nambu. Generalized hamiltonian dynamics. *Phys. Rev. D*, 7:2405–2412, Apr 1973.

[21] Alon Regev. A proof of catalan's convolution formula. *Integers*, 12(5):929–934, 2012.

[22] B.E. Sagan. *The Symmetric Group:Representations, Combinatorial Algorithms, and Symmetric Functions*. Graduate Texts in Mathematics. Springer US, 1991.

[23] L.W. Shapiro. A catalan triangle. *Discrete Mathematics*, 14(1):83–90, 1976.

[24] Richard P. Stanley. *Catalan Numbers*. Cambridge University Press, 2015.

[25] R.P. Stanley. *Enumerative Combinatorics: Volume 1*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1997.

[26] Jim Stasheff. Differential graded lie algebras, quasi-hopf algebras and higher homotopy algebras. In Petr P. Kulish, editor, *Quantum Groups*, pages 120–137, Berlin, Heidelberg, 1992. Springer Berlin Heidelberg.

[27] Steven J. Tedford. Combinatorial interpretations of convolutions of the catalan numbers. 11(1):35–45, 2011.

[28] Ualbai Umirbaev and Viktor Zhelyabin. A dixmier theorem for poisson enveloping algebras. *Journal of Algebra*, 568:576–600, 2021.

[29] D Yau. *Colored Operads*. American Mathematical Society, Rhode Island, 2016.

[30] James J. Zhang. Non-noetherian regular rings of dimension 2. 1998.