

Understanding Teacher and Student Learning Situated in a
School-wide Implementation of Fractions Instruction

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ABSTRACT

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The current push to improve fractions instruction is in response to studies that have aptly demonstrated the deficits in student and teacher understanding of fractions (e.g., Chinnappan & Desplat, 2012; Kloosterman, 2010; Ma, 1999; Rittle-Johnson, Siegler, & Alibali, 2001). While these studies have built an argument of the weaknesses in our current systems for supporting student learning, we have not studied how schools focused on improvement coordinate teacher and student learning. This qualitative study was designed to advance our understanding of how to support teachers' instructional practices and students' learning of fractions by focusing on what teachers and students *do* know and understand and how both teacher and student learning were intentionally considered and coordinated. A case study of one elementary school where teachers implemented a school-wide approach to fractions instruction and were positioned as learners alongside their students provided the opportunity to study how students' thinking developed over multiple years and what teachers know and understand about teaching fractions through their participation in a school-wide effort to implement fractions instruction. Three major findings related to student learning emerged: (1) over time, students demonstrated more

sophisticated understandings of fractions, (2) at the start of each subsequent school year, the cohort of students entering a particular grade level brought with them more sophisticated strategies for partitioning and sharing as well as more accurate representations, and (3) the understanding and strategies for students with one year of instruction were noticeably less sophisticated than students with two or three years of instruction, whose understandings and strategies were much more aligned with one another. Related to teacher learning, I offer examples of three kinds of intellectual resources that teachers generated, drew upon, made meaning of, and coordinated as they engaged in teaching mathematics and learning to teach mathematics: mathematical knowledge for teaching, instructional visions, and understanding of trajectories of student learning. In addition, I examine how the school-based professional learning opportunities supported the development of mutual understanding among teachers and instructional leaders such that there was instructional alignment across the school. I offer four findings from looking across the three professional learning contexts: (1) I argue that all three learning contexts provided opportunities for teachers and instructional leaders to develop mutual understanding around various kinds of intellectual resources, (2) I argue that the interactions in which teachers generated and used various intellectual resources varied across the different settings but worked together to support the development of mutual understanding, (3) I argue that mutual understanding both allowed the coach to develop unit plans that had meaning to the teachers and supported them to teach fractions in ways that were aligned within classrooms at the same grade level and across classrooms at different grade levels, and (4) I argue that teachers were positioned as both “listeners” and “sources” and that because of these different positionings, participants have different perspectives that shape the interpretations and functions of their contributions.

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CHAPTER 1: INTRODUCTION

When Lamon (2007) summarized six decades of research in the domain of rational numbers and ratio and proportion, she also put forth a framework for future research focused on “unraveling the complexities of teaching and learning these topics” (p. 629). The teaching of fractions remains among the most complex and challenging topics in elementary school mathematics—both for students and teachers. Initially, researchers used a cognitive approach and focused on individual students’ fraction strategies and ideas, typically absent of the classroom context (e.g., Behr, Wachsmuth, Post, & Lesh, 1984). More recently, as an attempt to better understand how to support students’ fraction learning in the classroom, researchers have focused their attention on teachers’ knowledge of fractions and their instruction as a way of improving student achievement (e.g., Ma, 1999; Siegler, et al., 2010). However, a majority of these studies of both student and teacher understanding have focused on the deficits in participants’ knowledge and inadequacy of current fractions’ instruction (e.g., Chinnappan & Desplat, 2012; Kloosterman, 2010; Ma, 1999; Rittle-Johnson, Siegler, & Alibali, 2001). While these studies have built an argument of the weaknesses in our current systems for supporting student learning, we have not studied how schools focused on improvement coordinate teacher and student learning.

This study was designed to advance our understanding of how to support teachers’ instructional practices and students’ learning of fractions by focusing on what teachers and students *do* know and understand and how both teacher and student learning were intentionally considered and coordinated. Using the case of a school where teachers have implemented a school-wide approach to fractions instruction and have been positioned as learners alongside

their students, I asked: *How did a school-wide approach to fractions instruction shape teacher and student learning?*

In this dissertation, I will report on a case study of the teacher and student learning situated in one school that implemented a school-wide approach to fractions. As part of this study, I analyzed how students' understanding of fractions changed over time as well as what teachers learned about fractions and teaching fractions. I also analyzed how three specific professional learning contexts supported teachers in developing mutual understanding around fractions instruction.

The remainder of this dissertation is organized into six chapters. In Chapter 2, I describe the conceptual framing and relevant literature for this study. In Chapter 3, I describe the setting and methods. Chapters 4, 5, and 6 present findings related to student learning, teacher learning, and how mutual understanding was developed. Chapter 7 offers a discussion of the findings across the case in relation to the literature presented in Chapter 2.

CHAPTER 2: CONCEPTUAL FRAMING & REVIEW OF THE LITERATURE

Conceptual Framing

In this study, I used a situative approach to understanding teacher and student learning. As Greeno (2006) describes, “The defining characteristic of a situative approach is that instead of focusing on individual learners, the main focus of analysis is on activity systems: complex social organizations containing learners, teachers, curriculum materials, software tools, and the physical environment” (p. 79). This approach reframes learning as something that is always and necessarily situated; that knowing and learning are constructed through participation in a community’s discourse and practices, and are shaped by the contexts and activities in which they occur (Greeno, Collins, & Resnick, 1996). Individual cognition and semiotic structures of information are considered as they are understood, used, and generated by people in their joint activity with one another. By focusing my analysis on the activity system of teachers’ collective work in a school, my goal was to develop an understanding of concepts and principles that explained how and why activities in the school resulted in changes in the discourse and practices of both teachers and students around fractions instruction, and what those changes were.

Greeno (2006) describes three ways in which analyses that adopt a situative approach differ from those using solely cognitive or interactional approaches. First, he describes the data needed to infer properties of information structures. Situative analyses include “records of interaction” (such as field notes or transcripts of a meeting or classroom instruction) rather than solely using “verbal reports” (such as thinking aloud protocols used by individual participants). The records of interaction capture “the evidence that participants provide each other through their collaborative discourse, [which] informs them about their understanding, goals, intentions,

and expectations, and it provides evidence to the researcher about semiotic structures that are being generated and used” (p. 86).

A second difference is related to how semiotic structures, or structures of information, are generated and used in learning activities. Instead of analyzing these processes at an individual level, a situative perspective considers *meaning* to be a relation between participants’ “joint actions of achieving mutual understanding” and “the states of affairs or ideas that the participants themselves interpret their statements to be referring to” (p. 86). In particular, I drew on van de Sande and Greeno’s (2012) work using the concept of *framing* as a way of analyzing records of interaction for evidence of mutual understanding. Framing refers to the way in which participants understand the activity in which they are engaged and they conceptualize mutual understanding as being dependent upon participants having *cognitive framings* that are sufficiently aligned. Cognitive framing includes both participants’ understanding of the kinds of knowledge that are relevant to their participation in an activity, the kinds of knowledge they need to construct in order to succeed in an activity, and the ways) in which participants organize information within a situation or problem they are working on.

A third difference is related to how information structures are conceptualized. Individual cognitive approaches typically attend to information structures as representations that connect concepts with each other. A situative approach instead focuses the researcher on both representations *and* representational practices, emphasizing that representation is both mental and socially distributed in practices. Greeno (2006) describes, “situativity treats representations as a relation between signs and aspects of situations, resulting from interpretations by people in their activity...mental representations are only relevant to the extent that they refer indexically to ongoing activity...Researchers need not assume that these functional relations are actually

represented in individual learner's minds unless there is evidence of an explicit representation" (pp. 86-7).

So, while there has been much research that has used a cognitive approach to characterize mental representations of students' fraction knowledge and teachers' professional knowledge, we know very little about how these mental representations are generated, negotiated, and interpreted by students and teachers in the context of their activity systems. It is important to note, however, that the cognitive approaches to studying students' fraction knowledge and teachers' professional knowledge are quite useful when adopting a situative perspective because studying knowledge remains relevant. However, we are also concerned with the practices that are influenced by knowledge as well as the situations in which it is embedded (Borko & Putnam, 1996). Applying this idea to this study, I analyzed ideas in the context of their use by studying how ideas are generated, negotiated, and taken up as teachers interact and participate in joint activity.

Though Greeno does not use the term "identity" as he discusses the positioning of participants, others who adopt a situative perspective point to the importance of attending to identity as part of a situative analysis (e.g., Peressini, Borko, Romagnano, Knuth, & Willis, 2004; Gresalfi & Cobb, 2011; Nolen, Horn, & Ward, 2015). Using a situative perspective to conceptualize identity, Gresalfi and Cobb (2011) describe it as "the set of practices and expectations that shape participation in particular contexts" (pp. 273-4). In particular, they describe the process of identifying in a particular context as the relation between two elements: the *normative identity* (established in a specific context) and individuals' *personal identities* (that develop as individuals participate in the practices of that context). So, in addition to attending to how structures of information are generated and used, I attended to the practices and

expectations that shaped teachers' participation as they worked together in their school. This conceptual framework for studying the learning of teachers and students in one school coordinates multiple domains of knowledge and beliefs, situated in, and interwoven through, teaching and mathematical practices developed and enacted over time.

Literature Review

The purpose of this case study was to understand how one school implemented a school-wide approach to fractions instruction and how that shaped teacher and student learning. Though I used a situative approach to understanding teacher and student learning, previous research focusing on individual cognition and semiotic structures of information was quite relevant as I considered how these ideas were understood, used, and generated within the activity system. In this section, I first synthesize literature that describes what it means for students to learn mathematics with understanding as well as what this implies for mathematics instruction. I then review literature around what teachers need to know and understand in order to teach fractions as well as school-wide efforts to support teacher learning and improve instruction.

What Does It Mean for Students to Learn Mathematics and Understand Fractions?

The research on children's thinking about fractions extends back at least six decades. However, a majority of that research was more clinical in nature. In the last two decades, as there has been an increased focus on the relationship between instruction and student learning, researchers have started attending to student thinking in the context of classrooms. In this first section, I review literature related to what it means to learn mathematics and what it means to understand fractions.

Learning mathematics with understanding. To articulate the overarching goal for what we hope students will achieve in classrooms, I draw on the conceptual work of other

mathematics education researchers. Carpenter & Lehrer (1999) describe a goal of *learning mathematics with understanding*. Munter (2014) expands on this notion describing how this requires both the development of multiple strands of mathematical proficiency (Kilpatrick, Swafford, & Findell, 2001) as well as forms of mathematical practice (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010).

Mathematical proficiency. Fifteen years ago, the National Research Council introduced the concept of *mathematical proficiency* in their 2001 report, *Adding It Up: Helping Children Learn Mathematics*. They suggested that mathematical proficiency encompasses five “strands”:

1. conceptual understanding (comprehension of mathematical concepts, operations, and relations),
2. procedural fluency (skills in carrying out procedures flexibly, fluently, and appropriately),
3. strategic competence (ability to formulate, represent, and solve mathematical problems),
4. adaptive reasoning (capacity for logical thought, reflection, explanation, and justification), and
5. productive disposition (habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy). (National Research Council, 2001, p. 116)

However, they are quite clear that the strands are not isolated ideas that develop independent from one another. Instead, the report describes how the five strands are interwoven and interdependent. Simply focusing on just one or two of the strands will not support students to become mathematically proficient.

The notion of “mathematical proficiency” has been widely accepted in the mathematics education community. For example, the National Mathematics Advisory Panel (2008) explicitly addresses mathematical proficiency with regards to fractions in their report, *Foundations for Success*, “A major goal for K–8 mathematics education should be *proficiency* with fractions...for such proficiency is foundational for algebra and, at the present time, seems to be severely underdeveloped. Proficiency with whole numbers is a necessary precursor for the study of

fractions, as are aspects of measurement and geometry. These three areas—whole numbers, fractions, and particular aspects of geometry and measurement—are the Critical Foundations of Algebra” (p. xvii, italics added).

Mathematical practices. The Common Core State Standards (CCSS; NGACBP & CCSSO, 2010) are the most recent effort to articulate goals for student learning that include both *what* and *how*. In addition to content standards, the CCSS include eight Standards for Mathematical Practice (SMPs) that describe *how* students ought to engage with mathematical content and with each other in learning mathematics:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

The SMPs are the same across all grade levels (K-12) and are closely related to both NCTM’s Mathematics Process Standards (National Council of Teachers of Mathematics, 2000) and the NRC’s Strands of Mathematical Proficiency (National Research Council, 2001). However, mathematical practices cannot be developed absent of mathematical content. In the next section, I review literature that describes what students are expected to know and understand about fractions by the time they leave elementary school (i.e., fifth grade).

Understanding fractions. In the sections that follow, I use the previously described *learning mathematics with understanding* to review literature that is specific to children’s fractions ideas. Specifically, I discuss the notion of developing rational number sense and summarize literature around multiple aspects of understanding what fractions mean.

Rational number sense. In *Everybody Counts: A Report to the Nation on the Future of Mathematics Education* (National Research Council, 1989), the authors contend that the “major objective of elementary school mathematics should be to promote *number sense*” (p. 46, italics added). While it is difficult to pin down a single definition of number sense, there is a set of characteristics that are repeatedly used, sometimes with varying terminology, to describe someone “having number sense.” These characteristics include flexible use of numbers when doing mental computations, using a variety of invented or efficient strategies for calculations, using number magnitude to make estimations, judging reasonableness of answers, and facility in moving between multiple representations (NCTM, 1989).

However, Sowder (1992) argues that, “It is possible to have good number sense for whole numbers, but not for fractions” (p. 6) which leads to the question, “What is *rational* number sense?” Cramer, Post, and delMas (2002) refer to “number sense for fractions” and describe it as “a well-internalized concept of the ‘bigness’ of rational numbers” (p. 129) and go on to explain that without a conceptual foundation of rational numbers, students cannot operate on fractions in a meaningful way. Using a slightly different term, Empson and Levi (2011) use the phrase “relational understanding of fractions” to describe when students can express a fraction in terms of other fractions and operations on those fractions. And, Lamon (2007) describes “rational number sense” as having “an intuitive feel for the relative sizes of rational numbers and the ability to estimate, to think qualitatively and multiplicatively, to solve proportions and to solve problems, to move flexibly between interpretations and representations, to make sense, and to make sound decisions and reasonable judgments” (p. 636).

While both of these descriptions are similar to that offered by the NCTM for number sense, “the concept of ‘bigness’ of rational numbers” (as described by Cramer et al., 2002) and

“the intuitive feel for the relative sizes of rational numbers” (as described by Lamon, 2007) are challenging concepts for students. Many researchers have suggested that challenges students encounter related to rational numbers are associated with their whole number knowledge and the overgeneralization of that knowledge (Lamon, 1999; Ni & Zhou, 2005; Post, Cramer, Behr, Lesh, & Harel, 1993). For example, students may use their knowledge and understanding of comparing 8 and 6 to incorrectly conclude that $\frac{1}{8}$ is greater than $\frac{1}{6}$. However, much of this research has taken an individual cognitive approach, most often using cognitive interviews with individual students—or it has focused on a very narrow set of fraction ideas (such as “ordering fractions” or “adding fractions”). In a handful of studies, particular curricular approaches have been studied with goals to “help children develop better overall conceptions of the rational number system as a whole and the way its various components fit together—not just better understandings of one or another of these components in isolation” (Moss & Case, 1999, p. 124).

While this research helps point towards the need for focusing on the development of more global cognitive processes and less on specific components or attainment of individual tasks, ideas, or strategies, the results of the curriculum interventions have not consistently supported any particular approach. If anything, these studies suggest that there may be multiple approaches to high quality fractions curriculum. However, while a single best approach does not exist, the cognitive studies done with children are quite extensive related to a number of concepts, including students’ partitioning strategies (e.g., Lamon, 1999); comparing, ordering, and equivalence of fractions (e.g., Behr et al., 1984); addition and subtraction with fractions (e.g., Hackenberg, 2007); and multiplication and division with fractions (e.g., Mack, 2001). Though much of this work is absent of the classroom context, they report on the thinking of many children and describe in great detail how children make senses of these concepts, common

strategies that children use, and the kinds of errors that students make as well as the underlying logic of the errors. In the next three sections, I will summarize the research related to conceptual understanding of what fractions are and how students develop that understanding. This includes how students develop understandings of partitioning strategies, the use of fraction language, and the use of symbolic notation for fractions.

Developing partitioning strategies. Partitioning can be defined as “the process of dividing an object or objects into a number of disjoint and exhaustive parts. This means that the parts are not overlapping and that everything is included in one of the parts. When we use the word in relation to fractions, it is with the additional stipulation that this parts must be of the same size” (Lamon, 1999, p. 76). Many researchers have suggested that partitioning activities are crucial for building rational number sense (Piaget, Inhelder, & Szeminska, 1960; Pothier & Sawada, 1983; Kieren, 1976; Streefland, 1991) though most of the existing research has focused on partitioning of area models versus number lines or sets of objects. A common analogy is that partitioning is as important to developing an understanding of fractions in the same way that counting is fundamental to developing an understanding of whole numbers (Behr & Post, 1992; Pothier & Sawada, 1983). To further this point, Lamon (1999) claims that partitioning is key to understanding and generalizing many important fraction concepts such as identifying “fair shares,” fractional parts of an objects, and fractional parts of sets of objects; comparing and ordering fractions; locating fractions on number lines; understanding the density of rational numbers; evaluating and finding equivalent fractions; and operating with fractions.

Using partitioning as a starting point for fractions is not a novel idea, but often students are introduced to fractions through worksheets where a shape is pre-partitioned and one or more of the pieces are shaded (Empson & Levi, 2011). This approach is problematic for a number of

reasons. First, students often engage with the part and the whole separately—as two different whole numbers that do not necessarily have a relationship. (e.g., seeing $\frac{3}{4}$ as 3 and 4). This is problematic as it can lead to inappropriately using whole number reasoning instead of reasoning with a fraction as a single quantity (Behr et al., 1984; Saxe, Gearhart, & Seltzer, 1999). Second, students are not experiencing the act of partitioning and seeing the results of making additional cuts—the idea that the more cuts you make, the smaller the pieces become. Finally, students may not actively engage with the ideas that fractions must have equal-size pieces—but that the equal-size pieces do not need to be congruent. One hurdle that students may have to overcome is the belief that partitions must result in pieces that are both the same size and the same shape. Students often have difficulty recognizing fractional parts as equal in size if the pieces are not congruent (same size and shape) (Bezuk & Bieck, 1993). However, understanding this is an important part of developing a precise understanding of what is meant by “equal parts” (Common Core Standards Writing Team, 2013).

Though the Common Core State Standards do not suggest or comment on particular strategies that students might use to partition, the existing research supports the notions that partitioning strategies develop over a predictable trajectory and that developing more efficient and sophisticated partitioning strategies is an important conceptual milestone (Empson & Levi, 2011; Lamon, 1996; Pothier & Sawada, 1983).

Empson & Levi (2011) developed a research-based trajectory of students’ partitioning strategies designed to support teachers’ understanding of how students’ partitioning strategies develop. Their trajectory offers five classifications (see Table 1) for students’ strategies. Their trajectory built on previous work by Pothier & Sawada (1983) and Lamon (1996). One significant addition is the last two classifications (Ratio and Multiplicative Coordination)

Table 1. Children’s strategies for solving equal sharing problems. (Adapted from Empson & Levi, 2011, p. 25.)

Problem: 6 children are sharing 8 small sandwiches. They are sharing so each child gets the same amount. How many sandwiches will one child get?	
Strategy Name	Strategy Description
Non-Anticipatory Sharing	Child does not think in advance of both number of sharers and amount to be shared. For example, child splits each sandwich into halves because halves are easy to make. Gives each person $\frac{1}{2}$. Child may or may not decide to split the last two sandwiches into sixths. <i>Each person gets $\frac{1}{2}$ of a sandwich and “2 little pieces,” if the last two sandwiches are split.</i>
Additive Coordination: Sharing One Item at a Time	Child represents each sandwich. Splits first sandwich into sixths because that is the number of sharers. Each person gets 1 sixth piece. Repeats process until all 8 sandwiches are shared. <i>Each person gets $\frac{8}{6}$ sandwiches altogether.</i>
Additive Coordination: Sharing Groups of Items	Child represents each sandwich. Realizes that 6 pieces can be created by splitting 2 sandwiches each into thirds. Each person gets $\frac{1}{3}$. Child moves on to another group of items and continues similarly until all the sandwiches are used up. <i>Each person gets $\frac{4}{3}$ sandwiches.</i> OR Child represents each sandwich. Realizes there are more sandwiches than people and gives each person a whole sandwich. Child moves on to remaining 2 sandwiches and divides into sixths or thirds. <i>Each person gets $1\frac{2}{6}$ or $1\frac{1}{3}$ sandwiches.</i>
Ratio (Repeated Halving, Factors)	Child may or may not represent all of the sandwiches and people. Uses knowledge of repeated halving or multiplication factors to transform the problem into a simpler problem, 3 children sharing 4 sandwiches. Solves the simpler problem. <i>Each child gets $\frac{4}{3}$ sandwiches.</i>
Multiplicative Coordination	Child does not need to represent each sandwich. Child understands that a things shared by b people is $\frac{a}{b}$, so 8 sandwiches shared by 6 people means <i>each person gets $\frac{8}{6}$ sandwiches.</i>

because they support the notion that developing more efficient and sophisticated partitioning strategies is a conceptual milestone for students. In other words, the goal is for student to use their early partitioning experiences to understand the impact of partitioning and generalize fraction concepts—and ultimately, students’ imagined partitions will suffice (Behr & Post, 1992).

It is also worth noting that studies examining students’ partitioning strategies have identified additional “factors” that impact students’ strategies. For instance, the numbers in a

sharing situation can impact students' strategies (Pothier & Sawada, 1983). Lamon (1996) found that "strategies were heavily influenced by social practice related to the commodity being shared" (p. 170). For example, in a situation where 3 people are sharing 3 pizzas, a student may give each person a third of each pizza (versus the more economical strategy of giving each person a whole pizza) because it is a common social practice to make sure each person gets some of each pizza (since they often have different toppings). The various shapes of objects also played a role in how students partitioned things. For example, a licorice stick, which is typically long and thin, is often divided by making many cuts of the same direction to cut it into smaller "sticks." Rarely are licorice sticks cut vertically in real life. A student could represent the licorice stick with a circle model or a rectangle with different dimensions, but this also requires the student to create a more abstract model. Similar kinds of abstraction may be needed if the object to share is something that is not traditionally cut when shared—for example, a bottle of soda shared with 6 people.

Developing an understanding of fraction language. The terms (e.g., "halves" and "thirds") and notation used for fractional parts are conventions and are not part of children's intuitive knowledge of fractions (Lampert, 1990—as cited in Ball, 1993; Empson & Levi, 2011). Ball's (1993) classroom-based research provides a nice illustration of this as a student used the invented (and logical) term "twoths" to refer to "halves." Because these terms are not intuitive, teachers must introduce the language and notation—and the logic it follows. In addition to building on students' thinking, math education researchers suggest that the line of questioning and specific language can make a difference. For example, instead of focusing on a more traditional question like "How many pieces is the whole brownie cut into?" which focuses on a number of pieces, Empson and Levi (2011) suggest that teachers ask questions that support

students in connecting the size of a piece and its name (e.g., “How many of these parts fit into the whole brownie?”). This supports children to look at the relationship between the part and the whole in order to determine the value and name of the fraction. They also suggest that children have many experiences focused on developing understanding of fractions and using fraction language prior to learning about fraction notation. For example, they suggest that teachers initially spell out fraction terms (e.g., write “2 fourths” instead of “ $2/4$ ”). The CCSS support this developmental trajectory; in Grades 1 and 2, the standards call for students to “describe the shares using the words halves, fourths, and quarters, and use the phrases half of, fourth of, and quarter of” (Common Core Standards Writing Team, 2013, p. 16).

Developing an understanding of symbolic notation for fractions. Often, children come to classrooms with informal knowledge about fraction notation—especially common fractions such as $1/2$ and $1/4$. However, using and understanding written notation for fractions is widely acknowledged as being challenging for students (Saxe, Taylor, & McIntosh, 2005; Hiebert, 1988). However, only a handful of studies have examined the developmental patterns in students’ use of notation or the role that instruction plays in this development. Most of the studies that do exist are case studies of individual students. One claim is that students may overgeneralize their prior knowledge of previously learned symbol systems (specifically whole numbers) as they attempt to make sense of symbolic representations that are new to them (Hiebert, 1988; Hiebert & Carpenter, 1992). Mack’s (1995) teaching experiment with 7 third- and fourth-graders suggests that these overgeneralizations may also include overgeneralizing new symbols systems to previously learned systems (e.g., in a representation such as “3 - $5/8$,” students referred to “3” as three eighths). Ball’s (1993) classroom-based research found that students initially made sense of fraction notations in ways that are not normative. For example,

one student interpreted $3/4$ as “‘make groups of three’ and then take all but one group (i.e., $3/4$ is one less than $4/4$)” (p. 174). Building on these initial studies, Saxe and his colleagues (2005) attempted a more systematic analysis with 384 students. They found that students’ knowledge of conventional notation and reference (the conceptual understanding of using any notational form—even unconventional, student-created notation—to represent mathematical ideas) develop somewhat independently. For example, they found that some students used conventional forms to refer to relations that were not part-whole relations whereas other students used unconventional notation to refer to part-whole relations.

Initial work in this area suggests that instruction must engage students in investigating the connections between notation and meaning (Hiebert, 1988). In order to better understand the role of instruction in developmental shifts in notation, Saxe et al. (2005) also analyzed classroom practice data. Their findings were in line with previous studies in that they found that classroom practices that build on students’ thinking were more likely to support shifts toward normative uses of notation. Mack (1995) reported success with a similar approach in her teaching experience; after asking students to supply specific interpretations of a particular symbolic representation, students stopped integrating symbolic representations for whole numbers and fractions and began recording appropriate symbolic representations.

My contribution. Though there has been much research focused on student learning of fractions, this study contributes to what we know in three significant ways. First, this study examines how student understanding develops across multiple school years. Often research on student thinking is limited to one point in time or multiple points within a single school year. Second, the analysis of student understanding of fractions attends to multiple features of students’ understanding, including their partitioning strategies, their use of fraction language and

symbolic notation, and their use of representation. Finally, student learning is analyzed within the context of a school-wide implementation of fractions instruction. Often, studies of student thinking occur absent of instruction or are limited to a single unit of instruction.

Returning to the goal described previously of students developing “rational number sense” or a “relational understanding of fractions,” the next section of the literature review considers the kinds of instructional approaches that might support such learning opportunities for students.

What is “High Quality Mathematics Instruction”?

Much of the work in math education over the last twenty years has converged on an increasingly clearer image of high quality mathematics instruction that gives all students access to developing an understanding of significant mathematical ideas (Munter, 2014). Different terms have been used to describe this image, including *ambitious* (Lampert, Beasley, Ghouseini, Kazemi, & Franke, 2010), *complex* (Boaler & Staples, 2008; Cohen & Lotan, 1997), and *responsible* (Ball, 2010), but common features include similar core features and descriptions of these core features. Three seminal pieces describe the aspects of mathematics classrooms that are essential for supporting children in learning mathematics with understanding. In the first, Hiebert and colleagues (1997) identify five dimensions and core features: the nature of classroom tasks, the role of the teacher, the social culture of the classroom, mathematical tools as learning supports, and equity and accessibility. In the second piece, Carpenter and Lehrer (1999) discuss three central dimensions of mathematics instruction: tasks or activities students engage in and the problems they solve, classroom normative practices, and tools that represent mathematical ideas and problem situations. In the third piece, Franke, Kazemi, & Battey (2007) describe three important dimensions: supporting discourse for doing and learning mathematics, establishing

norms for doing and learning mathematics, and building relationships for doing and learning mathematics. Looking across these three pieces, many of the aspects overlap. For example, all three pieces address the nature of classroom tasks. Two of the pieces (Hiebert, et al., 1997 and Franke, et al., 2007) include aspects related to the role of the teacher and the social culture and norms of the classroom. Munter (2014) drew on these seminal pieces to conceptualize a “vision of high quality mathematics instruction” (VHQMI), which can be described as a set of images of ideal mathematical instructional practices for which teachers strive.

His empirical work, which involved more than 900 interviews with teachers, principals, mathematics coaches, and district leaders, led him to define VHQMI in terms of three related dimensions of classroom instruction: role of the teacher, classroom discourse, and mathematical tasks though his initial conceptual work included other dimensions supported by literature such as equity and accessibility, social culture and norms, and mathematical tools as learning supports.

In the sections that follow, I will provide a brief summary of the dimensions that are discussed most often in regards to high quality mathematics instruction, including: the role of the teacher, classroom discourse, the nature of classroom tasks, social culture and norms, mathematical tools, and equity and accessibility (see Figure 1).

Role of teacher. Although the role of the teacher is theoretically described in similar ways by researchers, different sub-dimensions have been emphasized. Hiebert et al. (1997) described how teachers are responsible for selecting and creating tasks with goals in mind, sharing mathematical conventions, sharing alternative methods, and articulating ideas in students’ methods. Franke et al. (2007) emphasize the teacher’s role in building relationships with students which involves, among other things, getting to know students’ mathematical

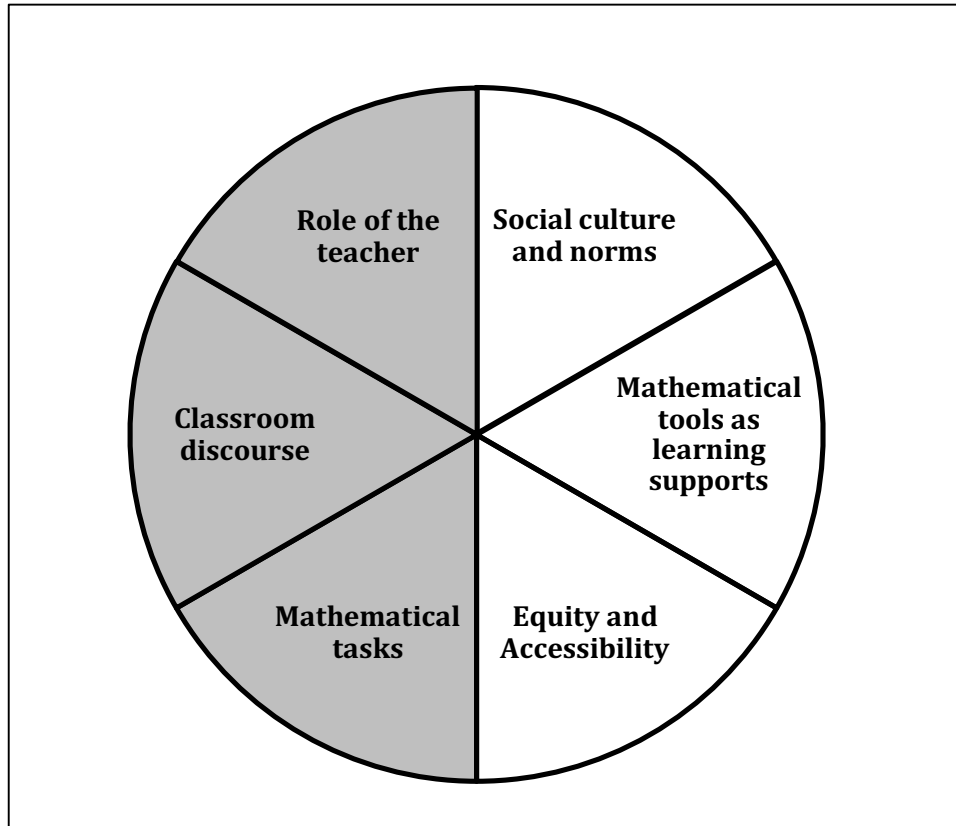


Figure 1. Dimensions of high quality mathematics instruction.

thinking, which they describe as “knowing the details of how one’s students are making sense of particular mathematical ideas and at the same time knowing in general how students develop in relation to particular mathematical ideas” (p. 242). In Munter’s (2014) work, he identified three dominant ways that the role of teacher was characterized: conceptions of typical activity structure, attribution of mathematical authority, and influence on classroom discourse.

Classroom discourse. Increasing opportunities for students to engage in mathematical discourse has been a key component of reform efforts in mathematics education (Carpenter & Lehrer, 1999). There has been an emphasis on teachers facilitating whole-class conversations, including student-to-student talk, with conceptual (versus calculations) orientations in order to create a discourse community (Lampert, 1990) or a math-talk learning community (Hufferd-Ackles, Fuson, & Sherin, 2004). Hufferd-Ackles and her colleagues examined how one teacher

built a discourse community, which they describe as “a community in which individuals assist one another’s learning of mathematics by engaging in meaningful mathematical discourse” (p. 81). Their study revealed four components of developing discourse communities. First, there is a shift from the teacher as the sole asker of questions to both students and the teacher as questioners. Second, there is an increase in students explaining and articulating their mathematical ideas. Third, there is a shift in the source of mathematical ideas, moving from the teacher as the source of all math ideas to students’ ideas influencing the direction of the lesson and discourse. Finally, students become increasingly responsible for evaluating their own ideas and those of their peers. However, as Munter (2014) points out, such discourse communities are dependent upon students having opportunities to engage in rich mathematical work, which is typically initiated by the nature of the classroom tasks.

The nature of classroom tasks. Hiebert et al. (1997) describe three characteristics of high-quality tasks: (1) tasks should encourage reflection and communication, (2) tasks should allow students to use tools, and (3) tasks should leave behind important residue (e.g., insights into the structure of mathematics). One of the larger studies in mathematics education, the QUASAR project, included a component that examined mathematical tasks and defined four levels of tasks. Lower-level tasks are classified as *memorization* or *procedures without connections (to understanding, meaning, or concepts)*. Higher-level tasks are classified as *procedures with connections (to understanding, meaning, or concepts)* or *doing mathematics*. The highest level tasks, classified as *doing mathematics*, require complex and non-algorithmic thinking; require students to explore and understand the nature of mathematical concepts, processes, or relationships; demand self-monitoring of one’s own cognitive processes, and

require students to analyze and actively examine the task throughout the process of solving it (Stein, Smith, Henningsen, & Silver, 2000).

Social culture and norms. Math education researchers have increasingly focused their studies on classroom contexts and we have a much better understanding of the kinds of social contexts that support students to learn mathematics with understanding. Franke et al. (2007) argue that teachers must attend to both the social norms (norms that govern interactions in general) and sociomathematical norms (norms that guide interactions around mathematics) that are developed in classrooms as these norms have consequences for both what students learn as well as the dispositions that students develop around what it means to do mathematics. Hiebert et al. (1997) suggest a number of norms that seem to support students to learn mathematics with understanding: “doing” mathematics involves collaboration with peers; information and solution methods become available through communication; create cognitive conflict (rather than avoid it); discussions are about methods and ideas; students choose their own methods and share them with others; use mistakes are sites for learning; and correctness is determined by the logic of mathematics (versus correctness determined by feedback from the teacher).

Mathematical tools as learning supports. Hiebert et al. (1997) and Carpenter and Lehrer (1999) describe how tools can be used to support students to learn mathematics with understanding. They argue that meaning must be constructed for and with tools. Being shown how to use tools in procedural ways will not support students to develop meaning of the tool or meaning for its’ use. Also important is that different kinds of meaning and use can be developed for a given tool. Tools can be used to keep records, communicate in various ways, or students can use the tools to support thinking.

Equity and accessibility. Though equity and accessibility are frequently addressed as part of the other dimensions (e.g., developing a discourse community is one way to support more children to learn with understanding), Hiebert et al. (1997) identify equity and accessibility as a dimension of its own. They contend that every learner can learn mathematics with understanding. In order for this to happen, they suggest that teachers create classroom environments that take into account the uniqueness of each child so that each has “the opportunity to engage in and reflect on tasks that are mathematically problematic in a social community where his or her thinking is discussed and valued” (p. 66). In a similar manner to other math educators, they relate this dimension to the previously described dimensions and suggest that together these dimensions work to enable all students to learn with understanding.

My contribution. This study contributes to the growing number of studies that provide images of what high quality mathematics instruction looks like. However, much of the existing literature describes reform at the classroom level or at a much broader level—that of multiple classrooms (among many schools) or one or more school districts. This study focuses on how high quality mathematics instruction is characterized within one elementary school. Second, this study focuses on images of high quality mathematics instruction in relation to fractions content. Most often, studies of high quality mathematics instruction do not consider particular strands of mathematics, but instead consider mathematics more generally.

As the image of high quality mathematics instruction described in this section has taken hold in the field of math education, researchers have focused on studying and articulating instructional practices that work towards developing the dimensions just described. This includes things like how to pose cognitively demanding tasks so that they are accessible to all students and remain cognitively demanding (Jackson, Shahan, Gibbons, & Cobb, 2012), and facilitating

whole class discussions that include eliciting and responding to student reasoning, orienting students to each other's ideas and to the mathematics, and positioning students competently (Chapin, O'Conner, & Anderson, 2003; McDonald, Kazemi, & Kavanagh, 2013; Staples, 2007; Stein, et al., 2000). However, developing these instructional practices often requires significant learning for teachers as well as instructional leaders that support and evaluate teachers (Stein, Smith, & Silver, 1999; Thompson & Zeuli, 1999). In the next section, I review literature that has examined what teachers need to know and be able to do in order to engage in high quality fractions instruction with students.

What Do Teachers Need to Know and Understand to Teach Fractions?

Traditionally, it was thought that teachers needed to know whatever mathematics was in their curriculum as well as some additional years of college mathematics. However, in recent years, as there has been focus on creating professional standards and alignment of teacher preparation with these standards, there has been a greater focus on what constitutes professional knowledge. In the sections that follow, I review literature that speaks to the kinds of intellectual resources that teachers draw upon as they engage in the complex work of teaching. This includes *mathematical knowledge for teaching, knowledge and understanding of student learning trajectories, and personal identities for teaching.*

Mathematical knowledge for teaching. Thirty years ago, Shulman (1986, 1987) and his colleagues suggested that teaching requires a special kind of knowledge, which they termed *pedagogical content knowledge (PCK)*. In the three decades since this concept was put forth, it has generally been accepted across content areas, but it has not necessarily been used consistently. In the field of mathematics, Ball, Thames, and Phelps (2008) built upon Shulman's notion of PCK by developing a practice-based theory of content knowledge for teaching

mathematics, called *Mathematical Knowledge for Teaching (MKT)*. They define *MKT* as the “mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students” and they describe six domains; three that comprise *subject matter knowledge* and three that comprise *pedagogical content knowledge*.

Subject matter knowledge. Ball et al. (2008) identified three forms of subject matter knowledge that they suggest need to be uncovered, mapped, organized, and included as part of preparation and professional development for teachers. These include:

- *common content knowledge (CCK)*: CCK is “the mathematical knowledge and skill used in settings other than teaching.” (p. 399)
- *specialized content knowledge (SCK)*: SCK is “a specialized form of pure subject matter knowledge— ‘pure’ because it is not mixed with knowledge of students or pedagogy and is thus distinct from the pedagogical content knowledge identified by Schulman and his colleagues and ‘specialized’ because it is not needed or used in settings other than mathematics teaching” (p. 396).
- *knowledge at the mathematical horizon*: This kind of knowledge is described as “an awareness of how mathematical topics are related over the span of mathematics included in the curriculum...it also includes the vision useful in seeing connections to much later mathematical ideas” (p. 403).

Pedagogical content knowledge. Ball et al. (2008) built on Shulman’s notion of PCK by sub-dividing it into three domains, which include:

- *knowledge of content and students (KCS)*: KCS is “knowledge that combines knowing about students and mathematics” (p. 401) in relation to one another. For example, teachers need to be able to anticipate students’ responses and what will be confusing or challenging.
- *knowledge of content and teaching (KCT)*: KCT “combines knowing about teaching and knowing about mathematics” (p. 401). Knowing how to sequence particular content for instruction and evaluating the particular advantages and disadvantages of particular representations are both examples of KCT.
- *knowledge of content and curriculum*: This draws on Shulman’s (1986) notion of “curricular knowledge” which is described as “the full range of programs designed for the teaching of particular subjects and topics at a given level, the variety of instructional materials available in relation to these programs, and the set of characteristics that serve as both the indications and contraindications for the use of particular curriculum or program materials in particular circumstances” (p. 10).

Though this conceptual approach to defining knowledge for teaching results in different domains of knowledge, Ball and her colleagues (2008) do not suggest that these are used independently of one another. Empirical studies of MKT, which will be discussed in more detail in the next section, suggest that MKT is multi-dimensional. They also suggest that these categories will likely need refinement and revision as the field continues to make progress in this area.

What we have learned about teachers' MKT. Researchers in math education have responded to Schulman's notion of PCK and Ball, et al.'s notion of MKT by designing studies intended to understand or measure teacher's knowledge of various mathematical concepts. Fractions are no exception. In the last seven years, there have been a handful of studies that have focused on teachers' knowledge in relation to one or more fraction ideas. This includes fraction equivalence (Ding, 2007), multiplying fractions (Izsak, 2008), and dividing fractions (Chinnappan & Desplat, 2012; Ma, 1999). Other studies attempted to take broader perspective of fractions by attending to multiple fraction ideas, such as all four operations (Chinnappan & Forrester, 2014; Depaepe et al., 2015; Izsak, Orrill, Cohen, & Brown, 2010; Izsak, Jacobson, de Araujo, & Orrill, 2012; Rosli, Han, Capraro, & Capraro, 2013) and the inclusion of things like flexibility and transfer (Lin, Becker, Ko, & Byun, 2013; Newton, 2008).

A majority of these studies used paper and pencil assessments of teacher's knowledge, often using a combination of both content-focused items and pedagogical-focused items. Sometimes, the assessments were accompanied by interviews. The analyses of these measures typically focused on what teachers *did not* understand as well as analyses of their errors. I did not find any studies that attempted to analyze teacher responses in order to identify what they *did* understand.

Additionally, a majority of the studies were situated in a pre-service teacher preparation context (Chinnappan & Forrester, 2014; Depaepe et al., 2015; Lin et al., 2013; Newton, 2008; Rosli et al., 2013; Thanheiser, Browning, Moss, Watanabe, Garza-Kling, 2010; van de Kieboom, 2013; Van Steenbrugge, Lesage, Valcke, & Desoete, 2014). The handful of studies focused on in-service teacher knowledge varied greatly in their foci. In the only study of elementary teachers, Chinnappan & Desplat (2012) conducted a case study of four teachers (two “experienced” teachers and two “less experienced teachers) and found that all teachers’ subject matter knowledge was conceptually weak and that teachers had limited knowledge about how they could help students who might have difficulty with the kinds of problems they encountered in the study. The remaining studies focused on middle school teachers. Izsak and his colleagues have focused on developing instruments that measure mathematical knowledge needed for teaching arithmetic with fractions, decimals, and proportions. In particular, their instruments emphasize the knowledge needed to reason about such arithmetic when numbers are embedded in problem situations (Bradshaw, Izsák, Templin, & Jacobson, 2014; Izsak, et al., 2010; Izsak, et al., 2012). One additional study examined if teachers’ math knowledge contributed to their use of curriculum materials. Sleep and Eskelson (2012) found that while teachers’ MKT was necessary, there were two additional factors—teachers’ orientations toward mathematics and mathematics teaching, and teachers’ goals for student learning both contributed to teachers’ use of curriculum materials.

When Shulman’s notion of pedagogical content knowledge was introduced 30 years ago, the math education research community quickly took up the idea. However, even with Ball, Thames, & Phelps’ work to conceptualize and study Mathematical Knowledge for Teaching in the last seven years, we have much to understand—particularly when it comes to what teachers

need to know and understand to teach fractions so that all students have opportunities to develop an understanding of significant mathematical ideas. In particular, we need to go beyond identifying what teachers *don't* know or understand and analyzing their errors to studying what teachers *do* understand—particularly in contexts where students are demonstrating sophisticated understanding of fractions. We also need studies that include elementary school teachers in in-service settings. With the majority of fraction instruction occurring in grades 3, 4, and 5 (NGACBP & CCSSO, 2010), we need research that studies teachers in these grade levels. Additionally, while pre-service settings are often convenient and accessible for researchers as well as obvious sites of teacher learning, preservice programs are relatively short when compared to the time educators spend in in-service settings. This, coupled with the previous review of the literature that points towards an imbalance of research favoring pre-service settings, indicates that more research needs to happen in schools with practicing teachers.

Knowledge and understanding of student learning trajectories. Within mathematics education, significant work has happened around developing cognitive trajectories of student learning (Simon, 1997; Simon, Tzur, Heinz, & Kinzel, 2004) and supporting teachers to engage with research-based information about students' trajectories (Carpenter, Fennema, & Franke, 1997). The data from a number of studies of the Cognitively Guided Instruction (CGI) project have consistently provided links between teacher learning and student outcomes (Carpenter et al., 1997; Carpenter, Fennema, Franke, Levi, & Empson, 1999; Carpenter, Franke, Jacobs, & Fennema, 1998; Fennema et al., 1996; Fennema, Franke, Carpenter, & Carey, 1993; Franke, Carpenter, Levi, & Fennema, 2000). One prominent feature of CGI is that teachers develop a detailed understanding of the development of students' mathematical thinking within particular mathematical domains (such as whole number).

Personal identities and normative identities. Research related to teacher learning has suggested that improvement of instructional practice is not limited to developing deep understanding of content (Hill, Rowan, & Ball, 2005). Improving practices also involves reconceptualizing what it means to teach mathematics (Cobb, McClain, Lamberg, and Dean, 2003; Franke et al., 2000; Kazemi & Franke, 2004). Cobb, Gresalfi, and Hodge (2009) describe a relationship between *normative* and *personal identities for teaching*. The *normative identity for teaching* is the identity established in a school that “constitutes the basis upon which a teacher is recognized as a certain kind of teacher by school leaders and other teachers” (Gresalfi & Cobb, 2011, p. 274). It is *not* an individual or personal notion, but is instead established collectively within a given context. The *personal identity for teaching* is developed by each individual in a particular context and “concerns the extent to which he or she identifies with others’ expectations for competent teaching in that context” (p. 275). There are three cases of personal identities that Gresalfi and Cobb (2011) elaborate upon:

- *Obligations-to-others:* In this case, the teacher attempt to merely comply with the expectations of other members of a professional teaching community.
- *Obligations-to-oneself:* In this case, a teacher might come to value what counts as competent teaching and be motivated to develop instructional practices of this type.
- *Opposition:* In this case, the teacher might oppose what counts as competent teaching by developing contrary instruction practices, in the process being recognized as less competent by others members of the school community.

Gresalfi and Cobb (2011) used this framework to analyze one aspect of teacher learning—how teachers developed personal identities for teaching—in the contexts of a professional development program and their own schools—each of which had different normative identities for mathematics teaching.

My contribution. This study contributes to our understanding of what teachers know and understanding in multiple ways. First, it goes beyond identifying what teachers *don't* know and

characterizes both what teachers know and how that knowledge is used as they engage in the work of teaching. To do this, I went beyond the typical methods used to study teachers' mathematical knowledge for teaching (paper-and-pencil assessments) and used records of interaction to analyze MKT in the context of the mathematical tasks of teaching. Second, by using a situative perspective, this study attends to the complex nature of teaching, the various intellectual resources used by teachers (including their personal identities), and the meaning made within the context of their work. Finally, the study involves elementary school teachers, who are considered generalists and are responsible for planning and instructing multiple content areas. Most studies have focused on middle school teachers, who often only teach mathematics and often have had more extensive preparation for teaching mathematics.

The work of teaching is complex and what teachers need to know and understand reflects that complexity. In the next section, I review the literature focusing on school-wide efforts to support teacher learning and improve instruction.

School-Wide Efforts to Improve Instruction

As a field, math education research has attended carefully to supporting individual teachers in improving instruction (Kazemi, 2008). However, reform efforts have increasingly focused on larger-scale efforts which has challenged the more traditional model of teachers working individually with minimal collaboration or coherence beyond single classrooms and pushed math education researchers to consider how theories and frameworks from organizational learning can inform frameworks for developing high quality mathematics instruction where schools are the unit of change. Organizational learning and school reform literature have identified a number of organization conditions that are important to school development that supports collective teacher learning including: job-embedded professional development that

focuses on student learning and is situated in practice, professional learning communities, and support from school-based leadership (e.g., principals and instructional coaches).

Job-embedded professional development. A number of studies and national policy reports over the last 25 years have worked to identify the essential characteristics of professional learning for teachers (see Hawley & Valli, 1999; Hiebert, 1999; Garet, Porter, Desimone, Birman, & Yoon, 2001 for more extensive reviews). Hawley and Valli (1999) argued that “if innovations are to take root at the school level, colleagues must develop a shared understanding of the purposes, rationale, and processes involved in the innovation and believe that they can make a difference for students...Teacher efficacy is enhanced within teachers have opportunities to see new strategies modeled, practice them, engage in peer coaching, acclimate students to new ways of learning, and use new teaching and learning strategies regularly and appropriately” (p. 130). Two features that are recurring in literature on job-embedded teacher learning opportunities are (1) focusing on students and their learning and (2) deprivatizing practice by creating opportunities for teachers to intentionally engage in practice with one another.

Focus on students and their learning. The first design principle put forth by Hawley & Valli (1999) for “effective” professional development was related to goals and student performance: “Professional development should be driven by analyses of the difference between goals and standards for student learning and student performance” (p. 139). They argue that such analyses will help determine what educators *need* to learn (versus what they *want* to learn) and will help keep professional development student-centered. Grossman, Wineberg, & Woolworth (2000) also contend that students must be a central consideration in the work of professional communities, but they also acknowledge the challenge in supporting teacher learning in

communities that “maintain focus on students while also creating a structure for teachers to engage as learners with the subject matters they teach” (p. 14).

Deprivatize practice by intentionally engaging in practice together. Often teachers have limited professional networks and teach in schools where their instruction is highly privatized (Cobb et al., 2003). These norms of privacy—that work against teachers learning together in the workplace—are usually reinforced by the structure of the school day where teachers’ conversations with one another are restricted to brief moments before or after school, during lunch, or in the hallway (Grossman et al., 2000). In order to develop opportunities for teacher learning in professional communities as described by Lave and Wenger (1991) and Cochran-Smith and Lytle (1999; 2009), opportunities for collective participation—as groups of teachers from the same school, department, or grade level—need to be intentionally created (Garet et al., 2001).

Professional learning communities. To date, building and sustaining *professional learning communities* (PLCs) of teachers has been the primary mechanism for supporting teachers’ generative learning for high quality mathematics instruction (Kazemi, 2008). Though the term is used to describe various combinations of individuals, one typical PLC structure in elementary schools is that of grade-level teams. PLCs as described by DuFour (2004) is a model for teacher collaboration guided by three core principles: ensuring that students learn, creating structures that promote a collaborative culture, and focusing on results.

School-based leadership. The kinds of changes in math instruction that are envisioned by reform efforts will be difficult to make without support and guidance (Borko, 2004). While teachers can receive “support and guidance” from their colleagues, most of the literature clearly identifies that school-based leadership is responsible for taking on this role (Ball & Cohen, 1999;

Hawley & Valli, 1999). In elementary schools, school-based leadership is typically in the form of instructional coaches and principals. Research on school-based coaching has shown quite a bit of variance in the work that coaches actually do—and who they work with (Gibbons, in press). In many cases, coaching is “by (teacher) invitation” so that only a subset of teachers in a school work with an instructional coach. The role of the principal as “instructional leader” is also a recurring theme in school reform literature (Bryk, Sebring, Allensworth, Luppescu, & Easton, 2010). The literature also points to ways that instructional leaders can support school-wide teacher learning, including: making time for teacher learning (Cochran-Smith & Lyle, 1999; Cobb et al., 2003), fostering coherence (Hawley & Valli, 1999), and supporting the development of norms (Borko, 2004; Grossman et al., 2000).

My contribution. This study adds to our understanding of school-wide efforts to improve instruction by using a situative perspective to understand both teacher and student learning and how they are coordinated with one another. Typically, studies focus on either teacher learning or student learning, but not both. Second, this study focuses on how an instructional approach was implemented and sustained school-wide for multiple years, whereas most studies focus on a single classroom for one school year.

CHAPTER 3: METHODS

This study used a qualitative case study approach to understand (1) how a school-wide approach to fractions instruction was implemented and sustained and (2) how the process and experience shaped the learning of the teachers and students.

Three sub-questions guided the analysis:

- 1) How did what students know and understand about fractions change over the first three years of a school-wide approach to fractions instruction?
- 2) Through their participation in the implementation of the school-wide approach to fractions instruction, what did teachers learn?
- 3) How did the professional learning opportunities support the development of collective understanding around fractions teaching and learning?

In this study, there were multiple “phenomena” being studied, and therefore, multiple units of analysis. In order to answer the first two sub-questions about the learning of students and teachers, I used “learners-in-contexts” as my unit of analysis (Nolen et al., 2015). This allowed me to embed the teachers and students in context, which was necessary since I was concerned with learning as “stretched across” learners and contexts (versus the studying individual learners and their learning). In order to answer the third sub-question, I used a different unit of analysis: the *activity system* that existed around the approach to fractions instruction. Greeno (2006) describes an activity system as a “complex social [organization] containing learners, teachers, curriculum materials, software tools, and the physical environment” (p. 79). By using a case study approach, I was able to use multiple units of analysis to investigate the multiple variables of potential importance and provide a rich and holistic account of the phenomena as a result (Merriam, 2009). In the remainder of this chapter, I will describe the context of Hilltop Elementary, the data set, the questions that guided my analysis and my analytic methods.

Context of this Study: Setting, Participants, and Timeframe

A *purposeful* sampling procedure was used to select the school for this study (Patton, 2003). The unique features of Hilltop Elementary include (a) the use of a school-wide approach to fractions instruction, that was (b) sustained over multiple years, with (c) improved student learning outcomes that outpaced peers at both the district and state level. There were two levels of sampling in this study: (1) the setting (selection of the school) and (2) the participants (selection of the teacher and student participants within the school). In the next section, I elaborate on the sampling strategies for the setting and participants, as well as the decisions for bounding the timeframe of the study.

Study Setting: Hilltop Elementary

The case for this study was Hilltop Elementary, an urban elementary school in the Seattle area. In the spring of 2011, Hilltop's student achievement scores on the state's standardized test were among the lowest 5% of Title 1 schools. As a result of consistently low student achievement scores, the school was mandated to enter a process of school improvement and was subsequently awarded a three-year School Improvement Grant (SIG). The grant had multiple requirements that resulted in a number of changes prior to the 2011-2012 school year. These changes included a new principal and full-time math coach (both who had instructional coaching experience prior to coming to Hilltop), an additional 30 minutes added to every school day and 5 additional school days per year, and new evaluation requirements tied to student achievement. Teachers were also given the option to stay at Hilltop or relocate to another school in the district. The grant also led to a partnership with math educators at a local university, with a focus on improving math learning opportunities for students. At the end of the three-year SIG, the growth

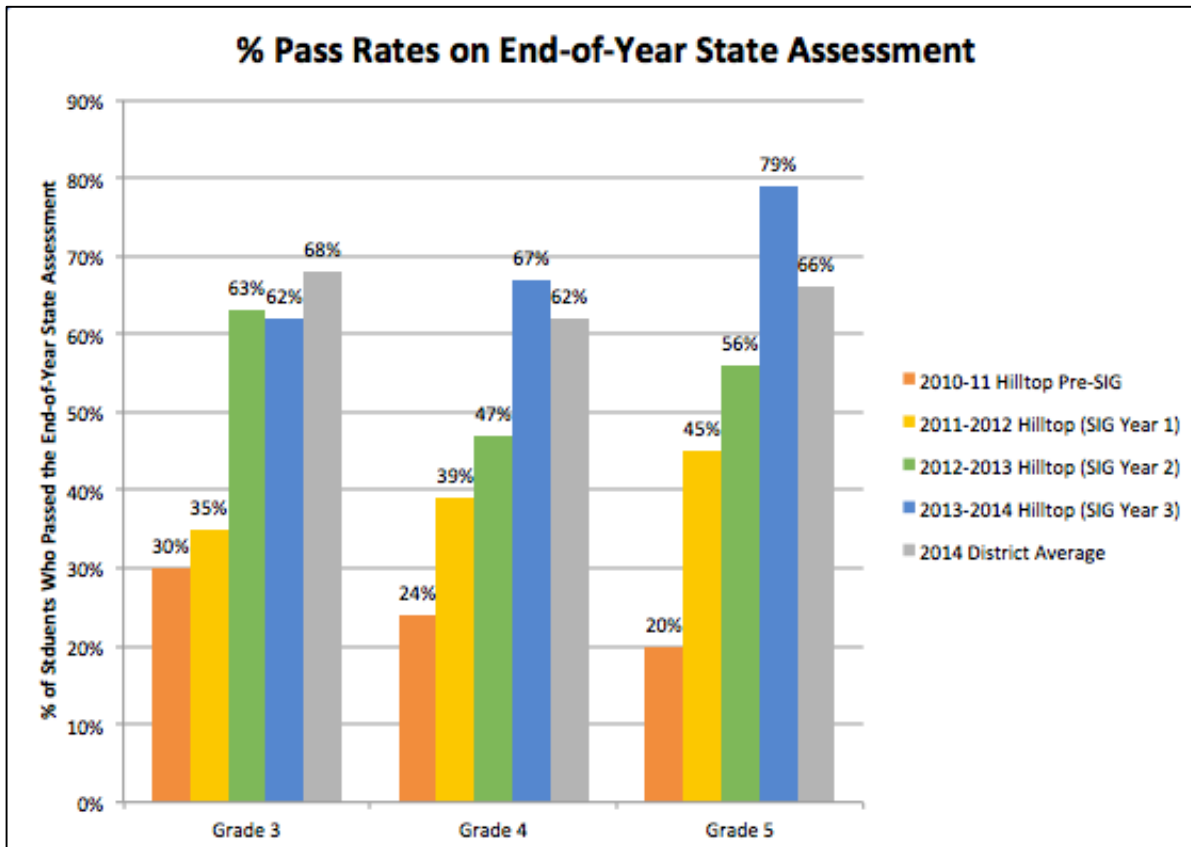


Figure 2. Student achievement scores prior to the SIG and the three years of the SIG.

of student achievement scores outpaced those of the district and the state, and at the end of Year 3, students' scores were at or above those of the district and the state (see Figure 2).

My relationship with Hilltop Elementary. My initial role with Hilltop Elementary was as part of the university team that partnered with the school to support them in reorganizing their school to support teacher and student learning. The first two years of the grant, I was also a staff member at Hilltop Elementary, where I worked as a part-time math facilitator & coach. During the three-year SIG period (August 2011 to June 2014), the university team met regularly with the Hilltop instructional leaders and teachers to work on curriculum and to co-design and co-facilitate professional development days. The data collected over those three years (e.g., unit plans, math lab agendas, and student interviews) are artifacts from the partnership. During the fourth and fifth years of school-wide reform (beginning Fall 2014), our relationship with the

school changed from guidance to study of their success as the grant ended. The interviews with teachers and the math coach were collected as part of this research relationship.

Mathematics instruction at Hilltop Elementary. At the start of the 2011-2012 school year, the master schedule established 75-minutes of mathematics instruction daily for all grade levels and this continued across the three years of the grant. The first year of the SIG, teachers used the district pacing guides to guide their instruction. The pacing guides coordinated a number of curriculum materials (though *Investigations in Number, Data, and Space* was the primary resource) into “units of instruction” with daily lesson plans. The pacing guides had previously been aligned to state standards by district staff and teachers. During that first year, the math coach began revising the pacing guides to incorporate her and teachers’ learning about mathematics and teaching mathematics. The math coach, with input from teachers, continued to modify the unit plans each year. Her revisions are in response to students and what the teachers are noticing about their thinking as well as in response to new learning about mathematics and how to teach it. For example, in the second year, she added Empson & Levi’s (2011) partitioning trajectory to the fractions pacing guides and in the third year, she added instructional activities connected to the daily lessons in all pacing guides.

Fractions instruction at Hilltop Elementary. During the first year of the SIG, the math coach and teachers at Hilltop adopted a new approach to teaching fractions following the recommendations from Empson & Levi (2011). This approach, sometimes called “an equal sharing approach” involves introducing the concept fractions through the context of sharing, introducing language and notation as the need arises during the sharing problems, and using strategies based on reasoning about number relationships to compare and operate with fractions.

Teacher learning at Hilltop Elementary. Supporting teachers’ learning was an additional component of the SIG. The instructional leaders at Hilltop strategically created a number of spaces and opportunities for teachers to learn with one another. This included both physical spaces (such as the Teachers’ Classroom¹) and temporal spaces such as Math Labs, weekly team meetings, and one-on-one coaching.

Math labs. Math Labs began in the first year of the SIG. They were school-based professional learning opportunities where teachers were released from their classrooms during the school day. Math Labs were co-facilitated by the university-based math educators and the school-based math coach. Initially, teachers attended a total of 8 full-day math labs with colleagues from their own grade level as well as one other grade level. Over the five years, the structure changed based on the school’s resources (e.g., the grant funds decreased each year) and goals for teacher learning (e.g., math labs were shortened to half days so teachers could also have a half-day literacy lab on the same day) (see Table 2).

In math labs, teachers engaged in activities that supported them in learning about mathematical content, instructional practices, and children’s thinking. Their work was organized around a “learning cycle” with four phases: learning together, co-planning a lesson, trying it out,

Table 2. Structure of Math Labs across five years.

School Year	Number of Labs Attended by Each Teacher	Structure of a Typical Lab
2011-12 (SIG Year 1)	8	Full Day; 2 grade levels per lab (e.g., Grade 2 & 3 together)
2012-13 (SIG Year 2)	6	Full Day; 1-2 grade levels per lab
2013-14 (SIG Year 3)	4	Full Day; 1 grade level per lab
2014-2015	4	Half Day; 1 grade level per lab
2015-2016	4	Half Day; 1 grade level per lab

¹ The “Teacher’s Classroom” was a classroom that was set-up and designated for teacher learning. All three instructional coaches had desks in this room and there were adult-size tables and chairs. The space was reserved for math labs and other professional development sessions.

and debriefing together. In the “learning phase,” teachers learned about new content such as “talk moves,” research-based student learning trajectories (such as those from CGI), new representations for mathematical content, and the Common Core mathematical content and practice standards. In the second phase, “preparing,” the group decided what they wanted to investigate together in a classroom based on what they learned in the first phase and/or had been noticing in their classrooms. They co-planned a lesson and one or more people volunteered or were invited to take the lead (though because everyone had contributed to the planning, it was still “the group’s” lesson). In the third phase, the group “enacted” the lesson in a classroom. The enactment was experimental in nature and was not intended to be a model lesson. They were often trying out a new task or attempting to uncover what ideas and strategies the students had about a particular mathematical idea. One important routine used as part of the classroom visits was “teacher time outs.” Similar to a time out in a sporting event, it involves pausing the lesson in the moment and talking briefly together as teachers about an instructional decision.² For example, a teacher may have thought of a great follow-up question or had an idea about how to represent a students’ thinking. Since the lesson was “co-owned,” teacher time outs allowed teacher to work on instruction together in the moment. In the fourth phase “Reflect,” the group debriefed the classroom experience including what they learned about students’ thinking, mathematics, and teaching.

Weekly PLC meetings. At the beginning of the SIG, grade level teams (called PLCs at Hilltop) met weekly. These meetings typically alternated their focus between mathematics and literacy. During the first year, the instructional leaders and teachers made structural changes to the weekly PLC meetings. These changes included adding a second weekly meeting so that

² See Gibbons, Hintz, Kazemi, & Hartmann (2015) for an in-depth analysis of the Teacher Time Out routine.

teachers were had dedicated time focused on both math and literacy on a weekly basis. Also, the PLC meetings were scheduled so that the math coach could attend all six math PLCs in a given week. Finally, the principal also joined each grade level for either their math or literacy PLC. In the PLCs with a math focus, the math coach typically brought a loose agenda and facilitated the meeting, but was very responsive to teachers' needs and teachers reported that they frequently contributed to the agenda both prior to the meetings or in the moment.

One-on-one coaching.

Participating in one-on-one coaching was an expectation for all teachers at Hilltop Elementary. A master schedule was created so that all teachers in a given grade level taught mathematics at the same time. The math blocks were also spread across the day so that the math coach could support math instruction across the day and spend a majority of her time in classrooms. The math coach supported teachers in a variety of ways during this time. She modeled new instructional activities or practices, she co-taught using teacher time outs, and she huddled with teachers during student work time to analyze and make sense of student thinking.

Participants

Teacher participants. The sampling of participants in this study was complicated by mobility among both teachers and students—both within school years and between school years. Each school year, there were teachers that left Hilltop, teachers that were hired at Hilltop, and teachers that changed grade levels. In order to include as many teachers as possible who had recent experience teaching fractions, I invited all of the current Grade 3, 4, and 5 teachers as well as the Special Education and English Language teachers. Nine (out of a possible ten) teachers participated in the interviews. Table 3 shows the study participants, their teaching experience

Table 3. Study participants.

Participant	Teaching Experience	SIG Years			Year 4	Year 5
		Year 1	Year 2	Year 3		
Layla	15 years before the SIG (all at Hilltop)	Grade 3	Grade 3	Grade 3	Grade 3	Grade 3
Leslie	8 years before the SIG (all at Hilltop)	Grade 4	Grade 4	Grade 3	Grade 5	Grade 5
Anna	4 years before the SIG (all at Hilltop)	Grade 1	Grade 1	Grade 1	Grade 4	Grade 3
Kyle	1 year before starting at Hilltop in Year 1 of the SIG	Grade 2	Grade 2	Grade 2	Grade 2	Grade 4
Becky	0 years before starting at Hilltop in Year 1 of the SIG	Grade 4	Grade 4	Grade 4	Grade 4	Grade 4
Tonya	3 years before starting at Hilltop				Grade 4	Grade 5
Keri	0 years before starting at Hilltop				Grade 3	Grade 3
Erin	3 years before starting at Hilltop in Year 2 of the SIG		Resource Teacher	Resource Teacher	Resource Teacher	Resource Teacher
Kathleen	16 years before starting at Hilltop in Year 2 of the SIG		English Language Teacher	English Language Teacher	English Language Teacher	English Language Teacher
Tara	20 years teaching and 5 years coaching before starting at Hilltop in Year 1 of the SIG	Math Coach	Math Coach	Math Coach	Math Coach	Math Coach

prior to the SIG or coming to Hilltop and their teaching assignments during the five years included in this study.

Student participants. Hilltop Elementary serves a diverse population. Approximately 87% of all students received free or reduced-price lunch, 29% of students were classified as transitional bilingual, and 13% qualified for special education services. Approximately half of the families reported speaking a language other than English at home. The school population, as reported to the state, consisted of 50% Black (and we estimate 20% of Black students spoke Somali or Amharic at home), 17% Hispanic, 13% Asian/Pacific Islander, 12% White, and 7% two or more races.

All students, kindergarten through fifth grade, were interviewed at five different points in time: Fall of SIG Year 1, Spring of SIG Year 1, Fall of SIG Year 2, Spring of SIG Year 2, and Spring of SIG Year 3. I analyzed the student interviews by cohort which I defined as a group of

Table 4. Timeline of interviews and fractions instruction for each cohort.

	Year 1 Fall		Year 1 Spring	Year 2 Fall		Year 2 Spring		Year 3 Spring
Cohort 1	Grade 5	Instruction	Grade 5		Instruction		Instruction	
Cohort 2	Grade 4		Grade 4	Grade 5		Grade 5		
Cohort 3	Grade 3		Grade 3	Grade 4		Grade 4		
Cohort 4				Grade 3		Grade 3		
Cohort 5						Grade 3		

Table 5. Number of interviews for each cohort at each time point.

	Year 1 Fall		Year 1 Spring	Year 2 Fall		Year 2 Spring		Year 3 Spring
Cohort 1	53	Instruction	58		Instruction		Instruction	
Cohort 2	58		53	51		50		
Cohort 3	78		75	72		70		
Cohort 4				51		47		
Cohort 5						71		

students in the same grade level at the same time. Using this definition, the student interviews I analyzed included five different cohorts of students. Table 4 shows a timeline of the interviews and instruction for each of the five cohorts whose interviews I analyzed and Table 5 shows the number of interviews per cohort at each time point.

Students in Cohort 1 were fifth-graders at Hilltop Elementary during the first year of the School Improvement Grant. These students participated in, at most, one year of the school-wide approach to fractions instruction and two interviews (Fall of Grade 5 and Spring of Grade 5). 73% of students in Cohort 1 participated in both interviews and 27% participated in only one interview.

Students in Cohort 2 were fourth-graders at Hilltop Elementary during the first year of the School Improvement Grant. These students participated in, at most, two years of the school-wide approach to fractions instruction and four interviews (Fall of Grade 4, Spring of Grade 4, Fall of Grade 5, and Spring of Grade 5). 39% of students in Cohort 2 participated in all four

interviews; 7% participated in three interviews; 32% participated in two interviews; and 22% participated in one interview.

Students in Cohort 3 were third-graders at Hilltop Elementary during the first year of the School Improvement Grant. These students participated in, at most, three years of the school-wide approach to fractions instruction and five interviews (Fall of Grade 3, Spring of Grade 3, Fall of Grade 4, Spring of Grade 4, and Spring of Grade 5). 26% of students in Cohort 3 participated in all five interviews; 9% participated in four interviews; 14% participated in three interviews; 34% participated in two interviews; and 17% participated in one interview.

Students in Cohort 4 were second-graders at Hilltop Elementary during the first year of the School Improvement Grant. These students participated in, at most, four years of the school-wide approach to fractions instruction and five interviews (Fall of Grade 2, Spring of Grade 2, Fall of Grade 3, Spring of Grade 3, and Spring of Grade 3). However, because this study did not include Grade 2 interviews, at most, three sets of interviews (Fall of Grade 3, Spring of Grade 3, and Spring of Grade 4) were used for this analysis. 52% of students in Cohort 4 participated in all three interviews used for analysis; 18% participated in two interviews; and 30% participated in one interview.

Students in Cohort 5 were first-graders at Hilltop Elementary during the first year of the School Improvement Grant. These students participated in, at most, five years of the school-wide approach to fractions instruction and five interviews (Fall of Grade 1, Spring of Grade 1, Fall of Grade 2, Spring of Grade 2, and Spring of Grade 2). However, because this study did not include Grade 1 and 2 interviews, only one set of interviews (Spring of Grade 3) was used for this analysis.

If students were absent, “make-up” interviews were typically scheduled later that same week or the following week. Excluded from this study were (1) students in the Grades 3-5 self-contained special education classroom since the interviews were modified for these students and (2) students who we were not able to fully interview without additional translation support.

Timeframe

The beginning of the study was naturally bounded by the start of the School Improvement Grant (SIG) in August 2011. This was the point in time when the partnership between Hilltop and university math educators began, the first year of implementing a new approach to teaching fractions, and was also when the first set of student interviews were collected. The study spanned five academic school years, including the three years of the SIG and two years post-SIG. Though the last set of student interviews were collected in May 2014, the interviews with the math coach and teachers were conducted in Winter 2016 and naturally included teachers’ learning in the time since the grant ended.

Data Sources and Collection

The research question for this case study sought to understand how a school-wide approach to fractions instruction was implemented and how it shaped teacher and student learning. As a way of approaching this question, I decomposed it into three sub-questions:

- 1) How did what students know and understand about fractions change over three years of a school-wide approach to instruction?
- 2) Through their participation in the implementation of the school-wide approach to fractions instruction, what did teachers learn about teaching fractions?
- 3) How did the professional learning opportunities support the development of collective understanding around fractions teaching and learning?

In order to answer these questions, I used multiple data sources. The various data sources frequently complemented one another and provided opportunities to triangulate across participants and data sources in order to develop an in-depth understanding of the activity system

and how it shaped teacher and student learning. The different data collection methods and sources include (1) audio and video recorded observations of Math Labs, PLC meetings, and one-on-one coaching, (2) semi-structured interviews with teachers and the math coach, (3) documents (such as “unit plans” that guided teachers’ fraction instruction as well as detailed facilitator plans for Math Labs), and (4) cognitive style interviews with students. Because some of the data are artifacts from the partnership with the school, which was not at the time set up as a formal research partnership, there is variance across years and settings (i.e., we do not have recordings from every moment in every Math Lab, but we do have recordings from within multiple labs across each year). The artifacts I do have provided insight into interactions in multiple contexts, among multiple actors, and across the 5 years included in this study.

Video and Audio Recordings of Math Labs, PLC Meetings, and One-on-One Coaching

There were 121 video and/or audio recordings across the five years documenting work on fractions in the context of Math Labs, PLCs, and one-one-one coaching. These data sources were of primary importance to this study because they provided an opportunity to analyze collaborative discourse, or group thinking made visible: “The evidence that participants provide each other through their collaborative discourse informs them about their understanding, goals, intentions, and expectations, and it provides evidence to the researcher about semiotic structures that are being generated and used” (Greeno, 2006, p. 86). After an initial review of the full set of data, I selected a subset of this data. When making these selections, I considered the following criteria:

- Across all 3 professional learning settings (Math Labs, PLCs, and coaching)
- Across Grades 3, 4 and 5
- Data that I collected
- Data that included extensive field notes (my own or from other researchers on our team)
- Data that included teachers who also participated in my interviews

Table 6 provides an overview of learning events that were included in this analysis.

Table 6. Learning events included in analysis.

Learning Event	Year	Grade Level	Participants
Math Lab on 12/10/14	4	4	Math Coach, Principal, Grade 4 teachers, English Language Teacher
Math Lab on 12/2/15	5	5	Math Coach, Principal, Grade 5 teachers, English Language Teacher
Math Lab on 12/9/15	5	4	Math Coach, Principal, Grade 5 teachers, English Language Teacher
Math Lab on 1/4/16	5	3	Math Coach, Principal, Grade 3 teachers, Resource Room Teacher
PLC on 12/11/14	4	4	Math Coach, Grade 4 teachers
PLC on 1/8/15	4	4	Math Coach, Grade 4 teachers
PLC on 1/8/15	4	3	Math Coach, Grade 3 teachers
PLC on 1/22/15	4	4	Math Coach, Grade 4 teachers, English Language Teacher
PLC on 1/29/15	4	4	Math Coach, Grade 4 teachers
PLC on 1/29/15	4	3	Math Coach, Two Grade 2 teachers
PLC on 2/5/15	4	4	Math Coach, Grade 4 teachers, English Language Teacher
Coaching on 1/8/15	4		Math Coach, Two Grade 4 teachers
Coaching on 1/12/15	4	4	Math Coach, Two Grade 4 teachers

Semi-Structured Teacher and Coach Interviews

Interviews were used to complement the records of interaction. Interviews provided a unique opportunity to capture participants' experiences and the meaning they make of those experiences in their own words, including rich, thick descriptions from their own points of view and perspective (Creswell, 2007). This included capturing how teachers view their work, the terminology they use, and the complexities of their individual perceptions and experiences (Patton, 2003). In particular, by using semi-structured interviews (instead of, for example, highly structured interviews), I had the opportunity to clarify statements, probe for additional information, and respond to new ideas that emerge (Merriam, 1998). I used the interview protocols in Appendix A to initiate and guide the interviews, but also asked questions that were responsive to what I heard from participants in the moment.

Teacher interviews. The teacher interviews ranged from 38 to 92 minutes in length, with an average length of 62 minutes. Interviews were audio recorded and I took field notes during the interview. Following each interview, I created an analytic memo recording initial interpretations related to teachers' perceptions of fractions instruction at their school, their experiences related to teaching fractions and the meaning they made of those experiences, as well as how they characterize their identity as a teacher of mathematics.

Math coach interviews. I interviewed the math coach twice, approximately six weeks apart. The first interview was 64 minutes in length and occurred prior to the teacher interviews; the second interview was 54 minutes in length and occurred after the teacher interviews. This interviews focused on understanding the properties of the activity system (such as the participants, the tools, the informational structures, and the practices of the participants), including how things have changed over time—and why. It also focused on understanding the coach's perception of fractions instruction at their school, her experiences related to teaching fractions and the meaning she made of those experiences, as well as how she characterizes her identity as a math coach. I also analyzed two interviews done by colleagues in Year 4. The first was a 45-minute interview done two days prior to the 12/10/14 Math Lab and included Tara's thinking about the upcoming math lab. The second interview was a 55-minute interview done at the conclusion of the Year 4 school year and included questions about her vision of high quality mathematics instruction.

Student interviews. Early in our partnership with Hilltop Elementary, we designed a school-wide, interview-style assessment in order to better understand why the mathematics test scores had been consistently very low. By listening to students explain their thinking and reasoning, we were able to collect data around students' understanding of the problem, strategy

use, and accuracy. Ginsberg (1981) makes the case for why interviews, in addition to observations, are important for understanding students' ideas:

It would be extremely valuable to observe, without interfering, how children add in the natural setting or make spontaneous use of concepts of infinity. But such observations, however desirable, are exceedingly difficult to make, partly because much of children's thought is private and partly because the occasions on which the thought is public are few and far between. (p. 5)

All students, kindergarten through fifth grade, were interviewed at five different points in time: Fall of SIG Year 1, Spring of SIG Year 1, Fall of SIG Year 2, Spring of SIG Year 2, and Spring of SIG Year 3. This study used only the "equal sharing" task that was posed to Grade 3, 4, and 5 students at all five points in time (see Appendix C). I excluded interviews with students in the Grades 3-5 self-contained special education classrooms since their interviews and items were modified. When interviewing students, assessors followed a protocol (but not a script) and for the equal sharing item, typically asked students about their partitioning strategy, their drawing, and how to say or write their answer (or sometimes both).

Addressing limitations of interview methods. Although interviews have many strengths as a data source, they also have limitations (Yin, 1998). First, not all participants are equally cooperative, articulate, and perceptive. To address this limitation, I invited all teachers who fit the sampling criteria in order to have a broad range of responses. Second, interviews require skill and technique on the part of the researcher. Prior to interviewing teachers, I piloted the protocol and practiced my interviewing skills by interviewing a former Hilltop teacher who did not fit the sampling criteria. Third, interviews are often critiqued for being biased tools of data gathering since they are the result of an interaction between and interviewer, an interviewee, and the context in which they take place (Fontana & Frey, 2003; Rubin & Rubin, 2011; Schwandt, 1997). To address this limitation, I triangulated interview data with information from other

sources—such as interviews from other participants, video and audio recordings of various learning events, the unit plans for fractions instruction, and math lab agendas.

Document Analysis

While Yin (1998) cautions that “documents must be carefully used and should not be accepted as literal recordings of events that have taken place,” (p. 81) the data that is found in such documents can provide descriptive information, verify emerging hypotheses, advance new categories and hypotheses, and track change and development (Merriam, 2009). This study primarily included two kinds of documents: the unit plans that teachers used to guide fractions instruction and artifacts (including detailed agendas) from math labs.

Unit plans for fractions instruction. The first year of the SIG, the math coach developed new unit plans for fractions instruction drawing on multiple resources. Each subsequent year, the math coach revised the unit plans for fractions instruction. The unit plans include an overview of the unit (with related Common Core content and practice standards, math vocabulary, and assessments to be used) as well as a tentative day-by-day plan including the big idea or focus question, an instructional activity, and the components of the main lesson for each day.

Math lab agendas and associated artifacts. Prior to each of the math labs, the math coach and/or university math educator created an agenda for the day. These agendas document the goals for teacher learning and how learning opportunities were structured. Some of the agendas also include facilitators’ reflections and/or field notes documenting conversations held during the day. So, while the agendas themselves are not a record of what actually occurred, some of them include additional notes that act as a record of sorts (i.e., while the notes were taken for purposes other than research, they are a record of the events from the perspective of the

math coach or university math educator). For this study, only the agendas (and any artifacts included in the agenda, such as handouts or readings) that include work on fractions were included.

Data Analysis

The purpose of this qualitative case study was to understand how a school-wide approach to fractions instruction shaped teacher and student learning and supported the development of collective understanding. In order to accomplish this, I developed a set of analytic questions for my sub-questions:

- 1. How did what students know and understand about fractions change over the first three years of a school-wide approach to fractions instruction?**
 - a. In what ways did students' partitioning strategies, fraction representations, fraction language, and written fraction notation change?
 - b. How did the "instructional start points" for each grade level change from year to year?
 - c. How did students' understanding differ for students who were at the school for various amounts of instruction (i.e., students who were new to the school as fifth graders versus students who were at the school for 3 years of instruction)?
- 2. Through their participation in the implementation of the school-wide approach to fraction instruction, what did teachers learn about teaching fractions?**
 - a. What do teachers understand about fractions content? (MKT)
 - b. What visions of fraction instruction are described by teachers? (VOI)
 - c. What do teachers understand about how students' understanding of fractions develops?
 - d. How do teachers describe their personal identities as math teachers? What is the prevailing normative identity regarding teaching mathematics? What is the relationship between personal identities and normative identities?
- 3. How did the professional learning opportunities support the development of collective understanding around fractions teaching and learning?**

These questions guided my analysis as I looked for broad themes and categories related how the activity system shaped teacher and student learning (sub-questions #1 and 2) and the properties of the activity system and the principles of coordination between those various components that supported the development of collective understanding (Greeno, 2006) (sub-question #3). The analytic questions were informed by my conceptual framing and review of the literature and

were revised numerous times throughout data collection and analysis. Additionally, as in most interpretive studies, the data analysis in this study was ongoing and iterative (Merriam, 2009).

Part 1: Analyzing Student Learning

Table 7 provides a summary of my analytic process for sub-question #1. I began by coding the student interviews using a coding scheme that I modified and expanded from our school-university partnership. The modifications drew on my review of the literature as well as an initial round of coding where I coded the interviews from six classes across various time points and grade levels. This initial round of coding helped clarify existing codes as well as identify new codes. Once I reached a point where my codes were exhaustive across two class sets, I coded the entire set of student interviews using my final codebook. I provide additional coding details in the next four sections.

Understanding the problem as a sharing situation. For this aspect, I looked for evidence that the student understood the problem as a sharing situation. This typically appeared in students' drawings or in their verbal explanations. Other codes in this part included "Added" (used when students added the number of sharers and the number of objects together), "Multiplied" (used when students multiplied the number of sharers and the number of objects together), and "Did not understand" (used when the student provided no evidence of understanding the problem as a sharing situation and did not add or multiply number of sharers and the number of objects).

Partitioning and sharing strategies. I used Empson & Levi's (2011) classification of partitioning and sharing strategies with one modification. When coding students' partitioning and sharing strategies for interviews that took place during the second and third years of the SIG, I found that many students initially used a "Share Groups of Items" strategy (i.e., they gave each

Table 7. Overview of process for analyzing student learning.

Question	Analyzed for...	Relevant Data	Approach to Analysis
<p><u>Question 1:</u> How did what students know and understand about fractions change over the first three years of a school-wide approach to fractions instruction?</p>	<p>In what ways did students' partitioning strategies, fraction representations, fraction language, and written fraction notation change?</p>	<p>Cognitive-Style Student Interviews</p> <ul style="list-style-type: none"> • Student work and field notes from interviews were used to understand the meaning students made of the problem and fractions, how students solved the problem, and what kinds of language and notation students used. 	<p>Phase 1:</p> <ul style="list-style-type: none"> • Each student interview was coded for a number of aspects: <ul style="list-style-type: none"> ○ Understanding of the problem as a sharing situation ○ Use of (verbal) fraction language ○ Use of symbolic notation ○ Drawing/Representation ○ Partitioning/Sharing strategy ○ Kinds of errors <p>Phase 2:</p> <p><i>Part A:</i> Once all student interviews were coded, I created pivot tables and then various summary tables organized by grade level and year. These summary tables were examined across the three years for changes in:</p> <ul style="list-style-type: none"> • Understanding of the problem as a sharing situation • partitioning strategies, • drawing/representation accuracy and errors, • use of fraction language, • use of symbolic notation, and • error patterns. <p><i>Part B:</i> I also used the pivot tables to create summary tables to examine how the “instructional start points” varied over the three years.</p> <p><i>Part C:</i> I created additional pivot tables and summary tables based on how long students were at Hilltop in order to examine how students’ responses varied based on the amount of Hilltop fractions instruction received.</p>
	<p>How did the “instructional start points” for each grade level change from year to year?</p>	<p>Teacher and Math Coach Interviews</p> <ul style="list-style-type: none"> • The teacher and math coach interviews supplemented my analysis of the student interviews. In particular, they were used to triangulate findings from the student interviews. 	
	<p>How did students’ understanding differ for students who were at the school for various amounts of instruction (i.e., students who were new to the school as fifth graders versus students who were at the school for 3 years of instruction)?</p>	<p>Unit Plans</p> <ul style="list-style-type: none"> • The unit plans supplemented my analysis of the student interviews. In particular, they were used to triangulate the findings from the student interviews, especially how “instructional start points” changed from year to year. 	

person a whole as their first step). However, students who started this way, did not necessarily use this same strategy when there were fewer objects than people. As a result, I found that I needed to make a distinction between a student's *primary partitioning strategy* and a student's *secondary partitioning strategy*. A student's primary partitioning strategy is the strategy that the student uses at the start of the problem, when the number of objects to share was greater than the number of people. For students whose primary partitioning strategy was to "Share Groups of Items," I also coded a secondary partitioning strategy, which was the student's strategy for sharing when the number of objects to share was fewer than the number of people. In addition to coding students' strategies, I also coded the actual partitions students made (e.g., "all sandwiches cut into sixths" or "6 sandwiches left as wholes & 2 sandwiches cut into thirds").

Language and notation. Paying attention to student's use of fraction language and notation was an area of learning for the interviewers. During the first round of interviews, many students were unable to provide answers because they did not attempt or understand the problem, got stuck, or didn't have a way to represent their answer other than the visual representation they created. In the subsequent rounds of interviews, as an increasing number of students used language and notation to represent their thinking, we realized that we needed to ask students for both the language and notation—not just one of the forms. As a result, the data set for language and notation are not as complete as the other features of students' responses. However, by developing a broader coding scheme, I was still able to analyze students' use of language and notation. After my initial round of coding which included separating language and notation, I developed a new coding scheme and recoded the students' responses into one of three categories: both language *and* notation, either language *or* notation, or neither language *nor* notation (see Table 8).

Table 8. Language and notation coding scheme.

Code	Code was used when...
Both language AND notation	<ul style="list-style-type: none"> • The interviewer asked for both language and notation, and both were accurate.
Either language OR notation	<ul style="list-style-type: none"> • The interviewer documented either language or notation and it was accurate, or • The interviewer documented both language and notation and one of them was accurate.
Neither language NOR notation	<ul style="list-style-type: none"> • The interviewer documented neither language nor notation. When this occurred, it was because the student did not have an answer. • The interviewer documented language or notation and it was inaccurate. • The interviewer documented language and notation and both were inaccurate.

Student’s drawings. I coded two aspects of students’ drawings. First, I coded whether or not the drawing was accurate. Second, I coded any errors in the drawing (e.g., “unequal sixths” or “incorrect number of people”). Occasionally, in coding students’ drawings, I needed to rely on assessors’ notes to help interpret whether inaccuracies were related to the student’s strategy or fine motor development.

Once all student interviews were coded, I created pivot tables for each of the twelve aspects. The pivot tables were used to create a variety of summary tables and graphs that I used to look for changes over time, as well as patterns and trends in student thinking.

Part 2: Analyzing Teacher Learning

Table 9 provides an overview of my analysis process for sub-question #2. I began by reviewing the collection of video and audio recordings and made decisions about the subset of data to include. I used Studiocode to code the recordings of the interviews, Math Labs, PLC meetings, and one-on-one coaching visits and Microsoft Word to code unit plans and Math Lab agendas. During the initial reading and open coding, I asked questions of the data such as, “What informational structures are being used, generated, and understood?”, “How are these structures

Table 9. Overview of process for analyzing teacher learning.

Question	Analyzed for...	Relevant Data	Approach to Analysis
<p>2. Through their participation in the implementation of the school-wide approach to fractions instruction, what did teachers learn about teaching fractions?</p>	<p>What do teachers understand about fractions content?</p>	<p>Teacher and Math Coach Interviews <i>and</i> Video/Audio Recordings of Math Labs, PLC Meetings, and One-on-one Coaching The interviews were coded for various kinds of intellectual resources in order to understand the meaning that is made and how it is used in various contexts (e.g., math labs, PLCs, and classroom instruction):</p> <ul style="list-style-type: none"> • Mathematical knowledge for teaching, • Visions of high quality fractions instruction, and • Student learning trajectories for fraction ideas and strategies. <p>These data sources were also coded for aspects of teacher’s personal identities as well as the normative identity of the school.</p> <p>Unit Plans & Math Lab Agendas The unit plans were analyzed to understand the kinds of knowledge that might be expected as teachers participate in math labs or use unit plans, including:</p> <ul style="list-style-type: none"> • Mathematical Knowledge for Teaching, • Visions of high quality fractions instruction, • Student learning trajectories for fraction ideas and strategies, and • Tool and routine use. <p>These data sources were also examined for aspects of the school’s normative identity.</p>	<p>Phase 1:</p> <ul style="list-style-type: none"> • Reviewed collection of video & audio data and made sampling decisions. <p>Phase 2:</p> <ul style="list-style-type: none"> • Used Studiocode to code recordings of interviews, math labs, PLCs, and one-on-one coaching. • Use Microsoft Word to code unit plans, and math lab agendas. • Revised and refined code list throughout coding. <p>Phase 3:</p> <ul style="list-style-type: none"> • Created summary files for each participant and learning activity (e.g., math lab, PLC meeting, coaching). • Created summary files for each of the high level/parent codes. <p>Phase 4:</p> <ul style="list-style-type: none"> • Within each participant and learning activity, I analyzed excerpts looking for patterns and created summary tables that illustrated the various kinds of intellectual resources that teachers used, understood, and generated. • I also analyzed the parent code summary files for evidence of knowledge and understanding that was shared among multiple teachers. • Analyzed excerpts from “records of interaction” that had been coded as “mutual understanding” to identify what was being mutually understood and how that was occurring in the interaction.
	<p>What visions of fractions instruction are described by teachers?</p>		
	<p>What do teachers understand about how students’ understanding of fractions develops?</p>		
	<p>How do teachers describe their personal identities as math teachers? What is the prevailing normative identity regarding teaching mathematics? What is the relationship between personal identities and normative identities?</p>		

being used—and for what purposes?”, “How are people participating in this context?”, and “What are the personal identities and normative identities communicated through interaction?” Throughout the coding process, I revised and refined the code list (see Appendix B for final codebook). Once the data was coded, I created summary files for each participant and learning activity (e.g., each of the four math labs had its own summary file). I used these summary files to create a summary file for each of my parent codes (e.g., one summary file for MKT, one file for VOI, etc.). This was followed by examining the various summary files for “patterns, themes, and regularities as well as contrasts, paradoxes, and irregularities” in the coded data (Coffey & Atkinson, 1996, p. 47). The patterns and themes identified during focused coding were used to make assertions about the meaning made by teachers—related to mathematical knowledge for teaching, visions of high quality instruction, how student understanding develops, and how they identify as teachers of mathematics (Emerson et al., 1995). As assertions were drafted, I revisited the various data sources to look for both confirming and disconfirming evidence. When disconfirming evidence was found, I revised the assertion or brought additional perspectives to the evidence in an attempt to better understand the data.

Part 3: Analyzing How Professional Learning Opportunities Supported the Development of Mutual Understanding

Table 10 provides an overview of my analysis process for sub-question #3. Analysis began with open coding of the teacher interviews, math labs, PLCs, and one-on-one coaching in Studiocode. I used an analytic approach described by van de Sande & Greeno (2012) to identify episodes where participants initially lacked mutual understanding but then achieved mutual understanding. Once these episodes were identified, I reviewed them multiple times to look for initial patterns, themes, and “critical events” that might help characterize what was mutually

Table 10. Overview of process for analyzing mutual understanding.

Question	Relevant Data	Approach to Analysis
<p>How did the professional learning opportunities support the development of collective understanding around fractions teaching and learning?</p>	<p>Video/Audio Recordings of Math Labs, PLC Meetings, and One-on-one Coaching The records of interaction were used to analyze what mutual understandings existed among teachers and how the various professional learning opportunities supported teachers to develop mutual understanding.</p> <p>Teacher and Math Coach Interviews The interviews supplemented the analysis of the records of interaction. I analyzed them for teachers’ descriptions of their interactions with colleagues that involved the development of mutual understanding.</p> <p>Agendas for Fractions-Based Math Labs <i>and</i> Unit Plans for Fractions Instruction These documents were used to supplement my analysis of records of interaction. In particular, they were useful in understanding:</p> <ul style="list-style-type: none"> • the normative principles for teaching and learning to teach, • which structures, tools, and routines were used and how they were used, and • teacher learning opportunities—both what got worked on and how it was worked on. 	<p>Phase 1:</p> <ul style="list-style-type: none"> • Used Studiocode to review recordings of interviews, math labs, PLCs, and one-on-one coaching to identify episodes where participants initially lacked mutual understanding but then achieve mutual understanding <p>Phase 2:</p> <ul style="list-style-type: none"> • Watched episodes multiple times to identify the <i>what</i> (MKT, VOI, etc.) was being working on and <i>how</i> it was being worked on in the interaction. <p>Phase 3:</p> <ul style="list-style-type: none"> • Selected representative excerpts from across settings • Developed narratives from the various excerpts that explain and illustrate <i>what</i> is mutually understood among teachers and <i>how</i> mutual understanding was developed.

understood and how mutual understanding developed in interaction. Reviewing data for “identifying significant moments” or “critical events” is a common strategy used in qualitative research (Powell, Francisco, & Maher, 2003). “Events” are identified as “connected sequences of utterances and actions that, within the context of...[the] research questions, require explanation by [the researcher(s)], by the learners, or both” (p. 416) and an event is called critical when “it demonstrates a significant or contrasting change from previous understanding, a conceptual leap from earlier understanding” (p. 416). Next, I created an analytic memo that captured my interpretation of each episode, including *what* kinds of knowledge were used and developed as well as *how* mutual understanding was developed. I selected episodes that were both representative of the data and from across the various professional learning settings and developed narratives that illustrated and explained what was mutually understood by teachers and how the mutual understanding was developed.

Limitations of the Study

Every study has limiting conditions. Some of the limiting conditions in this study are related to the common critiques of qualitative research methodology in general and others are inherent in the study's design. One concern is that of researcher bias, which is the idea that my prejudices and attitudes biased the collection, interpretation, and reporting of the data (Bogdan & Biklen, 1982). One of the key limitations of this study is the issue of subjectivity and potential bias regarding my own participation as a staff member of the elementary school as well as a member of the math educator team from the university.

A related limitation is that teachers and the math coach may have had difficulty adjusting to my role of interviewer, a phenomenon referred to as *participant reactivity* (Maxwell, 2005). Because the math coach and many of the teachers know me, their responses may have been influenced or affected. They may have tried overly hard to cooperate and offer responses they perceived I was seeking or might be helpful. Alternatively, because of familiarity with me, the math coach and teachers may have been more guarded and less candid.

Through careful thought and planning, I attempted to account for and minimize the impact of these limitations. First, I acknowledged my research agenda and documented my assumptions prior to beginning the study. Coding schemes were presented to my advisor and committee (as part of my study proposal) and were shared with colleagues for feedback. Coded documents and transcripts were shared with colleagues and two research groups. To address the issue of participant reactivity, I reflected regularly on how and in what ways I might have been influencing participants. Additionally, I considered how to address issues of trustworthiness. The follow section describes the design features that are intended to increase the study's trustworthiness.

Issues of Trustworthiness

In quantitative research, studies are deemed trustworthy based on their validity and reliability. In qualitative research, Guba and Lincoln (1998) argue that trustworthiness should be assessed differently and use the terms *credibility*, *dependability*, *confirmability*, and *transferability*.

Increasing credibility. Establishing that a study's findings are credible (or accurate) given the data presented is the responsibility of the researcher, but this can be a challenge as qualitative case studies are limited by the sensitivity and integrity of the investigator (Merriam, 2009). To enhance the methodological credibility of this study, I triangulated data sources as well as data collection methods. By collecting data from multiple sources using multiple methods, the data yielded a fuller and richer picture of what teachers and students learned. I employed multiple strategies to increase the interpretive credibility of this study. First, I documented my assumptions prior to beginning the study. Second, I searched for discrepant evidence and employed a peer review process (Lincoln & Guba, 1985) to look for variation in my understandings and seek instances that might challenge my expectations and the emergent findings. Finally, I offered a formal member check to the math coach and a subset of the participating teachers. I invited them to review my written descriptions and claims. Since the math coach has spent many hours engaging in fractions work with teachers and students, she is in a unique position to provide valuable feedback on my analysis and interpretations.

Increasing dependability. The qualitative researcher is also responsible for establishing that findings are consistent and dependable with the data collected (Merriam, 2009). I employed a technique suggested by Lincoln and Guba (1985) called an "audit trail" that documented and

described in detail how I collected data, developed codes and procedures, used coding schemes consistently, and made decisions throughout the study (as cited in Merriam, 2009).

Increasing confirmability. *Confirmability* implies that that findings are the result of the research and not the biases and subjectivity of the researcher. The audit trail described previously as a strategy for increasing dependability, also served as a method for increasing confirmability as it will provide an opportunity for readers to assess the findings of the study.

Increasing transferability. With the notions of transferability, a number of methods researchers, such as Lincoln and Guba (1985) suggest that “the burden of proof lies less with the original investigator than with the person seeing to make an application elsewhere” (as cited in Merriam, 2009, p. 224). However, this does not let the researcher off the hook; it is the researcher’s responsibility to provide “sufficient descriptive data to make transferability possible (Merriam, 2009, p. 225). In order to increase transferability, I have provided a highly descriptive, detailed presentation of the context, participants, and findings, sometimes called a “rich, thick description” (Merriam, 2009).

Conclusion

This study and the methods used were designed to advance our understanding of how to support teachers’ instructional practices and students’ learning of fractions by focusing on what teachers and students *do* know and understand and how both teacher and student learning were intentionally considered and coordinated school-wide. The contributions this study offers are many. First, this study offers multi-faceted images of students’ fraction understanding as it develops over multiple years within the context of a school-wide implementation of fractions instruction. Second, this study contributes to our understanding of what high-quality fractions instruction looks like in elementary schools as well as the kinds of knowledge that teachers draw

upon and what meaning they make of that knowledge as they develop their instructional practice. Finally, this study adds to our understanding of multi-year school-wide efforts to improve instruction by using a situative perspective to understand both teacher and student learning and how they are coordinated with one another.

CHAPTER 4: AN ANALYSIS OF STUDENT LEARNING

This chapter presents findings related to how the school-wide approach to fractions instruction at Hilltop Elementary shaped student learning over a three-year period. The analytic questions that guided this portion of the analysis were:

- In what ways did students' partitioning strategies, fraction representations, fraction language, and written fraction notation change?
- How did the "instructional start points" for each grade level change from year to year?
- How did students' understanding differ for students who were at the school for various amounts of instruction?

This chapter is organized in three sections. In the first section, I present the findings from an analysis of students' fraction knowledge and understanding across three years. For each of three cohorts, I examined five aspects of their knowledge and understanding related to the equal sharing problem that was included as part of the interview. This part of the analysis of student learning illustrates that over the three years, students developed more sophisticated strategies for partitioning and sharing, created more accurate representations of their partitions, and used more accurate fraction language and notation. In the second section, I present the findings from an analysis of the instructional start points of each school year. The main finding of this section is that at the start of each subsequent school year, the cohort of students entering a particular grade level brought more sophisticated strategies for partitioning and sharing and more accurate representations. Fluency with fraction language and notation also increased over time, but with less consistency than partitioning strategies and representations. In the third part, I present findings from an analysis examining student learning based on the amount of instruction students received. The understanding and strategies for students with one year of instruction were noticeably less sophisticated than students with two or three years of instruction, whose understandings and strategies were much more aligned.

Part 1: Change in Students' Knowledge and Understanding

For each cohort, I analyzed five aspects of their knowledge and understanding related to the equal sharing problem that was included as part of the interview. The five aspects included:

- students' understanding of the problem as a sharing situation,
- students' partitioning and sharing strategies,
- students' representations,
- students' use of language, and
- students' use of written notation.

The sections that follow examine how cohorts of students' knowledge and understanding changed over a three-year period.

Understanding an Equal Sharing Problem as a Sharing Situation

One of the first aspects I analyzed was whether students understood the equal sharing problem as a sharing situation. If students' work or verbal explanation demonstrated any evidence of sharing or distributing sandwiches, it was coded as "understanding the problem." Figure 3 shows that even in Fall of Year 1 (prior to the start of the school-wide approach to fractions instruction), a majority of students understood the problem as a sharing problem. This finding is not particularly striking considering what we know about the accessibility of sharing contexts, and is what we hoped would be the case when we designed the task. Our goal was to create a task that would provide entry for many students while also offering the opportunity for us to uncover students' understanding of fraction concepts and strategy use.

In subsequent interviews, the number of students who understood the problem as a sharing situation increased to 93% or higher across all interviews for all cohorts. This is not particularly surprising because in designing the interview instrument, we selected problems that would be accessible to students in order to uncover their mathematical thinking, understanding, and strategies.

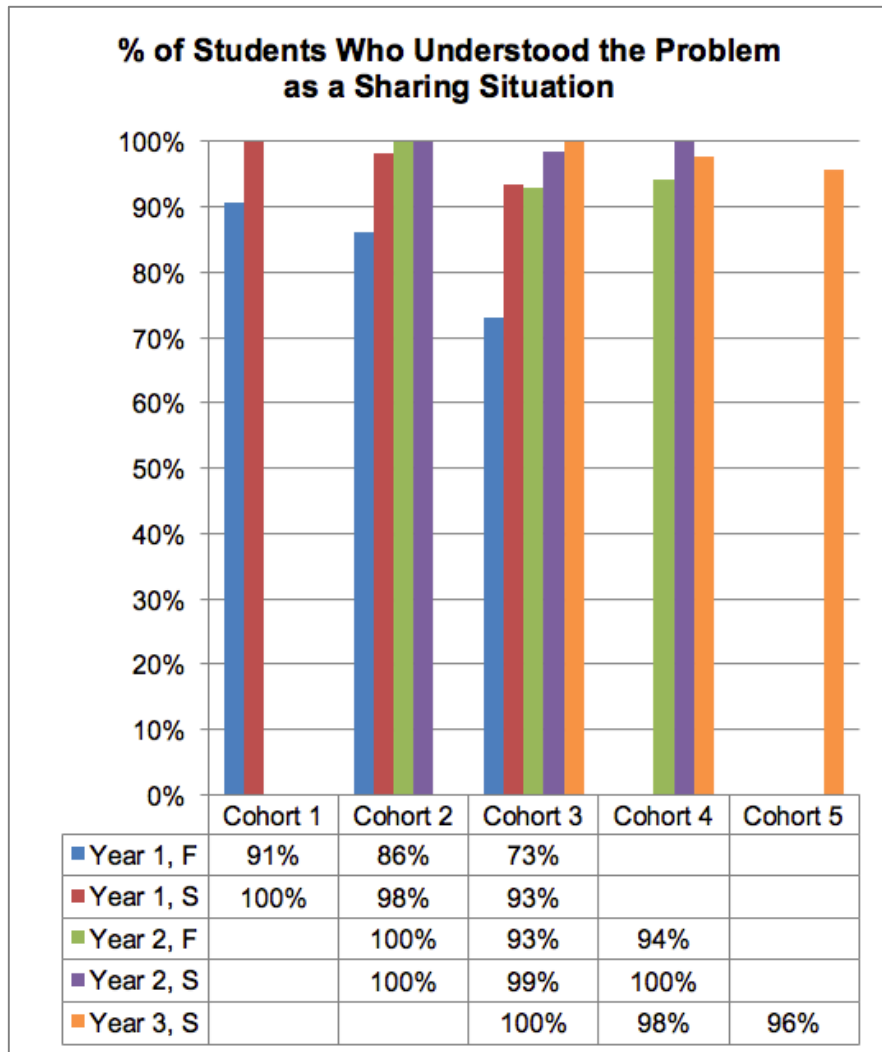


Figure 3. Percentage of students who understood the problem as a sharing situation.

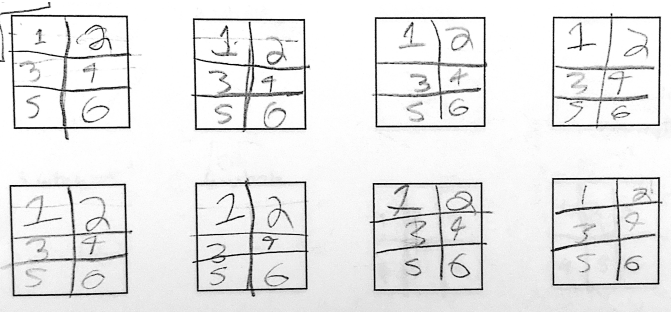
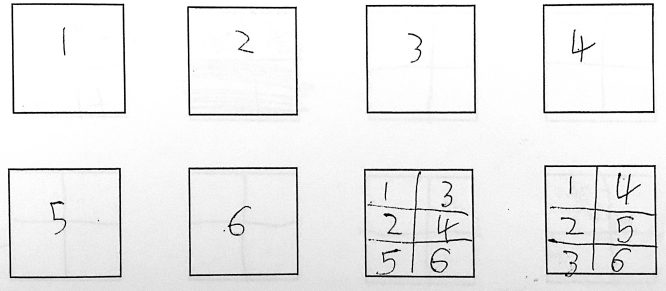
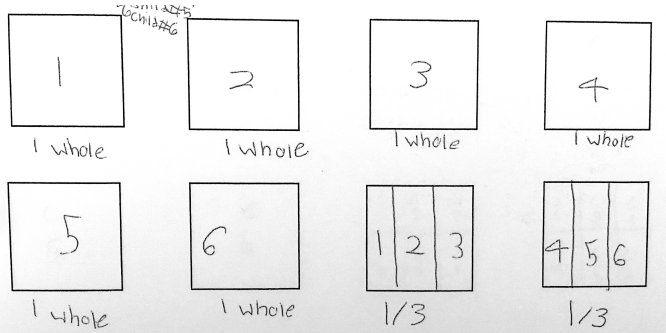
One striking trend of student’s understanding of the problem as a sharing situation is that it did not also mean that students also had a strategy to partition and share, create an accurate representation, or use fraction language and notation accurately. The next section will more closely examine how students’ partitioning and sharing strategies changed over time.

Students’ Partitioning and Sharing Strategies

When coding students’ partitioning and sharing strategies, I found that I needed to make a distinction between a student’s *primary partitioning strategy* (PPS) and a student’s *secondary partitioning strategy* (SPS). A student’s primary partitioning strategy is the strategy that the

student uses at the start of the problem, when the number of objects to share was greater than the number of people. For students whose primary partitioning strategy was to “Share Groups of Items,” I also coded a secondary partitioning strategy, which was the student’s strategy for sharing when the number of objects to share was fewer than the number of people. Table 11 provides examples of how primary and secondary partitioning strategies were used to code the student interviews.

Table 11. Examples of PPS and SPS code combinations.

PPS & SPS Code Combinations	Examples of Student Work
PPS: Share One at a Time SPS: None	
PPS: Share Groups of Items SPS: Share One at a Time	
PPS: Share Groups of Items SPS: Share Groups of Items	

Though I analyzed the portioning and sharing strategies for all five cohorts, I focused specifically on Cohorts 2, 3, and 4 when looking specifically at how students' strategies changed over time. The students in these three cohorts had two or more years of fractions instruction and three or more interviews.

Primary partitioning strategies (PPS). The graphs and tables in Figure 4 show the primary partitioning strategies used by students in Cohorts 2, 3, and 4. The graphs illustrate that students' primary partitioning strategies changed in similar ways across all three cohorts. First, the percentage of students who did not attempt or understand the problem dropped below 8% after each cohort's first year of fraction instruction. This can be seen by looking across the "Did Not Attempt/Und" row for each of the three cohorts. Second, the percentage of students who used *Coordinated Sharing* strategies increased between the fall and spring interviews for all three cohorts. For Cohort 2, this can be seen by comparing the top two sections of the first two bars with each other and then the top two sections of the last two bars with each other. For Cohort 3, this can be seen by comparing the top two sections of the first two bars with each other and then the top two sections of the third and fourth bars with each other. For Cohort 4, this can be seen by comparing the top two sections of the first two bars.

Due to mobility of students during and between school years, it is not surprising that there were students who used *No Coordination* and *Non-Anticipatory Sharing* strategies across all interviews (including Spring interviews) since their interviews may have taken place prior to their participation in fractions instruction at Hilltop Elementary.

Secondary partitioning strategies (SPS). The graphs and tables in Figure 5 show the secondary partitioning strategies for students in Cohorts 2, 3, and 4 that *Shared Groups of Items* as their primary sharing strategy. So each bar represents a subset of the cohort and the size of

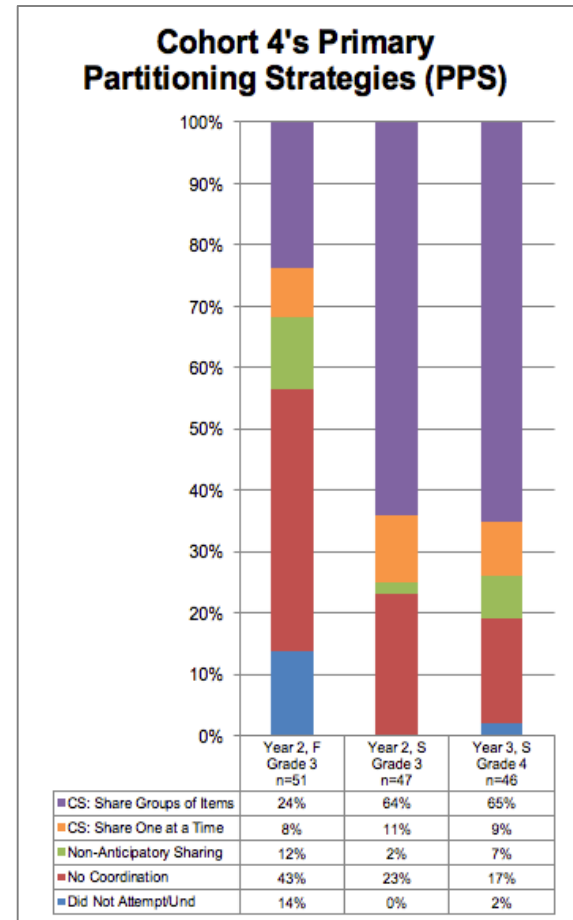
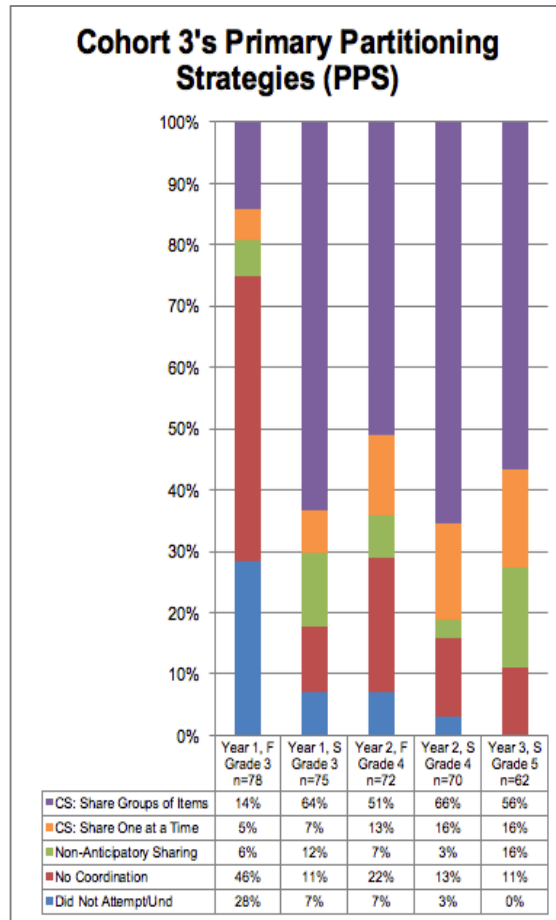
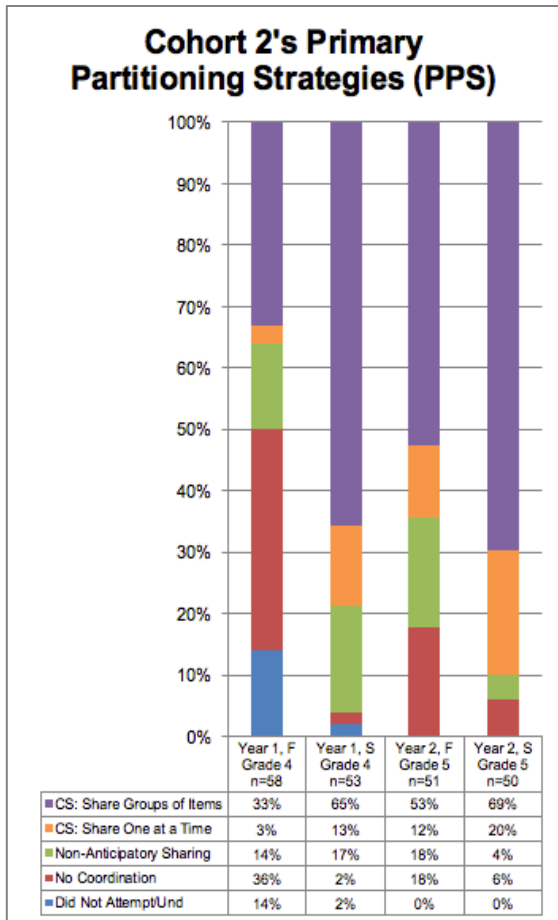


Figure 4. Primary partitioning strategies for Cohorts 2, 3, and 4.

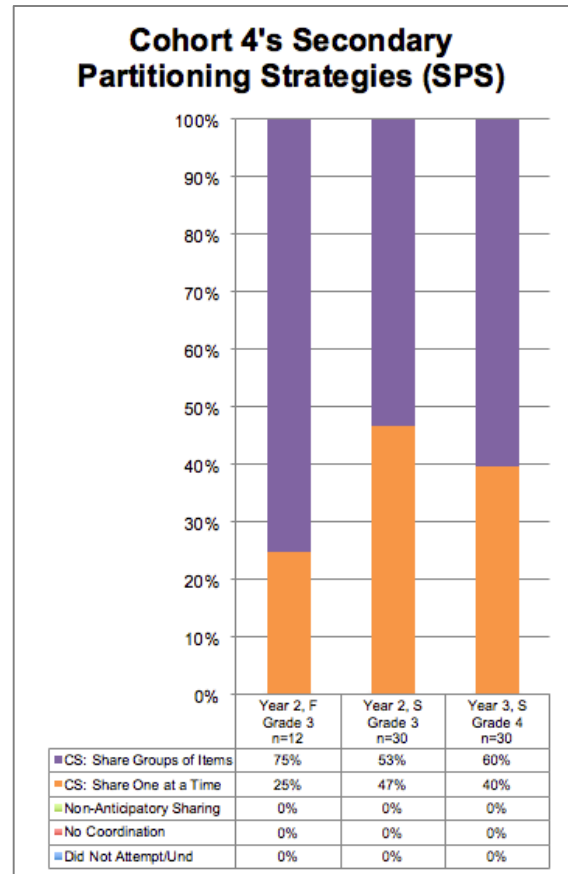
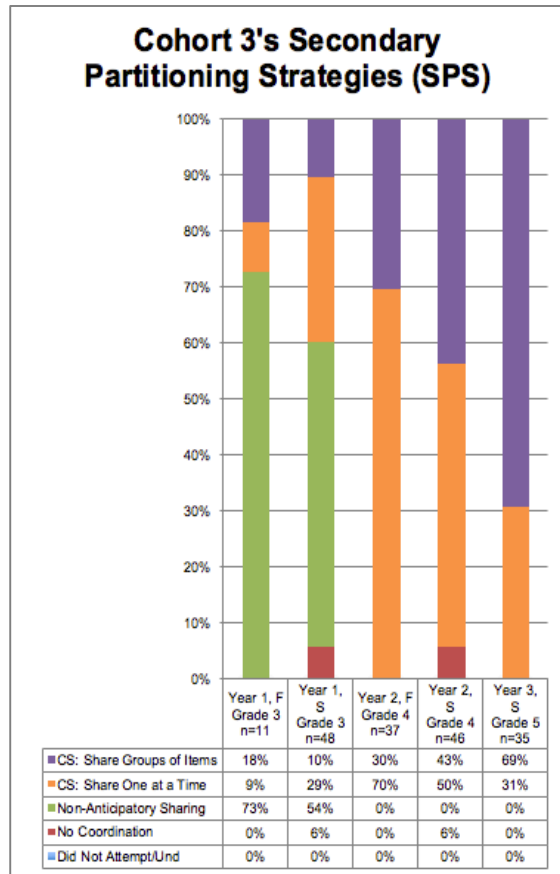
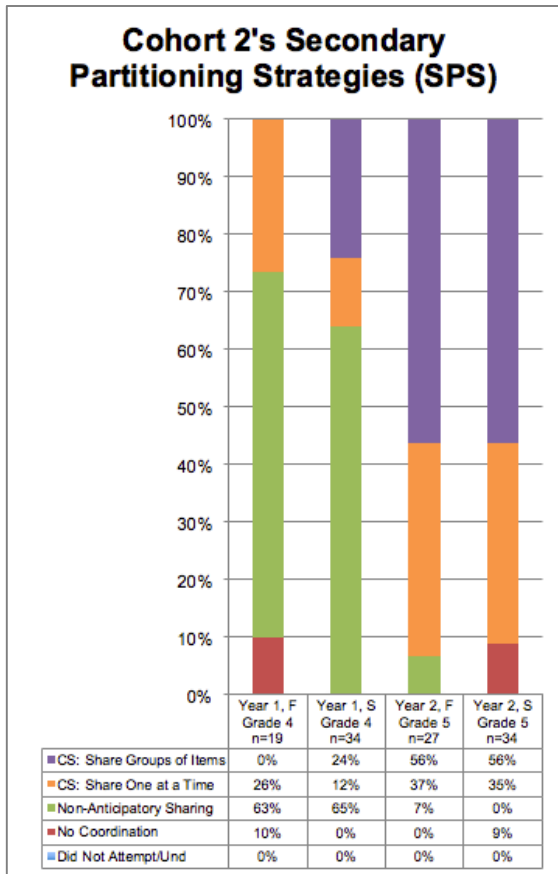


Figure 5. Secondary partitioning strategies for Cohorts 2, 3, and 4.

each subset is listed in the chart under each graph. In the first year of the grant (represented by Cohorts 2 & 3' first two bars), most students *Did Not Coordinate* or used a *Non-Anticipatory* secondary partitioning strategy. However, in the second and third years of the grant (represented by Cohort 2's last two bars, Cohort 3's last three bars, and all three of Cohort 4's bars), more than 90% of students used a *Coordinated Sharing* strategy. It is interesting and relevant to note that the revision of the interview task between Years 1 and 2 may be partially responsible for this shift.³ However, we see that when presented with this situation, students were able to use *Coordinated Sharing* strategies. One implication of this is that different strategies may be revealed when numbers change.

Representations of Partitions

Part of the task required students to create a representation of their partitions using paper and pencil. The task included pre-drawn circles (in Year 1) or squares (in Years 2 and 3) to represent the cookies (Year 1) or sandwiches (Years 2 and 3). When analyzing students' drawings of their partitions, three findings emerged. First, only one-third of students were able to create an accurate visual representation by the end of the first year. This can be seen by looking at the dark blue and red bars in Figure 6. However, by the end of the second year, approximately two-thirds of students in each cohort were able to create an accurate visual representation (as seen by looking at the green, purple, and light blue bars). This increase is likely related to two changes. First, as part of their math lab and PLC work, teachers discussed how different visual representations would support student understanding.

³ In Year 1, students whose strategies were coded for an *SPS* would have had to share 4 remaining items among 6 people whereas in Years 2 and 3, students would have had to share 2 remaining items among 6 people. In Year 1, the greater number of items created a situation where repeated halving was more likely to occur and in Years 2 and 3, the fewer number of items created a situation where students could not use repeated halving.

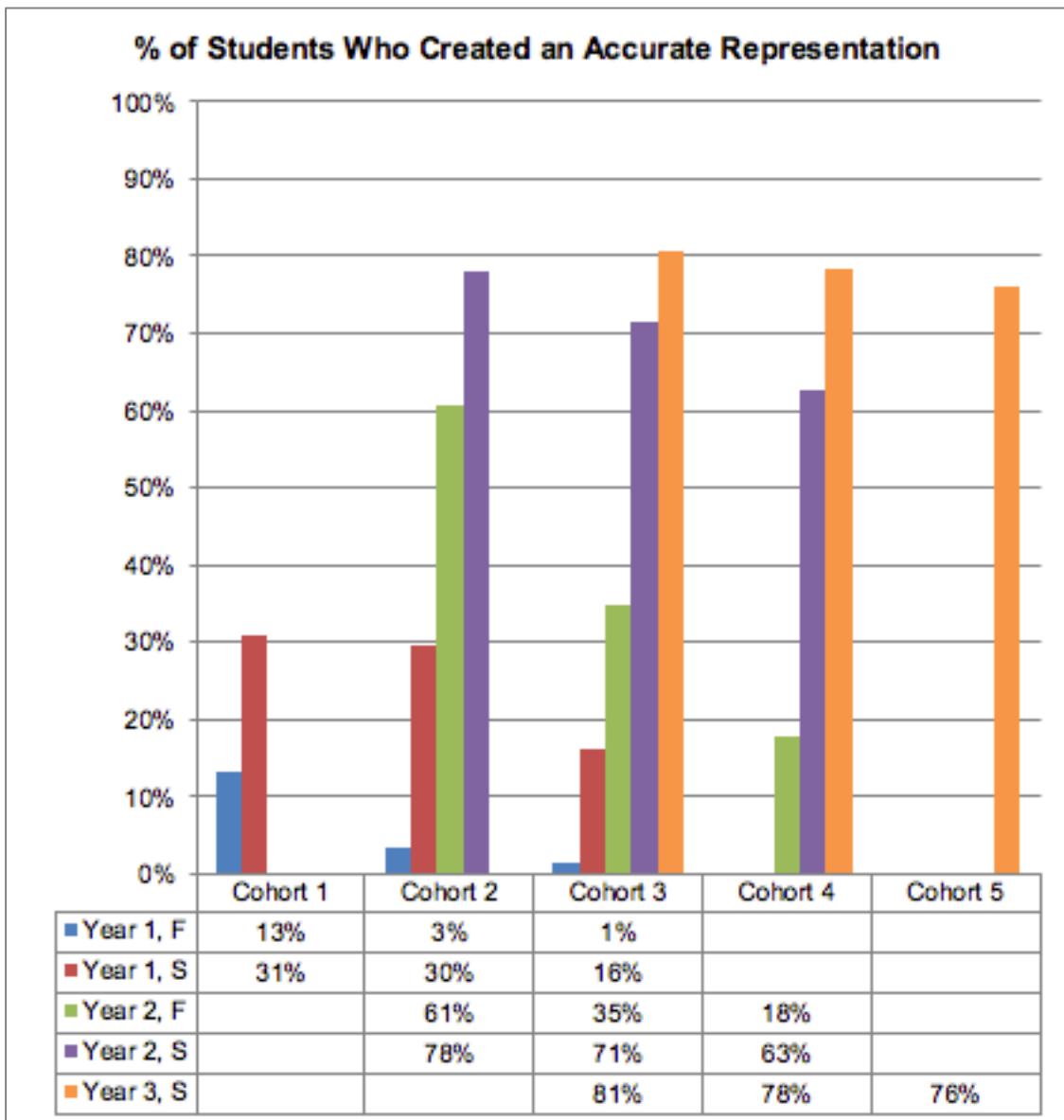


Figure 6. Percentage of students who created an accurate representation of their partitions.

Second, the increase between the red and green bars for Cohort's 2 and 3 may be related to a change in the task. In Year 1, students were given circles to use as part of their visual representation. In Years 2 and 3, we revised the task and gave them squares. The decision to make this change was made collectively by the university math educators and the math coach at Hilltop. After interviewing students and engaging in fractions instruction the first year, we realized that circles were difficult to partition into sixths, partially because there is only one way

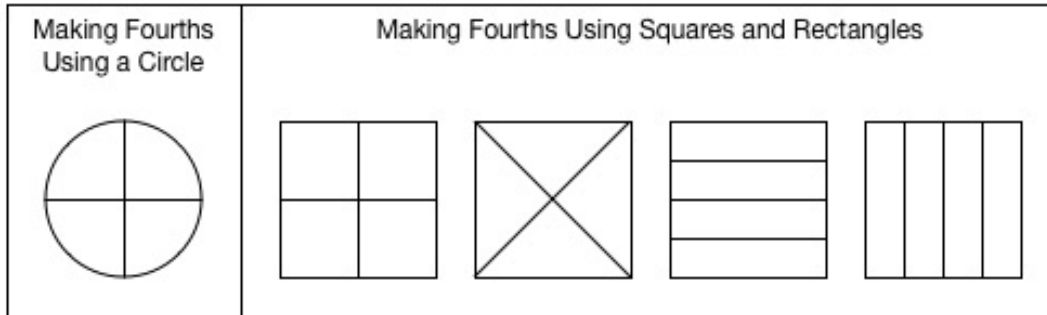


Figure 7. Making fourths with circles and squares/rectangles.

to accurately make the partitions. With squares, there were multiple ways students could approach the partitioning (see Figure 7).

Drawing Errors. An analysis of students' drawing errors shows that over time, for Cohorts 2, 3, and 4, both errors related to coordinating the number of people and the number of items and errors related to the size of the piece decreased in frequency (see Figure 8). Cohort 2's coordination errors decreased by 15% over two years and their size of piece errors decreased by 31%. Cohort 3's coordination errors decreased by 30% over three years and their size of the piece errors decreased by 17%. Cohort 4's coordination errors decreased by 23% over two years and their size of piece errors decreased by 30%.

Coordination errors. Coordination errors included things such as representing an incorrect number of people, an incorrect number of sandwiches, and having leftovers. Of these three, having leftovers was most common across cohorts and various interview time points. The decrease in these kinds of errors may have been related teachers' use of problem launching, where they focused on helping kids understand what the numbers represented as well as the ideas of having equal shares and no leftovers. In Math Labs, teachers discussed what ideas from sharing problems needed to be unpacked to support student sense making (everyone gets equal size pieces, the number of objects, the number of people). Also, the unit plans for all three grade levels across all 5 years started with sharing problems so students had multiple experiences

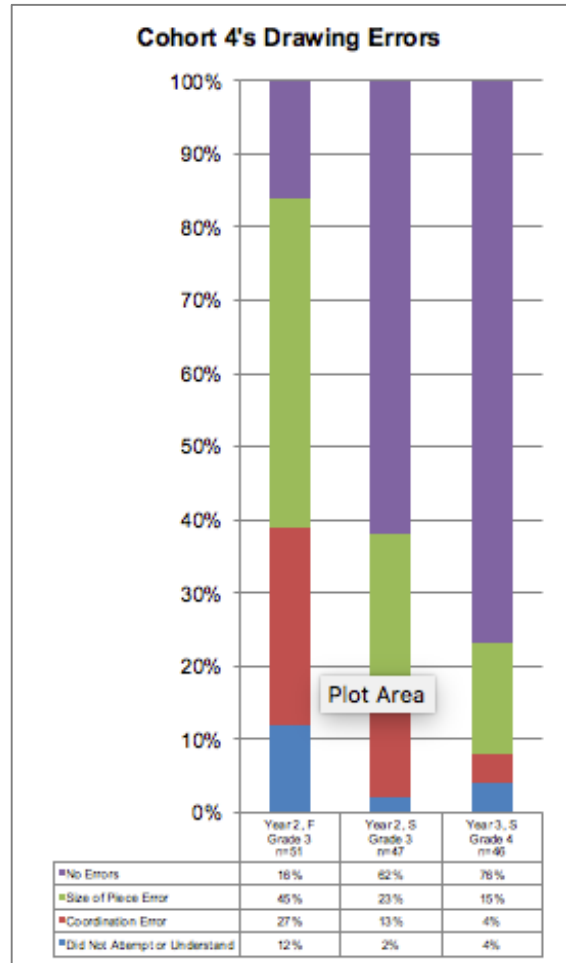
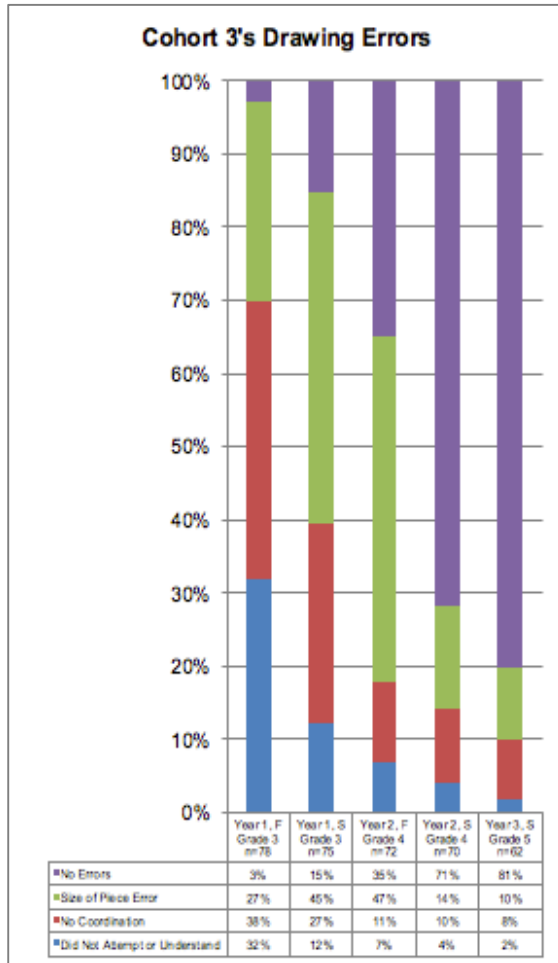
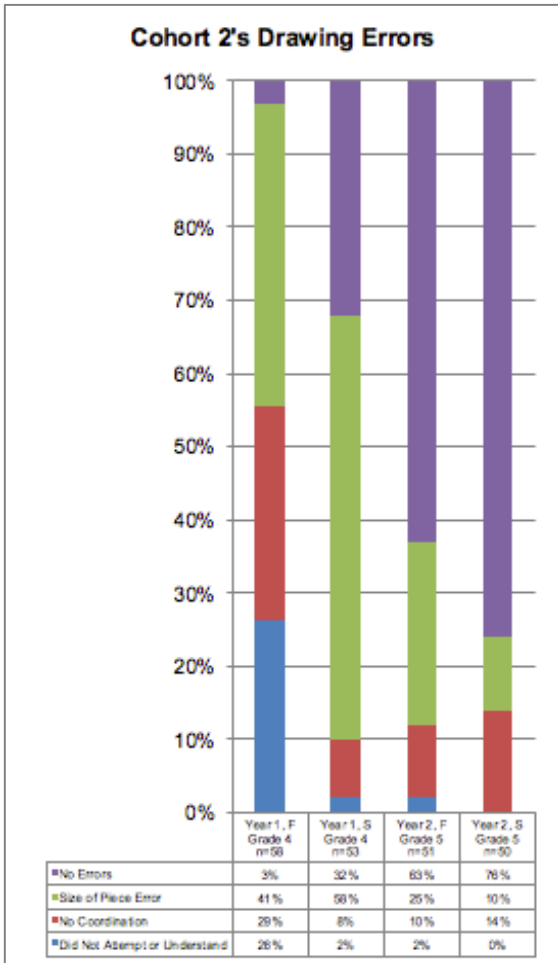


Figure 8. Drawing errors for Cohorts 2, 3, and 4.

across time solving this kind of problem. Also, when students made these kinds of drawing errors, they were typically unable to answer using correct language or notation.

Size of the piece errors. Size of the piece errors included things such as (see Table 12):

- making partitions that resulted in an accurate number of pieces that were not the same size,
- making partitions of one size, but calling it a different size piece, and
- making partitions that resulted in unequal portions.

The first of these, making partitions that resulted in an accurate number of pieces that were not the same size was the most common error made across all cohorts and interview time points. In early interviews, this happened most often when students partitioned an item into sixths. In later interviews, this kind of error occurred with both thirds and sixths. The second error, making pieces of one size, but calling it a different size piece was seen by two to four students at each time point. Most commonly, students drew eighths and called them “sixths,” but students also drew sevenths and called them sixths or drew tenths and called them twelfths. The final kind of error classified as a “size of the piece” error involved making partitions that resulted in unequal

Table 12. Examples of "size of piece" errors.

<p>a) Made partitions that resulted in an accurate number of pieces that were not the same size.</p>	<p>b) Made pieces of one size, but called it an inaccurate name.</p>	<p>c) Made partitions that resulted in unequal portions.</p>

portions, seemed related to the repeated halving partitioning strategy. Most often, the unequal portions resulted from partitioning the last two sandwiches so that one had 2 half-size pieces and one had 4 fourth-size pieces. Interestingly, size of the piece errors did not necessarily prevent students from using accurate language or notation.

Fraction Language and Notation

In Fall of Year 1, few students in Cohorts 2 (2%) and 3 (0%) were able to use accurate fraction language or notation to represent the answer to the sharing problem (see Figure 9). In Fall of Year 2, when Cohort 4 was in Grade 3, 16% of students were able to use accurate fraction language or notation. This slight shift between Year 1 and 2 may be a result of the second grade team's introduction to fractions in Year 1. When comparing fall interviews to spring interviews, we see large increases in the percentages of students who used accurate language or notation. Cohort 2 increased from 2% to 73% in Year 1 and 49% to 64% in Year 2. Cohort 3 increased from 0% to 48% in Year 1, 43% to 59% in Year 2, and by the end of Year 3, 74% of students used accurate fraction language or notation. Cohort 4 increased from 16% to 53% in Year 2 and by the end of Year 3, 74% of students used accurate fraction language or notation.

As described in more detail in the next two chapters, fraction language and notation were a key component of the school-wide approach to fractions instruction, as well as a component that shaped teacher learning. So, it is not surprising to see that students' use of fraction language and notation became more accurate over time. However, it is interesting to note that the percentages for Cohorts 2 and 3 decreased between spring interviews and the fall interviews the next school year. Since I do not have fall interviews for Year 3, I am not able to analyze this particular data to see if this occurs in other years. But, this trend is supported by teachers' discussions during Math Labs in Years 4 and 5 where they discussed that "students have a lot to

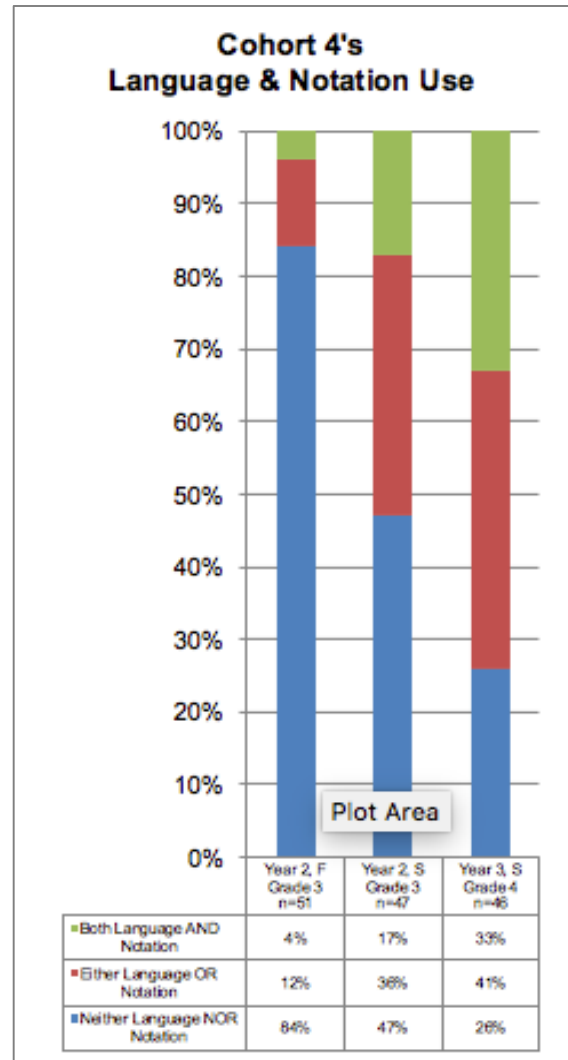
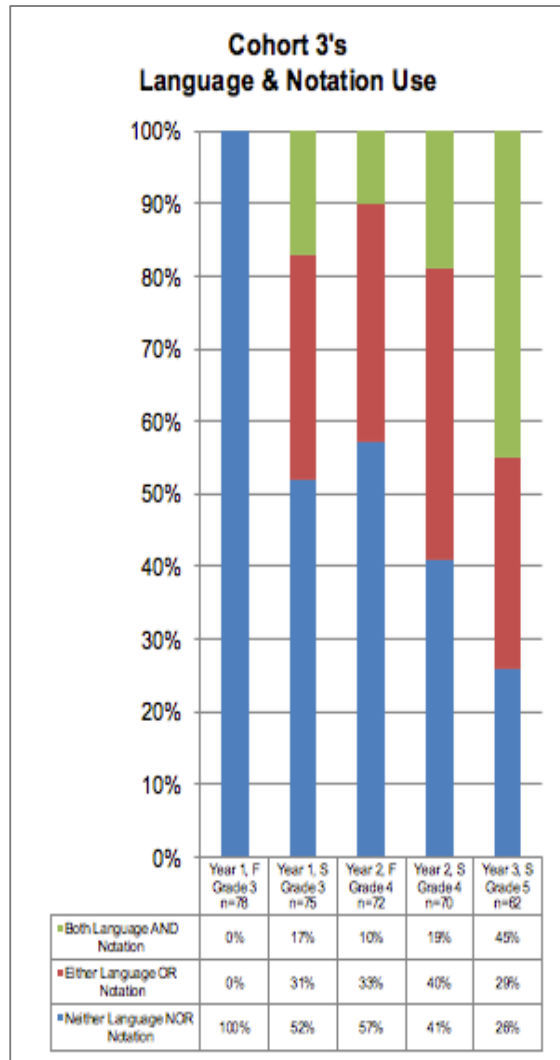
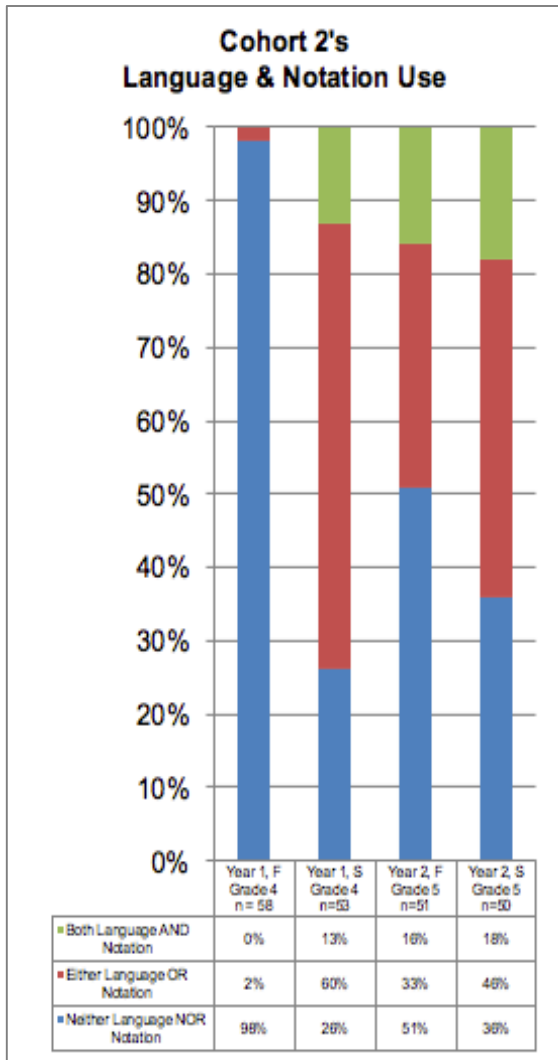


Figure 9. Use of accurate fraction language and notation by Cohorts 2, 3, and 4.

work with...they just need some help remembering the language. It came back really quickly [during our classroom visit] today.” This trend is not particularly surprising based on what we know from the literature—that fraction language and notation are conventions that are not intuitive for children.

Part 2: Change in Instructional Start Points

One of the things that Tara (the math coach) and many teachers commented on during their interviews was the “moving” or changing instructional start point for students. What they meant by that was that each year, students entered a particular grade level “knowing more” than the previous cohort of students. Using the student interview data, I analyzed the instructional start points for two grade levels (Grades 4 and 5) to see if the instructional start points changed—as well as how they changed. Figure 10 shows the subsets of data I analyzed for each grade level. One important point to note is that the analysis described in this chapter up to this point looked at changes within cohorts of students. I now shift to analyzing across different cohorts of students.

Though one may critique this approach because it uses Spring interviews from the previous grade level to determine instructional start points (instead of Fall interviews, which would be closer in time to the actual instruction), it is appropriate for this analysis for multiple reasons. First, this analysis is focused on understanding the broader trends occurring over time in

Grade 4 Instructional Start Points:

- Year 1 Instructional Start Point: Analyzed Grade 4 interviews from Fall of Year 1
- Year 2 Instructional Start Point: Analyzed Grade 3 interviews from Spring of Year 2
- Year 3 Instructional Start Point: Analyzed Grade 3 interviews from Spring of Year 3
- Year 4 Instructional Start Point: Analyzed Grade 3 interviews from Spring of Year 4

Grade 5 Instructional Start Points:

- Year 1 Instructional Start Point: Analyzed Grade 5 interviews from Fall of Year 1
- Year 2 Instructional Start Point: Analyzed Grade 4 interviews from Spring of Year 2
- Year 3 Instructional Start Point: Analyzed Grade 4 interviews from Spring of Year 3
- Year 4 Instructional Start Point: Analyzed Grade 4 interviews from Spring of Year 4

Figure 10. Interviews used for instructional start point analysis.

the school. Second, this is representative of how the coach and teachers used the data in their own instructional planning. The Spring interviews were used to develop high level instructional plans for the subsequent school year (and then the Fall interviews were taken into account as the plans were refined). The first set of graphs (see Figure 11) shows the change in instructional start points for students’ understanding of the problem, the accuracy of their visual representation, and the accuracy of their use of language and notation.

Understanding the Problem as a Sharing Situation

For both Grades 4 and 5, most students entered the respective grade level understanding that the problem involved a sharing situation. The data showed a slight increase in the percentage of students across the first three years for fourth graders and across the first two years for fifth

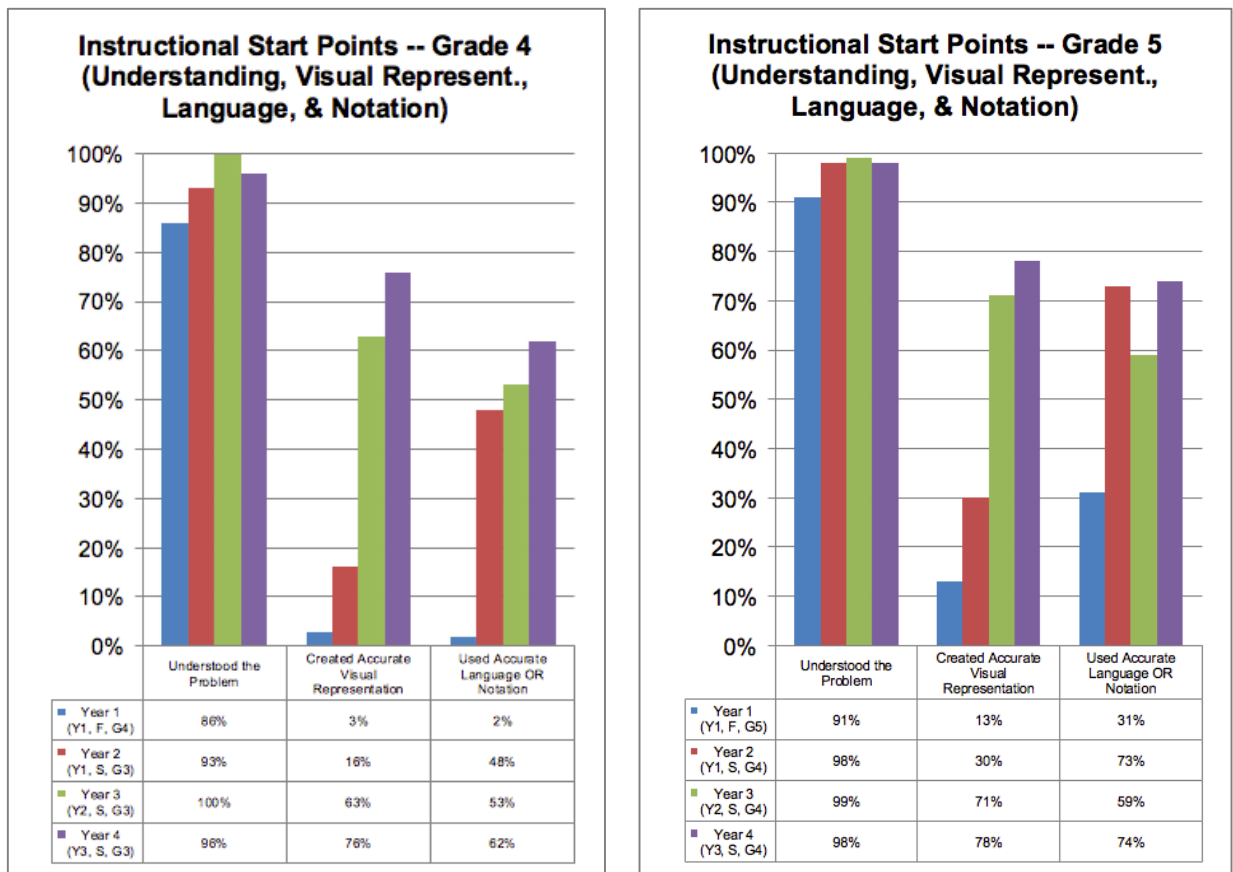


Figure 11. Instructional start points for Grades 4 and 5 (understanding, visual representations, language, and notation).

graders. After the initial increases, the percentage of students understanding the problem as a sharing situation remained at 96% or higher.

Students' Use of Visual Representations

A majority of the students in the first two cohorts of students to enter Grades 4 and 5 needed instructional support in creating accurate visual representations of the sharing situation. In Year 1, 3% of Grade 4 students and 13% of Grade 5 students were able to create accurate visual representations and in Year 2, 16% of Grade 4 students and 30% of Grade 5 students were able to do so. However, beginning in Year 3, this shifted dramatically so that a majority of students entered Grades 4 and 5 able to create accurate visual representations (63% for Grade 4, Year 3; 76% for Grade 4, Year 3; 71% for Grade 5, Year 3; and 78% for Grade 5, Year 4).

Students' Use of Notation and Language

In Year 1, 2% of students entering Grade 4 and 31% of students entering Grade 5 were able to use accurate fraction language or notation to represent the answer to the sharing problem. This instructional start point aspect changed quickly however. At the start of Year 2, 48% of students entering Grade 4 and 73% of students entering Grade 5 were able to use accurate fraction language or notation. This kind of difference has implications for how the teachers at Hilltop approached instruction. For example, in Year 1, when only one student in Grade 4 (which is the equivalent of 2% of students) uses accurate fraction language or notation, teachers' planning requires that they consider how to introduce language and notation to all students. In contrast, in Year 2, when roughly half (48%) of students enter Grade 4 using accurate language and notation, the instruction can draw and build upon what these students know and understand in order to support the learning of the other students who are still developing an understanding of the language and notation conventions.

One interesting irregularity in the Year 2 instructional start point data is the relationship between students' accuracy with visual representations and their accuracy with language and notation. Year 2 is the only time point where students' accuracy with language and notation is higher than their accuracy with the visual representation, and this is true for both Grades 4 and 5. This may be related to the fact that the interview task used a circle area model for in the data analyzed for Years 1 and 2 and a square area model in the data analyzed for Years 3 and 4. However, underlying this is an interesting implication—that errors in students' representations do not necessarily prevent them from using language and notation to accurately represent their answer (see Figure 12).

Partitioning Strategies

The graphs in Figure 13 illustrate how instructional start points related to students' primary and secondary partitioning and sharing strategies changed. In Year 1, for both Grades 4 and 5, less than half of students used Coordinated Sharing (Sharing Groups of Items and Sharing One at a Time) strategies as their primary sharing strategy (36% in Grade 4 and 43% in Grade 5). And, for those who did, less than half of them (26% in Grade 4 and 30% in Grade 5) used a

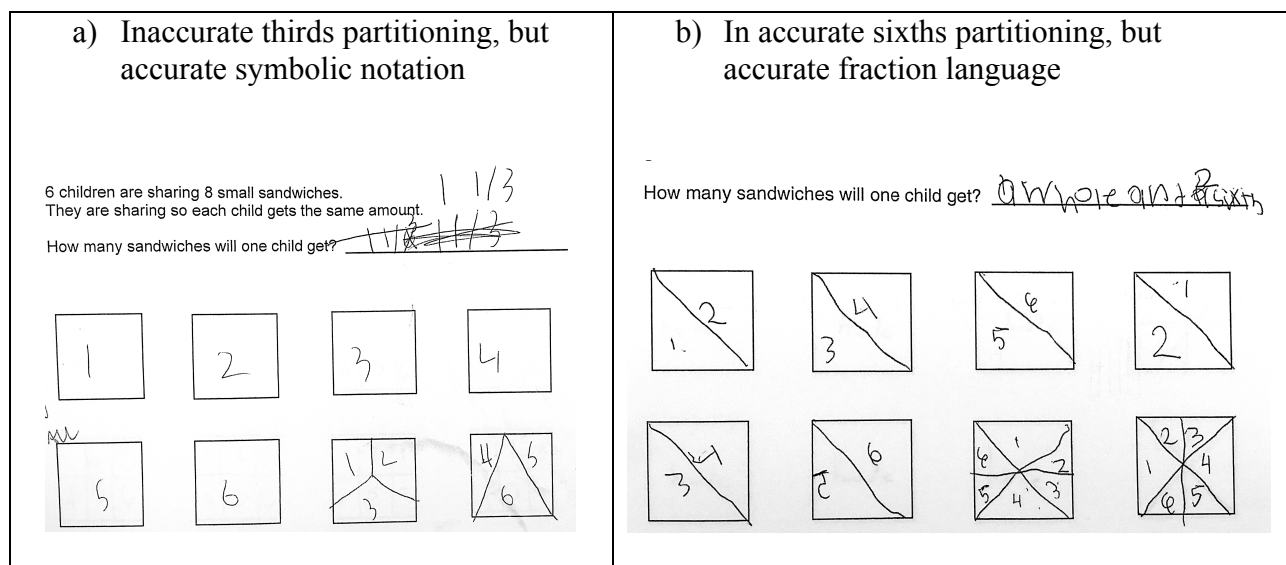


Figure 12. Accurate language or notation despite inaccurate partitions.

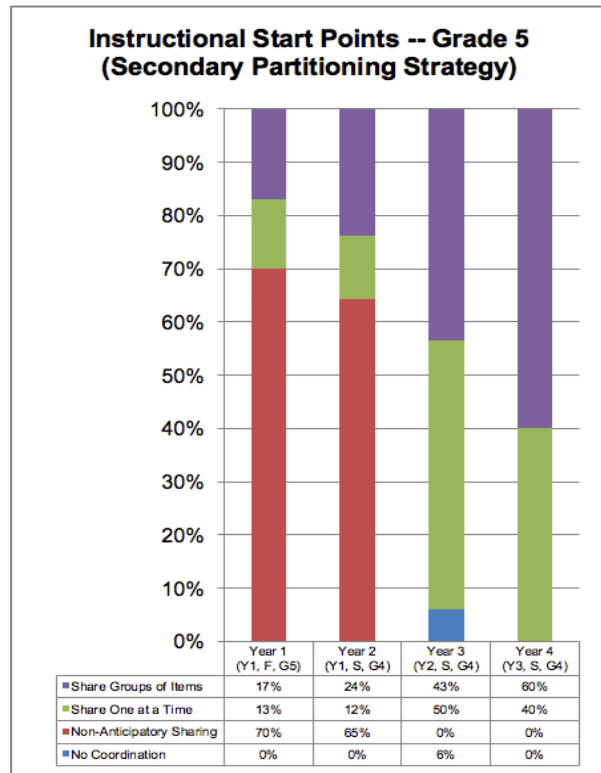
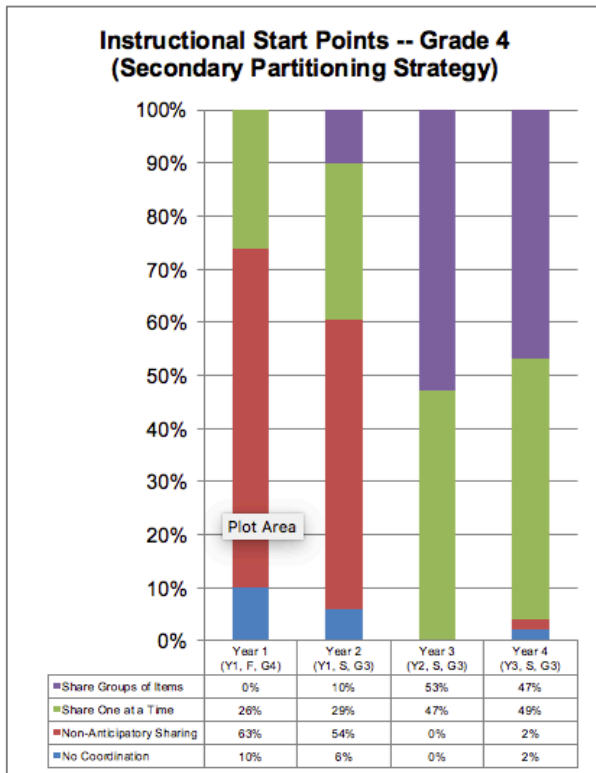
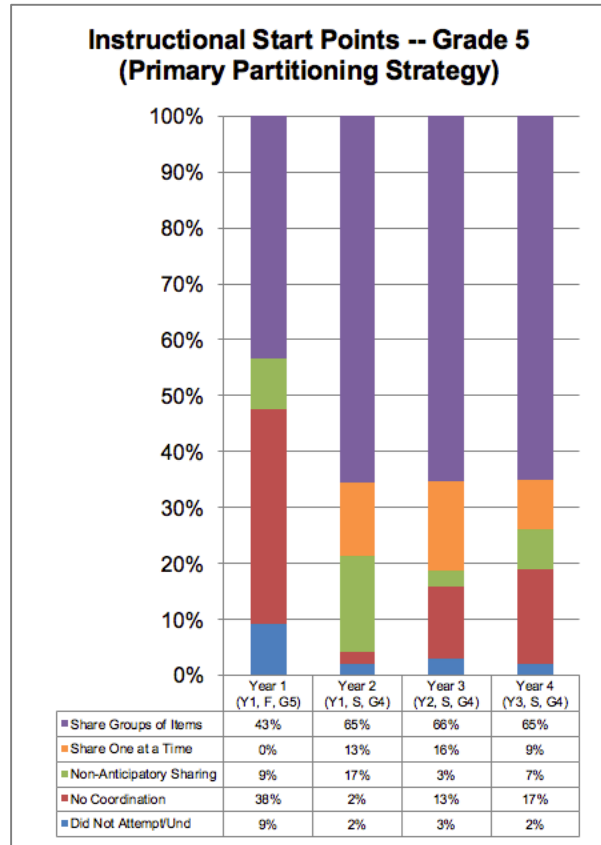
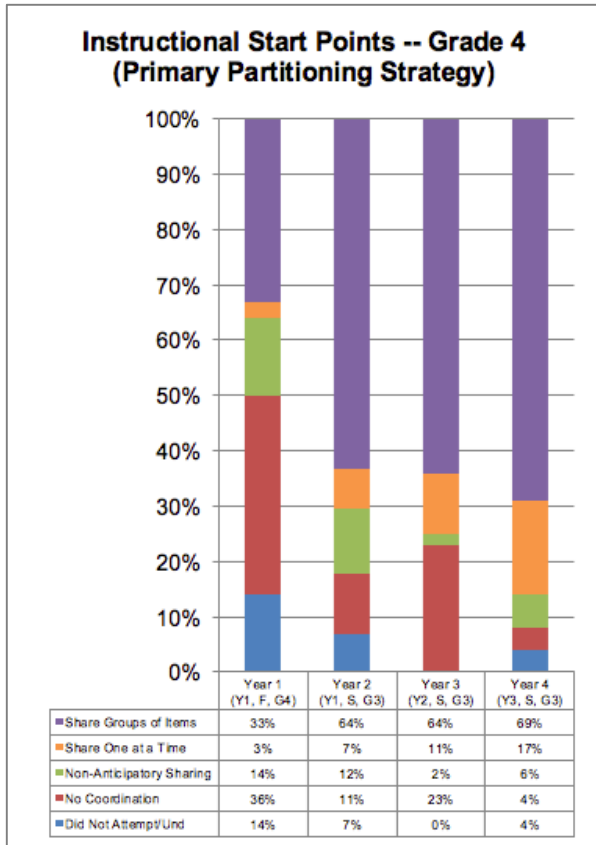


Figure 13. Instructional start points for Grades 4 and 5 (primary and secondary partitioning strategies).

Coordinated Sharing strategy as their secondary partitioning strategy. For Year 2's instructional start point, more students used Coordinated Sharing strategies as their primary partitioning strategies (71% in Grade 4 and 78% in Grade 5) in comparison to Year 1, but still less than half of those students (39% in Grade 4 and 36% in Grade 5) used a Coordinated Sharing strategy as their secondary partitioning strategy. Year 3's instructional start point for primary partitioning strategies is similar to Year 2 (75% in Grade 4 and 82% in Grade 5 used Coordinated Sharing strategies). However, the instructional start point with regards to these students' secondary partitioning strategy changed drastically (100% in Grade 4 and 93% in Grade 5). Year 4's instructional start point with regards to student's partitioning strategies is similar to Year 3. A majority of students use coordinated sharing strategies for both their primary and secondary partitioning strategies.

Implications of Changes in Instructional Start Points

The analysis of Grade 4 and Grade 5 instructional start points across four consecutive school years illustrates how subsequent cohorts of students' understanding of fractions became more sophisticated each year. At Hilltop, the coach and teachers noticed the shifts between cohorts of students and responded by revising unit plans and planning classroom visits that would provide opportunities to would provide additional information about a class of student's instructional start point.

Part 3: A Comparison of New and Returning Students

Since we interviewed students over a three-year period, I was able to analyze how students' understanding and strategies varied based on the length of time the students had been at Hilltop Elementary. For this part of the analysis, I focused on Cohort 3 students because they had the potential for three years of fractions instructions: as third-graders in Year 1, as fourth-graders

Table 13. Sub-groups of students based on amount of fractions instruction.

Sub-Group	Interview Participation	Number of Students in Sub-Group
One Year of Fraction Instruction at Hilltop (Grade 5 only)	<ul style="list-style-type: none"> • Spring, Year 3, Grade 5 	12
Two Years of Fractions Instruction at Hilltop (Grades 4 and 5)	<ul style="list-style-type: none"> • Spring, Year 3, Grade 5 • Spring, Year 2, Grade 4 <p><i>may have also participated in:</i></p> <ul style="list-style-type: none"> • Fall, Year 2, Grade 4 	15
Three Years of Fractions Instruction at Hilltop (Grades 3, 4, and 5)	<ul style="list-style-type: none"> • Spring, Year 3, Grade 5 • Spring, Year 2, Grade 4 • Fall, Year 2, Grade 4 • Spring, Year 1, Grade 3 <p><i>may have also participated in:</i></p> <ul style="list-style-type: none"> • Fall, Year 1, Grade 3 	35

in Year 2, and as fifth-graders in Year 3. I divided this cohort into three sub-groups as shown in Table 13: one year of fraction instruction at Hilltop, two years of fraction instruction at Hilltop, and three years of fraction instruction at Hilltop. A student's group was assigned based on the interview in which the student participated.

To examine how students' understanding may have been similar or different in relation to their participation in fractions instruction at Hilltop, I used the same characteristics as the previous analyses (understanding the problem as a sharing situation, creation of a visual representation, partitioning strategies, and use of fraction language and notation) to analyze the subgroup's responses in Spring of Year 3 (at the end of Grade 5).

All twelve students that were at Hilltop for a single year of fractions instruction (Grade 5) understood the problem as a sharing situation, which mirrored the understanding of the other two groups. However, the group created less accurate visual representations, used less accurate fraction language and notation, and use more emergent strategies in comparison to the other two

subgroups. Figure 14 shows that 50% of this subgroup was able to create an accurate visual representation and 42% of the subgroup used accurate language or notation. The distribution of responses in the other two groups was different from this group—yet, very similar to each other. 87% of students with two years of instruction and 89% of students with three years of instruction created accurate visual representations; 80% of students with two years of instruction and 83% of students with three years of instruction used accurate fraction language or notation.

The differences in students' partitioning strategies for the three sub-groups were not as extreme as those with visual representations and language and notation. The percentage of students that used a Coordinated Partitioning Strategy (such as Sharing Groups of Items or Sharing One at a Time) was 59% for the sub-group with one year of instruction, 66% for the sub-group with two years of instruction, and 78% for the sub-groups with three years of instruction (see Figure 15). Furthermore, when looking at the secondary partitioning strategies of these students, a majority of students in all three sub-groups used the Share Groups of Items strategy (see Figure 16).

Previous research helps us consider why the length of instruction seems to influence particular characteristics of students' responses more than others. Understanding sharing situations and partitioning objects in order to share them fairly is a familiar context that is intuitive for many students. So, students are more likely to be successful with these parts of the task regardless of the length of their fractions instruction—or where it took place. The increased use of more sophisticated partitioning strategies with increased instructional time at Hilltop does indicate that instruction plays a role in supporting students for whom sharing is unfamiliar and not intuitive.

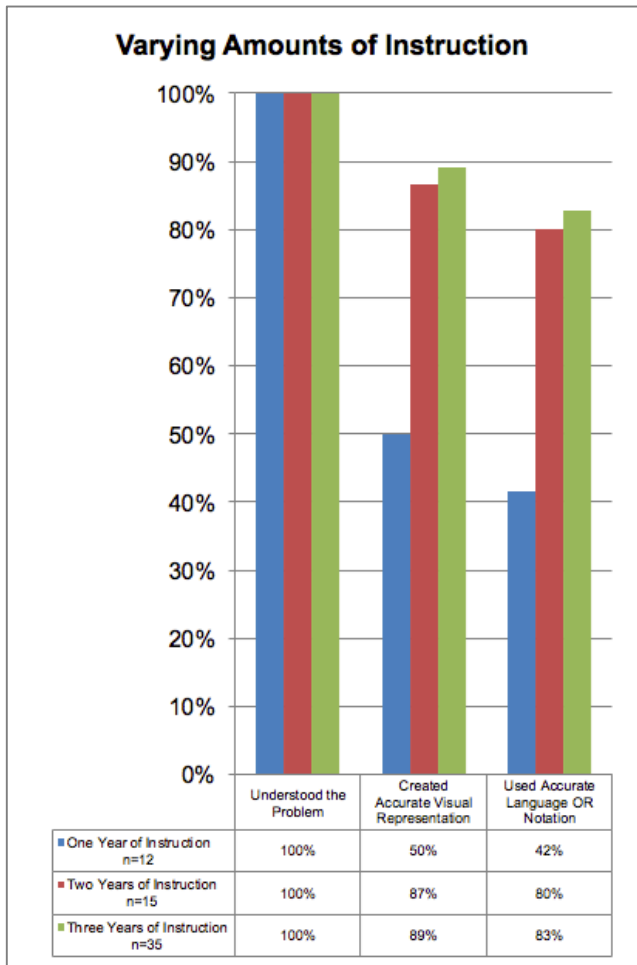


Figure 14. Students' understanding, representation, and language/notation with varying amounts of instruction.

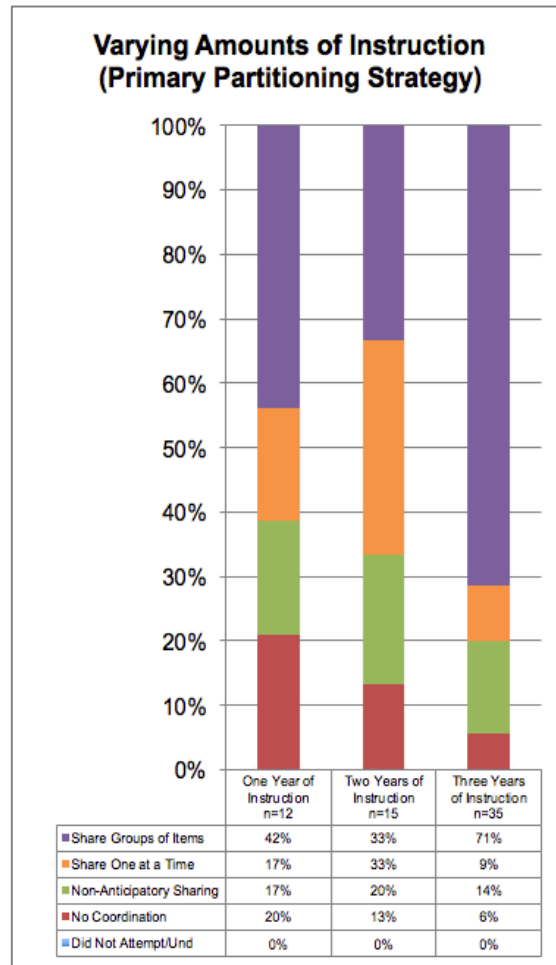


Figure 15. Students' primary partitioning strategies with varying amounts of instruction.

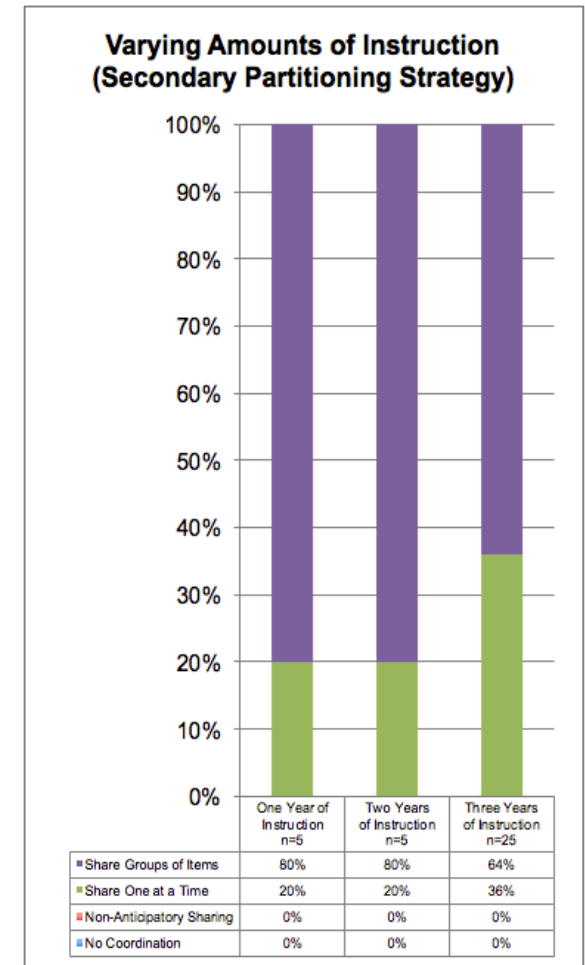


Figure 16. Students' secondary partitioning strategies with varying amounts of instruction.

On the contrary, creating accurate visual diagrams and using fraction language and notation are not necessarily intuitive for children and typically require thoughtful instruction in order for students to both understand the underlying logic and apply it accurately. The data analyzed here raise a question for future research of how long it might take students to develop this understanding. In this case, a majority of students in their second and third years of fraction instruction at Hilltop created accurate visual diagrams and used fraction language and notation accurately, but this was not true for students in their first year of fractions instruction at Hilltop (with likely two years of fractions instruction at other schools). If the trajectory for students developing a deep understanding of fractions spans multiple school years, it seems paramount that students' experiences are cohesive and not disjointed.

CHAPTER 5: AN ANALYSIS OF TEACHER LEARNING

One of the aims of this study is to contribute to our understanding of what teachers *do* know and understand and how that knowledge is used as they engage in the work of teaching. As teachers engage in the complex work of teaching, they draw upon, make meaning of, and coordinate various kinds of intellectual resources. Through my review of the literature, I identified four different kinds of intellectual resources that teachers develop and coordinate as they engage in teaching mathematics and learning to teach mathematics.

The analytic questions that guided this portion of the analysis were:

- What do teachers understand about fractions content? (MKT)
- What visions of fraction instruction are described by teachers?
- What do teachers understand about how students' understanding of fractions develops?
- How do teachers describe their personal identities as math teachers? What is the prevailing normative identity regarding teaching mathematics? What is the relationship between personal identities and normative identities?

In this chapter, I draw primarily from the teacher interviews to offer characterizations and illustrations of the various intellectual resources that teachers at Hilltop Elementary developed and used as they participated in the implementation of the school-wide approach to fractions instruction. My coding scheme for intellectual resources drew on the following literature:

- content knowledge for teaching mathematics (Ball, Thames, & Phelps, 2008),
- visions of high quality mathematics instruction (Munter, 2014)
- understanding trajectories of student learning (Carpenter et al., 1999; Empson & Levi, 2011), and
- teachers' personal identities related to teaching and learning to teach mathematics (Gresalfi & Cobb, 2011).

Content Knowledge for Teaching Mathematics

Mathematical knowledge for teaching (MKT) is one intellectual resource that teachers' draw upon as they engage in teaching and learning to teach. For the purpose of this study, I focused specifically on subject matter knowledge and borrowed the term *content knowledge for*

teaching mathematics (CKTM), which includes both common content knowledge and specialized content knowledge. In the sections that follow, I begin by describing teachers' reflections upon their own understanding of fractions prior to teaching at Hilltop. I follow this up with four illustrations that characterize how CKTM was used as teachers engaged in the work of teaching.

Teachers' Understanding of Fractions Prior to Teaching at Hilltop During the SIG

During my interviews with teachers, many of them discussed their own understanding of fractions content prior to teaching at Hilltop during the SIG. Most teachers described having a very procedural understanding. For example, Erin reflected on her memories of adding and subtracting fractions prior to teaching at Hilltop: "Before I just found common denominators because it's something that you do...you can't add fractions that have different denominators. Would I have been able to explain to you why? No." Leslie recalled a rhyme she learned for dividing fractions, "Yours is not to question why, just invert and multiply" adding "I had no idea why though." These examples illustrate how a majority of the interviewed teachers described their own content knowledge prior to teaching at Hilltop during the SIG.

However, there were two exceptions. One exception was Anna who "always loved fractions" and felt like she had an "unspoken understanding of how numbers work." The other exception was Becky, who shared in detail how challenging fractions had been for her as both a child and an adult. She explained how at the first fractions math lab during her first year, the group had "an open discussion" about their fraction experiences as students. She recalled sharing with Tara, Julie, and her colleagues:

I didn't know the difference between a fourth and a fifth. When I cooked, I would have to convert. I would go online and convert, okay, how many fourths are in a whole cup? That kind of thing...Sometimes I would wonder. It would tell me I need to get a third but I only have a half cup. Which is bigger? That kind of thing. Not even being able to really

compare. The only thing I remember learning about fractions is something about cross multiplying might be the only thing I remember. Or finding the least common multiple. That multiplication was very procedural. I didn't know how it worked. I just knew that I could multiply and how to make the denominator the same.

Though Anna and Becky's understanding of fractions are not necessarily representative, their characterizations of their content knowledge help illustrate the range of fraction understanding that can exist within a school, or even a grade level team.

Teachers' Understanding of Fractions After Teaching at Hilltop

In the four illustrations that follow, I offer examples of the kinds CKTM that teachers developed and used in the context of teaching and learning to teach fractions. In comparing the illustrations that follow with those previously described, the ways in which teachers talked about and used their CKTM after teaching at Hilltop were much more sophisticated than prior to the SIG or teaching at Hilltop. Their sophistication is evident in the illustrations that follow as they use mathematical language and notation that supports sense making, explain fraction ideas with conceptual understanding, and link representations to underlying ideas and other representations.

Illustration #1: Using mathematical language and notation. In the first year of the SIG, Tara (the math coach) focused on developing a common language for fractions across the school. She read Empson & Levi's (2011) book and the section related to "naming fractions" resonated with her. She worked with the university math educators and teams of teachers in math labs and PLCs to develop common language for defining and talking about what numerators and denominators represent, as well as a line of questioning that would introduce students to fraction language in the context of sharing problems, as well as the logic that the language follows. Figure 17 summarizes the language and questions that were developed collectively.

School-wide Language: (Example from the Grade 3, Year 4 Unit Plan)

- Naming a fractional quantity depends on the size of the part compared to the whole, rather than the number of parts into which the whole is cut.
- The numerator refers to the number of pieces in the share.
- The denominator refers to the size of the piece relative to the whole.
- The size of the part determines the fraction's name. (i.e., "How many of these parts fit into the whole cookie?")

Key Questions and Ideas: (From a Grade 3, Year 4 Math Lab Agenda)

- What's the size of the piece?
- How many equal size pieces fit into the whole? (When four equal size pieces fit into the whole, we call the pieces "fourths".)
- What's the size of the piece?
- How many fourth-size pieces?

Figure 17. Examples of school-wide fraction language.

The school-wide approach to talking about fractions was a consistent theme across all data sources, all participants, and all five years. Kyle remembered working on language the first year and laughed as he remembered having it written on a sticky note that he kept on the side of his easel so he could refer to it during instruction. He contrasted that with how natural it is for him to use the fraction language now: "I feel like that language is very easy for me to use now whereas before I would always have to think about what I was going to say before I said it or write down what I was going to say specifically about each fraction." Leslie, who has moved between all three intermediate grade levels and was in the middle of her second year of teaching 5th grade fractions at the point of our interview talked about how intentionally she prepares for the language that is part of her fractions instruction:

I was looking through the lessons ahead of time and really being cognizant of the language that I need to use when I talk about fractions of whole numbers. So, I actually practice that before I teach it (chuckles) because I don't want to say the wrong thing and mess them up. I feel a real responsibility to not screw up fractions for fifth graders, for who it's the diving board for everything else they're going to do.

Layla provided a comparison of the kind of language she used prior to the SIG (“Whereas in the past, I would just say three-sixths and move on.”) to her language at the time of the interview (“Now, I really stop and have them share, what does the 6 mean? It's the size of the piece. It's a sixth-size piece. And what does the 3 mean? It's the number of pieces.”).

In addition to developing common language, Tara guided the development of an instructional sequence for introducing fraction terminology and symbolic representation. In grade 3, fractions are initially represented using written words to represent the size of the piece. This is typically in the context of equal sharing problems since that is how fractions are introduced. For example, $\frac{3}{4}$ would be written initially as “3 fourth-sized pieces”. Then, as students develop understanding and fluency with the naming conventions, teachers shorten it to “3 fourths” and eventually introduce the symbolic representation of “ $\frac{3}{4}$.” This progression typically occurs over the first week of fractions instruction in Grade 3, but builds on students’ experiences in Grades 1 and 2, where symbolic notation is generally not used.

Illustration #2: Articulating mathematical explanations. Articulating mathematical explanations is another form of CKTM that was described by teachers and worked on during Math Labs and PLC meetings. In my interview with Erin, she said that being introduced to the denominator as “the size of the piece” helped “everything make more sense.” When I pressed for an example of what she meant, she explained:

It helped me understand why we find common denominators when we add or subtract. So if I have $\frac{1}{4}$ and I want to add it to $\frac{1}{3}$, if I'm visualizing that, I know that fourth-sized pieces are smaller than third-size pieces, so if I want to combine them together, there's no way of knowing what the final amount is going to be because the sizes of the piece are different. So, I need to find a way to make it so they have the same size piece.

This is an example of Erin articulating a mathematical explanation for why common denominators are used to add (and subtract) fractions. Though teachers at Hilltop press students

to provide explanations, there are times where they need to be able to articulate mathematical explanations themselves—both as they work with their colleagues and as they work with students.

Illustration #3: Recognizing what is involved in using a particular representation.

There are two representations that are primarily used with fractions at Hilltop Elementary. One is an area model and the other is a number line (see Figure 18). The area model is introduced initially to students because it provides a variety of partitioning possibilities. The number line is introduced in third grade after students have had substantial experience with the area model (which is introduced in first grade). Both of these models were new to teachers in some way or another. Though teachers were familiar with area models, they described experiences where the area model was pre-partitioned and pre-shaded. Partitioning was a new feature for all of the teachers. As the coach, university math educators, and teachers gained more experience with partitioning area models, squares and rectangles became the preferred shape because they offered more partitioning possibilities (though circles were not explicitly avoided) (see Figure 19). This learning also influenced a revision to the CGI assessment between Years 1 and 2 (see Appendix C).

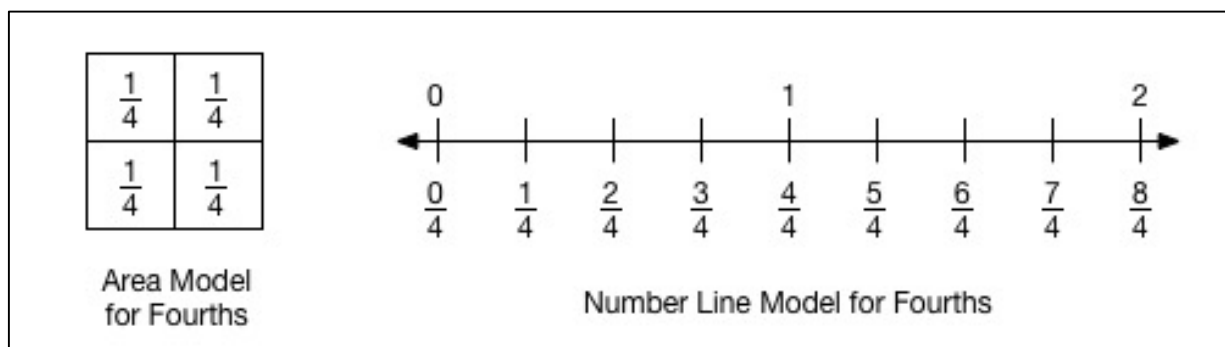


Figure 18. Fraction models used at Hilltop.

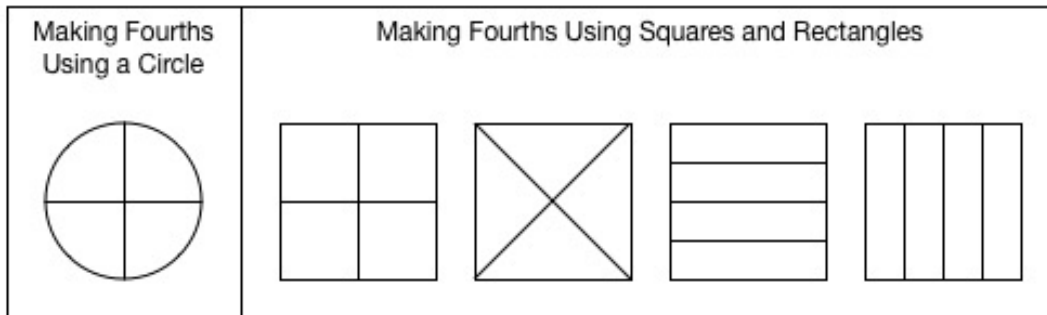


Figure 19. Partitioning circles and squares/rectangles.

With regards to number lines, Layla explained, “We didn’t do any number lines before. I had no knowledge of them, but I’ve learned that seeing it as movement along the number line is important to understand what $\frac{1}{3}$ means on a number line.” The “movement on the number line” is something that the Grade 3 team worked on in their fractions Math Lab in Year 5. They discussed how $\frac{1}{3}$ isn’t just a point on a number line, but that it represents the distance from 0. They puzzled together about how to support kids in understanding that. Together they tried constructing written representations and brainstormed how they might use a bead moving along a “partitioned” (folded) pipe cleaner.

Layla also shared a question that she was currently puzzling over and planned to take to Tara and her Grade 3 teammates. She is wondering about choosing a particular representation in relation to the context of a problem: “For example, I’m thinking that area models are more appropriate for contexts that involve sharing food, but number lines are better for problems involving distance. But would you ever want to switch that for any reason? Like use a number line for sharing food or an area model for distance? Are there times when it doesn’t matter?” Layla’s questions point towards the mathematical understanding needed to make decisions about what representations to use when, for what reasons, and in what ways.

Illustration #4: Linking representations to underlying ideas and to other representations. Linking representations to underlying ideas and to other representations is

another form of CKTM that Hilltop teachers used as part of their approach to fractions instruction. In Grades 3 and 4, students are introduced to equivalent fractions by comparing two different solutions to the same sharing problem. For example, there are multiple solutions for the problem: *6 children are sharing 8 small sandwiches. They are sharing so each child gets the same amount. How many sandwiches will one child get?*” Figure 20 shows three possible ways students might solve this problem. A teacher might put two or more of these up next to each other and ask students a question like, “Could these both (all) be correct? Is that possible?” However, in order to use these solutions to introduce the concept of equivalent fractions, teachers must be able to link two (or more) representations of the same problem to one another as well as the underlying idea of equivalence.

Leslie described a time that she put an area model and a number line next to one another to talk about similarities and differences in partitioning strategies. In particular, her goal was to help students use what they knew about partitioning an area model to help them develop more accurate partitions on a number line.

Erin (the resource room teacher, who pushes in to multiple classrooms on a daily basis) made an important distinction about how representations are often used at Hilltop. She described how she typically sees teachers doing the work of representing the students’ thinking (as the students’ explain)—as opposed to students creating the representation or showing student work.

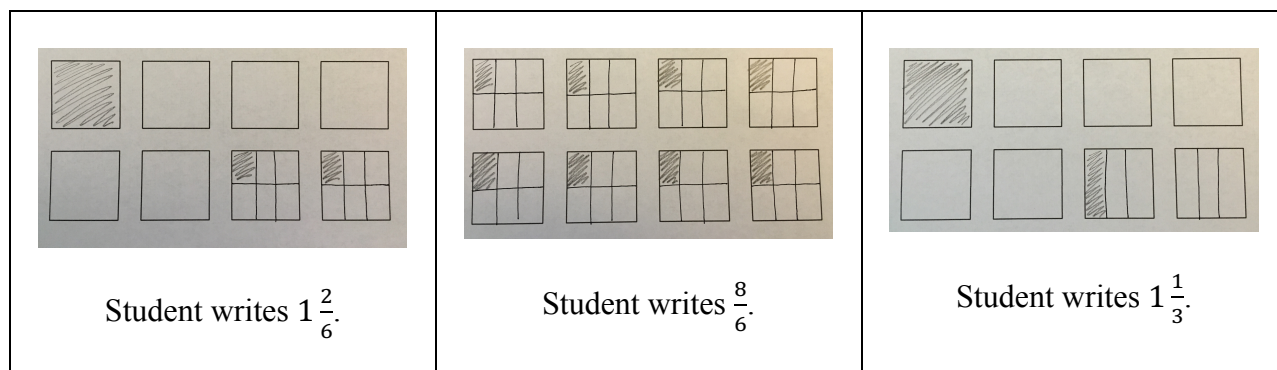


Figure 20. Three (of many) possible solutions for 6 children sharing 8 sandwiches.

This is a shift that has occurred over time for two reasons. One, students are able to see the creation of the representation, versus just the final product. And two, teachers can make decisions in the moment about how to represent students' ideas so that they are more accessible to the class as a whole. For example, teachers can make informed decisions about the pacing, the colors used to draw attention to particular relationships, and how things are labeled.

Instructional Vision

In addition to teachers' CKTM, I explored teachers' vision of high-quality mathematics instruction by asking them: "If you were asked to go into a teacher's classroom for one or more fractions lessons, what would you look for to see if the instruction was of high quality? What would the teacher be doing? What kinds of problems or tasks would you expect to find? Please describe what classroom discussion would look and sound like if instruction were of high quality." When Munter (2014) developed the rubrics for vision of high quality math instruction (VHQMI), he used a sample of 128 interview transcripts from administrators, math coaches, and teachers across 30 schools in four school districts that had recently begun implementing initiatives working towards ambitious teaching in math classrooms. Though his review of the literature indicated that instructional visions have many dimensions, only three were prevalent in his participants' responses: the role of the teacher, classroom discourse, and mathematical tasks.

In my analysis, I included the three dimensions from the VHQMI rubrics, but also included three additional dimensions: social culture and norms, mathematical tools as learning supports, and equity and accessibility. Since a primary goal of the SIG work at Hilltop Elementary had been to support teachers to develop ambitious teaching practices and the rise in students' achievement scores across grade levels and years provided evidence of change, I included the additional dimensions (see Figure 21). One aim of this portion of the analysis is to

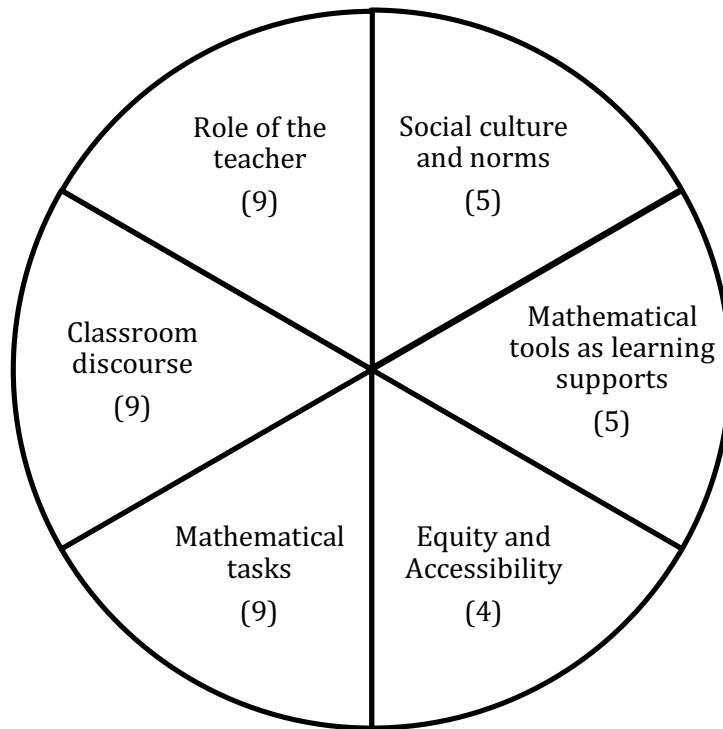


Figure 21. Dimensions of VHQMI and number of teachers that discussed each dimension during interview.

build on Munter’s initial work around VHQMI because as he points out, “articulations of a number of arguably crucial dimensions of mathematics classroom learning and teaching cannot be directly assessed with the instrument.”

As I illustrate teachers’ instructional visions in relation to the six dimensions, I will also argue that their instructional visions are both relatively sophisticated in relation to the images in Munter’s rubrics as well as the literature and well-aligned with one another.

Views of the Role of the Teacher

The VHQMI rubrics characterize the most sophisticated visions of the teachers’ role as being the “more knowledgeable other,” which is described as “proactively supporting students’ learning through coparticipation” and “designing learning environments that support problematizing mathematical ideas, giving students mathematical authority, holding students accountable to others and to shared disciplinary norms, and providing students with relevant

resources.” When analyzing Hilltop teachers’ characterizations of the teacher’s role, I used Munter’s (2014) three categories: *influencing classroom discourse*, *attribution of mathematical authority*, and *conception of typical activity structure* (Munter, 2014). The descriptions of five of the teachers included aspects from all three of these categories, whereas descriptions of four of the teachers included aspects from only the first two categories.

Influencing classroom discourse. All nine teachers described an instructional vision where the teacher’s role involved influencing classroom discourse. In particular, four teachers (Becky, Kyle, Layla, and Erin) described how they would expect to see teachers listening carefully to kids’ ideas and strategies. Teachers’ instructional visions also included teachers using talk moves (Becky, Tonya, Leslie) and asking lots of questions (Tonya, Leslie, Anna, Erin). These will be unpacked further in the “Views of the Role of Classroom Discourse” section that follows.

Attribution of mathematical authority. All nine teachers also discussed students’ roles with respect to the mathematics being learned as part of their instructional vision. Five teachers expressed that they expected students to be the ones solving problems and finding answers. Leslie explained, “I expect there to be questions from students...the teacher is not the person that answers questions or solves problems.” Teachers also described various strategies for eliciting students’ ideas and using their ideas as the source for whole group investigation and discussion. Tonya describes doing quick interviews with her students as they are working independently, “not necessarily to redirect their work, but just to figure out what it is they are thinking about.” Four teachers (Becky, Tonya, Kyle, Anna) described selecting and sequencing students’ strategies or work as part of a whole group summary discussion and three teachers (Anna, Erin,

Layla) talked about representing and recording students' thinking as part of a whole group discussion.

Conception of activity structure. Six teachers (Becky, Tonya, Leslie, Layla, Keri, & Kathleen) discussed activity structure as part of what they would look for as part of high-quality instruction. All six mentioned or described a launch-explore-summarize lesson structure (Madsen-Nason & Lappan, 1987). Tonya described the three phases most explicitly:

Launching a story problem means really digging into what is this asking us, do you-- getting the kids oriented to this task, and starting to generate ideas in their heads about how they might solve it. Sometimes we'll put tape over the values in the word problem so they don't start solving in their heads on the rug, but they are (instead) thinking what is this asking us to do...An exploration would be kids working--often in pairs, but sometimes by themselves--in 5th grade, we're moving more and more to them working by themselves. And just working on maybe the whole time, that problem, or maybe a couple of problems that are really similar to that. And the teacher during that time walks around and is taking notes. And then...the kids come back (to the carpet)...during the summary at the end, kids really having the chance to present their work and their strategy and have other students ask them questions about it...kids analyzing one another's work.

One additional comment that Becky made about activity structure was that she described typically starting lessons with an instructional activity that was aligned with the day's lesson.

Views of the Role of Classroom Discourse

The VHQMI rubrics characterize the most sophisticated visions of classroom discourse as involving whole class discussions that include student-to-student talk that is conceptually oriented and includes "articulating/refining conjectures and arguments for explaining mathematical phenomena." I used two of Munter's discourse categories to analyze the teachers' characterizations of the role of classroom discourse: patterns and structure of talk and nature of classroom talk (Munter, 2014). The patterns and structure of talk includes teachers' description of who talks to who (e.g., teacher to student, student to student), in what settings (e.g., small groups, whole groups), and the role of the teacher in such conversations (e.g., centralized, de-

centralized). The nature of classroom talk includes when teachers' description of classroom talk specifies what the talk should be about or what it should consist of (e.g., questions, explanations, mathematical arguments). Though some talked more in depth about one category than the other, descriptions from all nine teachers included aspects from both categories.

Patterns and structure of talk. Three teachers (Becky, Layla, and Keri) referred to “talk moves” when describing their instructional visions related to classroom discourse. Talk moves are a set of discourse moves that they teachers were introduced to in the first year of the SIG and include things like wait time, turn and talk, repeating, revoicing, and adding on (Chapin, O'Connor, & Anderson, 2009).

Two teachers (Tonya and Kyle) mentioned that the teacher would be asking questions to the whole group. However, it does not appear that the teachers' instructional visions involved the teacher-posed questions resulting in an IRE dialogue pattern (Mehan, 1979). Instead, eight of the nine teachers specifically mentioned using “turn and talks” and both Becky and Tonya described how the teacher would strategically call on students based on what the teacher heard while listening to students during the turn and talk. Moreover, five teachers (Becky, Kyle, Anna, Keri, Kathleen) described students listening to one another in order to agree, disagree, and ask questions of one another.

Nature of classroom talk. The teachers' instructional visions around the nature of classroom talk expected students to move beyond providing calculational steps and answers to more conceptually-based discussion involving as Tonya described, “arguing and grappling with ideas,” “making conjectures,” and “generalizing.” Five of the nine teachers indicated that kids should be explaining and defending their thinking by providing evidence (Becky, Tonya, Layla, Keri, Kathleen) and using academic language (Keri). Six teachers also gave examples of the

kinds of questions they expected to hear teachers ask (e.g., If this is how it works, why doesn't it work this way? Why do you think this works? Why do you think this doesn't work? Why fifths?). Leslie also gave examples of the kinds of questions that students might ask each other (e.g., How come you did that? I think you did this instead of this. What do you think?).

Four teachers (Becky, Tonya, Anna, Layla) also indicated that incorrect answers would play a role in the discussion. Tonya explained she looks for “who’s going to offer us something really juicy to think about?...an answer that’s kind of in process, and not correct, but something we could revise together.” Layla, too, expressed comfort in having incorrect answers emerge during whole group discussions: “I’m okay when kids give me answers that are not correct because we can talk about it.” She contrasted this to before the SIG when she described that she would “just say no, that’s not correct. And then give the correct answer and move on.”

Views of the Role of Mathematical Tasks

The VHQMI rubrics describe sophisticated visions of mathematical tasks as visions that emphasize tasks that have potential for students to engage in explaining, generalizing, and making connections between strategies and representations. In describing their instructional visions related to the role of mathematical tasks, teachers did mention a few specific tasks, but most of their comments about mathematical tasks were more general, describing task features and what tasks should offer with respect to student learning opportunities. Three different Instructional Activities (IAs) were mentioned: True-False Number Sentences (Becky), Analyzing Visuals (Tonya, Kyle, and Anna), and Number Strings (Leslie). Kyle also mentioned the more general term “number talks”, which is often used to describe a suite of IAs, and Layla mentioned “equal sharing problems,” as a task that would be used as part of high quality fractions instruction.

Five different features of tasks were described by teachers. Tonya and Kyle both raised the issue of having diversity in tasks. Kyle explained, “I don’t think there’s one kind. There’s a lot of breadth in good problems...it’s good to have exposure to lots of different things.” One specific way that Layla and Kathleen described diversity in tasks was through using a variety of contexts. However, a diversity in tasks should not come at the expense of tasks being “connected,” an idea discussed by both Becky and Kyle. In describing her vision, Becky said she expects IAs to be connected to the bigger lesson or objective for the day. Kyle explained that high quality instruction would “have activities that tie together and [the teacher] knowing how to use [the task] and when to use it is super important.” Three additional task features mentioned by teachers included having multiple approaches for solving (Becky, Layla, and Erin), offering opportunities for independent practice (Leslie and Keri), and being challenging (Anna).

A final way that teachers talked about tasks involved discussion of the kinds of mathematical practices in which tasks should engage students. These practices include explaining their thinking (Becky), making connections across strategies (Tonya), modeling mathematics (Anna), and thinking about problem situations (Anna).

During the interviews, I did not specifically ask about teachers’ instructional visions related to the role of social culture and norms, mathematical tools as learning supports, and equity and accessibility. However, visions related to these three dimensions were discussed by a number of the teachers. The exclusion of teachers from the next three sections does not imply that these three dimensions are not part of their instructional visions. Instead, it means that these dimensions did not come up unprompted during the interview with that teacher.

Views of Social Culture and Norms

Math education researchers have increasingly focused their studies on classroom contexts and we have a much better understanding of the kinds of social contexts that support students to learn mathematics with understanding. Hiebert et al. (1997) have suggested a number of norms that seem to support students to learn mathematics with understanding: “doing” mathematics involves collaboration with peers; information and solution methods become available through communication; create cognitive conflict (rather than avoid it); discussions are about methods and ideas; students choose their own methods and share them with others; use mistakes are sites for learning; and correctness is determined by the logic of mathematics (versus correctness determined by feedback from the teacher).

More than half of the Hilltop teachers interviewed discussed the social culture and norms of a classroom as part of their instruction visions. As part of her instructional vision for classroom discourse, Becky described how “kids sit on the rug next to someone who they can share idea with for whole group discussions.” In addition to one other teacher (Layla) explicitly mentioning kids sitting on the rug for whole group instruction, all nine Grade 3, 4, or 5 classrooms (where the interviews took place) had an area of the classroom with an area rug and easel where whole group discussions take place.

Three of the teachers (Becky, Leslie, and Layla) talked about how students use non-verbal hand signals to communicate with the teacher and other students. Layla explained how “having their thumbs at their chest” (instead of raising hands to indicate kids are ready to share) and “giving them wait time and having the other kids respect that” creates an environment where “kids don’t feel rushed.” Leslie described a new signal that teachers had developed earlier that school year. She explained that kids had started using a “slashing motion under their chins” when

they disagreed with something said by another student and teachers wanted to replace it with something more positive so they decided as a school that they would teach students to “scratch your head or chin” to show “I’m wondering about that” or “I have a question for you.”

Leslie and Layla also commented on the how students respond to challenging problems. Layla said about her third-grade students, “they persevere and know it's okay to struggle.” And Leslie elaborated on how her fifth-grade students have developed a playful and supportive response to challenges that emerge:

We have a saying in our class this year. The struggle is real. Because every day, they look at a lesson and they go and struggle with that on their own and they're supposed to struggle. I tell them, it's hard. You're going to have to really think about it and make sense of it. They expect a struggle, so they don't give up. And when somebody pops up and says, "This is hard," someone else will pop up and say, "The struggle is real!"

Views of Mathematical Tools as Learning Supports

Hiebert et al. (1997) and Carpenter and Lehrer (1999) described how tools can be used to support students to learn mathematics with understanding. They argued that meaning must be constructed for and with tools. Being shown how to use tools in procedural ways will not support students to develop meaning of the tool or meaning for its’ use. Also important is that different kinds of meaning and use can be developed for a given tool. Tools can be used to keep records, communicate in various ways, or students can use the tools to support thinking.

Though a majority of the teachers frequently mentioned models and representations as part of their instructional visions, they did not necessarily elaborate on how they were used as learning supports. The mathematical tools that were explicitly described as supporting learning included “representations,” “context,” and “hands-on materials.”

Becky and Tonya both discussed the use of representations. Becky explained that having more than one representation as part of the whole class discussion would provide opportunities

for more students “to connect.” Tonya described the role of direct modeling. Direct modeling strategies involve explicitly modeling all the parts and the action of a given situation. She explained teachers should let students “live in direct modeling land for a while when dealing with fractions, so they really understand why they are doing all this work.” In her statement, “this work” refers to students finding equivalent fractions as part of their work adding and subtracting fractions with unlike denominators.

Three teachers described how context was used as a learning support, which Tonya points out is atypical when she explains, “I think the context, interestingly enough... ends up being way more supportive than a naked number sentence.” She also reflected on her own experience as a student, where story problems were dreaded and typically were given as “challenges.” Karen also suggested adding context as a way of supporting student understanding and Leslie explained that sometimes contexts are needed to support students when they are working on number strings.

Views of Equity and Accessibility

Though equity and accessibility are frequently addressed as part of the other dimensions (e.g., developing a discourse community is one way to support more children to learn with understanding), Hiebert et al. (1997) identify equity and accessibility as a dimension of its own. They contend that every learner can learn mathematics with understanding. In order for this to happen, they suggest that teachers create classroom environments that take into account the uniqueness of each child so that each has “the opportunity to engage in and reflect on tasks that are mathematically problematic in a social community where his or her thinking is discussed and valued” (p. 66). In a similar manner to other math educators, they relate this dimension to the

previously described dimensions and suggest that together these dimensions work to enable all students to learn with understanding.

Four Hilltop teachers described more explicitly ideas about equity and access as part of their instructional vision. Both Tonya and Kathleen shared that they think about increasing access for their EL students. Tonya described “making sure our EL kids have access to the language that [the task is] talking about or that it's using” and Kathleen (the EL teacher) explained how turn and talks provide opportunities for students to practice language regardless of their different [English Language] levels, “maybe there are some kids that can support with full sentences and give evidence while other kids are at least turning to their partner and they're practicing some of the language.”

Becky described two ways her instructional vision focuses on issues of equity. First she described that the teacher should not always go to the same student “who you know has the right answer.” She went further, asserting that discussions need to include “multiple different kids showing their thinking.” Erin (the resource room teacher) described how she has seen many of the Hilltop general education teachers create more equitable and accessible opportunities for her special education students by encouraging all students to “choose a strategy that works for you.”

Trajectories of Student Learning

At the beginning of the first year, all teachers at Hilltop Elementary were introduced to Cognitively Guided Instruction, including trajectories of strategies that children invent to solve addition, subtraction, multiplication, and division problems involving both single- and multi-digit problems. In her interview, Leslie recalled learning about the “CGI trajectories” for the first time:

I remember my first or second year [of teaching] wishing I had something that would tell me where kids were and what it was they didn't know yet. And I didn't know what that

was. My experience as a paraeducator [prior to becoming a classroom teacher] was that you could pinpoint kids' reading development and I wondered, "Where is that for math?" So that we know what it is that we need to do for them next. Instead of just going on and leaving them behind. When the first year of the grant came along and we learned about the different levels when we were doing the CGI trajectory, I was just like, "Oh my, it's like angels singing. I've been looking for this for four years. It was thrilling. I was like, I know exactly what I need to learn now.

For Leslie, the CGI trajectories reminded her of the kinds of tools she had for thinking about kids' reading development, but was missing for kids' mathematics development. Kyle was also at Hilltop Elementary when the CGI trajectories were introduced for the first time. He talked extensively throughout his interview about the role that the CGI trajectories had played in his work as a second-grade teacher:

So much of it is being an observer of what the students are doing. For me, understanding the trajectory of where the students are coming from and where I need to get them to is super important. So not just the concepts, but the trajectory in which they understand the concepts have been most helpful for me. And that's where the observation comes in for me. If I see a student and they're doing something, I have an idea of what their next steps would be in order to better understand the concept that we're working on. If there's a student who's doing something a particular way, I know what questions to ask them and where to push them in order for them to be more efficient with what they are doing or more accurate, or a strategy that allows them to be more accurate. I have to have the knowledge of that trajectory and I think that's definitely been at the core of my teacher tools.

For Kyle, the trajectories were a tool that helped him understand where students were coming from as well as helped him think about advancing their strategies. The trajectories also pushed him to pay close attention to student thinking and how students were making sense of problems and the mathematics.

Trajectories for Adding and Subtracting Fractions

Although the trajectories that Leslie and Kyle refer to are specific to students' strategies for whole numbers, the notion of students' understanding and strategies became an organizing framework for other content at Hilltop Elementary. When I interviewed Kyle, he had recently

moved to fourth grade after teaching second grade for four years. The content standards in second grade includes extensive work with addition and subtraction of numbers to 1000 and align closely with the CGI trajectories for multi-digit strategies. When sharing about a recent conversation with Tara (the math coach), he described how his understanding of the trajectories for whole number addition and subtraction provided a framework for thinking about how students' strategies for adding and subtracting fractions develops. Kyle had noticed that when adding two mixed numbers (for example: $1\frac{1}{2} + 2\frac{1}{4}$), students were first adding the wholes ($1 + 2 = 3$), then adding the fractional parts ($\frac{1}{2} + \frac{1}{4} = \frac{3}{4}$), and finally, combining the two amounts ($3 + \frac{3}{4} = 3\frac{3}{4}$). However, this did not align with what he remembered reading in the Grade 4 roadmap: "I thought by looking through the pacing guide that one of the focuses was more toward converting mixed numbers into improper fractions" so he raised the issue at a PLC meeting, which he described:

So then we talked about it and drew a really nice parallel with what I was doing with addition and subtraction in second grade strategies in general. We thought, this is like breaking [whole numbers] up in to place value, when you are adding the hundreds, tens, and ones separately and then combining it. And I felt like that gave me the context to put the fractions into. So I knew that when it came to subtracting, that strategy would become difficult if your second fraction was larger than the first fraction by itself and that's where kids would run into a lot of issues. And so then from there, it clicked and I realized I should push them to be more comfortable using a number line or just taking away in chunks and keeping their first number whole because it will help them when it comes to subtracting. So far, we haven't given them numbers that they have to regroup yet, but when it does, it will be a strategy that will be helpful for them. If they are comfortable with that strategy now, when it comes to crossing the whole, hopefully they will latch on to that easier because they know the strategy already.

In this passage, Kyle is using what he knows about the trajectory for kids' strategies for whole number addition and subtraction to generate a trajectory for kids' strategies for addition and subtraction involving fractions. In his initial interpretation of the roadmap, he had come to expect more of a procedural strategy involving converting mixed numbers to into improper fractions,

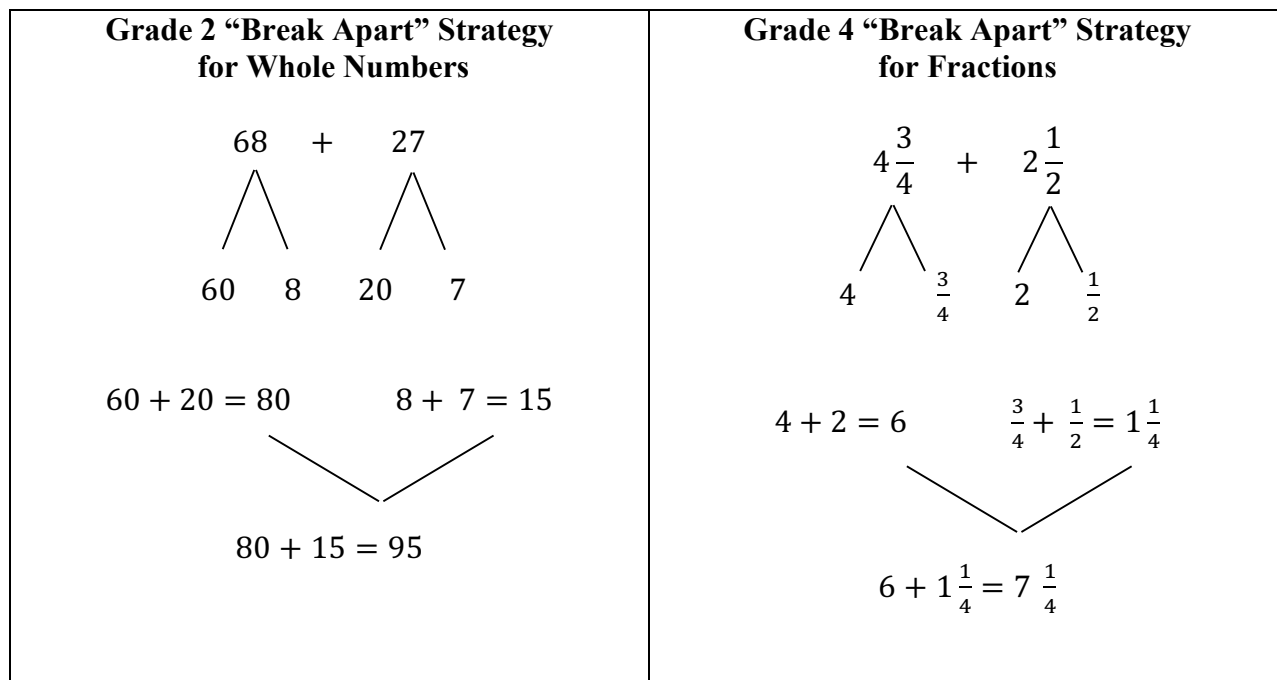


Figure 22. "Break apart" strategies for whole numbers and fractions, connected by Kyle.

but through conversation with Tara during the Grade 4 weekly PLC meeting, he connected the strategy kids were using (combining wholes and fractions) to a strategy he was familiar with from Grade 2 (breaking up into place value) (see Figure 22).

Leslie and Tonya also talked about a trajectory for adding and subtracting with fractions. Tonya described how it’s important to “let them live in direct modeling land for a while when dealing with fractions.” Direct modeling strategies involve explicitly modeling all the parts and the action of a given situation. It is at the beginning of the trajectory and where many students start out when solving problems. She explained that students “really understand why they’re doing [things like] changing denominators and converting back into a fraction that’s greater than one” when they first spend time direct modeling. She added, “When we don’t let kids model with fractions, you’re doing all of this stuff that feels really abstract and you’re not grounded in why.” Here, Tonya is using the mental trajectory she has developed to justify instructional decisions that are intended to support the development of students’ conceptual understanding.

Additional Trajectories for Fractions

Though not discussed in interviews, my analysis of math labs revealed two additional trajectories that teachers used as part of their work together. The first was Empson & Levi's (2011) trajectory of partitioning strategies and the second was a trajectory related to how students named fractional portions.

Empson and Levi's trajectory was primarily used by teachers when they examined student work during the math lab. For example, in the fourth grade lab, Tara (the math coach), gave a brief overview of the trajectory since two of the teachers were new to the grade level. Anna had moved from first grade and Tonya was new to Hilltop. Then, Tara asked teachers to "see if you can find an example of each of these strategies as you dig through your papers." As the teachers worked in partners to classify students' strategies, they asked questions of one another and to Tara, working together to make meaning and develop mutual understanding of how students partition.

The second trajectory used in labs involved how students learned to use fractional language and was described by Tara. She explained that, "typically what we'll see is that they'll sometimes call things wholes, then pieces or parts, then they'll name everything as halves before they actually have the name for it." The trajectory that Tara is describing is illustrated by the student work in Figure 23.

Normative and Personal Identities

Gresalfi and Cobb's concept of identity describes it as "the set of practices and expectations that shape participation in particular contexts" (pp. 273-4). In particular, they describe the process of identifying in a particular context as the relation between two elements:

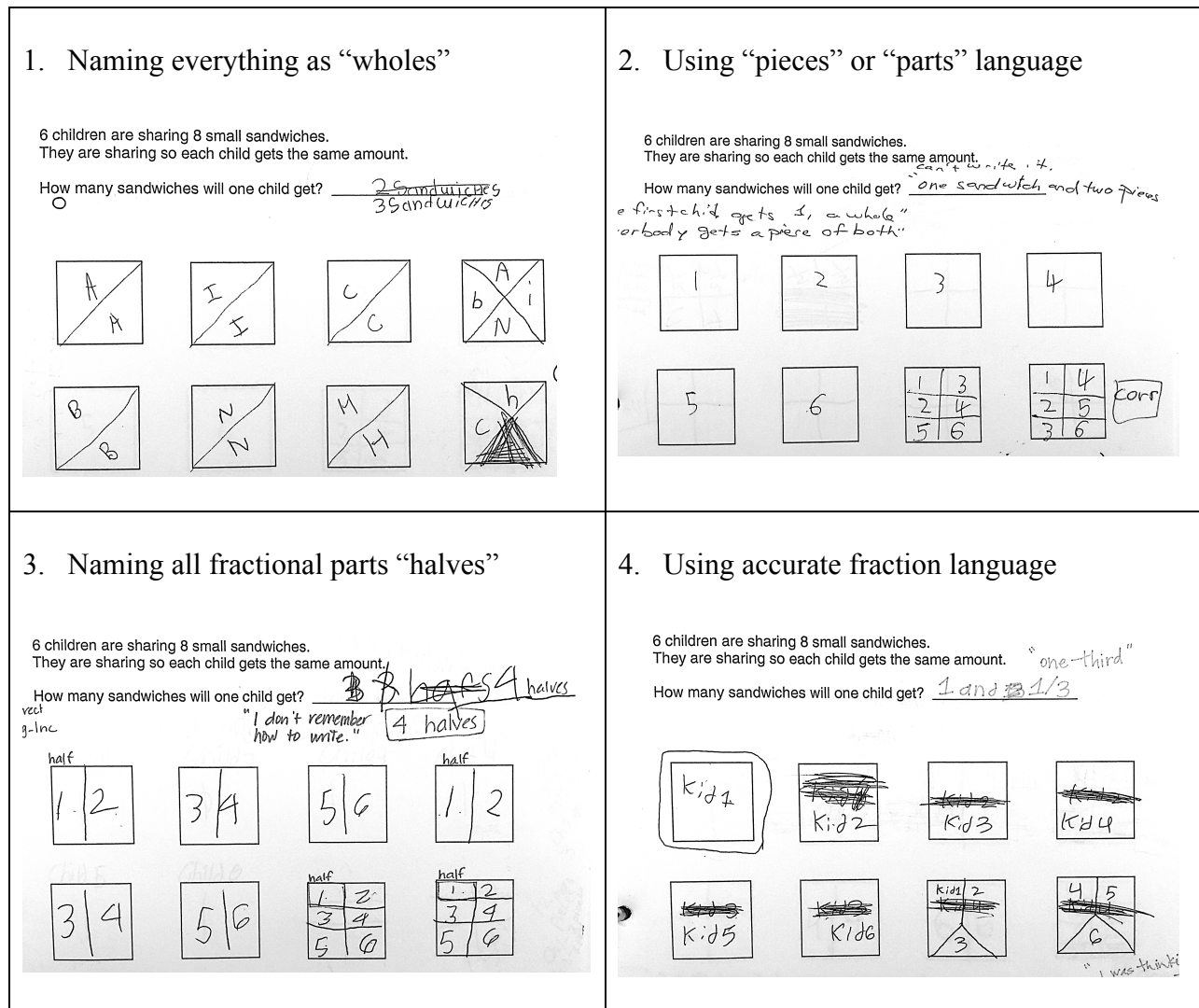


Figure 23. Fraction language trajectory described by Tara.

the normative identity (established in a specific context) and individuals' personal identities (that develop as individuals participate in the practices of that context). In the sections that follow, I will characterize the normative identity for teaching and learning to teach at Hilltop Elementary as well as teacher's personal identities—both as they started as teachers at Hilltop and after the fifth year of the SIG.

Normative Identity

The normative identity for teaching mathematics and learning to teach mathematics at Hilltop Elementary appears to be co-established by Julie (the principal), Tara (the math coach),

and teachers. Analysis of math labs, PLCs, and interviews with the teachers and math coach included multiple excerpts where a “set of practices and expectations” were stated explicitly, discussed, and/or enacted. In the sections that follow, I will characterize the set of practices and expectations around both the normative identity for teaching mathematics and the normative identity for learning to teach mathematics at Hilltop Elementary.

A normative identity for teaching mathematics. The normative identity for teaching mathematics overlaps in many ways with teachers’ instructional visions related to the role of the teacher, classroom discourse, social culture and norms, and equity and accessibility. For example, in a Grade 4 math lab in Year 4, Julie and Tara both convey an expectation that teachers need to go beyond examining a student’s written work by listening to the student’s thinking and watching them work in order to really assess a student’s understanding. The group is puzzling over a piece of student work that has some errors that are surprising to the group based on what they know about the student. Julie says, “We need to watch him work. We need to see what happened...now we know who we need to learn a little more about. In that case, I don’t think we can tell from there. He knew they got more than one. He knew they each needed to get an equal amount. He has some pieces.” Tara followed this up by suggesting, “He would be an interesting one to give it one on one to and see what he does.” In these statements, Julie and Tara are conveying their expectation that teachers understand the thinking of their students and engage in a variety of practices to accomplish that (reviewing student work, interviewing and observing them one-on-one when needed).

Another aspect of normative identity was described by Keri during her interview. She described, “...there’s an expectation [from the principal and math coach] of how we lead our math discussions on the carpet here... the importance of actually being able to explain what

you're thinking instead of memorizing the right answers.” Here, Keri is describing expectations for social culture & norms (“math discussions on the carpet”) as well as the nature of the classroom discourse (“being able to explain what you’re thinking instead of memorizing the right answers”).

Julie (the principal) also communicated expectations around equity and accessibility of learning opportunities for students. Becky described a PLC interaction from her first year where she was pushed to be in more in alignment with her teammates: “At a PLC, Jessica came and said I think we need to cut off homework at a certain time so you guys can all start math at the same time.” Becky had been spending a majority of her math block reviewing homework from the night before and cutting her instructional time short as a result. An expectation for “coherence and alignment” was again communicated, school-wide this time, during school-wide professional development prior to the start of the second SIG year. Julie told the teachers, “Equity means to me that every child walks into a classroom and their thinking is valued, they are seen as a contributing member, and the teacher has a plan for the child.” She followed this with her expectation for the coming school year: “This year is about getting people more in alignment. Now, it’s about alignment. No matter what classroom, child will experience the same instruction. This year, is about getting everyone all to an equitable level.”

A normative identity for learning to teach mathematics. A normative identity for learning to teach mathematics started to develop during the first year of the SIG. Partway through the school year, a quote was written in the corner of the whiteboard in the Teacher’s Classroom: “You can’t look good and get better at the same time.” This idea became a mantra for teachers during math labs as they tried out new things together well as in their classrooms on a daily basis. It was an idea that was championed by both Julie and Tara. Julie acknowledged this

in a Grade 4 Math Lab where teachers did a classroom visit with a true-false comparison with fractions. The students were finishing a multiplication and division (with whole numbers) unit and had not yet started their fractions unit, which had made it difficult to anticipate students' responses during the planning time of the math lab. After debriefing the classroom visit, Julie said to the group:

All of you, I have to compliment you. It's hard to go in cold to [figure out] what do they know. It's not the next in a series where you really can predict. All of your revisions, that was great. You're modeling it for them. They need to know that we don't know everything just like you don't know everything and we're going to revise our thinking just like we expect you to revise your thinking, so thank you for modeling that.

Later in the lab, Julie checked in with Tonya, who was new to Hilltop Elementary in Year 4 and for whom this was only her second math lab. After Tonya explained why she had hesitated to call a Teacher Time Out, Julie responded, "Feel free. Everybody's been there. You have to try it a couple times to know it's safe... [this needs to be a space that is] productive for you as a learner...it is super important."

Teachers' Personal Identities

In order to understand teachers' personal identities related to teaching mathematics, I asked, "How would you describe yourself as a math teacher right now?" and in order to understand how their personal identities may have developed while teaching at Hilltop, I followed up with, "Would you have used that same description x years ago (prior to coming to Hilltop or the grant starting)? How would that description have been different?"

Personal identities prior to teaching at Hilltop during the SIG. As teachers reflected on their identities prior to teaching at Hilltop, there were three ways they talked about identifying: first, as a learner of mathematics; second, more specifically as a learner of fractions; and third, as a teacher of mathematics.

Identifying as learners of mathematics. Describing herself as a learner of mathematics, Leslie said, “I passed all those college classes, but I didn’t really understand the process of understanding math.” She described a tension between having successfully completed college-level mathematics courses but not having a conceptual understanding of that mathematics. Anna described a slightly different identity as a learner of mathematics: “I always liked math, but had more of an unspoken understanding of how numbers worked.” She explained how this affected her identity as a teacher of mathematics: “I didn’t understand how to communicate my own understanding to kids or how to elicit that from the kids.” So, while she describes an “understanding” that seemed to be missing for Leslie, that understanding did not necessarily translate to her identity as a teacher.

Identifying as learners of fractions. Erin, Tonya and Becky both talked specifically about their identity as learners of fractions. Tonya reflected, “I know as a kid for me, it was incredibly abstract. I don’t remember modeling fractions at all.” Erin was a quite direct: “When I first started here at Hilltop, I hated fractions. Becky recalled her response in Year 1 when Tara (the math coach) announced at a math lab that their next unit would be fractions: “I remember Tara saying, okay the next unit is fractions and I remember me and [another teacher] were like, ‘Fractions, oh no, great. Because I did not have any good experiences with fractions when I was in school.’” For these teachers, they did not positively identify as learners of fractions, and in the case of at least Becky, this influenced her identity as a teacher of fractions (“Fractions, oh no, great.”) when she started at Hilltop Elementary.

Identifying as teachers of mathematics. Becky was not the only teacher that did not identify positively as a teacher of mathematics or fractions. Layla describes, “Before the grant, I wouldn’t have had my kids explain what a fraction was...I assumed I was making sure they

understood, but I didn't stop to have kids share their ideas. I would just move on when the correct answer came out." Erin and Keri also described identifying as teachers who tended to emphasize procedures and skills prior to coming to Hilltop Elementary. Erin described her teaching as "very structural and skill-based" and specifically said about teaching fractions, "I dreaded fractions units." Keri reflected, "I don't think I would have valued (discussion and explaining) before coming here. I liked the procedure of things and less than messy, so I don't think I would have valued that." In terms of deciding what to teach prior to the SIG, Leslie described "just following the lessons in the curriculum" and Kyle recalled, "I literally followed what was in the teaching guides." One interesting analytic note is that these characterizations of teachers' personal identities are a reflection upon their personal identities from two years prior (for Tony and Keri), four years prior (for Erin and Kathleen), and five years prior (for Becky, Leslie, Kyle, Layla, and Anna) as opposed to how they actually identified at those time points. A comment from Leslie is helpful in understanding why this distinction is important and one cannot assume that teachers would have given similar responses had I asked them the same question two, four, or five years ago: "I would say I was below 10% [confident] back then...At the time, I probably would have said I was doing okay, but now that I can compare what I have learned in the last few years to that, I was just reading the curriculum basically and didn't really have a deep understanding on my own of what I was trying to have the kids understand.

Personal Identities After the SIG. The ways in which teachers characterized their personal identities at the time of the interview were strikingly different from prior to teaching at Hilltop during the SIG. In the sections that follow, I provide an analysis of teachers' personal identities related to teaching mathematics and learning to teach mathematics.

Personal identities related to teaching mathematics. Though there were differences in the details and language of teachers' characterizations of how they currently see themselves as a teacher of mathematics, two predominant themes emerged. First, teachers described an increase in their confidence as teachers of mathematics. Four (Becky, Leslie, Layla, and Kathleen) of the nine teachers included this as part of their description. However, the reasons they connect to their increased confidence vary. Becky connected her increased confidence to a better understanding of content. Kathleen connected it with both increased content knowledge ("knowing the information") and pedagogical knowledge ("how to teach"). Leslie related her increased confidence to language ("I have words for things that I didn't when I started teaching.") and Layla associated her confidence with taking on a stance as a learner ("I'm confident teaching fractions now because I know it's okay to make a mistake as a teacher. You can show kids you revise and you move on.").

A second theme that emerged involved teachers' stance towards developing students' conceptual understanding of mathematics. Five teachers (Erin, Keri, Kyle, Layla, and Anna) described how supporting the development students' conceptual understanding was a priority. Keri explained, "What's important to me is that my kids are the ones explaining the whys. I like when my kids get the right answers, but I like it more when they explain and realize that they don't [necessarily] know the right answer." Similarly, Erin (the resource room teacher) described how she thinks about conceptual understanding in relation to her special education students: "I really feel like all the kids I work with can access the grade level content and it doesn't need to be easier for them. That's something specifically in math that I've pushed for. I really try my best to make sure my students are understanding the information conceptually which has been a turn

from how I taught before [coming to Hilltop]. It was very structural and skill-based, so that's been a shift in my perspective in my take as a teacher.”

Personal identities related to learning to teach mathematics. Three of the teachers (Kyle, Layla, and Tonya) also identified as learners when characterizing how they saw themselves as teachers of mathematics. Tonya elaborated on this notion:

I feel very much like a math learner with my students. Even doing the pre-work (prior to teaching) to be grounded in the concepts that we want the kids to grapple with before the moment of teaching, I feel like I'm definitely learning, even if not because of the particular ideas they've offered, but because I'm trying to learn about what kinds of patterns to kids make when they're learning this.

Kyle took a similar stance as someone that learns from his students. He described, “I am an observer of what the students are doing.” He elaborated and explained how his observations inform his teaching: “Now, it feels a little bit more organic in the sense that I am taking what students are doing and I'm asking them questions...I guess I see myself more as a facilitator in their thinking.” Layla’s stance as a learner connects back to her increased confidence. She explains how her working on practice together with her colleagues in math labs has supported the development of this stance: “Working at Hilltop and practicing together has taught me, we’re okay making mistakes. We all make mistakes. And we're not perfect, just because we're teachers.”

Alignment between Hilltop’s Normative Identities and Personal Identities

When comparing the normative identity that has developed at Hilltop Elementary with the personal identities characterized by teachers in Year 5, there are many ways in which they are aligned and support the notion of mutual understanding. When looking at normative and personal identities related to teaching mathematics, the normative expectations that teachers facilitate whole group discussions involving students explaining and justifying is aligned with

teachers' personal identities valuing the development of students' conceptual understanding. When looking at normative expectations related to learning to teach mathematics, there is a strong theme of viewing teachers as learners who are going to make mistakes and should embrace revision, which aligns strongly with teachers' personal identities as learners alongside their students.

In this chapter, I offered examples of four kinds of intellectual resources that teachers generated, drew upon, made meaning of, and coordinated as they engaged in teaching mathematics and learning to teach mathematics: CKTM, instructional visions, understanding of trajectories of student learning, and teachers' personal identities. In particular, I provide four illustrations of what teachers *do* know and understand about fractions and CKTM as they engaged in various mathematical tasks of teaching. I also argued that teachers had developed relatively sophisticated instructional visions that were aligned with one another. With regards to understanding trajectories of student learning, I explained how teachers used their cognitive framing of trajectories for operating with whole numbers to organize their understanding of students' fraction ideas, including operating with fractions and using fraction language. Finally, I offered an analysis of the normative identity established by the instructional leaders at Hilltop Elementary and how teachers' personal identities were not initially in alignment with this normative identity, but shifted over time to come into alignment. The various intellectual resources paired with the many dimensions of these intellectual resources point towards the complexity of teaching and teacher learning. In the next chapter, I offer an analysis of how the various teacher learning opportunities supported teachers in developing mutual understanding across these various intellectual resources.

CHAPTER 6: AN ANALYSIS OF THE MUTUAL UNDERSTANDING AROUND TEACHING AND LEARNING FRACTIONS

In the previous chapter, I described the kinds of mutual understanding in relation to fractions teaching and learning that have developed among teachers at Hilltop Elementary. This chapter builds on that analysis and examines *how* the school-based professional learning opportunities supported this development of mutual understanding among teachers and instructional leaders. I sought to understand how three specific professional learning structures (Math Labs, PLCs, and one-on-one coaching) provided opportunities for mutual engagement and supported the development of mutual understanding such that there was instructional alignment within classrooms at the same grade level and across classrooms at different grade levels. I offer four findings from looking across the three professional learning contexts:

1. All three learning contexts provided opportunities for teachers and instructional leaders to develop mutual understanding around Content Knowledge for Teaching Mathematics (CKTM), instructional visions, and student learning trajectories.
2. The interactions in which teachers generated and used various kinds of knowledge varied across the different settings but worked together to support the development of mutual understanding. Math Labs, which were less frequent, offered extended (day-long) opportunities for teachers to engage with one another around the big mathematical ideas involved in a particular unit of instruction as well as instructional vision and the development of student understanding. PLCs, which were shorter but more frequent, offered opportunities for teachers to build on the understandings developed in Math Labs, by focusing on day-to-day classroom instruction and records of student thinking. One-on-one coaching, which was also shorter but more frequent, typically focused more on the needs of each individual teacher.
3. Mutual understanding both allowed the coach to develop unit plans that had meaning to the teachers and supported them to teach fractions in ways that were aligned within classrooms at the same grade level and across classrooms at different grade levels.
4. Teachers were positioned as both “listeners” and “sources” and because of these different positionings, participants have different perspectives that shape the interpretations and functions of their contributions.

I begin by providing a brief overview of an “equal sharing” approach to fractions instruction and a description of the unit plans that Tara (the math coach) initially created based on an “equal

sharing approach” (Empson & Levi, 2011). I follow that with examples from each of the three professional learning contexts that characterize the kinds of knowledge that got worked on and the nature of the interactions that supported the development of mutual understanding among teachers and instructional leaders.

An “Equal Sharing” Approach

In the first year of the SIG, Tara created unit plans based on the big ideas in Empson & Levi’s (2011) book, “Extending Children’s Mathematics, Fractions and Decimals: Innovations in Cognitively Guided Instruction (CGI).” This approach to fractions instruction builds on the broader ideas of CGI, including:

- Posing problems for children to solve using their own strategies,
- Choosing or writing problems that elicit a variety of valid strategies and insights,
- Adjusting problem difficulty so that children can use what they understand to solve problems,
- Sequencing problems and numbers choices in developmentally appropriate ways,
- Asking probing questions to clarify and extend children’s thinking,
- Conducting discussions of students’ strategies so that students can make new mathematical connections, and
- Identifying the important mathematics in children’s thinking. (p. 227)

These ideas were introduced at the beginning of the first year of the SIG in relation to understanding and operating with whole numbers and Tara felt it was important that Hilltop’s fractions instruction build on the same ideas. Empson & Levi’s approach does this and connects these ideas specifically to fractions content. Their approach includes:

- Developing initial fraction concepts by connecting them to children’s intuitive understanding of sharing; allowing children to construct their own representations of sharing situations.
- Introducing fraction language and then notation as a way of representing partitioning situations that students are already familiar with.
- Using understanding of relationships between fractions to build their understanding of fraction equivalence and order.
- Using the fundamental properties of whole-number operations as the basis for learning to add, subtract, multiply, and divide fractions.

Hilltop's Unit Plans

Unit plans were important tools for teachers at Hilltop Elementary. All nine teachers referenced their use of shared unit plans during interviews. The unit plans were organized into two sections: the Overview and the Daily Lessons. The Overview included a description of what students should know and understand by the end of the unit, big mathematical ideas, the related Common Core State Standards (both content and practice standards), vocabulary, materials needed, and common assessments. See Figure 24 and Figure 25 for examples from the Year 4, Grade 4 overview⁴. Two of the fractions unit plans also included one of the trajectories for students' strategies from the Empson & Levi book. The Grade 3 unit plan had a trajectory for students' partitioning strategies and the Grade 4 unit plan had a trajectory for students' strategies to solve multiple groups problems. The Daily Lessons were organized into groups of 3-6 lessons that were connected to a big idea (or set of big ideas). For each group of lessons, the unit plan included the anticipated date of that lesson, the objective(s) for that group of lessons, an Instructional Activity for most days, the main lesson for the day, and suggestions for the classroom's workshop time. See Figure 26 for an example of a group of lessons from the Year 4, Grade 4 unit plan for fractions.

The unit plans did not include detailed lesson plans like a teacher's guide typically does, which raised the question of how the school-based professional learning opportunities supported the development of mutual understanding such that the coach could develop unit plans that both had meaning for teachers and supported them to teach fractions in ways that were aligned within classrooms at the same grade level and across classrooms at different grade levels. In the sections that follow, I will illustrate how Math Labs, PLCs, and one-on-one coaching supported teachers

⁴ The overview was 3 pages. Page 1 is shown as Figure 24, page 2 was a continuation of page 1 and included additional Common Core standards, and page 3 is shown as Figure 25.

<p>By the end of the unit students should be able to</p> <ul style="list-style-type: none"> ✓ Generate equivalent fractions and explain using visual models. ✓ Simplifying fractions and convert a mixed number to a fraction greater than one (WA PE only) ✓ Compare fractions and decimals. ✓ Add and subtract fractions with like denominators. ✓ Solve word problems involving addition and subtraction of fractions. ✓ Multiply a fraction by a whole number.
<p>Big Ideas About Fractions</p> <ul style="list-style-type: none"> • The numerator is the counting number. It refers to <i>how many</i> parts we have or our talking about. Key question: “How many fourth-sized pieces are there?” • The denominator tells what is being counted. It refers to the size of the piece. Key questions: “How many these equal sized pieces fit into the whole?” “What’s the size of the piece?” <p>If students struggle with naming fractions:</p> <ul style="list-style-type: none"> • Use words, rather than symbolic language: “Three fourth-sized pieces.” • Support naming the denominator: Since four of these equal sized pieces fit into the whole, we call the size of these pieces fourths. <p>Comparing Fractions</p> <ul style="list-style-type: none"> • Same size pieces (same denominators): If the size of the piece is the same, then we think about how many pieces we have. • Same number of pieces (same numerator): If we have the same number of pieces, we need to think about the size of the pieces. $3/4 > 3/8$ because fourths are bigger pieces than eighths. • More or less than a half or a whole • Closeness to a half or a whole
<p>Common Core Standards (The standards that are * will be assessed on MBA 2—March 1-5)</p> <p>4.NF.1—Explain why a fraction a/b is equivalent to a fraction $(n \times a)/(n \times b)$ by using visual fraction models, with attention to how the number and size of the parts differ even though the two fractions themselves are the same size. Use this principle to recognize and generate equivalent fractions.</p> <p>*4.NF.2—Compare two fractions with different numerators and different denominators, e.g. by creating common denominators or numerators, or by comparing to a benchmark fraction such as $1/2$. Recognize that comparisons are valid only when the two fractions refer to the same whole. Record the results of comparisons with symbols $>$, $=$, or $<$, and justify the conclusions, e.g., by using a visual fraction model.</p> <p>Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.</p> <p>4.NF.3 Understand a fraction a/b with $a > 1$ as a sum of fractions $1/b$.</p> <p>a. Understand addition and subtraction of fractions as joining and separating parts referring to the same whole.</p>

Figure 24. Unit plan overview, part 1.

<p>Vocabulary:</p> <p>fraction, partition, equal shares, compare, numerator, denominator, whole, number line, region/area model, decimal, tenths, hundredths, order, greater than ($>$), less than ($<$), equal to ($=$), equivalent, mixed number, simplify, fraction greater than one (improper fraction)</p>	<p>Materials:</p> <ul style="list-style-type: none"> • Investigations Unit 6 • Additional Lessons
<p>Common Assessments:</p> <p>Investigation Unit 6 Session 2.6 Grade 4 Unit 6 Fractions Part 1 End of Unit assessment</p>	<p>10 Minute Math:[*]</p> <p>About How Much? (see description), Close Fractions (see description), Correct Share (see description), Comparing Fraction Sets (see description), Missing Equivalencies (see description)</p>

Figure 25. Unit plan overview, part 2.

Big Idea: Understand equivalence and use equivalence to order and compare fractions

Date	Objectives/Focus Questions	HT Instructional Activities	Lessons	Workshop
Jan. 23	<ul style="list-style-type: none"> • Find equivalent fractions to a given fraction using visual models or a number strategy. • Use equivalent fractions to compare fractions with different numerators and different denominators. 	Launch first problem	HT Equivalence Day 1: Sharing Problem	<ul style="list-style-type: none"> • Sharing problems • Factors and multiples
27		<p>HT: Open Number Sentences</p> $\frac{1}{2} = \frac{\square}{4} = \frac{\square}{8} = \frac{\square}{16}$ <p>Represent with an area model. What patterns do you notice?</p>	HT Equivalence Day 2: More Sharing Problems Using an area model to find equivalent fractions	<ul style="list-style-type: none"> • Sharing problems • Factors and multiples
28		<p>HT: Open Number Sentences</p> $\frac{1}{4} = \frac{\square}{8} = \frac{\square}{16} = \frac{\square}{32}$ <p>Represent with an area model. What patterns do you notice?</p>	HT Equivalence Day 3: Finding Equivalent Fractions Students use an area model to find equivalent fractions.	<ul style="list-style-type: none"> • Multiplicative comparison problems
29		See lesson	HT Equivalence Day 4: Developing the Algorithm for Equivalent Fractions This lesson has been adapted from the District Lesson—Slicing Squares	<ul style="list-style-type: none"> • Additional equivalence problems • Multiplicative comparison problems

Figure 26. Example of a group of lessons in a unit plan.

in developing mutual understanding around CKTM, instructional visions, and understanding of trajectories of student learning.

A Closer Look at How Professional Learning Opportunities at Hilltop Elementary Supported the Development of Mutual Understanding

In the sections that follow, I analyze a Math Lab, a PLC meeting, and two one-on-one coaching interactions that occurred during the Grade 4 fractions unit in Year 4 (the first year after the SIG ended). Examining these interactions during year 4, I contend, provides insight into the structures that were developed at the school. That year, there were three Grade 4 teachers: Becky, Anna, and Tonya. Becky was in her 4th year of teaching fractions at the same grade level at Hilltop. Anna had taught at Grade 1 at Hilltop since before the SIG, but moved to Grade 4 in Year 4, so it was her first year teaching a fractions unit (though she did have experience using sharing problems in Grade 1). Tonya was new to Hilltop in Year 4, but had taught a Grade 5 fractions unit the previous year at a different school. There were three additional people at both the Math Lab and the PLC meeting analyzed here: Tara (the math coach), Julie (the principal), and Kathleen (the English Language teacher).

Math Labs

Math Labs were organized around a “learning cycle” with four phases: learning together, co-planning a lesson, trying it out, and debriefing together. In the sections that follow, I provide an illustration of the nature of the interactions that support the development of mutual understanding among teachers and instructional leaders and an analysis of the kinds of knowledge that get worked on together.

Phase 1: Learning together. The Math Lab included in this analysis was held the morning of December 10th, 2014. Teachers were almost finished with a lengthy unit on

multiplying and dividing multi-digit whole numbers and were preparing to start their fractions unit upon returning from winter break three and a half weeks later. The teachers gathered in the “Teacher’s Classroom⁵” a few minutes before 8:30am and the lab started as soon as everyone was in the room. The Math Lab started with approximately 15 minutes spent examining two Common Core content standards (one from Grade 3 and one from Grade 4; both related to comparing fractions) and three Common Core practice standards (Practices 3, 4, and 6) (see Figure 27). They discussed the differences between the Grade 3 and 4 expectations for students and how the Grade 4 standard built on the Grade 3 standard. During this conversation, Tara described what students had worked on in Grade 3 the previous school year and Julie emphasized the importance of supporting students to develop “conceptual understanding of comparing fractions.” She was explicit stated, “We don’t want kids to just develop a procedure” for comparing fractions.

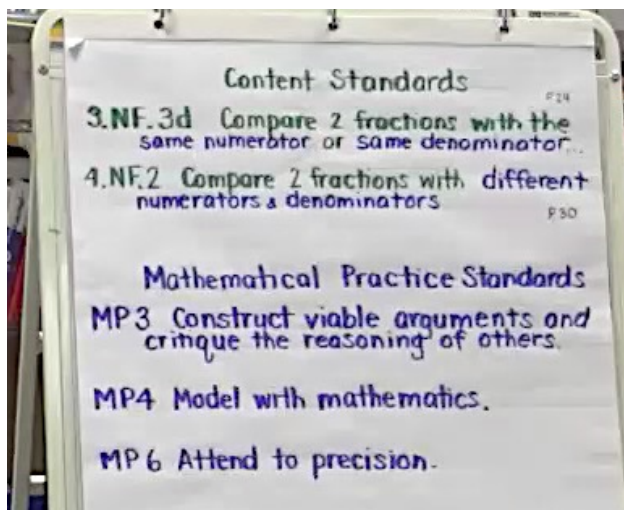


Figure 27. Common Core Standards for Grade 4 Math Lab.

⁵ The “Teacher’s Classroom” was a classroom that was set-up and designated for teacher learning. All three instructional coaches had desks in this room and there were adult-size tables and chairs. The space was reserved specifically for teacher learning (including Math Labs and other professional development sessions).

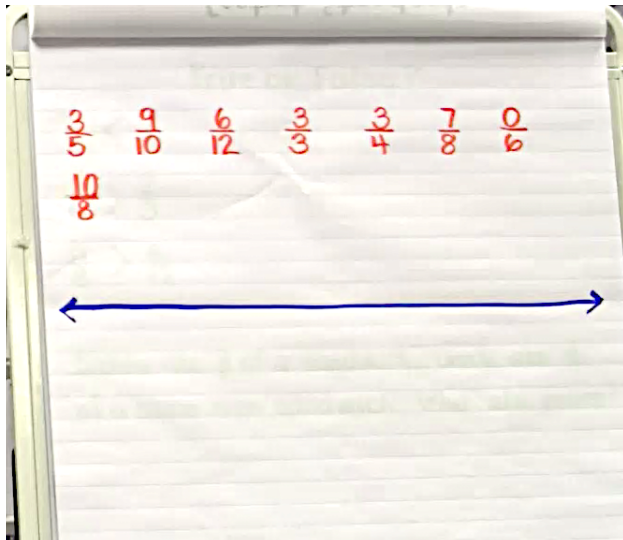


Figure 28. Fraction ordering task.

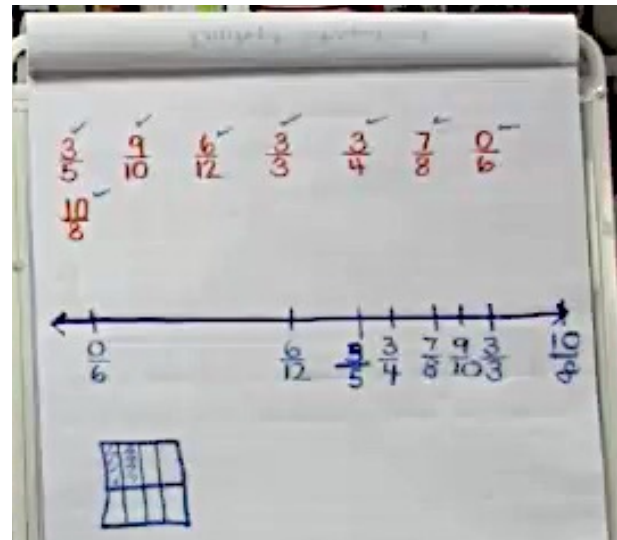


Figure 29. Tara's recording of teachers' collective solution.

Tara also engaged the teachers in doing mathematics themselves. She hung a poster with a list of fractions and an empty number line (see Figure 28) and asked the teachers to place the fractions on the number line. After teachers worked independently for a few minutes, Tara facilitated a whole group discussion, pressing teachers to justify why they placed each fraction where they did, and recording their solution (see Figure 29). Through this discussion, a number of different comparing strategies emerged and teachers were surprised by both the variety and particular strategies they “never would have thought of,” including using a common numerator and converting the fractions to decimals or percents.

The interactions in this phase of the Math Lab supported teachers in developing mutual understanding related to teachers’ CKTM and instructional visions. By analyzing Common Core standards and comparing the standards across two grade levels, teachers were developing mutual understanding around what their fourth-graders likely worked on the previous year as third-graders (especially since Tara is able to offer additional insight into that) and what their fourth-graders needed to work towards by the end of Grade 4. Teachers’ interactions (including sharing their own strategies and listening to their colleagues’ strategies) during the ordering fractions

task provided additional opportunities for the development of mutual understanding of both the meanings of the standards as well as their understanding of various strategies for comparing and ordering fractions. Tara’s selection of the ordering task and facilitation of the whole group discussion that followed both supported teachers in developing mutual understanding around instructional visions, specifically related to the role of tasks and the role of discourse. In particular, she provided an image of a task that had multiple entry points and offered opportunities for the Math Lab participants to use a variety of strategies as well as an image of discourse that included press for mathematical justification.

Phase 2: Co-planning a lesson. In the second phase of the Math Lab, the group co-planned the lesson for their upcoming classroom visit. This started with Tara introducing the goals for teachers’ learning that day (see Figure 30). It is important that these goals are made explicit because it makes clear what the intended outcomes are for teachers’ knowledge and instruction.

<p><u>Goals for Math Lab on 12/10/14:</u></p> <ul style="list-style-type: none">• Plan and implement an instructional activity focused on comparing fractions.• Examine how students make sense of fractions and the strategies they use to compare fractions.• Analyze the CGI data—Use it to inform instruction.• Identify the big ideas of the fractions unit.

Figure 30. Goals for teacher learning.

Tara also introduced the purpose of the classroom visit that day, which was “to explore student understanding of fractions and their reasoning as they compare fractions.” In preparation for the planning, she had made a poster with two possible tasks (see Figure 31) as a launching point for their planning discussion.

<p>Option 1: True or False?</p> $\frac{3}{8} < \frac{7}{8}$ $\frac{2}{6} > \frac{2}{3}$ $\frac{5}{8} > \frac{6}{12}$	<p>Option 2: Comparison Word Problems</p> <p>Sahra ate $\frac{3}{8}$ of a sandwich. Jack ate $\frac{7}{8}$ of a sandwich. Who ate more sandwich?</p>
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Figure 31. Tara's task suggestions for the classroom visit.

Option 1 was a series of True-False Statements and Option 2 was a comparison word problem. The teachers spent some time discussing their preferences, relating their choices back to the goal that Tara articulated for the classroom visit. The three classroom teachers preferred the True-False Statements. Tonya suggested she thought the True-False Statement was more “rigorous” and thought the teachers would learn more from it. Anna wondered if the words from the story problem “would get in the way” of understanding what kids knew. Kathleen shared that she thought the context in the word problems might support kids to make sense of the fractions because “words support the visualization and understanding.” After a few minutes of discussion, the group agreed to try the True-False Statements and Tara passed out a planning template so each teacher could record their shared plan as it developed.

During the co-planning time, the group spent time thinking through each of the three statements, planning and discussing multiple aspects of the lesson, including:

- **What to do if the first problem was too hard:** Initially teachers thought kids would find the first statement to be easy, but Tara pressed them, “What if it’s not easy to solve?” Teachers decided they would use a context or use an area model to represent the fractions if needed.
- **What questions to ask students to uncover their thinking:** They agreed to use questions like, “How do you know that?”, “What did you see in your head when you thought about $\frac{2}{3}$?”, and “What do we know about $\frac{6}{12}$?”
- **How to record students’ ideas:** The group pretty quickly decided to use an area model, which appeared to be a model with which they were all familiar since it was not discussed at length.

- **What they wanted kids to notice and understand:** In the first True-False statement, they wanted students to notice and understand that the pieces were the same size, but there were a different number of pieces; in the second statement, they wanted students to notice and understand that there were the same number of pieces, but they were different sizes; and in the third statement, they wanted students to notice the relationships between $\frac{1}{2}$ and each of the two fractions in the statement.
- **What they wanted to press kids on:** They wanted students to use the explicit fraction language that had been introduced in earlier grades in order to reason about the size of the pieces and the number of the pieces as part of their explanations.

Throughout the planning conversation, Tara made a public record of some of the important decisions that were made about language and representation (see Figure 32).

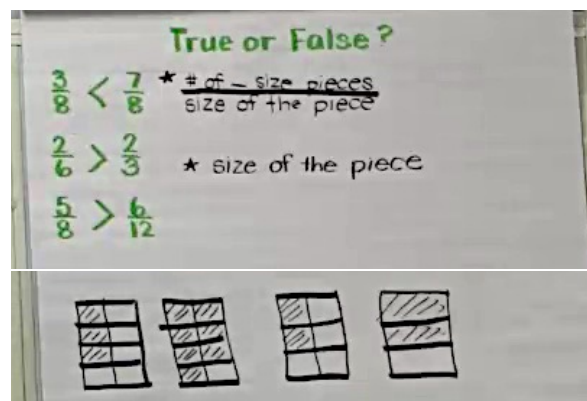


Figure 32. Public record created by Tara during co-planning conversation.

During the co-planning portion of the Math Lab, teachers had opportunities to develop mutual understanding around CKTM, instructional visions, and student learning trajectories through a number of different interactions. Tara’s goals for teacher learning (in Figure 30) suggest that she was attending to all three kinds of knowledge. The first goal (planning and implementing an instructional activity) was related to all three as can be seen in their planning conversations. Their discussion around which task to use supported the development of mutual understanding around instructional vision, particularly around the role of tasks, including what makes a task rigorous and the role of context within tasks. Their subsequent planning conversation supported the development of mutual understanding around what representation to use and in what ways, which is related to both CKTM and instructional vision. The group also

spent time anticipating student thinking, drawing on and developing mutual understanding of student learning trajectories.

The second goal for teacher learning, which was related to the purpose of the classroom visit that Tara articulated (“to explore student understanding of fractions and their reasoning as they compare fractions”) supported teachers to develop mutual understanding around how students develop understanding of fraction ideas. The third and fourth goals were worked on after the debriefing time when they analyzed student work from the recent CGI assessment and continued the next day in their PLC meeting when they discussed the big ideas in their unit plan for fractions. Though illustrations of these last two activities were not included as part of this analysis, the examination of student work likely offers opportunities for teachers to develop mutual understanding around trajectories of student understanding and the identification of big ideas in the fractions unit plan supports teachers to develop mutual understanding around CKTM and instructional vision.

Phase 3: Trying it out. Right before heading into Anna’s classroom, three of the teachers agreed to “pass the pen,” a routine that involves sharing facilitation of the lesson. Kathleen volunteered to facilitate the discussion of the first True-False Statement; Becky agreed to facilitate the second; and Anna said she would try the third. Tara also reminded the group that they could “jump in” to the lesson and call a “Teacher Time Out” (TTO). Since the classroom visits were meant to be a learning opportunity for teachers and were experimental in nature, a routine called “Teacher Time Out⁶” developed as a way of pausing instruction so that the adults in the room could think aloud with one another about their instructional decisions in the moment. The classroom visit lasted approximately 36 minutes and included 11 TTOs, three of which

⁶ See Gibbons, Hintz, Kazemi, & Hartmann (2015) for an in-depth analysis of the Teacher Time Out routine.

involved revisions of representations or language. I zoom in on these interactions in order to characterize how teaching together with students present offers a unique setting for teachers to engage in developing mutual understanding.

TTO example #1: Labeling pieces in an area model. Becky took the lead on facilitating students' discussion about the second True-False Statement ($\frac{2}{6} > \frac{2}{3}$). Students quickly came to the conclusion that the statement was false. When pressed for their reasoning, students suggested drawing area models for each fraction and described what those would look like. Becky had just finished creating these representations (see Figure 33) when Julie (the principal) threw a question out to the students: "What do you call the size of those pieces?"

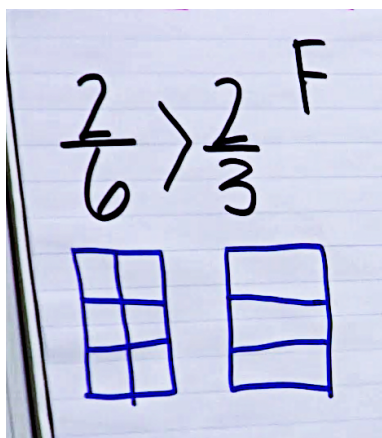


Figure 33. Area models created by Becky.

Becky pointed to the model for sixths and students replied, "Sixths." Julie followed up by asking, "Could we label those pieces? How much of the whole is [Becky] pointing to?" Students replied chorally, "One-sixth" and Becky wrote "1/6" in the top left piece. Then Julie asked, "Could we label the rest?" and Becky said, "I think so. That makes sense." and she asked the students, "What is this one called?" as she pointed to the top right piece. The students gave a choral response, "Two-sixths" and Becky wrote "2/6" in that piece (see Figure 34).

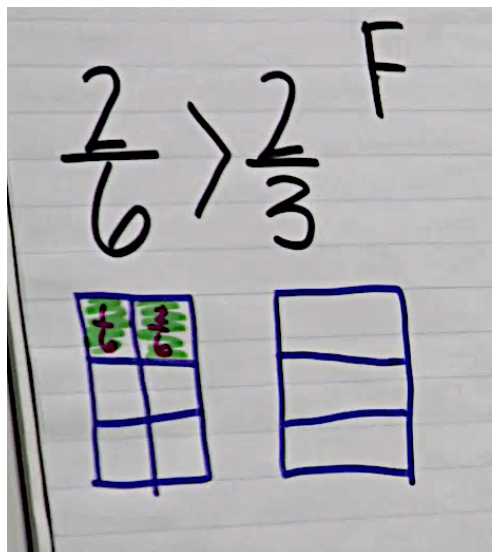


Figure 34. Becky's initial labeling of sixths.

Julie said, “Wait, wait, wait. That’s also—” and Tonya jumped in, “—one-sixth.” Becky paused and looked at what she had written and said, “Oooh, it's also a sixth. You're absolutely right. Becky revised the "2/6" to "1/6" (see Figure 35) and said, “This is one-sixth (points to first piece). Thank you for helping me remember.”

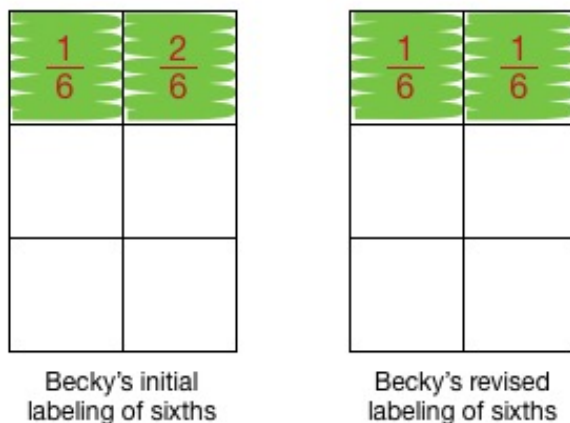


Figure 35. Becky's initial and revised labeling of sixths.

In this TTO, we initially see misalignment between Becky’s understanding of labeling pieces of an area model and Julie’s (and Tonya’s) understanding of labeling pieces of an area model. Becky’s labeling represented an “accumulation” of the pieces and following the students’ lead, she recorded the fractions that would be said if one were to count the pieces (one-sixth,

two-sixths, etc.) whereas Julie’s and Tonya’s responses indicated that their understanding of how to label an area model involved labeling the size of each piece. In this moment, it appears that this is not necessarily a new idea to Becky as she quickly aligns her understanding with Julie and Tonya and replies, “Thank you for helping me remember.” This episode illustrates how teachers had opportunities to work together on their CKTM, specifically related to recognizing what is involved in using a particular representation.

TTO example #2: Seeing relationships within representations. Later in the lesson, Anna had created a representation for the third True-False statement, ($\frac{5}{8} > \frac{6}{12}$) (see Figure 36).

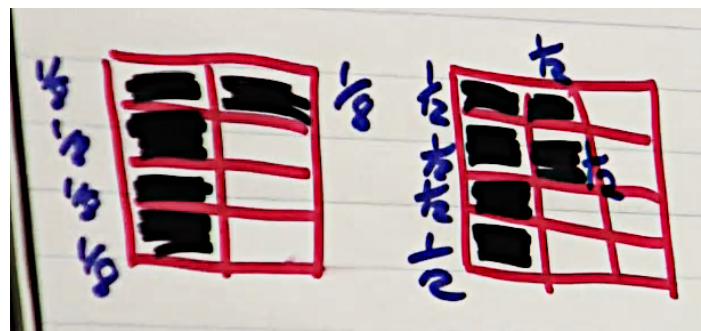


Figure 36. Anna's area models for $\frac{5}{8}$ and $\frac{6}{12}$.

She asked the students to focus on just the $\frac{6}{12}$ and wondered if there was anything they knew about that fraction. One student said he knew that $\frac{6}{12}$ was equal to $\frac{1}{2}$. Anna also asked students if there was anything they knew about $\frac{5}{8}$ and asked them to turn and talk with a neighbor.

During this turn and talk, Julie made a suggestion to Anna:

[Anna], could you represent them both top and bottom? I think it will be a little bit easier to see. As you draw them, either both vertical or both horizontal, so they see the relationship to a half. Do you see what I'm saying?

Anna redrew the twelfths using the same partitioning, but shaded it differently (see Figure 37).

She asked, “So would it help if I did this?”

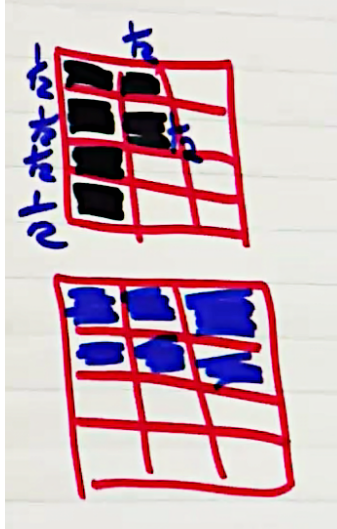


Figure 37. Anna's revision for 6/12.

Julie replied, “Yeah, and then draw the other one horizontally as well.” Anna redrew the eighths, using the same partitioning, but also shaded it differently (see Figure 38). Julie prompted, “Now shade the top half” and Anna said, “Ohh, I see. Like that. Does that help?” Anna’s final representation is shown in Figure 39.

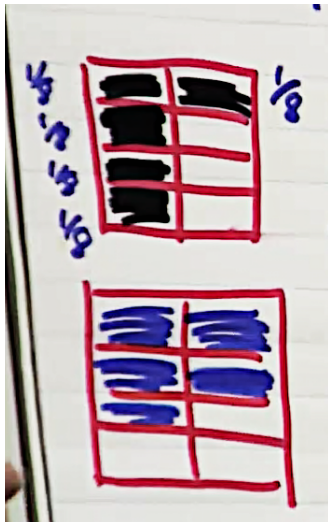


Figure 38. Anna's revision for 5/8.

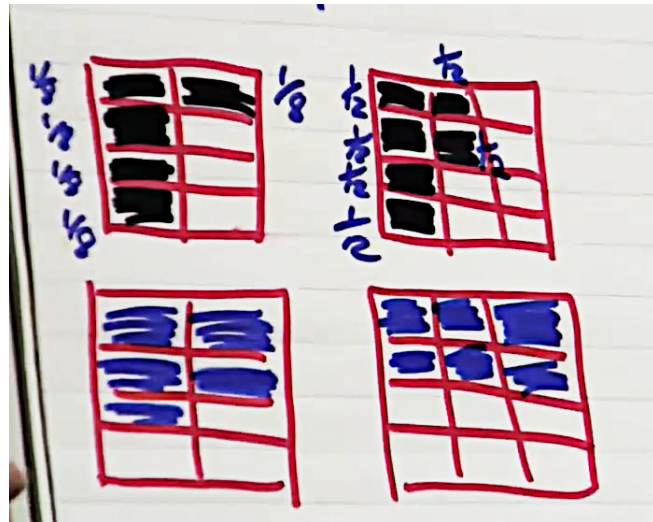


Figure 39. Anna's final revision to both 5/8 and 6/12.

In this TTO, we see Julie and Anna work together to revise Anna’s initial representation in order to make the fractions’ relationships to one-half more explicit. Interestingly, this was the only True-False statement for which the group had not discussed the representation during the

planning time (see Figure 32). So, it is unclear whether Julie had come into the classroom visit with a different mental image from what Anna initially created or if she revised her mental image in the moment after seeing Anna's representation. Either way, the enactment of the lesson with students provided an opportunity for this interaction to occur and the development of mutual understanding around linking representations to underlying ideas, which is related to both CKTM and instructional vision (particularly using mathematical tools as instructional supports).

TTO example #3: Precision of language. After bringing students back together from the turn and talk in example #2 and explaining how and why she revised her representation, Anna brought students back to thinking about the comparison. She asked students, "What one is bigger?" and after a few seconds, she followed that up with, "What one do I have more pieces of?" Right away, a number of kids gasped as if they had an "aha". However, when she invited a choral response, the responses were split between $\frac{5}{8}$ and $\frac{6}{12}$. Julie signaled Anna and she called a TTO. Julie said, "More pieces versus more. The answer shifts." Anna looked at her with confusion and said, "Yeah, because they are looking at the number of pieces." Julie responded, "But you said more pieces one of the times you asked." Anna replied, "Oh! That's why so many kids went 'Oh!'" and she jumped back into the lesson, revising and clarifying her question for the students. Julie and Anna's discussion in this TTO was focused on the two questions that Anna posed to students ("What one is bigger?" and "What one do I have more pieces of?"). When she asked her question the second time, she changed the wording and unintentionally changed the meaning. Her first question ("What one is bigger?") asked students to compare the shaded portions of $\frac{5}{8}$ and $\frac{6}{12}$ and results in an answer of $\frac{5}{8}$. However, her second question ("What one do I have more pieces of?") asked students to compare the number of shaded pieces and results in an answer of $\frac{6}{12}$.

In this TTO, we see an opportunity for the group to develop mutual understanding around precision of language. By seeing Anna's two questions side-by-side, the group was able to see how these two questions resulted in different answers from students despite sounding very similar. Though teachers had planned their language carefully and discussed it at length, the opportunity to see students' responses to teachers' use of language likely supported the development of their mutual understanding around this particular aspect of CKTM.

The three TTO illustrations from the classroom visit provide images of a unique way of teachers and instructional leaders interacting with one another. First, it is rare for teachers to engage in teaching with colleagues present. Second, it is rare for teachers and principals to be in classrooms together outside of evaluative contexts. However, these moments point towards opportunities to mutually engage around CKTM, instructional visions, and understanding of student learning trajectories that may not have occurred otherwise.

Phase 4: Debriefing together. After finishing the lesson and taking a short break, the group reconvened in the Teacher's Classroom and Tara asked the group for their initial thoughts. Their conversation immediately turned to what the students already seemed to know and understand about fractions, including students' ideas about what the numerator and denominator mean, the language they were using to explain those terms, their understanding that the whole matters, and their ability to visualize a variety of fractions. Towards the end of the debrief, the conversation shifted to talk about instruction and places where they will want to "linger a bit more." Anna suggested that she wanted to spend more time "thinking about the leftovers" because she noticed that when they looked at the number of missing pieces, they seemed to forget to consider the size of the piece. Tara suggested that using a context may support kids in attending to both the number of "leftover" pieces as well as the size of them. In this part of the

debrief, there were opportunities to develop mutual understanding around how students' understanding of fractions is developing (e.g., it is hard to consider both the number of pieces and the size of the piece together) and around instructional visions (e.g., using context as a learning support).

Much of the debrief discussion focused on making sense of what students understood and said during the classroom visit. This included revisiting ideas that emerged in the whole group discussion and replaying in detail what particular students shared during turn and talks. For example, Tonya replayed a turn and talk interaction she had with one student who had been sitting next to her:

He said, they're both. It's not true or false. It's both. He articulated that an eighth is going to be a bigger piece than a twelfth, but he wasn't willing to choose which one was bigger because five-eighths and six-twelfths is only one [piece] away [from each other]. It was too close for him to easily picture. Whereas two-sixths and two-thirds he was like, oh I can picture two-thirds. It's really big, it's most of the thing. And two-sixths is only a tiny bit. But he couldn't decide [true or false for five-eighths and six-twelfths] because he didn't have the strategy to compare it to a half. And six-twelfths wasn't a half for him immediately.

Almost everyone in the group replayed an interaction with one or more students. Through this discussion, teachers were supported in developing mutual understanding about students' understanding of what fractions mean, use of language, and use of strategies to compare fractions.

Towards the end of the debrief, the group revisited the role of contextualizing. Tara asked if any students contextualized the problem on their own. She shared how two students next to her created a story about a cake to make sense of the two fractions. None of the rest of the group observed students contextualizing on their own, but both Julie and Anna reported that they introduced a context as a way of supporting students during turn and talks and it seemed to help

in both cases. This particular part of the debrief supported teachers in developing mutual understanding around instructional visions (specifically, using context as a learning support).

The four phases of the Math Lab offered opportunities for teachers to interact in different ways with one another around various kinds of knowledge. However, much of the work in Math Labs focused on big ideas of the unit plan and fractions content as well as developing an initial understanding of student thinking (drawing both on research and the shared experience with students during the classroom visit). The next section examines the weekly PLC meetings, which built on the Math Lab experiences and provided additional, different opportunities for teachers to develop mutual understanding.

Weekly PLC Meetings

On January 22, 2015, a few weeks after the Math Lab and approximately three weeks into their fractions unit, Tara met with the Grade 4 teachers and Kathleen (the English Language teacher) in their weekly PLC meeting. Tara started this meeting by asking, “What are you noticing? What do your students know about fractions?” The teachers shared a number of things their students had been successful with including using number lines, naming fractions, talking about the numerator and denominator with meaning, and converting fractions greater than 1 (such as $\frac{10}{8}$) into mixed numbers (such as $1\frac{2}{8}$) by “reasoning about the relationship to division.” Tara’s opening question offered an opportunity for teachers to develop mutual understanding around how students’ fraction understanding was developing across the grade level in relation to their first two weeks of instruction.

Partway through the conversation, Becky asked Anna and Tonya, “Have you had anyone do [multiple groups problems] as a multiplication equation?” and she elaborated on how a few students wanted to use the algorithm for multiplying a whole number and a fraction. The other

two teachers had not observed this in their classrooms, but Tara followed this up by asking the group, “When the algorithm pops up, what do you want to say? What language do you want to use?” After some discussion, the group decided they would connect the algorithm to a familiar representation, likely an area model. This part of the PLC meeting provided an opportunity for teachers to develop mutual understanding related to both CKTM and instructional vision. In particular, teachers agreed that they were going to link the multiplication algorithm to a visual representation that was familiar to students as a way of supporting them to make meaning of that particular algorithm.

Towards the end of their meeting, the group shifted to discuss their upcoming lessons and Anna (who was new to Grade 4) asked a few questions about how to read multiplication equations with fractions in them. She asked the group, “What about saying 4 groups of one-sixth (for $4 \times \frac{1}{6}$)? After some discussion and trying out some different ways of saying it, they came to an agreement on the language they would use and then finished up the meeting talking through some scheduling logistics. Anna’s question here illustrates an opportunity for the group to develop mutual understanding around using common language for fractions, which was something that was also worked on during the Math Lab (as described in the third TTO example).

In their interviews, teachers described various ways in which Math Labs and PLCs were connected. Anna described Math Labs as “an extended PLC” where you get “more time to really dig into things with your team.” Tonya explained that Math Labs gave “the big picture” and PLCs gave teachers a chance to work on “the day-to-day stuff that comes up.” By talking about their personal experiences with the day-to-day teaching and learning in their own classrooms, they had opportunities to further develop mutual understanding beyond the Math Lab.

Tara also described how her role in PLCs was to intentionally support teachers in developing mutual understanding. Since Tara often visited multiple classrooms for the same lesson, she was able to ask questions about inconsistencies she noticed. She explained:

I'll bring that up in PLCs. I noticed your kids are using an area model. Your kids are using a number line. I wonder why that is. [The teachers] will talk about the advantages and disadvantages. It's very much about thinking about the content together across the grade level. Sharing knowledge. I view myself as facilitating a lot of their knowledge sharing. Some of it is pure content, because we've shifted to the Common Core...It's a way of checking on the content knowledge, filling in some pieces that they might not be aware of. Introducing a new IA, because if it doesn't come up [in] a lab, the only time to talk about it is during a PLC. Sometimes it's professional development, sometimes it's looking at data, sometimes it's sharing what teachers know across the grade level. Sometimes it's problem solving together and other times, it's gathering info about how things are playing out across classrooms.

Here, Tara is articulating that PLCs were spaces where teachers came together to develop mutual understanding around content (e.g., Common Core), instructional visions (e.g., learning new tasks like instructional activities), and student understanding (e.g., looking at data).

In my interviews with teachers and Tara, they described one additional way that PLCs supported the development of mutual understanding. Four of the nine teachers recalled how discussions in PLCs often involved discussions about the unit plans and frequently resulted in revisions to the unit plan. For example, Becky remembered a discussion in Year 2 or 3 about Multiple Groups problems seeming more intuitive for students than adding and subtracting fractions, so the Grade 4 teachers that year wondered if they should come first. Since the unit plans are “living documents” and revised before, during, and after the unit, the group modified the unit plan for the following year. Tonya provided a different example where the Year 5, Grade 5 team revised the unit plan mid-unit in order to be more intentional about connecting the algorithm at the end of the unit to their earlier work in the unit.

One-on-one Coaching

Visiting teachers' classrooms was a regular part of Tara's work. She used classroom visits as a way to support the development of mutual understanding around CKTM, instructional vision, and student learning trajectories. She explained in an interview, "In the classroom, it's tailored very much to what a particular teacher needs, as well as sometimes I'll be going through two to three classrooms on a given day and I'm watching for alignment."

During my interviews with teachers, many of them talked about the nature of those visits. Anna said, "I love when Tara comes into my classroom during math because I can just do a teacher time out anytime I'm not sure about something. There's so many times I say to my kids, 'I wish [Tara] was here right now.'" On occasion, Tara visited a single classroom for an entire math block, but more often she moved between same-grade classrooms on the same day because it allowed her to "watch for alignment" as well as pay attention to how kids were "making sense of the mathematics" both within a single classroom but also across the grade level. On January 12, 2015, Tara visited two of the Grade 4 classrooms. They were on their fifth day of fractions instruction and both classes were working on the same set of multiple groups problems when Tara went into the classrooms.

Tonya's classroom. Upon entering the classroom, Tara sat on the floor with Tonya's students. They were in the middle of a whole group discussion about a multiple groups problem: *Mrs. A ran $\frac{3}{4}$ of a mile every day after school. Her goal is to run 9 miles. How many days does she need to run to reach her goal?* One goal of the lesson that day was to use a number line to represent the problem. (See Figure 40 for an example of the public record created during the whole group discussion). Shortly after sitting down, she used a silent signal to suggest to Tonya that it was a good moment to do a turn and talk. After the turn and talk, Tonya asked the

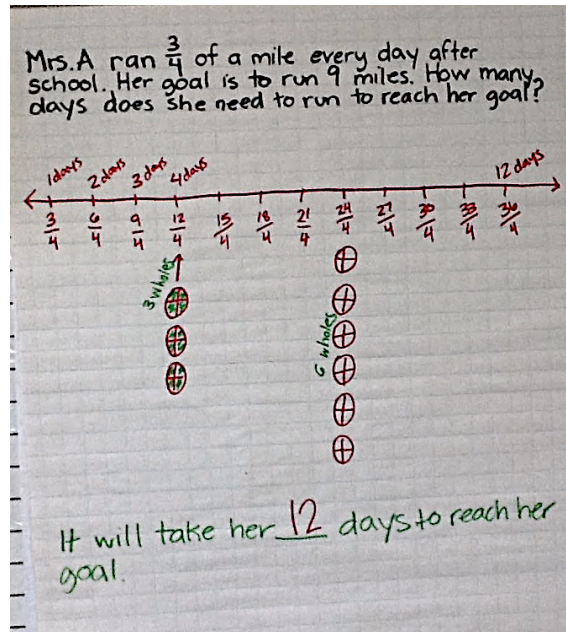


Figure 40. Public record from classroom visit.

students, “Will we be done?” and Tara added on, “What will the answer sound like?” After students suggested both “days” and “miles,” Tonya asked the students to turn and talk about whether the answer would be number of days or number of miles. Tara left as the number talk wrapped up and she headed next door to Anna’s classroom.

In this example of one-on-one coaching, Tara is supporting a teacher that is new to Hilltop in developing mutual understanding around instructional vision, particularly related to the role of classroom discourse. After visiting Tonya’s classroom, she explained that since Tonya was new to Hilltop, but not fractions content, Tara’s visits to Tonya’s classroom tended to involve supporting Tonya in learning “the Hilltop way.” She explained that this included ways of facilitating whole group discussions and using talk moves (e.g., the turn and talk she suggested during the classroom visit) and pressing on students to make sense of problems and explain their thinking (e.g., when Tara asks students, “What will the answer sound like?”).

Anna’s classroom. When Tara entered Anna’s classroom, the class had just finished the same number talk as Tonya’s class and students were working in pairs on a set of five different

multiple groups problems. Kathleen (the English Language teacher) was also in Anna's room working with students. Anna was working with a student and when Tara joined her, Anna explained what was happening, "They want to multiply the numbers together, but they don't understand how it works... So I told them to all go back and draw a picture. That [a picture] might help them." Tara pointed to a couple of other examples of student work on the table and said, "And some of it is differentiating between when is a number line helpful and when is an area model helpful? So even having that conversation with them about how do you decide what to use? We want the representations to support them rather than get in the way."

After their brief check-in, Tara and Anna each checked in with a few other students before Anna gathered the whole class on the rug for a whole group summary discussion about one of the multiple groups problem students worked on: "*Ty has 8 candy bars. He eats $\frac{1}{6}$ of a bar each day. How many days will it take him to eat all of the candy bars?*" Tara sat on the rug with students at the back of the group while Anna facilitated the conversation. The first strategy that emerged involved drawing eight candy bars (using an area model) and partitioning each bar into sixths. As a group, they counted the number of sixths in all eight candy bars by ones and ended up with 48 days (see Figure 41). At this point, Tara initiated a Teacher Time Out and posed a question to the students, "Did anyone do a representation like that but they counted [the number of days] differently?" A number of students indicated that they had and they shared alternative ways of finding the total number of sixths. One student shared that she counted by sixes because each candy bar had six (sixths) and it was faster. A second Teacher Time Out was initiated by Anna right after this exchange, where she asked Tara if she should also represent it using a number line. They quickly decided against it and Anna wrapped up the discussion at that point.

In Anna’s classroom, we see Tara support Anna in ways that are different from Tonya. Since Anna has been at Hilltop since before the SIG, she “knows the Hilltop way.” However, she recently moved from first grade, so Tara’s coaching involved supporting the development of Anna’s CKTM. In particular, Tara supported Anna to recognize what is involved in using particular representations in relation to particular contexts. She also helped Anna think about how to support students in making their own decisions about using representations.

How Teachers Are Positioned Across the Various Learning Spaces

Using positional framing to examine the interactions across Math Labs, PLCs, and one-on-one coaching experiences, we see evidence of teachers being positioned as both “listeners” and “sources.” Positional framing refers to a pattern that includes a listener and a source and suggests that participants with these different positionings have different perspectives that shape the interpretations and functions of their contributions (and the listener and source roles can shift among participants within an interaction). For example, in the PLC described previously, we see Becky turn to her two Grade 4 colleagues and ask them, “Have you had anyone do [multiple groups problems] as a multiplication equation?” In this instance, Becky positions herself as the listener and her two grade level colleagues (both of whom have less experience teaching Grade 4 at Hilltop) as sources. Another example can be seen in the planning portion of the Math Lab when Tara presses the group to consider, “What if [the first one is] not easy to solve?” The teachers discussed a couple of options and then decided they would use a context or use an area

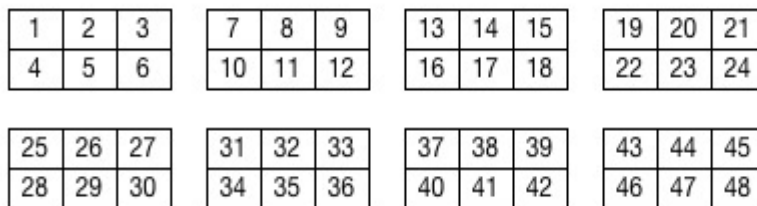


Figure 41. Model used in Anna's class for 8 candy bars, $\frac{1}{6}$ eaten each day.

model to represent the fractions if needed. In this interaction, Tara positions herself as a listener and the other group members as a source. Tara likely had some ideas for what to do, but instead of inserting them as a source, she chose to position the teachers as sources. This pattern which was typical at Hilltop is in contrast with interactions that are typical in professional development settings where teachers are exclusively (or primarily) positioned as listeners and “receive” information from the expert, or source.

Unit Plans as Tools from and for Mutual Understanding

The initial unit plans that Tara created drew on her understanding of the “equal sharing” approach from the Empson & Levi book. However, since the unit plans were living documents, they changed frequently in response to the mutual understanding that instructional leaders and teachers were developing around CKTM, instructional visions, and trajectories of student learning. In this way, the unit plans were a tool that was continuously developed and revised *from* mutual understanding.

However, the unit plans were also a tool used *for* developing mutual understanding. Erin (the resource room teacher) and Kathleen (the English Language teacher) both talked in their interviews about the unit plans being tools they relied on as they prepared for their “push-in” support of students. They were also able to use the plans from previous grade levels to help them think about intervention opportunities for their students since the unit plans were well-aligned across grade levels. Kyle, Leslie, and Anna described how appreciative they were when they changed grade levels. Because the unit plans were formatted in the same way and used similar Instructional Activities and language, they were able to pick up and use the unit plans with relative ease. They primarily described needing support around some of the new content ideas.

Keri and Layla both described how the unit plans kept teachers “on a similar schedule” so they could have relevant and timely discussions in their weekly PLC meetings.

In the chapter prior to this, I argued that over time, teachers developed mutual understanding in relation to Content Knowledge for Teaching Mathematics, visions of high quality math instruction, and understanding of student learning trajectories. In this chapter, I examined how the school-based professional learning opportunities (Math Labs, PLCs, and one-on-one coaching) provided opportunities for mutual engagement and supported the development of mutual understanding among teachers over time. I maintain that each of the structures offered unique opportunities for teachers to develop mutual understanding. For example, Math Labs and one-on-one coaching both provided opportunities for teachers to see things “come to life.” They have the opportunity to try out the things they talk about together and learn alongside students. Math Labs also tended to focus on the big mathematical ideas of a unit, whereas PLCs and one-on-one coaching tended to focus on the day-to-day lesson specifics. This analysis suggests that it was the coordination of these various professional learning opportunities that led to the development of mutual understanding that allowed the coach to develop unit plans that both had meaning to the teachers and supported them to teach fractions in ways that are aligned within classrooms at the same grade level and across classrooms at different grade levels. One limitation of this analysis is that I am unable to make claims about the alignment of teachers’ practice. Although the alignment of teachers’ instructional visions along with Tara’s intentional work supporting teachers to align practice across the school would suggest that this may also be the case, additional analysis is required to make claims about the alignment of teachers’ actual practice at Hilltop Elementary.

CHAPTER 7: DISCUSSION

In this final chapter, I begin by reviewing the major findings of this dissertation, as previously discussed in Chapters 4 through 6. I then make connections back to the literature reviewed in Chapter 2 in order to offer a broader discussion of the claims I've made in addressing the overarching research question, which was "How did a school-wide approach to fractions instruction shape teacher and student learning?" Then, I discuss limitations of this study and implications for practice and further research.

A Review of the Findings

In Chapter 4, I argued that over the three years of the SIG, students demonstrated more sophisticated understandings of fractions. By examining five aspects of students' knowledge and understanding related to an equal sharing problem, I found that students in three different cohorts developed more sophisticated strategies for partitioning and sharing, created more accurate representations of their partitions, and used more accurate fraction language and notation. In a second analysis of the student interviews, I found that at the start of each subsequent school year, the cohort of students entering a particular grade level brought with them more sophisticated strategies for partitioning and sharing as well as more accurate representations. Fluency with fraction language and notation also increased over time, but with less consistency than partitioning strategies and representations. In a third analysis of the student interviews, I found that the understanding and strategies for students with one year of instruction were noticeably less sophisticated than students with two or three years of instruction, whose understandings and strategies were much more aligned with one another.

In Chapter 5, I offered examples of four kinds of intellectual resources that teachers generated, drew upon, made meaning of, and coordinated as they engaged in teaching

mathematics and learning to teach mathematics: CKTM, instructional visions, understanding of trajectories of student learning, and teachers' personal identities. In particular, I provide four illustrations of what teachers *do* know and understand about fractions and CKTM as they engaged in various mathematical tasks of teaching. I also argued that teachers had developed relatively sophisticated instructional visions that were aligned with one another. With regards to understanding trajectories of student learning, I explained how teachers used their cognitive framing of trajectories for operating with whole numbers to organize their understanding of students' fraction ideas, including operating with fractions and using fraction language. Finally, I offered an analysis of the normative identity established by the instructional leaders at Hilltop Elementary and how teachers' personal identities were not initially in alignment with this normative identity, but shifted over time to come into alignment. The various intellectual resources paired with the many dimensions of these intellectual resources point towards the complexity of teaching and teacher learning.

In Chapter 6, I examined how the school-based professional learning opportunities (Math Labs, PLCs, and one-on-one coaching) supported the development of mutual understanding among teachers and instructional leaders such that there was instructional alignment within classrooms at the same grade level and across classrooms at different grade levels. I offered four findings from looking across the three professional learning contexts. First, I argued that all three learning contexts provided opportunities for teachers and instructional leaders to develop mutual understanding around Content Knowledge for Teaching Mathematics (CKTM), instructional visions, and student learning trajectories. Second, I argued that the interactions in which teachers generated and used various intellectual resources varied across the different settings but worked together to support the development of mutual understanding. Math Labs, which were less

frequent, offered extended (day-long) opportunities for teachers to engage with one another around the big mathematical ideas involved in a particular unit of instruction as well as instructional vision and the development of student understanding. PLCs, which were shorter but more frequent, offered opportunities for teachers to build on the understandings developed in Math Labs, by focusing on day-to-day classroom instruction and records of student thinking. One-on-one coaching, which was also shorter but more frequent, typically focused more on the needs of each individual teacher. Third, I argued that mutual understanding both allowed the coach to develop unit plans that had meaning to the teachers and supported them to teach fractions in ways that were aligned within classrooms at the same grade level and across classrooms at different grade levels. Finally, I argued that teachers were positioned as both “listeners” and “sources” and that because of these different positionings, participants have different perspectives that shape the interpretations and functions of their contributions (and the listener and source roles can shift among participants within an interaction).

Connecting this Case Back to the Literature: What are the Contributions?

This study drew on literature from multiple fields including learning theory, mathematics education, teacher learning, and school reform. In the following section, I connect the findings of this study back to these bodies of literature. First, I discuss how the use of a situative perspective to analyze the coordination of student and teacher learning within a school is a novel approach to understanding multiple learners-in-context (Nolen et al., 2015). Second, I describe how the findings related to students’ learning build on the existing literature related to students’ fraction knowledge and learning. Third, I discuss how this study contributes to the growing body of literature related to conceptualizing and illustrating high quality mathematics instruction. This is followed by a discussion of how this study uses new methods for analyzing what teachers know

and understanding about teaching fractions. Finally, I connect back to the school reform literature, specifically thinking about how this study contributes to discussions about reform at the school-level that takes place over multiple school years.

School-Wide Efforts to Improve Instruction: Using a Situative Approach to Analyzing the Coordination of Teacher and Student Learning

By using a single school as a case and purposeful sampling of a school where both teacher and student learning opportunities were intentionally considered and coordinated, this study is a contribution to the growing body of school reform literature. As a field, math education research has attended carefully to supporting teachers in improving instruction (e.g., Ball & Cohen, 1999; Borko, 2004; Cochran-Smith & Lytle, 1999; Kazemi, 2008). However, reform efforts have increasingly focused on larger-scale efforts (e.g., the MIST project) which has challenged the more traditional model of teachers working individually with minimal collaboration or coherence beyond single classrooms and pushed math education researchers to consider how theories and frameworks from organizational learning can inform frameworks for developing high quality mathematics instruction where schools are the unit of change. Organizational learning and school reform literature have identified a number of organizational conditions that are important to school development that supports collective teacher learning including: job-embedded professional development that focuses on student learning and is situated in practice, professional learning communities, and support from school-based leadership (e.g., principals and instructional coaches) (e.g., Hawley & Valli, 1999; Hiebert, 1999; Garet, Porter, Desimone, Birman, & Yoon, 2001).

This study builds on this body of literature by analyzing both teacher and student learning situated in a context where instructional leaders were attending to and coordinating both.

Typically, one kind of learning (either that of teachers or that of students) is the focus of a study while the other is relegated to the context or background. However, this study used a situative approach to reframe learning as something that is always and necessarily situated; that knowing and learning are constructed through participation in a community's discourse and practices, and are shaped by the contexts and activities in which they occur (Greeno et al., 1996). Using this approach to study teacher and student learning across five years contributes to our understanding of organizational learning (Boreham & Morgan, 2004). In particular, the case of Hilltop Elementary suggests that there were normative ways of engaging around teaching and learning that were sustained even when there was mobility among students and teachers. Among students, there was ~30% mobility within each school year. Among teachers, there was movement at each grade level between each school year. This suggests that the mutual understanding and coherence that developed can be attributed to organizational learning of the school as well as the experience teachers had over time.

This study also offers illustrations of what it looks like for an organization to learn as well as the structures and tools that support organizational learning. In particular, this analysis offers an illustration of what it looks like for a school to develop a shared professional discourse around mathematics instruction and specifically, fractions teaching and learning. With regards to structures that support organizational learning, we gain some insight from Tonya and Keri, who both started teaching at Hilltop in Year 4. In their interviews, they both described what it was like to enter a space where teachers had spent three years developing a professional discourse environment with a particular set of norms and ways of doing things. Additionally, the instructional leadership team has informally talked about getting better at recruiting teachers who will be a good fit with the existing professional discourse environment. As for tools, this study

illustrates how tools such as unit plans, which are common in many schools and districts, can both support the development of mutual understanding as well as be a storehouse of shared knowledge for a particular community.

By using a situative perspective, this study also contributes to our understanding of how individual cognition and semiotic structures of information are considered as they are understood, used, and generated by people in their joint activity with one another. While there has been much research that has used a cognitive approach to characterize mental representations of students' fraction knowledge and teachers' professional knowledge, we know very little about how these mental representations are generated, negotiated, and interpreted by students and teachers in the context of their activity systems. By drawing on a situative perspective, this study analyzed students' and teachers' ideas in the context of their use by examining how ideas were generated, negotiated, and taken up as participants interacted and participated in joint activity.

Specifically, there were three different spaces (Math Labs, PLC meetings, and classroom-based coaching) where teachers were continually supported to learn. These were also places where teachers were invited and allowed to be curious, ask their own questions, and collectively explore new ideas and practices with one another. Also important in this particular activity system was the unit plans that were created by the coach, used by the teachers, and refined collectively on an ongoing basis. The unit plans became storehouses for the mutual understanding that was developed among teachers—both within grade levels and across grade levels.

Student learning, while not necessarily occurring in the same exact spaces as teacher learning, was happening simultaneously and was certainly situated in the same activity system as the teachers' learning. Though I did not include extensive analysis of classroom-based

instruction, one could imagine how the changes seen in students' fraction understanding were related to learning opportunities in mathematics that were stronger and more coherent over time.

What Does It Mean for Students to Learn Mathematics and Understand Fractions?

This study built on the existing mathematics education literature focused on students' understanding of fractions in three ways. First, I make contributions towards our understanding of what it means for students to develop a conceptual understanding of what fractions mean. Though many researchers have suggested that partitioning activities are crucial for building rational number sense (Piaget, Inhelder, & Szeminska, 1960; Pothier & Sawada, 1983; Kieren, 1976; Streefland, 1991) and the existing research supports the notions that partitioning strategies develop over a predictable trajectory and that developing more efficient and sophisticated partitioning strategies is an important conceptual milestone (Empson & Levi, 2011; Lamon, 1996; Pothier & Sawada, 1983), we do not necessarily have a clear understanding of what else is involved as students develop a conceptual understanding of fractions, nor do we understand how these pieces might be coordinated by students. In this study, I analyzed students' responses to an equal sharing problem for multiple features in order to better understand what is involved in students' understandings of fractions. The findings from multiple cohorts of students in this study suggest that a conceptual understanding of fractions involves complex coordination between partitioning and the use of both fraction language and symbolic notation.

A second way in which this study builds on the existing literature is by examining how student understanding developed across multiple years. A majority of the work to date has included analysis of individual student thinking at a single point in time (e.g., Lamon, 1996; Pothier & Sawada, 1983) or over the course of a single unit of instruction (typically a four- to six-week period) (e.g., Ball, 1993; Mack, 1995). Finally, this study was situated within the

context of a school-wide implementation of a new approach to teaching fractions. While a number of studies exist that examine particular approaches to teaching fractions, none were school-wide nor were any across multiple years (e.g., Moss & Case, 1999; Siegler et al., 2007).

What is “High Quality Mathematics Instruction”?

This study also contributed to the growing body of research that provide images of what high quality mathematics instruction looks like by both building on existing work as well as contributing new ideas. First, this study examines high quality mathematics instruction situated in a single school. Traditionally, we have seen researchers zoom in to focus on images of high quality instruction situated either in single classrooms (e.g., Cohen & Lotan, 1995) or zoom out to examine high quality instruction at scale (multiple classrooms among many schools or even multiple school districts) (e.g., the MIST project⁷). The handful of studies that have focused on high-quality mathematics instruction in elementary schools have focused primarily on the teacher and student learning that occurs within a classroom’s activity system (e.g., Carpenter et al., 1997; Cobb, 2002; Simon & Schifter, 1991). By examining the efforts within a single school where teacher and student learning were intentionally considered and coordinated over multiple years, I was able to carefully document and analyze teachers’ visions of instruction.

Second, this study builds on Munter’s (2014) work around visions of high quality mathematics instruction (VHQMI). In creating the VHQMI rubrics, Munter drew on his extensive review of the literature to characterize the most sophisticated levels but rarely found evidence of these levels in his interview data (which is not entirely surprising since his participants were in the first three years of district-wide reform). However, using his rubrics to

⁷ MIST stands for Middle-school Mathematics and the Institutional Setting of Teaching and is a project based out of Vanderbilt University. This research project has numerous research publications, which can be found at: http://peabody.vanderbilt.edu/departments/tl/teaching_and_learning_research/mist/mist_dissemination.php

analyze the instructional visions of the Hilltop teachers provides evidence that the most sophisticated visions are realistic and attainable.

This study also contributes evidence that there are additional dimensions that should be considered as part of teachers' instructional visions. Munter discusses that there are likely additional dimensions because he limited his rubric development to the dimensions that appeared in his data. In particular, this study provides additional images of three dimensions (the role of the teacher, classroom discourse, and mathematical tasks) and new images of three additional dimensions (social culture and norms, mathematical tools as learning supports, and equity and accessibility). Considering the findings in this study in relation to Munter's findings, it is interesting to consider why the three additional dimensions appeared as part of teachers' instructional visions at Hilltop. Though the roles of the university mathematics educators, principal, and coach were not a focus of this study, their work likely influenced how these dimensions of teachers' instructional visions developed. As seen in the findings and evidence in Chapters 5 and 6, the instructional leaders at Hilltop Elementary established a normative identity for teaching and learning to teach at Hilltop Elementary which likely supported teachers in developing these additional dimensions of instruction vision.

What Do Teachers Need to Know and Understand to Teach Fractions?

Up until this point, studies of teacher knowledge have focused on identifying what teachers don't know and understand (e.g., Hill et al., 2005; Ma, 1999). Studies have also typically focused on one particular kind of intellectual resource (e.g., Ball et al., 2008; Carpenter et al., 1997; Munter, 2014). In this study, I attended to the complex nature of teaching and analyzed various intellectual resources used by teachers, the meaning made within the context of their work, and how these intellectual resources were coordinated. This study offers both

characterizations of what teachers *do* know and understand about fractions and teaching fractions and attends to multiple intellectual resources as they coexist and are coordinated in the context of teachers' work.

This study also makes a contribution to the methods for studying teachers' Mathematical Knowledge for Teaching (MKT). Traditionally, studies have used paper-and-pencil assessments to measure teachers' MKT by using a combination of both content-focused items and pedagogical-focused items. Sometimes, the assessments were also accompanied by interviews. However, the resulting analyses of these measures typically focused on what teachers *did not* understand as well as analyses of their errors. This study offers a new approach to examining the MKT that is involved in teaching mathematics—one that involves analysis of records of teachers' interaction with one another situated in the contexts where they teach and learn.

One additional contribution this study makes is through the inclusion of elementary school teachers who situated in their own classrooms and school, who are considered generalists and are responsible for planning and instructing multiple content areas. Most studies of teachers' fraction knowledge have focused on preservice teachers (e.g., Chinnappan & Forrester, 2014; Depaepe et al., 2015) or middle school teachers (e.g., Bradshaw et al., 2014; Izsak, et al., 2012), who often teach only mathematics and have had more extensive preparation for teaching mathematics.

Limitations of this Study

While this study offers an in-depth analysis of teacher and student learning related to fractions at one school across multiple years, it does not generalize past that specific context. In other words, another school could not simply pick up the tools created at Hilltop (such as the unit

plans), begin using them, and expect to see the same outcomes without also providing opportunities for teachers to developing meaning through interaction with one another.

This study is also limited in that it focused on teacher and student learning, but there were other learners in this context. The university-based math educators and school-based principal and math coach were also learners during this time and their work likely played a role in how teachers developed mutual understanding as well as the normative identities for learning mathematics, learning to teach mathematics, and teaching mathematics at Hilltop.

Finally, this dissertation did not study change in teachers' instructional practice, but rather, it attempted to study what students' learned, what teachers' learned, and how the school was organized to support the development of mutual understanding. While I offered findings on how the thinking of cohorts of students changed over time, I did not trace the learning of individual students nor did I trace the learning of individual students in relation who their teacher was each school year. In relation to teacher learning, I offered findings related to how teachers' content knowledge, instructional visions, and understanding of student learning trajectories changed over time, but I did not examine how teachers' instructional practice may have also changed.

Implications for Practice and Research

As is often true when working on mathematical tasks, starting with one question often leads to many more. This dissertation study is no different as there are a number of implications for practice and research that have emerged.

First, with regards to students making sense of fractions, additional work needs to be done that examines how students coordinate various aspects of fractions (quantity, verbal, symbolic, etc.). While this study examines various aspects in the context of an equal sharing

problem, additional work needs to be done in the context of other tasks (comparing, equivalence, operating with fractions, etc.). For example, if presented with a True-False statement such as $\frac{4}{1} = \frac{8}{2}$, how would kids say that equation and how would they represent it? What would it sound like for students to reason about that statement and how would they coordinate their understanding of the size of the piece and number of pieces?

Second, in my interviews with teachers, many of the teachers reported that they had built on their knowledge and understanding of the CGI trajectories for operating with whole numbers to construct mental trajectories for students' work with fractions. Documenting these trajectories as well as students' strategies would be a valuable extension of the CGI work on whole numbers and Empson & Levi's (2011) trajectory for partitioning strategies.

Third, Hilltop is a school where teachers have developed their content knowledge, instructional visions, and understanding of how students learn simultaneously. This structures in place provided opportunities for teachers to work on different kinds of knowledge simultaneously and in concert with one another—which is ideal since it is also how they are used. Separating them out and working on one kind of knowledge independently of the others likely does a disservice to teachers. However, coordinating them within the context of professional learning opportunities requires skilled facilitators that understand the complex nature of teaching and learning to teach. This too is an area that is in need of additional studies to better understand this work.

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APPENDIX A: INTERVIEW PROTOCOL FOR TEACHERS

The following interview protocol guided interviews with the **teachers** of the elementary school.

Identity

I'd like to start by asking you about your perspective on how you view yourself as a teacher of mathematics.

Interview Question

1. How would you describe yourself as math teacher now?
 - a. Would you have used that same description x years ago (prior to coming to this school or the grant starting)?
 - b. How comfortable are you teaching fractions? How is this different or the same as x years ago?

Vision of Quality Math Instruction and Instructional Practices⁸

Next, I'd like to ask you a few questions about your view of high quality mathematics instruction.

Interview Question

2. If you were asked to observe another teacher's classroom for one or more **fraction** lessons, what would you look for to decide whether the mathematics instruction is high quality?
 - a. What are some of the things you would expect to find the teacher actually doing in the classroom for instruction to be of high quality?
 - b. What kinds of problems or mathematical tasks would you expect to find the teacher actually working on for instruction to be of high quality?
 - i. Can you please describe a _____ [*use the word or phrase--e.g., "task" or "problem" --that the participant used for "task"*] that you would consider to be of high quality?
 - c. Can you please describe what classroom discussion would look and sound like if instruction were of high quality?
 - i. Would you expect to see the entire class participating in a single discussion, or would students be talking primarily in small groups?
 - d. Is there anything else you would look for?
 - i. *If so*, what?
 - ii. Why?
3. Do you think that others share your views (about high quality math instruction)? How

⁸ The questions in this section are adapted from the MIST project at Vanderbilt University.

do you know?

Professional Learning Opportunities

Now, I'd like to ask you some questions about math labs.

Interview Question

4. Can you tell me about your recent experiences with math labs?
 - a. How, if at all, has your participation in math labs changed?
5. How, if at all, has participation in math labs influenced your mathematics instruction specific to fractions?
6. Thinking specifically about math labs that focused on fractions:
 - a. What, if anything, have you learned about fractions?
 - i. How has this learning changed how you teach, plan for instruction, or assess students?
 - b. There are a number of resources used in math labs and I'd like to ask about how they are used and whether or not they are important to you. *Show list of resources (Unit Plans, Common Core Standards, Instructional Activities, student work, assessment data, Empson & Levi book).* Could you rank these tools from the most important/useful tool to the least important/useful tool?
 - i. Could you talk about how each tool is useful (or not)? How is it used (or not)?
 - c. Are there other resources that are important to you that we haven't yet talked about? How would those fit into your ranking?

Grade Level Teams

Next, I'm going to ask a few questions about the opportunities you have to collaborate with your grade level colleagues in your school.

Interview Question

7. Can you tell me what your PLC meetings are like?
Probe for:
 - Who leads these meetings?
 - Who attends these meetings? *For anyone that is not a teacher, ask: What does s/he do in these meetings?*
 - How often do you meet?
 - What happens in these meetings? What do you typically do in these meetings?
 - *Ask teacher to explain what they do in the activity (e.g., lesson planning-- Can you describe to me how you actually go about lesson planning? What does it look like?)*
 - Do you have to prepare in any way for the meeting?
 - What types of materials or resources do you use in these meetings?

- (Anticipated responses: unit plans, student work, assessment data, lesson plans)
 - Can you describe how you use _____ in the meeting (e.g., what you look at in student work)?
- a. How, if at all, does participation in this meeting/activity influence your mathematics instruction?
 - b. What would be different if you didn't have these meetings?

Unit Plans and Instructional Activities

Now I'm going to ask some questions about planning for instruction.

Interview Question

8. Take me through a typical lesson planning session.
 - a. Who, if anyone, do you plan with?
 - b. What materials do you use when you plan for **fractions** instruction? (Anticipated responses: unit plans, CCSS, CGI trajectories, CGI assessments, IA planning protocols)
 - i. *If the teacher does not say one of the 5 anticipated responses, ask about it specifically. For each material the teacher mentions, ask:*
 1. Can you give me an example of how you typically use _____ when planning for **fractions** instruction?
 2. Have you always used _____ as part of your planning?
 - a. *If yes*, how is _____ useful to you?
 - b. *If no*, why did you start using it?
 - c. Has the way you planned changed since you first started at this school? How so? Ask for a specific example.

9. How, if it does, does student work influence your planning practices?
 - a. What kinds of student work? (Anticipated responses: exit slips, CGI data).
 - b. Could you describe an example of when you used student work to inform your planning for fractions instruction?

Assessment

The next few questions ask about your experiences with assessing and monitoring student learning.

Interview Question

10. Thinking about when you've taught a fractions unit here at this school, can you describe how you monitored student learning?
 - a. Describe the task(s). What did the task(s) help you understand about students' understanding of fractions ideas?
 - b. Did you have conversations with anyone about the task(s) or what you learned

- about student thinking?
 - i. If so, who? What did you talk about?
- c. How, if at all, does what you learn from assessments influence your planning and teaching?
 - i. Could you describe a specific example?

11. Next, I'm going to ask about your experiences with the CGI assessment.
- a. What is your role in collecting the CGI data for your class?
 - i. Has this role changed over time?
 - b. Let's look together at the fraction problem. How, if at all, does this question help you understand your students' fraction ideas?
 - i. What's your understanding of why we ask this particular question? Do you wonder, do you still have questions?
 - c. How, if at all, has participation in the school-wide CGI assessments influenced your mathematics instruction specific to fractions?

Teacher Relationships and Networks

Last, I'm going to ask you about whom you seek out in your school for advice about fractions instruction.

Interview Question

12. How often do you seek out advice from your math coach about issues related to math teaching?
- a. *(If the teacher does seek help from the math coach)* What kinds of things do you typically discuss with your coach?
(Anticipated responses: task selection, math content, student work, meeting needs of particular students)
 - i. Can you give an example of a conversation you had with your coach about fractions?
 - ii. Do you typically initiate these exchanges by requesting assistance from your math coach or does she approach you?
 - b. *(If the teacher doesn't seek help from the math coach)* Can you tell us why you don't typically ask your math coach for assistance?

13. Since you've been at this school (*or since the start of the grant*), is there anyone else in this school that you have gone to for advice about fractions instruction?
- a. What role does that person play?
 - b. What do you typically talk about with that person?
 - i. Can you give me a specific example of a conversation you had recently with that person?

14. Do you seek out advice or ideas from anyone or anywhere outside your school?
- a. If so, who or where?
 - b. Can you give me a specific example of an idea or advice that came from outside your school?

Closing Questions

Just a couple more questions as we finish up.

Interview Question

15. *(For teachers that were at the school in August 2012)* Do you remember creating posters portraying how you think about math instruction and math learning in August 2012?

(Look at poster together.) Take a couple of minutes to remind yourself what you created.

- a. Can you describe what you put on your poster, and why?
- b. Would you revise or add anything to your poster to show how you think about math instruction and math learning? If so, what? Why?

16. Is there anything else would you like to tell me about your experience teaching fractions here at this school?

APPENDIX B: INTERVIEW PROTOCOL FOR MATH COACH

The following interview protocol guided interviews with the **coach** of the elementary school.

Identity

I'd like to start by asking you about your perspective on your role as math coach.

Interview Question

1. How would you describe yourself as math coach now?

Listen for:

- a. Would you have used that same description 5 years ago, when you first started at this school?
- b. How comfortable are you teaching fractions? How is this different or the same as 5 years ago?

Vision of Quality Math Instruction and Instructional Practices

Next, I'd like to ask you a few questions about your view of high quality mathematics instruction.

Interview Question

2. If you were asked to observe a teacher's classroom for one or more **fraction** lessons, what would you look for to decide whether the mathematics instruction is high quality?
 - a. What are some of the things you would expect to find the teacher actually doing in the classroom for instruction to be of high quality?
 - b. What kinds of problems or mathematical tasks would you expect to find the teacher actually working on for instruction to be of high quality?
 - i. Can you please describe a _____ [use the word or phrase--e.g., "task" or "problem" --that the participant used for "task"] that you would consider to be of high quality?
 - c. Can you please describe what classroom discussion would look and sound like if instruction were of high quality?
 - i. Would you expect to see the entire class participating in a single discussion, or would students be talking primarily in small groups?
 - d. Is there anything else you would look for?
 - i. *If so, what?*
 - ii. *Why?*

3. Do you think that others share your views (about high quality math instruction)? How do you know?

Professional Learning Opportunities

Now, I'd like to ask you some questions about math labs.

Interview Question

4. Can you tell me about your recent experiences with math labs?
 - a. How, if at all, has your participation in math labs changed over time?
5. Thinking about fractions math labs:
 - a. What have been your goals for teacher learning? How have you decided on these goals?
 - b. How have you worked towards these goals?
 - c. When you reflect on fractions math labs, are there particular moments that have been particularly successful?
 - d. When you reflect on fractions math labs, are there particular moments that have been particularly challenging?
 - e. There are a number of resources used in math labs and I'd like to ask about how they are used and whether or not they are important to you. *Show list of resources (Unit Plans, Common Core Standards, Instructional Activities, student work, assessment data, Empson & Levi book).* Could you rank these tools from the most important/useful tool to the least important/useful tool?
 - i. Could you talk about how each tool is useful (or not)? How is it used (or not)?
 - f. Are there other resources that are important to you that we haven't yet talked about? How would those fit into your ranking?
 - g. How do you think teachers would rank these tools/resources?
6. Thinking now about your own learning:
 - a. What, if anything, have you learned about fractions?
 - b. Can you describe any specific experiences where you learned something about fractions? (Probe for what was learned, who was part of the experience, the nature of the experience, etc.)

Grade Level Teams

Next, I'm going to ask a few questions about the opportunities you have to collaborate with your grade level colleagues in your school.

Interview Question

7. Can you tell me what PLC meetings are like?
Probe for:
 - Who leads these meetings?
 - Who attends these meetings? *For anyone that is not a teacher, ask: What does s/he do in these meetings?*
 - How often do they meet?
 - What happens in these meetings? What do you typically do in these meetings?
 - *Ask teacher to explain what they do in the activity (e.g., lesson planning-- Can you describe to me how you actually go about lesson planning?*

What does it look like?)

- Do you have to prepare in any way for the meeting?
- What types of materials or resources do you use in these meetings?
 - (Anticipated responses: unit plans, student work, assessment data, lesson plans)
 - Can you describe how you use _____ in the meeting (e.g., what you look at in student work)?
- a. How, if at all, does participation in this meeting/activity influence your work as a coach?
- b. What would be different if you didn't have these meetings?

Unit Plans, Instructional Activities, and Assessment

Now I'm going to ask some questions about the unit plans, instructional activities, and assessment.

Interview Question

8. I know you revise the unit plans each year. Can you talk a little bit about that process?
Listen/probe for: descriptions of the revisions, reasons for revisions (e.g., teacher feedback, student work), any evidence that revision accomplished the intended purpose
- a. Can you describe one or more specific revisions you've made to the fractions' unit plans?
9. Thinking about the unit plans for fractions, what IAs have been particularly helpful in supporting students to develop an understanding of fraction ideas and strategies? (Probe for: how she/they decided which IAs were useful)
10. Thinking about the unit plans for fractions instruction, what opportunities exist for teachers to assess student understanding?
- a. Describe one (or more) task(s). What did the task(s) help you/teachers understand about students' understanding of fractions ideas?
11. Next, I'm going to ask about your experiences with the CGI assessment.
- a. I'm familiar with the process we used during the three years of the grant. How has the process been modified since then?
 - b. Can you talk about the modifications you've made to the fraction task(s). Why do you ask that particular question(s)? How does the current version of the task help you understand students' fraction ideas?
 - c. How, if at all, has participation in the school-wide CGI assessments influenced your work as a math coach specific to fractions?

Teacher Relationships and Networks

Last, I'm going to ask you about how you support individual teachers who might seek out advice about fractions instruction.

Interview Question

12. How often do teachers seek out advice from you about issues related to math teaching?

a. What kinds of things do teachers typically discuss with you?

(Anticipated responses: task selection, math content, student work, meeting needs of particular students)

i. Can you give an example of a conversation you had with a teacher about fractions?

ii. Do teachers typically initiate these exchanges by requesting assistance from you or do you approach them?

13. Since you've been at this school, who have you gone to for advice about fractions instruction?

a. What do you typically talk about with _____?

i. Can you give me a specific example of a conversation you had recently with that person?

Closing Questions

Just one last question.

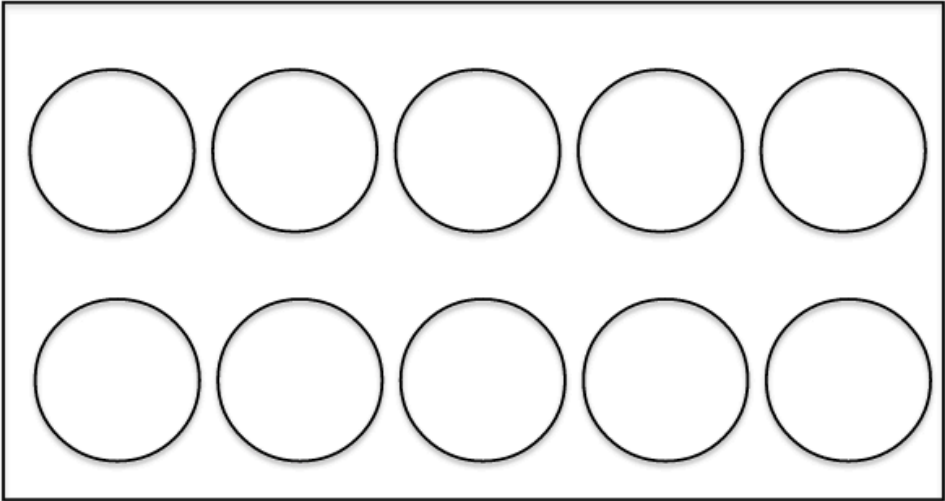
Interview Question

14. Is there anything else would you like to tell me about fractions instruction at this school?

APPENDIX C: EQUAL SHARING TASKS FROM STUDENT INTERVIEWS

Task Used in Year 1:

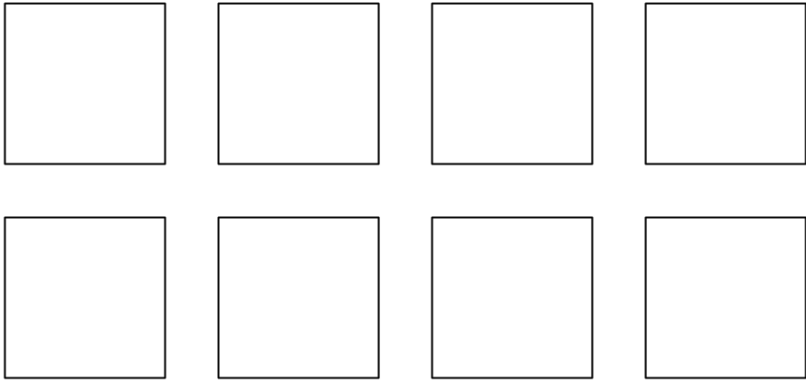
6 children are sharing 10 large cookies. They are sharing so each child gets the same amount. How much cookie will one child get?



The diagram consists of a large rectangular frame containing two rows of five circles each, representing 10 large cookies.

Task Used in Years 2 & 3:

6 children are sharing 8 small sandwiches. They are sharing so each child gets the same amount. How many sandwiches will one child get? _____



The diagram consists of a large rectangular frame containing two rows of four squares each, representing 8 small sandwiches.

APPENDIX D: CODEBOOKS

INITIAL SET OF HIGH-LEVEL, DESCRIPTIVE CODES

(Used for Recordings of Teacher and Coach Interviews, Math Labs, PLC Meetings, & One-on-one Coaching)

Main Codes	Definition	Examples
CONTENT	Fraction content topics	Meaning of Fractions, Equivalence, Ordering and Comparing, Operations, Decimals
TOOLS	Tools used by teachers to prepare for, enact, or reflect on fractions instruction; tools used by students during fractions instruction; includes representations	Curriculum resources, talk moves, number lines
ROUTINES	Routines used by teachers in their work with each other or students	Instructional Activities (e.g., choral counting, true/false number sentences); Teacher Time Out
MKT	Aspects of Teachers' Mathematical Knowledge for Teaching	Common Content Knowledge, Specialized Content Knowledge, PCK
VOI	Aspects of Teachers' Vision of Instruction	Role of the Teacher, Mathematical Tasks, Role of Classroom Discourse
IDENTITY	Aspects of normative and personal identities for teaching and learning to teach	Normative & personal identities for learning mathematics, teaching mathematics, and learning to teach mathematics
STUDENTS' THINKING	When teachers describe goals for student(s), replay an interaction with student(s), or anticipate student thinking	
MUTUAL UNDERSTANDING	Episodes where there was initially misalignment among participants' framing	

Codebook for Fractions Problem

Equal Sharing Problem:

6 children are sharing 8 sandwiches. They are sharing so each child gets the same amount. How many sandwiches will one child get?

Codes that can be used in any category: DNA: Did not attempt item DNF: Did not finish item

What is being coded?	Codes	Definitions	Examples
Did the child UNDERSTAND the problem as a sharing situation?	Y N_Add N_Mult N_Other	The child understood the problem as a sharing situation. The child added the two numbers together. The child multiplied the two numbers together. Evidence of some other inaccurate understanding.	Shared sandwiches with people Added to get 14. Multiplied to get 48. Could not retell the story.

What is being coded?	Codes	Definitions	Examples
Was the DRAWING correct?	Y N N/A	What the child DREW is an accurate way to partition What the child DREW is an inaccurate way to partition Did not create a drawing	N/A
What kind of DRAWING errors were made?	None Inc_# of People Inc_# of Sandwiches Had_Leftovers Unsure_Remaining Unsure_Sixths Unsure_Thirds Unequal_Portions Unequal_Sixths Unequal_Thirds Unequal_Twelfths Unequal_Eightheenths InsteadOfs N/A DNU	No Errors Drawing included an incorrect number of people Drawing included an incorrect number of sandwiches Does not use all of the sandwiches Child was unsure of how to part. the remaining amount Child was unsure of how to draw thirds Child was unsure of how to draw sixths Drawing has sharing/part that results in unequal shares Drawing includes partitions that result in unequal sixths Drawing includes partitions that result in unequal thirds Drawing includes partitions that result in unequal twelfths Drawing includes partitions that result in unequal 18ths Student intending one size of piece, but drew a diff one Did not create a drawing Did not understand the problem (rep. doesn't match story)	n/a Represents 4 people instead of 6 Draws 4 additional sandwiches Shares 6 of the 8 sandwiches; has 2 left Says, "I don't know how to share the last 2." Says, "I don't know how to make sixths." Says, "I don't know how to make thirds." 2 people get 2 sandwiches & 4 people get 1. Cuts remaining 2 into 4 and 8 unequal pieces Cuts one in fourths, then cuts 2 fourths in half Cuts one in half, then cuts a half in half 8ths instead of 6ths; 7ths instead of 6ths n/a draws 8 in each box (8 groups of 8)

What is being coded?	Codes/Definitions	Examples
Describe the Partitions	<p>Correct: All cut into thirds All cut into sixths 6 left as wholes & 1 cut into sixths & 1 cut into twelfths 6 left as wholes & 2 cut into thirds 6 left as wholes & 2 cut into sixths 6 cut into halves & 2 cut into thirds 6 cut into halves & 2 cut into sixths 6 cut into fourths & 2 cut into sixths 6 cut into sixths & 2 cut into thirds Only used a portion of the sandwiches and then generalized.</p> <p>Incorrect: All cut into halves All cut into fourths All cut into eighths All cut into sevenths 6 left as wholes & 2 cut into unequal pieces 6 left as wholes & 2 cut into halves 6 left as wholes & 2 cut into fourths 6 left as wholes & 2 cut into eighths 6 cut into halves & 2 cut into unequal pieces 6 cut into halves & 2 cut into fourths 6 cut into halves & 2 cut into eighths 4 left as wholes & 4 cut into halves Other</p> <p>Other: n/a</p>	

What is being coded?	Codes	Definitions	Examples
What PARTITIONING and SHARING STRATEGY was used?	0 - Did Not Attempt	Child does not attempt to partition the sandwiches	
	1 – No Coordination Between Sharers and Shares (<i>use this code if it's not clear which subcode applies</i>)		
	1a – Incorrect number of people	Child uses incorrect number of people	Shares all of the sandwiches with 4 people; each person gets 2.
	1b – Had leftovers	Child does not exhaust the sharing material; has leftovers	Gives each person 1 sandwich and “throws away” the last 2.
	1c – Did not create equal shares		Gives each person 1 & then gives 4 of the people half a sandwich to use up the last 2.
	2 – Non-Anticipatory Sharing: Child does not start out with a plan about how to share everything equally and completely. Students work it out as they go.		
	2a – Repeated halving	Repeatedly halves (or fourths) with coordination at end; often involves halves, fourths, eighths, etc. at end	Cuts 6 sandwiches in half; cuts last two into fourths.
	2b – Trial and Error	Makes multiple attempts to partition and share, revising the cuts until one works.	Cuts all sandwiches in halves; erases and cuts all sandwiches in fourths; erases and cuts all sandwiches in eighths; erases and cuts all sandwiches in sixths.
	3 - Share One At A Time	Child splits one sandwich into the number of sharers (sixths, in this case). Each person gets 1 (sixth) piece. Repeats this process until all (8) sandwiches are shared.	Splits each sandwich into sixths; gives each person 8 sixths.
	4 - Share Groups of Items	Child shares groups of items (such as 2 sandwiches cut in thirds, 3 sandwiches cut into halves, or 6 wholes) until sandwiches are gone.	Splits 2 sandwiches into thirds and gives each person a third. Does this for every 2 sandwiches until all sandwiches are used up. Each person gets 4 thirds.
	5 - Ratio	Child uses knowledge of repeated halving or multiplication factors to transform the problem into a simpler problem (e.g., 3 children sharing 4 sandwiches). Solves the simpler problem.	“6 children sharing 8 sandwiches is like 3 kids sharing 4 sandwiches...” (and solves 4 sandwiches shared with 3 kids).
6 - Multiplicative Coordination	Child does not use visual. Child understands that a things shared by b people is a/b , so 8 sandwiches shared by 6 people means that each person gets $8/6$ of a sandwich.	“8 sandwiches shared by 6 people means that each person gets $8/6$ of a sandwich.”	

What is being coded?	Codes	Coding Directions	Examples
VERBAL: What did the child SAY for the answer?	Text string	Record exactly what the child SAID If the child did not say an answer, enter N/A	“one whole and one-third size piece”; “six pieces” N/A
Was the VERBAL answer correct?	Y N N/A	What the child SAID was an accurate total What the child SAID was an inaccurate answer If the child did not say an answer, enter N/A	eight sixths one-sixth of each sandwich
VERBAL: What kind of language was used?	Pieces Language Whole Number Only FL_Inc FL_Corr Not Sure N/A	Expresses answer as an amount of pieces, slices, parts Expresses answer as a whole number of sandwiches Uses fractional language, but not correct Uses correct fractional language If the child says they don’t know how to say it If the child did not say an answer, enter N/A	8 pieces 8 sandwiches 8 halves 8 sixths; one and one third “I don’t know how to say that in fractions.” N/A
What did the child WRITE for the answer?	Text string	Record exactly what the child WROTE If the child did not say an answer, enter N/A	“1 1/3”; “8/6”; “6”; “6 pieces” N/A
WRITTEN: Did the child use accurate symbolic notation?	Y N N/A	What the child WROTE was a correct answer What the child WROTE was an incorrect answer If the child did not say an answer, enter N/A	
Did the VERBAL answer and the WRITTEN answer match?	Y N N/A	The verbal and written answers matched The verbal and written answers did not match Did not have a verbal and/or written answer	