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Jessica Merhej

On the Geometry of Rectifiable Sets with Carleson and
Poincaré-type Conditions

Jessica Merhej

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Reading Committee:

Tatiana Toro, Chair

Thomas Duchamp

Hart Smith

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University of Washington

Abstract

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Jessica Merhej

Chair of the Supervisory Committee:
Professor Tatiana Toro
Mathematics

A central question in geometric measure theory is whether geometric properties of a set translate into analytical ones. In 1960, E. R. Reifenberg proved that if an n -dimensional subset M of \mathbb{R}^{n+d} is well approximated by n -planes at every point and at every scale, then M is a locally bi-Hölder image of an n -plane. Since then, Reifenberg's theorem has been refined in several ways in order to ensure that M is a bi-Lipschitz image of an n -plane. In this thesis, we show that a Carleson condition on the oscillation of the tangent planes of an n -Ahlfors regular rectifiable subset M of \mathbb{R}^{n+d} satisfying a Poincaré-type inequality is sufficient to prove that M is contained inside a bi-Lipschitz image of an n -dimensional affine subspace of \mathbb{R}^{n+d} . We also show that this Poincaré-type inequality encodes geometrical information about M ; namely it implies that M is quasiconvex.

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DEDICATION

To my wonderful husband, Tima.

Chapter 1

INTRODUCTION

1.1 History, motivation, and the importance of bi-Lipschitz parametrizations

The Plateau problem has played a fundamental role in the development of geometric measure theory and geometric analysis. In dimension two, it was solved (independently) by Douglas and Radó (see [Rad30] and [Dou31]) in 1930. It took time to make sense of the question in higher dimensions. Reifenberg [Rei60] approached the question of regularity for solutions to the Plateau problem in 1960. His initial tool was the *topological disk theorem*. In recent years, there has been renewed interest in this result and its proof. Roughly speaking, the topological disk theorem states that if an n -dimensional subset M of \mathbb{R}^{n+d} is well approximated by an n -plane at every point and at every scale, then locally, M is a bi-Hölder image of the unit ball in \mathbb{R}^n . To be more precise, we state the theorem here:

Theorem 1.1.1. (*Topological Disk Theorem*) [Rei60] [DT12] *For all choices of integers $n > 0$ and $d > 0$, and $0 < \tau < 10^{-1}$, we can find $\epsilon > 0$ such that the following holds: Let $M \subset \mathbb{R}^{n+d}$ be a closed, n -dimensional set that contains the origin, and suppose that for $x \in M \cap B_{10}(0)$ and $0 < r \leq 10$ we can find an n -dimensional affine subspace $P(x, r)$ of \mathbb{R}^{n+d} that contains x such that*

$$\text{dist}(y, P(x, r)) \leq \epsilon r \quad \text{for } y \in M \cap B_r(x), \quad (1.1)$$

and

$$\text{dist}(y, M) \leq \epsilon r \quad \text{for } y \in P(x, r) \cap B_r(x). \quad (1.2)$$

Then, there exists a bijective mapping $g : \mathbb{R}^{n+d} \rightarrow \mathbb{R}^{n+d}$ such that

$$|g(x) - x| \leq \tau \quad \text{for } x \in \mathbb{R}^{n+d}, \quad (1.3)$$

$$\frac{1}{4}|x - y|^{1+\tau} \leq |g(x) - g(y)| \leq 3|x - y|^{1-\tau}, \quad (1.4)$$

for $x, y \in \mathbb{R}^{n+d}$ such that $|x - y| \leq 1$, and if we set $P = P(0, 10)$,

$$M \cap B_1(0) = g(P) \cap B_1(0). \quad (1.5)$$

A set satisfying inequalities (1.1) and (1.2) is said to be an ϵ -Reifenberg flat set and the map g constructed in the theorem above is called a *Reifenberg parametrization* of M . Semmes [Sem91a, Sem91b] uses a Reifenberg-type parametrization to get good parametrizations of chord arc surfaces with small constant. David, De Pauw, and Toro [DDPT08] give a generalization of Reifenberg's theorem in \mathbb{R}^3 . The works by David [Dav09, Dav10], partially generalizing Taylor's [Tay76] results rely on the Reifenberg-type parametrization constructed in [DDPT08]. In [Tor95], Toro refines Reifenberg's condition in order to guarantee the existence of better parametrizations, and so do David and Toro in [DT12]. David and Toro [DT99] also use Reifenberg-type parametrization to get snowflake-like embeddings of flat metric spaces, a work related to the results Cheeger and Colding [CC97] who use a Reifenberg-type parametrization to parametrize the limits of manifolds with Ricci curvature bounded from below. Colding and Naber improve this latter result in [CN13]. Moreover, Naber and Valtorta [NV15a, NV15b] use a variation of Reifenberg's parametrization to study the regularity of stationary and minimizing harmonic maps.

A question which motivated many of the papers mentioned above, is whether the map g in Theorem 1.1.1 is K -bi-Lipschitz, that is whether there exists a constant $K \geq 1$, such that for all $x, y \in \mathbb{R}^{n+d}$, we have

$$K^{-1}|x - y| \leq |g(x) - g(y)| \leq K|x - y|. \quad (1.6)$$

Notice that in Theorem 1.1.1, the smaller ϵ is, the closer the bi-Hölder exponent is to 1, that is, the closer the map g is to being bi-Lipschitz. Also, it is known that any Lipschitz domain

with sufficiently small Lipschitz constant is Reifenberg flat, for a suitable choice of ϵ depending on the Lipschitz constant. However, the converse is not true in general. In fact, the Von Koch snowflake (with sufficiently small angle) is an example of a Reifenberg flat set which is not Lipschitz (see [Tor97]). Finding bi-Lipschitz parametrizations of sets is a central question in areas of geometry and metric analysis. For example, Lipschitz functions in metric spaces play the role played by smooth functions in smooth manifolds. Moreover, many concepts in metric analysis (for instance metric dimensions) are invariant under bi-Lipschitz mappings. Another example where Lipschitz and bi-Lipschitz mappings are of utmost importance is the theory of rectifiability in geometric measure theory. An n -dimensional rectifiable subset of \mathbb{R}^{n+d} , up to a set of measure zero, is a set contained in a countable union of Lipschitz images of \mathbb{R}^n . Rectifiable sets are a measure theoretic generalization of smooth surfaces that provide the appropriate setting to study geometric variational problems. For a set to be rectifiable, it does not necessarily have to be smooth, but it inherits some characteristics of smooth surfaces. In particular, rectifiable sets are characterized by having *approximate tangent planes* almost everywhere. Moreover, if a set is a bi-Lipschitz image of \mathbb{R}^n in the ambient space \mathbb{R}^{n+d} for some $k \geq 1$, then the set is uniformly rectifiable, where the latter is a quantitative version of rectifiability.

So, it is very interesting to know what conditions guarantee that the map g in Theorem 1.1.1 is bi-Lipschitz. David and Toro [DT12] give several results, each providing sufficient conditions on the set M so that g is bi-Lipschitz. One of the conditions involves the Jones numbers

$$\beta_\infty(x, r) = \frac{1}{r} \inf_P \left\{ \sup \{ \text{dist}(y, P); y \in B_r(x) \cap M \} \right\}, \quad (1.7)$$

where $x \in M \cap B_{10}(0)$, $0 < r \leq 10$, and the infimum is taken over all n -dimensional affine subspaces P of \mathbb{R}^{n+d} , passing through x .

It is not surprising that the β_∞ numbers play a role here. They were introduced by Jones

in the Traveling Salesman Problem [Jon90], and then used by Bishop and Jones in [BJ94] and [BJ90], and by Lerman and many others in the context of Lipschitz or nearly Lipschitz parametrizations (see [DS93, DS91, Jon89, Jon91, Lég99, Paj97]).

Now, consider the function $J_\infty(x) = \sum_{k \geq 0} \beta_\infty^2(x, 10^{-k})$, where $x \in M \cap B_{10}(0)$. David and Toro prove [DT12] that if a set M is ϵ -Reifenberg flat, and if the function J_∞ is uniformly bounded on $M \cap B_{10}(0)$, then M is a bi-Lipschitz image of an n -dimensional affine subspace in \mathbb{R}^{n+d} . They also prove the same result while considering the possibly smaller ¹ numbers β_1 -numbers

$$\beta_1(x, r) = \inf_P \frac{1}{r^n} \int_{M \cap B_r(x)} \frac{\text{dist}(y, P)}{r} d\mathcal{H}^n(y), \quad (1.8)$$

where $x \in M \cap B_{10}(0)$, $0 < r \leq 10$, \mathcal{H}^n is the n -dimensional Hausdorff measure, and the infimum this time, is taken over all n -dimensional affine subspaces P of \mathbb{R}^{n+d} , passing through $B_r(x)$, (and not necessarily through x). One can think of the β_1 -numbers as a weak version of the β_∞ numbers. Analogous to the function J_∞ , consider the function $J_1(x) = \sum_{k \geq 0} \beta_1^2(x, 10^{-k})$, where $x \in M \cap B_{10}(0)$. Then, David and Toro prove

Theorem 1.1.2. *(see Theorem 1.4 in [DT12]) Suppose that n , d , and M are as in Theorem 1.1.1. Let $\epsilon > 0$ small enough, depending on n and d . Assume that for every $x \in M \cap B_{10}(0)$ and for every $0 < r \leq 10$, we can find an n -dimensional affine subspace $P(x, r)$ of \mathbb{R}^{n+d} that contains x such that (1.1) and (1.2) hold. Moreover, suppose there exists a positive number N such that for all $x \in M \cap B_{10}(0)$, we have $J_1(x) := \sum_{k \geq 0} \beta_1^2(x, 10^{-k}) \leq N$. Then, the mapping g provided by Theorem 1.1.1 is K -bi-Lipschitz, that is, (1.6) holds, with the bi-Lipschitz constant K depending only on n , d , and N .*

It was very interesting to find a condition involving the β_1 -numbers sufficient to guarantee a local bi-Lipschitz parametrization of M (from Theorem 1.1.2), since a previous result by

¹In the case where M is locally Ahlfors regular, we have $\beta_1(x, r) \leq \beta_\infty(x, r)$.

David and Semmes [DS91] stated that a Carleson condition on the β_1 -numbers

$$\int_{M \cap B_r(x)} \int_0^r \beta_1^2(x, r) \frac{dt}{t} d\mathcal{H}^n(y) \leq C_0 r^n,$$

where $x \in M$, $0 < r \leq 1$ and C_0 is a constant that depends only on n and d , is a necessary condition for M to be (locally) a bi-Lipschitz image of an n -plane (see [DT12], remark 15.6). Carleson-type conditions which are sufficient for M to admit a bi-Lipschitz parametrization have been studied (see [Tor95]). In [Tor95], Toro studies a Carleson-type condition on the Reifenberg flatness (equations (1.1) and (1.2)) which yields a bi-Lipschitz parametrization. As a corollary, she obtains an interesting result for a special type of *chord arc surfaces with small constant*, that is CASSC.

Definition 1.1.3. Let M be a connected C^2 hyper-surface in \mathbb{R}^{n+1} such that $M \cup \{\infty\}$ is a C^2 hyper-surface in $\mathbb{R}^{n+1} \cup \{\infty\}$. Let $\nu(x)$ denote a choice of unit normal to M . Let $\|\nu\|_*$ denote the BMO norm of ν , that is

$$\|\nu\|_* = \sup_{x \in M, r > 0} \frac{1}{\mathcal{H}^n(M \cap B_r(x))} \int_{M \cap B_r(x)} |\nu(y) - \nu_{x,r}| d\mathcal{H}^n(y),$$

where $\nu_{x,r} = \int_{M \cap B_r(x)} \nu(y) d\mathcal{H}^n(y) = \frac{1}{\mathcal{H}^n(M \cap B_r(x))} \int_{M \cap B_r(x)} \nu(y) d\mathcal{H}^n(y)$ denotes the average of the unit normal ν on the ball $B_r(x)$.

Suppose that there exists $\gamma > 0$ small enough such that $\|\nu\|_* \leq \gamma$ and the following holds

$$|\langle x - y, \nu_{x,r} \rangle| \leq \gamma r \quad \forall x \in M, 0 < r \leq 1 \text{ and } y \in M \cap B_r(x).$$

Then, M is called a *chord arc surface with small constant*.

Thus, CASSC are C^2 hyper-surfaces in \mathbb{R}^{n+1} that have small BMO norm, and at every point x and scale r , they are close to the n -plane whose normal is $\nu_{x,r}$. These hyper-surfaces were introduced by Semmes [Sem91a]. He proves that they can be locally parametrized by a $C^{0,\alpha}$ homeomorphism, for any $\alpha < 1$. It is then natural to ask if they admit a local bi-Lipschitz parametrization. In [Tor95], Toro proves the following theorem about CASSC:

Theorem 1.1.4. (see Corollary 5.1 in [Tor95]) Suppose M is a CASSC. There exists $\delta > 0$ and $\epsilon > 0$, depending only on n such that if $\|\nu\|_* \leq \delta$ and

$$\int_0^\infty \sup_{x \in M} \left(\int_{M \cap B_{2r}(x)} |\nu(y) - \nu_{x,2r}|^p \right)^{\frac{2}{p}} \frac{dr}{r} \leq \epsilon^2, \quad (1.9)$$

for some $p > n$, then M admits a local K -bi-Lipschitz parametrization, with the bi-Lipschitz constant K depending on ϵ , δ , and the dimension n .

1.2 The main results and the structure of this thesis

In this thesis, we consider an n -dimensional subset M of \mathbb{R}^{n+d} where the oscillation of the tangent planes of M satisfy a Carleson-type condition, and prove the existence of a local bi-Lipschitz parametrization for M . In particular, the restriction of this result to co-dimension 1 surfaces M generalizes Theorem 1.1.4, as both the regularity condition imposed on the hyper-surface M , and the Carleson-type condition on the oscillation of the unit normals of M are relaxed.

So what conditions do we want to start with? We consider n -dimensional rectifiable sets $M \subset \mathbb{R}^{n+d}$, which are Ahlfors regular, and satisfy a Poincaré-type inequality. As mentioned earlier, rectifiable sets are characterized by having *approximate tangent planes* almost everywhere (see Definitions 2.0.3 and 2.0.4 for precise definitions of rectifiability and approximate tangent planes). Thus, studying the behavior of the tangent planes of M in order to get regularity information about M is very appropriate. Denote by μ the n -Hausdorff measure restricted to M , that is, $\mu = \mathcal{H}^n \llcorner M$, and suppose that M is n -Ahlfors regular (see Definition 2.11 for the definition of n -Ahlfors regular sets). We note that CASSC are in particular rectifiable and n -Ahlfors regular (see [Sem91a]). The Poincaré-type inequality we consider on M is the following:

For all $x \in M$, $r > 0$, and f a locally Lipschitz function on \mathbb{R}^{n+d} , we have

$$\int_{B_r(x)} |f(y) - f_{x,r}| d\mu(y) \leq c_P r \left(\int_{B_{2r}(x)} |\nabla^M f(y)|^2 d\mu(y) \right)^{\frac{1}{2}}, \quad (1.10)$$

where c_P denotes the Poincaré constant that appears here, which is a constant depending only on n and d , $f_{x,r}$ is the average of the function f on $B_r(x)$ (see (4.1) for precise definition), and $\nabla^M f(y)$ denotes the tangential derivative of f (see (4.2) for the definition the tangential gradient).

We remark that Semmes proved in [Sem91c] that the Poincaré-type inequality (1.10), for $d = 1$, is satisfied by CASSC. In fact, this is the motivation behind our asking that the rectifiable set M satisfies this Poincaré-type inequality. This inequality is different from the usual Poincaré inequality on Euclidean space (see [EG92] p. 141). For instance, in (1.10), the average of the oscillation of f is bounded by its tangential derivative and not the usual derivative; moreover, the ball on the right hand side of (1.10) has twice the radius of the ball on the left hand side of (1.10), which is not the case in the usual Poincaré inequality. However, (1.10) fits perfectly with the Poincaré inequality that Riemannian manifolds with Ricci curvature bounded from below satisfy (see [HK00] p.46) once we take the metric g to be the pullback of the Euclidean metric to the manifold. Semmes' proof that CASSC satisfy (1.10) strongly depends on the fact that the surface is chord arc, and in particular, smooth. Here, we assume this inequality, and prove in the last chapter that not all rectifiable sets satisfy (1.10). In fact, we prove that this Poincaré-type inequality (1.10) gives connectivity information about M .

Notation:

Fix $x \in M$ and $r > 0$. Let $y \in M \cap B_r(x)$ such that the approximate tangent plane $T_y M$ of M at the point y exists, and denote by $\pi_{T_y M}$ the orthogonal projection of \mathbb{R}^{n+d} on $T_y M$. Using the standard basis of \mathbb{R}^{n+d} , $\{e_1, \dots, e_{n+d}\}$, we can view $\pi_{T_y M}$ as an $(n+d) \times (n+d)$ matrix whose j^{th} column is the vector is $\pi_{T_y M}(e_j)$. Thus, we denote $\pi_{T_y M}$ by the matrix

$(a_{ij}(y))_{ij}$. Finally, let $A_{x,r} = ((a_{ij})_{x,r})_{ij}$, be the matrix whose ij^{th} entry is the average of the function a_{ij} in the ball $B_r(x)$.

We are ready to state the main result of this thesis:

Theorem 1.2.1. *Let $M \subset B_1(0)$ be an n -Ahlfors regular rectifiable set containing the origin, and let $\mu = \mathcal{H}^n \llcorner M$ be the Hausdorff measure restricted to M . Assume that M satisfies the Poincaré-type inequality (1.10). There exists $\epsilon_0 > 0$ that depends only on n and d , such that if*

$$\int_0^1 \left(\int_{B_r(x)} |\pi_{T_y M} - A_{x,r}|^2 d\mu \right) \frac{dr}{r} < \epsilon_0^2 \quad \text{for } x \in M \cap B_{\frac{1}{10^3}}(0), \quad (1.11)$$

where $|\pi_{T_y M} - A_{x,r}|$ denotes the Frobenius norm² of $\pi_{T_y M} - A_{x,r}$, then there exists a bijective K -bi-Lipschitz map $g : \mathbb{R}^{n+d} \rightarrow \mathbb{R}^{n+d}$ where the bi-Lipschitz constant K depends only on n and d , and an n -dimensional plane Σ_0 , with the following properties:

$$g(z) = z \quad \text{when } d(z, \Sigma_0) \geq 2, \quad (1.12)$$

and

$$|g(z) - z| \leq C_2 \epsilon_0 \quad \text{for } z \in \mathbb{R}^{n+d}, \quad (1.13)$$

where C_2 is a constant depending only on n and d . Moreover,

$$g(\Sigma_0) \text{ is a } C_2 \epsilon_0\text{-Reifenberg flat set,} \quad (1.14)$$

and

$$M \cap B_{\frac{1}{10^3}}(0) \subset g(\Sigma_0). \quad (1.15)$$

It is worth mentioning here that Theorem 1.2.1 states that M is (locally) *contained in* a bi-Lipschitz image of an n -plane instead of M being exactly a (local) bi-Lipschitz image

² $|\pi_{T_y M} - A_{x,r}|^2 = \text{trace}((\pi_{T_y M} - A_{x,r})^2) = \sum_{i,j=1}^{n+d} |a_{ij}(y) - (a_{ij})_{x,r}|^2$

of an n -plane, as proved in Theorems 1.1.1, 1.1.2, and 1.1.4. This is very much expected, since when we drop the assumption of Reifenberg flatness on M , we have to deal with the fact that M might be full of holes. However, if we assume, in addition to the hypothesis of Theorem 1.2.1, that M is Reifenberg flat, then we do obtain that M is in fact (locally) a bi-Lipschitz image of an n -plane. We show this later in the thesis as a corollary to Theorem 1.2.1.

In the special case when M has co-dimension 1, we define the (generalized) unit normal to M at a point $y \in M$ to be the unit normal to the approximate tangent plane at that point. Thus, M admits a generalized unit normal at almost every point³. In this case, assuming that there exists a choice of unit normal ν to M such that the following Carleson-type condition⁴ on the oscillation of ν is satisfied

$$\sup_{x \in M \cap B_{\frac{1}{10^3}}(0)} \int_0^1 \left(\int_{B_r(x)} |\nu(y) - \nu_{x,r}|^2 d\mu \right) \frac{dr}{r} < \epsilon_0^2, \quad (1.16)$$

is the same as assuming condition (1.11). We also show this later as a corollary to Theorem 1.2.1. Notice, however, that (1.16) is a more relaxed condition than (1.9) in Theorem 1.1.4. Thus, Theorem 1.2.1, for the special case when M has co-dimension 1, does in fact generalize Theorem 1.1.4.

The thesis is structured as follows: in chapter 2, we record several definitions and preliminaries. In chapter 3, we prove some linear algebra lemmas needed to prove Theorem 1.2.1. In chapter 4, we prove Theorem 1.2.1, as well as the corollaries mentioned above in the two paragraphs that follow the statement of Theorem 1.2.1. The proof of Theorem 1.2.1 is done in several steps. First, we define the α -numbers

³Of course, there are two choices for the direction of the unit normal at every point where it exists.

⁴The Carleson-type condition imposed will guarantee a coherent choice of the unit normals.

$$\alpha(x, r) := \left(\int_{B_r(x)} |\pi_{T_y M} - A_{x,r}|^2 d\mu \right)^{\frac{1}{2}},$$

where $x \in M$, and $0 < r \leq 1$.

These α -numbers play the role that β_1 -numbers played for Theorem 1.1.2 (see Theorem 4.2.2). Then we prove Theorem 1.2.1 using the α -numbers, while handling the issue that M might be have many holes.

In chapter 5, we show that the Poincaré-type inequality satisfied by M is interesting by itself, as it encodes some geometric information about M . In fact, we show that if a rectifiable set M satisfies (1.10), then M is quasiconvex. A set is quasiconvex if any two points in the set are connected by a rectifiable curve, contained in the set, whose length is comparable to the distance between the two points.

Chapter 2

PRELIMINARIES

Throughout this thesis, our ambient space is \mathbb{R}^{n+d} . $B_r(x)$ denotes the open ball center x and radius r in \mathbb{R}^{n+d} , while $\bar{B}_r(x)$ denotes the closed ball center x and radius r in \mathbb{R}^{n+d} . $d(.,.)$ denotes the distance function from a point to a set. \mathcal{H}^n is the n -Hausdorff measure. Finally, c denotes a constant that depends on n and d only, and might vary from line to line.

We begin by recalling the definition of a Lipschitz and a bi-Lipschitz function:

Definition 2.0.1. Let $M \subset \mathbb{R}^{n+d}$. A function $f : M \rightarrow \mathbb{R}$ is called *Lipschitz* if there exists a constant $K > 0$, such that for all $x, y \in M$ we have

$$|f(x) - f(y)| \leq K |x - y|. \quad (2.1)$$

The smallest such constant is called the *Lipschitz constant* and is denoted by $lipf$.

Definition 2.0.2. A function $f : \mathbb{R}^{n+d} \rightarrow \mathbb{R}^{n+d}$ is called *K -bi-Lipschitz* if there exists a constant $K > 0$, such that for all $x, y \in \mathbb{R}^{n+d}$ we have

$$K^{-1}|x - y| \leq |f(x) - f(y)| \leq K |x - y|.$$

Next, we introduce the class of *n -rectifiable* sets:

Definition 2.0.3. Let $M \subset \mathbb{R}^{n+d}$ be an \mathcal{H}^n -measurable set. M is said to be *countably n -rectifiable* if

$$M \subset M_o \cup \left(\bigcup_{i=1}^{\infty} f_i(A_i) \right),$$

where $\mathcal{H}^n(M_o) = 0$, and $f_i : A_i \rightarrow \mathbb{R}^{n+d}$ is Lipschitz, and $A_i \subset \mathbb{R}^n$, for $i = 1, 2, \dots$

n -rectifiable sets are characterized in terms of approximate tangent spaces which we now define:

Definition 2.0.4. If M is an \mathcal{H}^n -measurable subset of \mathbb{R}^{n+d} . We say that the n -dimensional subspace $P(x)$ is the *approximate tangent space of M at x* , if

$$\lim_{h \rightarrow 0} h^{-n} \int_M f(h^{-1}(y-x)) d\mathcal{H}^n(y) = \int_{P(x)} f(y) d\mathcal{H}^n(y) \quad \forall f \in C_c^1(\mathbb{R}^{n+d}, \mathbb{R}). \quad (2.2)$$

Remark 2.0.5. Notice that if it exists, $P(x)$ is unique. From now on, we shall denote the tangent space of M at x by $T_x M$.

The following theorem gives the important characterization of n -rectifiable sets in terms of approximate tangent spaces:

Theorem 2.0.6. (see [Sim83]; Theorem 11.6¹)

Suppose M is an \mathcal{H}^n -measurable subset of \mathbb{R}^{n+d} . Then M is countably n -rectifiable if and only if the approximate tangent space $T_x M$ exists for \mathcal{H}^n -a.e. $x \in M$.

We also need to define the notion *Reifenberg flatness*:

Definition 2.0.7. Let M be an n -dimensional subset of \mathbb{R}^{n+d} . We say that M is ϵ -Reifenberg flat for some $\epsilon > 0$, if for every $x \in M$ and $0 < r \leq \frac{1}{10^4}$, we can find an n -dimensional affine subspace $P(x, r)$ of \mathbb{R}^{n+d} that contains x such that

$$d(y, P(x, r)) \leq \epsilon r \quad \text{for } y \in M \cap B_r(x),$$

and

$$d(y, M) \leq \epsilon r \quad \text{for } y \in P(x, r) \cap B_r(x).$$

Remark 2.0.8. Notice that the above definition is only interesting if ϵ is small, since any set is 1-Reifenberg flat.

¹See the proof of the *only if* part p. 62, to realize that Theorem 2.0.6 here is a special case of Theorem 11.6 in [Sim83].

In the proof of our theorems, we need to measure the distance between two n -dimensional planes. We do so in terms of normalized local Hausdorff distance:

Definition 2.0.9. Let x be a point in \mathbb{R}^{n+d} and let $r > 0$. Consider two closed sets $E, F \subset \mathbb{R}^{n+d}$ such that both sets meet the ball $B_r(x)$. Then,

$$d_{x,r}(E, F) = \frac{1}{r} \text{Max} \left\{ \sup_{y \in E \cap B_r(x)} \text{dist}(y, F) ; \sup_{y \in F \cap B_r(x)} \text{dist}(y, E) \right\}$$

is called the normalized Hausdorff distance between E and F in $B_r(x)$.

Finally, we recall the definition of an n -Ahlfors regular measure and an n -Ahlfors regular set:

Definition 2.0.10. Let $M \subset \mathbb{R}^{n+d}$ be a closed, \mathcal{H}^n -measurable set, and let $\mu = \mathcal{H}^n \llcorner M$ be the n -Hausdorff measure restricted to M . We say that μ is n -Ahlfors regular if for every $x \in M$ and $0 < r < 1$, we have

$$C^{-1} r^n \leq \mu(B_r(x)) \leq C r^n, \tag{2.3}$$

where C is a constant depending only on n and d . In such a case, the set M is called an n -Ahlfors regular set.

Chapter 3

LINEAR ALGEBRA DIGRESSION

To prove our main theorem, we need the following three linear algebra lemmas. Since they are independent results, let us digress a little bit and prove them here.

Lemma 3.0.1. *Let V be an n -dimensional subspace of \mathbb{R}^{n+d} . Denote by π_V the orthogonal projection on V . Then, there exists a $\delta_0 > 0$, depending only on n and d , such that for any $\delta \leq \delta_0$, and for any linear operator L on \mathbb{R}^{n+d} such that*

$$\|\pi_V - L\| \leq \delta, \quad (3.1)$$

where $\|\cdot\|$ denotes the induced operator norm, L has exactly n eigenvalues $\lambda_1, \dots, \lambda_n$ such that

$$|\lambda_j| \geq 1 - (n + d)\delta \geq \frac{3}{4}, \quad \forall j \in \{1, \dots, n\}, \quad (3.2)$$

and exactly d eigenvalues $\lambda_{n+1}, \dots, \lambda_{n+d}$, such that

$$|\lambda_j| \leq (n + d)\delta \leq \frac{1}{4}, \quad \forall j \in \{n + 1, \dots, n + d\}. \quad (3.3)$$

Proof. Since π_V is an orthogonal projection, then there exists an orthonormal basis $\{w_1, \dots, w_{n+d}\}$ of \mathbb{R}^{n+d} such that the matrix representation of π_V in this basis is

$$\pi_V = \begin{pmatrix} Id_n & 0 \\ 0 & 0 \end{pmatrix}$$

where Id_n denotes the $n \times n$ identity matrix. Let $\delta < \delta_0$ (with δ_0 to be determined later), and suppose L is as in the statement of the lemma. Let $L = (l_{ij})_{ij}$ be the matrix representation of L in the basis $\{w_1, \dots, w_{n+d}\}$. Then, by (3.1), we have

$$|\pi_V w_j - L w_j|^2 \leq \delta^2, \quad \forall j \in \{1 \dots n + d\},$$

that is,

$$|1 - l_{jj}|^2 + \sum_{i \neq j} |l_{ij}|^2 \leq \delta^2, \quad \forall j \in \{1 \dots n\}, \quad (3.4)$$

and

$$\sum_{i=1}^{n+d} |l_{ij}|^2 \leq \delta^2, \quad \forall j \in \{n+1 \dots n+d\}. \quad (3.5)$$

Now, for each $j \in \{1 \dots n+d\}$, consider the closed disk D_j in the complex plane, of center $(l_{jj}, 0)$ and radius $R_j = \sum_{i \neq j} |l_{ij}|$. Notice that by (3.4), (3.5), and the fact that $\delta < \delta_0$, we have

$$|1 - l_{jj}| \leq \delta \leq \delta_0, \quad \forall j \in \{1 \dots n\}, \quad (3.6)$$

$$|l_{jj}| \leq \delta \leq \delta_0, \quad \forall j \in \{n+1 \dots n+d\}, \quad (3.7)$$

and

$$R_j \leq (n+d-1)\delta \leq (n+d-1)\delta_0, \quad \forall j \in \{1 \dots n+d\}. \quad (3.8)$$

Choosing δ_0 such that $(n+d-1)\delta_0 \leq \frac{1}{8}$, we can guarantee that $\bigcup_{j=1}^n D_j$ is disjoint from

$\bigcup_{j=n+1}^{n+d} D_j$. Thus, by the Gershgorin circle theorem (see [LeV07], p.277-278), $\bigcup_{j=1}^n D_j$ contains

exactly n eigenvalues of L , and $\bigcup_{j=n+1}^{n+d} D_j$ contains exactly d eigenvalues of L . The lemma follows from (3.6), (3.7) and (3.8). \square

Notation:

Throughout the proof of the next lemma, V denotes an affine subspace of \mathbb{R}^{n+d} of dimension k , $k \in \{0, \dots, n-1\}$, and $N_\delta(V)$ denotes the δ -neighbourhood of V , that is,

$$N_\delta(V) = \{x \in \mathbb{R}^{n+d} \text{ such that } d(x, V) < \delta\}.$$

Also, $c(n, d, k)$ denotes a constant that depends only on n , d , and k , and might vary from line to line.

Lemma 3.0.2. *Let M be an n -Ahlfors regular subset of \mathbb{R}^{n+d} , and let $\mu = \mathcal{H}^n \llcorner M$ be the Hausdorff measure restricted to M . There exists a constant $C_0 \leq \frac{1}{2}$ depending only on n and d , such that the following is true. Fix $x_0 \in M$, $r_0 < 1$ and let $r = C_0 r_0$. Then, for every V , an affine subspace of \mathbb{R}^{n+d} of dimension $0 \leq k \leq n - 1$, there exists $x \in M \cap B_{r_0}(x_0)$ such that $x \notin N_{11r}(V)$ and $B_r(x) \subset B_{2r_0}(x_0)$.*

Proof. Fix $x_0 \in M$, $r_0 < 1$, and $k \in \{0, \dots, n - 1\}$. Let V be an affine k -dimensional subspace of \mathbb{R}^{n+d} . Consider $N_{11r}(V)$, where $r < r_0$ is to be determined later. The set $\mathcal{A} := \{B_{\frac{r}{5}}(x), x \in M \cap N_{11r}(V) \cap B_{r_0}(x_0)\}$ forms a cover for $M \cap N_{11r}(V) \cap B_{r_0}(x_0)$, and thus by Vitali's theorem, there exists a finite disjoint subset of \mathcal{A} , say $\mathcal{A}' := \{B_{\frac{r}{5}}(x_i)\}_{i=1}^N$, such that

$$M \cap N_{11r}(V) \cap B_{r_0}(x_0) \subset \bigcup_{i=1}^N B_r(x_i). \quad (3.9)$$

Let us start by getting an upper bound for the number of balls N , needed to cover $M \cap N_{11r}(V) \cap B_{r_0}(x_0)$. Notice that

$$\bigcup_{i=1}^N B_{\frac{r}{5}}(x_i) \subset B_{r+r_0}^k(a) \times B_{12r}^{n+d-k}(a), \quad (3.10)$$

where $a = \pi_V(x_0)$, the orthogonal projection of x_0 on V , $B_{r+r_0}^k(a) = V \cap B_{r+r_0}(a)$, and $B_{12r}^{n+d-k}(a) = V^\perp \cap B_{12r}(a)$ where V^\perp is the affine subspace, perpendicular to V and passing through a .

In fact, take $x \in \bigcup_{i=1}^N B_{\frac{r}{5}}(x_i)$. Then there exists $x_i \in M \cap B_{r_0}(x_0) \cap N_{11r}(V)$, with $i \in \{1, \dots, N\}$ such that $|x - x_i| \leq \frac{r}{5}$. Now, write x as $x = (\pi_V(x), \pi_{V^\perp}(x))$. On one hand, we have

$$\begin{aligned} |\pi_V(x) - a| &= |\pi_V(x) - \pi_V(x_0)| \\ &\leq |\pi_V(x) - \pi_V(x_i)| + |\pi_V(x_i) - \pi_V(x_0)| \\ &\leq |x - x_i| + |x_i - x_0| \leq r + r_0, \end{aligned} \quad (3.11)$$

where in the last step we used the facts that $x_i \in B_{r_0}(x_0)$ and $|x - x_i| \leq \frac{r}{5}$. On the other hand,

$$\begin{aligned} |\pi_{V^\perp}(x) - a| &\leq |\pi_{V^\perp}(x) - \pi_{V^\perp}(x_i)| + |\pi_{V^\perp}(x_i) - a| \\ &\leq |x - x_i| + 11r \leq 12r, \end{aligned} \quad (3.12)$$

where in the step before the last we used the fact that $x_i \in N_{11r}(V)$, and in the last step we used that $|x - x_i| \leq \frac{r}{5}$. Combining (3.11) and (3.12), we get (3.10).

Since the balls in \mathcal{A}' are disjoint, then by taking the Lebesgue measure on each side of (3.10), and using the fact that $r < r_0$, we get

$$\begin{aligned} N\omega_{n+d} \left(\frac{r}{5}\right)^{n+d} &\leq \omega_k (r_0 + r)^k \omega_{n+d-k} (12r)^{n+d-k} \\ &\leq c(n, d, k) (r_0 + r)^k r^{n+d-k} \\ &\leq c(n, d, k) r_0^k r^{n+d-k} \end{aligned} \quad (3.13)$$

where ω_l denotes the Lebesgue measure of the unit ball in \mathbb{R}^l . Thus,

$$N \leq c(n, d, k) r_0^k r^{-k}. \quad (3.14)$$

Now, we want to use the fact that μ is Ahlfors regular to compare the μ -measures of the sets $N_{11r}(V) \cap B_{r_0}(x_0)$ and $B_{r_0}(x_0)$. On one hand, since μ is lower Ahlfors regular and $x_0 \in M$, we have by (2.3)

$$\mu(B_{r_0}(x_0)) \geq C^{-1} r_0^n, \quad (3.15)$$

where C is a (fixed) constant depending on n and d . On the other hand, by (3.9), the fact that μ is upper Ahlfors regular and $x_i \in M$ for all $i \in \{1, \dots, N\}$, and by (3.14), we get

$$\begin{aligned} \mu(N_{11r}(V) \cap B_{r_0}(x_0)) &= \mu(M \cap N_{11r}(V) \cap B_{r_0}(x_0)) \\ &\leq \sum_{i=1}^N \mu(B_r(x_i)) \\ &\leq C N r^n \\ &\leq c(n, d, k) r_0^k r^{n-k}, \end{aligned} \quad (3.16)$$

where C is the constant from (2.3). Let us denote by c_1 the constant $c(n, d, k)$ we get from (3.16). From now till the end of the proof, c_1 will stand for exactly this constant. Hence, (3.16) becomes

$$\mu(N_{11r}(V) \cap B_{r_0}(x_0)) \leq c_1 r_0^k r^{n-k}, \quad (3.17)$$

where c_1 depends on n , d , and k . Thus, if we pick r such that

$$r^{n-k} < \frac{C^{-1}}{c_1} r_0^{n-k}, \quad (3.18)$$

then

$$c_1 r_0^k r^{n-k} < C^{-1} r_0^n. \quad (3.19)$$

Comparing (3.19) with (3.15) and (3.17), we get

$$\mu(N_{11r}(V) \cap B_{r_0}(x_0)) < \mu(B_{r_0}(x_0)),$$

and thus, there exists a point $x \in M \cap B_{r_0}(x_0)$ such that $x \notin N_{11r}(V)$.

Notice that the proof of the lemma would have been done if the statement allowed for $r = c(n, d, k)r_0$ where $c(n, d, k)$ is a constant depending on n , d , and k (see(3.18)). In fact, we have shown that for every $k \in \{0, \dots, n-1\}$, and for every V , an affine k -dimensional subspace of \mathbb{R}^{n+d} , there is a constant $c(n, d, k)$, such that if $r \leq c(n, d, k)r_0$, then we can find a point $x \in M \cap B_{r_0}(x_0)$ such that $x \notin N_{11r}(V)$.

Now, take $r = C_0 r_0$ where $C_0 := \min\{c(n, d, 0), \dots, c(n, d, n-1)\}$. First, notice that C_0 is a constant depending only on n and d . Moreover, when V is an affine k -dimensional subspace of \mathbb{R}^{n+d} , $k \in \{0, \dots, n-1\}$, we have $r = C_0 r_0 \leq c(n, d, k)r_0$. Thus, there exists a point $x \in M \cap B_{r_0}(x_0)$ such that $x \notin N_{11r}(V)$. Without loss of generality, we can assume that $C_0 \leq \frac{1}{2}$. The fact that $B_r(x) \subset B_{2r_0}(x_0)$ follows directly from the fact that $r < r_0$, and the proof is done. \square

Remark 3.0.3. Let us note here that as stated in the lemma above, the dimension of the affine subspace V is allowed to be 0. In fact, if V is a single point, say $V = \{y_0\}$, then $N_\delta(V) = B_\delta(y_0)$, and the proof follows exactly as above. Moreover, the dimension k of V has $n - 1$ as an upper bound. This is because the lemma fails for $k = n$ (take $M = V = \mathbb{R}^n$ and let $x_0 = 0$).

Lemma 3.0.4. *Fix $R > 0$, and let $\{u_1, \dots, u_n\}$ be n vectors in \mathbb{R}^{n+d} . Suppose there exists a constant $K_0 > 0$ that depends only on n and d , such that*

$$|u_j| \leq K_0 R \quad \forall j \in \{1, \dots, n\}. \quad (3.20)$$

Moreover, suppose there exists a constant $0 < k_0 < K_0$, that depends only on n and d , such that

$$|u_1| \geq k_0 R, \quad (3.21)$$

and

$$u_j \notin N_{k_0 R}(\text{span}\{u_1, \dots, u_{j-1}\}) \quad \forall j \in \{2, \dots, n\}. \quad (3.22)$$

Then, for every vector $v \in V := \text{span}\{u_1, \dots, u_n\}$, v can be written uniquely as

$$v = \sum_{j=1}^n \beta_j u_j, \quad (3.23)$$

where

$$|\beta_j| \leq K_1 \frac{1}{R} |v|, \quad \forall j \in \{1, \dots, n\} \quad (3.24)$$

with K_1 being a constant depending only on n and d .

Proof. Since the vectors $\{u_1, \dots, u_n\}$ are linearly independent (by (3.22)), then by the Gram-Schmidt process, we construct n orthonormal vectors, $\{e_1, \dots, e_n\}$ such that

$$\text{span}\{u_1, \dots, u_j\} = \text{span}\{e_1, \dots, e_j\} \quad \forall j \in \{1, \dots, n\}, \quad (3.25)$$

and

$$u_j = \sum_{i=1}^j b_{ij} e_i \quad \forall j \in \{1, \dots, n\}. \quad (3.26)$$

Let us first consider $j = 1$. By (3.26), (3.20), (3.21), and the fact that e_1 is a unit vector, we have

$$u_1 = b_{11} e_1 \quad \text{with } k_0 R \leq |b_{11}| \leq K_0 R. \quad (3.27)$$

For $j = 2$, (3.26), (3.22), and (3.25) tell us that

$$u_2 = b_{12} e_1 + b_{22} e_2,$$

with

$$u_2 \notin N_{k_0 R}(\text{span}\{u_1\}) = N_{k_0 R}(\text{span}\{e_1\}).$$

This means that

$$|b_{22}| = d(u_2, \text{span}\{e_1\}) \geq k_0 R. \quad (3.28)$$

Moreover, from (3.20) and the fact that the $\{e_1, e_2\}$ is a set of orthonormal vectors, we have

$$|u_2| = \sqrt{(b_{12})^2 + (b_{22})^2} \leq K_0 R,$$

that is

$$|b_{i2}| \leq K_0 R \quad i \in \{1, 2\}.$$

Continuing in a similar manner, we get for every $j \in \{1, \dots, n\}$,

$$|b_{jj}| = d(u_j, \text{span}\{e_1, \dots, e_{j-1}\}) \geq k_0 R, \quad (3.29)$$

and

$$|b_{ij}| \leq K_0 R \quad \forall i \in \{1, \dots, j\}. \quad (3.30)$$

Let B be the $n \times n$ matrix whose j -th column is u_j written in the orthonormal basis $\{e_1, \dots, e_n\}$. Notice that by construction, B is an upper triangular matrix, whose ij -th entry is b_{ij} , for every $i \leq j$. Moreover, B is invertible (since all its diagonal entries are non-zero by (3.29)), and is the change of basis matrix from the basis $\{u_1, \dots, u_n\}$ to the basis $\{e_1, \dots, e_n\}$.

Now, consider a vector $v \in V := \text{span}\{u_1, \dots, u_n\} = \text{span}\{e_1, \dots, e_n\}$. Denoting by v_u and v_e the representation of the vector v in the bases $\{u_1, \dots, u_n\}$ and $\{e_1, \dots, e_n\}$ respectively, let us set

$$v = \sum_{j=1}^n \beta_j u_j = \sum_{j=1}^n \alpha_j e_j \quad (3.31)$$

We know that $v_e = B \cdot v_u$, that is

$$v_u = B^{-1} \cdot v_e. \quad (3.32)$$

Substituting (3.31) in equality (3.32), we get

$$(\beta_1, \dots, \beta_n) = B^{-1} \cdot (\alpha_1, \dots, \alpha_n). \quad (3.33)$$

Let us recall here that

$$B^{-1} = \frac{1}{\det(B)} \text{adj}(B), \quad (3.34)$$

where $\text{adj}(B)$ is the adjoint matrix of B . Now, if we denote by $(\text{row})_l$, the l -th row of $\text{adj}(B)$, $l \in \{1 \dots n\}$, then by (3.30) and unravelling the definition of $\text{adj}(B)$, we get

$$|(\text{row})_l| \leq \sqrt{n} K_0^{n-1} (n-1)! R^{n-1} \quad \forall l \in \{1 \dots n\}. \quad (3.35)$$

Moreover, since B is an upper triangular matrix, whose j -th diagonal entry is b_{jj} , then by (3.29)

$$\det(B) = b_{11} \dots b_{nn} \geq k_0^n R^n. \quad (3.36)$$

We are now ready to get an upper bound on the β_j 's: From (3.33) and (3.34), we can see that for every $j \in \{1, \dots, n\}$

$$\beta_j = \frac{1}{\det(B)} (\text{row})_j \cdot (\alpha_1, \dots, \alpha_n). \quad (3.37)$$

Thus, by (3.37), (3.36), (3.35), (3.31), and the fact that $\{e_1, \dots, e_n\}$ are an orthonormal set of vectors, we get

$$\begin{aligned} |\beta_j| &\leq \frac{1}{k_0^n R^n} |(\text{row})_j| |(\alpha_1, \dots, \alpha_n)| \\ &\leq \frac{1}{k_0^n R^n} \sqrt{n} K_0^{n-1} (n-1)! R^{n-1} |v| \\ &= K_1 \frac{1}{R} |v|, \end{aligned} \quad (3.38)$$

where K_1 is a constant depending only on n and d . This completes the proof of the lemma.

□

Chapter 4

M IS CONTAINED IN A BI-LIPSCHITZ IMAGE OF AN N-PLANE

4.1 Notations and highlights of the proof of Theorem 1.2.1

Throughout the rest of the thesis, M denotes an n -Ahlfors regular rectifiable subset of \mathbb{R}^{n+d} and $\mu = \mathcal{H}^n \llcorner M$ denotes the Hausdorff measure restricted to M . The average of a function f on the ball $B_r(x)$ is denoted by

$$f_{x,r} = \int_{B_r(x)} f d\mu(y) = \frac{1}{\mu(M \cap B_r(x))} \int_{B_r(x)} f d\mu(y). \quad (4.1)$$

Finally, for a locally Lipschitz function f on \mathbb{R}^{n+d} , $\nabla^M f(y)$ denotes the tangential derivative of f at the point $y \in M$. More precisely,

$$\nabla^M f(y) = \pi_{T_y M}(\nabla f(y)), \quad (4.2)$$

where $\pi_{T_y M}$ is the orthogonal projection of \mathbb{R}^{n+d} on $T_y M$, and ∇f is the usual gradient of f .

The main goal of this chapter is to prove Theorem 1.2.1, the main theorem of this thesis. Recall that Theorem 1.2.1 states that if the Carleson-type condition (1.11) on the oscillation of the tangent planes to M is satisfied, and if M satisfies the Poincaré-type condition (1.10), then M lives inside a bi-Lipschitz image of an n -dimensional plane. Section 4.2 of this chapter is dedicated to the proof of this theorem, which is done in several steps: First, we define what we call the α -numbers

$$\alpha(x, r) := \left(\int_{B_r(x)} |\pi_{T_y M} - A_{x,r}|^2 d\mu \right)^{\frac{1}{2}}, \quad (4.3)$$

where $x \in M$, and $0 < r \leq \frac{1}{10}$, $\pi_{T_y M}$ has $(a_{ij}(y))_{ij}$ as its matrix representation in the standard basis of \mathbb{R}^{n+d} , and $A_{x,r} = ((a_{ij})_{x,r})_{ij}$ is the matrix whose ij^{th} entry is the average of the

function a_{ij} in the ball $B_r(x)$. These numbers are the most important ingredient to proving our theorem. In Lemma 4.2.1, we show that the Carleson condition (1.11) implies that these numbers are small. Moreover, for every point $x \in M$, the series $\sum_{k=1}^{\infty} \alpha^2(x, 10^{-k})$ is finite. Then, in Theorem 4.2.2, we show that the Poincaré-type inequality allows us to construct an n -plane $P_{x,r}$ at every point $x \in M$ and every scale $0 < r \leq \frac{1}{20}$ where the distance (in integral form) from $M \cap B_r(x)$ to $P_{x,r}$ is bounded by $\alpha(x, 2r)$. This means, by Lemma 4.2.1, that those distances are small, and for a fixed point x , when we add these distances at the scales 10^{-k} for $k \in \mathbb{N}$, this series is finite. Theorem 4.2.2 is the key point that allows us to use the bi-Lipschitz parametrization that G. David and T. Toro construct in [DT12]. In fact, what they do is construct approximating n -planes, and prove that at any two points that are close together, the two planes associated to these points at the same scale, or at two consecutive scales are close in the Hausdorff distance sense. From there, they construct a bi-Hölder parametrization for M . Then, they show that the sum of these distances at scales 10^{-k} for $k \in \mathbb{N}$ is finite (uniformly for every $x \in M$). This is what is needed for their parametrization to be bi-Lipschitz (see Theorem 4.2.4 below and the definition before it). Thus, the rest of this section is devoted to using Theorem 4.2.2 in order to prove the compatibility conditions between the approximating planes mentioned above, while handling the issue that our set M might be full of holes.

We end this chapter with section 4.3 where we prove two corollaries to Theorem 1.2.1 (the ones mentioned in the introduction). In Corollary 4.3.1, we show that if we assume, in addition to the hypothesis of Theorem 1.2.1, that M is ϵ_0 -Reifenberg flat, then (locally) M is exactly the bi-Lipschitz image of an n -plane. In other words, the containment in (1.15) becomes an equality. In Corollary 4.3.2, we show how in the special case when M has co-dimension 1, (1.11) translates into a Carleson-type condition on the oscillation of the unit normals to M .

4.2 The proof of Theorem 1.2.1

Let us begin with the lemma that explores the Carleson condition (1.11).

Lemma 4.2.1. *Let $M \subset B_1(0)$ be an n -Ahlfors regular rectifiable set containing the origin, and let $\mu = \mathcal{H}^n \llcorner M$ be the Hausdorff measure restricted to M . Let $\epsilon > 0$, and suppose that*

$$\int_0^1 \left(\int_{B_r(x)} |\pi_{T_y M} - A_{x,r}|^2 d\mu \right) \frac{dr}{r} < \epsilon^2, \quad \forall x \in M. \quad (4.4)$$

Then, for every $x \in M$, we have

$$\sum_{k=1}^{\infty} \alpha^2(x, 10^{-k}) \leq c\epsilon^2, \quad (4.5)$$

where the α -numbers are as defined in (4.3) and c is a constant that depends only on n and d . Moreover, for every $x \in M$ and $0 < r \leq \frac{1}{10}$, we have

$$\alpha(x, r) \leq c\epsilon, \quad (4.6)$$

where c is a constant that depends only on n and d .

Proof. Let $\epsilon > 0$ and suppose that (4.4) holds. By the definition of the Frobenius norm, (4.4) becomes

$$\sum_{i,j=1}^{n+d} \int_0^1 \left(\int_{B_r(x)} |a_{ij}(y) - (a_{ij})_{x,r}|^2 d\mu \right) \frac{dr}{r} < \epsilon^2, \quad \forall x \in M, \quad (4.7)$$

where $\pi_{T_y M} = (a_{ij}(y))_{ij}$ and $A_{x,r} = ((a_{ij})_{x,r})_{ij}$.

Fix $x \in M$, and fix $i, j \in \{1, \dots, n+d\}$. For all $a \in \mathbb{R}$, and for all $0 < r_0 \leq 1$, we have

$$\int_{B_{r_0}(x)} |a_{ij}(y) - (a_{ij})_{x,r_0}|^2 d\mu \leq \int_{B_{r_0}(x)} |a_{ij}(y) - a|^2 d\mu, \quad (4.8)$$

since the average $(a_{ij})_{x,r_0}$ of a_{ij} in the ball $B_{r_0}(x)$ minimizes the integrand on the right hand side of (4.8).

To prove (4.5), we start by showing

$$\sum_{k=1}^{\infty} \int_{B_{10^{-k}}(x)} |a_{ij}(y) - (a_{ij})_{x,10^{-k}}|^2 d\mu \leq c \sum_{k=0}^{\infty} \int_{10^{-k-1}}^{10^{-k}} \int_{B_r(x)} |a_{ij}(y) - (a_{ij})_{x,r}|^2 d\mu \frac{dr}{r}, \quad (4.9)$$

where c is a constant depending only on n and d .

Fix $k \in \mathbb{N}$ and let r be such that

$$10^{-k-1} < r \leq 10^{-k}, \quad \text{that is} \quad \frac{1}{10^{-k}} \leq \frac{1}{r} < \frac{1}{10^{-k-1}}. \quad (4.10)$$

Using (4.8) for $a = (a_{ij})_{x,r}$ and $r_0 = 10^{-k-1}$, (4.10), and the fact that μ is Ahlfors regular, we get

$$\begin{aligned} \int_{B_{10^{-k-1}}(x)} |a_{ij}(y) - (a_{ij})_{x,10^{-k-1}}|^2 d\mu &\leq \int_{B_{10^{-k-1}}(x)} |a_{ij}(y) - (a_{ij})_{x,r}|^2 d\mu \\ &\leq c \frac{1}{\mu(B_{10^{-k}}(x))} \int_{B_r(x)} |a_{ij}(y) - (a_{ij})_{x,r}|^2 d\mu \\ &\leq c \int_{B_r(x)} |a_{ij}(y) - (a_{ij})_{x,r}|^2 d\mu, \end{aligned} \quad (4.11)$$

where c is a constant depending only on n and d .

Dividing both sides of (4.11) by r and then integrating from 10^{-k-1} to 10^{-k} , we get

$$\int_{10^{-k-1}}^{10^{-k}} \int_{B_{10^{-k-1}}(x)} |a_{ij}(y) - (a_{ij})_{x,10^{-k-1}}|^2 d\mu \frac{dr}{r} \leq c \int_{10^{-k-1}}^{10^{-k}} \int_{B_r(x)} |a_{ij}(y) - (a_{ij})_{x,r}|^2 d\mu \frac{dr}{r}. \quad (4.12)$$

Using (4.10) on the left hand side of (4.12) gives us

$$\frac{1}{10^{-k}} \int_{10^{-k-1}}^{10^{-k}} dr \int_{B_{10^{-k-1}}(x)} |a_{ij}(y) - (a_{ij})_{x,10^{-k-1}}|^2 d\mu \leq c \int_{10^{-k-1}}^{10^{-k}} \int_{B_r(x)} |a_{ij}(y) - (a_{ij})_{x,r}|^2 d\mu \frac{dr}{r},$$

and thus

$$\int_{B_{10^{-k-1}}(x)} |a_{ij}(y) - (a_{ij})_{x,10^{-k-1}}|^2 d\mu \leq c \int_{10^{-k-1}}^{10^{-k}} \int_{B_r(x)} |a_{ij}(y) - (a_{ij})_{x,r}|^2 d\mu \frac{dr}{r}. \quad (4.13)$$

Taking the sum over k from 0 to ∞ on both sides of (4.13), we get

$$\sum_{k=0}^{\infty} \int_{B_{10^{-k-1}}(x)} |a_{ij}(y) - (a_{ij})_{x,10^{-k-1}}|^2 d\mu \leq c \sum_{k=0}^{\infty} \int_{10^{-k-1}}^{10^{-k}} \int_{B_r(x)} |a_{ij}(y) - (a_{ij})_{x,r}|^2 d\mu \frac{dr}{r},$$

that is,

$$\sum_{k=1}^{\infty} \int_{B_{10^{-k}}(x)} |a_{ij}(y) - (a_{ij})_{x,10^{-k}}|^2 d\mu \leq c \sum_{k=0}^{\infty} \int_{10^{-k-1}}^{10^{-k}} \int_{B_r(x)} |a_{ij}(y) - (a_{ij})_{x,r}|^2 d\mu \frac{dr}{r} \quad (4.14)$$

hence finishing the proof of (4.9).

But, it is trivial to check that

$$\sum_{k=0}^{\infty} \int_{10^{-k-1}}^{10^{-k}} \int_{B_r(x)} |a_{ij}(y) - (a_{ij})_{x,r}|^2 d\mu \frac{dr}{r} = \int_0^1 \left(\int_{B_r(x)} |a_{ij}(y) - (a_{ij})_{x,r}|^2 d\mu \right) \frac{dr}{r}. \quad (4.15)$$

Thus, plugging (4.15) in (4.14), we get

$$\sum_{k=1}^{\infty} \int_{B_{10^{-k}}(x)} |a_{ij}(y) - (a_{ij})_{x,10^{-k}}|^2 d\mu \leq c \int_0^1 \left(\int_{B_r(x)} |a_{ij}(y) - (a_{ij})_{x,r}|^2 d\mu \right) \frac{dr}{r}. \quad (4.16)$$

Since (4.16) is true for every $i, j \in \{1, \dots, n+d\}$, we can take the sum over i and j on both sides of (4.16), and using the definition of the Frobenius norm together with (4.3) and (4.4), we get

$$\sum_{k=1}^{\infty} \alpha^2(x, 10^{-k}) \leq c\epsilon^2,$$

which is exactly (4.5).

To prove inequality (4.6), fix $x \in M$ and $0 < r \leq \frac{1}{10}$. Then, there exists $k \geq 1$ such that

$$10^{-k-1} < r \leq 10^{-k}, \quad \text{that is} \quad \frac{1}{10^{-k}} \leq \frac{1}{r} < \frac{1}{10^{-k-1}}. \quad (4.17)$$

Now, fix $i, j \in \{1, \dots, n+d\}$. Using inequality (4.8) for $a = (a_{ij})_{x,10^{-k}}$ and $r_0 = r$, (4.17), and the fact that μ is Ahlfors regular, we get (by the same steps used to get (4.11)) that

$$\int_{B_r(x)} |a_{ij}(y) - (a_{ij})_{x,r}|^2 d\mu \leq c \int_{B_{10^{-k}}(x)} |a_{ij}(y) - (a_{ij})_{x,10^{-k}}|^2 d\mu. \quad (4.18)$$

Summing over i and j on both sides of (4.18), and using the definition of the Frobenius norm together with (4.3), we get

$$\alpha^2(x, r) \leq c \alpha^2(x, 10^{-k}), \quad (4.19)$$

where c is a constant depending only on n and d . Taking the square root on both sides of (4.19) and using (4.5) finishes the proof of (4.6) \square

As we mentioned before, the construction of the bi-Lipschitz map relies heavily on finding good approximating n -planes to M . By that, we mean that for a point $x \in M$, and a scale $0 < r < \frac{1}{20}$, we would like to find an n -plane $P(x, r)$ (not necessarily passing through x) such that $M \cap B_r(x)$ is close to $P(x, r)$. In the following theorem, with the help of Lemma 3.0.1 and the Poincaré-type inequality, we construct a plane $P_{x,r}$ that turns out to be, up to a small translation (as we see later), the plane $P(x, r)$ that we aim to get.

Theorem 4.2.2. *Let $M \subset B_1(0)$ be an n -Ahlfors regular rectifiable set containing the origin, and let $\mu = \mathcal{H}^n \llcorner M$ be the Hausdorff measure restricted to M . Assume that M satisfies the Poincaré-type inequality (1.10). There exists an $\epsilon_1 > 0$, depending only on n and d , such that for every $0 < \epsilon \leq \epsilon_1$, if*

$$\int_0^1 \left(\int_{B_r(x)} |\pi_{T_y M} - A_{x,r}|^2 d\mu \right) \frac{dr}{r} < \epsilon^2, \quad \forall x \in M, \quad (4.20)$$

then for every $x \in M$ and $0 < r \leq \frac{1}{20}$, there exists an affine n -dimensional plane $P_{x,r}$ such that

$$\int_{B_r(x)} \frac{d(y, P_{x,r})}{r} d\mu(y) \leq c \alpha(x, 2r), \quad (4.21)$$

where c is a constant depending only on n and d .

Proof. Fix $x \in M$ and $r \leq \frac{1}{20}$. Let $\epsilon \leq \epsilon_1$ (with ϵ_1 to be determined later) such that (4.20) is satisfied. By (4.3), (4.6) from Lemma 4.2.1, and the fact that $2r \leq \frac{1}{10}$, we have

$$\int_{B_{2r}(x)} |\pi_{T_y M} - A_{x,2r}|^2 d\mu = \alpha^2(x, 2r) \leq c\epsilon^2. \quad (4.22)$$

From (4.22) and the fact that M is rectifiable (so approximate tangent planes exist μ -a.e. (see Theorem 2.0.6)), it is easy to check that there exists $y_0 \in B_{2r}(x) \cap M$ such that $T_{y_0} M$ exists, and

$$|\pi_{T_{y_0} M} - A_{x,2r}| \leq \alpha(x, 2r) \leq c_1 \epsilon,$$

where c_1 is a (fixed) constant depending only on n and d . Comparing the operator norm with the Frobenius norm (the operator norm is at most the Frobenius norm), we get

$$\|\pi_{T_{y_0} M} - A_{x,2r}\| \leq \alpha(x, 2r) \leq c_1 \epsilon \leq c_1 \epsilon_1. \quad (4.23)$$

Let δ_0 be the constant from Lemma 3.0.1, and choose $\epsilon_1 \leq \frac{\delta_0}{c_1}$. Then, (4.23) becomes

$$\|\pi_{T_{y_0} M} - A_{x,2r}\| \leq \alpha(x, 2r) \leq \delta_0,$$

and by Lemma 3.0.1 (with $\delta = \alpha(x, 2r)$, $V = T_{y_0} M$, and $L = A_{x,2r}$), we deduce that $A_{x,2r}$ has exactly n eigenvalues such that $\lambda_{x,2r}^1, \dots, \lambda_{x,2r}^n$ such that $|\lambda_{x,2r}^i| \geq 1 - c\alpha(x, 2r)$, for all $i \in \{1, \dots, n\}$, and exactly d eigenvalues $\lambda_{x,2r}^{n+1}, \dots, \lambda_{x,2r}^{n+d}$ such that

$$|\lambda_{x,2r}^i| \leq c\alpha(x, 2r) \quad \forall i \in \{n+1, \dots, n+d\}. \quad (4.24)$$

Since $A_{x,2r}$ is a real symmetric matrix, $n+d$ eigenvectors of the matrix $A_{x,2r}$, say $v_{x,2r}^1, \dots, v_{x,2r}^{n+d}$ (each corresponding to exactly one of the $n+d$ eigenvalues mentioned above) can be chosen to be orthonormal. Thus, $v_{x,2r}^1, \dots, v_{x,2r}^{n+d}$ are linearly independent vectors of unit length, such that

$$A_{x,2r} v_{x,2r}^i = \lambda_{x,2r}^i v_{x,2r}^i \quad \forall i \in \{1, \dots, n+d\}. \quad (4.25)$$

Let us now focus our attention on the last d eigenvectors and eigenvalues. For $i \in \{n + 1, \dots, n + d\}$, consider the function f_i on \mathbb{R}^{n+d} defined by

$$f_i(y) = \langle y, v_{x,2r}^i \rangle, \quad y \in \mathbb{R}^{n+d}.$$

Notice that f_i is a smooth function on \mathbb{R}^{n+d} , and for every point $y \in M$ where the tangent plane $T_y M$ exists, (which, again, is almost everywhere in M), we have

$$|\nabla^M f_i(y)| \leq |\pi_{T_y M} - A_{x,2r}| + |\lambda_{x,2r}^i|. \quad (4.26)$$

In fact,

$$\nabla^M f_i(y) = \pi_{T_y M}(\nabla f(y)) = \pi_{T_y M}(v_{x,2r}^i) = (\pi_{T_y M} - A_{x,2r})(v_{x,2r}^i) + A_{x,2r}v_{x,2r}^i.$$

Thus, using the definition of the operator norm, the fact that $v_{x,2r}^i$ has unit length, (4.25), and the fact that the operator norm of a matrix is at most its Frobenius norm we get

$$\begin{aligned} |\nabla^M f_i(y)| &\leq |\pi_{T_y M} - A_{x,2r}| + |A_{x,2r}v_{x,2r}^i| \\ &\leq \|\pi_{T_y M} - A_{x,2r}\| + |\lambda_{x,2r}^i| \leq |\pi_{T_y M} - A_{x,2r}| + |\lambda_{x,2r}^i|. \end{aligned}$$

Now, applying the Poincaré inequality to the function f_i and the ball $B_r(x)$, and using (4.26), we get

$$\begin{aligned} \frac{1}{r} \int_{B_r(x)} \left| \langle y, v_{x,2r}^i \rangle - \int_{B_r(x)} \langle z, v_{x,2r}^i \rangle d\mu(z) \right| d\mu(y) \\ \leq c_P \left(\int_{B_{2r}(x)} (|\pi_{T_y M} - A_{x,2r}| + |\lambda_{x,2r}^i|)^2 d\mu(y) \right)^{\frac{1}{2}}, \quad (4.27) \end{aligned}$$

where c_P is a constant depending only on n and d .

But $v_{x,2r}^i$ is a constant vector, so (4.27) can be rewritten as

$$\begin{aligned} \frac{1}{r} \int_{B_r(x)} \left| \langle y, v_{x,2r}^i \rangle - \left\langle \int_{B_r(x)} z d\mu(z), v_{x,2r}^i \right\rangle \right| d\mu(y) \\ \leq c_P \left(\int_{B_{2r}(x)} (|\pi_{T_y M} - A_{x,2r}| + |\lambda_{x,2r}^i|)^2 d\mu(y) \right)^{\frac{1}{2}}, \quad (4.28) \end{aligned}$$

that is,

$$\begin{aligned}
\frac{1}{r} \int_{B_r(x)} \left| \left\langle y - \int_{B_r(x)} z d\mu(z), v_{x,2r}^i \right\rangle \right| d\mu(y) \\
\leq c_P \left(\int_{B_{2r}(x)} (|\pi_{T_y M} - A_{x,2r}| + |\lambda_{x,2r}^i|)^2 d\mu(y) \right)^{\frac{1}{2}} \\
\leq c \left(\left(\int_{B_{2r}(x)} |\pi_{T_y M} - A_{x,2r}|^2 \right)^{\frac{1}{2}} + |\lambda_{x,2r}^i| \right)
\end{aligned} \tag{4.29}$$

where c is a constant depending only on n and d .

Using (4.24) and (4.3), (4.29) becomes

$$\frac{1}{r} \int_{B_r(x)} \left| \left\langle y - \int_{B_r(x)} z d\mu(z), v_{x,2r}^i \right\rangle \right| d\mu(y) \leq c \left(\int_{B_{2r}(x)} |\pi_{T_y M} - A_{x,2r}|^2 \right)^{\frac{1}{2}}. \tag{4.30}$$

Since (4.30) is true for every $i \in \{n+1, \dots, n+d\}$, we can take the sum over i on both sides of (4.30) to get

$$\frac{1}{r} \sum_{i=n+1}^{n+d} \int_{B_r(x)} \left| \left\langle y - \int_{B_r(x)} z d\mu(z), v_{x,2r}^i \right\rangle \right| d\mu(y) \leq c \left(\int_{B_{2r}(x)} |\pi_{T_y M} - A_{x,2r}|^2 \right)^{\frac{1}{2}} \tag{4.31}$$

where c is a constant depending only on n and d .

We are now ready to choose our plane $P_{x,r}$. Take $P_{x,r}$ to be the n -plane passing through the point $c_{x,r} := \int_{B_r(x)} z d\mu(z)$, the centre of mass of μ in the ball $B_r(x)$, and such that $P_{x,r} - c = \text{span}\{v_{x,2r}^1, \dots, v_{x,2r}^n\}$. In other words, $(P_{x,r} - c_{x,r})^\perp = \text{span}\{v_{x,2r}^{n+1}, \dots, v_{x,2r}^{n+d}\}$. Here $(P_{x,r} - c_{x,r})^\perp$ denotes the d -plane of \mathbb{R}^{n+d} perpendicular to the n -plane $P_{x,r} - c_{x,r}$.

For $y \in B_r(x)$, we have that

$$d(y, P_{x,r}) = d(y - c_{x,r}, P_{x,r} - c_{x,r}) = \left| \sum_{i=n+1}^{n+d} \langle y - c_{x,r}, v_{x,2r}^i \rangle v_{x,2r}^i \right| \leq \sum_{i=n+1}^{n+d} |\langle y - c_{x,r}, v_{x,2r}^i \rangle| \tag{4.32}$$

Dividing by r and taking the average over $B_r(x)$ on both sides of (4.32), and using the definition of $c_{x,r}$, we get

$$\begin{aligned} \int_{B_r(x)} \frac{d(y, P_{x,r})}{r} d\mu(y) &\leq \frac{1}{r} \sum_{i=n+1}^{n+d} \int_{B_r(x)} \left| \left\langle y - \int_{B_r(x)} z d\mu(z), v_{x,2r}^i \right\rangle \right| d\mu(y) \\ &\leq c \left(\int_{B_{2r}(x)} |\pi_{T_y M} - A_{x,2r}|^2 d\mu \right)^{\frac{1}{2}}, \end{aligned}$$

where the last inequality comes from (4.31).

Thus, by the definition of $\alpha(x, 2r)$ (see (4.3)), we get (4.21) and the proof is done. \square

To start the proof of Theorem 1.2.1, we want to use the construction of the bi-Lipschitz map given by David and Toro in their paper [DT12]. For that, we need to introduce what we call a **coherent collection of balls and planes**. Here we follow the steps given by David and Toro (see [DT12], chapter 2).

First, set $r_k = 10^{-k-4}$ for $k \in \mathbb{N}$, and let ϵ be a small number (will be chosen later) that depends only on n and d . Choose a collection $\{x_{jk}\}$, $j \in J_k$ of points in \mathbb{R}^{n+d} , so that

$$|x_{jk} - x_{ik}| \geq r_k \quad \text{for } i, j \in J_k, i \neq j. \quad (4.33)$$

Set $B_{jk} := B_{r_k}(x_{jk})$ and $V_k^\lambda := \bigcup_{j \in J_k} \lambda B_{jk} = \bigcup_{j \in J_k} B_{\lambda r_k}(x_{jk})$, for $\lambda > 1$.

We also ask for our collection $\{x_{jk}\}$, $j \in J_k$ and $k \geq 1$ to satisfy

$$x_{jk} \in V_{k-1}^2 \quad \text{for } k \geq 1 \text{ and } j \in J_k. \quad (4.34)$$

Suppose that our initial net $\{x_{j_0}\}$ is close to an n -dimensional plane Σ_0 , that is

$$d(x_{j_0}, \Sigma_0) \leq \epsilon \quad \forall j \in J_0. \quad (4.35)$$

For each $k \geq 0$ and $j \in J_k$, suppose you have an n -dimensional plane P_{jk} , passing through x_{jk} such that the following compatibility conditions hold:

$$d_{x_{i_0}, 100r_0}(P_{i_0}, \Sigma_0) \leq \epsilon \quad \text{for } i \in J_0, \quad (4.36)$$

$$d_{x_{i_k}, 100r_k}(P_{i_k}, P_{j_k}) \leq \epsilon \quad \text{for } k \geq 0 \text{ and } i, j \in J_k \text{ such that } |x_{i_k} - x_{j_k}| \leq 100r_k, \quad (4.37)$$

and

$$d_{x_{i_k}, 20r_k}(P_{i_k}, P_{j, k+1}) \leq \epsilon \quad \text{for } k \geq 0 \text{ and } i \in J_k, j \in J_{k+1} \text{ such that } |x_{i_k} - x_{j, k+1}| \leq 2r_k. \quad (4.38)$$

We can now define a **coherent collection of balls and planes**:

Definition 4.2.3. A **coherent collection of balls and planes**, (in short a CCBP), is a triple $(\Sigma_0, \{B_{jk}\}, \{P_{jk}\})$ where the properties (4.33) up to (4.38) above are satisfied, with a prescribed ϵ that is small enough, and depends only on n and d .

Theorem 4.2.4. (see Theorems 2.4 in [DT12]) *There exists $\epsilon_2 > 0$ depending only on n and d , such that the following holds: If $\epsilon \leq \epsilon_2$, and $(\Sigma_0, \{B_{jk}\}, \{P_{jk}\})$ is a CCBP (with ϵ), then there exists a bijection $g : \mathbb{R}^{n+d} \rightarrow \mathbb{R}^{n+d}$ with the following properties:*

$$g(z) = z \quad \text{when } d(z, \Sigma_0) \geq 2, \quad (4.39)$$

and

$$|g(z) - z| \leq C_1 \epsilon \quad \text{for } z \in \mathbb{R}^{n+d}, \quad (4.40)$$

where C_1 is a constant depending only on n and d .

Moreover, $g(\Sigma_0)$ is a $C_1\epsilon$ -Reifenberg flat set that contains the accumulation set

$$\begin{aligned} E_\infty &= \{x \in \mathbb{R}^{n+d}; x \text{ can be written as} \\ &\quad x = \lim_{m \rightarrow \infty} x_{j(m),k(m)}, \text{ with } k(m) \in \mathbb{N}, \\ &\quad \text{and } j(m) \in J_{k_m} \text{ for } m \geq 0 \text{ and } \lim_{m \rightarrow \infty} k(m) = \infty\}. \end{aligned}$$

In [DT12], David and Toro give a sufficient condition for g to be bi-Lipschitz that we want to use in our proof. However, in order to state this condition, we need some technical details from the construction of the map g from Theorem 4.2.4. So, let us briefly discuss the construction here: David and Toro defined a mapping f whose goal is to push a small neighbourhood of Σ_0 towards a final set, which they proved to be Reifenberg flat. They obtained f as a limit of the composed functions $f_k = \sigma_{k-1} \circ \dots \circ \sigma_0$ where each σ_k is a smooth function that moves points near the planes P_{j_k} at the scale r_k . More precisely,

$$\sigma_k(y) = y + \sum_{j \in J_k} \theta_{jk}(y) [\pi_{jk}(y) - y], \quad (4.41)$$

where $\{\theta_{jk}\}_{j \in J_k, k \geq 0}$ is a partition of unity with each θ_{jk} supported on $10B_{jk}$, and π_{jk} denotes the orthogonal projection from \mathbb{R}^{n+d} onto the plane P_{jk} .

Since f in their construction was defined on Σ_0 , g was defined to be the extension of f on the whole space.

Corollary 4.2.5. (see Proposition 11.2 in [DT12]) *Suppose we are in the setting of Theorem 4.2.4. Define the quantity*

$$\epsilon'_k(y) = \sup\{d_{x_{im}, 100r_m}(P_{jk}, P_{im}); j \in J_k, i \in J_m, m \in \{k, k-1\}, \text{ and } y \in 10B_{jk} \cap 11B_{im}\} \quad (4.42)$$

for $k \geq 1$ and $y \in V_k^{10}$, and $\epsilon'_k(y) = 0$ when $y \in \mathbb{R}^{n+d} \setminus V_k^{10}$ (when there are no pairs (j, k) as above). If there exists $N > 0$ such that

$$\sum_{k=0}^{\infty} \epsilon'_k(f_k(z))^2 < N, \quad \forall z \in \Sigma_0, \quad (4.43)$$

then the map g constructed in Theorem 4.2.4 is K -bi-Lipschitz, where the bi-Lipschitz constant K depends only on n , d , and N .

We are finally ready to prove Theorem 1.2.1.

Proof of Theorem 1.2.1:

Proof. Let $\epsilon_0 > 0$ (to be determined later), and suppose that (1.11) holds. Let ϵ_2 be the constant from Theorem 4.2.4. We would like to apply Theorem 4.2.4 for $\epsilon = \epsilon_2$, and then Corollary 4.2.5. So our first goal is to construct a CCBP, and we do that in several steps: Let us start with a collection $\{\tilde{x}_{jk}\}$, $j \in J_k$ of points in $M \cap B_{\frac{1}{10^3}}(0)$ that is maximal under the constraint

$$|\tilde{x}_{jk} - \tilde{x}_{ik}| \geq \frac{4r_k}{3} \quad \text{when } i, j \in J_k \text{ and } i \neq j. \quad (4.44)$$

Of course, we can arrange matters so that the point 0 belongs to our initial maximal set, at scale r_0 . Thus, $0 = \tilde{x}_{i_0,0}$ for some $i_0 \in J_0$. Notice that for every $k \geq 0$, we have

$$M \cap B_{\frac{1}{10^3}}(0) \subset \bigcup_{j \in J_k} \bar{B}_{\frac{4r_k}{3}}(\tilde{x}_{jk}). \quad (4.45)$$

Later, we choose

$$x_{jk} \in M \cap B_{\frac{r_k}{6}}(\tilde{x}_{jk}), \quad j \in J_k. \quad (4.46)$$

By (4.45) and (4.46), we can see

$$M \cap B_{\frac{1}{10^3}}(0) \subset \bigcup_{j \in J_k} \bar{B}_{\frac{4r_k}{3}}(\tilde{x}_{jk}) \subset \bigcup_{j \in J_k} B_{\frac{3r_k}{2}}(x_{jk}). \quad (4.47)$$

Let us prove that such a collection $\{x_{jk}\}$, $j \in J_k$ satisfies (4.33) and (4.34):

To see (4.33), we proceed by contradiction. Suppose $|x_{jk} - x_{ik}| < r_k$ for some $i, j \in J_k$, with $i \neq j$. Then, by (4.46),

$$|\tilde{x}_{jk} - \tilde{x}_{ik}| \leq |\tilde{x}_{jk} - x_{jk}| + |x_{jk} - x_{ik}| + |x_{ik} - \tilde{x}_{ik}| < \frac{r_k}{6} + r_k + \frac{r_k}{6} = \frac{4r_k}{3}$$

which contradicts (4.44). This proves (4.33).

To see (4.34), fix $x_{j,k+1}$ with $k \geq 0$ and $j \in J_{k+1}$. By construction and (4.47), we have

$$\tilde{x}_{j,k+1} \in M \cap B_{\frac{1}{10^3}}(0) \subset \bigcup_{i \in J_k} B_{\frac{3r_k}{2}}(x_{ik}). \quad (4.48)$$

Using (4.46) and (4.48), we get

$$x_{j,k+1} \in \bigcup_{i \in J_k} B_{2r_k}(x_{ik}) = V_k^2.$$

Thus, (4.34) is satisfied.

Next, we choose our planes P_{jk} and our collection $\{x_{jk}\}$, for $k \geq 0$ and $j \in J_k$.

Fix $k \geq 0$ and $j \in J_k$. Let ϵ_1 be the constant from Theorem 4.2.2. For

$$\epsilon_0 \leq \epsilon_1, \quad (4.49)$$

we apply Theorem 4.2.2 to the point \tilde{x}_{jk} (by construction $\tilde{x}_{jk} \in M$) and radius $120r_k$ (notice that $120r_k \leq \frac{1}{20}$) to get an n -plane $P_{\tilde{x}_{jk}, 120r_k}$, denoted in this proof by P'_{jk} for simplicity reasons, such that

$$\int_{B_{120r_k}(\tilde{x}_{jk})} \frac{d(y, P'_{jk})}{120r_k} d\mu \leq c \alpha(\tilde{x}_{jk}, 240r_k). \quad (4.50)$$

Thus, by (4.50) and the fact that μ is Ahlfors regular, there exists $x_{jk} \in M \cap B_{\frac{r_k}{6}}(\tilde{x}_{jk})$ such that

$$\begin{aligned} d(x_{jk}, P'_{jk}) &\leq \int_{B_{\frac{r_k}{6}}(\tilde{x}_{jk})} d(y, P'_{jk}) d\mu \\ &\leq c \int_{B_{120r_k}(\tilde{x}_{jk})} d(y, P'_{jk}) d\mu \leq c \alpha(\tilde{x}_{jk}, 240r_k) r_k. \end{aligned} \quad (4.51)$$

Let P_{jk} be the plane parallel to P'_{jk} and passing through x_{jk} . From (4.51) and the fact that the two planes are parallel, it is clear that

$$d_{\tilde{x}_{jk}, 240r_k}(P_{jk}, P'_{jk}) \leq c \alpha(\tilde{x}_{jk}, 240r_k). \quad (4.52)$$

Moreover, for every $y \in B_{120r_k}(\tilde{x}_{jk})$, we have by the triangle inequality and (4.52)

$$\begin{aligned} d(y, P_{jk}) &\leq d(y, P'_{jk}) + c d_{\tilde{x}_{jk}, 240r_k}(P_{jk}, P'_{jk}) r_k \\ &\leq d(y, P'_{jk}) + c \alpha(\tilde{x}_{jk}, 240r_k) r_k. \end{aligned} \quad (4.53)$$

Dividing both sides of (4.53) by $120r_k$ and taking the average over $B_{120r_k}(\tilde{x}_{jk})$, we get

$$\int_{B_{120r_k}(\tilde{x}_{jk})} \frac{d(y, P_{jk})}{120r_k} d\mu \leq \int_{B_{120r_k}(\tilde{x}_{jk})} \frac{d(y, P'_{jk})}{120r_k} d\mu + c \alpha(\tilde{x}_{jk}, 240r_k), \quad (4.54)$$

which by (4.50) becomes

$$\int_{B_{120r_k}(\tilde{x}_{jk})} \frac{d(y, P_{jk})}{120r_k} d\mu \leq c \alpha(\tilde{x}_{jk}, 240r_k), \quad (4.55)$$

where c is a constant depending only on n and d .

To summarize what we did so far, we have chosen n -dimensional planes P_{jk} for $k \geq 0$ and $j \in J_k$ where each P_{jk} passes through x_{jk} , and satisfies (4.55).

We want to get our CCBP with ϵ_2 . Thus, we proceed by proving (4.36), (4.37), and (4.38), with $\epsilon = \epsilon_2$ starting with (4.37) and (4.38). We prove (4.37) and (4.38) simultaneously here. So, let us fix $k \geq 0$ and $j \in J_k$; let $m \in \{k, k-1\}$ and $i \in J_m$ such that

$$|x_{jk} - x_{im}| \leq 100r_m. \quad (4.56)$$

We want to show that P_{jk} and P_{im} are close together. To do that, we construct n linearly independent vectors that “effectively” span P_{jk} , that is, these vectors span P_{jk} , and they

are far away from each other (in a uniform quantitative manner). Then, we show that P_{im} is close to each of these vectors. This idea is very similar to the “effectively” spanning idea found in [NV15a] (see p. 26-28).

Let us start by proving the existence of such vectors in the following claim. Here is where we use lemma 3.0.2.

Claim: Denote by π_{jk} is the orthogonal projection of \mathbb{R}^{n+d} on the plane P_{jk} . Then, there exists $r = cr_k$ (where $c \leq \frac{1}{2}$ is a constant depending only on n and d), and a sequence of $n + 1$ balls $\{B_r(y_l)\}_{l=0}^n$, such that

1. $\forall l \in \{0, \dots, n\}$, we have $y_l \in M$ and $B_r(y_l) \subset B_{2r_k}(\tilde{x}_{jk})$.
2. $q_1 - q_0 \notin B_{5r}(0)$, and $\forall l \in \{2, \dots, n\}$, we have $q_l - q_0 \notin N_{5r}(\text{span}\{q_1 - q_0, \dots, q_{l-1} - q_0\})$,

where $q_l = \pi_{jk}(p(y_l))$ and $p(y_l) = \int_{B_r(y_l)} z d\mu(z)$ is the centre of mass of μ in the ball $B_r(y_l)$.

We prove this claim by induction:

For $l = 0$, take $y_0 = \tilde{x}_{jk}$ (recall that both k and j are fixed here). In this case, item 1 is trivial, and item 2 is not applicable. Thus, we have our points y_0 , $p(y_0)$, and q_0 .

Now, let $r = C_0 r_k$ as in Lemma 3.0.2, where we have applied the lemma on $x_0 = \tilde{x}_{jk}$ and $r_0 = r_k$. Recall that the constant C_0 we get from Lemma 3.0.2 is as desired (that is $C_0 \leq \frac{1}{2}$ depending only on n and d .) For $i = 1$, we apply Lemma 3.0.2 for $V = \{\tilde{x}_{jk}\}$, to get a point $y_1 \in M \cap B_{r_k}(\tilde{x}_{jk})$ such that $y_1 \notin B_{11r}(\tilde{x}_{jk})$ and $B_r(y_1) \subset B_{2r_k}(\tilde{x}_{jk})$. So item 1 is satisfied, and now we have our points $p(y_1)$ and q_1 .

For item 2, we need to prove that

$$|q_1 - q_0| \geq 5r. \tag{4.57}$$

In fact, we have for $l \in \{0, 1\}$, by the definition of $p(y_l)$, Jensen's inequality applied on the convex function $\phi(\cdot) = d(\cdot, P_{jk})$, the fact that μ is Ahlfors regular, $B_r(y_l) \subset B_{2r_k}(\tilde{x}_{jk})$, $r = C_0 r_k$, and (4.55), that

$$\begin{aligned} d(p(y_l), P_{jk}) &= d\left(\int_{B_r(y_l)} z d\mu(z), P_{jk}\right) \\ &\leq \int_{B_r(y_l)} d(z, P_{jk}) d\mu(z) \\ &\leq c \int_{B_{120r_k}(\tilde{x}_{jk})} d(z, P_{jk}) d\mu(z) \leq c \alpha(\tilde{x}_{jk}, 240r_k) r_k, \end{aligned} \quad (4.58)$$

where c is a constant depending only on n and d .

Also, by the definition of the center of mass, we know that

$$|y_l - p(y_l)| \leq r \quad l \in \{0, 1\}. \quad (4.59)$$

Thus, by the triangle inequality, (4.59), and (4.58), we get for $l \in \{0, 1\}$

$$\begin{aligned} |y_l - q_l| &\leq |y_l - p(y_l)| + |p(y_l) - q_l| \\ &= |y_l - p(y_l)| + d(p(y_l), P_{jk}) \\ &\leq r + c \alpha(\tilde{x}_{jk}, 240r_k) r_k. \end{aligned} \quad (4.60)$$

Notice now, that by (1.11), (4.6) in Lemma 4.2.1, the fact that $\tilde{x}_{jk} \in M \cap B_{\frac{1}{10^3}}(0)$ and $240r_k \leq \frac{1}{10}$, we have

$$\alpha(\tilde{x}_{jk}, 240r_k) \leq c \epsilon_0, \quad (4.61)$$

with c a constant depending only on n and d .

Plugging (4.61) in (4.60), and using the fact that $r = C_0 r_k$, we get for $l \in \{0, 1\}$

$$|y_l - q_l| \leq r + c \epsilon_0 r_k = r + c \epsilon_0 r. \quad (4.62)$$

Let us denote by c_1 the constant c from the last step of (4.62). Then, rewriting (4.62), we get $|y_l - q_l| \leq r + c_1 \epsilon_0 r$. For ϵ_0 such that $c_1 \epsilon_0 < 1$, we get

$$|y_l - q_l| \leq 2r \quad l \in \{0, 1\}. \quad (4.63)$$

We are now ready to prove (4.57):

Let us proceed by contradiction. Suppose that $|q_1 - q_0| < 5r$, then by (4.63), we get

$$\begin{aligned} |y_1 - y_0| &\leq |y_1 - q_1| + |q_1 - q_0| + |y_0 - q_0| \\ &\leq 2r + 5r + 2r = 9r. \end{aligned}$$

But $y_1 \notin B_{11r}(\tilde{x}_{jk}) = B_{11r}(y_0)$ by construction. Thus, we get a contradiction, and (4.57) is proved.

For our induction step, assume the statement is true for $l - 1$, and let's prove it for l . Consider the $(l - 1)$ -dimensional affine subspace

$$V^{l-1} = \text{span}\{q_1 - q_0, \dots, q_{l-1} - q_0\} + q_0.$$

Notice that our last induction process is when we have n points and want to construct the $(n + 1)^{st}$ point. Thus, $l - 1 \leq n - 1$, and we can apply Lemma 3.0.2, on the subspace V^{l-1} , to get a point $y_l \in M \cap B_{r_k}(\tilde{x}_{jk})$ such that $y_l \notin N_{11r}(V^{l-1})$. So, we have that

$$y_l - q_0 \notin N_{11r}(\text{span}\{q_1 - q_0, \dots, q_{l-1} - q_0\}). \quad (4.64)$$

Item 1 is clearly true. To prove item 2, we show that

$$q_l - q_0 \notin N_{5r}(\text{span}\{q_1 - q_0, \dots, q_{l-1} - q_0\}). \quad (4.65)$$

In fact, by the exact same calculations as above (see (4.58), (4.59), and (4.63)), we see that

$$d(p(y_l), P_{jk}) \leq c\alpha(\tilde{x}_{jk}, 240r_k)r_k, \quad (4.66)$$

$$|y_l - p(y_l)| \leq r, \quad (4.67)$$

and

$$|y_l - q_l| \leq 2r. \quad (4.68)$$

Let us now prove (4.65) by contradiction:

Suppose that $q_l - q_0 \in N_{5r}(\text{span}\{q_1 - q_0, \dots, q_{l-1} - q_0\})$, then, using (4.68), we get

$$\begin{aligned} d(y_l - q_0, \text{span}\{q_1 - q_0, \dots, q_{l-1} - q_0\}) & \\ & \leq d(y_l - q_0, q_l - q_0) + d(q_l - q_0, \text{span}\{q_1 - q_0, \dots, q_{l-1} - q_0\}) \\ & = |y_l - q_l| + d(q_l - q_0, \text{span}\{q_1 - q_0, \dots, q_{l-1} - q_0\}) \\ & \leq 2r + 5r = 7r < 11r. \end{aligned}$$

which is a contradiction by (4.64). Thus, induction process is complete, and so is the proof of the claim ■

From the construction in the claim above, notice that

$$P_{jk} - q_0 = \text{span}\{q_1 - q_0, \dots, q_n - q_0\}. \quad (4.69)$$

Also, by (4.58) and (4.66), we have $\forall l \in \{0, \dots, n\}$

$$d(p(y_l), P_{jk}) \leq c \alpha(\tilde{x}_{jk}, 240r_k) r_k, \quad (4.70)$$

and by (4.63), and recalling that $y_0 = \tilde{x}_{jk}$ we have

$$|y_0 - q_0| = |\tilde{x}_{jk} - q_0| \leq 2r. \quad (4.71)$$

Let us remember that our goal is to prove that P_{jk} and P_{im} are close to each other. In the

claim, we constructed an “effective” spanning set for P_{jk} , $\{q_1 - q_0, \dots, q_n - q_0\}$. Now, we can get a nice upper bound on the distance from each q_l to P_{im} , for $l \in \{0, \dots, n\}$.

In fact, by the definition of the center of mass, Jensen’s formula, the fact that μ is Ahlfors regular, $B_r(y_l) \subset B_{120r_m}(\tilde{x}_{im})$ (see item 1, (4.56), (4.46), and recall that $r < r_k \leq r_m$), $r = C_0 r_k$ and $r_m \in \{r_k, 10r_k\}$, and (4.55) for P_{im} , to get that for every $l \in \{0, \dots, n\}$

$$\begin{aligned} d(p(y_l), P_{im}) &= d\left(\int_{B_r(y_l)} z d\mu(z), P_{im}\right) \\ &\leq \int_{B_r(y_l)} d(z, P_{im}) d\mu(z) \\ &\leq \int_{B_{120r_m}(\tilde{x}_{im})} d(z, P_{im}) d\mu(z) \leq c \alpha(\tilde{x}_{im}, 240r_m) r_m. \end{aligned} \quad (4.72)$$

Combining (4.70) and (4.72), we get by the triangle inequality that for every $l \in \{0, \dots, n\}$

$$\begin{aligned} d(q_l, P_{im}) &\leq |q_l - p(y_l)| + d(p(y_l), P_{im}) \\ &= d(p(y_l), P_{jk}) + d(p(y_l), P_{im}) \\ &\leq c (\alpha(\tilde{x}_{jk}, 240r_k) r_k + \alpha(\tilde{x}_{im}, 240r_m) r_m). \end{aligned} \quad (4.73)$$

We are finally ready to compute the distance between P_{jk} and P_{im} . Let $y \in P_{jk} \cap B_\rho(x_{im})$ where $\rho = \{20r_m, 100r_m\}$. By (4.69), y can be written uniquely as $y - q_0 = \sum_{l=1}^n \beta_l (q_l - q_0)$, that is

$$y = q_0 + \sum_{l=1}^n \beta_l (q_l - q_0). \quad (4.74)$$

We want to apply Lemma 3.0.4, for $u_l = q_l - q_0$, $R = r$, and $v = y - q_0$. In fact, (3.21) and (3.22) are satisfied directly from item 2 for $k_0 = 5$. To see (3.20), note that by (4.68), (4.63), the fact that y_0 and $y_i \in B_{2r_k}(\tilde{x}_{jk})$ from item 1, for $l \in \{1, \dots, n\}$, and $r = C_0 r_k$, we have

$$|q_l - q_0| \leq |q_l - y_l| + |y_l - y_0| + |y_0 - q_0| \leq 4r + 2r_k \leq c_0 r, \quad (4.75)$$

where c_0 is a (fixed) constant depending only on n and d .

For $K_0 = c_0$ where K_0 the constant in the statement of Lemma 3.0.4, we get by Lemma 3.0.4 that

$$|\beta_l| \leq K_1 \frac{1}{r} |y - q_0| \quad \forall l \in \{1, \dots, n\}. \quad (4.76)$$

However, by (4.56), (4.46), (4.71), and remembering that $r < r_k \leq r_m$, we have

$$\begin{aligned} |y - q_0| &\leq |y - x_{im}| + |x_{im} - x_{jk}| + |x_{jk} - \tilde{x}_{jk}| + |\tilde{x}_{jk} - q_0| \\ &\leq \rho + 100r_m + \frac{r_k}{6} + 2r \leq 203r_m. \end{aligned} \quad (4.77)$$

But $r = C_0 r_k$, and $r_m = \{r_k, 10r_k\}$, and thus, combining (4.76) and (4.77), we get

$$|\beta_i| \leq c, \quad (4.78)$$

c being a constant that depends only on n and d .

So, using (4.74), (4.78), and (4.73), we get

$$\begin{aligned} d(y, P_{im}) &\leq \left(1 + \sum_{l=1}^n |\beta_l|\right) d(q_0, P_{im}) + \sum_{l=1}^n |\beta_l| d(q_l, P_{im}) \\ &\leq c \left(d(q_0, P_{im}) + \sum_{l=1}^n d(q_l, P_{im}) \right) \\ &\leq c \left(\alpha(\tilde{x}_{jk}, 240r_k) r_k + \alpha(\tilde{x}_{im}, 240r_m) r_m \right) \end{aligned} \quad (4.79)$$

Thus,

$$d_{x_{im}, \rho}(P_{jk}, P_{im}) \leq c \left(\alpha(\tilde{x}_{jk}, 240r_k) + \alpha(\tilde{x}_{im}, 240r_m) \right) \quad \rho \in \{20r_m, 100r_m\}. \quad (4.80)$$

And so, our planes P_{jk} and P_{im} are close. In fact, by (4.61), we know that

$$\alpha(\tilde{x}_{jk}, 240r_k) \leq c \epsilon_0. \quad (4.81)$$

Similarly, we have

$$\alpha(\tilde{x}_{im}, 240r_m) \leq c\epsilon_0. \quad (4.82)$$

Plugging (4.81) and (4.82) in (4.80), we get

$$d_{x_{im}, \rho}(P_{jk}, P_{im}) \leq c\epsilon_0 \quad \rho \in \{20r_m, 100r_m\}. \quad (4.83)$$

where c is a constant depending only on n and d .

So, we have shown that there exists two constants c_2 and c_3 , each depending only on n and d , such that

$$d_{x_{ik}, 100r_k}(P_{ik}, P_{jk}) \leq c_2\epsilon_0 \quad \text{for } k \geq 0 \text{ and } i, j \in J_k \text{ such that } |x_{ik} - x_{jk}| \leq 100r_k, \quad (4.84)$$

and

$$d_{x_{ik}, 20r_k}(P_{ik}, P_{j, k+1}) \leq c_3\epsilon_0 \quad \text{for } k \geq 0 \text{ and } i \in J_k, j \in J_{k+1} \text{ such that } |x_{ik} - x_{j, k+1}| \leq 2r_k. \quad (4.85)$$

For

$$c_2\epsilon_0 \leq \epsilon_2 \quad \text{and} \quad c_3\epsilon_0 \leq \epsilon_2, \quad (4.86)$$

we get (4.37) and (4.38).

We now prove (4.36). Recall that $0 = \tilde{x}_{i_0, 0}$ for some $i_0 \in J_0$. Choose Σ_0 to be the plane $P_{i_0, 0}$ described above (recall that $P_{i_0, 0}$ passes through $x_{i_0, 0}$, where $r_0 = 10^{-4}$). Then, what we need to prove is

$$d_{x_{j_0}, 100r_0}(P_{j_0}, P_{i_0, 0}) \leq \epsilon_2 \quad \text{for } j \in J_0. \quad (4.87)$$

Fix $j \in J_0$, and take the corresponding x_{j_0} . Since by construction $|\tilde{x}_{j_0}| < \frac{1}{10^3}$ and since (4.46) says that $|x_{j_0, 0} - \tilde{x}_{j_0, 0}| \leq \frac{r_0}{6}$, then, we have

$$|x_{j_0}| \leq \frac{r_0}{6} + \frac{1}{10^3}, \quad j \in J_0. \quad (4.88)$$

Moreover, by (4.46) and the fact that $0 = \tilde{x}_{i_0,0}$, we have

$$|x_{i_0,0} - \tilde{x}_{i_0,0}| = |x_{i_0,0}| \leq \frac{r_0}{6}. \quad (4.89)$$

Combining (4.88) and (4.89), and using the fact that $r_0 = 10^{-4}$ we get

$$|x_{j_0} - x_{i_0,0}| \leq \frac{r_0}{6} + \frac{1}{10^3} + \frac{r_0}{6} \leq \frac{r_0}{6} + 10r_0 + \frac{r_0}{6} \leq 100r_0. \quad (4.90)$$

Thus, by (4.37) for $x_{ik} = x_{j_0}$, $P_{ik} = P_{j_0}$, and $P_{jk} = P_{i_0,0}$, we get exactly (4.87), hence finishing the proof for (4.36).

It remains to prove (4.35) with $\epsilon = \epsilon_2$, that is

$$d(x_{j_0}, P_{i_0,0}) \leq \epsilon_2, \quad \text{for } j \in J_0. \quad (4.91)$$

By Markov's inequality, we know that

$$\mu \left(x \in B_{120r_0}(\tilde{x}_{i_0,0}); d(x, P_{i_0,0}) \geq \alpha^{\frac{1}{2}}(\tilde{x}_{i_0,0}, 240r_0) \right) \leq \frac{1}{\alpha^{\frac{1}{2}}(\tilde{x}_{i_0,0}, 240r_0)} \int_{B_{120r_0}(\tilde{x}_{i_0,0})} d(y, P_{i_0,0}) d\mu.$$

But since $\tilde{x}_{i_0,0} = 0$, and by using (4.55) with the fact that μ is Ahlfors regular, and (1.11) with (4.6) from Lemma 4.2.1 and the fact that $240r_0 \leq \frac{1}{10}$, we get

$$\begin{aligned} \mu \left(x \in B_{120r_0}(0); d(x, P_{i_0,0}) \geq \alpha^{\frac{1}{2}}(0, 240r_0) \right) &\leq \frac{1}{\alpha^{\frac{1}{2}}(0, 240r_0)} \int_{B_{120r_0}(0)} d(y, P_{i_0,0}) d\mu \\ &= \frac{\mu(B_{120r_0}(0))}{\alpha^{\frac{1}{2}}(0, 240r_0)} \int_{B_{120r_0}(0)} d(y, P_{i_0,0}) \\ &\leq c \alpha^{\frac{1}{2}}(0, 240r_0) \leq c \epsilon_0^{\frac{1}{2}}. \end{aligned} \quad (4.92)$$

Now, take a point $z \in M \cap B_{120r_0}(0)$. We consider two cases:

Either

$$d(z, P_{i_0,0}) \leq \alpha^{\frac{1}{2}}(0, 240r_0) \quad (4.93)$$

or

$$d(z, P_{i_0,0}) > \alpha^{\frac{1}{2}}(0, 240r_0). \quad (4.94)$$

In the first case, combining (4.93) with (1.11) and (4.6), we get

$$d(z, P_{i_0}) \leq c \epsilon_0^{\frac{1}{2}}. \quad (4.95)$$

In case of (4.94), let ρ be the biggest radius such that

$$B_\rho(z) \subset \left\{ x \in B_{120r_0}(0); d(x, P_{i_0}) > \alpha^{\frac{1}{2}}(0, 240r_0) \right\}.$$

Now, since $z \in M$ and μ is Ahlfors regular, we get using (4.92) that

$$C \rho^n \leq \mu(B_\rho(z)) \leq c \epsilon_0^{\frac{1}{2}}, \quad (4.96)$$

where C is the Ahlfors constant, depending only on n and d , and c is the constant (depending only on n and d), from the last step of (4.92). Thus, relabelling, (4.96) becomes

$$\rho \leq c \epsilon_0^{\frac{1}{2n}}, \quad (4.97)$$

where c is a constant depending only on n and d .

On the other hand, since ρ is the biggest radius such that $B_\rho(z) \subset \left\{ x \in B_{120r_0}(0); d(x, P_{i_0}) > \alpha^{\frac{1}{2}}(0, 240r_0) \right\}$, then there exists $x_0 \in \partial B_\rho(z)$ such that

$$d(x_0, P_{i_0}) \leq \alpha^{\frac{1}{2}}(0, 240r_0). \quad (4.98)$$

Thus, by (4.98), (4.97) and (1.11) together with (4.6), we get

$$\begin{aligned} d(z, P_{i_0}) &\leq |z - x_0| + d(x_0, P_{i_0}) \\ &= \rho + d(x_0, P_{i_0}) \leq c \epsilon_0^{\frac{1}{2n}} + \alpha^{\frac{1}{2}}(0, 240r_0) \leq c \epsilon_0^{\frac{1}{2n}} + \epsilon_0^{\frac{1}{2}} = c \epsilon_0^{\frac{1}{2n}}. \end{aligned} \quad (4.99)$$

Combining (4.95) and (4.99), we get that

$$d(z, P_{i_0}) \leq c_4 \epsilon_0^{\frac{1}{2n}} \quad \text{for } z \in M \cap B_{120r_0}(0), \quad (4.100)$$

where c_4 is a (fixed) constant depending only on n and d .

We are now ready to prove (4.91). Fix $j \in J_0$, and take the corresponding x_{j_0} . Since by construction $|\tilde{x}_{j_0}| < \frac{1}{10^3}$ and since (4.46) says that $|x_{j_0,0} - \tilde{x}_{j_0,0}| \leq \frac{r_0}{6}$, then, remembering that $r_0 = 10^{-4}$, we have

$$|x_{j_0}| \leq \frac{r_0}{6} + \frac{1}{10^3} \leq 11r_0, \quad j \in J_0.$$

Thus,

$$x_{j_0} \in M \cap B_{11r_0}(0) \subset M \cap B_{120r_0}(0). \quad (4.101)$$

For $z = x_{j_0}$ in (4.100), and for $c_4 \epsilon_0^{\frac{1}{2n}} \leq \epsilon_2$, we get

$$d(x_{j_0}, P_{i_0,0}) \leq \epsilon_2 \quad j \in J_0, \quad (4.102)$$

which is exactly (4.91).

We finally have our CCBP. Now, by the proof of Theorem 4.2.4 (see paragraph above (4.41)) we get the smooth maps σ_k and $f_k = \sigma_{k-1} \circ \dots \circ \sigma_0$ for $k \geq 0$, and then the map $f = \lim_{k \rightarrow \infty} f_k$ defined on Σ_0 , and finally the map g that we want. Moreover, by Theorem 4.2.4, we know that $g : \mathbb{R}^{n+d} \rightarrow \mathbb{R}^{n+d}$ is a bijection with the following properties:

$$g(z) = z \quad \text{when } d(z, \Sigma_0) \geq 2, \quad (4.103)$$

$$|g(z) - z| \leq C_1 \epsilon_2 \quad \text{for } z \in \mathbb{R}^{n+d}, \quad (4.104)$$

and

$$g(\Sigma_0) \text{ is a } C_1 \epsilon_2\text{-Reifenberg flat set.} \quad (4.105)$$

Fix ϵ_0 such that (4.49), (4.86), the line before (4.63), and the line before (4.102) are all satisfied. Notice that by the choice of ϵ_0 , we can write $\epsilon_0 = c_5 \epsilon_2$, where c_5 is a constant depending only on n and d . Hence, from (4.103), (4.104), (4.105), we directly get (1.12), (1.13), and (1.14).

We next show that

$$M \cap B_{\frac{1}{10^3}}(0) \subset g(\Sigma_0). \quad (4.106)$$

Fix $x \in M \cap B_{\frac{1}{10^3}}(0)$. Then, by (4.47), we see that for all $k \geq 0$, there exists a point x_{jk} such that $|x - x_{jk}| \leq \frac{3r_k}{2}$, and hence $x \in E_\infty \subset g(\Sigma_0)$ (E_∞ is the set defined in Theorem 4.2.4). Since x was an arbitrary point in $M \cap B_{\frac{1}{10^3}}(0)$, (4.106) is proved.

We still need to show that g is bi-Lipschitz. By Corollary 4.2.5, it suffices to show (4.43). In order to do that, we need the following inequality from [DT12] (see inequality (6.8) page 27 in [DT12]¹).

$$|f(z) - f_k(z)| \leq r_k \quad \text{for } k \geq 0 \text{ and } z \in \Sigma_0. \quad (4.107)$$

Let $z \in \Sigma_0$, and choose $\bar{z} \in M \cap B_{\frac{1}{10^3}}(0)$ such that

$$|\bar{z} - f(z)| \leq 2d(f(z), M \cap B_{\frac{1}{10^3}}(0)). \quad (4.108)$$

Fix $k \geq 0$, and consider the index $m \in \{k, k-1\}$ and the indices $j \in J_k$ and $i \in J_m$ such that $f_k(z) \in 10B_{jk} \cap 11B_{im}$. We show that

$$d_{x_{im}, 100r_m}(P_{jk}, P_{im}) \leq c\alpha(\bar{z}, r_{k-4}) \quad \text{for } k \geq 1. \quad (4.109)$$

where c is a constant depending only on n and d .

Notice that by (4.108) and (4.107), and since $\tilde{x}_{jk} \in M \cap B_{\frac{1}{10^3}}(0)$, $|\tilde{x}_{jk} - x_{jk}| \leq \frac{r_k}{6}$, and

¹Inequality (6.8) in [DT12] has a $C\epsilon$ in front of r_k ; however, ϵ was later chosen so that $C\epsilon \leq 1$ which gives us our inequality (4.107) above.

$f_k(z) \in 10B_{jk}$, we have

$$\begin{aligned}
|\bar{z} - f_k(z)| &\leq |\bar{z} - f(z)| + |f(z) - f_k(z)| \\
&\leq 2d(f(z), M \cap B_{\frac{1}{10^3}}(0)) + |f(z) - f_k(z)| \\
&\leq 2d(f_k(z), M \cap B_{\frac{1}{10^3}}(0)) + 3|f(z) - f_k(z)| \\
&\leq 2|f_k(z) - \tilde{x}_{jk}| + 3r_k \\
&\leq 2|f_k(z) - x_{jk}| + |\tilde{x}_{jk} - x_{jk}| + 3r_k \\
&\leq 20r_k + \frac{r_k}{6} + 3r_k \leq 24r_k.
\end{aligned} \tag{4.110}$$

Thus,

$$B_{240r_k}(\tilde{x}_{jk}) \subset B_{r_{k-4}}(\bar{z}). \tag{4.111}$$

In fact, for $a \in B_{240r_k}(\tilde{x}_{jk})$, we have by (4.46), the fact that $f_k(z) \in 10B_{jk}$, and (4.110), that

$$\begin{aligned}
|a - \bar{z}| &\leq |a - \tilde{x}_{jk}| + |\tilde{x}_{jk} - x_{jk}| + |x_{jk} - f_k(z)| + |f_k(z) - \bar{z}| \\
&\leq 240r_k + \frac{r_k}{6} + 10r_k + 24r_k \leq r_{k-4}.
\end{aligned}$$

Similarly, we can show that

$$B_{240r_m}(\tilde{x}_{im}) \subset B_{r_{k-4}}(\bar{z}). \tag{4.112}$$

Thus, by (4.111) and (4.112), we have

$$B_{240r_m}(\tilde{x}_{im}) \cup B_{240r_k}(\tilde{x}_{jk}) \subset B_{r_{k-4}}(\bar{z}). \tag{4.113}$$

But, writing $\pi_{T_y M} = (a_{pq}(y))_{pq}$, and using the definition of the Frobenius norm, together with (4.8) for $a = (a_{pq})_{\bar{z}, r_{k-4}}$, (4.113), and the fact that μ is Ahlfors regular

$$\begin{aligned}
\alpha^2(\tilde{x}_{jk}, 240r_k) &= \int_{B_{240r_k}(\tilde{x}_{jk})} |\pi_{T_y M} - A_{\tilde{x}_{jk}, 240r_k}|^2 d\mu \\
&= \sum_{p,q=1}^{n+d} \int_{B_{240r_k}(\tilde{x}_{jk})} |a_{pq}(y) - (a_{pq})_{\tilde{x}_{jk}, 240r_k}|^2 d\mu \\
&\leq \sum_{p,q=1}^{n+d} \int_{B_{240r_k}(\tilde{x}_{jk})} |a_{pq}(y) - (a_{pq})_{\bar{z}, r_{k-4}}|^2 d\mu \\
&\leq c \sum_{p,q=1}^{n+d} \int_{B_{r_{k-4}}(\bar{z})} |a_{pq}(y) - (a_{pq})_{\bar{z}, r_{k-4}}|^2 d\mu \\
&= \int_{B_{r_{k-4}}(\bar{z})} |\pi_{T_y M} - A_{\bar{z}, r_{k-4}}|^2 d\mu = c \alpha^2(\bar{z}, r_{k-4}),
\end{aligned}$$

and thus,

$$\alpha(\tilde{x}_{jk}, 240r_k) \leq c \alpha(\bar{z}, r_{k-4}). \quad (4.114)$$

Similarly, we can show that

$$\alpha(\tilde{x}_{im}, 240r_m) \leq c \alpha(\bar{z}, r_{k-4}). \quad (4.115)$$

Plugging (4.114) and (4.115) in (4.80) for $\rho = 100r_m$, we get

$$d_{x_{im}, 100r_m}(P_{jk}, P_{im}) \leq c \alpha(\bar{z}, r_{k-4}), \quad \forall k \geq 1. \quad (4.116)$$

where c is a constant depending only on n and d .

This finishes the proof of (4.109).

Hence, we have shown that $\epsilon'_k(f_k(z)) \leq c \alpha(\bar{z}, r_{k-4})$ for every $k \geq 1$, that is

$$\epsilon'_k(f_k(z))^2 \leq c \alpha^2(\bar{z}, r_{k-4}), \quad \forall k \geq 1 \quad (4.117)$$

Summing both sides of (4.117) over $k \geq 0$, and using (4.5) in Lemma 4.2.1 together with the fact that $\bar{z} \in M \cap B_{\frac{1}{10^3}}(0)$, we get

$$\sum_{k=0}^{\infty} \epsilon'_k(f_k(z))^2 \leq 1 + c \sum_{k=10}^{\infty} \alpha^2(\bar{z}, r_{k-4}) \leq 1 + c \epsilon_0^2 := N. \quad (4.118)$$

Inequality (4.43) is proved, and our theorem follows. □

4.3 Corollaries to Theorem 1.2.1

As mentioned before, in this section we prove two corollaries to Theorem 1.2.1. We begin with Corollary 4.3.1 which states that if we assume, in addition to the hypothesis of Theorem 1.2.1, that M is ϵ_0 -Reifenberg flat, then (locally) M is exactly the bi-Lipschitz image of an n -plane.

Corollary 4.3.1. *Let $M \subset B_1(0)$ be an n -Ahlfors regular rectifiable set containing the origin, and let $\mu = \mathcal{H}^n \llcorner M$ be the Hausdorff measure restricted to M . Assume that M satisfies the Poincaré-type inequality (1.10). There exists $\epsilon_0 > 0$ that depends only on n and d , such that if (1.11) is satisfied, and if for every $x \in M$ and $r < 1$ there is an n -plane $Q_{x,r}$, passing through x such that*

$$d(y, Q_{x,r}) \leq \epsilon_0 r \quad \forall y \in M \cap B_{10r}(x) \quad (4.119)$$

and

$$d(y, M) \leq \epsilon_0 r \quad \forall y \in Q_{x,r} \cap B_{10r}(x), \quad (4.120)$$

then there exists a bijective K -bi-Lipschitz map $g : \mathbb{R}^{n+d} \rightarrow \mathbb{R}^{n+d}$ where the bi-Lipschitz constant K depends only on n and d , and an n -dimensional plane Σ_0 , such that (1.12) and (1.13) hold, and

$$M \cap B_{\frac{1}{10^6}}(0) = g(\Sigma_0) \cap B_{\frac{1}{10^6}}(0). \quad (4.121)$$

Proof. Let ϵ_2 be as in Theorem 4.2.4, and let $\epsilon_0 \leq \epsilon \leq \epsilon_2$ (ϵ_0 and ϵ to be determined later). Going through the exact same steps as in the proof of Theorem 1.2.1, but with ϵ instead of ϵ_2 , we get a bijective map $g : \mathbb{R}^{n+d} \rightarrow \mathbb{R}^{n+d}$ such that (1.12) holds,

$$|g(z) - z| \leq C_1 \epsilon \quad \text{for } z \in \mathbb{R}^{n+d}, \quad (4.122)$$

and

$$M \cap B_{\frac{1}{10^3}}(0) \subset g(\Sigma_0). \quad (4.123)$$

Note that we have not fixed ϵ_0 and ϵ yet. However, we know that the above holds for $\epsilon_0 \leq \epsilon \leq \epsilon_2$ with inequalities (4.49), (4.86), the line before (4.63), and the line before (4.102) satisfied with ϵ instead of ϵ_2 .

Now, we want to show that

$$g(\Sigma_0) \cap B_{\frac{1}{10^6}}(0) \subset M. \quad (4.124)$$

To do that, we first need to show that for every $k \geq 0$ and for every $j \in J_k$, $M \cap B_{120r_k}(\tilde{x}_{jk})$ is close to P_{jk} and that the n -planes P_{jk} and $Q_{jk} := Q_{x_{jk}, r_k}$ are close to each other (in the Hausdorff distance sense). Let us begin by showing that for every $k \geq 0$ and for every $j \in J_k$,

$$d(z, P_{jk}) \leq \epsilon r_k \quad \forall z \in M \cap B_{120r_k}(\tilde{x}_{jk}), \quad (4.125)$$

Notice that we have already shown (4.125) for $k = 0$ and $j = i_0$ in Theorem 1.2.1 (see (4.100) and the line before (4.102)). Recall that P_{i_0} is the plane Σ_0 and $\tilde{x}_{i_0} = 0$, and we prove (4.100) in order to show that the net $\{x_{j_0}\}_{j \in J_0}$ is close to Σ_0 (see (4.91)). In fact, using the exact same steps used to prove (4.100) (see starting inequality (4.91) till inequality (4.100)), but for P_{jk} instead of P_{i_0} and \tilde{x}_{jk} instead of $\tilde{x}_{i_0} = 0$, we directly get (4.125).

Now, let us show that P_{jk} and Q_{jk} are close together, that is

$$d_{x_{jk}, 5r_k}(P_{jk}, Q_{jk}) \leq 3\epsilon r_k. \quad (4.126)$$

Since P_{jk} and Q_{jk} are n -planes, it is enough to show

$$\sup_{y \in Q_{jk} \cap B_{5r_k}(x_{jk})} d(y, P_{jk}) \leq 3\epsilon r_k. \quad (4.127)$$

Let $y \in Q_{jk} \cap B_{5r_k}(x_{jk})$. By (4.120), we get that $d(y, M) \leq \epsilon_0 r_k$, and thus, there exists $y' \in M$ such that $|y - y'| \leq 2\epsilon_0 r_k$. Recalling that $x_{jk} \in M \cap B_{\frac{r_k}{6}}(\tilde{x}_{jk})$ (see (4.46)), we get

$$|y' - \tilde{x}_{jk}| \leq |y' - y| + |y - x_{jk}| + |x_{jk} - \tilde{x}_{jk}| \leq 2\epsilon_0 r_k + 5r_k + \frac{r_k}{6} \leq 120r_k,$$

that is $y' \in B_{120r_k}(\tilde{x}_{jk})$. Hence, by (4.125), we get that $d(y', P_{jk}) \leq \epsilon r_k$, and using the fact that $\epsilon_0 \leq \epsilon$, we get

$$d(y, P_{jk}) \leq |y - y'| + d(y', P_{jk}) \leq 3\epsilon r_k,$$

which finishes the proof of (4.127) and in particular (4.126).

Before starting the proof of (4.124), let us recall a little bit how the map g was defined. In the proof of Theorem 4.2.4 (see paragraph above (4.41)) David and Toro constructed the smooth maps σ_k and $f_k = \sigma_{k-1} \circ \dots \circ \sigma_0$ for $k \geq 0$, and then defined the map $f = \lim_{k \rightarrow \infty} f_k$ on Σ_0 , and finally the map g was the extension of f to the whole space.

In order to prove (4.124), we will need the following inequality from [DT12] (see proposition 5.1 page 19 in [DT12])

$$d(f_k(z), P_{jk}) \leq c\epsilon r_k, \quad \forall z \in \Sigma_0, k \geq 0 \text{ and } j \in J_k \text{ such that } f_k(z) \in 10B_{jk} \quad (4.128)$$

where c is a constant depending only on n and d . We are finally ready to prove (4.124). Let $w \in g(\Sigma_0) \cap B_{\frac{1}{10^6}}(0)$, and let $d := d(w, M)$. We would like to prove that $d = 0$ (recall that M is closed by assumption). Let $z \in \Sigma_0$ such that $w = g(z)$. Notice that by (4.122), we have

$$|w - z| = |g(z) - z| \leq C_1\epsilon. \quad (4.129)$$

Recalling that $\Sigma_0 = P_{i_0}$, $\tilde{x}_{i_0} = 0$, $r_0 = \frac{1}{10^4}$, and that $x_{jk} \in B_{\frac{r_k}{6}}(\tilde{x}_{jk})$ (see (4.46)), we get

$$|z - x_{i_0}| \leq |z - w| + |w - \tilde{x}_{i_0}| + |\tilde{x}_{i_0} - x_{i_0}| \leq C_1\epsilon + \frac{1}{10^6} + \frac{r_0}{6} \leq 3r_0 \quad (4.130)$$

for ϵ such that $C_1\epsilon \leq r_0 = \frac{1}{10^4}$. Thus, $z \in P_{i_0} \cap B_{5r_0}(x_{i_0})$, and by (4.126), there is a point $z' \in Q_{i_0}$ such that $|z - z'| \leq 6\epsilon r_0$. Moreover,

$$|z' - x_{i_0}| \leq |z' - z| + |z - x_{i_0}| \leq 6\epsilon r_0 + 3r_0 \leq 10r_0, \quad (4.131)$$

for $\epsilon < 1$. Thus, $z' \in Q_{i_0 0} \cap B_{10r_0}(x_{i_0 0})$, and by (4.120), we get that $d(z', M) \leq \epsilon_0 r_0$. Combining (4.129), the line before and the line after (4.131), and the fact that $\epsilon_0 \leq \epsilon$, we get

$$d = d(w, M) \leq |w - z| + |z - z'| + d(z', M) \leq C_1 \epsilon + 6\epsilon r_0 + \epsilon_0 r_0 = c_6 \epsilon r_0 \leq \frac{r_0}{10}, \quad (4.132)$$

for ϵ such that $c_6 \epsilon \leq \frac{1}{10}$.

We proceed by contradiction. Suppose $d > 0$, then there exists $k \geq 0$ such that $r_{k+1} < d \leq r_k$. Notice that since $w = g(z)$, $z \in \Sigma_0$, and the maps g and f agree on Σ_0 , then by (4.107), we have

$$|w - f_k(z)| \leq c \epsilon r_k \leq r_k. \quad (4.133)$$

Now, by the definition of d , there exists $\xi \in M$ such that $|\xi - w| \leq \frac{3}{2}d$. Using (4.132) and the fact that $r_0 = \frac{1}{10^4}$, we get

$$|\xi| \leq |\xi - w| + |w| \leq \frac{3}{2} \frac{r_0}{10} + \frac{1}{10^6} \leq \frac{1}{10^3}, \quad (4.134)$$

and thus by (4.47), there exists $j \in J_k$ such that $\xi \in B_{\frac{3}{2}r_k}(x_{jk})$. Moreover,

$$|f_k(z) - x_{jk}| \leq |f_k(z) - w| + |w - \xi| + |\xi - x_{jk}| \leq c \epsilon r_k + \frac{3}{2}r_k + \frac{3}{2}r_k \leq 4r_k.$$

Since both k and j are now fixed, consider the n -plane P_{jk} . Inequality (4.128) tell us that $d(f_k(z), P_{jk}) \leq c \epsilon r_k$. Let $y \in P_{jk}$ such that $|y - f_k(z)| \leq c \epsilon r_k$. Then, by (4.133), the line below it, the line below (4.134), and recalling that $d \leq r_k$, we get

$$|y - x_{jk}| \leq |y - f_k(z)| + |f_k(z) - w| + |w - \xi| + |\xi - x_{jk}| \leq c_7 \epsilon r_k + 3r_k \leq 5r_k \quad (4.135)$$

for ϵ such that $c_7 \epsilon \leq 1$. Thus, $y \in P_{jk} \cap B_{5r_k}(x_{jk})$, and by (4.126) there exists $y' \in Q_{jk}$ such that $|y - y'| \leq 6\epsilon r_k$. But then, $|y' - x_{jk}| \leq |y - y'| + |y - x_{jk}| \leq 10r_k$; thus $y' \in Q_{jk} \cap B_{10r_k}(x_{jk})$ and by (4.120) we get that $d(y', M) \leq \epsilon_0 r_k$. Finally, using (4.133), the two lines before (4.135), and the three lines below it, we get

$$d = d(w, M) \leq |w - f_k(z)| + |f_k(z) - y| + |y - y'| + d(y', M) \leq c \epsilon r_k = c_8 \epsilon r_k \leq r_{k+1} \quad (4.136)$$

for ϵ such that $c_8\epsilon \leq \frac{1}{10}$, which contradicts the fact that $d > r_{k+1}$. This finishes the proof of (4.124).

Fix $\epsilon < \epsilon_2 < 1$ such that the lines after (4.130), (4.132), (4.135), and (4.136) hold, and then fix $\epsilon_0 \leq \epsilon$ such that (4.49), (4.86), and the lines before (4.63) and (4.102) are satisfied (with ϵ instead of ϵ_2). Writing $\epsilon_0 = c_9 \epsilon$ and replacing in (4.122), we get (1.13). The proof that g is bi-Lipschitz is the same as from Theorem 1.2.1. \square

In the special case when M has co-dimension 1, (1.11) translates a Carleson-type condition on the oscillation of the unit normals to M . We show this in the following Corollary:

Corollary 4.3.2. *Let $M \subset B_1(0) \subset \mathbb{R}^{n+1}$ be an n -Ahlfors regular rectifiable set containing the origin, and let $\mu = \mathcal{H}^n \llcorner M$ be the Hausdorff measure restricted to M . Assume that M satisfies the Poincaré-type inequality (1.10) with $d = 1$. There exists $\epsilon_0 > 0$ that depends only on n , such that if there exists a choice of unit normal ν to M where*

$$\sup_{x \in M \cap B_{\frac{1}{10^3}}(0)} \int_0^1 \left(\int_{B_r(x)} |\nu(y) - \nu_{x,r}|^2 d\mu \right) \frac{dr}{r} < \epsilon_0^2, \quad (4.137)$$

then there exists a bijective K -bi-Lipschitz map $g : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ where the bi-Lipschitz constant K depends only on n , and an n -dimensional plane Σ_0 , with the following properties:

$$g(z) = z \quad \text{when } d(z, \Sigma_0) \geq 2,$$

and

$$|g(z) - z| \leq C_2 \epsilon_0 \quad \text{for } z \in \mathbb{R}^{n+1},$$

where C_2 is a constant depending only on n . Moreover,

$$g(\Sigma_0) \text{ is a } C_2 \epsilon_0\text{- Reifenberg flat set,}$$

and

$$M \cap B_{\frac{1}{10^3}}(0) \subset g(\Sigma_0).$$

Proof. Suppose that (4.137) holds for some choice of unit normal ν to M . We show that (4.137) is in fact exactly inequality (1.11). Fix $x \in M$ and $0 < r < 1$ and let $y \in M \cap B_r(x)$ be a point where the approximate tangent plane $T_y M$ (and thus the unit normal $\nu(y)$) exists. Denote by $T_y M^\perp$ the subspace perpendicular to $T_y M$. Then, using the matrix representation of $\pi_{T_y M}$ in the standard basis of \mathbb{R}^{n+1} , and the fact that $\pi_{T_y M^\perp} = Id_{n+1} - \pi_{T_y M}$ where Id_{n+1} is the $(n+1) \times (n+1)$ identity matrix, one can easily see that

$$|\pi_{T_y M} - A_{x,r}|^2 = |\pi_{T_y M^\perp} - B_{x,r}|^2, \quad (4.138)$$

where $\pi_{T_y M^\perp} = (b_{ij}(y))_{ij}$ and $B_{x,r} = Id_{n+d} - A_{x,r} = ((b_{ij})_{x,r})_{ij}$.

Now, we want to express the right hand side of (4.138) using a different basis than the standard basis of \mathbb{R}^{n+1} . For any choice of orthonormal basis $\{\nu_1(y), \dots, \nu_n(y)\}$ of $T_y M$, we have that $\{\nu_1(y), \dots, \nu_n(y), \nu(y)\}$ is an orthonormal basis for \mathbb{R}^{n+1} . The matrix representation of $\pi_{T_y M^\perp}$ with $\{\nu_1(y), \dots, \nu_n(y), \nu(y)\}$ as a basis for the domain \mathbb{R}^{n+1} and the standard basis for the range \mathbb{R}^{n+1} , is the $(n+1) \times (n+1)$ matrix whose last column is $\nu(y)$ while the other columns are all zero. Thus, with this choice of bases and matrix representations, $B_{x,r}$ becomes the matrix whose last column is $\nu_{x,r}$ while the other column are all zero². Hence, using (4.138), we get that

$$|\pi_{T_y M} - A_{x,r}|^2 = |\pi_{T_y M^\perp} - B_{x,r}|^2 = |\nu(y) - \nu_{x,r}|^2. \quad (4.139)$$

Since (4.139) is true for any $y \in B_r(x)$, and since x and r are arbitrary, then,

$$\begin{aligned} \sup_{x \in M \cap B_{\frac{1}{10^3}}(0)} \int_0^1 \left(\int_{B_r(x)} |\nu(y) - \nu_{x,r}|^2 d\mu \right) \frac{dr}{r} \\ = \sup_{x \in M \cap B_{\frac{1}{10^3}}(0)} \int_0^1 \left(\int_{B_r(x)} |\pi_{T_y M} - A_{x,r}|^2 d\mu \right) \frac{dr}{r}, \end{aligned}$$

and the proof is done □

²Note that considering this choice of bases and matrix representations is only valid in co-dimension 1, as otherwise $B_{x,r}$ will not be well defined. This is because in higher co-dimensions, there is no unique way of choosing the unit normals that span the normal plane, whereas there is one choice (modulo direction) in co-dimension 1.

Chapter 5

QUASICONVEXITY OF M **5.1 Background, notations, and Theorem 5.1.5**

In this chapter, we show that the Poincaré-type inequality (1.10) that M satisfies encodes some geometric information about M . More precisely, consider the metric measure space (M, d_0, μ) , where $M \subset B_1(0)$ is an n -Ahlfors regular rectifiable set in \mathbb{R}^{n+d} , $\mu = \mathcal{H}^n \llcorner M$ is the Hausdorff measure restricted to M , and d_0 is the restriction of the standard Euclidean distance in \mathbb{R}^{n+d} to M (which is obviously a metric on M). Our goal in this chapter is to show that if M satisfies the Poincaré-type inequality (1.10), then (M, d_0, μ) is quasiconvex.

Definition 5.1.1. A metric space (X, d) is quasiconvex if there exists a constant $\kappa \geq 1$ such that for any two points x and y in X , there exists a rectifiable curve γ in X , joining x and y , such that $\text{length}(\gamma) \leq \kappa d(x, y)$.

E. Cartagena, J. Jaramillo, and N. Shanmugalingam proved a very nice theorem in their paper [DCJS13], concerning the quasiconvexity of metric measure spaces supporting some kind of generalized Poincaré inequality. To state that theorem, we first need to recall the notions of a doubling measure and a *local Lipschitz constant function* on a metric measure space (X, d, ν) .

Definition 5.1.2. Let (X, d, ν) be a metric measure space. We say that ν is a doubling measure if there is a constant $\kappa_0 > 0$ such that

$$\nu(B_{2r}^X(x)) \leq \kappa_0 \nu(B_r^X(x)),$$

where $x \in X$, $r > 0$, and $B_r^X(x)$ denotes the metric ball in X , center x , and radius r .

Definition 5.1.3. Let f be a Lipschitz function on a metric measure space (X, d, ν) . The local Lipschitz constant function of f is defined as follows

$$\text{Lip}f(x) = \lim_{r \rightarrow 0} \sup_{y \in B_r^X(x), y \neq x} \frac{|f(y) - f(x)|}{d(y, x)}, \quad x \in X, \quad (5.1)$$

where $B_r^X(x)$ denotes the metric ball in X , center x , and radius r .

Notation: Let us note here that for any Lipschitz function f , $\text{lip}f$ denotes the usual Lipschitz constant (see sentence below (2.1)), whereas $\text{Lip}f(\cdot)$ stands for the local Lipschitz constant function defined above.

Theorem 5.1.4. (see [DCJS13], Theorem 3.6) Let (X, d, ν) be a complete metric measure space, with ν a doubling measure. Let \mathcal{B} be the collection of all balls in X . Assume that for every Lipschitz function f , there exists a functional $a_f : \mathcal{B} \rightarrow [0, \infty)$ such that for each $B \in \mathcal{B}$, we have

$$\int_B |f(y) - f_B| d\nu(y) \leq a_f(B), \quad (5.2)$$

where $f_B := \int_B f d\nu$. If the functional $f \rightarrow a_f$ satisfies

(*) There exists a constant C_3 , depending only on n and d , such that whenever f is a Lipschitz function on X with $\|\text{Lip}f\|_{L^\infty(X)} \leq 1$, then

$$a_f(B) \leq C_3 \text{radius}(B) \quad \text{for all } B \in \mathcal{B}, \quad (5.3)$$

then (X, d, ν) is quasiconvex.

We want to use Theorem 5.1.4 to prove the main theorem of this section:

Theorem 5.1.5. Let (M, d_0, μ) be the metric measure space where $M \subset B_1(0)$ is n -Ahlfors regular rectifiable set in \mathbb{R}^{n+d} , $\mu = \mathcal{H}^n \llcorner M$ is the Hausdorff measure restricted to M , and d_0 is the restriction of the standard Euclidean distance in \mathbb{R}^{n+d} to M . Suppose that M satisfies the Poincaré-type inequality (1.10). Then (M, d_0, μ) is quasiconvex.

The next section is devoted to proving Theorem 5.1.5.

5.2 The proof of Theorem 5.1.5

We begin this section with the following lemma which is needed to prove Theorem 5.1.5. This lemma appears in [KT99] (p.379, Lemma 2.1), but for the sake of completion, we include the proof here.

Lemma 5.2.1. *Let M be an n -Ahlfors regular rectifiable subset of \mathbb{R}^{n+d} , and let $\mu = \mathcal{H}^n \llcorner M$ be the Hausdorff measure restricted to M . Let x be a point in M such that the approximate tangent plane $T_x M$ at x exists. Consider a sequence $\{h_i\}_{i \in \mathbb{N}}$ of positive real numbers such that $h_i \xrightarrow{i \rightarrow \infty} 0$, and for every $i \in \mathbb{N}$, let $M_i = \frac{M - x}{h_i}$. Then, for every $a \in T_x M$, there exists a sequence $\{a_i\}_{i \in \mathbb{N}}$, with $a_i \in M_i$ for all $i \in \mathbb{N}$, such that $a_i \xrightarrow{i \rightarrow \infty} a$.*

Proof. Let x , $\{h_i\}_{i \in \mathbb{N}}$, $\{M_i\}_{i \in \mathbb{N}}$, and a be as stated above. We first notice that it suffices to prove that $d(a, M_i) \xrightarrow{i \rightarrow \infty} 0$. In fact, suppose the latter is satisfied. For every $i \in \mathbb{N}$, let $a_i \in M_i$ such that $|a_i - a| \leq 2d(a, M_i)$. Since $|a_i - a| \leq 2d(a, M_i) \xrightarrow{i \rightarrow \infty} 0$, then, our sequence $\{a_i\}_{i \in \mathbb{N}}$, with $a_i \in M_i$ for all $i \in \mathbb{N}$, is such that $a_i \xrightarrow{i \rightarrow \infty} a$.

So, let's prove that $d(a, M_i) \xrightarrow{i \rightarrow \infty} 0$. We proceed by contradiction. Suppose that $\lim_{i \rightarrow \infty} d(a, M_i) \neq 0$. Then, there exists an $\epsilon_0 > 0$, and a subsequence $\{M_{i_k}\}_{k \in \mathbb{N}}$ of $\{M_i\}_{i \in \mathbb{N}}$, such that $d(a, M_{i_k}) \geq \epsilon_0$ for every $k \in \mathbb{N}$. Thus,

$$B_{\frac{\epsilon_0}{2}}(a) \cap M_{i_k} = \emptyset, \quad \forall k \in \mathbb{N}. \quad (5.4)$$

Now, let $\varphi \in C_c^\infty(\mathbb{R}^{n+d})$ be a non-negative function on \mathbb{R}^{n+d} , such that $\varphi = 1$ on $B_{\frac{\epsilon_0}{4}}(a)$ and $\varphi = 0$ on $B_{\frac{\epsilon_0}{2}}^c(a)$. By the definition of the approximate tangent plane $T_x M$ at x , we know that

$$\lim_{k \rightarrow \infty} \frac{1}{h_{i_k}^n} \int_M \varphi\left(\frac{y-x}{h_{i_k}}\right) d\mathcal{H}^n(y) = \int_{T_x M} \varphi(y) d\mathcal{H}^n(y). \quad (5.5)$$

Let us calculate the left hand side of (5.5). Fix $k \in \mathbb{N}$. Then, for $y \in M$, we have $\frac{y-x}{h_{i_k}} \in M_{i_k}$ which by (5.4) implies that $\frac{y-x}{h_{i_k}} \notin B_{\frac{\epsilon_0}{2}}(a)$. However, we have chosen φ such that $\text{spt}(\varphi) \subset B_{\frac{\epsilon_0}{2}}(a)$. Hence, we get

$$\frac{1}{h_{i_k}^n} \int_M \varphi \left(\frac{y-x}{h_{i_k}} \right) d\mathcal{H}^n(y) = 0. \quad (5.6)$$

Since (5.6) holds for all $k \in \mathbb{N}$, then by plugging (5.6) in (5.5), we get

$$\int_{T_x M} \varphi(y) d\mathcal{H}^n(y) = 0. \quad (5.7)$$

Now, remembering that $\varphi = 1$ on $B_{\frac{\epsilon_0}{4}}(a)$ and $\varphi \geq 0$, and using (5.7), we get

$$\omega_n \left(\frac{\epsilon_0}{4} \right)^n = \mathcal{H}^n(B_{\frac{\epsilon_0}{4}}(a) \cap T_x M) = \int_{B_{\frac{\epsilon_0}{4}}(a) \cap T_x M} \varphi(y) d\mathcal{H}^n(y) = 0.$$

This is a contradiction, and thus the proof is done \square

Now, let us turn our focus back to proving Theorem 5.1.5. As we mentioned before, we want to apply Theorem 5.1.4 to prove Theorem 5.1.5. In fact, we want to apply Theorem 5.1.4 to the metric measure space (M, d_0, μ) . To do that, we show that the hypotheses of Theorem 5.1.5 imply those of Theorem 5.1.4, and then use the Poincaré-type inequality (1.10) from Theorem 5.1.5 to define those functionals a_f mentioned in Theorem 5.1.4, and prove that the conditions (5.2) and (*) are satisfied.

So, our first step is to show that the hypotheses of Theorem 5.1.5 imply those of Theorem 5.1.4. Let (M, d_0, μ) be the metric measure space where $M \subset B_1(0)$ is n -Ahlfors regular rectifiable set in \mathbb{R}^{n+d} , $\mu = \mathcal{H}^n \llcorner M$ is the Hausdorff measure restricted to M , and d_0 is the restriction of the standard Euclidean distance in \mathbb{R}^{n+d} to M . First, notice that since M is a closed and bounded subset of \mathbb{R}^{n+d} , then M is complete.

Now, let \mathcal{B} be the collection of all metric balls in (M, d_0, μ) , and take $B \in \mathcal{B}$. Let $x \in M$ be the center of B , and $r > 0$ its radius. Denote such a ball by $B_r^M(x)$. It is trivial to see

that

$$B_r^M(x) = B_r(x) \cap M, \quad (5.8)$$

where $B_r(x)$ is the euclidean ball in \mathbb{R}^{n+d} of center $x \in M$ and radius $r > 0$. Using (5.8) and the fact that μ is Ahlfors regular, it is easy to check that μ is in fact doubling. Hence, we are in the setting of Theorem 5.1.4.

We want to use the Poincaré-type inequality (1.10) to define functionals a_f that satisfy (5.2) and (*). Comparing these inequalities and conditions, it is not surprising that the functionals we end up defining are related to both $|\nabla^M f|$ and $Lipf(\cdot)$ when both of these functions are well defined. The following proposition gives us the relation between those two latter functions.

Proposition 5.2.2. *Let (M, d_0, μ) be the metric measure space where $M \subset B_1(0)$ is n -Ahlfors regular rectifiable set in \mathbb{R}^{n+d} , $\mu = \mathcal{H}^n \llcorner M$ is the Hausdorff measure restricted to M , and d_0 is the restriction of the standard Euclidean distance in \mathbb{R}^{n+d} to M . Let f be a Lipschitz function on M . Then,*

$$|\nabla^M \bar{f}(x)| \leq n Lipf(x) \quad \mu\text{-almost every } x \in M,$$

where \bar{f} is a Lipschitz extension of f to the whole space \mathbb{R}^{n+d} , with $f = \bar{f}$ on M , and $lip\bar{f} \leq lipf$.

Proof. Let f be a Lipschitz function on M . Note that using the metric we have on M , we recall from (5.1) and (5.8) that

$$Lipf(x) = \lim_{r \rightarrow 0} \sup_{y \in B_r(x) \cap M, y \neq x} \frac{|f(y) - f(x)|}{|y - x|}, \quad x \in M. \quad (5.9)$$

The fact that f extends to a Lipschitz function \bar{f} defined on \mathbb{R}^{n+d} , with $f = \bar{f}$ on M , and $lip\bar{f} \leq lipf$ is well known. Fix $x \in M$ such that the approximate tangent plane $T_x M$ exists. We prove that

$$|\nabla^M \bar{f}(x)| \leq n Lipf(x). \quad (5.10)$$

Since M is rectifiable, then, by Theorem 2.0.6, μ - a.e. point in M admits an approximate tangent plane. Thus, by proving (5.10), we would have proved the theorem.

Let $\{\tau_1(x), \dots, \tau_n(x)\}$ be an orthonormal basis for $T_x M$. We claim that

$$| \langle \nabla \bar{f}(x), \tau_j(x) \rangle | \leq Lip f(x) \quad \forall j \in \{1, \dots, n\}. \quad (5.11)$$

To see this, fix $j \in \{1, \dots, n\}$. Consider a sequence $\{h_i\}_{i \in \mathbb{N}}$ of positive numbers, such that $h_i \xrightarrow{i \rightarrow \infty} 0$. By Rademacher's theorem, we have

$$\lim_{i \rightarrow \infty} \frac{|\bar{f}(x + h_i \tau_j(x)) - \bar{f}(x) - h_i \langle \nabla \bar{f}(x), \tau_j(x) \rangle|}{h_i} = 0. \quad (5.12)$$

For simplicity, let us use the notation $\epsilon_{i,j}$ for the quantity inside the limit in the left hand side of (5.12). Thus, we get

$$\lim_{i \rightarrow \infty} \epsilon_{i,j} = 0. \quad (5.13)$$

Now, from the definition of $\epsilon_{i,j}$, we have

$$\left| \frac{|\bar{f}(x + h_i \tau_j(x)) - \bar{f}(x)|}{h_i} - | \langle \nabla \bar{f}(x), \tau_j(x) \rangle | \right| \leq \epsilon_{i,j}, \quad \forall i \in \mathbb{N},$$

that is,

$$| \langle \nabla \bar{f}(x), \tau_j(x) \rangle | \leq \frac{|\bar{f}(x + h_i \tau_j(x)) - \bar{f}(x)|}{h_i} + \epsilon_{i,j}, \quad \forall i \in \mathbb{N}. \quad (5.14)$$

Let us now focus on the first summand of (5.14). We want to show that

$$\limsup_{i \rightarrow \infty} \frac{|\bar{f}(x + h_i \tau_j(x)) - \bar{f}(x)|}{h_i} \leq Lip f(x).$$

The reason why this inequality is not straight forward is that for $i \in \mathbb{N}$, the point $x + h_i \tau_j$ is not necessarily in M (recall from (5.9), $Lip f(x)$ only considers the points y that are in M and do not coincide with x). To remedy this, we need to move the points $x + h_i \tau_j$, $i \in \mathbb{N}$ just a little bit, to get a sequence of points $\{y_i\}_{i \in \mathbb{N}}$ that (just like the sequence $\{x + h_i \tau_j\}_{i \in \mathbb{N}}$) still

approaches the point x and does not coincide with it, but unlike the sequence $\{x + h_i \tau_j\}_{i \in \mathbb{N}}$, lives in M .

We proceed to constructing the sequence $\{y_i\}_{i \in \mathbb{N}}$. Since $\tau_j(x) \in T_x M$, then by Lemma 5.2.1, there exists a sequence $\{a_i\}_{i \in \mathbb{N}}$, with $a_i \in \frac{M - x}{h_i}$ for all $i \in \mathbb{N}$, such that $a_i \xrightarrow{i \rightarrow \infty} \tau_j(x)$. Writing

$$a_i = \frac{y_i - x}{h_i} \quad \forall i \in \mathbb{N}, \quad (5.15)$$

we get a sequence $\{y_i\}_{i \in \mathbb{N}}$, with $y_i \in M$ for all $i \in \mathbb{N}$, such that

$$\lim_{i \rightarrow \infty} \left| \frac{y_i - x}{h_i} - \tau_j(x) \right| = 0, \quad (5.16)$$

that is,

$$\lim_{i \rightarrow \infty} \left| \frac{y_i - x - h_i \tau_j(x)}{h_i} \right| = 0. \quad (5.17)$$

Notice that from the definition of the a_i 's in (5.15), and recalling that $\lim_{i \rightarrow \infty} a_i = \tau_j(x)$, $\tau_j(x)$ is a unit vector, and $\lim_{i \rightarrow \infty} h_i = 0$, we can easily see that

$$\lim_{i \rightarrow \infty} |y_i - x| = \lim_{i \rightarrow \infty} h_i |a_i| = 0. \quad (5.18)$$

Moreover, from (5.16) and the fact that $\tau_j(x)$ is a unit vector, we have

$$\lim_{i \rightarrow \infty} \left| \frac{y_i - x}{h_i} \right| = \lim_{i \rightarrow \infty} |a_i| = 1. \quad (5.19)$$

Thus, by (5.19), there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$, we have $|y_i - x| \geq \frac{h_i}{2}$, that is $y_i \neq x$, for all $i \geq i_0$. However, since all the limits and inequalities from (5.13) till (5.19) still hold when we restrict i to $i \geq i_0$, then without loss of generality, we can assume that

$$y_i \neq x \quad \forall i \in \mathbb{N}. \quad (5.20)$$

To sum up, $\{y_i\}_{i \in \mathbb{N}}$ is a sequence of points in M that approaches the point $x \in M$, and does not coincide with it.

Now, for $i \in \mathbb{N}$, we can write

$$\frac{|\bar{f}(x + h_i \tau_j(x)) - \bar{f}(x)|}{h_i} \leq \frac{|\bar{f}(x + h_i \tau_j(x)) - \bar{f}(y_i)|}{h_i} + \frac{|\bar{f}(y_i) - \bar{f}(x)|}{h_i}. \quad (5.21)$$

Rewriting the first term of the right hand side of (5.21) and remembering that \bar{f} is Lipschitz, we have

$$\begin{aligned} \frac{|\bar{f}(x + h_i \tau_j(x)) - \bar{f}(y_i)|}{h_i} &= \frac{|\bar{f}(x + h_i \tau_j(x)) - \bar{f}(y_i)|}{|y_i - x - h_i \tau_j(x)|} \cdot \frac{|y_i - x - h_i \tau_j(x)|}{h_i} \\ &\leq \text{lip} \bar{f} \frac{|y_i - x - h_i \tau_j(x)|}{h_i}. \end{aligned} \quad (5.22)$$

(note that in case $y_i - x - h_i \tau_j = 0$, (5.22) is satisfied trivially).

Also by rewriting the second term of the right hand side of (5.21) (using(5.20)), and remembering that the points y_i and x are in M , and that $\bar{f} = f$ on M , we get

$$\frac{|\bar{f}(y_i) - \bar{f}(x)|}{h_i} = \frac{|\bar{f}(y_i) - \bar{f}(x)|}{|y_i - x|} \cdot \frac{|y_i - x|}{h_i} = \frac{|f(y_i) - f(x)|}{|y_i - x|} \cdot \frac{|y_i - x|}{h_i}. \quad (5.23)$$

Plugging (5.22) and (5.23) in (5.21), we get

$$\frac{|\bar{f}(x + h_i \tau_j(x)) - \bar{f}(x)|}{h_i} \leq \text{lip} \bar{f} \frac{|y_i - x - h_i \tau_j(x)|}{h_i} + \frac{|f(y_i) - f(x)|}{|y_i - x|} \cdot \frac{|y_i - x|}{h_i}. \quad (5.24)$$

Since (5.24) holds for all $i \in \mathbb{N}$, then by taking the \limsup on both sides of (5.24), we get using (5.17) and (5.19) that

$$\limsup_{i \rightarrow \infty} \frac{|\bar{f}(x + h_i \tau_j(x)) - \bar{f}(x)|}{h_i} \leq \limsup_{i \rightarrow \infty} \frac{|f(y_i) - f(x)|}{|y_i - x|}. \quad (5.25)$$

But, using (5.18), (5.20), and remembering that $y_i \in M$, it is easy to check that

$$\limsup_{i \rightarrow \infty} \frac{|f(y_i) - f(x)|}{|y_i - x|} \leq Lipf(x). \quad (5.26)$$

Thus, plugging (5.26) back in (5.25), we get

$$\limsup_{i \rightarrow \infty} \frac{|\bar{f}(x + h_i \tau_j(x)) - \bar{f}(x)|}{h_i} \leq Lipf(x). \quad (5.27)$$

Finally, taking \limsup on both sides of (5.14), and using (5.27) and (5.13), we get

$$|\langle \nabla \bar{f}(x), \tau_j(x) \rangle| \leq Lipf(x) \quad \forall j \in \{1, \dots, n\},$$

hence finishing the proof of (5.11).

We are now ready to prove (5.10) which in turn, as mentioned earlier, finishes the proof of the theorem. Using (5.11) in the last step, we have

$$\begin{aligned} |\nabla^M \bar{f}(x)| &= |p_{T_x M}(\nabla \bar{f}(x))| \\ &= \left| \sum_{j=1}^n \langle \nabla \bar{f}(x), \tau_j(x) \rangle \tau_j(x) \right| \leq \sum_{j=1}^n |\langle \nabla \bar{f}(x), \tau_j(x) \rangle| \leq n Lipf(x), \end{aligned}$$

and the proof is done \square

Corollary 5.2.3. *Let (M, d_0, μ) be the metric measure space where $M \subset B_1(0)$ is n -Ahlfors regular rectifiable set in \mathbb{R}^{n+d} , $\mu = \mathcal{H}^n \llcorner M$ is the Hausdorff measure restricted to M , and d_0 is the restriction of the standard Euclidean distance in \mathbb{R}^{n+d} to M . Assume that M satisfies the Poincaré-type inequality (1.10). Let f be a Lipschitz function on M . Then, for every $x \in M$, and radius $r > 0$, we have*

$$\int_{B_r^M(x)} |f(y) - f_{B_r^M(x)}| d\mu(y) \leq C_4 r \left(\int_{B_{2r}^M(x)} (Lipf(y))^2 d\mu(y) \right)^{\frac{1}{2}}, \quad (5.28)$$

where C_4 is a constant depending only on n and d .

Proof. Let f , x , and r be as described above. Since f is Lipschitz on M , we can extend it to a Lipschitz function \bar{f} defined on \mathbb{R}^{n+d} , with $f = \bar{f}$ on M , and $lip\bar{f} \leq lipf$. By construction, \bar{f} is Lipschitz and thus locally Lipschitz on \mathbb{R}^{n+d} . Thus, we can apply the Poincaré-type inequality (1.10) to \bar{f} at the point x and radius r to get

$$\int_{B_r(x)} |\bar{f}(y) - \bar{f}_{x,r}| d\mu(y) \leq c_P r \left(\int_{B_{2r}(x)} |\nabla^M \bar{f}(y)|^2 d\mu(y) \right)^{\frac{1}{2}}. \quad (5.29)$$

Using the fact that $\bar{f} = f$ on M for the left hand side of (5.29), and Proposition 5.2.2 for the right hand side of (5.29), the latter becomes

$$\int_{B_r(x)} |f(y) - f_{x,r}| d\mu(y) \leq c r \left(\int_{B_{2r}(x)} (Lipf(y))^2 d\mu(y) \right)^{\frac{1}{2}}, \quad (5.30)$$

where c is a constant depending only on n and d .

Denoting by C_4 the constant c that appears in (5.30), (5.28) follows directly from (5.30), (5.8), and the fact that $\mu = \mathcal{H}^n \llcorner M$ \square

We are finally ready to put the pieces together and prove Theorem 5.1.5. Remember that our aim is to define the functionals $f \rightarrow a_f$ that satisfy (5.2) and (*).

Proof of Theorem 5.1.5:

Proof. We have already argued that (M, d_0, μ) is a complete metric measure space, with d_0 being the restriction of the standard Euclidean distance in \mathbb{R}^{n+d} to M , and $\mu = \mathcal{H}^n \llcorner M$. Denote by \mathcal{B} be the collection of all metric balls in (M, d_0, μ) . Let f be a Lipschitz function on M , and define the functional a_f to be

$$a_f(B) = C_4 \text{radius}(B) \left(\int_{2B} (Lipf(y))^2 d\mu(y) \right)^{\frac{1}{2}}, \quad \text{for all } B \in \mathcal{B}$$

where C_4 is the constant from Corollary 5.2.3, depending only on n and d .

It is clear that $a_f(B) \in [0, \infty)$ since $\|Lip f\|_{L^\infty(M)} \leq lip f$ for any Lipschitz function f on M . Also, by the few lines before (5.8) and by (5.28) in Corollary 5.2.3, we have that for every $B \in \mathcal{B}$

$$\int_B |f(y) - f_B| d\mu(y) \leq a_f(B),$$

and thus (5.2) is satisfied.

What is left is to check that condition (*) is satisfied. Suppose f is a Lipschitz function such that $\|Lip f\|_{L^\infty(M)} \leq 1$. Then, from the definition of a_f , we get that

$$a_f(B) \leq C_4 \text{radius}(B) \quad \text{for all } B \in \mathcal{B}.$$

Thus, (5.3) holds for $C_3 := C_4$, and condition (*) is satisfied. Hence, by applying Theorem 5.1.4 to the metric measure space (M, d_0, μ) , we get that (M, d_0, μ) is quasiconvex, and the proof is done

□

BIBLIOGRAPHY

- [BJ90] Christopher J. Bishop and Peter W. Jones, *Harmonic measure and arclength*, Ann. of Math. (2) **132** (1990), no. 3, 511–547. MR 1078268 (92c:30026)
- [BJ94] ———, *Harmonic measure, L^2 estimates and the Schwarzian derivative*, J. Anal. Math. **62** (1994), 77–113. MR 1269200 (95f:30034)
- [CC97] Jeff Cheeger and Tobias H. Colding, *On the structure of spaces with Ricci curvature bounded below. I*, J. Differential Geom. **46** (1997), no. 3, 406–480. MR 1484888 (98k:53044)
- [CN13] Tobias Holck Colding and Aaron Naber, *Lower Ricci curvature, branching and the bilipschitz structure of uniform Reifenberg spaces*, Adv. Math. **249** (2013), 348–358. MR 3116575
- [Dav09] Guy David, *Hölder regularity of two-dimensional almost-minimal sets in \mathbb{R}^n* , Ann. Fac. Sci. Toulouse Math. (6) **18** (2009), no. 1, 65–246. MR 2518104 (2010g:49077)
- [Dav10] ———, *$C^{1+\alpha}$ -regularity for two-dimensional almost-minimal sets in \mathbb{R}^n* , J. Geom. Anal. **20** (2010), no. 4, 837–954. MR 2683770 (2011h:49061)
- [DCJS13] Estibalitz Durand-Cartagena, Jesús A. Jaramillo, and Nageswari Shanmugalingam, *First order Poincaré inequalities in metric measure spaces*, Ann. Acad. Sci. Fenn. Math. **38** (2013), no. 1, 287–308. MR 3076810
- [DDPT08] Guy David, Thierry De Pauw, and Tatiana Toro, *A generalization of Reifenberg’s theorem in \mathbb{R}^3* , Geom. Funct. Anal. **18** (2008), no. 4, 1168–1235. MR 2465688 (2010d:49057)
- [Dou31] Jesse Douglas, *Solution of the problem of Plateau*, Trans. Amer. Math. Soc. **33** (1931), no. 1, 263–321. MR 1501590
- [DS91] G. David and S. Semmes, *Singular integrals and rectifiable sets in \mathbf{R}^n : Beyond Lipschitz graphs*, Astérisque (1991), no. 193, 152. MR 1113517 (92j:42016)
- [DS93] Guy David and Stephen Semmes, *Analysis of and on uniformly rectifiable sets*, Mathematical Surveys and Monographs, vol. 38, American Mathematical Society, Providence, RI, 1993. MR 1251061 (94i:28003)

- [DT99] G. David and T. Toro, *Reifenberg flat metric spaces, snowballs, and embeddings*, Math. Ann. **315** (1999), no. 4, 641–710. MR 1731465 (2001c:49067)
- [DT12] Guy David and Tatiana Toro, *Reifenberg parameterizations for sets with holes*, Mem. Amer. Math. Soc. **215** (2012), no. 1012, vi+102. MR 2907827
- [EG92] Lawrence C. Evans and Ronald F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992. MR 1158660 (93f:28001)
- [HK00] Piotr Hajlasz and Pekka Koskela, *Sobolev met Poincaré*, Mem. Amer. Math. Soc. **145** (2000), no. 688, x+101. MR 1683160 (2000j:46063)
- [Jon89] Peter W. Jones, *Square functions, Cauchy integrals, analytic capacity, and harmonic measure*, Harmonic analysis and partial differential equations (El Escorial, 1987), Lecture Notes in Math., vol. 1384, Springer, Berlin, 1989, pp. 24–68. MR 1013815 (91b:42032)
- [Jon90] ———, *Rectifiable sets and the traveling salesman problem*, Invent. Math. **102** (1990), no. 1, 1–15. MR 1069238 (91i:26016)
- [Jon91] ———, *The traveling salesman problem and harmonic analysis*, Publ. Mat. **35** (1991), no. 1, 259–267, Conference on Mathematical Analysis (El Escorial, 1989). MR 1103619 (92c:42013)
- [KT99] Carlos E. Kenig and Tatiana Toro, *Free boundary regularity for harmonic measures and Poisson kernels*, Ann. of Math. (2) **150** (1999), no. 2, 369–454. MR 1726699 (2001d:31004)
- [Lég99] J. C. Léger, *Menger curvature and rectifiability*, Ann. of Math. (2) **149** (1999), no. 3, 831–869. MR 1709304 (2001c:49069)
- [LeV07] Randall J. LeVeque, *Finite difference methods for ordinary and partial differential equations*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2007.
- [NV15a] Aaron Naber and Daniele Valtorta, *Rectifiable-reifenberg and the regularity of stationary and minimizing harmonic maps*, preprint, arXiv: 1504.02043, 2015.
- [NV15b] ———, *The singular structure and regularity of stationary and minimizing harmonic maps*, preprint, arXiv: 1505.03428, 2015.

- [Paj97] Hervé Pajot, *Conditions quantitatives de rectifiabilité*, Bull. Soc. Math. France **125** (1997), no. 1, 15–53. MR 1459297 (98m:28014)
- [Rad30] Tibor Radó, *On Plateau's problem*, Ann. of Math. (2) **31** (1930), no. 3, 457–469. MR 1502955
- [Rei60] E. R. Reifenberg, *Solution of the Plateau problem for m -dimensional surfaces of varying topological type*, Bull. Amer. Math. Soc. **66** (1960), 312–313. MR 0117614 (22 #8391)
- [Sem91a] Stephen Semmes, *Chord-arc surfaces with small constant. I*, Adv. Math. **85** (1991), no. 2, 198–223. MR 1093006 (93d:42019a)
- [Sem91b] ———, *Chord-arc surfaces with small constant. II. Good parameterizations*, Adv. Math. **88** (1991), no. 2, 170–199. MR 1120612 (93d:42019b)
- [Sem91c] ———, *Hypersurfaces in \mathbf{R}^n whose unit normal has small BMO norm*, Proc. Amer. Math. Soc. **112** (1991), no. 2, 403–412. MR 1065093 (91i:53006)
- [Sim83] Leon Simon, *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University, vol. 3, Australian National University, Centre for Mathematical Analysis, Canberra, 1983. MR 756417 (87a:49001)
- [Tay76] Jean E. Taylor, *The structure of singularities in soap-bubble-like and soap-film-like minimal surfaces*, Ann. of Math. (2) **103** (1976), no. 3, 489–539. MR 0428181 (55 #1208a)
- [Tor95] Tatiana Toro, *Geometric conditions and existence of bi-Lipschitz parameterizations*, Duke Math. J. **77** (1995), no. 1, 193–227. MR 1317632 (96b:28006)
- [Tor97] ———, *Doubling and flatness: geometry of measures*, Notices Amer. Math. Soc. **44** (1997), no. 9, 1087–1094. MR 1470167 (99d:28010)