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Self-shrinking Solutions to Mean Curvature Flow

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Abstract

Self-shrinking Solutions to Mean Curvature Flow

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We construct new examples of self-shrinking solutions to mean curvature flow. We first construct an immersed and non-embedded sphere self-shrinker. This result verifies numerical evidence dating back to the 1980's and shows that the rigidity results for constant mean curvature spheres in \mathbb{R}^3 and minimal spheres in S^3 do not hold for sphere self-shrinkers. Then, in joint work with Stephen Kleene, we construct infinitely many complete, immersed self-shrinkers with rotational symmetry for each of the following topological types: the sphere, the plane, the cylinder, and the torus.

We also prove rigidity theorems for self-shrinking solutions to geometric flows. In the setting of mean curvature flow, we show that the round sphere is the only embedded sphere self-shrinker with rotational symmetry. In addition, we show that every entire high codimension self-shrinker graph is a plane under a convexity assumption on the angles between the tangent plane to the graph and the base n -plane. Finally, in joint work with Peng Lu and Yu Yuan, we show that every complete entire self-shrinking solution on complex Euclidean space to the Kähler-Ricci flow is generated from a quadratic potential.

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DEDICATION

This thesis is dedicated to the memory of Professor William T. Guy, Jr.

Chapter 1

INTRODUCTION

In this thesis, we construct new examples of self-shrinking solutions to mean curvature flow, and we prove rigidity theorems for self-shrinking solutions to geometric flows.

1.1 Constructions

The mean curvature flow is a quasilinear parabolic equation with applications to geometric analysis ([33], [62], [47]), general relativity ([31], [45]), and topology ([19]). It arises in materials science as a model for the motion of grain boundaries in an annealing metal ([59], [60], [53]). Recently, mean curvature flow was used to explain the non-coalescence of oppositely charged fluid droplets ([58], [7], [39]).

The simplest examples of solutions to the mean curvature flow are the homothetic solutions. In [43], Huisken derived a monotonicity formula for the mean curvature flow, which he used to show that the blow-up of a solution to mean curvature flow at a type I singularity behaves asymptotically like a self-shrinking solution (also see [46]). The self-shrinking solutions to the mean curvature flow are ancient solutions to the flow, which correspond to surfaces, called *self-shrinkers*, that satisfy a quasilinear elliptic equation.

Self-shrinkers are important as they provide precious examples of solutions to mean curvature flow and they also characterize the asymptotic behavior of the flow at a given singularity. The self-shrinker equation is a quasilinear eigenvalue type equation involving the mean curvature. More precisely, an immersion F from an n -dimensional

manifold M into \mathbb{R}^{n+1} is a self-shrinker if

$$\Delta_g F = -\frac{1}{2}F^\perp, \quad (1.1)$$

where g is the metric on M induced by the immersion, Δ_g is the Laplace-Beltrami operator, and \perp is projection into the normal space of $F(M)$. The classification and construction of self-shrinkers has been an active topic in geometric analysis since the 1980's.

In Chapter 2, we introduce the equations for self-shrinkers and self-shrinkers with rotational symmetry, and we present some preliminary results from [24] and [27]. An arclength parametrized curve $\Gamma(s) = (x(s), r(s))$ in the upper half plane $\{(x, r) : x \in \mathbb{R}, r > 0\}$ is the profile curve of a rotational self-shrinker if and only if the angle $\alpha(s)$ solves

$$\dot{\alpha}(s) = \frac{x(s)}{2} \sin \alpha(s) + \left(\frac{n-1}{r(s)} - \frac{r(s)}{2} \right) \cos \alpha(s), \quad (1.2)$$

where $\dot{x}(s) = \cos \alpha(s)$ and $\dot{r}(s) = \sin \alpha(s)$. This differential equation is the geodesic equation for the conformal metric $g_{Ang} = r^{2(n-1)} e^{-(x^2+r^2)/2} (dx^2 + dr^2)$ on the upper half plane ([3], pp. 7-9). In the final three sections of Chapter 2, we study the shooting problem (1.2) from the axis of rotation: $\Gamma_\xi(0) = (\xi, 0)$, $\dot{\Gamma}(0) = (0, 1)$. We show that this problem is well-defined, and we prove comparison results for solutions, which involves the study of a Legendre type differential equation.

Examples of self-shrinkers in \mathbb{R}^{n+1} , $n \geq 2$, include the sphere of radius $\sqrt{2n}$, the plane, and the cylinder of radius $\sqrt{2(n-1)}$. In 1989, Angenent [3] constructed an embedded torus ($S^1 \times S^{n-1}$) self-shrinker and provided numerical evidence for the existence of an immersed and non-embedded sphere self-shrinker. In 1994, Chopp [12] provided numerical evidence for the existence of a number of self-shrinkers, including compact, embedded self-shrinkers of genus 5 and 7. Recently, Nguyen [54]–[56] and Kapouleas, Kleene, and Møller [49] used desingularization constructions to produce examples of complete, non-compact, embedded self-shrinkers with high genus in \mathbb{R}^3 . Møller [52] also used desingularization techniques to construct compact, embedded,

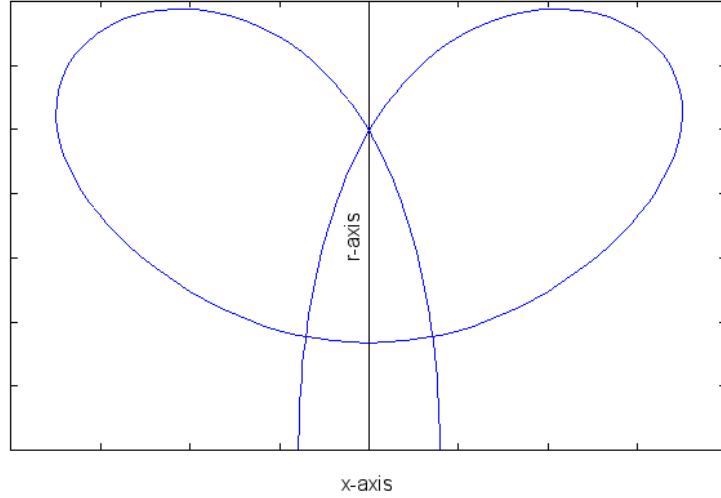


Figure 1.1: A geodesic whose rotation about the x -axis is an immersed sphere self-shrinker.

high genus self-shrinkers in \mathbb{R}^3 . In [50], Kleene and Møller constructed a family of non-compact, asymptotically conical ends with rotational symmetry, called trumpets, that interpolate between the plane and the cylinder.

In Chapter 3, we construct an immersed and non-embedded S^n self-shrinker in \mathbb{R}^{n+1} , $n \geq 2$, with a rotational symmetry.

Theorem. [24] *For $n \geq 2$, there exists an immersion $F : S^n \rightarrow \mathbb{R}^{n+1}$ satisfying $\Delta_g F = -\frac{1}{2}F^\perp$, and F is not an embedding.*

Numerical evidence for the existence of self-shrinkers dates back to the 1980's ([3], [12]); however, this is the first rigorous construction of an immersed S^n self-shrinker different from the “round sphere” of radius $\sqrt{2n}$. The existence of an S^2 self-shrinker different from the round sphere shows that the uniqueness results for constant mean curvature spheres in \mathbb{R}^3 (see Hopf [40]) and for minimal spheres in S^3

(see Almgren [2]) do not hold for self-shrinkers. One of the challenges of constructing complete self-shrinkers with rotational symmetry is that the geodesic flow for this metric seems to be nonintegrable (see [4], p. 5).¹ The basic idea of the construction is to study the shooting problem (1.2) with $\Gamma_\xi(0) = (\xi, 0)$ and $\dot{\Gamma}_\xi(0) = (0, 1)$. The goal is to find $\xi_0 > 0$ so that the geodesic Γ_{ξ_0} crosses over the r -axis, turns around, and intersects the r -axis perpendicularly. Then, using the symmetry of (1.2) with respect to reflections across the r -axis, the geodesic Γ_{ξ_0} will travel along its reflection until it intersects the x -axis at the point $(-\xi_0, 0)$, and the rotation of Γ_{ξ_0} about the x -axis will be an immersed and non-embedded sphere self-shrinker (see Figure 1.1). Writing the profile curve of a rotational self-shrinker as a graph over the r -axis, the differential equation (1.2) takes the form of a second order quasilinear eigenvalue type equation, and the construction follows from a detailed analysis of this equation. Precise estimates on the behavior of Γ_ξ are established for small $\xi > 0$, including direct arguments that show how Γ_ξ crosses the r -axis, turns around, and crosses back over the r -axis. A continuity argument is then used to find the desired $\xi_0 \in (0, \sqrt{2n})$. The direct crossing arguments are different from the limiting argument used in [3], and they can also be used to give a different construction of Angenent's torus self-shrinker.

In Chapter 4, in joint work with Stephen Kleene, we construct an infinite number of complete, immersed self-shrinkers with rotational symmetry for each of the rotational topological types: the sphere, the plane, the cylinder, and the torus.

Theorem. [27] *There are infinitely many complete, immersed self-shrinkers in \mathbb{R}^{n+1} , $n \geq 2$, for each of the following topological types: the sphere (S^n), the plane (\mathbb{R}^n), the cylinder ($\mathbb{R} \times S^{n-1}$), and the torus ($S^1 \times S^{n-1}$).*

As preparation for the construction, we first give a detailed description of the basic shape and limiting properties of the geodesics for the metric g_{Ang} . We prove

¹In the one-dimensional case, the self-shrinking solutions to the curve shortening flow have been completely classified (see Gage and Hamilton [34], Grayson [36], Abresch and Langer [1], Epstein and Weinstein [32], and Halldorsson [37]).

that the Euclidean curvature of a non-degenerate geodesic segment, written as a graph over the axis of rotation, can vanish at no more than two points, and we also show the different ways in which a family of geodesic segments can converge to a geodesic that exits the upper half plane.² Then, after establishing the asymptotic behavior of geodesics near the plane, the cylinder, and Angenent's torus, we use induction arguments to construct infinitely many self-shrinkers near each of these self-shrinkers. A new feature of the construction is the use of the Gauss-Bonnet formula to control the shapes of geodesics that almost exit the upper half plane.

1.2 Rigidity results

In contrast to the known examples are several rigidity theorems for self-shrinkers. In 1990, Huisken [43] showed that the sphere of radius $\sqrt{2n}$ is the only compact, mean-convex self-shrinker in \mathbb{R}^{n+1} , $n \geq 2$. In their recent study of generic singularities of the mean curvature flow, Colding and Minicozzi [15] showed that the only F -stable self-shrinkers with polynomial volume growth in \mathbb{R}^{n+1} , $n \geq 2$, are the sphere of radius $\sqrt{2n}$ and the plane. In 1989, Ecker and Huisken [30] showed that an entire self-shrinker graph with polynomial volume growth must be a plane in their study of the mean curvature flow of entire graphs. Afterwards, Lu Wang [63] showed that an entire self-shrinker graph has polynomial volume growth.

In Chapter 5, we prove a rigidity result for embedded sphere self-shrinkers with rotational symmetry.

Theorem. [26] *For $n \geq 2$, an embedded S^n self-shrinker in \mathbb{R}^{n+1} with rotational symmetry must be the sphere of radius $\sqrt{2n}$ centered at the origin.*

In [26], we showed that an embedded sphere self-shrinker with rotational symmetry must be mean-convex, and the rigidity result followed from Huisken's theorem [43]

²A geodesic can only exit the upper half plane through the axis of rotation, along the plane or the cylinder, or along one of the trumpets constructed by Kleene and Møller in [50].

for compact, mean-convex self-shrinkers. The proof we give in Chapter 5 uses the comparison results from Chapter 2 to bypass some of the preparation in [26]. We note that in an independent work [50], Kleene and Møller showed that the sphere of radius $\sqrt{2n}$, the plane, and the cylinder of radius $\sqrt{2(n-1)}$ are the only embedded, rotationally symmetric self-shrinkers of their respective topological type.

In Chapter 6, we use maximum principle arguments to prove a rigidity result for entire, high codimension self-shrinker graphs under a convexity assumption on the angles between the tangent plane to the graph and the base n -plane.

Theorem. [25] *Let $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ be a solution of*

$$g^{ij} \frac{\partial^2 \Phi}{\partial x^i \partial x^j} - \frac{1}{2} (\nabla_{\mathbb{R}^n} \Phi \cdot x - \Phi) = 0, \quad (1.3)$$

where g is the induced metric on the graph of Φ . If the eigenvalues of $(d\Phi)^T d\Phi$ satisfy $\lambda_i \lambda_j \leq 1$, $i \neq j$,³ then Φ is linear.

The heuristic idea behind the proof is to study the angle $\theta_1 = \arctan \sqrt{\lambda_1}$, where λ_1 is the largest eigenvalue. Adapting a maximum principle argument from Chau, Chen, and Yuan [10] we show that θ_1 achieves its maximum so that λ_1 and the volume element \sqrt{g} are bounded. The quantity $\log \sqrt{g}$ satisfies an elliptic differential inequality involving the second fundamental form, and another application of the maximum principle argument from [10] shows that $\log \sqrt{g}$ is constant. An analysis of the elliptic differential inequality satisfied by $\log \sqrt{g}$ establishes the pointwise vanishing of the second fundamental form. We note that in an independent work [21], Ding and Wang proved this result and other rigidity results in their study of high codimension self-shrinkers.

Finally, in Chapter 7, in joint work with Peng Lu and Yu Yuan, we use maximum principle techniques to show that every complete, entire self-shrinking solution on

³Under the equivalent geometric condition $\theta_i + \theta_j \leq \frac{\pi}{2}$, $i \neq j$, the square of the distance function to the base n -plane in the Grassmanian $G_{n,k}$ is convex (see Jost and Xin [48]).

complex Euclidean space to the Kähler-Ricci flow must be generated from a quadratic potential.

Theorem. [28] *Suppose u is an entire smooth pluri-subharmonic solution on \mathbb{C}^m to the complex Monge-Ampère equation*

$$\ln \det(u_{\alpha\bar{\beta}}) = \frac{1}{2}x \cdot Du - u. \quad (1.4)$$

Assume the corresponding Kähler metric $g = (u_{\alpha\bar{\beta}})$ is complete. Then u is quadratic.

Assuming a specific completeness assumption on the decay of the metric this result was proved in [10]. Similar rigidity results for self-shrinking solutions to Lagrangian mean curvature flows were obtained in [41], [10], and [22]. The idea of the proof, as in [10], is to force the phase $\ln \det(u_{\alpha\bar{\beta}})$ to attain its global maximum at a finite point. The new observation is that the radial derivative of the phase, which is the negative of the scalar curvature of the metric, is non-positive, so that the phase achieves its maximum at the origin. Under the assumption of completeness, we give a direct elliptic argument for the non-negativity of the scalar curvature.⁴ Since the phase satisfies an elliptic equation without zeroth order terms, the strong maximum principle establishes the constancy of the phase, and the homogeneity of the self-similar terms on the right hand side of (1.4) leads to the quadratic conclusion for the potential.

⁴The non-negativity of the scalar curvature was proved by B.-L. Chen [11], as the induced metric $u_{\alpha\bar{\beta}}(x/\sqrt{-t})$ is a complete ancient solution to the Ricci flow.

Chapter 2

**PRELIMINARY RESULTS FOR SELF-SHRINKERS
WITH ROTATIONAL SYMMETRY**

In this chapter, we introduce the self-shrinker equation and prove preliminary results for self-shrinkers with rotational symmetry.

2.1 The self-shrinker equation

An immersion F from an n -dimensional manifold M into \mathbb{R}^{n+1} is a self-shrinker if it satisfies the quasilinear elliptic equation

$$\Delta_g F = -\frac{1}{2}F^\perp, \quad (2.1)$$

where g is the metric on M induced by the immersion, Δ_g is the Laplace-Beltrami operator, and \perp is projection into the normal space of $F(M)$. Writing $F(M)$ locally at a point $F(p)$ as a graph over its tangent plane shows that $\Delta_g F(p)$ is normal to $F(M)$ at $F(p)$. It follows that

$$\Delta_g F = \langle \Delta_g F, \mathbf{N} \rangle \mathbf{N} = \langle g^{ij} F_{x_i x_j}, \mathbf{N} \rangle \mathbf{N} = g^{ij} A_{ij} \mathbf{N},$$

where x_i are local coordinates and $A_{ij} = \langle F_{x_i x_j}, \mathbf{N} \rangle$ is the second fundamental form. Therefore, the mean curvature of $F(M)$ is given by $\Delta_g F$, and we see that (2.1) is an equation of mean curvature type.

Examples of self-shrinkers in \mathbb{R}^{n+1} include the sphere of radius $\sqrt{2n}$ centered at the origin, the plane through the origin, the cylinder with an axis through the origin and radius $\sqrt{2(n-1)}$, and an embedded torus ($S^1 \times S^{n-1}$) constructed by Angenent in [3]. When F is a self-shrinker, the family of submanifolds $M_t = \sqrt{-t}F(M)$ is a solution to the mean curvature flow for $t \in (-\infty, 0)$. It is a consequence of Huisken's

monotonicity formula [43] that a solution to the mean curvature flow behaves asymptotically like a self-shrinker at a type I singularity (also see [46]).

Self-shrinkers can also be viewed as minimal surfaces for the conformal metric $e^{-|x|^2/(2n)}(dx_1^2 + \dots + dx_{n+1}^2)$ on \mathbb{R}^{n+1} . Let $F : M \rightarrow \mathbb{R}^{n+1}$ be an immersion, and consider a normal variation $F(p, \varepsilon) = F(p) + \varepsilon f(p)\mathbf{N}(p)$. Computing the first variation, we have

$$\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \int_M e^{-\frac{|F(p,\varepsilon)|^2}{4}} d\text{vol}_{g(\varepsilon)} = \int_M f \left[-\frac{1}{2} \langle F, \mathbf{N} \rangle - H \right] d\text{vol}_g,$$

where $H = \langle \Delta_g F, \mathbf{N} \rangle$ is the mean curvature. It follows that F is a self-shrinker if and only if it is a minimal surface for the conformal metric $e^{-|x|^2/(2n)}(dx_1^2 + \dots + dx_{n+1}^2)$.

If F is a surface of revolution: $F(s, \omega) = x(s)e_1 + r(s)\omega$, where $e_1 = (1, 0, \dots, 0)$ and $\omega \in S^{n-1}$, then computing the above first variation (which can be reduced to the case where $f = f(s)$) shows that F is a self-shrinker if and only if

$$\frac{1}{2}(x\dot{r} - r\dot{x}) + \frac{n-1}{r}\dot{x} + \frac{-\dot{x}\ddot{r} + \dot{r}\ddot{x}}{\dot{x}^2 + \dot{r}^2} = 0. \quad (2.2)$$

Comparing this to the geodesic equation for the metric $g_{Ang} = r^{2(n-1)}e^{-\frac{x^2+r^2}{2}}(dx^2 + dr^2)$ on the upper half plane $\mathbb{H} = \{r > 0\}$, we see that $(x(s), r(s))$ is the profile curve of a self-shrinker with rotational symmetry if and only if it is a geodesic for this metric.

2.2 The differential equations

It follows from equation (2.2) that an arclength parametrized curve $\Gamma(s) = (x(s), r(s))$ is the profile curve of a self-shrinker if and only if the angle $\alpha(s)$ solves

$$\dot{\alpha}(s) = \frac{x(s)}{2} \sin \alpha(s) + \left(\frac{n-1}{r(s)} - \frac{r(s)}{2} \right) \cos \alpha(s), \quad (2.3)$$

where $\dot{x}(s) = \cos \alpha(s)$ and $\dot{r}(s) = \sin \alpha(s)$. This is the geodesic equation for the conformal metric $g_{Ang} = r^{2(n-1)}e^{-(x^2+r^2)/2}(dx^2 + dr^2)$ on $\mathbb{H} = \{(x, r) : x \in \mathbb{R}, r > 0\}$. For $(x_0, r_0) \in \mathbb{H}$ and $\alpha_0 \in \mathbb{R}$, we let $\Gamma[x_0, r_0, \alpha_0]$ denote the unique solution to (2.3)

satisfying

$$\Gamma[x_0, r_0, \alpha_0](0) = (x_0, r_0), \quad \dot{\Gamma}[x_0, r_0, \alpha_0](0) = (\cos(\alpha_0), \sin(\alpha_0)),$$

and we define $\underline{\Gamma}$ to be the space of all curves $\Gamma[x_0, r_0, \alpha_0]$.

There are several particular curves of interest belonging to $\underline{\Gamma}$, namely the embedded ones. The known embedded curves are the semi-circle $\sqrt{2n}(\cos(s), \sin(s))$, the lines $(0, s)$ and $(s, \sqrt{2(n-1)})$, and a closed convex curve discovered by Angenent in [3]. We will refer to these curves as the sphere, the plane, the cylinder, and Angenent's torus (since the rotations of these curves about the x -axis respectively generate a sphere S^n , a plane \mathbb{R}^n , a cylinder $\mathbb{R} \times S^{n-1}$, and a torus $S^1 \times S^{n-1}$).

It will be useful to view a curve $\Gamma \in \underline{\Gamma}$ from three different perspectives: as a graph $(x, u(x))$ over the x -axis, as a graph $(f(r), r)$ over the r -axis, and as a geodesic for the metric g_{Ang} . The differential equations satisfied by $u(x)$ and $f(r)$ place limitations on the oscillatory behavior of Γ , and we will use these equations to describe the basic shape of the curves in $\underline{\Gamma}$. In addition, we will use the continuity properties of geodesics and the Gauss-Bonnet formula to establish convergence properties for the curves in $\underline{\Gamma}$.

When $\Gamma \in \underline{\Gamma}$ is given as $(x, u(x))$, the function $u(x)$ satisfies the differential equation

$$\frac{u''}{1 + (u')^2} = \frac{xu'}{2} - \frac{u}{2} + \frac{n-1}{u}. \quad (2.4)$$

This equation can be derived either directly from (2.1) or by using the geodesic equation (2.3). Differentiating (2.4), we have

$$\frac{u'''}{1 + (u')^2} = \frac{2u'(u'')^2}{(1 + (u')^2)^2} + \frac{xu''}{2} - \frac{n-1}{u^2}u'. \quad (2.5)$$

Similarly, when Γ is given as $(f(r), r)$, we have

$$\frac{f''}{1 + (f')^2} = \left(\frac{r}{2} - \frac{n-1}{r} \right) f' - \frac{f}{2} \quad (2.6)$$

and

$$\frac{f'''}{1+(f')^2} = \frac{2f'(f'')^2}{(1+(f')^2)^2} + \left(\frac{r}{2} - \frac{n-1}{r}\right) f'' + \frac{n-1}{r^2} f'. \quad (2.7)$$

We note that applying the Gauss-Bonnet formula (see [23], p.274) to a simple, compact region R in \mathbb{H} whose boundary is the piecewise smooth union of geodesic segments with external angles $\theta_0, \dots, \theta_k$, gives the formula

$$\int_R \left(1 + \frac{n-1}{r^2}\right) dx dr = 2\pi - \sum_{i=0}^k \theta_i. \quad (2.8)$$

2.3 Basic shape of a geodesic written as a graph over the x -axis

Let u be a solution to (2.4). If u has a local maximum (minimum) at a point x_0 , then $u(x_0) \geq \sqrt{2(n-1)}$ ($\leq \sqrt{2(n-1)}$) with equality if and only if $u \equiv \sqrt{2(n-1)}$ is the cylinder. Also, if both u' and u'' vanish at the same point, then u must be the cylinder. Using (2.5), when u is not the cylinder, we see that u' and u''' have opposite signs at points where $u'' = 0$, so that the zeros of u'' are separated by zeros of u' . It follows that a solution to (2.4) is either the cylinder, or it has a sinusoidal shape that oscillates between maxima above the cylinder and minima below the cylinder. The following result uses the oscillatory nature of solutions to (2.4) to show that a maximally extended solution must intersect the r -axis.

Proposition 1. *A maximally extended solution to (2.4) intersects the r -axis.*

Proof. Let $u : (a, b) \rightarrow \mathbb{R}$ be a maximally extended solution to (2.4). Suppose that $b < \infty$. We claim that u cannot oscillate too much near b . To see this, suppose to the contrary that u'' vanishes in every neighborhood of b . Then there exists an increasing sequence $x_k \rightarrow b$ that alternates between maxima and minima of u . Applying the continuity of the differential equation (2.4) to the cylinder solution, we see that there is $\varepsilon > 0$ so that $|u(x_k) - \sqrt{2(n-1)}| > \varepsilon$; otherwise, we can extend u past b . Then the graph of $u(x)$ contains geodesic segments (defined as graphs over the r -axis on the fixed neighborhood $|r - \sqrt{2(n-1)}| < \varepsilon$) that converge to the curve $\Gamma[b, \sqrt{2(n-1)}, \pi/2]$.

When $b \neq 0$, this forces the graph of $u(x)$ to become non-graphical (near b), and when $b = 0$ this forces u to extend past b (see Lemma 1). Thus, we have shown there is a neighborhood of b in which u'' does not vanish.

Next, we show that $b > 0$. We know that u'' does not vanish in a neighborhood of b . Examining equation (2.4) and using Lemma 2, we see that $u'(x)$ and $u''(x)$ must have the same sign when x is near b . If $\lim_{x \rightarrow b} u(x) \in (0, \infty)$, then $\lim_{x \rightarrow b} |u'(x)| = \infty$ (since u is maximally extended), and it follows from (2.4) that $b \geq 0$. In fact, $b > 0$, since the graph of u is not the plane. If $\lim_{x \rightarrow b} u(x)$ is 0 (or ∞), then u' and u'' are both negative (or both positive), and the term $\frac{n-1}{u} - \frac{u}{2}$ has the correct sign to force $b > 0$. Similar arguments may be applied to the left end point a to complete the proof of the lemma. \square

The following two lemmas were used in the proof of Proposition 1.

Lemma 1. *Let $f_k(r)$ be a sequence of maximally extended solutions to (2.6) defined on the neighborhood $|r - \sqrt{2(n-1)}| \leq \varepsilon$, for some $\varepsilon > 0$. Suppose $f_k(\sqrt{2(n-1)})$ is an increasing sequence that converges to $b < \infty$, and $f'_k(\sqrt{2(n-1)}) \rightarrow 0$. If $b \neq 0$, then the graph of f_k cannot be written as a graph over the x -axis for k sufficiently large. If $b = 0$, then f_k must vanish at some point for k sufficiently large.*

Lemma 2. *Let u be a solution to (2.4) defined on a finite interval (x_1, x_2) . If $u' < 0$ and $u'' > 0$ on (x_1, x_2) , then $\lim_{x \rightarrow x_2} u(x) > 0$.*

Before we prove Lemma 1 and Lemma 2, we prove some properties of solutions to (2.6).

Lemma 3. *There exists $M_1 > \sqrt{2(n-1)}$ with the following property: Let f be a solution to (2.6) with $f(\sqrt{2(n-1)}) > 0$. Suppose $f'(r) \leq 0$ when $r \geq \sqrt{2(n-1)}$. Then $f(r) < 0$ whenever $r > M_1$ and $f(r)$ is defined.*

Proof. Notice that $f''(r) < 0$ when $r \geq \sqrt{2(n-1)}$, $f(r) > 0$, and $f'(r) \leq 0$. Also, $f'''(r) \leq 0$ when $r \geq \sqrt{2(n-1)}$, $f'(r) \leq 0$, and $f''(r) < 0$. The idea of the proof is

to use this concave down behavior to force f to be negative when r is large enough. Choose $r > \sqrt{2(n-1)}$ so that $f > 0$ on $[\sqrt{2(n-1)}, r]$. Then

$$f''(r) \leq f''(\sqrt{2(n-1)}) \leq \frac{f''(\sqrt{2(n-1)})}{1 + f'(\sqrt{2(n-1)})^2} = -\frac{1}{2}f(\sqrt{2(n-1)}),$$

where we have used $f''' \leq 0$ on $[\sqrt{2(n-1)}, r]$, $f''(\sqrt{2(n-1)}) < 0$, and equation (2.4).

Integrating twice from $\sqrt{2(n-1)}$ to r , we have

$$f(r) \leq f(\sqrt{2(n-1)}) \left[1 - \frac{1}{4}(r - \sqrt{2(n-1)})^2 \right].$$

Choose $M_1 = 2 + \sqrt{2(n-1)}$. Then $f(r) < 0$ whenever $r > M_1$ and $f(r)$ is defined. \square

Next, we prove a lemma about solutions to (2.6), which shows that a positive, increasing, concave down solution cannot be defined on an interval of the form $(m, \sqrt{2(n-1)}]$, for arbitrarily small m .

Lemma 4. *There exists $m_1 > 0$ with the following property: Let f be a solution to (2.6) with $f(\sqrt{2(n-1)}) > 0$. Suppose $f'(r) > 0$ and $f''(r) < 0$ when $r < \sqrt{2(n-1)}$. Then $f(r) < 0$ whenever $r < m_1$ and $f(r)$ is defined.*

Proof. The idea of the proof is to use the f'/r term to force f to be negative when r is small. We break the proof up into two steps.

Step 1: Estimate f' in terms of f at some point less than $\sqrt{2(n-1)}$. Without loss of generality, we assume that $f(1)$ is defined and positive. Using equation (2.7), we see that $f'''(r) > 0$ when $r < \sqrt{2(n-1)}$ (since $f'(r) > 0$ and $f''(r) < 0$). Then, for $r < \sqrt{2(n-1)}$, we see that $f''(r) \leq f''(\sqrt{2(n-1)})$. Using equation (2.6) and the positivity of f' , we have $f''(\sqrt{2(n-1)}) \leq -\frac{1}{2}f(\sqrt{2(n-1)}) \leq -\frac{1}{2}f(1)$. Therefore, $f''(r) \leq -\frac{1}{2}f(1)$. Integrating from 1 to $\sqrt{2(n-1)}$, we arrive at the estimate

$$f'(1) \geq \frac{\sqrt{2(n-1)} - 1}{2} f(1).$$

Step 2: Estimate $f(r)$ for $r < 1$. Suppose $f > 0$ on $[r, 1]$. Then using $f' > 0$, $f'' < 0$, and (2.6), we have

$$\frac{f''(r)}{f'(r)} \leq -\frac{n-1}{r}.$$

Integrating from r to 1,

$$f'(r) \geq \frac{f'(1)}{r^{n-1}} \geq c_n \frac{f(1)}{r^{n-1}},$$

and integrating again

$$f(r) \leq f(1) \left[1 - c_n \int_r^1 \frac{1}{t^{n-1}} dt \right].$$

Choose m_1 so that $\int_{m_1}^1 \frac{1}{t^{n-1}} dt \geq 1/c_n$. Then $f(r) < 0$ whenever $r < m_1$ and $f(r)$ is defined. \square

Now we prove Lemma 1 and Lemma 2.

Proof of Lemma 1. Let $f_k(r)$ be a sequence of solutions to (2.6) defined on the neighborhood $|r - \sqrt{2(n-1)}| \leq \varepsilon$. Suppose $f_k(\sqrt{2(n-1)})$ is an increasing sequence that converges to $b < \infty$, and $f'_k(\sqrt{2(n-1)}) \rightarrow 0$. Let $f(r)$ denote the solution to (2.6) with $f(\sqrt{2(n-1)}) = b$ and $f'(\sqrt{2(n-1)}) = 0$. Then f_k converges smoothly to f , and we note that $f''(\sqrt{2(n-1)}) = -b/2$.

If $b \neq 0$, then $f''(\sqrt{2(n-1)}) \neq 0$, and for sufficiently large k , we have $f''_k \neq 0$ in a neighborhood of $\sqrt{2(n-1)}$ and $f'_k(r_k) = 0$ for some r_k near $\sqrt{2(n-1)}$. Therefore, f_k cannot be written as a graph over the x -axis for k sufficiently large.

If $b = 0$, then $f \equiv 0$, and for sufficiently large k , we may assume that the domain of f_k contains the interval $[m_1, M_1]$. Now, depending on the sign of $f'_k(\sqrt{2(n-1)})$, either $f'_k(r) \geq 0$ on $r \geq \sqrt{2(n-1)}$, or $f'_k(r) < 0$ and $f''_k(r) < 0$ on $r \leq \sqrt{2(n-1)}$. Applying Lemma 3 and Lemma 4, we conclude that f_k crosses the r -axis for k sufficiently large. \square

Proof of Lemma 2. It is sufficient to show that a solution $f(r)$ to (2.6) defined on (r_1, r_2) with $f \leq M$, $f' < 0$, and $f'' > 0$ satisfies $r_1 > 0$. Let $\alpha(r) = -(\pi/2 + \arctan f'(r))$ so that $\alpha(r) \in (-\pi/2, 0)$ and

$$\frac{d}{dr}(\log \cos \alpha(r)) = \frac{r}{2} - \frac{n-1}{r} - \frac{f(r)}{2f'(r)} \leq \frac{r}{2} - \frac{n-1}{r} + \frac{M}{2(-f'(r_2))}.$$

Integrating from r_1 to r_2 ,

$$\log \left(\frac{\cos \alpha(r_2)}{\cos \alpha(r_1)} \right) \leq \frac{(r_2)^2}{4} + (n-1) \log \left(\frac{r_1}{r_2} \right) + \frac{Mr_2}{2(-f'(r_2))}.$$

Therefore,

$$r_1 \geq r_2 \left[\cos \alpha(r_2) e^{-\frac{(r_2)^2}{4} + \frac{Mr_2}{2f'(r_2)}} \right]^{1/(n-1)}.$$

□

2.4 Basic shape of a geodesic written as a graph over the r -axis

In this section, we study the basic shape of the profile curve of a self-shrinker with rotational symmetry when it is written as a graph over the r -axis. Let f be a solution to (2.6). If f has a local maximum (minimum) at a point r_0 , then $f(r_0) \geq 0$ (≤ 0) with equality if and only if $f \equiv 0$ is the plane. Also, if both f' and f'' vanish at the same point, then f must be the plane. Using (2.7), when f is not the plane, we see that f' and f''' have the same sign at points where $f'' = 0$, and this restricts the oscillatory behavior of f . The following results place limitations on the oscillation of f . In fact, the concavity of f can change at most once.

Lemma 5. *Let f be a solution to (2.6). Suppose $f''(r_0) \leq 0$ and $f'(r_0) < 0$. Then $f''(r) < 0$ for $r > r_0$.*

Proof. First, we claim that $f''(r) < 0$ for r close to and greater than r_0 . When $f''(r_0) < 0$, this follows from continuity. When $f''(r_0) = 0$, we use equation (2.7) and the assumption $f'(r_0) < 0$ to see that $f'''(r_0) < 0$, and the claim follows.

Now, suppose to the contrary that $f''(r) = 0$ for some $r > r_0$. Let r_1 be the first r greater than r_0 for which $f''(r) = 0$. Then $f''(r) < 0$ for $r \in (r_0, r_1)$, $f'(r_1) < 0$, and $f'''(r_1) \geq 0$. However, it follows from (2.7) that $f'''(r_1) < 0$, which is a contradiction. Therefore, $f''(r) < 0$ for $r > r_0$. □

Lemma 6. *Let f be a solution to (2.6). Suppose $f'(r_0) = 0$ and $f(r_0) > 0$. Then $f'' < 0$.*

Proof. It follows from (2.6) that $f''(r_0) < 0$ (we will see in the next section that this also holds when $r_0 = 0$). Then $f'(r) < 0$ and $f''(r) < 0$ for r close to and greater than r_0 , and it follows from Lemma 5 that $f''(r) < 0$ when $r > r_0$. Arguing as we did in Lemma 5, suppose $f''(r) = 0$ for some $r < r_0$, and let r_1 be the first such r less than r_0 . Then $f''(r) < 0$ for $r \in (r_1, r_0)$, $f'(r_1) > 0$, and $f'''(r_1) < 0$. However, it follows from (2.7) that $f'''(r_1) > 0$, which is a contradiction. Therefore, we have $f''(r) < 0$ for $r < r_0$, which completes the proof. \square

2.5 Shooting from the axis of rotation

In this section we study solutions to (2.3) with $\Gamma(0) = (x_0, 0)$ and $\dot{\Gamma}(0) = (0, 1)$. Though the metric g_{Ang} and (2.3) are degenerate at the boundary $\{r = 0\}$, we will show in Proposition 2 that there is still a smooth one parameter family of solutions to the shooting problem:

$$Q[x_0](0) = (x_0, 0), \quad \dot{Q}[x_0](0) = (0, 1). \quad (2.9)$$

We note that the geodesics $Q[\sqrt{2n}]$ and $Q[0]$ are the profile curves for the sphere and the plane, respectively. From Lemma 6, we know that the concavity of the first graphical component of $Q[x_0]$, written as a graph over the r -axis, does not change. In fact, we will show that a solution to (2.6) with $f(0) > 0$ is concave down, $\lim_{r \rightarrow r_*} f'(r) = -\infty$ for some $r_* < \infty$, and the limit $\lim_{r \rightarrow r_*} f(r)$ is finite.

We begin this section by showing the initial value problem (2.9) is well-defined.

Proposition 2. *For any $b \in \mathbb{R}$, there exists $A = A(b) > 0$ and a unique analytic function f defined on $[0, 1/A]$ so that $f(0) = b$, $f'(0) = 0$, and f is a solution to (2.6). Moreover, f and f' depend continuously on b .*

Proof. We mention two proofs of the proposition. First, using a power series argument specific to the equation (2.6), we established the existence, uniqueness, and continuity results as stated in the proposition. This argument is presented here. Afterwards,

Robin Graham informed us of the general reference [6], where the Cauchy problem for singular systems of partial differential equations was studied. Applying Theorem 2.2 from [6] also shows that (2.6) has a unique analytic solution in a neighborhood of 0. We would like to thank Robin Graham for this reference.

To prove the proposition, we look for solutions of the form:

$$f(r) = \sum_{i=0}^{\infty} a_i r^i.$$

If we assume that we can differentiate the power series term by term so that

$$f'(r) = \sum_{i=0}^{\infty} (i+1)a_{i+1}r^i$$

and

$$f''(x) = \sum_{i=0}^{\infty} (i+2)(i+1)a_{i+2}r^i,$$

then the condition that f satisfies (2.6):

$$f'' = -\frac{1}{2}f + \frac{1}{2}rf' - \frac{n-1}{r}f' - \frac{1}{2}f(f')^2 + \frac{1}{2}r(f')^3 - \frac{n-1}{r}(f')^3,$$

is a condition on the coefficients $\{a_i\}$. Namely, $a_0 = f(0)$, $a_1 = 0$, and

$$\begin{aligned} (m+2)(m+1)a_{m+2} &= -\frac{1}{2}a_m + \frac{1}{2}ma_m - (n-1)(m+2)a_{m+2} \\ &\quad - \frac{1}{2} \sum_{i+j+k=m} (i+1)(j+1)a_{i+1}a_{j+1}a_k \\ &\quad + \frac{1}{2} \sum_{i+j+k=m-1} (i+1)(j+1)(k+1)a_{i+1}a_{j+1}a_{k+1} \\ &\quad - (n-1) \sum_{i+j+k=m+1} (i+1)(j+1)(k+1)a_{i+1}a_{j+1}a_{k+1}. \end{aligned}$$

The previous equation simplifies to:

$$\begin{aligned} (m+2)(m+n)a_{m+2} &= \frac{1}{2}(m-1)a_m - \frac{1}{2}a_0 \sum_{i+j=m} (i+1)(j+1)a_{i+1}a_{j+1} \quad (2.10) \\ &\quad + \frac{1}{2} \sum_{i+j+k=m-1} (i+1)(j+1)k \cdot a_{i+1}a_{j+1}a_{k+1} \\ &\quad - (n-1) \sum_{i+j+k=m+1} (i+1)(j+1)(k+1)a_{i+1}a_{j+1}a_{k+1}. \end{aligned}$$

Claim 1. $a_{2m+1} = 0$

Proof. This follows from the above formula for a_m and induction. We know $a_1 = 0$. Suppose $a_{2i+1} = 0$ for all $i < m$ and consider a_{2m+1} . Every term in the expression for a_{2m+1} contains a term of the form a_{2i+1} , and thus $a_{2m+1} = 0$, which proves the claim. \square

In order to construct a solution of (2.6) we need $\sum a_i r^i$ to be a convergent power series, and hence we need an estimate on the coefficients a_{2m} .

Claim 2. For each $M > 0$, there exists $A = A(M) > 0$ so that if $|a_0| \leq M$, then

$$|a_{2m}| \leq \frac{A^{2m-1}}{(2m)^3}.$$

Proof. Fix $M > 0$, and use (2.10) to choose $A > 1$ so that the estimate holds for $m = 1, 2, \dots, 4n$. Arguing inductively, suppose $a_{2i} \leq \frac{A^{2i-1}}{(2i)^3}$ when $i \leq m$, and consider a_{2m+2} . From (2.10), we have

$$\begin{aligned} (2m+2)(2m+n)a_{2m+2} &= \frac{1}{2}(2m-1)a_{2m} \\ &\quad - \frac{1}{2}a_0 \sum_{\substack{i+j=2m \\ i,j \text{ odd}}} (i+1)(j+1)a_{i+1}a_{j+1} \\ &\quad + \frac{1}{2} \sum_{\substack{i+j+k=2m-1 \\ i,j,k \text{ odd}}} (i+1)(j+1)k \cdot a_{i+1}a_{j+1}a_{k+1} \\ &\quad - (n-1) \sum_{\substack{i+j+k=2m+1 \\ i,j,k \text{ odd}}} (i+1)(j+1)(k+1)a_{i+1}a_{j+1}a_{k+1}. \end{aligned} \tag{2.11}$$

To estimate the terms with sums, we need the following inequality:

$$\sum_{\substack{i+j=2N \\ i,j \text{ odd}}} \frac{1}{(i+1)^2} \frac{1}{(j+1)^2} \leq \frac{2}{(2N+2)^2}. \tag{2.12}$$

This inequality follows from the identity:

$$\sum_{\substack{i+j=2N \\ i,j \text{ odd}}} \frac{1}{(i+1)^2} \frac{1}{(j+1)^2} = \frac{2}{(2N+2)^2} \sum_{\substack{i=1 \\ i \text{ odd}}}^{2N-1} \frac{1}{(i+1)^2} + \frac{4}{(2N+2)^3} \sum_{\substack{i=1 \\ i \text{ odd}}}^{2N-1} \frac{1}{i+1}.$$

Applying (2.12) twice, we have the following inequality:

$$\sum_{\substack{i+j+k=2N-1 \\ i,j,k \text{ odd}}} \frac{1}{(i+1)^2} \frac{1}{(j+1)^2} \frac{1}{(k+1)^2} \leq \frac{4}{(2N+2)^2}. \quad (2.13)$$

Now, we can use (2.12), (2.13), and the inductive hypothesis to estimate each term on the right hand side of formula (2.11):

$$\begin{aligned} \frac{1}{2}(2m-1)|a_{2m}| &\leq \frac{1}{2} \frac{2m-1}{(2m)^3} A^{2m-1} \leq \left[\frac{m+1}{(2m)^2} \right] \frac{A^{2m+1}}{2m+2}, \\ \frac{1}{2} \left| a_0 \sum_{\substack{i+j=2m \\ i,j \text{ odd}}} (i+1)(j+1)a_{i+1}a_{j+1} \right| &\leq \left[\frac{1}{2m+2} \right] \frac{A^{2m+1}}{2m+2}, \\ \frac{1}{2} \left| \sum_{\substack{i+j+k=2m-1 \\ i,j,k \text{ odd}}} (i+1)(j+1)k \cdot a_{i+1}a_{j+1}a_{k+1} \right| &\leq \left[\frac{1}{m+1} \right] \frac{A^{2m+1}}{2m+2}, \\ (n-1) \left| \sum_{\substack{i+j+k=2m+1 \\ i,j,k \text{ odd}}} (i+1)(j+1)(k+1)a_{i+1}a_{j+1}a_{k+1} \right| &\leq \left[\frac{2(n-1)}{m+2} \right] \frac{A^{2m+1}}{2m+2}. \end{aligned}$$

Applying these estimates to (2.11) when $m \geq 4n$, we see that

$$\begin{aligned} (2m+2)(2m+n)|a_{2m+2}| &\leq \left[\frac{m+1}{(2m)^2} + \frac{1}{2m+2} + \frac{1}{m+1} + \frac{2(n-1)}{m+2} \right] \frac{A^{2m+1}}{2m+2} \\ &\leq \frac{A^{2m+1}}{2m+2}, \end{aligned}$$

so that $a_{2m+2} \leq \frac{A^{2m+1}}{(2m+2)^3}$, which proves the claim. \square

The estimates on the coefficients a_i imply that the power series $f(r) = \sum a_i r^i$ is an analytic function on $[0, 1/A]$. By the previous discussion, $f(r)$ is the unique analytic solution to (2.6) in $(0, 1/A]$. Furthermore, since the coefficients a_i depend continuously on a_0 , the solution $f(r)$ depends continuously on the initial height $f(0) = b$, which completes the proof of the proposition. \square

For the remainder of this section, we prove results about the behavior of solutions to (2.6) with $f(0) > 0$. As we mentioned above, it follows from Lemma 6 that $f'' < 0$. In [63], Lu Wang proved that an entire self-shrinker graph must be a plane. We will prove a generalization of this result in Chapter 6. It follows that f cannot be defined on all of $[0, \infty)$, and therefore by the existence theory for differential equations there must be a point $r_* < \infty$ so that f blows-up at x_* (blows-up in the sense that either $|f|$ or $|f'|$ goes to ∞ as r goes to r_*). Since $f'' < 0$ and $r_* < \infty$, it follows that $\lim_{r \rightarrow r_*} f'(r) = -\infty$.

Lemma 7. *Let f be a solution to (2.6) with $f(0) > 0$, and let r_* be the blow-up point of f . Then $\sqrt{2(n-1)} < r_* < \infty$.*

Proof. We know from the previous discussion that $r_* < \infty$. To see that $r_* > \sqrt{2(n-1)}$, we first suppose to the contrary that $r_* < \sqrt{2(n-1)}$. Then using (2.6), we see that $\lim_{r \rightarrow r_*} f''(r) = \infty$, which contradicts the fact that $f'' < 0$. Therefore, $r_* \geq \sqrt{2(n-1)}$. It now follows from the existence of the cylinder self-shrinker that r_* is strictly greater than $\sqrt{2(n-1)}$. To see this, suppose to the contrary that $r_* = \sqrt{2(n-1)}$. Since $f'' < 0$, we know that $f(r) > 0$ for $r \in [0, r_*)$, and therefore there exists $x_* \geq 0$ so that $\lim_{r \rightarrow r_*} f(r) = x_*$. Near the point $(\sqrt{2(n-1)}, x_*)$, we can write the geodesic corresponding to f as a graph over the x -axis of a function $u(x)$. Then u is a solution to (2.4) with $u(x_*) = \sqrt{2(n-1)}$ and $u'(x_*) = 0$. It follows that $u(x) \equiv \sqrt{2(n-1)}$, which contradicts the fact that f is defined for $r < \sqrt{2(n-1)}$. \square

Lemma 8. *Let f be a solution to (2.6) with $f(0) > 0$, and let r_* be the blow-up point of f . Then $\lim_{r \rightarrow r_*} f(r) > -\infty$.*

Proof. We know from Lemma 7 that $\sqrt{2(n-1)} < r_* < \infty$. Fix $0 < \delta < 1$ so that $r_* - \delta > \sqrt{2(n-1)}$, and let $m > 0$ be such that $(\frac{1}{2}r - \frac{n-1}{r}) \geq m$ when $r \in (r_* - \delta, r_*)$. Choose $M > 0$ so that $m \geq \frac{3}{2M^2}$ and $M \geq -f'(x_* - \delta)$.

For $\varepsilon > 0$, define $\phi_\varepsilon(r)$ on $(r_* - \delta, r_* - \varepsilon)$ by

$$\phi_\varepsilon(r) = \frac{M}{\sqrt{(r_* - \varepsilon) - r}}.$$

Then

$$\phi_\varepsilon''(r) = \frac{3}{2M^2} \phi_\varepsilon(r)^2 \phi_\varepsilon'(r) \leq \left(\frac{1}{2}r - \frac{n-1}{r} \right) \phi_\varepsilon(r)^2 \phi_\varepsilon'(r),$$

for $r \in (r_* - \delta, r_* - \varepsilon)$. We will use the function ϕ_ε to show that $-f'$ blows-up no faster than $M/\sqrt{r_* - r}$. Let $\psi(r) = -f'(r)$. Then $\psi \geq 0$, $\psi' > 0$, and by (2.7),

$$\psi''(r) \geq \left(\frac{1}{2}r - \frac{n-1}{r} \right) \psi(r)^2 \psi'(r),$$

when $r \geq \sqrt{2(n-1)}$. We will show $\psi \leq \phi_\varepsilon$. By construction,

$$\psi(r_* - \delta) \leq M < \phi_\varepsilon(r_* - \delta)$$

and

$$\psi(r_* - \varepsilon) < \lim_{r \rightarrow (r_* - \varepsilon)} \phi_\varepsilon(x).$$

Therefore, if $\psi > \phi_\varepsilon$ at some point, then $\psi - \phi_\varepsilon$ achieves a positive maximum at some point $r_0 \in (r_* - \delta, r_* - \varepsilon)$. This leads to $(\psi - \phi_\varepsilon)'(r_0) = 0$ and $(\psi - \phi_\varepsilon)''(r_0) \leq 0$. Consequently,

$$0 \geq (\psi - \phi_\varepsilon)''(r_0) \geq \left(\frac{1}{2}r_0 - \frac{n-1}{r_0} \right) \psi'(r_0) (\psi(r_0)^2 - \phi_\varepsilon(r_0)^2) > 0,$$

which is a contradiction.

It follows that $\psi \leq \phi_\varepsilon$ on $(r_* - \delta, r_* - \varepsilon)$. Taking $\varepsilon \rightarrow 0$, we conclude that

$$f'(r) \geq \frac{-M}{\sqrt{r_* - r}},$$

for $r \in (r_* - \delta, r_*)$. Therefore, $\lim_{r \rightarrow r_*} f(r) > -\infty$. \square

Remark 1. *At this point, we can give a basic description of solutions to the shooting problem: For $b > 0$, let f_b denote the solution to (2.6) with $f_b(0) = b$ and $f_b'(0) = 0$. Then f_b is decreasing and concave down, and there exists a point $r_*^b \in (\sqrt{2(n-1)}, \infty)$ so that f_b is defined on $[0, r_*^b)$ and $\lim_{r \rightarrow r_*^b} f_b'(r) = -\infty$. Furthermore, there exists a point $x_*^b \in (-\infty, b)$ so that $f_b(r_*^b) = x_*^b$.*

We end this section by showing that a solution to (2.6) with initial height less than $\sqrt{2n}$ must cross the x -axis before it reaches its blow-up point. A theorem of Huisken [42], [43] says that the sphere of radius $\sqrt{2n}$ is the only compact, convex self-shrinker. Combining Huisken's theorem with Lemma 3 and Remark 1, we have the following result.

Lemma 9. *Let f be a solution to (2.6) with $0 < f(0) < \sqrt{2(n-1)}$, and let r_* be the blow-up point of f . Then $f(r_*) < 0$.*

Proof. Let M_1 be the constant from Lemma 3. Using the continuity of the differential equation (2.6) and comparing solutions to the plane $f_0(r) \equiv 0$, there exists $\varepsilon > 0$ so that $r_* > M_1$ whenever f is a solution to (2.6) with $0 < f(0) < \varepsilon$. Applying Lemma 3, we see that $f(r_*) < 0$ whenever $0 < f(0) < \varepsilon$.

To see that the lemma holds for all $0 < f(0) < \sqrt{2n}$, suppose to the contrary that there exists some $b \in (0, \sqrt{2n})$ so that $f_b(r_*) \geq 0$. Let b_0 be the smallest $b > 0$ for which $f_b(r_*) \geq 0$. Then $\varepsilon \leq b_0 < \sqrt{2n}$ and $f_{b_0}(r_*^{b_0}) = 0$. It follows that the geodesic corresponding to f_{b_0} intersects the r -axis perpendicularly. By symmetry of (2.3), this geodesic reflects across the r -axis. Since $f_{b_0}'' < 0$, this geodesic is also a convex curve, and therefore it is the profile curve for a compact, convex self-shrinker. By Huisken's theorem for compact, convex self-shrinkers, we conclude that $b_0 = \sqrt{2n}$, which is a contradiction. \square

2.6 Comparison results

In this section, we prove comparison results for the geodesics $Q[x_0]$ defined by (2.9). The main application of these results is that $Q[x_0]$ first intersects the r -axis outside of the sphere when $0 < x_0 < \sqrt{2n}$, and it first intersects the r -axis inside the sphere when $x_0 > \sqrt{2n}$.

Let f and g be solutions to (2.6). We are interested in the shooting problem where $f'(0) = g'(0) = 0$ and $g(0) > f(0) > 0$. In particular, we will consider the case where

g is the sphere ($g(r) = \sqrt{2n - r^2}$) and f is an inner-quarter sphere. In this setting, we know from the previous section that f is decreasing, concave down, and it crosses the r -axis before its slope blows-up. We want to show that f crosses the r -axis outside of the sphere.

We will use the following identities at 0 for solutions to (2.6):

$$f'(0) = 0, \quad f''(0) = -\frac{1}{2n}f(0), \quad f'''(0) = 0,$$

$$f^{(iv)}(0) = -\frac{3}{4n(n+2)} \left(\frac{f(0)^3}{n^2} + f(0) \right).$$

These identities follow from the power series expansion in Proposition 2 or from l'Hôpital's rule applied to (2.6) and its derivatives.

Lemma 10. *If f is a solution to (2.6) with $f(0) < \sqrt{2n}$, then f must intersect the sphere before it crosses the r -axis.*

Proof. Suppose f does not intersect g (the sphere) before it crosses the r -axis. Let $r_0 > 0$ be the point where f crosses the r -axis. Then $g > f$ on $[0, r_0)$.

We consider the function $v = \frac{f}{g}$, which satisfies

$$v' = \frac{f' - vg'}{g}$$

and

$$v'' = \frac{f'' - vg''}{g} - 2\frac{g'}{g}v'.$$

Now, using l'Hôpital's rule and the above identities at 0, we have $v'(0) = 0$ and $v''(0) = 0$. Similarly, $v'''(0) = 0$, and $v^{(iv)}(0) = -\frac{3}{4n^3(n+2)}v(0)[f(0)^2 - g(0)^2] > 0$, where we use the non-linear dependence of $f^{(iv)}(0)$ on $f(0)$ in the last equality. It follows that v is increasing near 0. Since $v(0) > 0$ and $\lim_{r \rightarrow r_0} v(r) = 0$ (where we use l'Hôpital's rule if $r_0 = \sqrt{2n}$), we see that v must achieve its supremum over $(0, r_0)$ at an interior point, say \bar{r} .

By assumption, $0 < v < 1$ in $(0, r_0)$, and in particular $0 < v(\bar{r}) < 1$. We compute $v''(\bar{r})$. Using $v'(\bar{r}) = 0$, we have at \bar{r} :

$$vg'' = (1 + (g')^2) \left[\left(\frac{r}{2} - \frac{n-1}{r} \right) f' - \frac{1}{2}f \right]$$

so that

$$f'' - vg'' = ((f')^2 - (g')^2) \left[\left(\frac{r}{2} - \frac{n-1}{r} \right) f' - \frac{1}{2}f \right] = (v^2 - 1)(g')^2 \frac{f''}{1 + (f')^2}.$$

Therefore, at \bar{r} :

$$v'' = \frac{f'' - vg''}{g} = (v^2 - 1) \frac{(g')^2}{g} \frac{f''}{1 + (f')^2} > 0,$$

which contradicts the fact that v has a maximum at \bar{r} . \square

Lemma 11. *If f is a solution to (2.6) with $f(0) < \sqrt{2n}$, then f can only intersect the sphere once before it crosses the r -axis.*

Proof. Suppose f intersects g (the sphere) at two points before it crosses the r -axis: say r_1 and r_2 , where $0 < r_1 < r_2$. Since $g(0) > f(0)$, we may assume that $f > g$ on (r_1, r_2) . We may also assume that $r_2 < \sqrt{2n}$ (otherwise, f would intersect the r -axis perpendicularly at $\sqrt{2n}$, contradicting Huisken's theorem).

We consider the function $w = \frac{f'}{g'}$, which satisfies

$$w' = \frac{f'' - wg''}{g'}$$

and

$$w'' = \frac{f''' - wg'''}{g'} - 2\frac{g''}{g'}w'.$$

Now, using l'Hôpital's rule and the above identities at 0, we have $w(0) = \frac{f''(0)}{g''(0)} = \frac{f(0)}{g(0)} < 1$ and $w'(0) = 0$. Furthermore, using the non-linear dependence of $g^{(iv)}(0)$ on $g(0)$, we have $w''(0) = \frac{1}{2n^2(n+2)}w(0)[f(0)^2 - g(0)^2] < 0$. It follows that w is decreasing near 0. By assumption, we have $w(r_2) \geq 1$, and consequently, w must achieve its infimum on $(0, r_2)$ at an interior point, say \bar{r} .

Recall

$$f''' = (1 + (f')^2) \left[\frac{2f'(f'')^2}{(1 + (f')^2)^2} + \left(\frac{r}{2} - \frac{n-1}{r} \right) f'' + \frac{n-1}{r^2} f' \right].$$

Using $w'(\bar{r})$, we have at \bar{r} :

$$\begin{aligned} \frac{2f'(f'')^2}{1 + (f')^2} - w \frac{2g'(g'')^2}{1 + (g')^2} &= 2f' \left[\frac{(f'')^2}{1 + (f')^2} - \frac{(g'')^2}{1 + (g')^2} \right] \\ &= 2f'(g'')^2 \left[\frac{w^2 - 1}{(1 + (f')^2)(1 + (g')^2)} \right]. \end{aligned}$$

Also, at \bar{r} :

$$\left[\left(\frac{r}{2} - \frac{n-1}{r} \right) f'' + \frac{n-1}{r^2} f' \right] - w \left[\left(\frac{r}{2} - \frac{n-1}{r} \right) g'' + \frac{n-1}{r^2} g' \right] = 0$$

and

$$\begin{aligned} (f')^2 \left[\left(\frac{r}{2} - \frac{n-1}{r} \right) f'' + \frac{n-1}{r^2} f' \right] - w (g')^2 \left[\left(\frac{r}{2} - \frac{n-1}{r} \right) g'' + \frac{n-1}{r^2} g' \right] \\ = w (g')^2 (w^2 - 1) \left[\left(\frac{r}{2} - \frac{n-1}{r} \right) g'' + \frac{n-1}{r^2} g' \right]. \end{aligned}$$

Therefore, at \bar{r} :

$$\begin{aligned} w'' &= \frac{f''' - wg'''}{g'} \\ &= \left\{ \frac{2w(g'')^2}{(1 + (f')^2)(1 + (g')^2)} + wg' \left[\left(\frac{1}{2}r - \frac{n-1}{r} \right) g'' + \frac{n-1}{r^2} g' \right] \right\} (w^2 - 1) \\ &= \left\{ \frac{2w(g'')^2}{(1 + (f')^2)(1 + (g')^2)} - wg' \frac{r}{(2n - r^2)^{3/2}} \right\} (w^2 - 1), \end{aligned}$$

where we used $g(r) = \sqrt{2n - r^2}$ in the last equality. Since $w(\bar{r}) < w(0) < 1$, we have $w''(\bar{r}) < 0$, which contradicts the fact that w has a minimum at \bar{r} . \square

We have the following consequence of Lemma 10 and Lemma 11.

Proposition 3. *When $0 < x_0 < \sqrt{2n}$, the geodesic $Q[x_0]$ intersects the sphere exactly once before it crosses the r -axis.*

When $f(0) > \sqrt{2n}$, similar arguments show that f blows-up at a point $r_* < \sqrt{2n}$. Assuming $f(r_*) > 0$, this follows from the proofs of Lemma 10 and Lemma 11. A proof that $f(r_*) > 0$ when $f(0) > \sqrt{2n}$ is given in the following section, where we study the linearized rotational self-shrinker differential equation near the sphere. In fact, we show the first graphical component of $Q[x_0]$ written as a graph over the x -axis has a local maximum and no local minima in the first quadrant of \mathbb{H} when $x_0 > \sqrt{2n}$. Therefore, we have the following result.

Proposition 4. *When $x_0 > \sqrt{2n}$, the geodesic $Q[x_0]$ intersects the sphere exactly once before it crosses the r -axis.*

2.7 A Legendre type differential equation

In this section, we study the behavior of the first graphical component of the geodesic $Q[x_0]$ written as a graph over the x -axis when x_0 is close to $\sqrt{2n}$. For simplicity, we use the term ‘quarter sphere’ and the notation Q to refer to this first graphical component. Following the analysis in Appendix A of [49], where the $n = 2$ case is treated, we show that the linearization of the rotational self-shrinker differential equation near the sphere is a Legendre type differential equation. An analysis of this differential equation shows that outer-quarter spheres in the first quadrant intersect the r -axis with positive slope, and inner-quarter spheres in the first quadrant intersect the r -axis with negative slope.

Writing the rotational self-shrinker differential equation in polar coordinates $\rho = \rho(\phi)$, where $\rho = \sqrt{x^2 + r^2}$ and $\phi = \arctan(r/x)$, we have

$$\rho'' = \frac{1}{\rho} \left\{ \rho'^2 + (\rho^2 + \rho'^2) \left[n - \frac{\rho^2}{2} - (n-1) \frac{\rho'}{\rho \tan \phi} \right] \right\}. \quad (2.14)$$

In these coordinates, the sphere corresponds to the constant solution $\rho = \sqrt{2n}$. We note that this equation has a singularity when $\phi = 0$ due to the $1/\tan \phi$ term.

Making the substitution $\psi = 1 - \cos \phi$, we can write equation (2.14) as

$$\begin{aligned} \psi \frac{d^2 \rho}{d\psi^2} &= \frac{1}{\rho(2-\psi)} \left(\rho^2 + \psi(2-\psi) \left(\frac{d\rho}{d\psi} \right)^2 \right) \left[n - \frac{\rho^2}{2} - (n-1) \frac{(1-\psi)}{\rho} \frac{d\rho}{d\psi} \right] \\ &\quad + \frac{\psi}{\rho} \left(\frac{d\rho}{d\psi} \right)^2 - (1-\psi) \frac{d\rho}{d\psi}, \end{aligned} \quad (2.15)$$

which has the form of the singular Cauchy problem studied in [6] (where ψ is the time variable). Applying Theorem 2.2 in [6] to (2.15), shows that the solution $\rho(\phi, \epsilon)$ to (2.14) with $\rho(0, \epsilon) = \sqrt{2n} + \epsilon$ and $\frac{d\rho}{d\phi}(0, \epsilon) = 0$ depends smoothly on (ϕ, ϵ) in a neighborhood of $(0, 0)$. It then follows from the smooth dependence on initial conditions (away from the singularity at $\phi = 0$) for solutions to (2.14) that $\rho(\phi, \epsilon)$ is smooth when $\phi \in [0, \pi/2]$ and ϵ is close to 0.

In order to understand the behavior of $\rho(\phi, \epsilon)$ when ϵ is close to 0, we study the linearization of the rotational self-shrinker differential equation near the sphere $\rho(\phi, 0) = \sqrt{2n}$. We define w by

$$w(\phi) = \left. \frac{d}{d\epsilon} \right|_{\epsilon=0} \rho(\phi, \epsilon).$$

Then w satisfies the (singular) linear differential equation:

$$w'' = -\frac{n-1}{\tan \phi} w' - 2nw, \quad (2.16)$$

with $w(0) = 1$ and $w'(0) = 0$. We will show that $w(\pi/2) < 0$ and $w'(\pi/2) < 0$.

Lemma 12. *Let w be the solution to (2.16) with $w(0) = 1$ and $w'(0) = 0$. Then $w(\pi/2) < 0$ and $w'(\pi/2) < 0$.*

Proof. We begin by making the substitution $\xi = \cos \phi$, which turns (2.16) into the following Legendre type differential equation:

$$(1 - \xi^2) \frac{d^2 w}{d\xi^2} = n\xi \frac{dw}{d\xi} - 2nw, \quad (2.17)$$

with the initial conditions at $\xi = 1$:

$$w(1) = 1, \quad \frac{dw}{d\xi}(1) = 2.$$

To prove the lemma, we need to show that $w = w(\xi)$ satisfies $w(0) < 0$ and $\frac{dw}{d\xi}(0) > 0$.

Taking derivatives of (2.17) we have the following second order differential equations:

$$(1 - \xi^2) \frac{d^3 w}{d\xi^3} = (n + 2) \xi \frac{d^2 w}{d\xi^2} - n \frac{dw}{d\xi}, \quad (2.18)$$

$$(1 - \xi^2) \frac{d^4 w}{d\xi^4} = (n + 4) \xi \frac{d^3 w}{d\xi^3} + 2 \frac{d^2 w}{d\xi^2}. \quad (2.19)$$

It follows from (2.17) and (2.18) that

$$\frac{d^2 w}{d\xi^2}(1) = \frac{2n}{n + 2}, \quad \frac{d^3 w}{d\xi^3}(1) = -\frac{4n}{(n + 2)(n + 4)}.$$

We also note that the differential equation (2.19) for $\frac{d^2 w}{d\xi^2}$ satisfies a maximum principle.

An analysis of the possible values of $\frac{d^2 w}{d\xi^2}(0)$ and $\frac{d^3 w}{d\xi^3}(0)$ shows that $\frac{d^2 w}{d\xi^2}(0) > 0$ and $\frac{d^3 w}{d\xi^3}(0) < 0$ are the only conditions that agree with the initial conditions at $\xi = 1$. For example, if $\frac{d^2 w}{d\xi^2}(0) < 0$ and $\frac{d^3 w}{d\xi^3}(0) > 0$, then the conditions $\frac{d^2 w}{d\xi^2}(1) > 0$ and $\frac{d^3 w}{d\xi^3}(1) < 0$ imply that $\frac{d^2 w}{d\xi^2}$ achieves a positive maximum on $[0, 1]$ at an interior point, which contradicts the maximum principle.

Since $\frac{d^2 w}{d\xi^2}(0) > 0$ and $\frac{d^3 w}{d\xi^3}(0) < 0$, it follows from (2.17) and (2.18) that $w(0) < 0$ and $\frac{dw}{d\psi}(0) > 0$. Regarding $w = w(\phi)$ as a function of ϕ , this says that $w(\pi/2) < 0$ and $w'(\pi/2) < 0$, which proves the lemma. \square

Now we can prove the assertion made at the beginning of the previous section.

Proposition 5. *The first graphical component of an outer-quarter sphere in the first quadrant intersects the r -axis with positive slope, and the first graphical component of an inner-quarter sphere in the first quadrant intersects the r -axis with negative slope.*

Proof. It follows from Lemma 12 that the proposition is true for the quarter sphere $Q[x_0]$ when x_0 is close to $\sqrt{2n}$. In fact, when x_0 is close to $\sqrt{2n}$, we know that $Q[x_0]$ is C^2 close to the sphere in the first quadrant, and applying Lemma 12 we have the following description of the shape of $Q[x_0]$ in the first quadrant: If $Q[x_0]$ is an inner-quarter sphere, then it is strictly convex and monotone in the first quadrant, and if

$Q[x_0]$ is an outer-quarter sphere, then it is strictly convex with a local maximum in the first quadrant.

To prove the proposition in general, we first consider the case of inner-quarter spheres. Suppose to the contrary that some inner-quarter sphere intersects the r -axis with non-negative slope, and let x_0 be the first $x_0 < \sqrt{2n}$ with this property. It follows from the previous description of the shape of quarter spheres near the sphere that $Q[x_0]$ is convex in the first quadrant and intersects the r -axis perpendicularly. Such a quarter sphere corresponds to a (closed) convex self-shrinker that is not the sphere, which contradicts Huisken's theorem for mean-convex self-shrinkers [43].

Next, we consider the case of outer-quarter spheres. As in the previous case, suppose to the contrary that some outer-quarter sphere intersects the r -axis with non-positive slope, and let x_0 be the first $x_0 > \sqrt{2n}$ with this property. It again follows from the previous description of the shape of quarter spheres near the sphere that $Q[x_0]$ intersects the r -axis perpendicularly; however, $Q[x_0]$ may not be convex in the first quadrant. We claim that the self-shrinker corresponding to $Q[x_0]$ is mean-convex. Writing $Q[x_0]$ as the graph $(x, u(x))$, where u is a solution to (2.4), it is sufficient to show that $\Psi(x) = xu' - u$ does not vanish for $0 \leq x < x_0$. Since $\Psi(0) < 0$ and $\Psi(x_0) = -\infty$, we need to show that $\Psi < 0$. If Ψ has a non-negative maximum at some point $x_1 \in (0, x_0)$, then

$$0 = \Psi'(x_1) = x_1 u''(x_1) = x_1 (1 + u'(x_1)^2) \left[\frac{\Psi(x_1)}{2} + \frac{n-1}{u(x_1)} \right] > 0,$$

which is a contradiction. Therefore, $\Psi < 0$ and the self-shrinker corresponding to $Q[x_0]$ is mean-convex, which contradicts Huisken's theorem for mean-convex self-shrinkers.

We conclude that the first graphical component of an outer-quarter sphere $Q[x_0]$ with $x_0 > \sqrt{2n}$ intersects the r -axis with positive slope, and the first graphical component of an inner-quarter sphere $Q[x_0]$ with $0 < x_0 < \sqrt{2n}$ intersects the r -axis with negative slope. \square

Chapter 3

AN IMMERSED SPHERE SELF-SHRINKER

In this chapter, we construct an immersed and non-embedded sphere self-shrinker.

Theorem 1. [24] *For $n \geq 2$, there exists an immersion $F : S^n \rightarrow \mathbb{R}^{n+1}$ satisfying $\Delta_g F = -\frac{1}{2}F^\perp$, and F is not an embedding.*

This result verifies numerical evidence from [3], and it is the first rigorous construction of an immersed S^n self-shrinker in \mathbb{R}^{n+1} (different from the “round sphere” of radius $\sqrt{2n}$). Moreover, the existence of an immersed and non-embedded sphere self-shrinker shows that the uniqueness results for constant mean curvature spheres in \mathbb{R}^3 (see Hopf [40]) and for minimal spheres in S^3 (see Almgren [2]) do not hold for sphere self-shrinkers. The construction we present in this chapter follows the original proof from [24]; however, here we use the notation introduced in Chapter 2.

The basic idea of the proof of Theorem 1 is to find a solution $Q[x_0]$ to the geodesic shooting problem (2.9) from Section 2.5 with self-intersections whose rotation about the x -axis is topologically S^n . In fact, using the symmetry of the geodesic equation with respect to reflections across the r -axis, it is sufficient to find a geodesic $Q[x_0]$ that intersects the r -axis perpendicularly (see Figure 1.1). Using comparison arguments, we give a detailed description of the first two ‘branches’ of the geodesics $Q[x_0]$ when $x_0 > 0$ is small. Then, following the approach of Angenent in [3], we use a continuity argument to find an initial condition that corresponds to a geodesic whose rotation about the x -axis is an immersed and non-embedded S^n self-shrinker.

In Section 3.1, we study the “first branch” of $Q[x_0]$ written as a graph over the r -axis. In the upper half plane \mathbb{H} , this first branch is given by the curve $(f(r), r)$, where f is the solution to (2.6) with $f(0) = x_0$. We know from Chapter 2 that f

is concave down and decreasing, and it blows up at some finite point (x_*, r_*) with $r_* > \sqrt{2(n-1)}$. We show $r_* \rightarrow \infty$ and $f(r_*) \rightarrow 0$ as $f(0) \rightarrow 0$.

In Section 3.3, we study the “second branch” of $Q[x_0]$ written as a graph over the r -axis. We write this second branch as the curve $(g(r), r)$, where g is the unique solution to (2.6) with $g(r_*) = x_*$ and $\lim_{r \rightarrow r_*} g'(r) = \infty$. Let $r_{**} \geq 0$ be the point so that g is a maximally extended solution to (2.6) on (r_{**}, r_*) . We show for small initial height $f(0)$ that g achieves a negative minimum at a point $r_m \in (r_{**}, r_*)$, g is concave up, $r_{**} > 0$, and $0 < g(r_{**}) < \infty$.

To prove $g(r_{**}) > 0$, we give a direct argument, which shows how the singular term in (2.6) forces g to cross the r -axis when r_{**} is small. This direct crossing argument is different from the limiting argument used by Angenent in [3], and the analysis of g in this section leads to a different construction of Angenent’s torus self-shrinker. We also note that Møller [52] constructed a torus self-shrinker with explicit estimates on the cross-sections, which he used to construct high genus compact, embedded self-shrinkers.

Finally, in Section 3.4 we finish the proof of the Theorem 1. We let f_b be the solution to (2.6) with $f_b(0) = b$, and define g_b , r_*^b , r_{**}^b , and r_m^b as above. Following Angenent’s argument in [3], we consider the initial height b_0 given by

$$b_0 = \sup\{\tilde{b} : \forall b \in (0, \tilde{b}], \exists r_m^b \in (r_{**}^b, r_*^b) \text{ so that } g'_b(r_m^b) = 0 \text{ and } g_b(r_{**}^b) > 0\}.$$

Using continuity arguments we show that g_{b_0} intersects the r -axis perpendicularly at $r_{**}^{b_0}$. Thus, the geodesic $Q[b_0]$ is a smooth curve in the upper half plane that intersects the x -axis perpendicularly at precisely two points (see Figure 1.1), and the rotation of this curve about the x -axis is an immersed and non-embedded S^n self-shrinker in \mathbb{R}^{n+1} .

3.1 The first branch

In this section we study solutions to (2.6) with $f(0) > 0$ and $f'(0) = 0$. For $b > 0$, let f_b denote the unique solution to (2.6) with $f_b(0) = b$. We know from Chapter 2 that f_b is decreasing and concave down, and there exists a point $r_*^b \in (\sqrt{2(n-1)}, \infty)$ so that f_b is defined on $[0, r_*^b)$ and $\lim_{r \rightarrow r_*^b} f_b'(r) = -\infty$. There also exists a point $x_*^b \in (-\infty, b)$ so that $f_b(r_*^b) = x_*^b$.

The main result of this section is the following proposition, which provides estimates for r_*^b and x_*^b when $b > 0$ is small.

Proposition 6. *For $b > 0$, let f_b denote the solution to (2.6) with $f_b(0) = b$ and $f_b'(0) = 0$. Let r_*^b denote the point where f_b blows-up, and let $x_*^b = f_b(r_*^b)$.*

There exists $\bar{b} > 0$ so that if $b \in (0, \bar{b}]$, then

$$r_*^b \geq \sqrt{\log \frac{2}{\pi b^2}},$$

$$\frac{-4(n+1)}{\sqrt{\log \frac{2}{\pi b^2}}} \leq x_*^b < 0,$$

and there exists a point $r_0^b \in [\sqrt{2n}, 2\sqrt{n}]$ so that $f_b(r_0^b) = 0$.

Before we prove the proposition, we prove some results about solutions to (2.6) when the initial height is small. Let f be the solution to (2.6) with $f(0) = b$ and $f'(0) = 0$.

Claim 3. *If $b < \sqrt{\frac{2}{3\pi}} \cdot \frac{1}{e^{2n}}$, then $r_* > 2\sqrt{n}$ and $|f'(r)| \leq \frac{\sqrt{3}}{3}$ for $r \in [0, 2\sqrt{n}]$.*

Proof. Since $f'(0) = 0$, $f'' < 0$, and $\lim_{r \rightarrow r_*} f'(r) = -\infty$, we know there exists

$\bar{r} \in (0, r_*)$ so that $f'(\bar{r}) = -\frac{\sqrt{3}}{3}$. For $r \in (0, \bar{r})$, we have

$$\begin{aligned}
\frac{d}{dr} \left(e^{-\frac{r^2}{2}} f'(r) \right) &= e^{-\frac{r^2}{2}} f''(r) - r e^{-\frac{r^2}{2}} f'(r) \\
&\geq \frac{2}{1 + f'(r)^2} e^{-\frac{r^2}{2}} f''(r) - r e^{-\frac{r^2}{2}} f'(r) \\
&= 2e^{-\frac{r^2}{2}} \left[\left(\frac{1}{2}r - \frac{n-1}{r} \right) f'(r) - \frac{1}{2}f(r) \right] - r e^{-\frac{r^2}{2}} f'(r) \\
&= -2e^{-\frac{r^2}{2}} \frac{n-1}{r} f'(r) - e^{-\frac{r^2}{2}} f(r) \\
&\geq -e^{-\frac{r^2}{2}} f(r).
\end{aligned}$$

Integrating from 0 to \bar{r} ,

$$-\frac{\sqrt{3}}{3} e^{-\frac{\bar{r}^2}{2}} \geq - \int_0^{\bar{r}} e^{-\frac{r^2}{2}} f(r) dr \geq -b \int_0^{\bar{r}} e^{-\frac{r^2}{2}} dr \geq -b \sqrt{\frac{\pi}{2}}.$$

When $b < \sqrt{\frac{2}{3\pi}} \cdot \frac{1}{e^{2n}}$ we have $e^{-\frac{\bar{r}^2}{2}} < e^{-2n}$, and therefore $\bar{r} > 2\sqrt{n}$. \square

Claim 4. *If $|f'(r)| \leq \frac{\sqrt{3}}{3}$ for $r \in [0, \sqrt{2(n-1)}]$, then $f'''(r) < 0$ for $r \in (0, r_*)$.*

Proof. From the power series expansion for f at $r = 0$, we know that $f'''(0) = 0$ and $f^{(iv)}(0) < 0$. Therefore, $f'''(r) < 0$ when $r > 0$ is near 0. Also, using (2.7), we see that $f'''(r) < 0$ when $r \geq \sqrt{2(n-1)}$. Suppose $f'''(r) = 0$ for some $r > 0$. Then there exists $\bar{r} \in (0, \sqrt{2(n-1)})$ so that $f'''(\bar{r}) = 0$ and $f'''(r) < 0$ for $r \in (0, \bar{r})$. It follows that $f^{(iv)}(\bar{r}) \geq 0$. Notice that $r f''(r) - f'(r)$ is decreasing and hence negative on $(0, \bar{r})$. Then, using the equation involving $f^{(iv)}$, obtained from differentiating (2.7), and the assumption that $|f'(\bar{r})| \leq \frac{\sqrt{3}}{3}$, we see that

$$\frac{f^{(iv)}(\bar{r})}{1 + f'(\bar{r})^2} = 2(f''(\bar{r}))^3 \frac{1 - 3(f'(\bar{r}))^2}{(1 + f'(\bar{r})^2)^3} + \frac{1}{2} f''(\bar{r}) + 2(n-1) \frac{\bar{r} f''(\bar{r}) - f'(\bar{r})}{(\bar{r})^3} < 0,$$

which is a contradiction. \square

Claim 5. *If $|f'(x)| \leq \frac{\sqrt{3}}{3}$ for $r \in [0, 2\sqrt{n}]$ and $b < \frac{1}{\sqrt{2n}}$, then $\frac{r f'(r) - f(r)}{\sqrt{1 + f'(r)^2}}$ is non-increasing on $[0, 2\sqrt{n}]$.*

Proof. Looking at the derivative of $\frac{rf'(r)-f(r)}{\sqrt{1+f'(r)^2}}$:

$$\frac{d}{dr} \left(\frac{rf'(r) - f(r)}{\sqrt{1+f'(r)^2}} \right) = f''(r) \frac{r + f(r)f'(r)}{(1+f'(r)^2)^{3/2}},$$

we see that it is enough to show $r + f(r)f'(r) \geq 0$. Since $r + f(r)f'(r)$ equals 0 when $r = 0$, it is sufficient to show $1 + f(r)f''(r) + f'(r)^2 \geq 0$ on $(0, 2\sqrt{n}]$. For $r \in (0, 2\sqrt{n}]$, assuming $|f'(r)| \leq \frac{\sqrt{3}}{3}$ and $b < \frac{1}{\sqrt{2n}}$, we have

$$\begin{aligned} f''(r) &= (1+f'(r)^2) \left[\left(\frac{1}{2}r - \frac{n-1}{r} \right) f'(r) - \frac{1}{2}f(r) \right] \\ &\geq \frac{4}{3} \left[-\sqrt{n} \frac{\sqrt{3}}{3} - \frac{b}{2} \right] \geq -2, \end{aligned}$$

and it follows that $1 + f(r)f''(r) + f'(r)^2 \geq 0$. □

Lemma 13. *Suppose $r_* > 2\sqrt{n}$ and there exists a point $r_0 \in [\sqrt{2n}, 2\sqrt{n}]$ so that $f(r_0) = 0$. Then, for $r \in [r_0, r_*)$,*

$$f(r) > \frac{4n}{r} f'(r).$$

Proof. Let $\Phi(r) = \frac{r}{4n}f(r) - f'(r)$. We want to show $\Phi(r) > 0$. We know that $\Phi(r_0) = -f'(r_0) > 0$. We also have

$$\begin{aligned} \frac{r}{4n}f(r) &= \frac{r}{4n} \int_{r_0}^r f'(t) dt \\ &> \frac{1}{4n} r(r - r_0) f'(r). \end{aligned}$$

Since $r_0 \geq \sqrt{2n}$, we see that $\Phi(r) > 0$ when $r \leq 2\sqrt{2n}$.

Suppose $\Phi(r) = 0$ for some $r \in [r_0, r_*)$. Then $r > 2\sqrt{2n}$, and there exists a point $\bar{r} \in (2\sqrt{2n}, r_*)$ so that $\Phi(\bar{r}) = 0$ and $\Phi(r) > 0$ for $r \in [r_0, \bar{r})$. This implies that

$\Phi'(\bar{r}) \leq 0$ and $\frac{\bar{r}}{4n}f(\bar{r}) = f'(\bar{r})$. Since $\bar{r} > 2\sqrt{2n}$ and $f(\bar{r}) < 0$, we have

$$\begin{aligned}
\Phi'(\bar{r}) &= \frac{1}{4n}f(\bar{r}) + \frac{\bar{r}}{4n}f'(\bar{r}) - f''(\bar{r}) \\
&\geq \frac{1}{4n}f(\bar{r}) + \frac{\bar{r}}{4n}f'(\bar{r}) - \frac{f''(\bar{r})}{1+f'(\bar{r})^2} \\
&= \frac{1}{4n}f(\bar{r}) + \frac{\bar{r}}{4n}f'(\bar{r}) - \left[\left(\frac{1}{2}\bar{r} - \frac{n-1}{\bar{r}} \right) f'(\bar{r}) - \frac{1}{2}f(\bar{r}) \right] \\
&= f(\bar{r}) \left(\frac{1}{4n} + \frac{\bar{r}^2}{16n^2} - \left[\left(\frac{\bar{r}}{2} - \frac{n-1}{\bar{r}} \right) \frac{\bar{r}}{4n} - \frac{1}{2} \right] \right) \\
&= f(\bar{r}) \left(\frac{3}{4} - \frac{2n-1}{16n^2}\bar{r}^2 \right) > 0,
\end{aligned}$$

which is a contradiction. \square

Proof of Proposition 6. Let $b > 0$, and let f be the solution to (2.6) with $f(0) = b$ and $f'(0) = 0$. We know there exists a point $r_* \in (\sqrt{2(n-1)}, \infty)$ and a point $x_* \in (-\infty, b)$ so that $\lim_{r \rightarrow r_*} f'(r) = -\infty$ and $f(r_*) = x_*$. We assume $b < \sqrt{\frac{2}{3\pi}} \cdot \frac{1}{e^{2n}}$ (and also $b < \frac{1}{\sqrt{2n}}$). By Claim 3, we know that $r_* > 2\sqrt{n}$ and $|f'(r)| \leq \frac{\sqrt{3}}{3}$ for $r \in [0, 2\sqrt{n}]$. Then by Claim 4 we have $f''' < 0$ on $(0, r_*)$. Integrating this inequality from 0 to r repeatedly, we see that

$$f(r) < b \left(1 - \frac{r^2}{4n} \right).$$

Since $r_* > 2\sqrt{n}$, it follows that there exists $r_0 \in (0, 2\sqrt{n})$ so that $f(r_0) = 0$.

To estimate r_0 from below, we write equation (2.6) in the form

$$\frac{d}{dr} \left(\frac{r^{n-1}f'(r)}{\sqrt{1+f'(r)^2}} \right) = \frac{1}{2}r^{n-1} \cdot \frac{rf'(r) - f(r)}{\sqrt{1+f'(r)^2}}. \quad (3.1)$$

It follows from Claim 5 that $\frac{rf'(r)-f(r)}{\sqrt{1+f'(r)^2}} \geq \frac{r_0f'(r_0)}{\sqrt{1+f'(r_0)^2}}$, for $r \in [0, r_0]$, and thus by integrating (3.1) from 0 to r_0 , we get

$$\begin{aligned}
\frac{r_0^{n-1}f'(r_0)}{\sqrt{1+f'(r_0)^2}} &= \int_0^{r_0} \frac{1}{2}r^{n-1} \cdot \frac{rf'(r) - f(r)}{\sqrt{1+f'(r)^2}} dr \\
&\geq \frac{r_0f'(r_0)}{\sqrt{1+f'(r_0)^2}} \int_0^{r_0} \frac{1}{2}r^{n-1} dr.
\end{aligned}$$

Therefore, $1 \leq \frac{(r_0)^2}{2n}$ and we see that $r_0 \geq \sqrt{2n}$. This proves the last statement in the proposition.

Next, we want to slightly refine the estimate from Claim 3 to establish a lower bound for r_* in terms of b . This will simplify the constants that appear in the following calculations. Let $r_1 \in (0, r_*)$ be the point where $f'(r_1) = -1$. Using the same argument we used in the proof of Claim 3, we integrate the inequality

$$\frac{d}{dr} \left(e^{-\frac{r^2}{2}} f'(r) \right) \geq -e^{-\frac{r^2}{2}} f(r)$$

from 0 to r_1 to conclude that $-e^{-\frac{(r_1)^2}{2}} \geq -b\sqrt{\frac{\pi}{2}}$ and therefore,

$$r_1 \geq \sqrt{\log \frac{2}{\pi b^2}}.$$

Since $r_0 \in [\sqrt{2n}, 2\sqrt{n}]$, it follows from Lemma 13 that $f(r) > \frac{4n}{r} f'(r)$. In particular, at r_1 , we have

$$f(r_1) > -\frac{4n}{r_1} \geq -\frac{8}{\sqrt{\log \frac{2}{\pi b^2}}}.$$

We will extend this estimate for $f(r_1)$ to an estimate for $f(r_*) = x_*$. We assume $b \leq \sqrt{\frac{2}{\pi e^{16n}}}$ so that $r_1 \geq 4\sqrt{n}$ (and $\frac{3n-1}{r} < \frac{r}{4}$ when $r \geq r_1$). For $r \geq r_1$, we have

$$\begin{aligned} f''(r) &\leq f'(r)^2 \frac{f''(r)}{1 + f'(r)^2} \\ &= f'(r)^2 \left[\left(\frac{1}{2}r - \frac{n-1}{r} \right) f'(r) - \frac{1}{2}f(r) \right] \\ &< f'(r)^3 \left(\frac{1}{2}r - \frac{3n-1}{r} \right) \\ &\leq \frac{1}{4}r f'(r)^3, \end{aligned}$$

where we have used that $r \geq 4\sqrt{n}$ and $f(r) > \frac{4n}{r} f'(r)$.

Integrating the previous inequality from r to r_* , implies

$$f'(r)^2 \leq \frac{4}{(r_*)^2 - r^2},$$

for $r \geq r_1$. Since $f'(r) < 0$, we have

$$f'(r) \geq -\frac{2}{\sqrt{(r_*)^2 - r^2}} \geq -\frac{1}{\sqrt{r_* + r_1}} \cdot \frac{2}{\sqrt{r_* - r}}, \quad (3.2)$$

for $r \in [r_1, r_*)$. At r_1 , this tells us that

$$-\frac{\sqrt{r_* - r_1}}{\sqrt{r_* + r_1}} \geq -\frac{2}{r_* + r_1}.$$

Finally, integrating (3.2) from r_1 to r_* , we have

$$f(r_*) - f(r_1) \geq -\frac{4}{\sqrt{r_* + r_1}} \cdot \sqrt{r_* - r_1},$$

and therefore

$$\begin{aligned} f(r_*) &\geq f(r_1) - \frac{4}{\sqrt{r_* + r_1}} \cdot \sqrt{r_* - r_1} \\ &\geq -\frac{4n}{r_1} - \frac{8}{r_* + r_1} \\ &\geq -\frac{4(n+1)}{r_1} \geq -\frac{4(n+1)}{\sqrt{\log \frac{2}{\pi b^2}}}, \end{aligned}$$

which completes the proof of the proposition. \square

3.2 Connecting the first and second branches

Fix $b > 0$, and let $Q[b]$ be the geodesic from the shooting problem (2.9). Then the first branch of $Q[b]$ is given by the curve $(f(r), r)$ where f is the solution to (2.6) with $f(0) = b$. At the blow-up point (x_*, r_*) , we may write $Q[b]$ as a graph over the x -axis: $(x, u(x))$, where u is the solution to (2.4) with $u(x_*) = r_*$ and $u'(x_*) = 0$. It follows from (2.4) and the property $r_* > \sqrt{2(n-1)}$ that $u''(x_*) < 0$. Therefore, we see that the geodesic $Q[b]$ turns around at the point (x_*, r_*) and travels back towards the x -axis. This second branch of the geodesic $Q[b]$ may be written as $(g(r), r)$, where g is the unique solution to (2.6) with $g(r_*) = x_*$ and $\lim_{r \rightarrow r_*} g'(r) = \infty$. (We note that f is the unique solution to (2.6) with $f(r_*) = x_*$ and $\lim_{r \rightarrow r_*} f'(r) = -\infty$.)

We remark that as we vary the initial condition $f(0) = b$, the solution g and its derivatives vary continuously. This follows from the smooth dependence on the parameter b for the family of solutions $\{Q[b]\}$ to the shooting problem (2.3). However, this continuity only works in compact subsets of the upper half plane. For instance, we can choose $f_b(0) = b > 0$ so small that f_b is arbitrarily close to the plane $f_0(r) \equiv 0$ in a compact subset of the upper half plane, but without the analysis from the previous section, we cannot make any conclusions about the behavior of f_b outside this compact set. Even if we know that f_b blows-up at the finite point (x_*^b, r_*^b) and the second branch g_b exists, we cannot use the continuity of the differential equation to deduce information about the behavior of g_b without further arguments. In the next section we study the behavior of g_b when the initial height $b > 0$ is small. It will follow from the analysis in that section that g_b converges to the plane as $b \rightarrow 0$.

3.3 The second branch

In this section we study the curve $(g(r), r)$, introduced in the previous section, as it travels from (x_*, r_*) toward the x -axis. For $b > 0$, let f denote the solution to (2.6) with $f(0) = b$ and $f'(0) = 0$, let r_* denote the point where f blows-up, and let $x_* = f(r_*)$. Also, let g denote the unique solution to (2.6) with $g(r_*) = x_*$ and $\lim_{r \rightarrow r_*} g'(r) = \infty$, and let $r_{**} \in [0, r_*)$ be the point where g blows-up, or if no such point exists, set $r_{**} = 0$. We will show for small $b > 0$ that g achieves a negative minimum at a point $r_m \in (r_{**}, r_*)$, g is concave up, $r_{**} > 0$, and $0 < g(r_{**}) < \infty$.

3.3.1 Basic shape of the second branch

We begin by observing that $g''(r) > 0$ for r close to r_* . This follows from equation (2.6), the property $\lim_{r \rightarrow r_*} g'(r) = \infty$, and the fact $r_* > \sqrt{2(n-1)}$. It then follows from the results in Chapter 2 that either (1.) $g' > 0$ on (r_{**}, r_*) , or (2.) $g'' > 0$ on (r_{**}, r_*) and there exists a point $r_m \in (r_{**}, r_*)$ so that $g'(r_m) = 0$.

Next, we prove some properties of g that are consequences of equations (2.6) and (2.7) and the fact that $\lim_{r \rightarrow r_*} g'(r) = \infty$.

Claim 6. *If there exists $r_m \in (r_{**}, r_*)$ so that $g'(r_m) = 0$, then $g(r) < 0$ whenever $r \in (r_{**}, r_m)$ and $r \geq \sqrt{2(n-1)}$.*

Proof. In this case, we have $g'' > 0$ on (r_{**}, r_*) . If $g(r) = 0$ for some $r \in (r_{**}, r_m)$, then $g'(r) < 0$, and using (2.6), we see that $r < \sqrt{2(n-1)}$. \square

Claim 7. $r_{**} < \sqrt{2(n-1)}$.

Proof. First, suppose there exists a point $r_m \in (r_{**}, r_*)$ so that $g'(r_m) = 0$ and $g'' > 0$ on (r_{**}, r_*) . By Claim 6, we know that $g(r) < 0$ when $r \in (r_{**}, r_m)$ and $r \geq \sqrt{2(n-1)}$. Also, if we let $M = -g(r_m)$, then $g(r) \geq -M$ when $r \in (r_{**}, r_*)$. Now, for any $\varepsilon > 0$, there exists a constant $m_\varepsilon > 0$ so that $\frac{1}{2}r - \frac{n-1}{r} > m_\varepsilon$ for $r \geq \sqrt{2(n-1)} + \varepsilon$, and using (2.6), we have $g'(r) > -\frac{M}{2m_\varepsilon}$ when $r \geq \sqrt{2(n-1)} + \varepsilon$. Therefore, $|g(r)|$ and $|g'(r)|$ are uniformly bounded for $r \geq \sqrt{2(n-1)} + \varepsilon$ and away from r_* . By the existence theory for differential equations $r_{**} < \sqrt{2(n-1)} + \varepsilon$. Taking $\varepsilon \rightarrow 0$, we have $r_{**} \leq \sqrt{2(n-1)}$. To see that $r_{**} < \sqrt{2(n-1)}$, suppose $r_{**} = \sqrt{2(n-1)}$. Then there exists $x_{**} \in [-M, 0]$ so that $\lim_{r \rightarrow r_{**}} g(r) = x_{**}$. It follows that near the point (x_{**}, r_{**}) the curve $(g(r), r)$ can be written as $(x, v(x))$ where v is a solution to (2.4) with $v(x_{**}) = \sqrt{2(n-1)}$ and $v'(x_{**}) = 0$. By the uniqueness of solutions to (2.6) we deduce that v is the constant function $v(x) = \sqrt{2(n-1)}$, which is a contradiction.

In the second case, $g' > 0$ on (r_{**}, r_*) . If $r_{**} = 0$, we are done. Otherwise, $r_{**} > 0$ and consequently $\lim_{r \rightarrow r_{**}} g'(r) = \infty$. In this case, g'' must be negative at some point, and applying Lemma 5, we see that $g'' < 0$ near x_{**} . If $g < 0$ near r_{**} , then it follows from (2.6) that $r_{**} < \sqrt{2(n-1)}$. On the other hand, if $g \geq 0$, then $|g|$ is uniformly bounded (since $g \leq x_*$) and arguing as we did in the first case, we see that $r_{**} < \sqrt{2(n-1)}$. \square

Lemma 14. $\lim_{r \rightarrow r_{**}} g(r) < \infty$.

Proof. Suppose $\lim_{r \rightarrow r_{**}} g(r) = \infty$. Since $g' > 0$ near r_* , there exists a point $r_m \in (r_{**}, r_*)$ so that $g'(r_m) = 0$. Then $g'' > 0$ on (r_{**}, r_*) , and $g(r_m) < 0$. It follows that there exists a point $r_\ell \in (r_{**}, r_m)$ so that $g(r_\ell) = 0$ with $g'(r_\ell) = -m < 0$. By Claim 6, we know there exists $\delta > 0$ so that $r_\ell + \delta < \sqrt{2(n-1)}$, and in particular $(\frac{1}{2}r - \frac{n-1}{r}) \leq -\frac{1}{M}$ for some $M > 0$, when $r \leq r_\ell$.

Let $\psi = -g'$. Then $\psi > 0$ when $r \in (r_{**}, r_\ell)$ and $\psi' < 0$. Using (2.7), we have

$$\psi'' \geq -\frac{1}{M}\psi' \cdot \psi^2,$$

when $r \in (r_{**}, r_\ell)$. Fix $\varepsilon > 0$, and let

$$\phi_\varepsilon(r) = \frac{m\sqrt{r_\ell - (r_{**} + \varepsilon)} + \sqrt{3M}}{\sqrt{r - (r_{**} + \varepsilon)}}.$$

Then

$$\phi_\varepsilon'' = -\frac{3}{2} \frac{1}{(m\sqrt{r_\ell - (r_{**} + \varepsilon)} + \sqrt{3M})^2} \phi_\varepsilon' \cdot \phi_\varepsilon^2 \leq -\frac{1}{M} \phi_\varepsilon' \cdot \phi_\varepsilon^2,$$

for $r \in (r_{**} + \varepsilon, r_\ell)$.

Now, $\phi_\varepsilon(r_\ell) > \psi(r_\ell)$ and $\phi_\varepsilon(r_{**} + \varepsilon) > \psi(r_{**} + \varepsilon)$. Suppose $\psi > \phi_\varepsilon$ at some point in $(r_{**} + \varepsilon, r_\ell)$, then $\psi - \phi_\varepsilon$ must achieve a positive maximum in $(r_{**} + \varepsilon, r_\ell)$. At such a point

$$0 \geq (\psi - \phi_\varepsilon)'' \geq -\frac{1}{M}\psi'(\psi^2 - \phi_\varepsilon^2) > 0,$$

which is a contradiction. Therefore, $\phi_\varepsilon \geq \psi$. Taking $\varepsilon \rightarrow 0$, we have the estimate

$$\psi(r) \leq \frac{m\sqrt{r_\ell - r_{**}} + \sqrt{3M}}{\sqrt{r - r_{**}}},$$

for $r \in (r_{**}, r_\ell)$. Integrating from r to r_ℓ ,

$$g(r) - g(r_\ell) \leq 2 \left(m\sqrt{r_\ell - r_{**}} + \sqrt{3M} \right) \left(\sqrt{r_\ell - r_{**}} - \sqrt{r - r_{**}} \right).$$

Since $g(r_\ell) = 0$, we have

$$\lim_{r \rightarrow r_{**}} g(r) \leq 2 \left(m\sqrt{r_\ell - r_{**}} + \sqrt{3M} \right) \left(\sqrt{r_\ell - r_{**}} \right),$$

which is a contradiction. □

Now that we know g is bounded from above, we can show that $r_{**} > 0$ when there exists $r_m \in (r_{**}, r_*)$ so that $g'(r_m) = 0$.

Claim 8. *Suppose there exists $r_m \in (r_{**}, r_*)$ so that $g'(r_m) = 0$. Then $r_{**} > 0$.*

Proof. If there exists $r_m \in (r_{**}, r_*)$ so that $g'(r_m) = 0$, then $g'' > 0$ and there exists $\epsilon > 0$ and $r_\epsilon \in (r_{**}, r_m)$ so that $\frac{g'(r_\epsilon)}{\sqrt{1+g'(r_\epsilon)^2}} = -\epsilon$. Also, by Lemma 14, there exists $M \geq 0$ so that $g < M$.

Let $\theta(r) = \arctan g'(r)$. Then

$$\frac{d}{dr} (\log \sin \theta(r)) = \frac{1}{2}r - \frac{n-1}{r} - \frac{g(r)}{2g'(r)} \leq \frac{1}{2}r - \frac{n-1}{r} - \frac{M}{2g'(r_\epsilon)},$$

for $r \in (r_{**}, r_\epsilon)$. Integrating the inequality from r to r_ϵ :

$$\log \left(\frac{\sin \theta(r_\epsilon)}{\sin \theta(r)} \right) \leq \frac{1}{4}(r_\epsilon)^2 + (n-1) \log \left(\frac{r}{r_\epsilon} \right) + \frac{Mr_\epsilon}{2(-g'(r_\epsilon))}.$$

Since $\sin \theta(r_\epsilon) = \frac{g'(r_\epsilon)}{\sqrt{1+g'(r_\epsilon)^2}} = -\epsilon$ and $\sin \theta(r_{**}) = -1$, we have

$$\epsilon \leq \left(\frac{r_{**}}{r_\epsilon} \right)^{n-1} e^{\frac{1}{4}(r_\epsilon)^2 + \frac{Mr_\epsilon}{2(-g'(r_\epsilon))}},$$

which proves the claim. □

3.3.2 Behavior of the second branch for small $b > 0$

Fix $b \in (0, \bar{b}]$, where \bar{b} is defined in Proposition 6. Let f denote the solution to (2.6) with $f(0) = b$ and $f'(0) = 0$, let r_* denote the point where f blows-up, and let $x_* = f(r_*)$. Also, let g denote the unique solution to (2.6) with $g(r_*) = x_*$ and $\lim_{r \rightarrow r_*} g'(r) = \infty$. We know there is a point $r_{**} \in [0, \sqrt{2(n-1)})$ so that g is defined on (r_{**}, r_*) and either blows-up as $r \rightarrow r_{**}$ or $r_{**} = 0$.

Claim 9. *Suppose $r_* \geq 2\sqrt{2n}$. Then there exists a point $r_m \in [r_* - 2, r_*)$ so that $g'(r_m) = 0$.*

Proof. Suppose $g'(r) > 0$ for $r \in [r_* - 2, r_*)$. When $r_* \geq 2\sqrt{2n}$, we have $r_* - 2 \geq \sqrt{2(n-1)}$ and $g(r_*) < 0$. Using (2.6), we see that $g'' > 0$ in $[r_* - 2, r_*)$ and $g''(r_* - 2) \geq -\frac{1}{2}g(r_* - 2)$. Then, using (2.7), we have $g''' > 0$ in $[r_* - 2, r_*)$. Integrating from $r_* - 2$ to r , we get

$$g(r) \geq g(r_* - 2) \left[1 - \frac{1}{4}(r - (r_* - 2))^2 \right],$$

for $r \in [r_* - 2, r_*)$. This tells us that $g(r_*) \geq 0$, which is a contradiction. \square

Let $\bar{b} > 0$ be given as in the conclusion of Proposition 6 so that if $b \in (0, \bar{b}]$, then $f \leq 0$ for $r \in [2\sqrt{2}, r_*)$. We also assume that \bar{b} is chosen so small that $r_* > 2\sqrt{2n}$ and $x_* \geq -\frac{1}{8\sqrt{n}} \geq -\frac{n+1}{2\sqrt{n}}$ when $b \in (0, \bar{b}]$.

Lemma 15. *If $b \in (0, \bar{b}]$, then $2x_* \leq g(r) < 0$ for $r \in [2\sqrt{n}, r_*]$.*

Proof. Let $u(x)$ denote the curve that connects f and g so that u is a solution to (2.4) with $u(x_*) = r_*$ and $u'(x_*) = 0$. Using (2.4) and (2.5), we have $u''(x_*) = \frac{n-1}{r_*} - \frac{1}{2}r_*$ and $u'''(x_*) = \frac{1}{2}x_*\left(\frac{n-1}{r_*} - \frac{1}{2}r_*\right)$ so that

$$u(x) = r_* + \frac{1}{2} \left(\frac{n-1}{r_*} - \frac{r_*}{2} \right) (x - x_*)^2 + \frac{x_*}{12} \left(\frac{n-1}{r_*} - \frac{r_*}{2} \right) (x - x_*)^3 + \mathcal{O}(|x - x_*|^4)$$

as $x \rightarrow x_*$. Since $r_* > \sqrt{2(n-1)}$ and $x_* < 0$, the coefficient of the $(x - x_*)^3$ term is positive. Now, for $r < r_*$ and near r_* , we can find $s, t > 0$ so that

$$u(x_* + t) = r = u(x_* - s),$$

and it follows from the Taylor expansion formula for $u(x)$ that $t > s$.

To prove the lemma, we consider the function $\delta(r) = f(r) + g(r)$. For $r \in [2\sqrt{n}, r_*)$, since $f(r) \leq 0$ and $g(r) < 0$, we have

$$\delta(r) < 0.$$

Also, by the previous paragraph,

$$\delta(r) = (x_* + t) + (x_* - s) > 2x_* = \delta(r_*),$$

for $r < r_*$ and near r_* .

Now we are in position to show that $\delta > 2x_*$ on $[2\sqrt{n}, r_*)$. Suppose $\delta(r) = 2x_*$ for some $r \in [2\sqrt{n}, r_*)$. It follows from the previous discussion that δ achieves a negative maximum at some point $\bar{r} \in (2\sqrt{n}, r_*)$. At this point we have $\delta''(\bar{r}) \leq 0$ and

$$\frac{\delta''(\bar{r})}{1 + f'(\bar{r})^2} = -\frac{1}{2}\delta(\bar{r}) > 0,$$

which is a contradiction. Therefore, $\delta > 2x_*$ on $[2\sqrt{n}, r_*)$. Since $f(r) \leq 0$ on $[2\sqrt{n}, x_*)$, this completes the proof of the lemma. \square

Claim 10. *Let $b \in (0, \bar{b}]$. If $g < 0$ on (r_{**}, r_*) , then*

$$r_{**} \leq \frac{8(n-1)}{\pi-1}(-x_*).$$

Proof. By our assumptions on \bar{b} , we know that $r_m > 2\sqrt{n}$, $g'' > 0$, and $g(2\sqrt{n}) \geq 2x_*$. Using equation (2.6), we get the estimate $g'(2\sqrt{n}) > \frac{\sqrt{n}}{n+1}g(2\sqrt{n}) \geq \frac{2\sqrt{n}}{n+1}x_*$ so that $g'(2\sqrt{n}) > -1$. For $r \in (r_{**}, 2\sqrt{n})$, we have

$$\frac{d}{dr}(\arctan g'(r)) = \left(\frac{1}{2}r - \frac{n-1}{r}\right)g'(r) - \frac{1}{2}g(r) \leq -\frac{n-1}{r_{**}}g'(r) - \frac{1}{2}g(2\sqrt{n}).$$

Integrate from r_{**} to $2\sqrt{n}$,

$$\arctan g'(2\sqrt{2}) + \frac{\pi}{2} \leq \left(\frac{n-1}{r_{**}} + \sqrt{n}\right)(-g(2\sqrt{n})).$$

Since $g'(2\sqrt{n}) > -1$, and $g(2\sqrt{n}) \geq 2x_*$, this becomes

$$\frac{\pi}{4} \leq \left(\frac{n-1}{r_{**}} + \sqrt{n}\right)2(-x_*).$$

Rearranging this inequality to estimate r_{**} and using $x_* \geq -\frac{1}{8\sqrt{n}}$, we have

$$r_{**} \leq \frac{8(n-1)}{\pi-1}(-x_*).$$

\square

Proposition 7. *There exists $\tilde{b} > 0$ so that for $b \in (0, \tilde{b}]$ there is a point $r_m^b \in (r_{**}^b, r_*^b)$ so that $g'_b(r_m^b) = 0$, and $0 < g_b(r_{**}^b) < \infty$.*

Proof. Fix $b \in (0, \bar{b}]$, and let $g = g_b$. By Claim 9 we know there exists $r_m > 2\sqrt{n}$ so that $g'(r_m) = 0$. It follows that $g'' > 0$ on (r_{**}, r_*) and $g' < 0$ on (r_{**}, r_m) . From Lemma 14, we know that $g(r_{**}) < \infty$.

Suppose $g(r_{**}) \leq 0$ so that $g < 0$ in (r_{**}, r_m) . By Claim 10 we know that $r_{**} < 1$ for b sufficiently small. Using the equation (2.7) for g''' , we see that $g''' < 0$ in $(r_{**}, \sqrt{2(n-1)})$. We know that $g''(\sqrt{n}) \geq -\frac{1}{2}g(\sqrt{n})$, and thus $g''(r) \geq -\frac{1}{2}g(\sqrt{n})$, for $r \in (r_{**}, \sqrt{n}]$. Integrating from 1 to \sqrt{n} and using that g is decreasing on $(1, \sqrt{n})$, we have

$$g'(1) \leq \frac{\sqrt{n}-1}{2}g(1).$$

Let $r \in (r_{**}, 1)$. Under the above assumptions, we may write (2.6) as $\frac{g''(r)}{g'(r)} \leq (\frac{1}{2}r - \frac{n-1}{r})$, and integrating this inequality from r to 1, we get $g'(r) \leq \frac{g'(1)}{r^{n-1}}e^{-\frac{1}{4}}$. Integrating again,

$$g(r) \geq g(1) - g'(1)e^{-\frac{1}{4}} \int_r^1 \frac{1}{t^{n-1}} dt.$$

Combining this with $g'(1) \leq \frac{\sqrt{n}-1}{2}g(1)$, we see that

$$0 > g(r) \geq g(1) \left(1 - \frac{\sqrt{n}-1}{2}e^{-\frac{1}{4}} \int_r^1 \frac{1}{t^{n-1}} dt \right),$$

and we have a contradiction for small $r_{**} > 0$. Therefore, $0 < g_b(r_{**}^b) < \infty$ when $b > 0$ is sufficiently small. \square

3.4 Construction of an immersed sphere self-shrinker

In this section, we complete the proof of Theorem 1. We consider the set

$$\{\tilde{b} : \forall b \in (0, \tilde{b}], \exists r_m^b \in (r_{**}^b, r_*^b) \text{ so that } g'_b(r_m^b) = 0 \text{ and } g_b(r_{**}^b) > 0\}.$$

By Proposition 7, we know that this set is non-empty. Following Angenent's argument in [3], we let b_0 be the supremum of this set:

$$b_0 = \sup\{\tilde{b} : \forall b \in (0, \tilde{b}], \exists r_m^b \in (r_{**}^b, r_*^b) \text{ so that } g'_b(r_m^b) = 0 \text{ and } g_b(r_{**}^b) > 0\}.$$

Since $g_b(r) = -\sqrt{2n - r^2}$ when $b = \sqrt{2n}$, we know that $b_0 \leq \sqrt{2n}$. We want to show g_{b_0} intersects the r -axis perpendicularly at $r_{**}^{b_0}$.

Claim 11. $b_0 < \sqrt{2n}$.

Proof. We will prove it is impossible to have $g'_{b_0} > 0$ and $g''_{b_0} \geq 0$ in $(r_{**}^{b_0}, r_*^{b_0})$. In particular, this will show $b_0 \neq \sqrt{2n}$.

Suppose $g'_{b_0} > 0$ and $g''_{b_0} \geq 0$ in $(r_{**}^{b_0}, r_*^{b_0})$. Then $r_{**}^{b_0} = 0$, and there exists $m > 0$ so that $g_{b_0}(x) \leq -m$ for $r \in (0, 1]$. Let b_n be an increasing sequence that converges to b_0 , and let g_n be the solution to (2.6) corresponding to the initial height b_n . Fix $\varepsilon > 0$. By continuity of the geodesics $Q[b]$, there exists $N = N(\varepsilon) > 0$ so that for $n > N$, we have $r_{**}^n < \varepsilon$ and $g_n(\varepsilon) < -m$. We know that g_n intersects the r -axis at some point $r_\ell^n < \varepsilon$, and we write a portion of the curve $(g(r), r)$ as the curve $(x, v(x))$, where $x \in [-m/2, 0]$ and v is a solution to (2.4). In fact, we have the estimates $0 < v_n(x) < \varepsilon$ and $v'_n(x) \leq 0$. Using (2.4), we get an estimate for $v''_n(x)$ when $x \in [-m/2, 0]$:

$$v''_n(x) \geq \frac{n-1}{\varepsilon} - \frac{1}{2}\varepsilon \geq \frac{n-1}{2\varepsilon},$$

for small $\varepsilon > 0$. Integrating repeatedly from x to 0:

$$v_n(x) \geq v_n(0) + v'_n(0)x + \frac{n-1}{4\varepsilon}x^2 \geq \frac{n-1}{4\varepsilon}x^2.$$

Taking $x = -m/2$, we have $v_n(-m/2) \geq \frac{(n-1)m^2}{16\varepsilon}$. This implies that the point $\bar{r} \in (r_{**}^n, \varepsilon)$ for which $g_n(\bar{r}) = -m/2$ satisfies $\bar{r} \geq \frac{(n-1)m^2}{16\varepsilon}$. Therefore, $\frac{(n-1)m^2}{16\varepsilon} \leq \bar{r} < \varepsilon$, which is a contradiction when $\varepsilon > 0$ is sufficiently small. \square

Claim 12. *There exists $r_m^{b_0} \in (r_{**}^{b_0}, r_*^{b_0})$ so that $g'_{b_0}(r_m^{b_0}) = 0$.*

Proof. Suppose not. Then $g'_{b_0} > 0$ in $(r_{**}^{b_0}, r_*^{b_0})$, and we must have $g''_{b_0} < 0$ near $r_{**}^{b_0}$. Let b_n be an increasing sequence that converges to b_0 , and let g_n be the solution to (2.6) corresponding to the initial height b_n . Since each curve g_n crosses the r -axis we know that $g''_n > 0$, which contradicts the fact that $g''_{b_0} < 0$ near $r_{**}^{b_0}$. \square

Since there exists $r_m^{b_0} \in (r_{**}^{b_0}, r_*^{b_0})$ with $g'_{b_0}(r_m^{b_0}) = 0$, we know that $g''_{b_0} > 0$ on $(r_{**}^{b_0}, r_*^{b_0})$ and $r_{**}^{b_0} > 0$. We also know that there is a point $x_{**}^{b_0}$ with $|x_{**}^{b_0}| < \infty$ so that $g_{b_0}(r_{**}^{b_0}) = x_{**}^{b_0}$.

Claim 13. $x_{**}^{b_0} = 0$.

Proof. We know that $r_{**}^{b_0} > 0$ and $|x_{**}^{b_0}| < \infty$. We also know that there exists $r_m^{b_0} \in (r_{**}^{b_0}, r_*^{b_0})$ so that $g'_{b_0}(r_m^{b_0}) = 0$. It follows from the continuous dependence of g_b on the initial height that g_b exists and $g'_b(r_m^b) = 0$ for some $r_m^b \in (x_{**}^b, x_*^b)$ with $g'_b(r_m^b) = 0$ when b is sufficiently close to b_0 .

If $x_{**}^{b_0} > 0$, then for b sufficiently close to b_0 , the curve g_b has a finite blow-up point (x_{**}^b, r_{**}^b) with $x_{**}^b > 0$, which contradicts the definition of b_0 . On the otherhand, if $x_{**}^{b_0} < 0$, then for b sufficiently close to b_0 , the curve g_b has a finite blow-up point (x_{**}^b, z_{**}^b) with $z_{**}^b < 0$, but this also contradicts the definition of b_0 . Therefore, $x_{**}^{b_0} = 0$. \square

It follows that the geodesic $Q[b_0]$ intersects the r -axis perpendicularly at the point $(0, r_{**}^{b_0})$, where $r_{**}^{b_0} \in (0, \sqrt{2(n-1)})$. Now we can describe the geodesic in the upper half plane whose rotation about the x -axis is an immersed and non-embedded S^n self-shrinker:

Proof of Theorem 1. Let $Q[b_0]$ be the geodesic solution to (2.3) obtained by shooting perpendicularly to the x -axis at the point $(b_0, 0)$ in the upper half plane. In this chapter, we showed that $Q[b_0]$ follows the curve $(f_{b_0}(r), r)$ from the x -axis, across the r -axis, to the point $(x_*^{b_0}, r_*^{b_0})$, and then it follows the curve $(g_{b_0}(r), r)$ until it intersects the r -axis perpendicularly at $(0, r_{**}^{b_0})$. Since the geodesic equation is symmetric with respect to reflections across the r -axis, we see that $Q[b_0]$ continues along the reflected curves $(-g_{b_0}(r), r)$ and $(-f_{b_0}(r), r)$ until it exits the upper half plane, where it intersects the x -axis perpendicularly at the point $(-b_0, 0)$. Notice that $(f_{b_0}(r), r)$ and $(-f_{b_0}(r), r)$ intersect transversely at $(x_*^{b_0}, r_*^{b_0})$. In fact, given the convex shapes

of f and g and the fact that $r_{**}^{b_0} < r_*^{b_0}$, the geodesic $Q[b_0]$ has two additional self-intersections (see Figure 1.1). Since the immersed and non-embedded geodesic $Q[b_0]$ intersects the x -axis perpendicularly at two points, we conclude that its rotation about the x -axis is an immersed and non-embedded S^n self-shrinker. \square

Chapter 4

CONSTRUCTION OF IMMERSED SELF-SHRINKERS

In this chapter, we construct an infinite number of complete, immersed self-shrinkers with rotational symmetry for each of the rotational topological types: the sphere, the plane, the cylinder, and the torus. This is a joint work with Stephen Kleene.

Theorem 2. [27] *There are infinitely many complete, immersed self-shrinkers in \mathbb{R}^{n+1} , $n \geq 2$, for each of the following topological types: the sphere (S^n), the plane (\mathbb{R}^n), the cylinder ($\mathbb{R} \times S^{n-1}$), and the torus ($S^1 \times S^{n-1}$).*

The self-shrinkers we construct in this chapter have rotational symmetry, and they correspond to geodesics for a conformal metric on the upper half plane: geodesics whose ends either intersect the axis of rotation perpendicularly or exit through infinity, and closed geodesics with no ends (see Appendix A). The heuristic idea of the construction is to first study the behavior of geodesics near two known self-shrinkers and then use continuity arguments to find self-shrinkers between them. In order to implement this heuristic, we first give a detailed description of the basic shape and limiting properties of the geodesics. We prove that the Euclidean curvature of a non-degenerate geodesic segment, written as a graph over the axis of rotation, can vanish at no more than two points, and we also show the different ways in which a family of geodesic segments can converge to a geodesic that exits the upper half plane. Then, after establishing the asymptotic behavior of geodesics near the plane, the cylinder, and Angenent's torus, we use induction arguments to construct infinitely many self-shrinkers near each of these self-shrinkers. A new feature of the construction is the

use of the Gauss-Bonnet formula to control the shapes of geodesics that almost exit the upper half plane.

We note that in the one-dimensional case, the self-shrinking solutions to the curve shortening flow have been completely classified (see Gage and Hamilton [34], Grayson [36], Abresch and Langer [1], Epstein and Weinstein [32], and Halldorsson [37]). One difficulty in higher dimensions ($n \geq 2$) is the presence of the $(n-1)/r$ term in the geodesic equation (2.3), which allows the Euclidean curvature of a geodesic to change sign and forces a geodesic intersecting the axis of rotation $\{r = 0\}$ to do so perpendicularly.

We also note that the existence of immersed S^2 self-shrinkers shows that the uniqueness results for constant mean curvature spheres in \mathbb{R}^3 (see Hopf [40]) and for minimal spheres in S^3 (see Almgren [2]) do not hold for self-shrinkers. In addition, Alexandrov's moving plane method does not seem to have a direct application to the self-shrinker equation.

4.1 Preliminary results

The self-shrinkers we construct have rotational symmetry about a line through the origin in \mathbb{R}^{n+1} , $n \geq 2$, and can be described by curves in the upper half of the (x, r) -plane. Recall from Chapter 2 that an arclength parametrized curve $\Gamma(s) = (x(s), r(s))$ is the profile curve of a self-shrinker if and only if the angle $\alpha(s)$ solves

$$\dot{\alpha}(s) = \frac{x(s)}{2} \sin \alpha(s) + \left(\frac{n-1}{r(s)} - \frac{r(s)}{2} \right) \cos \alpha(s),$$

where $\dot{x}(s) = \cos \alpha(s)$ and $\dot{r}(s) = \sin \alpha(s)$. Equation (2.3) is the geodesic equation for the conformal metric $g_{Ang} = r^{2(n-1)} e^{-(x^2+r^2)/2} (dx^2 + dr^2)$ on $\mathbb{H} = \{(x, r) : x \in \mathbb{R}, r > 0\}$. For $(x_0, r_0) \in \mathbb{H}$ and $\alpha_0 \in \mathbb{R}$, we let $\Gamma[x_0, r_0, \alpha_0]$ denote the unique solution to (2.3) satisfying

$$\Gamma[x_0, r_0, \alpha_0](0) = (x_0, r_0), \quad \dot{\Gamma}[x_0, r_0, \alpha_0](0) = (\cos(\alpha_0), \sin(\alpha_0)),$$

and we define $\underline{\Gamma}$ to be the space of all curves $\Gamma[x_0, r_0, \alpha_0]$.

There are several particular curves of interest belonging to $\underline{\Gamma}$, namely the embedded ones. The known embedded curves are the semi-circle $\sqrt{2n}(\cos(s), \sin(s))$, the lines $(0, s)$ and $(s, \sqrt{2(n-1)})$, and a closed convex curve discovered by Angenent in [3]. We will refer to these curves as the sphere, the plane, the cylinder, and Angenent's torus (since the rotations of these curves about the x -axis respectively generate a sphere S^n , a plane \mathbb{R}^n , a cylinder $\mathbb{R} \times S^{n-1}$, and a torus $S^1 \times S^{n-1}$). For convenience, we denote the sphere, the plane, and the cylinder curves by \mathcal{S} , \mathcal{P} , and \mathcal{C} , respectively. In Chapter 5 we present the result from [26] that the sphere of radius $\sqrt{2n}$ is the only embedded S^n self-shrinker with a rotational symmetry. In an independent work [50], Kleene and Møller showed that the sphere of radius $\sqrt{2n}$, the plane, and the cylinder of radius $\sqrt{2(n-1)}$ are the only embedded, rotationally symmetric self-shrinkers of their respective topological type. It is unknown if Angenent's torus is the only embedded, rotationally symmetric $S^1 \times S^{n-1}$ self-shrinker.

In Chapter 2, we showed that there is a smooth one parameter family of initial value problems, which we denote by $Q[x_0]$, satisfying

$$Q[x_0](0) = (x_0, 0), \quad \dot{Q}[x_0](0) = (0, 1).$$

The degeneracy of (2.3) reflects the imposed axial symmetry of our surfaces, and amounts to the fact that the tangent space of a smooth axially symmetric surface at the axis of symmetry is a perpendicular plane. We note that $Q[\sqrt{2n}]$ and $Q[0]$ are the sphere and the plane, respectively. In Chapter 3, we showed that there is $0 < x_1 < \sqrt{2n}$ so that $Q[x_1]$ is the profile curve of an immersed sphere self-shrinker.

It will be useful to view a curve $\Gamma \in \underline{\Gamma}$ from three different perspectives: as a graph $(x, u(x))$ over the x -axis, as a graph $(f(r), r)$ over the r -axis, and as a geodesic for the metric g_{Ang} . The differential equations satisfied by $u(x)$ and $f(r)$ place limitations on the oscillatory behavior of Γ , and we will use these equations to describe the basic shape of the curves in $\underline{\Gamma}$. In addition, we will use the continuity properties of geodesics

and the Gauss-Bonnet formula to establish convergence properties for the curves in $\underline{\Gamma}$.

This preliminary section is divided into three parts. First, we introduce some terminology and recall some known results about self-shrinkers. Then, we study the shape of solutions to (2.4). Finally, we use the Gauss-Bonnet formula (2.8) to prove some convergence results for geodesics.

4.1.1 Definitions and background

Our construction of immersed self-shrinkers follows from the study of the geometry of geodesic segments that are maximally extended as graphs over the x -axis. The plane and the cylinder are degenerate geodesics in the sense that their Euclidean curvature vanishes at every point. We will refer to a geodesic whose Euclidean curvature is not identically 0 as *non-degenerate*. We denote by $\underline{\Lambda}$ the space of non-degenerate geodesic segments that are maximally extended as graphs over the x -axis: $\Lambda \in \underline{\Lambda}$ if and only if $\Lambda \neq \mathcal{P}, \mathcal{C}$ is the graph of a maximally extended solution to (2.4). We note that the plane and the cylinder are the only geodesics that are not the union of elements of $\underline{\Lambda}$.

First, we describe the decomposition of a non-degenerate geodesic into the union of elements of $\underline{\Lambda}$. Given a non-degenerate geodesic of the form $\Gamma[x_0, r_0, \alpha_0]$, where $\cos(\alpha_0) \neq 0$, there exists a unique maximally extended solution $u : (a, b) \rightarrow \mathbb{R}$ to (2.4) with $u(x_0) = r_0$ and $u'(x_0) = \tan(\alpha_0)$. We define $\Lambda[0](\Gamma[x_0, r_0, \alpha_0]) \in \underline{\Lambda}$ to be the graph of u . If $b < \infty$ and $u(b) > 0$ (see Lemma 17), then the geodesic $\Gamma[x_0, r_0, \alpha_0]$ can be continued past the point $(b, u(b))$, and we denote this next maximally extended geodesic segment by $\Lambda[1](\Gamma[x_0, r_0, \alpha_0])$. In general, when it is defined, we use $\Lambda[k](\Gamma[x_0, r_0, \alpha_0]) \in \underline{\Lambda}$, $k \in \mathbb{Z}$, to denote the k^{th} maximally extended geodesic segment encountered in the parametrization of $\Gamma[x_0, r_0, \alpha_0]$, so that we get the (possibly finite) decomposition

$$\Gamma[x_0, r_0, \alpha_0] = \cdots \cup \Lambda[-1](\Gamma[x_0, r_0, \alpha_0]) \cup \Lambda[0](\Gamma[x_0, r_0, \alpha_0]) \cup \Lambda[1](\Gamma[x_0, r_0, \alpha_0]) \cup \cdots .$$

When $\cos(\alpha_0) = 0$, we define $\Lambda[k](\Gamma[x_0, r_0, \alpha_0])$ similarly.

Next, we introduce a topology on $\underline{\Lambda}$. Since every $\Lambda \in \underline{\Lambda}$ intersects the r -axis exactly once (see Proposition 1), there exists a unique pair $(r_\Lambda, \alpha_\Lambda) \in \mathbb{R}^+ \times (-\pi/2, \pi/2)$ such that

$$\Lambda = \Lambda[0](\Gamma[0, r_\Lambda, \alpha_\Lambda]).$$

Then $\underline{\Lambda}$ carries a topology induced by the natural distance function d :

$$d(\Lambda_1, \Lambda_2) = |r_{\Lambda_2} - r_{\Lambda_1}| + |\alpha_{\Lambda_2} - \alpha_{\Lambda_1}|.$$

By the continuity of geodesics, we know that a sequence $\Lambda_i \in \underline{\Lambda}$ converges smoothly to $\Lambda_\infty \in \underline{\Lambda}$ on compact subsets of \mathbb{H} if and only if $d(\Lambda_i, \Lambda_\infty) \rightarrow 0$.

To give a detailed description of the shape of a geodesic, we need to identify the points where its Euclidean curvature vanishes. For a C^2 curve γ in the upper half plane, we define the *degree* of γ to be the cardinality of the set where its Euclidean curvature vanishes, and we denote it by $deg(\gamma)$. In Section 4.1.2 we show that each geodesic segment $\Lambda \in \underline{\Lambda}$ satisfies $deg(\Lambda) \leq 2$ (see Proposition 10). We denote the space of all degree k curves in $\underline{\Lambda}$ by $\underline{\Lambda}(k)$, so that we have the following decomposition of $\underline{\Lambda}$:

$$\underline{\Lambda} = \bigcup_{k=0}^2 \underline{\Lambda}(k).$$

Writing a geodesic segment $\Lambda \in \underline{\Lambda}(k)$ as the graph of a maximally extended solution $u : (a, b) \rightarrow \mathbb{R}$ to (2.4), we know that u'' has a fixed sign near b (since u'' vanishes at k points). Therefore, we can decompose $\underline{\Lambda}(k)$ into the subsets $\underline{\Lambda}(k, +)$ and $\underline{\Lambda}(k, -)$, depending on the sign of u'' near its right end point. That is, we define $\underline{\Lambda}(k, +)$ to be the subset of $\underline{\Lambda}(k)$ consisting of maximally extended geodesic segments that are concave up near their right end points, and we define $\underline{\Lambda}(k, -)$ similarly.

In Section 2.8 we show that the boundaries of the sets $\underline{\Lambda}(k)$ in the (non-complete) topology on $\underline{\Lambda}$ consist of curves which exit the upper half plane either through the x -axis or through infinity (see Proposition 13). We refer to the elements of $\underline{\Lambda}$ that

exit the upper half plane either through the x -axis or through infinity as *half-entire graphs*, and we denote the set of all half-entire graphs by \underline{H} . The geodesics $Q[x_0]$ defined above correspond to a family of half-entire graphs that exit through the x -axis, namely the geodesic segments $\Lambda[0](Q[x_0])$. Using the linearization of the rotational self-shrinker differential equation near the sphere, Huisken's theorem on mean-convex self-shrinkers, and a comparison result for solutions to (2.6), we can prove the following result.

Proposition 8. *Let $Q = \Lambda[0](Q[x_0])$. Then $r_Q > \sqrt{2n}$ and $\alpha_Q < 0$ when $0 < x_0 < \sqrt{2n}$, and $r_Q < \sqrt{2n}$ and $\alpha_Q > 0$ when $x_0 > \sqrt{2n}$.*

Proof. The proof of this proposition follows from the results in Chapter 2. Using the linearization of the rotational self-shrinker differential equation near the sphere and Huisken's theorem, we have $\alpha_Q < 0$ when $0 < x_0 < \sqrt{2n}$, and $\alpha_Q > 0$ when $x_0 > \sqrt{2n}$ (see Proposition 5). Using the comparison results for quarter spheres (see Proposition 3 and Proposition 4), we know that Q intersects the sphere exactly once in the first quadrant, so that $r_Q > \sqrt{2n}$ when $0 < x_0 < \sqrt{2n}$, and $r_Q < \sqrt{2n}$ when $x_0 > \sqrt{2n}$. \square

A second family of half-entire graphs was constructed by Kleene and Møller (see Theorem 3 in [50]). They showed that for each fixed ray through the origin $r_\sigma(x) = \sigma x$, $\sigma > 0$, there exists a unique (non-entire) solution u_σ to (2.4), called a trumpet, asymptotic to r_σ so that: u_σ is defined on $[0, \infty)$; $u_\sigma(0) < \sqrt{2(n-1)}$; and $u_\sigma > r_\sigma$, $0 < u'_\sigma < \sigma$, and $u''_\sigma > 0$ on $[0, \infty)$. In addition, they showed that any solution to (2.4) defined on an interval (a, ∞) must be either be a trumpet u_σ or the cylinder $u \equiv \sqrt{2(n-1)}$. An immediate consequence of this last result is that the cylinder is the only entire solution to (2.4).

Now, we introduce some notation for the previously discussed half-entire graphs:

- *Inner-quarter spheres:* The set \underline{I}^+ of inner-quarter spheres in the first quadrant is the collection of curves of the form $I_x := \Lambda[0](Q[x])$, for $0 < x < \sqrt{2n}$. Each

$I \in \underline{I}^+$ intersects the r -axis above the sphere with negative slope:

$$r_I > \sqrt{2n}, \quad \alpha_I < 0.$$

- *Outer-quarter spheres:* The set \underline{Q}^+ of outer-quarter spheres in the first quadrant is the collection of curves of the form $O_x := \Lambda[0](Q[x])$, for $x > \sqrt{2n}$. Each $O \in \underline{Q}^+$ intersects the r -axis below the sphere with positive slope:

$$r_O < \sqrt{2n}, \quad \alpha_O > 0.$$

- *Trumpets:* The set \underline{T}^+ of trumpets in the first quadrant is the collection of the graphs of u_σ , where u_σ are the trumpets from [50]. Each $T \in \underline{T}^+$ intersects the r -axis below the cylinder with a positive slope:

$$r_T < \sqrt{2(n-1)}, \quad \alpha_T > 0.$$

The sets of half-entire graphs in the second quadrant: \underline{I}^- , \underline{Q}^- , and \underline{T}^- are defined similarly.

We also introduce the sets:

$$\underline{I} = \underline{I}^+ \cup \underline{I}^- \cup \{\mathcal{S}\}, \quad \underline{Q} = \underline{Q}^+ \cup \underline{Q}^- \cup \{\mathcal{S}\}, \quad \underline{T} = \underline{T}^+ \cup \underline{T}^-.$$

In Proposition 11, we show that the space \underline{H} of half-entire graphs is the union of the sets \underline{I} , \underline{Q} , and \underline{T} .

4.1.2 The shape of graphical geodesics

In this section, we study the shape of solutions to (2.4). This involves proving several results that place limitations on the possible behavior of these solutions. The main results in this section are Proposition 10, which shows that the Euclidean curvature $u''/(1+(u')^2)^{3/2}$ of a solution to (2.4) vanishes at no more than two points, and Proposition 11, which addresses the classification and the shapes of half-entire graphs.

Let u be a solution to (2.4). Recall, if u has a local maximum (minimum) at a point x_0 , then $u(x_0) \geq \sqrt{2(n-1)}$ ($\leq \sqrt{2(n-1)}$) with equality if and only if $u \equiv \sqrt{2(n-1)}$ is the cylinder. Also, if both u' and u'' vanish at the same point, then u must be the cylinder. Using (2.5), when u is non-degenerate¹, we see that u' and u''' have opposite signs at points where $u'' = 0$, so that the zeros of u'' are separated by zeros of u' . Therefore, a non-degenerate solution to (2.4) has a sinusoidal shape that oscillates between maxima above the cylinder and minima below the cylinder.

In the next part of this section, we show that the Euclidean curvature of a maximally extended non-degenerate solution to (2.4) can only vanish at a finite number of points.

Proposition 9. *Let $u : (a, b) \rightarrow \mathbb{R}$ be a non-degenerate maximally extended solution to (2.4). Then u'' can only vanish at a finite number of points.*

Proof. When $b < \infty$, it follows from the proof of Proposition 1 that $b > 0$ and there is a neighborhood of b in which u'' does not vanish. Suppose $b = \infty$, so that u is a solution to (2.4) on (a, ∞) , we know that u is either a trumpet or the cylinder. Since u is non-degenerate, it is a trumpet, and $u'' > 0$ on $[0, \infty)$. We conclude that $b > 0$ and u'' does not vanish in a neighborhood of b . Similar arguments may be applied to the left end point a to complete the proof of the proposition. \square

We make note of the following result, which was established during the proofs of Proposition 1 and Proposition 9.

Lemma 16. *Let $u : (a, b) \rightarrow \mathbb{R}$ be a maximally extended non-degenerate solution to (2.4). Then u' and u'' do not vanish in a neighborhood of b (or a). Moreover, u' and u'' have the same sign (have different signs) in this neighborhood.*

Proof. The lemma is true when $b = \infty$ by Theorem 3 in [50]. We assume that $b < \infty$. We also assume, from the proof of Proposition 1, that u'' does not vanish in

¹A solution to (2.4) is non-degenerate if it is not the cylinder.

a neighborhood of b . Using Lemma 2, we know that u' and u'' must have the same sign when u exits through the x -axis at b . If u exits through infinity (which does not happen when $b < \infty$), then given the previously described sinusoidal shape of u , we must have $u'(x), u''(x) \rightarrow \infty$ as $x \rightarrow b$. Finally, when $0 < \lim_{x \rightarrow b} u(x) < \infty$, it follows that $\lim_{x \rightarrow b} |u'(x)| = \infty$, and u' and u'' have the same sign near b . \square

Now that we've finished the proof of Proposition 9, we want to study the oscillatory behavior of solutions to (2.4). We begin by showing that a maximally extended solution $u : (a, b) \rightarrow \mathbb{R}$ to (2.4) cannot exit through infinity when b is finite.

Lemma 17. *Let $u : (a, b) \rightarrow \mathbb{R}$ be a maximally extended solution to (2.4). If b (or a) is finite, then $\lim_{x \rightarrow b} u(x) < \infty$ (or $\lim_{x \rightarrow a} u(x) < \infty$).*

Proof. We know from Lemma 16 that the $u'(x)$ and $u''(x)$ have the same sign (and do not vanish) as x approaches b . When u' and u'' are both negative near b , so that u is decreasing, the lemma holds. When u' and u'' are both positive near b , we will show that $\lim_{x \rightarrow b} u(x) < \infty$. To see this, suppose to the contrary that $\lim_{x \rightarrow b} u(x) = \infty$. Then there is a point $x_1 > 0$ for which $u(x_1) = x_1 u'(x_1)$. We consider the function $\Psi(x) = xu' - u$ (from Lemma 1 in [50]). If $x > 0$, then $\Psi' = xu'' > \frac{1}{2}x(1 + (u')^2)\Psi$, and hence $\Psi(x) > 0$ when $x > x_1$. Therefore, $u'(x) > 0$ and $u''(x) > 0$ for $x > x_1$. We note that $u''/(1 + (u')^2) \geq (n - 1)/u$ when $x > x_1$.

We will use a third derivative argument to show $u(b) < \infty$. Let $\psi = u'$. By the previous discussion, we have $\psi > 0$ and $\psi' > 0$ on (x_1, b) , and $\psi(x_1) = \frac{u(x_1)}{x_1}$. Using equation (2.5), for $x > x_1$, we have

$$\psi'' \geq \frac{1}{2}x_1\psi'\psi^2,$$

where we also used $u''/(1 + (u')^2) \geq (n - 1)/u$ when $x > x_1$.

Now, for small $\varepsilon > 0$, consider the function

$$\phi_\varepsilon(x) = \frac{M}{\sqrt{b - \varepsilon - x}}.$$

We choose $M > 0$ so that $\frac{u(x_1)}{x_1} \leq \frac{M}{\sqrt{b-x_1}}$ and $\frac{3}{M^2} \leq x_1$. Then

$$\phi_\varepsilon'' \leq \frac{1}{2}x_1\phi_\varepsilon'\phi_\varepsilon^2,$$

and $\phi_\varepsilon(x_1) > \psi(x_1)$. Suppose $\phi_\varepsilon - \psi$ is negative at some point in $(x_1, b - \varepsilon)$. Since $\phi_\varepsilon(x_1) > \psi(x_1)$ and $\phi_\varepsilon(b - \varepsilon) = \infty$, we know that $\phi_\varepsilon - \psi$ achieves a negative minimum at some point $x_0 \in (x_1, b - \varepsilon)$. Computing $(\phi_\varepsilon - \psi)''$ at x_0 , we arrive at a contradiction:

$$0 \leq (\phi_\varepsilon - \psi)''(x_0) \leq \frac{1}{2}x_1\phi_\varepsilon'(\phi_\varepsilon^2 - \psi^2)(x_0) < 0.$$

Therefore, $\psi \leq \phi_\varepsilon$. Taking $\varepsilon \rightarrow 0$ and integrating we see that u is bounded from above at b . \square

Next, we show that a solution to (2.4) must be convex when it perpendicularly intersects the r -axis below the cylinder.

Lemma 18. *Let $u : [0, b) \rightarrow \mathbb{R}$ be a solution to (2.4) satisfying*

$$u(0) < \sqrt{2(n-1)}, \quad u'(0) = 0.$$

Then $u''(x) > 0$ for $x \in [0, b_t)$.

Proof. Let $u_t : (a_t, b_t) \rightarrow \mathbb{R}$ be the maximally extended solution to (2.4) satisfying $u_t(0) = t$, $u_t'(0) = 0$. When $t < \sqrt{2(n-1)}$, we have $u''(0) > 0$, and if u_t is not strictly convex, then u_t has a sinusoidal shape and obtains a local maximum at a first point $y_t > 0$. In particular, there is a first point $z_t > 0$ such that

$$u_t(z_t) = \sqrt{2(n-1)}.$$

Examining equation (2.4), we conclude that $u_t'' > 0$ on $[0, z_t]$. Applying Lemma 4 to u_t written as a graph over the r -axis, we see that z_t cannot exist when $t < m_1$, and therefore $u_t'' > 0$ on $[0, b_t)$ for small $t > 0$.

We will use continuity to show that $u_t'' > 0$ on $[0, b_t)$ for all $0 < t < \sqrt{2(n-1)}$. Let $t_0 > 0$ be the first initial height for which this property does not hold. By continuity,

we know that $u''_{t_0} \geq 0$ on $[0, b_{t_0})$, and thus u_{t_0} does not have a sinusoidal shape with a local maximum at some first point. Therefore, we must have $t_0 \geq \sqrt{2(n-1)}$ (and hence $t_0 = \sqrt{2(n-1)}$), which completes the proof of the lemma. \square

Using the continuity of geodesics, we can prove a more general version of the previous lemma. By considering the family of shooting problems: $u_t(t) = r_0$, $u'_t(t) = 0$, for $t \in [0, x_0]$, where $r_0 < \sqrt{2(n-1)}$ and $x_0 > 0$, we can show that the solution $u_t : [t, b_t) \rightarrow \mathbb{R}$ to (2.4) is strictly convex on $[t, b_t)$. For the continuity argument to work we assume u_t is maximally extended at b_t and use the facts that $b_t < \infty$ and $\lim_{x \rightarrow b_t} u_t(x) < \infty$. We have the following result.

Lemma 19. *Let $u : [x_0, b) \rightarrow \mathbb{R}$ be a solution to (2.4) with*

$$u(x_0) < \sqrt{2(n-1)}, \quad u'(x_0) = 0,$$

where $x_0 > 0$. Then $u''(x) > 0$ for $x \in [x_0, b)$.

The following lemma shows that solutions which intersect the r -axis below the cylinder with negative slope are convex in the first quadrant.

Lemma 20. *Let $u : [0, b) \rightarrow \mathbb{R}$ be a solution to (2.4) satisfying*

$$u(0) < \sqrt{2(n-1)}, \quad u'(0) < 0.$$

Then $u''(x) > 0$ for $x \in [0, b)$.

Proof. We assume that u is maximally extended at b . Since $u(0) < \sqrt{2(n-1)}$ and $u'(0) < 0$, we know that $b < \infty$, and $u''(x) > 0$ as long as $u' < 0$ on $[0, x)$. It then follows (from Lemma 16) that u has a minimum somewhere on $[0, b)$. Appealing to Lemma 19 applied to the first minimum on $[0, b)$ proves the lemma. \square

Slightly adapting the proof of Lemma 20 we can show that solutions to (2.4) which intersect the r -axis between the cylinder and the sphere with negative slope are degree 1 curves in the first quadrant.

Lemma 21. *Let $u : [0, b) \rightarrow \mathbb{R}$ be a solution to (2.4), maximally extended at b , satisfying*

$$\sqrt{2(n-1)} \leq u(0) \leq \sqrt{2n}, \quad u'(0) < 0.$$

Then there is a point $x_0 \in [0, b)$ so that $u''(x) \leq 0$ for $x \in [0, x_0]$, and $u''(x) > 0$ for $x \in (x_0, b)$. Furthermore, there is a point $x_1 > x_0$ for which $u'(x_1) = 0$.

Proof. We consider the following family of shooting problems: For $t \in (0, u(0)]$, let $u_t : [0, b_t) \rightarrow \mathbb{R}$ be the solution to (2.4) with $u_t(0) = t$ and $u'_t(0) = u'(0)$. We assume that u_t is maximally extended at b_t , and we will use the facts that $b_t < \infty$ and $\lim_{x \rightarrow b_t} u_t(x) < \infty$, which follow from Theorem 3 in [50] and Lemma 17.

When $t < \sqrt{2(n-1)}$, we know (from the proof of Lemma 20) that u_t has a unique local minimum at a point $x_1^t \in (0, b_t)$ (the uniqueness follows from Lemma 19). We claim that this property is true for all $t \leq u(0)$. Suppose to the contrary that u_{t_*} does not have a local minimum in $(0, b_{t_*})$ for some first $t_* \leq u(0)$. Then, as t increases to t_* , the points $p_t = (x_1^t, u_t(x_1^t))$ must exit the first quadrant of the upper half plane. Since $u'_t(0) = u'(0) < 0$ and $u_t(0) \leq u(0)$, we know that the points p_t are bounded away from the r -axis. By continuity, since $b_{t_*} < \infty$, we know that x_1^t is bounded when t is less than and close to t_* . We also know that $u_t(x_1^t) \leq u(0)$ when $t < t_*$, and it follows that the points p_t cannot exit the first quadrant through infinity. Finally, since $u_{t_*}(0) \leq \sqrt{2n}$ and $u'_{t_*}(0) < 0$ we know from Proposition 8 that the graph of u_{t_*} is not a quarter sphere, and by continuity the points p_t are bounded away from the x -axis. Therefore, the points p_t cannot exit the first quadrant, which is a contradiction. We conclude that u_t has a unique local minimum at a point $x_1^t \in (0, b_t)$ for all $t \leq u(0)$.

Now, we can describe the behavior of u_t when $\sqrt{2(n-1)} \leq t \leq u(0)$. We know that u_t has a local minimum at x_1^t , and applying Lemma 19 we have $u''_t(x) > 0$ for $x \geq x_1^t$. Since $u'_t(0) < 0$, it follows from the sinusoidal shape of u_t that $u'_t = 0$ at exactly one point. Using $u''_t(0) \leq 0$ and $u''_t(x_1^t) > 0$, we see that $u''_t(x_0^t) = 0$ at some first point $x_0^t \in [0, x_1^t)$. Again appealing to the sinusoidal shape of u_t , we conclude

that $u_t''(x) > 0$ for $x \in (x_0^t, x_1^t)$, which finishes the proof of the lemma. \square

Now, we can show that a solution to (2.4) does not oscillate too much. We state the result in terms of the degree.

Proposition 10. *Let $\Lambda \in \underline{\Lambda}$ be a maximally extended geodesic segment. Then*

$$\deg(\Lambda) \leq 2.$$

Moreover, the only maximally extended geodesic segments with degree 2 are type $(2, +)$.

Proof. Let $\Lambda \in \underline{\Lambda}$ be a maximally extended geodesic segment. Given the sinusoidal shape of Λ , we know that Λ alternates between maxima and minima, and its Euclidean curvature vanishes exactly once between any successive maximum and minimum. Now, it follows from Lemma 18 and Lemma 19 that Λ remains convex after a minimum in the first quadrant (including the r -axis), and a similar statement holds in the second quadrant. In particular, Λ can have at most two minima (one in each quadrant). Also, Λ can have at most one maximum; otherwise Λ would oscillate ‘after’ a minimum, which cannot occur. It follows that $\deg(\Lambda) \leq 2$. Moreover, $\deg(\Lambda) = 2$ if and only if Λ has two minima, in which case Λ is type $(2, +)$. \square

The next proposition shows that the quarter spheres and trumpets account for all the half-entire graphs, and it classifies them into their different types.

Proposition 11. *The space \underline{H} of half-entire graphs is the union of the sets \underline{I} , \underline{Q} , and \underline{T} . Moreover, the elements of \underline{I}^+ are type $(0, -)$, the elements of \underline{Q}^+ are type $(1, -)$, and the elements of \underline{T}^+ are type $(0, +)$.*

Proof. We know that an element of $\underline{\Lambda}$ that exits the upper half plane through infinity must be a trumpet. We also know that a half-entire graph that exits through the x -axis must do so perpendicularly (use equation (2.4) and Lemma 16). Therefore, the space \underline{H} of half-entire graphs is the union of the sets \underline{I} , \underline{Q} , and \underline{T} .

Now we address the types of the half-entire graphs in the first quadrant. Let $Q[x_0]$ denote the geodesic satisfying $Q[x_0](0) = (x_0, 0)$ and $\dot{Q}[x_0](0) = (0, 1)$. For small $x_0 > 0$, it was shown in Chapter 3 that $\Lambda[0](Q[x_0])$ is type $(0, -)$ with a maximum in the second quadrant and a finite left end point. Arguing by continuity and using Huisken's theorem for mean-convex self-shrinkers, we see that this property persists for $0 < x_0 < \sqrt{2n}$. Therefore, the curves in \underline{I}^+ are type $(0, -)$. When $x_0 > \sqrt{2n}$, we know that $\Lambda[0](Q[x_0])$ intersects the r -axis below the sphere with positive slope, and it follows from Lemma 20 and Lemma 21 that $\Lambda[0](Q[x_0])$ is type $(1, -)$. Therefore, the curves in \underline{O}^+ are type $(1, -)$. Finally, it follows from Lemma 19 that the curves in \underline{T}^+ are type $(0, +)$ since $r_T < \sqrt{2(n-1)}$ and $\alpha_T > 0$ for $T \in \underline{T}^+$. \square

4.1.3 Applications of the Gauss-Bonnet formula

In this section we use the Gauss-Bonnet formula (2.8) to prove some convergence results for geodesics. Applying the Gauss-Bonnet formula to a region with a piecewise geodesic boundary shows that the region cannot enclose a 'large' area. In addition, if the region is near the x -axis, then it must enclose a 'small' area. Using this heuristic, we show that a family of geodesics converging to a half-entire graph will converge to the half-entire graph as the geodesics leave and return to the upper half plane (see Proposition 12). We also use the Gauss-Bonnet formula to describe the boundaries of the sets $\underline{\Lambda}(k, \pm)$.

We begin by showing that a geodesic cannot interpolate between two different half-entire graphs in the first quadrant.

Lemma 22. *Let $\Gamma_i \in \underline{\Gamma}$ be a sequence of geodesics with at least 2 graphical components. Let $u_i = \Lambda[0](\Gamma_i)$ and $v_i = \Lambda[1](\Gamma_i)$, and suppose the sequences u_i and v_i converge to the half-entire graphs u_∞ and v_∞ . Then $u_\infty = v_\infty$. The conclusion also holds when u_∞ is the cylinder.*

Proof. First, suppose u_∞ and v_∞ are both quarter spheres. Let p and q denote the

right end points of u_∞ and v_∞ , respectively. If $u_\infty \neq v_\infty$, then $p \neq q$, and there exists $\delta > 0$ so that $|p - q| > 2\delta$. Then for small $\varepsilon > 0$, we claim there exists a rectangle R of the form: $x_0 \leq x \leq x_0 + \delta$, $\varepsilon/2 \leq r \leq \varepsilon$ so that, for large i , the rectangle R is contained in a simple region bounded by Γ_i and the r -axis. To see this, we assume without loss of generality that the x -coordinate of p is less than the x -coordinate of q . Given the sinusoidal shapes of u_i and v_i , we know from Lemma 19 that $\Lambda[0](\Gamma_i)$ is type $(k_0, +)$ and hence $\Lambda[1](\Gamma_i)$ is type $(k_1, -)$, for some k_0 and k_1 . Using the continuity of the differential equation (2.4), we see that Γ_i follows along u_∞ (getting arbitrarily close to p), then follows the x -axis (getting arbitrarily close to q), and then travels back to the r -axis along v_∞ . This proves the claim. To conclude the proof of the lemma in this case, we observe that $\int_R r^{-2} dx dr = \delta/\varepsilon$, and an application of the Gauss-Bonnet formula (2.8) shows that this is impossible when ε is small.

Second, suppose u_∞ and v_∞ are both trumpets. Then there exist rays $r_\sigma(x) = \sigma x$ and $r_\tau(x) = \tau x$ so that u_∞ and v_∞ are asymptotic to r_σ and r_τ , respectively. If $u_\infty \neq v_\infty$, then $\sigma \neq \tau$. Now, the wedge between r_σ and r_τ has infinite area, and the same is true for the area of the wedge outside any compact set. Arguing as in the first case and using the property that the trumpets are asymptotic to the rays, we can show there is a simple region bounded by Γ_i and the r -axis that encloses arbitrarily large area as $i \rightarrow \infty$. An application of the Gauss-Bonnet formula shows that this is impossible. The proof is similar when one of the trumpets is the cylinder.

Finally, suppose u_∞ is a quarter sphere and v_∞ is a trumpet or a cylinder. It follows from the sinusoidal shape of u_i and Lemma 19 that $\Lambda[0](\Gamma_i)$ is type $(k_0, +)$ for some k_0 . Then, arguing as in the previous cases, we can show there is a simple region bounded by Γ_i and the r -axis that encloses arbitrarily large area as $i \rightarrow \infty$, and an application of the Gauss-Bonnet formula shows that this is impossible. \square

Next, we prove a lemma that describes the shape of a solution $u(x)$ to (2.4) when $u(0)$ is small or large.

Lemma 23. *Let m_1 and M_1 be the constants defined in Lemma 4 and Lemma 3. If $u : (a, b) \rightarrow \mathbb{R}$ is a maximally extended solution to (2.4), then*

1. *If $u(0) < m_1$, then the graph of u is in $\underline{\Lambda}(0, +)$.*

2. *If $u(0) > M_1$, then the graph of u is in $\underline{\Lambda}(0, -)$.*

Moreover, $a, b \rightarrow 0$ as $u(0) \rightarrow 0$ or $u(0) \rightarrow \infty$.

Proof. First, we treat the case where $u(0) < m_1$. If $u'(0) < 0$, then it follows from Lemma 20 that $u''(x) > 0$ for $x \geq 0$. When $u'(0) \geq 0$, it follows from (2.4) that $u''(x) > 0$ for $x \geq 0$ as long as $u < \sqrt{2(n-1)}$ on $[0, x]$. In both cases, we observe that a portion of the geodesic $(x, u(x))$ may be written as a graph over the r -axis: $(f(r), r)$, where f is a solution to (2.6). We claim that $u < \sqrt{2(n-1)}$ on $[0, b]$. To see this, suppose to the contrary that $u(x) \geq \sqrt{2(n-1)}$ for some $x > 0$. Then we may choose f so that $f(r) > 0$, $f'(r) > 0$, and $f''(r) < 0$ when $m_1 \leq r \leq \sqrt{2(n-1)}$. Applying Lemma 4 shows that this is impossible, and therefore $u < \sqrt{2(n-1)}$ on $[0, b]$. In particular, we have $u'' > 0$ on $[0, b]$.

Now, we estimate b in terms of $u(0)$. If, say, $u(b) \leq 3u(0)$, then $u(x) \leq 3u(0)$ on $[0, b]$, and we can write equation (2.4) as

$$\frac{d}{dx} (\arctan u') = \frac{xu' - u}{2} + \frac{n-1}{u} \geq -\frac{u(0)}{2} + \frac{n-1}{3u(0)},$$

where we have used $xu' - u$ is increasing on $(0, b)$. Integrating from 0 to b , we have

$$\pi \geq \left(\frac{n-1}{3u(0)} - \frac{u(0)}{2} \right) b,$$

and thus $b \rightarrow 0$ as $u(0) \rightarrow 0$. In general, if $u(b) = Au(0)$, where $A > 1$, then

$$\pi \geq \left(\frac{n-1}{Au(0)} - \frac{u(0)}{2} \right) b. \quad (4.1)$$

Applying the Gauss-Bonnet formula to the triangle T with vertices $(0, u(0))$, $(0, u(b))$, and $(b, u(b))$, which is enclosed by the geodesic corresponding to u and the r -axis, we

have

$$4\pi \geq \int_T \frac{n-1}{r^2} dx dr = \frac{(n-1)b}{(A-1)u(0)} \left[\log A + \frac{1}{A} - 1 \right].$$

If A is sufficiently large, then

$$4\pi \geq \frac{b \log A}{u(b)} \geq \frac{b \log A}{\sqrt{2(n-1)}}. \quad (4.2)$$

It follows that $b \rightarrow 0$ as $u(0) \rightarrow 0$: Fix $\varepsilon > 0$, and choose $u(0) < \varepsilon e^{-1/\varepsilon}$. If $A < e^{1/\varepsilon}$, then $Au(0) < \varepsilon$ and (4.1) implies $b \lesssim \varepsilon$. If $A \geq e^{1/\varepsilon}$, then $\log A \geq 1/\varepsilon$ and (4.2) implies $b \lesssim \varepsilon$.

Second, we treat the case where $u(0) > M_1$. Since $u(0) > \sqrt{2(n-1)}$, we know that $u''(0) < 0$ and $b < \infty$. When $u'(0) \leq 0$, it follows from (2.4) that $u''(x) < 0$ for $x \geq 0$ as long as $u > \sqrt{2(n-1)}$ on $[0, x]$. When $u'(0) > 0$, we can use the sinusoidal shape of u (and Lemma 16) to conclude that the first zero of u' occurs before the first zero of u'' , and similar reasoning shows that $u''(x) < 0$ as long as $u > \sqrt{2(n-1)}$ on $[0, x]$. Now, the decreasing portion of the geodesic $(x, u(x))$ can be written as a graph over the r -axis: $(f(r), r)$, where f is a solution to (2.6) and $f(r) > 0$ and $f'(r) \leq 0$ when $u(b) \leq r \leq M_1$. Applying Lemma 3 shows $u(b) > \sqrt{2(n-1)}$. In particular, we have $u'' < 0$ on $[0, b)$.

To estimate b in terms of $u(0)$, we note that the function f from the above paragraph is defined on $[\sqrt{2(n-1)}, M_1]$. Using the shape of the graph of u and equations (2.6) and (2.7) we know that $f > 0$ and $f'' < 0$ on $[\sqrt{2(n-1)}, M_1]$. Then the region bounded by the geodesic corresponding to u and the r -axis contains the triangle T with vertices $(0, M_1)$, $(0, \sqrt{2(n-1)})$, and $(b, u(b))$. Using the Gauss-Bonnet formula, we have $4\pi \geq \int_T dx dr = [M_1 - \sqrt{2(n-1)}]b/2$, so that $b \rightarrow 0$ as $u(b) \rightarrow \infty$.

Finally, the same arguments apply to the left end point a . □

Next, we prove a lemma which restricts the domain of a solution to (2.4) that intersects the r -axis with steep negative slope.

Lemma 24. *Let u be a maximally extended solution to (2.4) defined on the interval (a, b) . Then $b \rightarrow 0$ as $u'(0) \rightarrow -\infty$.*

Proof. Fix $\varepsilon > 0$. We will show there is $L > 0$ so that $b \lesssim \varepsilon$ when $u'(0) \leq -L$. By Lemma 23, there exist positive constants m and M so that $b < \varepsilon$ when $u(0) < m$ or $u(0) > M$, so we may assume that $m \leq u(0) \leq M$. There are two cases to consider, depending on the shape of u .

Case 1: $u'(0) \leq -L$ and $u'' < 0$ on $[0, b)$. Since $u(0) \leq M$, $u'(0) \leq -L$, and $u'' < 0$, we know that $x \leq M/L$ whenever $u(x)$ is defined (integrate $u' \leq -L$ from 0 to x). Therefore $b \leq M/L$, and we may choose $L > M/\varepsilon$.

Case 2: $u'(0) \leq -L$ and $u''(x) \geq 0$ for some $x \geq 0$. For large enough L (depending only on M and ε), using the continuity of the differential equation (2.6), we know there is a point $(c, u(c))$ so that $c < \varepsilon$ and $u(c) < \varepsilon$. By choosing $L > M/\varepsilon$, we may assume that $u''(c) > 0$ (see Case 1). Furthermore, by allowing for $c < 2\varepsilon$, we may assume $u'(c) \geq -1$. We work with $\varepsilon e^{-1/\varepsilon}$ in place of ε : We assume $c, u(c) < \varepsilon e^{-1/\varepsilon}$. Arguing as in the proof of Lemma 23, we write equation (2.4) as

$$\frac{d}{dx} (\arctan u'(x)) = \frac{xu'(x) - u(x)}{2} + \frac{n-1}{u(x)} \geq -\varepsilon e^{-1/\varepsilon} + \frac{n-1}{u(x)},$$

where we have used $xu(x)' - u(x)$ is increasing when $x \geq c$, along with the estimates on c , $u(c)$, and $u'(c)$. If $u(b) = Au(c)$, for some $A > 1$, then integrating from c to b , we have

$$\pi \geq \left(\frac{n-1}{Au(c)} - \varepsilon e^{-1/\varepsilon} \right) (b-c). \quad (4.3)$$

Applying the Gauss-Bonnet formula to the triangle T with vertices $(c, u(c))$, $(c, u(b))$, we have

$$4\pi \geq \int_T \frac{n-1}{r^2} dx dr = \frac{(n-1)(b-c)}{(A-1)u(c)} \left[\log A + \frac{1}{A} - 1 \right],$$

so that

$$4\pi \geq \frac{(b-c) \log A}{u(b)} \geq \frac{(b - \varepsilon e^{-1/\varepsilon}) \log A}{\sqrt{2(n-1)}}, \quad (4.4)$$

when A is sufficiently large. If $A < e^{1/\varepsilon}$, then $Au(c) < \varepsilon$ and (4.3) implies $b \lesssim \varepsilon$. If $A \geq e^{1/\varepsilon}$, then $\log A \geq 1/\varepsilon$ and (4.4) implies $b \lesssim \varepsilon$. If $u(b) \leq u(c)$, then

$$\pi \geq \left(\frac{n-1}{u(c)} - \varepsilon e^{-1/\varepsilon} \right) (b-c),$$

and we also have $b \lesssim \varepsilon$. □

The following result will be used in the proof of Lemma 26 to restrict the domain of a solution to (2.4) that intersects the r -axis with steep positive slope.

Lemma 25. *Let u be a maximally extended solution to (2.4) defined on the interval (a, b) . If u has a local maximum at $x_1 > 0$ and $u(x_1) > \max\{u(0), \sqrt{2(n-1)}\} + 2$, then $b < 2x_1$.*

Proof. This lemma follows from the proofs of Claim 9 and Lemma 15. Those results show that $u(b) > u(x_1) - 2$, and $u(x_1 - s) \geq u(x_1 + s)$ when $s > 0$. Since $u(b) > u(x_1) - 2 > u(0)$ and $u'' < 0$ on $[0, b)$, we have $b < 2x_1$. For convenience, we include proofs of these two facts.

Part 1: $u(b) > u(x_1) - 2$. The graph $(x, u(x))$ for $x > x_1$ can be written as the graph $(f(r), r)$, where f is a solution to (2.6). Now $f > 0$ and $f' < 0$ in a neighborhood of $u(x_1)$ (when $f(r)$ is defined), and using equations (2.6) and (2.7), we also have $f'' < 0$ and $f''' < 0$. Assuming $f' < 0$, these inequalities hold when $r \geq \sqrt{2(n-1)}$. Repeatedly integrating $f''' < 0$ from r to $u(x_1)$, we have $0 < [1 - (u(x_1) - r)^2/4]f(r)$, so that $f'(r) = 0$ for some $r > u(x_1) - 2$; hence $u(b) > u(x_1) - 2$.

Part 2: $u(x_1 - s) \geq u(x_1 + s)$ when $s > 0$. Since $u'(x_1) = 0$, using (2.6) and (2.7), we have

$$u'''(x_1) = \frac{x_1}{2} \left(\frac{n-1}{u(x_1)} - \frac{u(x_1)}{2} \right).$$

Let $\delta(s) = u(x_1 + s) - u(x_1 - s)$. Then $\delta(0) = \delta'(0) = \delta''(0) = 0$ and $\delta'''(0) = 2u'''(x_1) < 0$. It follows that $\delta(s) < 0$ for small $s > 0$. We will show that $\delta(s) < 0$ when $s > 0$. Let f be as in Part 1, and let g be the solution to (2.6) corresponding to the graph

of $u(x)$ for $x \leq x_1$. We note that there exists $0 < t < s$ so that $u(x_1 + t) = u(x_1 - s)$ when $s > 0$ is small. Setting $h(r) = f(r) + g(r)$, we have $h(r) = 2x_1 + t - s < 2x_1$ so that $h(r) < 2x_1$ when $r < u(x_1)$ is close to $u(x_1)$. We claim that $h(r) < 2x_1$ for $r \in (u(b), u(x_1))$. To see this, suppose that $h(r) = 2x_1$ for some $r \in (u(b), u(x_1))$. Then h achieves a positive local minimum at some point $r_0 \in (u(b), u(x_1))$. At r_0 we have $h(r_0) > 0$, $h'(r_0) = 0$, and $h''(r_0) \geq 0$, so that

$$0 \leq \frac{h''(r_0)}{1 + (f'(r_0))^2} = \left(\frac{r}{2} - \frac{n-1}{r} \right) 2f'(r_0) - \frac{h(r_0)}{2} < 0,$$

which is a contradiction. Therefore, $h(r) < 2x_1$ for $r \in (u(b), u(x_1))$. Finally, to see $\delta(s) < 0$ when $s > 0$, we suppose to the contrary that $\delta(s) = 0$ for some $s > 0$. Set $r = u(x_1 + s) = u(x_1 - s)$. Then

$$2x_1 > h(r) = (x_1 + s) + (x_1 - s) = 2x_1,$$

which is a contradiction. We conclude that $\delta(s) < 0$ when $s > 0$. \square

Now, we prove an estimate for the second graphical component of a geodesic whose first graphical component is close to a half-entire graph in the first quadrant.

Lemma 26. *Let $\Gamma_i \in \underline{\Gamma}$ be a sequence of geodesic curves with at least 2 graphical components. Let $u_i = \Lambda[0](\Gamma_i)$ and $v_i = \Lambda[1](\Gamma_i)$. Suppose the sequence u_i converges to u_∞ , where u_∞ is a half-entire graph in the first quadrant or the cylinder. Then there exist positive constants m , M , and L , depending on u_∞ , so that $m \leq v_i(0) \leq M$ and $|v'_i(0)| \leq L$.*

Proof. By choosing u_i sufficiently close to u_∞ , we may assume that the right end point of u_i is bounded away from the r -axis. Applying Lemma 23 and Lemma 24, we see that there are positive constants m , M , and L so that $m \leq v_i(0) \leq M$ and $v'_i(0) \geq -L$. We want to find an upper bound for $v'_i(0)$.

Fix small $\varepsilon > 0$, and choose $C > 4\pi/\varepsilon$ so that $u_\infty < C$ on $[0, 2\varepsilon]$. By continuity we also assume that $u_i < C$ on $[0, 2\varepsilon]$. If $v'_i(0)$ is sufficiently large, then $v_i(x_0) = 2C$

for some $x_0 < \varepsilon$. We claim that v_i has a local maximum at some point in $(0, 2\varepsilon)$. Suppose to the contrary that v_i has no local maximum in $(0, 2\varepsilon)$. Then the rectangle $R: x_0 \leq x \leq x_0 + \varepsilon, C \leq r \leq 2C$ is contained in a simple region bounded by the geodesic Γ_i and the r -axis. Applying the Gauss-Bonnet formula we arrive at a contradiction, which proves the claim. It follows from Lemma 25 that the right end point of v_i is less than 4ε . Since the right end point of u_i is bounded away from the r -axis, we conclude that $v'_i(0)$ has an upper bound. \square

Combining the previous results, we have the following proposition, which deals with the convergence of geodesics to half-entire graphs.

Proposition 12. *Let $\Gamma_i \in \underline{\Gamma}$ be a sequence of geodesics with at least $(k+2)$ graphical components, and suppose that the graphs $\Lambda[k](\Gamma_i)$ converge to a half-entire graph in the first quadrant Λ_0 . Then, either $\Lambda[k+1](\Gamma_i)$ or $\Lambda[k-1](\Gamma_i)$ converge to Λ_0 . The conclusion also holds when Λ_0 is the cylinder.*

Proof. Without loss of generality, we may assume that k is even. Under this assumption, the right end point of $\Lambda[k](\Gamma_i)$ is also the right end point of $\Lambda[k+1](\Gamma_i)$. To simplify notation, we let $u_i = \Lambda[k](\Gamma_i)$, $v_i = \Lambda[k+1](\Gamma_i)$, and $u_\infty = \Lambda_0$. Here we are identifying a solution to (2.4) with its graph.

With the above notation, the solutions u_i converge to the half-entire graph u_∞ . To prove the proposition, we need to show that the sequence of initial conditions $(v_i(0), v'_i(0))$ converges to $(u_\infty(0), u'_\infty(0))$. It is sufficient to show that every subsequence of $(v_i(0), v'_i(0))$ has a subsequence converging to $(u_\infty(0), u'_\infty(0))$.

We know from Lemma 26 that every subsequence of $(v_i(0), v'_i(0))$ has a convergent subsequence. Let v_∞ be the solution to (2.4) corresponding to such a convergent subsequence. Notice that v_∞ is a half-entire graph in the first quadrant (otherwise, v_∞ has a right end point in the upper half plane, and by continuity u_i cannot converge to u_∞). It follows from Lemma 22 that $v_\infty = u_\infty$. \square

function, we have $x_0 \rightarrow \infty$ as $\epsilon \rightarrow 0$. We note that $u_{0,\epsilon}(x_1) < \sqrt{2(n-1)}$ so that $u_{0,\epsilon}$ intersects the cylinder between x_0 and x_1 . Applying the Gauss-Bonnet formula to the region contained below the graph of $u_{0,\epsilon}$ and above the cylinder, we arrive at a contradiction (since the area of this region approaches ∞ as $\epsilon \rightarrow 0$). We conclude that $u_{0,\epsilon}$ is globally convex (and $T \notin \underline{\Lambda}(0, +)$). Applying Proposition 12 to Γ_ϵ we see that $u_{1,\epsilon}(x) \rightarrow u_{0,0}(x)$ as $\epsilon \rightarrow 0$, and examining the possible types of curves, we see that the graph of $u_{1,\epsilon}$ must be degree 1 for small ϵ . This says that T is in the boundary of $\underline{\Lambda}(0, +)$ and $\underline{\Lambda}(1, -)$. In case (b), we similarly conclude that the graph of $u_{0,\epsilon}$ is type $(1, -)$ and the graph of $u_{1,\epsilon}$ is type $(0, +)$ for small ϵ . In both cases we get that T is in the boundary of $\underline{\Lambda}(0, +)$ and $\underline{\Lambda}(1, -)$. A similar result holds when $T \in \underline{T}^-$.

Next, we consider the half-entire graphs in \underline{I} . Let I be an inner-quarter sphere (or the sphere). By performing a similar perturbation as above, we obtain curves Γ_ϵ with $\Lambda[0](\Gamma_\epsilon) \notin \underline{H}$ converging to I as $\epsilon \rightarrow 0$. If $I \in \underline{I}^+$, then it is type $(0, -)$, and an argument similar to the one in the trumpet case shows that $\Lambda[0](\Gamma_\epsilon)$ and $\Lambda[1](\Gamma_\epsilon)$ are type $(0, -)$ and type $(1, +)$ (or type $(1, +)$ and type $(0, -)$). It follows that \underline{I}^+ is contained in both $\partial\underline{\Lambda}(0, -)$ and $\partial\underline{\Lambda}(1, +)$. A similar result holds for \underline{I}^- . Also, since the sphere \mathcal{S} is the limit of elements in \underline{I}^+ (and \underline{I}^-), we see that \mathcal{S} is in $\partial\underline{\Lambda}(0, -)$, $\partial\underline{\Lambda}(1, -)$, and $\partial\underline{\Lambda}(1, +)$.

Lastly, we consider the outer-quarter spheres. Arguing as we did for the inner-quarter spheres, we have \underline{O}^+ is contained in both $\partial\underline{\Lambda}(1, -)$ and $\partial\underline{\Lambda}(2, +)$, and a similar result holds for \underline{O}^- . We note that $\mathcal{S} \in \partial\underline{\Lambda}(2, +)$.

Finally, by considering the possible limiting shapes of different types of curves and using the continuity of geodesics, we can complete the proof of the proposition. For instance, the limit of type $(1, +)$ curves can only be type $(0, +)$, type $(0, -)$, or type $(1, +)$, and by continuity such a limit cannot be in \underline{T}^+ , \underline{I}^- or \underline{O}^+ . \square

Several convergence results follow from Proposition 13. For instance, the geodesic

limit of type $(1, -)$ curves whose right end points remain bounded away from the r -axis in a compact subset of \mathbb{H} , must either be a type $(1, -)$ curve or a type $(0, -)$ curve. In particular, if these curves converge to a half-entire graph, then it must be in \underline{I}^- . We collect some of these results in the following corollary.

Corollary 1. *Let Λ_t be a family of geodesic segments in $\underline{\Lambda}(k, \pm)$ whose right end points p_t remain bounded away from the r -axis in a compact subset of \mathbb{H} . If $p_t \rightarrow p_\infty$, then there exists $\Lambda_\infty \in \underline{\Lambda}$ so that $\Lambda_t \rightarrow \Lambda_\infty$ both as geodesics and in the topology defined on $\underline{\Lambda}$. Moreover, if Λ_∞ is a half-entire graph, then the following statements hold:*

1. *If $\Lambda_t \in \underline{\Lambda}(0, +)$, then $\Lambda_\infty \in \underline{T}^-$,*
2. *If $\Lambda_t \in \underline{\Lambda}(0, -)$, then $\Lambda_\infty \in \underline{I}^-$,*
3. *If $\Lambda_t \in \underline{\Lambda}(1, +)$, then $\Lambda_\infty \in \underline{Q}^- \cup \underline{T}^-$,*
4. *If $\Lambda_t \in \underline{\Lambda}(1, -)$, then $\Lambda_\infty \in \underline{I}^-$,*
5. *If $\Lambda_t \in \underline{\Lambda}(2, +)$, then $\Lambda_\infty \in \underline{Q}^-$.*

4.2 Shooting problems

Our construction of immersed self-shrinkers involves the study of two shooting problems for the geodesic equation (2.3). In one of the shooting problems, we shoot perpendicularly from the x -axis and study the geodesics $Q[x_0] = \Gamma[x_0, 0, \pi/2]$. In the other shooting problem, we shoot perpendicularly from the r -axis and study the geodesics $\Gamma[0, r_0, 0]$. In both cases, the goal is to find a geodesic whose k^{th} component is a half-entire graph. The rotation of such a geodesic about the x -axis is a self-shrinker. In addition, we note that a geodesic, from one of these shooting problems, whose k^{th} component intersects the r -axis perpendicularly also corresponds to a self-shrinker.

In Section 4.1.3, we showed that the boundaries of the sets $\underline{\Lambda}(k, \pm)$ are half entire graphs (see Proposition 13 and Corollary 1). It follows that whenever a continuous

family of geodesic segments in $\underline{\Lambda}$ changes type, it must move through a half-entire graph. Therefore, we can construct self-shrinkers by finding solutions to the shooting problems whose components eventually have different types. In order to construct infinitely many self-shrinkers in this way, we first establish the asymptotic behavior of geodesics near the plane, the cylinder, and Angenent's torus.

4.2.1 Behavior of geodesics near the plane

To begin, we consider the continuous family of geodesics $Q[t] = \Gamma[t, 0, \pi/2]$ obtained by shooting perpendicularly from the x -axis. By Proposition 11, we know the types of the geodesic graphs $\Lambda[0](Q[t])$, and we are interested in describing the shapes of the graphs $\Lambda[k](Q[t])$ when $t > 0$ is small. The following two lemmas are consequences of several results from Section 4.1.

Lemma 27. *Let $\Gamma = \Gamma[0, r_0, \alpha_0]$ be a geodesic with $r_0 \in (m_1, \sqrt{2(n-1)})$ and $\alpha_0 \in (-\pi/2, 0)$. Then $\Lambda[1](\Gamma)$ exists, and for α_0 sufficiently close to $-\pi/2$, we have $\Lambda[1](\Gamma)$ is type $(0, -)$ with $r_{\Lambda[1](\Gamma)} \in (\sqrt{2n}, M_1)$ and $\alpha_{\Lambda[1](\Gamma)} \in (-\pi/2, 0)$. Moreover, $\alpha_{\Lambda[1](\Gamma)} \rightarrow -\pi/2$ as $\alpha_0 \rightarrow -\pi/2$.*

Proof. Let $u : (a, b) \rightarrow \mathbb{R}$ denote the maximally extended solution to (2.4) whose graph is the geodesic segment $\Lambda[0](\Gamma)$. By assumption $u(0) < \sqrt{2(n-1)}$ and $u'(0) < 0$, and it follows from the work in Section 4.1 that $b < \infty$ and $0 < u(b) < \infty$, and $u'' > 0$ on $[0, b)$ (see Section 4.1.1, Lemma 17, and Lemma 20). Since b and $u(b)$ are finite, we conclude that $\Lambda[1](\Gamma)$ exists.

When $\alpha_0 \rightarrow -\pi/2$, we have $u'(0) \rightarrow -\infty$, and it follows from Lemma 24 that $b \rightarrow 0$. We note that $\Lambda[0](\Gamma)$ achieves its minimum over $[0, b)$ at an interior point. By the continuity of equation (2.6), this minimum value approaches 0 as $\alpha_0 \rightarrow -\pi/2$. Applying Lemma 4, we have $u(b) < \sqrt{2(n-1)}$.

Now, let f denote the solution to (2.6) with $f(u(b)) = b$ and $f'(u(b)) = 0$. Then f is concave down, and using the continuity of equation (2.6), we see that the domain of

f approaches $(0, \infty)$ as $\alpha_0 \rightarrow -\pi/2$. Applying Lemma 3 we conclude that f crosses the r -axis below M_1 , and the slope at this point approaches 0 as $\alpha_0 \rightarrow -\pi/2$. In addition, when b is small, the comparison arguments used in the proof of Lemma 10 in Section 2.6 show that f crosses the sphere at least once in the first quadrant. Also, when b is small, the slope of $f(r)$ may be chosen small for $r \in [u(b), \sqrt{2n}]$ (since f is close to the plane), and we see that f crosses the sphere exactly once in the first quadrant. Therefore, $r_{\Lambda[1](\Gamma)} \in (\sqrt{2n}, M_1)$ and $\alpha_{\Lambda[1](\Gamma)} \rightarrow -\pi/2$ as $\alpha_0 \rightarrow -\pi/2$.

Finally, let v denote the solution to (2.4) whose graph is the geodesic segment $\Lambda[1](\Gamma)$. We claim that v has a maximum in the second quadrant and $v'' < 0$ when b is small. To see this, we first note that v intersects the r -axis above the cylinder with negative slope when b is small. Therefore, v is not a trumpet and it has a maximum in the second quadrant. Now, this maximum value goes to ∞ as b goes to 0, and arguing as in Part 1 in the proof of Lemma 25 and also using the convexity of f from the previous paragraph, we conclude that $v'' < 0$ for sufficiently small b . Therefore, $\Lambda[1](\Gamma)$ is type $(0, -)$ when α_0 is sufficiently close to $-\pi/2$. \square

Lemma 28. *Let $\Gamma = \Gamma[0, r_0, \alpha_0]$ be a geodesic with $r_0 \in (\sqrt{2n}, M_1)$ and $\alpha_0 \in (0, \pi/2)$. Then $\Lambda[1](\Gamma)$ exists, and for α_0 sufficiently close to $\pi/2$, we have $\Lambda[1](\Gamma)$ is type $(0, +)$ with $r_{\Lambda[1](\Gamma)} \in (m_1, \sqrt{2(n-1)})$ and $\alpha_{\Lambda[1](\Gamma)} \in (0, \pi/2)$. Moreover, $\alpha_{\Lambda[1](\Gamma)} \rightarrow \pi/2$ as $\alpha_0 \rightarrow \pi/2$.*

Proof. Let $u : (a, b) \rightarrow \mathbb{R}$ denote the maximally extended solution to (2.4) whose graph is the geodesic segment $\Lambda[0](\Gamma)$. By assumption $u(0) > \sqrt{2n}$ and $u'(0) > 0$, and it follows from the work in Section 4.1 that $b < \infty$, $0 < u(b) < \infty$ (see Section 4.1.1 and Lemma 17). Since b and $u(b)$ are finite, we conclude that $\Lambda[1](\Gamma)$ exists. In addition, using the sinusoidal shape of u , we note that u achieves a local maximum at some point $x_1 > 0$.

When $\alpha_0 \rightarrow \pi/2$, we have $u'(0) \rightarrow \infty$, and it follows from the continuity of equation (2.6), that $u(x_1) \rightarrow \infty$. Using Part 1 in the proof of Lemma 25, we have

$\Lambda[1](\Gamma)$ is type $(0, +)$ and $u(b) > u(x_1) - 2$ for α_0 sufficiently close to $\pi/2$. The triangle with vertices $(0, \sqrt{2(n-1)})$, $(0, \sqrt{2n})$, and $(x_1, u(x_1))$ is contained in a simple region bounded by Γ and the r -axis, and it follows from the Gauss-Bonnet formula (2.8) that $x_1 \rightarrow 0$ as $u(x_1) \rightarrow \infty$. Applying Lemma 25, we see that $b \rightarrow 0$ as $\alpha_0 \rightarrow \pi/2$.

Now, let f denote the solution to (2.6) with $f(u(b)) = b$ and $f'(u(b)) = 0$. By choosing b sufficiently small, we may assume that $u(b) > \sqrt{2n}$. Then f is concave down and $f'(\sqrt{2n}) > 0$. Using equation (2.6) we have $f'(\sqrt{2n}) < \sqrt{n/2}f(\sqrt{2n})$. Then using $f(\sqrt{2n}) < b$ and the continuity of equation (2.6), at the point $r = \sqrt{2n}$, we see that the domain of f approaches $(0, \infty)$ as $\alpha_0 \rightarrow \pi/2$. Applying Lemma 4 we conclude that f crosses the r -axis above m_1 , and the slope at this point approaches 0 as $\alpha_0 \rightarrow \pi/2$. Therefore, $r_{\Lambda[1](\Gamma)} \in (m_1, \sqrt{2n})$, and $\alpha_{\Lambda[1](\Gamma)} \rightarrow \pi/2$ as $\alpha_0 \rightarrow \pi/2$. \square

Now, we can describe the asymptotic behavior of the geodesics $Q[t]$ near the plane.

Proposition 14. *For each $N > 0$, there exists $\varepsilon > 0$ so that whenever $0 < t < \varepsilon$, the geodesic segment $\Lambda[k](Q[t])$ exists for $0 \leq k \leq N$. Moreover, $\Lambda[k](Q[t])$ is type $(0, -)$ when k is even, and $\Lambda[k](Q[t])$ is type $(0, -)$ when k is odd.*

Proof. We know that $\Lambda[0](Q[t])$ is type $(0, -)$ when $t < \sqrt{2n}$. We also know that $r_{\Lambda[0](Q[t])} \in (\sqrt{2n}, M_1)$ and $\alpha_{\Lambda[0](Q[t])} \in (-\pi/2, 0)$. By continuity, we have $\alpha_{\Lambda[0](Q[t])} \rightarrow -\pi/2$ as $t \rightarrow 0$. Then, applying Lemma 28, we see that $\Lambda[1](Q[t])$ exists, and for t sufficiently close to 0, we have $\Lambda[1](Q[t])$ is type $(0, +)$ with $r_{\Lambda[1](Q[t])} \in (m_1, \sqrt{2(n-1)})$ and $\alpha_{\Lambda[1](Q[t])} \in (-\pi/2, 0)$. Moreover, $\alpha_{\Lambda[1](Q[t])} \rightarrow -\pi/2$ as $t \rightarrow 0$. Applying Lemma 27 to $\Lambda[1](Q[t])$ shows that $\Lambda[2](Q[t])$ exists, and for t sufficiently close to 0, we have $\Lambda[2](Q[t])$ is type $(0, -)$ with $r_{\Lambda[2](Q[t])} \in (\sqrt{2n}, M_1)$ and $\alpha_{\Lambda[2](Q[t])} \in (-\pi/2, 0)$. Moreover, $\alpha_{\Lambda[2](Q[t])} \rightarrow -\pi/2$ as $t \rightarrow 0$. The proposition follows from repeated applications of Lemma 28 and Lemma 27. \square

4.2.2 Behavior of geodesics near the cylinder

Next, we study the continuous family of geodesics $\Gamma_t = \Gamma[0, t, 0]$ obtained by shooting perpendicularly from the r -axis. By Lemma 18 we know that $\Lambda[0](\Gamma_t)$ is type $(0, +)$ when $t < \sqrt{2(n-1)}$. The following lemma about geodesics near the cylinder will be used to describe the shape of Γ_t when t is close to $\sqrt{2(n-1)}$.

Lemma 29. *Let $\Gamma = \Gamma[0, r_0, \alpha_0]$ with $r_0 < \sqrt{2n}$ and $\alpha_0 \in (-\pi/2, 0)$. Then $\Lambda[1](\Gamma)$ exists, and for (r_0, α_0) sufficiently close to $(\sqrt{2(n-1)}, 0)$, the geodesic segment $\Lambda[1](\Gamma)$ is type $(1, -)$ with $\alpha_{\Lambda[1](\Gamma)} \in (0, \pi/2)$. Moreover, $\Lambda[1](\Gamma)$ converges to the cylinder as $(r_0, \alpha_0) \rightarrow (\sqrt{2(n-1)}, 0)$.*

Proof. By the work in Section 4.1, we know that $\Lambda[0](\Gamma)$ is type $(k_0, +)$ and it has a finite right end point. Therefore, $\Lambda[1](\Gamma)$ exists and has type $(k_1, -)$. Applying Proposition 12, we see that $\Lambda[1](\Gamma)$ converges to the cylinder as $(r_0, \alpha_0) \rightarrow (\sqrt{2(n-1)}, 0)$. In particular, $r_{\Lambda[1](\Gamma)} \rightarrow \sqrt{2(n-1)}$. Using Lemma 21 and the continuity of geodesics, we observe that a maximally extended graphical geodesic segment, which intersects the r -axis perpendicularly between the sphere and the cylinder, is type $(2, +)$. Combining this observation with the work in Section 4.1.2 shows that $\alpha_{\Lambda[1](\Gamma)} \in (0, \pi/2)$, and consequently $\Lambda[1](\Gamma)$ is type $(1, -)$. \square

Now, we can describe the asymptotic behavior of the geodesics $\Gamma[0, r_0, 0]$ near the cylinder.

Proposition 15. *Let $\Gamma_t = \Gamma[0, t, 0]$. For each $N > 0$, there exists $\varepsilon > 0$ so that whenever $\sqrt{2(n-1)} - \varepsilon < t < \sqrt{2(n-1)}$, the geodesic segment $\Lambda[k](\Gamma_t)$ exists for $0 \leq k \leq N$. Moreover, $\Lambda[k](\Gamma_t)$ is type $(1, -)$ when k is odd, and $\Lambda[k](\Gamma_t)$ is type $(1, +)$ when $k \geq 2$ is even.*

Proof. By Lemma 18, $\Lambda[0](\Gamma_t)$ is type $(0, +)$. Arguing as in the proof of Lemma 29, we see that $\Lambda[1](\Gamma_t)$ exists, and for t sufficiently close to $\sqrt{2(n-1)}$, the geodesic segment $\Lambda[1](\Gamma_t)$ is type $(1, -)$ with $\alpha_{\Lambda[1](\Gamma_t)} \in (0, \pi/2)$. Moreover, $\Lambda[1](\Gamma_t)$ converges

to the cylinder as $t \rightarrow \sqrt{2(n-1)}$. The proposition follows from repeated applications of Lemma 29. \square

4.2.3 Behavior of geodesics near Angenent's torus

We continue the study of the geodesics $\Gamma_t = \Gamma[0, t, 0]$ by illustrating two procedures for constructing self-shrinkers. We prove the result due to Angenent [3] that there is an embedded torus self-shrinker, and we also prove the result from [24] that there is an immersed sphere self-shrinker.

Consider the geodesics $\Gamma_t = \Gamma[0, t, 0]$, where $t < \sqrt{2(n-1)}$. From Lemma 18 we know that $\Lambda[0](\Gamma_t)$ is type $(0, +)$, and from the work in Section 4.1, we know that $\Lambda[1](\Gamma_t)$ exists. Proposition 15 tells us that $\Lambda[1](\Gamma_t)$ is type $(1, -)$ when t is close to $\sqrt{2(n-1)}$. Moreover, $\Lambda[1](\Gamma_t)$ has a local maximum in the first quadrant. When t is close to 0, it follows from the proof of Lemma 27 that $\Lambda[1](\Gamma_t)$ is type $(0, -)$ with a local maximum in the second quadrant.

There are two notable differences in the geodesics $\Lambda[1](\Gamma_t)$ when t is close to $\sqrt{2(n-1)}$ and when t is close to 0. One difference is the location of the local maximum, and the other difference is the curve type. As t decreases from $\sqrt{2(n-1)}$ to 0, there is a first initial height $t = r_{Ang}$ for which the local maximum of $\Lambda[1](\Gamma_t)$ intersects the r -axis. (More rigorously, let r_{Ang} denote the infimum of the set of $r < \sqrt{2(n-1)}$ with the property that for $t > r$, the maximum of $\Lambda[1](\Gamma_t)$ occurs in the first quadrant.) Then $r_{Ang} > 0$, and consequently $\Gamma_{r_{Ang}}$ is a closed geodesic whose rotation about the x -axis is an embedded torus self-shrinker. We will refer to $\Gamma_{r_{Ang}}$ as Angenent's torus.

Notice that $\Lambda[1](\Gamma_{r_{Ang}})$ is type $(0, -)$. In particular, as t decreases from $\sqrt{2(n-1)}$ to r_{Ang} , the geodesic segments $\Lambda[1](\Gamma_t)$ change type. Therefore, $\Lambda[1](\Gamma_t)$ must be in $\partial\mathbb{A}(1, -)$ for some $t < \sqrt{2(n-1)}$. By continuity, the right end points of $\Lambda[1](\Gamma_t)$ remain bounded away from the r -axis in a compact subset of \mathbb{H} when t is between, say, $\sqrt{2(n-1)} - \varepsilon$ and r_{Ang} , and applying Corollary 1 we see that there is $r_1 > r_{Ang}$ so

that $\Lambda[1](\Gamma_{r_1}) \in \underline{I}^-$. The rotation of the geodesic Γ_{r_1} about the x -axis is an immersed sphere self-shrinker.

We end this section with a description of the behavior of geodesics near Angenet's torus $\Gamma_{r_{Ang}}$.

Proposition 16. *Let $\Gamma_t = \Gamma[0, t, 0]$. For each $N > 0$, there exists $\varepsilon > 0$ so that whenever $r_{Ang} < t < r_{Ang} + \varepsilon$, the geodesic segment $\Lambda[k](\Gamma_t)$ exists for $0 \leq k \leq N$. Moreover, $\Lambda[k](\Gamma_t)$ is type $(0, +)$ when k is even and $\Lambda[k](\Gamma_t)$ is type $(0, -)$ when k is odd.*

Proof. The proposition follows from the continuity of geodesics and the convexity of $\Gamma_{r_{Ang}}$. \square

4.3 Construction of self-shrinkers

In this section we construct an infinite number of sphere and plane self-shrinkers near the plane (Theorem 3), an infinite number of sphere and tori self-shrinkers near the cylinder (Theorem 4), and an infinite number of sphere and cylinder self-shrinkers near Angenet's torus (Theorem 5). It follows from the asymptotic behavior of the trumpets at infinity (Theorem 3 in [50]) that the plane and cylinder self-shrinkers we construct are complete and have polynomial volume growth. In particular, by Colding and Minicozzi's theorem for complete self-shrinkers with polynomial volume growth in [15], the self-shrinkers we construct in this section are not F -stable.

Theorem 3. *There is a decreasing sequence $t_0 > t_1 > \dots$ so that the rotation of the geodesic $Q[t_k]$ about the x -axis is an S^n self-shrinker when k is even and a complete \mathbb{R}^n self-shrinker when k is odd. Moreover, $Q[t_k]$ is the union of $k + 1$ maximally extended geodesic segments.*

Proof. The proof is by induction. For the base case, we define $t_0 = \sqrt{2n}$ so that $Q[t_0] = \mathcal{S}$ is the sphere. We note that $\Lambda[0](Q[t])$ is type $(0, -)$ for $0 < t < t_0$ (this

follows from Proposition 11). We also note that $\Lambda[1](Q[t])$ exists for $0 < t < t_0$, since the sphere is the only profile curve corresponding to an embedded S^n self-shrinker.

Continuing the base case, we know that there is $\varepsilon > 0$ so that the left end point of $\Lambda[1](Q[t])$ remains bounded away from the r -axis in a compact subset of \mathbb{H} for $\varepsilon \leq t \leq t_0 - \varepsilon$. When t is close to 0 it follows from Proposition 14 that $\Lambda[1](Q[t])$ is type $(0, +)$. Applying Proposition 12 and Proposition 13 to the left end point of $\Lambda[0](Q[t_0])$ shows that $\Lambda[1](Q[t])$ is either type $(1, -)$ or type $(2, +)$ when t is close to $t_0 = \sqrt{2n}$. In particular, by choosing ε small enough, we see that the geodesic segments $\Lambda[1](Q[t])$ change type as t increases from ε to $t_0 - \varepsilon$. It follows that $\Lambda[1](Q[t])$ is a half-entire graph for some t between ε and $t_0 - \varepsilon$. We define t_1 to be the first $t > 0$ such that $\Lambda[1](Q[t])$ is a half-entire graph. Then $t_1 < t_0$, and by Corollary 1 the geodesic segment $\Lambda[1](Q[t_1])$ is a trumpet in the first quadrant.

For the inductive case, we assume that $t_N < t_{N-1} < \dots < t_0$ are defined: t_k is the first $t > 0$ such that $\Lambda[k](Q[t])$ is a half-entire graph. We also assume that $\Lambda[i](Q[t_k])$ is type $(0, -)$ when $i \leq k$ is even, and it is type $(0, +)$ when $i \leq k$ is odd. In addition, we assume that $\Lambda[k](Q[t_k])$ is an inner-quarter sphere in the second quadrant when k is even, and it is a trumpet in the first quadrant when k is odd. Now, suppose N is odd. Then $\Lambda[N+1](Q[t])$ exists for $0 < t < t_N$, and there is $\varepsilon > 0$ so that the right end point of $\Lambda[N+1](Q[t])$ remains bounded away from the r -axis in a compact subset of \mathbb{H} for $\varepsilon \leq t \leq t_N - \varepsilon$. When t is close to 0 it follows from Proposition 14 that $\Lambda[N+1](Q[t])$ is type $(0, +)$. Applying Proposition 12 and Proposition 13 shows that $\Lambda[N+1](Q[t])$ is type $(1, -)$ when t is close to t_N . In particular, by choosing ε small enough, we may assume that the geodesic segments $\Lambda[N+1](Q[t])$ change type as t increases from ε to $t_N - \varepsilon$. It follows that $\Lambda[N+1](Q[t])$ is a half-entire graph for some t between ε and $t_N - \varepsilon$. We define t_{N+1} to be the first $t > 0$ such that $\Lambda[N+1](Q[t])$ is a half-entire graph. Then $t_{N+1} < t_N$, and by Corollary 1 the geodesic segment $\Lambda[N+1](Q[t_{N+1}])$ is an inner-quarter sphere in the second quadrant. This completes the inductive case when N is odd. When N is even, the argument is similar

to the construction of $Q[t_1]$ from $Q[t_0]$. \square

Theorem 4. *There is an increasing sequence $t_0 < t_1 < \dots < \sqrt{2(n-1)}$ so that the rotation of the geodesic $\Gamma[0, t_k, 0]$ about the x -axis is an $S^1 \times S^{n-1}$ self-shrinker when k is even and an S^n self-shrinker when k is odd. Moreover, $\Gamma[0, t_k, 0]$ is the union of $k + 2$ distinct maximally extended geodesic segments.*

Proof. The proof is by induction; it is similar to the proof of Theorem 3. Let $\Gamma_t = \Gamma[0, t, 0]$. By Lemma 18, we know that $\Lambda[0](\Gamma_t)$ is type $(0, +)$ when $t < \sqrt{2(n-1)}$. It follows from Proposition 15 that for each $N > 0$, there exists $\varepsilon > 0$ so that whenever $\sqrt{2(n-1)} - \varepsilon < t < \sqrt{2(n-1)}$, the geodesic segment $\Lambda[k](\Gamma_t)$ exists for $0 \leq k \leq N$. Moreover, $\Lambda[k](\Gamma_t)$ is type $(1, -)$ when k is odd, and $\Lambda[k](\Gamma_t)$ is type $(1, +)$ when $k \geq 2$ is even.

For the base case, we define t_0 to be the largest $t < \sqrt{2(n-1)}$ such that $\Lambda[1](\Gamma_t)$ intersects the r -axis perpendicularly. Then $t_0 = r_{Ang}$ and Γ_{t_1} is the closed embedded convex curve constructed in Section 4.2.3.

Continuing the base case, since $\Lambda[1](\Gamma_t)$ is type $(1, -)$ when t is close to $\sqrt{2(n-1)}$ and type $(0, -)$ when t is close to t_0 , there is some t between $\sqrt{2(n-1)}$ and t_0 such that $\Lambda[1](\Gamma_t)$ is a half-entire graph. We define t_1 to be the largest $t < \sqrt{2(n-1)}$ such that $\Lambda[1](\Gamma_t)$ is a half-entire graph. By Corollary 1, the half-entire graph $\Lambda[1](\Gamma_{t_1})$ is an inner-quarter sphere in the second quadrant. Notice that $\Lambda[0](\Gamma_{t_1})$ is type $(0, +)$, $\Lambda[1](\Gamma_{t_1}) \in \underline{I}^-$, and $\Lambda[-1](\Gamma_{t_1}) \in \underline{I}^+$ so that Γ_{t_1} is the union of 3 maximally extended geodesic segments.

For the inductive case, we assume that $t_{2N-1} > t_{2N-3} > \dots > t_1$ are defined: t_{2k-1} is the largest $t < \sqrt{2(n-1)}$ such that $\Lambda[k](\Gamma_t)$ is a half-entire graph. We also assume that $\Lambda[N](\Gamma_{t_{2N-1}})$ is an inner-quarter sphere. Suppose $\Lambda[N](\Gamma_{t_{2N-1}})$ is an inner-quarter sphere in the first quadrant. By Proposition 12 and Proposition 13 we have $\Lambda[N+1](\Gamma_t)$ is type $(0, -)$ with a local maximum in the second quadrant when $t > t_{2N-1}$ is close to t_{2N-1} . It follows that the type of $\Lambda[N+1](\Gamma_t)$ changes as t

decreases from $\sqrt{2(n-1)}$ to t_{2N-1} , so we can define $t_{2N+1} > t_{2N-1}$ to be the largest $t < \sqrt{2(n-1)}$ such that $\Lambda[N+1](\Gamma_t)$ is a half-entire graph. Then $\Lambda[N+1](\Gamma_{t_{2N+1}})$ is an inner-quarter sphere in the second quadrant (since it is the limit of type $(1, -)$ geodesic segments). Therefore, the geodesic $\Gamma_{t_{2N+1}}$ is the union of $2N+3$ maximally extended geodesic segments, and its rotation about the x -axis is an immersed S^n self-shrinker. Furthermore, since the local maximums of $\Lambda[N+1](\Gamma_{t_{2N+1}})$ and $\Lambda[N+1](\Gamma_t)$ for t near t_{2N-1} are in different quadrants, there exists t_{2N} between t_{2N-1} and t_{2N+1} so that $\Lambda[N+1](\Gamma_{t_{2N}})$ intersects the r -axis perpendicularly. Then $\Gamma_{t_{2N}}$ is the union of $2N+2$ maximally extended geodesic segments, and its rotation about the x -axis is an immersed $S^1 \times S^{n-1}$ self-shrinker. This completes the inductive case. \square

Theorem 5. *There is a decreasing sequence $t_0 > t_1 > \dots > r_{Ang}$ so that the rotation of the geodesic $\Gamma[0, t_k, 0]$ about the x -axis is a complete $\mathbb{R}^1 \times S^{n-1}$ self-shrinker when k is even and an S^n self-shrinker when k is odd. Moreover, $\Gamma[0, t_k, 0]$ is the union of $2k+1$ maximally extended geodesic segments.*

Proof. The proof is by induction; it is similar to the proofs of Theorem 3 and Theorem 4, and we provide a sketch. Let $\Gamma_t = \Gamma[0, t, 0]$. Given $N > 0$, there exists $\varepsilon > 0$ so that whenever $r_{Ang} < t < r_{Ang} + \varepsilon$, the geodesic segment $\Lambda[k](\Gamma_t)$ exists for $0 \leq k \leq N$. Moreover, $\Lambda[k](\Gamma_t)$ is type $(0, +)$ when k is even, and it is type $(0, -)$ when k is odd. For the base case, we define $t_0 = \sqrt{2(n-1)}$, so that $\Gamma_{t_0} = \mathcal{C}$ is the cylinder. For the general case, we define t_k to be the first $t > r_{ang}$ such that $\Lambda[k](\Gamma_t)$ is a half-entire graph. Then $\Lambda[k](\Gamma_{t_k})$ is either a trumpet in the first quadrant or an inner-quarter sphere in the second quadrant, depending on whether it is the limit of type $(0, +)$ curves or $(0, -)$ curves, respectively. \square

Chapter 5

EMBEDDED SPHERE SELF-SHRINKERS WITH ROTATIONAL SYMMETRY

In this chapter, we present a rigidity result for embedded sphere self-shrinkers with rotational symmetry.

Theorem 6. [26] *For $n \geq 2$, an embedded S^n self-shrinker in \mathbb{R}^{n+1} with rotational symmetry must be the sphere of radius $\sqrt{2n}$ centered at the origin.*

In [26], we showed that an embedded sphere self-shrinker with rotational symmetry must be mean-convex, and the rigidity result followed from a theorem of Huisken [43], which says that the round sphere is the only compact, mean-convex self-shrinker. The proof we give in this chapter uses the comparison results from Chapter 2 to bypass some of the preparation in [26]. Since the proofs of the comparison results from Chapter 2 also use Huisken's theorem, the presentation in this chapter is essentially the same as that in [26].

5.1 Preliminary results

Let $F : S^n \rightarrow \mathbb{R}^{n+1}$ be an embedding satisfying $\mathbf{H} = -\frac{1}{2}F^\perp$. Let $M = F(S^n)$ denote the image of S^n in \mathbb{R}^{n+1} , and suppose M has rotational symmetry.

Claim 14. *Let M be a compact, embedded self-shrinker with a rotational symmetry. Then M is rotationally symmetric with respect to a line through the origin.*

Proof. Let $p_0 \in M$, and let $\ell(t) = he_n + te_{n+1}$ denote the axis of symmetry for M . Then $p_0 = (r_0\omega_0, 0) + \ell(t_0)$ for some $(r_0, t_0, \omega_0) \in [0, \infty) \times \mathbb{R} \times S^{n-1}$, and $(r_0\omega, 0) + \ell(t_0) \in M$ for all $\omega \in S^{n-1}$.

Let p_0 be the furthest point on M from the origin. Then $p_0 = p_0^\perp$, and $|p^\perp| \leq |p_0^\perp|$ for all $p \in M$. Let $p = (r_0\omega, 0) + \ell(t_0)$, for some $\omega \in S^{n-1}$. By the rotational symmetry of M , we have $|\mathbf{H}(p)| = |\mathbf{H}(p_0)|$, and since M is a self-shrinker, we have $|p^\perp| = |p_0^\perp| = \text{const}$. Therefore,

$$|p_0| = |p_0^\perp| = |p^\perp| \leq |p| \leq |p_0|,$$

and we conclude that $|p| = |p_0|$. Plugging different values of ω into the formula $|p| = \text{const}$, we see that $r_0h = 0$.

Finally, suppose to the contrary that $h \neq 0$. Then p_0 cannot be on the axis of rotation. To see this, we observe that (1.) the normal at p_0 is in the direction of the origin, (2.) if $h \neq 0$, then the axis of rotation does not pass through the origin, and (3.) if p is on the axis of rotation, then the normal at p points along the axis of rotation. Therefore, $r_0 \neq 0$, which contradicts the fact that $r_0h = 0$. Therefore, $h = 0$ and the axis of rotation passes through the origin. \square

Since M is a self-shrinker with rotational symmetry about a line through the origin, the analysis from Chapter 2 can be used to study M . Let Γ denote the profile curve of M in the upper half plane. Then Γ is a simple curve that intersects the axis of rotation at exactly two points. In the next section we will show that M has to be the sphere of radius $\sqrt{2n}$. The heuristic idea of the proof is to show that any two solutions to the shooting problem (2.9) must intersect each other transversely (except in one special case, which corresponds to the sphere of radius $\sqrt{2n}$).

Before we prove Theorem 6, we use the comparison results from Chapter 2 to show the profile curve Γ cannot intersect the x -axis at two points in the same quadrant.

Lemma 30. *The profile curve Γ cannot intersect the x -axis at two points in the same quadrant.*

Proof. Since M is compact, we know that Γ does not intersect the x -axis at the origin (otherwise, M would be a plane). Suppose to the contrary that Γ intersects

the positive x -axis at two different points. Then we can find solutions f and g to (2.6) with $f(0) < g(0)$ so that the geodesics $Q[f(0)]$ and $Q[g(0)]$ from the shooting problem (2.9) do not intersect transversely.

However, it follows from the proof of Lemma 10 in Section 2.6 that g blows-up before f does. Using the description of the geodesics $Q[x_0]$ established in Section 2.6, we see that $Q[f(0)]$ and $Q[g(0)]$ must intersect transversely. \square

5.2 Proof of rigidity

In this section, we complete the proof that an embedded S^n self-shrinker with rotational symmetry must be the sphere of radius $\sqrt{2n}$.

Let $F : S^n \rightarrow \mathbb{R}^{n+1}$ be an embedded self-shrinker with rotational symmetry. Let Γ be the profile curve of M in the upper half plane. Then Γ is a simple curve that intersects the x -axis at precisely two points. By Lemma 30, we know that Γ intersects the x -axis at a point $(x_+, 0)$ with $x_+ > 0$ and at a point $(x_-, 0)$ with $x_- < 0$. Let Γ^+ be the arc from $(x_+, 0)$ to the r -axis, and let $(0, r_+)$ be the point where Γ^+ intersects the r -axis. Similarly, we can define an arc Γ^- from $(x_-, 0)$ to the r -axis. Let $(0, r_-)$ be the point where Γ^- intersects the r -axis.

If either $x_+ = \sqrt{2n}$ or $x_- = -\sqrt{2n}$, then we are done. Suppose to the contrary that $x_+ \neq \sqrt{2n}$ and $x_- \neq -\sqrt{2n}$. It follows from the Proposition 3 and Proposition 4 in Section 2.6 that Γ^+ and Γ^- can only intersect if they are both inner-quarter spheres or both outer-quarter spheres. Moreover, the angles of these geodesics are such that Γ^+ would intersect Γ^- transversely if $r_+ = r_-$.

Therefore, we may assume without loss of generality that $r_- < r_+$. Let S be the simple closed curve formed by Γ^+ , the line segment $[0, x_+] \times \{0\}$, and the line segment $\{0\} \times [0, r_+]$. As we travel along Γ^- , we enter S at the point $(0, r_-)$. Since Γ is a simple curve, we see that there is a curve that starts at $(0, r_-)$, enters the interior of S , and first leaves S at some point $(0, r_0)$ with $r_0 < r_+$. In other words, there exists a curve γ in the upper half plane such that $\gamma(0) = (0, r_-)$, $\gamma(1) = (0, r_0)$, and $\gamma(t)$ is

in the region enclosed by S for $t \in (0, 1)$. In addition, we note that γ does not intersect Γ^+ .

We write $\gamma = (\gamma_1, \gamma_2)$. Let t_0 be the point where γ_1 achieves its maximum on $[0, 1]$. Since the maximum of γ_1 is positive, this occurs at some $t_0 \in (0, 1)$. Let f be the curve which describes Γ in a neighborhood of $\gamma(t_0)$, and let $\bar{r} = \gamma_2(t_0)$. Then f is a solution of (2.6) and $f'(\bar{r}) = 0$. Comparing the curve $(f(r), r)$ with Γ_+ , and using the argument in the proof of Lemma 10, we see that Γ must have a transverse intersection, which is a contradiction. Therefore, $x_+ = \sqrt{2n}$ and $x_- = -\sqrt{2n}$, which proves the theorem.

Chapter 6

HIGH CODIMENSION SELF-SHRINKER GRAPHS

In this chapter, we derive a Bernstein result for high codimension self-shrinker graphs under a convexity assumption on the angles between the tangent plane to the graph and the base n -plane. More precisely, we show if $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is a solution to

$$g^{ij} \frac{\partial^2 \Phi}{\partial x^i \partial x^j} - \frac{1}{2} (\nabla_{\mathbb{R}^n} \Phi \cdot x - \Phi) = 0, \quad (6.1)$$

where g is the induced metric on the graph of Φ , and the eigenvalues of $(d\Phi)^T d\Phi$ satisfy the convexity condition $\lambda_i \lambda_j \leq 1$, $i \neq j$, then Φ is linear.

When $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is a solution to (6.1), its graph satisfies

$$\Delta_g F + \frac{1}{2} F^\perp = 0, \quad (6.2)$$

where $F(x) = (x, \Phi(x))$. Conversely, if F is a solution to (6.2) that can be written as the graph of Φ , then Φ satisfies (6.1). Solutions to (6.2) are (high codimension) self-shrinkers, and they arise naturally in the study of singularities of mean curvature flow.

During their study of the mean curvature flow of hypersurfaces, Ecker and Huisken showed that a codimension one self-shrinker graph of at most polynomial volume growth must be linear. Their integral argument, given in the appendix of [30], relied on an equation involving the volume element of the graph and the second fundamental form. In a recent paper, Lu Wang used self-similarity and the maximum principle for mean curvature flow to show that self-shrinkers have linear growth, in turn, establishing the rigidity of entire codimension one self-shrinker graphs; see [63]. In this chapter, we prove the following result for high codimension self-shrinker graphs.

Theorem 7. [25] *Let $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ be a solution to*

$$g^{ij} \frac{\partial^2 \Phi}{\partial x^i \partial x^j} - \frac{1}{2} (\nabla_{\mathbb{R}^n} \Phi \cdot x - \Phi) = 0,$$

where g is the induced metric on the graph of Φ . If the eigenvalues of $(d\Phi)^T d\Phi$ satisfy $\lambda_i \lambda_j \leq 1$, $i \neq j$, then Φ is linear.

The heuristic idea behind the proof of Theorem 7 is to study the angle $\theta_1 = \arctan \sqrt{\lambda_1}$, where λ_1 is the largest eigenvalue of $(d\Phi)^T d\Phi$. We show that θ_1 satisfies the self-shrinker inequality $g^{ij} \frac{\partial^2 \theta_1}{\partial x^i \partial x^j} - \frac{1}{2} (\nabla_{\mathbb{R}^n} \theta_1) \cdot x \geq 0$, assuming the convexity condition $\lambda_i \lambda_j \leq 1$, $i \neq j$ (under the equivalent geometric condition $\theta_i + \theta_j \leq \frac{\pi}{2}$, $i \neq j$, the square of the distance function to the base n -plane in the Grassmanian $G_{n,k}$ is convex; see Jost and Xin [48]). This subharmonicity of θ_1 in this high codimension setting is a pleasant surprise. (A different combination $\log \sqrt{1 + \lambda^2}$ of the largest eigenvalue of $D^2 u$, where the scalar u is a solution to the special Lagrangian σ_2 -equation in dimension three, was found to be strongly subharmonic in Warren and Yuan [64].) Once we know θ_1 satisfies the self-shrinker inequality, we show θ_1 achieves its maximum, adapting a maximum principle argument from Chau, Chen, and Yuan [10], where the rigidity of Lagrangian self-shrinker graphs was proved. This tells us that λ_1 and the volume element \sqrt{g} are bounded. Next, we derive an inequality involving $f = \log \sqrt{g}$ and the second fundamental form, which we use along with the maximum principle argument from [10] to show f is constant. We use the inequality again to establish the pointwise vanishing of the second fundamental form, which proves Φ is linear.

In the dimension one and codimension one cases, our argument works without assuming any conditions on the eigenvalues of $(d\Phi)^T d\Phi$. (In particular, we give a different proof of the codimension one result from [30] and [63].) However, as the existence of minimal Hopf cones suggests, since minimal cones are self-shrinker graphs, an additional assumption such as ours may be needed to prove a Bernstein result in arbitrary codimension.

6.1 Preliminary results

Let M be an n -dimensional manifold. A smooth one-parameter family of immersions $F : M \times I \rightarrow \mathbb{R}^{n+k}$ is a (high codimension) solution to mean curvature flow if

$$\Delta_{g(t)}F - \frac{\partial F}{\partial t} = 0,$$

where $g(t)$ is the induced metric on $M_t = F(M, t)$ and $\Delta_{g(t)}$ is the Laplace-Beltrami operator. If the submanifold M_t can be written as the graph of $U(\cdot, t)$, where $U : \mathbb{R}^n \times I \rightarrow \mathbb{R}^k$ is a one-parameter family of functions, then U satisfies the evolution equation

$$g^{ij} \frac{\partial^2 U}{\partial x^i \partial x^j} - \frac{\partial U}{\partial t} = 0, \quad (6.3)$$

where $(g^{ij}) = (g_{ij})^{-1}$ and $g_{ij} = \delta_{ij} + \frac{\partial U}{\partial x^i} \cdot \frac{\partial U}{\partial x^j}$. In this case, $\Phi(x) = U(x, -1)$ satisfies (6.1). Conversely, if $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is a solution to (6.1), then $U(x, t) = \sqrt{-t}\Phi(\frac{x}{\sqrt{-t}})$ is a solution to (6.3).

Throughout this chapter, $U : \mathbb{R}^n \times I \rightarrow \mathbb{R}^k$ will be a solution to (6.3). The eigenvalues of $(dU)^T dU$ will be denoted by λ_i and listed in decreasing order: $\lambda_1 \geq \dots \geq \lambda_n \geq 0$. Notice that $\lambda_i = 0$ for $i > k$. The angles $\arctan \sqrt{\lambda_i}$ will be denoted by θ_i . These angles measure the distance between the tangent plane to the graph of $U(\cdot, t)$ and the base n -plane $\mathbb{R}^n \times \{0\}$. The volume element of the graph of $U(\cdot, t)$ will be denoted by $\sqrt{g}(t)$. In terms of the eigenvalues, we have $\sqrt{g} = \prod_{i=1}^n \sqrt{1 + \lambda_i}$.

In Section 6.1.1, we compute the evolution equations for λ_1 , θ_1 , and $\log \sqrt{g}$. Under the assumption $\lambda_i \lambda_j \leq 1$, $i \neq j$, we derive evolution inequalities for these functions. In Section 6.1.2, we restrict to the case of self-shrinker graphs and use a maximum principle argument to show the volume element is bounded when $\lambda_i \lambda_j \leq 1$, $i \neq j$.

6.1.1 Evolution equations

When λ_1 is a distinct eigenvalue of $(dU)^T dU$, we can implicitly compute its evolution equation from the characteristic polynomial. To do this, we fix a point (p_0, t_0) and

make orthonormal coordinate transformations on \mathbb{R}^n and \mathbb{R}^k so that $U_i^\alpha = \delta_{i\alpha}\sqrt{\lambda_i}$. Notice that $g^{ij}\frac{\partial^2}{\partial x^i\partial x^j}$ and equation (6.3) are invariant under orthonormal coordinate transformations. At (p_0, t_0) , we have $g_{ij} = \delta_{ij}(1 + \lambda_i)$, $g^{ij} = \delta_{ij}\frac{1}{1+\lambda_i}$, and $(g_{ij})_a = U_{ia}^j\sqrt{\lambda_j} + U_{ja}^i\sqrt{\lambda_i}$.

Lemma 31. *Let λ_1 be the largest eigenvalue of $(dU)^T dU(p_0, t_0)$. If λ_1 is distinct, then in the above coordinates, at the point (p_0, t_0) , λ_1 satisfies*

$$\begin{aligned} g^{ij}\frac{\partial^2\lambda_1}{\partial x^i\partial x^j} - \frac{\partial\lambda_1}{\partial t} &= 2\sum_{i=1}^n\sum_{\alpha=1}^k\frac{(U_{1i}^\alpha)^2}{1+\lambda_i} \\ &+ 2\sum_{i=1}^n\sum_{r=2}^n\frac{1}{1+\lambda_i}\frac{1}{\lambda_1-\lambda_r}\left(U_{1i}^r\sqrt{\lambda_r} + U_{ri}^1\sqrt{\lambda_1}\right)^2 \\ &+ 4\sum_{i=1}^n\sum_{j=1}^{\min\{n,k\}}\frac{1}{1+\lambda_i}\frac{1}{1+\lambda_j}\sqrt{\lambda_1}\sqrt{\lambda_j}U_{ij}^1U_{i1}^j. \end{aligned} \quad (6.4)$$

Moreover, if $\lambda_i\lambda_j \leq 1$, $i \neq j$, then at (p_0, t_0) , we have

$$g^{ij}\frac{\partial^2\lambda_1}{\partial x^i\partial x^j} - \frac{\partial\lambda_1}{\partial t} \geq 2\sum_{i=1}^n\frac{(U_{1i}^1)^2}{1+\lambda_i} + 4\frac{\lambda_1}{1+\lambda_1}\sum_{i=1}^n\frac{(U_{1i}^1)^2}{1+\lambda_i}. \quad (6.5)$$

Proof. If $M(x, t) = (m_{ab}(x, t))$ is a symmetric $n \times n$ matrix and λ_1 is a distinct eigenvalue of $M(p_0, t_0)$, then we may implicitly define the eigenvalue $\lambda_1(x, t)$ in a neighborhood of (p_0, t_0) . If we assume that $M(p_0, t_0)$ is diagonal, then at (p_0, t_0) we have

$$\frac{\partial(\lambda_1)}{\partial x^i} = \frac{\partial m_{11}}{\partial x^i}, \quad \frac{\partial(\lambda_1)}{\partial t} = \frac{\partial m_{11}}{\partial t},$$

and

$$\frac{\partial^2(\lambda_1)}{\partial x^i\partial x^j} = \frac{\partial^2 m_{11}}{\partial x^i\partial x^j} + 2\sum_{r=2}^n\frac{1}{\lambda_1-\lambda_r}\frac{\partial m_{1r}}{\partial x^i}\frac{\partial m_{1r}}{\partial x^j}.$$

If $M = (dU)^T dU$, so that $m_{ab} = \sum_{\alpha=1}^k U_a^\alpha U_b^\alpha$, then

$$\frac{\partial m_{ab}}{\partial x^i} = \sum_{\alpha=1}^k U_{ai}^\alpha U_b^\alpha + U_a^\alpha U_{bi}^\alpha,$$

$$\frac{\partial^2 m_{ab}}{\partial x^i\partial x^j} = \sum_{\alpha=1}^k U_{aij}^\alpha U_b^\alpha + U_{ai}^\alpha U_{bj}^\alpha + U_{aj}^\alpha U_{bi}^\alpha + U_a^\alpha U_{bij}^\alpha,$$

$$\frac{\partial m_{ab}}{\partial t} = \sum_{\alpha=1}^k (g^{ij} U_{ij}^\alpha)_a U_b^\alpha + U_a^\alpha (g^{ij} U_{ij}^\alpha)_b.$$

At (p_0, t_0) , we have

$$\frac{\partial m_{11}}{\partial x^i} = 2\sqrt{\lambda_1} U_{1i}^1, \quad \frac{\partial m_{1r}}{\partial x^i} = \sqrt{\lambda_r} U_{1i}^r + \sqrt{\lambda_1} U_{ri}^1,$$

$$\frac{\partial^2 m_{11}}{\partial x^i \partial x^j} = 2\sqrt{\lambda_1} U_{1ij}^1 + 2 \sum_{\alpha=1}^k U_{1i}^\alpha U_{1j}^\alpha,$$

$$\frac{\partial m_{11}}{\partial t} = -2 \frac{1}{1+\lambda_i} \frac{1}{1+\lambda_j} \sqrt{\lambda_1} U_{ij}^1 \left(\sqrt{\lambda_j} U_{i1}^j + \sqrt{\lambda_i} U_{j1}^i \right) + 2 \frac{1}{1+\lambda_i} \sqrt{\lambda_1} U_{i1}^1.$$

Plugging these expressions into the equations for λ_1 at (p_0, t_0) , we see that

$$\begin{aligned} g^{ij} \frac{\partial^2(\lambda_1)}{\partial x^i \partial x^j} - \frac{\partial(\lambda_1)}{\partial t} &= 2 \sum_{i=1}^n \sum_{\alpha=1}^k \frac{(U_{1i}^\alpha)^2}{1+\lambda_i} \\ &\quad + 2 \sum_{i=1}^n \sum_{r=2}^n \frac{1}{1+\lambda_i} \frac{1}{\lambda_1 - \lambda_r} \left(\sqrt{\lambda_r} U_{1i}^r + \sqrt{\lambda_1} U_{ri}^1 \right)^2 \\ &\quad + 4 \sum_{i=1}^n \sum_{j=1}^{\min\{n,k\}} \frac{1}{1+\lambda_i} \frac{1}{1+\lambda_j} \sqrt{\lambda_1} \sqrt{\lambda_j} U_{ij}^1 U_{i1}^j, \end{aligned}$$

which proves (6.4).

Now, suppose $\lambda_i \lambda_j \leq 1$, $i \neq j$. Then $\frac{(1+\lambda_1)\sqrt{\lambda_r}}{(1+\lambda_r)\sqrt{\lambda_1}} \leq 1$. We rewrite (6.4) as

$$g^{ij} \frac{\partial^2(\lambda_1)}{\partial x^i \partial x^j} - \frac{\partial(\lambda_1)}{\partial t} = 2 \sum_{i=1}^n \frac{(U_{1i}^1)^2}{1+\lambda_i} + 4 \frac{\lambda_1}{1+\lambda_1} \sum_{i=1}^n \frac{(U_{1i}^1)^2}{1+\lambda_i} + \text{I},$$

where

$$\begin{aligned} \text{I} &= 2 \sum_{i=1}^n \sum_{\alpha=2}^k \frac{(U_{1i}^\alpha)^2}{1+\lambda_i} + 2 \sum_{i=1}^n \sum_{r=2}^n \frac{1}{1+\lambda_i} \frac{1}{\lambda_1 - \lambda_r} \left(\sqrt{\lambda_r} U_{1i}^r + \sqrt{\lambda_1} U_{ri}^1 \right)^2 \\ &\quad + 4 \sum_{i=1}^n \sum_{j=2}^{\min\{n,k\}} \frac{1}{1+\lambda_i} \frac{1}{1+\lambda_j} \sqrt{\lambda_1} \sqrt{\lambda_j} U_{ij}^1 U_{i1}^j. \end{aligned}$$

To prove (6.5), we need to show I is nonnegative. Since $\frac{(1+\lambda_1)\sqrt{\lambda_r}}{(1+\lambda_r)\sqrt{\lambda_1}} \leq 1$, we have

$$\begin{aligned}
& \frac{(U_{1i}^r)^2}{1+\lambda_i} + \frac{1}{1+\lambda_i} \frac{1}{\lambda_1 - \lambda_r} \left(\sqrt{\lambda_r} U_{1i}^r + \sqrt{\lambda_1} U_{ri}^1 \right)^2 \\
& + 2 \frac{1}{1+\lambda_i} \frac{1}{1+\lambda_r} \sqrt{\lambda_1} \sqrt{\lambda_r} U_{ir}^1 U_{i1}^r \\
& = \frac{1}{1+\lambda_i} \frac{1}{\lambda_1 - \lambda_r} \lambda_1 (U_{1i}^r)^2 + \frac{1}{1+\lambda_i} \frac{1}{\lambda_1 - \lambda_r} \lambda_1 (U_{ri}^1)^2 \\
& + 2 \frac{1+\lambda_1}{(1+\lambda_r)(\lambda_1 - \lambda_r)} \frac{1}{1+\lambda_i} \sqrt{\lambda_1} \sqrt{\lambda_r} U_{1i}^r U_{ri}^1 \\
& = \frac{1}{1+\lambda_i} \frac{1}{\lambda_1 - \lambda_r} \lambda_1 \left((U_{1i}^r)^2 + 2 \frac{(1+\lambda_1)\sqrt{\lambda_r}}{(1+\lambda_r)\sqrt{\lambda_1}} U_{1i}^r U_{ri}^1 + (U_{ri}^1)^2 \right) \\
& \geq 0
\end{aligned}$$

for $r = 2, \dots, \min\{n, k\}$, which shows I is nonnegative. \square

Lemma 32. Let $\theta_1 = \arctan \sqrt{\lambda_1}$. If λ_1 is distinct and $\lambda_i \lambda_j \leq 1$, $i \neq j$, then

$$g^{ij} \frac{\partial^2 \theta_1}{\partial x^i \partial x^j} - \frac{\partial \theta_1}{\partial t} \geq 0. \quad (6.6)$$

Proof. We fix a point (p_0, t_0) , and choose coordinates as in Lemma 31. At the point (p_0, t_0) , we have

$$\begin{aligned}
g^{ij} \frac{\partial^2 \theta_1}{\partial x^i \partial x^j} - \frac{\partial \theta_1}{\partial t} & = -2 \frac{1}{1+\lambda_1} \frac{1}{\sqrt{\lambda_1}} \frac{1}{1+\lambda_1} \sum_{i=2}^n \frac{1}{1+\lambda_i} \lambda_1 (U_{1i}^1)^2 \\
& - \frac{1}{1+\lambda_1} \frac{1}{\sqrt{\lambda_1}} \sum_{i=2}^n \frac{1}{1+\lambda_i} (U_{1i}^1)^2 \\
& + \frac{1}{2} \frac{1}{1+\lambda_1} \frac{1}{\sqrt{\lambda_1}} \left(g^{ij} \frac{\partial^2 \lambda_1}{\partial x^i \partial x^j} - \frac{\partial \lambda_1}{\partial t} \right),
\end{aligned}$$

and (6.6) follows from (6.5). \square

Now, we consider the function $f = \log \sqrt{g}$, where $\sqrt{g}(t)$ is the volume element of the graph of $U(\cdot, t)$. We have

$$\begin{aligned}
\frac{\partial f}{\partial x^i} & = g^{ab} U_{ai}^\alpha U_b^\alpha, \\
\frac{\partial^2 f}{\partial x^i \partial x^j} & = -g^{ar} \left(U_{ri}^\beta U_s^\beta + U_r^\beta U_{si}^\beta \right) g^{sb} U_{aj}^\alpha U_b^\alpha \\
& + g^{ab} U_{aij}^\alpha U_b^\alpha + g^{ab} U_{aj}^\alpha U_{bi}^\alpha,
\end{aligned}$$

$$\begin{aligned} \frac{\partial f}{\partial t} &= -g^{ij}g^{ar} \left(U_{ri}^\beta U_s^\beta + U_r^\beta U_{si}^\beta \right) g^{sb} U_{ab}^\alpha U_j^\alpha \\ &\quad + g^{ij}g^{ab} U_{iab}^\alpha U_j^\alpha, \end{aligned}$$

$$\begin{aligned} g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial f}{\partial t} &= -g^{ij}g^{ar} \left(U_{ri}^\beta U_s^\beta + U_r^\beta U_{si}^\beta \right) g^{sb} U_{aj}^\alpha U_b^\alpha \\ &\quad + g^{ij}g^{ab} U_{aj}^\alpha U_{bi}^\alpha + g^{ij}g^{ar} \left(U_{ri}^\beta U_s^\beta + U_r^\beta U_{si}^\beta \right) g^{sb} U_{ab}^\alpha U_j^\alpha, \end{aligned}$$

where we sum α, β from 1 to k and all other indices from 1 to n .

We fix a point (p_0, t_0) , and choose coordinates as in Lemma 31. At (p_0, t_0) , we see that

$$\begin{aligned} g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial f}{\partial t} &= -\frac{1}{1+\lambda_i} \frac{1}{1+\lambda_a} \frac{1}{1+\lambda_b} \lambda_b (U_{ai}^b)^2 \\ &\quad + \frac{1}{1+\lambda_i} \frac{1}{1+\lambda_a} (U_{ai}^\alpha)^2 \\ &\quad + \frac{1}{1+\lambda_i} \frac{1}{1+\lambda_a} \frac{1}{1+\lambda_b} \sqrt{\lambda_b} \sqrt{\lambda_i} U_{ai}^b U_{ab}^i, \end{aligned} \tag{6.7}$$

where $\sqrt{\lambda_b} U_{ai}^b = 0$ when $b > k$.

Let $A(t)$ denote the second fundamental form of the graph of $U(\cdot, t)$. At (p_0, t_0) , we have

$$|A|^2 = g^{ij}g^{ab} A_{ia}^\alpha A_{jb}^\alpha = \frac{1}{1+\lambda_i} \frac{1}{1+\lambda_a} \frac{1}{1+\lambda_a} (U_{ia}^\alpha)^2.$$

Substituting into (6.7), we observe that

$$g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial f}{\partial t} = |A|^2 + \frac{1}{1+\lambda_i} \frac{1}{1+\lambda_a} \frac{1}{1+\lambda_b} \sqrt{\lambda_b} \sqrt{\lambda_i} U_{ai}^b U_{ab}^i. \tag{6.8}$$

Lemma 33. *If $\lambda_i \lambda_j \leq 1$, $i \neq j$, then*

$$g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial f}{\partial t} \geq 0, \tag{6.9}$$

where equality holds if and only if $|A| = 0$.

Proof. We choose coordinates as in Lemma 31. It follows from (6.7) that

$$\begin{aligned}
g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{\partial f}{\partial t} &\geq \sum_{\alpha \leq \min\{n, k\}} \frac{1}{1 + \lambda_a} \frac{1}{1 + \lambda_\alpha} (U_{a\alpha}^\alpha)^2 \\
&+ \sum_{\alpha \text{ or } i > \min\{n, k\}} \frac{1}{1 + \lambda_i} \frac{1}{1 + \lambda_a} \frac{1}{1 + \lambda_\alpha} (U_{ai}^\alpha)^2 \\
&+ \sum_{i < b \leq \min\{n, k\}} \frac{[(U_{ai}^b)^2 + 2\sqrt{\lambda_b} \sqrt{\lambda_i} U_{ai}^b U_{ab}^i + (U_{ab}^i)^2]}{(1 + \lambda_i)(1 + \lambda_a)(1 + \lambda_b)},
\end{aligned}$$

where we sum a from 1 to n . When $\lambda_i(x)\lambda_j(x) \leq 1$, $i \neq j$, all the terms on the right hand side are nonnegative, which proves (6.9). If $|A| = 0$, then f is constant and equality holds in (6.9). Conversely, if equality holds in (6.9), then (1.) $U_{a\alpha}^\alpha = 0$ for $\alpha \leq \min\{n, k\}$, (2.) $U_{ai}^\alpha = 0$ for α or $i > \min\{n, k\}$, and (3.) $U_{ai}^b = -U_{ab}^i$ for $i < b \leq \min\{n, k\}$. Notice that $U_{ii}^b = 0$ when (1.)-(3.) hold. We claim that (1.)-(3.) imply $|A| = 0$. To see this we need to show $U_{ai}^b = 0$ whenever a, b, i are distinct and all less than or equal to $\min\{n, k\}$. In this case, by (3.), we have

$$U_{ai}^b = -U_{ab}^i = -U_{ba}^i = U_{bi}^a = U_{ib}^a = -U_{ia}^b,$$

which implies $U_{ai}^b = 0$. □

6.1.2 Self-shrinker graphs

Suppose $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is a solution to

$$g^{ij} \frac{\partial^2 \Phi}{\partial x^i \partial x^j} - \frac{1}{2} (\nabla_{\mathbb{R}^n} \Phi \cdot x - \Phi) = 0, \quad (6.10)$$

where g is the induced metric on the graph of Φ . In this case, the graph of Φ is a self-shrinker, and it follows from Lemma 32 that

$$g^{ij} \frac{\partial^2 \theta_1}{\partial x^i \partial x^j} - \frac{1}{2} (\nabla_{\mathbb{R}^n} \theta_1) \cdot x \geq 0 \quad (6.11)$$

whenever λ_1 is distinct and $\lambda_i \lambda_j \leq 1$, $i \neq j$.

Proposition 17. *If $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^k$ is a solution to (6.10) and the eigenvalues of $(d\Phi)^T d\Phi$ satisfy $\lambda_i \lambda_j \leq 1$, $i \neq j$, then $\sqrt{g} \leq C < \infty$.*

Proof. Since $\sqrt{g} = \prod_{i=1}^n \sqrt{1 + \lambda_i}$, it suffices to show $\lambda_1(x) \leq C'$ for some $C' < \infty$. Suppose $\lambda_1(x_0) > 1$ for some x_0 with $|x_0| > \sqrt{2n}$. If no such x_0 exists, then we may take $C' = 1 + \max_{\overline{B_{\sqrt{2n}}}} \lambda_1$. We set $\delta = \frac{1}{2}|x_0|^2 - n + 1$ and fix $\varepsilon > 0$. Let $w(x) = \theta_1(x) - \varepsilon|x|^{1+\delta} - \max_{\partial B_{r_0}} \theta_1$, where $r_0 = [2(n-1+\delta)]^{1/2}$, and consider the open set $\Omega_\varepsilon = (B_{\pi/\varepsilon} \setminus \overline{B_{r_0}}) \cap \{\lambda_1 > 1\}$. The barrier $b(x) = \varepsilon|x|^{1+\delta} + \max_{\partial B_{r_0}} \theta_1$ was used in [10], where the rigidity of Lagrangian self-shrinkers was proved. When $|x| \geq r_0$, since $(g^{ij}) \leq I$ and $\delta \geq 1$, the barrier $b(x)$ satisfies $g^{ij} \frac{\partial^2 b}{\partial x^i \partial x^j} - \frac{1}{2}(\nabla_{\mathbb{R}^n} b) \cdot x \leq 0$. We note that λ_1 is distinct in Ω_ε . Using (6.11) we have $g^{ij} \frac{\partial^2 w}{\partial x^i \partial x^j} - \frac{1}{2}(\nabla_{\mathbb{R}^n} w) \cdot x \geq 0$ in Ω_ε . Since $w \leq 0$ on $\partial\Omega_\varepsilon$, it follows from the weak maximum principle that $w(x) \leq 0$ for all $x \in \Omega_\varepsilon$. Letting $\varepsilon \rightarrow 0$, we conclude that $\theta_1(x) \leq \max_{\partial B_{r_0}} \theta_1$ for $|x| \geq r_0$. We may take $C' = \max_{\overline{B_{r_0}}} \lambda_1$. \square

6.2 Proof of rigidity

Let Φ be a solution to (6.1), and suppose $\lambda_i \lambda_j \leq 1$, $i \neq j$. It follows from Lemma 33 that $g^{ij} \frac{\partial^2 f}{\partial x^i \partial x^j} - \frac{1}{2} \nabla_{\mathbb{R}^n} f \cdot x \geq 0$. Fix $\varepsilon > 0$, and let $w(x) = f(x) - \varepsilon|x|^2 - \max_{\partial B_{\sqrt{2n}}} f$. When $|x| \geq \sqrt{2n}$, the barrier function $b(x) = \varepsilon|x|^2 + \max_{\partial B_{\sqrt{2n}}} f$ satisfies $g^{ij} \frac{\partial^2 b}{\partial x^i \partial x^j} - \frac{1}{2}(\nabla_{\mathbb{R}^n} b) \cdot x \leq 0$. By Proposition 17, we know that f is bounded. Let $C < \infty$ be such that $f \leq C$, and let $\Omega_\varepsilon = B_{\sqrt{C/\varepsilon}} \setminus \overline{B_{\sqrt{2n}}}$. Then $g^{ij} \frac{\partial^2 w}{\partial x^i \partial x^j} - \frac{1}{2}(\nabla_{\mathbb{R}^n} w) \cdot x \geq 0$ in Ω_ε . Since $w \leq 0$ on $\partial\Omega_\varepsilon$, it follows from the weak maximum principle that $w \leq 0$ for all $x \in \Omega_\varepsilon$. Letting $\varepsilon \rightarrow 0$, we conclude that $f(x) \leq \max_{\partial B_{\sqrt{2n}}} f$ for $|x| \geq \sqrt{2n}$, and therefore f achieves its global maximum at a finite point in \mathbb{R}^n . An application of the strong maximum principle, shows f is constant. Once we know f is constant, it follows from Lemma 33 that $|A| = 0$.

Chapter 7

**SELF-SHRINKING SOLUTIONS TO THE
KÄHLER-RICCI FLOW**

In this chapter, we prove a rigidity result for entire self-shrinking solutions to the Kähler-Ricci flow on \mathbb{C}^n . This is a joint work with Peng Lu and Yu Yuan.

Theorem 8. [28] *Suppose u is an entire smooth pluri-subharmonic solution on \mathbb{C}^m to the complex Monge-Ampère equation*

$$\ln \det(u_{\alpha\bar{\beta}}) = \frac{1}{2}x \cdot Du - u. \quad (7.1)$$

Assume the corresponding Kähler metric $g = (u_{\alpha\bar{\beta}})$ is complete. Then u is quadratic.

Any solution to (7.1) leads to a self-shrinking solution $v(x, t) = -tu(x/\sqrt{-t})$ to a parabolic complex Monge-Ampère equation

$$v_t = \ln \det(v_{\alpha\bar{\beta}}) \quad (7.2)$$

in $\mathbb{C}^m \times (-\infty, 0)$, where $z^\alpha = x^\alpha + \sqrt{-1}x^{m+\alpha}$. Conversely, if a solution to (7.2) has the form $v(x, t) = -tu(x/\sqrt{-t})$, then u is a solution to (7.1). Note that the above equation of v is the potential equation of the Kähler-Ricci flow $\partial_t g_{\alpha\bar{\beta}} = -R_{\alpha\bar{\beta}}$. In fact, the corresponding metric $(u_{\alpha\bar{\beta}})$ is a shrinking Kähler-Ricci (non-gradient) soliton.

Assuming a certain inverse quadratic decay of the metric (a specific completeness assumption), Theorem 8 has been proved in [10]. Similar rigidity results for self-shrinking solutions to Lagrangian mean curvature flows were obtained in [41], [10], and [22].

The idea of our argument, as in [10], is still to force the phase $\ln \det(u_{\alpha\bar{\beta}})$ in equation (7.1) to attain its global maximum at a finite point. As this phase satisfies an

elliptic equation without the zeroth order terms (see (7.4) below), the strong maximum principle implies the constancy of the phase. Consequently, the homogeneity of the self-similar terms on the right hand side of equation (7.1) leads to the quadratic conclusion for the solution.

However, the difficulty of the above argument lies in the first step: Here we cannot construct a barrier as in [10], which requires the specific inverse quadratic decay of the metric, to show the phase achieves its maximum at a finite point. The new observation is that the radial derivative of the phase, which is the negative of the scalar curvature of the metric (7.5), is in fact non-positive; hence the phase value at the origin is its global maximum. The non-negativity of the scalar curvature is a result of B.-L. Chen [11], as the induced metric $g(x, t) = (u_{\alpha\bar{\beta}}(x/\sqrt{-t}))$ is a complete ancient solution to the (Kähler-)Ricci flow. Here we provide a direct elliptic argument for the non-negativity of the scalar curvature for the complete self-shrinking solutions (in Section 7.2, after necessary preparation in Section 7.1, where a pointwise approach to Perelman's upper bound of the Laplacian of the distance [57] is also included). Heuristically one sees the minimum of the scalar curvature is non-negative from its inequality (7.7); it is definitely so if the minimum is attained at a finite point. Note that a thorough study of the lower bound of scalar curvatures of gradient Ricci solitons is presented in [14, Chap.27].

7.1 Preliminary results

For the potential u of the Kähler metric $g = (g_{\alpha\bar{\beta}}) = (u_{\alpha\bar{\beta}})$ on \mathbb{C}^m , we denote the phase by $\Phi = \ln \det(u_{\alpha\bar{\beta}})$. Then the Ricci curvature is given by $R_{\alpha\bar{\beta}} = -\frac{\partial^2 \Phi}{\partial z^\alpha \partial \bar{z}^\beta}$. The “complex” scalar curvature is $R = g^{\alpha\bar{\beta}} R_{\alpha\bar{\beta}}$ (R is one-half of the usual “real” scalar curvature). Let $\rho(x)$ denote the Riemannian distance from x to 0 in (\mathbb{C}^m, g) . For a solution u to (7.1), we derive the following equations and inequalities for those geometric quantities.

7.1.1 Equation for phase Φ

Since u is a solution to (7.1), the phase satisfies the equation $\Phi = \frac{1}{2}x \cdot Du - u$. Taking two derivatives,

$$-R_{\alpha\bar{\beta}} = \frac{\partial^2 \Phi}{\partial z^\alpha \partial \bar{z}^\beta} = \frac{1}{2}x \cdot Du_{\alpha\bar{\beta}}. \quad (7.3)$$

Differentiating $\Phi = \ln \det(u_{\alpha\bar{\beta}})$,

$$D\Phi = g^{\alpha\bar{\beta}} Du_{\alpha\bar{\beta}}.$$

Combining these equations, we get

$$g^{\alpha\bar{\beta}} \frac{\partial^2 \Phi}{\partial z^\alpha \partial \bar{z}^\beta} = \frac{1}{2}x \cdot D\Phi. \quad (7.4)$$

In particular, we have the important relation

$$R = -\frac{1}{2}x \cdot D\Phi. \quad (7.5)$$

7.1.2 Inequality for scalar curvature R

Differentiating $R = -\frac{1}{2}x \cdot D\Phi$ twice and using $R_{\alpha\bar{\beta}} = -\frac{\partial^2 \Phi}{\partial z^\alpha \partial \bar{z}^\beta}$,

$$\frac{\partial^2 R}{\partial z^\alpha \partial \bar{z}^\beta} = -\frac{\partial^2 \Phi}{\partial z^\alpha \partial \bar{z}^\beta} - \frac{1}{2}x \cdot D \frac{\partial^2 \Phi}{\partial z^\alpha \partial \bar{z}^\beta} = R_{\alpha\bar{\beta}} + \frac{1}{2}x \cdot DR_{\alpha\bar{\beta}}. \quad (7.6)$$

Also, differentiating $R = g^{\alpha\bar{\beta}} R_{\alpha\bar{\beta}}$,

$$DR = -g^{\alpha\bar{\gamma}} Du_{\bar{\gamma}\delta} g^{\delta\bar{\beta}} R_{\alpha\bar{\beta}} + g^{\alpha\bar{\beta}} DR_{\alpha\bar{\beta}}.$$

Hence by (7.3),

$$\begin{aligned} \frac{1}{2}x \cdot DR &= -g^{\alpha\bar{\gamma}} \left(\frac{1}{2}x \cdot Du_{\bar{\gamma}\delta} \right) g^{\delta\bar{\beta}} R_{\alpha\bar{\beta}} + g^{\alpha\bar{\beta}} \frac{1}{2}x \cdot DR_{\alpha\bar{\beta}} \\ &= g^{\alpha\bar{\gamma}} (R_{\bar{\gamma}\delta}) g^{\delta\bar{\beta}} R_{\alpha\bar{\beta}} + g^{\alpha\bar{\beta}} \frac{1}{2}x \cdot DR_{\alpha\bar{\beta}}. \end{aligned}$$

Coupled with (7.6), we get

$$g^{\alpha\bar{\beta}} \frac{\partial^2 R}{\partial z^\alpha \partial \bar{z}^\beta} - \frac{1}{2}x \cdot DR = R - g^{\alpha\bar{\gamma}} g^{\delta\bar{\beta}} R_{\alpha\bar{\beta}} R_{\bar{\gamma}\delta} \leq R - \frac{1}{m}R^2,$$

or equivalently

$$g^{\alpha\bar{\beta}} \frac{\partial^2 R}{\partial z^\alpha \partial \bar{z}^\beta} \leq \frac{1}{2}x \cdot DR + R - \frac{1}{m}R^2. \quad (7.7)$$

7.1.3 Inequality for distance ρ

Fix a point $x \in \mathbb{C}^m$, and let $\rho = \rho(x)$. We assume that x is not in the cut locus of 0. Since (\mathbb{C}^m, g) is complete, there is a (unique) unit speed minimizing geodesic $\chi : [0, \rho] \rightarrow \mathbb{C}^m$ from 0 to x . We introduce a vector field $X(\tau)$ along $\chi(\tau)$ defined by $X = \chi^\alpha \frac{\partial}{\partial z^\alpha} + \chi^{\bar{\beta}} \frac{\partial}{\partial \bar{z}^\beta}$, where we regard $\chi \in \mathbb{C}^m$ as a tangent vector. Note that $X(0) = 0$ and $X(\rho) = x^i \frac{\partial}{\partial x^i}$.

We proceed to compute the directional derivative $x \cdot D\rho(x)$ using the metric g :

$$\begin{aligned} x \cdot D\rho(x) &= \langle X(\rho), \nabla_g \rho \rangle_g = \langle X(\rho), \dot{\chi}(\rho) \rangle \\ &= \int_0^\rho \frac{d}{d\tau} \langle X(\tau), \dot{\chi}(\tau) \rangle d\tau = \int_0^\rho \langle \nabla_\tau X(\tau), \dot{\chi}(\tau) \rangle d\tau, \end{aligned}$$

where the tangent vector $\dot{\chi}(\tau) = \frac{d}{d\tau} \chi$ and for simplicity of notation we have dropped the subscript g in the inner product $\langle \cdot, \cdot \rangle_g$. To calculate the above integrand, we first compute the covariant derivative of X along χ :

$$\begin{aligned} \nabla_\tau X &= \dot{\chi}^\alpha \frac{\partial}{\partial z^\alpha} + \dot{\chi}^{\bar{\beta}} \frac{\partial}{\partial \bar{z}^\beta} + \chi^\alpha \nabla_{\dot{\chi}} \frac{\partial}{\partial z^\alpha} + \chi^{\bar{\beta}} \nabla_{\dot{\chi}} \frac{\partial}{\partial \bar{z}^\beta} \\ &= \dot{\chi} + \chi^\alpha \Gamma_{\gamma\alpha}^\mu \dot{\chi}^\gamma \frac{\partial}{\partial z^\mu} + \chi^{\bar{\beta}} \Gamma_{\delta\bar{\beta}}^{\bar{\nu}} \dot{\chi}^\delta \frac{\partial}{\partial \bar{z}^\nu}. \end{aligned}$$

Then using the identity $\Gamma_{\gamma\alpha}^\mu g_{\mu\bar{\beta}} = u_{\gamma\alpha\bar{\beta}}$ (for a Kähler potential) and (7.3), we have

$$\langle \nabla_\tau X, \dot{\chi} \rangle = 1 + X \cdot Du_{\alpha\bar{\beta}} \dot{\chi}^\alpha \dot{\chi}^{\bar{\beta}} = 1 - 2R_{\alpha\bar{\beta}} \dot{\chi}^\alpha \dot{\chi}^{\bar{\beta}}.$$

Therefore, we have the formula:

$$x \cdot D\rho(x) = \rho(x) - \int_0^\rho 2R_{\alpha\bar{\beta}} \dot{\chi}^\alpha \dot{\chi}^{\bar{\beta}} d\tau. \quad (7.8)$$

We have the following estimate for the Laplacian of the distance function ρ .

Lemma 34. *Suppose $\text{Ric} \leq K$ on $B_g(0, \rho_0)$ for $\rho_0 > 0$. If $\rho(x) > \rho_0$ and x is not in the cut locus of 0, then*

$$g^{\alpha\bar{\beta}} \frac{\partial^2 \rho}{\partial z^\alpha \partial \bar{z}^\beta}(x) \leq \left[\frac{2m-1}{2\rho_0} + \frac{K}{3}\rho_0 \right] + \frac{1}{2} x \cdot D\rho(x) - \frac{1}{2} \rho(x). \quad (7.9)$$

Proof. The mean curvature H of the geodesic sphere $\partial B_g(0, \rho)$ with respect to the normal $\nabla_g \rho$ equals $\frac{-1}{2m-1} \Delta_g \rho$. As calculated in [9, p.52], H satisfies the following differential inequality

$$H_\rho \geq H^2 + \frac{1}{2m-1} Ric(\nabla_g \rho, \nabla_g \rho).$$

Let $H = \frac{-1}{\rho} + b$. Since the Riemannian metric g is asymptotically Euclidean as $\rho \rightarrow 0$, we know b is bounded for small ρ (in fact $O(\rho)$). We then have a corresponding inequality for b :

$$\frac{1}{\rho^2} + b_\rho \geq \frac{1}{\rho^2} - 2\frac{1}{\rho}b + b^2 + \frac{1}{2m-1} Ric(\nabla_g \rho, \nabla_g \rho),$$

and consequently

$$(\rho^2 b)_\rho \geq \rho^2 b^2 + \frac{\rho^2}{2m-1} Ric(\nabla_g \rho, \nabla_g \rho) \geq \frac{\rho^2}{2m-1} Ric(\nabla_g \rho, \nabla_g \rho).$$

Integrating along $\chi(\tau)$, we arrive at

$$b(\chi(\rho_0)) \geq \frac{1}{\rho_0^2} \int_0^{\rho_0} \frac{\tau^2}{2m-1} Ric(\dot{\chi}, \dot{\chi}) d\tau.$$

Then for $\rho \geq \rho_0$,

$$\begin{aligned} H(\chi(\rho)) &= H(\chi(\rho_0)) + \int_{\rho_0}^{\rho} H_\rho d\tau \\ &\geq H(\chi(\rho_0)) + \int_{\rho_0}^{\rho} \frac{1}{2m-1} Ric(\dot{\chi}, \dot{\chi}) d\tau \\ &\geq \frac{-1}{\rho_0} + \frac{1}{\rho_0^2} \int_0^{\rho_0} \frac{\tau^2}{2m-1} Ric(\dot{\chi}, \dot{\chi}) d\tau + \int_{\rho_0}^{\rho} \frac{1}{2m-1} Ric(\dot{\chi}, \dot{\chi}) d\tau. \end{aligned}$$

Substituting back $\Delta_g \rho = -(2m-1)H$ and recalling $x = \chi(\rho)$, we obtain

$$\Delta_g \rho \leq \frac{2m-1}{\rho_0} - \int_0^{\rho} Ric(\dot{\chi}, \dot{\chi}) d\tau + \int_0^{\rho_0} \left(1 - \frac{\tau^2}{\rho_0^2}\right) Ric(\dot{\chi}, \dot{\chi}) d\tau, \quad (7.10)$$

when $\rho \geq \rho_0$. In fact, this estimate was first derived in [57, Sec.8] (by a second variation argument).

Note that the Riemannian Laplacian $\Delta_g = 2g^{\alpha\bar{\beta}} \frac{\partial^2}{\partial z^\alpha \partial \bar{z}^\beta}$ and $Ric(\dot{\chi}, \dot{\chi}) = 2R_{\alpha\bar{\beta}} \dot{\chi}^\alpha \dot{\chi}^{\bar{\beta}}$.

Then (7.9) follows from combining (7.8) with (7.10). \square

To prove Theorem 8 we will also need an inequality for $\rho^y(x) = \text{distance from } x \text{ to } y \text{ in } (\mathbb{C}^m, g)$. Following the previous argument and using $(X - y^i \frac{\partial}{\partial x^i})$ instead of X for the vector field along χ , we have

Lemma 35. *Suppose $\text{Ric} \leq K$ on $B_g(y, \rho_0)$. Fix $A > \rho_0$, and let $x \in B_g(y, A)$. If $\rho^y(x) > \rho_0$ and x is not in the cut locus of y , then*

$$g^{\alpha\bar{\beta}} \frac{\partial^2 \rho^y}{\partial z^\alpha \partial \bar{z}^\beta}(x) \leq \left[\frac{2m-1}{2\rho_0} + \frac{K}{3}\rho_0 \right] + \frac{1}{2}x \cdot D\rho^y(x) - \frac{1}{2}\rho^y(x) + C_0 A|y|, \quad (7.11)$$

where the constant C_0 only depends on the ‘‘Euclidean’’ norms of $Du_{\alpha\bar{\beta}}$ and g^{-1} in $B_g(y, A)$.

Proof. Arguing as above, let χ be the unit speed minimizing geodesic from y to x . Using $(X - y^i \frac{\partial}{\partial x^i})$ for the vector field along χ (note that $X(0) = y^i \frac{\partial}{\partial x^i}$), we have

$$(x - y) \cdot D\rho^y(x) = \rho^y(x) - \int_0^{\rho^y(x)} 2R_{\alpha\bar{\beta}} \dot{\chi}^\alpha \dot{\chi}^{\bar{\beta}} d\tau - \int_0^{\rho^y(x)} y \cdot Du_{\alpha\bar{\beta}} \dot{\chi}^\alpha \dot{\chi}^{\bar{\beta}} d\tau.$$

The conclusion of the lemma follows, as above, by combining this equation with (7.10). \square

7.2 Proof of rigidity

First we prove the scalar curvature $R \geq 0$ on complete (\mathbb{C}^m, g) . Choose a cut-off function ϕ such that $\phi \equiv 1$ on $[0, 1]$, $\phi \equiv 0$ on $[2, \infty)$, $\phi' \leq 0$, $|\phi'| \leq C_1 \phi^{1/2}$, and $|\phi'' - 2(\phi')^2/\phi| \leq C_2 \phi^{1/2}$. For any small $\rho_0 > 0$, $K = K(\rho_0)$ can be chosen so that $\text{Ric} \leq K$ on $B_g(0, 2\rho_0)$. Fixing $A > \rho_0$, we derive an effective negative lower bound (7.12) for R on $B_g(0, A)$. Set $\tilde{R} = \phi(\rho/A)R$. If $R < 0$ at some point in $B_g(0, 2A)$, then \tilde{R} achieves a negative minimum at some point $p \in B_g(0, 2A)$, as $\overline{B_g(0, 2A)}$ is compact for each $A > 0$ by the completeness of (\mathbb{C}^m, g) . We consider two cases.

Case 1: p is not in the cut locus of 0. Then ρ is smooth near p , and we have

$$\Delta_g \tilde{R} = \left(\frac{\phi''}{A^2} + \frac{\phi'}{A} \Delta_g \rho \right) R + \phi \Delta_g R + 2\langle \nabla \phi, \nabla R \rangle,$$

where we have used $|\nabla\rho| = 1$. In order to have a linear differential inequality for \tilde{R} with smooth coefficients (even for the Lipschitz function ρ), we rewrite

$$\begin{aligned} \langle \nabla\phi, \nabla R \rangle &= \left\langle \nabla\phi, \frac{\nabla\tilde{R}}{\phi} - \frac{\nabla\phi}{\phi^2}\tilde{R} \right\rangle = \left\langle \nabla\frac{\tilde{R}}{R}, \frac{\nabla\tilde{R}}{\phi} \right\rangle - \frac{|\nabla\phi|^2}{\phi^2}\tilde{R} \\ &\stackrel{|\nabla\rho|=1}{=} \frac{|\nabla\tilde{R}|^2}{\tilde{R}} - \left\langle \frac{\nabla R}{R}, \nabla\tilde{R} \right\rangle - \frac{(\phi')^2}{A^2\phi}R \\ &\stackrel{\tilde{R}<0}{\leq} - \left\langle \frac{\nabla R}{R}, \nabla\tilde{R} \right\rangle - \frac{(\phi')^2}{A^2\phi}R. \end{aligned}$$

Using $\Delta_g = 2g^{\alpha\bar{\beta}}\frac{\partial^2}{\partial z^\alpha\partial\bar{z}^\beta}$, the inequalities (7.9) and (7.7) for ρ and R , and the inequalities for ϕ , we get

$$\begin{aligned} g^{\alpha\bar{\beta}}\frac{\partial^2\tilde{R}}{\partial z^\alpha\partial\bar{z}^\beta} &\stackrel{\phi'R>0}{\leq} \frac{1}{2A^2} \left[\phi'' - \frac{2(\phi')^2}{\phi} \right] R + \frac{\phi'R}{A} \left[\left(\frac{2m-1}{2\rho_0} + \frac{K}{3}\rho_0 \right) + \frac{1}{2}x \cdot D\rho(x) \right] \\ &\quad + \phi \left(\frac{1}{2}x \cdot DR + R - \frac{1}{m}R^2 \right) - \left\langle \frac{\nabla R}{R}, \nabla\tilde{R} \right\rangle \\ &\leq \frac{C_2}{2A^2}\phi^{1/2}|R| + \frac{C_1}{A} \left(\frac{2m-1}{2\rho_0} + \frac{K}{3}\rho_0 \right) \phi^{1/2}|R| - \frac{\phi R^2}{m} \\ &\quad + \tilde{R} + \frac{1}{2}x \cdot D\tilde{R} - \left\langle \frac{\nabla R}{R}, \nabla\tilde{R} \right\rangle \\ &\leq \frac{C(m, \rho_0)}{A^2} + \tilde{R} + b(x) \cdot D\tilde{R}, \end{aligned}$$

where $b(x)$ is a smooth function and $C(m, \rho_0)$ is a constant that depends only on m , ρ_0 , C_1 , and C_2 . Since \tilde{R} achieves its minimum at p , we have $\tilde{R}(p) \geq -\frac{C(m, \rho_0)}{A^2}$ and $R \geq -\frac{C(m, \rho_0)}{A^2}$ on $B_g(0, A)$.

Case 2: p is in the cut locus of 0. Then ρ is not smooth at p , and we argue using Calabi's trick [9, p.53] of approximating ρ from above by smooth functions (cf. [13, pp.453-456]). For completeness, we include the argument here. Let χ be a unit speed geodesic from 0 to p that minimizes length, and define $\rho_\varepsilon = \rho^{\chi(\varepsilon)} + \varepsilon$, where $\rho^{\chi(\varepsilon)}$ is the distance to $\chi(\varepsilon)$. Then $\rho_\varepsilon(p) = \rho(p)$ and $\rho_\varepsilon \geq \rho$ near p . Since p is not in the cut locus of $\chi(\varepsilon)$, we know that ρ_ε is smooth near p . Let $\tilde{R}_\varepsilon = \phi(\rho_\varepsilon/A)R$. Then \tilde{R}_ε is smooth near p . Furthermore, since ϕ is decreasing and $R < 0$ near p , the above

properties of ρ_ε show that $\tilde{R}_\varepsilon(p) = \tilde{R}(p)$ and $\tilde{R}_\varepsilon \geq \tilde{R}$ near p . It follows that \tilde{R}_ε has a local minimum at p . Arguing as we did in Case 1, and using Lemma 35 to estimate ρ_ε , we have

$$g^{\alpha\bar{\beta}} \frac{\partial^2 \tilde{R}_\varepsilon}{\partial z^\alpha \partial \bar{z}^\beta} \leq \frac{C(m, \rho_0)}{A^2} + \tilde{R}_\varepsilon + b(x) \cdot D\tilde{R}_\varepsilon + \frac{\phi' R}{A} [C_0 2A |\chi(\varepsilon)|],$$

where $b(x)$ is a smooth function, $C(m, \rho_0)$ is a constant that depends only on m, ρ_0, C_1 , and C_2 , and C_0 is a constant depending on the ‘‘Euclidean’’ norms of $Du_{\alpha\bar{\beta}}$ and g^{-1} in $\overline{B_g(0, 2A)}$. Note that we may choose C_0 independent of (small) ε . At p we have

$$\tilde{R}_\varepsilon(p) \geq -\frac{C(m, \rho_0)}{A^2} - \frac{\phi' R}{A}(p) [C_0 2A |\chi(\varepsilon)|].$$

Taking $\varepsilon \rightarrow 0$, we arrive at the inequality $\tilde{R}(p) \geq -\frac{C(m, \rho_0)}{A^2}$, which shows $R \geq -\frac{C(m, \rho_0)}{A^2}$ on $B_g(0, A)$.

Combining Case 1 and Case 2, we have shown that

$$R \geq -\frac{C(m, \rho_0)}{A^2} \text{ on } B_g(0, A). \quad (7.12)$$

Taking $A \rightarrow \infty$, we arrive at $R \geq 0$ on \mathbb{C}^m .

Now, we finish the proof of Theorem 8. Since $R \geq 0$, it follows from the equation $R = -g^{\alpha\bar{\beta}} \frac{\partial^2 \Phi}{\partial z^\alpha \partial \bar{z}^\beta} = -\frac{1}{2}x \cdot D\Phi$ that Φ achieves its global maximum at the origin. Applying the strong maximum principle to equation (7.4) we conclude that Φ is constant. Using $\frac{1}{2}x \cdot Du - u = \Phi$, we have

$$\frac{1}{2}x \cdot D[u + \Phi(0)] = u + \Phi(0).$$

Finally, it follows from Euler’s homogeneous function theorem that smooth $u + \Phi(0)$ is a homogeneous order two polynomial.

Remark. In fact, one sees that the Lipschitz function \tilde{R} (on the set where $\tilde{R} < 0$) is a subsolution to

$$g^{\alpha\bar{\beta}} \frac{\partial^2 \tilde{R}}{\partial z^\alpha \partial \bar{z}^\beta} \leq \frac{C(m, \rho_0)}{A^2} + \tilde{R} + b(x) \cdot D\tilde{R}$$

in the viscosity sense by using the same trick of Calabi. It follows from the comparison principle (cf. [20, p.18]) that the same negative lower bounds for \tilde{R} and hence R can be derived.

BIBLIOGRAPHY

- [1] U. Abresch, J. Langer, *The normalized curve shortening flow and homothetic solutions*, J. Differential Geom. 23 (1986), 175–196.
- [2] F.J. Almgren, Jr., *Some interior regularity theorems for minimal surfaces and an extension of Bernstein's theorem*, Ann. of Math. (2) 84 (1966) 277–292.
- [3] S. Angenent, *Shrinking doughnuts*, Nonlinear diffusion equations and their equilibrium states, 3 (Gregynog, 1989), Birkhäuser Boston Inc., Boston, MA, 1992, 21–38.
- [4] S. Angenent, *Some recent results on mean curvature flow*, Recent advances in partial differential equations (El Escorial, 1992), 1–18, RAM Res. Appl. Math., 30, Masson, Paris, 1994.
- [5] T. Aubin, *Some nonlinear problems in Riemannian geometry*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 1998.
- [6] M.S. Baouendi, C. Goulaouic, *Singular nonlinear Cauchy problems*, J. Differential Equations 22 (1976), 268–291.
- [7] J.C. Bird, W.D. Ristenpart, A. Belmonte, H.A. Stone, *Critical angle for electrically driven coalescence of two conical droplets*, Phys. Rev. Lett. 103 (2009) 164502.
- [8] K. Brakke, *The motion of a surface by its mean curvature*, Princeton University Press, Princeton, New Jersey, 1978.
- [9] E. Calabi, *An extension of E. Hopf's maximum principle with an application to Riemannian geometry*, Duke Math. J. 25 (1958), 45–56.
- [10] A. Chau, J. Chen, Y. Yuan, *Rigidity of entire self-shrinking solutions to curvature flows*, J. Reine Angew. Math. 664 (2012), 229–239.
- [11] B.-L. Chen, *Strong uniqueness of the Ricci flow*, J. Differential Geom. 82 (2009), no. 2, 363–382.

- [12] D. Chopp, *Computation of self-similar solutions for mean curvature flow*, Experiment. Math. 3 (1994), no. 1, 1–15.
- [13] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, L. Ni, *Ricci flow: techniques and applications, part I: geometric aspects*, Math. Surv. Monogr. 135. Amer. Math. Soc., 2007.
- [14] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, L. Ni, *Ricci flow: techniques and applications, part IV: long time solutions and related topics*, to appear.
- [15] T.H. Colding, W.P. Minicozzi, *Generic mean curvature flow I: generic singularities*, Ann. of Math. (2) 175 (2012), no. 2, 755–833.
- [16] T.H. Colding, W.P. Minicozzi, *Smooth compactness of self-shrinkers*, Comment. Math. Helv. 87 (2012), no. 2, 463–475.
- [17] T.H. Colding, T. Ilmanen, W.P. Minicozzi, *Rigidity of generic singularities of mean curvature flow*, preprint. Available at arXiv:1304.6356.
- [18] T.H. Colding, T. Ilmanen, W.P. Minicozzi, B. White, *The round sphere minimizes entropy among closed self-shrinkers*, preprint. Available at arXiv:1205.2043.
- [19] T.H. Colding, W.P. Minicozzi, E.K. Pedersen, *Mean curvature flow as a tool to study topology of 4-manifolds*, preprint. Available at arXiv:1208.5988.
- [20] M.G. Crandall, H. Ishii, P. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. 27 (1992), no. 1, 1–67.
- [21] Q. Ding, Z. Wang, *On the self-shrinking systems in arbitrary codimensional spaces*, preprint. Available at arXiv:1012.0429.
- [22] Q. Ding, Y.L. Xin, *The rigidity theorems for Lagrangian self shrinkers*, J. Reine Angew. Math. DOI: 10.1515/crelle-2012-0081.
- [23] M.P. DoCarmo, *Differential geometry of curves and surfaces*, Prentice-Hall, Englewood Cliffs, New Jersey, 1976.
- [24] G. Drugan, *An immersed S^2 self-shrinker*, to appear in Trans. Amer. Math. Soc.
- [25] G. Drugan, *A Bernstein result for high codimension self-shrinker graphs*, preprint. Available at <http://www.math.washington.edu/~drugan/papers>.

- [26] G. Drugan, *Embedded S^2 self-shrinkers with rotational symmetry*, preprint. Available at <http://www.math.washington.edu/~drugan/papers>.
- [27] G. Drugan, S.J. Kleene, *Immersed self-shrinkers*, preprint. Available at arXiv:1306.2383.
- [28] G. Drugan, P. Lu, Y. Yuan, *Rigidity of complete entire self-shrinking solutions to Kähler-Ricci flow*, to appear in Int. Math. Res. Not. doi:10.1093/imrn/rnu051
- [29] K. Ecker, *Regularity theory for mean curvature flow*, Progress in Nonlinear Differential Equations and their Applications, 57. Birkhäuser Boston Inc., Boston, MA, 2004.
- [30] K. Ecker, G. Huisken, *Mean curvature evolution of entire graphs*, Ann. Math. 130 (1989), no. 3, 453–471.
- [31] K. Ecker, G. Huisken, *Parabolic methods for the construction of spacelike slices of prescribed mean curvature in cosmological spacetimes*, Comm. Math. Phys. 135 (1991), no. 3, 595–613.
- [32] C.L. Epstein, M.I. Weinstein, *A stable manifold theorem for the curve shortening equation*, Comm. Pure Appl. Math. 40 (1987), no. 1, 119–139.
- [33] M.E. Gage, *Curve shortening makes convex curves circular*, Invent. Math. 76 (1984), 357–364.
- [34] M. Gage, R.S. Hamilton, *The heat equation shrinking convex plane curves*, J. Differential Geom. 23 (1986), no. 1, 69–96.
- [35] D. Gilbarg, N. Trudinger, *Elliptic partial differential equations of second order*, Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.
- [36] M. Grayson, *The heat equation shrinks embedded plane curves to round points*, J. Differential Geom. 26 (1987), no. 2, 285–314.
- [37] H.P. Halldorsson, *Self-similar solutions to the curve shortening flow*, Trans. Amer. Math. Soc. 364 (2012), no. 10, 5285–5309.
- [38] P. Hartman, *Ordinary differential equations*, John Wiley & Sons, Inc., New York-London-Sydney, 1964.

- [39] S. Helmenstorfer, *A model for the behavior of fluid droplets based on mean curvature flow*, preprint. Available at arXiv:1102.3546.
- [40] H. Hopf, *Differential geometry in the large*, Lecture Notes in Mathematics, 1000. Springer-Verlag, Berlin, 1989.
- [41] H. Huang, Z. Wang, *On the entire self-shrinking solutions to Lagrangian mean curvature flow*, Calc. Var. Partial Differential Equations 41 (2011), no. 3-4, 321–339.
- [42] G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differential Geom. 20 (1984), 237-266.
- [43] G. Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, J. Differential Geom. 31 (1990), no. 1, 285–299.
- [44] G. Huisken, *A distance comparison principle for evolving curves*, Asian J. Math. 2 (1998), no. 1, 127-133.
- [45] G. Huisken, S.T. Yau, *Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature*, Invent. Math. 124 (1996), 281-311.
- [46] T. Ilmanen, *Singularities of mean curvature flow of surfaces*, 1995. Available at <http://www.math.ethz.ch/~ilmanen/papers/pub.html>
- [47] T. Ilmanen, B. White, *Sharp lower bounds on density for area-minimizing cones*, preprint. Available at arXiv:1010.5068.
- [48] J. Jost, Y.L. Xin, *Bernstein type theorems for higher codimension*, Calc. Var. 9 (1999), 277–296.
- [49] N. Kapouleas, S.J. Kleene, N.M. Møller, *Mean curvature self-shrinkers of high genus: Non-compact examples*, preprint. Available at arXiv:1106.5454.
- [50] S.J. Kleene, N.M. Møller, *Self-shrinkers with a rotational symmetry*, to appear in Trans. Amer. Math. Soc.
- [51] H.B. Lawson Jr., R. Osserman, *Non-existence, non-uniqueness and irregularity of solutions to the minimal surface system*, Acta Math. 139 (1977), 1–17.

- [52] N.M. Møller, *Closed self-shrinking surfaces in R^3 via the torus*, preprint. Available at arXiv:1111.7318.
- [53] W.W. Mullins, *Two-dimensional motion of idealized grain boundaries*, J. Appl. Phys. 27, no. 8 (1956), 900–904.
- [54] X.H. Nguyen, *Construction of complete embedded self-similar surfaces under mean curvature flow. Part I*, Trans. Amer. Math. Soc. 361 (2009), no. 4, 1683–1701.
- [55] X.H. Nguyen, *Construction of complete embedded self-similar surfaces under mean curvature flow. Part II*, Adv. Differential Equations 15 (2010), no. 5–6, 503–530.
- [56] X.H. Nguyen, *Construction of complete embedded self-similar surfaces under mean curvature flow. Part III*, preprint. Available at arXiv:1106.5272.
- [57] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, preprint. Available at arXiv:math/0211159.
- [58] W.D. Ristenpart, J.C. Bird, A. Belmonte, F. Dollar, H.A. Stone, *Non-coalescence of oppositely charged drops*, Nature 461 (2009), 377–380.
- [59] C.S. Smith, *Introduction to grains, phases, and interfaces-an interpretation of microstructure*, Trans. Amer. Inst. of Min. and Met. Eng. 175 (1948), 15–51. (Reprinted in Met. and Mat. Trans. A vol. 41A, 2010, 1063–1100.)
- [60] C.S. Smith, *Grain shapes and other metallurgical applications of topology*, Metal Interfaces, American Society for Metals, Cleveland, 1952.
- [61] G. Tian, *Canonical metrics in Kähler geometry*, Notes taken by Meike Akveld, Lectures in Mathematics ETH Zrich, Birkhuser Verlag, Basel, 2000.
- [62] P. Topping, *Mean curvature flow and geometric inequalities*, J. Reine Angew. Math. 503 (1998), 47–61.
- [63] L. Wang, *A Bernstein type theorem for self-similar shrinkers*, Geom. Dedicata 151 (2011), 297–303.
- [64] M. Warren, Y. Yuan, *Hessian estimates for the sigma-2 equation in dimension 3* Comm. Pure Appl. Math. 62 (2009), 305–321.

Appendix A
PICTURES OF GEODESICS

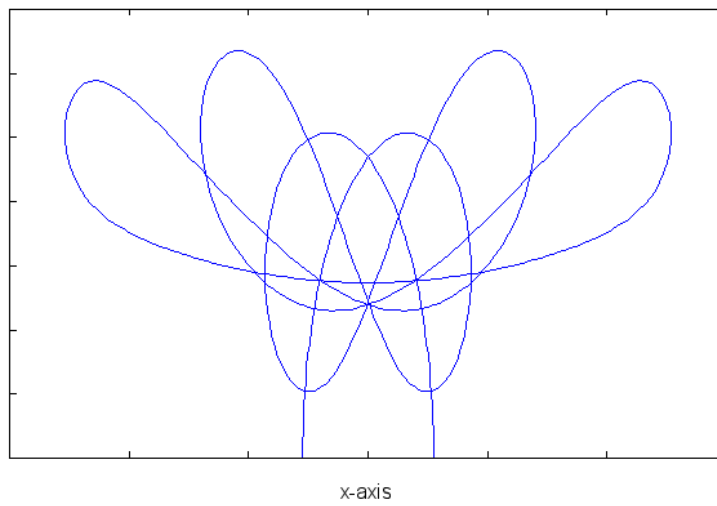


Figure A.1: A geodesic with several self-intersections whose rotation about the x -axis is an immersed sphere self-shrinker.

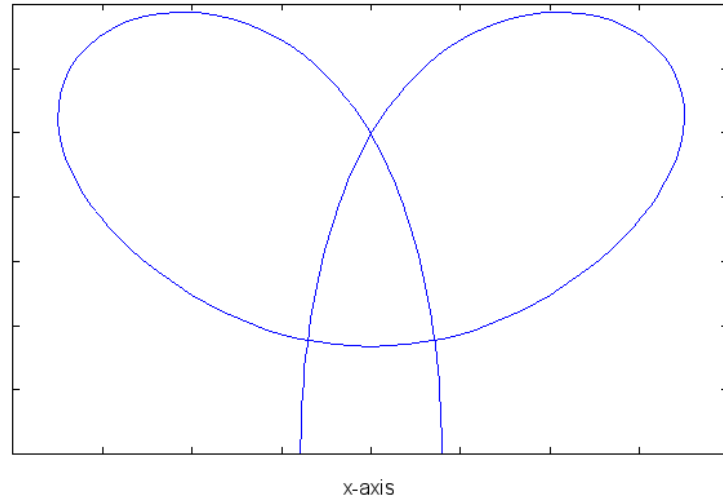


Figure A.2: A geodesic whose rotation about the x -axis is an immersed sphere self-shrinker.

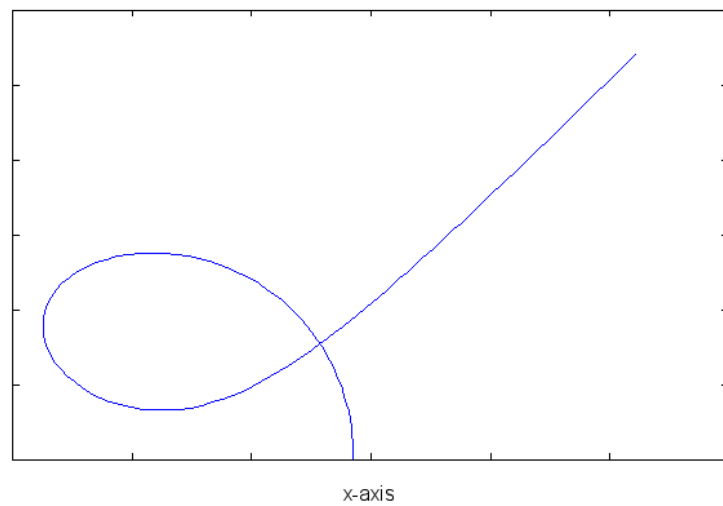


Figure A.3: A geodesic whose rotation about the x -axis is an immersed plane self-shrinker.

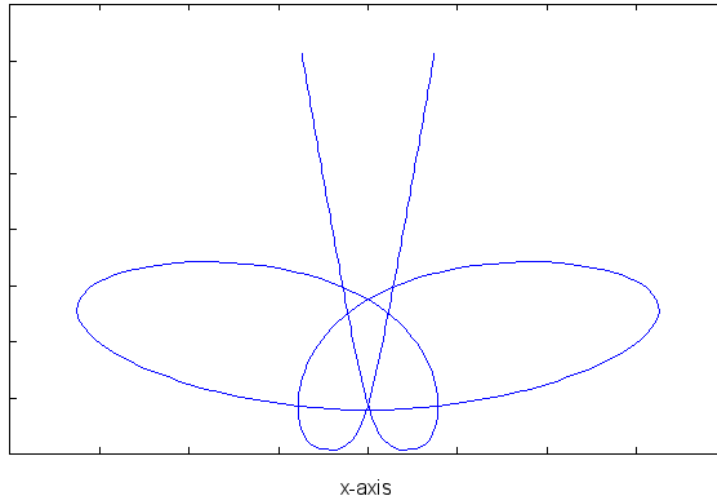


Figure A.4: A geodesic whose rotation about the x -axis is an immersed cylinder self-shrinker.

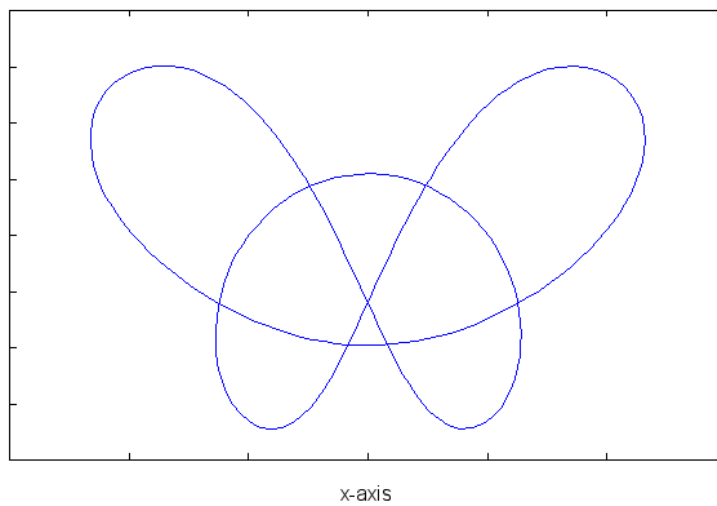


Figure A.5: A geodesic whose rotation about the x -axis is an immersed torus self-shrinker.

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