The C^*-algebra of a finite T_0 topological space

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Abstract

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We are concerned with the following motivating question: how can one extend the classical Gelfand-Naimark theorem to the simplest non-Hausdorff topological spaces? Our model space is a finite T_0 topological space, or equivalently, a finite poset. We construct a faithful functor from the category of finite posets with injective morphisms to the category $\mathcal{C}^*$, whose objects are C^*-algebras and whose morphisms are isomorphism classes of Hilbert C^*-bimodules. Then we show in various ways how the construction of this functor fails to extend to the category of finite posets.

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Chapter 1

INTRODUCTION

1.1 Commutative C^*-algebras and Gelfand's Theorem

A continuous map $f : X \to Y$ of topological spaces is *proper* if the inverse image of every compact set is compact. Define LCHaus to be the category of locally compact Hausdorff topological spaces with proper continuous maps as morphisms; define $\mathsf{C}C^*\text{-}\mathsf{Alg}$ to be the category of commutative C^*-algebras *with unit-preserving *-homomorphisms.* Then Gelfand's theorem states that there is a duality of categories such that

1. $X \rightsquigarrow C_0(X)$, and
2. $f : X \to Y \rightsquigarrow f^* : C_0(Y) \to C_0(X)$.
3. $C_0(X \times Y) \cong C_0(X) \otimes C_0(Y)$, where $\overline{\otimes}$ denotes the tensor product of the nuclear C^*-algebras $C_0(X)$ and $C_0(Y)$.
4. $C_0(X \sqcup Y) \cong C_0(X) \oplus C_0(Y)$

The adjoint functor is $\mathrm{Spec} : \mathsf{C}C^*\text{-}\mathsf{Alg} \to \mathsf{LCHaus}$, the maximal ideal spectrum.

The goal of this thesis is to extend this classical result in some way to non-Hausdorff spaces. Our toy model is the simplest kind of non-Hausdorff space: a finite T_0-space. The category of finite T_0-spaces is equivalent to the category of finite posets with order preserving maps (cf. Appendix A). To each finite T_0-space X, or equivalently each finite poset X, we associate a (usually noncommutative) concrete C^*-algebra $\mathcal{A}_X$ of bounded operators acting on a Hilbert space $\mathcal{H}_X$. If X is a finite Hausdorff space then $\mathcal{A}_X \cong C(X)$.

1.2 The main results

Theorem 1.1. *The C^*-algebras $\mathcal{A}_X$ have the following properties:*

1. *$\mathcal{A}_X$ is an approximately finite dimensional C^*-algebra.*
2. *The unitary equivalence classes of irreducible representations of $\mathcal{A}_X$ are in bijection with the points in X. We write $\mathcal{O}_x$ for the irreducible representation corresponding to x and $\mathfrak{p}_x = \operatorname{Ann}(\mathcal{O}_x)$.*
3. *Every representation $\pi : \mathcal{A}_X \to \mathcal{B}(\mathcal{H})$ on a separable Hilbert space $\mathcal{H}$ is unitarily equivalent to a direct sum of countably many irreducible representations.*
4. *The function $X \to \operatorname{Prim}(\mathcal{A}_X)$, $x \mapsto \mathfrak{p}_x$, is a homeomorphism from X to the primitive spectrum of $\mathcal{A}_X$ endowed with the Jacobson topology.*
5. *If Z is a closed subspace of X then there is a surjective C^*-algebra homomorphism $\mathcal{A}_X \to \mathcal{A}_Z$. We write $\mathcal{I}_Z$ for the kernel of this homomorphism.*
6. *If Y and Z are closed subspaces of X then $\mathcal{I}_{Y\cap Z} = \mathcal{I}_Y + \mathcal{I}_Z$ and $\mathcal{I}_{Y\cup Z} = \mathcal{I}_Y \cap \mathcal{I}_Z$.*
7. *The function $Z \mapsto \mathcal{I}_Z$ is an anti-isomorphism from the lattice of closed subspaces of X to the lattice of closed two-sided ideals in $\mathcal{A}_X$.*
8. *If Z is a closed subspace of X, then $\mathcal{I}_Z = \cap_{x\in Z}\mathfrak{p}_x$.*
9. *Every irreducible closed subspace of X has a generic point and, if $\overline{x}$ denotes the closure of $\{x\}$ in X, then $\mathfrak{p}_x = \mathcal{I}_{\overline{x}}$.*
10. *The ideal $\mathcal{I}_Z$ is strongly Morita equivalent to $\mathcal{A}_{X-Z}$.*
11. *$\mathcal{A}_{X\times Y} \cong \mathcal{A}_X \otimes \mathcal{A}_Y$, the C^*-tensor product of the (nuclear) C^*-algebras $\mathcal{A}_X$ and $\mathcal{A}_Y$.*

12. $\mathcal{A}_{X\sqcup Y} \cong \mathcal{A}_X \oplus \mathcal{A}_Y$.

The algebra $\mathcal{A}_X$ is defined in §2.1.1 as a particular subalgebra of the bounded operators on the Hilbert space $\mathcal{H}_X := \ell^2(X_\infty)$ where X_∞ denotes the discrete set of descending sequences $x_1x_2\dots$ where $x_i \geq x_{i+1}$.

There is a category $\mathcal{C}^*$ whose objects are C^*-algebras and whose morphisms are iso-classes of Hilbert C^*-bimodules (cf. Appendix B). Let FPos denote the category of finite posets and order-preserving morphisms. There is a functor $\mathcal{A}$ from the full subcategory of FPos consisting of injective morphisms to the category $\mathcal{C}^*$ such that $\mathcal{A}(X) = \mathcal{A}_X$. Note that for a fixed poset Y, an injective map $f : X \to Y$ is not completely determined by the values of the function; it also depends on the ordering on X.

The rest of this thesis is a result of various attempts to extend the functor $\mathcal{A}$ to the full category of posets. All such attempts were unsuccessful. We view this as evidence to support the conjecture that no such functor exists.

There is a category HS of Hilbert-Schmidt operators, and a functor $E : \mathsf{FPos} \to \mathsf{HS}$. On objects, $E(X)$ is a certain ring of Hilbert-Schmidt operators on $\mathcal{H}_X$; in fact, $E(X)$ is an ideal in $\mathcal{A}_X$ that is not closed in the operator norm. It is shown that since E is a functor, for maps $f : X \to Y$ and $g : Y \to Z$ of finite posets there is an $\mathcal{A}_Y$-$\mathcal{A}_X$-bimodule E_f and an $\mathcal{A}_Z$-$\mathcal{A}_Y$-bimodule E_g such that $E_g \otimes_{\mathcal{A}_Y} E_f \cong E_{gf}$ as $\mathcal{A}_Z$-$\mathcal{A}_X$ bimodules. The functor E has the following properties:

Theorem 1.2. *1. If $i_x : \{x\} \to X$ is the inclusion of a point, then $E_{i_x} \cong \mathcal{O}_x$*

2. If $\tau : X \to \{*\}$ is the terminal map, then $E_\tau \cong \mathcal{H}_X^*$.

3. If $f : X \to Y$ is continuous and $x \in X$, then $E_f \otimes_{\mathcal{A}_X} \mathcal{O}_x \cong \mathcal{O}_{f(x)}$.

1.3 Other constructions from the literature

1.3.1 Incidence Algebras

A similar philosophy underlies non-commutative algebraic geometry and the case of finite posets fits nicely into that framework. Associated to a finite poset is its *incidence algebra*, $\mathbb{I}_X$, which is a finite-dimensional algebra over $\mathbb{C}$. Given an order-preserving map $f : X \to Y$ between finite posets, there is an associated $\mathbb{I}_Y$-$\mathbb{I}_X$-bimodule $\mathbb{I}_f$ and these bimodules have the property that $\mathbb{I}_g \otimes_{\mathbb{I}_Y} \mathbb{I}_f \cong \mathbb{I}_{gf}$ as $\mathbb{I}_Z$-$\mathbb{I}_X$ bimodules for all order-preserving maps $f : X \to Y$ and $g : Y \to Z$ [1].

Even better, in [11], Ladkani shows that the category of right $\mathbb{I}_X$-modules is equivalent to the category of sheaves of $\mathbb{C}$-vector spaces on X, $\mathsf{Sh}(X)$. Under this equivalence, the diagram

$$\begin{tikzcd}
\mathsf{Mod}(\mathbb{I}_Y) \arrow[r, "-\otimes \mathbb{I}_f"] \arrow[d] & \mathsf{Mod}(\mathbb{I}_X) \arrow[d] \\
\mathsf{Sh}(Y) \arrow[r, "f^{-1}"'] & \mathsf{Sh}(X).
\end{tikzcd}$$

commutes whenever $f : X \to Y$ is a continuous map, where f^{-1} is the inverse image (or pullback) functor.

1.3.2 C^*-algebras with finite spectrum

For noncommutative C^*-algebras, there is typically not a bijection between irreducible representations and primitive ideals. The question of which topological spaces can arise as the primitive spectra of C^*-algebras and the problem of constructing C^*-algebras with prescribed primitive spectra have a long history.

Given a finite T_0 topological space X it has long been known how to construct C^*-algebras whose primitive spectra are homeomorphic to X, e.g., [5],[7].

Theorem 1.3 (Bratteli)**.** *[6] Let X be a topological space. There is an approximately finite-dimensional C^*-algebra (or AF-algebra) A such that* $\mathrm{Prim}(A) \cong X$ *if and only if X has the following properties:*

1. *X is T_0;*

2. *X has at most countably many closed subsets;*

3. *for every directed set Λ and every decreasing sequence of closed subsets $Z_i \subset X$, $i \in \Lambda$, the intersection of all Z_i is equal to some Z_j;*

4. *every closed irreducible subspace of X has a generic point; i.e., if Z is a closed subset of X and is not the union of two proper closed subsets, then $Z = \overline{x}$ for some $x \in X$.*

Corollary 1.4. *If X is a finite T_0 topological space, then $X \cong \operatorname{Prim} A$ for some AF algebra A.*

Behnke and Leptin [5] constructed all C*-algebras having only a finite number of irreducible representations. But none of these previous results address the question of functoriality.

A functor from a subcategory of finite posets to von Neumann regular rings is constructed in [3]. The subcategory consists of *complete* morphisms of posets, those injective maps that preserve the relation of immediate predecessor. The restriction on the maps between posets is such that they induce homomorphisms between the associated algebras. Our construction appears unrelated to that in [3].

Nevertheless, the results in [3] prompt one to ask how the monoid of isomorphism classes of finitely generated projective $\mathcal{A}_X$-modules is related to the abelian monoid generated by X. We do not consider that question.

1.3.3 The use of bimodules as morphisms

The category $\mathcal{C}^*$ is a typical replacement for C^*-Alg in the literature when one is looking for functorial behavior. For example in [13, p. 103], Landsman defines a functor from a suitable category of smooth Lie groupoids to $\mathcal{C}^*$. The objects of his category are Lie groupoids, and his morphisms are regular (left principal and right proper) bibundles, $G \leftarrow M \rightarrow H$. Here,

M is a smooth manifold equipped with left and right actions from G and H, respectively, together with compatible smooth maps $\tau : M \to G_0$ and $\sigma : M \to H_0$. These technical assumptions on the morphisms (i.e. left principal and right proper) are chosen so that there is a functor to $\mathcal{C}^*$.

Although our algebra $\mathcal{A}_X$ can be described as the C^*-algebra of a groupoid, Landsman's functor does not apply to our situation since we do not consider smooth manifolds.

Chapter 2

THE AF-ALGEBRA, $\mathcal{A}_X$

In this chapter, X is a fixed finite poset. We define a concrete C^*-algebra $\mathcal{A}_X$, i.e. a closed sub-algebra of bounded operators on a separable Hilbert space, $\mathcal{H}_X$. This construction has several nice properties, as we will eventually prove:

1. There is an isomorphism of posets $X \cong \operatorname{Prim}(\mathcal{A}_X)$ onto the primitive spectrum;
2. The set of unitary equivalence classes of irreducible representations of $\mathcal{A}_X$ is also isomorphic to X; the Hilbert space $\mathcal{H}_X$ decomposes as a finite direct sum, one for each equivalence class.

2.1 The definition of $\mathcal{A}_X$

We define

$$\begin{aligned} X_n &:= \{x_1 \ldots x_n \mid x_i \in X \text{ and } x_i \geq x_{i+1} \text{ for all } i \geq 1\}, \quad n \geq 1 \\ X_\infty &:= \{x_1 x_2 \ldots \mid x_i \in X \text{ and } x_i \geq x_{i+1} \text{ for all } i \geq 1\}, \end{aligned}$$

and maps

$$\begin{aligned} s &: X_n \to X, && s(x_1 \ldots x_n) := x_1, \\ t &: X_n \to X, && t(x_1 \ldots x_n) := x_n, \\ s &: X_\infty \to X, && s(x_1 x_2 \ldots) := x_1, \\ t &: X_\infty \to X, && t(x_1 x_2 \ldots) := x \quad \text{where } x \text{ is the unique element in} \\ & && \qquad X \text{ such that } x_n = x \text{ for all } n \gg 1. \end{aligned}$$

We define $X_0 := \varnothing$.

We will denote elements of X_∞ by lower-case bold-face letters. Let $\mathbf{x} = x_1x_2\ldots \in X_\infty$ and $p \in X_n$. If $t(p) \geq s(\mathbf{x})$, we define

$$p\mathbf{x} \;:=\; p_1\ldots p_n x_1 x_2 \ldots \in X_\infty, \qquad \text{and} \qquad pX_\infty \;:=\; \{p\mathbf{x} \mid \mathbf{x} \in X_\infty,\ t(p) \geq s(\mathbf{x})\}.$$

The set pX_∞ consists of the sequences in X_∞ whose first n entries agree with p. If $p \in X_n$ we say p has length n and write $|p| = n$.

If $x \in X$ we write x^∞ for the infinite sequence $xxx\ldots$, and x^n for the sequence $x\ldots x$ of length n. For each $x \in X$, let $[x] := \{\mathbf{x} \in X_\infty \mid t(\mathbf{x}) = x\}$. Thus X_∞ is the disjoint union of the subsets $[x]$, $x \in X$. With the appropriate identifications, there is an orthogonal decomposition

$$\ell^2(X_\infty) = \bigoplus_{x\in X} \ell^2([x]).$$

This decomposition appears again in 2.4. We note also that X_∞ is the disjoint union of the subsets xX_∞, $x \in X$. There is a corresponding orthogonal decomposition

$$\ell^2(X_\infty) = \bigoplus_{x\in X} \ell^2(xX_\infty).$$

There is an equivalence of categories (even an isomorphism) between finite posets and finite T_0-topological spaces (see Appendix A). Briefly, sets of the form $\downarrow x = \{x' \leq x \mid x' \in X\} = U_x$ are a base of open subsets; each such set U_x is a minimal neighborhood of $x \in X$. Note that $xX_\infty = (U_x)_\infty$.

2.1.1 Notation

If $p, q \in X_n$ and $t(p) = t(q)$, we adopt the convention that

$$pq^*\mathbf{x} \;=\; \begin{cases} p\mathbf{x}' & \text{if } \mathbf{x} = q\mathbf{x}' \\ \text{is undefined} & \text{if } \mathbf{x} \notin qX_\infty. \end{cases}$$

In (2.2-1), we define a linear operator that is also denoted pq^*.

Define

$$\mathcal{H}_X := \ell^2(X_\infty).$$

The set of characteristic functions

$$\chi_{\mathbf{x}} : X_\infty \to \{0,1\}, \qquad \chi_{\mathbf{x}}(\mathbf{x}') := \delta_{\mathbf{x},\mathbf{x}'}, \quad \mathbf{x} \in X_\infty,$$

is an orthonormal basis for $\mathcal{H}_X$.

2.2 The linear operators $pq^* \in \mathsf{B}(\mathcal{H}_X)$, and $\mathcal{A}_X$

If $p, q \in X_n$ and $t(p) = t(q)$ we define the operator

$$pq^* : \mathcal{H}_X \to \mathcal{H}_X \quad \text{by} \quad pq^*(\chi_{\mathbf{x}}) \ := \begin{cases} \chi_{pq^*\mathbf{x}} & \text{if } \mathbf{x} \in qX_\infty \\ 0 & \text{if } \mathbf{x} \notin qX_\infty, \end{cases} \tag{2.2-1}$$

and

$$\mathcal{A}_X := \overline{\operatorname{span}}\{pq^* \mid p, q \in X_n,\, n \geq 0,\, t(p) = t(q)\} \ \subseteq \ \mathsf{B}(\mathcal{H}_X).$$

Proposition 2.1. *Let X be a finite T_0 topological space. The algebra $\mathcal{A}_X$ is a unital C^*-subalgebra of $\mathsf{B}(\mathcal{H}_X)$ and the following properties hold.*

1. *If pq^* and rs^* are elements of $\mathcal{A}_X$ then*

$$(pq^*)(rs^*) = \begin{cases} p(sr')^* & \text{if } q = rr' \\ (pq')s^* & \text{if } r = qq' \\ 0 & \text{if } qX_\infty \cap rX_\infty = \varnothing. \end{cases}$$

2. *The set $\{xx^* \mid x \in X\}$ consists of mutually orthogonal projections in $\mathcal{A}_X$ and*

$$\sum_{x \in X} xx^* = \mathrm{id}_{\mathcal{H}_X},$$

the identity operator on $\mathcal{H}_X$.

3. *If $pq^* \in \mathcal{A}_X$, then $(pq^*)^* = qp^*$.*

Proof. Each pq^* is a bounded operator of norm 1 because it sends the orthonormal basis $\{\chi_{\mathbf{x}} \mid \mathbf{x} \in X_\infty\}$ to a subset of itself.

(1) This is a "bookkeeping" exercise: one simply examines the possible values of $(pq^*)(rs^*)(\chi_{\mathbf{x}})$ for $\mathbf{x} \in X_\infty$. (2) follows easily from (1).

(3) Let $\sum_{\mathbf{y}\in X_\infty} \lambda_{\mathbf{y}}\chi_{\mathbf{y}} \in \mathcal{H}_X$. Then

$$pq^*\left(\sum_{\mathbf{y}\in X_\infty} \lambda_{\mathbf{y}}\chi_{\mathbf{y}}\right) = \sum_{\mathbf{y}\in qX_\infty} \lambda_{\mathbf{y}}\chi_{pq^*\mathbf{y}} = \sum_{\mathbf{x}\in pX_\infty} \lambda_{qp^*\mathbf{x}}\chi_{\mathbf{x}}. \tag{2.2-2}$$

The calculation

$$\begin{aligned}
\left\langle pq^*\left(\sum_{\mathbf{x}\in X_\infty} \lambda_{\mathbf{x}}\chi_{\mathbf{x}}\right), \sum_{\mathbf{y}\in X_\infty} \mu_{\mathbf{y}}\chi_{\mathbf{y}}\right\rangle &= \left\langle \sum_{\mathbf{x}\in pX_\infty} \lambda_{qp^*\mathbf{x}}\chi_{\mathbf{x}}, \sum_{\mathbf{y}\in X_\infty} \mu_{\mathbf{y}}\chi_{\mathbf{y}}\right\rangle \\
&= \sum_{\mathbf{x}\in pX_\infty} \lambda_{qp^*\mathbf{x}}\overline{\mu_{\mathbf{x}}} \\
&= \sum_{\mathbf{y}\in qX_\infty} \lambda_{\mathbf{y}}\overline{\mu_{pq^*\mathbf{y}}} \\
&= \left\langle \sum_{\mathbf{x}\in X_\infty} \lambda_{\mathbf{x}}\chi_{\mathbf{x}}, \sum_{\mathbf{y}\in qX_\infty} \mu_{pq^*\mathbf{y}}\chi_{\mathbf{y}}\right\rangle \\
&= \left\langle \sum_{\mathbf{x}\in X_\infty} \lambda_{\mathbf{x}}\chi_{\mathbf{x}}, qp^*\left(\sum_{\mathbf{y}\in X_\infty} \mu_{\mathbf{y}}\chi_{\mathbf{y}}\right)\right\rangle
\end{aligned}$$

shows that qp^* is adjoint to pq^*. □

If $pq^* \in \mathcal{A}_X$, then $(pq^*)(qp^*) = pp^*$ and $(pp^*)^2 = pp^*$ so pq^* is a partial isometry. With the appropriate identifications, pq^* is an isometry $\ell^2(qX_\infty) \to \ell^2(pX_\infty)$.

In particular, if $x \in X$, then xx^* is the identity map $\ell^2(xX_\infty) \to \ell^2(xX_\infty)$ and vanishes on $\ell^2(yX_\infty)$ for $y \neq x$.

2.3 $\mathcal{A}_X$ *is an AF-algebra*

Proposition 2.2. *The algebra $\mathcal{A}_X$ is an AF-algebra: it is the norm closure of the ascending union of the finite dimensional *-subalgebras $\mathcal{A}^1 \subset \mathcal{A}^2 \subset \cdots$ where*

$$\mathcal{A}^n := \operatorname{span}\{pq^* \in \mathcal{A}_X \mid p, q \in X_n\}.$$

Proof. The elements xx^*, $x \in X$, form a basis of mutually orthogonal idempotents for $\mathcal{A}_1$ and their sum is $\mathrm{id}_{\mathcal{H}_X}$. Thus

$$\mathcal{A}^1 = \bigoplus_{x\in X} \mathbb{C}xx^*.$$

By Proposition 2.1(1), $\mathcal{A}^n$ is closed under composition, i.e. multiplication, and under the operation of taking adjoints. Thus, $\mathcal{A}^n$ is a C*-subalgebra of $\mathcal{A}_X$.

If $pq^* \in \mathcal{A}^n$, then

$$pq^* = \sum_{t(p)\geq x} (px)(qx)^*.$$

Thus $\mathcal{A}^n \subset \mathcal{A}^{n+1}$. By definition, $\mathcal{A}_X$ is the closure of the union of the $\mathcal{A}^n$s. ☐

Because it is a finite dimensional C^*-algebra, $\mathcal{A}^n$ is a finite direct sum of matrix algebras. We will make these summands explicit by representing $\mathcal{A}^n$ on $\ell^2(X_n)$.

Let $u \in X_n$. Define $\chi_u : X_n \to \mathbb{C}$ by $\chi_u(v) := \delta_{u,v}$. The set $\{\chi_u \mid u \in X_n\}$ is an orthonormal basis for $\ell^2(X_n)$.

Lemma 2.3. *The Hilbert space $\ell^2(X_n)$ is a faithful representation of $\mathcal{A}^n$ with respect to the the map $\pi : \mathcal{A}^n \to \mathsf{B}(\ell^2(X_n))$ defined by*

$$\pi(pq^*)(\chi_u) := \delta_{q,u}\chi_p \qquad \text{for each } u \in X_n. \tag{2.3-1}$$

Furthermore, $\{pq^* \in \mathcal{A}_X \mid p, q \in X_n\}$ is a basis for $\mathcal{A}^n$.

Proof. Since

$$\pi(pq^*)\pi(rs^*)(\chi_u) := \pi(pq^*)(\delta_{s,u}\chi_r) = \delta_{s,u}\delta_{q.r}\chi_p$$

and

$$\pi((pq^*)(rs^*))(\chi_u) = \pi(\delta_{q,r}ps^*)(\chi_u) = \delta_{q,r}\delta_{s,u}\chi_p,$$

π is an algebra homomorphism. It follows from (2.3-1) that $\{\pi(pq^*) \mid p, q \in X_n,\ t(p) = t(q)\}$ is linearly independent. Hence $\ker(\pi) = \{0\}$. Therefore $\{pq^* \in \mathcal{A}_X \mid p, q \in X_n\}$ is a basis for $\mathcal{A}^n$. ☐

If $p, q, r, s \in X_n$ and $t(p) = t(q)$ and $t(r) = t(s)$, then

$$(pq^*)(rs^*) = \delta_{q,r} ps^*. \tag{2.3-2}$$

For each $x \in X$, define $X_{n-1}x := \{p \in X_n \mid t(p) = x\}$. Since

$$X_n = \bigsqcup_{x \in X} X_{n-1}x, \tag{2.3-3}$$

there is an orthogonal decomposition

$$\ell^2(X_n) = \bigoplus_{x \in X} \ell^2(X_{n-1}x) \tag{2.3-4}$$

Lemma 2.4. *Equation (2.3-4) is the orthogonal decomposition of $\ell^2(X_n)$ into its irreducible $\mathcal{A}^n$-subrepresentations. The corresponding decomposition of $\mathcal{A}^n$ as a direct sum of simple C^*-algebras is*

$$\mathcal{A}^n = \bigoplus_{x \in X} B_x \tag{2.3-5}$$

where $B_x = \operatorname{span}\{pq^* \mid p, q \in X_{n-1}x\}$. The restriction of π to B_x is a C^*-algebra isomorphism $B_x \to \mathsf{B}(\ell^2(X_{n-1}x))$.

Proof. It follows from (2.3-2) that B_x is a C^*-subalgebra of $\mathcal{A}^n$. It follows from (2.3-3) and Lemma 2.3 that $\mathcal{A}^n$ is the direct sum of the B_x, $x \in X$. Equation (2.3-2) implies that $B_x B_{x'} = 0$ if $x \neq x'$. Hence each B_x is a two-sided ideal of $\mathcal{A}^n$.

Let $u, v \in X_{n-1}x$. Since $\pi(uv^*)(\chi_v) = \chi_u$, $\ell^2(X_{n-1}x)$ is an irreducible representation of $\pi(B_x)$. If $w \in X_n - X_{n-1}x$ then $\pi(uv^*)(\chi_w) = 0$; it follows easily that each $\ell^2(X_{n-1}x)$ is an irreducible $\mathcal{A}^n$-submodule of $\ell^2(X_n)$. □

Proposition 2.5. *[6, Lemma 3.1]. If $\mathcal{I}$ is a closed two-sided ideal in an AF-algebra $\mathcal{A} = \overline{\cup_{n \geq 0} \mathcal{A}^n}$, then*

$$\mathcal{I} = \overline{\bigcup_{n \geq 0} (\mathcal{I} \cap \mathcal{A}^n)}.$$

Corollary 2.6. *If $\mathcal{I}$ is a closed ideal of $\mathcal{A}_X$, then $\mathcal{I} = \overline{\operatorname{span}}\{pq^* \in \mathcal{A}_X \mid pq^* \in \mathcal{I}\}$.*

Proof. By Proposition 2.5, it suffices to show that every ideal $\mathcal{J} \subseteq \mathcal{A}^n$ is equal to $\operatorname{span}\{pq^* \mid p, q \in X_n,\ t(p) = t(q),\ pq^* \in \mathcal{J}\}$. It is certainly true that each B_x is spanned by the elements pq^* that are contained in it. Since every ideal in $\mathcal{A}^n$ is a direct sum of various B_x's the same is true for every ideal in $\mathcal{A}^n$. □

The next result is the analogue of the fact that if X and Y are compact Hausdorff spaces, then $C(X \times Y) \cong C(X) \otimes C(Y)$; here, $\otimes$ is the completion of the algebraic tensor product with respect to the unique C^*-norm. Indeed, since commutative C^*-algebras are nuclear they admit unique cross-norms.

Theorem 2.7. *If X and Y are finite T_0-topological spaces, then*

1. $(X \times Y)_\infty \cong X_\infty \times Y_\infty$ *via* $(x, y)_i \mapsto (x_i) \times (y_i)$;
2. $\mathcal{H}_{X\times Y} \cong \mathcal{H}_X \otimes \mathcal{H}_Y$ *via* $\chi_{(x,y)_i} \mapsto \chi_{x_i} \otimes \chi_{y_i}$;
3. $\mathcal{A}_{X\times Y} \cong \mathcal{A}_X \otimes \mathcal{A}_Y$ *via* $(p_x, p_y)(q_x, q_y)^* \mapsto p_x q_x^* \otimes p_y q_y^*$.

Note that since $\mathcal{A}_X$ is nuclear, there is a unique C^*-norm on the algebraic tensor product; $\mathcal{A}_X \otimes \mathcal{A}_Y$ is the completion with respect to this norm.

Proof. (1) Every sequence in $(X \times Y)_\infty$ has the form

$$p_1 \geq p_2 \geq \cdots .$$

Necessarily, $p_i = (x, y)_i$ for some $x \in X$ and $y \in Y$ and $p_i \geq p_{i+1}$ if and only if $x_i \geq x_{i+1}$ and $y_i \geq y_{i+1}$. This establishes the given bijection $(X \times Y)_\infty \to Y_\infty \times X_\infty$.

If S and T are two sets, then there is a bounded $\mathbb{C}$-linear isomorphism from $\ell^2(S \times T) \to \ell^2(T) \otimes \ell^2(S)$ such that $\chi_{(s,t)} \mapsto \chi_s \otimes \chi_t$. Letting $S = Y_\infty$ and $T = X_\infty$ we see that $\ell^2(X_\infty \times Y_\infty) \cong \ell^2(X_\infty) \otimes \ell^2(Y_\infty) = \mathcal{H}_X \otimes \mathcal{H}_Y$. Using (1), we have that $\ell^2((X \times Y)_\infty) \cong \ell^2(X_\infty \times Y_\infty)$, and hence the result. The map described in the statement is the resulting map.

Under the explicit isomorphism in (2), the action of the operator $(p_x, p_y)(q_x, q_y)^* \in \mathcal{A}^n_{X\times Y}$ on $\mathcal{H}_{X\times Y}$ corresponds to the action of the operator $p_y q_y^* \otimes p_x q_x^* \in \mathcal{A}^n_Y \otimes \mathcal{A}^n_X$ on $\mathcal{H}_Y \otimes \mathcal{H}_X$. Thus, $\mathcal{A}_{X\times Y}$ is the same concrete C^*-algebra of operators on $\mathcal{H}_X \otimes \mathcal{H}_Y$ as $\mathcal{A}_X \otimes \mathcal{A}_Y$. □

2.4 Irreducible representations of $\mathcal{A}_X$ and their annihilators

In this section we describe the representations of $\mathcal{A}_X$ on Hilbert spaces. A Hilbert-space representation of a C^*-algebra $\mathcal{A}$ is a separable Hilbert space $\mathcal{H}$ together with a continuous map $\lambda : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ such that

$$\langle \lambda(a)h_1, h_2\rangle = \langle h_1, \lambda(a^*)h_2\rangle, \qquad \text{for all } a \in \mathcal{A},\ h_1, h_2 \in \mathcal{H}.$$

The representation is non-degenerate if $\lambda(A)\mathcal{H}$ is dense in $\mathcal{H}$ (by convention, $\lambda(A)\mathcal{H}$ is the linear span of $\{\lambda(a)h \mid a \in \mathcal{A},\ h \in \mathcal{H}\}$).

For each $x \in X$, define

$$\begin{aligned} \mathcal{O}_x &:= \{\xi \in \ell^2(X_\infty) \mid \xi(\mathbf{x}) = 0 \text{ if } t(\mathbf{x}) \neq x\} \qquad\qquad \text{and} \\ \mathfrak{p}_x &:= \operatorname{Ann}(\mathcal{O}_x) = \{a \in \mathcal{A}_X \mid a(\xi) = 0 \text{ for all } \ \xi \in \mathcal{O}_x\}. \end{aligned}$$

It is clear that $\mathcal{O}_x$ is a closed subspace of $\ell^2(X_\infty)$. The set $\{\chi_{\mathbf{x}} \mid t(\mathbf{x}) = x\}$ is an orthonormal basis for $\mathcal{O}_x$. Extension by zero is an isomorphism $\ell^2\big([x]\big) \cong \mathcal{O}_x$ of Hilbert spaces.

Because X is finite, every sequence in X_∞ is eventually constant. There is therefore a finite partition

$$X_\infty = \bigsqcup_{x\in X} \{\mathbf{x} \in X_\infty \mid t(\mathbf{x}) = x\}$$

and a corresponding direct-sum decomposition

$$\mathcal{H}_X \ = \ \ell^2(X_\infty) \ = \ \bigoplus_{x\in X} \mathcal{O}_x$$

of $\mathcal{H}_X$ into mutually orthogonal closed subspaces. This decomposition is the same as that in 2.1.

Proposition 2.8. *For $x \in X$, $\mathcal{O}_x$ is an irreducible representation of $\mathcal{A}_X$. Furthermore, $\mathcal{O}_x$ is unitarily equivalent to $\mathcal{O}_{x'}$ if and only if $x = x'$ if and only if $\mathfrak{p}_x = \mathfrak{p}_{x'}$.*

Proof. For the duration of this proof, we write $\mathcal{A}$ for $\mathcal{A}_X$.

Let $\xi \in \mathcal{O}_x$ and $pq^* \in \mathcal{A}$. We will show that $pq^*(\xi) \in \mathcal{O}_x$. Let $\mathbf{x} \in X_\infty$. Then

$$pq^*(\xi)(\mathbf{x}) = \begin{cases} \xi(pq^*\mathbf{x}) & \text{if } \mathbf{x} \in qX_\infty \\ 0 & \text{if } \mathbf{x} \notin qX_\infty. \end{cases}$$

If $\mathbf{x} \in qX_\infty$, then $t(pq^*\mathbf{x}) = t(\mathbf{x})$ so $\xi(pq^*\mathbf{x}) = 0$ if $t(\mathbf{x}) \neq x$; since $pq^*(\xi)$ vanishes on $\mathcal{O}_{x'}$ for all $x' \neq x$, $pq^*(\xi) \in \mathcal{O}_x$. Thus, $\mathcal{O}_x$ is an $\mathcal{A}$-subrepresentation of $\mathcal{H}_X$.

Claim: $\mathcal{O}_x$ is generated by χ_{x^∞}. Proof: Certainly $\chi_{x^\infty} \in \mathcal{O}_x$. Suppose $t(\mathbf{x}) = x$. Then $\mathbf{x} = pq^*x^\infty$ for some $pq^* \in \mathcal{A}$ whence $pq^*(\chi_{x^\infty}) = \chi_{\mathbf{x}}$. Therefore $\chi_{\mathbf{x}} \in \mathcal{A}\chi_{x^\infty}$. Since $\mathcal{O}_x = \overline{\text{span}}\{\chi_{\mathbf{x}} \mid t(\mathbf{x}) = x\}$, $\mathcal{O}_x = \overline{\mathcal{A}\chi_{x^\infty}}$. $\Diamond$

To show $\mathcal{O}_x$ is irreducible we consider a non-zero element $\xi \in \mathcal{O}_x$, say

$$\xi = \sum_{\mathbf{x} \in [x]} \lambda_{\mathbf{x}} \chi_{\mathbf{x}}.$$

Choose $\mathbf{x}$ such that $\lambda_{\mathbf{x}} \neq 0$. There is some p in some X_n such that $\mathbf{x} = px^\infty$. Since

$$(px)(px)^*(\xi) = \lambda_{px^\infty}\chi_{px^\infty} = \lambda_{\mathbf{x}}\chi_{px^\infty},$$

$\chi_{px^\infty} \in \mathcal{A}\xi$. Therefore $\mathcal{A}\xi = \mathcal{O}_x$.

Let x and x' be distinct elements in X. Without loss of generality we can, and do, assume that $x \not\geq x'$. If $t(\mathbf{x}) = x'$, then $s(\mathbf{x}) \neq x$ so $xx^*(\chi_{\mathbf{x}}) = 0$. Hence xx^* annihilates $\mathcal{O}_{x'}$. But xx^* does not annihilate $\mathcal{O}_x$ because $xx^*(\chi_{x^\infty}) = \chi_{x^\infty}$. Hence $\mathfrak{p}_x \neq \mathfrak{p}_{x'}$. Since their annihilators are different $\mathcal{O}_x$ is not isomorphic to $\mathcal{O}_{x'}$. $\square$

Proposition 2.19 below shows that $\{\mathcal{O}_x \mid x \in X\}$ is a complete set of irreducible representations of $\mathcal{A}_X$ up to unitary equivalence.

Proposition 2.9. *1. Fix $x \in X$ and define $e := \sum_{y \not\geq x} yy^*$. Then*

$$\mathfrak{p}_x \;=\; \overline{\text{span}}\{pq^* \in \mathcal{A}_X \mid t(p) \not\geq x\} \;=\; \mathcal{A}e\mathcal{A}.$$

2. The function $X \to \text{Prim}(X)$, $x \mapsto \mathfrak{p}_x$, is an injective order-preserving map of posets.

Proof. (1) Let $pq^* \in \mathcal{A}_X$ and $\mathbf{x} \in X_\infty$. Then $pq^*(\chi_{\mathbf{x}}) = 0$ if and only if $\mathbf{x} \notin qX_\infty$. Therefore,

$$\begin{aligned} pq^* \in \mathfrak{p}_x \quad &\Leftrightarrow \quad \{\mathbf{x} \in X_\infty \mid t(\mathbf{x}) = x\} \subseteq X_\infty - qX_\infty \\ &\Leftrightarrow \quad \{\mathbf{x} \in X_\infty \mid t(\mathbf{x}) = x\} \cap qX_\infty = \varnothing \\ &\Leftrightarrow \quad t(q) \ngeq x. \end{aligned}$$

It now follows from Corollary 2.6 that $\mathfrak{p}_x = \overline{\text{span}}\{pq^* \in \mathcal{A}_X \mid t(p) \ngeq x\}$.

In particular, $yy^* \in \mathfrak{p}_x$ if $y \ngeq x$. Therefore $\mathfrak{p}_x$ contains e and $\mathcal{A}e\mathcal{A}$.

Since $\{yy^* \mid y \in X\}$ is a set of mutually orthogonal projections, $\mathcal{A}e\mathcal{A}$ is equal to the two-sided ideal generated by $\{yy^* \mid y \ngeq x\}$. If $t(p) \ngeq x$, then

$$pq^* = \sum_{t(p) \geq y} (py)(qy)^* = \sum_{t(p) \geq y} (py)e(qy)^*$$

so $pq^* \in \mathcal{A}e\mathcal{A}$. Hence $\mathfrak{p}_x \subseteq \mathcal{A}e\mathcal{A}$.

(2) It follows from (1) that $\mathfrak{p}_x \neq \mathfrak{p}_y$ if $x \neq y$. It also follows from (1) that $\mathfrak{p}_x \subseteq \mathfrak{p}_y$ if $x \leq y$. The map $x \mapsto \mathfrak{p}_x$ is therefore injective and order-preserving. $\square$

Proposition 2.16 shows that the map $X \to \text{Prim}(\mathcal{A}_X)$, $x \mapsto \mathfrak{p}_x$, is surjective and hence a homeomorphism of topological spaces.

Lemma 2.10. *Let $x \in X$. The following conditions are equivalent:*

1. $\dim(\mathcal{O}_x) = 1$*;*
2. $\dim(\mathcal{O}_x) < \infty$*;*
3. *x is a maximal element in X, i.e., if $x' \in X$ and $x' \geq x$, then $x' = x$;*
4. $\overline{x} = \{x\}$.

Proof. (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (3) If x is not maximal choose $y \in X$ such that $y > x$. For each $n \geq 0$, let $\mathbf{x}_n$ be the element $y \dots y x^\infty$ where there are n y's. Each $\mathbf{x}_n$ belongs to $[x^\infty]$ so $\{\chi_{\mathbf{x}_n} \mid n \geq 0\}$ is an infinite linearly independent subset of $\mathcal{O}_x$. Thus $\dim(\mathcal{O}_x) = \infty$.

(3) $\Rightarrow$ (1) If x is maximal then $[x^\infty] = \{x^\infty\}$, so $\mathcal{O}_x = \ell^2([x^\infty]) = \mathbb{C}$.

(3) $\Leftrightarrow$ (4) This follows from the fact that $\overline{x} = \{y \in X \mid x \leq y\}$. $\square$

2.5 The ideal of $\mathcal{A}_X$ "vanishing" on a closed subspace of X

We use quotation marks around the word "vanishing" because the ideal associated to a closed subspace $Z \subseteq X$ is defined to be the ideal that annihilates, i.e., vanishes on, the irreducible representations $\mathcal{O}_z$ for all $z \in Z$.

This point of view is consistent with the classical results about ideals in $C(X)$, the C*-algebra of functions on a locally compact Hausdorff space X: the ideal of functions vanishing on a closed subspace $Z \subseteq X$ consists of the functions that annihilate the irreducible representations $C(X)/\mathfrak{m}_z$, $z \in Z$, where $\mathfrak{m}_z$ is the ideal of functions vanishing at the point z.

If Z is a closed subspace of X we define

$$\mathcal{I}_Z := \operatorname{Ann}\left(\bigoplus_{z \in Z} \mathcal{O}_z\right) = \bigcap_{z \in Z} \mathfrak{p}_z.$$

In particular, $\mathcal{I}_\varnothing = \mathcal{A}_X$ and $\mathcal{I}_X = \{0\}$.

Since $\mathfrak{p}_x \subseteq \mathfrak{p}_y$ if and only if $x \leq y$, $\mathfrak{p}_x = \mathcal{I}_{\overline{x}}$.

Proposition 2.11. *Let Y and Z be closed subspaces of X. Then*

1. $\mathcal{I}_Y \cap \mathcal{I}_Z = \mathcal{I}_{Y \cup Z}$;
2. $\mathcal{I}_Y + \mathcal{I}_Z = \mathcal{I}_{Y \cap Z}$;
3. $\mathcal{I}_Z = \overline{\mathit{span}}\{pq^* \in \mathcal{A}_X \mid t(p) \notin Z\}$;
4. $\mathcal{I}_Z = \mathcal{A}_X e \mathcal{A}_X = \mathcal{I}_Z e \mathcal{I}_Z$ *where* $e = \sum_{x \notin Z} xx^*$.

Proof. (1) This follows from the fact that $\mathcal{H}_Y + \mathcal{H}_Z = \mathcal{H}_{Y\cup Z}$.

(3) We have

$$\begin{aligned}
\mathcal{I}_Z &= \overline{\text{span}}\{pq^* \in \mathcal{A}_X \mid pq^* \in \mathcal{I}_Z\} && \text{by Corollary 2.6}\\
&= \overline{\text{span}}\{pq^* \in \mathcal{A}_X \mid pq^* \in \mathfrak{p}_z \text{ for all } z \in Z\}\\
&= \overline{\text{span}}\{pq^* \in \mathcal{A}_X \mid t(p) \ngeq z \text{ for any } z \in Z\} && \text{by Proposition 2.9(1)}\\
&= \overline{\text{span}}\{pq^* \in \mathcal{A}_X \mid t(p) \notin Z\}.
\end{aligned}$$

(2) By (3),

$$\begin{aligned}
\mathcal{I}_Y + \mathcal{I}_Z &= \overline{\text{span}}\{pq^* \in \mathcal{A}_X \mid \text{either } pq^* \notin Y \text{ or } pq^* \notin Z\}\\
&= \overline{\text{span}}\{pq^* \in \mathcal{A}_X \mid pq^* \notin Y \cap Z\}
\end{aligned}$$

which is equal to $\mathcal{I}_{Y\cap Z}$ by (3).

(4) The projections xx^*, $x \in X - Z$, that add up to e are mutually orthogonal projections so an ideal contains e if and only if it contains $\{xx^* \mid x \in X - Z\}$. Also $xx^* = e(xx^*) = (xx^*)e$.

Let $x \in X - Z$. By (3), $xx^* \in \mathcal{I}_Z$. Hence $e \in \mathcal{I}_Z$. Therefore $\mathcal{A}_X e \mathcal{A}_X \subseteq \mathcal{I}_Z$.

On the other hand, suppose that $p, q \in X_n$ and $x = t(p) = t(q) \notin Z$. Then pq^*, $p(x^n)^*$, and $x^n q^*$, all belong to $\mathcal{I}_Z$ and

$$pq^* = \big(p(x^n)^*\big)(xx^*)\big(x^n q^*\big) = \big(p(x^n)^*\big)e\big(x^n q^*\big)$$

so $pq^* \in \mathcal{I}_Z e \mathcal{I}_Z$. It follows that $\mathcal{I}_Z \subseteq \mathcal{I}_Z e \mathcal{I}_Z$. The two equalities in (4) follow. $\square$

Theorem 2.12. *Let Z be a closed subspace of X.*

1. *The function $\varphi : \mathcal{H}_Z \to \mathcal{H}_X$ defined by $\varphi(\chi_{\mathbf{z}}) := \chi_{\mathbf{z}}$ is an isometry from $\mathcal{H}_Z$ onto $\oplus_{x\in Z}\mathcal{O}_x$.*

2. *If we identify $\mathcal{H}_Z$ with the closed subspace $\oplus_{x\in Z}\mathcal{O}_x$ of $\mathcal{H}_X$, then the action of $\mathcal{A}_X$ on the subrepresentation $\oplus_{x\in Z}\mathcal{O}_x$ of $\mathcal{H}_X$ is a surjective C^*-algebra homomorphism $\Phi : \mathcal{A}_X \to \mathcal{A}_Z$ and $\ker(\Phi) = \mathcal{I}_Z$.*

3. In particular, $\mathcal{A}_X/\mathcal{I}_Z \cong \mathcal{A}_Z$*.*

Proof. (1) Let $\mathbf{x} \in X_\infty$. If $t(\mathbf{x}) \in Z$, then $s(\mathbf{x})$ is in Z, since Z is closed and $s(\mathbf{x}) \geq t(\mathbf{x})$. Therefore $\{\chi_{\mathbf{x}} \mid \mathbf{x} \in Z_\infty\}$ is an orthonormal basis for the closed subspace $\oplus_{x\in Z}\mathcal{O}_x \subseteq \mathcal{H}_X$. Since $\{\chi_{\mathbf{z}} \mid \mathbf{z} \in Z_\infty\}$ is an orthonormal basis for $\mathcal{H}_Z$, φ is an isometry.

(2) By Proposition 2.8, $\oplus_{x\in Z}\mathcal{O}_x$ is an $\mathcal{A}_X$-subrepresentation of $\mathcal{H}_X$ so the action of $\mathcal{A}_X$ on it is, in effect, a C*-algebra homomorphism

$$\Phi : \mathcal{A}_X \to \mathsf{B}\left(\bigoplus_{x\in Z}\mathcal{O}_x\right) \cong \mathsf{B}(\mathcal{H}_Z).$$

It is obvious that $\ker(\Phi) = \operatorname{Ann}\big(\oplus_{x\in Z}\mathcal{O}_x\big) = \mathcal{I}_Z$.

The fact that the image of Φ is $\mathcal{A}_Z$ is essentially a tautology: if $pq^* \in \mathcal{A}_Z$ with p and q in Z_n there is also an element in $\mathcal{A}_X$ labelled pq^* and $\Phi(pq^*) = pq^*$. □

From now on, if Z is a closed subspace of X remake the identification

$$\mathcal{H}_Z := \bigoplus_{z\in Z}\mathcal{O}_z \subseteq \mathcal{H}_X.$$

Lemma 2.13. *If x is a minimal element in X then $X - \{x\}$ is closed and $\mathcal{I}_{X-\{x\}} = \mathcal{A}_X(xx^*)\mathcal{A}_X$ is a minimal non-zero two-sided ideal in $\mathcal{A}_X$.*

Proof. Since x is minimal, $\{x\}$ is open and $X - \{x\}$ is closed. By Proposition 2.11(4), $\mathcal{A}_X(xx^*)\mathcal{A}_X = \mathcal{I}_{X-\{x\}}$. Let $pq^* \in \mathcal{A}_X(xx^*)\mathcal{A}_X$. Then $t(p) = x$ by Proposition 2.11(3). Let $n = |p|$. Then $\big(x^np^*\big)\big(pq^*\big)\big(q(x^n)^*\big) = x^n(x^n)^* = xx^*$ where the last equality follows from the fact that x is minimal. Hence $xx^* \in \mathcal{A}_X(pq^*)\mathcal{A}_X$ for all $pq^* \in \mathcal{A}_X(xx^*)\mathcal{A}_X$. It follows that $\mathcal{A}_X(xx^*)\mathcal{A}_X$ is a minimal non-zero two-sided ideal. □

Theorem 2.14. *The map $Z \mapsto \mathcal{I}_Z$ is a lattice anti-isomorphism from the set of closed subspaces of X to the set of closed two-sided ideals in $\mathcal{A}_X$.*

Proof. We argue by induction on $|X|$. The result is obvious when $|X| = 1$ because then $\mathcal{A}_X \cong \mathbb{C}$. Suppose the result holds for T_0 spaces of cardinality $< |X|$.

By Proposition 2.11(3), the function $Z \mapsto \mathcal{I}_Z$ from closed subspaces of X to closed two-sided ideals of $\mathcal{A}_X$ is injective. We will now show this function is surjective.

Let $\mathcal{I}$ be a non-zero closed two-sided ideal in $\mathcal{A}_X$. Let pq^* be a non-zero element in $\mathcal{I}$. Let x be a minimal element in X such that $t(p) \geq x$. Then $(px)(qx)^* \in \mathcal{I}_{X-\{x\}}$ and

$$(px)(qx)^* \;=\; \big[(px)(px)^*\big](pq^*)\big[(qx)(qx)^*\big] \;\in\; \mathcal{A}_X(pq^*)\mathcal{A}_X \;\subseteq\; \mathcal{I}.$$

Therefore $\mathcal{I}$ contains the minimal non-zero ideal $\mathcal{I}_{X-\{x\}}$.

Let $\Phi : \mathcal{A}_X \to \mathcal{A}_{X-\{x\}}$ be the surjective homomorphism in Theorem 2.12. Then $\Phi(\mathcal{I})$ is an ideal in $\mathcal{A}_{X-\{x\}}$ so there is a closed subspace Z of $X-\{x\}$ such that $\Phi(\mathcal{I})$ is generated by $\{yy^* \mid y \in (X-\{x\})-Z\}$. Hence $\mathcal{I}$ is generated by $\{yy^* \mid y \in X-Z\}$. Since x is minimal in X, Z is also a closed subspace of X so $\mathcal{I} = \mathcal{I}_Z$.

This completes the proof that the function $Z \mapsto \mathcal{I}_Z$ from closed subspaces of X to closed two-sided ideals of $\mathcal{A}_X$ is surjective and hence bijective. □

Lemma 2.15. *Let $Z \subseteq X$ be a closed subset with corresponding ideal $\mathcal{I}_Z$. Then*

$$\operatorname{Ann}_{\mathcal{A}_X}(\mathcal{I}_Z) = \{\phi \in \mathcal{A}_X \mid \phi \cdot \mathcal{I}_Z = 0\} = \mathcal{I}_{\overline{X-Z}}.$$

Proof. Since $\operatorname{Ann}_{\mathcal{A}_X}(\mathcal{I}_Z)$ is a closed two-sided ideal, it is of the form $\mathcal{I}_C$ for some closed subset $C \subseteq X$. It remains to show that C is the smallest closed subset in X containing $X-Z$.

Note that $\mathcal{I}_Z = \operatorname{Ann}_{\mathcal{A}_X}(\bigoplus_{x\in Z}\mathcal{O}_x)$, and likewise for C. For each $x \in X$, $\mathcal{O}_x$ is a simple $\mathcal{A}_X$-module; thus,

$$\mathcal{I}_Z \cdot \mathcal{O}_x = \begin{cases} \mathcal{O}_x & \text{if } x \notin Z \\ 0 & \text{if } x \in Z. \end{cases}$$

Since $\mathcal{I}_C \cdot \mathcal{I}_Z = 0$,

$$\mathcal{I}_C(\mathcal{I}_Z \cdot \mathcal{H}_X) = \mathcal{I}_C\Big(\bigoplus_{x\in X-Z}\mathcal{O}_x\Big) = 0, \qquad \text{and thus,} \quad X - Z \subseteq C.$$

This shows that C is a closed subset containing $X - Z$. On the otherhand, if C' is a closed subset of X that contains $X - Z$ then $\mathcal{I}_{C'} \cdot \mathcal{I}_Z$ annihilates $\mathcal{H}_X$. Since $\mathcal{H}_X$ is a faithful $\mathcal{A}_X$-module, $\mathcal{I}_{C'} \cdot \mathcal{I}_Z = 0$. It follows that $\mathcal{I}_{C'} \subseteq \operatorname{Ann}_{\mathcal{A}_X}(\mathcal{I}_Z) = \mathcal{I}_C$. By the *anti*-isomorphism of lattices, we have $C \subseteq C'$. □

Proposition 2.16. *The map* $X \to \operatorname{Prim}(\mathcal{A}_X)$*,* $x \mapsto \mathfrak{p}_x = \mathcal{I}_{\overline{x}}$*, is an isomorphism of posets.*

Proof. Let $\mathfrak{p}$ be a primitive ideal in $\mathcal{A}_X$. Then $\mathfrak{p} = \mathcal{I}_Z$ for some closed subspace $Z \subseteq X$. A primitive ideal is never the intersection of two strictly larger ideals so, by Proposition 2.11(1), Z is not the union of two strictly smaller closed subspaces. But Z is also the union of the closures of its individual points so is the closure of a single point. Hence $\mathfrak{p} = \mathcal{I}_{\overline{x}}$ for some $x \in X$. It now follows from Propositions 2.9(1) and 2.11(3) that $\mathfrak{p} = \mathfrak{p}_x$. □

Remark 2.17. When $\operatorname{Prim}(\mathcal{A}_X)$ is given the Jacobson topology and X is viewed as a T_0 topological space, the map $X \to \operatorname{Prim}(\mathcal{A}_X)$ defined above is a homeomorphism.

Proposition 2.18. *Let* Z *be a closed subspace of* X *and define*

$$e := \sum_{x \in X - Z} xx^*.$$

1. *There is a* C^**-algebra isomorphism* $e\mathcal{I}_Z e \cong \mathcal{A}_{X-Z}$*.*

2. $\mathcal{I}_Z$ *is strongly Morita equivalent to* $\mathcal{A}_{X-Z}$*.*

Proof. The result is clear for $Z = \varnothing$ and $Z = X$ so we assume that $\varnothing \subsetneq Z \subsetneq X$. Let $U = X - Z$. For each $x \in X$, xx^* is the orthogonal projection of $\ell^2(X_\infty)$ onto $\ell^2(xX_\infty)$. Because X_∞ is the disjoint union of the sets xX_∞, $x \in X$, there is an orthogonal decomposition

$$\ell^2(X_\infty) = \bigoplus_{x \in X} \ell^2(xX_\infty).$$

If $\mathbf{x} \in X_\infty$ and $s(\mathbf{x}) \in U$, then $\mathbf{x} \in U_\infty$ because U is an open subset of X. Therefore

$$U_\infty = \bigsqcup_{x \in U} xX_\infty$$

and e is the orthogonal projection of $\ell^2(X_\infty)$ onto

$$\ell^2(U_\infty) = \bigoplus_{x \in U} \ell^2(xX_\infty).$$

(1) If $p, q \in U_n$ with $n \geq 1$ and $t(p) = t(q)$, then the notation pq^* denotes an element of $\mathcal{A}_X$ and an element of $\mathcal{A}_U$. The element pq^* in $\mathcal{A}_X$ vanishes on $\ell^2(X_\infty - U_\infty)$ and sends $\ell^2(U_\infty)$ to itself; the restriction of the action of the element $pq^* \in \mathcal{A}_X$ to $\ell^2(U_\infty)$ is the same as the action of the element $pq^* \in \mathcal{A}_U$.

Let $\Phi : \mathcal{A}_U \to e\mathcal{I}e$ be the map $\Phi(1) = e$ and $\Phi(pq^*) = pq^*$ when $p, q \in U_n$ with $n \geq 1$. In $\mathcal{A}_X$, $e(pq^*)e = pq^*$ so $\Phi(\mathcal{A}_U) \subseteq e\mathcal{I}e$. It is clear that Φ is a homomorphism of C^*-algebras. However,

$$e\mathcal{I}e \;=\; \overline{\operatorname{span}}\{pq^* \in \mathcal{A}_X \mid s(p), s(q) \in U\} \;=\; \overline{\operatorname{span}}\{pq^* \in \mathcal{A}_X \mid p, q \in U_n,\, n \geq 1\}$$

so the image of Φ is $e\mathcal{I}e$. Hence $\Phi : \mathcal{A}_U \to e\mathcal{I}e$ is an isomorphism.

(2) This follows from (1) and Proposition 2.11(4) which shows $\mathcal{I}_Z = \mathcal{A}_X e \mathcal{A}_X$. ☐

Proposition 2.19. *The set* $\{\mathcal{O}_x \mid x \in X\}$ *is a complete set of irreducible representations of* $\mathcal{A}_X$ *up to unitary equivalence.*

Proof. Let $(\pi, \mathcal{H})$ be an irreducible representation of $\mathcal{A}_X$. Let $\mathcal{I} = \ker(\pi)$. Then $\mathcal{I} = \mathcal{I}_Z$ for some closed subspace $Z \subseteq X$, so $(\pi, \mathcal{H})$ is a faithful irreducible representation of $\mathcal{A}_Z$. It therefore suffices to prove that $(\pi, \mathcal{H})$ is unitarily equivalent to some $\mathcal{O}_x$ under the additional hypothesis that $\ker(\pi) = 0$. We therefore assume that $\ker(\pi) = 0$.

Let x be a minimal element of X and write $\mathcal{J}$ for $\mathcal{A}_X(xx^*)\mathcal{A}_X$. Since xx^* does not annihilate χ_{x^∞} it does not annihilate $\mathcal{O}_x$. If ξ is a non-zero element in $\mathcal{O}_x$, then $\mathcal{J}\xi = \mathcal{J}\mathcal{A}\xi = \mathcal{J}\mathcal{O}_x$; since $\mathcal{J}\mathcal{O}_x$ is stable under the left action of $\mathcal{A}_X$ on $\mathcal{H}_X$, $\mathcal{J}\mathcal{O}_x$ is a $\mathcal{A}_X$-subrepresentation of $\mathcal{O}_x$; but $\mathcal{O}_x$ is irreducible so $\mathcal{O}_x = \mathcal{J}\mathcal{O}_x = \mathcal{J}\xi$. Hence $\mathcal{O}_x$ is an irreducible representation of $\mathcal{J}$.

Since $\mathcal{J} = \mathcal{I}_{X-\{x\}}$, it is strongly Morita equivalent to $\mathcal{A}_{\{x\}}$ which is equal to $\mathbb{C}$. Hence $\mathcal{J}$ has a unique irreducible representation up to unitary equivalence. So as a representation of $\mathcal{J}$, $(\pi|_{\mathcal{J}}, \mathcal{H})$ is unitarily equivalent to $\mathcal{O}_x$.

Since $\mathcal{J}$ is Morita equivalent to $\mathbb{C}$, $\pi(\mathcal{J}) = \mathsf{K}(\mathcal{O}_x)$. Let $\rho : \mathcal{A}_X \to \mathsf{B}(\mathcal{O}_x)$ be the representation on $\mathcal{O}_x$ obtained by restricting the action of $\mathcal{A}_X$ on $\mathcal{H}_X$. Let $\alpha : \mathsf{B}(\mathcal{H}) \to \mathsf{B}(\mathcal{O}_x)$ be the isomorphism such that $\alpha\rho(b) = \pi(b)$ for all $b \in \mathcal{J}$. If $a \in \mathcal{A}_X$ and $b \in \mathcal{J}$, then

$$\alpha\rho(a)\pi(b) = \alpha\rho(a)\alpha\rho(b) = \alpha\rho(ab) = \pi(ab) = \pi(a)\pi(b)$$

so $(\alpha\rho(a) - \pi(a))\pi(b) = 0$ for all $b \in \mathcal{J}$. Hence $(\alpha\rho(a) - \pi(a))\mathsf{K}(\mathcal{O}_x) = 0$. But $\mathsf{K}(\mathcal{O}_x)$ is an essential ideal in $\mathsf{B}(\mathcal{O}_x)$ so $\alpha\rho(a) = \pi(a)$ for all $a \in \mathcal{A}_X$. □

We remark that for every integer $n \geq 1$, $\{pp^* \mid p \in X_n\}$ is complete set of mutually orthogonal idempotents.

Theorem 2.20 (Jensen)**.** *A separable C^*-algebra A is* scattered *if and only if $\hat{A}$ is countable.*

Corollary 2.21. *Let X be a finite T_0 space. Since $\hat{\mathcal{A}}_X \cong X$ is finite, $\mathcal{A}_X$ is a scattered C^*-algebra. Thus every representation of $\mathcal{A}_X$ is unitarily equivalent to the direct sum of countably many irreducible representations $\{\mathcal{O}_x\}$.*

Proof. By Proposition 2.19, $\hat{\mathcal{A}}_X \cong X$ is finite, and thus countable. Jensen's theorem proves that $\mathcal{A}_X$ is scattered. □

2.6 Summary

We have proved Theorem 1.1. We pause to compare that theorem to the classical case. Suppose that X is a locally compact Hausdorff space and let $C_0(X)$ be the commutative C^*-algebra of complex-valued continuous functions that vanish at infinity.

1. The unitary equivalence classes of irreducible representations of $C_0(X)$ are in bijection with the points in X, via $x \mapsto \mathbb{C}_x = C_0(X)/\mathfrak{m}_x$, where $\mathfrak{m}_x = \{f \in C(X) \mid f(x) = 0\}$.

2. If $i_Z : Z \subseteq X$ is a closed subspace of X, then there is a surjective C^*-algebra homomorphism $C_0(X) \to C_0(Z)$ defined by pre-composing with i_Z. Write $\mathcal{I}_Z$ for the kernel of this homomorphism.

3. The function $Z \mapsto \mathcal{I}_Z$ is an anti-isomorphism from the lattice of closed subspaces of X to the lattice of closed two-sided ideals in $C_0(X)$.
4. If Z is a closed subspace of X then $\mathcal{I}_Z := \bigcap_{z \in Z} \mathfrak{m}_z$.
5. The ideal $\mathcal{I}_Z$ is isomorphic to $C_0(X - Z)$.
6. $C_0(X \times Y) \cong C_0(X) \overline{\otimes} C_0(Y)$.
7. $C_0(X \sqcup Y) \cong C_0(X) \oplus C_0(Y)$.

Chapter 3

THE FUNCTOR $\mathcal{A} : \mathsf{FPosInj} \to \mathcal{C}^*$

Definition 3.1. Let $f : X \to Y$ be an injective map of posets, and let $\iota_f : \mathcal{H}_X \to \mathcal{H}_Y$ be the linear operator sending $\chi_{\mathbf{x}} \mapsto \chi_{f(\mathbf{x})}$. Define

$$\mathcal{A}_f = \mathcal{A}_Y \iota_f \mathcal{A}_X,$$

the closed $\mathcal{A}_Y$-$\mathcal{A}_X$-subbimodule of $\mathsf{B}(\mathcal{H}_X, \mathcal{H}_Y)$ generated by ι_f. Define $\mathcal{A}_f^n$ to be the span of operators of the form $pq^* \circ \iota_f \circ ab^*$, where $pq^* \in \mathcal{A}_Y^n$ and $ab^* \in \mathcal{A}_X^n$.

Lemma 3.2. *Let $f : X \to Y$ be an injective map of posets.*

1. *$\mathcal{A}_f$ contains $\sum_{n \geq 1} \mathcal{A}_f^n$ as a dense subset.*

2. *$\mathcal{A}_f = \overline{span}\{pf(b)^* \circ \iota_f \circ bb^* \mid p \in Y_n,\ b \in X_n,\ f(t(b)) = t(p)\}$.*

3. *The adjoint of the operator $pf(b)^* \circ \iota_f \circ bb^* : \mathcal{H}_X \to \mathcal{H}_Y$ is*

$$(bb^*)^* \circ \iota_f^* \circ (pf(b)^*)^* = bb^* \circ \pi_f \circ f(b)p^*,$$

where $\pi_f : \mathcal{H}_Y \to \mathcal{H}_X$ is the linear operator such that

$$\chi_{\mathbf{y}} \mapsto \sum_{f(\mathbf{x})=\mathbf{y}} \chi_{\mathbf{x}}.$$

Proof. (1) By definition, the linear span of elements of the form $\phi \circ \iota_f \circ \psi$, $\phi \in \mathcal{A}_Y$ and $\psi \in \mathcal{A}_X$, is dense in $\mathcal{A}_f$. Since $\mathcal{A}_Y$ and $\mathcal{A}_X$ are ultramatricial algebras, there are sequences $\phi_n \in \mathcal{A}_Y^n$ and $\psi_n \in \mathcal{A}_X^n$ such that $\phi_n \to \phi$ and $\psi_n \to \psi$. Thus,

$$\begin{aligned}
\|\phi \circ \iota_f \circ \psi - \phi_n \circ \iota_f \circ \psi_n\| &= \|\phi \circ \iota_f \circ \psi - \phi_n \circ \iota_f \circ \psi + \phi_n \circ \iota_f \circ \psi - \phi_n \circ \iota_f \psi_n\| \\
&\leq \|\phi \circ \iota_f \circ \psi - \phi_n \circ \iota_f \circ \psi\| + \|\phi_n \circ \iota_f \circ \psi - \phi_n \circ \iota_f \psi_n\| \\
&\leq \|\phi - \phi_n\| \|\iota_f \circ \psi\| + \|\phi_n \circ \iota_f\| \|\psi - \psi_n\| \to 0.
\end{aligned}$$

Thus, the closure of $\sum_{n\geq 1}\mathcal{A}_f^n$ contains all elements of the form $\phi\circ\iota_f\circ\psi$; since it is also closed under addition, it is equal to $\mathcal{A}_f$.

(2) Note that $\mathcal{A}_f$ contains the span of elements of the form $pq^*\circ\iota_f\circ ab^*$ as a dense subset. If $f(a)\neq q$ and $\chi_{\mathbf{x}}\in\mathcal{H}_X$ then

$$pq^*\circ\iota_f\circ ab^*(\chi_{\mathbf{x}}) = \chi_{pq^*\cdot f(ab^*\cdot\mathbf{x})} = 0.$$

In otherwords, the operator $pq^*\circ\iota_f\circ ab^* = 0$ unless $q = f(a)$.

Compute that

$$pf(a)^*\circ\iota_f\circ ab^* = pf(b)^*\circ\iota_f\circ bb^*$$

by checking that this holds for the basis elements $\chi_{\mathbf{x}}\in\mathcal{H}_X$. The result follows at once.

(3) Observe that

$$\langle\pi_f(\chi_{\mathbf{y}}),\chi_{\mathbf{x}}\rangle = \delta_{f(\mathbf{x}),\mathbf{y}} = \langle\chi_{\mathbf{y}},\iota_f(\chi_{\mathbf{x}})\rangle, \qquad \text{for all } \mathbf{y}\in Y_\infty \text{ and } \mathbf{x}\in X_\infty;$$

Since $\{\chi_{\mathbf{y}}\mid\mathbf{y}\in Y_\infty\}$ and $\{\chi_{\mathbf{x}}\mid\mathbf{x}\in X_\infty\}$ are Hilbert bases for $\mathcal{H}_Y$ and $\mathcal{H}_X$, we have that

$$\langle\pi_f(\psi),\phi\rangle = \langle\psi,\iota_f(\phi)\rangle, \qquad \text{for all } \psi\in\mathcal{H}_Y \text{ and } \phi\in\mathcal{H}_X,$$

i.e. $\iota_f^* = \pi_f$; since ι_f is bounded, π_f is bounded. □

See Appendix B.1 for the notion of a concrete Hilbert module.

Corollary 3.3. *If $f : X\to Y$ is an injective map of posets, then $\mathcal{A}_f$ is a concrete Hilbert $\mathcal{A}_X$-module and the left-action of $\mathcal{A}_Y$ is by adjointable operators. Thus, $\mathcal{A}_f$ is a Hilbert C^*-bimodule.*

Proof. We show that $u^*v\in\mathcal{A}_X$ for every $u,v\in\mathcal{A}_f$. It is necessary to show that $\pi_f\circ\iota_f\in\mathcal{A}_X$. Since f is injective,

$$\pi_f\circ\iota_f(\chi_{\mathbf{x}}) = \sum_{f(\mathbf{x})=f(\mathbf{x}')}\chi_{\mathbf{x}'} = \chi_{\mathbf{x}};$$

thus $\pi_f\circ\iota_f = \mathrm{id}_{\mathcal{H}_X}$. Now consider $u = qf(b)^*\circ\iota_f\circ bb^*, v = pf(a)^*\circ\iota_f\circ aa^*\in\mathcal{A}_f$. Then

$$u^*v = \delta_{pq}bb^*\circ\pi_f\circ f(b)f(a)^*\circ\iota_f\circ aa^* = \delta_{pq}ba^*\in\mathcal{A}_X.$$

This shows that $u^*v \in \mathcal{A}_X$ whenever $u, v \in \mathcal{A}_f^n$. The general result follows from continuity of the adjoint map and composition.

The left-action of $\mathcal{A}_Y$ is adjointable since $\mathcal{A}_Y$ is a concrete C^*-algebra acting on $\mathcal{H}_Y$. □

Theorem 3.4. *If $f : X \to Y$ and $g : Y \to Z$ are injective maps of posets, then*

$$\mathcal{A}_{g\circ f} = \mathcal{A}_g \otimes_{\mathcal{A}_Y} \mathcal{A}_f,$$

where the tensor product is the balanced tensor product of C^*-bimodules.

Proof. Using Lemma B.6 in the Appendix, compute that

$$\begin{aligned}
\mathcal{A}_g \otimes_{\mathcal{A}_Y} \mathcal{A}_f &= \mathcal{A}_g \circ \mathcal{A}_f = \overline{\operatorname{span}}\{\phi \circ \psi \mid \phi \in \mathcal{A}_g,\ \psi \in \mathcal{A}_f\} \\
&= \overline{\operatorname{span}}\{\phi \circ \psi \mid \phi \in \mathcal{A}_g^n,\ \psi \in \mathcal{A}_f^n,\ n > 0\} \\
&= \overline{\operatorname{span}}\{qg(c)^* \circ \iota_g \circ cc^* \circ bf(a)^* \circ \iota_f \circ aa^* \mid a \in X_n, bf(a) \in \mathcal{A}_Y^n,\ c \in Y_n,\ qg(c)^* \in \mathcal{A}_Y^n,\ n > 0\} \\
&= \overline{\operatorname{span}}\{qg(c)^* \circ \iota_g \circ cf(a)^* \circ \iota_f \circ aa^* \mid a \in X_n, cf(a) \in \mathcal{A}_Y^n,\ qg(c)^* \in \mathcal{A}_Y^n,\ n > 0\} \\
&= \overline{\operatorname{span}}\{qg(f(a))^* \circ \iota_{gf} \circ aa^* \mid a \in X_n, cf(a) \in \mathcal{A}_Y^n,\ qg(c)^* \in \mathcal{A}_Y^n,\ n > 0\} \\
&= \mathcal{A}_{g\circ f}.
\end{aligned}$$

□

Theorem 3.5. *Let X be a finite poset with open subset $j : U \to X$ and closed subset $i : Z \to X$.*

1. *There is an isomorphism of Hilbert C^*-bimodules $\Phi : \mathcal{A}_i \to \mathcal{A}_Z$ defined by*

$$\Phi(pa^* \circ \iota_i \circ aa^*) = pa^*;$$

2. *There is an isomorphism Hilbert C^*-bimodules $\Psi : \mathcal{A}_j \to \mathcal{A}_X e$, where $e = \sum_{x\in U} xx^*$; Ψ is defined by*

$$\Psi(pa^* \circ \iota_j \circ aa^*) = pa^*.$$

Corollary 3.6. *$\mathcal{A} : \mathsf{FPosInj} \to \mathcal{C}^*$ is a functor such that $\mathcal{A}(X) = \mathcal{A}_X$.*

$\mathcal{A}_f$ *must be an* $\mathcal{A}_Y$*-*$\mathcal{A}_X$*-bimodule*

Proposition 3.7. *If $Z \subseteq X$ is closed and does not contain any minimal elements of X, then the quotient $\mathcal{A}_X/\mathcal{I}_Z$ admits no $\mathcal{A}_X$-valued inner-product. Thus, $\mathcal{A}_X/\mathcal{I}_Z$ cannot be made a (right) Hilbert $\mathcal{A}_X$-module.*

Proof. Consider an $\mathcal{A}_X$-valued inner-product on $\mathcal{A}_X/\mathcal{I}_Z$, and let J be the closure of the image of this inner-product. Then J is a closed two-sided ideal in $\mathcal{A}_X$ by Lemma B.3; we will show that $\mathcal{J} = 0$, contradicting that the inner-product is positive-definite.

For $r \in \mathcal{A}_X$, write $\overline{r}$ for $r + \mathcal{I}_Z \in \mathcal{A}_X/\mathcal{I}_Z$. We note that J is spanned by elements of the form

$$\langle \overline{r}, \overline{s} \rangle = \langle \overline{1} \cdot r, \overline{1} \cdot s \rangle = r^* \langle \overline{1}, \overline{1} \rangle s.$$

Then $J \subseteq \mathrm{Ann}_{\mathcal{A}_X}(\mathcal{I}_Z)$ because

$$r^* \langle \overline{1}, \overline{1} \rangle s \cdot \mathcal{I}_Z \subseteq r^* \langle \overline{1}, \overline{1} \rangle \mathcal{I}_Z = r^* \langle 1_Y, \overline{\mathcal{I}_Z} \rangle = r^* \langle 1_Y, 0 \rangle = \{0\}.$$

By Lemma 2.15, $\mathrm{Ann}_{\mathcal{A}_Y}(\mathcal{I}_Z) = \mathcal{I}_C$, where C is the smallest closed subset containing $X - Z$.

Since Z contains no minimal elements of X, $X - Z$ contains all minimal elements of X. Since C is closed in X and contains all the minimal elements of X, $C = X$ and $\mathcal{I}_C = 0$. Thus, $J = 0$. □

Since $\mathcal{A}_X/\mathcal{I}_Z \cong \mathcal{A}_Z$ as C^*-algebras, and $\mathcal{A}_Z$ is a right $\mathcal{A}_Z$-module using the standard inner-product, the quotient $\mathcal{A}_X/\mathcal{I}_Z$ can be given the structure of a right Hilbert $\mathcal{A}_Z$-module. Thus, we must choose $\mathcal{A}_f$ to be an $\mathcal{A}_Y$-$\mathcal{A}_X$-bimodule.

3.1 Problems extending $\mathcal{A}$ to FPos

There are two problems for extending the definition of $\mathcal{A}_f$ when $f : X \to Y$ is not injective. The first is that ι_f may not be bounded. We briefly isolate the equivalent property of f such that ι_f is a bounded operator.

Definition 3.8. A strictly monotone morphism $f : X \to Y$ of posets is a function such that $x < x' \Rightarrow f(x) < f(x')$.

Note that the strictly monotone morphisms form a subcategory of the category of posets containing $\mathsf{FPosInj}$ as a subcategory.

Lemma 3.9. *If $f : X \to Y$ is a continuous map of finite posets, then the following are equivalent:*

1. *The linear operator $\iota_f : \mathcal{H}_X \to \mathcal{H}_Y$ is bounded;*
2. *f is strictly monotone;*
3. *There is a uniform bound (depending only on X) for the cardinality of the fibers $f^{-1}(f(\mathbf{x}))$, $\mathbf{x} \in X_\infty$.*

Proof. (1) implies (2): If f is not strictly monotone, then there is $x, x \in X$, $x < x'$ such that $f(x) \not< f(x')$. Since f is order-preserving, $f(x) = f(x')$. Let $h = \sum_{n=1}^{\infty} \frac{1}{n} \chi_{(x')^n x^\infty}$. This is square-summable: $\|h\|^2 = \pi^2/6$. However, $\iota_f(h) = \sum_{n=1}^{\infty} \frac{1}{n} \chi_{f(x)^\infty}$ is a divergent sum. Thus ι_f is not bounded.

(2) $\Rightarrow$ (3): Consider $\mathbf{x} \in X_\infty$; $\mathbf{x}$ has the form

$$\mathbf{x} = x_1^{m_1} x_2^{m_2} x_3^{m_3} \cdots x_{n-1}^{m_{n-1}} xxx \cdots x \cdots$$

where $1 \leq n \leq \text{height}(X)$, $x_1 > x_2 > \cdots > x_{n_1} > x$, and each exponent $m_i \geq 1$. Consider $\mathbf{x}' \in f^{-1}(f(\mathbf{x}))$, i.e. $f(\mathbf{x}'_i) = f(x_i)$, for all $i \geq 1$. Since $\mathbf{x}'_i \geq \mathbf{x}'_{i+1}$, (2) implies that $\mathbf{x}'_i = \mathbf{x}'_{i+1}$ if and only if $x_i = x_{i+1}$. So $\mathbf{x}' \in X_\infty$ must have the form

$$\mathbf{x}' = y_1^{m_1} y_2^{m_2} y_3^{m_3} \cdots y_{n-1}^{m_{n-1}} yyy \cdots y \cdots$$

where $y_1 > y_2 > \cdots > y_{n-1} > y$, $f(y_i) = f(x_i)$. We also have from (2) that either $y_i = x_i$ or they are incomparable. Let width(X) be the cardinality of the largest subset of X consisting of incomparable elements. The number of possible choices for each y_j is not greater than width(X); thus,

$$|f^{-1}(f(\mathbf{x}))| \leq (\mathrm{width}(X))^{(\mathrm{height}(X))} < \infty.$$

Since this bound is independent of $\mathbf{x}$, the result is proved.

(3) ⇒ (1) If $|f^{-1}(f(\mathbf{x}))| < M$, then $\|\iota_f(\chi)\| \leq M\|\chi\|$; indeed, if $\chi = \sum_{\mathbf{x}\in X_\infty} \lambda_{\mathbf{x}}\chi_{\mathbf{x}}$, then

$$\iota_f(\chi) = \sum_{\mathbf{x}\in X_\infty} \lambda_{\mathbf{x}}\chi_{f(\mathbf{x})} = \sum_{\mathbf{y}\in Y_\infty} \left(\sum_{f(\mathbf{x})=\mathbf{y}} \lambda_{\mathbf{x}} \right) \chi_{\mathbf{y}};$$

since $\left|\sum_{f(\mathbf{x})=\mathbf{y}} \lambda_{\mathbf{x}}\right|^2 < M^2 \sup_{f(\mathbf{x})=\mathbf{y}} |\lambda_{\mathbf{x}}|^2$, it follows that

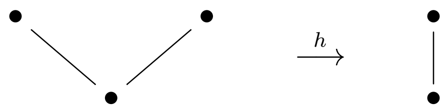
$$\|\iota_f(\chi)\|^2 \leq M^2 \sum_{\mathbf{x}\in X_\infty} |\lambda_{\mathbf{x}}|^2 = M^2\|\chi\|^2,$$

so $\|\iota_f\|_\infty \leq M$. □

Question 3.10. The property *strictly monotone* is order-theoretic. Is there a topological characterization of that property? (cf Theorem A.14)

Remark 3.11. If $f : X \to Y$ is strictly monotone, then height$(X) \leq$ height(Y). If $f : X \to Y$ is an injective morphism of posets then it is strictly monotone. This notion does not imply that X or Y are ranked.

Example 3.12. The following "height" map $h : X \to Y$ is strictly monotone but not injective:



Theorem 3.13. *Let $f : X \to Y$ be a strictly monotone map of posets. Then the following are equivalent:*

1. *f is injective;*

2. *$\mathcal{A}_f$ is a concrete Hilbert C^*-bimodule over $\mathcal{A}_Y$ and $\mathcal{A}_X$.*

Proof. (1) $\Rightarrow$ (2) Suppose that f is injective. The concrete $\mathcal{A}_X$-valued inner product on $\mathcal{A}_f$ is defined as follows: if $\phi, \psi \in \mathcal{A}_f$ then $\langle \phi, \psi \rangle = \phi^*\psi$; we show that $\phi^*\psi \in \mathcal{A}_X$.

From the lemma we know that for $aa^*, bb^* \in \mathcal{A}_X^n$, $pf(a)^*, qf(b)^* \in \mathcal{A}_Y^n$,

$$\begin{aligned}(pf(a)^* \circ \iota_f \circ aa^*)^* \circ (qf(b)^* \circ \iota_f \circ bb^*) &= (aa^* \circ \pi_f \circ f(a)p^*) \circ (qf(b)^* \circ \iota_f \circ bb^*)\\ &= \delta_{pq} ab^* \in \mathcal{A}_X^n,\end{aligned}$$

since $p = q$ implies that $t(f(a)) = t(f(b))$; the injectivity of f then implies that $t(a) = t(b)$.

By linearity, it follows that if $\phi, \psi \in \mathcal{A}_f^n$, then $\phi^*\psi \in \mathcal{A}_X$. If $\phi, \psi \in \mathcal{A}_f$, then there are sequence $\phi_n, \psi_n \in \mathcal{A}_f^n$ such that $\phi_n \to \phi$ and $\psi_n \to \psi$. Then

$$\begin{aligned}\|\phi^*\psi - \phi_n^*\psi_n\| &= \|\phi^*\psi - \phi_n^*\psi + \phi_n^*\psi - \phi_n^*\psi_n\|\\ &\leq \|\phi^*\psi - \phi_n^*\psi\| + \|\phi_n^*\psi - \phi_n^*\psi_n\|\\ &\leq \|\phi\|\|\psi - \psi_n\| + \|\phi_n\|\|\psi - \psi_n\| \to 0.\end{aligned}$$

Since $\phi_n^*\psi_n \in \mathcal{A}_X \subseteq \mathsf{B}(\mathcal{H}_X)$ for each n, and $\mathcal{A}_X$ is closed, it follows that $\phi^*\psi \in \mathcal{A}_X$.

(2) $\Rightarrow$ (1) If f is not injective, then $\iota_f^*\iota_f = \pi_f\iota_f$ does not preserve tails-equivalence classes. Indeed, if $f(x) = f(x')$, and $x \neq x'$, then

$$\pi_f\iota_f(\chi_{x^\infty}) = \pi_f(\chi_f(x)^\infty) = \sum_{f(\mathbf{x}')=f(x)^\infty} \chi_{\mathbf{x}'}.$$

The sum on the right contains $\chi_{(x')^\infty}$ for example. Thus, $\iota_f^*\iota_f \notin \mathcal{A}_X$, and so $\mathcal{A}_f$ is not a concrete Hilbert $\mathcal{A}_X$-module. $\square$

Example 3.14. In Example 3.12, $\iota_h : \mathcal{H}_X \to \mathcal{H}_Y$ does not generate a Hilbert C^*-bimodule over $\mathcal{A}_Y$ and $\mathcal{A}_X$ because $\langle \iota_h, \iota_h \rangle \notin \mathcal{A}_X$. Indeed,

$$\langle \iota_h, \iota_h \rangle(\chi_{1^\infty}) = \iota_h^*\iota_h(\chi_{1^\infty}) = \chi_{1^\infty} + \chi_{2^\infty};$$

this shows that $\langle \iota_h, \iota_h \rangle$ does not preserve tail-equivalence classes.

Finally, we have the following negative result:

Theorem 3.15. *Suppose $\mathcal{A} : \mathsf{FPos} \to \mathcal{C}^*$ is a functor such that $\mathcal{A}_{f:X\to Y} \subseteq \mathsf{B}(\mathcal{H}_X, \mathcal{H}_Y)$ is a concrete Hilbert C*-bimodule for every map $f : X \to Y$ of finite posets. Then $\mathcal{A}_{i_Z} \ncong \mathcal{A}_X/\mathcal{I}_Z$ for some finite poset X and some closed subset $Z \subseteq X$.*

Lemma 3.16. *Let $f : X \to Y$ be a map of posets, and let $V \subseteq \mathsf{B}(\mathcal{H}_X, \mathcal{H}_Y)$ be a concrete Hilbert $\mathcal{A}_X$-module. If $\phi \in V$, and $\mathcal{O}_x$ and $\mathcal{O}_{x'}$ are distinct simple summands of $\mathcal{H}_X$, then $\phi(\mathcal{O}_x) \perp \phi(\mathcal{O}_{x'})$.*

Proof of Lemma. Since $\phi \in V$, $\phi^*\phi \in \mathcal{A}_X$. Then, $\phi^*\phi(\mathcal{O}_x) \subseteq \mathcal{O}_x$ so

$$\{0\} \supseteq \langle \mathcal{O}_x, \mathcal{O}_{x'} \rangle \supseteq \langle \phi^*\phi(\mathcal{O}_x), \mathcal{O}_{x'} \rangle = \langle \phi(\mathcal{O}_x), \phi(\mathcal{O}_{x'}) \rangle. \qquad \square$$

Proof of Theorem. We proceed by contradiction. Let $X = \{0, 1, 2\}$, ordered so that $0 < 1$ and $0 < 2$, but 1 and 2 are incomparable. Let $\tau : X \to \{*\}$ be the terminal map.

Claim: $\mathcal{A}_\tau \subseteq \mathsf{B}(\mathcal{O}_x, \mathbb{C}) = \mathcal{O}_x^*$ for a unique $x \in X$. **Proof**: Consider $0 \neq \phi \in \mathcal{A}_\tau \subseteq \mathsf{B}(\mathcal{H}_X, \mathbb{C})$. Since the codomain of ϕ is one dimensional, the previous lemma implies that there is a unique $x \in X$ such that $\phi(\mathcal{O}_x) \neq 0$. Furthermore, if $\psi \in \mathcal{A}_\tau$ is any other element, then $\psi(\mathcal{O}_{x'}) = 0$ for $x' \neq x$ (otherwise, the sum $\phi + \psi \in \mathcal{A}_\tau$ would be an element such that $(\phi + \psi)(\mathcal{O}_x)$ and $(\phi + \psi)(\mathcal{O}_{x'})$ are not orthogonal, contradicting the previous lemma). It follows that $\mathcal{A}_\tau \subseteq \mathsf{B}(\mathcal{O}_x, \mathbb{C}) = \mathcal{O}_x^*$. Since $\mathcal{O}_x$ is a simple left $\mathcal{A}_X$-module, $\mathcal{A}_\tau = \mathcal{O}_x^*$, or $\mathcal{A}_\tau = 0$. Consider the composition $\{*\} \to X \to \{*\}$, where the first map $i_1 : \{*\} \to X$ sends $* \mapsto 1$ and the second is the terminal map. The composition is $\mathrm{id}_{\{*\}}$, and $\mathcal{A}_{\{*\}} = \mathbb{C}$. It follows from functoriality that $\mathcal{A}_\tau \neq 0$. Thus, $\mathcal{A}_\tau = \mathcal{O}_x^*$ for some $x \in X$. This proves the claim.

If $x \neq 1$, then

$$\mathcal{A}_\tau \otimes_{\mathcal{A}_X} \mathcal{A}_{i_1} = \mathcal{O}_x^* \otimes_{\mathcal{A}_X} \mathcal{A}_X/\mathcal{I}_{\{1\}} = 0,$$

since $\mathcal{O}_x^* \cdot \mathcal{I}_{\{1\}} = \mathcal{O}_x^*$. If $x = 1$, repeat this argument with the composition $\tau \circ i_2$, where $i_2 : \{*\} \to X$ sends $* \mapsto 2$. But then we have

$$\mathcal{A}_{id_{\{*\}}} = \mathcal{A}_{\tau \circ i_x} = \mathbb{C} \neq 0 = \mathcal{A}_\tau \otimes_{\mathcal{A}_X} \mathcal{A}_{i_x}, \qquad x = 1 \text{ or } x = 2,$$

contradicting that $\mathcal{A}$ is a functor. $\square$

Remark 3.17. It is conceivable that there is a functor $\mathcal{A} : \mathsf{FPos} \to \mathcal{C}^*$ such that $\mathcal{A}_{f:X\to Y} \subseteq \mathsf{B}(\mathcal{H}, \mathcal{K})$ is a concrete Hilbert C^*-bimodule for each $f : X \to Y$. In this case, $\mathcal{H}$ is a Hilbert space that $\mathcal{A}_X$ acts on so it must be isomorphic to direct sums of $\mathcal{O}_x$'s. It is unclear what to pick for $\mathcal{K}$ if we cannot pick $\mathcal{H}_Y$. The author has made several attempts choosing, e.g. $\mathcal{K} = \mathcal{H}_Y \otimes \mathcal{H}_X$, without success.

3.2 The ultramatricial algebra, S_X

Definition 3.18. Let X be a finite poset.

1. Define $H_X = \operatorname{span}_{\mathbb{C}} \{\chi_{\mathbf{x}} \mid \mathbf{x} \in X_\infty\}$;

2. Define $S_X = \operatorname{span} \{pq^* \mid t(p) = t(q), p, q \in X_n\} \subseteq \operatorname{End}_{\mathbb{C}}(H_X)$¿

H_X is a dense linear subspace of $\mathcal{H}_X$, and the range of each operator $pq^*|_{H_X} \in \mathcal{A}_X$ is contained in H_X. The closure of S_X under the operator norm is $\mathcal{A}_X$. Since S_X is also closed under taking adjoints, we say that S_X is a pre-C^*-algebra.

To avoid the problem of $\iota_f : \mathcal{H}_X \to \mathcal{H}_Y$ being unbounded when $f : X \to Y$ is not injective, consider the restriction $\iota_f|_{H_X} : H_X \to H_Y$. When there is no danger of confusion, we use the same symbol for ι_f and its restriction.

Definition 3.19. Let $f : X \to Y$ be a morphism of finite posets. Define

$$S_f = S_Y \circ \iota_f \circ S_X \subseteq \operatorname{Hom}_{\mathbb{C}}(H_X, H_Y),$$

the S_Y-S_X-bimodule generated by ι_f.

Lemma 3.20. *If $f : X \to Y$, and $g : Y \to Z$ are maps of finite posets, then we have the following equality of unbounded operators:*

$$\iota_g \circ \iota_f = \iota_{g\circ f} : \mathcal{H}_X \to \mathcal{H}_Z.$$

Theorem 3.21.

1. *If X is a finite poset then $S_{\mathrm{id}_X} = S_X$;*

2. *If $f : X \to Y$ and $g : Y \to Z$ are morphisms of finite posets, then*

$$\Phi : S_g \otimes_{S_Y} S_f \to S_{gf} : \quad \phi \otimes \psi \mapsto \phi \circ \psi$$

is an isomorphism.

Proof. (1) This is obvious since $\iota_{\mathrm{id}_X} = 1_{S_X}$. (2) It is clear that Φ is well-defined. To show that Φ is surjective we note that $S_{gf} = \varinjlim_n S^n_{gf}$, and each S^n_f is spanned by elements of the form $pgf(b)^* \circ \iota_f \circ bb^*$. Since

$$\Phi(pgf(b)^* \circ \iota_g \circ f(b)f(b)^* \otimes_{S_Y} f(b)f(b)^* \circ \iota_f \circ bb^*) = pgf(b)^* \circ \iota_f \circ bb^*,$$

Φ is surjective.

Φ restricts to the map $\Phi_n : S^n_g \otimes_{S_Y} S^n_f \to S^n_{gf}$. To prove injectivity of Φ it is enough to show that Φ_n has a left-inverse. Define $\Psi_n : S^n_{gf} \to S^n_g \otimes_{S_Y} S^n_f$ via

$$ugf(a)^* \circ \iota_{gf} \circ aa^* \mapsto ugf(a)^* \circ \iota_g \circ f(a)f(a)^* \otimes_{S_Y} f(a)f(a)^* \circ \iota_f aa^*.$$

The next few computations establish that $\Psi_n \circ \Phi_n = \mathrm{id}$.

For $aa^* \in S^n_X$, $qf(a)^*, pp^* \in S^n_Y$ and $ug(p)^* \in S^n_Z$, consider the element

$$x = ug(p)^* \circ \iota_g \circ pp^* \otimes_{S_Y} qf(a)^* \circ \iota_f \circ aa^* \in S^n_g \otimes_{S_Y} S^n_f.$$

Claim: $x = \Psi_n \circ \Phi_n(x) = \delta_{pq} ugf(a)^* \circ \iota_g \circ f(a)f(a)^* \otimes_{S_Y} f(a)f(a)^* \circ \iota_f \circ aa^*$

Proof: The second equality is obvious; we establish the first. If $q \neq p$, then $x = 0$ in the tensor product, since

$$\begin{aligned} pp^* \otimes_{S_Y} qf(a)^* &= pp^* \cdot pp^* \otimes_{S_Y} qf(a)^* \\ &= pp^* \otimes_{S_Y} pp^* \cdot qf(a)^* \\ &= \delta_{pq} pp^* \cdot pp^* \otimes_{S_Y} pf(a)^*. \end{aligned}$$

Since either $p = f(a)$ or else

$$\begin{aligned} pp^* \otimes_{S_Y} pf(a)^* &= pp^* \otimes_{S_Y} (pf(a)^* + f(a)p^*) \cdot f(a)f(a)^* \\ &= pp^* \cdot (pf(a)^* + f(a)p^*) \otimes_{S_Y} f(a)f(a)^* \\ &= pf(a)^* \otimes_{S_Y} pf(a)^*, \end{aligned}$$

we have

$$x = \delta_{pq} ug(p)^* \circ \iota_g \circ pf(a)^* \otimes_{S_Y} f(a)f(a)^* \circ \iota_f \circ aa^*.$$

Since $ug(p)^* \circ \iota_g \circ pf(a)^* = ugf(a)^* \circ \iota_g \circ f(a)f(a)^*$,

$$x = \delta_{pq} ugf(a)^* \circ \iota_g \circ f(a)f(a)^* \otimes_{S_Y} f(a)f(a)^* \circ \iota_f \circ aa^*.$$

Since elements of the form x span $S_g^n \otimes_{S_Y^n} S_f^n$, the result is proved. □

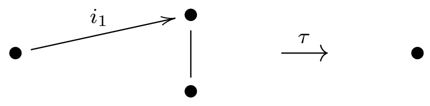
Corollary 3.22. *Let $f : X \to Y$ be a map of finite posets.*

1. *If $f : X \to Y$ is strictly monotone then $\overline{S}_f = \mathcal{A}_f$.*
2. *If $f : X \to Y$ is not strictly monotone then $S_f \not\subseteq \mathsf{B}(\mathcal{H}_X, \mathcal{H}_Y)$.*

One might consider the $\mathcal{A}_Y$-$\mathcal{A}_X$-bimodule obtained by intersecting $S_Y \iota_f S_X$ with $\mathsf{B}(\mathcal{H}_X, \mathcal{H}_Y)$ and taking the closure in $\mathsf{B}(\mathcal{H}_X, \mathcal{H}_Y)$. We have the following negative result.

Theorem 3.23. *The association $f \rightsquigarrow \overline{(S_Y \iota_f S_X) \cap \mathsf{B}(\mathcal{H}_X, \mathcal{H}_Y)}$, from finite posets to bimodules is not functorial.*

Proof. Let $X = \{0, 1\}$ be the Sierpinski space, i.e. $\{1\}$ is the only closed singleton. Consider the inclusion of the closed point, followed by the terminal map:



Then $\mathcal{A}_{i_1} \cong \mathcal{A}_X / \mathcal{I}_{\{1\}} \cong \mathbb{C}$. And the composition $\tau \circ i_1 = \mathrm{id}_{\{1\}}$, so $\mathcal{A}_{\tau \circ i_1} \cong \mathbb{C}$. Observe that

$$\overline{\iota_\tau \circ S_X \cap \mathsf{B}(\mathcal{H}_X, \mathbb{C})} = \mathcal{O}_0^*.$$

Indeed, no operator of the form $\iota_\tau \circ (1^n)(1^n)^*$ is bounded on $\mathcal{H}_X$, as can be checked by computing the value on

$$\xi = \sum_{m=1}^{\infty} \frac{1}{m} \chi_{1^{n+m}0^\infty}.$$

On the other hand, each operator of the form pq^*, $t(p) = t(q) = 0$ is compact, and so we obtain precisely $\mathcal{O}_0^*$.

However, $\mathcal{O}_0^* \otimes_{\mathcal{A}_X} \mathcal{A}_X/\mathcal{I}_1 = 0$. Since this is not isomorphic to $\mathcal{A}_{\{1\}} = \mathbb{C}$, the association is not functorial. $\square$

Summary

We have shown that there is a faithful functor $\mathcal{A} : \mathsf{FPosInj} \to \mathcal{C}^*$ that satisfies the following good properties:

1. If $i_Z : Z \to X$ is the inclusion of a closed subset, then $\mathcal{A}_{i_Z} \cong \mathcal{A}_X/\mathcal{I}_Z$ as Hilbert $\mathcal{A}_X$-$\mathcal{A}_Z$-bimodules; further, $\mathcal{A}_X/\mathcal{I}_Z \cong \mathcal{A}_Z$ as C^*-algebras.

2. If $j_U : U \to X$ is the inclusion of an open subset, then $\mathcal{A}_{j_U} \cong \mathcal{A}_X e$, as Hilbert $\mathcal{A}_X$-$\mathcal{A}_U$-bimodules, where $e = \sum_{u \in U} uu^*$. The bimodule $\mathcal{A}_X e$ is the one implementing the Morita equivalence between $\mathcal{I}_{X-U}$ and $\mathcal{A}_U$.

We have also shown that $S : \mathsf{FPos} \to \mathsf{BMod}$ is a faithful functor on the larger category of finite posets with monotone maps. When $f : X \to Y$ is injective, there is a norm on S_f such that $\overline{S}_f = \mathcal{A}_f$.

Chapter 4

THE FUNCTOR $E : \mathsf{FPos} \to \mathsf{HS}$

In this chapter we describe a functor $E : \mathsf{FPos} \to \mathsf{HS}$ from the category of finite posets with all order-preserving maps to the category of Hilbert-Schmidt operators. The faithful functor E has the following properties:

Proposition 4.1. *1. If X is a finite poset, then $E_{\mathrm{id}_X} = E_X$ is a closed sub-algebra of the Hilbert-Schmidt operators on $\mathcal{H}_X$. E_X is a (non-closed) two-sided ideal in $\mathcal{A}_X$, and is thus an $\mathcal{A}_X$-$\mathcal{A}_X$-bimodule;*

2. If $\iota_x : \{x\} \to X$ is the inclusion of a point, then $E_{\iota_x} \cong \mathcal{O}_x$;

3. If $\tau : X \to \{*\}$ is the terminal map, then $E_\tau \cong \mathcal{H}_X^*$;

4. If $f : X \to Y$ is any map of finite posets, then $E_f \otimes_{\mathcal{A}_X} \mathcal{O}_x \cong \mathcal{O}_{f(x)}$.

Some Notation

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Write $\mathsf{B}(\mathcal{H}_1, \mathcal{H}_2)$ for the Banach space of bounded operators $\mathcal{H}_1 \to \mathcal{H}_2$ and $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ for the Banach space of compact operators $\mathcal{H}_1 \to \mathcal{H}_2$. A composition of bounded (resp. compact) operators is bounded (resp. compact) so $\mathsf{B}(\mathcal{H}_1)$ and $\mathsf{B}(\mathcal{H}_2)$ are rings, and $\mathsf{B}(\mathcal{H}_1, \mathcal{H}_2)$ is an $\mathsf{B}(\mathcal{H}_2)$-$\mathsf{B}(\mathcal{H}_1)$-bimodule.

4.1 The linear transformations $\theta_{\mathbf{y},\mathbf{x}}$

Let X, and Y be finite T_0 topological spaces, $\mathbf{x} \in X_\infty$, and $\mathbf{y} \in Y_\infty$. Define the linear transformation

$$\theta_{\mathbf{y},\mathbf{x}} : \mathcal{H}_X \to \mathcal{H}_Y \qquad \text{by} \qquad \theta_{\mathbf{y},\mathbf{x}}(\xi) = \xi(\mathbf{x})\chi_{\mathbf{y}}.$$

Lemma 4.2. *With the above notation,*

1. $\theta_{\mathbf{y},\mathbf{x}}(-) = \langle \chi_{\mathbf{x}}, - \rangle \chi_{\mathbf{y}}$*;*

2. *The set* $\{\theta_{\mathbf{y},\mathbf{x}} \mid \mathbf{x} \in X_\infty, \, \mathbf{y} \in Y_\infty\}$ *is linearly independent;*

3. $\theta_{\mathbf{y},\mathbf{x}} \circ \theta_{\mathbf{w},\mathbf{z}} = \delta_{\mathbf{x},\mathbf{w}} \theta_{\mathbf{y},\mathbf{z}}$*;*

4. $\theta^*_{\mathbf{y},\mathbf{x}} = \theta_{\mathbf{x},\mathbf{y}}$*;*

5. $\theta_{\mathbf{y},\mathbf{x}} \in \mathsf{K}(\mathcal{H}_X, \mathcal{H}_Y)$*;*

6. *if* $x = t(\mathbf{x})$ *and* $y = t(\mathbf{y})$*, then* $\theta_{\mathbf{y},\mathbf{x}}$ *sends* $\mathcal{O}_x$ *to* $\mathcal{O}_y$*.*

Proof. (1) This is obvious. (2) This follows from the relation $\theta_{\mathbf{y},\mathbf{x}}(\chi_{\mathbf{w}}) = \delta_{\mathbf{x},\mathbf{w}} \chi_{\mathbf{y}}$. (3) For each $\xi \in \mathcal{H}_X$,

$$\theta_{\mathbf{y},\mathbf{x}} \circ \theta_{\mathbf{w},\mathbf{z}}(\xi) = \theta_{\mathbf{y},\mathbf{x}}(\xi(\mathbf{z})\chi_{\mathbf{w}}) = \xi(\mathbf{z})\chi_{\mathbf{w}}(\mathbf{x})\chi_{\mathbf{y}} = \delta_{\mathbf{x},\mathbf{w}} \theta_{\mathbf{y},\mathbf{z}}(\xi).$$

(4) Let $\eta \in \mathcal{H}_X$ and $\xi \in \mathcal{H}_Y$. Using the inner product on $\ell^2(X_\infty)$, we have

$$\langle \eta, \theta_{\mathbf{x},\mathbf{y}}(\xi) \rangle = \langle \eta, \xi(\mathbf{y})\chi_{\mathbf{x}} \rangle = \xi(\mathbf{y}) \sum_{\mathbf{w} \in X_\infty} \overline{\eta(\mathbf{w})} \chi_{\mathbf{x}}(\mathbf{w}) = \overline{\eta(\mathbf{x})} \xi(\mathbf{y})$$

and, using the inner product on $\ell^2(Y_\infty)$,

$$\langle \theta_{\mathbf{y},\mathbf{x}}(\eta), \xi \rangle = \langle \eta(\mathbf{x})\chi_{\mathbf{y}}, \xi \rangle = \overline{\eta(\mathbf{x})} \sum_{\mathbf{z} \in Y_\infty} \overline{\chi_{\mathbf{y}}(\mathbf{z})} \xi(\mathbf{z}) = \overline{\eta(\mathbf{x})} \xi(\mathbf{y}).$$

Hence $\theta^*_{\mathbf{x},\mathbf{y}} = \theta_{\mathbf{y},\mathbf{x}}$

(5) The image of $\theta_{\mathbf{x},\mathbf{y}}$ is $\mathbb{C}\chi_{\mathbf{x}}$ so $\operatorname{rank}(\theta_{\mathbf{x},\mathbf{y}}) = 1$.

(6) Since $\{\chi_{\mathbf{w}} \mid t(\mathbf{w}) = y\}$ is a Hilbert space basis for $\mathcal{O}_y$ it suffices to observe that $\theta_{\mathbf{x},\mathbf{y}}(\chi_{\mathbf{w}}) = \delta_{\mathbf{y},\mathbf{w}} \chi_{\mathbf{x}} \in \mathcal{O}_x$. ☐

Each operator $\theta_{\mathbf{y},\mathbf{x}} : \mathcal{H}_X \to \mathcal{H}_Y$ is a rank 1 partial isometry. The closure of $\operatorname{span}_{\mathbb{C}}\{\theta_{\mathbf{y},\mathbf{x}} \mid \mathbf{x} \in X_\infty, \, \mathbf{y} \in Y_\infty\}$ in the operator-norm topology is $\mathsf{K}(\mathcal{H}_X, \mathcal{H}_Y)$. This is typically not a Hilbert space.

4.2 Hilbert-Schmidt Operators

There is an inner-product on $\operatorname{span}_{\mathbb{C}}\{\theta_{\mathbf{x},\mathbf{y}}\}$, defined by

$$\langle S, T\rangle := \sum_{\mathbf{x}\in X_\infty} \langle S\chi_{\mathbf{x}}, T\chi_{\mathbf{x}}\rangle_{\mathcal{H}_Y} = \operatorname{tr}(S^*T).$$

The set $\{\theta_{\mathbf{y},\mathbf{x}}\}$ is an orthonormal basis for this inner-product so the norm-completion is a Hilbert space.

Definition 4.3. Given two Hilbert spaces $\mathcal{H}_X$ and $\mathcal{H}_Y$, define

$$\mathsf{HS}(\mathcal{H}_X, \mathcal{H}_Y) = \overline{\operatorname{span}}\left\{\theta_{\mathbf{x},\mathbf{y}}\right\},$$

where the norm is the one induced from the inner-product above.

The set $\mathsf{HS}(\mathcal{H}_X, \mathcal{H}_Y)$ consists precisely of those bounded operators $\phi : \mathcal{H}_X \to \mathcal{H}_Y$ such that

$$\|\phi\|_{\mathsf{HS}}^2 = \sum_{\mathbf{x}\in X_\infty} \|\phi\chi_{\mathbf{x}}\|_{\mathcal{H}_Y}^2 = \operatorname{tr}(\phi^*\phi) < \infty.$$

The Hilbert-Schmidt inner-product is independent of the choice of orthonormal basis; we have chosen $\{\chi_{\mathbf{x}} \in \mathcal{H}_X \mid \mathbf{x} \in X_\infty\}$ here for convenience. For every unit vector $v \in \mathcal{H}_X$, there is an orthonormal basis $\{v = v_0, v_1, \ldots\}$ for $\mathcal{H}_X$ beginning with v; it follows that

$$\|Tv\|_\infty^2 \leq \sum_{n=0}^{\infty} \|Tv_n\|^2 = \|T\|_{\mathsf{HS}}^2,$$

thus, $\|T\|_\infty \leq \|T\|_{\mathsf{HS}}$. Since compact operators are operator norm limits of finite rank operators, $\mathsf{HS}(\mathcal{H}_X, \mathcal{H}_Y) \subset \mathsf{K}(\mathcal{H}_X, \mathcal{H}_Y)$.

Proposition 4.4. *Consider* $T \in \mathsf{HS}(\mathcal{H}_X, \mathcal{H}_Y)$.

1. $T^* \in \mathsf{HS}(\mathcal{H}_Y, \mathcal{H}_X)$ *and* $\|T^*\|_{\mathsf{HS}} = \|T\|_{\mathsf{HS}}$.

2. *If* $\phi \in \mathsf{B}(\mathcal{H}_Y), \psi \in \mathsf{B}(\mathcal{H}_X)$ *then* $\|\phi \circ T \circ \psi\|_{\mathsf{HS}} \leq \|\phi\|_\infty \|T\|_{\mathsf{HS}} \|\psi\|_\infty$.

3. $\mathsf{HS}(\mathcal{H}_X, \mathcal{H}_Y)$ *is a* $\mathsf{B}(\mathcal{H}_Y)$-$\mathsf{B}(\mathcal{H}_X)$*-bimodule under the action of composition.*

Proof. (1) This follows from the calculation

$$\begin{aligned}\|T\|_{\mathsf{HS}}^2 = \sum_{\mathbf{x}\in X_\infty} \|T\chi_{\mathbf{x}}\|_{\mathcal{H}_Y}^2 = \sum_{\mathbf{y}\in Y_\infty}\sum_{\mathbf{x}\in X_\infty} |\langle T\chi_{\mathbf{x}}, \chi_{\mathbf{y}}\rangle|^2 = \sum_{\mathbf{x}\in X_\infty}\sum_{\mathbf{y}\in Y_\infty} |\langle \chi_{\mathbf{y}}, T^*\chi_{\mathbf{x}}\rangle|^2 \\ = \sum_{\mathbf{y}\in Y_\infty} \|T^*\chi_{\mathbf{y}}\|_{\mathcal{H}_X}^2 = \|T^*\|_{\mathsf{HS}}^2.\end{aligned}$$

(2) Compute that

$$\|\phi T\|_{\mathsf{HS}}^2 = \sum_{\mathbf{y}\in Y_\infty} \|\phi T\chi_{\mathbf{y}}\|^2 \leq \sum_{\mathbf{y}\in Y_\infty} \|\phi\|_\infty^2 \|T\chi_{\mathbf{y}}\|^2 = \|\phi\|_\infty^2 \|T\|_{\mathsf{HS}}^2.$$

Thus $\|\phi T\|_{\mathsf{HS}} \leq \|\phi\|_\infty \|T\|_{\mathsf{HS}}$.

Since $T\circ\psi = (\psi^*\circ T^*)^*$, (1) implies $\|T\circ\psi\|_{\mathsf{HS}} = \|\psi^*\circ T^*\|_{\mathsf{HS}}$; thus

$$\|T\circ\psi\|_{\mathsf{HS}} \leq \|\psi^*\|_\infty \|T\|_{\mathsf{HS}} = \|T\|_{\mathsf{HS}}\|\psi\|_\infty.$$

(3) follows immediately from (2). □

For the next definition we make use of the following notation: for $\mathcal{M}\subseteq \mathsf{HS}(\mathcal{H},\mathcal{H}')$ and $\mathcal{N}\subseteq \mathsf{HS}(\mathcal{H}',\mathcal{H}'')$,

$$\mathcal{N}\circ\mathcal{M} = \overline{\operatorname{span}}_{\mathbb{C}}\{n\circ m \mid m\in M, n\in N\}.$$

Definition 4.5. Let HS be the category whose objects are closed complex subalgebras $\mathcal{O}\subseteq \mathsf{HS}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, subject to the condition that $\mathcal{O}\circ\mathcal{O} = \mathcal{O}$. A morphism from $\mathcal{O}\subseteq\mathsf{HS}(\mathcal{H})$ to $\mathcal{O}'\subseteq\mathsf{HS}(\mathcal{H}')$ is a closed $\mathbb{C}$-linear subspace $\mathcal{M}\subseteq\mathsf{HS}(\mathcal{H},\mathcal{H}')$ such that $\mathcal{O}'\circ\mathcal{M} = \mathcal{M} = \mathcal{M}\circ\mathcal{O}$. If $\mathcal{M}\subseteq\mathsf{HS}(\mathcal{H},\mathcal{H}')$ and $\mathcal{N}\subseteq\mathsf{HS}(\mathcal{H}',\mathcal{H}'')$ are composable (i.e. the codomain of $\mathcal{M}$ is the domain of $\mathcal{N}$) then the composition is defined to be $\mathcal{N}\circ\mathcal{M}$.

4.3 The $\mathcal{A}_Y$-$\mathcal{A}_X$-bimodule E_f

Let $f : X\to Y$ be a continuous map between finite T_0-spaces. Define

$$E_f := \overline{\operatorname{span}}\{\theta_{\mathbf{y},\mathbf{x}} \mid f(t(\mathbf{x})) = t(\mathbf{y})\} \subseteq \mathsf{HS}(\mathcal{H}_X,\mathcal{H}_Y).$$

Proposition 4.6. *Let X and Y be finite posets, and let $f : X\to Y$ be an order-preserving map.*

1. *E_X is a Banach *-algebra and a Hilbert space;*

2. *E_f is an E_Y-E_X-bimodule;*

3. *If $pq^* \in \mathcal{A}_X$, $uv^* \in \mathcal{A}_Y$, $\mathbf{x} \in X_\infty$, $\mathbf{y} \in Y_\infty$, and $f(t(\mathbf{x})) = t(\mathbf{y})$, then*

$$(uv^*) \cdot \theta_{\mathbf{y},\mathbf{x}} \cdot (pq^*) = \begin{cases} \theta_{uv^*\mathbf{y},qp^*\mathbf{x}} & \textit{if } \mathbf{x} \in pX_\infty \textit{ and } \mathbf{y} \in vY_\infty \\ 0 & \textit{otherwise.} \end{cases}$$

4. *E_f is an $\mathcal{A}_Y$-$\mathcal{A}_X$-subbimodule of $\mathsf{HS}(\mathcal{H}_X, \mathcal{H}_Y)$.*

Proof. (1) $\mathsf{HS}(\mathcal{H}_X)$ is a Banach *-algebra and a Hilbert space, and $E_X \subseteq \mathsf{HS}(\mathcal{H}_X)$ is a closed sub-algebra. (2) This follows from the relation that $\theta_{\mathbf{z},\mathbf{y}} \circ \theta_{\mathbf{y}',\mathbf{x}} = \delta_{\mathbf{y},\mathbf{y}'}\theta_{\mathbf{z},\mathbf{x}}$. (3) By definition, $(uv^*) \cdot \theta_{\mathbf{y},\mathbf{x}} \cdot (pq^*)$ is the composition

$$\begin{tikzcd} \mathcal{H}_X \arrow[r, "pq^*"] & \mathcal{H}_X \arrow[r, "\theta_{\mathbf{y},\mathbf{x}}"] & \mathcal{H}_Y \arrow[r, "uv^*"] & \mathcal{H}_Y. \end{tikzcd}$$

If $\mathbf{z} \in qX_\infty$, then $\delta_{\mathbf{x},pq^*\mathbf{z}} = 1$ if and only if $\mathbf{z} = q\mathbf{z}'$ and $\mathbf{x} = p\mathbf{z}'$ if and only if $\mathbf{x} \in pX_\infty$ and $\delta_{qp^*\mathbf{x},\mathbf{z}} = 1$. If $\mathbf{z} \in X_\infty$, then

$$\begin{aligned}
(uv^* \circ \theta_{\mathbf{y},\mathbf{x}} \circ pq^*)(\chi_{\mathbf{z}}) &= \begin{cases} (uv^* \circ \theta_{\mathbf{y},\mathbf{x}})(\chi_{qp^*\mathbf{z}}) & \text{if } \mathbf{z} \in pX_\infty \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \delta_{\mathbf{x},qp^*\mathbf{z}} uv^*\big(\chi_{\mathbf{y}}\big) & \text{if } \mathbf{z} \in pX_\infty \\ 0 & \text{otherwise.} \end{cases} \\
&= \begin{cases} \delta_{qp^*\mathbf{x},\mathbf{z}} \chi_{uv^*\mathbf{y}} & \text{if } \mathbf{x} \in pX_\infty \text{ and } \mathbf{y} \in vY_\infty \\ 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \theta_{uv^*\mathbf{y},qp^*\mathbf{x}}(\chi_{\mathbf{z}}) & \text{if } \mathbf{y} \in vY_\infty \text{ and } \mathbf{x} \in pX_\infty \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

It follows that E_f is closed under the actions of $\mathcal{A}_X^n$ and $\mathcal{A}_Y^n$. For general $a \in \mathcal{A}_X$, choose a converging sequence $a_n \in \mathcal{A}_X^n$ and observe that

$$\|m \cdot a - m \cdot a_n\|_{\mathsf{HS}} \leq \|m\|_{\mathsf{HS}} \|a - a_n\|_\infty \to 0.$$

Thus, $m \cdot a \in E_f$ since E_f is closed and $a_n \cdot m \to a \cdot m$. The argument for the right-action is entirely similar. Then (4) follows at once, since the composition of a Hilbert-Schmidt operator and a bounded operator is again Hilbert-Schmidt. □

Theorem 4.7. *The association $f \rightsquigarrow E_f$ defines a functor $E : \mathsf{FPos} \to \mathsf{HS}$.*

Proof. Suppose that $f : X \to Y$ and $g : Y \to Z$ are order-preserving functions of finite posets. We will show that $E_g \circ E_f = E_{g \circ f}$.

Essentially by definition,

$$\begin{aligned} E_g \circ E_f &= \overline{\operatorname{span}}_{\mathbb{C}} \{\theta_{\mathbf{z},\mathbf{y}} \circ \theta_{\mathbf{y}',\mathbf{x}} \mid g(t(\mathbf{y})) = t(\mathbf{z}),\ f(t(\mathbf{x})) = t(\mathbf{y}')\} \\ &= \overline{\operatorname{span}}_{\mathbb{C}} \{\theta_{\mathbf{z},\mathbf{y}} \circ \theta_{\mathbf{y},\mathbf{x}} \mid g(t(\mathbf{y})) = t(\mathbf{z}),\ f(t(\mathbf{x})) = t(\mathbf{y})\} \\ &= \overline{\operatorname{span}}_{\mathbb{C}} \{\theta_{\mathbf{z},\mathbf{x}} \mid g(f(t(\mathbf{x})) = t(\mathbf{z})\} = E_{g \circ f}. \end{aligned}$$

□

4.4 Alternate Description of E_f

Let $y \in Y$. We write $\mathcal{O}_y^*$ for the Hilbert space dual of the Hilbert space $\mathcal{O}_y$. Since $\{\chi_{\mathbf{y}} \mid t(\mathbf{y}) = y\}$ is an orthonormal basis for $\mathcal{O}_y$, $\{\langle \chi_{\mathbf{y}}, - \rangle \mid t(\mathbf{y}) = y\}$ is a Hilbert space basis for $\mathcal{O}_y^*$. We will write

$$\chi_{\mathbf{y}}^* := \langle \chi_{\mathbf{y}}, - \rangle.$$

Let $x \in X$ and $y \in Y$. We write $\mathcal{O}_y \otimes \mathcal{O}_x^*$ for the Hilbert space tensor product of the Hilbert spaces $\mathcal{O}_y$ and $\mathcal{O}_x^*$. It is the completion of the algebraic tensor product of these two vector spaces with respect to the “obvious” sesquilinear form—see [16, Ch.IV, §1].

Proposition 4.8. *There is an isomorphism of Hilbert spaces (and E_Y-E_X bimodules)*

$$E_f = \bigoplus_{x \in X} \mathsf{HS}(\mathcal{O}_x, \mathcal{O}_{f(x)}) \cong \bigoplus_{x \in X} \mathcal{O}_{f(x)} \otimes \mathcal{O}_x^*,$$

defined by $\theta_{\mathbf{y},\mathbf{x}} \mapsto \chi_{\mathbf{y}} \otimes \chi_{\mathbf{x}}^*$.

Proof. Recall that $\mathcal{H}_X = \bigoplus_{x \in X} \mathcal{O}_x$. In the definition of E_f we considered only those $\theta_{\mathbf{y},\mathbf{x}}$ such that $f(t(\mathbf{x})) = t(\mathbf{y})$. Observe that $\theta_{\mathbf{y},\mathbf{x}}$ restricts to a rank one operator on $\mathcal{O}_x$ whose extension to $\mathcal{O}_x^{\perp}$ in $\mathcal{H}_X$ is 0. Since $\mathcal{O}_{f(x)}$ and $\mathcal{O}_x$ are left E_Y- and E_X-modules, the indicated decomposition is one of bimodules.

The result follows since the given map is a bijection on the orthonormal bases for $\mathsf{HS}(\mathcal{O}_x, \mathcal{O}_{f(x)})$ and $\mathcal{O}_{f(x)} \otimes \mathcal{O}_x^*$ which is inner-product preserving:

$$\langle \theta_{\mathbf{y},\mathbf{x}}, \theta_{\mathbf{y}',\mathbf{x}'} \rangle = \delta_{\mathbf{y},\mathbf{y}'} \delta_{\mathbf{x},\mathbf{x}'} = \langle \chi_{\mathbf{y}}, \chi_{\mathbf{y}'} \rangle \langle \chi_{\mathbf{x}}^*, \chi_{\mathbf{x}'}^* \rangle = \langle \chi_{\mathbf{y}} \otimes \chi_{\mathbf{x}}^*, \chi_{\mathbf{y}'} \otimes \chi_{\mathbf{x}'}^* \rangle. \qquad \square$$

Corollary 4.9. *Let $f : X \to Y$ and $g : Y \to Z$ be continuous maps of finite posets. By extending scalars from E_X to $\mathcal{A}_X$, E_Y to $\mathcal{A}_Y$, and E_Z to $\mathcal{A}_Z$ we have that*

$$E_g \otimes_{\mathcal{A}_Y} E_f \cong E_{g \circ f} \quad \text{as } \mathcal{A}_Z\text{-}\mathcal{A}_X\text{-bimodules.}$$

Proof. One can show that for $y \in Y$, $\mathcal{O}_y^* \otimes_{\mathcal{A}_Y} \mathcal{O}_y = \mathbb{C}$ and for y, y' distinct in Y, $\mathcal{O}_y^* \otimes_{\mathcal{A}_Y} \mathcal{O}_{y'} = 0$. By repeatedly using the isomorphism in the previous proposition,

$$\begin{aligned} E_g \otimes_{\mathcal{A}_Y} E_f &\cong \left(\bigoplus_{y \in Y} \mathcal{O}_{g(y)} \otimes \mathcal{O}_y^* \right) \otimes_{\mathcal{A}_Y} \left(\bigoplus_{x \in X} \mathcal{O}_{f(x)} \otimes \mathcal{O}_x^* \right) \\ &\cong \bigoplus_{x \in X} \mathcal{O}_{g \circ f(x)} \otimes \mathcal{O}_x^* \cong E_{g \circ f}. \end{aligned} \qquad \square$$

Corollary 4.10.

1. *If $i_x : \{x\} \to X$ is the inclusion of $x \in X$, $E_{i_x} = \mathsf{HS}(\mathbb{C}, \mathcal{O}_x) \cong \mathcal{O}_x$ as left $\mathcal{A}_X$-modules, via $\theta_{\mathbf{x},1} \mapsto \chi_{\mathbf{x}}$;*

2. *If $\tau : X \to \{*\}$ is the terminal map, $E_\tau = \bigoplus_{x \in X} \mathsf{HS}(\mathcal{O}_X, \mathbb{C}) \cong \mathcal{H}_X^*$ as right $\mathcal{A}_X$-module, via $\theta_{1,\mathbf{x}} \mapsto \chi_{\mathbf{x}}^*$.*

Corollary 4.11. *Let $f : X \to Y$ be a continuous function between finite T_0-topological spaces. If $x \in X$, then $E_f \otimes_{\mathcal{A}_X} \mathcal{O}_x \cong \mathcal{O}_{f(x)}$.*

Proof. Let $i_x : \{*\} \to X$ be the inclusion of x. There is an isomorphism $E_f \otimes_{\mathcal{A}_X} E_{i_x} \cong E_{f \circ i_x}$ of $\mathcal{A}_Y$-$\mathbb{C}$ bimodules, hence an isomorphism $E_f \otimes_{\mathcal{A}_X} \mathcal{O}_x \cong \mathcal{O}_{f(x)}$ of $\mathcal{A}_Y$-$\mathbb{C}$ bimodules. $\square$

Corollary 4.12. *If* $f, g : X \to Y$ *and* $f \neq g$*, then* E_f *is not isomorphic to* E_g *as bimodules, i.e. the functor* E *is faithful.*

Proof. Let $f, g : X \to Y$ be different continuous functions between finite T_0-topological spaces. There is a point $x \in X$ such that $f(x) \neq g(x)$. By Proposition 2.8, $\mathcal{O}_{f(x)} \ncong \mathcal{O}_{g(x)}$. But $\mathcal{O}_{f(x)} \cong E_f \otimes_{\mathcal{A}_X} \mathcal{O}_x$ and $\mathcal{O}_{g(x)} \cong E_g \otimes_{\mathcal{A}_X} \mathcal{O}_x$, so the $\mathcal{A}_Y$-$\mathcal{A}_X$-bimodules E_f and E_g are not isomorphic. $\square$

Proposition 4.13. *If* $f : X \to Y$ *and* $f' : X' \to Y'$ *are two maps of finite posets, then* $f \times f' : X \times X' \to Y \times Y'$ *is another map of finite posets and there is an isomorphism*

$$\phi : E_{f \times f'} \to E_f \otimes E_{f'} \qquad \theta_{(\mathbf{y},\mathbf{y'}),(\mathbf{x},\mathbf{x'})} \mapsto \theta_{\mathbf{y},\mathbf{x}} \otimes \theta_{\mathbf{y'},\mathbf{x'}}$$

of Hilbert spaces (and also $\mathcal{A}_Y \otimes \mathcal{A}_{Y'}$-$\mathcal{A}_X \otimes \mathcal{A}_{X'}$*-bimodules).*

Chapter 5

THE GROUPOID APPROACH

There is an alternate description of $\mathcal{A}_X$ due to Exel, Lopes and Renault. Our goal in this chapter is to provide an account of this, and define the Hilbert C^*-bimodule $\mathcal{A}_f$ when f is injective.

If S is a topological space, denote by $C(S)$ the $\mathbb{C}$-vector space of complex-valued continuous functions on S. If E is an equivalence relation on S, $C(S;E)$ denotes those complex-valued functions that are constant on E-equivalence classes. If S is a compact Hausdorff space then its coordinate ring is the commutative C^*-algebra $C(S)$. A proper equivalence relation $R \subseteq S \times S$ is one such that the quotient S/R is again Hausdorff. If R is proper then $C(S/R) \cong C(S;R)$. There are two canonical examples corresponding to the minimal and maximal equivalence relations on S. If $R = \{(s,s) \mid s \in S\}$ then $S/R \cong S$ and $C(S/R) \cong C(S)$. If $R = S \times S$, then $X/S \cong \{*\}$ and $C(X/S) \cong \mathbb{C}$.

Not every equivalence relation is proper. If $S = X$ is a finite poset where at least two elements are comparable and R is the tails-equivalence relation then $X_\infty/R \cong X$ is a finite non-Hausdorff space. Perhaps it was this example that motivated Exel, Lopes and Renault to develop a theory of "approximately proper equivalence relations." Define for each $n > 0$,

$$R_X^n = \{(\mathbf{x},\mathbf{y}) \in X_\infty \times X_\infty \mid \mathbf{x}_{>n} = \mathbf{y}_{>n}\};$$

define $R_X = \bigcup_{n>0} R_X^n$. Each R_X^n is a proper equivalence relation on X_∞; there is a groupoid C^*-algebra $C^*(R_X^n)$ for each $n > 0$, which is isomorphic to the compact operators on $C(X_\infty; R_X^n)$. There is a system directing the set $\{C^*(R_X^n) \mid n > 0\}$ such that $C^*(R_X) \cong \mathcal{A}_X$.

5.1 Groupoids

A groupoid is a set G together with a set $G^{(2)} \subseteq G \times G$ of composable pairs and a product map $(x, y) \mapsto xy$ from $G^{(2)} \to G$ and an involution $x \mapsto x^{-1}$ from $G \to G$ such that

1. if $(x, y) \in G^{(2)}$ and $(y, z) \in G^{(2)}$ then $(xy, z), (x, yz) \in G^{(2)}$ and $(xy)z = x(yz)$;
2. for each $x \in G$, $(x, x^{-1}) \in G^{(2)}$, and $(z, x) \in G^{(2)}$ implies $(zx)x^{-1} = z$.

If G is a groupoid, define maps $r, d : G \to G$ via $r(x) = xx^{-1}$ and $d(x) = x^{-1}x$; these are called the range and domain maps. They have a common image called the unit space of G, denoted $G^{(0)}$. If $u \in G^{(0)}$, then

$$ud(u) = r(u)u = u.$$

For each unit $u \in G^{(0)}$, there is a group $\mathsf{Iso}(u) = \{x \mid r(x) = d(x) = u\}$, called the isotropy group at u.

Definition 5.1. Let X be a finite poset. For each integer $n \geq 1$, define

$$R^n_X = \{(\mathbf{x}', \mathbf{x}) \in X_\infty \times X_\infty \mid \mathbf{x}'_{>n} = \mathbf{x}_{>n}\}.$$

Define $(R^n_X)^{(2)}$ to consist of pairs $(\mathbf{x}'', \mathbf{x}') \times (\mathbf{x}', \mathbf{x})$, with product map $(R^n_X)^{(2)} \times (R^n_X)^{(2)} \to R^n_X$

$$(\mathbf{x}'', \mathbf{x}') \times (\mathbf{x}', \mathbf{x}) = (\mathbf{x}'', \mathbf{x}).$$

Define an inverse map via $(\mathbf{x}', \mathbf{x})^{-1} = (\mathbf{x}, \mathbf{x}')$.

This defines a groupoid, R^n_X. The unit space $(R^n_X)^{(0)} = \{(\mathbf{x}, \mathbf{x}) \mid \mathbf{x} \in X_\infty\} \cong X_\infty$, and $r(\mathbf{x}', \mathbf{x}) = (\mathbf{x}', \mathbf{x}')$ and $d(\mathbf{x}', \mathbf{x}) = (\mathbf{x}, \mathbf{x})$. A groupoid is called *principal* if $g \mapsto (r(g), d(g)) \in G^{(0)} \times G^{(0)}$ is injective. R^n_X is a principal groupoid. Define a topology on R^n_X by declaring the following sets to be clopen: for $p, q \in X_m$,

$$Z(p, q) = \{(\mathbf{x}, \mathbf{y}) \in R^n_X \mid \mathbf{x}_{\leq m} = p,\ \mathbf{y}_{\leq m} = q\}.$$

This is the induced (or relative) topology from the product $X_\infty \times X_\infty$, where X_∞ is the totally disconnected space with clopen sets given by the "cylinder sets" $[p] = \{\mathbf{x} \in X_\infty \mid p = \mathbf{x}_{\leq |p|}\}$.

Note that R_X^n is a compact Hausdorff space and thus is locally compact. R_X^n is a topological groupoid because the inverse map is continuous (a homeomorphism) and the product map $R_X^{(2)} \times R_X^{(2)} \to R$ is continuous when the domain is equipped with the product topology (and $(R_X^n)^{(2)}$ has the induced topology from $R_X^n \times R_X^n$).

A locally compact groupoid is called r-discrete if the unit space $G^{(0)} \subseteq G$ is open. Notice that

$$(R_X^n)^{(0)} = \bigsqcup_{p \in X_n} Z(p,p)$$

is clopen in R_X^n; so R_X^n is r-discrete.

Definition 5.2. Let X be a finite poset. Define

$$R_X = \{(\mathbf{x}', \mathbf{x}) \in X_\infty \times X_\infty \mid \mathbf{x}'_{>n} = \mathbf{x}_{>n} \text{ for some } n \geq 0 \ \}.$$

Equip R_X with the product and inverse map as above, so that R_X^n is a subgroupoid of R_X for each $n \geq 1$. Topologize R_X with the inductive limit topology: a set $U \subseteq R_X$ is open if and only if $U \cap R_X^n$ is open in R_X^n for all $n \geq 1$.

Proposition 5.3. *If X is a a finite poset, then R_X is a principal r-discrete groupoid.*

Remark 5.4. If we had given R_X the induced topology from $X_\infty \times X_\infty$, then it would not have (necessarily) been r-discrete. This presents a challenge for defining a Haar system for R_X, and ultimately the groupoid C^*-algebra $C^*(R_X)$.

5.2 *Definition of* $C^*(R_X^n)$

This follows the construction of Exel and Renault [p. 202].

Definition 5.5. $C(R_X^n)$ is a $\mathbb{C}$-vector space which is an algebra when equipped with the following convolution product: for $h, k \in C(R_X^n)$,

$$(h \cdot k)|_{(\mathbf{x}', \mathbf{x})} = \sum_{\mathbf{x}'' \in [\mathbf{x}]_n} h(\mathbf{x}', \mathbf{x}'') k(\mathbf{x}'', \mathbf{x});$$

it is a *-algebra when given the involution $h^*(\mathbf{x}', \mathbf{x}) = \overline{h(\mathbf{x}, \mathbf{x}')}$; this is anti-multiplicative.

Proposition 5.6. *Let X be a finite poset.*

1. *For each $n \geq 1$, there is an inner-product on $C(X_\infty)$ with values in $C(X_\infty; R_X^n)$ defined by*
$$\langle \psi, \phi \rangle|_{\mathbf{x}'} = \frac{1}{\#[\mathbf{x}']_n} \sum_{\mathbf{x} \in [\mathbf{x}']_n} \overline{\psi(\mathbf{x})} \phi(\mathbf{x}).$$
2. *$C(X_\infty)$ is a pre-Hilbert module over $C(X_\infty; R_X^n)$ when equipped with this inner-product.*

Proof. (1) It is obvious that this value depends only on $[\mathbf{x}'] \in X_\infty / R_n$ and that the resulting function is continuous. The sesqui-linear form is positive definite since

$$\langle \phi, \phi \rangle|_{\mathbf{x}'} = \frac{1}{\#[\mathbf{x}']_n} \sum_{\mathbf{x} \in [\mathbf{x}']_n} |\phi(\mathbf{x})|^2,$$

which is zero if and only if $\phi([\mathbf{x}']) = \{0\}$; if this holds for each $\mathbf{x}' \in X_\infty$ then $\phi = 0$ is the zero function.

(2) The right-action of $C(X_\infty / R_X^n)$ is $\phi \cdot \alpha|_{\mathbf{x}} = \phi(\mathbf{x}) \eta([\mathbf{x}])$. For every $\mathbf{x}' \in X_\infty$, we have

$$\begin{aligned}
\langle\langle \psi, \phi \cdot \alpha \rangle|_{\mathbf{x}'} &= \frac{1}{\#[\mathbf{x}']_n} \sum_{\mathbf{x} \in [\mathbf{x}']_n} \overline{\psi(\mathbf{x})} \phi(\mathbf{x}) \alpha(\mathbf{x}) \\
&= \left(\frac{1}{\#[\mathbf{x}']_n} \sum_{\mathbf{x} \in [\mathbf{x}']_n} \overline{\psi(\mathbf{x})} \phi(\mathbf{x}) \right) \alpha(\mathbf{x}') \\
&= \left(\langle \psi, \phi \rangle \cdot \alpha \right) |_{\mathbf{x}'}.
\end{aligned}$$

Thus, $C(X_\infty)$ is a pre-Hilbert $C(X_\infty; R_X^n)$-module. □

Definition 5.7. Let X be a finite poset.

1. M^n denotes the completion of $C(X_\infty)$ under the norm induced by the $C(X_\infty; R_X^n)$-valued inner-product;
2. $\mathcal{L}(M^n)$ denotes the C^*-algebra of adjointable operators on M^n;

3. For $k \in C(R_X^n)$ and $\phi \in M^n$, define $k \cdot \phi \in M^n$ by

$$k \cdot \phi|_{\mathbf{x}'} = \sum_{\mathbf{x} \in [\mathbf{x}']_n} k(\mathbf{x}', \mathbf{x})\phi(\mathbf{x}).$$

Lemma 5.8. *Let X be a finite poset.*

1. *M^n is a left $C(R_X^n)$-module;*
2. *$C(R_X^n)$ acts adjointably on M^n.*

Proof. (1) For every $h, k \in C(R_X^n)$, we check that

$$(h \cdot k) \cdot \phi|_{\mathbf{x}'} = (h \cdot (k \cdot \phi))|_{\mathbf{x}'}.$$

Indeed,

$$\begin{aligned}
(h \cdot (k \cdot \phi))|_{\mathbf{x}'} &= \sum_{\mathbf{x}'' \in [\mathbf{x}']_n} h(\mathbf{x}', \mathbf{x}'')(k \cdot \phi)|_{\mathbf{x}''} \\
&= \sum_{\mathbf{x}'' \in [\mathbf{x}']_n} \sum_{\mathbf{x} \in [\mathbf{x}']_n} h(\mathbf{x}', \mathbf{x}'')k(\mathbf{x}'', \mathbf{x})\phi(\mathbf{x}) \\
&= \sum_{\mathbf{x} \in [\mathbf{x}']_n} \left(\sum_{\mathbf{x}'' \in [\mathbf{x}']_n} h(\mathbf{x}', \mathbf{x}'')k(\mathbf{x}'', \mathbf{x}) \right) \phi(\mathbf{x}) \\
&= \sum_{\mathbf{x} \in [\mathbf{x}']_n} (h \cdot k)|_{(\mathbf{x}', \mathbf{x})}\phi(\mathbf{x}) \\
&= (h \cdot k) \cdot \phi|_{\mathbf{x}'}.
\end{aligned}$$

(2) Compute that

$$\begin{aligned}\langle \psi, k\cdot\phi\rangle|_{\mathbf{x}'} &= \frac{1}{\#[\mathbf{x}']_n}\sum_{\mathbf{x}\in[\mathbf{x}']_n}\overline{\psi}(\mathbf{x})\left(\sum_{\mathbf{x}''\in[\mathbf{x}]_n} k(\mathbf{x},\mathbf{x}'')\phi(\mathbf{x}'')\right)\\ &= \frac{1}{\#[\mathbf{x}']_n}\sum_{\mathbf{x}\in[\mathbf{x}']_n}\sum_{\mathbf{x}''\in[\mathbf{x}']_n} k(\mathbf{x},\mathbf{x}'')\overline{\psi}(\mathbf{x})\phi(\mathbf{x}'')\\ &= \frac{1}{\#[\mathbf{x}']_n}\sum_{\mathbf{x}''\in[\mathbf{x}']_n}\sum_{\mathbf{x}\in[\mathbf{x}']_n} \overline{k^*(\mathbf{x}'',\mathbf{x})}\,\overline{\psi}(\mathbf{x})\phi(\mathbf{x}'')\\ &= \frac{1}{\#[\mathbf{x}']_n}\sum_{\mathbf{x}''\in[\mathbf{x}']_n}\overline{\left(\sum_{\mathbf{x}\in R_n(\mathbf{x}'')} k^*(\mathbf{x}'',\mathbf{x})\psi(\mathbf{x})\right)}\phi(\mathbf{x}'')\\ &= \langle k^*\cdot\psi, \phi\rangle|_{\mathbf{x}'}. \qquad\square\end{aligned}$$

Definition 5.9. Let X be a finite poset and $n > 0$ an integer. Define

$$C^*(R^n_X) = \overline{C(R^n_X)} \subseteq \mathcal{L}(M^n),$$

the closure of $C(R^n_X)$ in the operator norm topology.

5.2.1 Comparison of Norms

Recall that two norms $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ on a vector space V are called equivalent if there exist positive constants c_1, c_2 such that

$$c_1\|v\|_\alpha \le \|v\|_\beta \le c_2\|v\|_\alpha \qquad \text{for every } v\in V.$$

Equivalent norms on V induce homeomorphic completions.

Proposition 5.10. *Let X be a finite poset and $n > 0$ an integer.*

1. *The supremum norm and the $C(X_\infty; R^n_X)$-inner product norm on $C(X_\infty)$ are equivalent; indeed, for $\phi \in C(X_\infty)$,*

$$|X|^{-n/2}\,\|\phi\|_{sup} \le \|\phi\|_{M^n} \le \|\phi\|_{sup}$$

2. *$M^n = C(X_\infty)$;*

3. *The supremum and operator norms on $C(R^n_X)$ are equivalent; indeed, for $k \in C(R^n_X)$,*

$$|X|^{-n/2} \|k\|_{sup} \leq \|k\|_\infty \leq |X|^n \cdot \|k\|_{sup}$$

4. *$C^*(R^n_X) = C(R^n_X)$.*

Proof. (1) Compute

$$\begin{aligned}\frac{1}{|X|^n} \|\phi\|^2_{\text{sup}} &= \frac{1}{|X|^n} \sup\nolimits_{\mathbf{x} \in X_\infty} |\phi(\mathbf{x})|^2 \\ &\leq \sup\nolimits_{\mathbf{x}' \in X_\infty} \left\{ \frac{1}{\#[\mathbf{x}']_n} \sum_{\mathbf{x} \in [\mathbf{x}']_n} |\phi(\mathbf{x})|^2 \right\} \\ &= \|\phi\|^2_{M^n} \leq \|\phi\|^2_{\text{sup}},\end{aligned}$$

since $\#[\mathbf{x}']_n \leq |X|^n$, for every $\mathbf{x}' \in X_\infty$.

(3) We first show that for each $p \in X_n$, $\|1_{[p]}\|_{M^n} \leq 1$; indeed,

$$\begin{aligned}\|1_{[p]}\|^2_{M^n} = \|\langle 1_{[p]}, 1_{[p]} \rangle\|_{\text{sup}} &= \sup\nolimits_{\mathbf{x}' \in X_\infty} \{ \frac{1}{\#[\mathbf{x}']_n} \sum_{\mathbf{x} \in [\mathbf{x}']_n} |1_{[p]}(\mathbf{x})|^2 \} \\ &= \frac{1}{\#[\mathbf{x}']_n} \leq 1,\end{aligned}$$

where $\mathbf{x}'_n = t(p)$. It follows that

$$\|k\|_\infty \geq \|k \cdot 1_{[p]}\|_{M^n} \geq |X|^{-n/2} \|k \cdot 1_{[p]}\|_{\text{sup}} \geq |X|^{-n/2} |k(\mathbf{x}, p\mathbf{x}_{>n})|,$$

for every $p \in X_n$ and $\mathbf{x} \in X_\infty$, $\mathbf{x}_n = t(p)$. Thus, $|X|^{-n/2}\|k\|_{\text{sup}} \leq \|k\|_\infty$.

We will show that $\|k\|_\infty \leq |X|^n \|k\|_{\text{sup}}$. First compute that for every $\mathbf{x}' \in X_\infty$ and $\phi \in C(X_\infty)$,

$$\begin{aligned}\langle k \cdot \phi, k \cdot \phi \rangle|_{\mathbf{x}'} &= \frac{1}{\#[\mathbf{x}']_n} \sum_{\mathbf{x} \in [\mathbf{x}']_n} |(k \cdot \phi)(\mathbf{x})|^2 \\ &= \frac{1}{\#[\mathbf{x}']_n} \sum_{\mathbf{x} \in [\mathbf{x}']_n} \left| \sum_{\mathbf{x}'' \in [\mathbf{x}']_n} k(\mathbf{x}, \mathbf{x}'')\phi(\mathbf{x}'') \right|^2 \\ &\leq \|k\|^2_{\text{sup}} \cdot \|\phi\|^2_{\text{sup}} \cdot |X|^n \leq \|k\|^2_{\text{sup}} \cdot \|\phi\|^2_{M^n} \cdot |X|^{2n},\end{aligned}$$

where the first inequality holds since $\#[\mathbf{x}']_n \leq |X|^n$. It follows that

$$\|k\|_\infty^2 = \sup_{\|\phi\|_{M^n}\leq 1}\{\|\langle k \cdot \phi, k \cdot \phi\rangle\|_{\text{sup}}\} \leq \|k\|_{\text{sup}}^2 \cdot |X|^{2n},$$

so $\|k\|_\infty \leq |X|^n \, \|k\|_{\text{sup}}$. This completes the proof of (3).

$C(X_\infty)$ is already closed in the sup norm since X_∞ is compact; (1) shows that the sup norm and inner-product norm are equivalent, so $C(X_\infty)$ is closed in that norm. Thus (1) implies (2). A similar argument shows that (3) implies (4). □

Proposition 5.11. *Let X be a finite poset and $n > 0$ a positive integer. The commutative C^*-algebra $C(X_\infty)$ is a sub-algebra of $C^*(R_X^n)$ via the map sending $\phi \mapsto k_\phi$, where*

$$k_\phi(\mathbf{x}, \mathbf{y}) = \begin{cases} \phi(\mathbf{x}) & \textit{if } \mathbf{x} = \mathbf{y} \\ 0 & \textit{otherwise.} \end{cases}$$

In particular, $k_\phi \cdot \psi|_{\mathbf{x}'} = \phi(\mathbf{x}')\psi(\mathbf{x}')$, i.e. k_ϕ is left-multiplication by ϕ.

Proof. If $\phi \in C(X_\infty)$, k_ϕ is continuous because the diagonal $\{(\mathbf{x}, \mathbf{x}) \mid \mathbf{x} \in X_\infty\} \subseteq R_X^n$ is clopen. Compute that

$$k_\phi \cdot \psi|_{\mathbf{x}} = \sum_{\mathbf{y}\in[\mathbf{x}]_n} k_\phi(\mathbf{x}, \mathbf{y})\psi(\mathbf{y}) = \phi(\mathbf{x})\psi(\mathbf{x});$$

so that $k_\phi = L_\phi$ is left-multiplication by ϕ. It follows immediately that $\phi \rightsquigarrow k_\phi$ is an injective algebra homomorphism of C^*-algebras, and thus is also continuous (in fact, norm preserving). □

Remark 5.12. $C^*(R_X^n)$ is a unital C^*-algebra: if 1_{X_∞} is the constant function 1, then $k_{1_{X_\infty}}$ is the unit in $C^*(R_X^n)$.

5.3 $C^*(R_X^n) \cong \mathcal{K}(M^n)$

We will show that $C^*(R_X^n) \cong \mathcal{K}(M^n)$, the algebra of generalized compact operators on M^n.

Definition 5.13. Let X be a finite poset and $n > 0$ a positive integer.

1. Define $\zeta_n \in C(R_X^n)$ by $\zeta_n(\mathbf{x}', \mathbf{x}) = \frac{1}{\#[\mathbf{x}']_n} = \frac{1}{\#[\mathbf{x}]_n}$;

2. Define $\rho_n \in C(X_\infty)$ by $\rho_n(\mathbf{x}) = \frac{1}{\sqrt{\#[\mathbf{x}']_n}}$

Note that $\zeta_n(\mathbf{x}', \mathbf{x}) = \rho_n(\mathbf{x}')\rho_n(\mathbf{x})$. Also, ζ_n is a projection in $\mathcal{L}(M^n)$ since it is self-adjoint and idempotent:

$$\begin{aligned}\zeta_n \cdot \zeta_n|_{(\mathbf{x}',\mathbf{x})} &= \sum_{\mathbf{x}'' \in [\mathbf{x}]_n} \zeta_n(\mathbf{x}', \mathbf{x}'')\, \zeta_n(\mathbf{x}'', \mathbf{x}) = \sum_{\mathbf{x}'' \in [\mathbf{x}]_n} \frac{1}{\#[\mathbf{x}']_n} \cdot \frac{1}{\#[\mathbf{x}]_n} \\ &= \frac{1}{\#[\mathbf{x}']_n} = \zeta_n(\mathbf{x}', \mathbf{x}).\end{aligned}$$

Proposition 5.14. *Let X be a finite poset, and $n > 0$ a positive integer.*

1. *For each $\psi, \phi \in C(X_\infty)$,*
$$\theta_{\psi,\phi} = \psi \cdot \zeta_n \cdot \overline{\phi}.$$

2. $\mathcal{K}(M^n) = C^*(R_X^n) = \overline{\mathit{span}}\{\psi\, \zeta_n\, \phi \mid \psi,\ \phi \in C(X_\infty)\}$.

Proof. (1) For every $\phi' \in C(X_\infty)$, $\mathbf{x}' \in X_\infty$,

$$\begin{aligned}\theta_{\psi,\phi}(\phi')|_{\mathbf{x}'} = \psi(\mathbf{x}')\langle \phi, \phi' \rangle_{M^N}|_{\mathbf{x}'} &= \frac{1}{\#[\mathbf{x}']_n} \sum_{\mathbf{x} \in [\mathbf{x}']_n} \psi(\mathbf{x}')\overline{\phi(\mathbf{x})}\phi'(\mathbf{x}) \\ &= \sum_{\mathbf{x} \in [\mathbf{x}']_n} \psi(\mathbf{x}') \frac{1}{\#[\mathbf{x}']_n} \overline{\phi(\mathbf{x})}\phi'(\mathbf{x}) = \sum_{\mathbf{x} \in [\mathbf{x}']_n} \psi(\mathbf{x}')\zeta_n(\mathbf{x}', \mathbf{x})\overline{\phi}(\mathbf{x})\phi'(\mathbf{x}) \\ &= \sum_{\mathbf{x} \in [\mathbf{x}']_n} (\psi \cdot \zeta_n \cdot \overline{\phi})|_{(\mathbf{x}',\mathbf{x})}\phi'(\mathbf{x}) = (\psi \cdot \zeta_n \cdot \overline{\phi}) \cdot \phi'|_{\mathbf{x}'}.\end{aligned}$$

This proves (1).

(2) It follows from (1) that $\mathcal{K}(M^n) \subseteq \overline{\mathit{span}}\{\psi\, \zeta_n\, \phi \mid \psi,\ \phi \in C(X_\infty)\} \subseteq C^*(R_X^n)$. Now we prove that $C^*(R_X^n) \subseteq \mathcal{K}(M^n)$. Let $1_{[p]} \in C(X_\infty)$ denote the indicator on the cylinder set $[p]$, i.e.

$$1_{[p]}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} = p\mathbf{x}' \\ 0 & \text{otherwise.} \end{cases}$$

It is clear the set of functions $\{1_{[p]} \times 1_{[q]} \mid p, q \in X_m,\, t(p) = t(q),\, m > 0\}$ separates points in R_X^n. So by the Stone-Weierstrass approximation theorem,

$$\overline{\operatorname{span}}\{1_{[p]} \times 1_{[q]} \mid p, q \in X_m,\, t(p) = t(q),\, m > 0\} = C(R_X^n) = C^*(R_X^n).$$

Since

$$1_{[p]} \times 1_{[q]} = 1_{[p]} \frac{1}{\rho_n} \cdot \zeta_n \cdot 1_{[q]} \frac{1}{\rho_n} = \theta_{1_{[p]} \frac{1}{\rho_n}, 1_{[q]} \frac{1}{\rho_n}},$$

we conclude that that $C^*(R_X^n) \subseteq \mathcal{K}(M^n)$. ☐

5.4 $C^*(R_X)$ *and* $\mathcal{A}_X$

Definition 5.15. Let $\Phi_n : C^*(R_X^n) \to C^*(R_X^{n+1})$ denote the extension of k by zero to R_X^{n+1}.

Proposition 5.16. *Let X be a finite poset and $n > 0$ a positive integer.*

1. *For each $k \in C(R_X^n)$, $\Phi_n(k)$ is continuous on R_X^{n+1};*

2. *The action of k and $\Phi_n(k)$ on $C(X_\infty)$ are equal;*

3. *Φ_n is an injective *-homomorphism.*

Proof. (1) First observe that $R_X^n \subseteq R_X^{n+1}$; in fact, R_X^n is a clopen subset, since

$$R_X^n = \bigsqcup_{\substack{p,q \in X_n \\ t(p)=t(q)}} \bigsqcup_{z \leq t(p)} [pz] \times [qz] \subseteq R_X^{n+1}.$$

Thus, $\Phi_n(k)$ is continuous on R_X^{n+1}.

(2) Compute that

$$\begin{aligned} (\Phi_n(k) \cdot \phi)|_{\mathbf{x}' \in X_\infty} &= \sum_{\mathbf{x} \in [\mathbf{x}']_{n+1}} \Phi_n(k)(\mathbf{x}', \mathbf{x})\phi(\mathbf{x}) \\ &= \sum_{\mathbf{x} \in [\mathbf{x}']_n} k(\mathbf{x}', \mathbf{x})\phi(\mathbf{x}) \\ &= (k \cdot \phi)|_{\mathbf{x}' \in X_\infty}. \end{aligned}$$

This shows in particular that Φ_n is injective. Further, Φ_n is an algebra homomorphism since

$$\begin{aligned}\Phi_n(k \cdot l)|_{(\mathbf{x}',\mathbf{x})} &= \sum_{\mathbf{x}'' \in [\mathbf{x}]_n \cap [\mathbf{x}']_n} k(\mathbf{x}', \mathbf{x}'') l(\mathbf{x}'', \mathbf{x}) \\ &= \sum_{\mathbf{x}'' \in [\mathbf{x}]_{n+1}} \Phi_n(k)|_{(\mathbf{x}',\mathbf{x}'')} \Phi_n(l)|_{(\mathbf{x}'',\mathbf{x})} \\ &= \Phi_n(k) \cdot \Phi_n(l)|_{(\mathbf{x}',\mathbf{x})};\end{aligned}$$

Φ_n is a *-homomorphism since both $\Phi_n(k^*)|_{(\mathbf{x}',\mathbf{x})}$ and $\Phi_n(k)^*|_{(\mathbf{x}',\mathbf{x})}$ are equal to

$$\begin{cases} \overline{k(\mathbf{x}, \mathbf{x}')} & \text{if } \mathbf{x}' \in [\mathbf{x}]_n \\ 0 & \text{otherwise.} \end{cases}$$

Φ_n is injective because the restriction map $k \mapsto k|_{R_X^n}$ from $C^*(R_X^{n+1}) \to C^*(R_X^n)$ is a left-inverse of Φ_n. This completes the proof of (3). $\square$

Definition 5.17. Let X be a finite poset and $n > 0$ a positive integer.

1. Define $C^*(R_X) = \varinjlim_n C^*(R_X^n)$;

2. Define a $\mathbb{C}$-linear map $\Psi_n : \mathcal{A}_X^n \to C^*(R_X^n)$ by $\Psi_n(pq^*) = 1_{[p]} \times 1_{[q]} \in C^*(R_X^n)$

Theorem 5.18. *Let X be a finite poset.*

1. *The maps $\Psi_n : \mathcal{A}_X^n \to C^*(R_X^n)$ are injective *-homomorphisms that are compatible with the morphisms Φ_n directing $C^*(R_X)$;*

2. *$C^*(R_X)$ is isomorphic to $\mathcal{A}_X$; in particular, $C^*(R_X)$ is an AF-algebra.*

Proof. (1) For $pq^*, ab^* \in \mathcal{A}_X^n$,

$$\begin{aligned}
\Psi_n(pq^*) \cdot \Psi_n(ab^*) &= (1_{[p]} \times 1_{[q]}) \cdot (1_{[a]} \times 1_{[b]})|_{(\mathbf{x}',\mathbf{x})} \\
&= \sum_{\mathbf{x}'' \in [\mathbf{x}]_n} (1_{[p]} \times 1_{[q]})|_{(\mathbf{x}',\mathbf{x}'')} (1_{[a]} \times 1_{[b]})|_{(\mathbf{x}'',\mathbf{x})} \\
&= \begin{cases} 1 & \text{if } a = q,\ \mathbf{x} = b\mathbf{x}_{>n}, \text{ and } \mathbf{x}' = p\mathbf{x}_{>n} \\ 0 & \text{otherwise} \end{cases} \\
&= \delta_{aq} 1_{[p]} \times 1_{[b]}|_{(\mathbf{x}',\mathbf{x})} = \Psi_n(pq^* \cdot ab^*).
\end{aligned}$$

Thus Ψ_n is an algebra homomorphism. It is a *-homomorphism since $\Psi_n((pq^*)^*) = 1_{[q]} \times 1_{[p]} = \Psi_n(pq^*)^*$.

We check that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{A}_X^n \arrow[r] \arrow[d, "\Psi_n"'] & \mathcal{A}_X^{n+1} \arrow[d, "\Psi_{n+1}"] \\
C^*(R_X^n) \arrow[r, "\Phi_n"'] & C^*(R_X^{n+1})
\end{tikzcd}$$

The unlabeled arrow is the one given by the Cuntz-Krieger relations, e.g. $pq^* \mapsto \sum_{x \le t(p)} (px)(qx)^*$. Compute:

$$\Psi_{n+1}\left(\sum_{x \le t(p)} (px)(qx)^* \right) = \sum_{x \le t(p)} 1_{[px]} \times 1_{[qx]}$$

$$\Phi_n \circ \Psi_n(pq^*) = \Phi_n(1_{[p]} \times 1_{[q]}).$$

$$\begin{aligned}
\Phi_n(1_{[p]} \times 1_{[q]})|_{(\mathbf{x}',\mathbf{x})} &= \begin{cases} 1 & \text{if } \mathbf{x}' = px\mathbf{x}_{>n+1} \text{ and } \mathbf{x} = qx\mathbf{x}_{>n+1}, \\ & \qquad \text{for some } x \le t(p) \\ 0 & \text{otherwise.} \end{cases} \\
&= \sum_{x \le t(p)} 1_{[px]} \times 1_{[qx]}|_{(\mathbf{x}',\mathbf{x})}.
\end{aligned}$$

Since each map is linear, and the pq^* are a basis for $\mathcal{A}_X^n$, the diagram commutes. Since each Φ_n and Ψ_n is injective, the resulting map $\Psi : \mathcal{A}_X \to C^*(R_X)$ is injective.

On the otherhand, Ψ is surjective because the image contains

$$\operatorname{span}\left\{1_{[p]} \times 1_{[q]} \mid p, q \in X_m,\, t(p) = t(q),\, m > 0\right\}$$

which is dense in each $C^*(R_X^n)$, and thus in $C^*(R_X)$. This proves (2). $\square$

5.5 *Comparing with the Incidence Algebra* $\mathbb{I}_X$

Definition 5.19. As a $\mathbb{C}$-vector space, $\mathbb{I}_X = \{\phi : X \times X \to \mathbb{C} \mid \phi(y, x) = 0 \text{ if } x \not\leq y\ \}$. The multiplication is a convolution defined by

$$\phi * \psi(z, x) = \sum_{y \in X} \phi(z, y)\psi(y, x) = \sum_{x \leq y \leq z} \phi(z, y)\psi(y, x).$$

The resulting function is in $\mathbb{I}_X$ because the relation $\leq$ is transitive. The algebra is unital with unit $1(y, x) = \delta_{yx}$; this function is in $\mathbb{I}_X$ because the relation $\leq$ is reflexive.

One can view $\mathbb{I}_X$ as a sub-algebra of $C^*(R_X^1)$ as follows: to each $\phi \in \mathbb{I}_X$, let $\hat{\phi}$ denote the function

$$\hat{\phi}|_{(\mathbf{x}', \mathbf{x})} = \phi(\mathbf{x}'_1, \mathbf{x}_1).$$

Then $\hat{\phi} \in C(R_X^1)$ (in particular, it is continuous), and the association $\phi \rightsquigarrow \hat{\phi}$ is an injective unit-preserving $\mathbb{C}$-algebra map.

Note that $\mathbb{I}_X$ is not a C^*-algebra, e.g. it typically has no involution.

5.6 *Defining the Hilbert* $C^*(R_Y^n)$*-*$C^*(R_X^n)$*-bimodule,* $C^*(\Gamma_f^n)$

Definition 5.20. Let $f : X \to Y$ be a morphism of finite posets. Define $\Gamma_f^n = \{(\mathbf{y}, \mathbf{x}) \in Y_\infty \times X_\infty \mid \mathbf{y}_{\geq n} = f(\mathbf{x})_{\geq n}\}$, with the induced topology from $Y_\infty \times X_\infty$. Then Γ_f^n is a compact Hausdorff space; define a left action of $C^*(R_Y^n)$ and a right action of $C^*(R_X^n)$ on $C^*(\Gamma_f^n)$ as follows: for $h \in C^*(R_Y^n)$, $k \in C^*(R_X^n)$ and $\alpha \in C^*(\Gamma_f^n)$,

$$h \cdot \alpha(\mathbf{y}', \mathbf{x}') := \sum_{\mathbf{y} \in R_Y^n(\mathbf{y}')} h(\mathbf{y}', \mathbf{y})\alpha(\mathbf{y}, \mathbf{x}'), \qquad \alpha \cdot k(\mathbf{y}', \mathbf{x}') := \sum_{\mathbf{x} \in R_X^n(\mathbf{x}')} \alpha(\mathbf{y}', \mathbf{x})k(\mathbf{x}, \mathbf{x}').$$

Lemma 5.21. *Let* $f : X \to Y$ *be a continuous map of finite posets, and* $n > 0$ *a positive integer.*

1. $C(\Gamma_f^n)$ *is a* $C^*(R_Y^n)$*-*$C^*(R_X^n)$*-bimodule;*

2. *There is a* $C^*(R_X^n)$*-valued inner product on* $C(\Gamma_f^n)$ *defined by*

$$\langle \alpha, \beta \rangle|_{(\mathbf{x}', \mathbf{x})} = \sum_{\mathbf{y} \in [f(\mathbf{x})]_n} \overline{\alpha(\mathbf{y}, \mathbf{x}')} \beta(\mathbf{y}, \mathbf{x}).$$

Proof. (1) For $h \in C^*(R_Y^n)$, $k \in C^*(R_X^n)$ and $\alpha \in C(\Gamma_f^n)$, we show that $(h \cdot \alpha) \cdot k = h \cdot (\alpha \cdot k)$:

$$\begin{aligned}
(h \cdot \alpha) \cdot k|_{(\mathbf{y}', \mathbf{x}')} &= \sum_{\mathbf{x} \in [\mathbf{x}']_n} (h \cdot \alpha)(\mathbf{y}', \mathbf{x}) k(\mathbf{x}, \mathbf{x}') = \sum_{\mathbf{x} \in [\mathbf{x}]_n} \sum_{\mathbf{y}' \in [\mathbf{y}']_n} h(\mathbf{y}', \mathbf{y}) \alpha(\mathbf{y}, \mathbf{x}) k(\mathbf{x}, \mathbf{x}'), \\
h \cdot (\alpha \cdot k)|_{(\mathbf{y}', \mathbf{x}')} &= \sum_{\mathbf{y} \in [\mathbf{y}']_n} h(\mathbf{y}', \mathbf{y}) (\alpha \cdot k)(\mathbf{y}, \mathbf{x}') = \sum_{\mathbf{y} \in [\mathbf{y}']_n} \sum_{\mathbf{x} \in [\mathbf{x}']_n} h(\mathbf{y}', \mathbf{y}) \alpha(\mathbf{y}, \mathbf{x}) k(\mathbf{x}, \mathbf{x}');
\end{aligned}$$

the two sums are equal since the indexing set of each sum is finite and so the the order of summation can be changed.

The form $\langle -, - \rangle$ defined in the statement is clearly sesqui-linear. Observe that

$$\langle \beta, \alpha \rangle|_{(\mathbf{x}, \mathbf{x}')} = \sum_{\mathbf{y} \in [f(\mathbf{x})]_n} \overline{\beta(\mathbf{y}, \mathbf{x})} \alpha(\mathbf{y}, \mathbf{x}') = \overline{\langle \alpha, \beta \rangle}|_{(\mathbf{x}', \mathbf{x})};$$

so $\langle \alpha, \beta \rangle^* = \langle \beta, \alpha \rangle$. We must also show that the form is positive-definite. So consider $0 \neq \alpha \in C(\Gamma_f^n)$. Since $\alpha \neq 0$, there is some point $(\mathbf{y}, \mathbf{x}) \in \Gamma_f^n$ such that $\alpha(\mathbf{y}, \mathbf{x}) \neq 0$. Then

$$\langle \alpha, \alpha \rangle|_{(\mathbf{x}, \mathbf{x})} = \sum_{\mathbf{y} \in [f(\mathbf{x})]_n} |\alpha(\mathbf{y}, \mathbf{x})|^2 > 0,$$

so $\langle \alpha, \alpha \rangle \neq 0$. It remains to show that $\langle \alpha, \alpha \rangle \geq 0$. We establish this by showing that

$$\langle \langle \alpha, \alpha \rangle_{C^*(R_X^n)} \cdot \phi, \phi \rangle|_{\mathbf{x}'} \geq 0, \qquad \text{for every } \phi \in C(X_\infty),\ \mathbf{x}' \in X_\infty.$$

Compute:

$$\begin{aligned}
\langle\langle\alpha,\alpha\rangle_{C^*(R^n_X)}\cdot\phi,\phi\rangle|_{\mathbf{x}'} &= \sum_{\mathbf{x}\in[\mathbf{x}']_n}\overline{(\langle\alpha,\alpha\rangle\cdot\phi)\,|_{\mathbf{x}}}\cdot\phi(\mathbf{x}) \\
&= \sum_{\mathbf{x}\in[\mathbf{x}']_n}\overline{\left(\sum_{\mathbf{x}''\in[\mathbf{x}']_n}\langle\alpha,\alpha\rangle|_{(\mathbf{x},\mathbf{x}'')}\cdot\phi(\mathbf{x}'')\right)}\cdot\phi(\mathbf{x}) \\
&= \sum_{\substack{\mathbf{x}\in[\mathbf{x}']_n\\ \mathbf{x}''\in[\mathbf{x}']_n}}\overline{\left(\sum_{\mathbf{y}\in[f(\mathbf{x}')]_n}\overline{\alpha(\mathbf{y},\mathbf{x})}\alpha(\mathbf{y},\mathbf{x}'')\phi(\mathbf{x}'')\right)}\cdot\phi(\mathbf{x}) \\
&= \sum_{\mathbf{y}\in[f(\mathbf{x}')]_n}\sum_{\substack{\mathbf{x}\in[\mathbf{x}']_n\\ \mathbf{x}''\in[\mathbf{x}']_n}}\overline{\alpha(\mathbf{y},\mathbf{x}'')\phi(\mathbf{x}'')}\alpha(\mathbf{y},\mathbf{x})\phi(\mathbf{x}) \\
&= \sum_{\mathbf{y}\in[f(\mathbf{x}')]_n}\left|\sum_{\mathbf{x}\in[\mathbf{x}']_n}\alpha(\mathbf{y},\mathbf{x})\phi(\mathbf{x})\right|^2\geq 0.
\end{aligned}$$

□

Theorem 5.22. *Let $f : X \to Y$ be a map of finite posets. Then for each $n > 0$, $C(\Gamma^n_f)$ is a pre-Hilbert $C(R^n_X)$-module. Furthermore, the left action of $C(R^n_Y)$ is by adjointable operators on $C(\Gamma^n_f)$. Thus, the closure of $C(\Gamma^n_f)$ in the norm induced by the inner-product is a Hilbert C^*-bimodule over $C^*(R^n_Y)$ and $C^*(R^n_X)$.*

Proof. Suppose that $\alpha,\beta\in C(\Gamma^n_f)$ and $k\in C(R^n_X)$. We show that $\langle\alpha,\beta\cdot k\rangle = \langle\alpha,\beta\rangle k$. Indeed,

$$\begin{aligned}
\langle\alpha,\beta\cdot k\rangle|_{(\mathbf{x}',\mathbf{x})} &= \sum_{\mathbf{y}\in[f(\mathbf{x})]_n}\overline{\alpha(\mathbf{y},\mathbf{x}')}(\beta\cdot k)|_{(\mathbf{y},\mathbf{x})} \\
&= \sum_{\mathbf{y}\in[f(\mathbf{x})]_n}\overline{\alpha(\mathbf{y},\mathbf{x}')}\left(\sum_{\mathbf{x}''\in[\mathbf{x}]_n}\beta(\mathbf{y},\mathbf{x}'')k(\mathbf{x}'',\mathbf{x})\right) \\
&= \sum_{\mathbf{y}\in[f(\mathbf{x})]_n}\sum_{\mathbf{x}''\in[\mathbf{x}]_n}\overline{\alpha(\mathbf{y},\mathbf{x}')}\beta(\mathbf{y},\mathbf{x}'')k(\mathbf{x}'',\mathbf{x})
\end{aligned}$$

$$\begin{aligned}
\langle \alpha, \beta \rangle \cdot k|_{(\mathbf{x}', \mathbf{x})} &= \sum_{\mathbf{x}'' \in [\mathbf{x}]_n} \langle \alpha, \beta \rangle|_{(\mathbf{x}', \mathbf{x}'')} k(\mathbf{x}'', \mathbf{x}) \\
&= \sum_{\mathbf{x}'' \in [\mathbf{x}]_n} \left(\sum_{\mathbf{y} \in [f(\mathbf{x}'')]_n} \overline{\alpha(\mathbf{y}, \mathbf{x}')} \beta(\mathbf{y}, \mathbf{x}'') \right) k(\mathbf{x}'', \mathbf{x}) \\
&= \sum_{\mathbf{x}'' \in [\mathbf{x}]_n} \sum_{\mathbf{y} \in [f(\mathbf{x})]_n} \overline{\alpha(\mathbf{y}, \mathbf{x}')} \beta(\mathbf{y}, \mathbf{x}'') k(\mathbf{x}'', \mathbf{x}).
\end{aligned}$$

These two sums are equal, since the indexing sets are finite and thus the order of summation can be interchanged. This proves that $C(\Gamma_f^n)$ is a pre-Hilbert $C(R_X^n)$-module. Now we show that the left action of $C(R_Y^n)$ is adjointable, i.e.

$$\langle h \cdot \alpha, \beta \rangle = \langle \alpha, h^* \cdot \beta \rangle, \qquad \text{for every } h \in C(R_Y^n),\ \alpha, \beta \in C(\Gamma_f^n).$$

Indeed, compute that

$$\begin{aligned}
\langle h \cdot \alpha, \beta \rangle|_{(\mathbf{x}', \mathbf{x})} &= \sum_{\mathbf{y} \in [f(\mathbf{x})]_n} \overline{h \cdot \alpha|_{(\mathbf{y}, \mathbf{x}')}} \beta(\mathbf{y}, \mathbf{x}) \\
&= \sum_{\mathbf{y} \in [f(\mathbf{x})]_n} \overline{\sum_{\mathbf{y}' \in [\mathbf{y}]_n} h(\mathbf{y}, \mathbf{y}') \alpha(\mathbf{y}', \mathbf{x}')} \beta(\mathbf{y}, \mathbf{x}) \\
&= \sum_{\mathbf{y} \in [f(\mathbf{x})]_n} \sum_{\mathbf{y}' \in [\mathbf{y}]_n} \overline{h(\mathbf{y}, \mathbf{y}') \alpha(\mathbf{y}', \mathbf{x}')} \beta(\mathbf{y}, \mathbf{x}).
\end{aligned}$$

$$\begin{aligned}
\langle \alpha, h^* \cdot \beta \rangle|_{(\mathbf{x}', \mathbf{x})} &= \sum_{\mathbf{y} \in [f(\mathbf{x})]_n} \overline{\alpha(\mathbf{y}, \mathbf{x}')} (h^* \cdot \beta)|_{(\mathbf{y}, \mathbf{x})} \\
&= \sum_{\mathbf{y} \in [f(\mathbf{x})]_n} \overline{\alpha(\mathbf{y}, \mathbf{x}')} \sum_{\mathbf{y}' \in [\mathbf{y}]_n} h^*(\mathbf{y}, \mathbf{y}') \beta(\mathbf{y}', \mathbf{x}) \\
&= \sum_{\mathbf{y} \in [f(\mathbf{x})]_n} \sum_{\mathbf{y}' \in [\mathbf{y}]_n} \overline{h(\mathbf{y}', \mathbf{y}) \alpha(\mathbf{y}, \mathbf{x}')} \beta(\mathbf{y}', \mathbf{x}).
\end{aligned}$$

These are equal since, e.g. one can interchange $\mathbf{y}$ and $\mathbf{y}'$ in the second sum. □

5.6.1 Comparing norms on $C(\Gamma_f^n)$

There are two norms on $C(\Gamma_f^n)$, the sup norm and the one induced from the $C^*(R_X^n)$-valued inner-product.

Proposition 5.23. *Let $f : X \to Y$ be a map of finite posets, and $n > 0$ a positive integer.*

1. *For every $\alpha \in C(\Gamma_f^n)$, $|X|^{-n/4} \|\alpha\|_{sup} \leq \|\alpha\|_{C^*(R_X^n)} \leq |X|^{n/2} |Y|^{n/4} \|\alpha\|_{sup}$;*

2. *$C(\Gamma_f^n)$ is closed in the norm induced from the $C(R_X^n)$-valued inner-product.*

Proof. (1) Compute that

$$\begin{aligned}
\|\langle \alpha, \alpha\rangle \cdot \phi\|_{M^n}^2 &= \sup\nolimits_{\mathbf{x}' \in X_\infty} \frac{1}{\#[\mathbf{x}']_n} \sum_{\mathbf{x} \in [\mathbf{x}']_n} |(\langle \alpha, \alpha \rangle \cdot \phi)(\mathbf{x})|^2 \\
&= \sup\nolimits_{\mathbf{x}' \in X_\infty} \frac{1}{\#[\mathbf{x}']_n} \sum_{\mathbf{x} \in [\mathbf{x}']_n} \left| \sum_{\mathbf{x}'' \in [\mathbf{x}]_n} \langle \alpha, \alpha \rangle|_{(\mathbf{x}, \mathbf{x}'')} \cdot \phi(\mathbf{x}'') \right|^2 \\
&= \sup\nolimits_{\mathbf{x}' \in X_\infty} \frac{1}{\#[\mathbf{x}']_n} \sum_{\mathbf{x} \in [\mathbf{x}']_n} \left| \sum_{\mathbf{x}'' \in [\mathbf{x}]_n} \sum_{\mathbf{y} \in [f(\mathbf{x})]_n} \overline{\alpha(\mathbf{y}, \mathbf{x})} \alpha(\mathbf{y}, \mathbf{x}'') \phi(\mathbf{x}'') \right|^2 \\
&\leq \|\alpha\|_{\text{sup}}^4 \|\phi\|_{\text{sup}}^2 |X|^n |Y|^n \leq |X|^n |Y|^n \|\alpha\|_{\text{sup}}^4 (|X|^n \|\phi\|_{M^n}^2);
\end{aligned}$$

thus, $\|\alpha\|_{C^*(R_X^n)} = \|\langle \alpha, \alpha \rangle\|_\infty^{1/2} \leq |X|^{n/2} |Y|^{n/4} \|\alpha\|_{\text{sup}}$

We now verify the first inequality. Note that $(\langle \alpha, \alpha \rangle \cdot 1_{[(\mathbf{x}')_{\leq n}]})|_{\mathbf{x}'} = \sum_{\mathbf{y} \in [f(\mathbf{x}')]_n} |\alpha(\mathbf{y}, \mathbf{x}')|^2$, so that in particular,

$$|\alpha(\mathbf{y}, \mathbf{x}')|^2 \leq (\langle \alpha, \alpha \rangle \cdot 1_{[(\mathbf{x}')_{\leq n}]})|_{\mathbf{x}'}, \qquad \text{for every } (\mathbf{y}, \mathbf{x}') \in \Gamma_f^n.$$

Since $\|1_{[(\mathbf{x}')_{\leq n}]}\|_{M^n} \leq 1$, it follows that

$$\begin{aligned}
\|\alpha\|_{\text{sup}}^2 &\leq \sup\nolimits_{\|\phi\|_{M^n} \leq 1} \|\langle \alpha, \alpha \rangle \cdot \phi\|_{\text{sup}} \leq |X|^{n/2} \sup\nolimits_{\|\phi\|_{M^n} \leq 1} \|\langle \alpha, \alpha \rangle \cdot \phi\|_{M^n} \\
&= |X|^{n/2} \|\alpha\|_{C^*(R_X^n)}^2.
\end{aligned}$$

(2) Since the norms are equivalent, and $C(\Gamma_f^n)$ is closed under the sup norm, it is also closed under the inner-product norm. □

5.7 Defining $C^*(\Gamma_f)$ and comparison with $\mathcal{A}_f$

There is a set inclusion $\Gamma_f^n \subseteq \Gamma_f^{n+1}$; we note that this is the inclusion of a clopen subset since

$$\Gamma_f^n = \left(\bigsqcup_{\substack{p\in X_n, a\in Y_n \\ f(t(p))=t(a)}} \bigsqcup_{x\le t(p)} [af(x)] \times [px] \right) \cap \Gamma_f^{n+1}.$$

Thus, there is an extension by zero map $\Phi_n^f : C(\Gamma_f^n) \to C(\Gamma_f^{n+1})$.

Lemma 5.24. *Let $f : X \to Y$ be a map of finite posets. For each $\alpha \in C^*(\Gamma_f^n)$,*

1. *$\Phi_n^f(\alpha \cdot k) = \Phi_n^f(\alpha) \cdot \Phi_n(k)$, for each $k \in C^*(R_X^n)$;*
2. *$\Phi_n^f(h \cdot \alpha) = \Phi_n(h) \cdot \Phi_n^f(k)$, for each $h \in C^*(R_Y^n)$.*

Proof. (1) Compute that

$$\begin{aligned}
\Phi_n^f(\alpha)\cdot\Phi_n(k)|_{(\mathbf{y},\mathbf{x})} &= \sum_{\mathbf{x}'\in[\mathbf{x}]_{n+1}} \Phi_n^f(\alpha)|_{(\mathbf{y},\mathbf{x}')}\Phi_n(k)|_{(\mathbf{x}',\mathbf{x})} \\
&= \sum_{\mathbf{x}'\in[\mathbf{x}]_{n}} \Phi_n^f(\alpha)|_{(\mathbf{y},\mathbf{x}')}\Phi_n(k)|_{(\mathbf{x}',\mathbf{x})} \\
&= \begin{cases} \sum_{\mathbf{x}'\in[\mathbf{x}]_n} \alpha(\mathbf{y},\mathbf{x}')k(\mathbf{x}',\mathbf{x}) & \text{if } \mathbf{y}\in[f(\mathbf{x})]_n \\ 0 & \text{otherwise.} \end{cases} \\
&= \Phi_n^f(\alpha\cdot k)|_{(\mathbf{y},\mathbf{x})}
\end{aligned}$$

The proof of (2) is similar:

$$\begin{aligned}
\Phi_n(h)\cdot\Phi_n^f(\alpha)|_{(\mathbf{y},\mathbf{x})} &= \sum_{\mathbf{y}'\in[f(\mathbf{x})]_{n+1}} \Phi_n(h)|_{(\mathbf{y},\mathbf{y}')}\Phi_n^f(\alpha)|_{(\mathbf{y}',\mathbf{x})} \\
&= \sum_{\mathbf{y}'\in[f(\mathbf{x})]_{n}} \Phi_n(h)|_{(\mathbf{y},\mathbf{y}')}\Phi_n^f(\alpha)|_{(\mathbf{y}',\mathbf{x})} \\
&= \begin{cases} \sum_{\mathbf{y}'\in[f(\mathbf{x})]_n} h(\mathbf{y},\mathbf{y}')\alpha(\mathbf{y}',\mathbf{x}) & \text{if } \mathbf{y}\in[f(\mathbf{x})]_n \\ 0 & \text{otherwise.} \end{cases} \\
&= \Phi_n^f(h\cdot\alpha)|_{(\mathbf{y},\mathbf{x})}.
\end{aligned}$$

□

Definition 5.25. If $f : X \to Y$ is a map of posets such that for every $x'', x' \geq x$,

$$x'' \neq x' \quad \Rightarrow \quad f(x'') \neq f(x')$$

then f is called locally injective (on closed subsets).

Note: If $f : X \to Y$ is locally injective on closed subsets, then it is strictly monotone.

Theorem 5.26. *Let $f : X \to Y$ be a map of finite posets, and $n > 0$ a positive integer. Then*

$$\langle \Phi_n^f(\alpha), \Phi_n^f(\beta) \rangle = \Phi_n(\langle \alpha, \beta \rangle) \qquad \textit{for every } \alpha, \beta \in C^*(\Gamma_f^n)$$

if and only if $f : X \to Y$ is locally injective.

Proof. Suppose that f is locally injective on closed subsets. Compute

$$\begin{aligned}
\langle \Phi_n^f(\alpha), \Phi_n^f(\beta) \rangle|_{(\mathbf{x}',\mathbf{x})} &= \sum_{\mathbf{y} \in [f(\mathbf{x})]_{n+1}} \overline{\Phi_n^f(\alpha)}|_{(\mathbf{y},\mathbf{x}')} \Phi_n^f(\beta)|_{(\mathbf{y},\mathbf{x})} \\
&= \sum_{\mathbf{y} \in [f(\mathbf{x}')]_n \cap [f(\mathbf{x})]_n} \overline{\alpha(\mathbf{y},\mathbf{x}')} \beta(\mathbf{y},\mathbf{x}) \\
&= \begin{cases} \langle \alpha, \beta \rangle|_{(\mathbf{x}',\mathbf{x})} & \text{if } \mathbf{x}' \in [\mathbf{x}]_n. \\ 0 & \text{otherwise.} \end{cases} \\
&= \Phi_n(\langle \alpha, \beta \rangle)|_{(\mathbf{x}',\mathbf{x})}.
\end{aligned}$$

Indeed, the third equality follows from the local injectivity of f: since $\mathbf{x}'_{n+1} = \mathbf{x}_{n+1}$, either $\mathbf{x}'_n = \mathbf{x}_n$ or $f(\mathbf{x}')_n \neq f(\mathbf{x})_n$, so $[f(\mathbf{x}')]_n \cap [f(\mathbf{x})]_n = \emptyset$.

If f is not locally injective then there are points $x', x'' \geq x$ such that $x' \neq x''$ and $f(x') = f(x'') = y$. Let $\alpha = 1_{[y^n]} \times 1_{[(x')^n]}$ and $\beta = 1_{[y^n]} \times 1_{[(x'')^n]}$; let $\mathbf{x} = (x')^n x^\infty$, and let $\mathbf{x}' = (x'')^n x^\infty$. Then $(\mathbf{x}', \mathbf{x}) \notin R_X^n$, so $\Phi_n(\langle \alpha, \beta \rangle)|_{(\mathbf{x},\mathbf{x}')} = 0$. However, $(\mathbf{x}', \mathbf{x}) \in R_X^{n+1}$, and

$$\begin{aligned}
\langle \Phi_n^f(\alpha), \Phi_n^f(\beta) \rangle|_{(\mathbf{x}',\mathbf{x})} &= \sum_{\mathbf{y} \in [f(\mathbf{x})]_{n+1}} \overline{\Phi_n^f(\alpha)}|_{(\mathbf{y},\mathbf{x}')} \Phi_n(\beta)|_{(\mathbf{y},\mathbf{x})} \\
&= \sum_{\mathbf{y} \in [f(\mathbf{x})]_n \cap [f(\mathbf{x}')]_n} \overline{\alpha(\mathbf{y},\mathbf{x}')} \beta(\mathbf{y},\mathbf{x}) \\
&= \alpha(y^n f(x)^\infty, (x')^n x^\infty) \beta(y^n f(x)^\infty, (x'')^n x^\infty) = 1.
\end{aligned}$$

Thus, $\Phi_n(\langle\alpha,\beta\rangle) \neq \langle\Phi_n^f(\alpha),\Phi_n^f(\beta)\rangle$. □

Definition 5.27. If $f : X \to Y$ is a map of finite posets, define $C(f) := \varinjlim_n C^*(\Gamma_f^n)$, in the category of right $C(X_\infty)$-modules.

Corollary 5.28. *Let $f : X \to Y$ be a map of finite posets. Then*

1. *$C(f)$ is a $C^*(R_Y)$-$C^*(R_X)$-bimodule;*
2. *If $f : X \to Y$ is locally injective on closed subsets, then $C(f)$ is a pre-Hilbert $C^*(R_X)$-module.*

Definition 5.29. If $f : X \to Y$ is locally injective on closed subsets, define $C^*(f) := \overline{C(f)}$, the completion of $C(f)$ with respect to the norm induced from the $C^*(R_X)$-valued inner-product.

Theorem 5.30.

1. *If $f : X \to Y$ and $g : Y \to Z$ are two locally injective maps of finite posets, then*
$$C^*(g) \otimes_{C^*(R_Y)} C^*(f) \cong C^*(g \circ f).$$
2. *If $f : X \to Y$ is injective, then $\mathcal{A}_f \cong C^*(f)$ as Hilbert C^*-bimodules over $\mathcal{A}_Y$ and $\mathcal{A}_X$.*

Proof. (1) Define $\Xi_n : C^*(\Gamma_g^n) \otimes_{C^*(R_Y^n)} C^*(\Gamma_f^n) \to C^*(\Gamma_{g\circ f}^n)$ by

$$\Xi_n(\alpha\otimes\beta)|_{(\mathbf{z},\mathbf{x})} = \sum_{\mathbf{y}\in[f(\mathbf{x})]_n} \alpha(\mathbf{z},\mathbf{y})\beta(\mathbf{y},\mathbf{x}).$$

We check that Ξ_n is well-defined: if $h \in C^*(R_Y^n)$, then

$$\begin{aligned}\Xi_n(\alpha\cdot h\otimes\beta)|_{(\mathbf{z},\mathbf{x})} &= \sum_{\mathbf{y}\in[f(\mathbf{x})]_n} (\alpha\cdot h)|_{(\mathbf{z},\mathbf{y})}\beta(\mathbf{y},\mathbf{x}) = \sum_{\mathbf{y}\in[f(\mathbf{x})]_n}\sum_{\mathbf{y}'\in[\mathbf{y}]_n} \alpha(\mathbf{z},\mathbf{y}')h(\mathbf{y}',\mathbf{y})\beta(\mathbf{y},\mathbf{x}) \\ &= \sum_{\mathbf{y}'\in[f(\mathbf{x})]_n}\sum_{\mathbf{y}\in[f(\mathbf{x})]_n} \alpha(\mathbf{z},\mathbf{y}')h(\mathbf{y}',\mathbf{y})\beta(\mathbf{y},\mathbf{x}) = \sum_{\mathbf{y}'\in[f(\mathbf{x})]_n} \alpha(\mathbf{z},\mathbf{y}')(h\cdot\beta)|_{(\mathbf{y}',\mathbf{x})} \\ &= \Xi_n(\alpha\otimes h\cdot\beta)|_{(\mathbf{z},\mathbf{x})}.\end{aligned}$$

To show that Ξ_n is injective, it is enough to show that it preserves the inner-product (as it will then be an isometry). Compute

$$\begin{aligned}
\langle \Xi_n(\alpha' \otimes \beta'), &\Xi_n(\alpha \otimes \beta)\rangle|_{(\mathbf{x}',\mathbf{x})} \\
&= \sum_{\mathbf{z}\in[g\circ f(\mathbf{x})]_n} \overline{\Xi_n(\alpha' \otimes \beta')|_{(\mathbf{z},\mathbf{x}')}}\Xi_n(\alpha \otimes \beta)|_{(\mathbf{z},\mathbf{x})} \\
&= \sum_{\mathbf{z}\in[g\circ f(\mathbf{x})]_n} \overline{\left(\sum_{\mathbf{y}'\in[f(\mathbf{x}')]_n} \alpha'(\mathbf{z},\mathbf{y}')\beta'(\mathbf{y}',\mathbf{x}')\right)} \left(\sum_{\mathbf{y}\in[f(\mathbf{x})]_n} \alpha(\mathbf{z},\mathbf{y})\beta(\mathbf{y},\mathbf{x})\right) \\
&= \sum_{\mathbf{z}\in[g\circ f(\mathbf{x})]_n} \sum_{\substack{\mathbf{y}\in[f(\mathbf{x})]_n \\ \mathbf{y}'\in[f(\mathbf{x}')]_n}} \overline{\alpha'(\mathbf{z},\mathbf{y}')\beta'(\mathbf{y}',\mathbf{x}')}\alpha(\mathbf{z},\mathbf{y})\beta(\mathbf{y},\mathbf{x});
\end{aligned}$$

$$\begin{aligned}
\langle \alpha' \otimes \beta', \alpha \otimes \beta\rangle|_{(\mathbf{x}',\mathbf{x})} &= \langle \beta', \langle \alpha', \alpha\rangle \cdot \beta\rangle|_{(\mathbf{x}',\mathbf{x})} \\
&= \sum_{\mathbf{y}'\in[f(\mathbf{x})]_n} \overline{\beta'(\mathbf{y}',\mathbf{x}')}\left(\langle \alpha', \alpha\rangle \cdot \beta\right)|_{(\mathbf{y}',\mathbf{x})} \\
&= \sum_{\mathbf{y}'\in[f(\mathbf{x})]_n} \overline{\beta'(\mathbf{y}',\mathbf{x}')}\left(\sum_{\mathbf{y}\in[\mathbf{y}']_n} \langle \alpha', \alpha\rangle|_{(\mathbf{y}',\mathbf{y})}\beta(\mathbf{y},\mathbf{x})\right) \\
&= \sum_{\mathbf{y}'\in[f(\mathbf{x})]_n} \overline{\beta'(\mathbf{y}',\mathbf{x}')}\left(\sum_{\mathbf{y}\in[\mathbf{y}']_n} \left(\sum_{\mathbf{z}\in R_Z^n(g(\mathbf{y}'))} \overline{\alpha'(\mathbf{z},\mathbf{y}')}\alpha(\mathbf{z},\mathbf{y})\right)\beta(\mathbf{y},\mathbf{x})\right) \\
&= \sum_{\mathbf{y},\mathbf{y}'\in[f(\mathbf{x})]} \sum_{\mathbf{z}\in[g(\mathbf{y}')]_n} \overline{\alpha'(\mathbf{z},\mathbf{y}')\beta'(\mathbf{y}',\mathbf{x}')}\alpha(\mathbf{z},\mathbf{y})\beta(\mathbf{y},\mathbf{x}).
\end{aligned}$$

Since $\mathbf{x}' \in [\mathbf{x}]_n$, it follows that $f(\mathbf{x}') \in [f(\mathbf{x})]_n$, i.e. $[f(\mathbf{x}')]_n = [f(\mathbf{x})]_n$. Thus, the two sums are equal, and Ξ_n is an isometry.

Since Ξ_n is an isometry, it is in particular bi-Lipschitz; thus, to prove that it is surjective it is enough that the image contain a dense set. The image of Ξ_n contains e.g. $\{1_{[p]} \times 1_{[a]} \mid p \in Z_m, a \in X_m, t(g \circ f(a)) = t(p)\}$, since

$$\Xi_n((1_{[p]} \times 1_{[f(a)]}) \otimes (1_{[f(a)]} \times 1_{[a]})) = 1_{[p]} \times 1_{[a]}.$$

Thus the image contains $C(g \circ f)$, which is dense in $C^*(g \circ f)$.

Finally, we show that Ξ commutes with the maps directing the systems $C^*(g \circ f)$ and $C^*(g) \otimes_{C^*(R_Y)} C^*(f)$, i.e.

$$\Xi_{n+1}(\Phi_n^g(\alpha) \otimes \Phi_n^f(\beta)) = \Phi_n^{g\circ f} \circ \Xi_n(\alpha \otimes \beta);$$

indeed, for $(\mathbf{z}, \mathbf{x}) \in \Gamma_{g\circ f}^{n+1}$,

$$\begin{aligned}
\Xi_{n+1}(\Phi_n^g(\alpha) \otimes \Phi_n^f(\beta))|_{(\mathbf{z},\mathbf{x})} &= \sum_{\mathbf{y}\in[f(\mathbf{x})]_{n+1}} \Phi_n^g(\alpha)|_{(\mathbf{z},\mathbf{y})}\Phi_n^f(\beta)|_{(\mathbf{y},\mathbf{x})} \\
&= \sum_{\mathbf{y}\in[f(\mathbf{x})]_{n}} \alpha(\mathbf{z},\mathbf{y})\beta(\mathbf{y},\mathbf{x}) \\
&= \begin{cases} \Xi_n(\alpha \otimes \beta)|_{(\mathbf{z},\mathbf{x})} & \text{if } g \circ f(\mathbf{x})_{n+1} = \mathbf{z}_{n+1}, \\ 0 & \text{otherwise;} \end{cases} \\
&= \Phi_n^{g\circ f}(\Xi_n(\alpha \otimes \beta))|_{(\mathbf{z},\mathbf{x})}.
\end{aligned}$$

It follows that the isomorphisms Ξ_n induce an isomorphism $C^*(g) \otimes_{C^*(R_Y)} C^*(f) \to C^*(g \circ f)$.

(2) Since $\Psi_n^X : \mathcal{A}_X^n \to C^*(R_X^n)$ and $\Psi_n^Y : \mathcal{A}_Y^n \to C^*(R_Y^n)$ are algebra maps, by restriction $C(\Gamma_f^n)$ is an $\mathcal{A}_Y$-$\mathcal{A}_X$-bimodule. Let $\Psi_n^f : \mathcal{A}_f^n \to C(\Gamma_f^n)$ be the linear map defined by $\Psi_n^f(pf(a)^* \circ \iota_f \circ aa^*) = 1_{[p]} \times 1_{[a]}$. Then Ψ_n^f is an $\mathcal{A}_Y^n$-$\mathcal{A}_X^n$-bimodule morphism. We check that Ψ_n^f is an isometry: for $a, b \in X_n$, $p, q \in Y_n$ and $f(t(a)) = t(p)$, $f(t(b)) = t(q)$,

$$\begin{aligned}
\langle \Psi_n^f(pf(a)^* \circ \iota_f \circ aa^*), \Psi_n^f(qf(b)^* \circ \iota_f \circ bb^*)\rangle_{C(\Gamma_X^n)} &= \langle 1_{[p]} \times 1_{[a]}, 1_{[q]} \times 1_{[b]}\rangle \\
&= \delta_{pq} 1_{[a]} \times 1_{[b]} = \Psi_n^X(\delta_{pq} ab^*) \\
&= \Psi_n^X(\langle pf(a)^* \circ \iota_f \circ aa^*), qf(b)^* \circ \iota_f \circ bb^*\rangle_{\mathcal{A}_X^n}).
\end{aligned}$$

(Note that $ab^* \in \mathcal{A}_X^n$ since f is injective.) Since $\Psi_n^X : \mathcal{A}_X^n \to C(\Gamma_X^n)$ is injective, the map $\Psi_n : \mathcal{A}_f \to C(\Gamma_f^n)$ is an isometry.

Next, we check that the following square commutes:

$$\begin{tikzcd}
\mathcal{A}_f^n \arrow[r] \arrow[d, "\Psi_n"'] & \mathcal{A}_f^{n+1} \arrow[d, "\Psi_{n+1}"] \\
C^*(\Gamma_f^n) \arrow[r, "\Phi_n"'] & C^*(\Gamma_f^{n+1})
\end{tikzcd}$$

As in the previous case where $f = \mathrm{id}_X$, the top map is given by the Cuntz-Krieger relations, and so we have

$$\begin{tikzcd}
pf(a)^* \circ \iota_f \circ aa^* \arrow[r, mapsto] \arrow[d, mapsto] & \sum_{x \leq t(a)} (pf(x)) f(ax)^* \circ \iota_f \circ (ax)(ax)^* \arrow[d, mapsto] \\
1_{[p]} \times 1_{[a]} \arrow[r, mapsto] & \sum_{x \leq t(a)} 1_{[pf(x)]} \times 1_{[ax]}
\end{tikzcd}$$

The general result follows from taking linear combinations of elements in $\mathcal{A}_f^n$.

Thus, the maps Ψ_n induce a well-defined injective linear map Ψ of $\mathcal{A}_Y$-$\mathcal{A}_X$-bimodules. The image of Ψ_n contains

$$\{1_{[p]} \times 1_{[a]} \mid a \in X_m,\, p \in X_m, \text{ and } f(t(a)) = t(p)\},$$

which separates points in $\Gamma_f = \varinjlim_n \Gamma_f^n$, so the image must contain $C(f)$. Since Ψ preserves the inner-product and $\mathcal{A}_f$ is complete, the image of Ψ is complete and is thus $C^*(f)$. $\square$

5.8 Conditional Expectation

A conditional expectation is a positive linear retraction onto a sub-algebra which is a bimodule morphism.

Definition 5.31. Let $E_n : C(X_\infty) \to C(X_\infty)$ be the $\mathbb{C}$-linear map $E_n(\phi) = \langle 1_{X_\infty}, \phi \rangle_{M^n}$, i.e.

$$E_n(\phi)|_{\mathbf{x}'} = \frac{1}{[\mathbf{x}']_n} \sum_{\mathbf{x} \in [\mathbf{x}']_n} \phi(\mathbf{x}).$$

Lemma 5.32. *Let X be a finite poset and $n > 0$ a positive integer. Then $E_n : C(X_\infty) \to C(X_\infty; R_n)$ is a conditional expectation. Furthermore,*

1. *For every $m \geq n$, $E_m \circ E_n = E_n \circ E_m = E_m$;*

2. *For every $\phi, \psi' \in C(X_\infty)$, $E_{n+1}(\langle \phi, \psi' \rangle_{M^n}) = \langle \phi, \psi' \rangle_{M^{n+1}}$.*

Proof. If $\phi \geq 0$, then $\phi = \overline{\psi}\psi$, so $E_n(\phi) = \langle 1, \overline{\psi}\psi\rangle_{M^n} = \langle \psi, \psi\rangle_{M^n} \geq 0$. If $\alpha \in C(X_\infty; R_X^n)$ and $\phi \in C(X_\infty)$, then

$$E_n(\phi\alpha) = \langle 1, \phi\alpha\rangle_{M^n} = \langle 1, \phi\rangle_{M^n}\alpha = E_n(\phi)\alpha;$$

it is a bimodule morphism since $C(X_\infty)$ is commutative.

(1) We will show that $E_{n+1}E_n = E_nE_{n+1} = E_{n+1}$. The second equality follows from $C(X_\infty; R_X^{n+1}) \subseteq C(X_\infty; R_X^n)$. For the first equality, compute for $\phi \in C(X_\infty)$ and $\mathbf{x}' \in X_\infty$,

$$\begin{aligned} E_{n+1} \circ E_n(\phi)|_{\mathbf{x}'} &= \frac{1}{\#[\mathbf{x}']_{n+1}} \sum_{\mathbf{x} \in [\mathbf{x}']_{n+1}} E_n(\phi)|_{\mathbf{x}} \\ &= \frac{1}{\#[\mathbf{x}']_{n+1}} \sum_{\mathbf{x} \in [\mathbf{x}']_{n+1}} \sum_{\mathbf{x}'' \in [\mathbf{x}]_n} \frac{1}{\#[\mathbf{x}]_n}\phi(\mathbf{x}'') \\ &= \frac{1}{\#[\mathbf{x}']_{n+1}} \sum_{\mathbf{x}'' \in [\mathbf{x}']_{n+1}} \phi(\mathbf{x}'') = E_{n+1}(\phi)|_{\mathbf{x}'} \end{aligned}$$

indeed, $\phi(\mathbf{x}'')$ appears precisely $\#[\mathbf{x}'']_n$ times in the double summation. The result for $m \geq n$ is then proved by induction (the case $m = n$ follows since E_n is a retraction).

To prove (3), note that $\langle \phi, \psi'\rangle_{M^n} = \langle 1, \overline{\phi}\psi'\rangle_{M^n} = E_n(\overline{\phi}\psi')$, so

$$E_{n+1}(\langle \phi, \psi'\rangle_{M^n}) = E_{n+1} \circ E_n(\overline{\phi}\psi') = E_{n+1}(\overline{\phi}\psi') = \langle \phi, \psi'\rangle_{M^{n+1}}. \qquad \square$$

Definition 5.33. Let $f : X \to Y$ be a map of finite posets, and $n > 0$ a positive integer. Define $e_n^f \in C(\Gamma_f^n)$ via

$$e_n^f(\mathbf{y}, \mathbf{x}) = \rho_n(\mathbf{y})\rho_n(\mathbf{x}).$$

Theorem 5.34. *Let $f : X \to Y$ be a map of finite posets, and $n > 0$ a positive integer. Then*

$$C^*(\Gamma_f^n) = \overline{span}\{\psi\, e_n^f\, \phi \mid \psi \in C(Y_\infty),\, \phi \in C(X_\infty)\}.$$

Proof. Since $\{1_{[p]} \times 1_{[a]} \mid p \in Y_m,\, a \in X_m,\, t(p) = f(t(a)),\, m > 0\}$ separates points in Γ_f^n, it is enough to show that every such function $1_{[p]} \times 1_{[a]}$ has the form $\psi\, e_n^f\, \phi$. Compute that for

$(\mathbf{y}', \mathbf{x}') \in \Gamma_f^n$,

$$\left(\rho_Y^{-1/2} 1_{[p]}\right) \cdot e_n^f \cdot \left(\rho_X^{-1/2} 1_{[a]}\right) |_{(\mathbf{y}',\mathbf{x}')} = \rho_Y^{-1/2}(\mathbf{y}') 1_{[p]}(\mathbf{y}') e_n^f(\mathbf{y}', \mathbf{x}') 1_{[a]}(\mathbf{x}') \rho_X^{-1/2}(\mathbf{x}')$$
$$= (1_{[p]} \times 1_{[a]})|_{(\mathbf{y}',\mathbf{x}')}.$$

Thus $C^*(\Gamma_f^n) \subseteq \overline{span}\{\psi e_n^f \phi \mid \psi \in C(Y_\infty),\ \phi \in C(X_\infty)\}$. ☐

Theorem 5.35. *Let $f : X \to Y$ be a map of finite posets, $n > 0$ a positive integer.*

1. $e_n^f \cdot \zeta_n^X = e_n^f$, *and* $\zeta_n^Y \cdot e_n^f = e_n^f$;
2. $\langle e_n^f, e_n^f \rangle = \zeta_n^X \in C(R_X^n)$;
3. $\langle \psi e_n^f \phi, \psi' e_n^f \phi' \rangle_{C^*(R_X^n)} = f^*(E_n(\overline{\psi}\psi')) \cdot \overline{\phi} \zeta_n^X \phi'$.

Proof. The first two are elementary and their proofs are omitted. To prove (3), first compute

$$\langle e_n^f, \psi e_n^f \rangle|_{(\mathbf{x}',\mathbf{x})} = \sum_{\mathbf{y} \in [f(\mathbf{x})]_n} e_n^f(\mathbf{y}, \mathbf{x}') \psi(\mathbf{y}) e_n^f(\mathbf{y}, \mathbf{x}) = \zeta_n^X(\mathbf{x}', \mathbf{x}) \frac{1}{\#[f(\mathbf{x})]_n} \sum_{\mathbf{y} \in [f(\mathbf{x})]_n} \psi(\mathbf{y})$$
$$= f^*(E_n^Y(\psi)) \zeta_n|_{(\mathbf{x}',\mathbf{x})}.$$

Note that since $E_n(\overline{\psi}\psi') \in C(Y_\infty; R_Y^n)$, the pull-back $f^*(E_n(\overline{\psi}\psi')) \in C(X_\infty; R_X^n)$, and so commutes with ζ_n^X. Using properties of the $C^*(R_X^n)$-valued inner-product, we compute that

$$\langle \psi e_n^f \phi, \psi' e_n^f \phi' \rangle = \overline{\phi} \langle e_n^f, \overline{\psi} \psi' e_n^f \rangle \phi' = f^*(E_n(\overline{\psi}\psi')) \overline{\phi} \zeta_n^X \phi'.$$ ☐

5.9 Summary

In the previous chapters we constructed for each injective map $f : X \to Y$ of finite posets a concrete Hilbert C^*-bimodule $\mathcal{A}_f$ over $\mathcal{A}_Y$ and $\mathcal{A}_X$. Furthermore, the association $f \rightsquigarrow \mathcal{A}_f$ is functorial. The construction given cannot be extended to a larger class of morphisms.

In this chapter we have described a groupoid R_X and its groupoid C^*-algebra $C^*(R_X)$; this convolution algebra is isomorphic to $\mathcal{A}_X$. For each morphism $f : X \to Y$ of finite posets that is locally injective on closed subsets, there is a Hilbert C^*-bimodule $C^*(f)$ over

$C^*(R_Y)$ and $C^*(R_X)$. Furthermore, the association $f \rightsquigarrow C^*(f)$ is functorial, and the given construction cannot be extended to a larger class of morphisms. When the map f is injective, the resulting bimodule $C^*(f)$ is isomorphic to $\mathcal{A}_f$.

Appendix A

TOPOLOGICAL SPACES AND POSETS

Elementary Definitions

Definition A.1. A topological space (X, τ) is a set X, together with a topology $\tau \subseteq \mathcal{P}(X)$. The elements of τ are called *open* subsets of X, and τ satisfies the following properties:

1. $\emptyset$ and X are open subsets;
2. The arbitrary union of open subsets is open;
3. A finite intersection of open subsets is open.

A subset $C \subseteq X$ is called closed if the complement $X \setminus C$ is open. X is called a *discrete* space if $\tau = \mathcal{P}(X)$, or equivalently, if every singleton is open. We say that X has the indiscrete topology if $\tau = \{X, \emptyset\}$. These are the maximal and minimal topologies on X.

A morphism $f : X \to Y$ between topological spaces is a function that satisfies the following property: for every open subset $U \subseteq Y$, $f^{-1}(U)$ is an open subset of X. Such morphisms are called *continuous*. The category of topological spaces is denoted Top, and the full subcategory of finite topological spaces is denoted FTop.

Definition A.2. Let (X, τ) be a topological space.

- X is called T_0 if for each pair (x, y) of distinct elements of X, there is an open subset containing x but not y, or there is an open subset containing y but not x;
- X is called T_1 if for each pair (x, y) of distinct elements of X, there is an open subset containing x but not y;

- X is called T_2 or Hausdorff if for each pair (x, y) of distinct elements of X, there are open subsets containing x and y that are disjoint.

It is obvious that $T_2 \Rightarrow T_1 \Rightarrow T_0$. This following lemma partially motivates our study of finite T_0-spaces.

Lemma A.3. *Let X be a T_1 space.*

1. *Every finite subset of X is closed;*
2. *If X is finite, then X is discrete.*

Proof. (1) Consider a singleton $\{x\} \subseteq X$. Since X is T_1, for each $y \in X \setminus \{x\}$ there is an open set U_y that contains y but not x. Then the union $U = \bigcup_{y \in Y \setminus \{x\}} U_y = X \setminus \{x\}$ is an open set. So $\{x\}$ is closed. Since the finite union of closed sets is closed, the result is proved.

(2) Since X is finite, every subset of X is the complement of a finite subset, and thus every subset is open. □

In particular, if some finite set X is not discrete then some point $\{x\}$ is not closed; if $y \in \overline{x}$, then the constant sequence $s_n = x$ converges to $y \in X$.

Definition A.4. Let X be a set.

1. A *preorder* on X is a reflexive transitive relation on X.
2. A *partial order* on X is an anti-symmetric preorder.

A morphism of preordered sets is a *monotone* function, i.e. a function $f : X \to Y$ such that $x \leq x' \implies f(x) \leq f(x')$. The category of preordered sets is denoted Pre; the full sub-category of finite preordered sets is denoted FPre. If $(X, \leq)$ is a preorder, we write $x < y$ to mean that $x \leq y$ and $x \neq y$. One says that y covers x, denoted $x \lessdot y$, if $x < y$ and $x < z < y$ implies $x = z$ or $y = z$. One also says that x is an *immediate predecessor* of y.

Associated to every partially ordered set $(X, \leq)$ there is a directed graph called the *Hasse diagram* of X. The set of vertices is the set X, and there is a (directed) edge $x \to y$ if $x \lessdot y$.

A.1 *The isomorphism* $\mathsf{FTop} \cong \mathsf{FPre}$

Definition A.5. Given a topological space $(X, \mathcal{T})$, the *associated preorder* is $(X, \leq)$, where

$$x \leq y \qquad \text{if and only if} \qquad y \in \overline{\{x\}}.$$

Proposition A.6. *The association $(X, \mathcal{T}) \rightsquigarrow (X, \leq)$ is a functor from the category of topological spaces to the category of pre-ordered sets.*

Proof. If $f : X \to Y$ is a continuous maps, we check that the underlying set map $f : X \to Y$ is monotone. Suppose then that $x \leq x'$; by definition, this means that the constant sequence $s_n = x'$ converges to x. Since f is continuous, we have that $f(s_n) = f(x')$ converges to $f(x)$. Thus $f(x) \leq f(x')$ in Y, as required. □

Definition A.7. Given a preordered set $(X, \leq)$, the *associated topological space* is $(X, \mathcal{T})$ where the topology $\mathcal{T}$ is generated by the basis of open sets

$$U_x = \{x' \in X \mid x' \leq x\}.$$

In the associated space, U_x is the minimal open subset containing x; the topology is characterized by

$$x \leq y \iff U_x \subset U_y.$$

Proposition A.8. *The association $(X, \leq) \rightsquigarrow (X, \mathcal{T})$ sending a preorder to its associated topological space is a functor.*

Proof. Suppose that $f : X \to Y$ is a monotone map. We check that the underlying set map is continuous, by checking that $f^{-1}(U_y)$ is open in X for every y in Y. Indeed, we have

$$f^{-1}(U_y) = \bigcup_{f(x) \leq y} U_x. \qquad \square$$

Observe that the associated space to any preorder is a topology with minimal neighborhoods. It is equivalent to require that the arbitrary intersection of open sets is open. A topological space with this property is called an *Alexandroff space*.

Theorem A.9.

1. *The category of Alexandroff spaces is isomorphic to the category of preordered sets.*
2. *This isomorphism restricts to an isomorphism from the category of T_0 topological spaces to the category of posets.*

Proof. 1. If X is Alexandroff, the associated preorder is characterized by $x \leq y$ iff $U_x \subseteq U_y$, where U_x is the minimal open set containing x. Then the associated space of this preordered set has a basis of open sets given by $U_x = \{y \mid U_y \subseteq U_x\} = \{y \mid y \leq x\}$. Beginning instead with a preordered set X, the associated space is Alexandroff and has a basis $U_x = \{y \mid y \leq x\}$. Its associated preorder is given by $x \leq' x'$ iff $U_x \subseteq U_{x'}$. But

$$\begin{aligned} x \leq' x' &\Leftrightarrow U_x \subseteq U_{x'} \\ &\Leftrightarrow y \leq x \text{ implies } y \leq x' \\ &\Leftrightarrow x \leq x'. \end{aligned}$$

Thus the preorder $\leq'$ is the same as $\leq$.

2. Under the isomorphism, the anti-symmetry property $x \leq y$ and $y \leq x \implies x = y$ is equivalent to the property $U_x = U_y \implies x = y$. This is the T_0 condition for an Alexandroff space. □

For the remainder of this thesis, we will often conflate preordered sets with Alexandroff spaces using the isomorphism of categories. The previous theorem restricts to the subcategories of finite preorders and finite topological spaces as the isomorphism is the identity on the underlying sets.

A.2 The Kolmogorov Quotient

Given an arbitrary topological space X, there is an equivalence relation defined by $x \sim y$ if every open set containing x also contains y and conversely. The quotient space $\widetilde{X} := X/_{\sim}$ is

T_0 and is often called the *Kolmogorov quotient* of X. The inclusion functor from the (full) subcategory of T_0-spaces to the category of topological spaces has the Kolmogorov quotient as a left-adjoint. This persists when one restricts to the subcategory of Alexandroff spaces. Thus, Poset is a reflective subcategory of Pre.

Theorem A.10. *[14, Thm 4] The quotient map $q : X \to \widetilde{X}$ is a natural homotopy equivalence.*

Proof. Choosing a representative from each equivalence class, define a function $s : \widetilde{X} \to X$ such that $qs = \mathrm{id}_{\widetilde{X}}$. Then q is a homotopy equivalence if $sq : X \to X$ is homotopic to id_X.

Define $H : X \times I \to X$ by $H(x, 0) = sq(x)$ and $H(x, t) = x$ for t in $(0, 1]$. Then $H^{-1}(U) = U \times I$ is open for every open subset $U \subset X$; thus H is continuous and is therefore a homotopy from sq to id_X. □

Definition A.11. Given a topological space X, we define the poset $\mathcal{O}p(X)$ as the collection of open subsets of X ordered by inclusion.

Remark A.12. The poset $\mathcal{O}p(X)$ is a distributive lattice, and the association $X \rightsquigarrow \mathcal{O}p(X)$ is a duality between the category FPos and the category of finite distributive lattices (with so-called $\{0, 1\}$-homomorphisms). This is a special case of Birkhoff's representation theorem.

Proposition A.13. *If X and Y are Alexandroff spaces, then $\mathcal{O}p(X) \cong \mathcal{O}p(Y)$ if and only if $\widetilde{X} \cong \widetilde{Y}$.*

The finite T_0 spaces are the simplest non-trivial examples of non-Hausdorff spaces. Because of this proposition, sheaves on X are equivalent to sheaves on $\widetilde{X}$. From the point-of-view of sheaves, we lose nothing by restricting to finite posets rather than considering finite pre-ordered sets. The following theorem compares topological and order-theoretic properties of a function.

Theorem A.14. *Let $f : X \to Y$ be a function between finite T_0 topological spaces.*

1. *f is continuous if and only if $x \leq y \Rightarrow f(x) \leq f(y)$;*

2. *f is a quotient map onto its image if and only if $x \le y \Leftrightarrow f(x) \le f(y)$;*

3. *f is closed if and only if $y \ge f(x) \Rightarrow y = f(x')$ for some $x' \ge x$;*

4. *f is open if and only if $y \le f(x) \Rightarrow y = f(x')$ for some $x' \le x$;*

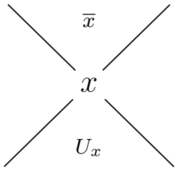
5. *f is proper if and only if f is continuous and closed.*

Opposites

Given an Alexandroff space X, one may interchange "open" and "closed" thereby sending X to X^{op}. This corresponds to sending a preordered set $(X, \le)$ to its opposite $X^\circ = (X, \ge)$. Thus, there are two possibilities for the isomorphism of categories; our choice agrees with that of McCord, Landi and Sorkin, i.e. $x \le y \iff U_x \subset U_y$. The opposite convention is used by S. Ladkani, and is succinctly characterized by

$$x \le y \iff x \in \overline{\{y\}}.$$

If X is a poset then our choice has the property that $X \hookrightarrow \mathcal{O}p(X) : x \to U_x$ represents X as a(n initial) subposet of $\mathcal{O}p(X)$. The following crude diagram shows how the open and closed sets U_x and $\overline{x}$ fit into the Hasse diagram:



If X and Y are Alexandroff spaces, then the cartesian product $X \times Y$ with the product topology is the categorical product. If $(X, \le)$ and $(Y, \le)$ are the associated preordered sets, then the categorical product is $X \times Y$ together with the the order

$$(x, y) \le (x', y') \quad \iff \quad x \le x' \text{ and } y \le y'.$$

It follows from the isomorphism of categories that this is the preorder associated to the topological space $X \times Y$.

A.3 Preordered sets as small categories

A preordered set $(X, \leq)$ is the same data as a *small category* $\mathcal{C}$ whose hom-sets consist of at most one element. The correspondence is given by

$$x \leq y \quad \text{if and only if} \quad \mathrm{Hom}_{\mathcal{C}}(x, y) \neq \emptyset.$$

The reflexive and transitive properties of $\leq$ correspond to the identity and composability properties of a category.

Using this language, a map of preordered sets $f : X \to Y$ is the data of a functor between the small categories X and Y. Given two maps $f, g : X \to Y$, a natural transformation $\eta : f \to g$ exists if and only if $f(x) \leq g(x)$ for every $x \in X$. We write $f \leq g$ in this case; this relation is reflexive and transitive, defining a preorder on $\mathrm{Hom}_{\mathsf{Pre}}(X, Y)$. If Y is a poset, then the relation on $\mathrm{Hom}_{\mathsf{Pre}}(X, Y)$ is anti-symmetric and so $\mathrm{Hom}_{\mathsf{Pre}}(X, Y)$ is a poset.

Proposition A.15. *For every triple of preordered sets X, Y, and Z, the map Φ of preordered sets*

$$\Phi : \mathrm{Hom}_{\mathsf{Pre}}(X \times Y, Z) \to \mathrm{Hom}_{\mathsf{Pre}}(X, \mathrm{Hom}_{\mathsf{Pre}}(Y, Z))$$

defined by $\Phi(g)(x) = g(x, -)$ is an isomorphism with inverse

$$\psi : \mathrm{Hom}_{\mathsf{Pre}}(X, \mathrm{Hom}_{\mathsf{Pre}}(Y, Z)) \to \mathrm{Hom}_{\mathsf{Pre}}(X \times Y, Z)$$

defined by $\psi(f)(x, y) = f(x)(y)$.

This shows that Pre has an internal hom-functor $\mathrm{Hom}_{\mathsf{Pre}}(X, Y)$; thus, Pre is a closed symmetric monoidal category. The same holds for Pos.

Combinatorial Terminology

A preordered set X is *totally ordered* if for each pair of distinct elements x, y in X, either $x \leq y$ or $y \leq x$, but not both. Any subset $S \subset X$ of a preordered set is again pre-ordered by restricting the relation.

Definition A.16. Let X be a pre-ordered set.

1. A *chain* C is a non-empty subset of X that is totally ordered.
2. A *quasi-chain* of length n is a finite sequence $(x_i)_{i=0}^n$ such that

$$x_i \geq x_{i+1} \qquad \text{for } 0 \leq i < n.$$

Definition A.17. Let X be a pre-ordered set, and define $[n] = \{0, \ldots, n\} \subseteq \mathbb{N}$ as the totally ordered set on $n+1$ elements. Define

$$X_n := \mathrm{Hom}_{\mathsf{Pre}}([n]^\circ, X).$$

Then X_n consists of quasi-chains of length n, i.e. non-increasing sequences $x_0 x_1 \ldots x_n$ in X. For a map $f : X \to Y$ of pre-ordered sets define

$$f_n = \mathrm{Hom}_{\mathsf{Pre}}([n]^\circ, f);$$

then $f_n : X_n \to Y_n$ given by $f(x_i) = (f(x_i))$.

Definition A.18. Given a preordered set X, we define ΔX to be the following *abstract simplicial complex.* The set of vertices is X and the set of simplices is the set of chains in X.

Example A.19. $\Delta[n]$ is the standard n-simplex.

Appendix B

C^*-ALGEBRAS, CATEGORIES, AND FUNCTORS

Definition B.1. A C^*-algebra $\mathcal{A}$ is a Banach *-algebra satisfying the C^*-identity: $\|a^*a\| = \|a\|^2$ for every $a \in \mathcal{A}$

A *-homomorphism of C^*-algebras is a continuous algebra homomorphism that intertwines the involutions. Denote by $C^*\text{-}\mathsf{Alg}$ the category whose objects are C^*-algebras and whose morphisms are *-homomorphisms. Let $\mathsf{C}C^*\text{-}\mathsf{Alg}$ denote the full subcategory of commutative C^*-algebras. A classical theorem of Gelfand states that the functor $X \rightsquigarrow C_0(X)$ is a duality from locally compact Hausdorff (LCH) spaces to $\mathsf{C}C^*\text{-}\mathsf{Alg}$.

If X, Y are LCH spaces and $f : X \to Y$ is a proper continuous map then the pull-back $f^* : C_0(Y) \to C_0(X)$ is a *-homomorphism of C^*-algebras. If $g : Y \to Z$ is another proper continuous map of LCH spaces then $(g \circ f)^* = g^* \circ f^* : C_0(Z) \to C_0(X)$. Unfortunately, this does not generalize well to noncommutative C^*-algebras as there are typically too few *-homomorphisms. Instead, we observe that $C_0(X)$ is a $C_0(Y)$-$C_0(X)$-bimodule via f^*, and

$${}_{(g\circ f)^*}C_0(X) \cong {}_{g^*}C_0(Y) \otimes_{C_0(Y)} {}_{f^*}C_0(X).$$

Thus, we will consider various subcategories of Bimod, the category whose objects are rings and whose morphisms are isomorphism classes of bimodules. In particular, a morphism ${}_BM_A \in \mathsf{Bimod}(A, B)$ is a B-A-bimodule. The composition of two morphisms is defined by tensor product:

$${}_BM_A \in \mathsf{Bimod}(A,B),\ {}_CN_B \in \mathsf{Bimod}(B,C) \qquad \Rightarrow \qquad N \circ M = N \otimes_B M \in \mathsf{Bimod}(A,C).$$

Define the subcategory $\mathcal{C}^*$, whose objects are C^*-algebras and whose morphisms are isoclasses of Hilbert C^*-bimodules.

Another theorem of Gelfand states that every C^*-algebra is isometrically *-isomorphic to a closed *-subalgebra $B \subseteq \mathsf{B}(\mathcal{H})$ of bounded operators on a Hilbert space. Such an algebra B is called a concrete C^*-algebra.

B.1 Background: Hilbert C^*-modules

This material is taken from [12, Ch. 1]. Let B be a C*-algebra. A B-valued inner product on a complex vector space V is a map $\langle -, - \rangle : V \times V \to B$ such that

1. $\langle x, \lambda y + \lambda' y' \rangle = \lambda \langle x, y \rangle + \lambda' \langle x, y' \rangle$ for all $x, y, y' \in V$ and $\lambda, \lambda' \in \mathbb{C}$;
2. $\langle x, y \rangle^* = \langle y, x \rangle$ for all $x, y \in V$;
3. $\langle x, x \rangle \geq 0$ for all $x \in V$;[1]
4. $\langle x, x \rangle = 0$ if and only if $x = 0$.

Given such data, the function $\|x\|_V := \sqrt{\|\langle x, x \rangle\|}$ is a norm on V. We give V the topology induced by this norm. The inner product is a continuous function in each component.

We call V a Hilbert B-space if it is complete with respect to the norm $\| \cdot \|_V$. A Hilbert B-space $\mathcal{H}$ is a right Hilbert B-module, or a Hilbert C*-module over B, if $\langle h, h'b \rangle = \langle h, h' \rangle b$ for all $b \in B$ and all $h, h' \in \mathcal{H}$. For example, a right ideal I in B is a right Hilbert B-module with respect to the inner product $\langle x, y \rangle = x^* y$.

Given a concrete C^*-algebra B, there is a notion of a concrete Hilbert B-module.

Definition B.2. Let $B \subseteq \mathsf{B}(\mathcal{H})$ be a concrete C^*-algebra. Let $V \subseteq \mathsf{B}(\mathcal{H}, \mathcal{K})$ be a closed right B-submodule for some Hilbert space $\mathcal{K}$, such that

$$u, v \in V \qquad \Rightarrow \qquad u^* v \in B,$$

---

[1] An element $b \in B$ is positive, written $b \geq 0$, if $b = b^*$ and the spectrum of b is in $\mathbb{R}_{\geq 0}$.

where $u^* : \mathcal{K} \to \mathcal{H}$ is the classical adjoint of $u : \mathcal{H} \to \mathcal{K}$, i.e. the unique bounded linear map such that $\langle u^*(\chi), \psi \rangle = \langle \chi, u(\psi) \rangle$ for every $\chi \in \mathcal{K}$ and $\psi \in \mathcal{H}$. Then there is a B-valued inner-product defined by

$$\langle u, v \rangle = u^* v,$$

which makes V a right Hilbert B-module. Such a V is called a concrete Hilbert B-module.

It is an interesting fact that every Hilbert B-module arises in this way [15, Theorem 3.1, p. 373].

Lemma B.3. *If V is a right Hilbert B-module then the closure of the image of the B-valued inner-product*

$$J := \overline{span}\{\langle a, b \rangle_B \mid a, b \in V\}$$

is a closed two-sided ideal in B.

Proof. By definition, J is a closed subspace of B. We show it is closed under multiplication. If $u, v \in V$ and $a, b \in B$, the element $a\langle u, v \rangle b \in J$ since $a\langle u, v \rangle b = \langle u \cdot a^*, v \cdot b \rangle$. The result follows at once. □

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be right Hilbert B-modules. A linear map $T : \mathcal{H}_1 \to \mathcal{H}_2$ is adjointable if there is a linear operator $T^* : \mathcal{H}_2 \to \mathcal{H}_1$ such that $\langle \xi, T\eta \rangle = \langle T^*\xi, \eta \rangle$ for all $\xi \in \mathcal{H}_1$ and $\eta \in \mathcal{H}_2$. An adjointable operator is a B-module homomorphism because

$$\langle \xi, T(\eta b) \rangle = \langle T^*\xi, \eta b \rangle = \langle T^*\xi, \eta \rangle b = \langle \xi, T\eta \rangle b = \langle \xi, (T\eta) b \rangle$$

for all $\xi \in \mathcal{H}_1$ whence $T(\eta b) = (T\eta)b$ for all $\eta \in \mathcal{H}_2$ and all $b \in B$. An adjointable operator is continuous because its graph is closed. Indeed, if $\eta_m \to \eta$ and $T\eta_m \to \zeta$, then for every $\xi \in \mathcal{H}_1$

$$\langle \xi, T\eta \rangle = \langle T^*\xi, \eta \rangle = \lim_{m\to\infty} \langle T^*\xi, \eta_m \rangle = \lim_{m\to\infty} \langle \xi, T\eta_m \rangle = \langle \xi, \zeta \rangle;$$

therefore, $T\eta = \zeta$ so the graph of T is closed.

We write $\mathcal{L}_B(\mathcal{H}_1, \mathcal{H}_2)$, or just $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ if B is clear form the context, for the set of adjointable maps $\mathcal{H}_1 \to \mathcal{H}_2$. If $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, then $T^* \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1)$. Composition of linear maps gives a map

$$\mathcal{L}(\mathcal{H}_2, \mathcal{H}_3) \times \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2) \to \mathcal{L}(\mathcal{H}_1, \mathcal{H}_3).$$

Therefore $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$ is an algebra. It is a unital algebra because the identity map $\mathcal{H} \to \mathcal{H}$ is adjointable.

Indeed, $\mathcal{L}(\mathcal{H})$ is a unital C*-algebra with respect to the involution $T \mapsto T^*$ and the norm $\|\xi\| = \sqrt{\|\langle \xi, \xi \rangle\|}$.

The space of compact operators, $\mathsf{K}(\mathcal{H}_1, \mathcal{H}_2) \subset \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is defined to be the operator norm closure of the span of the operators $\theta_{x,y}$ defined by

$$\theta_{x,y}(z) := x\langle y, z\rangle_{\mathcal{H}_1}$$

for $x \in \mathcal{H}_2$ and $y, z \in \mathcal{H}_1$. We note that

$$(\theta_{x,y})^* = \theta_{y,x}, \quad \theta_{w,x}\theta_{y,z} = \theta_{w\langle x,y\rangle,z} = \theta_{w,z\langle y,x\rangle}. \tag{B.1-1}$$

If $T \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$, then $\theta_{x,y}T = \theta_{x,T^*y}$. The compact operators $\mathsf{K}(\mathcal{H})$ form a C*-ideal in $\mathcal{L}(\mathcal{H})$.

Elements in $\mathsf{K}(\mathcal{H}_1, \mathcal{H}_2)$ viewed as linear operators between Banach spaces need not be compact operators in the usual sense.

Hilbert bimodules

Let B and A be C*-algebras and $\mathcal{H}$ a right Hilbert A-module. A *-homomorphism $\rho : B \to \mathcal{L}(\mathcal{H})$ is non-degenerate if $\rho(B)\mathcal{H}$ is dense in $\mathcal{H}$; in that case we say that ρ gives $\mathcal{H}$ a B-A-Hilbert bimodule structure. Equivalently, if $\mathcal{H}$ is a right Hilbert A-module with an B-A-bimodule structure such that $B \cdot \mathcal{H}$ is dense in $\mathcal{H}$ and $\langle b \cdot \xi, \eta\rangle = \langle \xi, b^* \cdot \eta\rangle$ for all $b \in B$ and $\xi, \eta \in \mathcal{H}$, then $\mathcal{H}$ is an B-A-Hilbert bimodule.

The asymmetry in the definition of an B-A-Hilbert bimodule, that the inner product on $\mathcal{H}$ takes values in A, is analogous to the different roles played by B and A in the definition of a homomorphism $A \to B$ of C*-algebras: A is a B-A-bimodule, but B is not an A-module.

If A is a C*-algebra and I is a (closed, two-sided) ideal in A then I is an A-A-Hilbert bimodule with respect to the inner product $\langle a, b\rangle = a^*b$ and the left-action given by left multiplication.

Example B.4. The set of bounded operators $\mathsf{B}(\mathcal{H}_1, \mathcal{H}_2)$ is an $\mathsf{B}(\mathcal{H}_2)$-$\mathsf{B}(\mathcal{H}_1)$-Hilbert bimodule with respect to the inner product $\langle u, v\rangle = u^*v$.

B.2 Tensor Product of Hilbert Spaces

This material is taken from [8]. Consider a complex vector space $\mathcal{H}$ equipped with a sesquilinear form $\langle\cdot,\cdot\rangle : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ that is linear in the second argument, skew-symmetric, and positive-definite. Then $\mathcal{H}$ is normed via $\|x\|^2 = \langle x, x\rangle$, and is a metric space via $d(x, y) = \|x - y\|$. If $\mathcal{H}$ is complete with respect to the norm then we say that $\mathcal{H}$ is a Hilbert space.

Recall that $\ell^2(X)$ is the set of square-summable functions $s : X \to \mathbb{C}$, i.e. those functions s such that $\int \overline{s} s d\mu < \infty$, where μ is the counting measure on X. Every Hilbert space $\mathcal{H}$ is isometric with $\ell^2(X)$ for some set X; the cardinality of X is the dimension of $\mathcal{H}$ and completely determines $\ell^2(X)$ up to Hilbert space isomorphism. A Hilbert space is separable if it contains a countable dense subset; this holds if it is either finite dimensional or isomorphic to $\ell^2(\mathbb{N})$.

Consider two Hilbert spaces $\ell^2(X)$ and $\ell^2(Y)$. The algebraic tensor product $\ell^2(X) \odot \ell^2(Y)$ has an inner-product $\langle \phi \otimes \psi, \phi' \otimes \psi'\rangle := \langle \phi, \phi'\rangle\langle \psi, \psi'\rangle$. Indeed, positive-definiteness is established by choosing an orthonormal basis for Y, say (ψ_i). Then one may write $t \in \ell^2(X) \odot \ell^2(Y)$ as $t = \sum_i \phi_i \otimes \psi_i$, so that

$$\langle t, t\rangle = \sum_{i,j} \langle \phi_i, \phi_j\rangle\langle \psi_i, \psi_j\rangle = \sum_i \langle \phi_i, \phi_i\rangle \geq 0.$$

The completion of $\ell^2(X) \odot \ell^2(Y)$ with respect to the induced norm defines $\ell^2(X) \otimes_{\mathbb{C}} \ell^2(Y)$, a Hilbert space; there is an isometry

$$\phi : \ell^2(X) \otimes_{\mathbb{C}} \ell^2(Y) \to \ell^2(X \times Y) : \quad \phi \otimes \psi(x, y) \mapsto \phi(x)\psi(y) \in \ell^2(X \times Y).$$

B.3 Tensor product of Banach Spaces

Recall that a Banach space is a normed complex vector space that is complete. If X and Y are Banach spaces, then a cross-norm c on the algebraic tensor product $X \odot Y$ is a norm such that for all $x \in X$, $y \in Y$, and $x' \in X'$, $y' \in Y'$ (the topological dual spaces)

1. $c(x \otimes y) = \|x\| \, \|y\|$;
2. $c'(x' \otimes y') = \|x'\| \, \|y'\|$.

There are two canonical cross-norms on $X \odot Y$, the projective cross-norm π and the injective cross-norm ε:

$$\pi(t) = \inf \left\{ \sum_{i=1}^{n} \|x_i\| \|y_i\| \mid \sum_{i=1}^{n} x_i \otimes y_i = t \right\}$$
$$\varepsilon(t) = \sup \left\{ |(x' \otimes y')(t)| \mid x' \in X', y' \in Y', \text{ and } \|x'\| = \|y'\| = 1 \right\}.$$

We are using the notation $X' = \mathcal{B}(X, \mathbb{C})$ for the continuous linear dual of X. Completing $X \odot Y$ with respect to either cross-norm defines $X \otimes_\pi Y$ the projective tensor product and $X \otimes_\varepsilon Y$, the injective tensor product.

Definition B.5. A nuclear Banach space X is a Banach space such that the natural map

$$X \otimes_\pi \ell^1(\mathbb{N}) \to X \otimes_\varepsilon \ell^1(\mathbb{N})$$

is an isomorphism.

If X is nuclear then $X \otimes_\pi Y \to X \otimes_\varepsilon Y$ is an isomorphism for every Banach space Y. Informally, there is a unique cross-norm on the algebraic tensor product.

B.4 Tensor product of C-algebras*

If A and B are C*-algebras, $A \odot B$ is a *-algebra. A C^*-norm on $A \odot B$ is a cross-norm such that

$$\|t^*t\| = \|t^2\|, \qquad \text{for all } t \in A \odot B.$$

In general, there are many inequivalent C^*-norms on $A \odot B$. The two canonical ones are the minimum norm and the maximum norm: for $t \in A \odot B$,

$$\begin{aligned} \|t\|_{min} &= \sup\{\|(\rho_A \otimes \rho_B)(t)\| : (\rho_A, \mathcal{H}_A), (\rho_B, \mathcal{H}_B) \text{ representations}\} \\ \|t\|_{max} &= \sup\{\|t\|_\beta : \|\cdot\|_\beta \text{ is a } C^*\text{-seminorm on } A \odot B \ \}. \end{aligned}$$

Every C^*-norm $\|\cdot\|$ on $A \odot B$ satisfies

$$\|t\|_{min} \le \|t\| \le \|t\|_{max}, \qquad \forall t \in A \odot B.$$

A C^*-algebra A is nuclear if $A \odot B$ admits a unique C^*-norm for every C^*-algebra, B. Finite-dimensional C^*-algebras are nuclear, and directed limits of nuclear C^*-algebras are nuclear. In particular, $\mathcal{A}_X$ is nuclear because it is an AF-algebra. Commutative C^*-algebras are also nuclear.

B.5 Tensor product of C^*-bimodules

Let $\mathcal{H}_1$ be an A-B-Hilbert bimodule and $\mathcal{H}_2$ a B-D-Hilbert bimodule. We define $\mathcal{H}_1 \otimes_B \mathcal{H}_2$ to be the A-D-Hilbert bimodule that is the completion of the algebraic tensor product $\mathcal{H}_1 \otimes_{B,\mathsf{alg}} \mathcal{H}_2$ with respect to the D-valued inner product

$$\left\langle \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \right\rangle := \left\langle \xi_2, \langle \xi_1, \eta_1 \rangle \cdot \eta_2 \right\rangle$$

where $\langle \xi_1, \eta_1 \rangle \cdot \eta_2$ denotes the result of the action on $\langle \xi_1, \eta_1 \rangle \in B$ on $\eta_2 \in \mathcal{H}_2$. We call $\mathcal{H}_1 \otimes_B \mathcal{H}_2$ the balanced tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$. This is also called the interior tensor product by some authors.

Lemma B.6. *Let B_1, B_2 and B_3 be concrete C^*-algebras acting on $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_3$. Suppose that M and N are concrete Hilbert C^*-bimodules over B_2, B_1 and B_3, B_2, respectively. Define $N \circ M \subseteq \mathsf{B}(\mathcal{H}_1, \mathcal{H}_3)$ to be the concrete Hilbert bimodule*

$$N \circ M := \overline{span}\{n \circ m \mid n \in N, m \in M\}.$$

Then the there is an isomorphism $\Phi : N \otimes_{B_2} M \to N \circ M$ from the balanced tensor product defined by

$$\Phi(n \otimes m) = n \circ m.$$

Proof. First, Φ is well-defined on the tensor product since

$$\Phi(n \cdot \phi \otimes m) = n \circ \phi \circ m = \Phi(n \otimes \phi \cdot m), \qquad n \in N,\, m \in M,\, \phi \in B_2.$$

It is obvious that Φ is a map of B_3-B_1-bimodules, since the actions are given by composition of functions. We show that Φ preserves the inner-product: for $n, n' \in N$ and $m, m' \in M$,

$$\begin{aligned}\langle n' \otimes m', n \otimes m\rangle_{B_1} &= \langle m', \langle n', n\rangle_{B_2} m\rangle_{B_1} \\ &= \langle m', (n')^* n \circ m\rangle_{B_1} = (m')^*(n')^* nm = (n'm')^*(nm). \\ \langle \Phi(n' \otimes m'), \Phi(n \otimes m)\rangle &= \langle n'm', nm\rangle = (n'm')^*(nm).\end{aligned}$$

It follows that Φ is an isometry, and thus is injective. Since the image of Φ clearly contains $n \circ m$ for all $n \in N$ and $m \in M$, the image of Φ is $N \circ M$. $\square$

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