# On $f$-vectors of polytopes and matroids

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## Abstract

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The $f$-vector of a simplicial complex is a fundamental invariant that counts the number of faces in each dimension. A natural question in the theory of simplicial complexes is to understand the relationship between the $f$-vector of the simplicial complex and the properties of the topological, algebraic and combinatorial structures associated to the simplicial complex. This dissertation studies this question for two different classes of simplicial complexes: simplicial polytopes and matroid independence complexes.

For the class of simplicial polytopes we study what are the possible $f$-vectors of polytopes that are good approximations of a convex body with smooth enough boundary. In particular, in Chapter 2, we settle a longstanding conjecture of Kalai asserting that good approximations of smooth convex body $K$ are far from extremal in the sense of the lower bound theorem in a precise way. We make the result quantitative in the case the convex body is of type $C^2$.

Little is known about $f$-vectors of matroid independence complexes. A full characterization is believed to be out of reach and several conjectures about properties of such vectors are wide open. A famous one is a conjecture of Stanley that predicts certain behavior based on the properties of the Stanley-Reisner ring of the matroid independence complex. The main goal of the second part of this document is to study this conjecture. In Chapter 3, we prove that the conjecture holds for rank 4 matroids by means of a new combinatorial method. In Chapter 4 we study the external activity complex of a matroid which is topologically simpler than the independence complex of the matroid and contains the information of $f$-vector. Chapter 5 introduces the notion of a quasi-matroidal class of ordered simplicial complexes, the notion is used to provide extensions of various properties of matroids, including a refinement of Stanley's conjecture that is proved to hold in a variety of cases, including Schubert matroids.

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# Chapter 1

# INTRODUCTION

Simplicial complexes are ubiquitous in mathematics. A simplicial complex can be viewed combinatorially as a family of subsets of a given set, geometrically as a collection of simplices glued together, topologically as a triangulation of a given topoogical space and algebraically as a squarefree monomial ideal of a polynomial ring. These different points of view provide a rich dictionary between combinatorics, geometry, topology, and commutative algebra that can be used in a variety of ways to solve problems of many different flavors.

In this dissertation we study the $f$-vectors of (some classes of) simplicial complexes: a fundamental invariant that is straightforward to define and that reveals fascinating connections in the dictionary mentioned above. The faces of an abstract (or combinatorial) simplicial complex are the members of the family of subsets of a given set. The dimension of a face is its cardinality minus one and the $f$-vector is a vector whose $i$-th entry is the number of faces of dimension $i$.

For a fixed class $\mathcal{F}$ of simplicial complexes, one may ask for a numerical characterization of the possible $f$-vectors of complexes in $\mathcal{F}$. This has been done successfully for various classes of simplicial complexes. When $\mathcal{F}$ is the class of all simplicial complexes, a numerical characterization is provided by the Kruskal-Katona theorem [62, 53]. On a more algebraic side, when the class $\mathcal{F}$ consists of simplicial complexes whose Stanley-Reisner rings over a field $\mathbb{F}$ are Cohen-Macaulay, the answer is provided by Stanley in [89]. On the geometric side of the story, if $\mathcal{F}$ denotes the class of simplicial polytopes, then the classification is given by the celebrated $g$-theorem, conjectured by McMullen [70] and spectacularly resolved by Stanley [90] (necessity) and Billera and Lee [14] (sufficiency). Many other classes of simplicial complexes have a complete numerical classification.

On the other hand of the story, various classes of complexes are still widely mysterious. For instance, the famous $g$-conjecture states that if $\mathcal{F}$ consists of complexes homeomorphic to spheres, then the family of possible $f$-vectors is the same as the family of $f$-vectors of simplicial polytopes. If $\mathcal{F}$ is the class of pure simplicial complexes, i.e. complexes in which all the maximal faces under inclusion have the same dimension, then there is not even a conjectural answer to the question.

The first part of this dissertation (Chapter 2) deals with the $f$-vector theory of simplicial polytopes, that is, simplicial complexes that are the boundary complexes of the convex hull of points in general position in $\mathbb{R}^d$. As explained above, the family of all $f$-vectors of simplicial polytopes is completely classified. However, little is known about the relationship between the metric properties of a polytope and its $f$-vector. The goal of Chapter 2 is to address this question from the point of view of approximation theory. We study polytopes that approximate convex bodies whose boundary is sufficiently smooth, settle an old conjecture of Kalai [51] about such approximations, and provide quantitative bounds for nice enough bodies. Our results show that there is an intimate relationship between the face numbers of a polytope and the metric structure of some of its realizations. This part is based on an article [3] written with Karim Adiprasito and Eran Nevo.

The second part of this dissertation (Chapters 3-5) focuses on Deloera-Kemper-Klee that the problem is completely out of reach with current methods. On the other hand, it is known that there are many restrictings to being the $f$-vector of a matroid, and it is reasonable to try to understand those restrictions and come up with more.

One of the main conjectures in this field is due to Stanley [89, Section 7]. It posits that the $f$-vectors of matroid independence complexes satisfy more restrictions than those implied by the classification of Cohen Macaulay complexes provided by Stanley. It is phrased in terms of the $h$-vector (a transformation of the $h$-vector that contains the same information as the $f$-vector) and order monomial ideals (families of monomials closed under divisibility). It predicts that, for an arbitrary matroid $M$ there is a multicomplex $\mathcal{O}(M)$ such that all maximal monomials of $\mathcal{O}(M)$ w.r.t divisibility have the same degree and that contains exactly

$h_i(M)$ monomials of degree $i$. See the Introductions of chapters 3 and 5 for more detailed discussions of the conjecture as well as its current status.

Chapter 3 introduces a technique coming from a well-known shelling order of a matroid independence complex that allows us to refine Stanley's conjecture into a purely combinatorial conjecture. One advantage of this approach is that in order to prove the new conjecture in rank $d$ it is enough to prove it for matroid with at most $2d$ elements. An algorithm is presented that resolves this new conjecture and hence Stanley's conjecture for rank-3 and rank-4 matroids. This provides new solid evidence in favor of Stanley's conjecture and the first proof for rank-4 matroids. Pure $O$-sequences of degree 3 are completely classified [26], while such a characterization for degree 4 is believed to be out of reach with current methods. This makes the rank-4 result significantly stronger than the rank-3 result. This Chapter is based on the article [54] written with Steven Klee.

Chapter 4 discusses the External Activity Complex of a matroid. This complex arised naturally in the work of Ardila and Boocher [7] where they studied studying initial ideals of the coordinate ring of the embedding of a linear space in a product of projective lines. We prove that the external activity complex is shellable and find various shelling orders that are related to shelling orders of the matroid independence complex. Furthermore, it is shown that the restriction sets of the shellings can be expressed in terms of internal activities. This yields a new way to interpret the $h$-vector of the independence complex of the matroid. The external activity complex turns out to be a cone over a contractible space which makes it easier to study, at least from a topological point of view. This chapter is based on joint work with Federico Ardila and Federico Castillo.

Chapter 5 discusses the notion of a quasi-matroidal class of ordered simplicial complexes. A family of pure ordered simplicial complexes is said to be quasi-matroidal if it is closed under a few standard operations on simplicial complexes, contains pure shifted complexes and has the property that if a fixed simplicial complex is on the family for every ordering of the groundset, then it is the independence complex of a matroid. We then present many examples of quasi-matroidal classes and highlight various similarities between the classes of shifted simplicial complexes and matroid independence complexes that were known, yet seemed quite mysterious. Quasi-matroidal classes provide a scenario to test statements about matroid independence complexes: translate the statement into the language of shifted complexes by finding an appropriate quasi-matroidal class where the statement in question can be extended. Solutions to the problem for shifted complexes are often easier and can hint a strategy for a general solution. Quasi-matroidal classes also provide tools to study matroid independence complexes by induction on the number of bases, a technique out of reach in matroid theory but desirable from a topological perspective. As examples of the latter ideas, we provide a generalization of the Tutte polynomial and nbc complexes in an appropriate quasi-matroidal class. Both generalizations satisfy many of the classical properties. Furthermore, we posit an extension of the main conjecture discussed in Chapter 3, and prove that it works for pure shifted complexes.

Each chapter is independent of the others: they can be read separately. The notation is consistent with the bigger picture that each chapter fits in. The first three chapters are accepted papers and the last one will be submitted promptly.

# Chapter 2

# A GEOMETRIC LOWER BOUND THEOREM

## *2.1 Introduction*

The combinatorial structure of polytopes was studied since antiquity and has been one of the major topics in algebraic and geometric combinatorics in the last few decades. The simplest combinatorial invariant of a $d$-polytope $P$ is the $f$-vector $(f_{-1}, f_0, \dots, f_{d-1})$, where $f_i$ is the number of $i$-dimensional faces of $P$. Understanding face numbers of polytopes is one of the oldest branches of mathematics.

The celebrated $g$-theorem, conjectured by McMullen [69], gives a complete characterization of the $f$-vectors of *simplicial* polytopes, namely polytopes all whose proper faces are simplices. It is conveniently phrased is terms of the $g$-vector, obtained by a linear transformation of the $f$-vector. Billera and Lee [14] proved sufficiency of the numerical conditions and Stanley [90] proved their necessity by relating the $g$-numbers to the primitive Betti numbers of the associated projective toric varieties. Some extremal cases in terms of the $g$-numbers are well understood; for instance polytopes with $g_k = 0$ are exactly the $(k-1)$-stacked polytopes, as stated in the Generalized Lower Bound Conjecture (GLBC) of McMullen-Walkup [72] and recently proved by Murai-Nevo [76]. However, away from the extremal primitive Betti vectors, the simplicial polytopes become much harder to understand.

An equally foundational subject in polytope theory is approximation theory. Polytopes are dense in the space of convex bodies with respect to several different metrics, and the question what is the minimal number of faces of a certain dimension that are needed to produce an approximation of a certain quality has been substantially studied; see Schneider [85], Gruber [41, 42], and finally Böröczky [19, 20], producing asymptotically tight answer for the *individual* face numbers for $C^2$-convex bodies.

In 1994 Kalai [51] posed a visionary conjecture that relates the entire $f$-vector of a simplicial polytope $P$ to its metric structure. Roughly speaking, Kalai conjectures that if $K$ is a convex body whose boundary is of type $C^1$ and $P$ is a simplicial polytope that is close to $K$ in the Hausdorff distance, then the $f$-vector of $P$ must be far away from extremal $f$-vectors in the sense of the $g$-theorem. Kalai states his conjecture using the $g$-vector and shadow functions $\partial^k$ (see [101, Section 8.5]):

**Conjecture 2.1.1** (Kalai [51])**.** *Let $K$ be a $C^1$-convex body in $\mathbb{R}^d$ and let $\{P_n\}_{n=1}^{\infty}$ be a sequence of simplicial polytopes that converges to $K$ in the Hausdorff metric. Then*

*(i) for every $1 \leqslant k \leqslant \left\lfloor \frac{d}{2} \right\rfloor$,*

$$\lim_{n\to\infty} g_k(P_n) = \infty,$$

*(ii) and for every $1 \leqslant k \leqslant \left\lfloor \frac{d}{2} \right\rfloor - 1$,*

$$\lim_{n\to\infty} \left(g_k - \partial^{k+1} g_{k+1}\right) = \infty.$$

The aim of this paper is to resolve part (i) of Conjecture 2.1.1 and provide a quantitative lower bound on the $g$-numbers in the case when the boundary of $K$ is of type $C^2$. This provides the first bridge between the approximation theory by convex polytopes and the Stanley-Reisner theory of convex polytopes. From the geometric point of view, it connects the geometry of the toric variety of the approximating polytope with the geometry of the underlying polytope. More specifically, this result shows that there is an intimate relation between the metric structure of some embeddings of a polytope and the primitive Betti numbers in the cohomology ring of the associated toric variety. On the other hand, our quantitative results generalize the theorems of Böröczky in the case when the approximating polytopes are simplicial.

Although in this paper we focus mainly on the Hausdorff metric, most of the results hold for other metrics, such as Schneider's metric, the Banach–Mazur distance, the symmetric difference distance, etc. as we rely on Böröczky's method [19] for the final approximation.

In [4] Adiprasito, Nevo and Samper provided a notion of higher chordality of simplicial complexes and showed that it generalizes the classical notion of chordal graphs. In [1] the Adiprasito introduced toric chordality, a powerful algebraic tool to study chordality in the stress-space of the simplicial complex as studied by Lee [64]. He related this algebraic notion of chordality to the higher chordality notions of [4] and derived, among many other results, a quantitative version of the GLBC in terms of the topological Betti numbers of induced subcomplexes. In this paper, we use this result to prove Kalai's lower bound conjecture in full generality (alternatively, for self-containedness, we use a weaker statement proved in the last section).

This paper is organized as follows: in Section 2.2 we provide the needed preliminaries, in Section 2.3 we give a simple proof of Conjecture 2.1.1(i) for the unit 4-ball, using framework rigidity arguments. These arguments are vastly generalized in Section 2.4 to prove Conjecture 2.1.1(i) in full generality, for $C^1$-convex bodies. In Section 2.5 we generalize Böröczky's results by giving asymptotically tight lower bounds on the $g$-numbers when approximating a $C^2$-convex body, in terms of its Hausdorff distance from the approximating simplicial polytope. We also observe that Conjecture 2.1.1(ii) holds for approximations by random polytopes.

## *2.2 Preliminaries*

### *2.2.1 Convex bodies*

A convex body $K$ in $\mathbb{R}^d$ is a convex compact subset of $\mathbb{R}^d$ with non-empty interior. The main example of a convex body is the closed unit ball $\overline{B_1(0)}$ in $\mathbb{R}^d$ with the standard metric. In general, every convex body is a convex embedding of $\overline{B_1(0)}$ in $\mathbb{R}^d$. The boundary of a convex body $K$ is denoted by $\partial K$, and $\mathbb{S}^{d-1} := \partial\overline{B_1(0)}$ denotes the standard unit sphere.

Endow $\mathbb{R}^d$ with the standard inner product denoted by $\langle\, , \rangle$. For an element $u \in \mathbb{S}^{d-1} \subseteq \mathbb{R}^d$ and a convex body $K$, let $c(u, K) = \max_{\{v \in K\}}\langle u, v\rangle$. Also, let $H^+(u, K) = \{s \in \mathbb{R}^d, \langle u, s\rangle \leqslant c(u, K)\}$ be a supporting halfspace of $K$ in direction $u$. It is well known that

$K = \bigcap_{u \in \mathbb{S}^{d-1}} H^+(u, K)$. The boundary of $H^+(u, K)$ is denoted by $H(u, K)$. For a point $x \in \partial K$ there is at least one point $u \in \mathbb{S}^{d-1}$ such that $x \in H(u, K)$. If this point $u$ is unique we say that $x$ is non-singular. Denote the unique such direction by $u(x)$, whenever $x$ is a non singular point. For every non singular point $x$ there exist neighborhoods $U_x \subseteq \partial K$ and $V_x \subseteq H(u(x), K)$ of $x$, where $V_x$ is convex, and a non-negative convex function $f_x : V_x \to \mathbb{R}$, such that, for every $v$ in $V_x$, the point $\varphi_x(v) = v - f_x(v)u(x)$ is an element of $U_x$ and the map $\varphi_x$ is a homeomorphism from $V_x$ to $U_x$.

Endowing $\overline{B_1(0)}$ with its standard differential structure, we say that a convex body $K$ is of *type* $C^k$ if it is the image of a $C^k$-embedding of $\overline{B_1(0)}$ in $\mathbb{R}^d$. Equivalently, the boundary $\partial K$ is a $C^k$-hypersurface in $\mathbb{R}^d$. If $k \geqslant 1$, and $K$ is a $C^k$-convex body, then every point $x \in \partial K$ is non-singular.

### *2.2.2 Polytopes and simplicial complexes*

A *polytope* $P$ is the convex hull of finitely many points in some Euclidean space; equivalently it is a bounded intersection of finitely many closed half-spaces. Polytopes are a very special class of convex bodies. A *face* of a polytope $P$ is the intersection of a supporting hyperplane of $P$ with $P$. The *dimension* of a face is the dimension of its affine span. Assume that $P$ is $d$-dimensional. The *$f$-vector* of $P$ is the vector $f_P := (f_{-1}, f_0, f_1, \ldots, f_{d-1})$ where $f_i$ is the number of $i$-dimensional faces of $P$ ($f_{-1} = 1$ for the empty face). A *simplex* is the convex hull of a set of affinely independent points, thus a $k$-dimensional simplex has $k + 1$ vertices. A polytope $P$ is *simplicial* if all proper faces of $P$ are simplices. We denote the set of proper faces of $P$ by $\partial P$ and call it the *boundary* of $P$.

A (geometric) *simplicial complex* $\Delta$ is a finite family of simplices such that (i) if $F$ is in $\Delta$ and $G$ is a face of $F$, then $G$ is also in $\Delta$, and (ii) for any two elements $F$ and $G$ of $\Delta$, $F \cap G$ is a face of both $F$ and $G$. Note that a polytope $P$ is simplicial if and only if the boundary of $P$ is a simplicial complex. The elements of a simplicial complex are also called *faces* and the *dimension* of a simplicial complex is the maximal dimension of a face. As in the case of polytopes we may define the $f$-vector of $\Delta$, $f_\Delta := (f_{-1}, f_0, \ldots, f_{d-1})$, to be the vector such that $f_i$ is the number of faces of dimension $i$, called *$i$-faces.* Thus, for $\Delta = \partial P$, $f_\Delta = f_P$.

The set of faces of $\Delta$ of dimension at most $i$ is a subcomplex called the *$i$-th skeleton* of $\Delta$ and denoted by $\Delta^{(i)}$. The set of 0-faces is denoted by $V(\Delta)$ and is called the set of *vertices* of $\Delta$; the 1-faces are called *edges.* When all faces of $\Delta$ that are maximal under inclusion have the same dimension $d$ we say $\Delta$ is *pure* and refer to its $d$-faces as *facets* and to its $(d-1)$-faces as *ridges.*

The *link* of a face $F$ of $\Delta$, denoted by $\mathrm{link}_\Delta(F)$, or $\mathrm{link}(F)$ for short, is the set of all faces $G$ of $\Delta$, such that $F \cap G = \varnothing$ and $G$ is contained in a face that contains $F$. It is straightforward (see [101, Proposition 2.4, page 55]) that for every face $F$ of a simplicial polytope $P$ the link of $F$ in $\partial P$ is combinatorially isomorphic to the boundary of some simplicial polytope. The link of a vertex is sometimes called a *vertex figure.* For a subset $W$ of the vertex set of $\Delta$, let $\Delta_W$ denote the *induced subcomplex* of $\Delta$ on $W$, namely the complex whose faces are the subsets of $W$ which are faces of $\Delta$.

For a simplicial complex $\Delta$, let $\tilde{H}_k(\Delta)$ be the reduced $k$-th (simplicial or singular) homology group over $\mathbb{Q}$ and let $\tilde{\beta}_k(\Delta) := \dim_{\mathbb{Q}} \tilde{H}_k(\Delta)$ be the $k$-th topological Betti number. We say that a cycle (either simplicial or singular) is not trivial if its homology class does not vanish. Simplicial cycles can be viewed as singular cycles.

For a simplex $\Gamma$ in $\mathbb{R}^d$ of dimension $< d$, and $v$ a point not in the affine span of $\Gamma$, let $v * \Gamma = \mathrm{conv}(v, \Gamma)$. The simplex $v * \Gamma$ is called the cone over $\Gamma$ with apex $v$.

A point set in $\mathbb{R}^d$ is *generic*, or in *general position*, if any $d+1$ of its points are affinely independent. An affine subspace is *generic* w.r.t. a collection of geometric simplices if it contains no vertex, and its parallels contain no edge, of these simplices.

### *2.2.3 f-vectors of simplicial polytopes*

The *$f$-polynomial* of a $d$-dimensional simplicial polytope $P$ is the generating function of the $f$-vector, given by the polynomial $f_P(x) = \sum_{j=0}^{d} f_{j-1} x^j$. Sometimes it is convenient to consider the *$h$-polynomial*, $h_P(x) := (1-x)^d f_P\left(\frac{x}{1-x}\right)$. The $h$-vector $(h_0, h_1, \ldots, h_d)$ of

$P$ is the vector of coefficients of the $h$-polynomial, that is, $h_P(x) = \sum_{i=0}^{d} h_i x^i$. Knowing the $h$-vector is equivalent to knowing the $f$-vector. The Dehn-Sommerville relations (see [56, Theorem 3.2 and Prop. 3.3]) assert that $h_i = h_{d-i}$ for a simplicial $d$-polytope $P$ and $0 \leqslant i \leqslant d$. It follows that the first half of the entries of the $f$-vector of $P$ determine the entire $f$-vector of $P$.

The celebrated classification by [14] and [90] of the $f$-vectors of simplicial $d$-polytopes is known as the $g$-theorem and is usually stated in terms of the $g$-vector $(g_0, g_1, \dots, g_{\lfloor \frac{d}{2} \rfloor})$, where $g_0 := h_0 = 1$ and $g_i = h_i - h_{i-1}$ for $1 \leqslant i \leqslant \lfloor \frac{d}{2} \rfloor$. To prove Conjecture 2.1.1(i) we only require the lower bound part of this theorem that states the nonnegativity of the $g_i$.

**Theorem 2.2.1.** *(g-theorem) An integer vector* $(g_0, g_1, \dots, g_{\lfloor \frac{d}{2} \rfloor})$ *is the g-vector of a simplicial d-polytope if and only if it is the Hilbert function of some graded commutative algebra finitely generated in degree 1. In particular,* $g_0 = 1$ *and* $g_k \geqslant 0$ *for* $1 \leqslant k \leqslant \lfloor \frac{d}{2} \rfloor$.

A numerical characterization of the Hilbert functions as in the $g$-theorem is due to Macaulay, using his shadow functions $\partial^k(\cdot)$, cf. [101, Section 8.5]. We will use them only in the last Remark 2.5.7 and in Conjecture 2.1.1(ii).

The following recent result of Adiprasito [1] generalizes the lower bound theorem, and will be crucial in our proof of Conjecture 2.1.1(i).

**Theorem 2.2.2** (The quantitative lower bound theorem)**.** *Let $P$ be a simplicial $d$-polytope with boundary complex $\Delta$, $k \leqslant \frac{d}{2}$, and let $W$ be any subset of the vertices, then:*

$$\tilde{\beta}_{d-k-1}(\Delta_W) \leqslant g_k(\Delta). \tag{2.1}$$

The proof uses a subtle approach via combinatorial Morse theory. We will therefore, for purposes of self-containedness, provide also a slightly weaker alternative lemma to the same effect based on the McMullen proof of the hard Lefschetz theorem, see Lemma 2.4.6.

### *2.2.4 Framework rigidity*

Let $G = (V, E)$ be a graph and let $\varphi : V \to \mathbb{R}^d$ be any map. We say that $\varphi$ is *rigid* if there exists $\varepsilon > 0$ such that if $\varphi' : V \to \mathbb{R}^d$ is such that $|\varphi(v) - \varphi'(v)| < \varepsilon$ for any $v \in V$ and $|\varphi(v) - \varphi(w)| = |\varphi'(v) - \varphi'(w)|$ for every $\{w, v\} \in E$, then $|\varphi(v) - \varphi(w)| = |\varphi'(v) - \varphi'(w)|$ for every $\{w, v\} \in \binom{V}{2}$. We say that $\varphi$ is *flexible* if it is not rigid.

The set of all maps $V \to \mathbb{R}^d$ forms a $d \cdot |V|$-dimensional vector space over $\mathbb{R}$ which can be endowed with the Lebesgue measure. A graph $G$ is *generically $d$-rigid* if almost every map $\varphi : V \to \mathbb{R}^d$ is rigid and *generically $d$-flexible* if almost every such map is flexible. It is known that every graph is either generically $d$-rigid or generically $d$-flexible.

Fix a vertex set $V$ and consider the family $\mathcal{R}(V, d) \subseteq 2^{\binom{V}{2}}$ of all the minimal under inclusion edge sets $E$ such that $G = (V, E)$ is a generically $d$-rigid graph. The collection $\mathcal{R}(V, d)$ is the set of bases of a matroid. In particular, the cardinality of any element of $\mathcal{R}(V, d)$ is an invariant denoted by $\rho(V, d)$.

Let $G = (V, E)$ be graph and let $\varphi : V \to \mathbb{R}^d$ be a map. A *stress*, w.r.t. $(G, \varphi)$, is a map $\omega : E \to \mathbb{R}$ such that for every vertex $v$:

$$\sum_{u:\ \{u,v\} \in E} \omega(\{u, v\})(\varphi(u) - \varphi(v)) = 0. \tag{2.2}$$

The family of stresses of $(G, \varphi)$ is a vector space; if $\varphi$ is generic and $G$ is generically $d$-rigid then this stress space has dimension $|E| - \rho(V, d)$.

Kalai [50] observed that for $d \geqslant 3$ the graph of a simplicial $d$-polytope $P$ is generically $d$-rigid, and used it to prove that the dimension of the stress space of this graph equals $g_2(\partial P)$. This provides an alternative proof of the lower bound theorem of Barnette [13], where the minimizers are those $P$ with $g_2(\partial P) = 0$. Kalai also showed that, for $d \geqslant 4$, $g_2(\partial P) = 0$ if and only if $P$ is *stacked*, namely it can be obtained from the $d$-simplex by repeatedly stacking a $d$-simplex over a facet of the polytope already constructed. Further, for $d \geqslant 5$ this happens if and only if every vertex link is stacked.

### *2.2.5 The Hausdorff metric*

For a point $x \in \mathbb{R}^d$ and $A \subset \mathbb{R}^d$ define $d(x, A) := \inf_{a \in A} |x - a|$ to be the distance from $x$ to $A$ in the usual Euclidean metric.

Let $A, B$ be two bounded subsets of $\mathbb{R}^d$. Define the Hausdorff distance between $A$ and $B$ by:

$$\delta^H(A, B) := \max \left\{ \sup_{a \in A} d(a, B),\ \sup_{b \in B} d(b, A) \right\}.$$

It is easy to verify that $\delta^H$ defines a metric on the space of compact subsets of $\mathbb{R}^d$, and thus restricts to a metric on the space of convex bodies in $\mathbb{R}^d$.

### *2.2.6 Approximation theory*

Every convex body $K$ can be approximated by polytopes in the Hausdorff metric. A natural question is what is the minimal number of vertices that achieves an approximation of distance $\varepsilon$. Assume that $K$ is of type $C^1$. Let $n(\varepsilon)$ be the minimal number of vertices of a polytope $P$ with $\delta^H(P, K) < \varepsilon$. It is clear that $n(\varepsilon)$ goes to infinity as $\varepsilon$ goes to 0.

If $K$ is $C^2$ then the asymptotic behavior of $n(\varepsilon)$ is well understood. Böröczky [19, Theorem A(9)] computed the asymptotic growth of $n(\varepsilon)$ explicitly, as follows:

**Theorem 2.2.3.** *If $K$ is a $C^2$-convex body then:*

$$\lim_{\varepsilon \to 0} n(\varepsilon) \varepsilon^{(d-1)/2} = 4^{\frac{1-d}{2}} \frac{\Theta_{d-1}}{V_{d-1}} \int_{\partial K} \sqrt{\kappa} d(\partial K), \tag{2.3}$$

*where $V_d$ is the volume of the unit d-ball, $\Theta_d$ is the covering density of $\mathbb{R}^d$ by unit d-balls, and $\kappa$ is the Gauss curvature.*

For our purposes, the important property of equation (2.3) is that the right-hand side is strictly bigger than 0 and bounded. In particular $n(\varepsilon)$ behaves roughly like $\varepsilon^{-\frac{d-1}{2}}$ for small enough $\varepsilon$.

### *2.3 Warm up: rigidity and Kalai's conjecture for the unit 4-ball*

This section is devoted to proving Kalai's conjecture for simplicial 4-dimensional polytopes approximating the unit 4-ball, using rigidity theory. We then vastly generalize the ideas demonstrated here to prove the general case in the next section.

As mentioned in Subsection 2.2.6, Conjecture 2.1.1(i) holds for $k = 1$ (for any $d$), so the first open case of this conjecture is $k = 2, d = 4$, and the most basic $C^1$-convex body to consider is the unit 4-ball. For the rest of the section, the support of a stress $\omega$ of an embedded graph $G$ is the set of vertices that belong to an edge $e$ of $G$ such that $w(e) \neq 0$.

**Lemma 2.3.1.** *Let $P$ be a generically embedded simplicial 4-polytope and let $v$ be a vertex of $P$. Assume that link($v$) is not stacked. Then there is a non-zero stress $w$ supported in $N_2(v) := \{u \in V(P) :\ d(u, v) \leqslant 2$ in the graph metric$\}$.*

*Proof.* We follow ideas of Kalai [50]. Recall that a simplicial 3-polytope is stacked if and only if its 1-skeleton is *chordal*, cf. [50, Theorem 8.5], namely, all its induced cycles have length 3. As link($v$) is not stacked, there exists an induced cycle $C = v_1, \ldots v_m$ of link($v$) with $m \geqslant 4$. There are two cases to consider:

i. $C$ is not induced in $\partial P$. Then for some $1 \leqslant i < j \leqslant m$ there is an edge $e = \{v_i, v_j\}$ in $\partial P$ that is not in link($v$). By the Cone Lemma in rigidity, cf. [98, Theorem 5], as the graph (1-skeleton) of link($v$) is generically 3-rigid, the graph $G$ of $v * \text{link}(v)$ is generically 4-rigid. Thus, $G \cup \{e\}$ supports a nonzero stress $w$. The vertex support of $w$ is contained in the vertices of $G \cup \{e\} \subseteq N_1(v)$, thus also in $N_2(v)$.

ii. $C$ is induced in $\partial P$. Consider the complex $\Delta = \bigcup_{i=3}^{m} v_i * \text{link}(v_i)$. By the Gluing Lemma in rigidity, cf. [10, Theorem 2], the graph $G = \Delta^{(1)}$ is generically 4-rigid (as all the cones are, and $v_i * \text{link}(v_i) \cap v_{i+1} * \text{link}(v_{i+1})$ contains a tetrahedron so this intersection has at least 4 vertices). The edge $e = \{v_1, v_2\}$ is not an edge of $\Delta$, but both $v_1$, $v_2$ are vertices of $\Delta$. Thus the given embedding of $G \cup \{e\}$ has a nonzero stress. This stress is supported in $N_2(v)$ as desired.

□

**Theorem 2.3.2.** *Kalai's Conjecture 2.1.1(i) holds for the unit* 4*-ball.*

*Proof.* Assume by contradiction that $g_2(P_n) \leqslant g-1$ for all $n$, for some positive integer $g$. Let $B$ denote the unit 4-ball.

Then there exist $\varepsilon_1 > 0$ and $g$ oriented hyperplanes $H_1, \ldots, H_g$ that intersect the interior of $B$ such that the corresponding negative sides intersected with $B$ are far from each other: $0 \notin H_i^-$ for $1 \leqslant i \leqslant g$ and for any $1 \leqslant i < j \leqslant g$, $\varepsilon_1 < \min\{d(x,y) : \ x \in H_i^- \cap B, \ y \in H_j^- \cap B\}$. Let $\delta = \min_{1 \leqslant i \leqslant g}(1 - d(0, H_i))$.

If a simplicial polytope $P$ well-approximates $B$, then all its edges must be short. Specifically, there exists $\frac{\delta}{2} > \varepsilon_2 > 0$ such that if $\delta^H(P, B) < \varepsilon_2$ then all edges of $P$ have length $< \min\{\frac{\delta}{2}, \frac{\varepsilon_1}{4}\}$. (A quantitative estimate will be given in Section 2.5, when we compute effective lower bounds on the $g$-numbers for $C^2$-convex bodies.)

By rescaling and slightly moving the vertices, w.l.o.g. we may assume the approximating polytopes $P_n$ are generically embedded and contained in $B$. Now, for $P \subseteq B$ as above, if in each cap $H_i^- \cap B$ there is a vertex $v_i$ of $\partial P$ whose link is not stacked, then by Lemma 2.3.1 there is a stress $w_i$ supported in $N_2(v_i)$. By the choice of $\varepsilon_2$, for all $1 \leqslant i < j \leqslant g$, $N_2(v_i) \cap N_2(v_j) = \varnothing$, and thus the $g$ stresses $w_i$ are linearly independent, yielding $g_2(P) \geqslant g$, a contradiction. It follows that there is $1 \leqslant i \leqslant g$ for which all vertices $v$ of $P$ in $H_i^- \cap B$ have stacked links. Denote $H = H_i$.

Note that for each such vertex $v$, $v * \mathrm{link}(v)$ has a unique stacked triangulation $\Sigma_v$ (i.e., one without new vertices or edges). The family $\Sigma = \cup_{v \in V(P) \cap H^-} \Sigma_v$ is a geometric simplicial complex: indeed if $u$ and $v$ are vertices and $\Gamma$ is a simplex in $\Sigma_v$ that contains $u$ and $v$, then the graph of $\Gamma$ is contained in $u * \mathrm{link}(u)$ and by the uniqueness of the stacked triangulation $\Gamma \in \Sigma_u$ as well. In the next Lemma 2.3.3 we will show that the geometric realization $|\Sigma|$ contains $H \cap P$ and thus has a point $x$ in the interior of some 4-simplex $\sigma \in \Sigma$ with $d(x, \partial B) \geqslant \delta$.

As all edges of $\sigma$ have length $< \frac{\delta}{2}$, all vertices of $\sigma$ are of distance $< \frac{\delta}{2}$ from $x$, and thus of

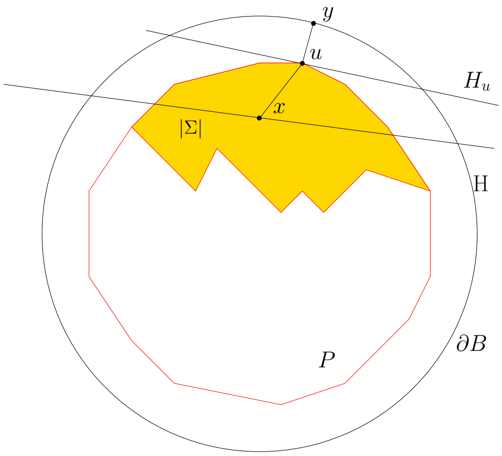


Figure 2.1: $\delta^H(P, B) \geqslant d(y, u) \geqslant d(x, \partial B) - d(x, u) \geqslant \varepsilon_2$.

distance $> \delta - \frac{\delta}{2} = \frac{\delta}{2}$ from $\partial B$ by the triangle inequality. For a vertex $u \in \sigma$ and a supporting hyperplane $H_u$ of $P$ at $u$, the point $y$ in $\partial B \cap H_u^-$ on the line orthogonal to $H_u$ through $u$ is of distance $> \frac{\delta}{2} > \varepsilon_2$ from $P$, a contradiction. $\square$

**Lemma 2.3.3.** *Let $P$ be a simplicial $d$-polytope and let $H$ be a generic oriented hyperplane that passes through the interior of $P$. For each vertex $v$ of $P$ in $H^-$, let $\Sigma'_v$ be a triangulation of link$(v)$ and let $\Sigma_v$ be the collection of simplices formed by coning the simplices of $\Sigma'_v$ with $v$. Let $\Sigma \subset P$ be the family of all simplices of $\Sigma_v$ for all $v \in H^-$ and assume that it is a geometric simplicial complex. Then, for every point $x \in P \cap H$ there is a simplex $\Gamma \in \Sigma$ that contains $x$.*

*Proof.* Let $|\Sigma|$ be the set of points that belong to some simplex of $\Sigma$, so $|\Sigma| \subseteq P$. We need to show that $P \cap H \subseteq |\Sigma| \cap H$. Since $H$ is generic it contains none of the nonempty faces of $P$ nor of $\Sigma$. Let $x \in P \cap H$ be generic, i.e. in general position, with respect to the vertices of $P \cap H$, and let $\ell$ be a generic line in $H$ through $x$. We claim that $|\Sigma| \cap \ell = P \cap \ell$. To establish this, note that $\ell \cap P$ is a closed line segment and admits a continuous parametrization $\gamma : [0, 1] \to \ell \cap P$.

Assume that there is $x \in P \cap \ell$ that is not in $|\Sigma|$. Notice that $\gamma(0)$ lies in the relative interior of a facet of $\partial P$, so this facet contains a vertex $y \in H^-$, by genericity of $H$. This facet is contained in a $d$-simplex of $\Sigma_y$, thus $\gamma([0,z))$ is contained in $|\Sigma|$ for some positive real $z$. Let $s = \inf\{t \in [0,1] \mid \gamma(t) \notin |\Sigma|\}$. Notice that $s \geqslant z > 0$. As $s > 0$, by compactness of $|\Sigma|$ we conclude $\gamma(s) \in |\Sigma|$. By genericity of $\ell$, $\gamma(s)$ is in the relative interior of a $d$- or a $(d-1)$-simplex of $\Sigma$.

The former case is clearly not possible: $\gamma(s)$ would be in the interior of $|\Sigma|$ and therefore in the interior of $|\Sigma| \cap \ell$, a contradiction. In the latter case we will show that $\gamma(s)$ is in the interior of $|\Sigma|$ unless it is in $\partial P$. The reason for this is the following: let $\Gamma$ be a $(d-1)$-simplex of $\Sigma$ that contains $\gamma(s)$. The ridge $\Gamma$ is contained in exactly two facets $F_1, F_2$ of the ball $\Sigma$ unless it is on the boundary of $P$; indeed, the boundary ridges of $\Sigma$ not on $\partial P$ do not contain the vertex $y \in H^-$ introduced in the preceding paragraph. If $\gamma(s)$ is not in $\partial P$ we obtain that $\gamma(s)$ is in the interior of $F_1 \cup F_2$, thus also in the interior of $|\Sigma|$. If $\gamma(s) \in \partial P$, then $s = 0$ or $s = 1$. The case $s = 0$ was discarded before. The case $s = 1$ says $P \cap l = |\Sigma| \cap l$.

It follows that $\ell \cap P \subseteq \ell \cap |\Sigma|$, so in particular $x \in |\Sigma| \cap H$. The set of generic points of $P \cap H$ is dense in $P \cap H$ and is contained in the closed set $|\Sigma| \cap H$. The desired inclusion follows. ☐

## *2.4 A proof of Kalai's conjecture for $C^1$-convex bodies*

Here we prove the first main result of the paper, that part (i) of Kalai's conjecture is true. The following lemma is due to Zalgaller [100], see also Schneider's book [86, Section 2.3, Theorem in Note 1, case $s = 1$].

**Lemma 2.4.1.** *Let $K$ be a convex body in $\mathbb{R}^d$ and let $\pi$ denote an orthogonal projection onto a $k$-dimensional subspace $H$, chosen uniformly at random from the $(d,k)$-Grassmannian. Then, with probability* 1*, all the affine subspaces that are orthogonal to $H$ and support $K$ do not contain a segment of $\partial K$. Thus, $\pi$ restricts to a homeomorphism from $K \cap \pi^{-1}(\partial\pi(K))$ to $\partial\pi(K)$.*

In particular, the preimage of $\partial\pi(K)$ under $\pi$ is, with probability 1, homeomorphic to a $(k-1)$-sphere; we denote this preimage by $\gamma_\pi$. Let $\gamma_\pi + \varepsilon := \gamma_\pi + \varepsilon\overline{B_1(0)}$ (Minkowski sum),

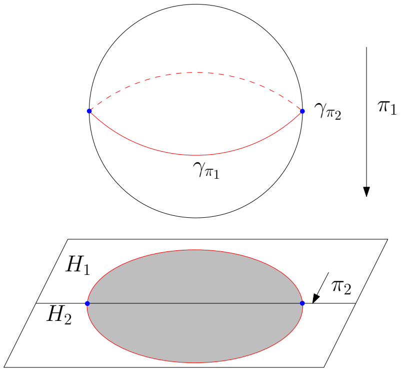


Figure 2.2: Projections $\pi_1$ and $\pi_2$ to subspaces $H_1 \supseteq H_2$ with the respective $\gamma_{\pi_1} \cong \mathbb{S}^1$ and $\gamma_{\pi_2} \cong \mathbb{S}^0$.

where $\overline{B_1(0)}$ is the ball of radius 1 in $\mathbb{R}^d$. Notice that in the Hausdorff metric

$$\lim_{\varepsilon \to 0} K \cap \pi^{-1}(\pi(\gamma_\pi + \varepsilon)) = \gamma_\pi. \tag{2.4}$$

Notice that if $\varepsilon$ is small enough, then there is a point $u \in \pi(K) \backslash \pi(\gamma_\pi + \varepsilon)$. Let $\hat{r} : \pi(\gamma_\pi + \varepsilon) \to \partial\pi(K)$ be the map that sends a point $x$ to the unique element $\hat{r}(x)$ of $\partial\pi(K)$ in the infinite ray from $u$ to $x$. The map $\hat{r}$ is a strong deformation retract if $\varepsilon$ is small enough. Now define $r : \gamma_\pi + \varepsilon \to \gamma_\pi$ by letting $r(x)$ be the unique point in $\gamma_\pi$ that projects to $\hat{r}(\pi(x))$. Then $r$ is a strong deformation retract whenever $\hat{r}$ is, that is, for every small enough $\varepsilon$.

**Lemma 2.4.2.** *Let $\varepsilon > 0$ be small enough so that the $\varepsilon$-neighborhood $\gamma_\pi + \varepsilon$ deformation retracts to $\gamma_\pi$. Then, every simplicial polytope $P \subseteq K$ sufficiently close to $K$ in the Hausdorff metric, has a subcomplex $\Delta \subseteq \partial P \cap (\gamma_\pi + \varepsilon)$ whose embedding into $\gamma_\pi + \varepsilon$ induces an isomorphism in homology.*

*Proof.* Let $\Delta := P \cap \pi^{-1}(\partial(\pi(P)))$. Then $\Delta$ is a subcomplex of $P$ and $\pi(\Delta) = \partial(\pi(P))$. By equation (2.4) there exists $\epsilon' > 0$ such that $\pi^{-1}(\pi(\gamma_\pi + \epsilon')) \subseteq \gamma_\pi + \varepsilon$. If $P$ is close enough to $K$ then $\partial\pi(P) \subseteq \pi(\gamma_\pi + \epsilon')$, thus equation (2.4) implies $\Delta$ is contained in $\gamma_\pi + \varepsilon$. Note that $\pi_{|\Delta}$ is a homotopy equivalence from $|\Delta|$ to $\pi(\gamma_\pi + \varepsilon)$.

Let $g : \pi(\gamma_\pi) \to \gamma_\pi$ be the inverse of $\pi$ restricted to $\gamma_\pi$, namely $g(x)$ is the point $\pi^{-1}(x) \cap K$.

Let $\iota$ denote the inclusion of $\Delta$ in $\gamma_\pi + \varepsilon$, then $r \circ \iota = g \circ \hat{r} \circ \pi_{|\Delta}$. The induced maps in homology of $r,\ g,\ \hat{r},\ \pi_{|\Delta}$ are clearly isomorphisms, so $\iota$ is an isomorphism too. $\square$

Until now, we have not yet used the $C^1$ property of $K$ in any way. Now we use the fact that all points of $\partial K$ are non-singular. (In fact, this property is equivalent to being $C^1$.)

Consider any non-singular convex body $K$, let $(\varepsilon_i)$ denote a sequence of real positive numbers tending to 0, and let $(P_i)$ denote a sequence of simplicial polytopes so that $\delta^H(K, P_i) < \varepsilon_i$ for all $i$.

**Lemma 2.4.3.** *With $K$, $(\varepsilon_i)$ and $(P_i)$ as above, for every $\varepsilon > 0$,*

$$\max\{\operatorname{diam}\sigma : \sigma \text{ is a face of } P_i,\ V(\sigma) \subset \gamma_\pi + \varepsilon_i,\ |\sigma| \not\subset \gamma_\pi + \varepsilon\} \xrightarrow{i\to\infty} 0.$$

*Proof.* Assume by contradiction that there are $\delta > 0$, a subsequence $(P_j)$ and faces $\sigma_j \in P_j$ such that $V(\sigma_j) \subset \gamma_\pi + \varepsilon_j,\ |\sigma_j| \not\subset \gamma_\pi + \varepsilon$, and $\operatorname{diam}(\sigma_j) \geqslant \delta$.

There are two vertices in $\sigma_j$ whose distance is $\geqslant \delta$ and by the triangle inequality every point in $\sigma_j$ is at least $\frac{\delta}{2}$ apart from one of them. Taking a point of $\sigma_j$ not in $\gamma_\pi + \varepsilon$ we obtain a line segment $e_j \subset \sigma_j$ of length at least $\frac{\delta}{2} > 0$ connecting two points $v_j$, $v_j'$ such that $e_j \not\subset \gamma_\pi + \varepsilon$ and $v_j$ is a vertex of $\sigma_j$.

By compactness, passing to a subsequence we can assume that there is convergence $e_j \xrightarrow{j\to\infty} e = [v, v']$ with $v \neq v'$ , and $v_j \xrightarrow{j\to\infty} v$, so $v \in \gamma_\pi$.

Since $P_j \xrightarrow{j\to\infty} K$ then $e \subset \partial K$. We claim that in fact $e$ must be contained in $\gamma_\pi$. Notice that for any point $x$ in $\gamma_\pi$, the hyperplane $T_x$ tangent to $\partial K$ at $x$ projects to the tangent space to $\partial(\pi(K))$ at $\pi(x)$. As $T_v$ is the unique tangent plane at $v$, since $K$ is nonsingular, $e \subset T_v$ (and $e \subset T_{v'}$) and therefore $e \subset \gamma_\pi$ by Lemma 2.4.1.

We conclude that for any fixed $\varepsilon > 0$, a large enough $j$ satisfies $|e_j| \subseteq e + \varepsilon \subseteq \gamma_\pi + \varepsilon$, a contradiction to the choice of $e_j$. $\square$

Combining Lemmas 2.4.2 and 2.4.3 gives:

**Corollary 2.4.4.** *For any non-singular convex body $K$ and every $\varepsilon$ small enough, there is $\varepsilon' > 0$ small enough such that for every simplicial polytope $P$ that is $\varepsilon'$-close to $K$ in the Hausdorff metric, the subcomplex $\Gamma \subseteq \partial P$ induced by the vertices of $P$ in $\gamma_\pi + \varepsilon'$ is contained in $\gamma_\pi + \varepsilon$, and this inclusion induces a surjection in homology.*

*Proof.* For small enough $\varepsilon > 0$, $\gamma_\pi + \varepsilon$ retracts to $\gamma_\pi$. By Lemma 2.4.3, there exists $\varepsilon' < \frac{\varepsilon}{2}$ such that, if $P$ is $\varepsilon'$-close to $K$, for the complex $\Gamma$ on the vertices of $P$ in $\gamma_\pi + \varepsilon'$, all edges of $\Gamma$ that are not contained in $\gamma_\pi + \varepsilon'$ are of length $< \frac{\varepsilon}{2}$, so for a subcomplex $\Delta \subset \partial P \cap (\gamma_\pi + \varepsilon')$ as in Lemma 2.4.2 there are embeddings $|\Delta| \hookrightarrow |\Gamma| \hookrightarrow \gamma_\pi + \varepsilon$. Consider the induced maps in homology: as the composition is a surjection in homology by Lemma 2.4.2, so is the second map. ☐

We are now ready to prove Kalai's conjecture:

**Theorem 2.4.5.** *Let $K$ be a $d$-dimensional $C^1$-convex body in $\mathbb{R}^d$ and let $g, k > 0$ be integers with $k \leqslant \frac{d}{2}$. There exists $\varepsilon > 0$ such that if $P$ is a simplicial polytope with $\delta^H(P, K) \leqslant \varepsilon$, then $g_k(P) > g$.*

*Proof.* Let $x$ be an extremal point of $K$. Then, by Lemma 2.4.1, there are

- ▻ a projection $\pi$ of $\mathbb{R}^d$ onto a $(d-k)$-dimensional subspace, and
- ▻ a ray $l$ emanating from $x$, $l \cap K = \{x\}$, such that the projection $\pi_l$ onto the orthogonal space to $l$ contains the range of $\pi$, and
- ▻ points $x \neq y_i \in l$ converging to $x$, and projective transformations $p_i$, each mapping $y_i$ to infinity along $l$,

such that

- ▻ each composition $\pi_i = \pi \circ p_i$ restricts to a homeomorphism from $\gamma_{\pi_i} := \pi_i^{-1}(\partial \pi_i(K)) \cap K$ to $\partial \pi_i(K)$; so each $\gamma_{\pi_i}$ is a $(d-k-1)$-cycle, and
- ▻ each $\pi_l \circ p_i$ restricts to a homeomorphism from $\gamma_{p_i} = (\pi_l \circ p_i)^{-1}(\partial \pi_l \circ p_i(K)) \cap K$ to $\partial(\pi_l \circ p_i)(K)$; so each $\gamma_{p_i}$ is a $(d-2)$-cycle containing $\gamma_{\pi_i}$.

Note that $x \notin \gamma_{\pi_i} := \gamma_i$ for all $i$, but $\gamma_i \to \{x\}$ in the Hausdorff measure. By passing to a subsequence of $(y_i)$, we may assume that $\gamma_i \cap \gamma_j = \varnothing$ for all $i \neq j$: indeed, $x \notin \gamma_i$ and for

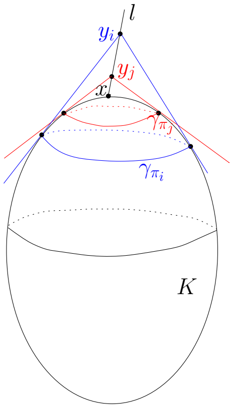


Figure 2.3: Suitable values of $y_i$ give disjoint $\gamma_{\pi_i}$.

every $\varepsilon > 0$ we have that $\gamma_j$ is contained in the open ball $B(x, \varepsilon)$ for sufficiently large $j$, so given $y_{i_j}$ we just need to pick $y_{i_{j+1}}$ so that $\gamma_{i_{j+1}} \subseteq B(x, d(x, \gamma_{i_j}))$.

Consider now the $(d-k-1)$-cycles $\gamma_1, \ldots \gamma_{g+1}$. For $\varepsilon > 0$ small enough, the neighborhoods $\gamma_i + \varepsilon$ are pairwise disjoint and, for each $i$, $\gamma_i + \varepsilon$ deformation retracts to $\gamma_i$. By Corollary 2.4.4, there is some $0 < \varepsilon' < \varepsilon$ such that the embedding of the induced complex $\Gamma_i$ on the vertices of $P$ in $\gamma_i + \varepsilon'$, into $\gamma_i + \varepsilon$, induces a surjection in homology.

It remains to show that for $\varepsilon$ small enough and for every $i \neq j$, there is no edge in $\partial P$ between a vertex of $\Gamma_i$ and a vertex of $\Gamma_j$. Once this is shown we get that the complex $\Gamma = \cup_{1 \leqslant i \leqslant g+1} \Gamma_i$ is an induced subcomplex of $\partial P$, with $\tilde{\beta}_{d-k-1}(\Gamma) = \sum_{j=1}^{g+1} \tilde{\beta}_{d-k-1}(\Gamma_i) \geqslant g + 1$, thus Theorem 2.2.2 finishes the proof.

Assume by contradiction there are approximating polytopes $(P_n)$ with $v_i(n) \in \Gamma_i(n) \subseteq P_n$, $v_j(n) \in \Gamma_j(n) \subseteq P_n$, and $v_i(n)v_j(n)$ an edge of $P_n$. Then there exist a subsequence $(P_{a_n})$ of $(P_n)$, a point $v_i \in \gamma_i$ with $v_i(a_n) \to v_i$ and a point $v_j \in \gamma_j$ with $v_j(a_n) \to v_j$. The segment $[v_i, v_j]$ is contained in $\partial K$. As $K$ is $C^1$, that is nonsingular, $[v_i, v_j]$ is contained in the unique hyperplane $H$ through $v_j$ that supports $K$. Thus, by the choice of $\pi_j$, also $v_i \in \gamma_j$, contradicting that $\gamma_j$ and $\gamma_i$ are disjoint. □

Let us remark that since the homology cycles $\gamma_i$ are represented by spheres, we can substitute, for self-containedness, the use of Theorem 2.2.2 in the proof of Theorem 2.4.5 by the following simple lemma.

Let $\Delta$ be a simplicial complex with a map $\varphi$ of its vertex set $V$ to $\mathbb{R}^d$, and let

$$\widetilde{\varphi} : V \longrightarrow \mathbb{R}^d \times \{1\} \ \subset \ \mathbb{R}^{d+1}$$

the homogenization of $\varphi$. An *(affine) $k$-stress* is a map $\omega$ from the $(k-1)$-dimensional faces of $\Delta$ to $\mathbb{R}$ such that, for every $(k-2)$-face $\tau$ of $\Delta$, the *Minkowski balancing condition* is satisfied, i.e.

$$\sum_{\sigma:\ \sigma\ (k-1)\text{-face},\ \tau\subset\sigma} \omega(\sigma)(\widetilde{\varphi}(\sigma\backslash\tau)) = 0 \quad \operatorname{mod} \operatorname{span}(\widetilde{\varphi}(\tau)),$$

namely, the sum on the left-hand side lies in the linear span of $\widetilde{\varphi}(\tau)$. We refer to Lee [64] for a comprehensive introduction to the subject of affine stresses.

As such, a stress on a graph is the same as a 2-stress (we will henceforth leave out the quantifier "affine“). Moreover, it turns out that $k$-stresses are special $(k-1)$-cycles in the simplicial chain complex of $\Delta$ with real coefficients, see Ishida [48] and Tay–Whiteley [94] for the associated homology theories and more background on the notions. We turn back to the problem at hand.

**Lemma 2.4.6.** *Let $\gamma$ denote a simplicial $(k-1)$-sphere on vertex set $W$. Assume $\gamma$ is realized as a subcomplex of the boundary $\Delta$ of a simplicial $d$-polytope $P$, where $k \leqslant \frac{d}{2}$. Assume furthermore that the fundamental class of $\gamma$ defines a nontrivial homology class in $\widetilde{H}_{k-1}(\Delta_W)$. Then the simplicial neighborhood*

$$\Gamma \ := \ \{\sigma \in \Delta : \exists \tau \in \Delta,\ \sigma \subset \tau,\ \tau \cap \gamma \neq \varnothing\}$$

*of $\gamma$ in $\Delta$ supports a $k$-stress homologous to the fundamental class of $\gamma$ in $\Gamma$.*

We refer the reader to the last section for a proof of the lemma. To finish the alternative proof of Theorem 2.4.5, it suffices to recall the central corollary from the hard Lefschetz theorem for projective toric varieties together with the fact that stresses supported on disjoint vertex sets are linearly independent:

**Theorem 2.4.7.** [71, Theorems 6.1 & 7.3 and p.431] *For any simplicial $d$-polytope $P$, and $k \leqslant \frac{d}{2}$,*

$$g_k(P) \;=\; \dim\{\textit{space of } k\textit{-stresses supported in } P\}.$$

## 2.5 Refined bounds for $C^2$-convex bodies

In the case that $K$ is of type $C^2$ the asymptotic growth of $g_k$ can be bounded below. We start by computing these bounds for approximations of the unit ball and then use tricks of Böröczky to pass to the case of general $C^2$-convex body. The idea is to use the quantitative lower bound Theorem 2.2.2 (or Lemma 2.4.6) again and to provide such bounds by finding many cycles in $\partial K$ that are disjoint and far from each other.

Let $\overline{B_1(0)}$ be the unit ball in $\mathbb{R}^d$. The following lemma is known:

**Lemma 2.5.1.** *For every sufficiently small $\varepsilon$ there is a subset $A$ of the boundary of $\overline{B_1(0)}$ with $|A| = \Omega(\varepsilon^{1-d})$ and distance $d(x,y) \geqslant \varepsilon$ for every pair of points $x, y \in A$.*

*Proof.* Pick an orthogonal basis of $\mathbb{R}^{d-1} \times \{0\}$ with vectors of length $\varepsilon$. Consider the intersection of the lattice generated by this basis and $\overline{B_1(0)}$ and lift it to the boundary of $\overline{B_1(0)}$ to obtain the set $A$. That $A$ works. □

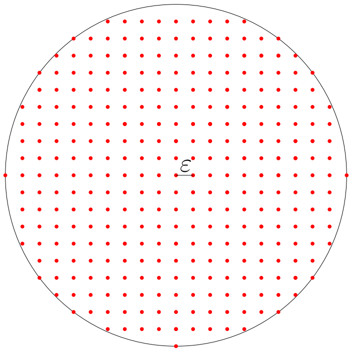


Figure 2.4: The set $A$ for $\mathbb{S}^2$ projected to $\mathbb{R}^2$.

We are now in a position to provide quantitative lower bounds for the $g$-numbers when approximating the unit ball.

**Theorem 2.5.2.** *Let $k \leqslant \frac{d}{2}$. If $\delta^H(P, \overline{B_1(0)})$ is small enough then*

$$g_k(P) = \Omega\left(\delta^H(P, \overline{B_1(0)})^{\frac{1-d}{2}}\right). \tag{2.5}$$

*Proof.* The idea is to intersect $P$ with $(d-k)$-dimensional affine subspaces, where each subspace is close to a different point of a set $A$ from the previous lemma. Then the induced complex on vertices of $P$ that are close enough to these intersections will have $\tilde{\beta}_{d-k-1} \geqslant |A|$ (with contribution of at least one $(d-k-1)$-cycle per intersection). Here are the details.

Let $\varepsilon > 0$ be sufficiently small so by Lemma 2.5.1 there is a set $A$ of points of the boundary of $\overline{B_1(0)}$ with cardinality $\Omega\left(\varepsilon^{\frac{1-d}{2}}\right)$ such that the $d(x,y) > 35\varepsilon^{\frac{1}{2}}$ for every $x, y \in A$.

For each $x \in A$ let $H_x$ be the affine hyperplane 'below $x$' such that $d(x,y) = 11\varepsilon^{\frac{1}{2}}$ for every $y \in \partial\overline{B_1(0)} \cap H_x$, and let $L_x$ be any $(d-k)$-dimensional subspace contained in $H_x$ that passes through the center $u(x)$ of the ball $H_x \cap \overline{B_1(0)}$.

Let $P$ be a simplicial polytope with $\delta^H(P, \overline{B_1(0)}) < \varepsilon$ and boundary complex $\Delta = \partial P$. By rescaling $P$ (multiplying by $(1+\varepsilon)^{-1}$) we obtain a polytope contained in $\overline{B_1(0)}$, combinatorially equivalent to $P$ and whose distance to $\overline{B_1(0)}$ is smaller than $2\varepsilon$, so it is enough to assume that $P \subseteq \overline{B_1(0)}$. If $\varepsilon$ is small enough, then the length of an edge $e \in \partial P$ is bounded above by $4\varepsilon^{\frac{1}{2}}$. To see this, apply the Pythagorean theorem to the triangle in the plane spanned by $e$ and the origin, whose vertices are the origin, the intersection point of the line spanned by $e$ and the line orthogonal to it through the origin, and the appropriate end point of $e$.

For each $x \in A$ let $W_x$ be the set of all vertices of $P$ contained in a face that intersects $L_x$. Then for any vertex $v \in W_x$, $d(v, L_x) \leqslant 4\varepsilon^{\frac{1}{2}}$, as it is bounded by the length of the longest edge of a face containing $v$ that intersects $L_x$.

Let $\Delta_W$ be the complex induced by the vertices in $W := \bigcup_{x \in A} W_x$. For points $x \neq y$ in $A$, and vertices $v \in W_x,\ u \in W_y$, the triangle inequality yields $|v - u| \geqslant \varepsilon^{\frac{1}{2}}(35 - 11 - 4 - 11 - 4) = 5\varepsilon^{\frac{1}{2}}$. As the longest edge in $P$ has length $\leqslant 4\varepsilon^{\frac{1}{2}}$, we conclude that $\Delta_W$ is the disjoint union of the subcomplexes $\Delta_{W_x}$, for all $x \in A$.

We claim that, for $\varepsilon > 0$ small enough, $\beta_{d-k-1}(\Delta_{W_x}) \geqslant 1$. The argument is similar to,

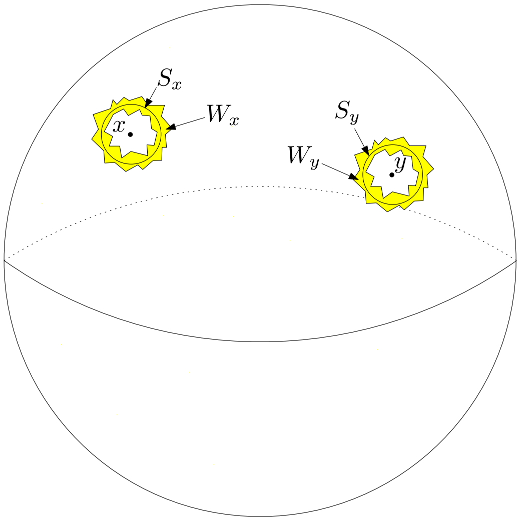


Figure 2.5: Any $W_x$ and $W_y$ are disjoint and far from each other.

and simpler than, the one we used in the proof of Theorem 2.4.5: let $S_x := L_x \cap \partial\overline{B_1(0)}$. Then clearly for small enough $\varepsilon > 0$ there exists $\varepsilon' > 0$ such that $S_x + \varepsilon'$ contains the strip $\partial\overline{B_1(0)} \cap (L_x + 4\varepsilon^{\frac{1}{2}})$ and is homotopy equivalent to $S_x$. Then the composition of the following maps induces an isomorphism in homology:

$$\partial P \cap L_x \hookrightarrow \Delta_{W_x} \hookrightarrow S_x + \varepsilon',$$

where both ends are nontrivial singular $(d-k-1)$-cycles. Thus $\tilde{\beta}_{d-k-1}(\Delta_{W_x}) \geqslant 1$.

It then follows from Theorem 2.2.2 that

$$g_k(P) \geqslant \beta_{d-k-1}(\Delta_W) = \sum_{x \in A} \beta_{d-k-1}(\Delta_{W_x}) \geqslant |A| = \Omega\left(\varepsilon^{\frac{1-d}{2}}\right). \tag{2.6}$$

□

In fact, instead of using Theorem 2.2.2, it suffices to use Lemma 2.4.6, as taking a $(k-1)$-sphere in each of the $S_x$'s gives $|A|$ pairwise disjoint $(k-1)$-spheres and thus they correspond to linearly independent $k$-stresses.

**Corollary 2.5.3.** *Let $E$ be an ellipsoid and let $k \leqslant \frac{d}{2}$. If $\delta^H(P,E)$ is small enough then*

$$g_k(P) = \Omega\left(\delta^H(P,E)^{\frac{1-d}{2}}\right). \tag{2.7}$$

*Proof.* There is an affine transformation that maps $E$ to $\overline{B_1(0)}$. Affine transformations map any polytope to a combinatorially equivalent polytope and the distances are preserved up to a constant, so the result follows from Theorem 2.5.2. □

**Theorem 2.5.4.** *Let $K$ be a $C^2$-convex body and let $k \leqslant \frac{d}{2}$. If $\delta^H(P,K)$ is small enough then*

$$g_k(P) = \Omega\left(\delta^H(P,K)^{\frac{1-d}{2}}\right). \tag{2.8}$$

*Proof.* Let $x$ be a point in $\partial K$ of positive curvature and let $E$ be the tangent conic to $\partial K$ at $x$ given by the Hessian of $\partial K$ at $x$. Then there is a neighborhood of $x$ in $\partial K$ that lies between $(1+\varepsilon)E - \varepsilon x$ and $(1-\varepsilon)E + \varepsilon x$. This follows from the fact that $K$ and $E$ have the same tangent space at $x$ and the same Hessian, thus the error in approximation is of the third order (see Schneider [86, Chapter 2.5]. The projections to the tangent plane at $x$ gives a homeomorphism between neighborhoods of $x$ in $\partial K$ and $\partial E$ that allows to transfer cycles in $\partial E$ of Lemma 2.5.3 to cycles in $\partial K$. Approximating those cycles give the desired lower bound as in Theorem 2.5.2 since the number of cycles in ellipsoids can be estimated locally up to a constant. □

**Remark 2.5.5.** *Using arguments of Böröczky [20], one can refine Theorem 2.5.4 to show that, for some constant $C$ independent of $K$, $k$ and $d$, we have*

$$g_k(P) \geqslant C \cdot \left(\int_{\partial K} \kappa(x)^{\frac{1}{d+1}}\right)^{-\frac{d+1}{d-1}} \cdot \left(d \cdot \delta^H(P,K)\right)^{\frac{1-d}{2}}$$

*where $\kappa(x)$ is the determinant of the second fundamental form.*

**Remark 2.5.6** (Tightness)**.** *Notice that, by Böröczky's [20, Theorem B], in an optimal $\varepsilon$-approximation of the unit $d$-ball $\overline{B_1(0)}$ by a polytope $P$, the number $f_k$ of $k$-faces is bounded above by $C\varepsilon^{\frac{1-d}{2}}$, where $C$ is a constant depending only on $\overline{B_1(0)}$. Since $g_k \leqslant f_{k-1}$, the lower* *bound is tight up to a constant. Again, this result holds for other distance notions as well, and extends to approximations of convex bodies with $C^2_+$ boundary, i.e convex bodies whose Gaussian curvature is positive at every boundary point.*

**Remark 2.5.7** (Upper bounds: Conjecture 2.1.1(ii) holds for random polytopes.)**.** *Bárány [12, Theorem 6, Corollary 2] showed that if $P_n$ is a polytope obtained from sampling $n$ points uniformly at random in a $C^2$-convex body $K$ then $\mathbb{E}(\delta^H(P_n, K)) = \Theta(\big(\frac{\log n}{n}\big)^{\frac{2}{d+1}})$. (Bárány assumed positive Gaussian curvature, but Böröczky's results show this assumption is not needed.) Furthermore, for any $0 \leqslant k \leqslant d-1$ he showed that $\mathbb{E}(f_k(P_n)) = \Theta(n^{\frac{d-1}{d+1}})$.*

*Combining this with Theorem 2.5.4 we conclude that part (ii) of Kalai's conjecture holds for* random *simplicial polytopes. Indeed, for $\frac{1}{m+1} \leqslant \delta^H(P, K) < \frac{1}{m}$ small enough, $g_k(P)$ is of order $m^{\frac{d-1}{2}+o(1)}$ for all $1 \leqslant k \leqslant \frac{d}{2}$, thus $\partial^k(g_k(P)) = O(m^{\frac{(d-1)(k-1)}{2k}+o(1)}) = o(g_{k-1}(P))$, and (ii) follows from (i).*

## 2.6 From induced homology cycles to affine stresses

The purpose of this section is to prove Lemma 2.4.6. Let us first observe a simpler and at first insufficient lemma that gets us almost to the goal.

**Lemma 2.6.1.** *Let $\gamma$ denote a simplicial $(k-1)$-sphere contained in the boundary of a simplicial $d$-polytope $P$, where $k \leqslant \frac{d}{2}$. Assume that $\gamma$ is an induced subcomplex in $\Delta = \partial P$. Then the simplicial neighborhood $\Gamma$ of $\gamma$ in $\Delta$ supports a $k$-stress homologous to the fundamental class of $\gamma$ as a cycle in $\Gamma$.*

*Proof.* The simplicial neighborhoods of vertices $v$ in $\Delta$ are denoted by $\mathrm{st}_\Delta v$, and their interiors are denoted by $\mathrm{st}^\circ_\Delta v$.

The proof of Lemma 2.6.1 is now the same as for induced cycles in the graph of $P$ by Kalai [50] that we used in Lemma 2.3.1: Consider the family

$$(U_v := \mathrm{st}^\circ_\Delta v)_{v \text{ vertex of } \gamma}.$$

This is a good cover of the open set $\cup_{v \text{ in } \gamma} U_v$. The generalized Mayer–Vietoris principle given by the Čech complex of this cover gives a double complex whose spectral sequences compute homology groups of the nerve $\mathcal{N}$ of $(U_v)$. Instead of applying this to compute the usual homology groups, however, we can also apply this to Ishida's chain complex [48] for stress groups, which is worked out in detail by Tay–Whiteley [94, Sections 10 & 12].

With this, we get straightforwardly a natural surjection

$$\left\{ i\text{-stresses of } \bigcup_{v \text{ vertex of } \gamma} \mathrm{st}_\Delta v \right\} \longrightarrow\!\!\!\!\rightarrow \ \widetilde{H}_{i-1}(\mathcal{N}) \tag{2.9}$$

where $i$ is the smallest integer $\leqslant \frac{d}{2}$ so that $\widetilde{H}_{i-1}(\mathcal{N})$ is nontrivial. But $\mathcal{N}$ is just the simplicial sphere $\gamma$ itself, because $\gamma$ is induced in $\Delta$. The claim follows. □

We now are left with the delicate task of extending this to a proof of Lemma 2.4.6.

***Proof of Lemma 2.4.6.*** Consider $\Delta = \partial P$ and $\Delta'$ the result of a barycentric subdivision of all faces intersecting $\gamma$, see Figure 2.6. Then the corresponding subdivision $\gamma'$ of $\gamma$ in $\Delta'$ is an induced subcomplex of $\Delta'$, and of the induced subdivision of $\Gamma$, denoted $\Gamma'$. We consider also the open simplicial complex

$$\check{\Gamma} \ := \ \{\sigma \in \Gamma : \sigma \cap \gamma \neq \varnothing\}$$

and its induced subdivision in $\Gamma'$, denoted by $\check{\Gamma}'$.

The barycentric subdivision is algebraically realized by iterative blowups of the toric variety, or, combinatorially, by stellar subdivisions of $\Delta$, performed at faces intersecting $\gamma$ in order of decreasing dimension. Following McMullen [71], this preserves the validity of the hard Lefschetz theorem. Hence we can apply Lemma 2.6.1 to conclude that the fundamental class of $\gamma'$ is naturally homologous to a stress $\widetilde{\gamma}$ in its simplicial neighborhood $\widetilde{\Gamma} \subset \Gamma'$.

Now, we have to be careful since a stellar subdivision in the blowup sequence may introduce stresses on its own. To control this, consider a simplicial $d$-polytope $X$ and its stellar subdivision $X'$ at a face $\sigma$. Following McMullen again, the stresses of $X'$ are decomposed

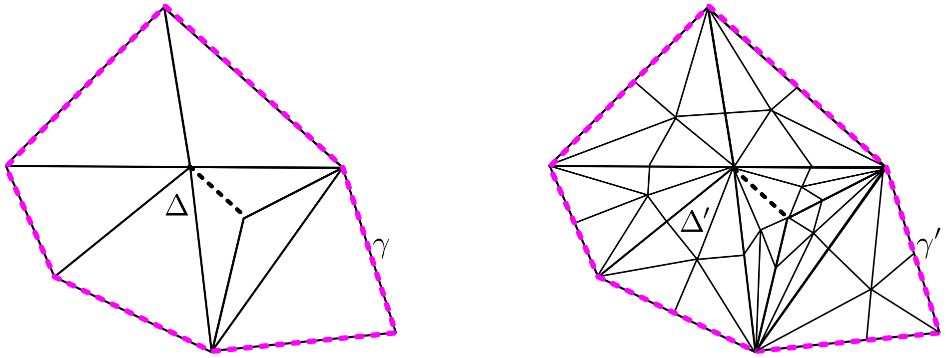


Figure 2.6: Barycentric subdivision of a simplicial complex $\Delta$ at faces intersecting a cycle $\gamma$. In the resulting complex $\Delta'$, the cycle $\gamma'$ subdividing $\gamma$ is an induced subcomplex. The dotted black edge is not subdivided in the process, as it is not incident to $\gamma$.

into pullbacks of stresses of $X$, and Gysin pullbacks of stresses in the face figure $X_\sigma$ of $\sigma$ in $X$, the latter of which are the "new" stresses to be controlled, see also [11, Thm.1.2(3)], [38, Section 6], and [2, Theorem 6.18] for a detailed presentation of the Gysin maps involved. It is straightforward to see that the latter stresses are naturally supported in the simplicial neighborhood of the link $\operatorname{link}_\Delta \sigma$ of $\sigma$.

We conclude that all newly created $k$-stresses in the transition from $\Delta$ to $\Delta'$, where $k \geqslant 2$, seen as $(k-1)$-cycles, are supported in $\Gamma'$. Moreover, they are zero-homologous as simplicial cycles in $\breve{\Gamma}'$.

But the stress $\widetilde{\gamma}$ generates a nontrivial homology class in $\breve{\Gamma}'$, so it is linearly independent of the newly created stresses. Hence, we may blow down again, which maps $\widetilde{\gamma}$ to a nontrivial stress supported in the simplicial neighborhood of $\gamma$, as desired. $\square$

## Chapter 3

# LEXICOGRAPHIC SHELLABILITY, MATROIDS AND PURE ORDER IDEALS

### *3.1 Introduction*

This chapter studies the $h$-vector theory of matroid independence complexes. These complexes were originally motivated by trying to abstract the notion of linear independence from linear algebra. They have appeared in areas as diverse as graph theory, algebraic geometry, commutative algebra, and optimization.

The $h$-vector also appears naturally in these contexts. Understanding $h$-vectors has driven a lot of research in the past few decades. One particular question is to understand which positive integer sequences appear as the $h$-vectors of matroids. Some conditions are known:

i the $h$-vector of any matroid is an $O$-sequence [89](this is because the Stanley-Reisner ring of a matroid is Cohen Macaulay),

ii the $h$-vector of a matroid satisfies the Brown-Colbourn inequalities [21], and

iii $h_i \leqslant h_{d-i}$ for $i \leqslant \frac{d}{2}$ [25].

However, there are many of integer vectors that satisfy these conditions but are not $h$-vectors of matroids. The problem of completely characterizing $h$-vectors of matroids is exceedingly hard, however Stanley [89] proposed a conjecture that narrows down the search. Recall that an integer sequence $\mathbf{a} := (a_0, a_1, \ldots, a_r)$ is a a pure $O$-sequence if there exists a pure multicomplex $\mathcal{O}$ of degree $r$ that has exactly $a_i$ monomials of degree $i$ for $0 \leqslant i \leqslant r$.

**Conjecture 3.1.1. (Stanley, 1977)** *The $h$-vector of an arbitrary matroid is a pure $O$-sequence.*

The conjecture remains open despite a tremendous amount of effort that has been put forth in trying to find a proof. Merino obtained the first positive result: he proved the conjecture for cographic matroids using the critical group of the associated graph [73]. Schweig [87] verified the conjecture for lattice path matroids. Oh [78] generalized this result to the case of cotransversal matroids by studying integer points of generalized permutohedra associated to bipartite graphs. Merino et al. [74] proved the conjecture for paving matroids by noting that all but the last entry of the $h$-vector is determined by the dimension and the number of vertices of the matroid. Hà, Stokes, and Zanello [43] established the conjecture for matroids of rank three by studying properties of the level Artinian algebras. De Loera, Kemper, and Klee [30] proved the conjecture combinatorially for matroids of rank 3 and corank 2 by studying the lattice of flats. They also computationally verified the conjecture for all matroids with at most 9 elements using a database that contains all such matroids. Constantinescu, Kahle and Varbaro [27] proved the conjecture for proper skeleta of matroids and rank $d$ matroids with $h_d \leqslant 5$ using again commutative algebra and level Artinian algebras. Their results show that a brute force approach to computationally disproving the conjecture is unfeasible with the current computational power. In other words, given a matroid $h$-vector that is a candidate as a counterexample to Stanley's conjecture, it is impossible to test all pure multicomplexes with the correct number of variables and maximal monomials against that $h$-vector in a reasonable amount of time.

Pure $O$-sequences have also been widely studied on their own. A lot of research on pure $O$-sequences has been driven by attempts to prove Stanley's conjecture. Notably, Hibi [44] gave a set of inequalities satisfied by pure $O$-sequences and proposed a weaker version of Stanley's conjecture. The weaker conjecture was later resolved by Chari [25] for a larger class of PS-ear decomposable simplicial complexes. Much more can be said about pure $O$-sequences. A good reference is [18].

However, there seems to be a lot of skepticism about the validity of Stanley's conjecture. The classes of matroids for which the conjecture is known to hold are either too restricted or too special. Cographic and cotransversal matroids account for a very small fraction of all matroids. Paving matroids have too much structure and rank 3/corank 2 matroids do not seem to capture the full set of features and pathologies encountered in matroid theory. For instance, the simplification of a rank 3 matroid is paving, so the behavior of rank 3 matroids is similar to that of paving matroids.

In this paper we formulate an alternative version of Conjecture 5.1.2 and prove that our conjecture holds for matroids of rank three and four. As matroids of rank four exhibit a less predictable behavior than those of rank three, we believe that our result provides non-trivial evidence for the validity of Stanley's conjecture as well as a new approach to proving the conjecture for all matroids. We use techniques from lexicographic shellability to get a simple decomposition of the $h$-vector that naturally gives rise to an inductive procedure to construct a pure multicomplex. In particular, instead of using the usual Tutte polynomial approach, we decompose the $h$-vector according to the independent sets disjoint from a fixed basis. We then propose a new approach in Conjecture 3.3.10 and show that it is sufficient to prove this new conjecture for a finite number of matroids of each rank.

Afterwards we study matroids of rank three and four. We describe algorithms that construct a multicomplex recursively and show that the output satisfies the conditions of our conjecture, provided both algorithms produce pure $O$-sequences for the finitely many cases that have to be considered in our conjecture. All the matroids necessary for the computations have at most 8 elements and have been classified up to isomorphism in [67], [66] and [68]. We then implemented the algorithms to computationally verify our conjecture for matroids of rank three and four in Sage [95]. We provide several examples of the output of the algorithms.

## 3.2 Preliminaries

A *matroid* $\Delta$ is a pair $(E, \mathcal{I})$ where $E$ is a finite set and $\mathcal{I} \subseteq 2^E$ satisfies the following axioms.

1. $\varnothing \in \mathcal{I}$

2. If $I \in \mathcal{I}$ and $I' \subseteq I$ then $I' \in \mathcal{I}$. (Alternatively, $\mathcal{I}$ is a simplicial complex.)

3. **(Extension axiom)** If $I, I' \in \mathcal{I}$ and $|I| < |I'|$ then there is $v \in I'$ such that $I \cup \{v\} \in \mathcal{I}$.

The set $E$ is called the *vertex* set of $\Delta$ and the elements of $\mathcal{I}$ are called the *independent sets* of $\Delta$. For $A \subseteq E$ define the *rank* of $A$, denoted by $\operatorname{rk}(A)$, to be the size of the maximum integer $k$ such that there is $I \subseteq A$ with $I \in \mathcal{I}$ and $|I| = k$. Abusing notation we write $\operatorname{rk}(\Delta) := \operatorname{rk}(E)$. Two matroids $\Delta = (E, \mathcal{I})$, $\Delta' = (E', \mathcal{I}')$ are *isomorphic* if there is a bijective map $f : E \to E'$ that induces a bijection from $\mathcal{I}$ to $\mathcal{I}'$.

A *basis* of a matroid is an independent set that is maximal under inclusion. We denote by $\mathcal{B}$ the set of bases of a matroid. It follows from the extension axiom that all the bases of a matroid have the same cardinality. A subset $\mathcal{B}$ of $2^E$ is the collection of bases of a matroid if and only if the following conditions hold (see Chapter 1 from [80]):

1. $\mathcal{B} \neq \varnothing$.

2. **(Exchange axiom)** For $B, B' \in \mathcal{B}$ and $x \in B - B'$ there is $y$ in $B' - B$ such that $(B - \{x\}) \cup \{y\} \in \mathcal{B}$.

Notice that the independent sets of the matroid can be recovered easily from the set of bases: a subset $A$ of $2^E$ is independent if and only if it is contained in some $B \in \mathcal{B}$. A *loop* of $\Delta$ is an element $e$ of $E$ such that $\{e\} \notin \mathcal{I}$. A *coloop* of $\Delta$ is an element $e \in E$ that is contained in every basis. We say that a matroid is a *cone* if it has a coloop.

It is sometimes useful to consider restrictions of matroids. For $A \subseteq E$ define $\Delta|_A := (A, \mathcal{I}|_A)$, where $\mathcal{I}|_A$ is the set of independent sets of $\mathcal{I}$ contained in $A$. It is easy to see that $\Delta|_A$ is a matroid. Denote the set of bases of $\Delta|_A$ by $\mathcal{B}|_A$. For a detailed introduction to the theory of matroids see [80].

A *shelling* of a matroid $\Delta$ is an ordering of the bases of $\Delta$, $B_1, \ldots, B_k$ in such a way that for each $1 \leqslant j < i \leqslant k$ there exists $1 \leqslant \ell < i$ and $x \in B_\ell$ such that $B_i \cap B_j \subseteq B_i \cap B_\ell = B_\ell - x$. Björner [17] showed that every matroid admits a particularly nice shelling. Let $\Delta$ be a matroid and order the vertices of $\Delta$ arbitrarily. Then the lexicographic ordering of the bases of $\Delta$ with respect to this vertex order is a shelling order for $\Delta$. This shelling in fact characterizes matroids inside the wider class of shellable simplicial complexes. Any shelling of this form is called a *lexicographic shelling*.

It is a well-know fact that if $B_1, B_2, \ldots, B_k$ is a shelling, then for every $B_i$ there is a subset $\mathcal{R}(B_i)$ of $B_i$ that is minimal with respect to not being contained in any $B_j$ with $j < i$; that is, $\mathcal{R}(B_i)$ is a set such that $A \subseteq B_i$ is not contained in $B_j$ for any $j < i$ if and only if $\mathcal{R}(B_i) \subseteq A$. These sets are called the *restriction sets* of the shelling and have a very rich combinatorial structure. When there is a possibility for ambiguity, we will write $\mathcal{R}(B, \Delta)$ to indicate that we are considering the restriction set of $B$ as a basis in the matroid $\Delta$.

As evidence of this combinatorial structure, define the *h-polynomial* of a matroid by

$$h(\Delta, x) := \sum_{i=1}^{k} x^{|\mathcal{R}(B_i)|}.$$

This polynomial has degree at most $d := \operatorname{rk}(\Delta)$. Let $h_i$ be the coefficient of $x^i$ in $h(\Delta, x)$. The vector $h(\Delta) := (h_0, h_1, \ldots, h_d)$ is called the *h-vector* of $\Delta$ and is a very important invariant of the matroid. Consider the polynomial

$$f(\Delta, x) = \sum_{j=0}^{d} f_j x^j := (1 + x) h\left(\Delta, \frac{x}{1+x}\right).$$

The coefficient $f_j$ of $x^j$ is equal to the number of independent sets of rank $j$ in $\Delta$ (see [17]). In particular, this implies that the $h$-polynomial is independent of the shelling order. One major problem in the theory of matroids is to understand the possible values that the $h$-vector can take. For more details about shellability and matroids see [17].

An *order ideal* or *multicomplex* $\mathcal{O}$ is a finite non empty collection of monomials in a finite set of variables such that if $m \in \mathcal{O}$ and $m' | m$ then $m' \in \mathcal{O}$. A monomial order $\mathcal{O}$ is called *pure* if all its maximal monomials have the same degree. For an order ideal $\mathcal{O}$ let $F_i = F_i(\mathcal{O})$ denote the number of monomials of degree $i$. Assume that $d$ is the maximum degree of a monomial in $\mathcal{O}$. The vector $F(\mathcal{O}) := (F_0, F_1, \ldots, F_d)$ is called the *F-vector* of $\mathcal{O}$. An *O-sequence* is an integer sequence $\mathbf{a} = (a_0, a_1, \ldots, a_d)$ such that there exists an order ideal $\mathcal{O}$ with $F(\mathcal{O}) = \mathbf{a}$. An $O$-sequence is *pure* if it comes from a pure order ideal.

Stanley [89] showed that the $h$-vector of any matroid (and more generally any Cohen-Macaulay simplicial complex) is an $O$-sequence and posited Conjecture 5.1.2. For a detailed explanation of the relationship between matroids, simplicial complexes, and commutative algebra see [91].

## 3.3 Restriction sets of lexicographic shellings

For a positive integer $n$ let $[n]$ denote the set of integers $\{1, 2, \dots n\}$. Let $\Delta = ([n], \mathcal{I})$ be a matroid that has $[d]$ as a basis. There is no loss of generality in assuming that $[d]$ is a basis in $\Delta$ since we can reorder the ground set of $\Delta$ without changing its combinatorial structure. The lexicographic order on the bases of $\Delta$ gives a shelling. From now on we assume that $\Delta$ is endowed with this shelling. It is clear that $[d]$ is the first basis of this shelling. For bases $B, B'$ of $\Delta$ we write $B < B'$ if $B$ is smaller than $B'$ in the lexicographic order (lex order, for short) induced by the natural order of $[n]$. Our goal now is to understand the set $\{\mathcal{R}(B) \mid B \in \mathcal{B}(\Delta)\}$.

**Lemma 3.3.1.** *Let $I$ be an independent set of $\Delta$ such that $I \cap [d] = \varnothing$. Then there exists a basis $B$ of $\Delta$ with the property that $\mathcal{R}(B) = I$.*

*Proof.* Let $B$ be the lexicographically smallest basis of $\Delta$ that contains $I$. We claim that $\mathcal{R}(B) = I$. First note that $\mathcal{R}(B) \subseteq I$ because $B$ is the first basis of the shelling order that contains $I$. On the other hand, if $v$ is an element of $I - R(B)$, then we can apply the extension axiom to $B - \{v\}$ and $[d]$ to obtain another basis $B'$. Then $\mathcal{R}(B) \subseteq B'$ and $B'$ is smaller than $B$ in the lex order, which is a contradiction. $\square$

**Lemma 3.3.2.** *Let $B$ be a basis of $\Delta$ and let $I = B - [d]$. Then $I \subseteq \mathcal{R}(B)$.*

*Proof.* Assume to the contrary that there is an element $v \in I - \mathcal{R}(B)$. Then use the extension axiom with $B - \{v\}$ and $[d]$ to find a basis $B' < B$ that contains $\mathcal{R}(B)$. This is a contradiction.

$\square$

Now we introduce two matroids that can be associated to a given independent set $I \in \Delta$ with $I \cap [d] = \varnothing$.

**Definition 3.3.3.** *Let $I$ be an independent set of $\Delta$ such that $I \cap [d] = \varnothing$. Let $\Gamma_I$ be the matroid whose independent sets are subsets $G$ of $[d]$ such that $G \cup I$ is an independent of $\Delta$. Let $\mathcal{B}_I$ be the set of bases of $\Gamma_I$. To simplify notation we write $\mathcal{B}_x := \mathcal{B}_{\{x\}}$. Furthermore, let $\Delta_I := \Delta|_{[d]\cup I}$.*

We note that $\Gamma_I$ and $\Delta_I$ are indeed matroids since $\Gamma_I = \text{link}_\Delta(I)|_{[d]}$, and links and restrictions of matroid independence complexes are also matroid independence complexes.

**Theorem 3.3.4.** *Let $I$ be an independent set of $\Delta$ and let*

$$U_I := \{\mathcal{R}(B,\Delta) - I \,:\, B \in \mathcal{B}, B - [d] = I\}.$$

*Let $\mathcal{B}_I = \{G_1, \dots G_\ell\}$ be the set of bases of $\Gamma_I$ ordered lexicographically with respect to the natural order on $[d]$, and let*

$$V_I = \{\mathcal{R}(G_i, \Gamma_I) \,|\, 1 \leqslant i \leqslant \ell\}.$$

*Then $U_I = V_I$.*

*Proof.* Note that $U_I$ and $V_I$ are finite sets of the same cardinality. Therefore, it is sufficient to prove that $V_I \subseteq U_I$. As in the statement of the theorem, we let $\{G_1, \dots, G_\ell\}$ denote the set of facets of $\Gamma_I$, ordered lexicographically.

Pick any index $1 \leqslant j \leqslant \ell$. Note that $G_j \cup I$ is a basis of $\Delta$. We need to show that $\mathcal{R}(G_j, \Gamma_I) \in U_I$. We claim that $\mathcal{R}(G_j \cup I, \Delta) = \mathcal{R}(G_j, \Gamma_I) \cup I$. This proves that $\mathcal{R}(G_j, \Gamma_I) \in U_I$ since it will show that $\mathcal{R}(G_j, \Gamma_I) = \mathcal{R}(G_j \cup I, \Delta) - I$.

It only remains to prove that $\mathcal{R}(G_j \cup I, \Delta) = \mathcal{R}(G_j, \Gamma_I) \cup I$. First, we will show $\mathcal{R}(G_j \cup I, \Delta) \subseteq \mathcal{R}(G_j, \Gamma_I) \cup I$. To do this, we will show that $G_j \cup I$ is the lexicographically smallest facet of $\Delta$ that contains $\mathcal{R}(G_j, \Gamma_I) \cup I$. Suppose to the contrary that $\mathcal{R}(G_j, \Gamma_I) \cup I \subseteq F$ for some basis $F$ in $\Delta$ that precedes $G_j \cup I$ lexicographically. Since $I \subseteq F$, we know $F - [d] \supseteq I$. It is sufficient to consider the case that $F - [d] = I$. Indeed, if $F - [d]$ properly contains $I$, then we can use the basis exchange axiom to replace all elements of $F - ([d] \cup I)$ with elements of $[d]$. This will give a new basis $F'$ that precedes $F$ (and hence also $G_j \cup I$) lexicographically with the property that $F' - [d] = I$ and $\mathcal{R}(G_j, \Gamma_I) \cup I \subseteq F'$.

Given a basis $F$ that precedes $G_j \cup I$ lexicographically and such that $F - [d] = I$, consider $G_s = F - I$. Then $G_s$ precedes $G_j$ lexicographically. Since $\mathcal{R}(G_j, \Gamma_I) \cup I \subseteq F$, we also know that $\mathcal{R}(G_j, \Gamma_I) \subseteq F - I = G_s$. But this means $\mathcal{R}(G_j, \Gamma_I)$ is contained in a facet of $\Gamma_I$ that precedes $G_j$ lexicographically. This contradicts the definition of the restriction set of a facet in a shelling order and therefore $\mathcal{R}(G_j \cup I, \Delta) \subseteq \mathcal{R}(G_j, \Gamma_I) \cup I$.

Finally, we will show that $\mathcal{R}(G_j \cup I, \Delta) \supseteq \mathcal{R}(G_j, \Gamma_I) \cup I$. We already know that $I \subseteq \mathcal{R}(G_j \cup I, \Delta)$, thus we only need to consider elements of $g \in \mathcal{R}(G_j, \Gamma_I)$. If $g \in \mathcal{R}(G_j, \Gamma_I)$, then there exists $i < j$ such that $G_j - \{g\} \subseteq G_i$. But this implies that $(G_j - \{g\}) \cup I \subseteq G_i \cup I$, and since $G_i < G_j$ we also have that $G_i \cup I < G_j \cup I$. Thus $g \in \mathcal{R}(G_j \cup I, \Delta)$, as desired.

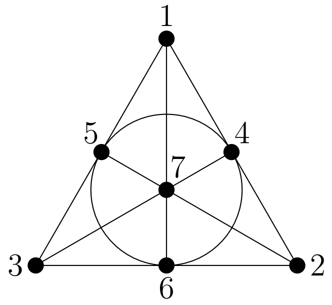
This concludes the proof of our claim that $\mathcal{R}(G_j \cup I, \Delta) = \mathcal{R}(G_j, \Gamma_I) \cup I$ and hence the proof of the theorem. □

**Corollary 3.3.5.** *Keeping the same notation as in Theorem 3.3.4 we have:*

$$h(\Delta, x) = \sum_{I \in \Delta|_{[n]-[d]}} x^{|I|} h(\Gamma_I, x). \tag{3.1}$$

We illustrate Theorem 3.3.4 in the following example.

**Example 3.3.6.** *Consider the Fano matroid with its ground set labeled as in the following illustration.*



*The independent sets of the Fano matroid correspond to sets of at most three points that do not lie on a line or the circle.*

*Let* $I = \{6\}$*. In this case,* $\Gamma_I$ *has facets* $\{\{1,2\},\{1,3\}\}$*. Under the lexicographic shelling on* $\Gamma_I$*, the corresponding restriction sets are* $V_I = \{\varnothing, \{3\}\}$*. Similarly, the facets of* $\Delta_I$ *that contain* 6 *are* $\{\{1,2,6\},\{1,3,6\}\}$*. To form the set* $U_I$*, we must compute the restriction set for each of these facets relative to the entire shelling order on* $\Delta$*. The facets of the Fano matroid that precede* $\{1,3,6\}$ *lexicographically are* $\{\{1,2,3\},\{1,2,5\},\{1,2,6\},\{1,3,4\},\{1,3,6\}\}$*. Since* $\{1,2,6\}$ *is the lexicographically smallest facet containing* $I = \{6\}$*, we have* $\mathcal{R}(\{1,2,6\}) = \{6\}$*. Since* $\{1,3\} \subseteq \{1,3,5\}$*,* $\{1,6\} \subseteq \{1,2,6\}$*, and* $\{1,3,6\}$ *is the lexicographically smallest face containing* $\{3,6\}$ *we have* $\mathcal{R}(\{1,3,6\}) = \{3,6\}$*. Since* $U_I$ *is obtained by removing* $I = \{6\}$ *from each of these restriction sets, it follows also that* $U_I = \{\varnothing, \{3\}\}$*.*

The following lemma is proved in [91, Theorem III.3.4], but we include a proof for the sake of completeness.

**Lemma 3.3.7.** *Let* $\Delta$ *be a rank* $d$ *matroid. Then* $h_d(\Delta) = 0$ *if and only if* $\Delta$ *is a cone.*

*Proof.* First suppose that $h_d(\Delta) = 0$. Let $B_1, B_2, \ldots, B_k$ be a shelling order and consider $\mathcal{R}(B_k)$. Since $h_d = 0$, there is $j \in B_k - \mathcal{R}(B_k)$. It follows that $\mathcal{R}(B_k) \subseteq (B_k - j)$. Thus the only basis of $\Delta$ that contains $B_k - \{j\}$ is $B_k$.

We claim that every basis of $\Delta$ contains $j$, and hence $j$ is a cone point. Indeed, suppose $B$ is a basis of $\Delta$ that does not contain $j$. Then the basis exchange axiom tells us there is an element $x \in B - B_k$ such that $B' := (B_k - \{j\}) \cup \{x\}$ is a basis of $\Delta$. However $B_k - \{j\} \subseteq B'$, meaning $B' = B_k$. This means $x \in B_k$ as well, which is a contradiction. Therefore, there is no such basis $B$, and hence $j$ belongs to every basis of $\Delta$.

Conversely, suppose $\Delta$ is a cone and let $u$ belong to every basis. Then $u$ is not contained in the restriction set of any basis, so no restriction set has size $d$. Therefore $h_d = 0$. ☐

**Lemma 3.3.8.** *Let* $\Delta$ *be a rank* $d$ *matroid with* $h_d(\Delta) \neq 0$*. If* $I$ *is an independent set disjoint from* $[d]$ *such that* $h_{d-|I|}(\Gamma_I) = 0$*, then there exists* $z \in [n] - ([d] \cup I)$ *such that* $I \cup \{z\} \in \Delta$*.*

*Proof.* We know that $\Delta$ is not a cone because $h_d(\Delta) \neq 0$. On the other hand, since $h_{d-|I|}(\Gamma_I) = 0$ we see that $|I| < d$ and $\Gamma_I$ is a cone. Hence there is $u \in [d]$ that belongs to every basis of $\Gamma_I$.

Since $\Delta$ is not a cone and $[d]$ is a basis of $\Delta$, there is $z \in [n] - [d]$ such that $B := ([d] - \{u\}) \cup \{z\}$ is a basis of $\Delta$. Now we can use the independent set extension axiom to extend $I$ to a basis of $\Delta$ by using elements from $B$. Call this new basis $B'$. It follows that $z \in B'$: if $z \notin B'$, then $B'$ would be constructed by adding only elements of $[d] - \{u\}$ to $I$. But there is no basis of $\Gamma_I$ that does not contain $u$. Therefore, $z \in B'$ and hence $I \cup \{z\} \subseteq B'$ is independent in $\Delta$. $\square$

We are now in a position to state our new approach to Stanley's conjecture.

**Definition 3.3.9.** *A* based matroid *is a pair* $(\Delta, B, <)$ *where* $\Delta$ *is a matroid,* $B$ *is a basis of* $\Delta$ *and* $<$ *is a total order of* $E(\Delta) - B$*. For an independent set* $I$*, such that* $I \cap B = \varnothing$*, let* $\Gamma_I$ *be the matroid whose elements are subsets* $U$ *of* $B$ *with* $U \cup I \in \Delta$*. Two based matroids* $(\Delta, B, <)$*,* $(\Delta', B', <')$ *are* isomorphic *if there is a matroid isomorphism* $f : \Delta \to \Delta'$ *such that* $f(B) = B'$ *and* $f$ *is order-preserving on* $E(\Delta) - B$*.*

**Conjecture 3.3.10.** *Let* $d > 1$ *be a fixed integer and let* $\mathcal{A}^d$ *be the family of based matroids of rank* $d$*. There exists a map* $\mathcal{F}$ *from* $\mathcal{A}^d$ *to the family of pure order ideals such that the following conditions hold for every based matroid* $(\Delta, B, <)$*.*

1. *The variables of* $\mathcal{F}(\Delta, B, <)$ *are* $\{x_i \,|\, i \in E(\Delta) - B\}$*.*

2. *Every monomial in* $\mathcal{F}(\Delta, B, <)$ *is supported on a set of the form* $\{x_i \,|\, i \in I\}$ *for some independent set* $I$ *of* $\Delta$ *with* $I \cap B = \varnothing$*.*

3. *For each independent set* $I$ *that is disjoint from* $B$*, there are exactly* $h_j(\Gamma_I)$ *monomials in* $\mathcal{F}(\Delta, B, <)$ *with degree* $|I| + j$ *and support* $\{x_i \,|\, i \in I\}$*.*

4. *For each independent set* $I$ *that is disjoint from* $B$*, the restriction of* $\mathcal{F}(\Delta, B, <)$ *to the variables* $\{x_i \,|\, i \in I\}$ *is* $\mathcal{F}(\Delta|_{B \cup I}, B, <)$*.*

5. *If $(\Delta', B', <')$ is a based matroid and $f : (\Delta, B, <) \to (\Delta, B, <')$ is an isomorphism, then $\mathcal{F}(\Delta, B, <)$ is naturally isomorphic to $\mathcal{F}(\Delta', B', <')$ by relabeling the index of each variable in $\mathcal{F}(\Delta, B, <)$ with its image under $f$.*

Notice that Conjecture 3.3.10 together with Corollary 3.3.5 implies Stanley's conjecture (Conjecture 5.1.2). To see this, note that for an arbitrary matroid $\Delta$ we can pick a basis $B$ and any order $<$ on $E(\Delta) - B$ and apply the conjecture to the based matroid $(\Delta, B, <)$.

Next, we make three remarks in defense of this conjecture as a reasonable approach to proving Stanley's conjecture.

First, for any independent set $I$ with $I \cap [d] = \varnothing$, notice that $\mathrm{rk}(\Gamma_I) = d - |I|$ and $|E(\Gamma_I)| \leqslant d$. By the Upper Bound Theorem [88], $h_j(\Gamma_I)$ is bounded by the number of monomials of degree $j$ in $|E(\Gamma_I)| - \mathrm{rk}(\Gamma_I) \leqslant |I|$ variables. This shows that condition 3 in the above conjecture cannot fail on account of $h_j(\Gamma_I)$ exceeding the number of monomials of degree $|I| + j$ supported on $x^I$.

Second, even if $\Delta$ is not a cone, one should expect that $\Gamma_I$ will be a cone for many of the independent sets $I \in \Delta$. Therefore, if the conjecture is true, one should not expect that each of the monomials supported on $x^I$ will all divide into a monomial of degree $d$ that is also supported on $x^I$. However, Lemma 3.3.8 indicates that in this case, we can expect each of the maximal monomials supported on $x^I$ to divide into a monomial of higher degree that is supported on $x^{I'}$ for some $I' \supset I$.

Finally, the order condition may seem strange at first. However, in the case of rank 3 and 4 matroids, it is used largely as a "tie-breaker" in the algorithms we define to construct pure $O$-sequences. Specifically, we need rules that allow us to distinguish independent sets $I, I'$ with $\Gamma_I = \Gamma_{I'}$. Based on evidence of the cases $d = 3, 4$, we believe the order may only be needed to distinguish independent sets $\{x\}$, $\{y\}$ with $h(\Gamma_{\{x\}}) = h(\Gamma_{\{y\}})$. Every possible ordering of the ground set of $\Delta$ has to be considered in order for condition 4 to hold, as in such restrictions we can get isomorphic matroids with same initial bases but a different underlying order.

We now prove a theorem that will be crucial for treating the cases of rank three and rank four matroids.

**Theorem 3.3.11.** *Let* $\mathcal{U}_d$ *be the subset of* $\mathcal{A}^d$ *consisting of matroids with at most* $2d$ *vertices. Assume that there is a map* $\mathcal{G}$ *from* $\mathcal{U}_d$ *to the family of pure order ideals such that* $\mathcal{G}(\Delta, B, <)$ *satisfies the conditions on* $\mathcal{F}(\Delta, B, <)$ *of Conjecture 3.3.10 for all* $\Delta \in \mathcal{U}_d$*. Then there exists a function* $\mathcal{F}$ *that satisfies the conditions of Conjecture 3.3.10 and such that* $\mathcal{F}|_{\mathcal{U}_d} = \mathcal{G}$*.*

*Proof.* Let $(\Delta, B, <)$ be a matroid of $\mathcal{A}^d$. For each independent set $I$ with $B \cap I = \varnothing$, the based matroid $(\Delta|_{B\cup I}, B, <)$ is in $\mathcal{U}_d$, so $\mathcal{G}(\Delta|_{B\cup I}, B, <)$ is well-defined. Consider the set of monomials:

$$\mathcal{F}(\Delta, B, <) := \bigcup_{\substack{I\in\Delta,\\ I\cap B=\varnothing}} \mathcal{G}(\Delta|_{B\cup I}, B, <).$$

We claim that $\mathcal{F}$ is the desired map. By definition of $\mathcal{F}$, the variables of $\mathcal{F}(\Delta, B, <)$ are $\{x_i \,|\, i \in E(\Delta) - B\}$ so condition 1 of Conjecture 3.3.10 is satisfied. As every monomial comes from some $\mathcal{G}(\Delta|_{B\cup I}, B, <)$, its support is a subset of $\{x_i \,|\, i \in I\}$ and hence condition 2 of Conjecture 3.3.10 holds. Notice that if $I' \subset I$ then $\mathcal{G}(\Delta|_{B\cup I'}, B, <) \subseteq \mathcal{G}(\Delta|_{B\cup I}, B, <)$ since the restriction of $\mathcal{G}(\Delta|_{B\cup I}, B, <)$ to the variables $\{x_i \,|\, i \in I'\}$ is $\mathcal{G}(\Delta|_{B\cup I'}, B, <)$ by assumption. From this we obtain that the monomials in $\mathcal{F}(\Delta, B, <)$ whose support is $\{x_i \,|\, i \in I\}$ form the set of such monomials in $\mathcal{G}(\Delta|_{I\cup B}, B, <)$. Since $\Gamma_I(\Delta) = \Gamma_I(\Delta|_{I\cup B})$, we conclude that there are $h_j(\Gamma_I)$ monomials of degree $|I| + d$ in $\mathcal{F}(\Delta, B, <)$ whose support is $\{x_i \,|\, i \in I\}$, and so condition 3 of Conjecture 3.3.10 is satisfied. Conditions 4 and 5 of Conjecture 3.3.10 are immediate from the definition. Therefore $\mathcal{F}(\Delta)$ satisfies conditions 1 to 5 of Conjecture 3.3.10.

We now show that $\mathcal{F}(\Delta, B, <)$ is a pure order ideal. Let $m$ be a monomial in $\mathcal{F}(\Delta, B, <)$. There is an independent set $I$ such that $m \in \mathcal{G}(\Delta|_{B\cup I}, B, <)$. Since $\mathcal{G}(\Delta|_{B\cup I}, B, <)$ is a multicomplex, all the divisors of $m$ are in $\mathcal{F}(\Delta, B, <)$, and so $\mathcal{F}(\Delta, B, <)$ is a multicomplex. Let $k = \max\{i \,|\, h_i(\Delta) > 0\}$. We claim that there is $I'$ such that $I \subset I'$ and $h_k(\Delta|_{B\cup I'}) > 0$. Removing coloops (all are contained in B), we can assume that $h_d(\Delta) > 0$.

Then by Lemma 3.3.8 there is $I'$ that contains $I$ such that $h_{d-|I'|}(\Gamma_{I'}) > 0$. Corollary 3.3.5 implies that $h_d(\Delta|_{B\cup I'}) \geqslant h_{d-|I'|}(\Gamma_{I'}) > 0$ as desired. Now $m$ is an element of a pure multicomplex $\mathcal{G}(\Delta|_{B\cup I'}, B, <) \supset \mathcal{G}(\Delta_{B\cup I}, B, <)$ of degree $k$, so it divides a monomial of degree $k$ supported in a subset of $I'$. It follows that $\mathcal{F}(\Delta, B, <)$ is pure, as claimed. $\square$

Notice that $\mathcal{U}_d$ contains only finitely many isomorphism classes of based matroids. Therefore, to verify our conjecture for matroids of a fixed rank we only have to construct the desired order ideal for finitely many matroids. In the following sections we construct algorithms that receive as their input matroids of rank 3 and 4 respectively, and output the desired order ideal. This can be used to explicitly construct $\mathcal{G}$ for $d = 3, 4$, which in turn implies Conjecture 3.3.10.

### *3.4 Stanley's conjecture for rank 3 matroids*

We begin by verifying Conjecture 3.3.10 for rank three matroids. For this, we develop an algorithm that constructs a pure order ideal for every matroid whose vertex set is ordered. We verify with a computer's aid that the algorithm produces a suitable $\mathcal{F}(\Delta, B, <)$ for all based matroids of rank 3 with at most 6 vertices and conclude from Theorem 3.3.11 that the conjecture holds for $d = 3$.

First we will give an overview to show how our algorithm will work. Let $\Delta$ be a matroid of rank $d$ on ground set $[n]$ such that $[d]$ is a basis of $\Delta$. Our goal is to construct a pure multicomplex $\mathcal{O}$ with the property that for each independent set $I \subseteq [n] - [d]$, $\mathcal{O}$ contains $h_j(\Gamma_I)$ monomials of degree $|I| + j$ supported on $x^I$. When $|I| = 1$, it is easy to verify that $h_j(\Gamma_I)$ is equal to either zero or one for all $j$ and that $|\mathcal{B}_I|$ counts the number of indices $j$ for which $h_j(\Gamma_I) = 1$. Thus if $I = \{x\}$, we add $\{x^t \mid 1 \leqslant t \leqslant |\mathcal{B}_x|\}$ to $\mathcal{O}$.

Next, we proceed to independent sets of the form $I = \{x, y\}$. In rank 3, the $h$-vector of $\Gamma_I$ can only be $(1, 0)$, $(1, 1)$, or $(1, 2)$ because $\Gamma_I$ can consist of either one, two, or all three of the vertices among $\{1, 2, 3\}$. In each case, $h_0(\Gamma_I) = 1$, so we add $xy$ to $\mathcal{O}$. When $h_1(\Gamma_I) = 1$, we have to make a choice of whether to add $x^2y$ or $xy^2$ to $\mathcal{O}$. This choice depends on $\mathcal{B}_x$

and $\mathcal{B}_y$. Specifically, if $|\mathcal{B}_x| < |\mathcal{B}_y|$, then we choose to add $xy^2$ to $\mathcal{O}$. For example, if $|\mathcal{B}_x| = 1$ and $|\mathcal{B}_y| = 2$, then $x^2 \notin \mathcal{O}$ so it does not make sense to add $x^2y$ to $\mathcal{O}$; but $y^2 \in \mathcal{O}$, and hence all divisors of $xy^2$ also belong to $\mathcal{O}$. Moreover, $xy^2$ serves as a maximal monomial that is divisible by both $x$ and $y^2$. Lemma 3.4.1 below will confirm that if $h_1(\Gamma_I) = 1$, then either $|\mathcal{B}_x| \geqslant 2$ or $|\mathcal{B}_y| \geqslant 2$ so that it will be possible to add either $x^2y$ or $xy^2$ to $\mathcal{O}$ while still preserving the structure of $\mathcal{O}$ as a multicomplex.

As a further remark, one could wonder what would happen if $I = \{x, y\}$, $h(\Gamma_I) = (1, 1)$, and $|\mathcal{B}_x| = |\mathcal{B}_y| = 2$. This would indeed be problematic as we would have $x^2$ and $y^2$ in $\mathcal{O}$, but neither $x^3$ nor $y^3$ in $\mathcal{O}$. Since $h(\Gamma_I) = (1, 1)$, we would only be allowed to add $x^2y$ or $xy^2$ to $\mathcal{O}$, but not both. Thus only one of the monomials $x^2$ or $y^2$ would divide a maximal monomial supported on $xy$, but we might not be able to guarantee that the other would ever divide into a maximal monomial. The following lemma will forbid such pathologies. When $h(\Gamma_I) = (1, 1)$, if $|\mathcal{B}_x| = 2$, then $|\mathcal{B}_y| = 3$. This means that $x^2$ belongs to $\mathcal{O}$, $x^3$ does not belong to $\mathcal{O}$, and $y^3$ belongs to $\mathcal{O}$. Thus it is natural to add $x^2y$ to $\mathcal{O}$ to maintain purity throughout the construction.

The following lemma shows that when $I = \{x, y\}$ as above, the relationship between $h(\Gamma_I)$, $\mathcal{B}_x$, and $\mathcal{B}_y$ is not arbitrary. The lemma can be proved directly through repeatedly applying the exchange axiom to small matroids of rank 3, but we have also verified it directly in Sage using the databases of small matroids from [66].

**Lemma 3.4.1.** *Let $\Delta$ be a matroid of rank 3 on the ground set $[n]$ for which $[3]$ is a basis. Let $I = \{x, y\} \subseteq [n] - [3]$ be an independent set. Then*

1. *If $h(\Gamma_I) = (1, 0)$, then there exists $z \in [n] - (\{x, y\} \cup [3])$ such that $\{x, y, z\} \in \Delta$.*

2. *If $h(\Gamma_I) = (1, 1)$, then one of the following holds:*

   *(a) $|\mathcal{B}_x| = 1$ and $|\mathcal{B}_y| \geqslant 2$,*

   *(b) $|\mathcal{B}_y| = 1$ and $|\mathcal{B}_x| \geqslant 2$,*

   *(c) $|\mathcal{B}_x| = 2$ and $|\mathcal{B}_y| = 3$,*

(d) $|\mathcal{B}_x| = 3$ and $|\mathcal{B}_y| \geqslant 2$.

3. If $h(\Gamma_I) = (1, 2)$, then $|\mathcal{B}_x| \geqslant 2$ and $|\mathcal{B}_y| \geqslant 2$.

In light of these motivating examples, we present the algorithm for constructing a natural pure $O$-sequence that can be associated to any rank 3 matroid.

**Algorithm 3.4.2. Constructing a pure degree 3 order ideal.**

**INPUT** : *A rank 3 matroid $\Delta$ whose vertex set is ordered*

**OUTPUT** :*A pure order ideal $\mathcal{O}$ whose $F$-vector is $h(\Delta)$.*

**OUTLINE:**

**STEP 0:** *Reorder the vertices of the matroid as follows. Pick the lexicographically smallest basis of $\Delta$ and relabel its elements as $\{1, 2, 3\}$. For the remaining vertices declare $x < y$ if $|\mathcal{B}_x| < |\mathcal{B}_y|$ or $\mathcal{B}_x <_{lex} \mathcal{B}_y$ and preserve the relative order of the vertices for which $|\mathcal{B}_x|$ is fixed.*

**STEP 1:** *Construct the family of independent sets $I$ with $I \cap \{1, 2, 3\} = \varnothing$ and partition them into four collections $A_0$, $A_1$, $A_2$, $A_3$, where $A_i$ contains all the elements of size $i$. Initialize a list of monomials $\mathcal{O}$ to the empty list.*

**STEP 2** *Add the monomial 1 to $\mathcal{O}$. It corresponds to the empty set, the only element of $A_0$.*

**STEP 3** *For each $I = \{x\} \in A_1$, add all the monomial $x^t$ to $\mathcal{O}$, where $1 \leqslant t \leqslant |\mathcal{B}_x|$.*

**STEP 4** *For $I = \{x, y\} \in A_2$ with $x < y$ we split into cases according to the values of $h(\Gamma_I)$, $|\mathcal{B}_x|$, and $|\mathcal{B}_y|$:*

1. *If $h(\Gamma_I) = (1, 0)$, then add the monomial $xy$ to $\mathcal{O}$.*

2. *If $h(\Gamma_I) = (1, 1)$ we again split into cases:*

    i. *If $|\mathcal{B}_x| = 1$ then add the monomial $xy^2$ to $\mathcal{O}$. ($x^2$ is not on the list, so we cannot add anything that is divisible by $x^2$).*

    ii. *Else add the monomial $x^2y$ to $\mathcal{O}$.*

*3. If $h(\Gamma_I) = (1,2)$, then add the monomials $xy$, $xy^2$, and $x^2y$ to $\mathcal{O}$.*

**STEP 5:** *For each $I = \{x,y,z\} \in A_3$ add the monomial $xyz$ to $\mathcal{O}$.*

For a rank 3 based matroid $(\Delta, B, <)$ let $\mathcal{F}(\Delta, B, <)$ be the output of the algorithm when we relabel the vertices of $\Delta$ to have $B$ as the smallest lexicographic basis and keep the relative order on $E(\Delta) - B$. By construction of the algorithm, conditions 1 through 5 from Conjecture 3.3.10 are satisfied. So if the output is a pure order ideal for each element of $\mathcal{U}_3$, then the proof of Conjecture 3.3.10 in the case of $d = 3$ follows by Theorem 3.3.11.

We have computationally verified that the output of this code produces a pure order ideal for each element of $\mathcal{U}_3$. A summary of our code is included in Section 3.6, and all of our code is available at [55]. Thus we have computationally verified that Stanley's Conjecture holds for matroids of rank three.

**Theorem 3.4.3.** *Conjecture 3.3.10 holds for $d = 3$.*

The algorithm does more than giving us a tool to check that the hypotheses of Theorem 3.3.11 indeed hold for $d = 3$, it explicitly constructs a valid map $\mathcal{F}$. Furthermore, if the vertices of the matroid are ordered it gives the pure order ideal even if the number of vertices is larger than 6. To illustrate the results of the algorithm we now present an example.

**Example 3.4.4.** *Once again, we consider the Fano matroid of Example 3.3.6. Under the natural ordering of the vertex set $\{1,2,3,4,5,6,7\}$, the Fano matroid has the following restricted h-vectors, which contribute the shown monomials to the corresponding monomial family. It can be easily verified that the resulting family of monomials forms a pure multicomplex.*

| *I* | $h(\Gamma_I)$ | *Monomials* |
|---|---|---|
| $\varnothing$ | $(1,0,0,0)$ | *1* |
| $\{4\}$ | $(1,1,0)$ | $x_4,\ x_4^2$ |
| $\{5\}$ | $(1,1,0)$ | $x_5,\ x_5^2$ |
| $\{6\}$ | $(1,1,0)$ | $x_6,\ x_6^2$ |
| $\{7\}$ | $(1,1,1)$ | $x_7,\ x_7^2,\ x_7^3$ |
| $\{4,5\}$ | $(1,2)$ | $x_4x_5,\ x_4x_5^2,\ x_4^2x_5$ |
| $\{4,6\}$ | $(1,2)$ | $x_4x_6,\ x_4x_6^2,\ x_4^2x_6$ |
| $\{4,7\}$ | $(1,1)$ | $x_4x_7,\ x_4^2x_7$ |
| $\{5,6\}$ | $(1,2)$ | $x_5x_6,\ x_5x_6^2,\ x_5^2x_6$ |
| $\{5,7\}$ | $(1,1)$ | $x_5x_7,\ x_5^2x_7$ |
| $\{6,7\}$ | $(1,1)$ | $x_6x_7,\ x_6^2x_7$ |
| $\{4,5,7\}$ | $(1)$ | $x_4x_5x_7$ |
| $\{4,6,7\}$ | $(1)$ | $x_4x_6x_7$ |
| $\{5,6,7\}$ | $(1)$ | $x_5x_6x_7$ |

## 3.5 Stanley's conjecture for rank 4 matroids

We now proceed to prove Conjecture 3.3.10 for $d = 4$. As for the rank 3 case, we give an algorithm that explicitly constructs a pure order ideal for every matroid of rank 4 whose vertex set is ordered. We verify that the output of the algorithm satisfies the conditions of Theorem 3.3.11 for all based matroids in $\mathcal{U}_4$, therefore proving Conjecture 3.3.10 for rank 4 matroids and also, as a result, Stanley's Conjecture 5.1.2. We first study some properties of rank four matroids.

### 3.5.1 Structural properties of rank 4 matroids

In order to motivate and explain why the algorithm in the next section is built as it is, we will now present a collection of structural results about the restricted $h$-vectors of matroid complexes. These lemmas admit theoretical proofs, but since it is enough to check them for matroids of rank 4 with seven elements, we are able to computationally verify the results in just a few seconds using Sage [55]. There are 374 such matroids to check.

Throughout this section, $\Delta$ is a rank 4 matroid such that $\{1,2,3,4\}$ is a basis of $\Delta$. All the lemmas are local, that is, all are concerned with an independent set $I$ disjoint from $\{1,2,3,4\}$ and the properties of the studied structure only depend on the matroid $\Delta_I$ that results from restricting the ground set of $\Delta$ to $\{1,2,3,4\} \cup I$. Further, the independent set $I$ is not a basis, hence it suffices to check the properties for matroids with at most 7 elements. The code to do the verification is in the document `verifyLemmas.sage` of [55].

As in the rank 3 case, if $I \subseteq [n]-[4]$ is an independent set, the structure of $h(\Gamma_I)$ depends heavily on the structure of $\{h(\Gamma_{I'}) \mid I' \subseteq I\}$. The following lemmas illustrate the structural relationships that will be essential to our algorithm.

**Lemma 3.5.1.** *Assume that* $I = \{x, y\}$.

1. *If* $h(\Gamma_I) = (1,1,0)$ *and* $|\mathcal{B}_x| = 1$, *then* $|\mathcal{B}_y| \geqslant 2$*:*
2. *If* $h(\Gamma_I) = (1,1,1)$ *then one of the following is true:*
   - *(a)* $\min\{|\mathcal{B}_x|, |\mathcal{B}_y|\} = 1$ *and* $\max\{|\mathcal{B}_x|, |\mathcal{B}_y|\} \geqslant 3$ *, or*
   - *(b)* $\min\{|\mathcal{B}_x|, |\mathcal{B}_y|\} \geqslant 3$.
3. *If* $h(\Gamma_I) = (1,2,0)$ *then* $\min\{|\mathcal{B}_x|, |\mathcal{B}_y|\} \geqslant 2$.
4. *If* $h(\Gamma_I) = (1,2,1)$ *then* $|\mathcal{B}_x| \in \{2,4\}$ *and* $|\mathcal{B}_y| \in \{2,4\}$.
5. *If* $h(\Gamma_I) = (1,2,2)$ *then* $\min\{|\mathcal{B}_x|, |\mathcal{B}_y|\} \geqslant 2$ *and* $\max\{|\mathcal{B}_x|, |\mathcal{B}_y|\} \geqslant 3$.
6. *If* $h(\Gamma_I) = (1,2,3)$ *then* $\min\{|\mathcal{B}_x|, |\mathcal{B}_y|\} \geqslant 3$.

**Lemma 3.5.2.** *Assume that* $I = \{x, y, z\}$.

1. *If* $h(\Gamma_I) = (1,1)$*, then* $\max\{|\mathcal{B}_x|, |\mathcal{B}_y|, |\mathcal{B}_z|\} \geqslant 2$.

2. *If* $h(\Gamma_I) = (1,2)$*, then at most one of* $|\mathcal{B}_x|, |\mathcal{B}_y|, |\mathcal{B}_z|$ *is equal to one.*

3. *If* $h(\Gamma_I) = (1,3)$*, then* $\min\{|\mathcal{B}_x|, |\mathcal{B}_y|, |\mathcal{B}_z|\} \geqslant 2$.

These lemmas serve as the motivation for the design of Algorithm 3.5.3. We believe that the main reason that the algorithm produces a pure $O$-sequence is hidden behind these inequalities. It seems to us that the key to proving Stanley's conjecture in higher rank is to understand how these structural inequalities generalize in higher dimensions.

### *3.5.2 The algorithm*

We now present the algorithm that will give the main result of the paper. The heuristic motivation for this algorithm is that we choose the lexicographically smallest possible set of monomials at each step subject to the requirements set forth by Conjecture 3.3.10. The lemmas from the previous section are the key motivation behind all of the necessary subcases. To simplify notation we write $\Gamma_{x,y,z} := \Gamma_{\{x,y,z\}}$, $\Gamma_{x,y} := \Gamma_{\{x,y\}}$, $\Gamma_x := \Gamma_{\{x\}}$ and $\mathcal{B}_x$ for the set of bases of $\Gamma_x$.

**Algorithm 3.5.3. Constructing a pure rank 4 multicomplex**

**INPUT** : *A rank 4 matroid* $\Delta$ *whose vertex set is ordered*

**OUTPUT** :*A pure multicomplex* $\mathcal{O}$ *whose* $F$*-vector is* $h(\Delta)$.

**STEP 0:** *Reorder the vertices of the matroid as follows. Pick the lexicographic smallest basis of* $\Delta$ *and relabel its elements as* $\{1,2,3,4\}$*. For the remaining vertices declare* $x < y$ *if* $|\mathcal{B}_x| < |\mathcal{B}_y|$ *and keep the original relative order of the vertices with* $|\mathcal{B}_x|$ *fixed. Relabel this ordered set as* $[n] - \{1,2,3,4\}$ *preserving the new order.*

**STEP 1:** *Construct the family of independent sets* $I$ *with* $I \cap \{1,2,3,4\} = \varnothing$ *and separate them into five groups* $A_0$*,* $A_1$*,* $A_2$*,* $A_3$*,* $A_4$*, where* $A_i$ *contains all the elements of size* $i$*. Initialize the list of monomials* $\mathcal{O}$ *to the empty list.*

**STEP 2:** *Add the monomial 1 to the list. It corresponds to the empty set, the only element* *of* $A_0$.

**STEP 3:** *For each* $I = \{x\} \in A_1$ *add the monomial* $x^t$, *for* $1 \leqslant t \leqslant |\mathcal{B}_x|$, *to* $\mathcal{O}$.

**STEP 4:** *For each* $I = \{x, y\} \in A_2$ *with* $x < y$, *we split in cases according to the h-vectors of* $\Gamma_{x,y}, \Gamma_x, \Gamma_y$ *using the following rules:*

1. *If* $h(\Gamma_{x,y}) = (1, 0, 0)$, *then add the monomial* $xy$ *to* $\mathcal{O}$.

2. *Else if* $h(\Gamma_{x,y}) = (1, 1, 0)$, *then there are two subcases to consider:*

   i. *If* $|\mathcal{B}_x| = 1$ *add the monomials* $xy$ *and* $xy^2$ *to* $\mathcal{O}$.

   ii. *Else add the monomials* $xy$ *and* $x^2y$ *to* $\mathcal{O}$.

3. *Else if* $h(\Gamma_{x,y}) = (1, 1, 1)$, *then there are two subcases to consider:*

   i. *If* $|\mathcal{B}_x| = 1$ *add the monomials* $xy$, $xy^2$ *and* $xy^3$ *to* $\mathcal{O}$;

   ii. *Else add the monomials* $xy$, $x^2y$ *and* $x^3y$ *to* $\mathcal{O}$.

4. *Else if* $h(\Gamma_{x,y}) = (1, 2, 0)$, *add the monomials* $xy$, $x^2y$ *and* $xy^2$ *to* $\mathcal{O}$.

5. *Else if* $h(\Gamma_{x,y}) = (1, 2, 1)$, *add the monomials* $xy$, $x^2y, xy^2, x^2y^2$ *to* $\mathcal{O}$.

6. *Else if* $h(\Gamma_{x,y}) = (1, 2, 2)$ *then there are two subcases to consider:*

   i. *If* $|\mathcal{B}_x| < 3$ *add the monomials* $xy$, $x^2y$, $xy^2$, $x^2y^2$ *and* $xy^3$ *to* $\mathcal{O}$.

   ii. *Else add the monomials* $xy\, x^2y, xy^2, x^3y$ *and* $xy^3$ *to* $\mathcal{O}$.

7. *Else, add the monomials* $xy, x^2y, xy^2, x^3y, x^2y^2$ *and* $xy^3$ *to* $\mathcal{O}$.

**STEP 5:** *For each* $I = \{x, y, z\} \in A_3$ *with* $x < y < z$, *we split in cases according to the values of* $h(\Gamma_{x,y,z})$, $h(\Gamma_{x,y})$, $h(\Gamma_{x,z})$ *and* $h(\Gamma_{y,z})$, $h(\Gamma_x)$, $h(\Gamma_y)$, $h(\Gamma_z)$.

1. *If* $h(\Gamma_{x,y,z}) = (1, 0)$, *then add the monomial* $xyz$ *to* $\mathcal{O}$.

2. *Else if* $h(\Gamma_{x,y,z}) = (1,1)$ *there are several subcases:*
    - i. *If* $h(\Gamma_{x,y}) = (1,0,0)$ *add the monomials* $xyz$ *and* $xyz^2$ *to* $\mathcal{O}$.
    - ii. *Else if* $h(\Gamma_{x,z}) = (1,0,0)$, *then add the monomials* $xyz$ *and* $xy^2z$ *to* $\mathcal{O}$.
    - iii. *Else if* $h(\Gamma_{y,z}) = (1,0,0)$, *then add the monomials* $xyz$ *and* $x^2yz$ *to* $\mathcal{O}$.
    - iv. *Else if* $h(\Gamma_{x,y}) = (1,1,0)$ *or* $h(\Gamma_{x,y}) = (1,1,1)$ *we split in two cases:*
        - a. *if* $|\mathcal{B}_x| = 1$ *add the monomials* $xyz$ *and* $xy^2z$ *to* $\mathcal{O}$*;*
        - b. *else add the monomials* $xyz$ *and* $x^2yz$ *to* $\mathcal{O}$.
    - v. *Else if* $h(\Gamma_{x,z}) = (1,1,0)$, *then*
        - a. *if* $|\mathcal{B}_x| = 1$ *add the monomials* $xyz$ *and* $xyz^2$ *to* $\mathcal{O}$*;*
        - b. *else add monomials* $xyz$ *and* $x^2yz$.
    - vi. *Else if* $h(\Gamma_{y,z}) = (1,1,0)$ *or* $h(\Gamma_{y,z}) = (1,1,1)$, *then*
        - a. *if* $|\mathcal{B}_y| = 1$ *add the monomials* $xyz$ *and* $xyz^2$*;*
        - b. *else add monomials* $xyz$ *and* $xy^2z$.
    - vii. *Else add the monomials* $xyz$ *and* $x^2yz$.

3. *Else if* $h(\Gamma_{x,y,z}) = (1,2)$, *we again have several cases:*
    - i. *If* $|\mathcal{B}_x| = 1$ *add* $xyz$, $xy^2z$ *and* $xyz^2$ *to* $\mathcal{O}$.
    - ii. *Else if* $|\mathcal{B}_y| = 1$ *add* $xyz, x^2yz$ *and* $xyz^2$ *to* $\mathcal{O}$.
    - iii. *Else if* $|\mathcal{B}_z| = 1$ *add* $xyz, x^2yz$ *and* $xy^2z$ *to* $\mathcal{O}$.
    - iv. *Else if* $h(\Gamma_{x,y}) = (1,1,0)$ *or* $h(\Gamma_{x,y}) = (1,1,1)$, *then*
        - a. *if* $|\mathcal{B}_x| > |\mathcal{B}_y|$ *add the monomials* $xyz$, $xy^2z$ *and* $xyz^2$ *to* $\mathcal{O}$*;*
        - b. *else add* $xyz$, $x^2yz$ *and* $xyz^2$ *to* $\mathcal{O}$.
    - v. *Else if* $h(\Gamma_{x,z}) = (1,1,0)$ *or* $h(\Gamma_{x,z}) = (1,1,1)$, *then*
        - a. *if* $|\mathcal{B}_x| > |\mathcal{B}_z|$ *add the monomials* $xyz$, $xy^2z$ *and* $xyz^2$ *to* $\mathcal{O}$*;*

```
            b. else add $xyz$, $x^2yz$ and $xy^2z$ to $\mathcal{O}$.
        vi. Else if $h(\Gamma_{y,z}) = (1,1,0)$ or $h(\Gamma_{y,z}) = (1,1,1)$, then
            a. if $|\mathcal{B}_y| > |\mathcal{B}_z|$ add the monomials $xyz$, $x^2yz$ and $xyz^2$ to $\mathcal{O}$;
            b. else add $xyz$, $x^2yz$, $xy^2z$ to $\mathcal{O}$
        vii. Else if $h(\Gamma_{x,y}) = (1,2,0)$, add the monomials $xyz$, $x^2yz$ and $xy^2z$ to $\mathcal{O}$.
        viii. Else if $h(\Gamma_{x,z}) = (1,2,0)$, add the monomials $xyz$, $x^2yz$ and $xyz^2$ to $\mathcal{O}$.
        ix. Else if $h(\Gamma_{y,z}) = (1,2,0)$, add the monomials $xyz$, $xy^2z$ and $xyz^2$ to $\mathcal{O}$.
        x. Else add the monomials $xyz$, $x^2yz$ and $xy^2z$ to $\mathcal{O}$.
    4. Else if $h(\Gamma_{x,y,z}) = (1,3)$, then add the monomials $xyz$, $x^2yz$, $xy^2z$ and $xyz^2$ to $\mathcal{O}$.
STEP 6: For each $I = \{w, x, y, z\} \in A_4$ add the monomial $wxyz$ to $\mathcal{O}$.
```

For a based matroid $(\Delta, B, <)$, let $\mathcal{F}(\Delta, B, <)$ be the monomial family output by Algorithm 3.5.3 when we input $\Delta$. We want to show that $\mathcal{F}(\Delta, B, <)$ is indeed a pure multicomplex. It is sufficient to show that claim holds as $(\Delta, B)$ ranges over the elements of $\mathcal{U}_4$. To do this we implemented the algorithm in Sage and verified computationally that it produces pure $O$-sequences for each element of $\mathcal{U}_4$ (up to ordered isomorphism). Matroids with 8 elements are classified in [67], [66] and [68]. We use the classification to produce all isomorphism classes of based matroids and get the desired multicomplex for small matroids.

### *3.5.3 Matroids with 8 elements*

We begin by establishing a lemma that is essential in reducing the number of computations that are required to prove Algorithm 3.5.3 works in general.

**Lemma 3.5.4.** *Let $\Delta$ and $\Delta'$ be rank 4 matroids whose ground set is $[k]$ for $5 \leqslant k \leqslant 8$. Assume the following assertions hold:*

1. *$\{1,2,3,4\}$ is a basis of both $\Delta$ and $\Delta'$.*

2. *For all $A \subseteq [k] - \{1,2,3,4\}$, $A$ is independent in $\Delta$ if and only if $A$ is independent in $\Delta'$.*

3. *For $I \subseteq [k] - \{1,2,3,4\}$ independent in $\Delta$, $h(\Gamma_I(\Delta)) = h(\Gamma_I(\Delta'))$.*

*Then $\mathcal{F}(\Delta, \{1,2,3,4\}, <) = \mathcal{F}(\Delta', \{1,2,3,4\}, <)$.*

*Proof.* Since $|\mathcal{B}_x(\Delta)| = \sum_{i=0}^{3} h_i(\Gamma_{\{x\}}(\Delta))$ all the cases considered in every step of the algorithm only depend on $h(\Gamma_I(\Delta))$ as $I$ ranges among the independent sets of $\Delta$ that do not intersect $\{1,2,3,4\}$. This proves the claim. $\square$

Lemma 3.5.4 together with the fact that every based matroid $(\Delta, B, <) \in \mathcal{U}_4$ is isomorphic to a based matroid $(\Delta', \{1,2,3,4\}, <)$ where $E(\Delta') = \{1,2,3,4,5,6,7,8\}$ and $<$ is the natural order imply that it is enough to follow the following procedure. We outline this computational procedure further in Section 3.6.

- Start with an empty list of *matroids-to-check* that saves the families of $h$-vectors of the form $(h(\Gamma_I) \,|\, I \cap \{1,2,3,4\} = \varnothing)$. For each such family of $h$-vectors we will store exactly one matroid with those restricted $h$-vectors.

- For each based matroid $(\Delta, \{1,2,3,4\})$ with vertex set $\{1,2,3,4,5,6,7,8\}$ determine if there is a matroid $(\Delta', \{1,2,3,4\})$ whose family of $h$-vectors $\{h(\Gamma'_I), \,|\, I \cap \{1,2,3,4\} = \varnothing\} = \{h(\Gamma_I), \,|\, I \cap \{1,2,3,4\}\}$. (Here, $\Gamma'_I := \mathrm{link}_{\Delta'}(I)|_{\{1,2,3,4\}}$.) If no such matroid exists, then add $\Delta$ and its list of restricted $h$-vectors to the list of *matroids-to-check.*

- Run Algorithm 3.5.3 on every matroid in the list of *matroids-to-check.*

- Verify that the output of the algorithm for every complex to check is indeed a pure order ideal.

The result of the test is positive. The total number of matroids in the list of matroids to verify is 9085. Below is and example of the outputs of the algorithm.

**Example 3.5.5.** *Let $\Delta$ be the dual matroid of the Fano matroid from Example 3.4.4, that is, the matroid whose bases are the complements of the bases of the Fano matroid with the labels as in the previous example. The restricted $h$-vectors and corresponding monomials in this case are shown in the following table. Once again, we can easily check that the resulting family of monomials is a pure multicomplex.*

| $I$ | $h(\Gamma_I)$ | *Monomials* |
|---|---|---|
| $\{5\}$ | $(1,1,1,0)$ | $x_5,\ x_5^2,\ x_5^3$ |
| $\{6\}$ | $(1,1,1,0)$ | $x_6,\ x_6^2,\ x_6^3$ |
| $\{7\}$ | $(1,1,1,0)$ | $x_7,\ x_7^2,\ x_7^3$ |
| $\{5,6\}$ | $(1,2,2)$ | $x_5x_6,\ x_5^2x_6,\ x_5x_6^2,\ x_5^3x_6,\ x_5x_6^3$ |
| $\{5,7\}$ | $(1,2,2)$ | $x_5x_7,\ x_5^2x_7,\ x_5x_7^2,\ x_5^3x_7,\ x_5x_7^3$ |
| $\{6,7\}$ | $(1,2,2)$ | $x_6x_7,\ x_6^2x_7,\ x_6x_7^2,\ x_6^3x_7,\ x_6x_7^3$ |
| $\{5,6,7\}$ | $(1,2)$ | $x_5x_6x_7,\ x_5^2x_6x_7,\ x_5x_6^2x_7$ |

## *3.6 Summary of code*

This section contains a summary of the computational approach undertaken to verify Conjecture 3.3.10 in ranks three and four. The code and data are available online at [55].

In order to computationally verify our results, we used the database of matroids in [67]. This gave us a representative of each isomorphism class of matroids of rank three (respectively four) on at most six (respectively eight) elements. The files `matroidsnXrY.sage` contain the bases of each isomorphism class of matroids of rank $Y$ on $X$ elements. Initially, the bases are stored as 0/1 lists that encode the $Y$-element subsets of $[X]$ under the reverse lexicographic order. The code in the file `constructMatroids.sage` converts each 0/1 list into a list of facets/bases of a simplicial complex.

For each such isomorphism class, any potential reordering of the ground set would give a different based matroid with a different initial basis under the lexicographic shelling order. For each based matroid, we computed the set of restricted $h$-vectors $\{h(\Gamma_I) \mid I \subseteq [n]\backslash[d]\}$. This analysis is also done in the file `constructMatroids.sage`. For each unique set of restricted $h$-vectors, we stored the corresponding ordered matroid and list of restricted $h$-vectors for further analysis. The restricted $h$-vectors and corresponding matroids are stored in two separate lists in the files `hvecsnXrY.sage`. From the original list of 1331 isomorphism classes of matroids of rank four on at most eight elements, we constructed a list of 9085 based matroids to be examined.

Now that we have saved a permanent record of all possible sets of restricted $h$-vectors $\{h(\Gamma_I) \mid I \subseteq [n]\backslash[d]\}$ for all matroids of rank three (or four) on at most six (or eight) elements, we are able to computationally verify Lemmas 3.4.1, 3.5.1, and 3.5.2 and implement our Algorithms 3.4.2 and 3.5.3. The Lemmas are verified using the code in `verifyLemmas.sage`. The files `constructMonomialsRank3.sage` and `constructMonomialsRank4.sage` contain implementations of the algorithms. Finally, we wrote code in the file `verifyPureOSequence.sage` to test whether a given list of monomials is a pure multicomplex. We ran these tests in the file `verifyAlgorithm.sage`, and the output verified that the family of monomials constructed by our algorithm was indeed a pure order ideal in each case. Finally we verified that the $F$-vector of the output of the algorithm coincides with the $h$-vector of the input. The $h$-vector of the input is computed using the standard `SimplicialComplex` class of Sage.

# Chapter 4

# THE TOPOLOGY OF THE EXTERNAL ACTIVITY COMPLEX OF A MATROID

## *4.1 Introduction*

**Wider context of this work.** Matroid theory is a combinatorial theory of independence which has its roots in linear algebra and graph theory, but which turns out to have deep connections with many fields. There are natural notions of independence in linear algebra, graph theory, matching theory, the theory of field extensions, and the theory of routings, among others. Matroids capture the combinatorial essence that those notions share.

A matroid can be described in many equivalent ways, arising from the many contexts where matroids are found: the bases, the circuits, the lattice of flats, and the matroid polytope, among others. One important approach, which is the most relevant one to this paper, has been to model a matroid in terms of a simplicial or polyhedral complex. In fact, most of these topological models arise naturally in algebraic and geometric contexts, and offer new tools to prove combinatorial theorems. A celebrated recent example is the proof by Huh [45] and Huh and Katz [47] of Rota's 1971 conjecture [84] that the coefficients of the characteristic polynomial of a linear matroid are unimodal. A key ingredient of this proof is the Bergman complex $\mathcal{B}(M)$ described below.

Let us describe a few constructions of this flavor, and provide a few references for the interested reader. The notion of shellability is a very useful unifying tool in this approach, as explained in [17].

- [81, 17] The *independence complex* or *matroid complex* $IN(M)$ is homotopy equivalent to a wedge of $T_M(0,1) = |\mu(M^*)|$ spheres of dimension $r(M) - 1$ if $M$ is coloopless. This complex is shellable, and its shelling polynomial is $T_M(x,1)$. The shellability of $IN(M)$

naturally leads to the important notions of internal and external activity of $M$.

- [99, 83, 24] The *broken circuit complex* $\overline{BC}_<(M)$ is, a cone over a space homotopy equivalent to a wedge of $|\beta(M)|$ spheres of dimension $r(M)-2$. It can be naturally embedded into $IN(M)$. It is shellable and its shelling polynomial is $T_M(x,0)$. The embedding is a combinatorial witness of such a result. Its face numbers equal the coefficients of the characteristic polynomial of $M$ up to sign.

- [36, 89, 79] the (proper part of the) *order complex of the lattice of flats* $\Delta(L_M\backslash\{\widehat{0},\widehat{1}\})$ is homotopy equivalent to a wedge of $T_M(1,0) = |\mu(M)|$ spheres of dimension $r(M)-2$. It is shellable. This is a motivating example for the theory of Cohen Macaulay posets. It also arises naturally in Orlik and Solomon's presentation of the cohomology of the complement of a complex hyperplane arrangement.

- [92, 9] The *Bergman complex* $\mathcal{B}(M)$ is the link of the origin in the *tropical linear space* $\mathrm{Trop}(M)$. It is not always simplicial. Though not obvious from its definition, $\mathcal{B}(M)$ is a coarsening of $\Delta(L_M\backslash\{\widehat{0},\widehat{1}\})$ and hence shares its topological properties. These complexes are fundamental objects in tropical geometry because $\mathrm{Trop}(M)$ is the tropical analog of a linear space.

The purpose of this paper is to describe a new member of this family.

- The *external activity complex* $\mathrm{Act}_<(M)$ is, after removing cone points, either contractible or a sphere of dimension $n+r-1-|AE(M)|$ where $AE(M)$ is the set of externally absolute elements. It contains a copy of $IN(M)$ as a subcomplex. It is shellable, and its shelling polynomial is $T_M(x,1)$. Its shellability is closely related to Las Vergnas's *active orders* on the bases of $M$.

Hence the external activity complex sheds new light on the shelling polynomial $T_M(x,1)$ of a matroid $M$. This is a subject of great attention thanks to Stanley's 1977 *h-vector conjecture*, one of the most intriguing open problems in matroid theory:

**Conjecture 4.1.1.** *[89] For any matroid $M$, there exists a set $X$ of monomials such that:*

*- if $m$ and $m'$ are monomials such that $m \in X$ and $m'|m$, then $m' \in X$,*

- *all the maximal monomials in $X$ have the same degree,*
- *there are exactly $h_i$ monomials of degree $i$ in $X$, where $\sum_i h_i x^{r-i} = T(x,1)$.*

This conjecture has been proved, using rather different methods, for several families: cographic matroids, [73], lattice path matroids [87], cotransversal matroids [78], paving matroids [**?**], and matroids up to rank 4 or corank 2 [30, 54]. The general case remains open.

**Motivation for this work.** The *external activity complex* $\mathrm{Act}_<(M)$ of a matroid is a simplicial complex associated to a matroid $M$ and a linear order $<$ on its ground set. This complex arose in work of the first author with Adam Boocher [7]. They started with a linear subspace $L$ of affine space $\mathbb{A}^n$ with a chosen system of coordinates. There is a natural embedding $\mathbb{A}^n \hookrightarrow (\mathbb{P}^1)^n$ into a product of projective lines, and they considered the closure $\widetilde{L}$ of $L$ in $(\mathbb{P}^1)^n$. They proved that many geometric and algebraic invariants of the variety $\widetilde{L}$ are determined by the matroid of $L$.

As is common in combinatorial commutative algebra, a key ingredient of [7] was to consider the initial ideals $\mathrm{in}_<\widetilde{L}$ under various term orders. These initial ideals are the Stanley-Reisner ideals of the external activity complexes $\mathrm{Act}_<(M)$ under the different linear orders $<$ of the ground set. This led them to consider and describe the complexes $\mathrm{Act}_<(M)$.

The ideals $\mathrm{in}_<\widetilde{L}$ are shown to be Cohen-Macaulay in [7], and the authors asked the stronger question: Are the external activity complexes $\mathrm{Act}_<(M)$ shellable? The purpose of this note is to answer this question affirmatively.

**Our results.** The facets of $\mathrm{Act}_<(M)$ are indexed by the bases $\mathcal{B}$ of $M$, and [7] suggested a possible connection between $\mathrm{Act}_<(M)$ and LasVergnas's *internal order* $<_{int}$ on $\mathcal{B}$. Suprisingly, we find that it is the *external/internal order* $<_{ext/int}$ on $\mathcal{B}$, also defined in [63], which plays a key role. Our main result is the following:

**Theorem 4.1.2.** *Let $M = (E, \mathcal{B})$ be a matroid, and let $<$ be a linear order on the ground set $E$. Any linear extension of LasVergnas's external/internal order $<_{ext/int}$ of $\mathcal{B}$ induces a shelling of the external activity complex $Act_<(M)$.*

As a corollary we obtain that these orders also shell the independence complex $IN(M)$, and in fact we show a stronger statement.

**Theorem 4.1.3.** *Any linear extension of the internal order* $<_{int}$ *gives a shelling order of the independence complex* $IN(M)$*.*

These theorems are as strong as possible in the context of LasVergnas's active orders. We also obtain the following enumerative corollary.

**Theorem 4.1.4.** *The* $h$*-vector of* $Act_<(M)$ *equals the* $h$*-vector of* $M$*.*

It is easy to see that $\mathrm{Act}_<(M)$ is a cone, and hence trivially contractible. It is more interesting to study the *reduced external activity complex* $\mathrm{Act}^{\bullet}_<(M)$, obtained by removing all the cone points of $\mathrm{Act}_<(M)$. Our main topological result is the following.

**Theorem 4.1.5.** *Let* $M$ *be a matroid and* $<$ *be a linear order on its ground set. The reduced external activity complex* $Act^{\bullet}_<(M)$ *is contractible if* $M$ *contains* $U_{1,3}$ *as a minor, and a sphere otherwise.*

In Proposition 4.6.5 we will see there is an embedding of the independence complex $IN(M)$ in $\mathrm{Act}^{\bullet}_<(M)$, and both complexes have the same $h$-vector. If $M$ is coloopless its independence complex is homotopy equivalent to a wedge of $|\mu(M^*)|$ spheres, while the external activity complex is contractible or a sphere. Thus $\mathrm{Act}^{\bullet}_<(M)$ can be seen as a topologically simpler model than $IN(M)$ for the matroid $M$.

The paper is organized as follows. In Section 4.2 we introduce the necessary definitions and preliminaries. In Section 4.3 we carry out an example in detail, and show that the hypotheses of Theorems 4.1.2 and 4.1.3 are best possible. In Section 4.4 we prove our main Theorem 4.1.2 on the shellability of the external activity complex $\mathrm{Act}_<(M)$, and Theorem 4.1.3, which gives many new shellings of the independence complex $IN(M)$. In Section 4.5 we show that $\mathrm{Act}_<(M)$ and $IN(M)$ have the same $h$-vector. Finally, in Section 4.6, we describe the topology of the reduced external activity complex in Theorem 4.1.5.

## 4.2 Preliminaries

In this section we collect the background information on matroids and shellability that we will need to prove our results.

### 4.2.1 Matroids

**Basic definitions.** A *simplicial complex* $\Delta = (E, \mathcal{I})$ is a pair where $E$ is a finite set and $\mathcal{I}$ is a non empty family of subsets of $E$, such that if $A \in \mathcal{I}$ and $B \subset A$, then $B \in A$. Elements of $\mathcal{I}$ are called *faces* of the complex. The maximal elements of $\mathcal{I}$ are called *facets.* A complex is said to be *pure* if all facets have the same number of elements.

The following is one of many equivalent ways of defining a matroid:

**Definition 4.2.1.** *A* matroid $M = (E, \mathcal{I})$ *is a simplicial complex such that the restriction of* $M$ *to any subset of* $E$ *is pure.*

Since there are several simplicial complexes associated to $M$, we will denote this one $IN(M) = (E, \mathcal{I})$. It is often called the *independence complex* of $M$.

The two most important motivating examples of matroids are the following.

- (Linear Algebra) Let $E$ be a set of vectors in a vector space, and let $\mathcal{I}$ consist of the subsets of $E$ which are linearly independent. Then $(E, \mathcal{I})$ is a *linear* matroid.

- (Graph Theory) Let $E$ be the set of edges of an undirected graph $G$, and let $\mathcal{I}$ consist of the sets of edges which contain no cycle. Then $(E, \mathcal{I})$ is a *graphic* matroid.

For any matroid $M = (E, \mathcal{I})$, it is customary to call the sets in $\mathcal{I}$ *independent.* The facets of a matroid are called *bases.* The set of all bases is denoted $\mathcal{B}$.

**Example 4.2.2.** *The simplest example of a matroid is the uniform matroid* $U_{k,n}$*, whose ground set is* $[n]$ *and whose independent sets are all the subsets of* $[n]$ *of cardinality at most* $k$*. The uniform matroid* $U_{1,3}$ *is going to play an important role later.*

The minimal non-faces of $M$, that is, the minimal dependent sets, are called *circuits.* The circuits of a matroid have a special structure [80]:

**Lemma 4.2.3** (Circuit Elimination Property)**.** *If* $\gamma_1$ *and* $\gamma_2$ *are circuits of a matroid and* $c \in \gamma_1 \cap \gamma_2$*, then there is a circuit* $\gamma_3$ *that is contained in* $\gamma_1 \cup \gamma_2 - c$*.*

**Duality.** Matroids have a notion of duality which generalizes orthogonal complements in linear algebra and dual graphs in graph theory.

Let $M$ be a matroid with bases $\mathcal{B}$. Then the set

$$\mathcal{B}^* = \{E - B \,:\, B \text{ is a basis of } M\}$$

is the collection of bases of a matroid $M^* = (E, \mathcal{B}^*)$, called the *dual matroid* $M^*$. The circuits of the dual matroid $M^*$ are called the *cocircuits* of $M$.

**Deletion, contraction, and minors.** We say that an element $e \in E$ is a *loop* of a matroid $M$ if it is contained in no basis; that is, if $\{e\}$ is a dependent set. Dually, $e$ is a *coloop* if it is contained in every basis of $M$.

The *deletion* $M\backslash e$ of a non-coloop $e \in E$ is the matroid on $E - e$ whose bases are the bases of $M$ that do not contain $e$. We also call this the *restriction* of $M$ to $E - e$. Dually, the *contraction* $M/e$ of a non-loop $e \in E$ is the matroid on $E - e$ whose bases are the subsets $B$ of $E - e$ such that $B \cup e$ is a basis of $M$.

It is easy to see that any sequence of deletions and contractions of different elements commutes. We say that a matroid $M'$ is a *minor* of a matroid $M$ if $M'$ is isomorphic to a matroid obtained from $M$ by performing a sequence of deletions and contractions.

**Fundamental circuits and cocircuits.** Given a basis $B$ and an element $e \in E - B$ there is a unique circuit contained in $B \cup e$, called the *fundamental circuit* of $e$ with respect to $B$. It is given by

$$\mathrm{Circ}(B, e) = \{x \in E \,:\, B \cup e - x \in \mathcal{B}\}\,.$$

Given a basis $B$ and an element $i \in B$ there is a unique cocircuit disjoint with $B - i$, called the *fundamental cocircuit* of $i$ with respect to $B$. It is given by

$$\mathrm{Cocirc}(B, i) = \{x \in E \,:\, B \cup x - i \in \mathcal{B}\}\,.$$

Note that the cocircuit $\mathrm{Cocirc}(B, i)$ in $M$ equals the circuit $\mathrm{Circ}(E - B, i)$ in the dual $M^*$.

**Basis activities.** Let $<$ be a linear order on the ground set $E$. For a basis $B$, define the sets:

$$\begin{aligned} EA(B) &= \{e \in E - B \,:\, \min\left(\mathrm{Circ}(B, e)\right) = e\} \\ EP(B) &= \{e \in E - B \,:\, \min\left(\mathrm{Circ}(B, e)\right) \neq e\} \end{aligned}$$

The elements of $EA(B)$ and $EP(B)$ are called *externally active* and *externally passive* with respect to $B$, respectively. Note that $EA(B) \uplus EP(B) = E - B$, where $\uplus$ denotes a disjoint union.

Dually, let

$$\begin{aligned} IA(B) &= \{i \in B \,:\, \min\left(\mathrm{Cocirc}(B, i)\right) = i\} \\ IP(B) &= \{i \in B \,:\, \min\left(\mathrm{Cocirc}(B, i)\right) \neq i\} \end{aligned}$$

The elements of $IA(B)$ and $IP(B)$ are called *internally active* and *internally passive* with respect to $B$, respectively. Note that $IA(B) \uplus IP(B) = B$. Also note that the internally active/passive elements with respect to basis $B$ in $M$ are the externally active/passive elements with respect to basis $E - B$ in $M^*$.

The following elegant result of Tutte [96] (for graphs) and Crapo [28] (for arbitrary matroids) underlies many of the results of [7] and this paper.

**Theorem 4.2.4.** *[28, Proposition 5.12] Let $M$ be a matroid on the ground set $E$ and let $<$ be a linear order on $E$.*

1. *Every subset $A$ of $E$ can be uniquely written in the form $A = B \cup X - Y$ for some basis*

*$B$, some subset $X \subseteq EA(B)$, and some subset $Y \subseteq IA(B)$. Equivalently, the intervals $[B - IA(B), B \cup EA(B)]$ form a partition of the poset $2^E$ of subsets of $E$ ordered by inclusion.*

2. *Every independent set $I$ of $E$ can be uniquely written in the form $I = B - Y$ for some basis $B$ and some subset $Y \subseteq IA(B)$. Equivalently, the intervals $[B - IA(B), B]$ form a partition of the independence complex $IN(M)$.*

The *Tutte polynomial* of $M$ is

$$T_M(x,y) = \sum_{B \text{ basis}} x^{|IA(B)|} y^{|EA(B)|}.$$

It follows from the work of Crapo and Tutte [28, 96] that this polynomial does not depend on the chosen order $<$. The Tutte polynomial is the most important matroid invariant, because it answers an innumerable amount of questions about the combinatorics, algebra, geometry, and topology of matroids and related objects. For more information, see [23].

**The external activity complex.** Let $M$ be a matroid on $E$. Let $\overline{E} = \{\overline{e} : e \in E\}$ be a second copy of $E$, and let $[[E]] = E \uplus \overline{E}$. This set of size $2|E|$ will be the ground set of the external activity complex of $M$. For each subset $S \subseteq E$ we write $\overline{S} := \{\overline{s} \mid s \in S\} \subset \overline{E}$. Therefore, each subset of $[[E]]$ can be written uniquely in the form $S_1 \cup \overline{S_2}$ for $S_1, S_2 \subseteq E$.

Our main object of study is the following.

**Theorem 4.2.5.** *[7] Let $M = (E, \mathcal{B})$ be a matroid and let $<$ be a linear order on $E$. $M$. There is a simplicial complex called the external activity complex $Act_<(M)$ on ground set $[[E]]$ such that*

1. *The facets are $F(B) := B \cup EP(B) \cup \overline{B \cup EA(B)}$ for every basis $B \in \mathcal{B}$.*

2. *The minimal non-faces are $S(\gamma) = c \cup \overline{\gamma - c}$ for every circuit $\gamma$, where $c$ is the $<$-smallest element of $\gamma$.*

The complement of the facet $F(B)$ in $[[E]]$ is $G(B) = EA(B) \cup \overline{EP(B)}$.

**Las Vergnas's three active orders.** Given a matroid $M = (E, \mathcal{B})$ and a total order $<$ on the ground set of $M$, LasVergnas introduced the following three *active orders*. In each case, he proved that there are several equivalent definitions.

**Definition 4.2.6.** *The* external order $<_{ext}$ *on* $\mathcal{B}$ *is characterized by the following equivalent properties for two bases* $A$ *and* $B$*:*

1. $A \leqslant_{ext} B$,
2. $A \subseteq B \cup EA(B)$,
3. $A \cup EA(A) \subseteq B \cup EA(B)$,
4. $B$ *is the lexicographically largest basis contained in* $A \cup B$.

*This poset is graded with* $r(B) = |EA(B)|$*. Adding a minimum element turns it into a lattice.*

**Definition 4.2.7.** *The* internal order $<_{int}$ *on* $\mathcal{B}$ *is characterized by the following equivalent properties for two bases* $A$ *and* $B$*:*

1. $A \leqslant_{int} B$,
2. $A - IA(A) \subseteq B$,
3. $A - IA(A) \subseteq B - IA(B)$,
4. $A$ *is the lexicographically smallest basis containing* $A \cup B$.

*This poset is graded with* $r(B) = r - |IA(B)|$*. Adding a maximum element turns it into a lattice.*

The internal and external orders are consistent in the sense that $A \leqslant_{int} B$ and $B \leqslant_{ext} A$ imply $A = B$. Therefore the following definition makes sense.

**Definition 4.2.8.** *The* external/internal order $<_{ext/int}$ *is the weakest order which simultaneously extends the external and the internal order. It is characterized by the following equivalent properties for two bases* $A$ *and* $B$*:*

1. $A \leqslant_{ext/int} B$,
2. $IP(A) \cap EP(B) = \varnothing$,

*This poset is a lattice. It is not necessarily graded.*

Note that Theorem 4.2.6.4 and 4.2.7.4 imply the following.

**Proposition 4.2.9.** *The lexicographic order* $<_{lex}$ *on* $\mathcal{B}$ *is a linear extension of the three posets* $<_{int}, <_{ext}$*, and* $<_{ext/int}$*. In symbols, any of* $A <_{int} B$*,* $A <_{ext} B$ *or* $A <_{ext/int} B$ *implies* $A <_{lex} B$*.*

### *4.2.2 Shellability and the h-vector*

**Shellability.** Shellability is a combinatorial condition on a simplicial complex that allows us to describe its topology easily. A simplicial complex is shellable if it can be built up by introducing one facet at a time, so that whenever we introduce a new facet, its intersection with the previous ones is pure of codimension 1. More precisely:

**Definition 4.2.10.** *Let* $\Delta$ *be a pure simplicial complex. A* shelling order *is an order of the facets* $F_1, \dots F_k$ *such for every* $i < j$ *there exist* $k < j$ *and* $f \in F_j$ *such that* $F_i \cap F_j \subseteq F_k \cap F_j = F_j - f$*. If a shelling order exists, then we call* $\Delta$ shellable.

Given a shelling order and a facet $F_j$, there is a subset $\mathcal{R}(F_j)$ such that for every $A \subseteq F_j$, we have $A \nsubseteq F_i$ for all $i < j$ if and only if $\mathcal{R}(F_j) \subseteq A$. Equivalently, when we add facet $F_j$ to the complex, the new faces that we introduce are precisely those in the interval $[\mathcal{R}(F_j), F_j]$. The set $\mathcal{R}(F_j)$ is called the *restriction set* of $F_j$ in the shelling.

**The $f$-vector and $h$-vector.** The *$f$-vector* of a $(d-1)$-dimensional simplicial complex $\Delta$ is $(f_0, \dots, f_d)$ where $f_i$ is the number of faces of $\Delta$ of size $i$. The *$h$-vector* $(h_0, \dots, h_d)$ is an equivalent way of storing this information; it is defined by the relation

$$f_0(x-1)^d + f_1(x-1)^{d-1} + \cdots + f_d(x-1)^0 = h_0x^d + h_1x^{d-1} + \cdots + h_dx^0.$$

This polynomial is also known as the *shelling polynomial* $h_\Delta(x)$, due to the following description of the $h$-vector for shellable complexes.

**Proposition 4.2.11.** *[17, Proposition 7.2.3] If $F_1, \ldots, F_k$ is a shelling order for a $(d-1)$-dimensional simplicial complex $\Delta$, then*

$$h_i := \left| \{ j \,:\, |\mathcal{R}(F_j)| = i \} \right| .$$

Note that it is not clear a priori that these numbers should be the same for any shelling order.

Understanding the topology of a shellable simplicial complex is easy once we know the last entry of the $h$-vector, thanks to the following result.

**Theorem 4.2.12.** *[61, Theorem 12.2(2)] Any geometric realization of a $(d-1)$-dimensional shellable simplicial complex $\Delta$ is homotopy equivalent to a wedge of $h_d$ spheres of dimension $d-1$. In particular, if $h_d = 0$, then every geometric realization of $\Delta$ is contractible.*

An important property for matroids is their shellability:

**Theorem 4.2.13.** *[17, Theorem 7.3.3] The lexicographic order $<_{lex}$ on the bases of a matroid $M$ gives a shelling order of the independence complex $IN(M)$. Furthermore, the restriction set of a basis $B$ in this shelling order is given by $IP(B)$.*

A straightforward consequence of the previous theorem is that the internal order poset is equal to the poset of bases of $M$ where the order is given by inclusion of restriction sets of the lexicographic shelling order.

## 4.3 Example

Before proving our theorems, we illustrate them in an example. Consider the graphic matroid given by the graph of Figure 4.1. Its bases are all the 3-subsets of [5] except $\{1,2,3\}$ and $\{1,4,5\}$. Under the standard order $1 < 2 < 3 < 4 < 5$ on the ground set, Table 4.1 records the basis activity of the various bases.

The resulting internal, external, and external/internal orders $<_{int}, <_{ext}, <_{ext/int}$ are shown in Figure 4.2. By Theorems 4.2.6, 4.2.7, and 4.2.8, these three orders are isomorphic to the

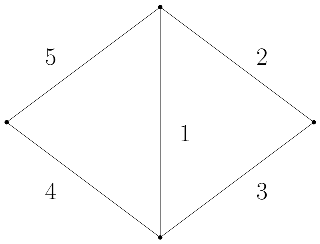


Figure 4.1: A graphic matroid.

| $B$ | $EP(B)$ | $EA(B)$ | $IP(B)$ | $IA(B)$ |
|---|---|---|---|---|
| 124 | 35 | $\varnothing$ | $\varnothing$ | 124 |
| 125 | 45 | $\varnothing$ | 5 | 12 |
| 134 | 25 | $\varnothing$ | 3 | 14 |
| 135 | 24 | $\varnothing$ | 35 | 1 |
| 234 | 5 | 1 | 23 | 4 |
| 235 | 4 | 1 | 235 | $\varnothing$ |
| 245 | 3 | 1 | 45 | 2 |
| 345 | $\varnothing$ | 12 | 345 | $\varnothing$ |

Table 4.1: The bases $B$ together with their sets of externally passive, externally active, internally passive, and internally active elements.

three families of sets $\{B \cup EA(B) : B \text{ basis}\}$, $\{B - IA(B) : B \text{ basis}\}$, and $\{B \cup EA(B) - IA(B) : B \text{ basis}\}$, partially ordered by containment.

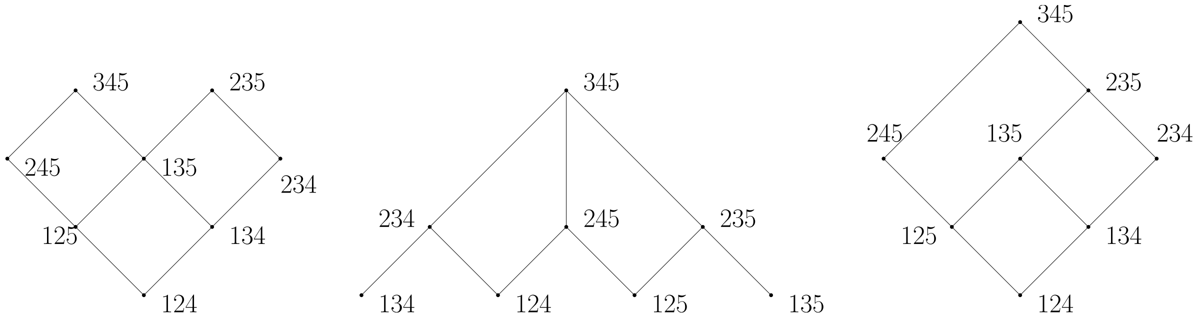


Figure 4.2: The active orders $<_{int}, <_{ext}$, and $<_{ext/int}$, respectively.

Table 4.1 lists the bases in lexicographic order $<_{lex}$, and this is a shelling order for the independence complex $IN(M)$ by Theorem 4.2.13. The restriction set for each basis $B$ is $\mathcal{R}(B) = IP(B)$. For example, when we add facet 134 in the third step of the shelling, this means that the new faces that appear are the four sets in the interval $[\mathcal{R}(134), 134] = [3, 134]$; that is, faces $3, 13, 34$, and $134$.

Our goal is to shell the external activity complex $\mathrm{Act}_<(M)$ whose facets, listed in Table 4.2, are the sets $F(B) = B \cup EP(B) \cup \overline{B \cup EA(B)}$. Since $\overline{1}, 3, 4$, and $5$ are in all facets of $\mathrm{Act}_<(M)$, we remove them, and shell the resulting *reduced external activity complex* $Act^{\bullet}_<(M)$. Our main result, Theorem 4.1.2, states that any linear extension of the external/internal order $<_{ext/int}$ gives a shelling order for this complex. For example, we may again consider the lexicographic order, which is indeed a linear extension of $<_{ext/int}$.

| $B$ | $F(B)$ | $F(B)^{\bullet}$ | $\mathcal{R}(F(B))$ |
|---|---|---|---|
| 124 | $12345\overline{124}$ | $12\overline{24}$ | $\varnothing$ |
| 125 | $12345\overline{125}$ | $12\overline{25}$ | $\overline{5}$ |
| 134 | $12345\overline{134}$ | $12\overline{34}$ | $\overline{3}$ |
| 135 | $12345\overline{135}$ | $12\overline{35}$ | $\overline{35}$ |
| 234 | $2345\overline{1234}$ | $2\overline{234}$ | $\overline{23}$ |
| 235 | $2345\overline{1235}$ | $2\overline{235}$ | $\overline{235}$ |
| 245 | $2345\overline{1245}$ | $2\overline{245}$ | $\overline{45}$ |
| 345 | $345\overline{12345}$ | $\overline{2345}$ | $\overline{345}$ |

Table 4.2: The bases $B$ of $M$, the corresponding facets $F(B)$ and $F(B)^{\bullet}$ of $\mathrm{Act}_<(M)$ and $\mathrm{Act}^{\bullet}_<(M)$, and their (shared) restriction set $\mathcal{R}(F(B))$ in the shelling.

For each basis $B$, Table 4.2 lists the corresponding facet $F(B)$ of $\mathrm{Act}_<(M)$, the corresponding facet $F(B)^{\bullet}$ of $\mathrm{Act}^{\bullet}_<(M)$, and the restriction set of the facet $F(B)$ in the shelling. This restriction set is $\mathcal{R}(F(B)) = \overline{IP(B)}$. For example, when we add facet $12\overline{34}$ to the complex $\mathrm{Act}^{\bullet}_<(M)$ in the third step of the shelling, the new faces that appear are the eight sets in the interval $[\mathcal{R}(12\overline{34}), 12\overline{34}] = [\overline{3}, 12\overline{34}]$.

Notice that we can embed $IN(M) \longrightarrow \mathrm{Act}^{\bullet}_<(M)$ by sending $1 \to 1, 2 \to \overline{2}, 3 \to \overline{3}, 4 \to$

$\overline{4}, 5 \rightarrow \overline{5}$. The latter complex has the same $h$-vector and is contractible. Therefore, it is no coincidence that the shellings of $IN(M)$ and $Act_<(M)$ are related. In fact, we will prove that any shelling order for $Act_<(M)$ is a shelling order for $IN(M)$. Theorem 4.1.2 then gives:

$$\text{any linear extension of } <_{ext/int} \text{ is a shelling order for } IN(M) \text{ and } Act_<(M). \tag{4.1}$$

We conclude this section with two examples showing that the linear extensions of the internal and external orders $<_{int}$ and of $<_{ext}$ are not necessarily shelling orders for $Act_<(M)$.

**Example 4.3.1.** *Consider any linear extension of* $<_{ext}$ *starting with* 124 *and* 135 *in that order, such as:*

$$124, 135, 125, 134, 234, 235, 245, 345.$$

*This is not a shelling order for* $IN(M)$ *because the second facet* 135 *intersects the first facet* 124 *in codimension 2. By Corollary 4.4.3 (or directly by inspection), this is not a shelling order for* $Act_<(M)$ *either. Therefore:*

$$\textit{a linear extension of } <_{ext} \textit{ need not be a shelling order for } IN(M) \textit{ or for } Act_<(M). \tag{4.2}$$

**Example 4.3.2.** *Consider the following linear extension of* $<_{int}$*:*

$$124, 125, 134, 135, 245, 345, 234, 235$$

*which gives the following order on the facets:*

$$12\overline{24},\ 12\overline{25},\ 12\overline{34},\ 12\overline{35},\ 2\overline{245},\ \overline{2345},\ 2\overline{234},\ 2\overline{235},$$

*This is a shelling of* $IN(M)$ *by Theorem 4.1.3. However, it is not a shelling of* $Act_<(M)$ *and* $Act^\bullet_<(M)$*. To see this, suppose we introduce the facets of* $Act^\bullet_<(M)$ *in the order above. When we introduce the sixth facet* $\overline{2345}$

*we introduce two new minimal faces:* $\overline{23}$ *and* $\overline{345}$*; so this is not a shelling order for*

*$Act_<(M)$. Hence*

$$\text{a linear extension of } <_{int} \text{ is a shelling order for } IN(M), \text{ but not necessarily for } Act_<(M). \tag{4.3}$$

In summary, combining (4.1), (4.2), and (4.3), we see that the hypotheses of Theorems 4.1.2 and 4.1.3 are as strong as possible in the context of LasVergnas's active orders.

## 4.4 Shellability of the external activity complex

In this section we prove our main result, which states that the external activity complex is shellable. We begin by proving two technical lemmas.

**Lemma 4.4.1.** *Let $M$ be a matroid on an ordered ground set, and let $A, C$ be bases of $M$. There exist $c \in EP(A) \cap C$ and $a < c$ such that $C - c \cup a$ is a basis if and only if $A \ngeq_{ext/int} C$ in LasVergnas's external/internal order.*

*Proof.* Given $c \in C$, we can find an element $a < c$ with $C - c \cup a \in \mathcal{B}$ if and only if $c \in IP(C)$. To find such an element $c$ with the additional condition that $c \in EP(A)$, we need $IP(C) \cap EP(A) \neq \varnothing$; this is equivalent to $A \ngeq_{ext/int} C$ in LasVergnas's external/internal order by Theorem 4.2.8.2. □

A total order $<$ on the set $\mathcal{B}$ of bases of $M$ induces an order on the set of facets $\{F(B) : B \in \mathcal{B}\}$ of the external activity complex $\mathrm{Act}_<(M)$. We now characterize the shelling orders on $\mathrm{Act}_<(M)$.

**Lemma 4.4.2.** *Let $\mathcal{B}$ be the set of bases of a matroid $M$. A total order $<$ on the set $\mathcal{B}$ induces a shelling of the external activity complex $Act_<(M)$ if and only if for any bases $A < C$ there exists a basis $B < C$ such that*

*(a) $B = X \cup b$ and $C = X \cup c$ for some $b \neq c$.*

*(b) $c \notin A$ and $c \in EA(B)$ if and only if $c \in EA(A)$.*

*(c) For any* $d \notin B \cup C = X \cup b \cup c$ *we have* $d \in EA(B)$ *if and only if* $d \in EA(C)$

*Proof.* By definition, $<$ induces a shelling order if for every $A < C$ there exist $B < C$ and $c^{\pm} \in F(C)$ (where $c^{\pm}$ equals $c$ or $\overline{c}$ for some $c \in E$) such that

$$F(A) \cap F(C) \subset F(B) \cap F(C) = F(C) - c^{\pm}.$$

Recalling that $G(D) = EA(D) \cup \overline{EP(D)}$ is the complement of $F(D)$ in $[[E]]$ for each basis $D$, this is equivalent to

$$G(A) \cup G(C) \supset G(B) \cup G(C) = G(C) \cup c^{\pm}.$$

Define the support of $S \subset [[E]]$ to be $\mathrm{supp}(S) = \{i \in E \ : \ i \in S \ \text{ or } \ \overline{i} \in S\}$. Notice that we have $\mathrm{supp}(G(D)) = E - D$ for any basis $D$. Then

$$|E| - |B \cap C| = |\mathrm{supp}(G(B) \cup G(C))| = |\mathrm{supp}(G(C) \cup c^{\pm})| = |E| - r + 1.$$

where $r$ is the rank of the matroid. This implies (a).

If (c) was not satisfied for some $d \notin B \cup C$, we would find both $d$ and $\overline{d}$ in $G(B) \cup G(C) = G(C) \cup c$, a contradiction. Finally, $c^{\pm}$ is in $G(A)$ and $G(B)$, which implies (b).

The converse follows by a very similar argument. ☐

**Corollary 4.4.3.** *If a total order* $<$ *on* $\mathcal{B}$ *induces a shelling of the external activity complex* $Act_{<}(M)$*, then it also induces a shelling of the independence complex* $IN(M)$*.*

*Proof.* Let $A < C$ and assume that $B < C$ satisfy conditions (a), (b), and (c) of Lemma 4.4.2. Since $\mathrm{supp}(G(D)) = E - D$ for every basis $D$, the containment $G(A) \cup G(C) \supset G(B) \cup G(C)$ gives $E - (A \cap C) \supset E - (B \cap C)$, which implies $A \cap C \subset B \cap C = X = C - c$. Hence the total order $<$ induces a shelling order of $IN(M)$. ☐

Now we are ready to prove our main theorem.

**Theorem 4.4.4.** *Let $M = (E, \mathcal{B})$ be a matroid, and let $<$ be a linear order on the ground set $E$. Any linear extension of LasVergnas's external/internal order $<_{ext/int}$ of $\mathcal{B}$ induces a shelling of the external activity complex $Act_<(M)$.*

*Proof.* We use the characterization of Lemma 4.4.2. Consider bases $A < C$; we will find the desired basis in two steps. We construct a basis $B$ and, if necessary, a second basis $B'$, and we will show that one of them satisfies the conditions (a),(b),(c) of Lemma 4.4.2.

**Step 1.** Since $A \not\geq_{ext/int} C$, we first use Lemma 4.4.1 to find $c \in EP(A) \cap C$ and a minimal element $b < c$ such that

$$B = X \cup b$$

is a basis, where $X = C - c$. The minimality of $b$ implies that $b$ is minimum in $\mathrm{Cocirc}(B, b)$, so $b \in IA(B)$. Therefore $B \backslash IA(B) \subseteq X \subseteq C$. Theorem 4.2.7 then implies that $B <_{int} C$, which in turn gives $B <_{ext/int} C$, and hence $B < C$.

Property (a) is clearly satisfied. By construction $c \notin A$ and $c \in EP(A)$. Since $b < c$ is in $\mathrm{Circ}(B, c)$, we have $c \in EP(B)$. Therefore (b) is also satisfied. Property (c) does not always hold; let us analyze how it can fail, and adjust $B$ accordingly if necessary.

Suppose (c) fails for an element $d \notin B \cup C$; call such an element a $\{B, C\}$ *external disagreement*. This means that $d$ is minimum in one of the fundamental circuits $\beta = \mathrm{Circ}(B, d)$ and $\gamma = \mathrm{Circ}(C, d)$ but not in the other one.

Since they have different minima, we have $\beta \neq \gamma$; so using circuit elimination, we can find a circuit $\alpha \subseteq \beta \cup \gamma - d$. This circuit must contain $b$ and $c$, or else it would be contained in basis $B$ or $C$. This implies that

$$b, c \in \alpha, \qquad b, d \in \beta, \qquad c, d \in \gamma.$$

It follows that $D = X \cup d = (B \cup d) - b$ is a basis. By the uniqueness of fundamental circuits, we must have

$$\alpha = \mathrm{Circ}(B, c) = \mathrm{Circ}(C, b), \quad \beta = \mathrm{Circ}(B, d) = \mathrm{Circ}(D, b), \quad \gamma = \mathrm{Circ}(C, d) = \mathrm{Circ}(D, c).$$

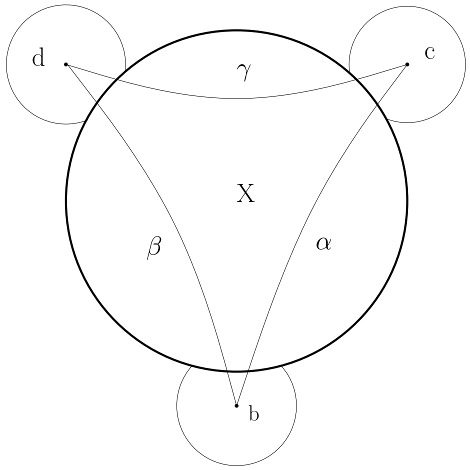


Figure 4.3: The bases $B = X \cup b$, $C = X \cup c$, and $D = X \cup d$ and the fundamental circuits $\beta, \gamma, \alpha$.

Taking into account that $b < c$, we consider three cases:

- **1.** $b < c < d$**:** Since $b \in \text{Circ}(B, d) = \beta$ and $c \in \text{Circ}(C, d) = \gamma$, $d$ is minimum in neither $\beta$ nor $\gamma$, a contradiction.

- **2.** $d < b < c$**:** The minimality of $b$ implies that $X \cup d = D$ is not a basis, a contradiction.

- **3.** $b < d < c$**:** Since $d$ is not minimum in $\text{Circ}(B, d) = \beta \ni b$, we have $d \in EP(B)$; so $d$ is a $\{B, C\}$ external disagreement if and only if $d \in EA(C)$.

We conclude that, under the above hypotheses,

$$d \text{ is a } \{B, C\} \text{ external disagreement} \iff X \cup d \text{ is a basis, } b < d < c, \text{ and } d \in EA(C). \tag{4.4}$$

If there are no $\{B, C\}$ external disagreements, $B$ is our desired basis. Otherwise, proceed as follows.

**Step 2.** Define the basis

$$B' = X \cup b'$$

where $b'$ is the largest $\{B, C\}$ external disagreement. We have $b < b' < c$ and $b' \in EA(C)$. It follows that $B' \subset C \cup EA(C)$, so $B' <_{ext} C$ by Theorem 4.2.6. This implies that $B' <_{ext/int} C$, which in turn gives $B' < C$. Now we claim that $B'$ satisfies conditions (a),(b),(c) of Lemma 4.4.2.

Property (a) is clearly satisfied. By construction $c \notin A$ and $c \in EP(A)$. Since $b' < c$ is in $\mathrm{Circ}(B', c)$, we have $c \in EP(B')$, so (b) holds. To show (c), assume contrariwise that $d' \notin X \cup b' \cup c$ is a $\{B', C\}$ external disagreement; that is, it is minimum in one of the fundamental circuits $\beta' = \mathrm{Circ}(B', d')$ and $\gamma' = \mathrm{Circ}(C, d')$ but not in the other.

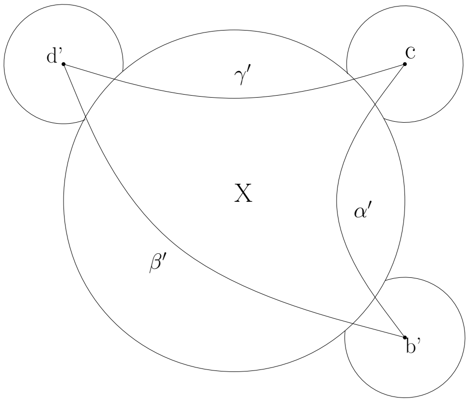


Figure 4.4: The bases $B' = X \cup b'$, $C = X \cup c$, and $D' = X \cup d'$ and the fundamental circuits $\beta', \gamma', \alpha'$.

As in Step 1, $D' = X \cup d'$ must be a basis, and we have circuits

$$\alpha' = \mathrm{Circ}(B', c) = \mathrm{Circ}(C, b'), \quad \beta' = \mathrm{Circ}(B', d') = \mathrm{Circ}(D', b'), \quad \gamma' = \mathrm{Circ}(C, d') = \mathrm{Circ}(D', c).$$

with

$$b', c \in \alpha', \qquad b', d' \in \beta', \qquad c, d' \in \gamma'.$$

Once again, in view of $b' < c$, we consider three cases:

- **Case 1** $b' < c < d'$**:** Since $b' \in \mathrm{Circ}(B, d') = \beta'$ and $c \in \mathrm{Circ}(C, d') = \gamma$, $d'$ is minimum in neither $\beta$ nor $\gamma$, a contradiction.

- **Case 2** $d' < b' < c$**:** If $d' \in EA(B')$ then $d' = \min \beta'$. Since $b' \in EA(C)$, we have $b' = \min \alpha'$. Because they have different minima, we have $\beta' \neq \alpha'$, so we can use circuit elimination to find a circuit $\gamma'' \subseteq (\alpha' \cup \beta') - b'$. Again, that circuit must contain $c$ and $d'$ or else it would be contained in $C$ or $D'$. Therefore, by the uniqueness of fundamental circuits, $\gamma'' = \gamma'$. Now, since $\gamma' \subseteq (\alpha' \cup \beta') - b'$ and $d' \in \gamma'$, we have $d' = \min \gamma'$ and $d' \in EA(C)$.

  Similarly, if $d' \in EA(C)$ then $d' = \min \gamma'$. Since $b' \in EA(C)$, we have $b' = \min \alpha'$. As above, we can conlcude that $\beta' \subseteq (\alpha' \cup \gamma') - c$ and $d' \in \beta'$, we have $d' = \min \beta'$ and $d' \in EA(B')$.

  In either case, we get a contradiction.

- **Case 3** $b' < d' < c$ Since $d'$ is not minimum in $\beta' = \mathrm{Circ}(B, d') \ni b'$, if $d'$ is a $\{B', C\}$ external disagreement, it must be minimum in $\gamma = \mathrm{Circ}(C, d')$; that is, $d' \in EA(C)$. We have $b < b' < d' < c$, and $X \cup d'$ is a basis. Therefore, recalling (4.4), $d'$ is also a $\{B, C\}$ external disagreement, contradicting the maximality of $b'$.

In conclusion, there are no $\{B', C\}$ external disagreements, and property (c) holds. Therefore the basis $B'$ has all required properties. $\square$

**Corollary 4.4.5.** *Any linear extension of the external/internal order* $<_{ext/int}$ *gives a shelling order for the independence complex* $IN(M)$.

*Proof.* This follows from Theorem 4.1.2 and Corollary 4.4.3. $\square$

In fact, we now prove a stronger result. We begin with a useful lemma.

**Lemma 4.4.6.** *Let* $I$ *be an independent set of* $M$ *and let* $C$ *be any basis that contains* $I$*. If* $B$ *is the lexicographically smallest basis that contains* $I$ *then* $B \leqslant_{int} C$.

*Proof.* By Theorem 4.2.7.4, we need to show that $B$ is the lexicographically smallest basis that contains $B \cap C$. To do so, assume there is a basis $A \supseteq B \cap C$ with $A <_{lex} B$ Then $A \supseteq B \cap C \supseteq I$, contradicting the minimality of $B$. $\square$

**Theorem 4.4.7.** *intshell Any linear extension of the internal order $<_{int}$ gives a shelling order of the independence complex $IN(M)$.*

*Proof.* Let $<$ be any linear extension of $<_{int}$, and let $A < C$ be bases, so $A \nsucc_{int} C$. We claim that there exists $B <_{int} C$ (and hence $B < C$) such that $A \cap C \subseteq B \cap C = C - c$ for some $c$ in $C$. This will prove the desired result.

To show this, let $D$ be the lexicographically smallest basis that contains $A \cap C$. Notice that $D \neq C$ because $A \nsucc_{int} C$, using Theorem 4.2.7.4. Let $d$ be smallest element in $D - C$ and let $c$ be any element of $C - D$ such that $C' = C - c \cup d$ is a basis. Also notice that $D <_{int} C$ by Lemma 4.4.6; and since $<_{lex}$ is a linear extension of $<_{int}$, we have $D <_{lex} C$. This gives $d = \min(D - C) < \min(C - D) \leqslant c$, and therefore $C' <_{lex} C$.

Put $X = C - c$ and let $B$ be the lexicographically smallest basis that contains $X$. Since $C'$ contains $X$, $B \leqslant_{lex} C' <_{lex} C$, so $B \neq C$. Therefore $B <_{int} C$ by Lemma 4.4.6. Also note that, since $c \notin D \supset A \cap C$ and $c \in C$, we must have $c \notin A$. This gives $A \cap C \subseteq C - c = X$, and therefore $A \cap C \subseteq B \cap C = X$. It follows that $B$ satisfies the desired properties. ☐

### 4.5 The $h$-vector

We now describe the restriction sets for the shellings of Theorem 4.1.2.

**Proposition 4.5.1.** *Let $<$ be any linear extension of $<_{ext/int}$, and regard it as a shelling order for $IN(M)$. Then the restriction set of each facet $C$ (which is a basis of $M$) is $IP(C)$.*

*Proof.* We need to show $IP(C)$ is the minimum subset of $C$ which is not a subset of a basis $B < C$.

To show that $IP(C)$ indeed has this property, assume that if $IP(C) \subseteq B$. Then by Theorem 4.2.7.2, we have $C \leqslant_{int} B$ and hence $C \leqslant B$, as desired.

To show minimality, let $U \subsetneq IP(C)$. By Theorem 4.2.4.2 we can find a basis $A$ such that $A - IA(A) \subseteq U \subseteq A$. This gives $A - IA(A) \subseteq U \subsetneq C - IA(C)$, which in light of Theorem 4.2.7.3 gives $A <_{int} C$, and hence $A < C$. Therefore $U$ is a subset of $A$ with $A < C$, as desired. ☐

**Proposition 4.5.2.** *Let $<$ be any linear extension of $<_{ext/int}$, and regard it as a shelling order for $Act_<(M)$. Then the restriction set of each facet $F(C)$ (where $C$ is a basis of $M$) is $\overline{IP(C)}$.*

*Proof.* We need to show $\overline{IP(C)}$ is the minimum subset of $F(C)$ which is not a subset of $F(B)$ for any basis $B < C$.

To show $\overline{IP(C)}$ does have this property, assume that $\overline{IP(C)} \subseteq F(B) = B \cup EP(B) \cup \overline{B \cup EA(B)}$ for some basis $B$. Then $IP(C) \subset B \cup EA(B)$, so $IP(C) \cap EP(B) = \varnothing$. By Theorem 4.2.8.2, $C <_{ext/int} B$ so $C < B$, as desired.

To show minimality, let $\overline{U} \subsetneq \overline{IP(C)}$, so $U \subsetneq IP(C)$. By Proposition 4.5.1, $U$ is contained in a basis $A < C$, and hence $\overline{U}$ is contained in $F(A)$ for that basis, as desired.

☐

As an immediate consequence, we obtain our main enumerative result.

**Theorem 4.5.3.** *hvector The h-vector of $Act_<(M)$ equals the h-vector of $M$.*

*Proof.* This follows from the previous two results, in light of Proposition 4.2.11.

☐

## *4.6 Topology*

The external activity complex $\mathrm{Act}_<(M)$ is a cone; for example, it is easy to see that every facet contains $\overline{\min E}$ and $\max E$. Therefore $\mathrm{Act}_<(M)$ is trivially contractible. It is more interesting to study the topology of the *reduced external activity complex* $\mathrm{Act}^\bullet_<(M)$, obtained by removing all cone points of $\mathrm{Act}_<(M)$. It turns out that Corollary 4.1.4 gives us enough information to describe it. First we need a few technical lemmas.

**Definition 4.6.1.** *Define a* loop *of a simplicial complex $\Delta$ to be an element $l$ of the ground set such that $\{l\}$ is not a face of $\Delta$.*

**Definition 4.6.2.** *An element $e$ of a matroid $M$ is* absolutely externally active *if it is externally active with respect to every basis not containing it, or* absolutely externally passive *if it is externally passive with respect to every basis not containing it.*

Let $AEA(M)$ and $AEP(M)$ be the respective sets of elements, and call the elements of $AE(M) = AEA(M) \cup AEP(M)$ *externally absolute.*

**Lemma 4.6.3.** *The set of cone points of $Act_<(M)$ is $AEP(M) \cup \overline{AEA(M)}$. The ground set of $Act^{\bullet}_<(M)$ is $\{e : e \notin AEP(M)\} \cup \{\overline{e} : e \notin AEA(M)\}$, and this simplicial complex has no loops.*

*Proof.* The first two statements are clear from the definitions. For the last one, if $e \notin AEP(M)$, then we can find a basis $B$ with respect to which $e$ is externally active, so $\{\overline{e}\} \subset F(B)$ is a face of $\mathrm{Act}^{\bullet}_<(M)$. Similarly, if $e \notin AEA(M)$, then we can find a basis $B$ with respect to which $e$ is externally passive, so $\{e\} \subset F(B)$ is a face of $\mathrm{Act}^{\bullet}_<(M)$. □

**Lemma 4.6.4.** *Let $M = (E, \mathcal{B})$ be a matroid. Every element $e \in E$ is externally absolute if and only if the circuits of $M$ are pairwise disjoint.*

*Proof.* The backward direction is a straightforward consequence of the definitions. To prove the forward direction, we proceed by contradiction. Assume that every element of $M$ is externally absolute, and that we have two circuits $\gamma_1$ and $\gamma_2$ with $\gamma_1 \cap \gamma_2 \neq \varnothing$ whose minimal elements are $c_1$ and $c_2$, respectively. Consider two cases.

1. If $c_1 = c_2$ then perform circuit elimination to get $\gamma_3 \subset \gamma_1 \cup \gamma_2 - c_1$. Let $c_3$ be the minimal element of $\gamma_3$; without loss of generality assume $c_3 \in \gamma_1$. Then $c_3$ is externally active for some basis, as testified by $\gamma_3$, and it is externally passive for another basis, as testified by $\gamma_1$. Hence $c_3$ is not absolute, a contradiction

2. If $c_1 \neq c_2$ and $c \in \gamma_1 \cap \gamma_2$, then perform circuit elimination with $c$ to get a circuit $\gamma_3 \subset \gamma_1 \cup \gamma_2 - c$. Let $c_3$ be the minimal element of $\gamma_3$; assume $c_3 \in \gamma_1$. If $c_3 = c_1$, then case 1 applies to circuits $\gamma_1$ and $\gamma_3$, and we get a contradiction. Otherwise, we must have $c_1 < c_3$ since $c_1 = \min \gamma_1$. Therefore $c_3$ is externally active for some basis, as testified by $\gamma_3$, and externally passive for another basis, as testified by $\gamma_1$, a contradiction. □

**Proposition 4.6.5.** *If a matroid is the disjoint union of circuits, then* $\mathrm{Act}^{\bullet}_{<}(M) \cong IN(M)$*. Otherwise,* $\mathrm{Act}^{\bullet}_{<}(M)$ *has a proper subcomplex which is isomorphic to* $IN(M)$*. The embedding may be chosen so that the image of facet* $B$ *of* $IN(M)$ *is a subset of the facet* $F(B)$ *of* $\mathrm{Act}^{\bullet}_{<}(M)$*.*

*Proof.* For every $e \in E$ let $e' = e$ if $e$ is absolutely externally active, and $e' = \overline{e}$ otherwise. The set $E' = \{e' : e \in E\}$ is a subset of the vertices of $\mathrm{Act}^{\bullet}_{<}(M)$ by Lemma 4.6.3. For every basis $B$ of $M$ the set $B' = \{b' : b \in B\}$ is a subset of $F(B)$, and hence a face of $\mathrm{Act}^{\bullet}_{<}(M)$. This gives the desired embedding of $IN(M)$ in $\mathrm{Act}^{\bullet}_{<}(M)$.

If $M$ is the disjoint union of circuits, then $E'$ equals the ground set of $\mathrm{Act}^{\bullet}_{<}(M)$, and $B'$ equals $F(B) \cap E'$ for all bases $B$, so this embedding is actually an isomorphism.

If $M$ is not the disjoint union of circuits, by Lemma 4.6.3, $E'$ is a proper subset of the ground set of $\mathrm{Act}^{\bullet}_{<}(M)$, so the embedding of $IN(M)$ is a proper subcomplex of $\mathrm{Act}^{\bullet}_{<}(M)$. □

**Lemma 4.6.6.** *If a matroid* $M$ *of rank* $r$ *is the disjoint union of circuits, then the independence complex* $IN(M)$ *is homeomorphic to an* $(r-1)$*-sphere.*

*Proof.* If $M$ is a single circuit (necessarily of size $r+1$), then $IN(M)$ is the boundary of an $r$-simplex, and hence an $(r-1)$-sphere.

If $M$ is the disjoint union of circuits $\gamma_1, \ldots, \gamma_k$ then $IN(M)$ is the join of $IN(\gamma_1), \ldots, IN(\gamma_k)$; that is, $IN(M) = IN(\gamma_1) \star \cdots \star IN(\gamma_k) = \{A_1 \cup \cdots \cup A_k : A_i \in IN(\gamma_i) \text{ for } 1 \leqslant i \leqslant k\}$. The result then follows from the fact that the join of two spheres $\mathbb{S}^k$ and $\mathbb{S}^l$ is homeomorphic to the sphere $\mathbb{S}^{k+l+1}$. [61, Chapter 2.2.2] □

proposition

The matroids with pairwise disjoint cycles have a nice characterization in terms of excluded minors.

**Lemma 4.6.7.** *A matroid* $M$ *contains two circuits with non empty intersection if and only if* $U_{1,3}$ *is a minor of* $M$*.*

*Proof.* First suppose that $M$ contains two intersecting circuits $\gamma$ and $\delta$ which intersect at $e$. Let $c \in \gamma - \delta$ and $d \in \delta - \gamma$. Restricting to $\gamma \cup \delta$ and then contracting every element except for $c, d$, and $e$, we obtain $U_{1,3}$ as a minor.

To show the converse consider any matroid $N$ and an element $e \in E$. Notice that every circuit of $N \backslash e$ is a circuit of $N$; and if $\gamma$ is a circuit of $N$, then either $\gamma$ or $\gamma \cup e$ is a circuit of $N$. It follows that if either $N \backslash e$ or $N/e$ have two overlapping circuits, so does $N$. Since $U_{1,3}$ has two overlapping circuits, so does every matroid containing it as a minor.

□

Now we are ready to prove our main topological result.

**Theorem 4.6.8.** *topo Let $M$ be a matroid and $<$ be a linear order on its ground set. The reduced external activity complex $Act^{\bullet}_{<}(M)$ is contractible if $M$ contains $U_{1,3}$ as a minor, and a sphere otherwise.*

*Proof.* Notice that if $M$ has a coloop $c$, then both $c$ and $\overline{c}$ are cone points of $\mathrm{Act}_{<}(M)$, and are invisible in $\mathrm{Act}^{\bullet}_{<}(M)$. Therefore we may assume that $M$ is coloop free.

Let $r$ be the rank of $M$, and let $d = \dim(\mathrm{Act}^{\bullet}_{<}(M)) = \dim(Act_{<}(M)) - |AE(M)| = n + r - 1 - |AE(M)|$. We consider two cases.

1. If $M$ is not the disjoint union of circuits, $|AE(M)| < n$ by Lemma 4.6.4, so $d > r - 1$. Clearly $h_d(\mathrm{Act}^{\bullet}_{<}(M)) = h_d(Act_{<}(M))$, Theorem 4.1.4 gives $h_d(Act_{<}(M)) = h_d(IN(M))$, and since $IN(M)$ is $(r-1)$-dimensional, $h_d(IN(M)) = 0$. Therefore, by Theorem 4.2.12, $\mathrm{Act}^{\bullet}_{<}(M)$ is contractible.

2. If $M$ is the disjoint union of circuits, then $\mathrm{Act}^{\bullet}_{<}(M) \cong IN(M)$ is a sphere invoking Proposition 4.6.5 and Lemma 4.6.6.

The result follows from Lemma 4.6.7. □

We conclude that the simplicial complex $\mathrm{Act}^{\bullet}_{<}(M)$ is a model for a matroid $M$ which is topologically simpler than the "usual" model $IN(M)$.

## *4.7 Questions*

- There should be "affine" analogs of the results of this paper. Geometrically, they should correspond to taking the closure of an affine subspace $L$ of $\mathbb{A}^n$ in $(\mathbb{P}^1)^n$, as opposed to a linear subspace, as explained in [7]. To a *morphism of matroids* $M \to M'$, one may associate an external activity complex $\mathrm{Act}_<(M \to M')$ [7] and active orders $<_{int}, <_{ext}, <_{ext/int}$ [63]. The analogous foundational results, such as Theorems 4.2.4, 4.2.6, 4.2.7, 4.2.8 hold there as well. [5, 63] Do our main theorems hold in that more general setting?

- Even though $\mathrm{Act}_<(M)$ only pays attention to the external activities of the bases of $M$, it is the external/internal order $<_{ext/int}$ which plays a crucial role in its shelling. This makes the following question from [7] even more natural: is $\mathrm{Act}_<(M)$ part of a larger (and well-behaved) simplicial complex which simultaneously involves the internal and external activities of the bases of M? Ideally we would like it to come from a natural geometric construction.

- Notice that for an ordered matroid $M$, every linear extension of the poset of restriction sets of the lexicographic shelling order of $IN(M)$ gives another shelling order with the same restriction sets. That means that every possible order of the facets that could give a shelling with the same restriction sets gives another shelling of $IN(M)$. Does this property say something more about the independence complex. Is there a wide class of examples of a shellable complex with a fix shelling order, such that every linear extension of the poset is again a shelling. Notice that 4.3.2 is an example that $Act_<(M)$ with the associated lexicographic shelling does not have this property.

# Chapter 5

# QUASI-MATROIDAL CLASSES OF ORDERED SIMPLICIAL COMPLEXES

## *5.1 Introduction*

The term cryptomorphism is an informal mathematical notion that was invented by Birkhoff [15] in order to capture the phenomenon that a class of objects can be described in several different ways that are not trivially equivalent. Matroids, as an abstract apparatus to study the notion of independence in mathematics, can be defined by a wide variety of axioms that are equivalent, yet have various distinct flavors. Classical matroid cryptomorphisms include, among others, the independence, circuit, basis exchange, submodularity, flat exchange, and closure axioms. Each such set of axioms provides a natural way to study matroids. Furthermore, there are many theorems in matroid theory that seem to be deeply connected to specific axioms: they are quite easy to prove from one point of view and quite hard from another one. For an introduction to the theory of matroids and many existing cryptomorphisms the reader is referred to the books of Oxley [80], Welsh [97] and the book chapters by Björner [17] and Ardila [6].

Various other cryptomorphisms of matroids have appeared over the years and have turned out to also be useful for many other purposes. Interesting examples come from the theory of simplicial complexes via purity of induced subcomplexes, commutative algebra via the Cohen-Macualayness of the Stanley-Reisner ring of the independence complex and all of its induced subcomplexes (see [91]), the theory of polytopes via the matroid basis polytope (see [40]) and optimization via the greedy algorithms working for varying weights (see [17]).

Many theorems about matroids appear to have an axiom or a natural set of axioms attached to them in the sense that those axioms play the key role in proving the desired property. For example, the fact that matroids are shellable follows naturally from the exchange axiom, and the theory of internal activities follows from the shellability property. On the other hand, the pure subcomplexes cryptomorphism seems to be a natural consequence of the independence axiom. Also, the behavior of nbc complexes and external activity theories appear to be governed by the circuit axiom. Following this heuristic line of thought, the behavior of the Tutte polynomial would have to be captured by the exchange axiom and the circuit axiom, as it has a natural interpretation in terms of internal and external activities.

Two particularly interesting cryptomorphisms come from the theory of ordered matroids. In particular, Björner [17] proved that a simplicial complex is the independence complex of a matroid if and only if, for every ordering of the vertex set, the induced lexicographic order on the facets is a shelling order. Another outstanding characterization, due to Gale [39], is the minimality property in the coordinatewise order, now called the Gale ordering. A family of $d$-element subsets of a fixed set $E$ is the set of bases of a matroid if and only if for every order of $E$ the minimal lexicographic facet is componentwise minimal, that is, if $b_1 < \cdots < b_r$ are the elements of the smallest basis in the lexicographic order and $b'_1 < \cdots < b'_r$ are the elements of any other basis, then $b_i \leqslant b'_i$ for all $i$.

The reason for the last two characterizations to be of a particular interest is the following: they both give a property of ordered simplicial complexes that has to be satisfied for *all* possible orderings of the groundset. It is standard in matroid theory, just as in linear algebra when one has a collection of vectors, to endow the groundset of the independence complex with a total order. For instance, the widely studied nbc complex (see for example [22]) of a matroid is an object that can only be constructed once an order for the groundset of the matroid is fixed. In fact, different orders of the groundset may give many non-isomorphic nbc complexes. Another example comes from the theory of the Tutte polynomial (see for [96, 28]), a bivariate polynomial with integer coefficients that can be associated to every matroid. The Tutte polynomial encodes all invariants of matroids that satisfy a linear deletion-contraction recurrence. It is known that the coefficients of the Tutte polynomial are non-negative integers, but a combinatorial interpretation of the coefficients of the polynomial is only known once an order for the groundset is fixed.

Additional motivation to study orderings of the groundset more carefully comes from the theory of shifted complexes. They form a remarkable class of simplicial complexes that became popular because of their simple, yet elegant and useful structure. Shifted complexes appear in the proof of the Kruskal-Katona theorem on face enumeration of simplicial complexes (see [62, 53]) and in Kalai's algebraic shifting theory [52] which does the same enumeration while keeping track of more refined invariants of the original simplicial complex. The definition of shifted simplicial complexes, i.e., ordered complexes in which big vertices can be replaced by small vertices without leaving the complex, appears to have the same flavor to that of matroid theory via the exchange axiom. However, the two classes of complexes are quite different: the former relies on a specific order of the groundset and contains many complexes that are not matroids, while most matroids on a fixed groundset are not shifted for any choice of ordering.

The similarities between the two classes are, however, quite striking. For example, assuming purity in the shifted class, both classes admit quite natural shelling orders (once matroids are ordered) and the combinatorial invariants read from both shelling orders behave quite similarly. Furthermore, both classes admit a very flexible theory of restrictions and contractions, both are closed under a certain type of duality, and in both cases the corresponding Gale orderings have a minimum. In addition, the intersection of both classes of complexes is remarkable: ordered complexes that are simultaneously shifted and matroid independence complexes are sometimes called Schubert matroids; they correspond to the matroids associated to generic points in Schubert strata of (framed) Grassmannian manifolds.

An even more remarkable and mysterious similarity comes from the theory of combinatorial Laplacians as introduced by Eckmann [35] and Friedman [37]. For a simplicial complex $\Delta$, let $(C_\bullet(\Delta), \partial)$ be the simplicial chain complex of $\Delta$ over $\mathbb{R}$ and let $(C_\bullet(\Delta), \delta)$ be the dual complex obtained by using the natural face basis in each degree of the chain complex. For every integer $k$, the operator $D_k := \delta\partial + \partial\delta$, called the *Laplacian* of $\Delta$, is a self-adjoint operator on $C_k(\Delta)$ that has non-negative real eigenvalues. It is then desirable to relate the spectral theory of $D_k$ to the combinatorial structure of $\Delta$ just as in spectral graph theory: graphs can be viewed as one-dimensional simplicial complexes and the classical spectral theory is a special case of this one.

It was shown in a series of papers ([60, 31, 34, 59, 32]) that the eigenvalues of the Laplacians of both matroid independence complexes and shifted simplicial complexes are integer numbers. Furthermore, the eigenvalues can be put into a bivariate generating function, called the spectral polynomial, that satisfies a special kind of recurrence similar to the deletion-contraction recurrence for matroids, except that it has an error correction term coming from relative topology. This is a very rare property: the Laplacian operators of most simplicial complexes on a fixed vertex set do not have integral spectra. This leads naturally to the following question that has been repetedly asked.

**Question 5.1.1** ([82, 33, 34]). *Is there a class of simplicial complexes that contains matroid independence complexes and shifted simplicial complexes, and explains the integral Laplacian phenomenon?*

Yet another reason for a more detailed study of ordered complexes comes from the theory of $f$-vectors of matroids. The $f$-vector $(f_0, \ldots, f_d)$ of a rank-$d$ simplicial complex $\Delta$ enumerates faces of each rank, i.e the entry $f_i$ counts the number of independent sets (or faces of the independence complex) of rank $i$. It is natural to ask for a characterization of the possible $f$-vectors of matroids. This question has been answered entirely and quite succesfully for other classes of simplicial complexes: for instance the class of all simplicial complexes [62, 53], the class of Cohen-Macaulay simplicial complexes [89], and the class of simplicial polytopes [14, 90]. The $h$-vector of a matroid is an invertible transformation of the $f$-vector that is sometimes more convenient. Thus an equivalent question is that of classifying the possible $h$-vectors of matroids. The advantage here is that the $h$-vector theory of a matroids has a combinatorial realization provided by the lexicographic shelling order of the bases of the matroid, after fixing one ordering of the groundset (see Björner [17] for details).

Even though the family of $h$-vectors of matroids is believed to be quite wild and hopeless to fully classify, there are several restrictions the possible values such a vector can take. An astonishing result of Adiprasito, Huh, and Katz [2], that builds on previous work of Huh and Katz [45, 47], proves that the $f$-vector of the nbc complex of a matroid is log concave, thus resolving a long standing conjecture due to Heron, Rota and Welsh. This imposes strong restrictions on the family of $f$-vectors of matroids.

One of the most intriguing questions in matroid theory concerns the $h$-vector of the independence complex of a matroid. Given an ordered rank-$d$ matroid, the lexicographic order on the bases is a shelling of the independence complex and the same holds for nbc bases. It follows that the independence complex and the nbc complex of a matroid are Cohen-Macaulay, thus the corresponding $h$-vectors are $O$-sequences. In other words, there is a family of monomials $\mathcal{O}$ closed under divisibility with exactly $h_i$ monomials of degree $i$ for every $i = 1, \ldots, d$. The known general constructions for $\mathcal{O}$ are not combinatorial and the stucture of $\mathcal{O}$ has little to do with the structure of the matroid. It is easy to find several $O$-sequences that are not $h$-vectors of matroids, thus one might ask if there are other conditions that $h$-vectors of matroids have to satisfy. In 1977 Stanley posited the following conjecture on $h$-vectors of matroids.

**Conjecture 5.1.2** ([89])**.** *The $h$-vector of a matroid independence complex is a* **pure** *$O$-sequence.*

Being a pure $O$-sequence simply means that there exists a multicomplex $\mathcal{O}$ that realizes the $h$-vector of the independence complex of the matroid with the additional property that all maximal monomials of $\mathcal{O}$ with respect to divisibility have the same degree. Stanley's conjecture has received a lot of attention in the last few decades and is known to hold for various special classes of matroids [73, 87, 78, 74, 30, 43, 27, 77, 54, 29]. See [54] or Dall [29] for details about the current status of the conjecture.

More is known about matroid $h$-vectors. Hibi [44] found a set of inequalities satisfied by pure $O$-sequences and Chari [25] provided a topological decomposition of the independence complex a matroid that implies Hibi's inequalities for the $h$-vector. Furthermore, Swartz [93] provided an algebraic version of these inequalities in the artinian reduction of the Stanley-Reisner ring of the independence complex. Juhnke-Kubitzke and Van Dinh [49] proved such inequalities for $h$-vectors of nbc complexes of representable matroids building on work of Huh [46].

In [54] Klee and the author conjectured a more refined version of Stanley's conjecture that predicts the existence of a multicomplex $\mathcal{O}$ whose combinatorial structure is related to the combinatorial structure of the underlying matroid. The idea is to use the shelling order: each monomial of $\mathcal{O}$ corresponds to a basis of the matroid and depends on the restriction set of the basis. There are two main obstructions to such an approach. The first one is that constructing a matroid by using the shelling order yields intermediate complexes that do not come from matroids. The second one is that the purity cannot be expected to hold during the whole process, which means that a substitution for purity is required in the inductive setting.

The main goal of this paper is to discuss three quasi-matroidal classes of complexes, i.e., classes of ordered simplicial complexes that contain all ordered matroids and all pure shifted complexes and such that a fixed simplicial complex that belongs to the quasi-matroidal class in question for every order of its vertex set is necessarily the independence complex of a matroid. Examples of quasi-matroidal classes are implicitly known in the literature. For instance, the class of ordered complexes with the property that the lexicographic order of the facets is a shelling order is an example. The class of ordered complexes for which the Gale ordering has a minimum is another example. Various new quasi-matroidal classes will be described in this paper. Three of these classes are deeply related to three classical cryptomorphisms: the independence, exchange, and circuit axioms. The three classes are pairwise different, enjoy some interesting properties of matroids and elucidate similarities between matroids and shifted complexes.

The following list summarizes our results on quasi-matroidal classes:

- Each class carries a piece of matroid theory, and thus effectively provides a way to classify some matroid properties according to the classical matroid properties that need to be extended to achieve analogous results.

- The independence quasi-matroidal class implies that many of induced subcomplexes are pure and provides a formal dependence relation between the independence axiom and the purity of induced subcomplexes.
- The exchange quasi-matroidal class turns out to be a subclass of the complexes that are shellable in lexicographic order. It provides a meaningful internal activity theory and is closed under what we call Gale truncations, which makes it a suitable class of complexes to do induction on the number of facets.
- The circuit quasi-matroidal class gives a good theory of fundamental circuits and a well behaved nbc complex theory.
- The intesection of the exchange and circuit quasi-matroidal classes imply that the nbc complex is pure shellable and that there is a well behaved Tutte polynomial whose coefficients can be interpreted combinatorially in terms of internal and external activities.
- The complexes belonging to the independence and exchange quasi-matroidal classes that also satisfy another technical restriction admit a reformulation of the conjecture of Klee and the author [54]. This conjecture turns the purity part of Stanley's conjecture into a poset theoretic restriction, which is suitable for induction on the number of bases (or facets).
- The new conjecture is satisfied by Gale truncations of matroids of rank up to four and Gale truncations of the internally perfect matroids defined by Dall [29]. Furthermore, in order to verify the conjecture for complexes of rank $d$, it suffices to verify it for complexes with no more than $2d-1$ vertices.
- It is also shown that the new conjecture is satisfied by pure shifted complexes. This provides a new proof of Stanley's conjecture for Schubert matroids.

The paper is organized as follows. Section 2 provides some background and definitions. Section 3 introduces the notion of a quasi-matroidal classes and studies some basic properties. Section 4 is devoted to the independence, exchange, and circuit quasi-matroidal classes and discusses some basic properties of each resulting class of complexes. Section 5 deals with the first facet property, a condition on the local structure of the ordered simplicial complexes that provides a lot of flexibility to play with combinatorial operations that preserve some quasi-matroidal classes. Section 6 discusses complexes in the intersection of the exchange and circuit quasi-matroidal classes and develops a theory of Tutte polynomials and nbc complexes. Section 7 gives the relaxation of the conjecture of Klee and the author for complexes with the first facet property that belong to the independence and exchange quasi-matroidal classes. It ends with a proof of the new conjecture for shifted complexes. Section 8 contains open problems, brief descriptions of future research projects and various comments about connections to the existing literature.

## *5.2 Preliminaries*

An *ordered simplicial complex* is a pair $\Psi = (E, \Delta)$ where $E$ is a totally ordered finite set and $\Delta \subseteq 2^E$ is a simplicial complex, that is, if $A \in \Delta$ and $B \subset A$, then $B \in \Delta$. Matroid terminology will be used throughout the paper. Elements of $\Delta$ are called *independent sets.* Maximal under inclusion independent sets are called *bases.* The set of bases is denoted by $\mathcal{B}$. A complex $\Psi$ is called *pure* if all the bases have the same cardinality. The smallest lexicographic basis is denoted by $B_0$. Minimal elements not in $\Delta$ are called *circuits.* The set of circuits is denoted by $\mathcal{C}$. The *rank* of an independent set is equal to its cardinality and the *rank* of a subset $A$ of $E$ is the maximum rank of an independent set contained in $A$. Abusing notation, define the rank of $\Psi$ to be the rank of $E$. The rank of $\Psi$ is usually denoted by $d$. Two ordered complexes $\Psi = (E, \Delta)$ and $\Psi' = (E', \Delta')$ are said to be isomorphic if $|E| = |E'|$ and the unique ordered bijection of $E$ and $E'$ induces a bijection between $\Delta$ and $\Delta'$.

A *loop* of $\Psi$ is an element of $E$ that is not in any basis. A *vertex* is an element of $E$ that is not a loop. The set of vertices of $\Psi$ is denoted by $V(\Psi)$. A *coloop* is an element of $\Psi$ that belongs to every basis. For $A \subset E$, define the *restriction* $\Psi|_A$ to be the pair $(A, \Delta|_A)$, where $\Delta_A = \{I \in \Delta : I \subseteq A\}$. The *deletion* $\Psi \backslash \{e\}$ of an element $e$ that is not a coloop is the restriction to $E \backslash \{e\}$. The *contraction* $\Psi / \{e\}$ of an element $e$ that is not a loop is the complex

$(E\backslash\{e\}, \mathrm{Link}_\Delta(e))$, where $\mathrm{Link}_\Delta(e) = \{I \in \Delta \,|\, e \notin I \text{ and } I \cup \{e\} \in \Delta\}$. The *contraction* $\Psi/I$ of an independent set $I$ is the complex that results from contracting the vertices of $I$ in any order. For an independent set $I$, let $B_{I,0}$ be the smallest lexicographic basis of $\Psi/I$. The complex $(\{e\}, \{\varnothing\})$ is denoted by $\Psi_{\text{loop}}$ and the complex $(\{e\}, \{\varnothing, \{e\}\})$ is denoted by $\Psi_{\text{coloop}}$.

Given two ordered complexes $\Psi = (E, \Delta)$ and $\Psi' = (E', \Delta')$, a *shuffle* $s(E, E')$ of $E$ and $E'$ is an ordered set with order-preserving inclusions $j : E \to X$ and $j' : E' \to X$, such that $j(E) \cap j'(E') = \varnothing$ and $j(E) \cup j(E') = s(E, E')$. Given a shuffle $s(E, E')$ of $E$ and $E'$, the join $\Psi *_{s(E,E')} \Psi' := (s(E, E'), \Delta * \Delta')$ is the complex whose independent sets are of the form $j(I) \cup j'(I')$ for some $I \in \Delta$ and $I' \in \Delta'$. If the ranks of $\Psi$ and $\Psi'$ are equal, the connected sum $\Psi \# \Psi'$ is the complex obtained by identifying $B_0$ and $B'_0$ via the unique order-preserving bijection. The rank-$k$ skeleton $\Psi^{(k)}$ of $\Psi$ is the the complex $(E, \Delta^{(k)})$ whose independent sets are the independent sets of $\Delta$ of rank at most $k$.

Let $B$ be a basis. An element $e \in E\backslash B$ is called *externally active* with respect to $B$ if there is a circuit $C \subset B \cup \{e\}$ such that $e$ is the smallest element of $C$. An element $e \in E\backslash B$ is *externally passive* if it is not externally active. The sets of externally active and passive elements of $B$ are denoted by $EA(B)$ and $EP(B)$ respectively. An element $b \in B$ is called *internally active* if $B$ is the smallest basis in lexicographic order that contains $B\backslash\{b\}$. Equivalently, there is no $b' < b$ that is not in $B$ and such that $(B\backslash\{b\}) \cup \{b'\} \in \mathcal{B}$. An element $b \in B$ is *internally passive* if it is not internally active. The sets of internally active and passive elements of $B$ are denoted by $IA(B)$ and $IP(B)$ respectively.

A *broken circuit* of $\Psi$ is a subset $D$ of $E$ that is of the form $C - \{c\}$, where $C$ is a circuit and $c$ is the smallest element of $c$. The *nbc complex* $\mathrm{nbc}(\Psi)$ of $\Psi$ is the complex $(E, \Gamma)$ whose bases are the bases of $\Psi$ that do not contain a broken circuit.

The *Gale ordering* of a pure ordered complex $\Psi$ is the poset $\mathrm{Gale}(\mathcal{B}, <_{\text{Gale}})$ defined by the following relation: $B <_{\text{Gale}} B'$ if and only if the elements of $B = \{b_1 < \cdots < b_d\}$ and $B' = \{b'_1 < b'_2 < \cdots < b'_d\}$ satisfy $b_i \leqslant b'_i$ for every $i$. Let $\mathcal{J}$ be an order ideal of $\mathrm{Gale}(\mathcal{B}, <_{\text{Gale}})$. The *Gale truncation* at $\mathcal{J}$ is the complex $\Psi[\mathcal{J}] := (E, \Delta[\mathcal{J}])$ whose bases are the elements of $\mathcal{J}$.

The *internal poset* $\mathrm{Int}(\Psi) = (\mathcal{B}, <_{\mathrm{int}})$ of $\Psi$ is defined by the relation $B_1 \leqslant_{\mathrm{int}} B_2$ whenever $IP(B_1) \subseteq IP(B_2)$.

An ordered complex $\Psi$ is an *ordered matroid* if and only if either of the following three equivalent axioms is satisfied:

i. **Independence axiom:** If $I_1$ and $I_2$ are independent sets such that $|I_1| > |I_2|$ then there is an element $i \in I_1 \backslash I_2$ such that $I_2 \cup \{i\} \in \Delta$.

ii. **Exchange axiom:** If $B_1$ and $B_2$ are bases and $b_1$ is an element in $B_1 \backslash B_2$, then there is $b_2 \in B_2 \backslash B_1$ such that $B_1 \backslash \{b_1\} \cup \{b_2\}$ is a basis.

iii. **Circuit axiom:** If $C_1$ and $C_2$ are circuits and $c \in C_1 \cap C_2$, then there is a circuit $C_3$ contained in $(C_1 \cup C_2) \backslash c$.

A simple consequence of matroid duality (e.g see [39]) is that $\mathrm{Gale}(\Psi)$ has a minimum and a maximum whenever $\Psi$ is an ordered matroid. Furthermore, Gale showed that this is a property that in fact characterizes matroids.

**Theorem 5.2.1.** [17] *A simplicial complex $\Delta$ is the independence complex of a matroid if and only if the the Gale poset of $\Psi = (E, \Delta)$ has a minimum for every ordering $E$ of the vertex set of $\Delta$.*

Notice that the order of $E$ is not used at all in the definition of a matroid. The *ordered uniform matroid* of rank $d$ over an ordered set $E$ is the complex $U_{E,d} = (E, X_d)$ whose bases are all the $d$-subsets of $E$.

An ordered complex $\Psi = (E, \Delta)$ is *shifted* if the following holds: if $B$ is a basis and $i < j \in E$ are such that $i \notin B$ and $j \in B$, then $B \backslash \{i\} \cup \{j\}$ is also a basis. Equivalently, $\Psi$ is a Gale truncation of $U_{E,d}$ for some $d$. The Gale ordering of a shifted complex is isomoprhic to an ordered ideal of Young's lattice of integer partitions.

The *$f$-vector* of a rank-$d$ complex $\Psi$ is the vector $(f_0, f_0, \ldots, f_d)$ where $f_i$ is the number of independent sets of rank $i$. Notice that the empty set is the only independent set of rank

0, thus $f_0 = 1$. The *h-vector* $(h_0, \ldots, h_d)$ of $\Psi$ is given by the following polynomial relation

$$h(\Psi, x) := \sum_{j=0}^{d} h_j x^j = \sum_{j=0}^{d} f_j t^j (1-t)^{d-j}. \tag{5.1}$$

The $h$-vector carries the same information as the $f$-vector and is sometimes more convenient, in particular, when studying simplicial complexes through the lens of commutative algebra.

A *shelling order* of a pure complex $\Psi$ is an order $B_1, \ldots, B_k$ of the bases such that for every $i < j$ there is $k \leqslant j$ and $b \in B_j$ such that $B_i \cap B_j \subseteq B_k \cap B_j = B_j \backslash \{b\}$. The complex $\Psi$ is said to be *shellable* if it admits a shelling order. The following property holds for every shelling order $B_1, \ldots,$ $B_k$ of a complex $\Psi$: For every $j = 1, \ldots, k$ there is a unique minimal subset $\mathcal{R}(B_j)$ of $B_j$ such that for every $i < j$ the set $\mathcal{R}(B_j)$ is not contained in $B_i$. It turns out that

$$h(\Psi, x) = \sum_{j=1}^{k} x^{|\mathcal{R}(B_j)|}. \tag{5.2}$$

It is known that both ordered matroids and pure shifted complexes are shellable. The lexicographic order of the bases is a shelling order. Again, this is another matroid defining property.

**Theorem 5.2.2.** [39] *A simplicial complex $\Delta$ is the independence complex of a matroid if and only if the the Gale poset of $\Psi = (E, \Delta)$ has a minimum for every ordering $E$ of the vertex set of $\Delta$.*

A pure ordered complex $\Psi = (E, \Delta)$ is *vertex decomposable* if and only if either one of the following holds:

- $\Psi$ has exaclty one basis.
- There exists a vertex $e$ of $\Psi$ such that $\Psi \backslash \{e\}$ is vertex decomposable of the same rank of $\Psi$ and $\Psi / \{e\}$ is vertex decomposable.

If $\Psi$ is a vertex decomposable with more than one basis, a vertex $e$ of $\Psi$ that satisfies the second condition of vertex decomposability is called a *shedding vertex.* It is a theorem of Billera and Provan [81] that every vertex decomposable complex is shellable. They furthermore showed that matroids are vertex decomposable: any vertex is a shedding vertex. Pure shifted complexes are also vertex decomposable: the largest vertex is always a shedding vertex.

For two sets $A, B$, let $A \triangle B$ be their symmetric difference, i.e, the set $(A \backslash B) \cup (B \backslash A)$. Whenever a subset of a small set is considered we omit parethenses and commas to simplify notation. For example, the subset $\{2, 4\}$ of $\{1, 2, 3, 4, 5\}$ is denoted by 24.

### *5.3 Quasi-matroidal classes of ordered complexes*

In this section $\mathcal{A}$ denotes a class of ordered complexes. We say that $\mathcal{A}$ is *closed under joins* if for every pair of complexes $\Psi$ and $\Psi'$ in $\mathcal{A}$ and every shuffle $s$ of their groundsets, the join $\Psi *_s \Psi'$ is a complex in $\mathcal{A}$. We say that $\mathcal{A}$ is *closed under deletions* if for every complex $\Psi$, the deletion of the largest element of the groundset yields a complex in $\mathcal{A}$. Finally, we say that $\mathcal{A}$ is *closed under contractions* if for every complex $\Psi$ the contraction of every independent set of $\Psi$ is a complex in $\mathcal{A}$. The following notion encapsulates the central type of objects we will study.

**Definition 5.3.1.** *A class $\mathcal{A}$ of ordered simplicial complexes is called a* quasi-matroidal *if the following conditions are satisfied:*

1. *Every ordered matroid is an object in $\mathcal{A}$.*
2. *If $\Delta$ is a simplicial complex with vertex set $X$ and for every order $E$ of $X$, the pair $(E, \Delta)$ is in $\mathcal{A}$, then $\Delta$ is a matroid independence complex.*
3. *Every pure shifted complex is in $\mathcal{A}$.*
4. *$\mathcal{A}$ is closed under joins, deletions and contractions.*

Theorems 5.2.1 and 5.2.2 provide two different examples of quasi-matroidal classes.

**Example 5.3.2.** *The following two classes are quasi-matroidal:*

1. *The class LEX of all pure ordered complexes closed under joins, deletions and contractions for which the lexicographic order on the bases is a shelling order.*

2. *The class GALE of all pure ordered complexes closed under joins, deletions and contractions, for which the Gale poset has a unique minimal basis.*

A slightly bigger class that contains both of the above classes is the following one:

**Example 5.3.3.** *Let PURE denote the class of all pure ordered complexes such that a complex $\Psi = (E, \Delta)$ is in PURE if and only if one of the following is satisfied:*

- *$\Psi$ has exactly one basis or,*
- *The deletion $\Psi/\{v\}$ is in PURE and has the same rank as $\Psi$ if $v$ denotes the largest non-coloop vertex of $\Psi$ and every contraction of $\Psi$ is in PURE.*

Pure shifted complexes as well as matroids are easily seen to be in PURE. In fact, PURE is a quasi-matroidal class due to the following classical theorem.

**Theorem 5.3.4.** [91] *A simplicial complex $\Delta$ is the independence complex of a matroid if and only if every induced subcomplex is pure.*

PURE explains some of the first pleasant similarities between pure shifted complexes and matroids.

**Theorem 5.3.5.** *Every ordered complex in PURE is vertex decomposable and hence shellable.*

While the classes in the examples above explain various similarities between shifted complexes and matroid independence complexes, they are too big and contain many complexes that are far from shifted complexes or matroids. For instance, all of them contain the complex $\Psi = ([4], \Delta)$ with bases $12, 13, 24$. This complex is the path with three edges, which is the canonical example of a complex whose Laplacian has non-integral spectra. Thus it is desirable to consider smaller quasi-matroidal classes, so that the complexes belonging to such classes share deeper structural properties with matroids and shifted complexes.

In order to do so, we introduce a few basic constructions of quasi-matroidal classes. Given two quasi-matroidal classes $\mathcal{A}$ and $\mathcal{A}'$, the class $\mathcal{A} \cap \mathcal{A}'$ of all complexes contained in both $\mathcal{A}$ and $\mathcal{A}'$ is clearly quasi-matroidal. A class $\mathcal{A}'$ is called a subclass of $\mathcal{A}$ if all the elements of $\mathcal{A}'$ are elements of $\mathcal{A}$.

### *5.4 Three quasi-matroidal classes*

The purpose of this section is to introduce three new quasi-matroidal classes that will be studied throughout the paper.

**Definition 5.4.1.** *Let $\Psi = (E, \Delta)$ be an ordered simplicial complex. The classes QI, QE and QC are defined by the following axioms.*

- **Quasi-independence Axiom (QI):** *$\Psi$ is pure and for every pair of independent sets $I_1, I_2$, if $|I_1| > |I_2|$ and $I_1 \backslash I \subseteq B_{I,0}$ for some $I \subseteq I_1 \cap I_2$, then there exists $e \in I_1 \backslash I_2$ such that $I_2 \cup \{e\}$ is independent.*
- **Quasi-exchange Axiom (QE):** *$\Psi$ is pure and for every pair $B_1$, $B_2$ of bases of $\Delta$, if $b_1 \in B_1 \backslash B_2$ satisfies $b_1 > \max B_2 \backslash B_1$, then there is $b_2 \in B_2 \backslash B_1$ such that $(B_1 \backslash \{b_1\}) \cup \{b_2\} \in \Delta$.*
- **Quasi-circuit Axiom (QC):** *If $C_1$, $C_2$ are distinct circuits $(E, \Delta)$ and $c \in C_1 \cap C_2$ such that $c < \max C_1 \triangle C_2$, then there is a circuit $C_3$ of $(E, \Delta)$ contained in $(C_1 \cup C_2) \backslash \{c\}$.*

The first goal is to show that the classes QI, QE, and QC are quasi-matroidal. It is straightforward to see that ordered matroids belong to the three classes. The second quasi-matroidal axiom is also straightforward: removing the conditions of the order in QI, QE, and QC yields the classic independence, exchange and circuit axioms of matroid theory. On the other hand it is an interesting exercise to show that shifted simplicial complexes satisfy the axioms.

**Theorem 5.4.2.** *If $\Psi$ is a shifted complex, then $\Psi$ belongs to QI, QE and QC.*

*Proof.* To prove that $\Psi$ belongs to QI notice that if $I$ is any independent set disjoint from $B_0$, then $B_{I,0}$ is the initial subset of $B_0$ of size $d - |I|$, where $d$ is the rank of $B_0$. Thus if $I_1$ and $I_2$ are independent sets that satisfy the conditions of QI with a witness $I \subseteq I_1 \cap I_2$. Then $I_1 \backslash I$ is a subset of the first $d - |I|$ elements of $B_0$ and $B_{I_2,0}$ consists of the first $d - |I_2|$ elements of $B_0$. Hence $B_{I_2,0} \subseteq B_{I,0}$. Then $B_{I_2,0} \cap (I_1 \backslash I) \neq \emptyset$. Otherwise, $|B_{I_2,0} \cup (I_1 \backslash I)| =$

$(d - |I_2|) + (|I_1| - |I|) > d - |I| = |B_{I,0}|$ which is a contradiction since both are subsets of $B_{I,0}$.

To prove that $\Psi$ belongs to QE, notice that if $B_1$ and $B_2$ are bases and $b_1 \in B_1 \backslash B_2$ is bigger than any element in $B_2 \backslash B_1$, then shiftedness of $\Psi$ allows to choose any $b_2 \in (B_2 \backslash B_1)$ to replace $b_1$ in $B_1$.

To prove that $\Psi$ belongs QC, let $C_1$ and $C_2$ be circuits of $\Psi$ and let $c \in C_1 \cap C_2$ satisfy the conditions for QC. Assume that $I := (C_1 \cup C_2) \backslash c$ is independent and let $c' = \max C_1 \triangle C_2$. By shiftedness, $I' := (I \backslash \{c'\}) \cup \{c\}$ is independent, but it also contains either $C_1$ or $C_2$, which is a contradiction. $\square$

It is clear that every quasi-matroidal class can be transformed into a matroid cryptomorphism. While matroid cryptomorphisms give rise to the same class, there are various quasi-matroidal classes each of which highlights different aspects of matroid theory. The following theorem shows that the three defined classes are indeed different.

**Theorem 5.4.3.** *The classes QI, QE and QC are all distinct, furthermore, no class is contained in another one.*

*Proof.* For every pair of axioms one has to provide examples of complexes that satisfy one but not the other axiom.

- The complex $\Psi_1 = (\{1, 2, 3, 4\}, \Delta_1)$ with bases $12, 13, 14, 34$ satisfies QI, QE, but the pair of circuits $23, 24$ contradicts QC.
- The complex $\Psi_2 = (\{1, 2, 3, 4\}, \Delta_2)$ with bases $14, 24, 23, 34$ satisfies QI, but the bases 14 and 23 show that it does not satisfy QE, and the circuits 13, 14 show that it does not satisfy QC.
- The complex $\Psi_3 = (\{1, 2, 3, 4\}, \Delta_3)$ with bases $12, 13, 23, 34$ satisfies QC, but fails QI and QE.
- The complex $\Psi_4 = (\{1, 2, 3, 4, 5\}, \Delta_4)$ with bases $13, 14, 23, 24, 25$ satisfies QE, but not QI or QC.

$\square$

Notice that the proof of the theorem shows that QC is not contained in QI$\cap$QE. On the other hand, it is a straightforward exercise in graph theory to check that a rank-two complex that belongs to QE$\cap$QC also belongs to QI. Nevertheless, the proof fails in rank-three and it is natural to ask the following question.

**Question 5.4.4.** *Are the classes QI$\cap$QE, QI$\cap$QC, QE$\cap$QC and QI$\cap$QE$\cap$QC all distinct?*

Regardless of the answer, the definitions provide various classes of simplicial complexes and it should come as no surprise that each of them shares some structural properties with the family of ordered matroids. The goal for the rest of this section is to start developing the first steps of a theory for these classes of complexes that includes some analogs of matroid properties as well as to provide various types of constructions that can be performed within a given class.

### *5.4.1 The quasi-independence class*

We now show that QI is a quasi-matroidal class with a little extra structure.

**Theorem 5.4.5.** *Let $\Psi = (E, \Delta)$ and $\Psi' = (E', \Delta')$ be ordered complexes in QI.*

*i. If $s(E, E')$ is a shuffle of $E$ and $E'$ then the join $\Psi *_{s(E,E')} \Psi'$ is in QI.*

*ii. If $v \in E \backslash B_0$ then the contraction $\Psi/\{v\}$ is in QI.*

*iii. If $v$ is the largest non-coloop vertex of $\Psi$, then $\Psi\backslash\{v\}$ is in QI.*

*iv. If $0 \leqslant k \leqslant rk(\Psi)$ then the skeleton $Skel_k(\Psi)$ is in QI.*

*v. If $rk(\Psi) = rk(\Psi')$ then the connected sum $\Psi\#_{\varphi}\Psi'$ is in QI.*

*Parts [i.], [ii.], and [iii.] imply that QI is a quasi-matroidal subclass of PURE.*

*Proof.* Part [i.] follows from the fact that joins preserve purity and commute with links, i.e., if $\Delta_1$ and $\Delta_2$ are complexes and $I_1, I_2$ are faces of $\Delta_1$ and $\Delta_2$ then $\mathrm{Link}_{\Delta_1 * \Delta_2}(I_1 \cup I_2) = \mathrm{Link}_{\Delta_1}(I_1) * \mathrm{Link}_{\Delta_2}(I_2)$.

For [ii.] notice that if $I_1$ , $I_2$ and $I \subset I_1 \cap I_2$ satisfy the hypotheses of the axiom in $\mathrm{Link}_{\Delta}(v)$, then $I_1 \cup \{v\}, I_2 \cup \{v\}$ and $I \cup \{v\}$ satisfy the hypotheses in $\Psi$. We may therefore use the QI axiom in $\Psi$ to obtain the result.

To prove part [iii.] notice that $I$ is an independent set in $\Psi$ that does not contain $v$ and then $B_{I,0}$ does not contain $v$. Indeed if $v \in B_{I,0}$, let $B = (I \cup B_{I,0})\backslash\{v\}$ and use the QI with $B$ and $B_0$, i.e, there is $u \in B_0\backslash B$ such that $B \cup \{u\}$ is independent. The vertex $u$ is not a coloop of $\Psi$ (it does not belong to the basis $B_{I,0} \cup I$) and is therefore smaller than $v$. It follows that $(B \cup \{u\})\backslash I$ is a basis of $\Psi/I$ smaller in lexicographic order than $B_{I,0}$, a contradiction.

Part [iv.] follows from the equality $\text{Link}_{\text{Skel}_k\Delta} I = \text{Skel}_{k-|I|}(\text{Link}_\Delta I)$ and the fact that the smallest lexicographic facet of $\text{Skel}_k\Psi$ is the smallest lexicographic $k$-face of $B_0$.

For part [v.] assume that $I_1, I_2$, and $I \subseteq I_1 \cap I_2$ satisfy the hypotheses in the connected sum. If $I_1$ and $I_2$ are independent in $\Psi$ and $\Psi'$, respectively, then $I_1 \cap I_2$ is a subset of $B_0 = B'_0$ (under the natural identification), and so $B_{I,0} = B_0\backslash I = B'_0\backslash I$. Hence $I_1 \subseteq B_0 = B'_0$ and we may apply the axiom for $\Psi'$. If both independent sets come from the same $\Psi$ or $\Psi'$, then the QI axiom can be applied as coming from $\Psi$ or $\Psi'$.

□

The following theorem shows that QI refines PURE in a special sense: a larger family of induced subcomplexes are pure for complexes in QI.

**Theorem 5.4.6.** *Let $\Psi = (E, \Delta)$ be a complex in QI and let $A \subseteq E$ be a subset such that rk$(A) = |B_0 \cap A|$. Then $\Psi|_A$ is pure.*

*Proof.* Notice that if $I$ is an independent set in $\Psi|_A$, it is possible to apply the QI axiom with $B_0 \cap A$ and $I$ to extend $I$ to an independent set $B$ of $\Psi|_A$. Then $|B| = |B_0 \cap A| = \text{rk}(A)$. It follows that $B$ is a basis of $\Psi_A$ and this implies purity. □

### *5.4.2 The Exchange axiom*

The theory of shellability of simplicial complexes is best studied in the language of facets and it is natural that it should follow from conditions on the set of bases of a simplicial complex. It turns out that QE is a suitable class to apply this technology.

**Theorem 5.4.7.** *Let $\Psi = (E, \Delta)$ and $\Psi' = (E', \Delta')$ be ordered complexes in QE.*

*i. If $s(E, E')$ is a shuffle of $E$ and $E'$ then the join $\Psi *_{s(E,E')} \Psi'$ is in QE.*

*ii. If $U \subset E$ is any subset such that $\Psi|_U$ is pure and of the same rank as $\Psi$, then the restriction $\Psi|_U$ is in QE.*

*iii. If $A \subseteq$ contains $B_0$ and all the elements smaller than the largest non-coloop of $B_0$, then the restriction $\Psi|_A$ is in QE.*

*iv. If $v$ is a vertex of $\Psi$ then the contraction $\Psi/v$ is in QE.*

*In particular, QE is a quasi-matroidal sublclass of PURE.*

*Proof.* Part [i.] follows from the fact that joins preserve purity, and bases in the join correspond to pairs of bases, coming from each complex. Then the QE axiom applies to either basis for each element that can be switched.

For [ii.] notice that if $B_1$ and $B_2$ are bases of $\Psi|_U$ then applying the QE axiom with $B_1$ and $B_2$ produces bases of $\Psi|_U$.

For [iii.] it suffices to show that $\Psi|_A$ is pure. Let $I$ be a face of $\Psi|_A$ and let $B$ be a basis that contains $F$. By QE applied to $B$ and $B_0$ it is possible to replace all vertices in $B\backslash A$ with elements of $B_0$, since $\min E - A > \max B_0$. The resulting basis is contained in $A$ and it still contains $F$. It follows that $\Psi|_A$ is pure.

For [iv.] notice that if $B_1$ and $B_2$ are bases of $\Psi/v$ then $B_1 \cup \{v\}$ and $B_2 \cup \{v\}$ are bases of $\Psi$ and the QE axiom applies. Notice that in this case $v$ is irrelevant since it belongs to both bases, thus the QE axiom holds in $\Psi/v$. $\square$

The QE axiom is quite rich from the perspective of combinatorial topology in particular, by using shellability. While Theorem 5.3.5 shows that $\Psi$ is shellable, the shelling orders provided by the vertex decomposition are obtained recursively, and consequently, the restriction sets are difficult to study. As the generating function of the restriction is equal to the $h$-polynomial of $\Psi$, it is desirable to have shelling with restriction sets that are easier to compute directly. Complexes in QE guarantee that this holds.

**Theorem 5.4.8.** *Let* $\Psi = (E, \Delta)$ *be a complex in QE. The lexicographic order on* $\mathcal{B}$ *is a shelling order. Under this shelling order* $\mathcal{R}(B) = IP(B)$.

*Proof.* Let $B_1 <_{lex} B_2$ be bases of $\Psi$. The goal is to find a basis $B_3 <_{lex} B_2$ such that $B_1 \cap B_3 \subseteq B_2 \cap B_3 = B_2 \backslash \{e\}$. The proof is by induction on $r - |B_1 \cap B_2|$. The base case is trivial. There are two cases to consider:

Case 1. The element $b := \max B_1 \triangle B_2$ is an element of $B_2$. By QE there is $b' \in B_1 \backslash B_2$ such that $B_3 = (B_2 \backslash \{b\}) \cup \{b'\}$ is a basis. That choice of $B_3$ works directly.

Case 2. The element $b := \max B_1 \triangle B_2$ is an element of $B_1$. By the QE axiom there is $b' \in B_2 \backslash B_1$ such that $B_1' = (B_1 \backslash \{b\}) \cup \{b'\}$ is a basis. Then $B_1 \cap B_2 \subset B_1' \cap B_2$ and by inductive hypothesis, there exists $B_3$ such that $B_1' \cap B_2 \subseteq B_2 \cap B_3 = B_2 \backslash \{e\}$, and so such $B_3$ does the job.

The second statement follows directly from the definition of the restriction set exactly as in [17]. ☐

**Remark 5.4.9.** *Theorem 5.4.8 shows that the class QE is contained in LEX and is much smaller in general. Graphs are rank-two simplicial complexes and shellable just means that they are connected. Given any connected graph* $\Delta$ *with with vertex set* $[n]$ *(viewed as a rank-two simplicial complex) such that the graph* $\Delta|_{[r]}$ *is connected for every* $1 \leqslant r \leqslant n$*, we get that the complex* $([n], \Delta)$ *is an element of LEX. However a substantial proportion of such graphs in not in QE. Therefore, QE is significantly smaller than LEX even if we just compare rank-two complexes.*

The following corollary is an immediate consequence of Theorem 5.4.8. It generalizes the result of Björner [17] and, as we will see in Section 5.7, it becomes more powerful when studied in this context.

**Corollary 5.4.10.** *If* $\Psi = (E, \Delta)$ *is in LEX (or in QE), then*

$$h(\Psi, x) = \sum_{B \in \mathcal{B}} x^{|IP(B)|} \tag{5.3}$$

There is another remarkable way to obtain complexes in QE from older ones which is less conventional in combinatorial topology. It implies that the partial steps of the construction of a complex in QE using the shelling of 5.4.8 are complexes in QE.

**Theorem 5.4.11.** *Let $\Psi = (E, \Delta)$ be a complex in QE and let $\mathcal{J}$ be an order ideal of Gale$(\Psi)$. Then $\Psi[\mathcal{J}]$ is in QE. Furthermore, for every basis $B$ of $\Psi[\mathcal{J}]$, the equality $IP(B, \Psi[\mathcal{J}]) = IP(B, \Psi)$ holds.*

*Proof.* The first part of the theorem follows directly from the fact that the lexicographic order is a linear extension of Gale$(\Psi)$ and the fact that an exchange produced by the QE axiom yields a new basis that is smaller in the lexicographic order than the original one.

The equality of passive sets even holds for general ordered complexes since the bases that witness external activity of $B$ are all smaller than $B$ in Gale$(\Psi)$ (they differ from $B$ by one element). $\square$

A useful property of complexes in QE concerns the strucure of Gale$(\Psi)$.

**Lemma 5.4.12.** *Let $\Psi = (E, \Delta)$ be a complex that satisfies QE. Then $B_0$ is the unique minimal basis of Gale$(\Psi)$. In particular, QE is a sublcass of GALE.*

*Proof.* It is straightforward that $B_0$ is minimal, since the lexicographic order is a linear extension of Gale$(\Psi)$. On the other hand, if $B$ is any other basis, apply the QE axiom with $B_0$ to get a basis $B'$. The exchange takes an element from $B$ and replaces it with a smaller vertex, thus $B' <_{\text{Gale}} B$. $\square$

Another useful tool is the internal poset of Las Vergnas. We will see that Int$(\Psi)$ is coarser than Gale$(\Psi)$ which will be handy when we study Stanley's conjecture. The following are some structural results.

**Lemma 5.4.13.** *Let $\Psi$ be a complex in QE and let $B$ be a basis. There exists an element $b \in IP(B)$ and a basis $B'$ such that $IP(B') = IP(B)\backslash\{b\}$. In particular, Int$(\Psi)$ is graded and the degree of a basis is the cardinality of its internally passive subset.*

*Proof.* Since the lexicographic order is a shelling and since the restriction sets that shellings are the the internally passive sets, the first part of the result follows from [17, Lemma 7.2.6]. The fact that the cardinality of the passive set provides a grading of $\mathrm{Int}(\Psi)$ is straightforward from the previous part. □

Finally, we establish the following relationship between the Gale poset and the Int poset for complexes in QE.

**Theorem 5.4.14.** *Let $\Psi$ be a complex in QE and let $B$ and $B'$ be bases such that $IP(B) \subseteq B'$. Then $B <_{Gale} B'$ and hence Gale$(\Psi)$ is a poset extension of Int$(\Psi)$.*

*Proof.* Fix $B$ and proceed by induction on $|B' \backslash B|$. The base case is $B = B'$, in which case there is nothing to show. Assume that the property holds for all bases $B''$ with $IP(B) \subseteq B''$ and $|B'' \backslash B| < |B' \backslash B|$. Next apply the QE axiom with $B$ and $B'$. Let $u = \max B \triangle B'$. Notice that $u \notin IP(B) \subseteq B \cap B'$. Hence if $u \in B$, then by the QE axiom there would be $b' \in B' \backslash B$ such that $B'' = (B \backslash \{u\}) \cup \{b'\}$ is a basis, but in this case $B'' <_{lex} B$ and $IP(B) \subseteq B''$ which is impossible. It follows that $u \in B'$ which in turns implies the existence of an element $b \in B$ such that $B'' := (B' \backslash \{u\}) \cup \{b\}$ is a basis. Then $B'' <_{\text{Gale}} B'$, $IP(B) \subseteq B''$ and $|B'' \backslash B| < |B' \backslash B|$. By the inductive hypothesis $B <_{\text{Gale}} B'' <_{\text{Gale}} B'$ as desired.

If $B$ and $B'$ are bases with $B$ smaller than $B'$ in $\mathrm{Int}(\Psi)$, then $IP(B) \subseteq IP(B') \subseteq B'$, thus $B <_{Gale} B'$, which shows that $\mathrm{Gale}(\Psi)$ is a poset extension of $\mathrm{Int}(\Psi)$. □

### *5.4.3 The Circuit axiom*

The theory of circuits in matroid theory is what allows for meaningful external activity theories to play a prominent role in the understanding of objects such as the broken circuit complex and Orlik-Solomon algebras. It is a dual notion to that of internal activity in matroid theory. Unfortunately, this is not the case in the quasi-matroidal setting and a careful separate study is required when it comes to duality. The first step toward understanding QC is, again, constructing new complexes from old ones.

**Theorem 5.4.15.** *Let $\Psi = (E, \Delta)$ and $\Psi' = (E', \Delta')$ be ordered complexes in QC.*

*i. If $s(E, E')$ is a shuffle of $E$ and $E'$, then the join $\Psi *_{s(E,E')} \Psi'$ is in QC.*

*ii. If $U \subset E$ is any subset, then $\Psi|_U$ is in QC.*

*iii. If $v$ is a vertex of $\Psi$, then contraction $\Psi/v$ is in QC.*

*In particular, QC is a quasi-matroidal class.*

*Proof.* Part [i.] follows easily from the fact that circuits in the join are either circuits in $\Psi$ or in $\Psi'$, thus if they intersect, they come from the same complex.

Part [ii.] is an easy consequence of the fact that the circuits of $\Psi|_U$ are the circuits of $\Psi$ that are contained in $U$.

Part [iii.] follows since the circuits of $\Psi/v$ are in natural bijection with circuits of $\Psi$ that contain $v$. □

**Theorem 5.4.16.** *Let $\Psi = (E, \Delta)$ be a complex in QC. If $B$ is a basis and $e \in E \backslash B$ is externally active, then there is a unique circuit contained in $B \cup \{e\}$.*

*Proof.* Assume there are two such circuits $C_1$, $C_2$. Since $e$ is externally active we may assume that $e = \min C_1$. Notice that $e$ is also in $C_2$, because the circuit cannot be a subset of $B$. Then $c < \max C_1 \cap C_2$ and the QC axiom implies that there is a circuit $C_3 \subseteq (C_1 \cup C_2) \backslash \{e\} \subseteq B$. □

Furthermore, the independent sets of $\mathrm{nbc}(\Psi)$ also have a simple description in terms of broken circuits.

**Lemma 5.4.17.** *Let $\Psi = (E, \Delta)$ be a pure complex in QC. An independent set $I$ of $\Psi$ is independent in nbc($\Psi$) if and only it contains no broken circuit.*

*Proof.* Since the bases of $\mathrm{nbc}(\Psi)$ contain no broken circuit, any independent set in $\mathrm{nbc}(\Psi)$ does not contain a broken circuit either. Thus it suffices to show that if $I \in \Delta$ contains no broken circuit, then $I$ is independent in $\mathrm{nbc}(\Psi)$. Let $B$ be the smallest lexicographic basis of $\Psi$ that contains $I$ and assume that $B$ contains a broken circuit $\tilde{C}$ where $C = \tilde{C} \cup \{c\}$ is the circuit that was broken. There is an element $c' \in \tilde{C} \backslash I$. By Theorem 5.4.16, $C$ is the unique circuit contained in $B \cup \{c\}$. Hence $B \backslash \{c'\} \cup \{c\}$ is a basis that is smaller in the lexicographic order and contains $I$, leading to a contradiction. $\square$

This endows QC with a theory of fundamental circuits analogous to that of matroids. That is, if $B$ is a basis and $e \in EA(B)$ then the fundamental circuit $Ci(B, e)$ is the unique circuit contained in $B \cup \{e\}$. Fundamental circuits play a key role in studying various aspects of the Tutte polynomial, which will be studied in a quasi-matroidal setting in Section 6.

### 5.5 The First Basis Property

The goal of this section is to introduce an important property that holds for matroids and shifted complexes and imposes additional structure on Gale truncations. It is a local condition for ordered complexes. Recall that for a vertex $v$ of an ordered complex, $B_{v,0}$ denotes the smallest lexicographic basis of $\Psi/v$.

**Definition 5.5.1.** *Let $\Psi = (E, \Delta)$ be an ordered complex and let $B_0$ be the first lexicographic basis. We say that $\Psi$ satisfies the First Basis Property (FBP) if either:*

*i. the rank of $\Psi$ is $1$, or*

*ii. $\Psi$ has exactly one basis, or*

*iii. for every vertex $v$ of $\Psi$ that is not in $B_0$, the contraction $\Psi/v$ satisfies FBP and $B_{v,0} \subset B_0$.*

Notice that shifted complexes satisfy the FBP. We now show that matroids satisfy the FBP. For this, we first recall the following result:

**Theorem 5.5.2.** [29, Corollary 2.3] *Let $\Psi = (E, \Delta)$ be an ordered matroid. Then $IA(B) \subseteq B_0$ for every basis $B$.*

This allows us to show the first main result of the section.

**Theorem 5.5.3.** *Ordered matroids satisfy the FBP.*

*Proof.* The proof is a direct induction on the rank of the matroid. The base case is trivial. Let $\Psi$ be a rank-$r$ ordered matroid and let $v$ be a vertex of $\Psi$ not in $B_0$. Then $\Psi/v$ is a matroid of a smaller rank, thus it satisfies the FBP by induction. Now let $B$ be the smallest lexicographic facet that contains $v$. Then $IP(B) \subseteq \{v\}$ and since $IP(B)$ is the restriction set of $B$ with respect to the lexicographic shelling order, it follows that $|IP(B)| > 0$, since $B \neq B_0$. Thus $IP(B) = \{v\}$ and $B \backslash \{v\} = IA(B) \subseteq B_0$ by Theorem 5.5.2. $\square$

As in the previous section the next step is to study how to construct new complexes from old. The proof is straightforward and is ommitted.

**Theorem 5.5.4.** *Let* $\Psi = (E, \Delta)$ *and* $\Psi' = (E', \Delta')$ *be complexes that satisfy the FBP.*

*i. If* $A \subseteq E$ *contains* $B_0$ *and* $\Psi|_A$ *is pure, then the restriction* $\Psi|_A$ *satisfies the FBP.*

*ii. If* $I$ *is an independent set of* $\Psi$*, then the contraction* $\Psi/I$ *satisfies the FBP.*

*iii. If* $s(E, E')$ *is a shuffle of* $E$ *and* $E'$*, then the join* $\Psi *_{s(E,E')} \Psi'$ *satisfies the FBP.*

*iv. If* $\mathcal{J}$ *is an order ideal of Gale*$(\Psi)$*, then the Gale truncation* $\Psi[\mathcal{J}]$ *satisfies the FBP.*

**Remark 5.5.5.** *Notice that Theorems 5.5.3 and 5.5.4 imply that if* $\mathcal{A}$ *is a quasi-matroidal class, then the class* $\mathcal{A}\cap$ *FBP of complexes in* $\mathcal{A}$ *that satisfy FBP is also a quasi-matroidal class.*

We will study the classes QE∩FBP, QI∩FBP and their intersection.

**Lemma 5.5.6.** *Assume that* $\Psi = (E, \Delta)$ *is a complex that satisfies the FBP and let* $I$ *be an independent set such that* $B_0 \cap I = \varnothing$*. Then* $B_{I,0} \subseteq B_0$*.*

*Proof.* If $v \in I$ then $\Psi/I = (\Psi/v)/(I/\{v\})$. Then by induction and the FBP condition $B_{I,0} \subseteq B_{v,0} \subseteq B_0$. $\square$

**Theorem 5.5.7.** *Let* $\Psi = (E, \Delta)$ *be a complex QE∩FBP, and let* $B$ *be a basis of* $\Psi$*. Then* $B \backslash B_0 \subseteq IP(B)$*.*

*Proof.* The proof goes by induction on the rank of $\Psi$. Let $v$ be the maximal vertex in $I := B \backslash B_0$. By QE applied with $B_0$ we obtain that $v \in IP(B)$. From $B_{v,0} \subset B_0$ it follows that $I - v \subseteq (B\backslash\{v\})\backslash B_{v,0}$. Thus for $v' \in I\backslash\{v\}$ and by induction $v' \in IP(B\backslash v, \Psi\backslash v)$. That is, there is $u < v'$ such that $B\backslash\{v, v'\} \cup \{u\}$ is a basis of $\Psi\backslash v$. Therefore $B\backslash\{v'\} \cup \{u\}$ is a facet of $\Psi$, and so $v' \in IP(B)$. ☐

Finally, the following result will provide flexibility in doing inductive arguments on the number of facets of complexes and in exploiting the whole power of the shelling orders.

**Theorem 5.5.8.** *Let $\Psi = (E, \Delta)$ be a complex in QI$\cap$FBP and let $\mathcal{J}$ be an order ideal of Gale$(\Psi)$. Then $\Psi[\mathcal{J}]$ satisfies QI and FBP.*

*Proof.* Since we saw that the FBP is preserved by Gale truncations, it suffices to show that $\Psi[\mathcal{J}]$ is in QI. Let $I_1$ and $I_2$ be independent sets in $\Psi[\mathcal{J}]$ with $|I_1| > |I_2|$ and $I \subseteq I_1 \cap I_2$ such that $I_1\backslash I \subseteq B_{I,0}$. Notice that $B_{I_2,0}$ is a basis of $\Psi[\mathcal{J}]$, because $I_2$ is independent in $\Psi[\mathcal{J}]$. By FPP $B_{I_2,0} \subseteq B_{I,0}$. It follows from the cardinality condition and the pigeon hole principle there is some $e \in (I_1\backslash I_2) \cap B_{I_2,0}$. ☐

**Remark 5.5.9.** *Notice that QI is, in general, not closed under Gale truncations. For example, the complex $\Psi_2$ in the proof of Theorem 5.4.3 fails to satisfy the FBP and it is easily seen that removing the top Gale facet makes the QI axiom fail.*

### *5.6 Tutte polynomials and nbc complexes*

In this section we develop a theory of Tutte polynomials (that is, a universal deletion-contraction invariant) for complexes in QE$\cap$QC. The aim is to get a theory of Tutte-Grothendieck invariants similar to that in matroid theory.

Let $\mathcal{S} :=$ QE$\cap$QC and let $R$ be a ring. An invariant $f$ is a map that associates to every complex $\Psi$ of $\mathcal{S}$ an element $f(\Psi) \in R$ in such a way that if $\Psi \cong \Psi'$, then $f(\Psi) = f(\Psi')$. A *Tutte-Grothendieck invariant*, *TG-invariant* for short, is an invariant that satisfies the following recurrence:

$$f(\Psi) = \begin{cases} f(\Psi|_{\{e\}})f(\Psi|_{E\backslash\{e\}}) & \text{if } e \text{ is a loop or a coloop,} \\ f(\Psi/\{e\}) + f(\Psi\backslash\{e\}) & \text{if } e \text{ is the largest non-coloop vertex of } \Psi. \end{cases} \tag{5.4}$$

Just as in matroid theory are many natural TG invariants for the class $\mathcal{S}$ and there is a TG invariant that rules them all.

**Definition 5.6.1.** *Let* $\Psi$ *be a complex in* $\mathcal{S}$*. The Tutte Polynomial of* $\Psi$ *is defined to be*

$$T(\Psi, x, y) := \sum_{B \in \mathcal{B}} x^{|IA(B)|} y^{|EA(B)|}. \tag{5.5}$$

**Theorem 5.6.2.** *The Tutte Polynomial is a TG invariant that is universal in the following sense: if* $f$ *is a TG-invariant on the class* $\mathcal{S}$*, then for every* $\Psi$ *in* $\mathcal{S}$ *we have that*

$$f(\Psi) = T(\Psi, f(\Psi_{\text{coloop}}), f(\Psi_{\text{loop}})). \tag{5.6}$$

*Proof.* Notice that $T(\Psi_{\text{coloop}}, x, y) = x$ and $T(\Psi_{\text{loop}}, x, y) = y$. Every loop is externally active and every coloop is internally active. Thus the recurrence works for loops and coloops. If $v$ is the largest non-coloop, then the facets are divided into two types.

a. $v \notin B$, that is, $B$ is a basis of $\Psi \backslash v$. Then $v$ is an internally passive element of $B$, and so $IA(B, \Psi) = IA(B, \Psi \backslash v)$. Circuits of $\Psi \backslash v$ are circuits of $\Psi$ that do not contain $v$, thus $EA(B, \Psi) = EA(B, \Psi \backslash v)$.

b. $v \in B$, that is, $B/\{v\}$ is a basis of $\Psi/v$. Then $v$ is internally passive in $B$ by QE with $B_0$ and it is straightforward that $IA(B, \Psi) = IA(B/\{v\}, \Psi/v)$. If $e \in E \backslash (B \backslash \{v\})$, then $v$ is not the smallest element of any circuit in $B \cup \{e\}$: $e$ is in any such circuit and if it is not a coloop then $e < v$. Hence $EA(B, \Psi) = EA(B/\{v\}, \Psi/v)$.

The result follows directly. The fact that evaluations of the Tutte polynomial gives rise to all TG invariants is a straightforward inductive argument. □

Next we provide a wealth of interesting invariants that satisfy such a recursion. The most prominent one comes from the theory of nbc complexes. They turn out to behave pretty similar to matroids. Recall from Corollary 5.4.10 that the $h$-polynomial is given by $h(\Psi, x) = \sum_{i=0}^{r} h_i x^i = \sum_{B \in \mathcal{B}} x^{|IP(B)|}$ for any rank-$r$ complex $\Psi$ in QE. If it also satisfies QC it is possible to compare the $h$-polynomial with the evaluation $T(\Psi, x, 1)$.

**Lemma 5.6.3.** *The $h$-vector is a TG invariant for the class $\mathcal{S}$ the following sense: if $\Psi$ is a complex in $\mathcal{S}$ then*

$$h(\Psi, x) = x^r T(\Psi, x^{-1}, 1)$$

*Proof.* Use the equations $|IP(B)| + |IA(B)| = r$ and $T(\Psi, x, 1) = \sum_{B \in \mathcal{B}} x^{|IA(B)|}$. ☐

This lemma implies that standard objects such as the (reverse) $f$-polynomial, the number of faces and the number of bases are evaluations of the Tutte polynomial. They are, however, not very surprising. A much stronger and interesting evaluation that has a rich combinatorial interpretation comes from the theory of nbc complexes. The characteristic polynomial of $\Psi$ can be defined as the standard evaluation of the Tutte polynomial and one might hope that there is a natural poset that replaces the lattice of flats to get the Möbius function interpretation of the characteristic polynomial. This question is particularly interesting in the class of shifted complex and it is the source of inspiration for the definition of another quasi-matroidal class.

**Theorem 5.6.4.** *Let $\Psi = (E, \Delta)$ be a complex in $\mathcal{S}$. Then the lexicographic order of the bases of nbc($\Psi$) is a shelling order of nbc($\Psi$).*

*Proof.* The first paragraph of Björner's argument in [17, Lemma 7.3.2] is replaced by Theorem 5.4.8. The second paragraph applies verbatim to get the result. ☐

Now we show that the polynomial $f(\Psi, x) = \sum_{B \in \mathcal{B}(\mathrm{nbc}(\Psi))} x^{|IA(B, \mathrm{nbc}(\Psi))|}$ is a TG invariant. It is easy check that $f(\Psi_{\mathrm{coloop}}, x) = x$ and $f(\Psi_{\mathrm{loop}}, x) = 0$. This implies that $f(\Psi, x) = T(\Psi, x, 0)$ just as in the classical case.

**Theorem 5.6.5.** *The polynomial invariant $f$ is a TG invariant. Consequently, if $\Psi$ is a complex in $\mathcal{S}$ and $B$ is an nbc basis of $\Psi$, then $IA(B, \Psi) = IA(B, nbc(\Psi))$.*

*Proof.* If there is a loop $v$, then nbc($\Psi$) is the void complex since $\varnothing$ is a broken circuit; since $f(\Psi_{\mathrm{loop}}) = 0$, the recurrence holds directly. If $v$ is a coloop, then it is externally active for every basis, thus $IA(B, \mathrm{nbc}(\Psi)) = \{v\} \cup IA(B\backslash\{v\}, \mathrm{nbc}(\Psi)|_{E-v})$. Assume then that $\Psi$ has no loops (otherwise $\mathrm{nbc}(\Psi)$ is trivial).

Let $v$ be the largest non-coloop vertex of $\Psi$ and let $B$ be a basis of $\mathrm{nbc}(\Psi)$. There are two cases:

Case 1. $v \notin B$. Then $u \in IA(B, \mathrm{nbc}(\Psi))$ if and only if $B$ is the smallest lexicographic basis that contains $B\backslash\{u\}$. This, however, is equivalent to a statement in $\Psi\backslash v$ since adding $v$ to $B\backslash\{u\}$ yields a basis bigger than $B$ in lexicographic order. Thus $IA(B, \mathrm{nbc}(\Psi)) = IA(B, \mathrm{nbc}(\Psi\backslash v))$.

Case 2. $v \in B$. First we claim that $v \in IP(B, \mathrm{nbc}(\Psi))$. Note that $v \in IP(B, \Psi)$, because it is the largest vertex and we can use the quasi-exchange axiom with $B_0$ and $B$. Let $B'$ be the smallest lexicographic basis of $\Psi$ that contains $B\backslash v$. We claim that $B'$ is a basis of $\mathrm{nbc}(\Psi)$. Note that $v \notin B'$ since applying QE with $B$ and $B_0$ allows us to remove $v$ to get a smaller basis. Let $b \in B'\backslash B$ and assume, for the sake of contradiction, that there is a broken circuit $\gamma = C - c \subseteq B'$. Since $B$ is an nbc basis, $\gamma$ is not contained in $B$, thus $b \in C - c$ and $c < b$. Notice that $c$ is externally active and hence $C$ is the unique circuit in $B' \cup C$ by Theorem 5.4.16. Therefore $(B'\backslash\{b\}) \cup \{c\}$ is a basis that contains $B\backslash\{v\}$ and is smaller in lexicographic order than $B'$, which contradicts the choice of $B'$.

Next we notice that $B\backslash\{v\}$ is a basis of $\mathrm{nbc}(\Psi/v)$. To prove this assume that $u \in EA(B\backslash\{v\}, \Psi/v)$ and let $C$ be the unique circuit of $\Psi/v$ contained in $(B\backslash\{v\}) \cup \{u\}$. Then one out of $C$ and $C \cup \{v\}$ is a circuit in $\Psi$. Note that $u$ is a vertex of $\Psi$ and since it belongs to a circuit, it is not a coloop. It follows that $u$ is the smallest element in $C \cup \{v\}$ and hence in the corresponding circuit $\hat{C}$ that contains it. This circuit $\hat{C}$ is contained $B \cup \{u\}$, and so $u \in EA(B, \Psi)$. This is a contradiction, since $B$ is an nbc basis in $\Psi$.

Furthermore, notice that if $B'$ is a basis of $\mathrm{nbc}(\Psi/\{v\})$, then $B' \cup \{v\}$ is a basis of $\Psi$. If there is some element $u$ in $EA(B' \cup \{v\}, \Psi)$, then there is a circuit $C \subseteq B' \cup \{u, v\}$ for which $u$ is the minimal element and $C \cap B'$ is a circuit in $\Psi/v$ for which $u$ is the minimal element, contradicting the fact that $B'$ is an nbc basis of $\Psi/v$.

The previous two paragraphs imply that $\mathrm{nbc}(\Psi/v) = \mathrm{nbc}(\Psi)/v$. Now notice that $u \in IA(B, \mathrm{nbc}(\Psi))$ if and only if $B$ is the smallest lexicographic basis of $\mathrm{nbc}(\Psi)$ containing $B\backslash\{u\}$, and since $v \in B\backslash\{u\}$ this is equivalent to saying that $B\backslash\{v\}$ is the smallest lexicographic basis of $\mathrm{nbc}(\Psi)/v = \mathrm{nbc}(\Psi/v)$ that contains $B\backslash\{u, v\}$. This, in turn, equivelent to saying that $u$ is an element of $IA(B\backslash\{v\}, \mathrm{nbc}(\Psi/v))$.

Computing the activities polynomial yields:

$$\begin{aligned} f(\Psi, x) &= \sum_{B \in \mathcal{B}(\mathrm{nbc}(\Psi))} x^{|IA(B, \mathrm{nbc}(\Psi))|} \\ &= \sum_{v \in B \in \mathcal{B}(\mathrm{nbc}(\Psi))} x^{|IA(B, \mathrm{nbc}(\Psi))|} + \sum_{v \notin B \in \mathcal{B}(\mathrm{nbc}(\Psi))} x^{|IA(B, \mathrm{nbc}(\Psi))|} \\ &= \sum_{B' \in \mathcal{B}(\mathrm{nbc}(\Psi/v))} x^{|IA(B', \mathrm{nbc}(\Psi/v))|} + \sum_{B'' \in \mathcal{B}(\mathrm{nbc}(\Psi\backslash v))} x^{|IA(B'', \mathrm{nbc}(\Psi\backslash v))|} \\ &= f(\Psi/v, x) + f(\Psi\backslash v, x). \end{aligned}$$

It follows that $f(\Psi, x) = T(\Psi, x, 0)$. Notice that $T(\Psi, x, 0) = \sum_{B \in \mathcal{B}, EA(B) = \varnothing} x^{|IA(B,\Psi)|}$. Also, it is straightforward that $IA(B, \Psi) \subseteq IA(B, \mathrm{nbc}(\Psi))$ for every basis of $\mathrm{nbc}(B)$ and hence, by the polynomial equality, $IA(B, \Psi) = IA(B, \mathrm{nbc}(\Psi))$ □

In standard terms the previous theorem can be rewritten as follows:

**Corollary 5.6.6.** *If $\Psi$ is a rank-$r$ complex in QE∩QC, then*

$$h(nbc(\Psi), x) = x^r T(\Psi, x^{-1}, 0). \tag{5.7}$$

## *5.7 A refinement of Stanley's conjecture*

The purpose of this section is to propose an extension of Stanley's $h$-vector conjecture to the setting complexes in QI∩QE∩FBP. The lexicographic order of the bases of an ordered matroid has a family of well understood restriction sets whose size generating function gives the $h$-vector of the independence complex. Techniques from combinatorial topology suggest that a plausible approach to understanding the properties of such structures is by recursive construction adding one basis at a time. However, removing bases from the matroid, even in the correct order suggested by the shelling, produces complexes that are not matroidal. One of the advantages of working in the more general setting is that this problem disappears. Removing facets consistently with the shelling order preserves the validity of the axioms and gives tools to prove theorems by induction on the number of bases.

For the remainder of this section, unless stated otherwise, $\Psi = (E, \Delta)$ denotes a rank-$r$ complex in QE$\cap$QI$\cap$FBP. The goal is to mimic conjecture 3.10 from [54] and build a finer conjecture in this larger class. In order to do that it is necessary to introduce some extra notation. Fix a subset $A$ of $E$ and consider the splitting of an independent $I$ of $\Delta$ in the two sets $I \cap A$ and $I \backslash A$. This induces a partition of the independent sets according to their part that is not in $A$. For any $F \in \Delta|_{E-A}$, let $\Psi_{A,I} := (\Psi/I)|_A = (A, \Delta_{A,I})$. The set of independents of $\Delta$ is the (disjoint) union of the independents of the $\Delta_{A,I}$ sets and writing this fact in terms of $h$-polynomials we derive the following lemma.

**Lemma 5.7.1.** *Let $\Psi = (E, \Delta)$ be an arbitrary pure ordered complex. For an arbitrary $A \subset E$, we have that*

$$h(\Psi, x) = \sum_{I \in \Delta|_{E\backslash A}} (1-x)^{(d-|I|)-rk\,\Psi_{A,I}} h(\Psi_{A,I}, x). \tag{5.8}$$

*In particular, if $rk\,\Psi_{A,I} = d - |I|$ for every $I \in \Delta|_{E\backslash A}$, then*

$$h(\Psi, x) = \sum_{I \in \Delta|_{E\backslash A}} x^{|I|} h(\Delta_{A,I}, x). \tag{5.9}$$

*Proof.* The following identity for the $f$-polynomial follows from the splitting of the faces:

$$f(\Psi, x) = \sum_{I \in \Delta|_{E\backslash A}} x^{|I|} f(\Psi_{A,I}, x). \tag{5.10}$$

Transforming this into an $h$-vector equality yields the result. □

If the complex $\Psi$ is either in QE or QI, then the refined decomposition of equation (5.9) holds in many cases:

**Corollary 5.7.2.** *The refined decomposition of the $h$-vector holds for $\Psi$ and $A$ in the following cases:*

*i. $\Psi$ is in QE and $A$ contains all coloops and all elements smaller than or equal to the smallest non-coloop of $B_0$.*

*ii. $\Psi$ is in QI and $A$ contains $B_0$.*

*Proof.* For each case it suffices to show that, for any independent set $I$ disjoint from $A$, there is a basis $B$ containing $I$ such that $B \backslash I \subseteq A$: it implies that the restricted contractions have the correct rank.

i. If $B'$ is a basis that contains $I$, then every element of $B$ that is not contained in $A$ or $I$ allows to apply QE with $B'$ and $B_0$ to obtain $B$.

ii. Apply QI with $B_0$ and $I$ to obtain $B$.

$\square$

A subset $A$ that satisfying the conditions of Corollary 5.7.2 is called *admissible*. The following theorem gives a combinatorial interpretation of Corollary 5.7.2.

**Theorem 5.7.3.** *Assume that $\Psi = (E, \Delta)$ is in QE and $A$ is an admissible subset of $E$. If $B$ is a basis with $B \backslash B_0 = I$, then*

$$IP(B) = I \cup IP(B \backslash I, \Psi_{A,I}). \tag{5.11}$$

*Finally, if $\Psi$ is also in QI$\cap$FBP, and if $A$ contains $B_0$, then equation (5.11) holds as well.*

*Proof.* Notice first that in both cases, $I \subseteq IP(B)$: If $\Psi$ satisfies only QE then every element of $I$ can be exchanged using QE with $B$ and $B_0$. If $\Psi$ also satisfies QI and FBP this is simply Theorem 5.5.7.

Thus in both cases it follows that $IA(B, \Psi) = IA(B, \Psi) \backslash A \subseteq IA(B \backslash I, \Psi_{A,I})$: if $b \in IA(B, \Psi)$ then $B$ is the smallest lexicographic basis of $\Psi$ that contains $B \backslash \{b\}$, and so $B \backslash I$ is the smallest lexicographic basis of $\Psi / I$ that contains $\mathcal{B} \backslash (I \cup \{b\})$.

Passing to the passive sets yields that $F \cup IP(B \backslash I, \Psi/I) \subseteq IP(B, \Psi)$. The equality follows by the double computation of the $h$-polynomial of $\Psi$ using Corollaries 5.4.10 and 5.7.2, i.e.,

$$h(\Delta, x) = \sum_{B \in \mathcal{B}} x^{|IP(B)|} = \sum_{I \in \Delta|_{E \backslash A}} x^{|I|} h(\Delta_{A,I}, x). \tag{5.12}$$

Both sums can be regarded as a sum over the bases and the inclusion of passive sets above forces all the equalities. □

Similar results were used in [54] to provide a combinatorial conjecture that implies Stanley's conjecture for matroids. The next goal is to extend the combinatorial conjecture to the class of complexes satisfying QE, QI and FBP. It becomes feasible to do induction on the number of bases by taking order ideals of the Gale posets. Such order ideals preserve QE, QI and FBP, but general order ideals of matroids are not matroids. We expect that this technique will have many applications and hope it could eventually lead to a full resolution of Stanley's conjecture.

The main idea for the stronger conjecture is to use the $h$-vector decomposition and the relations between Int($\Psi$) and Gale($\Psi$). Recall, from Lemma 5.4.13, that Int($\Psi$) is graded by the size of the internally passive sets, hence its rank generating function coincides with the $h$-polynomial of $\Psi$. One way to attack Stanley's conjecture is to construct one monomial of degree $|IP(B)|$ for every basis of an ordered matroid to obtain a suitable multicomplex from a combinatorial rule. A candidate for the poset of such a multicomplex under divisibility could be Int($\Psi$). This, however, does not work in general: Int($\Psi$) fails to have enough relations to be the face poset of a multicomplex. However, one might hope that it is possible to add some extra relations to Int($\Psi$) to obtain the face poset of a multicomplex with the same rank generating function. This is the point where the inductive step should come: by Theorem 5.4.14 the Int poset of any Gale truncation of $\Psi$ is an order ideal of Int($\Psi$).

The following conjecture is a strengthening of Stanley's conjecture about $h$-vectors of matroids. It aims to construct the multicomplex using the very rich combinatorial structure of complexes satisfying QE, QI and FBP.

**Conjecture 5.7.4.** *There exists a map $\mathcal{F}$ from the class of ordered simplicial complexes in QE$\cap$ QI$\cap$FBP to the class of multicomplexes, such that for each $\Psi$ the multicomplex $\mathcal{F}(\Psi)$ satisfies the following:*

*i. The set of variables appearing in $\mathcal{F}(\Psi)$ is $\{x_i : i$ is a vertex of $\Delta$ not contained in $B_0\}$.*

*ii. The monomials of $\mathcal{F}(\Psi)$ are in bijection with bases of $\Psi$, in such a way that:*

  *(a) the monomial associated to the basis $B$ under this bijection is denoted by $m_B$,*

  *(b) the degree of $m_B$ is equal to $|IP(B)|$,*

  *(c) the support of $m_B$, i.e., the set $\{e \in E : x_e | m_B\}$, is equal to $B \backslash B_0$.*

*iii. The poset $\mathcal{M}(\Psi) = (\mathcal{B}, <_m)$, where we write $B <_m B'$ if and only if $m_B | m_{B'}$, is an extension of Int$(\Psi)$ and is extended by Gale$(\Psi)$.*

*iv. If $A \subseteq E$ contains $B_0$, then $\mathcal{F}(\Psi|_A) \subseteq \mathcal{F}(\Psi)$.*

Roughly speaking, the conjecture predicts that there is a natural way to extend the poset Int$(\Psi)$ by adding some relations of Gale$(\Psi)$. We now show that this conjecture in fact implies Stanley's conjecture.

**Theorem 5.7.5.** *Conjecture 5.7.4 implies Stanley's conjecture.*

*Proof.* Assume conjecture 5.7.4 holds and let $\Psi$ be an ordered matroid with multicomplex $\mathcal{F}(\Psi)$. We have to show that $\mathcal{F}(\Psi)$ is pure. $\mathcal{M}(\Psi)$ is an extension of Int$(\Psi)$ that preserves the grading and all the maximal elements of Int$(\Psi)$ have rank $k \leqslant r$ in Int$(\Psi)$. It follows that all maximal elements of $\mathcal{M}(\Psi)$ have degree $k$ which is the requirement for purity. $\square$

One particular feature of this conjecture is that it can be reduced to a statement about finitely many objects in a fixed rank.

**Theorem 5.7.6.** *Assume that the map $\mathcal{F}$ of Conjecture 5.7.4 exists for the class of all rank $r$ ordered complexes QI$\cap$QE$\cap$FBP with $|E| \leqslant 2r - 1$. Then Conjecture 5.7.4 holds for rank $d$ complexes.*

*Proof.* Assume that Conjecture 5.7.4 holds for all complexes $\Psi$ with $|E(\Psi)| \leqslant 2r - 1$ and let $\Psi$ be any ordered complex of rank $r$ satisfying QE$\cap$QI$\cap$FBP. For each basis $B$ disjoint from

$B_0$ let $\mathbf{x}_B := \prod_{b \in B} x_b$ and consider the set of monomials

$$\mathcal{F}(\Psi) := \{x_B | B \in \mathcal{B},\ B \cap B_0 = \varnothing\} \cup \left( \bigcup_{I \in \Delta|_{E-B_0},\, |I| \leqslant d-1} \mathcal{F}(\Psi|_{B_0 \cup I}) \right). \tag{5.13}$$

The multicomplexes in the union are defined since they correspond to complexes whose groundset has at most $2d-1$ elements, so they exist by assumption. Notice that there is one monomial for each basis $B$ of $B \in \mathcal{B}(\Psi)$. If $B_0 \cap B \neq \varnothing$, then $B$ can be found in any restriction $\Psi|_{B_0 \cup I}$ for any $I$ containing $B \backslash B_0$ and $m_B$ comes from $\mathcal{F}(\Psi|_{B_0 \cup I})$. The choice of $I$ is irrelevant: condition *iv.* of the conjecture is satisfied, hence $m_B$ is the same monomial for all such complexes.

Condition *i.* is straightforward to verify. Restrictions preserve internally passive sets, and so conditions *ii.* and *iv.* follow. To prove condition *iii.* define the *weak Gale order* $<_{wG}$ of $\Psi$ to be defined by $B_1 <_{wG} B_2$ if there is an independent $I$ such that $B_1 < B_2$ in the Gale order of $\Psi|_{B_0 \cup I}$. To check property *iii.* it is enough to see that $wGale(\Psi) := (\mathcal{B}, <_{wG})$ is between $\text{Int}(\Psi)$ and $\text{Gale}(\Psi)$. □

It can be checked that Conjecture 5.7.4 refines Conjecture 3.11 in [54]. The main difference is that the one in [54] is formulated for very special kinds of ordered matroids. The restrictions on the orders can be replaced with the FBP. Consequently, the results of [54] show that Conjecture 5.7.4 holds for matroids of rank at most 4 by providing an algorithm that constructs $\mathcal{F}(\Psi)$. The proof is computer aided. Even better, a careful and straightforward analysis of the algorithm shows that the conjecture holds also for Gale truncations of matroids of rank at most 4.

Internally perfect matroids defined by [29] satisfy the conjecture and are a potential candidate for a big class of matroids satisfying Conjecture 5.7.4. In particular, $\text{Int}(\Psi)$ is the poset of divisibility of a pure multicomplex for internally perfect matroids, i.e., for internally perfect matroids the equality $\mathcal{M}(\Psi) = \text{Int}(\Psi)$ holds.

Considerations about Gale trunctations open the doors to do induction on the number of bases of the complex: one can construct the multicomplex for order ideals of Gale and there is at most one monomial that is not achieved by this technique. A good setting to guess how to do this type of construction is the class of pure shifted complexes; recall that these complexes belong to QI∩QI∩FBP and are closed under Gale truncations.

**Theorem 5.7.7.** *Conjecture 5.7.4 holds for shifted complexes.*

*Proof.* Recall that a rank-$d$ complex $\Psi = ([n], \Delta)$ is shifted if and only if Gale($\Psi$) is an order ideal of Young's lattice of partitions contained in a $d \times (n-d)$ box. A basis of $\Psi$ with vertices $v_d > v_{d-1} > \cdots > v_1$ corresponds to the partition $\lambda(B) = \lambda_d \geqslant \lambda_{d-1} \geqslant \cdots \geqslant \lambda_1$, where $\lambda_i = v_i - i$ (note that $\lambda_i$ might be equal to zero for some values of $i$).

The goal is to construct a monomial $m_\lambda$ for each partition $\lambda := \lambda(B)$. The following two properties are crucial in that process.

- $IA(B) = [m]$, where $m$ is the largest number such that $[m] \subseteq B$. This means that $\deg m_\lambda = |IP(B)| = \ell(\lambda)$, where $\ell(\lambda)$ denotes the length of $\lambda$.
- The set of variables of $m_\lambda$ are indexed by the set $B \backslash [d]$. In the partition such variables correspond to rows that intersect the main diagonal of the $d \times (n-d)$ box. It follows that the size of the support of $m_\lambda$ is equal to the side length of the Durfee square of $\lambda$, that is the maximal side length of a square formed by boxes that fits inside the Young diagram of the partition.

Notice that in small cases the construction is straightforward (see Figure 1). Let Dur($\lambda$) denote the partition obtained from $\lambda$ by removing the rows of the Durfee square. Assume that the side length of the Durfee square is $k$ and let $i_1 < \cdots < i_k$ be vertices of $B$ corresponding to the first $k$ rows. Now Dur($\lambda$) is a partition fitting in a $(d-k) \times k$ box. Hence by induction it has a monomial $x_{d-k+1}^{\alpha_1} x_{d-k+2}^{\alpha_2} \ldots x_d^{\alpha_k}$ of degree $\ell(\lambda) - k$ (here some $\alpha_i$ may be zero). Let $m_\lambda := x_{i_1}^{\alpha_1+1} x_{i_2}^{\alpha_2+1} \ldots x_{i_k}^{\alpha_k+1}$.

Let $\mathcal{F}(\Psi) = \{m_{\lambda(B)} \,|\, \text{B is a basis of } \Psi\}$. We claim that $\mathcal{F}(\Psi)$ is the desired multicomplex. To show that this is a multicomplex, it is necessary to show that all the divisors of $m_\lambda$ are in $\mathcal{F}(\Psi)$ if $m_\lambda$ is. It suffices to show that all the divisors $m$ of $m_\lambda$ of degree one less than the degree of $m_\lambda$ are in $\mathcal{F}(\Psi)$. If the supports of $m$ and $m_\lambda$ are equal, then induction works directly passing to $\mathrm{Dur}(\lambda)$. Otherwise let $\tilde{m}$ be the monomial resulting from dividing either $m$ or $m_\lambda$ by each one of its variables once (both choices give the same monomial). Let $e+k-1$ denote the degree of $m$, where $k-1$ is the number of variables of $m$. Our goal is to find $\mu$ such that $m = m_\mu$. The Durfee square of $\mu$ must have side length $k-1$. Let $j_1, \dots, j_{k-1}$ be the variables of $m$ and let $\tilde{m} = x_{j_1}^{\alpha_1} \dots x_{j_{k-1}}^{\alpha_{k-1}}$, with $\alpha_i \geqslant 0$ and $\sum \alpha_i = e$. By induction the monomial $x_{e+1}^{\alpha_1} \dots x_{e+k-1}^{\alpha_{k-1}}$ comes from a diagram $\tilde{\mu} \subseteq e \times (k-1)$. Thus we can construct $\mu$ by putting the first $k-1$ rows of $\mu$ in order to get the right support and putting $\tilde{\mu}$ below. It is straightforward to check that $\tilde{\mu} \subseteq \mathrm{Dur}(\lambda)$: notice that, as diagrams contained in the $e \times k$ box, the monomial associated to $\mu$ is $\tilde{m}$ and the monomial associated to $\mathrm{Dur}(\lambda)$ is $x_{e+1}^{\alpha_1} \dots x_{e+j-1}^{\alpha_{j-1}} x_{e+j+1}^{\alpha_j} \dots x_{e+k}^{\alpha_{k+1}}$ for some $1 \leqslant j \leqslant k-1$, thus $\mathrm{Dur}(\mu)$ is obtained from $\tilde{\mu}$ by adding a box to the first $k-1-j$ rows. In this construction, it is also straightforward that $IP(B_\mu) \subseteq IP(B_\lambda)$ by direct computation and the previous observation.

Thus conditions *i.*, *ii.* and *iii.* of Conjecture 7.4 are satisfied. Condition *iv.* is straighforward.

There is a way to visualize $m_\lambda$ in the combinatorial structure of the Young diagram. The construction will be called the *bouncing light construction*. Put mirrors on the vertical boundaries of the Young diagram of $\lambda$. The left-hand side mirrors reflect lines parallel to the $x$-axis in the direction of the diagonal and the right-hand side mirrors reflect lines coming in the direction of the diagonal to lines parallel to the $x$-axis. Put a light on the right-hand side of each row of the Durfee square and shoot the light parallel to the $x$-axis. For each $i_j$ let $\beta_j$ be the number of times that the light bounces off the left wall. Then a straightforward induction shows that $m_\lambda = x_{i_1}^{\beta_1} x_{i_2}^{\beta_2} \dots x_{i_k}^{\beta_k}$. The only thing we need to prove is that mirrors of all boxes are reached. This can be done by induction on $\ell(\lambda)$ by passing from $\lambda$ to $\mathrm{Dur}(\lambda)$. $\square$

We end this section with two examples that illustrate the constructions.

**Example 5.7.8.** *Consider the order ideal of the Young lattice fitting into a* $3 \times 3$ *box presented in Figure 5.2. Depicted in the figure are the Young diagrams. To the left of the diagram are listed the vertices of the corresponding facet with the variable vertices highlighted in red.*

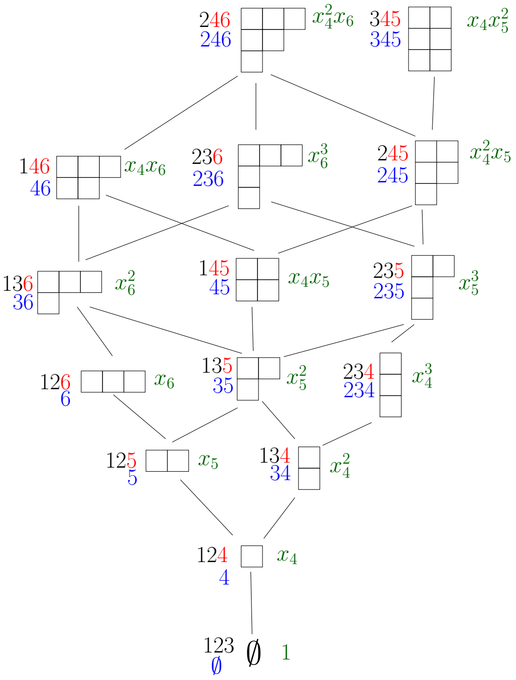


Figure 5.1: An order ideal of the Young lattice

*Below the facets are the internally passive sets written in blue. To the right of the diagram is the corresponding monomial written in green.*

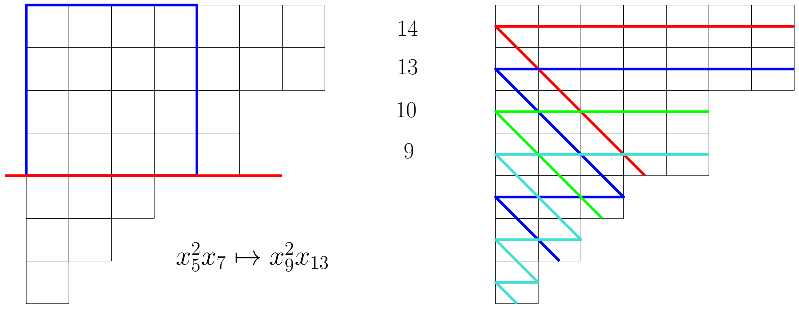


Figure 5.2: Two ways to construct: inductively on the left and the bouncing light construction on the right.

**Example 5.7.9.** *Consider the partition* $(7, 7, 5, 5, 3, 2, 1)$*. It fits into the* $7 \times 7$ *box. It corresponds to the facet* $\{14, 13, 10, 9, 7, 6, 4\}$ *in any complex it belongs to and the monomial associated is* $x_{14}x_{13}^2x_{10}x_9^3$*. The first four rows give the variables.*

- *The left hand side of the figure shows the inductive construction. The blue square is the Durfee square and the top four rows give the answer. The remaining partition* $(3, 2, 1)$ *should be thought as fitting into the* $3 \times 4$ *box. Hence it has a monomial in a subset of the variables* $x_5$, $x_6$, $x_7$, $x_8$ *and then we do an ordered substitution of variables:* $x_5 \mapsto x_9$, $x_6 \mapsto x_{10}$, $x_7 \mapsto x_{13}$, $x_8 \mapsto x_{14}$.
- *The right hand side gives the construction of the bouncing light. The variables* $x_{14}$, $x_{13}, x_{10}, x_9$ *correspond to colors red, blue, green* and *light blue respectively.*

## 5.8 Questions, remarks and future directions

### *5.8.1 Questions*

- It would be interesting to find rich families of examples of complexes that are in one of QI, QE or QC that cannot be constructed from matroids and shifted complexes using standard operations. As shown in Theorem 5.4.3 such examples exist, but are typically found by ad hoc methods and, at present, there are no constructions of infinite families of examples.

- Notice that Theorem 5.7.6 is slightly stronger than the analogue theorem proved in by Klee and the author in [54]. In particular, the old proof required complexes with $2d$ elements in order to solve rank-$d$. This new reduction opens the door to the case of rank-5 matroids since it suffices to construct $\mathcal{F}$ for all rank-5 matroids with 9 elements and those are fully classified in [68]. Is there an algorithm similar to that by Klee and Samper that solves Conjecture 5.7.4 for rank 5 matroids?
- Is it possible to use similar ideas to study oriented matroids. What is the correct definition of an oriented quasi-matroidal class of complexes?

### *5.8.2 Remarks*

- The class QE is related to the class of squarefree weakly polymatroidal ideals of Hibi and Kokubo [58] that has been widely studied in commutative algebra. In particular, the Stanley Reisner ring of a complex in QE is weakly polymatroidal (after choosing the right conventions for the order). Mohammadi and Moradi [75] showed that weakly polymatroidal ideals have linear quotients, which is an algebraic analogue for shellability. In fact, complexes that are weakly polymatroidal satisfy a similar axiom to the QE axiom, except that the element $b_1$ to be removed from a basis has to be the largest in the symmetric difference. Based on empirical information the flexibility of allowing various exchanges helps with constructions. However, we have not been able to find an example of a weakly polymatroidal complex that is not in QE.
- Other relaxations of matroid theory have been considered in the literature. In particular, Lenz [65] considered collections of bases that satisfy what he calls the forward exchange axiom.

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