Visibility and Invisibility in Inverse Problems

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A dissertation
submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy

University of Washington

2020

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Abstract

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In this thesis, we describe two projects dealing with two different aspects of inverse problems. The first project concerns *visibility*, i.e., the problem of reconstructing the internal properties of a medium from external measurements. Specifically, we study the inverse problem for the conductivity equation

$$\operatorname{div}(\gamma \nabla u) = 0$$

in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$. The goal is to reconstruct the positive scalar function γ from its Dirichlet-to-Neumann map Λ_γ, which maps the Dirichlet data of solutions to the conductivity equation to their Neumann data. For dimensions $n \geq 3$, injectivity of the map $\gamma \mapsto \Lambda_\gamma$ was proved for $\gamma \in C^2$ by Sylvester and Uhlmann in 1987. Later in 1988, Nachman provided a constructive procedure for computing γ from Λ_γ. In the same year, Alessandrini proved log-type stability estimates for $\|\gamma_1 - \gamma_2\|_{L^\infty}$ in terms of the operator norm of $\Lambda_{\gamma_1} - \Lambda_{\gamma_2}$. Since then, the problems of injectivity, stability and reconstruction for less regular conductivities has seen considerable interest. Here, we show the validity of Nachman's reconstruction procedure for $\gamma \in W^{3/2,2n}(\Omega)$ such that $\gamma \equiv 1$ near $\partial\Omega$, and derive log-type stability estimates under the slightly stronger assumption that $\gamma_1, \gamma_2 \in W^{2-s,n/s}(\Omega)$ for some $0 < s < 1/2$.

In the second part of the thesis, we consider the problem of making an object *invisible* with respect to electromagnetic measurements made on its boundary. The idea is to enclose a

region $D \subset \mathbb{R}^3$ by an "invisibility cloak" with carefully designed electromagnetic parameters so that electromagnetic measurements on the boundary of the cloak are indistinguishable from measurements that would be obtained in empty space, regardless of the contents of D. For the conductivity equation, such a cloak was designed by Greenleaf, Lassas and Uhlmann in 2003, based on the transformation properties of γ under change of coordinates. The cloaking parameters obtained in this way are anisotropic and degenerate at the boundary between the cloak and the object. This presents a serious difficulty for theoretical analysis as well as practical implementation, and was later addressed by Greenleaf, Kurylev, Lassas and Uhlmann in 2008. The problem of degeneracy was addressed by using regular approximations to the coordinate transformation involved in the construction of the degenerate cloak, so that one obtains an *approximate cloak* of arbitrary accuracy. Using inverse homogenization techniques, these regularized cloaks were further approximated by cloaks whose parameters are both non-degenerate and isotropic. In this thesis, we construct a similar non-degenerate and isotropic approximate cloak for the time-harmonic Maxwell's equations. This was a joint work with Dr. Tuhin Ghosh.

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GLOSSARY

PDE: Partial Differential Equation

EIT: Electrical Impedance Tomography

CGO: Complex Geometrical Optics

Ω: A bounded open set in $\mathbb{R}^n$, where $n \geq 2$

$\partial\Omega$: The boundary of Ω

Λ_γ: The Dirichlet-to-Neumann map for the conductivity equation with conductivity γ

Λ_q: The Dirichlet-to-Neumann map for the Schrödinger equation with potential q

H^s: The standard L^2 based Sobolev space of order $s \in \mathbb{R}$

$W^{s,p}$: The Bessel potential space of functions f such that $(I - \Delta)^{s/2} f \in L^p$

C^α: The space of Hölder continuous functions of order $\alpha \in [0, 1]$

$C^{m,\alpha}$: Space of m times continuously differentiable functions all of whose partial derivatives up to order $m \in \mathbb{N}$ are in C^α $(0 \leq \alpha \leq 1)$

$C^{m+\alpha}$: Same as $C^{m,\alpha}$

C^s_*: The Zygmund space of order $s \geq 0$ on $\mathbb{R}^n$

$\mathcal{S}(\mathbb{R}^n)$: Space of Schwartz functions in $\mathbb{R}^n$

$\mathcal{S}'(\mathbb{R}^n)$: Space of tempered distributions in $\mathbb{R}^n$

L^2_δ: L^2 space of functions on $\mathbb{R}^n$ with respect to the measure $(1 + |x|^2)^{\delta/2}\, dx$

H^m_δ: Sobolev space of functions whose derivatives up to order m are in L^2_δ

m_V: The multiplication operator $\varphi \mapsto V\varphi$

$\Re(z)$: The real part of a complex number z

$\Im(z)$: The imaginary part of a complex number z

$\operatorname{div} F$: Divergence of a vector field F

$\operatorname{curl} F$: Curl of a vector field F in $\mathbb{R}^3$

$H(\operatorname{div}, \Omega)$: Sobolev space of L^2 vector fields F on $\Omega \subset \mathbb{R}^n$ such that $\operatorname{div} F \in L^2$

$H(\operatorname{curl}, \Omega)$: Sobolev space of L^2 vector fields F on $\Omega \subset \mathbb{R}^3$ such that $\operatorname{curl} F \in L^2$

div_S, curl_S,: The surface divergence and curl operators along a surface $S \subset \mathbb{R}^3$

$H^s(\operatorname{div}, \partial\Omega)$: Space of tangential vector fields $F \in H^s(\partial\Omega)$ such that $\operatorname{div}_{\partial\Omega} F \in H^s(\partial\Omega)$

$H^s(\operatorname{curl}, \partial\Omega)$: Space of tangential vector fields $F \in H^s(\partial\Omega)$ such that $\operatorname{curl}_{\partial\Omega} F \in H^s(\partial\Omega)$

$\Lambda_{\epsilon,\mu}$: The impedance map for the time-harmonic Maxwell's equations with magnetic permeability μ and complexified electric permittivity ϵ

Y: the unit cube $[0,1]^N \subset \mathbb{R}^N$

ACKNOWLEDGMENTS

Firstly, I would like to express my deepest gratitude to my advisor, Professor Gunther Uhlmann for his continuous support, mentorship, patience, and encouragement throughout my PhD years. It was an honor and a privilege to learn from and work with such an incredible mathematician and human being. I could not have hoped for a more understanding and supportive PhD advisor. I have benefited greatly from his guidance, both professionally and personally.

I would like to thank Profs. Hart Smith, Yu Yuan, Ulrich Hetmaniuk and Marina Meilă for serving as members of my PhD supervisory committee. I am particularly grateful to Profs. Hart Smith and Yu Yuan for helpful discussions during my PhD and comments on my thesis. I would also like to thank my undergraduate advisor Prof. Venky Krishnan, who first introduced me to inverse problems and set me on my path to graduate school.

I am grateful to my friends and colleagues Drs.Tuhin Ghosh, Karthik Iyer, Benjamin Palacios and Yiran Wang for many helpful discussions and suggestions. Dr. Tuhin Ghosh was my collaborator on one of the projects discussed in this thesis and I learnt a lot by working with him.

Special thanks to my longtime friends Siddharth Bharat, Sachit, Kevin Lui, and Nikolas Eptaminitakis, who have enriched my life in so many ways. I could not have finished this thesis without their support and companionship.

Finally, and most importantly, I would like to thank my parents for their unconditional love and support.

DEDICATION

to my mother

Chapter 1

INTRODUCTION

Inverse problems are concerned with recovering the unknown internal physical parameters of an object (ex., the electrical conductivity inside a patient's body, the speed of sound in the interior of the earth, etc.) using external measurements. Such problems lie at the foundations of several contemporary technological developments, including various techniques of medical imaging (X-ray Computed Tomography, Magnetic Resonance Imaging, Ultrasound imaging, etc.), geophysical imaging and detection of oil and mineral deposits, creation of astrophysical images from telescope data, detection of cracks and interfaces inside objects, RADAR and SONAR imaging, etc.

In general, we may conceptualize an inverse problem as follows: Let p and m denote the physical parameter(s) to be recovered and the recorded measurements respectively. We assume that p and m belong to some function spaces X and Y. Suppose we have a known *forward operator* $F : X \to Y$ that maps p to what the recorded measurements m would be if p were the true parameter. Usually, the operator F will be highly non-linear. The observed measurements m_{obs} are modeled by

$$m_{obs} = F(p) + \epsilon$$

where ϵ is a small error term (arising either from the limited precision of measurements, or some inherent randomness in the underlying physical phenomena). The goal of the inverse problem is to estimate the parameter p given the measurement m_{obs}. There are several aspects of this problem that need to be considered:

1. **Uniqueness.** Assuming no noise and fully accurate measurements, do different parameters necessarily lead to different measurements? In other words, does $F(p_1) = F(p_2)$

imply $p_1 = p_2$?

2. **Existence/ Range Characterization.** Given an arbitrary element m of the measurement space Y, does there necessarily exist a $p \in X$ such that $F(p) = m$? If not, is there a way of characterizing all elements in the range of F?

3. **Reconstruction.** Given a set of measurements $m = F(p)$, is there an explicit formula or an algorithm for computing p from m? What if the measurements include an error term?

4. **Stability.** If $F(p_1) \approx F(p_2)$ in some sense, does it follow that $p_1 \approx p_2$ in some sense? In other words, does a small change in measurements necessarily imply only a small change in the underlying physical parameters?

5. **Partial data.** What if we have access to only a subset of the measurements (for ex., if measurements are taken on only a part of the exterior)?

6. **Numerics.** For all of the questions above, we work within a continuous physical model. But in practice, the measurements taken and all the computations made are discrete. How do we design an efficient numerical algorithm whose discrete computations lead to approximately correct reconstructions?

In most inverse problems, the underlying physical process is modeled by a partial differential equation (PDE), whose coefficients encode the internal physical parameters to be recovered. The external measurements are related to the solutions of this PDE. Thus, the problem becomes one of recovering the coefficients of the PDE given some information about its solution set.

While inverse problems are concerned with making visible the hidden interior physical parameters of a medium, one can also consider the problem of making the interior of an object *invisible* with respect to exterior measurements. The idea is to surround the object to be hidden by a medium (called the *invisibility cloak*) with carefully designed physical parameters so that any measurements made on the exterior of the cloak are indistinguishable from measurements one would obtain in empty space, regardless of the values of the physical parameters inside the cloaked object. In recent years, there has been a wave of serious theoretical proposals for designing such an invisibility cloak [20]. The particular approach to cloaking that has received the most attention is that of *transformation optics*, which attempts to use the invariance properties of the underlying PDE to design the physical parameters of the cloak, so that exterior measurements are identical to what would be obtained in empty space.

In this thesis, we present the results of two projects related to inverse problems and invisibility. The first project, concerning *visibility* for the inverse conductivity problem, is discussed in Chapter 2. Next in Chapter 3, we will describe a project concerning *invisibility* for time-harmonic Maxwell's equations. We present an overview of these two chapters in the sections below.

1.1 Reconstruction of rough conductivities from boundary measurements

The inverse conductivity problem, also called the Calderón problem, is one of the most well studied inverse problems. First proposed by Alberto Calderón in 1980 [9], it forms the basis of a widely used imaging technique called Electrical Impedance Tomography (EIT), where one attempts to determine the electrical conductivity inside an object using voltage and current measurements on its boundary. The conductive medium is modeled as a bounded open set $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary. The electrical conductivity of the medium at a point $x \in \Omega$ is denoted by $\gamma(x)$. If the conductivity is isotropic (i.e., same in all directions), γ is given by a positive scalar function on Ω. On the other hand, if the conductivity is anisotropic, γ is given by a symmetric positive definite $n \times n$ matrix valued function on Ω. If there are no sources or sinks of current inside Ω, Ohm's Law states that the electric potential u in Ω satisfies the PDE

$$\mathrm{div}(\gamma\nabla u) = 0.$$

Therefore, if we apply an electric potential f on the boundary $\partial\Omega$, the induced electric potential u inside Ω is the unique solution of the Dirichlet boundary value problem

$$\begin{cases} \mathrm{div}(\gamma\nabla u) = & 0 \text{ in } \Omega \\ u = & f \text{ on } \partial\Omega. \end{cases}$$

The inverse problem is to determine the function γ from the knowledge of the Dirichlet-to-Neumann map $\Lambda_\gamma : u|_{\partial\Omega} \to \gamma\partial_\nu u|_{\partial\Omega}$, where ν is the outward unit normal vector field on $\partial\Omega$. Physically, $\gamma\partial_\nu u|_{\partial\Omega}$ is the induced electric current flux at the boundary.

Calderón solved a linearized version of the problem, by proving that the linearization of the map $\gamma \mapsto \Lambda_\gamma$ is injective at $\gamma \equiv 1$. In the seminal paper [36], Sylvester and Uhlmann showed the injectivity of $\gamma \mapsto \Lambda_\gamma$ for C^2 isotropic conductivities γ in dimensions $n \geq 3$. The authors first use a Liouville transform to reduce the problem to an inverse problem for the Schrödinger equation: Let $q \in L^\infty(\Omega)$ be such that the following boundary value problem is well-posed:

$$\begin{cases} (-\Delta + q)v = & 0 \qquad \text{in } \Omega \\ v = & f \qquad \text{on } \partial\Omega. \end{cases}$$

The Dirichlet-to-Neumann map for q is defined by $\Lambda_q(f) = \partial_\nu v|_{\partial\Omega}$. The problem of determining γ from Λ_γ was reduced to the problem of determining q from Λ_q where $q = \Delta\sqrt{\gamma}/\sqrt{\gamma}$. Next, the authors proved that $\Lambda_{q_1} = \Lambda_{q_2} \Rightarrow q_1 \equiv q_2$ using the so-called Complex Geometrical Optics (CGO) solutions to $(-\Delta + q_i)v = 0$ defined globally in $\mathbb{R}^n$. Later in 1988, Nachmann [32] provided a constructive procedure for computing q (resp., γ) from Λ_q (resp., Λ_γ) when $q \in L^\infty$ (resp., $\gamma \in C^2$). The procedure is based on the observation that CGO solutions satisfying certain decay conditions are uniquely determined by their restrictions to $\partial\Omega$. In turn, these restrictions can be characterized as the unique solutions of certain boundary integral equations on $\partial\Omega$. In the same year, Alessandrini proved logarithmic stability estimates for $\|\gamma_1 - \gamma_2\|_{L^\infty}$ in terms of the operator norm of $\Lambda_{\gamma_1} - \Lambda_{\gamma_2}$ for $\gamma_1, \gamma_2 \in H^s(\Omega)$, $s > n/2 + 2$.

It is natural to ask what are the minimum regularity assumptions on γ under which uniqueness and the reconstruction procedure continue to hold. This question is also of practical importance. For example, it was pointed out in [10] that if q arises from a Gaussian random field satisfying certain conditions, almost every instantiation of q belongs to a Sobolev space of fixed negative regularity. For the case $n \geq 3$, uniqueness has been proved under successively lower regularity assumptions on γ: for $\gamma \in C^{3/2+}$ in [5], for $C^{3/2}$ in [35], for the Bessel potential space $W^{3/2,2n+}$ in [6], for $\gamma \in W^{3/2+,2}$ in [34] and for $\gamma \in C^1$ or $\gamma \in C^{0,1}$ with $\|\nabla \log \gamma\|_{L^\infty}$ small in [25]. The smallness condition was removed in [11]. It was conjectured by Brown in [6] that uniqueness holds for $\gamma \in W^{1,n}$ for all $n \geq 3$. This was proved for $n = 3, 4$ in [24]. Reconstruction has also been achieved for conductivities satisfying the assumptions of [25] (namely, $\gamma \in C^1$ or $\gamma \in C^{0,1}$ with $\|\nabla \log \gamma\|_{L^\infty}$ small) in [13].

In Chapter 2, we will prove the validity of Nachman's reconstruction procedure for $\gamma \in W^{3/2,2n}(\Omega)$ that are $\equiv 1$ in a neighborhood of $\partial\Omega$ [37]. Note that elements of $W^{3.2,2n}$ need not even be Lipschitz. Therefore, this result is a new contribution to the study of the reconstruction problem for rough conductivities. Moreover, we will also prove a log-type stability estimate for $\|\gamma_1 - \gamma_2\|_{C^\alpha(\overline{\Omega})}$ $(0 < \alpha < 1)$ in terms of the operator norm of $\Lambda_{\gamma_1} - \Lambda_{\gamma_2}$, under the assumption that $\gamma_1, \gamma_2 \in W^{2-s,n/s}(\Omega)$ for some $s \in (0, 1/2)$ and $\gamma_1, \gamma_2 \equiv 1$ near $\partial\Omega$.

1.2 Approximate isotropic cloak for time-harmonic Maxwell's equations

A region in space is said to be cloaked if any external measurements made on the region are indistinguishable from what would be obtained in empty space, regardless of the contents of the interior of the region. It is well known that uniqueness no longer holds for the Calderón problem when the conductivities are allowed to be anisotropic. Indeed, given a diffeomerphism $\Phi : \overline{\Omega} \to \overline{\Omega}$ that fixes the boundary $\partial\Omega$, one can use the transformation properties of the equation $\mathrm{div}(\gamma \nabla u) = 0$ to *push forward* the conductivity γ to a new conductivity $\Phi_*\gamma$ so that

$$\Lambda_\gamma = \Lambda_{\Phi_*\gamma}.$$

In 2003, Greenleaf, Lassas and Uhlmann [22, 23] extended this idea to theoretically construct an invisibility cloak for the conductivity equation. By using a degenerating coordinate transform Φ, they pushforwarded $\gamma_0 \equiv 1$ in $\Omega \setminus \{p\}$ to an anisotropic conductivity in $\Omega \setminus \overline{D}$, where $p \in \Omega$ and $D \Subset \Omega$ with smooth boundary. Then, using a removable singularity argument, they showed that $\Lambda_\gamma = \Lambda_{\gamma_0}$ whenever $\gamma = \Phi_* \gamma_0$ in $\Omega \setminus \overline{D}$, regardless of the value of γ inside D. This technique has come to be known as *transformation optics*. This method was later used to design invisibility cloaks for the time-harmonic Maxwell's equations in [17]. However, the material parameters in these constructions were anisotropic and *singular* in the sense of degenerating near the boundary between the cloak and the cloaked region. This presented a serious challenge to practical implementation using metamaterials, as well as making the theoretical analysis difficult. The problem of singular parameters was dealt with by constructing regular approximations to the ideal cloak [29, 18]. Nevertheless, these approximations still used anisotropic parameters.

To deal with the problem of anisotropic parameters, Greenleaf, Kurylev, Lassas and Uhlmann introduced an approach called *Isotropic Transformation Optics* [19, 21] based on inverse homogenization. It is a well known phenomenon in homogenization theory [1, 38, 3] that isotropic parameters with periodic microstructures can be used to approximate the behavior of anisotropic parameters in the small-scale limit. In [19, 21], the authors constructed explicit isotropic parameters for the conductivity equation that forms an approximate invisibility cloak in the small-scale limit. In Chapter 3, we describe a similar construction by the author and T. Ghosh that forms an approximate isotropic cloak for time-harmonic Maxwell's equations [14]. The chief obstacle is the fact that unlike the conductivity equation, the bilinear form associated with Maxwell's equations is not coercive. We overcome this problem by adding another approximation step - where a small parameter is introduced to make the bilinear form coercive and then letting this parameter go to 0. It turns out that as long as we take limits in the right order, the boundary measurements for this 3-step approximate cloaking construction converge to the boundary measurements that would be obtained in empty space.

Chapter 2

THE CALDERÓN PROBLEM FOR ROUGH CONDUCTIVITIES

Let Ω be a bounded domain in $\mathbb{R}^n\,(n \geq 2)$ with Lipschitz boundary, and let γ be a positive real-valued function on Ω. Assume that γ satisfies the uniform ellipticity condition

$$0 < c < \gamma(x) < c^{-1} \qquad \text{for a.e. } x \in \Omega.$$

We interpret $\gamma(x)$ as the electrical conductivity of the medium at the point $x \in \Omega$. If there are no sources or sinks of current in Ω, a voltage potential f applied at the boundary $\partial\Omega$ induces a potential u inside Ω that solves the Dirichlet Boundary Value Problem

$$\begin{cases} \operatorname{div}(\gamma\nabla u) = & 0 \text{ in } \Omega \\ u = & f \text{ on } \partial\Omega. \end{cases} \tag{2.1}$$

We know from standard elliptic PDE theory that given any $f \in H^{1/2}(\partial\Omega)$, there exists a unique $u = u_f \in H^1(\Omega)$ that satisfies (2.1). The Dirichlet-to-Neumann map of γ, denoted by Λ_γ, is formally defined as the map that sends

$$f \mapsto \gamma \frac{\partial u_f}{\partial \nu}\bigg|_{\partial\Omega},$$

where ν is the outward pointing unit normal vector field on $\partial\Omega$. Physically, $\gamma\partial_\nu u_f(x)$ measures the outward flux density of electric current through $\partial\Omega$ at the point $x \in \partial\Omega$.

Consider the case where $\partial\Omega$ is smooth, $\gamma \in C^\infty(\overline{\Omega})$, and $f \in C^\infty(\partial\Omega)$. Then $u_f \in C^\infty(\overline{\Omega})$ by elliptic regularity and therefore, $\Lambda_\gamma f = \gamma\partial_\nu u_f$ is a pointwise well-defined smooth function on $\partial\Omega$. If g is another smooth function on $\partial\Omega$ and $v_g \in C^\infty(\overline{\Omega})$ satisfies $v_g|_{\partial\Omega} = g$, then by the Divergence theorem,

$$\begin{aligned} \langle \Lambda_\gamma f, g\rangle &= \int_{\partial\Omega} \gamma \frac{\partial u_f}{\partial \nu} v_g \; d\sigma \\ &= \int_{\Omega} \operatorname{div}\left(\gamma v_g \nabla u_f\right) \; d\sigma \\ &= \int_{\Omega} \operatorname{div}\left(\gamma \nabla u_f\right) v_g \; dx + \int_{\Omega} \gamma \nabla u_f \cdot \nabla v_g \; dx \\ &= \int_{\Omega} \gamma \nabla u_f \cdot \nabla v_g \; dx \qquad (\text{since } \operatorname{div}(\gamma \nabla u_f) \equiv 0 \text{ in } \Omega). \end{aligned} \tag{2.2}$$

Here, $d\sigma$ denotes the surface measure on $\partial\Omega$. Observe that (2.2) is well defined whenever $u_f, v_g \in H^1(\Omega)$. Moreover, for any bounded Lipschitz domain Ω, there exists a bounded extension operator $\mathcal{E} : H^{1/2}(\partial\Omega) \to H^1(\Omega)$ that satisfies $(\mathcal{E}g)|_{\partial\Omega} = g$ for all $g \in H^{1/2}(\partial\Omega)$. Therefore, for any $f \in H^{1/2}(\partial\Omega)$,

$$g \mapsto \int_{\Omega} \gamma \nabla u_f \cdot \nabla \mathcal{E} g \; dx$$

is a continuous linear functional on $H^{1/2}(\partial\Omega)$. By identifying the dual of $H^{1/2}(\partial\Omega)$ with $H^{-1/2}(\partial\Omega)$, we may define $\Lambda_\gamma f$ as the unique element of $H^{-1/2}(\partial\Omega)$ that satisfies

$$\langle \Lambda_\gamma f, g\rangle = \int_{\Omega} \gamma \nabla u_f \cdot \nabla v_g \; dx \qquad \text{for all } g \in H^{1/2}(\partial\Omega), \tag{2.3}$$

where $v_g \in H^1(\Omega)$ is any extension of g (including, in particular, $v_g = \mathcal{E}g$). By construction, this definition agrees with the pointwise definition of $\Lambda_\gamma f$ when f is smooth. Moreover, the a-priori estimate

$$\|u_f\|_{H^1(\Omega)} \lesssim \|f\|_{H^{1/2}(\partial\Omega)}$$

implies that

$$\begin{aligned} |\langle \Lambda_\gamma f, g\rangle| &= \left| \int_{\Omega} \gamma \nabla u_f \cdot \nabla \mathcal{E} g \; dx \right| \\ &\lesssim \|u_f\|_{H^1(\Omega)} \|\mathcal{E} g\|_{H^1(\Omega)} \\ &\lesssim \|f\|_{H^{1/2}(\partial\Omega)} \|g\|_{H^{1/2}(\partial\Omega)}. \end{aligned}$$

This shows that Λ_γ is a bounded linear map from $H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$.

Remark 1. Here we have used the notation $\alpha \lesssim \beta$ to mean that there exists a constant $C > 0$ independent of α and β such that $\alpha \leq C\beta$ for all admissible α, β.

The map Λ_γ encodes the set of all possible voltage and current measurements that can be made at the boundary. The Calderón problem is concerned with recovering the conductivity function γ from the knowledge of the map Λ_γ. In this chapter, we will examine this problem for bounded Lipschitz domains in $n \geq 3$ dimensions and conductivities γ that are $\equiv 1$ in a neighborhood of $\partial\Omega$ and belong to the fractional order Sobolev space $W^{3/2,2n}(\Omega)$.

Definition 2.1. For any $s \in \mathbb{R}$ and $p \in (1, \infty)$, the Bessel potential space $W^{s,p}(\mathbb{R}^n)$ is defined as the Banach space of all distributions f on $\mathbb{R}^n$ such that $(I - \Delta)^{s/2} f \in L^p(\mathbb{R}^n)$, and equipped with the norm

$$\|f\|_{W^{s,p}} = \|(I - \Delta)^{s/2} f\|_{L^p}.$$

When s is a nonnegative integer, this agrees with the usual definition of the Sobolev space $W^{s,p}$ as the space of all distributions on $\mathbb{R}^n$ all of whose partial derivatives up to order s are in $L^p(\mathbb{R}^n)$.

Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $s \in \mathbb{R}$ and $p \in (1, \infty)$. We define $W^{s,p}(\Omega)$ as the space of $W^{s,p}(\mathbb{R}^n)$ functions resticted to Ω, i.e.,

$$W^{s,p}(\Omega) := \{u|_\Omega : u \in W^{s,p}(\mathbb{R}^n)\}$$

and equipped with the norm

$$\|f\|_{W^{s,p}(\Omega)} = \inf\{\|u\|_{W^{s,p}(\mathbb{R}^n)} : u|_\Omega = f\}.$$

We will also consider the spaces $W^{s,p}_{\text{comp}}(\Omega)$, which we define by

$$W^{s,p}_{\text{comp}}(\Omega) := \{u \in W^{s,p}(\mathbb{R}^n) : \text{supp } u \Subset \Omega\}.$$

The main result of this chapter is as follows:

Theorem 2.3 (T., 2020 [37]). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$. Let $\gamma \in W^{3/2,2n}(\Omega)$ be a positive real valued function satisfying*

$$0 < c < \gamma(x) < c^{-1} \qquad \text{for a.e. } x \in \Omega \tag{2.4}$$

and $\gamma \equiv 1$ in a neighborhood of $\partial\Omega$. Then,

(a) There is a constructive procedure for computing the conductivity function γ from the knowledge of the map $\Lambda_\gamma : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$. Moreover,

(b) We have the following stability estimate: Let $\gamma_j \in W^{3/2,2n}(\Omega)$, $j = 1, 2$, be such that $\gamma_j \equiv 1$ near $\partial\Omega$ and satisfy the ellipticity bound (2.4). Suppose in addition that $\|\gamma_j\|_{W^{2-s,n/s}(\Omega)} \leq M$ for some $0 < s < 1/2$, and let $0 \leq \alpha < 1$. Then there exist $C = C(\Omega, n, c, M, s, \alpha) > 0$ and $0 < \sigma = \sigma(n, s, \alpha) < 1$ such that

$$\|\gamma_1 - \gamma_2\|_{C^\alpha(\overline{\Omega})} \leq C \left(|\log \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}\to H^{-1/2}}|^{-\sigma} + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}\to H^{-1/2}} \right). \tag{2.5}$$

Calderón's approach to recovering γ was based on the weak definition of the Dirichlet-to-Neumann map. Observe that for any $u \in H^1(\Omega)$ satisfying $\operatorname{div}(\gamma\nabla u) = 0$ in Ω and an arbitrary $v \in H^1(\Omega)$, (2.3) implies that

$$\langle \Lambda_\gamma(u|_{\partial\Omega}), v|_{\partial\Omega} \rangle = \int_\Omega \gamma \nabla u \cdot \nabla v \; dx.$$

If $\widetilde{\gamma}$ is another such conductivity and $u, \widetilde{u} \in H^1(\Omega)$ satisfy $\operatorname{div}(\gamma\nabla u) = \operatorname{div}(\widetilde{\gamma}\nabla\widetilde{u}) = 0$, then this implies the identity

$$\langle (\Lambda_\gamma - \Lambda_{\widetilde{\gamma}})(u|_{\partial\Omega}), \widetilde{u}|_{\partial\Omega} \rangle = \int_\Omega (\gamma - \widetilde{\gamma}) \nabla u \cdot \nabla \widetilde{u} \; dx. \tag{2.6}$$

Now consider the problem of showing that the map $\gamma \mapsto \Lambda_\gamma$ is injective (on some space of admissible conductivities γ). If $u, \widetilde{u}$ are as above and $\Lambda_\gamma = \Lambda_{\widetilde{\gamma}}$, we get

$$0 = \int_\Omega (\gamma - \widetilde{\gamma}) \nabla u \cdot \nabla \widetilde{u} \; dx.$$

Thus, if the set

$$V = \operatorname{span}\{\nabla u \cdot \nabla \widetilde{u} : \operatorname{div}(\gamma \nabla u) = \operatorname{div}(\widetilde{\gamma} \nabla \widetilde{u}) = 0 \text{ in } \Omega\}$$

is dense in $L^2(\Omega)$ (for example), then we can conclude that $\gamma = \widetilde{\gamma}$ almost everywhere.

When $\gamma = \widetilde{\gamma} = 1$, Calderón observed that given any $\zeta \in \mathbb{C}^n$ satisfying $\zeta \cdot \zeta = \sum_{j=1}^n \zeta_j^2 = 0$ the function $e^{x\cdot\zeta}$ satisfies

$$\operatorname{div}(\nabla e^{x\cdot\zeta}) = \Delta e^{x\cdot\zeta} = (\zeta \cdot \zeta) e^{x\cdot\zeta} = 0.$$

Given $\xi \in \mathbb{R}^n$, choose $\eta \in \mathbb{R}^n$ such that $\eta \cdot \xi = 0$ and $|\eta| = |\xi|$. Define

$$\zeta_1 = \frac{1}{2}(\eta - i\xi), \qquad \zeta_2 = \frac{1}{2}(-\eta - i\xi).$$

Then it is easy to see that $\zeta_j \cdot \zeta_j = 0$, $j = 1, 2$. Therefore,

$$\nabla u_1 \cdot \nabla u_2 = (\zeta_1 \cdot \zeta_2) e^{-ix\cdot\xi} = -\frac{1}{2}|\xi|^2 e^{-ix\cdot\xi} \in V.$$

Thus V contains all complex exponentials of the form $e^{-ix\cdot\xi}$, $\xi \in \mathbb{R}^n$, which immediately implies that V is dense in L^2. Calderón [9] used a slightly more sophisticated version of this argument to prove uniqueness for a linearized version of the problem near $\gamma \equiv 1$. Uniqueness for the full nonlinear problem with smooth conductivities was first proved by Sylvester and Uhlmann [36] using the so-called Complex Geometrical Optics (CGO) solutions of $\operatorname{div}(\gamma_j \nabla u_j) = 0$. These are solutions of the form

$$u_\zeta(x) = e^{x\cdot\zeta}(1 + r_\zeta(x))$$

where $\zeta \in \mathbb{C}^n$ is such that $\zeta \cdot \zeta = 0$ and $r_\zeta \to 0$ in some sense as $|\zeta| \to \infty$. A key step in constructing such solutions is to use a standard relationship between the divergence form operators $\operatorname{div}(\gamma\nabla)$ and Schrödinger operators of the form $-\Delta + q$, where q is a scalar function.

2.1 Reduction to the Schrödinger Equation

Using the Leibniz rule, it is easy to see that $\operatorname{div}(\gamma \nabla u) = 0$ iff

$$-\Delta u - \nabla \log \gamma \cdot \nabla u = 0.$$

Since the coefficient $\nabla \log \gamma$ is a gradient, we can use a Liouville transform to reduce this equation to a zeroth order perturbation of the Laplace equation. Observe that for any function ϕ, we can formally write

$$\begin{aligned} e^{-\phi}\Delta(e^{\phi}u) &= e^{-\phi}\nabla(e^{\phi}\nabla u + e^{\phi}u\nabla\phi) \\ &= \Delta u + 2\nabla\phi\cdot\nabla u + |\nabla\phi|^2 u + \Delta\phi u. \end{aligned}$$

Choosing $\phi = \frac{1}{2}\log\gamma$, we get

$$-\Delta u - \nabla\log\gamma\cdot\nabla u = -\gamma^{-1/2}\Delta(\gamma^{1/2}u) + \left(\frac{1}{4}|\nabla\log\gamma|^2 + \frac{1}{2}\Delta\log\gamma\right)u.$$

Therefore, u satisfies $\mathrm{div}(\gamma\nabla u) = 0$ iff $v := \gamma^{1/2}u$ satisfies

$$(-\Delta + q)v = 0, \tag{2.7}$$

where $q = \left(\frac{1}{4}|\nabla\log\gamma|^2 + \frac{1}{2}\Delta\log\gamma\right)$. Taking $u \equiv 1$, we see that q can also be written as

$$q = \gamma^{-1/2}\Delta\gamma^{1/2}.$$

The next proposition shows that under our assumptions on γ, q belongs to a Sobolev space of negative regularity.

Proposition 2.4. *Let $\gamma \in W^{3/2,2n}(\Omega)$ be such that*

$$0 < c < \gamma(x) < c^{-1} \qquad a.e.\ on\ \Omega$$

and $\gamma \equiv 1$ on a neighborhood of $\partial\Omega$. Extend γ to all of $\mathbb{R}^n$ by defining $\gamma \equiv 1$ on $\mathbb{R}^n \setminus \Omega$ and define $q = \Delta\sqrt{\gamma}/\sqrt{\gamma} = \frac{1}{4}|\nabla\log\gamma|^2 + \frac{1}{2}\Delta\log\gamma$. Then $q \in W^{-1/2,2n}_{comp}(\mathbb{R}^n)$.

Proof. That q is compactly supported in Ω follows from the fact that $\gamma \equiv 1$ outside a compact subset of Ω. Next, we recall the Sobolev embedding properties of Bessel potential spaces: if $0 < p_1 < p_2 < \infty$ and $-\infty < s_2 < s_1 < \infty$ are such that

$$s_1 - \frac{n}{p_1} = s_2 - \frac{n}{p_2},$$

then we have a continuous embedding of $W^{s_1,p_1}(\mathbb{R}^n) \hookrightarrow W^{s_2,p_2}(\mathbb{R}^n)$ (ref. [39], Theorem 2.7.1). Therefore, we have

$$\begin{aligned}
\|q\|_{W^{-1/2,2n}} &\lesssim \|\Delta \log \gamma\|_{W^{-1/2,2n}} + \||\nabla \log \gamma|^2\|_{W^{-1/2,2n}} \\
&\lesssim \|\log \gamma\|_{W^{3/2,2n}} + \||\nabla \log \gamma|^2\|_{L^n} \quad (\text{as } L^n(\mathbb{R}^n) \hookrightarrow W^{-1/2,2n}(\mathbb{R}^n)) \\
&= \|\log \gamma\|_{W^{3/2,2n}} + \|\nabla \log \gamma\|^2_{L^{2n}} \\
&\lesssim \|\log \gamma\|_{W^{3/2,2n}} + \|\log \gamma\|^2_{W^{1,2n}} \\
&\lesssim \|\log \gamma\|_{W^{3/2,2n}} + \|\log \gamma\|^2_{W^{3/2,2n}} \quad (\text{as } W^{3/2,2n}(\mathbb{R}^n) \hookrightarrow W^{1,2n}(\mathbb{R}^n))
\end{aligned}$$

where the last embedding estimate follows from the monotonicity properties of $W^{s,p}$ spaces (ref. [39], Proposition 2.3.2/2). Now, it suffices to show that $\|\log \gamma\|_{W^{3/2,2n}}$ is finite. Consider a smooth bounded sub-domain $\Omega' \subset \Omega$ such that

$$\text{supp } (\log \gamma) \subset \Omega' \subset \overline{\Omega'} \subset \Omega.$$

Next, choose a bounded function $F : \mathbb{R} \to \mathbb{R}$ that satisfies $F(x) = \log x$ on $[c, c^{-1}]$ and has bounded continuous derivatives up to order 2. We will use the fact that for any $s \geq 1$, $1 < p < \infty$ and $f \in C^{[s]+1}(\mathbb{R})$ that has bounded derivatives up to order $[s] + 1$, the composition map $u \mapsto f \circ u$ maps $W^{s,p}(\Omega') \cap W^{1,sp}(\Omega')$ continuously into $W^{s,p}(\Omega')$ [4]. Notice that $W^{3/2,2n}(\Omega') \hookrightarrow W^{1,3n}(\Omega')$ by Sobolev embedding (ref. [39], Theorem 3.3.1(ii)). Therefore,

$$\|\log \gamma\|_{W^{3/2,2n}(\Omega')} = \|F \circ \gamma\|_{W^{3/2,2n}(\Omega')} < \infty.$$

Finally, observe that since $\log \gamma \equiv 0$ near $\partial\Omega'$, $\log \gamma \in W_0^{3/2,2n}(\Omega')$ (i.e., the closure of $C_c^\infty(\Omega')$ in $W^{3/2,2n}(\Omega')$). Since extension by 0 is a continuous map from $W_0^{3/2,2n}(\Omega') \to W^{3/2,2n}(\mathbb{R}^n)$ (ref. [39], Section 3.4.3, Corollary and Remark 2) we also have $\|\log \gamma\|_{W^{3/2,2n}} \lesssim \|\log \gamma\|_{W^{3/2,2n}(\Omega')}$. Putting all these bounds together, we get

$$\|q\|_{W^{-1/2,2n}} \lesssim \|\log \gamma\|_{W^{3/2,2n}(\Omega')} + \|\log \gamma\|^2_{W^{3/2,2n}(\Omega')} < \infty$$

which completes the proof. □

Therefore, in order to work with the PDE (2.7), we first need to clarify what it means to multiply a function in H^1 by a function in $W^{-1/2,2n}_{\text{comp}}(\mathbb{R}^n)$. We begin with an important estimate that was proved in [10].

Lemma 2.5. *Let $s > 0$ and $p \in (2, \infty)$ be such that $p \geq n/s$. Then for all $f, g \in \mathcal{S}(\mathbb{R}^n)$, we have*

$$\|fg\|_{W^{s,p'}} \lesssim \|f\|_{H^s}\|g\|_{H^s},$$

where $1/p + 1/p' = 1$.

Proof. The Kato-Ponce inequality (ref. [16, 27], also called fractional Leibniz rule) tells us that

$$\|fg\|_{W^{s,p'}} \lesssim \|f\|_{H^s}\|g\|_{L^r} + \|f\|_{L^r}\|g\|_{H^s}$$

where $1/r = 1/p' - 1/2 = 1/2 - 1/p$. Now applying the Sobolev embedding theorem, we get

$$\|fg\|_{W^{s,p'}} \lesssim \|f\|_{H^s}\|g\|_{H^t} + \|f\|_{H^t}\|g\|_{H^s},$$

where $1/2 - t/n = 1/r$, or equivalently, $t = n/2 - n/r = n/p$. Now the estimate follows from the fact that $t \leq s$. □

Given any $V \in W^{-1/2,2n}(\mathbb{R}^n)$, define the multiplication operator $m_V : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ by duality as follows:

$$\langle m_V(\varphi), \psi\rangle := \langle V, \varphi\psi\rangle, \qquad \varphi, \psi \in \mathcal{S}(\mathbb{R}^n).$$

Corollary 2.6. *The multiplication operator m_V satisfies the bounds*

$$|\langle m_V(\varphi), \psi\rangle| = |\langle V, \varphi\psi\rangle| \lesssim \|\varphi\|_{H^{1/2}}\|\psi\|_{H^{1/2}}, \qquad \varphi, \psi \in \mathcal{S}(\mathbb{R}^n).$$

Consequently, m_V extends to a bounded linear operator from $H^{1/2}(\mathbb{R}^n) \to H^{-1/2}(\mathbb{R}^n)$.

Proof. Using Lemma 2.5 and the fact that $W^{-1/2,2n}(\mathbb{R}^n)$ is the L^2-dual of $W^{1/2,2n/(2n-1)}$ ([39], Theorem 2.11.2), we get

$$|\langle V, \varphi\psi\rangle| \leq \|V\|_{W^{-1/2,2n}}\|\varphi\psi\|_{W^{1/2,2n/(2n-1)}} \leq \|V\|_{W^{-1/2,2n}}\|\varphi\|_{H^{1/2}}\|\psi\|_{H^{1/2}}.$$

This proves the result. □

Remark 2. We will prove a more precise estimate in Theorem 2.17.

Remark 3. Henceforth, by a slight abuse of notation, we will use qu and $m_q(u)$ interchangeably whenever the latter is well defined.

Thus we are lead to consider the boundary value problem

$$\begin{cases} -\Delta u + qu = & 0 \text{ in } \Omega \\ u = & f \text{ on } \partial\Omega. \end{cases} \tag{2.8}$$

Proposition 2.7. *(The Fredholm Alternative) Suppose* $q \in W^{-1/2,2n}_{comp}(\Omega)$. *Then exactly one of the following must be true:*

(i) For any $f \in H^{1/2}(\partial\Omega)$, *there exists a unique* $u \in H^1(\Omega)$ *that satisfies* (2.8). *Moreover, there exists* $C = C(q, \Omega) > 0$ *such that*

$$\|u\|_{H^1(\Omega)} \leq C\|f\|_{H^{1/2}(\partial\Omega)}, \qquad \text{for all } f \in H^{1/2}(\partial\Omega).$$

(ii) There exists $u \in H^1(\Omega), u \neq 0$ *such that*

$$\begin{cases} (-\Delta + m_q)u = & 0 \text{ in } \Omega \\ u = & 0 \text{ on } \partial\Omega. \end{cases}$$

That is, 0 *is a Dirichlet eigenvalue of* $(-\Delta + m_q)$ *on* Ω.

Proof. It follows from Corollary 2.6 that m_q maps $H^1(\Omega) \to H^{-1/2}_{\text{comp}}(\Omega)$. Composing this with the compact inclusion $H^{-1/2}_{\text{comp}}(\Omega) \hookrightarrow H^{-1}(\Omega)$, we see that $m_q : H^1(\Omega) \to H^{-1}(\Omega)$ is compact as well. Moreover, since $\Delta : H^1_0(\Omega) \to H^{-1}(\Omega)$ is invertible, $(-\Delta + m_q) : H^1_0(\Omega) \to H^{-1}(\Omega)$ is a Fredholm operator. Therefore the result follows from standard Fredholm theory. □

As in the case of the conductivity equation, if 0 is not a Dirichlet eigenvalue of $(-\Delta+q)$, we define the Dirichlet-to-Neumann map $\Lambda_q : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$ by duality: Given $f \in H^{1/2}(\partial\Omega)$, let $u \in H^1(\Omega)$ be the unique solution of (2.8). Then $\Lambda_q f$ is defined as the unique element of $H^{-1/2}(\partial\Omega)$ that satisfies

$$\langle \Lambda_q f, g \rangle = \int_{\partial\Omega} \Lambda_q(f) g \, d\sigma = \int_\Omega \nabla u \cdot \nabla v \, dx + \langle m_q u, v \rangle_{L^2(\Omega)} \qquad \text{for all } g \in H^{1/2}(\partial\Omega),$$

where $v \in H^1(\Omega)$ is any function such that $v|_{\partial\Omega} = g$. If f, q and $\partial\Omega$ were sufficiently smooth, this would coincide with the pointwise definition

$$\Lambda_q f = \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega}.$$

We also get the following integral identity: Let $q_1, q_2 \in W^{-1/2,2n}_{\text{comp}}(\Omega)$ be such that 0 is not a Dirichlet eigenvalue of $(-\Delta + m_{q_j})$ on Ω, $j = 1, 2$ and let $u_1, u_2 \in H^1(\Omega)$ be solutions of $(-\Delta + m_{q_j})u_j = 0$. Then

$$\int_{\partial\Omega} (\Lambda_{q_1} - \Lambda_{q_2}) u_1 \cdot u_2 \, d\sigma = \int_{\Omega} (m_{q_1} - m_{q_2}) u_1 \cdot u_2 \, dx \tag{2.9}$$

We are now ready to show that the boundary value problem for the conductivity equation

$$\begin{cases} \operatorname{div}(\gamma \nabla u) = & 0 \text{ in } \Omega \\ u = & f \in H^{1/2}(\partial\Omega) \end{cases} \tag{2.10}$$

can be reduced to the corresponding problem for the Schrödinger equation

$$\begin{cases} (-\Delta + q) w = & 0 \text{ in } \Omega \\ w = & f \text{ on } \partial\Omega. \end{cases} \tag{2.11}$$

Proposition 2.8. *Let $\gamma \in W^{3/2,2n}(\Omega)$ be such that*

$$0 < c < \gamma(x) < c^{-1} \qquad \text{a.e. on } \Omega$$

and $\gamma \equiv 1$ on a neighborhood of $\partial\Omega$. Extend γ to all of $\mathbb{R}^n$ by defining $\gamma \equiv 1$ on $\mathbb{R}^n \setminus \Omega$ and define $q = \Delta\sqrt{\gamma}/\sqrt{\gamma}$.

Then $u \in H^1(\Omega)$ solves (2.10) if and only if $w = \gamma^{1/2} u \in H^1(\Omega)$ solves (2.11). In particular, 0 is not a Dirichlet eigenvalue of $(-\Delta + m_q)$ on Ω and $\Lambda_q = \Lambda_\gamma$.

Proof. It suffices to show that for all $w \in H^1(\Omega)$,

$$\nabla \cdot (\gamma \nabla(\gamma^{-1/2} w)) = \gamma^{1/2} \left(\Delta w - q w\right).$$

Indeed, since $\gamma \in [c, c^{-1}]$ a.e.,

$$\begin{aligned} \gamma\nabla(\gamma^{-1/2}w) &= \gamma^{1/2}\nabla w - (\nabla\gamma^{1/2})w \quad \text{in } H^{-1}(\Omega) \\ \Rightarrow \nabla\cdot\gamma\nabla(\gamma^{-1/2}w) &= \gamma^{1/2}\Delta w + \nabla\gamma^{1/2}\cdot\nabla w - (\Delta\gamma^{1/2})w - \nabla\gamma^{1/2}\cdot\nabla w \\ &= \gamma^{1/2}(\Delta w - qw) \quad \text{in } H^{-2}(\Omega) \end{aligned}$$

and hence also in the sense of distributions. This along with the fact that $\gamma \equiv 1$ on $\partial\Omega$ implies that w solves (2.11) iff $u = \gamma^{-1/2}w$ solves (2.10).

□

Proposition 2.9. *Let γ be as in Theorem 2.3 and $q = \Delta\sqrt{\gamma}/\sqrt{\gamma}$. Then $\sqrt{\gamma}$ is the unique solution in $H^1(\Omega)$ of*

$$\begin{cases} (-\Delta + q)u = & 0 \text{ in } \Omega \\ u \equiv & 1 \text{ on } \partial\Omega. \end{cases}$$

Proof. $u = \sqrt{\gamma}$ is clearly a solution. Moreover, the solution is unique by Proposition 2.8 and Proposition 2.7. □

Therefore, if we can find a constructive procedure for computing q from the knowledge of $\Lambda_q = \Lambda_\gamma$, we can also reconstruct γ from the recovered function q. Thus, the inverse problem for the conductivity equation has been reduced to the inverse problem for the Schrödinger equation. In the next few sections, we will show how to reconstruct q from Λ_q. In fact, we will prove the following result:

Theorem 2.10 (T., 2020 [37]). *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$. Let $q \in W^{-1/2,2n}_{comp}(\Omega)$ be such that 0 is not a Dirichlet eigenvalue of the boundary value problem* (2.11). *Then,*

(a) There is a constructive procedure to compute q from the knowledge of the map Λ_q. Moreover,

(b) We have the following stability estimate: Let $q_j \in W^{-1/2,2n}_{comp}(\Omega)$, $j = 1,2$. *Suppose in addition that* $\|q_j\|_{W^{-s,n/s}} \le M$ *for some* $0 < s < 1/2$. *Then there exist* $C = C(\Omega, n, c, M, s) > 0$ *and* $0 < \sigma = \sigma(n,s) < 1$ *such that*

$$\|q_1 - q_2\|_{H^{-1}} \le C \left(|\log \|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}\to H^{-1/2}}|^{-\sigma} + \|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}\to H^{-1/2}} \right). \quad (2.12)$$

Theorem 2.3 will be proved as a consequence of the above theorem.

2.2 Complex Geometrical Optics Solutions

In this section, we will construct Complex Geometrical Optics (CGO) solutions to the Schrodinger equation

$$(-\Delta + q)u = 0 \qquad \text{in } \mathbb{R}^n$$

under the assumption that $q \in W^{-1/2,2n}_{\text{comp}}(\mathbb{R}^n)$. As we observed before, given any $\zeta \in \mathbb{C}^n$ satisfying $\zeta \cdot \zeta = 0$, $e^{x\cdot\zeta}$ is a harmonic function in $\mathbb{R}^n$. Viewing $(-\Delta + q)$ as a perturbation of the Laplacian, we look for solutions to $(-\Delta + q)u = 0$ of the form

$$u_\zeta(x) = e^{x\cdot\zeta}(1 + r_\zeta(x))$$

where r_ζ is *small* in a certain asymptotic sense. Solutions such as u_ζ are known as CGO solutions. We will show that such a solution exists whenever $|\zeta|$ is large enough and establish certain asymptotic bounds on r_ζ as $|\zeta| \to \infty$.

Observe that $u_\zeta(x) = e^{x\cdot\zeta}(1 + r_\zeta(x))$ solves $(-\Delta + q)u_\zeta = 0$ iff

$$\begin{aligned} -\Delta(e^{x\cdot\zeta} r_\zeta) + e^{x\cdot\zeta} q r_\zeta &= -e^{x\cdot\zeta} q \\ \Leftrightarrow (-\Delta_\zeta + m_q) r_\zeta &= -q \end{aligned} \quad (2.13)$$

where $\Delta_\zeta v := e^{-x\cdot\zeta}\Delta(e^{x\cdot\zeta} v) = (\Delta + 2\zeta\cdot\nabla)v$. The symbol of Δ_ζ is

$$p_\zeta(\xi) = -|\xi|^2 + 2i\zeta\cdot\xi,$$

which vanishes only on a co-dimension 2 sphere in $\mathbb{R}^n$. Therefore, we may define a right inverse G_ζ of Δ_ζ as the operator whose symbol is $p_\zeta(\xi)^{-1}$. To be precise, for any $f \in \mathcal{S}(\mathbb{R}^n)$, we define $G_\zeta f$ as the tempered distribution given by

$$G_\zeta f = \left(\frac{\widehat{f}(\xi)}{-|\xi|^2 + 2i\zeta \cdot \xi} \right)^\vee . \tag{2.14}$$

We look for solutions to (2.13) of the form $r_\zeta = G_\zeta s_\zeta$. We see that such an s_ζ should satisfy

$$(I - m_q G_\zeta) s_\zeta = q$$

where I denotes the identity operator. The strategy used by Sylvester and Uhlmann in [36] is to establish bounds on the operators m_q and G_ζ between appropriate function spaces such that the operator norm $\|m_q G_\zeta\| < 1$ for $|\zeta|$ large enough. If that is the case, the above equation has a unique solution given by the Neumann series

$$s_\zeta = \sum_{j=0}^{\infty} (m_q G_\zeta)^j q.$$

We will use the same bounds on G_ζ as the one proved in [36]. However, since we make lower regularity assumptions on q, we will need a different bound on the multiplication operator m_q. The same bound can also be obtained by a slight modification of the estimate in [6].

Let us begin by introducing some function spaces.

Definition 2.11. For $\delta \in \mathbb{R}$, we define the weighted L^2 space $L^2_\delta(\mathbb{R}^n)$ by the norm

$$\|u\|_{L^2_\delta} = \left(\int_{\mathbb{R}^n} (1 + |x|^2)^\delta |u(x)|^2 \, dx \right)^{1/2} .$$

For $m \in \mathbb{N}$, we define the corresponding weighted Sobolev spaces $H^m_\delta(\mathbb{R}^n)$ through the norms

$$\|u\|_{H^m_\delta} = \sum_{|\alpha| \le m} \|\partial^\alpha u\|_{L^2_\delta}.$$

Finally, notice that L^2_δ and $L^2_{-\delta}$ are dual to each other with respect to $\langle \cdot, \cdot \rangle_{L^2}$. Motivated by this fact, we define the negative order spaces $H^{-m}_\delta(\mathbb{R}^n)$ for $m \in \mathbb{N}$ as the dual spaces of $H^m_{-\delta}(\mathbb{R}^n)$.

We also introduce the following scaled Sobolev norms, which will provide a convenient way of summarizing the bounds on $G_\zeta u$ proved in [36].

Definition 2.12. Let $s \in \mathbb{R}, k \geq 1$. We define $H^{s,k}(\mathbb{R}^n)$ through the norms

$$\|u\|_{H^{s,k}} = \|(k^2 - \Delta)^{s/2} u\|_{L^2} = \frac{1}{(2\pi)^{n/2}} \left(\int (k^2 + |\xi|^2)^s |\widehat{u}(\xi)|^2 \, d\xi \right)^{1/2}.$$

Observe that $H^{s,k}(\mathbb{R}^n)$ and $H^{-s,k}(\mathbb{R}^n)$ are dual to each other with respect to $\langle \cdot, \cdot \rangle_{L^2}$. If $s \in \mathbb{N}$, then for $\delta \in \mathbb{R}$, we define $H^{s,k}_\delta(\mathbb{R}^n)$ through the norms

$$\|u\|_{H^{s,k}_\delta} = \sum_{|\alpha| \leq s} k^{s-|\alpha|} \|\partial^\alpha u\|_{L^2_\delta}.$$

For negative integers s, we define $H^{s,k}_\delta(\mathbb{R}^n)$ as the dual of $H^{-s,k}_{-\delta}(\mathbb{R}^n)$.

Note that while we could define $H^{s,k}_\delta(\mathbb{R}^n)$ even for non-integral values of s, this turns out to be unnecessary for our purposes. As in the case of the usual negative order Sobolev spaces, we have the following alternate characterization of $H^{-m,k}_\delta(\mathbb{R}^n)$ for $n \in \mathbb{N}$. The proof is very similar to the standard case and therefore is omitted.

Proposition 2.13. *Let $m \in \mathbb{N}, \delta \in \mathbb{R}, k \geq 1$. For every $u \in H^{-m,k}_\delta(\mathbb{R}^n)$, there exist $\{u_\alpha \in L^2_\delta(\mathbb{R}^n) : |\alpha| \leq m\}$ such that*

$$\langle u, v \rangle = \sum_{|\alpha| \leq m} \langle \partial^\alpha u_\alpha, v \rangle = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \langle u_\alpha, \partial^\alpha v \rangle_{L^2} \quad \forall v \in H^{m,k}_{-\delta}(\mathbb{R}^n).$$

Moreover, u_α can be chosen to satisfy

$$\sum_{|\alpha| \leq m} k^{-(m-|\alpha|)} \|u_\alpha\|_{L^2_\delta} = \|u\|_{H^{-m,k}_\delta}.$$

Let us also record the following simple fact for future use.

Lemma 2.14. *Let $m \in \mathbb{Z}, k \geq 1$ and $\delta, \eta \in \mathbb{R}$. For any $\varphi \in C^\infty_c(\mathbb{R}^n)$,*

$$\|\varphi u\|_{H^{m,k}_\delta} \lesssim_{\varphi, m, \delta, \eta} \|u\|_{H^{m,k}_\eta}, \qquad u \in H^{m,k}_\eta(\mathbb{R}^n).$$

Proof. For $m \geq 0$, this follows from the fact that for any multi-index α, $\partial^\alpha \varphi(x)(1+|x|^2)^{(\delta-\eta)/2}$ is bounded above. Now suppose $m < 0$. Let $v \in H^{-m,k}_{-\delta}(\mathbb{R}^n)$. Then

$$\begin{aligned} |\langle \varphi u, v \rangle_{L^2}| &= |\langle u, \varphi v \rangle| \\ &\leq \|u\|_{H^{m,k}_\eta} \|\varphi v\|_{H^{-m,k}_{-\eta}} \\ &\leq \|u\|_{H^{m,k}_\eta} \|v\|_{H^{-m,k}_{-\delta}}. \end{aligned}$$

Taking the supremum of the left hand side over all v with $\|v\|_{H^{-m,k}_{-\delta}} \leq 1$ gives us the desired result. ☐

Now, let us recall the bounds on G_ζ proved by Sylvester and Uhlmann.

Proposition 2.15 (Sylvester-Uhlmann, 1987 [36])**.** *Let $\zeta \in \mathbb{C}^n$ be such that $|\zeta| \geq 1$ and $\zeta \cdot \zeta = 0$, and let $0 < \delta < 1/2$. Then G_ζ satisfies*

$$\|G_\zeta u\|_{H^s_{-\delta}} \lesssim |\zeta|^{s-1} \|u\|_{L^2_\delta}, \qquad u \in L^2_\delta,\ k = 0, 1, 2.$$

As an easy corollary, we obtain the following estimate for G_ζ on negative-order Sobolev spaces:

Corollary 2.16. *Let $\zeta \in \mathbb{C}^n$ be such that $\zeta \cdot \zeta = 0$ and $k = |\zeta| \geq 1$, and let $0 < \delta < 1/2$. Then G_ζ maps $H^{-1,k}_\delta \to H^{1,k}_{-\delta}(\mathbb{R}^n)$ and satisfies the bound*

$$\|G_\zeta u\|_{H^{1,k}_{-\delta}} \lesssim k \|u\|_{H^{-1,k}_\delta}, \qquad u \in H^{-1,k}_\delta(\mathbb{R}^n). \tag{2.15}$$

Proof. Let $u \in H^{-1,k}_\delta(\mathbb{R}^n)$. Then by Proposition 2.13, there exist $u_0, u_1, \ldots, u_n \in L^2_\delta(\mathbb{R}^n)$ such that $u = u_0 + \sum_{j=1}^n \partial_j u_j$ and

$$k^{-1} \|u_0\|_{L^2_\delta} + \sum_{j=1}^n \|u_j\|_{L^2_\delta} = \|u\|_{H^{-1,k}_\delta}.$$

Now, by Proposition 2.15 and the fact that G_ζ commutes with $\partial_j, j = 1, \ldots, n$,

$$\begin{aligned}
\|G_\zeta u_0\|_{L^2_{-\delta}} &\lesssim k^{-1} \|u_0\|_{L^2_\delta} \lesssim \|u\|_{H^{-1,k}_\delta}, \\
\|G_\zeta \partial_j u_j\|_{L^2_{-\delta}} &\lesssim \|G_\zeta u_j\|_{H^1_{-\delta}} \lesssim \|u_j\|_{L^2_\delta} \lesssim \|u\|_{H^{-1,k}_\delta}, \\
\|\nabla G_\zeta u_0\|_{L^2_{-\delta}} &\lesssim \|G_\zeta u_0\|_{H^1_{-\delta}} \lesssim \|u_0\|_{L^2_\delta} \lesssim k \|u\|_{H^{-1,k}_\delta}, \\
\|\nabla G_\zeta \partial_j u_j\|_{L^2_{-\delta}} &\lesssim \|G_\zeta u_j\|_{H^2_{-\delta}} \lesssim k \|u_j\|_{L^2_\delta} \lesssim k \|u\|_{H^{-1,k}_\delta}.
\end{aligned}$$

Combining all the above inequalities, we get (2.15). ☐

Next, we establish $H^{s,k}$ bounds on the multiplication operator $m_q : f \mapsto qf$ when q is of negative Sobolev regularity. This result follows from a simple modification of Proposition 3.2 in [10]. Nevertheless, we present the full proof here for completeness.

Theorem 2.17. *Let* $s > 0$ *and* $\max\{2, n/s\} < p < \infty$. *Suppose* $V \in W^{-s,p}(\mathbb{R}^n)$. *Then for* $k \geq 1$,

$$\|Vf\|_{H^{-s,k}} \lesssim \omega(k)\|f\|_{H^s} \lesssim \omega(k)\|f\|_{H^{s,k}}, \qquad \forall f \in H^s(\mathbb{R}^n), \tag{2.16}$$

where ω *is a positive function such that* $\omega(k) \to 0$ *as* $k \to \infty$. *If in addition we have* $0 < s \leq 1$, *then*

$$\|Vf\|_{H^{-1,k}} \lesssim k^{-(1-s)}\omega(k)\|f\|_{H^1}, \quad \text{and} \tag{2.17}$$

$$\|Vf\|_{H^{-1,k}} \lesssim k^{-2(1-s)}\omega(k)\|f\|_{H^{1,k}}. \tag{2.18}$$

Proof. By duality, it suffices to prove that

$$|\langle Vf, g\rangle_{L^2}| = |\langle V, fg\rangle_{L^2}| \lesssim \omega(k)\|f\|_{H^s}\|g\|_{H^s_k} \qquad \text{for all } f, g \in \mathcal{S}(\mathbb{R}^n).$$

Let $W \in L^p(\mathbb{R}^n)$ be such that $V = (I - \Delta)^{s/2}W$. Then we have,

$$\langle V, fg\rangle = \langle (I - \Delta)^{s/2}W, fg\rangle = \langle W, (I - \Delta)^{s/2}(fg)\rangle.$$

Let $\varphi \in C_c^\infty(\mathbb{R}^n; [0, 1])$ be such that $\int_{\mathbb{R}^n} \varphi(x)dx = 1$. We consider the sequence of mollifiers $\varphi_\epsilon(x) := \epsilon^{-n}\varphi(x/\epsilon)$ and define $W_\epsilon := \varphi_\epsilon * W$. Choosing $t \in (s - n/p, s)$, we may write

$$\begin{aligned}\langle V, fg\rangle &= \langle W_\epsilon, (I - \Delta)^{s/2}(fg)\rangle + \langle W - W_\epsilon, (I - \Delta)^{s/2}(fg)\rangle \\ &= \langle (I - \Delta)^{t/2}W_\epsilon, (I - \Delta)^{(s-t)/2}(fg)\rangle + \langle W - W_\epsilon, (I - \Delta)^{s/2}(fg)\rangle.\end{aligned}$$

Now, by Hölder's inequality,

$$|\langle V, fg\rangle| \leq \|(I-\Delta)^{t/2}W_\epsilon\|_{L^q}\|(I-\Delta)^{(s-t)/2}(fg)\|_{L^{q'}} + \|W - W_\epsilon\|_{L^p}\|(I-\Delta)^{s/2}(fg)\|_{L^{p'}}, \tag{2.19}$$

where $q = n/(s-t)$ and p', q' are conjugate exponents of p, q respectively. Since $t > s - n/p$, we have $q > p$ and therefore by Young's convolution inequality,

$$\begin{aligned}\|(I - \Delta)^{t/2}W_\epsilon\|_{L^q} &= \|((I - \Delta)^{t/2}\varphi_\epsilon) * W\|_{L^q} \\ &\leq \|(I - \Delta)^{t/2}\varphi_\epsilon\|_{L^r}\|W\|_{L^p} \\ &= \|\varphi_\epsilon\|_{W^{r,t}}\|W\|_{L^p},\end{aligned}$$

where $1/p+1/r=1+1/q$. Now by Sobolev embedding, $W^{r,t}\hookrightarrow L^u$ where $1/u=1/r-t/n$. Moreover, it can be easily verified that $\|\varphi_\epsilon\|_{L^u}=\epsilon^{n(1-u)/u}\|\varphi\|_{L^u}$. Therefore,

$$\begin{aligned}\|(I-\Delta)^{t/2}W_\epsilon\|_{L^q} &\leq \|\varphi_\epsilon\|_{L^u}\|W\|_{L^p}\\ &\lesssim \epsilon^{-t+n/q-n/p}\|W\|_{L^p}.\end{aligned}$$

Also, by Lemma 2.5,

$$\begin{aligned}\|(I-\Delta)^{(s-t)/2}(fg)\|_{L^{q'}} &\lesssim \|f\|_{H^{s-t}}\|g\|_{H^{s-t}}, \quad \text{and}\\ \|(I-\Delta)^{s/2}(fg)\|_{L^{p'}} &\lesssim \|f\|_{H^s}\|g\|_{H^s}.\end{aligned}$$

Therefore, from (2.19), we get

$$\begin{aligned}|\langle V,fg\rangle| &\lesssim \epsilon^{-t+n/q-n/p}\|W\|_{L^p}\|f\|_{H^{s-t}}\|g\|_{H^{s-t}}+\|W-W_\epsilon\|_{L^p}\|f\|_{H^s}\|g\|_{H^s}\\ &\lesssim \epsilon^{-t+n/q-n/p}\|W\|_{L^p}\|f\|_{H^{s-t}}\|g\|_{H^{s-t,k}}+\|W-W_\epsilon\|_{L^p}\|f\|_{H^s}\|g\|_{H^{s,k}}\\ &\lesssim (\epsilon^{-t+n/q-n/p}k^{-t}\|W\|_{L^p}+\|W-W_\epsilon\|_{L^p})\|f\|_{H^s}\|g\|_{H^{s,k}}.\end{aligned}$$

Here we have used the easy estimate $\|h\|_{H^{s-t,k}}\lesssim k^{-t}\|h\|_{H^{s,k}}$ for any $h\in\mathcal{S}(\mathbb{R}^n)$. Note that $n/q-n/p\geq(s-t)-s=-t$. Now choose $\epsilon=k^{-1/4}$. Then we get

$$|\langle V,fg\rangle|\lesssim\omega(k)\|f\|_{H^s}\|g\|_{H^{s,k}}\lesssim\omega(k)\|f\|_{H^{s,k}}\|g\|_{H^{s,k}} \tag{2.20}$$

where $\omega(k)=k^{-t/2}\|W\|_{L^p}+\|W-W_{k^{-1/4}}\|_{L^p}\to 0$ as $k\to\infty$. This proves (2.16). Now (2.17) and (2.18) follow from the fact that if $0<s\leq 1$,

$$|\langle V,fg\rangle| \lesssim \omega(k)\|f\|_{H^s}\|g\|_{H^{s,k}}\lesssim\omega(k)k^{-(1-s)}\|f\|_{H^1}\|g\|_{H^{1,k}}, \tag{2.21}$$

$$|\langle V,fg\rangle| \lesssim \omega(k)\|f\|_{H^{s,k}}\|g\|_{H^{s,k}}\lesssim\omega(k)k^{-2(1-s)}\|f\|_{H^{1,k}}\|g\|_{H^{1,k}}. \tag{2.22}$$

□

If in addition, V is compactly supported, the multiplication operator m_V can be extended to $H^{s,k}_\delta$ spaces.

Corollary 2.18. *Let* $0 < s < 1$ *and* $q \in W^{-s,n/s}(\mathbb{R}^n)$ *be such that supp* q *is compact. Suppose* $\delta, \eta \in \mathbb{R}$*. Then* $m_q : f \mapsto qf$ *satisfies the norm bounds*

$$\|m_q f\|_{H_\delta^{-1,k}} \lesssim k^{-(1-s)}\omega(k)\|f\|_{H_\eta^1}, \tag{2.23}$$

$$\|m_q f\|_{H_\delta^{-1,k}} \lesssim k^{-2(1-s)}\omega(k)\|f\|_{H_\eta^{1,k}}. \tag{2.24}$$

where ω *is a positive function on* $[1,\infty)$ *that satisfies* $\omega(k) \to 0$ *as* $k \to \infty$.

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ be such that $\varphi \equiv 1$ on supp q. Then by (2.21), for all $f, g \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned} |\langle qf, g\rangle_{L^2}| = |\langle q, fg\rangle| &= |\langle q, (\varphi f)(\varphi g)\rangle| \\ &\lesssim \omega(k)k^{-(1-s)}\|\varphi f\|_{H^1}\|\varphi g\|_{H^{1,k}} \\ &\lesssim \omega(k)k^{-(1-s)}\|f\|_{H_\eta^1}\|g\|_{H_{-\delta}^{1,k}} \quad \text{by Lemma 2.14.} \end{aligned}$$

Now (2.23) follows by density and duality. (2.24) similarly follows from (2.22). □

With the bounds on m_q and G_ζ in hand, we are now ready to construct CGO solutions.

Theorem 2.19. *Let* $q \in W^{-s,n/s}(\mathbb{R}^n)$*,* $0 < s \leq 1/2$ *be such that supp* q *is compact. Fix* $\delta \in (0, 1/2)$*. Then there exists* $M > 0$ *such that for all* $\zeta \in \mathbb{C}^n$ *satisfying*

$$\zeta \cdot \zeta = 0, \qquad |\zeta| \geq M,$$

there exists a unique solution to

$$(-\Delta + m_q)u = 0 \quad \text{in } \mathbb{R}^n$$

of the form

$$u = u_\zeta(x) = e^{x\cdot\zeta}(1 + r_\zeta(x))$$

where $r_\zeta \in H_{-\delta}^{1,k}(\mathbb{R}^n)$*. Moreover, we have the norm bound*

$$\|r_\zeta\|_{H_{-\delta}^{1,k}} \lesssim |\zeta|^s.$$

Proof. As seen before, $u_\zeta = e^{x\cdot\zeta}(1 + r_\zeta)$ satisfies $(-\Delta + q)u = 0$ if and only if

$$(-\Delta_\zeta + q)r_\zeta = -q$$

where $\Delta_\zeta = e^{-\zeta\cdot x}\Delta e^{\zeta\cdot x}$. Looking for solutions of the form $r_\zeta = G_\zeta s_\zeta$, we see that such an s_ζ should satisfy

$$(I - m_q \circ G_\zeta)s_\zeta = q. \tag{2.25}$$

Let $k = |\zeta|$. It follows from Corollary 2.16 and (2.24) from Corollary 2.18 that

$$\begin{aligned} \|G_\zeta\|_{H^{-1,k}_\delta \to H^{1,k}_{-\delta}} &\lesssim k, \\ \|m_q\|_{H^{1,k}_{-\delta} \to H^{-1,k}_\delta} &\lesssim k^{-2(1-s)}\omega(k) \end{aligned}$$

where $\omega(k) \to 0$ as $k \to \infty$. Therefore, $\|m_q \circ G_\zeta\| \lesssim k^{-1+2s}\omega(k) \to 0$ as $k \to \infty$ and there exists $M > 0$ such that for $k = |\zeta| \geq M$,

$$\|m_q \circ G_\zeta\|_{H^{-1,k}_\delta \to H^{-1,k}_\delta} \leq \frac{1}{2}.$$

Moreover, $q \in H^{-1,k}_\delta(\mathbb{R}^n)$. Indeed, suppose $\varphi \in C^\infty_c(\mathbb{R}^n)$ is such that $\varphi \equiv 1$ on supp q. Clearly $q = q\varphi = m_q(\varphi)$. Applying Theorem 2.17 with $k = 1$, we get

$$\|q\|_{H^{-s}} = \|\varphi q\|_{H^{-s}} \lesssim \|\varphi\|_{H^s} \lesssim \|\varphi\|_{H^s}.$$

Therefore,

$$\begin{aligned} \|q\|_{H^{-1,k}_\delta} = \|\varphi q\|_{H^{-1,k}_\delta} &\lesssim \|q\|_{H^{-1,k}} \qquad \text{by Lemma 2.14} \\ &\lesssim k^{-(1-s)}\|q\|_{H^{-s,k}} \\ &\lesssim k^{-(1-s)}\|q\|_{H^{-s}} \\ &\lesssim k^{-(1-s)}\|\varphi\|_{H^s}. \end{aligned}$$

Thus, for all $|\zeta| = k \geq M$, (2.25) has a unique solution given by the Neumann series

$$s_\zeta = \sum_{j=0}^{\infty}(m_q \circ G_\zeta)^j q$$

and we have the estimates

$$\|s_\zeta\|_{H^{-1,k}_\delta} \lesssim \|q\|_{H^{-1,k}_\delta} \lesssim k^{-(1-s)}, \tag{2.26}$$

$$\|r_\zeta\|_{H^{1,k}_{-\delta}} = \|G_\zeta s_\zeta\|_{H^{1,k}_{-\delta}} \lesssim k^s. \tag{2.27}$$

This completes the proof. □

2.3 Reconstruction through Boundary Integral Equations

The next theorem shows that Λ_q uniquely determines the Fourier transform of q, and hence q itself. Notice that we make the crucial assumption that $n \geq 3$.

Theorem 2.20. *Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 3)$ with Lipschitz boundary and let $q \in W^{-1/2,2n}_{comp}(\Omega)$ be such that 0 is not a Dirichlet eigenvalue of $(-\Delta+q)$ in Ω. We let Λ_q and Λ_0 denote the Dirichlet-to-Neumann maps for $-\Delta+q$ and $-\Delta$ respectively. Fix a non-zero $\xi \in \mathbb{R}^n$. Then for $k > 0$ sufficiently large, there exist $\zeta_1, \zeta_2 \in \mathbb{C}^n$ with $\zeta_j \cdot \zeta_j = 0$ and $|\zeta_j| = k$, $j = 1, 2$, such that*

$$\lim_{k\to\infty} \langle (\Lambda_q - \Lambda_0)(u_{\zeta_1}|_{\partial\Omega}), e^{x\cdot\zeta_2}\rangle = \langle q, e^{-ix\cdot\xi}\rangle = \int_\Omega q e^{-ix\cdot\xi}\, dx$$

where u_{ζ_1} is the unique solution to $(-\Delta+q)u = 0$ of the form $u_\zeta = e^{x\cdot\zeta}(1+r_\zeta)$ constructed in Theorem 2.19.

Proof. Since $n \geq 3$, we can choose unit vectors $\alpha, \beta \in \mathbb{R}^n$ such that $\{\xi/|\xi|, \alpha, \beta\}$ forms an orthonormal set. Now define $\zeta_1, \zeta_2 \in \mathbb{C}^n$ by

$$\zeta_1 = \frac{k}{\sqrt{2}}\alpha + i\left(-\frac{\xi}{2} + \sqrt{\frac{k^2}{2} - \frac{|\xi|^2}{4}}\beta\right), \tag{2.28}$$

$$\zeta_2 = -\frac{k}{\sqrt{2}}\alpha + i\left(-\frac{\xi}{2} - \sqrt{\frac{k^2}{2} - \frac{|\xi|^2}{4}}\beta\right). \tag{2.29}$$

Observe that $\zeta_1 \cdot \zeta_1 = \zeta_2 \cdot \zeta_2 = 0$ and $|\zeta_1| = |\zeta_2| = k$. Therefore, by Theorem 2.19, for all k large enough, there exist solutions $u_{\zeta_1} = e^{\zeta_1 \cdot x}(1 + r_{\zeta_1}(x))$ of $(-\Delta+q)u = 0$ such that

$\|r_{\zeta_1}\|_{H^{1,k}_{-\delta}} \lesssim k^{1/2}$. Moreover, the fact that $\zeta_2 \cdot \zeta_2 = 0$ implies $\Delta e^{x\cdot\zeta_2} = 0$. Therefore, by the integral identity(2.9),

$$\begin{aligned}
\langle(\Lambda_q - \Lambda_0)(u_{\zeta_1}|_{\partial\Omega}), e^{x\cdot\zeta_2}\rangle &= \langle q, u_{\zeta_1} e^{x\cdot\zeta_2}\rangle \\
&= \langle q, e^{x\cdot(\zeta_1+\zeta_2)}(1 + r_{\zeta_1})\rangle \\
&= \langle q, e^{-ix\cdot\xi}\rangle + \langle q, e^{-ix\cdot\xi} r_{\zeta_1}\rangle.
\end{aligned}$$

Now choose a cut-off function $\varphi \in C_c^\infty(\mathbb{R}^n)$ be such that $\varphi \equiv 1$ on $\overline{\Omega} \supset$ supp q. Using the estimate (2.23) in Corollary 2.18, we get

$$\begin{aligned}
|\langle q, e^{-ix\cdot\xi} r_{\zeta_1}\rangle| &= |\langle q, e^{-ix\cdot\xi}\varphi r_{\zeta_1}\rangle| \\
&\lesssim k^{-1/2}\omega(k)\|e^{-ix\cdot\xi}\varphi\|_{H^1}\|r_{\zeta_1}\|_{H^{1,k}_{-\delta}} \\
&\lesssim k^{-1/2}\omega(k)k^{1/2} = \omega(k) \qquad \text{(Theorem 2.19)},
\end{aligned}$$

where $\omega(k) \to 0$ as $k \to \infty$. Therefore, we conclude that

$$\lim_{k\to\infty}\langle(\Lambda_q - \Lambda_0)(u_{\zeta_1}|_{\partial\Omega}), e^{x\cdot\zeta_2}\rangle = \langle q, e^{-ix\cdot\xi}\rangle = \widehat{q}(\xi).$$

□

This Theorem shows that if we can somehow determine $u_{\zeta_1}|_{\partial\Omega}$, then we can use our knowledge of the map Λ_q to recover $\widehat{q}(\xi)$ for all $\xi \neq 0$. Moreover, since q is compactly supported, $\widehat{q}$ is continuous and therefore, $\widehat{q}(0)$ can also be recovered by continuity. Finally, we can recover q from $\widehat{q}$ using the Fourier Inversion formula. Therefore, our next goal is to find a constructive procedure to determine $u_\zeta|_{\partial\Omega}$. We will be able to characterize $u_\zeta|_{\partial\Omega}$ as the unique solution of a certain boundary integral equation of Fredholm type. The method was first introduced by Nachman [32] in 1988. We will mostly follow the presentation and notation in [12].

We begin by fixing some notation. Throughout the rest of this chapter, we assume that Ω is a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 3$. We let Ω_+ denote the exterior domain $\mathbb{R}^n \setminus \overline{\Omega}$. We also let $\mathrm{tr}, \mathrm{tr}_+$ and tr_- denote the trace operators that map $u \mapsto u|_{\partial\Omega}$ acting on $H^1_{\text{loc}}(\mathbb{R}^n), H^1(\Omega_+)$ and $H^1(\Omega)$ respectively. Finally, we let $(\partial_\nu\cdot)|_{\partial\Omega}, (\partial_\nu\cdot)_+$ and $(\partial_\nu\cdot)_-$ denote the normal trace operators that map $H^1_{\text{loc}}(\mathbb{R}^n), H^1(\Omega_+)$ and $H^1(\Omega)$ into $H^{-1/2}(\partial\Omega)$ respectively.

Next, given any $\zeta \in \mathbb{C}^n$ such that $\zeta \cdot \zeta = 0$ and $|\zeta| \geq 1$, we define the operator K_ζ by

$$K_\zeta(f) = e^{x\cdot\zeta} G_\zeta(e^{-x\cdot\zeta} f)$$

where G_ζ is the right inverse of Δ_ζ defined in (2.14). By a slight abuse of notation, we also use $K_\zeta(x, y)$ denote the Schwartz kernel of the operator K_ζ.

Lemma 2.21. *The operator K_ζ maps $H^{-1}_{comp}(\mathbb{R}^n) \to H^1_{loc}(\mathbb{R}^n)$ and satisfies*

$$\Delta K_\zeta f = f \qquad \textit{for all } f \in H^{-1}_{comp}(\mathbb{R}^n).$$

Proof. Fix $0 < \delta < 1/2$. Clearly, $f \mapsto e^{-x\cdot\zeta} f$ maps $H^{-1}_{\text{comp}} \hookrightarrow H^{-1}_\delta(\mathbb{R}^n)$. Then by Proposition 2.15, $f \mapsto G_\zeta(e^{-x\cdot\zeta} f)$ takes H^{-1}_{comp} into $H^1_{-\delta}(\mathbb{R}^n)$. Finally, multiplication by $e^{x\cdot\zeta}$ takes $H^1_{-\delta}(\mathbb{R}^n) \to H^1_{\text{loc}}(\mathbb{R}^n)$, which proves that $K_\zeta : H^{-1}_{\text{comp}}(\mathbb{R}^n) \to H^1_{\text{loc}}(\mathbb{R}^n)$.
Moreover, by definition of K_ζ,

$$\Delta K_\zeta f = e^{x\cdot\zeta} \Delta_\zeta G_\zeta(e^{-x\cdot\zeta} f) = f, \quad \forall f \in H^{-1}_{\text{comp}}(\mathbb{R}^n)$$

since G_ζ is a right inverse of Δ_ζ. □

Definition 2.22. Let $\zeta \in \mathbb{C}^n$ be such that $\zeta \cdot \zeta = 0$ and $|\zeta| \geq 1$. We define the modified (or Fadeev-type) Single layer potential S_ζ for $\partial\Omega$ as the operator

$$S_\zeta = K_\zeta \circ \operatorname{tr}^* : H^{-1/2}(\partial\Omega) \to H^1_{\text{loc}}(\mathbb{R}^n),$$

where $\operatorname{tr}^* : H^{-1/2}(\partial\Omega) \to H^{-1}_{\text{comp}}(\mathbb{R}^n)$ is the adjoint map of $\operatorname{tr} : H^1_{\text{loc}}(\mathbb{R}^n) \to H^{1/2}(\partial\Omega)$. To be precise, for any $f \in H^{-1/2}(\partial\Omega)$, $\operatorname{tr}^*(f)$ is the unique element of $H^{-1}_{\text{comp}}(\mathbb{R}^n)$ that satisfies

$$\int_\Omega \operatorname{tr}^*(f) v \, dx = \int_{\partial\Omega} f \operatorname{tr}(v) \, d\sigma \qquad \text{for all } v \in H^1_{\text{loc}}(\mathbb{R}^n)$$

where $d\sigma$ is the standard surface measure on $\partial\Omega$.

We will show that $u_\zeta|_{\partial\Omega}$ can be characterized as the unique solution $f \in H^{1/2}(\partial\Omega)$ of the following Boundary Integral Equation:

$$(\mathrm{Id} + \mathrm{tr}\, S_\zeta(\Lambda_q - \Lambda_0))f = e^{x\cdot\zeta} \quad \text{on } \partial\Omega. \tag{2.30}$$

Theorem 2.23. *Let $q \in W^{-1/2,2n}_{comp}(\Omega)$ be such that 0 is not a Dirichlet eigenvalue of $(-\Delta+q)$ in Ω. Let $\zeta \in \mathbb{C}^n$ be such that $\zeta\cdot\zeta = 0$ and $|\zeta|$ is sufficiently large, and let $0 < \delta < 1/2$. Consider the following problems:*

$$(DE) \quad \begin{cases} (-\Delta + q)u = 0 \quad \text{in } \mathbb{R}^n, \\ e^{-x\cdot\zeta}u - 1 \in H^1_{-\delta}(\mathbb{R}^n). \end{cases}$$

$$(EP) \quad \begin{cases} (i) \quad \Delta\widetilde{u} = 0 \quad \text{in } \Omega_+, \\ (ii) \quad \widetilde{u} = u|_{\Omega_+} \quad \text{for some } u \in H^1_{loc}(\mathbb{R}^n), \\ (iii) \quad e^{-x\cdot\zeta}\widetilde{u} - 1 = r|_{\Omega_+} \quad \text{for some } r \in H^1_{-\delta}(\mathbb{R}^n), \\ (iv) \quad (\partial_\nu u)_+ = \Lambda_q(\mathrm{tr}_+ u) \quad \text{on } \partial\Omega. \end{cases}$$

$$(BIE) \quad \begin{cases} (Id + \mathrm{tr}\, S_\zeta(\Lambda_q - \Lambda_0))f = e^{x\cdot\zeta} \quad \text{on } \partial\Omega, \\ f \in H^{1/2}(\partial\Omega). \end{cases}$$

Each of these problems has a unique solution. Furthermore, they are equivalent in the following sense: If u solves (DE), $\widetilde{u} = u|_{\Omega_+}$ solves (EP) and conversely, if $\widetilde{u}$ solves (EP), there exists a solution u of (DE) such that $\widetilde{u} = u|_{\Omega_+}$. Also, if u solves (DE), $f := u|_{\partial\Omega}$ solves (BIE) and conversely, if f solves (BIE), there exists a solution u of (DE) such that $f = u|_{\partial\Omega}$.

Proof. (DE) can be rephrased as the problem of finding solutions of the form $u = e^{x\cdot\zeta}(1+r)$ to the equation

$$(-\Delta + q)u = 0 \quad \text{in } \mathbb{R}^n,$$

where $r \in H^1_{-\delta}(\mathbb{R}^n)$. Therefore, (DE) has a unique solution by Theorem 2.19 for $|\zeta|$ sufficiently large. Now we show that (DE) is equivalent to (EP) and (BIE).

$(DE) \Rightarrow (BIE)$**:** Let u be the solution of (DE) and let $f = u|_{\partial\Omega}$. Clearly, $u \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$ and hence $f = \mathrm{tr}(u) \in H^{1/2}(\partial\Omega)$. Now, fix $x \in \Omega_+$ and define the function v on Ω by

$v(y) = K_\zeta(x,y)$, $y \in \Omega$. Since $\Delta v = 0$ in Ω, v is smooth by elliptic regularity. Now, by Green's theorem,

$$\int_{\partial\Omega} (u\partial_\nu v - v\partial_\nu u) d\sigma = \int_\Omega (u\Delta v - v\Delta u).$$

We know that $\Delta v = 0$ and $\Delta u = qu$. Moreover, since u, v satisfy $(-\Delta + q)u = 0$ and $\Delta v = 0$ in Ω respectively, $\partial_\nu u = \Lambda_q(u|_{\partial\Omega})$ and $\partial_\nu v = \Lambda_0(v|_{\partial\Omega})$. Substituting these into the above identity, we get

$$\begin{aligned} \int_{\partial\Omega} u\Lambda_0(v|_{\partial\Omega})\, d\sigma - \int_{\partial\Omega} K_\zeta(x,y)\Lambda_q(f)(y)\, d\sigma(y) &= -\int_\Omega K_\zeta(x,y)(qu)(y)\, dy \\ \implies \int_{\partial\Omega} u\Lambda_0(v|_{\partial\Omega})\, d\sigma - S_\zeta \Lambda_q f(x) &= -K_\zeta(qu)(x). \end{aligned}$$

Next, by symmetry of Λ_0, $\int_{\partial\Omega} u\Lambda_0(v|_{\partial\Omega})\, d\sigma = \int_{\partial\Omega} v\Lambda_0(f)\, d\sigma = S_\zeta\Lambda_0(f)$. Therefore, the above equation becomes

$$S_\zeta(\Lambda_0 - \Lambda_q)f(x) = -K_\zeta(qu)(x), \quad x \in \Omega_+. \tag{2.31}$$

Now, we simplify the right hand side. By definition,

$$\begin{aligned} K_\zeta(qu) &= e^{x\cdot\zeta}G_\zeta(e^{-x\cdot\zeta}qu) = e^{x\cdot\zeta}G_\zeta(e^{-x\cdot\zeta}\Delta u) \\ &= e^{x\cdot\zeta}G_\zeta \circ \Delta_\zeta(e^{-x\cdot\zeta}u) = e^{x\cdot\zeta}G_\zeta \circ \Delta_\zeta(e^{-x\cdot\zeta}u - 1). \end{aligned}$$

But we know that $e^{-x\cdot\zeta}u - 1 \in H^1_{-\delta}(\mathbb{R}^n)$ and G_ζ is a right inverse of Δ_ζ on $H^1_{-\delta}(\mathbb{R}^n)$. Therefore we get $K_\zeta(qu) = e^{x\cdot\zeta}(e^{-x\cdot\zeta}u - 1) = u - e^{x\cdot\zeta}$ and

$$u(x) + S_\zeta(\Lambda_q - \Lambda_0)f(x) = e^{x\cdot\zeta}, \qquad x \in \Omega_+.$$

Taking traces along $\partial\Omega$ on both sides, we get $(\mathrm{Id} + \mathrm{tr}\, S_\zeta(\Lambda_q - \Lambda_0))f = e^{x\cdot\zeta}$ on $\partial\Omega$, as desired.

$(BIE) \Rightarrow (EP)$: Suppose f solves (BIE). Define

$$\widetilde{u} := e^{x\cdot\zeta} - S_\zeta(\Lambda_q - \Lambda_0)f.$$

Clearly, $\widetilde{u}|_{\partial\Omega} = f$ and $\Delta\widetilde{u} = 0$ on $\mathbb{R}^n \setminus \partial\Omega$. Moreover, (ii) follows from the mapping properties of S_ζ. Next, from the jump properties of single layer potentials, we get

$$(\partial_\nu \widetilde{u})_- - (\partial_\nu \widetilde{u})_+ = -(\Lambda_q - \Lambda_0)f.$$

Since $\Delta\widetilde{u} = 0$ in Ω, $(\partial_\nu \widetilde{u})_- = \Lambda_0(\widetilde{u}|_{\partial\Omega}) = \Lambda_0 f$. Therefore, $(\partial_\nu \widetilde{u})_+ = \Lambda_q f$ and we have verified (iv). Finally, we note that

$$e^{-x\cdot\zeta}\widetilde{u} - 1 = -e^{-x\cdot\zeta} S_\zeta(\Lambda_q - \Lambda_0) f = G_\zeta e^{-x\cdot\zeta}\,\mathrm{tr}^*\, h,$$

where $h = (\Lambda_0 - \Lambda_q) f \in H^{-1/2}(\partial\Omega)$. Since $e^{-x\cdot\zeta}\,\mathrm{tr}^*\, h \in H^{-1}(\mathbb{R}^n)$ is compactly supported, $e^{-x\cdot\zeta}\,\mathrm{tr}^*\, h \in H^{-1}_{\delta}(\mathbb{R}^n)$ by the usual arguments. Finally, since $G_\zeta : H^{-1}_{\delta}(\mathbb{R}^n) \to H^1_{-\delta}(\mathbb{R}^n)$, we conclude that $e^{-x\cdot\zeta}\widetilde{u} - 1 \in H^1_{-\delta}(\mathbb{R}^n)$.

$(EP) \Rightarrow (DE)$: Let $\widetilde{u}$ solve (EP) and let $v \in H^1(\Omega)$ be the solution of

$$\begin{cases} (-\Delta + q)v = & 0, \\ v|_{\partial\Omega} = & \mathrm{tr}_+\, \widetilde{u}. \end{cases}$$

Define u on $\mathbb{R}^n$ by

$$u(x) = \begin{cases} v(x) & \text{in } \Omega, \\ \widetilde{u}(x) & \text{in } \Omega_+. \end{cases}$$

We have $\mathrm{tr}_-(u) = \mathrm{tr}_+(u)$ by construction and $(\partial_\nu u)_- = \Lambda_q(\mathrm{tr}_+\, \widetilde{u}) = (\partial_\nu u)_+$ by EP (iv). Therefore, it follows that $u \in H^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $(-\Delta + q)u = 0$ in $\mathbb{R}^n$. Finally, $e^{-x\cdot\zeta}u - 1 \in H^1_{-\delta}(\mathbb{R}^n)$ because of EP(iii) and the fact that $u = \widetilde{u}$ on Ω_+. □

Let us conclude by showing that the Boundary Integral Equation (2.30) is indeed Fredholm.

Proposition 2.24. *Let $q \in W^{-1/2,2n}_{comp}(\Omega)$ be such that 0 is not a Dirichlet eigenvalue of $(-\Delta + m_q)$ on Ω. Then the operator*

$$\mathrm{tr}\, S_\zeta(\Lambda_q - \Lambda_0) : H^{1/2}(\partial\Omega) \to H^{1/2}(\partial\Omega)$$

is compact.

Proof. Let $P_q : H^{1/2}(\partial\Omega) \to H^1(\Omega)$ be the solution operator that maps $f \in H^{1/2}(\partial\Omega)$ to the unique solution $u \in H^1(\Omega)$ of

$$\begin{cases} (-\Delta + q)u = & 0, \\ u|_{\partial\Omega} = & f. \end{cases}$$

By the same argument as the one leading to (2.31), we have

$$\operatorname{tr} S_\zeta(\Lambda_q - \Lambda_0)f = -\operatorname{tr} K_\zeta \circ m_q \circ P_q(f), \qquad f \in H^{1/2}(\partial\Omega).$$

But the right hand side is compact since $m_q : H^1(\Omega) \to H^{-1}_{\text{comp}}(\Omega)$ is compact by Proposition 2.7. This proves the result. □

2.4 Stability Estimates

We conclude this chapter with proofs of the stability estimates (2.5) and (2.12). We will first prove the stability estimate for the Schrödinger equation, and use this estimate to prove the estimate for the conductivity equation.

Given any $q \in W^{-1/2,2n}_{\text{comp}}(\Omega)$, we define the set of Cauchy data for q as

$$\mathcal{C}_q = \left\{ \left(u|_{\partial\Omega}, \frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} \right) \in H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega) : (-\Delta + q)u = 0 \right\}.$$

If 0 is not a Dirichlet eigenvalue of $(-\Delta + q)$ on Ω, then $\mathcal{C}_q$ is precisely the graph of the Dirichlet-to-Neumann map Λ_q. However, unlike Λ_q, $\mathcal{C}_q$ is well defined even when 0 is a Dirichlet eigenvalue of $(-\Delta + q)$ on Ω. We will equip $H^{1/2}(\partial\Omega) \times H^{-1/2}(\partial\Omega)$ with the norm

$$\|(f, g)\|_{H^{1/2} \oplus H^{-1/2}} = (\|f\|^2_{H^{1/2}(\partial\Omega)} + \|g\|^2_{H^{-1/2}(\partial\Omega)})^{1/2}.$$

Given $q_1, q_2 \in W^{-1/2,2n}_{\text{comp}}(\Omega)$, we define the distance between their Cauchy data sets by

$$\operatorname{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) = \max \left\{ \sup_{(f_1,g_1) \in \mathcal{C}_{q_1}} \inf_{(f_2,g_2) \in \mathcal{C}_{q_2}} \frac{\|(f_1 - f_2, g_1 - g_2)\|_{H^{1/2} \oplus H^{-1/2}}}{\|(f_1, g_1)\|_{H^{1/2} \oplus H^{-1/2}}}, \right.$$
$$\left. \sup_{(f_2,g_2) \in \mathcal{C}_{q_2}} \inf_{(f_1,g_1) \in \mathcal{C}_{q_1}} \frac{\|(f_1 - f_2, g_1 - g_2)\|_{H^{1/2} \oplus H^{-1/2}}}{\|(f_2, g_2)\|_{H^{1/2} \oplus H^{-1/2}}} \right\}.$$

It can be verified that if $\mathcal{C}_{q_j}$ are in fact the graphs of the Dirichlet-to-Neumann maps Λ_{q_j},

$$\frac{\|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2} \to H^{-1/2}}}{\sqrt{1 + \|\Lambda_{q_1}\|^2_{H^{1/2} \to H^{-1/2}}}\sqrt{1 + \|\Lambda_{q_2}\|^2_{H^{1/2} \to H^{-1/2}}}} \leq \operatorname{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) \leq \|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2} \to H^{-1/2}}. \tag{2.32}$$

We will establish bounds on $\|q_1 - q_2\|_{H^{-1}}$ in terms of $\operatorname{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})$. The estimate (2.12) follows from the theorem below:

Theorem 2.25. *Let $0 < s < 1/2$ and $q_1, q_2 \in W^{-s,n/s}_{comp}(\Omega)$ satisfy the a-priori estimate*

$$\|q_j\|_{W^{-s,n/s}} \leq M, \qquad j = 1, 2.$$

Then there exists $C > 0$ and $\sigma = \sigma(n, s) \in (0, 1)$ such that

$$\|q_1 - q_2\|_{H^{-1}} \leq C(|\log\{dist(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})\}|^{-\sigma} + dist(\mathcal{C}_{q_1}, \mathcal{C}_{q_2})). \tag{2.33}$$

Before beginning the proof, we record the following simple estimate:

Lemma 2.26. *For any $f \in H^{s,k}(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$, we have the estimate*

$$\|e^{-ix\cdot\xi} f\|_{H^{s,k}} \lesssim (1 + |\xi|^2)^{s/2} \|f\|_{H^{s,k}}.$$

Proof. By definition,

$$\begin{aligned}
\|e^{-ix\cdot\xi} f\|^2_{H^{s,k}} &= \frac{1}{(2\pi)^n} \int |\widehat{e^{-ix\cdot\xi} f}(\eta)|^2 (k^2 + |\eta|^2)^s \, d\eta \\
&= \frac{1}{(2\pi)^n} \int |\widehat{f}(\eta + \xi)|^2 (k^2 + |\eta|^2)^s \, d\eta \\
&= \frac{1}{(2\pi)^n} \int |\widehat{f}(\eta)|^2 k^{2s} \left(1 + \frac{|\eta - \xi|^2}{k^2}\right)^s d\eta \\
&\lesssim \int |\widehat{f}(\eta)|^2 k^{2s} \left(1 + \frac{|\eta|^2}{k^2}\right)^s \left(1 + \frac{|\xi|^2}{k^2}\right)^s d\eta \quad \text{(Peetre's inequality)} \\
&\lesssim (1 + |\xi|^2)^s \int |\widehat{f}(\eta)|^2 (k^2 + |\eta|^2)^s \, d\eta \\
&\lesssim (1 + |\xi|^2)^s \|f\|^2_{H^{s,k}}
\end{aligned}$$

which completes the proof. □

Proof of Theorem 2.25. Let $u_1, u_2 \in H^1(\Omega)$ satisfy $(-\Delta + q_j)u_j = 0$ in Ω, $j = 1, 2$. By the integral identity (2.9), we have

$$\int_{\partial\Omega} \left(u_2 \frac{\partial u_1}{\partial \nu} - u_1 \frac{\partial u_2}{\partial \nu}\right) d\sigma = \int_\Omega (q_1 - q_2) u_1 u_2 \, dx.$$

Let $(f, g) \in \mathcal{C}_{q_1}$. Then there exists $v \in H^1(\Omega)$ such that $(-\Delta + q_1)v = 0$, and

$$v|_{\partial\Omega} = f, \qquad \frac{\partial v}{\partial \nu}\Big|_{\partial\Omega} = g.$$

Applying (2.9) again,

$$0 = \int_{\partial\Omega} (q_1 - q_1) u_1 v \, dx = \int_{\partial\Omega} \left(f \frac{\partial u_1}{\partial \nu} - u_1 g \right) d\sigma$$

and therefore,

$$\int_{\Omega} (q_1 - q_2) u_1 u_2 \, dx = \int_{\partial\Omega} \left((u_2 - f) \frac{\partial u_1}{\partial \nu} - u_1 \left(\frac{\partial u_2}{\partial \nu} - g \right) \right) d\sigma.$$

This implies

$$\begin{aligned} \left| \int_{\Omega} (q_1 - q_2) u_1 u_2 \, dx \right| &\leq \|u_2 - f\|_{H^{1/2}(\partial\Omega)} \left\| \frac{\partial u_1}{\partial \nu} \right\|_{H^{-1/2}(\partial\Omega)} + \|u_1\|_{H^{1/2}(\partial\Omega)} \left\| \frac{\partial u_2}{\partial \nu} - g \right\|_{H^{-1/2}(\partial\Omega)} \\ &\leq \left\| \left(u_1, \frac{\partial u_1}{\partial \nu} \right) \right\|_{H^{1/2} \oplus H^{-1/2}} \cdot \left\| \left(u_2 - f, \frac{\partial u_2}{\partial \nu} - g \right) \right\|_{H^{1/2} \oplus H^{-1/2}}. \end{aligned}$$

Taking supremum over all $(f, g) \in \mathcal{C}_{q_1}$, we get

$$\left| \int_{\Omega} (q_1 - q_2) u_1 u_2 \, dx \right| \leq \left\| \left(u_1, \frac{\partial u_1}{\partial \nu} \right) \right\| \cdot \operatorname{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) \cdot \left\| \left(u_2, \frac{\partial u_2}{\partial \nu} \right) \right\|. \tag{2.34}$$

Now, we let u_1, u_2 be the CGO solutions constructed in Theorem 2.19. Choose $k > 0$, $\xi \in \mathbb{R}^n \setminus \{0\}$ and let α, β be unit vectors in $\mathbb{R}^n$ such that $\{\alpha, \beta, \xi/|\xi|\}$ forms an orthonormal set. Define $\zeta_1, \zeta_2 \in \mathbb{C}^n$ as in (2.28)-(2.29) and let

$$\begin{aligned} u_1(x) &= u_{\zeta_1}(x) = e^{x \cdot \zeta_1}(1 + r_1(x)), \\ u_2(x) &= u_{\zeta_2}(x) = e^{x \cdot \zeta_2}(1 + r_2(x)). \end{aligned}$$

where r_j, $j = 1, 2$, satisfy (2.27). Observe that

$$\begin{aligned} \left\| \left(u_j, \frac{\partial u_j}{\partial \nu} \right) \right\|_{H^{1/2} \oplus H^{-1/2}} &\lesssim \|u_j\|_{H^1(\Omega)} \lesssim \|e^{x \cdot \zeta_j}\|_{C^1(\Omega)} \|1 + r_j\|_{H^1(\Omega)} \\ &\lesssim k e^{Rk}(1 + k^s) \qquad \text{where } R = \sup_{x \in \Omega} |x| \\ &\lesssim e^{Sk}, \qquad \text{for some } S > R. \end{aligned}$$

Substituting in (2.34), we get

$$\left| \int_{\Omega} (q_1 - q_2) u_1 u_2 \, dx \right| \lesssim e^{2Sk} \operatorname{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}).$$

Now consider

$$(\widehat{q}_1 - \widehat{q}_2)(\xi) = \int_\Omega (q_1 - q_2)e^{-ix\cdot\xi}\,dx = \int_\Omega (q_1 - q_2)(u_1u_2 - e^{-ix\cdot\xi}(r_1 + r_2 + r_1r_2))\,dx$$

which implies that

$$\begin{aligned} |(\widehat{q}_1 - \widehat{q}_2)(\xi)| \lesssim &\left|\int_\Omega (q_1 - q_2)u_1u_2\,dx\right| + |\langle q_1 - q_2, e^{-ix\cdot\xi}(r_1 + r_2)\rangle| \\ &+ |\langle m_{q_1-q_2}(e^{-ix\cdot\xi}r_1), r_2\rangle|. \end{aligned} \tag{2.35}$$

Choose a cut-off function $\varphi \in C_c^\infty(\mathbb{R}^n)$ such that $\varphi \equiv 1$ on $\overline{\Omega}$. By (2.16),

$$|\langle m_{q_j}(e^{-ix\cdot\xi}r_1), r_2\rangle| \lesssim \omega(k)\|e^{-ix\cdot\xi}\varphi r_1\|_{H^{s,k}}\|\varphi r_2\|_{H^{s,k}}$$

where $\omega(k) \to 0$ as $k \to \infty$. It is obvious from the proof of Theorem 2.17 that

$$\omega(k) \leq \max_{j=1,2} \|q_j\|_{W^{-s,n/s}} \leq M \qquad \text{for all } k \geq 1.$$

We also have $\|e^{-ix\cdot\xi}\varphi r_1\|_{H^{s,k}} \lesssim (1 + |\xi|^2)^{s/2}\|\varphi r_1\|_{H^{s,k}}$ by Lemma 2.26. Therefore,

$$\begin{aligned} |\langle m_{q_j}(e^{-ix\cdot\xi}r_1, r_2\rangle| &\lesssim M(1 + |\xi|^2)^{s/2}\|\varphi r_1\|_{H^{s,k}}\|\varphi r_2\|_{H^{s,k}} \\ &\lesssim k^{-2(1-s)}M(1 + |\xi|^2)^{s/2}\|r_1\|_{H^{1,k}_{-\delta}}\|r_2\|_{H^{1,k}_{-\delta}} \\ &\lesssim k^{-4\epsilon}M(1 + |\xi|^2)^{s/2} \qquad \text{by (2.27)}, \end{aligned}$$

where $\epsilon = 1 - 2s$. Next, again by (2.16), for $j, l = 1, 2$,

$$\begin{aligned} |\langle q_j, e^{-ix\cdot\xi}r_l\rangle| &= |\langle m_{q_j}(\varphi), e^{-ix\cdot\xi}\varphi r_l\rangle| \\ &\lesssim \omega(k)\|\varphi\|_{H^s}\|e^{-ix\dot{\xi}}\varphi r_l\|_{H^{s,k}} \lesssim M(1 + |\xi|^2)^{s/2}\|\varphi r_l\|_{H^{s,k}} \\ &\lesssim M(1 + |\xi|^2)^{s/2}k^{-1+s}\|r_l\|_{H^{1,k}_{-\delta}} \\ &\lesssim M(1 + |\xi|^2)^{s/2}k^{-2\epsilon} \qquad \text{by (2.27)}. \end{aligned}$$

Substituting all these bounds into (2.35), we get

$$|\widehat{q}_1(\xi) - \widehat{q}_2(\xi)| \lesssim e^{2Sk}\mathrm{dist}(\mathcal{C}_{q_1}, \mathcal{C}_{q_2}) + k^{-2\epsilon}M(1 + |\xi|^2)^{s/2}.$$

We therefore have

$$\begin{aligned}\|q_1-q_2\|_{H^{-1}}^2 &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(1+|\xi|^2)^{-1}|\widehat{q}_1(\xi)-\widehat{q}_2(\xi)|^2\,d\xi\\ &\lesssim \int_{|\xi|\le\rho}(1+|\xi|^2)^{-1}|\widehat{q}_1(\xi)-\widehat{q}_2(\xi)|^2\,d\xi+\int_{|\xi|>\rho}(1+|\xi|^2)^{-1}|\widehat{q}_1(\xi)-\widehat{q}_2(\xi)|^2\,d\xi\\ &\lesssim \rho^n e^{4Sk}\mathrm{dist}^2(\mathcal{C}_{q_1},\mathcal{C}_{q_2})+k^{-4\epsilon}M^2\rho^{n+2s-2}\\ &\quad+\frac{1}{(1+\rho^2)^{1-s}}\int(1+|\xi|^2)^{-s}(\widehat{q}_1^2(\xi)+\widehat{q}_2^2(\xi))\,d\xi\\ &\lesssim \rho^n e^{4Sk}\mathrm{dist}^2(\mathcal{C}_{q_1},\mathcal{C}_{q_2})+M^2k^{-4\epsilon}\rho^{n-2\epsilon-1}+M^2\rho^{-1-2\epsilon}.\end{aligned}$$

In order to make the last two terms small and of the same order in ρ, we choose

$$k=\rho^{\frac{n}{4\epsilon}},$$

which gives us

$$\begin{aligned}\|q_1-q_2\|_{H^{-1}}^2 &\lesssim \rho^n e^{4S\rho^{\frac{n}{4\epsilon}}}\mathrm{dist}^2(\mathcal{C}_{q_1},\mathcal{C}_{q_2})+\rho^{-1-2\epsilon} &(2.36)\\ &\lesssim e^{T\rho^{\frac{n}{4\epsilon}}}\mathrm{dist}^2(\mathcal{C}_{q_1},\mathcal{C}_{q_2})+\rho^{-1-2\epsilon} &(2.37)\end{aligned}$$

for fixed $T>4S$. Now choose

$$\rho=\left(\frac{1}{T}|\log\{\mathrm{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})\}|\right)^{\frac{4\epsilon}{n}}$$

so that when $\mathrm{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})<1$,

$$e^{T\rho^{\frac{n}{4\epsilon}}}\mathrm{dist}^2(\mathcal{C}_{q_1},\mathcal{C}_{q_2})=\mathrm{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2}).$$

Combining this with (2.37), we see that when $\mathrm{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})<1$,

$$\begin{aligned}\|q_1-q_2\|_{H^{-1}}^2 &\lesssim \mathrm{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})+|\log\{\mathrm{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})\}|^{-\frac{4\epsilon(1+2\epsilon)}{n}}\\ &\lesssim |\log\{\mathrm{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})\}|^{-\frac{4\epsilon(1+2\epsilon)}{n}}.\end{aligned}$$

This gives us (2.33) when $\mathrm{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})<1$ for $\sigma=4\epsilon(1+2\epsilon)/n=4(1-s)(1-2s)/n$. Moreover, (2.33) is trivially true when $\mathrm{dist}(\mathcal{C}_{q_1},\mathcal{C}_{q_2})>1$ since $\|q_j\|_{H^{-1}}\lesssim\|q_j\|_{W^{-s,n/s}}\le M$ for $j=1,2$. Therefore, the proof is complete. □

We now use this result to prove the stability estimate for the conductivity equation. We will use the fact that $W^{s,p}$ embeds into the Zygmund space C^t_* for $t = s - n/p$. For $m \in \mathbb{N}$ and $0 < \alpha < 1$, $C^{m+\alpha}_*$ is identical to the space $C^{m,\alpha}$. We refer the reader to [39] for more details about Zygmund spaces.

Theorem 2.27. *Let $0 < s < 1/2$ and $\gamma_1, \gamma_2 \in W^{2-s,n/s}(\Omega)$ be such that $\gamma_j \equiv 1$ in a neighborhood of $\partial\Omega$ and*

$$0 < c < \gamma_j(x) < c^{-1}, \quad \text{for a.e. } x \in \Omega,\ j = 1, 2.$$

Given any $\alpha \in (0,1)$, there exists $C > 0$ and $\sigma = \sigma(n, s, \alpha) \in (0,1)$ such that

$$\|\gamma_1 - \gamma_2\|_{C^\alpha(\overline{\Omega})} \leq C(|\log \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}\to H^{-1/2}}|^{-\sigma} + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}\to H^{-1/2}}). \tag{2.38}$$

Proof. As in Proposition 2.8, let us extend γ_j to all of $\mathbb{R}^n$ by defining $\gamma_j \equiv 1$ on $\mathbb{R}^n \setminus \Omega$, so that $\gamma_j - 1 \in W^{-s,n/s}_{\text{comp}}(\Omega)$. Note that this implies $\gamma_j \in C^{1+\epsilon}_* = C^{1,\epsilon}$ for $\epsilon = 1 - 2s$. Define $q_j = \gamma_j^{-1/2}\Delta\gamma_j^{1/2}$. Also choose a bounded domain U such that $\overline{\Omega} \subset U$ and ∂U is smooth. We observe that the function $v = \log\gamma_1 - \log\gamma_2$ solves the following elliptic boundary value problem:

$$\begin{cases} \nabla \cdot ((\gamma_1\gamma_2)^{1/2}\nabla v) = & 2(\gamma_1\gamma_2)^{1/2}(q_2 - q_1) \quad \text{in } U \\ v = & 0 \quad \text{on } \partial U. \end{cases}$$

Therefore, we have the estimate

$$\|\log\gamma_1 - \log\gamma_2\|_{H^1(U)} \lesssim \|q_1 - q_2\|_{H^{-1}(U)} \lesssim \|q_1 - q_2\|_{H^{-1}}.$$

Now consider the identities

$$\begin{aligned} \gamma_1 - \gamma_2 &= \left(\int_0^1 e^{t\log\gamma_1 + (1-t)\log\gamma_2} dt\right) \cdot (\log\gamma_1 - \log\gamma_2), \\ \nabla\gamma_1 - \nabla\gamma_2 &= \gamma_1\nabla\log\gamma_1 - \gamma_2\nabla\log\gamma_2 = \gamma_1(\nabla\log\gamma_1 - \log\gamma_2) + \frac{\gamma_1 - \gamma_2}{\gamma_2}\nabla\gamma_2. \end{aligned}$$

Together with the fact that $\gamma_j \in C^{1,\epsilon}$, these identities imply that

$$\|\gamma_1 - \gamma_2\|_{H^1(U)} \lesssim \|\log\gamma_1 - \log\gamma_2\|_{H^1(U)} \lesssim \|q_1 - q_2\|_{H^{-1}}. \tag{2.39}$$

Next, recall from Proposition 2.8 that $\Lambda_{\gamma_j} = \Lambda_{q_j}$. By (2.33), for $\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}\to H^{-1/2}} = \|\Lambda_{q_1} - \Lambda_{q_2}\|_{H^{1/2}\to H^{-1/2}} < 1/2$,

$$\|q_1 - q_2\|_{H^{-1}} \lesssim |\log \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}\to H^{-1/2}}|^{-\sigma}$$

which along with (2.39) implies

$$\|\gamma_1 - \gamma_2\|_{H^1(U)} \lesssim |\log \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}\to H^{-1/2}}|^{-\sigma}.$$

Now, given $\alpha \in (0,1)$, define $p = n/(1-\alpha)$. By Hölder's inequality and the fact that $\gamma_j, \nabla\gamma_j$ are bounded,

$$\|\gamma_1 - \gamma_2\|_{W^{1,p}(U)} \lesssim \|\gamma_1 - \gamma_2\|_{H^1(U)}^{2/p}.$$

Therefore, whenever $\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}\to H^{-1/2}} < 1/2$,

$$\begin{aligned} \|\gamma_1 - \gamma_2\|_{W^{1,p}(U)} &\lesssim |\log \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}\to H^{-1/2}}|^{-\sigma'} \\ &\lesssim |\log \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}\to H^{-1/2}}|^{-\sigma'} + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}\to H^{-1/2}} \end{aligned}$$

for $\sigma' = \frac{2\sigma}{p} = \frac{8(1-s)(1-2s)(1-\alpha)}{n^2}$. On the other hand, the above estimate is clearly true when $\|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}\to H^{-1/2}} \geq 1/2$ due to the fact that $\gamma_j \in W^{1,\infty}$. Therefore, in all cases, we have

$$\|\gamma_1 - \gamma_2\|_{W^{1,p}(U)} \lesssim |\log \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}\to H^{-1/2}}|^{-\sigma'} + \|\Lambda_{\gamma_1} - \Lambda_{\gamma_2}\|_{H^{1/2}\to H^{-1/2}}.$$

Finally, (2.38) follows from the fact that $W^{1,p}(U) \hookrightarrow C^\alpha_*(U) = C^\alpha(U) \hookrightarrow C^\alpha(\overline{\Omega})$. $\square$

Chapter 3

ISOTROPIC CLOAKING FOR MAXWELL'S EQUATIONS

3.1 Maxwell's Equations

Maxwell's equations are a system of PDEs that describe the behavior of electric and magnetic fields arising from a distribution of charges and currents. Let $\Omega \subset \mathbb{R}^3$ be an open set representing an electromagnetic medium in space. For every $x \in \Omega$, let $\widetilde{\epsilon}(x), \mu(x)$, and $\sigma(x)$ denote the electric permittivity, magnetic permeability and conductivity at the point x. We allow the possibility that these parameters are *anisotropic*, in which case, they are given by real positive semi-definite matrix valued functions on Ω. We also assume for simplicity that there are no free charges or free currents inside Ω. Then Maxwell's equations state that the electric field $\mathbf{E}(x,t)$ and the magnetic field $\mathbf{H}(x,t)$ inside Ω (and varying with time t) satisfy the equations

$$\begin{aligned} \operatorname{div} \widetilde{\epsilon}\mathbf{E} &= 0 \\ \operatorname{div} \mu\mathbf{H} &= 0 \\ \operatorname{curl} \mathbf{E} &= -\mu\frac{\partial \mathbf{H}}{\partial t} \\ \operatorname{curl} \mathbf{H} &= \sigma\mathbf{E} + \widetilde{\epsilon}\frac{\partial \mathbf{E}}{\partial t}. \end{aligned}$$

We will be particularly interested in solutions of the form $\mathbf{E}(x,t) = \Re(E(x)e^{-i\omega t})$ and $\mathbf{H}(x) = \Re(H(x)e^{-i\omega t})$. Here, $\omega > 0$ and E, H can take values in $\mathbb{C}^3$. It can be easily verified that such E, H must satisfy

$$\begin{cases} \operatorname{curl} E = i\omega\mu H, \\ \operatorname{curl} H = -i\omega\left(\widetilde{\epsilon} + \frac{i\sigma}{\omega}\right) E. \end{cases} \tag{3.1}$$

These are the so-called time harmonic Maxwell's equations at frequency ω. For convenience, we define $\epsilon = \widetilde{\epsilon} + i\sigma/\omega$. In this section, we will discuss some basic existence and uniqueness results for boundary value problems for this system of equations. Throughout this section, we will assume that the medium Ω is *regular*, i.e., there exists $c > 0$ such that

$$c|\xi|^2 \leq \sum_{i,j=1}^{3} \widetilde{\epsilon}_{ij}(x)\xi_i\xi_j \leq c^{-1}|\xi|^2, \tag{3.2}$$

$$c|\xi|^2 \leq \sum_{i,j=1}^{3} \mu_{ij}(x)\xi_i\xi_j \leq c^{-1}|\xi|^2, \tag{3.3}$$

and

$$0 \leq \sum_{i,j=1}^{3} \sigma_{ij}(x)\xi_i\xi_j \leq c^{-1}|\xi|^2 \tag{3.4}$$

for all $\xi \in \mathbb{R}^3$ and a.e. $x \in \Omega$. Under these regularity assumptions on ϵ and μ, it can be shown that the boundary value problem for the Maxwell equations is uniquely solvable in a certain Sobolev-type space, except for a discrete set of resonance frequencies ω. The material in this section is classical, so we omit most of the proofs. The reader can refer to any of [28, 33, 31, 8, 7, 15] for the details. We begin by introducing some Sobolev-type function spaces needed to establish this result.

Definition 3.1. Let $\Omega \subset \mathbb{R}^3$ be open. We define the function spaces $H(\mathrm{curl}, \Omega)$ and $H(\mathrm{div}, \Omega)$ by

$$\begin{aligned} H(\mathrm{curl}, \Omega) &= \{u = (u^1, u^2, u^3) \in (L^2(\Omega))^3 : \mathrm{curl}\, u \in L^2(\Omega)^3\}, \\ H(\mathrm{div}, \Omega) &= \{u = (u^1, u^2, u^3) \in (L^2(\Omega))^3 : \mathrm{div}\, u \in L^2(\Omega)\}, \end{aligned}$$

and equip them with the inner products

$$\begin{aligned} \langle u, w\rangle_{H(\mathrm{curl},\Omega)} &= \int_\Omega u \cdot \overline{w} + (\mathrm{curl}\, u) \cdot \overline{(\mathrm{curl}\, w)}, \\ \langle u, w\rangle_{H(\mathrm{div},\Omega)} &= \int_\Omega u \cdot \overline{w} + (\mathrm{div}\, u)\overline{(\mathrm{div}\, w)} \end{aligned}$$

respectively. As in the case of the usual Sobolev spaces, we also define $H_0(\mathrm{curl}, \Omega)$ and $H_0(\mathrm{div}, \Omega)$ as the closures of $C_c^\infty(\Omega)$ in $H(curl, \Omega)$ and $H(div, \Omega)$ respectively.

It is easy to show that $H(\mathrm{curl}, \Omega)$ and $H(\mathrm{div}, \Omega)$ are both Hilbert spaces. If in addition, Ω has smooth boundary, $C^\infty(\overline{\Omega})^3$ is dense in both of these spaces, and there is a canonical way of taking traces of these functions on the boundary $\partial\Omega$.

Definition 3.2. Let $\Omega \subset \mathbb{R}^3$ be an open set with smooth boundary. Let ν be the outward unit normal vector field on $\partial\Omega$. For $u \in C^\infty(\overline{\Omega})^3$, we define

$$\begin{cases} \mathrm{tr}_n\, u := \nu \cdot u \quad \text{on } \partial\Omega \\ \mathrm{tr}_T\, u := u - (\nu \cdot u)\nu = -\nu \times (\nu \times u) \quad \text{on } \partial\Omega \\ \mathrm{tr}_\tau\, u := \nu \times u \quad \text{on } \partial\Omega. \end{cases}$$

In order to characterize the kernels of these trace maps on $H(\mathrm{curl}, \Omega)$ and $H(\mathrm{div}, \Omega)$, we need the notions of intrinsic gradient, curl, and divergence on a smooth surface in $\mathbb{R}^3$. Let $\varphi \in C^\infty(\partial\Omega)$ and $u \in C^\infty(\partial\Omega)^3$ be smooth scalar and vector valued functions on $\partial\Omega$ respectively. We may extend φ and u to smooth functions in a neighorhood of $\partial\Omega$. We also have a unique extension of ν to a smooth unit vector field on a neighborhood of $\partial\Omega$. Then the *surface gradient*, *surface curl*, and *surface divergence* operators are defined by

$$\begin{aligned} \nabla_{\partial\Omega}\varphi &= \nu \cdot \nabla\varphi|_{\partial\Omega}, \\ \mathrm{curl}_{\partial\Omega}\, u &= \nu \cdot (\mathrm{curl}\, u)|_{\partial\Omega}, \qquad \text{and} \\ \mathrm{div}_{\partial\Omega}\, u &= \mathrm{div}(u_T)|_{\partial\Omega} = \mathrm{div}(u - (u \cdot \nu)\nu)|_{\partial\Omega} \end{aligned}$$

respectively. It can be verified that the surface curl and surface divergence of u are related by

$$\mathrm{curl}_{\partial\Omega}\, u = \mathrm{div}_{\partial\Omega}(\nu \times u).$$

Moreover, we have the following two integral identities:

$$\begin{aligned} \int_{\partial\Omega} (\mathrm{curl}_{\partial\Omega}\, u)\varphi\, d\sigma &= -\int_{\partial\Omega} u \cdot (\nu \times \nabla_{\partial\Omega}\varphi)\, d\sigma, \\ \int_{\partial\Omega} (\mathrm{div}_{\partial\Omega}\, u)\varphi\, d\sigma &= -\int_{\partial\Omega} u \cdot \nabla_{\partial\Omega}\varphi\, d\sigma \end{aligned}$$

where $d\sigma$ is the surface measure on $\partial\Omega$. These identities allow us to extend the definitions of surface curl and divergence to distributions $u \in \mathcal{D}'(\partial\Omega)$ by

$$\begin{cases} \langle Div_{\partial\Omega} u, \varphi\rangle & = -\langle u, \nabla_{\partial\Omega}\varphi\rangle \\ \langle Curl_{\partial\Omega} u, \varphi\rangle & = -\langle u, \nu \times \nabla_{\partial\Omega}\varphi\rangle \end{cases} \qquad \forall \varphi \in C^\infty(\partial\Omega).$$

Now consider the following Sobolev spaces of tangential vector fields on $\partial\Omega$:

Definition 3.3. For all $s \in \mathbb{R}$,

$$\begin{aligned} TH^s(\partial\Omega) &:= \{u \in (H^s(\partial\Omega))^3 : \ \mathrm{tr}_n\, u = 0 \text{ on } \partial\Omega\} \\ H^s(\mathrm{div}, \partial\Omega) &:= \{u \in TH^s(\partial\Omega) : \mathrm{div}_{\partial\Omega}\, u \in H^s(\partial\Omega)\} \\ H^s(\mathrm{curl}, \partial\Omega) &:= \{u \in TH^s(\partial\Omega) : \mathrm{curl}_{\partial\Omega}\, u \in H^s(\partial\Omega)\}. \end{aligned}$$

Theorem 3.4 (Trace Theorem)**.** *The normal trace operator* $\mathrm{tr}_n : u \mapsto \nu \cdot u|_{\partial\Omega}$ *extends to a continuous linear map from* $H(\mathrm{div}, \Omega) \to H^{-1/2}(\partial\Omega)$*. The tangential trace operators* $\mathrm{tr}_T : u \mapsto -\nu \times (\nu \times u|_{\partial\Omega})$ *and* $\mathrm{tr}_\tau : u \mapsto \nu \times u|_{\partial\Omega}$ *extend to continuous linear operators from* $H(\mathrm{curl}, \Omega) \to H^{-1/2}(\mathrm{curl}, \partial\Omega)$ *and* $H(\mathrm{curl}, \Omega) \to H^{-1/2}(\mathrm{div}, \partial\Omega)$ *respectively. The kernels of these operators are precisely the closures of* $C_c^\infty(\Omega)^3$ *in the corresponding domain spaces, that is,*

$$\ker \mathrm{tr}_n = H_0(\mathrm{div}, \Omega), \qquad \ker \mathrm{tr}_T = \ker \mathrm{tr}_\tau = H_0(\mathrm{curl}, \Omega).$$

Moreover, $\mathrm{tr}_n, \mathrm{tr}_T, \mathrm{tr}_\tau$ *are surjective and have bounded right inverses* $\eta_n : H^{-1/2}(\partial\Omega) \to H(\mathrm{div}, \Omega)$*,* $\eta_T : H^{-1/2}(\mathrm{curl}, \partial\Omega) \to H(\mathrm{curl}, \Omega)$ *and* $\eta_\tau : H^{-1/2}(\mathrm{div}, \partial\Omega) \to H(\mathrm{curl}, \Omega)$*, respectively.*

We may now state the existence and uniqueness theorem for boundary value problems for the time harmonic Maxwell's equations.

Theorem 3.5. *Suppose* $\widetilde{\epsilon}, \mu, \sigma$ *satisfy the conditions* (3.2)-(3.4)*. There exists a discrete* *subset $F \subset \mathbb{R}$ such that for all $\omega \in \mathbb{R} \setminus F$, the following boundary value problem:*

$$\begin{cases} \operatorname{curl} E = i\omega\mu H \\ \operatorname{curl} H = -i\omega\left(\widetilde{\epsilon} + \frac{i\sigma}{\omega}\right) E + J \\ \nu \times E|_{\partial\Omega} = f \in H^{-1/2}(div, \partial\Omega) \end{cases} \tag{3.5}$$

has a unique solution $(E, H) \in H(curl, \Omega) \times H(curl, \Omega)$ and satisfies the stability estimate

$$\|E\|_{H(\operatorname{curl},\Omega)} + \|H\|_{H(\operatorname{curl},\Omega)} \leq C(\omega)(\|f\|_{H^{-1/2}(\operatorname{div},\partial\Omega)} + \|J\|_{L^2(\Omega)}).$$

On the other hand, if $\omega \in F$, there exist non-zero solutions to the corresponding homogeneous system

$$\begin{cases} \operatorname{curl} E = i\omega\mu H \\ \operatorname{curl} H = -i\omega\left(\widetilde{\epsilon} + \frac{i\sigma}{\omega}\right) E \\ \nu \times E|_{\partial\Omega} = 0. \end{cases}$$

We call F the set of electromagnetic eigenvalues for $(\Omega; \epsilon, \mu)$.

The proof is based on the variational characterization of (3.5) and Fredholm theory. A complete proof can be found in [31, 28]. We conclude this section with a couple of important facts about these curl and divergence based Sobolev spaces.

Lemma 3.6 (Duality)**.** *The spaces $H^{-1/2}(\operatorname{div}, \partial\Omega)$ and $H^{-1/2}(\operatorname{curl}, \partial\Omega)$ are mutually adjoint with respect to the scalar product in $TL^2(\partial\Omega)$. Moreover, for any $u_1, u_2 \in H(\operatorname{curl}, \Omega)$, the following Stokes formula holds*

$$\int_\Omega (u_1 \cdot (\operatorname{curl} u_2) - u_2 \cdot (\operatorname{curl} u_1))\, dx = \langle \gamma_T\, u_2, \gamma_\tau u_1 \rangle$$

where $\langle , \rangle$ denotes the duality bracket between $H^{-1/2}(\operatorname{curl}, \partial\Omega)$ and $H^{-1/2}(\operatorname{div}, \partial\Omega)$ with respect to the $L^2(\partial\Omega)$ inner product.

The next theorem is a crucial ingredient of the proof of the homogenization result Theorem 3.17, as well as the proof of our Main Theorem in Section 5. We include a proof for completeness.

Theorem 3.7 (The Div-Curl Lemma [3]). *Suppose*

$$\begin{aligned} u^n &\rightharpoonup u \quad \textit{in } H(\text{curl}, \Omega) \ \textit{and} \\ w^n &\rightharpoonup w \quad \textit{in } H(\text{div}, \Omega) \end{aligned}$$

as $n \to \infty$. *Then*

$$u^n \cdot \overline{w}^n \to u \cdot \overline{w} \quad \textit{in } \mathcal{D}'(\Omega),$$

that is,

$$\int_\Omega u^n \cdot \overline{w}^n \varphi \to \int_\Omega u \cdot \overline{w} \varphi \qquad \forall \varphi \in C_c^\infty(\Omega) \quad \textit{as } n \to \infty. \tag{3.6}$$

Proof. Fix $\varphi \in C_c^\infty(\Omega)$ and choose $\psi \in C_c^\infty(\Omega)$ such that $\psi \equiv 1$ on supp φ. Then (3.6) is equivalent to

$$\int_\Omega (\varphi u^n) \cdot \overline{\psi v^n} \to \int_\Omega (\varphi u) \cdot \overline{\psi v}. \tag{3.7}$$

Extending φu and ψv by 0 outside Ω, we see that $\varphi u, \psi v \in H(\text{curl}, \mathbb{R}^3)$. Set $r^n = \mathcal{F}(\varphi u^n), r = \mathcal{F}(\varphi u)$ and $s^n = \mathcal{F}(\psi v^n), s = \mathcal{F}(\psi v)$, where $\mathcal{F}(f)$ denotes the Fourier transform of f. By taking the Fourier transform of (3.7), we get

$$I^n = \int_{\mathbb{R}^3} r^n \cdot \overline{s}^n d\xi \to \int_{\mathbb{R}^3} r \cdot \overline{s}\, d\xi \tag{3.8}$$

by the Plancherel theorem. Note that $\varphi u^n \to \varphi u$ and $\psi v^n \to \psi v$ in $\mathcal{E}'(\Omega)$ and consequently

$$\begin{aligned} r^n(\xi) &= \langle \varphi u^n, e^{-ix\cdot\xi} \rangle \to \langle \varphi u, e^{-ix\cdot\xi} \rangle = r(\xi), \quad \text{and} \\ s^n(\xi) &= \langle \psi v^n, e^{-ix\cdot\xi} \rangle \to \langle \psi v, e^{-ix\cdot\xi} \rangle = s(\xi) \end{aligned}$$

uniformly on compact subsets of $\mathbb{R}^3_\xi$. Therefore, for any $M > 0$,

$$\int_{|\xi|\le M} r^n \cdot \overline{s}^n \to \int_{|\xi|\le M} f \cdot \overline{s} \quad \text{as } n \to \infty. \tag{3.9}$$

Moreover, for all $j, k \in \{1, 2, 3\}$,

$$\begin{aligned} \xi_j r_k^n - \xi_k r_j^n &\rightharpoonup \xi_j r_k - \xi_k r_j \quad \text{in } L^2(\mathbb{R}^3), \text{ and} \\ \xi_j s_j^n &\rightharpoonup \xi_j s_j \quad \text{in } L^2(\mathbb{R}^3) \end{aligned}$$

and consequently, are bounded in $L^2(\mathbb{R}^3)$ uniformly with respect to n . Now, we observe that

$$\xi_k r^n \cdot \overline{s}^n = \sum_{j=1}^{3} (\xi_k r_j^n - \xi_j r_k^n)\overline{s}_j^n + r_k^n \xi_j \overline{s}_j^n.$$

Therefore, by the Cauchy-Schwarz inequality,

$$|\xi||r^n \cdot \overline{s}^n| \leq g(\xi) \in L^1(\mathbb{R}^3) \quad \text{for all } n.$$

Now,

$$\begin{aligned} \left| I^n - \int_{|\xi|\leq M} r^n \cdot \overline{s}^n d\xi \right| &\leq \int_{|\xi|\geq M} \frac{1}{|\xi|} |\xi||r^n \cdot \overline{s}^n| d\xi \\ &\leq \frac{\|g\|_{L^1}}{M} \end{aligned}$$

for all n. Therefore, by triangle inequality,

$$\left| I^n - \int r \cdot \overline{s} d\xi \right| \leq \frac{\|g\|_{L^1}}{M} + \left| \int_{|\xi|\leq M} r^n \cdot \overline{s}^n d\xi - \int_{|\xi|\leq M} r \cdot \overline{s} d\xi \right| + \left| \int_{|\xi\leq M} r \cdot \overline{s} d\xi - \int_{R^3} r \cdot \overline{s} d\xi \right|.$$

It now follows from (3.9) that by taking n, M large enough, the left hand side can be made arbitrarily small, thus proving the result. □

3.2 Cloaking for Maxwell's equations

In this section, we will rigorously define what is meant by an invisibility cloak, and summarize the constructions of the degenerating ideal cloak and the regularized approximate cloak for Maxwell's equations constructed in [17] and [2] respectively.

Suppose $\omega \in \mathbb{R}$ is not an eigenvalue for $(\Omega; \epsilon, \mu)$. Analogous to the Dirichlet-to-Neumann map for the conductivity and Schrödinger equations, we define the so-called *Impedance map* $\Lambda_{\epsilon,\mu} : H^{-1/2}(\text{div}, \partial\Omega) \to H^{-1/2}(\text{div}, \partial\Omega)$ by

$$\Lambda_{\epsilon,\mu}(f) = \nu \times H|_{\partial\Omega},$$

where (E, H) is the unique solution of (3.5). In other words, $\Lambda_{\epsilon,\mu}$ maps the tangential component of the electric field on the boundary to the tangential component of the magnetic field on the bounday. It follows from Theorem 3.5 that $\Lambda_{\epsilon,\mu}$ is a continuous linear map. It is the analogue of the Dirichlet-to-Nuemann map for Maxwell's equations and encodes the set of all the possible electromagnetic measurements that can be made at the boundary of Ω.

Definition 3.8. Let $D \Subset \Omega$ be a smooth bounded subdomain and let $\widetilde{\epsilon_c}, \mu_c$ and σ_c be real symmetric matrix valued measurable functions on $\Omega \backslash \overline{D}$. Define $\epsilon_c = \widetilde{\epsilon_c} + i\sigma_c/\omega$ as before. We say that $(\Omega \backslash \overline{D}; \epsilon_c, \mu_c)$ is an invisibility **cloak** for the region D, if for any *regular* permittivity, permeability and conductivity $\widetilde{\epsilon_a}, \mu_a, \sigma_a$ defined in Ω, we have

$$\Lambda_{\epsilon_e,\mu_e} = \Lambda_{I,I} \qquad \text{on } \partial\Omega,$$

where I is the 3×3 identity matrix and ϵ_e, μ_e are the electromagnetic parameters of the extended object:

$$(\epsilon_e, \mu_e) = \begin{cases} (\widetilde{\epsilon_c} + i\sigma_c/\omega, \mu_c) & \text{in } \Omega \setminus \overline{D} \\ (\widetilde{\epsilon_a} + i\sigma_a/\omega, \mu_a) & \text{in } D \end{cases}$$

In other words, the boundary measurements $\Lambda_{\epsilon_e,\mu_e}$ must be indistinguishable from the measurements one would get if Ω was a vacuum (which corresponds to $\epsilon = \mu = I$). We call $\Omega \setminus \overline{D}$ and D the cloaking region and the cloaked region respectively.

The following invariance property of Maxwell's equations forms the basis of the transformation optics method of constructing electromagnetic cloaks: Let $\widetilde{\Omega}$ be another smooth bounded domain and suppose there exists a smooth diffeomorphism $F : \Omega \to \widetilde{\Omega}$.

Lemma 3.9. *Suppose* $(E, H) \in H(\text{curl}, \Omega) \times H(\text{curl}, \Omega)$ *satisfies the time harmonic Maxwell's equations (3.1) in* Ω. *Define the EM fields*

$$\widetilde{E} = F_*E := (DF^t)^{-1} E \circ F^{-1}, \qquad \widetilde{H} = F_*H = (DF^t)^{-1} H \circ F^{-1}$$

on $\widetilde{\Omega}$. *Then* $(\widetilde{E}, \widetilde{H}) \in H(\text{curl}, \widetilde{\Omega}) \times H(\text{curl}, \widetilde{\Omega})$ *and satisfy*

$$\begin{cases} \text{curl}\, \widetilde{E} = i\omega\mu' \widetilde{H}, \\ \text{curl}\, \widetilde{H} = -i\omega \left(\widetilde{\epsilon}' + \frac{i\sigma'}{\omega}\right) \widetilde{E} \end{cases} \tag{3.10}$$

where $\widetilde{\epsilon}', \mu', \sigma'$ *are the* push-forwards *of* $\widetilde{\epsilon}, \mu, \sigma$ *under* F*, defined by*

$$
\begin{aligned}
\widetilde{\epsilon}' &= F_*\widetilde{\epsilon} := \left(\frac{1}{\det(DF)}(DF)\cdot\widetilde{\epsilon}\cdot(DF)^t\right)\circ F^{-1},\\
\mu' &= F_*\mu := \left(\frac{1}{\det(DF)}(DF)\cdot\mu\cdot(DF)^t\right)\circ F^{-1},\\
\sigma' &= F_*\sigma := \left(\frac{1}{\det(DF)}(DF)\cdot\sigma\cdot(DF)^t\right)\circ F^{-1}.
\end{aligned}
$$

A proof can be found in [30]. As an immediate consequence, we have the following corollary.

Corollary 3.10. *Let* $(\Omega;\epsilon,\mu)$ *be as before and suppose* $F:\overline{\Omega}\to\overline{\Omega}$ *is a diffeomorphism such that* $F|_{\partial\Omega} = Id$*. Then*

$$\Lambda_{\epsilon,\mu} = \Lambda_{F_*\epsilon,F_*\mu} \qquad \text{on } \partial\Omega.$$

Proof. Fix $f \in H^{-1/2}(\mathrm{div},\partial\Omega)$. Let (E,H) and $(\widetilde{E},\widetilde{H})$ be the solutions of (3.1) and (3.10) respectively, with the boundary condition

$$\nu\times E = \nu\times\widetilde{E} = f.$$

Note that $F|_{\partial\Omega} = Id$ and consequently, $(DF_x)|_{T_x\partial\Omega} = Id$ for all $x\in\partial\Omega$. Therefore

$$\Lambda_{F_*\epsilon,F_*\mu}f = \nu\times\widetilde{H}|_{\partial\Omega} = (DF^t)^{-1}(\nu\times H)\circ F^{-1}|_{\partial\Omega} = \nu\times H|_{\partial\Omega} = \Lambda_{\epsilon,\mu}f.$$

□

Remark 4. In Lemma 3.9 and Corollary 3.10, it is sufficient to assume that $\Omega,\widetilde{\Omega}$ are Lipschitz and that F is bi-Lipschitz.

We now summarize the singular ideal cloak and the regular approximate cloak constructions in [17] and [2]. Henceforth, we let B_r denote $\{x\in\mathbb{R}^3 : |x| < r\}$. Consider the map $F:\overline{B}_2\setminus\{0\}\to\overline{B}_2\setminus\overline{B}_1$ given by

$$F(x) = \left(1+\frac{|x|}{2}\right)\frac{x}{|x|},$$

which blows up the Origin to the cloaked region $\overline{B}_1$. The central idea of transformation optics-based electromagnetic cloaking is to use the invariance properties 3.9 and 3.10 to "hide" an arbitrary regular electromagnetic object in B_1 using the transformation F. Notice however, that F is not a regular change of coordinates, as $\det(DF) \to 0$ as $|x| \to 0$. Define the electromagnetic parameters in $\overline{B}_2 \setminus \overline{B}_1$ to be the pushforwards of the identity under the coordinate transformation F:

$$\widetilde{\epsilon}(x) = \widetilde{\mu}(x) = F_* I = \frac{(DF) \cdot I \cdot (DF)^t}{|\det DF|}(F^{-1}(x)), \quad 1 < |x| \leq 2.$$

A simple computation gives us an explicit formula for $\widetilde{\epsilon}$ and $\widetilde{\mu}$:

$$\widetilde{\epsilon} = \widetilde{\mu} = 2\frac{(|x|-1)^2}{|x|^2}\Pi(x) + 2(I - \Pi(x))$$

where $\Pi(x)$ is the projection map in the radial direction.

$$\Pi(x)w = \left(w \cdot \frac{x}{|x|}\right)\frac{x}{|x|}.$$

It is easy to see that one of the eigenvalues of ϵ (and μ), namely, the one corresponding to the radial direction goes to 0 as $|x| \to 1$. Therefore, the regularity conditions (3.2)-(3.4) no longer hold. Again consider the extended object

$$(B_2; \widetilde{\epsilon}_e, \widetilde{\mu}_e) := \begin{cases} (B_2 \setminus \overline{B}_1; \widetilde{\epsilon}, \widetilde{\mu}) & \text{in } B_2 \setminus \overline{B}_1 \\ (B_1; \epsilon_a, \mu_a) & \text{in } B_1 \end{cases}$$

where ϵ_a and μ_a are arbitrary. It was shown in [17] that distributional solutions $(\widetilde{E}, \widetilde{H})$ of the Maxwell equations corresponding to $\widetilde{\epsilon}_e$ and $\widetilde{\mu}_e$ satisfying certain "finite energy" conditions have the same Cauchy data as the solutions of the Maxwell equations in free space. More precisely, if $(E, H) \in H(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega)$ is the solution of

$$\begin{cases} \mathrm{curl}\, E = i\omega H, \\ \mathrm{curl}\, H = -i\omega E, \\ \nu \times E|_{\partial\Omega} = \nu \times \widetilde{E}|_{\partial\Omega} \end{cases} \tag{3.11}$$

then

$$\nu \times \widetilde{H}|_{\partial\Omega} = \nu \times H|_{\partial\Omega}.$$

Thus, $(B_2 \setminus \overline{B}_1; \widetilde{\epsilon}, \widetilde{\mu})$ forms a perfect cloak for the region B_1. However, the singular nature of the parameters of this cloak poses serious challenges to practical implementation as well as theoretical analysis. Therefore, it is natural to consider regular approximations to the singular change of variables. As a trade-off, we lose the perfect cloaking of the singular construction and instead obtain an *approximate* cloak.

Definition 3.11. Let $D \Subset \Omega$ be a smooth bounded subdomain and let $\widetilde{\epsilon_c}^\rho$, μ_c^ρ and σ_c^ρ be real symmetric matrix valued measurable functions on $\Omega \setminus \overline{D}$, indexed by a positive real number ρ. Define $\epsilon_c^\rho = \widetilde{\epsilon_c}^\rho + i\sigma_c^\rho/\omega$ as before. We say that $(\Omega \setminus \overline{D}; \epsilon_c^\rho, \mu_c^\rho)$ is an **approximate cloak** for the region D, if for any *regular* permittivity, permeability and conductivity $\widetilde{\epsilon_a}, \mu_a, \sigma_a$ defined in Ω, we have

$$\|\Lambda_{\epsilon_e^\rho, \mu_e^\rho} - \Lambda_{I,I}\|_{\mathcal{L}(H^{-1/2}(div,\partial\Omega), H^{-1/2}(div,\partial\Omega))} \to 0 \qquad \text{as } \rho \to 0+\,.$$

where I is the 3×3 identity matrix and, as before $\epsilon_e^\rho, \mu_e^\rho$ are the electromagnetic parameters of the extended object:

$$(\epsilon_e^\rho, \mu_e^\rho) = \begin{cases} (\widetilde{\epsilon_c^\rho} + i\sigma_c^\rho/\omega, \mu_c) & \text{in } \Omega \setminus \overline{D} \\ (\widetilde{\epsilon_a} + i\sigma_a/\omega, \mu_a) & \text{in } D \end{cases}$$

Here, $\|\cdot\|_{\mathcal{L}(H^{-1/2}(\mathrm{div},\partial\Omega), H^{-1/2}(\mathrm{div},\partial\Omega))}$ denotes the standard operator norm on bounded linear maps from $H^{-1/2}(\mathrm{div}, \partial\Omega)$ to itself.

Two different approximation schemes have been proposed in the literature for approximate cloaking. In [18], Greenleaf et. al. proposed that one use the transformation F to blow up a small ball B_ρ to a larger ball B_R, where $1 < R < 2$. On the other hand, the proposal of Kohn et. al. in [29] is to use a transformation similar to F to blow up B_ρ to B_1. Here we will work with the latter approximation scheme. However, it is known that the former approximation scheme gives a similar performance [30].

Now, for $0 < \rho < 1$, consider the bi-Lipschitz transformation $F_\rho : \overline{B}_3 \to \overline{B}_3$ given by

$$F_\rho(x) = \begin{cases} x & \text{for } 2 \le |x| \le 3, \\ \left(\frac{2(1-\rho)}{2-\rho} + \frac{|x|}{2-\rho}\right)\frac{x}{|x|} & \text{for } \rho < |x| \le 2, \\ \frac{x}{\rho} & \text{for } |x| \le \rho, \end{cases} \tag{3.12}$$

Notice that F_ρ dilates B_ρ to B_1 and retracts $\overline{B}_3 \backslash B_\rho$ to $\overline{B}_3 \backslash B_1$. Also note that we are working in a slightly larger ball B_3 rather than B_2 and that $F_\rho = \mathrm{Id}$ in the annulus $\overline{B}_3 \setminus B_2$. The advantage of this modification is that the pushed-forward electric permittivity and magnetic permeability will both be identity in a neighbourhood of the boundary of our domain, which will prove useful in the proof of our Main Theorem in Section 5.

Theorem 3.12 ([2])**.** *Given $0 < \rho < 1$, consider the electromagnetic object $(B_3, \epsilon_e^\rho, \mu_e^\rho)$ with electromagnetic parameters defined by*

$$(\epsilon_e^\rho, \mu_e^\rho) = \begin{cases} ((F_\rho)_* I, (F_\rho)_* I) & \text{in } B_3 \setminus B_1, \\ ((F_\rho)_*(1 + i\rho^{-2}/\omega)I, (F_\rho)_* I) & \text{in } B_1 \setminus B_{1/2}, \\ (\epsilon_a, \mu_a) & \text{in } B_{1/2} \end{cases} \tag{3.13}$$

where ϵ_a, μ_a are arbitrary and regular. Suppose also that $\omega \in \mathbb{R}$ is not an eigenvalue of $(B_3; I, I)$. Then there exists $\rho_0 > 0$ such that for all $0 < \rho < \rho_0$, ω is not an eigenvalue for $(B_3, \epsilon_e^\rho, \mu_e^\rho)$ for any choice of regular ϵ_a, μ_a. Moreover,

$$\|\Lambda_{\epsilon_e^\rho, \mu_e^\rho} - \Lambda_{I,I}\|_{\mathcal{L}(H^{-1/2}(\mathrm{div}, \partial B_3), H^{-1/2}(\mathrm{div}, \partial B_3))} \to 0 \quad \text{as } \rho \to 0+ .$$

Notice that apart from regularizing F, we have added a layer of high conductivity in the region $B_1 \setminus B_{1/2}$. This additional layer was shown to be essential in [30], since in the absence of this layer, one can have ϵ_a, μ_a that make ω an eigenvalue.

3.3 Homogenization

In this section, we recall the notion of H-convergence and present a homogenization result for Maxwell's equations.

Definition 3.13. Let $0 < \alpha < \beta < \infty$ and let $\Omega \subset \mathbb{R}^N$ be open. We define $\mathcal{M}_{\mathbb{R}}(\alpha, \beta; \Omega)$ to be the set of all real $N \times N$ matrix-valued measurable functions $A(x) = [a_{kl}(x)]_{1 \le k,l \le N}$ defined almost everywhere on Ω such that

$$\begin{aligned} (A(x)\xi, \xi) &= \sum_{k,l=1}^{N} a_{kl}(x)\xi_k\xi_l \geq \alpha|\xi|^2, \quad \text{and} \\ |A(x)\xi| &\leq \beta|\xi| \end{aligned}$$

for all $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

Analogously, in the complex case, we define $\mathcal{M}_{\mathbb{C}}(\alpha, \beta; \Omega)$ to be the set of all complex $N \times N$ matrix-valued measurable functions $A(x)$ defined almost everywhere on Ω such that

$$\begin{aligned} -\mathrm{i}\xi \cdot \left(A(x) - A(x)^\dagger\right) \overline{\xi} &\geq \alpha|\xi|^2, \quad \text{and} \\ |A(x)\xi| &\leq \beta|\xi| \end{aligned}$$

for all $\xi \in \mathbb{C}^N$ and a.e. $x \in \Omega$. Here $A(x)^\dagger = (\overline{A(x)})^t = \overline{A(x)^t}$ denotes the Hermitian conjugate of $A(x)$. Note that $A(x) \in \mathcal{M}_{\mathbb{C}}(\alpha, \beta; \Omega)$ implies that $\Re(A(x))$ (real part of the matrix $A(x)$) is symmetric, i.e. $\Re(A(x))_{kl} = \Re(A(x))_{lk}$ for all $1 \le k, l \le N$.

We will say $A \in \mathcal{M}(\alpha, \beta; \Omega)$ if either $A \in \mathcal{M}_{\mathbb{R}}(\alpha, \beta; \Omega)$ or $A \in \mathcal{M}_{\mathbb{C}}(\alpha, \beta; \Omega)$.

Next, we define the notion of H-convergence [1, 38]:

Definition 3.14. Let $A^n \in \mathcal{M}(\alpha, \beta; \Omega)$ for $n \in \mathbb{N}$ and $A^* \in \mathcal{M}(\widetilde{\alpha}, \widetilde{\beta}; \Omega)$. We say

$$A^n \xrightarrow{H} A^* \text{ or } H\text{-converges to } A^*$$

if for all test sequences $u^n \in H^1(\Omega)$ satisfying

$$\begin{aligned} u^n \rightharpoonup u \quad &\text{weakly in } H^1(\Omega), \quad \text{and} \\ -\operatorname{div}(A^n \nabla u^n) &\text{ is strongly convergent in } H^{-1}(\Omega). \end{aligned}$$

we have $A^n \nabla u^n \rightharpoonup A^* \nabla u$ in $L^2(\Omega)^N$. We call A^* the homogenized matrix for the sequence $\{A^n\}$.

To illustrate the utility of H-convergence, let us consider the following sequence of elliptic boundary value problems: Let A^n be a sequence of matrices in $\mathcal{M}(\alpha, \beta; \Omega)$ such that $A^n \xrightarrow{H} A^*$ and let u_n be the solutions of

$$\begin{cases} -\operatorname{div}(A^n \nabla u_n) = f \in H^{-1}(\Omega), \\ u_n|_{\partial\Omega} = 0. \end{cases}$$

It is easy to see that $\|u_n\|_{H^1(\Omega)} \leq C$ for some constant C independent of n. Therefore, there exists a subsequence, which we still denote by u_n that converges weakly to some $u \in H_0^1(\Omega)$. Now, the definition of H-convergence implies that $A^n \nabla u_n \rightharpoonup A^* \nabla u$ as $n \to \infty$. Consequently, $0 = -\operatorname{div}(A^n \nabla u_n) \rightharpoonup -\operatorname{div}(A^* \nabla u)$. Therefore, the weak limit u of u_n is in fact the solution of the following "homogenized" problem:

$$\begin{cases} -\operatorname{div}(A^* \nabla u) = f \quad \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

3.3.1 Homogenization with periodic micro-structures

For some types of sequences A^n, the homogenized matrix A^* can be explicitly computed. Let us consider the class of periodic micro-structures as an example. Let Y denote the unit cube $[0,1]^N$ in $\mathbb{R}^N$. Let $A(y) = [a_{kl}(y)]_{1 \leq k,l \leq N} \in \mathcal{M}(\alpha, \beta; Y)$ be such that $a_{kl}(y)$ are Y-periodic functions $\forall k, l = 1, 2.., N$, that is, $a_{kl}(y + z) = a_{kl}(y)$ whenever $z \in \mathbb{Z}^N$ and $y \in Y$. Now we set

$$A^n(x) = [a_{kl}^n(x)] = [a_{kl}(nx)], \qquad x \in [0, 1/n]^N$$

and extend it to all of $\mathbb{R}^N$ by $1/n$-periodicity. The restriction of A^n to Ω is known as a periodic micro-structure. Such micro-structures arise in the study of physical systems where the parameters vary periodically, with a period that is very small compared to the dimensions of the object under consideration.

In this classical case, the homogenized conductivity $A^* = [a^*_{kl}]$ is a constant matrix whose entries are given by [1, 3]

$$a^*_{kl} = \int_Y \sum_{i,j=1}^{N} a_{ij}(y) \frac{\partial}{\partial y_i}(\chi_k(y) + y_k) \frac{\partial}{\partial y_j}(\chi_l(y) + y_l) dy$$

where we define the χ_k through the so-called cell-problems. For each canonical basis vector e_k, χ_k is defined to be the unique solution of the conductivity problem in the periodic unit cell :

$$-\operatorname{div}_y(\, A(y)(\nabla_y \chi_k(y) + e_k)) = 0 \quad \text{in } \mathbb{R}^N, \quad y \mapsto \chi_k(y) \quad \text{is } Y\text{-periodic}$$

in the Sobolev space $H^1_{\#,0}(Y)$ defined by

$$H^1_{\#,0}(Y) := \left\{ f \in H^1_{\text{loc}}(\mathbb{R}^N) : \, y \mapsto f(y) \text{ is } Y\text{-periodic}, \int_Y f(y) dy = 0 \right\}.$$

We can generalize the above case to what are called *locally periodic* micro-structures. Let $A(x,y) = [a_{kl}(x,y)]_{1 \leq k,l \leq N} \in \mathcal{M}(\alpha, \beta; \Omega \times Y)$ be such that $a_{kl}(x,\cdot)$ are Y-periodic functions with respect to the second variable $\forall k, l = 1, 2, .., N$ and for almost every x in Ω. Now we set

$$A^n(x) = [a^n_{kl}(x)] = [a_{kl}(x, nx)]$$

Then the homogenized conductivity $A^*(x) = [a^*_{kl}(x)]$ is given by [3, 26]

$$a^*_{kl}(x) = \int_Y \sum_{i,j=1}^{N} a_{ij}(x,y) \frac{\partial}{\partial y_i}(\chi_k(x,y) + y_k) \frac{\partial}{\partial y_j}(\chi_l(x,y) + y_l) dy \tag{3.14}$$

where $\chi_k(x,\cdot) \in H^1_{\#,0}(Y)$ solves the following cell problem for almost every x in Ω:

$$-\operatorname{div}_y(\, A(x,y)(\nabla_y \chi_k(x,y) + e_k)) = 0 \quad \text{in } \mathbb{R}^N, \quad y \mapsto \chi_k(x,y) \quad \text{is } Y\text{-periodic.}$$

Notice that even if each $A^n(x)$ is a scalar matrix, the homogenized matrix $A^*(x)$ need not be scalar valued. This is the crucial property of homogenization that allows us to approximate anisotropic permittivities and permeabilities by isotropic ones. We note two important properties of H-convergence:

Proposition 3.15. *Suppose $A^n \xrightarrow{H} A^*$ in Ω. Then*

1. $A^n|_{\Omega'} \xrightarrow{H} A^*|_{\Omega'}$ *for all open sets* $\Omega' \subset \Omega$.

2. *For any matrix-valued function* $M(x)$, *if* $MA^n \in \mathcal{M}(\alpha, \beta, \Omega)$, *then* $MA^n \xrightarrow{H} MA^*$.

The proof can be found in [1].

3.3.2 Homogenization of the Maxwell system

Let $\{\epsilon^n\}, \{\mu^n\}$ be sequences such that $\epsilon^n, \mu^n \in \mathcal{M}_{\mathbb{C}}(\alpha, \beta; \Omega)$ $\forall n \in \mathbb{N}$ for some $0 < \alpha < \beta$. Then consider the following sequence of time-harmonic Maxwell equations at frequency $\omega > 0$:

$$\begin{cases} \operatorname{curl} E^n = i\omega\mu^n H^n \\ \operatorname{curl} H^n = -i\omega\epsilon^n E^n \\ \nu \times E^n|_{\partial\Omega} = f \in H^{-1/2}(\operatorname{div}, \partial\Omega) \end{cases} \tag{3.15}$$

It can be shown (see Proposition 3.16 below) that the above problem admits a unique solution $(E^n, H^n) \in H(\operatorname{curl}, \Omega) \times H(\operatorname{curl}, \Omega)$ which satisfies

$$\|E^n\|_{H(\operatorname{curl},\Omega)} + \|H^n\|_{H(\operatorname{curl},\Omega)} \leq C\|f\|_{H^{-1/2}(\operatorname{div},\partial\Omega)},$$

where the constant $C = C(\alpha, \beta, \omega)$ is independent of n. Therefore, E^n, H^n have subsequences (still denoted by E^n, H^n) such that

$$(E^n, H^n) \rightharpoonup (E, H) \text{ weakly in } H(\operatorname{curl}, \Omega) \times H(\operatorname{curl}, \Omega).$$

Our goal is to get the limiting equation for $(E, H) \in H(\operatorname{curl}, \Omega) \times H(\operatorname{curl}, \Omega)$.

Let us begin with the an existence-uniqueness result for a Maxwell system of the type (3.15):

Proposition 3.16. *Let $\epsilon, \mu \in \mathcal{M}_{\mathbb{C}}(\alpha, \beta; \Omega)$ for some $0 < \alpha < \beta$. Then the following Maxwell system in Ω at frequency $\omega > 0$*

$$\begin{cases} \operatorname{curl} E = i\omega\mu H \\ \operatorname{curl} H = -i\omega\epsilon E \\ \nu \times E|_{\partial\Omega} = f \in H^{-1/2}(\operatorname{div}, \partial\Omega) \end{cases} \tag{3.16}$$

admits a unique solution $(E, H) \in H(\operatorname{curl}, \Omega) \times H(\operatorname{curl}, \Omega)$ satisfying

$$\|E\|_{H(\operatorname{curl},\Omega)} + \|H\|_{H(\operatorname{curl},\Omega)} \leq C\|f\|_{H^{-1/2}(\operatorname{div},\partial\Omega)}, \tag{3.17}$$

where the constant $C = C(\alpha, \beta, \omega)$.

Remark 5. Note that Theorem 3.5 gives us existence and uniqueness when $\mu, \Re(\epsilon) \in \mathcal{M}_{\mathbb{R}}(\alpha, \beta; \Omega), \Im(\epsilon) \geq 0$ and for $\omega \in \mathbb{R} \setminus F$. Here we have instead assumed μ, ϵ to be in $\mathcal{M}_{\mathbb{C}}(\alpha, \beta; \Omega)$ and $\omega > 0$ is arbitrary.

Proof. Let us consider the equation satisfied by H

$$\operatorname{curl}\left(\frac{1}{i\omega}\epsilon^{-1}(x)(\operatorname{curl} H(x))\right) + i\omega\mu(x)H(x) = 0 \quad \text{in } \Omega.$$

The weak formulation of the above problem would be

$$-\int_{\Omega}\left((\operatorname{curl}\overline{w}) \cdot \frac{1}{i\omega}\epsilon^{-1}(\operatorname{curl} H) + i\omega\overline{w} \cdot \mu H\right) dx = \int_{\partial\Omega} (\nu \times E) \cdot \overline{w}\, dS, \quad \forall w \in H(\operatorname{curl}, \Omega). \tag{3.18}$$

Define the sesquilinear form $a : H(curl, \Omega) \times H(curl, \Omega) \to \mathbb{C}$ by

$$a(u, w) := -\int_{\Omega}\left((\nabla \times \overline{w}) \cdot \frac{1}{i\omega}\epsilon^{-1}(\nabla \times u) + i\omega\overline{w} \cdot \mu u\right) dx, \quad \forall u, w \in H(\operatorname{curl}, \Omega).$$

We note that, by writing $\overline{w} = \gamma_T\, \overline{w} + (\overline{w} \cdot \nu)\nu$ on $\partial\Omega$, the right hand side of (3.18) becomes

$$\begin{aligned} \int_{\partial\Omega} (\nu \times E) \cdot \overline{w}\, dS &= \int_{\partial\Omega} (\nu \times E) \cdot \gamma_T\, \overline{w}\, dS + \int_{\partial\Omega} (\nu \times E) \cdot (\overline{w} \cdot \nu)\nu\, dS, \quad w \in H(\operatorname{curl}, \Omega) \\ &= \int_{\partial\Omega} (\nu \times E) \cdot \gamma_T\, \overline{w}\, dS \quad (\text{since } (\nu \times E) \cdot \nu = 0 \text{ on } \partial\Omega). \end{aligned} \tag{3.19}$$

Then the weak formulation of the problem can be rewritten as

$$a(H,w) = \langle f, \gamma_T\, w\rangle, \quad \forall w \in H(\mathrm{curl}, \Omega) \tag{3.20}$$

where $\langle , \rangle$ denotes the duality bracket between $H^{-1/2}(\mathrm{div}, \partial\Omega)$ and $H^{-1/2}(\mathrm{curl}, \partial\Omega)$ (cf. Lemma 3.6) with respect to the $TL^2(\partial\Omega)$ inner product as in (3.19).

We now claim that $a(\cdot,\cdot)$ is coercive, that is,

$$\Re a(u,u) = -\int_\Omega \left(\frac{1}{i\omega}(\mathrm{curl}\,\overline{u}) \cdot \left((\epsilon^{-1})^\dagger - \epsilon^{-1}\right)(\mathrm{curl}\, u) + i\omega u^* \cdot \left(\mu - \mu^\dagger\right) u \right) \geq C||u||^2_{H(curl,\Omega)}. \tag{3.21}$$

Indeed, from the definition of $\mathcal{M}_\mathbb{C}(\alpha, \beta; \Omega)$ we have

$$\begin{cases} -i\omega\xi \cdot \left(\mu(x) - \mu(x)^\dagger\right)\overline{\xi} \geq C_1|\xi|^2 \\ -\frac{i}{\omega}\xi \cdot \left(\epsilon(x) - \epsilon^\dagger(x)\right)\overline{\xi} \geq C_2|\xi|^2 \end{cases} \quad \forall \xi \in \mathbb{C}^3, \quad \text{a.e. } x \in \Omega.$$

Now, for any $\zeta \in \mathbb{C}^3$,

$$-\frac{i}{\omega}(\epsilon^t\zeta)\cdot\left((\epsilon^{-1})^\dagger - \epsilon^{-1}\right)\overline{(\epsilon^t\zeta)} = -\frac{i}{\omega}\zeta\cdot\left(\epsilon - \epsilon^\dagger\right)\overline{\zeta} \geq C_2|\zeta|^2.$$

Setting $\zeta = (\epsilon^t)^{-1}\xi$, we conclude that

$$-\frac{i}{\omega}\xi\cdot((\epsilon^{-1})^\dagger - \epsilon^{-1})\overline{\xi} \geq C_3|\xi|^2, \qquad \forall \xi \in \mathbb{C}^3.$$

Therefore, by (3.21),

$$\Re a(u,u) \geq C_3\|\,\mathrm{curl}\, u\|^2_{L^2} + C_1\|u\|^2_{L^2} \geq \min\{C_1, C_3\}\|u\|^2_{H(\mathrm{curl},\Omega)},$$

which proves the claim. Moreover, since $|\mu\xi|, |\epsilon\xi| \leq \beta|\xi|$ for all $\xi \in \mathbb{C}^3$ we can see that the sesquilinear form a is continuous, i.e.,

$$|a(u,w)| \leq C||u||_{H(\mathrm{curl},\Omega)}||w||_{H(\mathrm{curl},\Omega)}.$$

From the standard trace theory (cf. Lemma 3.6 and Theorem 3.4) it is easy to see that the right-hand side of (3.20) is continuous on $H(\mathrm{curl}, \Omega)$, i.e.,

$$\begin{aligned} |\langle f, \gamma_T\, w\rangle| &\leq C||f||_{H^{-1/2}(\mathrm{div},\partial\Omega)}||w||_{H^{-1/2}(\mathrm{curl},\partial\Omega)} \\ &\leq C'||f||_{H^{-1/2}(\mathrm{div},\partial\Omega)}||w||_{H(\mathrm{curl},\Omega)}, \quad w \in H(\mathrm{curl},\Omega). \end{aligned} \tag{3.22}$$

Therefore, (3.20) has a unique solution $H \in H(\text{curl}, \Omega)$ by the Lax-Milgram theorem, and from (3.21),(3.22) it enjoys the norm estimate

$$||H||_{H(\text{curl},\Omega)} \leq C||f||_{H^{-1/2}(\text{div},\partial\Omega)}.$$

Finally, we set $E = \frac{i}{\omega}\epsilon^{-1}\,\text{curl}\, H$. It is easy to check that the weak curl of E is $i\omega\mu H$, and therefore (3.16) and (3.17) easily follow. □

We now prove the homogenization result for the Maxwell system (3.15). Our proof closely follows the method used in [3], though our assumptions on ϵ and μ are slightly different.

Theorem 3.17. *Let $\epsilon^n, \mu^n \in \mathcal{M}_{\mathbb{C}}(\alpha, \beta; \Omega)$ be such that*

$$\begin{cases} \mu^n \xrightarrow{H} \mu^*, \\ \epsilon^n \xrightarrow{H} \epsilon^* \end{cases} \quad \textit{in } \Omega, \ \textit{as } n \to \infty.$$

Suppose $(E^n, H^n) \in H(\text{curl}, \Omega) \times H(\text{curl}, \Omega)$ is the unique solution of the Maxwell system (3.15)*. Then, up to a subsequence,*

$$(E^n, H^n) \rightharpoonup (E, H) \ \textit{weakly in } H(\text{curl}, \Omega) \times H(\text{curl}, \Omega) \ \textit{as } n \to \infty.$$

where $(E, H) \in H(\text{curl}, \Omega) \times H(\text{curl}, \Omega)$ is the unique solution of the following homogenized time-harmonic Maxwell system:

$$\begin{cases} \text{curl}\, E = i\omega\mu^* H \\ \text{curl}\, H = -i\omega\epsilon^* E \\ \nu \times E|_{\partial\Omega} = f \end{cases} \tag{3.23}$$

Proof. Consider the equation satisfied by H^n:

$$\text{curl}\, \frac{1}{i\omega}(\epsilon^n)^{-1}\, \text{curl}\, H^n + i\omega\mu^n H^n = 0. \tag{3.24}$$

By the estimate (3.17), we have $\|H^n\|_{H(\text{curl},\Omega)} \leq C(\alpha, \beta, \Omega)$. Therefore, up to a subsequence,

$$H^n \rightharpoonup H \qquad \text{in } H(\text{curl}, \Omega).$$

Also, since $\|(\epsilon^n)^{-1}(\operatorname{curl} H^n)\|_{L^2} \leq C(\alpha, \beta, \Omega)$,

$$(\epsilon^n)^{-1} \operatorname{curl} H^n \rightharpoonup \mathfrak{h}_1 \qquad \text{in } (L^2(\Omega))^3. \tag{3.25}$$

Similarly, $\mu^n H^n$ is bounded in $L^2(\Omega)^3$. So assume

$$\mu^n H^n \rightharpoonup \mathfrak{h}_2 \qquad \text{weakly in } L^2(\Omega)^3. \tag{3.26}$$

Then, from the equation (3.24), we have

$$0 = \operatorname{curl} \frac{1}{i\omega}(\epsilon^n)^{-1} \operatorname{curl} H^n + i\omega\mu^n H^n \rightharpoonup \operatorname{curl} \frac{1}{i\omega}\mathfrak{h}_1 + i\omega\mathfrak{h}_2 \quad \text{in } (H(\operatorname{curl}, \Omega))' \tag{3.27}$$

Now, let $u^n \in H^1(\Omega)$ solve

$$-\operatorname{div}(\epsilon^n(x)\nabla u^n(x)) = F \text{ in } \Omega$$
$$u^n = 0 \text{ on } \partial\Omega$$

where $F \in H^{-1}(\Omega)$. Then we know that up to a subsequence, as $n \to \infty$, we have

$$u^n \rightharpoonup u \quad \text{weakly in } H^1(\Omega)$$
$$\epsilon^n \nabla u^n \rightharpoonup \epsilon^* \nabla u \quad \text{weakly in } L^2(\Omega)^3$$

where $u \in H^1(\Omega)$ solves

$$\begin{aligned} -\nabla \cdot (\epsilon^*(x)\nabla u(x)) &= F \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega. \end{aligned} \tag{3.28}$$

Let us consider the following identity:

$$((\epsilon^n)^{-1} \operatorname{curl} H^n) \cdot \epsilon^n \nabla u^n = (\operatorname{curl} H^n) \cdot \nabla u^n \quad \text{in } \Omega. \tag{3.29}$$

From (3.26) and (3.27), $(\epsilon^n)^{-1} \operatorname{curl} H^n \rightharpoonup \mathfrak{h}_1$ in $H(\operatorname{curl}, \Omega)$. Also from (3.25) and (3.28), $\epsilon^n \nabla u^n \rightharpoonup \epsilon^* \nabla u$ in $H(\operatorname{div}, \Omega)$. Therefore, by the Div-Curl lemma, it follows that

$$((\epsilon^n)^{-1} \operatorname{curl} H^n) \cdot \epsilon^n \nabla u^n \to \mathfrak{h}_1 \cdot \epsilon^* \nabla u \quad \text{in } \mathcal{D}'(\Omega).$$

Now, we find the limit of the right hand side of (3.29). Since, $\operatorname{div}(\operatorname{curl} H^n) = 0$ and $\operatorname{curl}(\nabla u^n) = 0$, we see that $\operatorname{curl} H^n \rightharpoonup \operatorname{curl} H$ in $H(\operatorname{div}, \Omega)$ and $\nabla u^n \rightharpoonup \nabla u$ in $H(\operatorname{curl}, \Omega)$. Therefore, again by the Div-Curl lemma,

$$\operatorname{curl} H^n \cdot \nabla u^n \to \operatorname{curl} H \cdot \nabla u \qquad \text{in } \mathcal{D}'(\Omega).$$

Now by equating the limits of the two sides of the equation, we get

$$\mathfrak{h}_1 \cdot \epsilon^* \nabla u = \operatorname{curl} H \cdot \nabla u \quad \text{in } \Omega.$$

In other words, $(\epsilon^* \mathfrak{h}_1 - \operatorname{curl} H) \cdot \nabla u = 0$, where $u \in H_0^1(\Omega)$ is the solution of (3.28). Since $(-\operatorname{div}(\epsilon^* \nabla))^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega)$ is an isomorphism, by varying $F \in H^{-1}(\Omega)$, u spans all of $H_0^1(\Omega)$. Therefore,

$$\begin{aligned} \epsilon^* \mathfrak{h}_1 &= \operatorname{curl} H \\ \Rightarrow \mathfrak{h}_1 &= (\epsilon^*)^{-1} \operatorname{curl} H. \end{aligned}$$

Next, we show that $\mathfrak{h}_2 = \mu^* H$. Let $w^n \in H^1(\Omega)$ be the unique solution of

$$\begin{aligned} -\operatorname{div}(\mu^n(x) \nabla w^n(x)) = G \text{ in } \Omega \\ w^n = 0 \text{ on } \partial\Omega \end{aligned}$$

where $G \in H^{-1}(\Omega)$. Since $\mu^n \xrightarrow{H} \mu^*$, up to a subsequence, as $n \to \infty$, we have

$$\begin{aligned} w^n \rightharpoonup w \quad \text{weakly in } H^1(\Omega) \\ \mu^n \nabla w^n \rightharpoonup \mu^* \nabla w \quad \text{weakly in } L^2(\Omega)^3 \end{aligned}$$

where $w \in H^1(\Omega)$ solves

$$\begin{aligned} -\operatorname{div}(\mu^*(x) \nabla w(x)) = G \text{ in } \Omega \\ w = 0 \text{ on } \partial\Omega. \end{aligned} \tag{3.30}$$

Consider the identity

$$\mu^n H^n \cdot \nabla w^n = H^n \cdot \mu^n \nabla w^n.$$

We have $\operatorname{div} \mu^n H^n = 0$. So $\mu^n H^n \rightharpoonup \mathfrak{h}_2$ weakly in $H(\div, \Omega)$. Similarly, since $\operatorname{curl} \nabla w^n = 0$, $\nabla w^n \rightharpoonup \nabla w$ weakly in $H(\operatorname{curl}, \Omega)$. So, applying the Div-Curl lemma again, we get

$$\mu^n H^n \cdot \nabla w^n \to \mathfrak{h}_2 \cdot \nabla w \qquad \text{in } \mathcal{D}'(\Omega).$$

From the right hand side, we have $H^n \rightharpoonup H$ in $H(\operatorname{curl}, \Omega)$ and that $\mu^n \nabla w^n \rightharpoonup \mu^* \nabla w$ in $H(\operatorname{div}, \Omega)$ by (3.30). So again by Div-Curl lemma, we have

$$H^n \cdot \mu^n \nabla w^n \to H \cdot \mu^* \nabla w \qquad \text{in } \mathcal{D}'(\Omega).$$

Now, by equating the two limits, we get

$$\begin{aligned} \mathfrak{h}_2 \cdot \nabla w &= H \cdot \mu^* \nabla w \\ \Rightarrow (\mathfrak{h}_2 - \mu^* H) \cdot \nabla w &= 0. \end{aligned}$$

Again, by varying G, w spans $H_0^1(\Omega)$. So we get $\mathfrak{h}_2 = \mu^* H$ in Ω. Thus, we have the following homogenized equation for $H \in H(\operatorname{curl}, \Omega)$:

$$\operatorname{curl} \frac{1}{i\omega} (\epsilon^*)^{-1} \operatorname{curl} H + i\omega \mu^* H = 0 \qquad \text{in } H(curl, \Omega).$$

As before, set $E = \frac{i}{\omega} (\epsilon^*)^{-1} \operatorname{curl} H$. Then we see that $(E^n, H^n) \rightharpoonup (E, H)$ and $(E, H) \in H(\operatorname{curl}, \Omega) \times H(\operatorname{curl}, \Omega)$ satisfies (3.23). $\square$

3.4 Approximate Isotropic Cloak

As we have seen, $(B_3 \setminus \overline{B}_1; \epsilon_c^\rho, \mu_c^\rho)$ forms an approximate electromagnetic cloak for the region B_1. Therefore, for any $f \in H^{-1/2}(\operatorname{div}, \partial B_3)$,

$$\Lambda_{\epsilon_e^\rho, \mu_e^\rho} f \to \Lambda_{I,I} f \quad \text{strongly in } H^{-1/2}(\operatorname{div}, \partial B_3)$$

as $\rho \to 0$, where $(B_3; \epsilon_e^\rho, \mu_e^\rho)$ is the extended object defined as before. We now want to construct *isotropic* sequences $\{\epsilon_m\}$ and $\{\mu_m\}$ such that

$$\Lambda_{\epsilon_m, \mu_m} f \to \Lambda_{I,I} f \quad \text{strongly as } m \to \infty.$$

We would like to do this by first constructing sequences of permittivities and permeabilities whose H-limits are ϵ_e^ρ and μ_e^ρ, and then passing to the limit as $\rho \to 0$. However, we will not be able to directly find sequences whose homogenized limits are $(\epsilon_e^\rho, \mu_e^\rho)$ since $\mu_e^\rho \notin \mathcal{M}_{\mathbb{C}}(\alpha, \beta; B_3)$ for any α, β. Therefore, we will go through a two-step process: first we fix a small parameter $\delta > 0$, and construct sequences ϵ_δ^n, μ^n such that

$$\begin{aligned}(1+i\delta)^{-1}\epsilon_\delta^n &\overset{H}{\longrightarrow} (1+i\delta)\epsilon_e^\rho,\\ (1+i\delta)\mu^n &\overset{H}{\longrightarrow} (1+i\delta)\mu_e^\rho.\end{aligned}$$

Note that both sides of the equations above are in $\mathcal{M}_{\mathbb{C}}(\alpha, \beta; B_3)$ for some α, β, so that Theorem 3.17 applies. In the next step, we let $\delta \to 0$. We will be able to show in Section 5 that

$$\lim_{\delta\to 0}\left(\lim_{n\to\infty} \Lambda_{\epsilon_\delta^n, \mu^n} f\right) = \Lambda_{\epsilon_e^\rho, \mu_e^\rho} f$$

where the limit is in the strong topology of $H^{-1/2}(\mathrm{div}, \partial B_3)$. We note however that the order in which we take the limits can not be interchanged. Finally, by a diagonal argument, by choosing sequences $n_m \to \infty, \delta_m \to 0$ and $\rho_m \to 0$, we can construct (ϵ_m, μ_m) such that

$$\Lambda_{\epsilon_m, \mu_m} f \to \Lambda_{I,I} f \quad \text{strongly as } m \to \infty.$$

In this section we shall give an explicit construction of the approximate isotropic cloak. Throughout this section we assume that $\rho > 0$ is a fixed parameter. Let us define

$$\gamma^* = \begin{cases} (F_\rho)_* I & \text{in } B_3 \setminus B_{1/2}, \\ 1 & \text{in } B_{1/2}, \end{cases}$$

where F_ρ is as defined in (3.12). More explictly,

$$\gamma^*(x) = \begin{cases} I & \text{for } 2 \le |x| \le 3 \\ \frac{1}{b}(I - \Pi(x)) + \frac{(|x|-a)^2}{b|x|^2}\Pi(x) & \text{for } 1 < |x| < 2 \\ \rho^{-1} I & \text{for } 1/2 < |x| < 1 \\ I & \text{for } |x| < 1/2 \end{cases} \tag{3.31}$$

where

$$a = \frac{2(1-\rho)}{2-\rho}, \qquad b = \frac{1}{2-\rho}$$

and $\Pi(x) = |x|^{-2}xx^t$ is the projection in the radial direction. Given arbitrary regular ϵ_a, μ_a in $B_{1/2}$, we note that the extended object is given by

$$(\epsilon_e^\rho, \mu_e^\rho) = \begin{cases} (\gamma^*, \gamma^*) & \text{in } B_3 \setminus B_1 \\ (\gamma^*(1 + i\rho^{-2}/\omega), \gamma^*) & \text{in } B_1 \setminus B_{1/2} \\ (\epsilon_a \gamma^*, \mu_a \gamma^*) & \text{in } B_{1/2}. \end{cases}$$

We will construct isotropic matrices of the form

$$\gamma^n(x) = \gamma(x, n|x|)I, \quad x \in B_3 \tag{3.32}$$

that H-converge to γ^*, where, $\gamma(x,y)$ is periodic in $y \in Y$. Recall that γ^* can be computed from $\gamma(x,y)$ using (3.14). From γ^n, it is straightforward to construct isotropic electric permittivities and magnetic permeabilities for the approximate cloak. Our construction is mostly based on [18]. Let us change to polar coordinates. Let $s = (r, \theta, \varphi)$ and $t = (r', \theta', \varphi')$ be the spherical polar coordinates corresponding to the two scales x and y respectively. Next we homogenize $\gamma(s,t)$ in the (r', θ', φ')-coordinates. Let e_1, e_2, e_3 denote the canonical basis vectors of $\mathbb{R}^3$ in r', θ' and φ' directions respectively. Then for almost every $s \in \Omega$, there exist unique solutions $\chi_k(s,t)$, $k = 1,2,3$ of the equation

$$\begin{aligned} &\text{div}_t(\gamma(s,t)(\nabla_t \chi_k(s,t) + e_k)) = 0 \text{ in } \mathbb{R}^N, \\ &t = (r', \theta' \varphi') \mapsto \chi_k(s,t) \quad \text{is 1-periodic in each of } r', \theta', \varphi'. \end{aligned} \tag{3.33}$$

which satisfies the condition

$$\int_Y \chi_k(s,t)dt = 0, \tag{3.34}$$

where $dt = dr'd\theta'd\varphi'$ and $Y = [0,1]^3$.

Since $\gamma(s,t)$ is independent of θ' and φ', (3.33) and (3.34) imply that $\chi_k = 0$ for $k = 2,3$. Now consider the equation (3.33) for χ_1:

$$\frac{\partial}{\partial r'}\left(\gamma(s,r')\frac{\partial \chi_1(s,t)}{\partial r'}\right) = -\frac{\partial \gamma(s,r')}{\partial r'}. \tag{3.35}$$

It is clear from the above equation that χ_1 is independent of θ', φ' as well. Moreover, from (3.35) we get

$$\frac{\partial \chi_1}{\partial r'} = -1 + \frac{C}{\gamma(s,t)}$$

where the constant C can be found by using the periodicity of χ_1 with respect to r' to be

$$C = \frac{1}{\int_0^1 \gamma^{-1}(s,t)dr'} := \underline{\gamma}(s).$$

Here $\underline{\gamma}(s)$ denotes the harmonic mean of $\gamma(s,t)$ in the second variable. Similarly, we let $\overline{\gamma}(s)$ denote the arithmetic mean of $\gamma(s,t)$ in the second variable:

$$\overline{\gamma}(s) = \int_0^1 \gamma(s,t)dr'.$$

Now, for γ^* to be the homogenized limit of γ^n, we must have

$$\gamma^*_{kl}(s) = \int_Y \gamma(s,r') \left(\frac{\partial \chi_k(s,t)}{\partial t_l} + \delta_{kl} \right) dt,$$

which simplifies to

$$\gamma^*(s) = \underline{\gamma}(s)\Pi(s) + \overline{\gamma}(s)(I - \Pi(s)), \tag{3.36}$$

where, as before, $\Pi(s) : \mathbb{R}^3 \to \mathbb{R}^3$ is the projection on to the radial direction. Comparing this with (3.31), we see that it suffices to construct a $\gamma(s,t)$ such that

$$\overline{\gamma}(s) = \begin{cases} 1 & \text{in } B_3 \setminus B_2 \\ \frac{1}{b} & \text{in } B_2 \setminus B_1 \\ \rho^{-1} & \text{in } B_1 \setminus B_{1/2} \\ 1 & \text{in } B_{1/2} \end{cases} \tag{3.37}$$

and

$$\underline{\gamma}(s) = \begin{cases} 1 & \text{in } B_3 \setminus B_2 \\ \frac{(|s|-a)^2}{b|s|^2} & \text{in } B_2 \setminus B_1 \\ \rho^{-1} & \text{in } B_1 \setminus B_{1/2} \\ 1 & \text{in } B_{1/2} \end{cases} \tag{3.38}$$

Let us now construct such a γ. Our construction is based on the one presented in [18]. However, we are able to work with a simpler construction since we do not need γ to be continuous. It is easy to define $\gamma(s,t)$ for $s \in B_1 \cup (B_3 \setminus B_2)$:

$$\gamma(s,t) = \begin{cases} 1 & \text{for } s \in B_3 \setminus B_2, t \in Y \\ \rho^{-1} & \text{for } s \in B_1 \setminus B_{1/2},\ t \in Y, \\ 1 & \text{for} s \in B_{1/2},\ t \in Y. \end{cases} \tag{3.39}$$

For $s \in B_2 \setminus B_1$, suppose we can write

$$\gamma(s,t) = \alpha(s)\chi_{(0,1/2)}(t) + \beta(s)\chi_{(1/2,1)}(t) \tag{3.40}$$

where α and β are positive functions of s. Then (3.37) and (3.38) translate to

$$\begin{cases} \overline{\gamma}(s) = \frac{\alpha(s)+\beta(s)}{2} = \frac{1}{b}, \\ \underline{\gamma}(s) = \frac{2\alpha(s)\beta(s)}{\alpha(s)+\beta(s)} = \frac{(|s|-a)^2}{b|s|^2}, \end{cases} \qquad \forall s \in B_2 \setminus B_1. \tag{3.41}$$

Eliminating β from the above equations, we get

$$2\alpha^2 - \frac{2}{b}\alpha + \frac{(|s|-a)^2}{b^2|s|^2} = 0.$$

This equation will have two positive roots so long as the descriminant is non-negative:

$$\begin{aligned} \frac{4}{b^2} - \frac{8(|s|-a)^2}{b^2|s|^2} &\geq 0 \\ \Leftrightarrow 4|s|^2 - 16a|s| + 8|a|^2 &\leq 0 \end{aligned}$$

which is equivalent to

$$(2-\sqrt{2})a \leq s \leq (2+\sqrt{2})a.$$

But this condition is true for any $0 < \rho < 1/2$, as

$$\begin{aligned} (2-\sqrt{2})a &= (2-\sqrt{2})\frac{2-2\rho}{2-\rho} < 2-\sqrt{2} < 1 \leq |s|, \quad \text{and} \\ (2+\sqrt{2})a &= (2+\sqrt{2})\left(1+\frac{\rho}{2-\rho}\right) > (2+\sqrt{2})\frac{2}{3} > 2 \geq |s|. \end{aligned}$$

Therefore, it follows that (3.41) can be solved for α and β to obtain positive functions of s on $B_2 \setminus B_1$. Finally, equations (3.39) and (3.40) define a $\gamma(s,t)$ with all the desired properties.

Construction of isotropic electromagnetic parameters :

We shall now construct the electromagnetic parameters of our approximate isotropic cloak. We continue to assume that the parameter $\rho > 0$ introduced in the definition of the approximate cloak (cf. Definition 3.11) is fixed. Recall that we defined

$$\gamma^n(x) = \gamma(x, n|x|) \quad \text{for all } x \in B_3, \quad n \in \mathbb{N}.$$

We define a sequence of isotropic non-singular *magnetic permeabilities* $\{\mu^n\}$ by

$$\mu^n := \varphi_1 \gamma^n \quad \text{in } B_3, \tag{3.42}$$

where

$$\varphi_1(x) = \begin{cases} 1 & \text{in } B_3 \setminus B_{\frac{1}{2}} \\ \mu_a & \text{in } B_{\frac{1}{2}} \end{cases}$$

and μ_a is an arbitrary permeability in $B_{\frac{1}{2}}$ as introduced in the Definition of cloaking (cf. Definition 3.8).

Next, we fix $\delta > 0$ and define sequences of isotropic non-singular *electric permittivities* $\{\widetilde{\epsilon}^n_\delta\}$ and isotropic *conductivities* $\{\sigma^n_\delta\}$ as follows:

$$\widetilde{\epsilon}^n_\delta := \Re(\epsilon^n_\delta) \quad \text{in } B_3 \quad \text{and} \quad \sigma^n_\delta := \omega \Im(\epsilon^n_\delta) \quad \text{in } B_3$$

where

$$\epsilon^n_\delta = \widetilde{\epsilon}^n_\delta + \frac{i}{\omega}\sigma^n_\delta := (1 + i\delta)^2 \left(1 + \frac{i}{\omega}\varphi_3\right)\varphi_2 \gamma^n \quad \text{in } B_3, \tag{3.43}$$

and

$$\varphi_2(x) = \begin{cases} 1 & \text{in } B_3 \setminus B_{\frac{1}{2}} \\ \widetilde{\epsilon}_a + \frac{i}{\omega}\sigma_a & \text{in } B_{\frac{1}{2}} \end{cases}$$

Here $\widetilde{\epsilon}_a, \sigma_a$ are arbitrary electric permittivity and conductivity respectively in $B_{\frac{1}{2}}$ as introduced in the definition of cloaking (cf. Definition 3.8) and

$$\varphi_3 = \rho^{-2} \chi_{B_1 \setminus B_{\frac{1}{2}}}.$$

Now consider the following system of Maxwell equations:

$$\begin{cases} \operatorname{curl} E_\delta^n = i\omega\mu^n H_\delta^n & \text{in } B_3 \\ \operatorname{curl} H_\delta^n = -i\omega\epsilon_\delta^n E_\delta^n & \text{in } B_3 \\ \nu \times E_\delta^n = f & \text{on } \partial B_3 \end{cases}$$

where $f \in H^{-1/2}(\text{div}, \partial B_3)$. H_δ^n satisfies the equation

$$\frac{1}{i\omega} \operatorname{curl}\left((\epsilon_\delta^n)^{-1} \operatorname{curl} H_\delta^n\right) + i\omega\mu_\delta^n H_\delta^n = 0 \quad \text{in } B_3. \tag{3.44}$$

Multiplying throughout by $(1+i\delta)$, we get

$$\frac{1}{i\omega} \operatorname{curl}\left(\left((1+i\delta)^{-1}\epsilon_\delta^n\right)^{-1} \operatorname{curl} H_\delta^n\right) + i\omega(1+i\delta)\mu^n H_\delta^n = 0 \quad \text{in } B_3.$$

The above equation can be written in the variational form as

$$a_\delta^n(H_\delta^n, w) = (1+i\delta)\int_{\partial B_3} f\cdot\overline{w} \qquad \forall w \in H(\text{curl}, B_3),$$

where a_δ^n is the sesquilinear form on $H(\text{curl}, B_3)$ defined by

$$a_\delta^n(u,w) = -\int_{B_3}\left\{\frac{1}{i\omega}\left((1+i\delta)^{-1}\epsilon_\delta^n)^{-1} \operatorname{curl} u\right)\cdot(\operatorname{curl}\overline{w}) + i\omega(1+i\delta)\mu^n u\cdot\overline{w}\right\}.$$

Let us define $\widehat{\mu}_\delta^n = (1+i\delta)\mu^n$. Then we see that

$$\begin{aligned} \left(\widehat{\mu}_\delta^n - (\widehat{\mu}_\delta^n)^\dagger\right) &= 2i\delta\varphi_1\gamma^n \\ \Rightarrow -\frac{i}{\omega}\xi\cdot\left(\widehat{\mu}_\delta^n - (\widehat{\mu}_\delta^n)^\dagger\right)\cdot\overline{\xi} &= \frac{2\delta}{\omega}(\xi\varphi_1\gamma^n\overline{\xi}) > \delta c|\xi|^2. \end{aligned}$$

for all $\xi \in \mathbb{C}^3$. Similarly, we define

$$\widehat{\epsilon}_\delta^n = (1+i\delta)^{-1}\epsilon_\delta^n$$

We find that

$$\begin{aligned} \widehat{\epsilon}_\delta^n - (\widehat{\epsilon}_\delta^n)^\dagger &= 2i\Im[(1+i\delta)^{-1}\epsilon_\delta^n] \\ &= 2i\Im\left[(1+i\delta)\left(1+\frac{i}{\omega}\varphi_3\right)\varphi_2\gamma^n\right] \\ &= 2i\left[\frac{\varphi_3}{\omega}+\delta\right]\varphi_2\gamma^n \\ \Rightarrow -i\omega\xi\cdot(\widehat{\epsilon}_\delta^n - (\widehat{\epsilon}_\delta^n)^\dagger)\cdot\overline{\xi} &= 2\omega\left[\frac{\varphi_3}{\omega}\xi\varphi_2\cdot\gamma^n\overline{\xi} + \delta\xi\cdot\varphi_2\gamma^n\cdot\overline{\xi}\right] \\ &\geq c\delta|\xi|^2 \end{aligned}$$

where c is some positive constant that depends on ω. Note that $\widehat{\mu}_\delta^n$ and $\widehat{\epsilon}_\delta^n$ are uniformly bounded in $L^\infty(B_3)$ and we take $0 < \delta < 1$. So all conditions for H-convergence are satisfied and we have

$$\begin{aligned} \widehat{\mu}_\delta^n &\xrightarrow{H} (1+i\delta)\varphi_1\gamma^* \quad \text{and} \\ \widehat{\epsilon}_\delta^n &\xrightarrow{H} (1+i\delta)\left(1+\frac{i}{\omega}\varphi_3\right)\varphi_2\gamma^* \quad \text{in } B_3. \end{aligned}$$

by Proposition 3.15. We note that

$$\begin{aligned} \mu^* &:= \varphi_1\gamma^* = \mu_e^\rho \quad \text{and} \\ \epsilon^* &:= \left(1+\frac{i}{\omega}\varphi_3\right)\varphi_2\gamma^* = \epsilon_e^\rho \end{aligned}$$

respectively, where $\epsilon_e^\rho, \mu_e^\rho$ are as in (3.13). Set

$$\epsilon_\delta^* := (1+i\delta)^2\left(1+\frac{i}{\omega}\varphi_3\right)\varphi_2\gamma^*.$$

We now apply Theorem 3.17 to conclude that

$$\begin{aligned} H_\delta^n &\rightharpoonup H_\delta \quad \text{and} \\ E_\delta^n &\rightharpoonup E_\delta \quad \text{weakly in } H(\mathrm{curl}, B_3) \end{aligned}$$

where $(H_\delta, E_\delta) \in H(\mathrm{curl}, B_3) \times H(\mathrm{curl}, B_3)$ is the solution of the following homogenized equation corresponding to (3.44):

$$\begin{gathered} \frac{1}{i\omega}\,\mathrm{curl}\left(\left((1+i\delta)^{-1}(\epsilon_\delta^*)\right)^{-1}\mathrm{curl}\, H_\delta\right) + i\omega(1+i\delta)\mu^* H_\delta = 0 \quad \text{in } B_3 \\ E_\delta = \frac{-1}{i\omega}(\epsilon_\delta^*)^{-1}\,\mathrm{curl}\, H_\delta \\ \nu \times E_\delta = f \quad \text{on } \partial B_3. \end{gathered}$$

Dividing the first equation by $1+i\delta$, we get

$$\begin{gathered} \frac{1}{i\omega}\,\mathrm{curl}\left((\epsilon_\delta^*)^{-1}\,\mathrm{curl}\, H_\delta\right) + i\omega\mu^* H_\delta = 0 \quad \text{in } B_3 \\ E_\delta = \frac{-1}{i\omega}(\epsilon_\delta^*)^{-1}\,\mathrm{curl}\, H_\delta \\ \nu \times E_\delta = f \quad \text{on } \partial B_3. \end{gathered}$$

Therefore, (E_δ, H_δ) solve the following homogenized Maxwell system:

$$\begin{cases} \operatorname{curl} E_\delta = i\omega\mu^* H_\delta \\ \operatorname{curl} H_\delta = -i\omega(\epsilon_\delta^*) E_\delta \\ \nu \times E_\delta|_{\partial B_3} = f \in H^{-1/2}(\operatorname{div}, \partial B_3) \end{cases} \tag{3.45}$$

Finally, we pass to the limit as $\delta \to 0$. Suppose E, H satisfy

$$\begin{cases} \operatorname{curl} E = i\omega\mu^* H \\ \operatorname{curl} H = -i\omega\epsilon^* E \\ \nu \times E|_{\partial B_3} = f. \end{cases} \tag{3.46}$$

Subtracting (3.46) from (3.45),

$$\begin{cases} \operatorname{curl}(E_\delta - E) = i\omega\mu_e^\rho (H_\delta - H) \\ \operatorname{curl}(H_\delta - H) = -i\omega\epsilon_e^\rho (E_\delta - E) + (2 + i\delta)\omega\delta\epsilon_e^\rho E_\delta \\ \nu \times (E_\delta - E)|_{\partial B_3} = 0. \end{cases}$$

Therefore, by Theorems 3.5 and 3.12, if ω is not an eigenvalue of $(B_3; I, I)$ and if $\rho > 0$ is small enough, there exists a constant $C > 0$ that is independent of δ, but dependent on ρ, such that

$$\|E_\delta - E\|_{H(\operatorname{curl}, B_3)} + \|H_\delta - H\|_{H(\operatorname{curl}, B_3)} \le C\delta \|E_\delta\|_{L^2}.$$

Next, we write $E_\delta = (E_\delta - E) + E$ and apply the triangle inequality to get

$$\begin{aligned} \|E_\delta - E\|_{H(\operatorname{curl}, B_3)} + \|H_\delta - H\|_{H(\operatorname{curl}, B_3)} &\le C\delta(\|E_\delta - E\|_{L^2} + \|E\|_{L^2}) \\ \Rightarrow (1 - C\delta)\|E_\delta - E\|_{H(\operatorname{curl}, B_3)} + \|H_\delta - H\|_{H(\operatorname{curl}, B_3)} &\le C\delta \|E\|_{L^2} \\ \Rightarrow \|E_\delta - E\|_{H(\operatorname{curl}, B_3)} + \|H_\delta - H\|_{H(\operatorname{curl}, B_3)} &\le C'\delta \|E\|_{L^2} \end{aligned}$$

for small enough δ. In conclusion, we have proved the following theorem:

Theorem 3.18. *Suppose ω is not an electromagnetic eigenvalue for $(B_3; I, I)$ and suppose $\rho > 0$ is small enough. Let $(E_\delta^n H_\delta^n), (E_\delta, H_\delta)$ be defined as above. Then for any fixed $\delta > 0$,*

$$\lim_{n\to\infty} E_\delta^n = E_\delta, \quad \lim_{n\to\infty} H_\delta^n = H_\delta \quad \textit{weakly in } H(\mathrm{curl}, B_3)$$

and as $\delta \to 0$,

$$\lim_{\delta\to 0} E_\delta = E, \quad \lim_{\delta\to 0} H_\delta = H \quad \textit{strongly in } H(\mathrm{curl}, B_3).$$

3.5 Convergence of the Impedance map

In this section we finally prove our main result, namely that our construction does in fact give an approximate isotropic cloak. Recall that we defined the impedance map $\Lambda_{\epsilon,\mu}$: $H^{-1/2}(\mathrm{div}, \partial\Omega) \to H^{-1/2}(\mathrm{div}, \partial\Omega)$ by setting $\Lambda_{\epsilon,\mu} f = \nu \times H|_{\partial\Omega}$, where (E, H) is the unique solution of the boundary value problem

$$\begin{cases} \mathrm{curl}\, E = i\omega\mu H & \text{in } \Omega, \\ \mathrm{curl}\, H = -i\omega\epsilon E & \text{in } \Omega, \\ \nu \times E|_{\partial\Omega} = f. \end{cases}$$

Main Theorem (Ghosh, T., 2018 [14])**.** *Suppose $\omega > 0$ is not an eigenvalue for $(B_3; I, I)$, and let ϵ_δ^n, μ^n be defined as in the previous section. Then, given any $f \in H^{-1/2}(div, \partial B_3)$, we have*

$$\lim_{\delta\to 0}\left(\lim_{n\to\infty} \Lambda_{\epsilon_\delta^n,\mu^n} f\right) = \Lambda_{\epsilon_e^\rho,\mu_e^\rho} f \quad \textit{strongly in } H^{-1/2}(\mathrm{div}, \partial B_3),$$

and as a consequence

$$\lim_{\rho\to 0}\left(\lim_{\delta\to 0}\left(\lim_{n\to\infty} \Lambda_{\epsilon_\delta^n,\mu^n} f\right)\right) = \Lambda_{I,I} f \quad \textit{strongly in } H^{-1/2}(\mathrm{div}, \partial B_3).$$

We emphasize once again that the order of the limits in the above equation can not be changed. Now, since $H_\delta \to H$ strongly in $H(\mathrm{curl}, B_3)$, it follows immediately by the continuity of the trace map $H(\mathrm{curl}, B_3) \to H^{-1/2}(\mathrm{div}, \partial B_3)$ that

$$\Lambda_{\epsilon_\delta^*,\mu^*} f = \nu \times H_\delta|_{\partial B_3} \to \nu \times H|_{\partial B_3} = \Lambda_{\epsilon_e^\rho,\mu_e^\rho} f$$

strongly in $H^{-1/2}(\mathrm{div}, \partial B_3)$ as $\delta \to 0$. So it suffices to prove the following theorem:

Theorem 3.19. *For a fixed $\delta > 0$,*

$$\Lambda_{\epsilon_\delta^n,\mu^n} f = \nu \times H_\delta^n|_{\partial B_3} \to \nu \times H_\delta|_{\partial B_3} = \Lambda_{\epsilon_\delta^*,\mu^*} f$$

strongly in $H^{-1/2}(\mathrm{div}, \partial B_3)$ as $n \to \infty$.

Proof. Let $\varphi \in C^\infty(\overline{B}_3)$ be a non-negative smooth function such that supp $\varphi \subset \overline{B}_3 \setminus \overline{B}_2$ and $\varphi \equiv 1$ in a neighbourhood of ∂B_3. We note that on the support of φ,

$$\epsilon_\delta^n = \epsilon_\delta^* = (1 + i\delta)^2, \quad \text{and} \tag{3.47}$$

$$\mu^n = \mu^* = 1. \tag{3.48}$$

It is clear by the continuity of the trace map that

$$\|\nu \times (H_\delta^n - H_\delta)|_{\partial B_3}\|_{H^{-1/2}(div,\partial B_3)} \leq C(\|\varphi(H_\delta^n - H_\delta)\|_{L^2} + \|\operatorname{curl}(\varphi(H_\delta^n - H_\delta))\|_{L^2}). \tag{3.49}$$

We will show that both the terms in the right hand side of the above equation will go to 0. The proof will follow a series of steps.

Step 1. For any $\psi \in C_c^\infty(B_3 \setminus \overline{B}_2)$, we claim that

$$\int \psi |E_\delta^n - E_\delta|^2 \to 0, \quad \int \psi |H_\delta^n - H_\delta|^2 \to 0 \qquad \text{as } n \to \infty.$$

Indeed, we already know $E_\delta^n - E_\delta \rightharpoonup 0$ in $H(curl, B_3 \setminus \overline{B}_2)$. Consequently, we also have $E_\delta^n - E_\delta \rightharpoonup 0$ in $L^2(B_3 \setminus \overline{B}_2)$. Moreover, $\nabla \cdot (E_\delta^n - E_\delta) = \frac{i}{\omega}(1 + i\delta)^{-2} \nabla \cdot \operatorname{curl}(H_\delta^n - H_\delta) = 0$ in $B_3 \setminus \overline{B}_2$. Therefore, $E_\delta^n - E_\delta \in H(div, B_3 \setminus \overline{B}_2)$ and for any $v \in H(div, B_3 \setminus \overline{B}_2)$,

$$\int (E_\delta^n - E_\delta) \cdot \overline{v} + \nabla \cdot (E_\delta^n - E_\delta) \nabla \cdot v \to 0$$

which implies that $E_\delta^n - E_\delta \rightharpoonup 0$ in $H(div, B_3 \setminus \overline{B}_2)$ as well. Therefore, by the Div-Curl lemma, $|E_\delta^n - E_\delta|^2 = (E_\delta^n - E_\delta) \cdot (E_\delta^n - E_\delta) \to 0$ in $\mathcal{D}'(B_3 \setminus \overline{B}_2)$. By similar arguments, we also have $|H_\delta^n - H_\delta|^2 \to 0$ in $\mathcal{D}'(B_3 \setminus \overline{B}_2)$. This completes the proof of Step 1.

Step 2. Next, we want to show that $\int \varphi^2 |E_\delta^n - E_\delta|^2 \to 0$.

Consider the space of functions

$$V = \{u \in H(curl, B_3) : \nabla \cdot u \in L^2(B_3), \nu \times u|_{\partial B_3} = 0\}$$

with the following inner product:

$$\begin{aligned}\langle u, w\rangle_V &= \langle u, w\rangle_{H(curl,B_3)} + \langle \nabla \cdot u, \nabla \cdot w\rangle_{L^2} \\ &= \langle u, w\rangle_{L^2} + \langle \operatorname{curl} u, \operatorname{curl} w\rangle_{L^2} + \langle \nabla \cdot u, \nabla \cdot w\rangle_{L^2}.\end{aligned}$$

It is well known V is a Hilbert space and that the inclusion $V \hookrightarrow L^2(B_3)^3$ is compact [33]. Now, we know that $\nu \times \varphi(E^n_\delta - E_\delta)|_{\partial B_3} = 0$ and $\varphi(E^n_\delta - E_\delta) \rightharpoonup 0$ in $H(curl, B_3)$. Next, consider

$$\nabla \cdot (\varphi(E^n_\delta - E_\delta)) = \varphi \nabla \cdot (E^n_\delta - E_\delta) + \nabla\varphi \cdot (E^n_\delta - E_\delta).$$

The first term on the right hand side vanishes identically since $\nabla \cdot (E^n_\delta - E_\delta) = 0$ on supp φ. Also, since $\varphi \equiv 1$ in a neighbourhood of ∂B_3, $\nabla\varphi \in C^\infty_c(B_3 \setminus \overline{B}_2)$. Therefore, from Step 1, $\nabla\varphi \cdot (E^n_\delta - E_\delta) \to 0$ strongly in $L^2(\Omega)$. As a consequence, for any $w \in V$,

$$\langle \varphi(E^n_\delta - E_\delta), w\rangle_V = \langle \varphi(E^n_\delta - E_\delta), w\rangle_{H(curl,B_3)} + \langle \nabla \cdot (\varphi(E^n_\delta - E_\delta)), \nabla \cdot w\rangle_{L^2} \to 0$$

as $n \to \infty$. Therefore, $\varphi(E^n_\delta - E_\delta) \rightharpoonup 0$ in V as well. By the compactness of the inclusion $V \hookrightarrow L^2(B_3)^3$, this further implies $\varphi(E^n_\delta - E_\delta) \to 0$ strongly in $L^2(B_3)^3$.

Step 3. We now show that $\int |\operatorname{curl}(\varphi(H^n_\delta - H_\delta)|^2 \to 0$.

Note that

$$\operatorname{curl}(\varphi(H^n_\delta - H_\delta)) = \nabla\varphi \times (H^n_\delta - H_\delta) + \varphi \operatorname{curl}(H^n_\delta - H_\delta).$$

As we have already observed, $\nabla\varphi \in C^\infty_c(B_3)$ and therefore by Step 1, the first term on the right hand side converges strongly to 0 in $L^2(B_3)$. On the other hand, $\operatorname{curl}(H^n_\delta - H_\delta) = -\frac{i}{\omega}(E^n_\delta - E_\delta)$ on supp φ. Therefore, by Step 2, the second term on the right hand side also converges to 0 strongly in L^2. This completes Step 3.

Step 4. The final step is to show that $\int \varphi^2 |H^n_\delta - H_\delta|^2 \to 0$.

Consider the variational equations satisfied by H_δ^n and H_δ: for all $w \in H(curl, \Omega)$,

$$\begin{aligned} -\int \frac{1}{i\omega}\{(\epsilon_\delta^n)^{-1}\,\mathrm{curl}\,H_\delta^n\}\cdot \mathrm{curl}\,\overline{w} + i\omega\mu^n H_\delta^n \cdot \overline{w} &= \int_{\partial\Omega} f\cdot\overline{w}, \quad \text{and} \\ -\int \frac{1}{i\omega}\{(\epsilon_\delta^*)^{-1}\,\mathrm{curl}\,H_\delta\}\cdot \mathrm{curl}\,\overline{w} + i\omega\mu^* H_\delta \cdot \overline{w} &= \int_{\partial\Omega} f\cdot\overline{w}. \end{aligned}$$

Taking the difference of these two equations and rearranging terms, we get

$$\int (\mu^n H_\delta^n - \mu^* H_\delta)\cdot\overline{w} = \frac{1}{\omega^2}\int \{(\epsilon_\delta^n)^{-1}\,\mathrm{curl}\,H_\delta^n - (\epsilon_\delta^*)^{-1}\,\mathrm{curl}\,H_\delta\}\cdot\mathrm{curl}\,\overline{w}.$$

Now let $w = \varphi^2(H_\delta^n - H_\delta)$. By equations (3.47) and (3.48),

$$\begin{aligned} \int \varphi^2 |H_\delta^n - H_\delta|^2 &= \omega^{-2}(1+i\delta)^{-2}\int \mathrm{curl}(H_\delta^n - H_\delta)\cdot\mathrm{curl}(\varphi(H_\delta^n - H_\delta)). \\ \Rightarrow \|\varphi(H_\delta^n - H_\delta)\|_{L^2}^2 &\le C\|H_\delta^n - H_\delta\|_{H(curl,\Omega)}\|\,\mathrm{curl}(\varphi(H_\delta^n - H_\delta))\|_{L^2}. \end{aligned}$$

It is clear from Step 3 that the right hand side of the above equation goes to 0 as $n \to \infty$. Therefore, we conclude that $\varphi(H_\delta^n - H_\delta) \to 0$ strongly in $L^2(B_3)^3$. Finally, applying the conclusions of Steps 3 and 4 to (3.49) gives us the desired result.

□

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