Research on Student Understanding of Quantum Mechanics as a Guide for Improving Instruction

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Abstract

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This dissertation reports on an investigation of student understanding of topics in quantum mechanics that are at a level appropriate for the third year of undergraduate study. Initially, the focus was on student ability to relate classical and quantum mechanics. This early research led naturally to the investigation of student understanding of time-dependence, which is the primary emphasis in this dissertation. Research on student learning of more advanced topics ensued. The conceptual and reasoning difficulties that were identified have informed the development of tutorials to improve student learning. Initial assessment of that curriculum has yielded encouraging results.

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Last but not least, words cannot adequately express my gratitude to my wife Nicole for moving from her beautiful little cassita in the shadow of a volcano in a world heritage site to rainy Seattle to support me in so many ways while I completed this dissertation.

DEDICATION

To my infant daughter

Ava

and her peers.

Would that this effort help form
a few rungs of a ladder enabling
you to climb upon the shoulders
of giants and reach for the stars.

Chapter 1

Introduction

The Physics Education Group at the University of Washington has been conducting research on the learning and teaching of physics for many years. These investigations have primarily taken place in the context of introductory university courses and in special courses for pre-college teachers. The careful documentation and characterization of student difficulties has proven invaluable in the design of instructional materials that have been shown to be effective in improving functional understanding of physics among undergraduate and graduate students and among pre-service and in-service K–12 teachers.[1] The present effort extends this same methodology into the junior-level quantum mechanics course. This dissertation describes a systematic investigation of student understanding of some basic concepts and representations in introductory non-relativistic quantum mechanics. In addition, it reports on preliminary efforts to design, implement, and assess curriculum to address some of the specific difficulties that were identified.

It is in the interest of physics faculty to ensure the quality and efficacy of courses for future physicists. Students in such courses will work in their labs, further their research, and eventually determine the future of their profession. Quantum mechanics courses, in particular, are often regarded as gateway courses without which students have no access to current research topics. This alone justifies an examination of student learning in upper-division courses. A study of student understanding of quantum mechanics, however, also has the potential to affect a wider population than that of future physicists. Periodically, a great deal of interest in the physics community is generated around the possibility of integrating concepts from quantum mechanics into the curriculum at ever earlier levels. At times, lively debate has surrounded this issue,[2] and some serious attempts have been made to bring it to fruition.[3] This study does not attempt to make a recommendation as to whether such a proposal is realistic. It does, however, through an investigation of student understanding of topics in quantum mechanics, provide some insight into what students learn in a third-year course on quantum mechanics. It will be shown that these students often have significant difficulties with this material. Furthermore, some of these difficulties point toward difficulties with underlying material. Thus, there are strong implications for the teaching of this material at an earlier level.

The particular focus of this dissertation is important both on physical and pedagogical grounds. Special emphasis has been placed on student understanding of stationary states, superpositions of stationary states, and the time evolution of both. A number of previous studies have focused on student understanding of stationary states.[4] Superpositions of stationary states allow for time dependent probability distributions. Other studies have investigated student difficulties with potentials[5] or with allowable states given a potential. Relatively little effort, however, has been devoted to the investigation of student understanding of the predictions of the Schrödinger equation for the time evolution of a system.[6]

The organization of this dissertation is guided by the physics under consideration rather than by a chronology of the work or by the identification of specific student difficulties. It is hoped that this organization will make the findings more accessible to instructors by placing student difficulties in the familiar physical frameworks in which the course is usually taught.

Chapter 2 describes the institutional context in which the research took place and the research methods used. It also describes prior research on student difficulties as well as curriculum development in modern physics and quantum mechanics that is not directly related to ideas discussed later in the dissertation. The remainder of the dissertation is divided into three parts. The first part focuses on student understanding of stationary states, superpositions of stationary states, and the time dependence of both. The second part describes difficulties students have in relating quantum mechanics and classical mechanics. The final part describes additional student difficulties with more advanced topics that build on the basic ideas discussed in the first two parts. These include: probability current, angular momentum, identical particles, perturbation theory, and formalism (*e.g.,* Dirac notation).

1.1 Notes to Chapter 1

1 See (McDermott, 1991) and (McDermott, 2001).

2 See (Jossem, 1966) for the recommendations of an interdisciplinary conference, and (Michels, 1964) for a related charge to the Commiission on College Physics, as well as the lively and more recent exchange in the following references. (Rigden, 1987, Stork, 1987, Payne, 1988, Pasachoff, 1988, Rex, 1988, Kowalski, 2001, Presto, 2001, George, 2001, Hobson, 2000, Howes, 2000)

3 See (Zollman et al., 2002) for a description of one such attempt aided by computer software.

4 See (Ambrose, 1999) and (Bao, 1999)

5 See (Jolly et al., 1998) or (Cataloglu, 2002).

6 See (Singh, 2006), (Singh, 2001),(Cataloglu, 2002),(Cataloglu, 2002), and (Sadaghiani, 2005)

Chapter 2

Context for Research

The research that forms the basis of this dissertation has been influenced by prior research in physics education, the particular tradition of physics education research at the University of Washington, and the structure of the course offerings at the University of Washington. These courses are typical of the way physics is taught at most colleges and universities in the United States. This chapter briefly reviews these influences and discusses the instructional setting in which the research and curriculum development took place.

2.1 Prior Research Relevant to Investigation

A number of authors have written about the teaching of quantum mechanics at various levels. Much of this literature has been produced by experienced teachers of the subject.[1] Such work, while often anecdotal, has served as a starting point to begin a systematic investigation of student understanding. Recently, there have also been a number of more systematic studies of student understanding of topics in quantum mechanics. Some have focused on high-school students, some on teachers, and others on university students. The topics have included the wave nature of particles,[2] tunneling,[3] and time dependence.[4] Other studies have focused on student understanding of material requisite for quantum mechanics although this purpose was not the incentive for the research. In particular, there is an extensive literature on student understanding of concepts in probability.[5] These investigations span a large range of ages from young children to adults. This prior work has relevance to particular sections of this dissertation. Descriptions of pertinent material from this body of work are included at the beginnings of the relevant parts of this dissertation.

2.2 Prior Physics Education Research and Curriculum Development at the University of Washington

Since the early 1970s, the Physics Education Group at the University of Washington has been engaged in an iterative cycle of research, curriculum development, and instruction with the goal of understanding and improving student learning in physics. To this end, the group has employed various methods that include interviews, informal observations, and written questions. In addition, the group has developed a number of instructional techniques that have proven effective. These include worksheets that incorporate guided inquiry, student interactions in small groups, and instructor-student semi-Socratic dialogues.

In the 1970s David Trowbridge,[6] who was then a graduate student in the Physics Education Group, designed several demonstrations that he used as the basis for individual interviews with students. These individual demonstration interviews, which were inspired by the work of Piaget, provided insight into the nature of student reasoning about velocity and acceleration. The group later extended this method fro probing student understanding to other topics. Insights into the nature of student reasoning that were gained in this way informed the development of written questions. Much of the early work took place in nonstandard physics courses. Most of these courses were intended to prepare prospective and practicing teachers to teach physics and physical science by inquiry but some were designed to strengthen the preparation in physics of under-prepared students aspiring to science-related careers.[7]

In the early 1990s, the group began utilizing its extensive research base on the learning and teaching of physics to improve the effectiveness of instruction in large introductory physics courses. At large universities, such courses typically involve hundreds (if not thousands) of students, scores of teaching assistants, and many faculty members. These courses also serve different purposes for different student populations. For some students, the course is a first step toward a physics major. For others, it forms the basis for the study of another science or engineering or fulfills a requirement for another major. Still others take the introductory physics course to add breadth to their education. For most students, it is their terminal course in physics. Due to the overlapping interests in, and the complexity of, such courses, revolutionary change in the structure or content of the course is very difficult to achieve. Incremental improvement is a more realistic goal.

Beginning in 1991, changes that may appear relatively minor were made to both the structure and emphasis of the introductory calculus-based course at the University of Washington. The largest of these changes was the replacement of one lecture each week by a small-group session led by two or three Teaching Assistants (TAs). During these small-group sessions, students work through curriculum in the form of worksheets called "tutorials" developed by the Physics Education Group. Accompanying this change in format was extensive research and evaluation of the tutorials, including preand post- testing. This research and evaluation continues to feed into the development and revision of the tutorials in an ongoing, iterative manner.

Many tutorials can be characterized by a strategy that begins by eliciting known conceptual and reasoning difficulties. The students are then guided through a process in which they confront and resolve their difficulties. Students in groups of three to five struggle together with the ideas in the tutorials. The role of the teaching assistants is to engage students in a semi-Socratic dialogue in which they supplement the questions on the worksheets by asking additional questions, rather than simply providing answers. The purpose is to guide students through their own intellectual efforts to an understanding of the materials. For students who have grasped the basic ideas, the teaching assistants ask challenging questions that deepen the experience. In this manner, the teaching assistants and other instructors are able to probe student understanding, and also to tailor instruction to the needs of individual students.

Starting in 2002, an opportunity arose to extend this same model of research, curriculum development, and instruction to the quantum mechanics sequence at the University of Washington. In the mid 1990s Brad Ambrose,[8] who was then a graduate student in the Physics Education Group, began an effort to explore student understanding of quantum mechanics. His investigations, some in the context of the junior-level quantum mechanics courses, focused on student ability to relate classical mechanics to quantum mechanics. His findings are discussed where they are most relevant to the present study.

Our research on student understanding of quantum mechanics has evolved. The emphasis in this dissertation is on quantum mechanical superposition. This new focus grew naturally out of the former effort, as we recognized that an understanding of superposition was necessary for students to analyze classical limits. We soon found, however, that both the importance of superposition and student difficulties with superposition extend far beyond their application to classical limits. This recognition motivated a more general investigation of student understanding of time dependence in quantum mechanics.

2.3 Instructional Context

The majority of the research discussed in this dissertation has taken place in the context of tutorial instruction in the junior-level quantum mechanics courses at the University of Washington. These courses, however, exist in the broader context of the undergraduate physics major. Below is a brief description of the courses related to quantum mechanics that are typically taken by a physics major. This summary is followed by an overview of the implementation of the tutorials in the junior-level quantum mechanics sequence, Physics 324 and 325.

2.3.1 Relevant background of students

Students majoring in physics at the University of Washington are encouraged to take a particular sequence of courses. Each year is divided into four quarters, but enrollment in the summer quarter is not required. The courses below represent the sequence suggested by the Physics Department during period of the research that is described.

During the first year, students are expected to take three quarters of introductory calculus-based physics, as well as three quarters of calculus. The introductory physics courses consist of three parts: lectures, laboratories, and tutorials. The lectures and labs are fairly typical of those found at other institutions. The tutorials that have been briefly described earlier in this chapter are an innovation designed by the Physics Education Group at the University of Washington. They take place once every week and are a required part of the course.

In their second, year students take courses that are an introduction to thermodynamics, statistical mechanics, electronics, experimental physics, and modern physics. Their mathematics courses include: linear algebra, differential equations, and advanced calculus. In addition, the Physics Department offers a course on mathematical methods in physics.

Typically, the third-year sequence includes electricity and magnetism as well as the first course devoted solely to quantum mechanics. Students have a choice. They can take the standard two quarter quantum mechanics course, Physics 324 and 325, designed to prepare them for graduate work in physics. Alternatively, students may take Physics 315, Applications of Modern Physics, for a one quarter introduction to quantum mechanics. In practice, some students take both Physics 315 and the Physics 324/325 sequence either sequentially or concurrently. Thus, by the time they complete their junior-year, physics majors at the University of Washington have had between two and four quarters of exposure to quantum mechanics. The courses providing this exposure are briefly described below.

Physics 225–modern physics

Students typically take modern physics during their second year. The content of this course varies somewhat with the instructor, but it usually contains introductions to special relativity and quantum mechanics. The official course description is as follows.

> PHYS 225 Modern Physics: Special theory of relativity, phenomena of modern physics with emphasis on photons, electrons, and atoms; introduction to quantum physics. Prerequisite: 2.0 in the third quarter of introductory physics which may be taken concurrently.[9]

Physics 315–applications of modern physics

The content of Physics 315 also depends on the instructor, but always contains a significant amount of quantum mechanics. Physics majors are required to take either this course or Physics 324. The course catalog describes the class as follows:

> PHYS 315 Applications of Modern Physics: Foundations of quantum physics, including Schroedinger equation, tunneling, atoms, spin, and applications. These include semiconductor devices, lasers, magnetic resonance imaging (MRI), quantum cryptography, atomic microscopes. Prerequisite: minimum grade of 2.0 in modern physics, minimum grade of 2.0 in sophomore level thermodynamics, minimum grade of 2.0 in the first quarter of elementary mathematical physics which may be taken concurrently.[10]

Physics 324 and 325–quantum mechanics

Physics 324 and 325 constitute the two-quarter junior-level quantum mechanics sequence intended for physics majors planning on studying physics in graduate school. The course catalog describes the two courses as follows:

> PHYS 324 Quantum Mechanics: First part of a two-quarter sequence. Introduction to non-relativistic quantum mechanics: need for quantum theory, Schroedinger equation, operators, angular momentum, the hydrogen atom, identical particles, and the periodic table. Prerequisite: advanced calculus; minimum grade of 2.0 in modern physics; minimum grade of 2.0 in the second quarter of elementary mathematical physics.

> PHYS 325 Quantum Mechanics: Continuation of PHYS 324. Introduction to non-relativistic quantum mechanics: perturbation theory, the variational principle, radiation; application of quantum mechanics to atomic physics, magnetic resonance, scattering, and various special topics. Prerequisite: PHYS 324.[11]

Typically these two courses are taught in the autumn and winter quarters for three consecutive years by the same member of the physics faculty. Physics

324 is also typically offered in the summer, but the summer course is usually taught by a different physics faculty member each year. Most of the research took place in these courses, which are described in greater detail in the next section.

2.4 Tutorial Implementation in Junior-level Quantum Mechanics

The Physics Education Group had conducted some research in the physics 324/325 sequence before 2002. Before that time, however, the course structure involved only lectures, three being given each week. In 2002, however, a unique opportunity developed that allowed us to expand our research and curriculum development efforts in this course. At that time a group of physics majors petitioned the Department to increase the credit for this course to make it commensurate with the amount of work required. The Department agreed with the stipulation that an extra hour of class time be required. Rather than adding an additional hour of lecture to the course, the Department chose to add a small-group section. The enrollment for the entire course typically ranges between 50 and 80 students. Thus, there are usually three or four of these sections.

When the small-group sections were added, the course instructor asked the Physics Education Group to help design appropriate instruction for these sections. The group began to develop, implement, test, and modify tutorials modeled on *Tutorials in Introductory Physics*. The purpose of these new materials was to address specific student difficulties that we had begun to identify through research.

The implementation of tutorials in the course has not been without challenges. At the onset, a few vocal students expressed a desire to use the extra session for additional practice solving homework problems. As a compromise, the instructor asked that some of the tutorials be modified so that they could be used for half of a class period, leaving another half for problem solving. This often required splitting the study of a particular concept into two smaller tutorials. Other tutorials were used for an entire class period in exchange for entire class periods devoted to problem solving.

At the end of the first year, a survey administered by the course instructor revealed that the students overwhelmingly found the tutorials to be a useful part of the course. Furthermore, students indicated that their estimation of the tutorials' benefit would improve if more time were allotted for tutorials. Figure 2.1 shows the results of this survey.

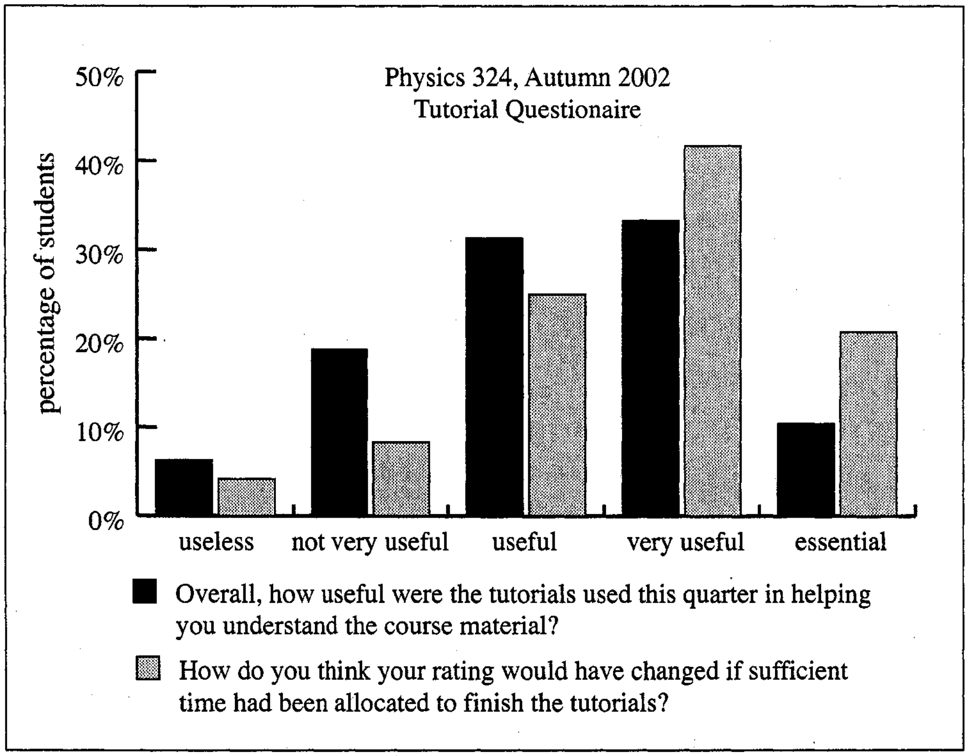


Figure 2.1 Students were asked to rate the usefulness of the tutorials. Then they were asked how their answer would change if sufficient time were provided to complete all the tutorials.

Given the positive results of the survey, the tutorials began to take a more prominent role in instruction. As time progressed, students began to ask for more sections to be devoted entirely to tutorials. An optional evening session led by a teaching assistant provided problem solving practice for those students who wanted it. These sessions were sporadically attended.

The tutorials have come to be both an expected and a desired part of the course by students and faculty alike. By Autumn 2005 and Winter 2006, a full sequence of tutorials was being implemented with the exception of exam weeks. The number of full and half tutorials implemented in Physics 324 and 325 is tabulated in Table 2.1.

Table 2.1 Tutorial instruction in Physics 324/325.

Course	Quarter	N	Number of partial tutorials	Number of full tutorials
Physics 324	Autumn 2002	82	2	2
Physics 325	Winter 2003	50	3 (3 topics)	0
Physics 324	Summer 2003	11	1	7
Physics 324	Autumn 2003	48	6 (3 topics)	2
Physics 325	Winter 2004	33	4 (2 topics)	0
Physics 324	Autumn 2004	52	4 (2 topics)	3
Physics 325	Winter 2005	44	8 (4 topics)	0
Physics 324	Autumn 2005	55	0	7
Physics 325	Winter 2006	41	0	7 (5 topics)
Physics 324	Autumn 2006	51	0	7

Teaching assistants must be prepared to use the tutorials for quantum mechanics effectively. Therefore, they are required to meet before each tutorial session with the course instructor and an experienced quantum mechanics TA. Together, all work through the upcoming tutorial. During this meeting they also develop questioning strategies to help guide students during the tutorial.

2.5 The Effect of Institutional Context on Research

The constraints of this course have influenced the type of research that we have been able to conduct. The constant changes in the time devoted to tutorials have made it difficult to isolate the effect of content modifications. Three additional factors concerning upper-division courses have also had a great impact on the design of our research program. When compared to introductory physics courses, advanced courses have a smaller enrollment, are offered less frequently, and build on a larger edifice of prior knowledge. The quantum mechanics course at the University of Washington has a maximum enrollment of about 80 students each year and is at times much smaller. The introductory calculus-based physics course, on the other hand, has about 1000 students enrolled each quarter. When compared with this course, the smaller number of students taking quantum mechanics gives rise to a larger variation in student responses from class to class. Moreover, since the quantum mechanics courses at the University of Washington are only offered once every year, access to students is limited and thus the opportunities for modifying and testing curriculum are few. Given these constraints, we have focused most of our research on the identification of common student difficulties and to a lesser extent on the precise measurement of their prevalence. We have posed many questions to a few students as opposed to asking many students a few questions. This approach has allowed us to view the same set of student difficulties in many different contexts, and thus gain more insight into student reasoning and the persistence of specific difficulties. Thus, by suitably modifying the research methods that the Physics Education Group has applied to the introductory course, we have been able to adapt to changes in the quantum mechanics course structure in our effort to investigate student understanding of physics at this more advanced level.

2.6 Notes to Chapter 2

1 See (Styer, 1996).
2 See (Ambrose, 1999).
3 See (Bao, 1999).
4 See (Singh, 2001).
5 See (Shaughnessy, 1992) for a review of this literature.
6 See (Trowbridge, 1979).
7 See (McDermott et al., 2006) or the sequence of three articles (McDermott et al., 1980b), (McDermott et al., 1980c), and (McDermott et al., 1980a).
8 See (Ambrose, 1999).
9 See (Washington, 2006).
10 See (Washington, 2006).
11 See (Washington, 2006).

Part I

Student Difficulties with Time Evolution in Quantum Mechanics

Chapter 3

Evidence for Persistent Difficulties with Time-Dependence in Quantum Mechanics

Typical instruction in introductory quantum mechanics has a strong focus on stationary states (*i.e.,* states associated with probability distributions that do not change over time). These states arise when the Schrödinger equation is separated into a time-dependent and a time-independent part. The stationary states are those associated with the time-independent part (often called the time-independent Schrödinger equation or the energy eigenvalue equation). Each has a time dependence given by $e^{-iE_nt/\hbar}$ where the E_n are the eigenvalues for the time-independent part. The probability distributions associated with these stationary states do not depend on time, hence their name. A primary reason for a strong focus on stationary states is that any state can be expressed as a superposition of stationary states.

Experienced instructors, however, have argued that an overemphasis on stationary states can leave students unable to reason about the time evolution of quantum mechanical states.[1] There is now documented evidence from physics education research that student difficulties with time dependence in quantum mechanics are present after traditional instruction and are persistent enough to affect student reasoning on final exams and even at the beginning of graduate courses on quantum mechanics.[2] In this chapter, we present evidence from research conducted at the University of Washington and elsewhere that illustrates the persistence of such difficulties. A detailed examination of these and related difficulties is presented in relevant portions of the dissertation.

3.1 Identification of and Persistence of Specific Difficulties

A particularly basic question that has been used by researchers and instructors alike demonstrates some student difficulties with time dependence and their persistence. In this question, students are given the initial state of a system as a sum of a few of the system's energy eigenstates. For example, $\Psi(x, 0) = a\phi_1(x) + b\phi_2(x)$ where a and b are constants (real or imaginary), and ϕ_1 and ϕ_2 are two stationary states with energies E_1 and E_2, respectively. The students are then asked to give the state at a later time, t. In this case, the correct answer is $\Psi(x, t) = a\phi_1(x)e^{-iE_1t/\hbar} + b\phi_2(x)e^{-iE_2t/\hbar}$. Versions of this question have been asked at various colleges and universities in a number of different physical contexts throughout instruction on quantum mechanics.

3.1.1 Results immediately following instruction

In the fall of 2004 Homeyra Sadaghiani asked such a question of 48 students in an upper-level quantum mechanics course at The Ohio State University.[3] She found that only one-third of the students could answer correctly.

3.1.2 Results on final exams

At the University of Washington questions of this format have been given on final examinations in junior-level quantum mechanics courses with traditional lecture instruction on time dependence. The courses include Physics 324 and 325 during 2002, 2003, 2004, and 2006. In all, 125 different students answered such questions. The fraction of students able to answer correctly varied from one- to two- thirds of a given class.

3.1.3 Results in graduate school

Assessments of student ability to answer this question have also been made at the graduate level. A survey of 202 beginning graduate students at seven different universities was conducted by Chandralekha Singh from the University of Pittsburgh.[4] She found that only 45% of these graduate students could correctly solve this type of problem.

3.1.4 Need for additional questions

In all of the research discussed above, the various researchers have presented data about the percentage of students who answered the question correctly, and the specific incorrect answers given by other students. In all cases, the incorrect responses were similar. The most common error is to treat the state as if it is an energy eigenstate and write $\Psi(x,t) = [a\phi_1(x) + b\phi_2(x)]e^{-iEt/\hbar}$.

Other than indicating the existence of a problem, the results of this research question are not detailed enough to give insight into the nature of student thinking. The incorrect answer given above, indicates a failure of many students to distinguish between stationary states and more general states of a system. The reasons for student thinking, however, are not revealed sufficiently by this question. In order to develop curriculum, we felt we first needed to design research questions to probe these issues in greater detail. The following example illustrates part of the reason we felt additional research was needed.

In the Winter Quarter of 2006, two questions similar to the one described above were given on the final exam for the Physics 315 course at the University of Washington. One version gave the initial wave function as $\Psi(x, 0) = \sqrt{\frac{2}{7}}\phi_1(x) + \sqrt{\frac{5}{7}}\phi_2(x)$ and asked students to give a mathematical expression for the time dependence of the state. The other version gave the initial wave function as $\psi(x) = \frac{1}{2}u_o(x) + \frac{\sqrt{3}}{2}u_1(x)$ and asked the students for the possible values for a measurement of energy.[5]

About 60% of students answered the first question above correctly, but only 30% of students answered correctly for the second, ostensibly identical, question. Although the number of students answering correctly varied widely, the types of errors that students made were similar. We interpret this result as an indication that student understanding of time-dependence in quantum mechanics is complex and possibly context dependent. We thus felt it needed to be probed through the use of more instruments than these questions alone.

3.2 | Additional Research

The remaining chapters in this part of the dissertation describe our efforts to investigate, in greater detail, student understanding of time dependence and closely related topics in quantum mechanics. By asking many questions that focus on a single idea but occur in different contexts in different parts of the course, we have been able to identify a number of student difficulties related to time evolution in quantum mechanics. We have also gained insight into their persistence. Several broad classes of errors emerged from the student responses.

- Beliefs about the time dependence of states that are unrelated to the formalism developed in a quantum mechanics course. (*e.g.*, Beliefs that in a time-independent potential states must decay, diffuse, or revive).

- Tendency to treat energy eigenstates as if they are the only valid states for a system.
- Tendency to treat all operators as if they have the same eigenfunctions (*e.g.,* momentum, position, and energy all have the same eigenfunctions).
- Tendency to treat any collapse of the wave function due to a measurement as if it makes all observables known.
- Tendency to treat certain special functions as if they are stationary states for all systems (*e.g.,* sinusoidal functions are stationary for all potentials).
- Tendency to treat the possible values for energy measurements of a given state as if they change with time for an isolated system.
- Tendency to treat superpositions of degenerate stationary states as if they were not stationary.

As will be described, these classes of difficulties are closely related. We have found this particular partitioning of difficulties to be useful in categorizing student conceptions. A given student, however, may not hold a consistent set of incorrect ideas, and their explanations may differ depending on the circumstances. In the following chapters, evidence for these broad classes of difficulties will be presented. In addition, descriptions of additional difficulties within these broad classes, along with descriptions of our efforts to address these difficulties with a tutorial curriculum, will be described.

The next chapter in this part discusses certain basic incorrect ideas that some students bring with them to the course and that often continue to affect their reasoning throughout the course. The roots of these difficulties seem to be outside the formalism of quantum mechanics. The subsequent chapters focus on specific classes of difficulties that students have in applying the model developed in non-relativistic quantum mechanics to physical problems. These are interspersed with chapters describing curriculum to address those difficulties. This part concludes with a preliminary assessment of the effectiveness of the curriculum.

3.3 Notes to Chapter 3

[1] See (Styer, 1996) and (Styer, 1990)

[2] See (Sadaghiani, 2005) and (Singh, 2006)

[3] See (Sadaghiani, 2005) It was presented in a multiple choice format just after the students had lecture instruction on the relevant material. for a detailed description of the multiple choice question used at The Ohio State University and student responses.

[4] See (Singh, 2006)

[5] Details of these questions can be found in the appendix.

Chapter
4

Overarching Student Conceptions about Time Dependence

Evidence was presented in Chapter 3 that many students lack the facility with time dependence in quantum mechanics to answer mathematical questions about the time evolution of a superposition of stationary states. As discussed, most of the relevant research has focused on student ability to write the time dependence mathematically. In these studies, the equations given by students were analyzed, but much student reasoning remained hidden. Little or no research has been conducted on student reasoning concerning time dependence using other representations (*e.g.*, graphical), or has probed student qualitative understanding. In this part of the dissertation, we focus on student reasoning about how stationary states and superpositions of stationary states evolve in time.

This chapter focuses on conceptions about time-dependence that students seem to bring to a junior-level quantum mechanics course and that persist throughout instruction. The chapter is divided into two sections. The first (Section 4.1) deals with student reasoning about the shape of wave functions as time evolves. The second (Section 4.2) concerns the effect of a misconception about statistics on student ability to reason about time dependence in quantum mechanics.

4.1 Student Predictions concerning the Shape of Wave Functions

At the beginning of our investigation, we expected that students would have some difficulties in applying to physical systems the Schrödinger equation along with the attendant postulates that together form the generally accepted model for non-relativistic quantum mechanics. We further expected that some of those difficulties would be associated with the time evolution of systems. We did not, however, realize the extent to which students' prior knowledge would impact their thinking about time dependence in quantum mechanics.

We found that many students tended to reason about time dependence in ways unrelated to the formalism that they had been taught. This became especially clear during interviews originally intended to probe student understanding of measurement. This finding was especially significant since these interviews were with students who had completed the entire quantum mechanics sequence.

4.1.1 Interviews that revealed the existence of alternate student conceptions of time dependence

During our initial set of interviews, we found that the ideas that students were using to answer questions concerning the time evolution of wave functions seemed to fall into three categories: a belief that any system will return to its initial state, a belief that all wave functions spread, and a belief that a system will drop into a stationary state. We term these ideas "revival," "diffusion," and "decay," respectively. Excerpts from interviews illustrating each of these categories are presented in what follows.

"Revival" of wave functions

Figure 4.1 shows an excerpt from an interview with a student who has completed the entire undergraduate quantum mechanics sequence at the University of Washington (*i.e.,* Physics 324 and Physics 325). The student and interviewer are discussing the effect of a position measurement that was just made on a system that had been in an energy eigenstate. The student correctly indicated that the wave function would look essentially like a delta function after the measurement. The student was then being asked about the subsequent time evolution of the wave function.

> **Student:** *Once you have measured this thing, then you have a delta function that just kind of bounces back and forth like a ball.*
> **Interviewer:** *OK*
> **Student:** *You already know where it's at, so you have it bouncing back and forth. But, at the same time, I'm thinking, "Well a long time later that thing is going to, you know, re-distribute into like a bell curve and that thing is going to be bouncing back and forth." We still don't know how it is bouncing back and forth, but eventually I think it is going back to this [points at a sketch of the initial energy eigenstate].*

Figure 4.1 Example of student reasoning based on wave function revival.

After the measurement, the wave function is a superposition of many stationary states. Thus, its time dependence is complex, but it will always be a superposition and never a single stationary state. Notice that the student focuses on two different time periods: one a short time after the measurement and the other a long time later. A short time after the measurement, the student believes that the peak of the wave function moves back and forth in the well. The student, however, believes that the wave function will slowly change back to the original state (in this case an energy eigenstate) after a sufficient amount of time has elapsed. The student fails to recognize that the wave function immediately after the measurement consists of a superposition of many energy eigenstates (*i.e.*, that the wave function can be written as $\Psi(x,t) = \sum c_n \psi_n e^{iE_n t/\hbar}$) and that each of these energy eigenstates contribute to the wave function. There is no mechanism for the energy eigenstates involved in the superposition to change after the measurement.

Figure 4.2 shows an excerpt from another interview on the same topic with a different student. This student also believes that the state will eventually return to the initial state, but adds other details. The student believes that once it returns to an eigenstate it will remain in an eigenstate because eigenstates are "stable." The student does not have a mechanism for how this happens.

The student makes no attempt to connect this prediction to the quantum mechanical model developed in class, rather it comes from memory. When pushed, the student concludes that information about the initial state must be stored somewhere if it is to return to that state. The student hypothesizes that perhaps such information is stored in the complex part of the wave function.

Both students failed to recognize that after the wave function was measured, it could be represented by a superposition of stationary states. The particular set of stationary states and their amplitudes that form the terms in the superposition do not change with time. The relative phases of those terms, and thus the shape of the wave function, however, will change with time.

"Diffusion" of wave functions

The excerpt shown in Figure 4.3 illustrates a different way in which students reasoned about the time evolution of wave functions during interviews. As

Student: *So, I'm going to say it goes back to its original.*
Interviewer: *So, it's that first excited state?*
Student: *Yeah.*
Interviewer: *Then, if I keep waiting what happens?*
Student: *It is just going to keep rotating around in the complex plane. Unless it is perturbed it is going to stay like this [points at the original wave function].*
Interviewer: *So, once it gets to this [the original] it stays here?*
Student: *Yeah, because it is stable.*
...
Interviewer: *Is there any mechanism? How does it work that it gets back to the original state? any principles?*
Student: *That is really the bothersome part. Is it magically? [long pause] I haven't ever really resolved that question. I think that I have just tried to remembered what it was so that I could do it on the tests. I think it has something to do with the complex nature of the wave, and something to do with the time dependence of the wave.*
Interviewer: *So that information is encoded in the complex nature in some way?*
Student: *Yeah.*
...
Student: *You still have the information of the original. It is not thrown away.*

Figure 4.2 Example of student reasoning based wave function revival.

was the case for the students discussed earlier, the student has completed the entire undergraduate quantum mechanics sequence at the University of Washington. As before, the student and interviewer have been discussing a system initially in an energy eigenstate upon which a position measurement has been made. The student has correctly drawn a spike to represent the state of the system immediately after the measurement.

This excerpt has some points in common with the earlier interviews in that the student considers the revival of the initial state as a possibility. Clearly, however, the idea of the wave function spreading out or diffusing dominates this student's thinking. Additionally, the student does not seem to recognize that the system cannot return to the original state.

> **Interviewer:** *So, let's say we have made that position measurement and you get your wave function like the one you have drawn, and then I wait a long time. If I wait a long time, what happens?*
> **Student:** *So then it spreads out. Now, I don't remember if it spreads out to the initial wave function or a superposition of different energy eigenstates. I know it spreads out though.*

Figure 4.3 Example of student reasoning based wave function "diffusion" or "revival."

Many other students expressed similar ideas in interviews. Figure 4.4 shows another example. The student in this interview is discussing the same situation with the interviewer as the previous student.

> **Interviewer:** *We measure it, and you said it looks like that spike. Then, what happens a long time later if I just wait?*
> **Student:** *...This will expand slowly. It will still have an area of one spread out over the whole well again.*
> **Interviewer:** *So when it spreads out, will it go toward something specific in the end? Will it go flat in the end? When you say it spreads out, what do you mean?*
> **Student:** *I assume it just has the shape of the function. It just kind of spreads out. It will still have a kind-of bell shape about it.*
> ...
> It is just going to keep going down the whole time even while it is oscillating. It is going to become like a flat function. ...It should be fairly flat except for the boundary conditions of it being zero at the sides.

Figure 4.4 Example of student reasoning based on wave function "diffusion."

The excerpt shown in Figure 4.4 again illustrates the idea of diffusion or spread of a wave function, but in this case the student suggest the wave function will be generally flat after sufficient time has elapsed. The student does not refer to the wave function as consisting of a superposition of stationary states whose components do not change with time.

It should be noted that this type of reasoning can be used both correctly and incorrectly as will be discussed later in this dissertation.

"Decay" of wave functions

Some students envision the time dependence of a wave function as a diffusion toward a flat probability distribution. Others see it as a decay toward a stationary state. In some interviews students expressed this idea explicitly. Figure 4.5 shows a small portion of one such student's reasoning while discussing the time dependence of an excited state in the infinite square well. As was the case with the previous students, this student has completed all undergraduate instruction on quantum mechanics.

> **Student:** *Its natural tendency is to want to be in its lowest energy state. So, there would be no reason for it to stay in the excited state for an infinitely long period of time when it has the lower one for it to go back into.*

Figure 4.5 A student using "decay" type reasoning in discussing the time dependence of an excited state in the infinite square well.

The student clearly believes the state should decay to the ground state even though there is no term present in the system's Hamiltonian to make this possible. In fact, decay was not covered in the course. The reasoning expressed in Figure 4.5 shows no indication of a mechanism, rather it is based on the initial and final states and their relative energies.

The interview shown in Figure 4.7 illustrates use of "decay" type reasoning by another student. Here the student and interviewer are discussing a system originally in a non-stationary state shown in Figure 4.6 upon which a position measurement is made.

Note that the student incorrectly associates the original state with the first excited state since both the initial state given to the student and the first excited state in the square well have probability distributions with two maxima. The student believes the system will evolve toward the ground state, thus exhibiting "decay" type reasoning.

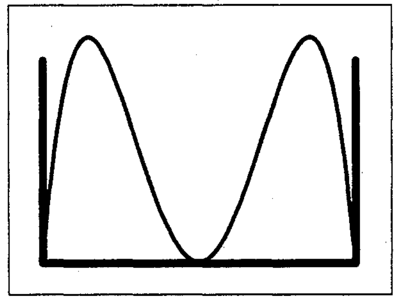


Figure 4.6 The initial state given to the student whose interview transcript is shown in Figure 4.7

In the excerpt the student uses language that makes it sound like the wave function is returning to its initial state, but when asked directly concludes that the wave function will evolve toward the ground state. For this student, in contrast to the previous student, the "decay" process is a gradual one that may have something to do with the physical act of measurement. It is interesting to note that later in this same interview this student was asked a different problem in the context of time-dependent perturbation theory and exhibited reasoning that was more consistent with wave function "revival." The categories of "revival," "diffusion," and "decay," while useful, do not necessarily divide students into mutually exclusive groups.

Summary

Several conclusions emerged from the interviews. Students were making incorrect predictions for the time evolution of wave functions. They seemed to recognize that a measurement of position tends to localize the probability distribution for a particle. They did not, however, recognize that after the measurement the wave function consists of a superposition of energy eigenstates that will remain as part of the wave function as long as the system is isolated. Students did not appear to have a mechanism upon which to base their predictions. Specifically, they were not using the formalism they were taught in class, but instead seemed to be stating from memory what

Student: *Well, I'm thinking that if it goes back I think it will go back to the ground state. ...Once the particle is here [after the position measurement] sort of stationary for a very brief time when you measure it. It will come gradually to the ground state because you stopped it. I think of a particle in an infinite square well as having its own way of moving around the well.*
Interviewer: *And how would you describe that?*
Student: *It would kind of just bounce back and forth between boundaries. If you stop it, then you don't have any thing sort of to kick it back up to the first excited state.*
Interviewer: *So there is no information about the original state? ...*
Student: *I think that it gradually goes back.*
Interviewer: *So you are saying it goes back to this [points at the original state]?*
Student: *I think it goes back to the ground state.*

Figure 4.7 Example of student reasoning based on a decay of the wave function.

they thought would happen. If pushed, sometimes students would create possible mechanisms to justify their predictions, but most neither did so on their own, nor appeared to be using these mechanisms to make their initial prediction. Students spontaneously distinguished short term and long term behavior of the wave function, and often errors in reasoning were not readily apparent until students described the long term behavior. Student responses could be roughly divided into three, sometimes overlapping categories that we term "revival," "diffusion," and "decay." These ideas were persistent. They were present after all instruction when the interviews took place.

Investigating student reasoning about the shape of wave functions as time evolves is complicated by the fact that professional physicists can correctly use the same terms to describe the behavior of wave functions that students use incorrectly. For example, atomic transitions are possible in nature. Thus, it is not incorrect to say one state decays to another. Students in introductory non-relativistic quantum mechanics, however, do not study Hamiltonians containing terms that would make such transitions possible. Students, however, still reason as if such transitions were required.

In addition, the Schrödinger equation for a free particle can be described as a diffusion equation with an imaginary constant. Some books and many instructors speak of wave packets diffusing.[1] While ideas about diffusion can be applied correctly to many situations they can be over-generalized.[2] While "diffusion" based reasoning may not be an appropriate description of a wave function after a measurement, we found that many students believe an arbitrary wave function in a potential well will always diffuse until the probability of finding the particle is equal everywhere. Others reason that it diffuses until the probability of finding the particle is zero everywhere.

In retrospect, we realized that the majority of the interviews were conducted through the use of somewhat complicated contexts. In the context of a position measurement, students must first understand that a position measurement leaves the system in a state described by a superposition of many stationary states. Then they must recognize that each state that contributes to that superposition will continue to contribute unless the system is perturbed.[3]

To address the issues elicited by the interviews, we designed written questions in simpler contexts to probe student reasoning and to measure the prevalence of incorrect ideas. In designing these questions we deliberately attempted to elicit these difficulties by using the conclusions that emerged from the interviews. The next section describes responses to these written questions. We also discuss results from other written questions that were found to elicit these difficulties.

4.1.2 Written questions to probe student conceptions in greater detail

Ideas associated with "revival," "diffusion," and "decay" were found in student responses to a wide variety of questions that were posed throughout the course. These ideas arose even in questions that were not intended to probe these conceptions. Certain questions, however, were better at eliciting these ideas than others. In what follows, a question that elicits "revival" type reasoning is described and then followed by several questions that elicit both "decay" and "diffusion" type reasoning.

Written question that elicits "revival" type reasoning

The "revival" response commonly seems to arise when students are shown a problem in which there is an initial state that is subjected to either a change in the potential or a measurement. Students tend to use these ideas especially if they are asked about the state of the system a long time later. The characteristic response is to claim that the system will slowly evolve back into its initial state.

The exam question shown in Figure 4.8 seems to elicit this sort of response even though its initial purpose was to probe student understanding of quantum mechanical formalism. Students were given an initial state on which a measurement of position is made. Among other things, they are asked if the value of the wave function at a particular position will be different a long time after the measurement from its value before the measurement. Students were expected to recognize that a measurement of position localizes the wave function. After the measurement the wave function is composed of many stationary states. Since each energy eigenstate has a different time dependence, the full wave function will continue changing with time no matter how much time elapses.

One-third of the 51 students taking this exam in Autumn quarter of 2004, reasoned that the wave function will return to its original shape. Examples of student responses are shown in Figure 4.9.

This type of reasoning permeates student responses throughout the course. It is often employed in unexpected places by only a few students at a time, but its presence seems to be relatively unaffected by instruction.

A particle is confined to a one-dimensional infinite square well of width L. At the instant shown at right the normalized wavefunction is entirely real.

Now assume a measurement of position has been made, and the particle has been found near $x_o = 0.8L$ with small uncertainty.

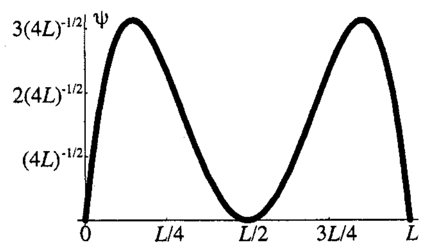


Figure 4.8 Students are given an initial wave function, told that a measurement of position has been made, and asked about the wave function immediately after and a long time after the measurement.

*The wave function will go back into its stationary state after a bit.**

Same as before the measurement, the wave function will slowly go back to original state

**[Note, the wave function was never in a stationary state.]*

Figure 4.9 Examples of student reasoning characteristic of the idea that altered states return to their initial conditions.

Written questions that elicit "decay" and "diffusion" type reasoning

In contrast to the previous question, the multi-part question discussed next was designed, specifically, to elicit student reasoning associated with "decay" and "diffusion" as well as other ideas related to time dependence.

Time evolution of a stationary state

In the Autumn quarters of 2005 and 2006, we asked 90 students in Physics 324 a multi-part question to probe their understanding of time dependence and to look for reasoning associated with "decay" and "diffusion." The first part of that question focused on their ideas about the time evolution of a single stationary state. At this point in the course the students had studied quantum mechanics in one-dimension. In addition, they had completed three homework assignments and taken their first examination.

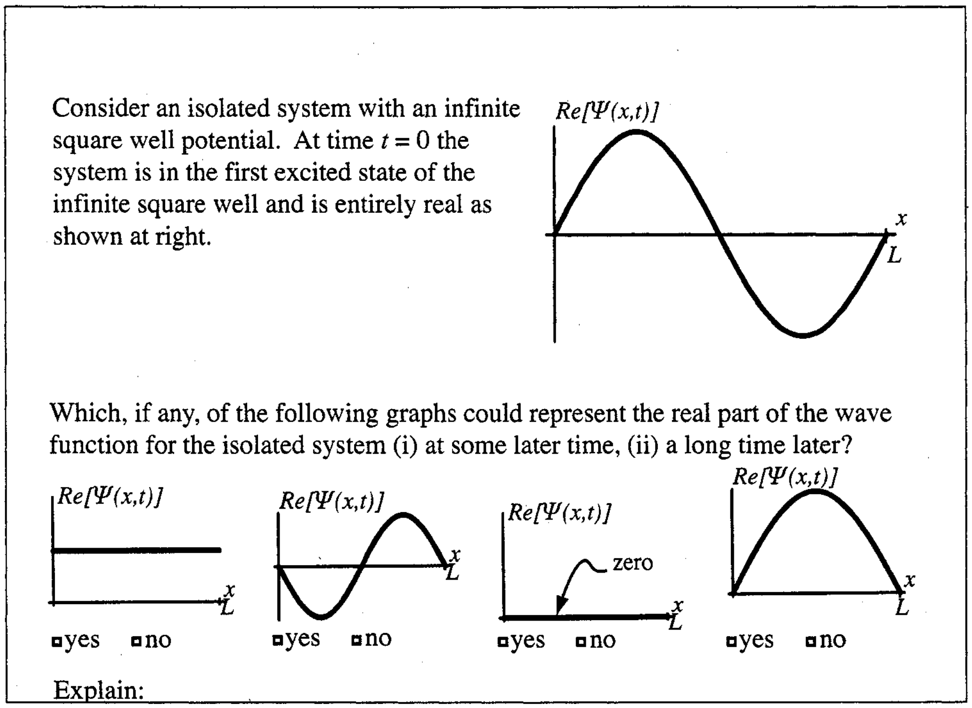
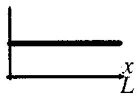
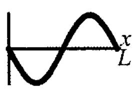
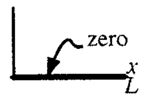


Figure 4.10 A question asked of Physics 324 students in Autumn 2005 and 2006 to probe their understanding of the time evolution of a single stationary state.

The first part of the question given to the students is shown in Figure 4.10. Students were shown a graph of the real part of a wave function versus position and told that it represents the first excited state of the infinite square well at some initial time. They were then presented with four graphs and asked which of the four could represent the real part of the wave function at (1) some later time and (2) a much later time. Students needed to recognize that the time dependence is given by $\Psi(x,t) = \Psi(x,0)e^{-iE_2t/\hbar}$ where E_2 is the energy of the first excited state. Then, they needed to deduce that only the second and third choices show possible graphs representing the real part of this wave function at any later time. The answers to parts (1) and (2) of the question are thus the same.

Table 4.1 Percentages of students selecting graphs as possible wave functions and percentage of correct responses for the question shown in Figure 4.10.

					**All correct choices (middle two)**
short time	15%	75%	30%	30%	15%
long time	25%	40%	35%	15%	5%
				Correct on both parts	5%

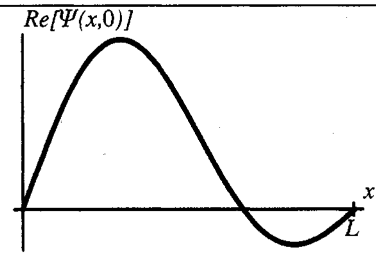
The percentage of students choosing each graph as a possibility is shown in Table 4.1. Only about 15% could correctly identify the second and third graphs as possible graphs of the real part of the wave function some time later. The percentage correct for all four graphs drops for Question (2) concerning a long time later. Only about 5% could answer both of the parts correctly even though they had received standard instruction on these topics.

These results indicate that very few students have a correct method to determine the time evolution of even the simplest of quantum mechanical states, namely a stationary state. It was evident from their responses that they were not applying the Schrödinger equation and the model being developed in their non-relativistic quantum mechanics course.

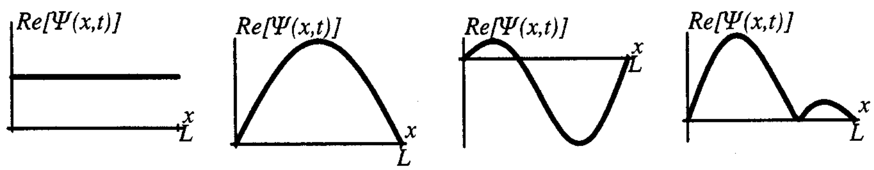
Time evolution of a simple superposition of stationary states

On a subsequent part of the same question (See Figure 4.11.) students were asked a similar question about the time dependence of a state they were told can be written as a superposition of two stationary states. In this case, students had to recognize that each stationary state has its own time dependent phase factor. Thus, there will be a time when the ground state component is entirely real and the first excited state component is entirely complex giving rise to the second graph shown. There will also be a time when the two components are out of phase by π giving rise to the third graph. The real part of the wave function, however, can never resemble the first or the fourth graph.

Consider another isolated system with an infinite square well potential. At time $t = 0$ the state of this system can be represented by the wave function shown at right. At $t = 0$, this function can be written as a linear combination of only the ground state (with energy E_1) and the first excited state (with energy E_2) of the infinite square well and is entirely real.



Which, if any, of the following graphs could represent the real part of the wave function for the isolated system (i) at some later time, (ii) a long time later?



□yes □no	□yes □no	□yes □no	□yes □no
Explain:	Explain:	Explain:	Explain:

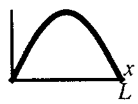
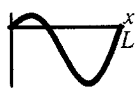
Figure 4.11 A question asked of Physics 324 students in Autumn 2005 and 2006 to probe their understanding of the time evolution of a superposition of stationary states.

As shown in Table 4.2 only about 10% of students could correctly answer this question. As in the analogous questions about a stationary state discussed previously, the performance was worse on the part of this question concerning the long term behavior of the wave function.

Summary of both questions

Overall 40% of students chose the ground state as a possibility in some part of their response for the part of the pretest in which the initial state was the

Table 4.2 Percentages of students selecting graphs as possible wave functions, and percentage of correct responses for the question shown in Figure 4.11.

					**All correct choices (middle two)**
short time	20%	50%	65%	20%	20%
long time	30%	20%	30%	10%	10%
				Correct on both parts	10%

first-excited state. Also 40% of student choose a non-zero flat line at some point in the entire pretest. This response, however, is incorrect for all parts of the pretest.

An analysis of student reasoning yields greater insight than the raw statistics. Surprisingly a full two-thirds of the students used incorrect reasoning based on the diffusion or decay of wave functions at some point in their answers to both questions.

In some cases, student reasoning was very explicit as is the case for the first two students whose reasoning is shown in Figure 4.12. Clearly both of these students are envisioning a decay type process. The first student's description appears remarkably similar to what students may have learned about atomic transitions in their introductory or modern physics courses. Here, however, these results have been over-generalized.

In many cases it was difficult to distinguish student reasoning based on diffusion from that based on decay. Some students used both as a basis for their choices. In fact, the second student whose response is shown in Figure 4.12 went on to reason that a long time later the wave function would diffuse to the constant non-zero line. The third student's reasoning is also somewhat surprising. The language of decay is used to support an evolution toward a non-zero flat line. Thus, it is clear that these categories are neither mutually exclusive nor easily identified by student answers without explanations.

It could drop from a p orbital to an s orbital. Probably gives off a photon. [selects the ground state]

because it could stay in the first state or it could drop to the ground state [selects the ground and first excited states]

Only the first [the non-zero flat line], decay over a long time gives a constant function.

Figure 4.12 Examples of student reasoning characteristic of the idea that arbitrary wave functions decay.

Some lines of reasoning related to diffusion that led to incorrect predictions are illustrated in Figure 4.13. Notice both students seem to be envisioning a continuous process by which the wave function changes from relatively localized to completely de-localized. There is no hint that this process will cycle back and forth or do anything other than remain completely de-localized once it reaches that condition.

The system would spread as time went on, so depending on how much time had passed, the system could spread by different amounts. Because the system would spread so much for a long time period, it would appear like a line.

As t goes the function will spread out between 0 and L. [After a long time] completely spread out [selects a flat, non-zero, line]

Figure 4.13 Examples of student reasoning characteristic of the idea that arbitrary wave functions diffuse.

Other students give more specific reasons for this diffusion that hint at a relationship to mathematical difficulties investigated in other studies.[4] The first two of the quotes shown in Figure 4.14 are from students who indicated that the real part of a wave function will evolve toward zero after a long time has elapsed. The third is from a student who indicated that the real part of a wave function will evolve toward a non-zero horizontal line.

> *Since $\psi \sim e^{\alpha t}$, as t grows the function should spread, so iii [the horizontal line at zero] is ψ as $t \to \infty$.*
>
> $e^{-iE_1 t/\hbar}$ *[as] t grows large* $\psi \to 0$
>
> *wave diffuses according to* e^{-ik}

Figure 4.14 Mathematical justification for the diffusion of wave functions.

It might be said that the first student applied the wrong mathematics to the physical situation, but correctly interpreted the mathematics; while the second two students applied the correct mathematics to the physical situation, but incorrectly interpreted the mathematics. It is important to note that none of these students are using the idea of diffusion to characterize the correct behavior of a wave function evolving in time as a physicist might.

Summary

There are many indications of reasoning based on "decay," "diffusion," or "revival" on written questions in other contexts. This idea also comes up in strongly in classroom interactions with students. The prevalence and persistence of these lines of reasoning suggest that many students believe that the wave function changes with time in a way that is not predicted by the Schrödinger equation. Many seek out various justifications for this conclusion rather than deducing the behavior of a system from the Schrödinger equation.

In addition to our own work, other researchers[5] have commented on student use of decay in their explanations in interviews and on written questions. Further explorations of this type of error have not, however, been previously reported.

This section describes ideas that students bring with them to a quantum mechanics course that affect their ability to reason about wave functions as time evolves. We have found that other pre-conceptions, which are neither associated with the shape of the wave function nor with the typical quantum mechanical formalism, can affect student reasoning about time dependence. One such example is discussed in the next section.

4.2 Reasoning Consistent with the Gambler's Fallacy

During interviews and during instruction on measurement and time dependence, some students seemed to be using reasoning commonly termed the gambler's fallacy. The gambler's fallacy is a widely known example of the "representativeness heuristic" identified by Tversky and Kahneman.[6] Simply put, it states that individuals act as if they believe that a particular possible event in a random process becomes more likely when it has not occurred in several trials. This contradicts the independence of events assumed to be random and uncorrelated. In physics this is especially important. Physical events with distinguishable particles are independent only in the absence of interactions.

We have initially noted this tendency during interviews. For example, some students were told to imagine an ensemble of identical systems each in a quantum mechanical superposition of two eigenstates. When asked about energy measurements made on such an ensemble, some reasoned that the probability of measuring the energy corresponding to the first state would depend on the number of systems in which that particular result had already been obtained. Namely, the probability of obtaining that result would be smaller if that energy had been obtained from a measurement on many other systems. Some of these students seemed to be associating this decrease or increase of probability with quantum mechanical time dependence.

We interpret the error above as, yet another, naïve mechanism for quantum mechanical time dependence (*i.e.,* a preconceived notion about time dependence not directly related to the formalism taught in class). After observing this tendency in a small number of students, we were interested in probing its prevalence. We developed and administered two questions for this purpose.

4.2.1 Student understanding of the statistical interpretation of superpositions

Probability and statistics are notoriously difficult subjects for students at all levels.[7] An extensive literature describing these difficulties exists both in mathematics education and in cognitive psychology. A number of the studies have direct bearing on our research. Shown below, are portions of one such published study followed by related questions that we have asked in quantum mechanics classes.

Question asked of psychology students in context of SAT scores

> The average SAT for all the high school students in a large school district is known to be 400. You have randomly picked 10 students for a study in educational achievement. The first student you picked had an SAT of 250.
>
> What do you expect the average SAT to be for the entire sample of 10?
>
> What do you expect the average SAT to be for the next 9 students, not including the 250?

Figure 4.15 Question asked of 205 undergraduate psychology students at the University of Massachusetts

The questions shown in Figure 4.15 were asked of 205 undergraduate psychology students at the University of Massachusetts.[8] In these two questions students are told that the average SAT score for a large population is 400. They are told that the first score in a randomly selected sample of 10 is 250.

Then, they are asked for the expected average of the entire sample of ten and for the expected average for the remaining nine.

To answer this question, one must recognize that the expected value for any single measurement is the population average. Since one data point in the sample is known to be less than the population average, then the expected sample average is less than the population average. The expected score for the remaining 9 students is the average SAT score for the entire district.

About 20% of students taking introductory psychology at the University of Massachusetts at Amherst answered correctly as shown in Table 4.3. About one-third said that the average in each case was 400 despite having known about the first measurement in the sample. This was interpreted as the "representative response" because students could have been reasoning that the sample, no matter how small, had to be representative of the population. Tversky and Kahneman label this a belief in a law of large numbers.

Another 10% of the students suggested that the mean for the sample of ten was 400, but the mean for nine would be greater than 400. This was interpreted to mean that the students expected the mean for ten to come out to 400 and needed to balance some values above 400 with the value of 250. A student exhibiting the "balancing" response again expects the whole sample to represent the population. In order to make this happen, they require the set of nine to be greater than the mean to balance the first value that was below the mean.

The researchers in this study also found what they labeled a "trend" response in which students predicted the mean of ten scores to be lower than the mean of nine scores which was itself below the mean. Thus the students see a downward trend. About 10% of students exhibited this response.

The researchers followed this study up with 31 interviews using both this question in the context of an SAT test and another in the context of an IQ test. During some of the interviews students were presented with the alternative ways of reasoning used by other students. Interestingly, no student changed to the correct answer. There was, however, some shift to the balancing idea, and the trend response became less prevalent.

Table 4.3 The results of the study conducted by Pollatsek *et al.*

Label	**Mean of 10 scores**	**Mean of 9 scores**	**Written** question	**Interviews**
Correct	< 400	= 400	44 (21%)	6 (19%)
Representative	= 400	= 400	68 (33%)	15 (48%)
Balancing	= 400	> 400	25 (12%)	6 (19%)
Trend	< 400	< 400	18 (9%)	2 (6%)
Unclassified			50 (24%)	2 (6%)
Totals			205	31

Question asked of quantum mechanics students in the context of SAT scores

We asked the first part of the question in the SAT context described above to 50 quantum mechanics students on their second day of class. Only about 40% percent chose the correct response (that the average for the entire sample of 10 students is expected to be less than 400). Over half (55%) answered 400. The students treated the subsequent measurements as depending on the results of the first consistent with the "balancing" response. Eight percent stated that the answer would be greater than 400.

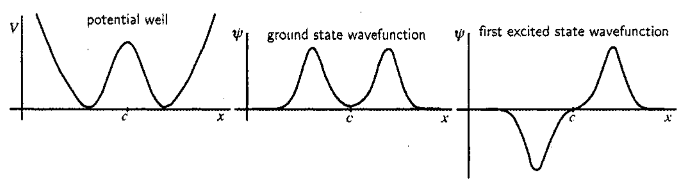
We have also asked thirteen in-service teachers both parts of the question. Six of those teachers or 45% clearly exhibited the "balancing" response.

Clearly, the students in the quantum mechanics course have many of the same problems as students in the University of Massachusetts study.[9] We were interested in seeing how these ideas played out a quantum mechanical context.

Similar questions asked in a quantum mechanical context

We asked two questions similar to the SAT question but in the context of a quantum mechanical description of a nitrogen atom in an ammonia molecule. The questions are shown in Figure 4.16.

The potential for a nitrogen atom in an ammonia molecule (NH_3) is shown below. The ground state and the first excited state wave functions are also shown.



Consider a large ensemble of identical systems s_1, s_2, s_3, s_4, etc., each in the ground state of this potential. Assume the position of the nitrogen atom in the first system has been measured and found to be to the left of $x = c$.
Suppose you now measure with small uncertainty the position x_2, x_3, x_4, etc. of the nitrogen atom in each of the next nine systems.

Would you expect the average of the values for all ten systems (i.e., the average of x_1–x_{10}) to be greater than, less than, or equal to $x = c$? Explain.

Would you expect the average of the values for the position of the nitrogen atom in only systems s_2–s_{10} to be greater than, less than, or equal to $x = c$? Explain.

Figure 4.16 A question concerning the nitrogen atom in an ammonia molecule asked of students in Physics 324 in Autumn of 2004 on their second exam.

Table 4.4 The results of the our study similar to Pollatsek *et al.*, but in a quantum mechanical context.

Label	Mean of 10 measurements	Mean of 9 measurements	
Correct	$x < c$	$x = c$	11 (23%)
Representative	$x = c$	$x = c$	22 (47%)
Balancing	$x = c$	$x > c$	13 (28%)
Trend	$x < c$	$x < c$	0 (0%)
Unclassified			1 (2%)
Totals			47

Students should recognize that the events are independent. Thus, the expected value of the unknown measurements is $x = c$. Since the first measurement is known to be to the left of c, the average of all ten should be less than c.

Initially we did not expect our upper division quantum mechanics students to exhibit these difficulties in as great a proportion as the introductory psychology students. The results, shown in Table 4.4, are surprisingly similar to the results obtained from the introductory psychology class especially when one considers the interview data. Thus, an assumption about the absence of common statistical misconceptions in populations of quantum mechanics students is unwarranted.

4.3 Summary

This chapter presented several lines of reasoning that students use to support incorrect predictions about the time evolution of wave functions. The first set of reasoning difficulties have proven to be particularly resistant to both standard and research based instruction. These we termed "revival," "decay," and "diffusion." We also explored a statistical difficulty known as the gambler's fallacy that some students may see as a mechanism for time dependence. Classroom observations suggest that some students look for mechanisms in quantum mechanics to support these pre-conceptions, thus hindering their ability to learn the accepted theory. We have presented evidence that these ideas, which students seem to bring with them to the classroom, are often largely unaffected by instruction in a standard quantum mechanics sequence.

Note that while most of the examples that have been presented have come from the context of the infinite square well, the types of student reasoning that have been discussed are not confined to this context. The square well, however, is a simple context with which the students become quite familiar. Thus, framing questions in the context of the square well eliminates some of the difficulties associated with the complexity of other potentials.

In this chapter, we have described certain naïve or pre-conceived ideas about time dependence in quantum mechanics. All of the patterns of reasoning described have been grouped in this chapter because they lie outside the normal content of a quantum mechanics course. In contrast to this chapter, the remainder of this part of the dissertation will focus on problems related to time-dependence that students have with the specifics of the model of quantum mechanics that is developed in the course. In addition, curriculum designed to address difficulties identified by research will be described, and preliminary assessment of that curriculum will be discussed.

4.4 Notes to Chapter 4

1 It should also be noted that "quantum state diffusion," as discussed in the literature, "is stochastic motion of an individual quantum system, in contrast to the determinism of Schrödinger dynamics." See (Percival, 1998) for an extensive discussion.

2 See (Mentrup, 2003) for examples of systems where wave functions cannot be said to diffuse.

3 See Section 7.3 for a more detailed description of student reasoning concerning the stationary states that make up a superposition.

4 See (Sadaghiani, 2005) for a discussion of student understanding of complex exponentials. In particular, some students ascribe the behavior of a decaying exponential function to a complex exponential function.

5 See (Singh, 2006) for a comment about student explanations based on decay.

6 See (Tversky, 1971).

7 See (Shaughnessy, 1992) and (Kahneman, 1972) for examples of research on student understanding of probability and statistics.

8 See (Pollatsek et al., 1984) for the study using this particular version of the questions.

9 Evidence from a study at the University of Maryland (Bao, 2002) indicates that about two-thirds of 18 students taking a quantum mechanics course for engineers answered similar, but not identical, questions about classical probability in a manner consistent with the gambler's fallacy.

Chapter 5

Identification of Difficulties Related to a Confusion between Stationary States and Superpositions of Stationary States

During this investigation, we found that on a wide variety of questions, students make errors that seem to be related to a confusion between the properties of stationary states and states that consist of a superposition of stationary states. For example, some students explicitly treat energy eigenstates as if they are the only valid states for a system. For others, this confusion is indirectly reflected in a tendency to treat all states as having certain features that are specific only to energy eigenstates. The converse is also true. In many cases, this confusion seems to contribute to a failure of students to recognize how the time dependence of general states can differ from that of stationary states. In this chapter, we illustrate some of the different ways in which students seem to regard general superpositions of states as having certain properties associated only with stationary states and vice versa.

5.1 Research by Other Investigators

Experienced teachers of quantum mechanics have likely observed that many students believe that energy eigenstates are the only allowed states. Daniel Styer at Oberlin College postulates several possible causes including: (1) a similarity of this conception to the correct statement that energy eigenvalues are the only possible energies, and (2) an overemphasis of the energy eigenvalue equation, $H\Psi = E\Psi$, (often called the Schrödinger equation in introductory presentations) over the Schrödinger equation, $H\Psi = i\hbar\frac{\partial\Psi}{\partial t}$.[1]

As part of a study at The Ohio State University by Homeyra Sadaghiani, 35 students were asked directly whether $\Psi(x, t = 0) = c_1\psi_1(x) + c_2\psi_2(x)$ is an eigenstate.[2] More than 40% of the students taking quantum mechanics in the fall of 2004 argued incorrectly that this was an energy eigenstate. Only about one-third of the students used correct reasoning to conclude that it was not an energy eigenstate.

Chandralekha Singh at the University of Pittsburgh found that undergraduate students in quantum mechanics courses at six different universities believed that an eigenstate of any operator is a stationary state.[3] A study conducted among beginning graduate students at several universities found that about one-third of the students associated a single time-dependent exponential with every state.[4] For example, they seemed to believe that the time dependence of a system initially in the state $\Psi(x, 0) = a\psi_1 + b\psi_2$, is given by $\Psi(x, t) = \Psi(x, 0)e^{-iEt/\hbar}$. They were thus treating superpositions of stationary states as having a time dependence associated with a single stationary state. Both the belief that all eigenstates are stationary and the belief that the time dependence of any state is given by a single time dependent phase are consistent with a confusion between stationary states are superpositions of stationary states.

The findings from our research will be shown to corroborate the results above. We will also show that the difficulties identified by other researchers are present among quantum mechanics students at the University of Washington. In addition, we have observed similar reasoning in new contexts. We have grouped our findings under headings that describe some common student difficulties.

5.2 Belief That Stationary States Are the Only Valid States

Typical instruction in introductory quantum mechanics focuses on stationary states. Results from the research discussed above, however, suggested the possibility that students may believe that stationary states are the only valid states. A pretest question was designed to explore this issue. Two versions of the question were developed for the junior-level quantum mechanics course at the University of Washington (Physics 324). One version was administered in Autumn quarter of 2003 and the other in Autumn quarter of 2004. The differences between the two questions were minor in format and wording. In both classes, students had received lecture instruction on the relevant topics, but they had not yet participated in any tutorials on time dependence. The basic question given to both classes is shown in Figure 5.1. (It should be noted that two-quarters of mathematical physics is a prerequisite for Physics 324, and this course includes Fourier analysis.) In that question students were shown a graphical representation of a superposition of stationary states and asked whether it was a valid state and if it was associated with a single energy.

Table 5.1 shows the percentage of students who correctly identified the wave function as valid, the percentage of students who correctly concluded that the wave function was not associated with a single energy, and the percentage of students who answered both parts correctly without using incorrect reasoning.[5] About 15% of the students who took this pretest did not recognize that the function in Figure 5.1 is a valid wave function. Only half of

A particle is in a quantum mechanical infinite square well. (1) Could the graph at right represent a possible wave function for the particle at some instant in time? (2) If so, is it associated with a single energy? Explain.

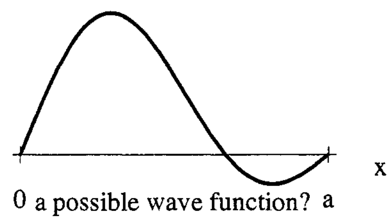


Figure 5.1 A question to probe whether students recognize the validity of a non-stationary state.

Table 5.1 Percentage of Physics 324 students in Autumn 2003 and 2004 who could (1) identify the superposition of stationary states in Figure 5.1 as a valid state, and (2) recognize that such a state cannot be associated with a single energy. The final column indicates students who could do both using no incorrect reasoning.

Context	**(1) Correctly identified valid wave function**	**(2) Correct not single energy**	**Correct on both no incorrect reasoning**
Autumn 2003 (N = 40)	80% (31)	45% (18)	40% (15)
Autumn 2004 (N = 31)	95% (29)	75% (23)	70% (21)
Total (N = 71)	85% (60)	60% (41)	50% (36)

the students who took the pretest, however, correctly identified the function as a valid wave function not associated with a single energy without using incorrect reasoning. Note that this analysis may overestimate student understanding, since some of the responses that were tabulated as correct with no incorrect reasoning had cursory or non-existent explanations.

The explanations that students did use to support both correct and incorrect answers, however, provided us with insight into their reasoning about general states. Examples related to various student difficulties will be presented below.

5.2.1 Belief that all states have a definite energy

As shown in Table 5.1, many of the students who answered the question in Figure 5.1 incorrectly thought that the wave function was associated with a single energy. About 30% gave answers consistent with every state being associated with a single energy. Figure 5.2 shows an example of one such student's reasoning.

> *It could represent a particle, it is sinusoidal in nature, & 0 at both ends of the well, even though it is not symmetric. I would associate it with the 2nd energy state because it has 1 full cycle. It is therefore 1 above the ground state.*

Figure 5.2 Example of student reasoning indicating that the wave function in Figure 5.1 is associated with a particular energy.

This student uses the shape of the wave function and the fact that it has only one zero crossing at the center to conclude that it is associated with the first excited state above the ground state. This student arrives at the correct answer for the first part and considers the fact that the function satisfies the boundary conditions. The comment about discrete eigenvalues, however, indicates that thinking about the energy eigenvalues and possibly the equation, $H\psi(x) = E\psi(x)$, forms part of the student's reasoning.

5.2.2 Belief that variations in the shape of any wave function must be reflected in variations in the potential

Another way in which students tend to confuse the properties of stationary and non-stationary states also presented itself in response to the question shown in Figure 5.1. Some students seem to believe that the wave function has a well-defined energy in different regions of space. An example of a student expressing this type of reasoning is shown in Figure 5.3.

No [it is not a valid wave function], because if the amplitudes are different then the potential must be different.

Figure 5.3 Example of student reasoning influenced by the energy eigenvalue problem on a question concerning the validity of a wave function.

This student reasons that a variation in the amplitude of a wave function can only be due to a variation in the potential. This type of reasoning is consistent with a failure to accept an indefinite energy. Instead, these students posit a definite energy for different parts of the well, reasoning locally in a way reminiscent of classical mechanics, but perhaps confusing potential, kinetic, and total energy. It should be noted that using the shape of the potential to reason about the allowed energy eigenstates is a valuable skill, and student difficulties associated with that task are discussed in Chapter 11. This is, then, another example where reasoning that is useful in dealing with energy eigenstates is incorrectly over-generalized and applied to non-stationary states.[6]

5.3 Tendency to Treat All States as Having a Single Time-dependent Phase

As discussed in Chapter 3, other researchers as well as ourselves have noticed a tendency of some students to treat all wave functions including superpositions of stationary states, as having a single time-dependent phase. This is an over-generalization on the part of students as this property is true only of stationary states. This tendency is related to the more general incorrect idea that energy eigenstates are the only valid quantum mechanical states.

Such reasoning can be elicited when students are asked to give, mathematically, the time dependence of a superposition of stationary states. An example from an exam question given on the final exam for Physics 315 in Winter of 2004 (shown in Figure 5.4) illustrates this error.

The system is a particle in a one-dimensional potential well. At $t = 0$ the wave function is:

$$\psi(x) = \frac{1}{2}u_o(x) + \frac{\sqrt{3}}{2}u_1(x)$$

where u_o and u_1 are the wave functions for the two lowest energy states with energies E_o and E_1. u_o and u_1 are real and orthonormal. Write $\Psi(x,t)$ in terms of u_o and u_1.

Figure 5.4 An example of the type of question that elicits a response with a single time-dependent exponential.

A common student response to this question is shown in Figure 5.5. Notice that the student only writes a single energy, which is consistent with the idea that all states are energy eigenstates.

$$\Psi(x,t) = [\frac{1}{2}u_o(x) + \frac{\sqrt{3}}{2}u_1(x)]e^{-iEt/\hbar}$$

Figure 5.5 An example of the type of question that elicits a response with a single time-dependent exponential.

Student responses to such questions are further explored in Chapter 9 in which the prevalence of incorrect responses among different populations are compared.

The preceding mathematical response has a close graphical analog in some responses to the problem described in Figure 5.1. Imagine that the wave function is plotted on three orthogonal axes with the real part, the imaginary part, and the spatial dimension each on an axis. Incorrectly associating a single time dependent phase with every wave function amounts to rotating the entire wave function around the spatial axes. Figure 5.6 illustrates such an incorrect procedure. Some student responses such as the one shown in Figure 5.7 reflect such incorrect graphical reasoning. We have also observed this tendency during instruction. The tutorial that students work through later guides students to reason correctly about the time dependence of wave functions using similar graphical representations.

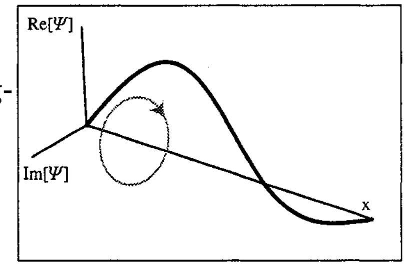


Figure 5.6 Associating a single time dependent phase with every state corresponds graphically to rotating every wave function about the spatial axes into and out of the real and imaginary planes.

> *Yes [it is valid], think of it as a frozen jump rope around the x-axis.*

Figure 5.7 Example of student reasoning graphically that all states have definite energies

5.4 Belief That Expectation Values Are Always Time-independent

Another way in which students seem to confuse the properties of stationary and non-stationary states is related to the time dependence of expectation values. For a stationary state, all expectation values are independent of time. For a superposition of stationary states, on the other hand, expectation values are generally time dependent. If students believe that the time dependence of all states is in the form of a single time dependent exponential or that a sum of stationary states is stationary, they are likely to conclude that all expectation values will be time independent.

Two questions that probe student beliefs about the time dependence of expectation values are given below. In one, the student is given a function of x that is not directly represented as a sum of stationary states. In the other question, students are shown a state that is explicitly a superposition of stationary states.

The first question is shown in Figure 5.8. This question was administered in Autumn quarter of 2005 to students in Physics 324 on their first exam. Twelve students who took Physics 324 in the summer of 2003 were also asked a similar question in which students also had to recognize the initial state as a superposition of stationary states.

A particle of mass, *m*, is placed in an infinite square well potential:

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 \le x \le L \\ \infty & x > L \end{cases}$$

The particle is prepared into an initial state (at time $t = 0$) given by:

$$\Psi(x,0) = \begin{cases} 0 & x < 0 \\ Nx & 0 \le x \le L \\ 0 & x > L \end{cases}$$

Will $\langle x \rangle$ change with time? Explain.

Figure 5.8 A question about expectation values that change with time. This question was asked of Physics 324 students on their first exam in Autumn of 2005. Students must deduce that the function given can be written as a superposition of stationary states.

To answer the question shown in Figure 5.8 correctly, students need to recognize that the wave function given is not a stationary state for the infinite square well potential described, but must be a superposition of stationary states.

Questions were also asked in contexts in which the superposition of stationary states was explicit. Such a question was administered in Autumn quarter of 2006 to students in Physics 324 on their first exam and shown

A particle of mass, m, is moving in a harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2x^2$. At $t = 0$ the system is prepared in an initial state

$$\Psi(x,0) = \frac{1}{\sqrt{2}}(\psi_1(x) + \psi_2(x))$$

Are $< x >$ and $< p >$ evolving with time? Explain.

Figure 5.9 A question about expectation values that change with time. This question was asked of Physics 324 students on their first exam in Autumn of 2006. Students are given the initial wave function explicitly as a superposition of stationary states.

in Figure 5.9. The question shown in Figure 5.9 gives the initial state as a superposition of stationary states, therefore students need not deduce that information. Otherwise, the reasoning required is similar to that required in the previous question.

Since both systems involve superpositions of stationary states, students should conclude that the expectation value for either position or momentum will, in general, change with time.

Table 5.2 Student performance on questions concerning the time dependence of expectation values.

Question	**N**	**expectation value is time-independent (incorrect)**
Figure 5.8 **Su 2003, Au 2005**	70	60% (41)
Figure 5.9 **Au 2006**	46	45% (20)

The results in Table 5.2 indicate that many students were unable to answer correctly. Of the 70 students who answered the question shown in Figure 5.8, which required that students first determine that the initial state was not stationary, 60% (41) incorrectly said that the expectation value did not depend on time. Of the 46 students who answered the question shown in Figure 5.8, 45% (20) incorrectly answered that the expectation value was time independent. Students made these errors despite having listened to lectures and completed homework on the relevant topics. Several common incorrect reasoning patterns emerged from the student responses.

Reasoning based on the use of naïve mechanisms for time dependence

On both the question shown in Figure 5.8 and the question shown in Figure 5.9, some students correctly answered that the expectation value of the position changes with time but used questionable reasoning. The reasoning that many students used to support correct answers was often difficult categorize as correct or incorrect. For example, the response shown in Figure 5.10 is, for the most part, correct. The student, however, is answering a question involving the superposition of two stationary states. Hence, the comment about the wave function becoming more dispersed potentially represents incorrect reasoning.

> *Yes, we know that only the energy of a particle is invariant in time. So, $< x >$ and $< p >$ must be evolving by the wave becoming more dispersed.*

Figure 5.10 Questionable reasoning supporting a correct response.

Other students gave reasoning that was clearly incorrect. Some of those students explicitly stated that the wave function would eventually become flat even though the question did not ask for the long-term time dependence. Frequently students seemed to be using one of the naïve mechanisms discussed in Chapter 4 (*e.g., diffusion, decay, or revival*).

Reasoning based on a belief that every state is associated with a single time dependent phase

No, [< x > will not change with time] because $\rho(x) = |\psi(x)|^2$ is time independent and only certain stationary states are allowed, which are time independent.

No, expectation values are time independent because the $e^{-iE_nt/\hbar}$ of $\psi(x,t) = \phi(x)e^{-iE_nt/\hbar}$ cancels during the $\psi^*(x,t)\psi(x,t)$ step.

No, because the wave function in an infinite square well is composed of stationary states which do not change with time. $< x(t) >= \int \psi^*(x,t)\hat{x}\psi(x,t)dx$, but since we are dealing with stationary states this always returns the expectation value for $< x >$, same as $\psi(x,0)$.

$< x >$ will not change with time because this is a stationary state and therefore not time-dependent. Since $< x >= \int \psi^*\psi dx$, the complex time exponential drops out when you multiply the wave function by its complex conjugate.

Figure 5.11 Examples of student reasoning concerning expectation values.

Of the 35 students who clearly explained their incorrect responses to the question shown in Figure 5.8, 60% (20) claimed that the complex exponential factor dropped out when calculating the square of the absolute value of the wave function. These students often wrote explanations consistent with the tendency to treat all wave functions as having a single time dependent phase. Some of these students went on to say that all probabilities or expectation values are always time-independent in quantum mechanics. These responses are consistent with associating a single energy with any wave function as described in Section 5.3. Nine students, or 25% of those with clear explanations, indicated that all states in the infinite square well are stationary. Illustrative student responses to this question are shown in Figure 5.11.

These same patterns of reasoning were also common in student responses to the question shown in Figure 5.9. Another way in which students justified their answers is given below.

5.5 Belief That a Sum of Stationary States Is Again Stationary

Some of the students who answered the question shown in Figure 5.8 indicated as part of their explanation that the sums of stationary states are stationary. For most responses in which students incorrectly claimed that the expectation value was time independent, it was not clear whether they believed that the initial wave function could be written as a superposition of stationary states or not. The question shown in Figure 5.9, however, clearly showed an initial state that is explicitly a superposition of stationary states. Thus, the 45% (20) of students who claimed that the expectation value did not depend on time on that question were doing so for a state they knew to be a superposition of stationary states. The reasoning given by many students was also consistent with a belief that a sum of stationary states is again stationary. An example is shown Figure 5.12.

> *No, because $\phi(x,t)$ is a superposition of stationary states, so the time dependence drops out for any expectation values.*

Figure 5.12 Example of student reasoning supporting the belief that a sum of stationary states is a stationary state.

5.6 Failure to Distinguish Mixed States from Superpositions

An additional difficulty students have in distinguishing between stationary states and superpositions of stationary states is differentiating between their interpretations. Many students have difficulty interpreting what is meant by a superposition of stationary states. In particular, some students seem to think of the superposition as reflecting a situation that more closely resembles a mixed state. Specifically, they view the square of the coefficients in each term of the superposition as reflecting the probability that the system actually is in that stationary state state. To them, the superposition, as a whole, reflects a lack of information about which stationary state actually describes the system.

Initially, this difficulty was noted during instruction, but it has also appeared in interviews and written questions. The interpretation of superpositions as being mixed states allows students to work with superpositions of stationary states while maintaining that energy eigenstates are the only allowed states. A superposition that is interpreted as an expression of ignorance as to which energy eigenstate the system is *really* in has a number of consequences. For example, all expectation values would be time independent. This distinction between mixed states and superpositions is important because it is at the heart of what makes classical and quantum mechanics different.[7]

After observing students who confused mixed states and superpositions during instruction, we conducted interviews to gain more insight into this problem. Figure 5.13 shows a dialogue between a student and an interviewer discussing a linear superposition of two of the stationary states in the infinite square well. Here the discussion is about the wave function, $\psi(x) = \sqrt{\frac{1}{4}}\phi(x)_1 + \sqrt{\frac{3}{4}}\phi(x)_2$, where $\phi(x)_1$ is the ground state of a particle in an infinite square well, and $\phi(x)_2$ is the first excited state. The interviewer has explicitly raised the issue of whether the state should be interpreted as a mixed state with a lack of knowledge of which state the system really occupies. The student agrees with this description and provides evidence to support it.

The idea that the square of the coefficient of a stationary state in a superposition of stationary states gives the probability that the system is actually in that stationary state seems to be tentative when probed directly. Many students seem to recall the correct response. Such a dialogue is illustrated in Figure 5.14.

Observations of students during interviews and in class indicate that many students have memorized the fact that a superposition is not simply a way

Interviewer: *Could it be that what this means is that I don't know what state it is in yet it really is in either the ground state or the first excited state, and I just don't which and one-fourth is the probability that it is actually in the ground state and three-quarters is the probability that it is actually in the first-excited state. I simply don't have knowledge of which one it is in? Is that a possible explanation for what is going on with this superposition?*
Student: *Yes, that is how I would explain that when you measure the energy you get one or the other because wouldn't you only be able to measure E_1 if it was in [that state]. If it had that energy, it would have to be in the state with that energy. Only the ground state can have the ground state energy so it would have to be in that state. I think. Yeah.*
Interviewer: *If I threw a coin up in the air and the coin lands on the table and I cover it up, then I know that either the coin is heads or the coin is tails. I don't know which one it is. Maybe I have weighted the coin so it is one-fourth probability that it is heads and three-fourths probability that it is tails. I know it is either heads or tails ...but I don't know what it is until I look. So are you saying that this wave function is expressing the same kind of uncertainty.*
Student: *The situation that we have set up is that it can be in either the ground or first excited state. Unless we make a measurement of the energy we don't know which state it is in. So, we draw a wave like this to represent the entire possibility of where it might be or what kind of state it might be in.*

Figure 5.13 Example of a student attributing the properties of a mixed state to a superposition.

Student: *[the coefficients] might tell me if I'm more likely to be in $\phi_1(x)$. If they are the same, I think I'm as likely to be in one of those states as the other.*
...[Later]
Interviewer: *...Do you mean that if I have something like this [a superposition], then it is really in one [state] or the other?*
Student: *No, I guess I don't believe that now that you have stated it that way. ...I don't really know what [the superposition] is.*

Figure 5.14 Example showing a student responding with the correct answer when probed directly. Again, the discussion surround a superposition of two stationary states.

of expressing lack of knowledge. They are, however, often unable to apply this idea in context.

We have also seen confusion between superpositions and mixed states in written questions and during instruction. In most written questions, it is difficult to distinguish this problem from other incorrect responses. A question asked of Physics 315 students in the Winter quarter of 2004, however, probed this idea directly. Figure 5.15 shows the question asked of these students after all instruction. In the context of a spin $\frac{1}{2}$ system, students are asked whether a specific mixture in which two-thirds of the particles are spin-up and one-third of the particles are spin-down is the same as a superposition given by: $\frac{1}{\sqrt{3}}(\sqrt{2}, 1)$ in the basis of the z component of the spin. If they do not believe they are identical they must propose an experiment to distinguish between the two.

To answer correctly, students should argue that the mixture and the superposition of states are different. They should recognize that a Stern-Gerlach device not oriented in the $\hat{z}$ direction could distinguish between the two cases. In the case of the mixture, a Stern-Gerlach device oriented in the $\hat{x}$ direction would have a 50% probability of measuring a particle as having spin-up in the $\hat{x}$ direction regardless if the particle originally had spin-up or spin-down in the $\hat{z}$ direction. If, on the other hand, all of the particles are described by the given superposition, that superposition can be written in the basis of the x component of the spin. After this has been done, it can be calculated that a Stern-Gerlach device oriented in the $\hat{x}$ direction would have a 97% probability of measuring a particle as having spin-up in the $\hat{x}$ direction and only a 3% chance of measuring spin-down.

As shown in Table 5.3, only slightly more than half of the students argued that they were not identical and only two-fifths proposed a good experiment to distinguish between the two. Thus, almost half the class did not distinguish between a mixed state and a superposition.

A beam of spin $\frac{1}{2}$ particles is a **mixture** with 2/3 of the particles having spin up and 1/3 spin down.
Another beam of particles is described by:

$$\chi_o = \frac{1}{\sqrt{3}}\begin{pmatrix}\sqrt{2}\\ 1\end{pmatrix}$$

Are the two beams identical? If so, give a convincing argument. If not, propose an experiment with a Stern-Gerlach devices with one blocked exit to distinguish between them.

Figure 5.15 A question asked of students in Physics 315 on their final exam in Winter quarter of 2004.

Table 5.3 Physics 315 student performance on a question requiring students to differentiate mixtures and superpositions.

Answers	percentage
No, they are not identical	55% (13)
Yes, they are identical	45% (11)
describes a correct procedure	40% (9)

5.7 Belief That Superpositions of Degenerate States Are Not Stationary

Often students who are able to solve problems concerning the time evolution of one-dimensional systems in quantum mechanics, have difficulties with multi-dimensional systems in which the energy eigenvalues are degenerate. In such systems, a superposition of stationary states could again be a stationary state provided the states that comprise the superposition are degenerate. Some students over-generalize from the one-dimensional case to incorrectly conclude that any superposition of stationary states is not stationary.

Figure 5.16 shows a question designed to probe student difficulties with superpositions of degenerate states. This question was asked of 76 students in Physics 324 in Autumn of 2005 and 2006. The question presents students with four wave functions labeled A–D for a three-dimensional harmonic oscillator. Each wave function is a superposition of two stationary states. In cases A and D the states that make up the superposition have the same energy. In cases B and C the states that make up the superposition have different energies. Students are asked whether each of the four wave functions represent allowed states, whether they change as time evolves, and whether their probability density changes as time evolves. In a correct response, a student recognizes that all four wave functions represent possible states, and that they all depend on time. Only states B and C, however, are associated with time-dependent probability densities because they represent sums of stationary states with different energies.

> Consider a three–dimensional harmonic oscillator described by the Hamiltonian $H = -\frac{\hbar^2}{2M}\nabla^2 + \frac{1}{2}M\omega^2 l^2$. In Cartesian co-ordinates the energy eigenfunctions are given by $\psi(x, y, z) = N\phi_{n_x}(x)\phi_{n_y}(y)\phi_{n_z}(z)$ with eigenvalues $E_{n_x n_y n_z} = \hbar\omega(\frac{3}{2}+n_x+n_y+n_z)$. In spherical co-ordinates the energy eigenfunctions are given by $\psi(r, \theta, \phi) = NR_{nl}(r)Y_l^m(\theta, \phi)$ with eigenvalues $E_{nl} = \hbar\omega(\frac{3}{2} + 2n + l)$.
>
> For each state below answer the following questions:
>
> (1) Is this an allowed state? (2) Does the wave function change as time evolves? (3) Does the probability density associated with the state change as time evolves?
>
> In each case assume N is the proper normalization.
>
> A. $\Psi(\vec{r}, t = 0) = N(\phi_1(x)\phi_0(y)\phi_0(z) + \phi_0(x)\phi_0(y)\phi_1(z))$
>
> B. $\Psi(\vec{r}, t = 0) = N(\phi_1(x)\phi_0(y)\phi_0(z) + \phi_1(x)\phi_0(y)\phi_1(z))$
>
> C. $\Psi(\vec{r}, t = 0) = N(\phi_1(x)\phi_1(y)\phi_0(z) + R_{11}(r)Y_1^1(\theta, \phi))$
>
> D. $\Psi(\vec{r}, t = 0) = N(\phi_2(x)\phi_1(y)\phi_0(z) + R_{11}(r)Y_1^1(\theta, \phi))$

Figure 5.16 A question given to 76 Physics 324 students in Autumn 2005 and 2006 that elicits the idea that any superposition of stationary states is not stationary.

Fewer than 10% of the 76 students gave correct answers for all four wave functions. Most of the students gave very little explanation, but their answers revealed significant difficulties. As shown in Table 5.4, fewer than half of the students answered correctly concerning the probability density for any of the wave functions. Clearly, many students do not distinguish between the behavior of systems composed of degenerate stationary states and those compose of non-degenerate stationary states.

Table 5.4 Physics 324 student performance on a question dealing with degenerate states in Autumn of 2005 and 2006 during the tenth week of the class.

Question	**correct all parts**	**correct concerning ρ**
A	30% (22)	50% (37)
B	40% (32)	50% (38)
C	20% (16)	35% (25)
D	10% (9)	25% (20)

Difficulties with superpositions of degenerate states continue into Physics 325. In preparation for studying degenerate perturbation theory 30 students in Physics 325 in the Winter quarter of 2007 were asked the question shown in Figure 5.17. In that question they were asked to consider four possible first excited states for a two-dimensional harmonic oscillator. They were asked to determine which, if any, of the wave functions were valid first excited states.

Each of the wave functions given were linear combinations of first excited state wave functions. Hence, they are all valid first excited state wave functions. Only 45% of the students, however, identified all four states as valid first excited states. Thus, difficulties with degenerate states persist throughout instruction.

A two–dimensional isotropic harmonic oscillator is described by the Hamiltonian $H = -\frac{\hbar^2}{2M}\nabla^2 + \frac{1}{2}M\omega^2 r^2$. The first two energy eigenfunctions for the one–dimensional harmonic oscillator are:

$$\varphi_o(x) = A_o e^{-\alpha x^2} \text{ and } \varphi_1(x) = A_1 x e^{-\alpha x^2} \text{ where } A_o = \left(\frac{M\omega}{\pi\hbar}\right)^{\frac{1}{4}}, A_1 = \left(\frac{4M^3\omega^3}{\pi\hbar^3}\right)^{\frac{1}{4}}, \text{ and } \alpha = \frac{M\omega}{2\hbar}.$$

The eigenvalues corresponding to these eigenfunctions are $\frac{1}{2}\hbar\omega$ and $\frac{3}{2}\hbar\omega$ respectively.

Which, if any, of the following wave functions are valid first excited states for the *unperturbed* oscillator? Explain

1. $\varphi_o(x)\varphi_1(y)$
2. $\varphi_o(y)\varphi_1(x)$
3. $(\varphi_o(x)\varphi_1(y) + \varphi_o(y)\varphi_1(x))/\sqrt{2}$
4. $(\varphi_o(x)\varphi_1(y) - \varphi_o(y)\varphi_1(x))/\sqrt{2}$

Figure 5.17 A question asked of 30 Physics 325 students in Winter quarter of 2007 concerning degenerate states.

5.8 Summary

This chapter has described a number of very closely related difficulties that can all be interpreted as a confusion between stationary states and superpositions of stationary states. These include:

- Belief that stationary states are the only valid states
- Tendency to treat superpositions of stationary states as having different well-defined energies in different regions of space
- Tendency to treat all states as having a single time-dependent phase
- Belief that a sum of stationary states is again stationary
- Belief that expectation values are always time-independent
- Failure to distinguish mixed states from superpositions
- Belief that superpositions of degenerate states are not stationary

These difficulties are all closely related and reflect difficult aspects of ideas that seem similar to an expert. We have described them as a confusion between stationary states and superpositions of stationary states, but they might be characterized in may different ways. For example, one might also relate them to an assumption that all states satisfy the energy eigenvalue equation, $H\Psi = E\Psi$ (also known as the time-independent Schrödinger equation). We do not, however, claim that these ideas form a consistent incorrect framework in the minds of students. We rather note that these ideas are present to one degree or another in the reasoning provided by many students in a typical quantum mechanics course. Because they arise frequently, their identification can help inform instruction. In Chapter 6 we discuss the development of a tutorial that gradually evolved to the point that it addresses the most common and persistent difficulties explicitly. In addition, we discuss preliminary work on a tutorial to address student difficulties with degenerate states.

5.9 Notes to Chapter 5

1 See (Styer, 1996) for Styer's list of misconceptions regarding quantum mechanics based on his observations of students, colleagues, writings and himself.

2 See section 6.3 of (Sadaghiani, 2005)

3 See (Singh, 2001) for the details of a test administered to 89 undergraduate students from six universities.

4 See (Singh, 2006) and also Chapter 9

5 It should be noted that this pretest was given in lecture during Autumn 2003 and in the tutorial sections in Autumn 2004. Attendance is not perfect in either the lecture or tutorial sections. So, some students did not take the pretest. In Autumn 2003, 40 of the 47 students who eventually took the final exam took the pretest. In Autumn 2004, 31 of the 52 students who eventually took the final exam took the pretest. Assuming that the more diligent students attend all their classes, makes plausible the discrepancy in the results between the two years. If the tabulated percentages are calculated using only the students who took the final exams, the difference between the results becomes statistically insignificant.

6 Note that students who answered the question in Figure 5.1 had worked through two tutorials that focusing on drawing qualitatively correct wave functions for highly excited energy eigenstates given a potential. It may be, in part, ideas from these tutorials that students are incorrectly overgeneralizing.

7 See (Feynman, 1965)

Chapter 6

Curriculum to Address Student Confusion of Stationary States and Superpositions of Stationary States

As illustrated in Chapter 5, after a standard course in quantum mechanics, many students fail to develop an appreciation of the significance of superpositions and its role in the time-dependence of quantum mechanical states. A failure to distinguish between the properties of stationary states and superposition of stationary states has many implications. It obscures an understanding of many aspects of the formalism of quantum mechanics, as well as inhibits students from understanding the fundamental ways in which theories of classical and quantum mechanics differ.

In the Summer quarter of 2003 we began work on a tutorial entitled *Time Dependence in Quantum Mechanics* that would have several objectives.

First, the tutorial should provide students with a mechanism for time dependence consistent with the Schrödinger equation and help them understand the model developed in a typical quantum mechanics course. In addition, it should, make explicit that there exist valid states other than stationary states. The tutorial would have exercises that explicitly try to address student difficulties described in Chapters 4 and 5.

The latest version of the tutorial is reproduced in full in Appendix A. It is the third tutorial in a sequence of about seven tutorials designed for a first quarter course on quantum mechanics. (The first two tutorials on relating classical to quantum mechanics are described in Chapter 12.) Students work through the tutorial *Time Dependence in Quantum Mechanics* only after they have had significant lecture instruction on one-dimensional quantum mechanics, including time-dependence. Students have also typically completed a number of homework problems that involve aspects of time dependence before they work through the tutorial.

The remainder of this chapter describes the exercises in the tutorial, *Time Dependence in Quantum Mechanics*, as well as some typical interactions between teaching assistants and students during tutorial sessions. In addition, a short discussion of the preliminary development of a tutorial on the time dependence of states composed of degenerate stationary states is discussed.

6.1 | Tutorial Instruction concerning Stationary States

The first section of the tutorial, *Time Dependence in Quantum Mechanics*, begins in the context of stationary states. Students are asked to sketch the first two stationary states for the infinite square well. An additional question asks whether it is possible to represent the wave function on a single two dimensional graph.

At this point, a discussion usually ensues among the students that often results in their representing the wave function by a three dimensional graph in which the three axes represent the real part of the wave function, the imaginary part of the wave function, and a spatial dimension. Occasionally a group will choose to represent a single wave function by two separate graphs: one for the real part, and one for the imaginary part.

Next, students are asked whether the first two stationary state wave functions change with time, and if so how.

At this point, students frequently debate whether or not to include an exponential time factor for a stationary state. This is usually resolved when students reach a question in the tutorial that asks whether their wave functions satisfy the full Schrödinger equation, $i\hbar\frac{\partial\Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x)\Psi(x,t)$. It comes as a surprise to many students to find that, although the spatial part satisfies the eigenvalue equation $H\psi(x) = E\psi(x)$, a stationary state does not satisfy the full Schrödinger equation without the exponential time piece. Many students are surprised at this time dependence, and at times this has brought out interesting discussions concerning the name "stationary" versus the name "static". The students who have drawn three dimensional graphs may notice that the mathematical time dependence corresponds to a rotation of the curve about the spatial axis. If students have not chosen to represent their wave function on a three dimensional graph, the teaching assistant will often ask them to consider this representation in addition to any other that they might have chosen.

Next, the tutorial asks students to graph the probability density associated with each state and whether such a graph can be plotted using a two dimensional graph. In addition, they are asked whether the probability density depends on time.

At this point, teaching assistants frequently ask students if there are any features in their graph of the wave functions that reflect the fact that the probability density does not depend on time. Many students who have drawn a three dimensional graph of the wave function notice that the probability density at any point is just the square root of the distance between the spatial axis and the wave function. This, then, does not change as the curve rotates.

The next question in the tutorial asks the students to calculate the time required for the first excited state to return to its original value.

To answer, students may sketch the ground state at this time and both states at twice this time. Students are led through the graphical consequences of their earlier statements about the mathematical time dependence of stationary states. For those who have not already determined that the curve rotates around the spatial axes in the three dimensional representation, this way of thinking about the time dependence of the state becomes apparent.

It is important that all group members have a clear understanding of the time dependence associated with stationary states at this point, because the tutorial goes on to combine these stationary states into a superposition.

6.2 Tutorial Instruction Concerning Superpositions of Stationary States

In Section II of the tutorial, students are shown the graph in Figure 6.1 and asked whether it could represent a possible wave function and, if so, whether it is an energy eigenstate or could be made from a sum of energy eigenstates. Although the graph shown in Figure 6.1 is the sum of the first two energy eigenstates for the infinite square well, students are not told this until later in the tutorial.

In answering the above question, students are also asked to consider the truth of the following two statements.

> The wave function is a sum of eigenstates, which makes it an eigenstate.
>
> The wave function is a solution of the Schrödinger equation. So, there must be a definite energy associated with it.

II. Linear superpositions of eigenstates

Consider the graph shown at right at time $t = 0$. (Assume $\text{Im}[\psi] = 0$.)

A. Could this graph be a possible wave function for the particle in the infinite square well? If so, is it an eigenstate, or could it be made from a sum of eigenstates? Explain.

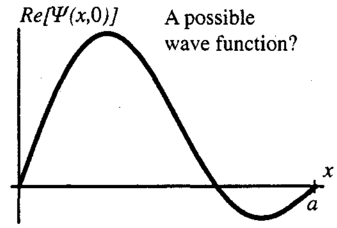


Figure 6.1 A superposition of the ground state and and first excited state in the infinite square well shown to students in the tutorial on time dependence.

These questions force students to confront the incorrect idea that all valid states are energy eigenstates. The second statement has proven especially compelling for students. Many must be led to test for themselves whether a superposition of two or more energy eigenstates satisfies the energy eigenvalue equation in order to resolve this difficulty. They are often surprised that after operating on a general state with the Hamiltonian, they cannot factor the result into a product of an energy and the original wave function.

To elicit their initial ideas, students are asked whether they believe the wave function changes as time progresses. Then they are given the following dialogue between two students.

> Student 1: *At times the wave function is like the ground state at other times it is like the first excited state. So, sometimes it has energy E_o and at other times E_1.*

> Student 2: *No the c_ns are always the same. So, the probabilities of measuring E_o or E_1 don't change even though it looks like ψ_o at times when $e^{-iE_1 t/\hbar} = 0$, and like ψ_1 at times when $e^{-iE_o t/\hbar} = 0$.*

Students are asked whether they agree or disagree with each statement.

In responding to the student dialogue, the students must carefully consider their thoughts concerning the time dependence of the wave function. This question begins to help address the difficulty of distinguishing a superposition from a mixed state. It also addresses a frequently observed mathematical difficulty, whereby students assume $e^{-iE_1t/\hbar}$ is zero for certain times. This assumption may confuse this issue if not confronted.

Eventually, most students correctly state that the wave function, given in Figure 6.1, changes with time. Groups generally spend a significant amount of time in discussion to determine that both student statements are incorrect. At this point, it is important that teaching assistants ensure that students articulate their reasons for disagreeing with the students in the student dialogue.

6.3 Tutorial Instruction Focusing on Time Dependent Probability Densities

In the last part of the tutorial students are told that the graph given in Figure 6.1 is an equally weighted superposition of the first two stationary states in the infinite square well. (Some may have already decided that this is likely the case.) They then are asked to make sketches of the probability density at several subsequent times.

The time evolution of this combined state is most easily visualized on a three dimensional plot. Some students need to be guided to use their descriptions of the time evolution of the ground and first excited states from the first section of the tutorial in creating a graph of the combined state. The realization that the graph can be created in this manner often gives students confidence. Finally they must, once again, interpret the square of the distance between the spatial axis and the wave function as the probability density. This exercise gives them practice in applying the mechanism for time dependence embodied in the Schrödinger equation to a simple state with a time dependent probability distribution.

Next, students are asked to consider what would happen a long time later. Usually group discussions lead them to rule out incorrect reasoning based on diffusion or decay.[1] Students are also led to conclude that a time dependent probability density is only possible with a superposition of stationary states. They also consider implications of this for systems which are most easily explained by classical mechanics.

6.4 Tutorial Instruction concerning Degenerate States in the Two-dimensional Harmonic Oscillator

A separate tutorial is being developed to address student difficulties concerning degenerate states in the two-dimensional harmonic oscillator. That tutorial is generally the last tutorial in the Physics 324 sequence. It follows a tutorial on uncertainty which is described in the second part of the dissertation. The focus of the tutorial is on a phenomena that students encounter after they began considering quantum mechanics in more than one dimension. Namely, it is possible to have superpositions of stationary states which are again stationary provided those states are degenerate. To a lesser degree the tutorial considers different co-ordinate systems. The research base behind this tutorial is less extensive than the previous tutorial described. The full tutorial is available in the appendix.

The tutorial begins with a consideration of a two-dimensional harmonic oscillator. Students are given the ground state and first excited state of the one-dimensional oscillator and asked to find the ground and first excited states for the two-dimensional oscillator in Cartesian co-ordinates. They are then asked about the characteristics of various superpositions of products of the one-dimensional solutions. Students must decide if these wave functions or their probability distributions depend on time. In addition, they must decide if they are associated with single energies. In this way they are led to the realization that stationary states can be made out of superpositions of degenerate states. Next they are asked to go through similar exercises using polar co-ordinates. Finally they relate states described in Cartesian to those described in polar co-ordinates.

6.5 Informal Assessment

Informal assessment of both tutorials has been made through observations and discussions during and following instruction. The results indicate that the tutorials are challenging and fruitful for students. Most seem to benefit from thinking more deeply about time dependence than they otherwise would have done. Formal assessment of the tutorials takes place through examination questions. The exams, however, typically occur after students have worked through a set of several tutorials. The other tutorials in this set are described in Chapter 8. Discussion of post-test results is, therefore, deferred until Chapter 9.

6.6 | Notes to Chapter 6

[1] See Chapter 4.

Chapter 7

Investigation of Student Understanding of Eigenstates and Measurements

One expected outcome of an introductory non-relativistic quantum mechanics course is for students to be able to determine the possibilities and probabilities of obtaining certain values as a result of a measurement on a system at any time given an initial state. This process, however, has a number of steps. One necessary step is for students to recognize that not all states are stationary states. According to the Schrödinger equation, there are a set of states called stationary states that have a simple time dependence. The time dependence of superpositions of stationary states can be built up using the time dependence of each individual stationary state in the superposition. Student understanding of the dynamics of isolated systems and the relationship between stationary states and superpositions of stationary states were discussed Chapter 4 and Chapter 5.

A deeper understanding, however, is required to predict the possible outcomes of an experiment on a quantum mechanical system. Once students recognize that only certain special states are stationary, they must recognize that these sets of states are generally different for different potentials. Student difficulties with this idea are discussed below in Section 7.1. Students must, in addition, recognize that while the stationary states are eigenstates of energy there are also eigenstates of other observables, and these eigenstates are in general distinct. An understanding of the model for measurement developed in a typical course is also needed. In that model, students are expected to associate a mathematical operator with each possible observable. Furthermore, they learn, that the only allowed results for a measurement are the eigenvalues of the corresponding operator, and a measurement leaves the system in an eigenstate of the operator corresponding to the eigenvalue measured. These types of difficulties associated with the collapse of the wave function are discussed in Section 7.2. In addition, students need to recognize that after a measurement an isolated system's time evolution is again governed by the Schrödinger equation. This idea will be discussed in Section 7.3.

7.1 Belief That Certain Special Functions Are Stationary States for All Potentials.

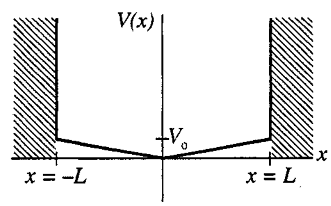
As discussed in Chapters 4 and 5, students have serious difficulties with time-dependence even in systems for which they are familiar with the stationary states (*e.g.,* the infinite square well). We have found that their difficulties are even more significant for systems with unfamiliar potentials. Informal classroom observations indicated that many students fail to recognize that different potentials have different stationary states. Instead, they have a tendency to write an arbitrary wave function as a superposition of the stationary states of the infinite square well, and construct time-dependent solutions by multiplying each term by a different complex time-dependent exponential. In this section we describe research in which we probed the prevalence and nature of these difficulties in greater detail.

7.1.1 Indications of difficulties in the context of perturbation theory

It was during an investigation of student difficulties with time-independent perturbation theory that it first became evident that some students thought of the stationary states for the infinite square well as stationary for all potentials. In the spring quarter of 2004 and 2006 students in the Physics 325 course were asked the question shown in Figure 7.1 on their second exam.

[7 pts] A particle is in the potential well shown at right. This well is only slightly perturbed from the infinite square well. At time $t = 0$ s the particle's wave function is given by

$$\psi(x) = \sqrt{1/L}\,\sin\left(\frac{\pi x}{L}\right).$$



1. If a measurement of the particle's energy were made at time $t = 0$ s, which of the following would be true?
   - There is one particular value of the energy that could be measured. It is equal to E_o, the ground state energy of the unperturbed well.
   - There is one particular value of the energy that could be measured. It is greater than E_o, the ground state energy of the unperturbed well.
   - There is one particular value of the energy that could be measured. It is less than E_o, the ground state energy of the unperturbed well.
   - There is not just one particular value of the energy that might be measured. Instead there are more than one possible values that might be obtained from a measurement of energy.

   Breifly explain.

Figure 7.1 A question in the context of time independent perturbation theory that revealed students reasoning that certain states are stationary regardless of the potential.

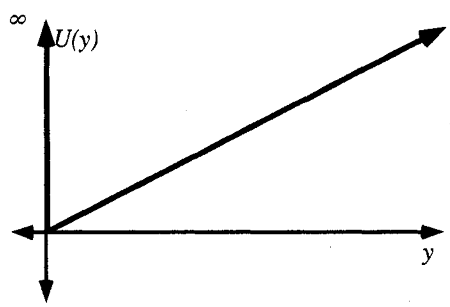
In that question students were given a potential well with a "V"-shaped bottom. They were told that the initial wave function for the system was identical to that of the infinite square well. They were then asked about the possible results of energy measurements. Since the initial state is not an energy eigenstate for the potential well, it must be possible to express the initial state as a linear combination of the energy eigenstates of the potential well. Thus, there are multiple possible results from a measurement of energy. Only 30% of the students answered correctly that more than one value is possible. Of the 70 students who took the exam, 10% thought the only possible measurement result was the ground state energy of the unperturbed well. Most of the students, about 65%, answered that there was only one possible value, but it was greater than the ground state energy of the infinite square well. Their answers indicate that they seem to believe that the ground state of the infinite square well is also a stationary state for the "V"-shaped well. It simply corresponds to a different energy.

The answers given by students are clearly incorrect, but may be influenced by the experience students have with time-independent perturbation theory. While learning time-independent perturbation theory, students are expected to calculate the approximate difference in energy between the ground state of the perturbed and unperturbed systems. This may lead them to believe that the energy of the ground state differs in the two cases, but the wave function does not. The context of time-independent perturbation theory made it difficult to separate more general student difficulties from those associated with perturbation theory. With that in mind, we designed similar questions to be asked earlier in the course, well before students encounter perturbation theory.

7.1.2 Probing prevalence in a simpler context

In Autumn quarter of 2005 and 2006, we administered a question designed to probe the prevalence of student beliefs that certain functions are stationary states for all potentials. This question was designed for the Physics 324 course in order to avoid the complications of perturbation theory discussed in the previous section. The question shown in Figure 7.2 shows students an initial wave function for a system having a potential similar to that of a ball bouncing elastically under the influence of gravity. The initial wave function given to students has the same shape as the ground state of the infinite square well. The students are asked whether the wave function and the probability distribution depend on time. They must then describe how they would express that time dependence mathematically or explain why it does not depend upon time.

Consider a quantum mechanical system with a potential similar to that of a ball making elastic collisions with a surface. Such a potential is shown at right. This system is prepared so that at $t = 0$ it is in a state described by a wave function identical to that of the ground state of the infinite square well. That is, $\Psi(y,0) = \sqrt{2/L}\,\sin(\pi y/L)$.



Does the wave function associated with this state depend on time? If so, describe how you would write down its time dependence. If not, explain why not.

Does the probability density associated with this state depend on time? If so, describe its time dependence. If not, explain why not.

Figure 7.2 A question to probe whether students consider certain states to be stationary regardless of the potential.

A correct answer to this question requires recognition that the ground state of the infinite square well is not an energy eigenstate of the "V"-shaped well shown. This can be concluded from the fact that the Schrödinger equation depends on the potential. The ground state of the infinite square well is, however, a possible wave function for this potential and can be written a sum of the energy eigenstates of the "V"-shaped well. Expressed in this way, the time dependent solution can be easily obtained by multiplying each term by the appropriate time-dependent complex exponential. So, the wave-function does depend on time, and its time dependence is more complicated than that of a single time-dependent complex exponential. The probability density will also depend on time. It is equal to the absolute value of the wave-function squared, and cross terms in this calculation retain time-dependence.

Only about 15% of the 93 students who took this question could answer correctly with correct reasoning—even after significant lecture instruction on this material. The most prevalent error, made by 55% of the students, was to multiply the initial wave function by a single time-dependent complex exponential factor. These students treated the initial wave function as if it were an energy eigenfunction of the "V"-shaped well (*e.g.*, $\Psi(x,t) = \Psi(x,0)e^{-iEt/\hbar}$). Most of these students then claimed that the probability density would not change with time because the time dependence cancels out. Again, this would be correct if the original wave function were an energy eigenfunction for the "V"-shaped well. An example of this type of reasoning is shown in Figure 7.3

These results indicate that most students do not recognize that an energy eigenfunction or stationary state is associated with a particular potential or class of potentials. Rather, they seem to regard certain functions as global energy eigenfunctions that apply to any potential.

> *Yes [it depends upon time], it should have a complex exponential time dependence like any eigenfunction.*

Figure 7.3 Example of student reasoning illustrating the idea that certain functions are energy eigenfunctions for all potentials.

Table 7.1 Prevalence of common responses to the question shown in Figure 7.2.

Response	**Prevalence**
correct with correct reasoning	15% (15)
"global eigenfunction" reasoning	55% (49)
classical reasoning	10% (8)
"diffusion" or "decay" reasoning	5% (5)
Total	($N = 93$)

The other most common errors made by the students could be classified into two categories. (See Table 7.1.) Ten percent of the students reasoned incorrectly basing their answers on classical mechanics. Many of these students seemed to be associate the system with a definite energy that was conserved and regarded energy conservation as a criterion for lack of time dependence. Reasoning such as this is typified by the student quote in figure Figure 7.4

> *No, there is no time dependence with the probability density because the collisions are elastic it remains the same.*

Figure 7.4 Example of student reasoning illustrating a misapplication of classical mechanics.

In addition, a few students explicitly reasoned as if wave functions diffuse or decay as discussed in Chapter 4. This was true despite the fact that this question was not designed to probe for "decay" or "diffusion" type of reasoning.

7.2 Difficulties Related to the Collapse of the Wave Function

In introductory quantum mechanics, students are expected to learn that measurements are associated with mathematical operators. When an observable is measured, the system "collapses" onto an eigenstate of the operator corresponding to the measurement that is made. In general the eigenstates of different operators are different provided they don't commute. In addition, the system will not stay in an eigenstate of an operator following a measurement unless the operator commutes with the Hamiltonian.

In this section, we present evidence of student difficulties with these last two points. We have found that some students seem to reason as if all operators have the same eigenstates. Thus, for them, any measurement collapses the wave function and permits the simultaneous measurement of all quantities.

Many students also seem reason that the eigenstates of any operator behave like energy eigenstates, and remain eigenstates for all time.

We have asked a number of questions to probe student thinking about the effect of a measurement on a system. These are discussed below in the context of the errors they elicit.

7.2.1 Belief that all operators have the same eigenfunctions

A number of written questions were designed to probe whether students formally believe that all operators have the same eigenfunctions.

A question designed to test whether or not students distinguish between energy and momentum eigenstates in the harmonic oscillator is shown in Figure 7.5. Students are given a state that they are told is a superposition of two energy eigenfunctions of the simple harmonic oscillator and asked if the energy eigenfunctions are also momentum eigenstates. As a hint, students are given the form of the momentum eigenstates, $e^{\pm ipx/\hbar}$.

This question was asked of 79 students on the final exam of Physics 324 in Autumn of 2002, but as shown in Table 7.2 only 55% of students correctly reasoned that the eigenfunctions for energy and momentum are distinct in the harmonic oscillator.

Problem 6. (30 pts) A particle is in a 1-dimensional harmonic oscillator potential, $V(x) = m\omega^2 x^2/2$. At time $t = 0$, the particle is described by the following wavefunction:

$$\Psi(x,0) = \frac{(1+i)}{\sqrt{10}}\Phi_0(x) + \frac{(2-2i)}{\sqrt{10}}\Phi_3(x)$$

where $\Phi_n(x)$ are the spatial energy eigenfunctions that satisfy the time independent Schrodinger equation: $\hat{H}\Phi_n(x) = E_n\Phi_n(x)$.

A.)

i.) (3 pts) Are $\Phi_n(x)$ also eigenstates of momentum? Explain. (Hint, momentum eigenfunctions have the form $e^{\pm ipx/\hbar}$.

Figure 7.5 A question asked of Physics 324 students in Autumn 2002 on their final examination to probe whether they believe that the eigenfunctions for energy and momentum are identical in the harmonic oscillator.

The incorrect answers indicate that the students do not identify the eigenstates of energy and momentum as being distinct in the harmonic oscillator. Some, like the student whose response is shown in Figure 7.6 incorrectly apply reasoning from classical mechanics. Others come to an incorrect conclusion from erroneous quantum mechanical calculations. Still others echo reasoning that we saw in Chapter 5. They do so by concluding that the energy eigenfunction is a momentum eigenstate because it can be made up of a sum of momentum eigenstates, and a sum of momentum eigenstates is again a momentum eigenstate.

Yes [eigenstates of momentum and energy are the same] $p = \sqrt{2Em}$
$\hat{p} = -\sqrt{2m\hat{H}}$ *so an eigenfunction of* $\hat{H}$ *will also be one of* $\hat{p}$.

Yes, $[H, p_x] = 0$.

Yes – Position & momentum are described by the same wave function.

$\phi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int e^{-ipx/\hbar}\psi(x,0)dx$ *Yes,* $\phi_n(x)$ *are also eigenstates of momentum because* $\phi_n(x)$ *can be written in terms of p (i.e.,* $e^{\pm ipx/\hbar}$*).*

Figure 7.6 Examples of student reasoning characteristic of the idea that eigenstates of momentum are identical to eigenstates of energy in the harmonic oscillator.

We have also asked students to compare the eigenstates of energy and position. An example of such a question is shown in Figure 7.7. Here students are told that a system in an infinite square well has been found to be in an energy eigenstate. Then they are asked if this system is also in an eigenstate of position. Students should recognize that the energy eigenfunctions for an infinite square well potential are sinusoidal inside the potential. They should also recognize that eigenstates of position are delta functions. Thus, an energy eigenstate cannot be a position eigenstate.

As shown in Table 7.2, only about half of the 85 Physics 324 students who were asked this question in the Autumn quarters of 2005 and 2006 could correctly answer that the system was not in an eigenstate of position.

A particle is in the infinite square well potential shown at right.

At time $t = 0$, the energy of the system is measured and found to be E_2, the first excited state energy.

A. Immediately after the measurement, is the system in an eigenstate of position? Explain.

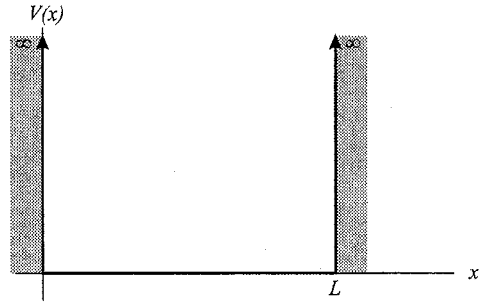


Figure 7.7 A question asked of 85 Physics 324 students in Autumn 2005 and 2006 during their seventh week of class to probe whether they believe that the eigenfunctions for energy and position are identical for the infinite square well.

Table 7.2 Physics 324 student performance on questions requiring a recognition that eigenstates of distinct operators are in general distinct.

	Autumn 2005 seventh week $N = 45$	**Autumn 2006 seventh week** $N = 40$	**Autumn 2002 final exam** $N = 79$
energy and momentum	–	–	55% correct
energy and position	50% correct	55% correct	–

Yes. For infinite square well energy eigenstates are position eigenstates. $[H, x] = 0$

Well ψ *collapses to eigenstate* $n = 2$ *which is like* $\frac{2}{L\pi}\sin(\frac{2\pi x}{L})e^{-iE_2 t/\hbar}$, *which is an eigenfunction of position in position basis. So yes it is in an eigenfunction of position for this basis.*

Yes because the system collapses to the measured value.

Yes, because it is a determinate state.

Yes. Each eigenstate of position corresponds to an energy state.

$\psi_n(x,t) = \psi(x)e^{-iE_n t/\hbar}$ *Yes, because the energy measurement tells us the wave function where it may be found.*

Figure 7.8 Examples of student reasoning characteristic of the idea that eigenstates of position are identical to eigenstates of energy in the infinite square well.

Upon examining student responses, some of the same difficulties emerge that were discussed previously. Figure 7.8 shows some examples of student reasoning. Some mathematical difficulties are again present, as illustrated by the first response. The second response shows a confusion between an eigenfunction of a particular operator and an expansion of a state in terms of a basis formed by an operators eigenfunctions. The remaining students seem to be expressing the idea that both the position and energy have definite values after any measurement. They seem to associate the "collapse of the wave function" with an event which makes all variables known. As one of the responses in Figure 7.8 indicates, many think the system is in a "determinate" state after the measurement where determinate means the values of all observables are known.

The results above indicate that only about half of the students in a given class distinguish between the eigenstates of different operators. In their responses, some students seem to indicate that after any measurement, all observables are knowable. Thus, a position measurement, for example, yields a wave function that has not only a definite position but also a definite momentum and energy as well. The following section discusses these ideas in greater detail.

7.2.2 Tendency to treat all observables as having definite values after any measurement

Written questions and interviews have revealed students reasoning as if all observables have definite values after a measurement. Student reasoning has primarily been probed in two different contexts. In the first context a system is initially prepared in an energy eigenstate. Then a measurement of momentum is made. Finally, students are asked about the results of possible energy measurements on the system. The second context is identical to the first except the momentum measurement is replaced by a position measurement.

Problem 6, continued from the previous page
B.) At some time, $t' > 0$, the energy of the particle is measured to be E_3.
i.) (4 pts) Write down the normalized wavefunction for the particle, $\Psi(x,t)$, for times $t > t'$ (*i.e.* after the measurement) in terms of the spatial wavefunctions, $\Phi_n(x)$.
ii.) (4 pts) After the measurement, is the probability density, $\rho(x,t)$, time-dependent? Explain.
C.) At an even later time, $t'' > t'$, the **momentum** of the particle is measured.
i.) (4 pts) After this momentum measurement is made, can the energy of the particle be represented by a single number (*i.e.* is the particle in an energy eigenstate)? Explain.

Figure 7.9 The continuation of a question asked of Physics 324 students in Autumn 2002 on their final examination to probe whether they believe that the eigenfunctions for energy and momentum are identical in the harmonic oscillator.

The question shown in Figure 7.9 continues the question about the harmonic oscillator shown in Figure 7.5. Students are told that the energy of a particle in that system is measured at time t' and found to be E_3 followed by a momentum measurement at time t''. They are asked if the energy of the particle can be represented by a single number after the momentum measurement. Even though this was on the final examination, 55% of the students thought the energy could be represented by a single number. Quotes that illustrate student responses are shown in Figure 7.10.

It is evident from the student quotes in Figure 7.10 that many students view the momentum measurement as collapsing the wave function onto an energy eigenstate. Some say this explicitly. Others use the word eigenstate with no qualifier suggesting that one need not specify an operator. Still others use slightly more sophisticated reasoning suggesting that the classical relationship between momentum and energy for a free particle implies that they must have the same eigenstates.

A number of written questions and interviews have focused on a similar context where a position measurement intervenes between two measurements of energy.

The interview transcript Figure 7.11 shows a student discussing a system with an interviewer. The system under consideration was initially in an energy eigenstate, and then a position measurement was made.

Yes. The wave function has already collapsed. The energy continues to be E_3.

Since energy and momentum commute, they are compatible operators and the measurement of one does not depend on the other ⇒ the fixed energy of the particle also fixes momentum.

Yes. By measuring the momentum, we are once again forcing the particle to choose an energy eigenstate ($E \cong \frac{p^2}{2m}$) & in an energy eigenstate, you only have a single energy, so it can be represented by one number.

($E = \frac{p^2}{2m}$) So, once momentum is measured, a definite Energy has been frozen. So therefore Energy can be a single # (or E eigenstate).

The wave function ψ has been collapsed by a measurement from the superposition to an eigenstate at time t'. From then on the particle is in that state. Therefore when the momentum of the particle is measured at time t'' the result will be an eigenvalue of the momentum operator that is a single number. $\hat{p}(\phi_3(x)e^{-iEt/\hbar}) = p_3(\phi_3(x)e^{-iEt/\hbar})$ [An arrow identifies p_3 as an eigenvalue.]

Yes, even though a new measurement is made, momentum, the previously measured energy puts the particle in an energy eigenstate.

Yes, Energy and momentum share the same basis and thus commute. (i.e., energy and momentum are well defined functions of one another.)

Yes. A measurement of the particle's momentum forces the wave function into an energy eigenstate.

Figure 7.10 Examples of student reasoning about the state of the system following a momentum measurement in the harmonic oscillator.

The student correctly described the shape of the wave function after a position measurement as a delta function, but then goes on to incorrectly conclude that the energy of the system is known after the measurement.

Figure 7.12 shows another interview excerpt that illustrates a different student's belief that any measurement causes the wave function collapse and makes all quantities definite. The situation is similar to that above except the system was originally in a superposition of stationary states. The student,

Student: *We have found its energy. There is another uncertainty principle that says, "We know the energy or we know the time." There is something that relates energy and time.*
Interviewer: *Ok*
Student: *So, when we find the position and we find the new wave function is this delta function then we know the energy.*

Figure 7.11 Example of student reasoning based on a global collapse of the wave function.

however, had incorrectly associated the original state with the first excited state because it has two humps.

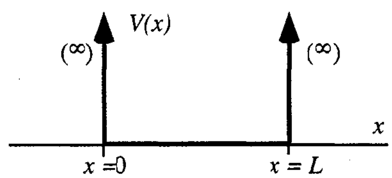
Student: *Once the particle is here [after the position measurement] sort of stationary for a very brief time when you measure it. It will come gradually to the ground state because you stopped it. I think of a particle in an infinite square well as having its own way of moving around the well.*
Interviewer: *And how would you describe that?*
Student: *It would kind of just bounce back and forth between boundaries. If you stop it, then you don't have any thing sort of to kick it back up to the first excited state. ...You place it at a certain point. ...You know it is there. You know it is not moving anywhere. Since it is not moving there shouldn't be any momentum with it. ...I think that the fact that you collapsed the wave function that the energy has to be automatically lowered. ...Any sort of measurement done on the particle can collapse the wave function.*

Figure 7.12 Example of student reasoning concerning the collapse of the wave function.

For this student the measurement of position stops the particle. Thus the position is known, the momentum is zero, and the energy is the lowest possible, that of the ground state. Note the last statement made by the student, "Any sort of measurement done on the particle can collapse the wave function." In this statement there is no indication of the wave function collapsing to a particular state. Instead, the student seems to believe that every system has a wave function that is either collapsed or not collapsed. If it is collapsed, all observables have definite values.

A particle in an infinite square well (shown at right) is prepared such that at time $t = 0$ it is in the ground state: $\psi(x) = \varphi_1(x)$.

At time $t_1 > 0$ you measure the *position* of the particle and find that at that time it is located at $x = 0.3L$.



Suppose a few minutes later, at a time t_2 you measured the *energy* of the particle. (Assume that, during the time interval $t_1 < t < t_2$, the particle remains isolated from the environment.)

Which of the following statements would best describe the result of your energy measurement? **Circle one and explain your reasoning.**

i. The value that you measure would *definitely* be the ground state energy.
ii. The value that you measure could *possibly* (but not necessarily) be the ground state energy.
iii. The value that you measure would *definitely not* be the ground state energy.

Figure 7.13 An example of a question asked of students in Physics 324 between the years of 1996 and 2006 requiring students to differentiate between position and energy eigenstates.

Between the years of 1996 and 2006 about 250 students were given versions of a written question that required them to differentiate between position and energy eigenstates. The original question was designed by Brad Ambrose to probe student understanding of compatible and incompatible observables. The questions were not identical, but there were only minor variations from year to year. An example is shown in Figure 7.13. All versions of the questions asked students to consider a particle in an infinite square well. They were either told that an energy measurement had been made which yielded a particular result (*e.g.*, the ground state or the first excited state energy), or they were given that the system was in a specified energy eigenstate as illustrated in Figure 7.13. Then they were told that a position measurement was made, and asked about the possible results of a subsequent energy measurement. Students needed to recognize that the position measurement leaves the system in a superposition of many energy eigenstates. Thus, a subsequent measurement of energy could yield many possible values.

Table 7.3 Results from questions requiring Physics 324 students to differentiate between position and energy eigenstates between 1996 and 2006.

Year	N	Correct	Same Result as before	other definite	other	not answered
2006	40	65% (26)	30% (12)	0	5% (1)	0
2005	46	55% (26)	30% (14)	0	5% (2)	10% (4)
2004	23	35% (8)	45% (10)	0	15% (3)	10% (2)
2003	33	35% (12)	40% (14)	0	15% (5)	5% (2)
2002	54	40% (22)	35% (19)	0% (1)	15% (8)	5% (4)
1997	28	35% (10)	30% (8)	0	5% (2)	30% (8)
1996	25	45% (11)	25% (6)	0	15% (4)	15% (4)
Total	**249**	**45% (115)**	**35% (83)**	**0% (1)**	**10% (25)**	**10% (24)**

As shown in Table 7.3, only 45% of the 256 students who answered this question could do so correctly. The most common incorrect response was to indicate that the second energy measurement would yield the same result as the first. This response was given by more than one-third of the students. Examples of student responses are show in in Figure 7.14.

if you have already measured the energy, the system stays in that energy eigenstate.

The particle remains isolated, so the energy value would not change.

The measurement collapses the wave function.

The particle will (intuitively) be in the ground state forever in the infinite square well. Its position will change but not its energy.

The energy does not depend on time – if the particle is in the ground state, it has $E = E_1 \ldots$ the particle will stay in the ground state.

Figure 7.14 Quotes from students who answered the questions incorrectly.

We interpret the results presented in this section as indicating a belief among a large fraction of students that any measurement collapses the wave function in such a way as to make all observables known. This interpretation is supported by the reasoning students used to explain their answers. This idea seems to be both prevalent and persistent.

The difficulties described above extend beyond the first quarter of quantum mechanics. A question that illustrates this persistence is shown in Figure 7.15, which was administered on the final exam of Physics 325 in the Winter of 2004. This problem was the first part of a question originally designed to probe student understanding of identical particles.

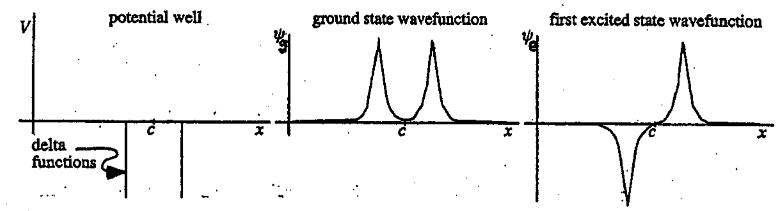
In this question students are shown the ground state and the first excited state for a system with a double delta function potential. They are told that the particle is measured to be on the left side of the well. Then they are asked to chose between several possible descriptions of the subsequent state of the system (*e.g.,* They are asked whether the particle could be in the ground state, the first excited state, or etc.). Since the measurement collapses the wave function onto a position eigenstate, the resulting state must be a superposition of states. The system cannot be in the ground state, or the first excited state, nor can $\langle x \rangle$ be time independent.

Even on the final exam of Physics 325 after two quarter of quantum mechanics fewer than 40% of the students answered this question correctly. About 40% stated that the system could be in one or more of the energy eigenstates.

7.3 Difficulties Related to Student Conceptions about Time Dependence of States

In Chapter 4 we discussed several types of reasoning that students bring with them to a quantum mechanics course (*e.g.,*"diffusion", "decay", and "revival")that are unrelated to the formalism that they will later learn. Evidence was presented that some students incorrectly use those types of reasoning throughout the course. In many cases they attempt to make the formalism that they learn support their incorrect ideas. This section focuses on

A particle is placed in the double delta function potential shown below. The two delta functions are close enough together to allow tunneling of the particle between the wells. Point c lies halfway between the delta functions. (Recall that the attractive delta function potential has only one bound state, leading to the ground and excited state wave functions shown below.)



A. (5 pts) If the position of particle is measured to be to the left of point c, then at a later time, which of the following statements are true? Explain.

a.) The particle may be in the ground state.
b.) The particle may be in the excited state.
c.) The particle will be in a superposition of the ground and excited state.
d.) The expectation value of the particle's position will not change with time.

Figure 7.15 A portion of an identical particle question asked of students in the Winter quarter of 2004 on the Physics 325 final exam.

(a,b,d) Measuring the position of the particle collapses the wave function so the expectation value will not change with time. Also since the wave function for both states has probability to the left of point c then either of them is a possible state. c can't be correct since there is nothing that shows that the particle will be in a superposition.

(a, b) Both statements make sense because $|\psi|^2$ is the same for either the ground or the first excited, and since you measured he thing, the wave function must collapse to an eigenstate, so c) is not correct, and the particle can move so d) is wrong.

(a, b) The first measurement collapses the particle's wave function to either the ground or excited state and the particle will stay in that state.

(d) $\langle x \rangle = \int_0^\infty \psi^* \delta() \psi dx$– the time dependent part will drop out.

(a, b, d) Both states have non-zero probabilities to the left of point c, and the measurement will collapse ψ to an energy eigenfunction.

Figure 7.16 Examples of student reasoning characteristic of the idea that any measurement collapses the wave function onto an eigenstate of energy.

students who treat the possible values for energy measurements on isolated systems as if they change with time. Some students seem to use this belief to support their pre-conceived ideas about the time-dependence of wave functions.

7.3.1 Results from interviews

During interviews and informal discussions with students, it often became clear that some were trying to use the formalism they learned in the course to justify their pre-conceived notions about the time-dependence of wave functions. An excerpt from one such interview is given in Figure 7.17. The student and interviewer were discussing a system originally in an energy eigenstate upon which a position measurement was made. The student expressed the belief that the system eventually returns to the original state using "revival" type reasoning as as discussed Chapter 4. Notice this student justifies this belief by allowing the energies and coefficients in an energy eigenstate expansion to change with time. A vague notion of information seems to guide these changes for this student.

> **Interviewer:** *So, you are suggesting that all of that [the revival of the original wave function] is encoded in those complex terms?*
> **Student:** *Well yeah. [writes $Ae^{iE_nt/\hbar}$] Like in the E_n part and also in the amplitude. ...These are your two sources of information. ...*
> **Interviewer:** *So, as time goes on do these two change with time?*
> **Student:** *Yes, it has to because this [the wave function after measurement] is different from here [the original].*
> ...
> If this is a linear combination of eigenstates, the time dependence would be like E_n would have to be like a function to kind of like cancel out the misshapes
> ...
> The E's and the A's are time dependent functions.

Figure 7.17 Example of student justifying their belief in wave function revival.

Figure 7.18 shows another example of a student justifying wave function revival by allowing the coefficients in a stationary state expansion to change with time. In this interview a system initially in a stationary state was being considered. After a measurement of position was made, the student was then asked to describe the long term time evolution of the system.

> **Student:** *...I guess my final answer would be that it goes back to the first excited state. ...*
> *It is not the same as the energy before because it is a sum of all the other energy eigenstates all the states in some way such that it ends up being a delta function.*
> *...*
> *It is a sum of a bunch of different eigenstates.*
> **Interviewer:** *Ok*
> **Student:** *Does it make sense that that thing can change? Sure, the coefficients can change in time.*
> **Interviewer:** *Ok*
> **Student:** *The coefficients $c(t)$.*
> **Interviewer:** *Which coefficients?*
> **Student:** *The coefficients of the eigenstates. ...So you have these coefficients that are going to change in time. [Writes $\sum c_i(t)\psi_i(x)$] ...*

Figure 7.18 Example of a student justifying a naïve belief in the revival of wave function with using time-dependent coefficients in an expansion in stationary states.

Another interview excerpt shown in Figure 7.19 illustrates a student justifying a belief that wave functions diffuse by allowing the coefficients in an energy eigenstate expansion to change with time.

> **Student:** *[writes $\Psi = A\psi_o e^{iE_o t} + B\psi_1 e^{iE_1 t} + C\psi_1 e^{iE_1 t}$] I was saying that you could possibly give a time dependence to these coefficients [points to A, B , and C] that would cause C to slowly decay down. As this one [A]goes down these [B and C] go up so that this [$\psi^*\psi = 1$] always stays true.*

Figure 7.19 Example of student belief that the coefficients in an energy eigenfunction expansion change in time.

It is often difficult to discern this type of reasoning from reading the brief responses that most students give to examination questions. We have, however, asked some questions that probe closely related issues. These are discussed below.

7.3.2 Tendency to treat the expectation value of the energy as a time-dependent quantity

Based on the results discussed above, we were interested in determining whether students who believe that the coefficients in a stationary state expansion change with time for an isolated system, would also conclude that the expectation value of the energy is time-dependent.

> The system is a particle in a one-dimensional potential well. At $t = 0$ the wave function is $\psi(x) = \frac{1}{2}u_o(x) + \frac{\sqrt{3}}{2}u_1(x)$ where u_o and u_1 are the wave functions for the two lowest energy states with energies E_o and E_1. u_o and u_1 are real and orthonormal.
>
> If you were to measure the energy of such a system, what is the probability of finding: $\frac{1}{4}(E_o + 3E_1)$, E_1, $\frac{1}{2}(E_o + E_1)$
>
> Write $\Psi(x,t)$ in terms of u_o and u_1.
>
> Evaluate $\langle H \rangle$ for the state $\psi(x)$.
>
> In each case indicate (*without calculations*) whether the quantity varies with time: $\langle H \rangle$, $\Psi^*(x,t)\Psi(x,t)$, $\int \Psi^*(x,t)\Psi(x,t)dx$

Figure 7.20 A question asked of 23 students in Physics 315 on their final exam in the Winter quarter of 2004.

In Winter of 2004 the Physics 315 class was given the question shown in Figure 7.20 on their final exam. The question required the students to answer whether the expectation value of the Hamiltonian for a system composed of a superposition of two energy eigenstates varied with time. The students were asked this question immediately after they were asked to evaluate the expectation value of the Hamiltonian for the superposition. To answer correctly, students could have reasoned that the superposition will always be made up of the same two energy eigenstates, and that the square of the absolute value of their coefficients will not change with time. Hence, the expectation value will not depend on time.

Five of the 23 students (about 20%) answered incorrectly that the expectation value of the Hamiltonian would vary with time. Five students also said that the integral of the probability distribution changes with time.

In 2006 the Physics 315 students were asked a very similar question, but they were not first asked to calculate $\langle H \rangle$. Twelve of the 30 students (about 40%) who answered this part of this question did so incorrectly. In addition, twelve answered the question about the integral of the probability distribution incorrectly.

The results are summarized in Table 7.4. Also given is the percentage of the students who took the exam and answered both questions correctly (that neither the expectation value of the energy nor the integral of the probability distribution change with time). Notice that for each question students were able to perform better when prompted by a calculation of $\langle H \rangle$, but had difficulty reasoning without this leading step.

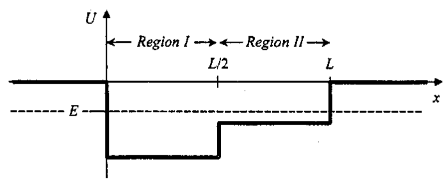
Table 7.4 Physics 315 student performance on questions that probe whether the expectation value of the energy changes over time.

incorrect statement	**Winter 2004**	**Winter 2006**
$\langle H \rangle$ varies with time	20%(5)	40%(12)
$\int \Psi^*(x,t)\Psi(x,t)dx$ varies with time	20%(5)	40%(12)
correct on both	60%(14)	30%(10)

These results indicate to us that even on the final examination students fail to understand that the expectation value of the energy does not depend on time for an isolated system.

7.3.3 Belief that possible values for energy measurements change with time

The same sorts of difficulties described in Section 7.3.2 have been seen in Physics 324. In both the Autumn quarter of 2002 and the Summer quarter of 2004 students were asked a pretest question concerning a particle in a one dimensional well which had two levels illustrated in Figure 7.21. The last part of that question asked the students about repeated measurements of the energy.



Suppose a different electron is placed in a potential well identical to the one above. A measurement of the energy is made and found to be E'. If another measurement is made some time later, which of the following will be true? Explain.

- The second energy measurement will yield the value E' with complete certainty.
- The second energy measurement could yield one of two possible values for the energy.
- The second energy measurement could yield one of three or more possible values for the energy.

Figure 7.21 A part of a question concerning a two-level well which appeared on a pretest asked of Physics 324 students in Autumn 2002 and Summer 2003.

Students were asked this question after standard lecture instruction on such topics, but before any specialized instruction. Only half of students in Autumn 2002 and one fifth of students on the Summer of 2003 could answer this question correctly. This is another example of students willingness to accept the possible values for energy measurements changing in an isolated system. In this case even when the system is originally in an energy eigenstate.

7.3.4 Persistence of difficulties

The types of difficulties discussed seem to be sufficiently robust that they arise in the second quarter of quantum mechanics, Physics 325. The topics in Physics 325 are more advanced, but nevertheless some of the same basic difficulties reappear. Figure 7.22 shows the second part of a question that students were asked about perturbation theory on their second exam of Physics 325 in the Winter of both 2004 and 2006. They were given an initial wave function and a one-dimensional potential well. They were first asked about a measurement at $t = 0$. (That question is shown in Figure 7.1) Next, they were asked to choose among several statements about the energy of the system shown in Figure 7.22. To answer correctly, students must recognize that for an isolated system, the same value or values for the energy are possible with the same relative probabilities regardless of the time the energy is measured.

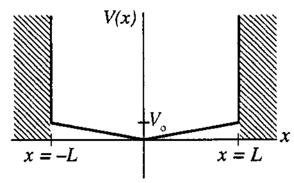
Even by their second exam in the second quarter of quantum mechanics during Winter Quarter 2006, only 60% of the 30 students taking the exam choose the third option in Figure 7.22. In Winter 2006 several other options were added to the question. The new options are shown in Figure 7.23.

This new set of options, while lengthy, attempted to probe student reasoning in more detail. A correct response included the third, sixth, and eighth choice. Only ten percent of the forty students who took this exam chose those three options. Only 25% of students chose only options where the possible values of energy measurements would not change for the isolated system.

Hence, many students reason in a manner consistent with the coefficients in a superposition of stationary states describing an isolated system changing

[6 pts] A particle is in the potential well shown at right. This well is only slightly perturbed from the infinite square well. At time $t = 0$ s the particle's wave function is given by

$$\psi(x) = \sqrt{\frac{1}{L}}\cos(\pi x / 2L)$$



First Part: [Question from Figure 7.1 concerning an energy measurement at $t = 0$.]

Second Part: Suppose, instead that the energy of the particle had been measured at a time $t > 0$ s. Which of the following is true?

- The number of possible values that might be obtained for a measurement of the energy is greater than it was in the previous part.
- The number of possible values that might be obtained for a measurement of the energy is less than it was in the previous part.
- The same value or values for the energy are possible as in the previous part, and they have the same relative probabilities.
- The same value or values for the energy are possible as in the previous part, but they have the different relative probabilities.

Figure 7.22 A part of a question concerning a perturbed well which appeared on the second exam and was asked of Physics 325 students in Winter 2004 and Winter 2006.

with time. This tendency may be a result of a desire for results to conform to pre-conceived ideas related to "diffusion", "decay", "revival", or the "gambler's fallacy".

7.4 Summary

This chapter has focused on three types of student difficulties all of which are related to student ability to identify, interpret, and use eigenstates in quantum mechanics as well as recognize their role in measurement. The first section explored a student belief that certain functions, typically sinusoids, are always eigenstates regardless of the system or operator under consideration. The second section detailed difficulties related to the col-

Suppose, instead that the energy of the particle had been measured at a time $t > 0$ s. Which of the following is true? Circle all that apply.

- Just after $t = 0$, the number of possible values that might be obtained for a measurement of the energy is greater than it was in the previous part.
- Just after $t = 0$, the number of possible values that might be obtained for a measurement of the energy is less than it was in the previous part.
- Just after $t = 0$, the same value or values for the energy are possible as in the previous part and they have the same relative probabilities.
- Just after $t = 0$, the same value or values for the energy are possible as in the previous part but they have different relative probabilities.
- Only one value for the energy is possible regardless of the time at which the measurement is made.
- Multiple values of the energy are possible regardless of the time at which the energy measurement is made.
- A long time after $t = 0$, only one value for the energy is possible.
- A long time after $t = 0$, multiple values for the energy are possible.

Figure 7.23 A part of a question with revised options concerning a perturbed well which appeared on the second exam and was asked of Physics 325 students in Winter 2006.

lapse of the wave function. These difficulties were shown to be intertwined with student understanding of eigenfunctions particularly their ability to differentiate between eigenfunctions of different operators. The third section focused on incorrect student reasoning concerning the time dependence of energy eigenfunctions or sums of eigenfunctions. This incorrect reasoning is often used to support student ideas about time-dependence discussed in Chapter 4. The next chapter focuses on attempts to address the student difficulties discussed in this chapter.

Chapter 8

Curriculum to Address Difficulties with Eigenstates and Measurement

The tutorials on quantum mechanics that were developed by the Physics Education Group prior to the research discussed in this dissertation included only a single tutorial on measurement. The research described in Chapters 4, 5, and 7 suggested to us that the issues involved were sufficiently complex as to warrant more than one tutorial. In Autumn 2004, we decided to design a new set of tutorials to focus on eigenstates, measurement and the related issue of time-dependence. The goals for the set of tutorials were to help students recognize that: (1) different potentials correspond to different sets of eigenstates, (2) a superposition of stationary states is not associated with a single energy, (3) the probability of measuring a particular value for the energy does not change with time in an isolated system, (4) the mathematical operation of the Hamiltonian acting on the wave function does not correspond to a measurement of the state, (5) a quantum mechanical superposition cannot be interpreted as simply a lack of knowledge about the state of the system. A summary of the evolution of the tutorial sequence is shown in Table 8.1 The set of tutorials and some of the motivation for their evolution is described below.

Table 8.1 Evolution of tutorials on time-dependence and measurement.

Course	**Tutorial**						
	MQM	**PCR**	**TDQ**	**EME**	**PME**	**UCQ**	**2DH**
Au 2002	X						
Su 2003	X	X					
Au 2003	X		X				
Au 2004	X		X				
Au 2005			X	X	X	X	X
Au 2006			X	X	X	X	X

MQM Original measurement tutorial (described in this chapter)
PCR Probability current (described in Chapter 14)
TDQ Time dependence in quantum mechanics (described in Chapter 6)
EME Energy measurements (described in this chapter)
PME Position, momentum, and energy measurements (described in this chapter)
UCQ Uncertainty: Classical and quantum (described in Chapter 13)
2DH Degenerate states in the two-dimensional harmonic oscillator (described in Chapter 6)

8.1 Measurements in Quantum Mechanics: Original Measurement Tutorial

The original tutorial on measurement had been designed early in the research on quantum mechanics by the Physics Education Group. Research had indicated that the tutorial effectively addressed some of the difficulties that had been identified prior to its development. The tutorial is briefly described below along with some of the motivation for changing the tutorial. This is followed by a description of the tutorials in the new sequence.

The original tutorial on measurement had two sections and was intended to be used as an interactive tutorial lecture. The first section focused on energy eigenstates. The second section focused on compatible and incompatible observables.

8.1.1 Energy eigenstates in the original measurement tutorial

Time dependence

The first section of the tutorial began by asking students to calculate $\hat{H}\Psi_1(x,t)$ for a single stationary state, $\Psi_1(x,t) = \psi_1(x)e^{-iE_1t/\hbar}$ in an unspecified one-dimensional potential well and to show that their result was consistent with Ψ_1 being an eigenstate of the Hamiltonian.

Students were intended to recognize that operating on the function with the Hamiltonian yields a constant multiplied by the same function. Thus, they could conclude that, by the definition of an eigenfunction, this state was an energy eigenstate.

Next students were asked to consider the simple superposition of two stationary states, $i\sqrt{\frac{1}{3}}\Psi_1(x,t) - \sqrt{\frac{2}{3}}\Psi_2(x,t)$. They were then asked if this superposition was an energy eigenstate.

Students were intended to use the same reasoning as before, but this time conclude that this state is not an energy eigenstate because it does not satisfy $\hat{H}\psi = E\psi$.

We found, however, that this exercise was difficult for students, despite being a very basic idea in quantum mechanics. The exercise, however, was useful for students, therefore, we decide to expand upon this idea and move it into the time dependence tutorial that was described in Chapter 6.

Measurement

In the original measurement tutorial students are next asked what results are possible for an energy measurement on the above superposition of two states both at an initial time and at a time after an initial measurement of energy had been made. In addition, they are asked to write down an expression for the wave function after the measurement and answer whether it is an energy eigenstate. They are also asked about repeated measurements of energy. Finally, they summarize their results by confronting two incorrect student statements.

In this sequence of questions students are intended to recognize that for the simple superposition of two stationary states, $i\sqrt{\frac{1}{3}}\Psi_1(x,t) - \sqrt{\frac{2}{3}}\Psi_2(x,t)$, possible results for an energy measurement are the energies corresponding to the states Ψ_1 and Ψ_2. A measurement collapses the wave function to one of these energy eigenstates. Subsequent measurements of energy on the system will always yield the same value.

Students usually had the most difficulty with the question about what happens at a later time. After the first energy measurement was made, some would answer that the system would oscillate from one eigenstate to another or that it would return to its initial state. These ideas were sometimes very strongly held as described in Chapter 4. In order to address these ideas more effectively, similar questions were moved forward to the time dependence tutorial described in Chapter 6 and also revisited in the tutorials described below.

Compatible and incompatible observables

The second section of the original measurement tutorial focused on compatible and incompatible measurements. Students were asked to consider a system initially in a stationary state for the infinite square well. They were then told a position measurement is made. The students were led to draw a possible wave function after the position measurement and to conclude that the resulting function is not an energy eigenstate but a superposition of many energy eigenstates. Then they were asked about a subsequent measurement of energy.

It was intended for students to recognize that the position measurement leaves the system in a superposition of many energy eigenstates. Thus, subsequent measurements of energy could have different possible results. These results are generally different than they would have if the position measurement were not made. This sequence of ideas led them to the idea of compatible and incompatible observables. The original tutorial went on by giving students practice with these same ideas using other observables.

Elements of the discussion of compatible and incompatible observables were retained in the new tutorial sequence. It was found, however, that students had additional difficulties with measurements of position, momentum and energy that were not being addressed in this tutorial.

8.1.2 Need for new tutorials

The original measurement tutorial did effectively address some student difficulties. Subsequent research, however, has revealed that some student difficulties related to those originally identified persisted after students had worked through the tutorial. To help address these issues, especially those that involved both time dependence and measurement, we reorganized and expanded the tutorial sequence. The tutorial on time dependence which has already been described was added (See Chapter 6). Also, the measurement tutorial was modified and expanded into two tutorials as described below.

8.2 New Tutorial Sequence

A new tutorial sequence gradually evolved from the original sequence. First, a tutorial called *Probability Current* discussed in Chapter 14 was used alongside the original measurement tutorial. Next a tutorial entitled *Time Dependence in Quantum Mechanics* was developed to replace the *Probability Current* tutorial. Later, the original measurements tutorial was replaced by two tutorials entitled *Energy Measurements* and *Position, Momentum, and*

Energy Measurements respectively. Finally a tutorial on *Degenerate States in the Two-Dimensional Harmonic Oscillator* was added to the sequence. In summary, the sequence of tutorials related to time dependence and measurement currently in use contains four tutorials. They are: *Time Dependence in Quantum Mechanics* discussed in Chapter 6, *Energy Measurements* discussed below, Position, Momentum, and Energy Measurements discussed below, and *Degenerate States in the Two-Dimensional Harmonic Oscillator* discussed in Chapter 6. It should be noted that other tutorials meant for the junior-level quantum mechanics sequence also involve time dependence and measurement but these topics are not the main focus of those other tutorials.

8.2.1 Energy measurements: the second tutorial in the new sequence

The tutorial on energy measurements has four sections. The first section focuses on the some of the differences between mixed states and superpositions. The second section concerns stationary states that are not sinusoidal. The third section requires students to consider measurements made on a system, and whether the possible results for such measurements depends upon time. The fourth section generalizes the results of the specific cases considered.

The nature of superpositions

The first section of the *Energy Measurements* tutorial begins by asking students about a superposition of two stationary states in the infinite square well. Students are asked whether they agree or disagree with the following statement.

> *The wave function given by* $\Psi(x,0) = \sqrt{\frac{1}{2}}\psi_1 + \sqrt{\frac{1}{2}}\psi_2$ *represents a lack of knowledge about the state of the system. The system is definitely in either the ground state or the first excited state. The wave*

function simply tells you that the probability is $\frac{1}{2}$ that the system is really in the ground state and $\frac{1}{2}$that it is really in the first excited state.

This question engenders animated group discussion about the nature of superpositions. Most groups rather quickly disagree with the statement. The next question (shown in Figure 8.1), however, requires them to think more deeply about the issues involved. It presents them with two ensembles of systems. In one ensemble there is a mixture of systems. Half are in one stationary state, and half are in another. In the second ensemble, all the systems are in a known superposition of the same two states. The coefficients in the superposition in the second ensemble are arranged so that each state is equally weighted. Students are asked if it is possible that the two situations give rise to different physical results. Generally groups take more time with this question than the first.[1]

> Suppose you have the following two ensembles of systems in a quantum mechanical infinite square well at time $t = 0$:
>
> (1) half of the systems are in state ψ_1 and half of the systems are in state ψ_2
>
> (2) all of the systems are in state $(\psi_1 + \psi_2)/\sqrt{2}$.
>
> At $t = 0$ would the probability of finding a particle on the left half of the well in a randomly selected system from ensemble (1) be *greater than, less than,* or *equal to* the probability of finding a particle on the left half of the well in a randomly selected system from ensemble (2)? Would your answer change as time evolves? Explain.

Figure 8.1 A question regarding the nature of superpositions in the energy measurements tutorial.

To answer the question, students need to recognize that at $t = 0$ the superposition gives rise to a probability density that is larger on one side of the well than the other, but the mixed state has a symmetric distribution. Thus, at $t = 0$ multiple measurements of position on each ensemble would yield different results statistically. In addition, as time evolves the probability distribution for the superposition shifts back and forth in the well, but the distribution for the mixed state remains fixed. This gives another way to differentiate between the two ensembles. The result gives a case in which the interpretation of a wave function given by the student at the beginning of this section cannot be valid.

After some discussion with members of their group, students are generally able to answer this question. Sometimes, it is helpful for teaching assistants to point out that the superposition given in this case is the same as the one students encountered in the time dependence tutorial. This allows the students to think more critically about the question since they know from prior experience that the probability distribution associated with the superposition is time dependent.

Energy eigenfunction are not all sinusoidal

The second section of the tutorial proceeds to focus on a particular wave function, $\sqrt{\frac{1}{L}}\cos\frac{\pi x}{2L}$, in the harmonic oscillator. This wave function happens to be identical in form to an energy eigenstate in the infinite square well. Students are first asked whether the probability density associated with this state changes with time. The students are then asked whether or not they agree with the following statement.

> *The state does depend on time. The full wave function is given by* $\Psi(x,t) = \sqrt{\frac{1}{L}}\cos\frac{\pi x}{2L}e^{iE_1 t/\hbar}$. *The probability density, however, does not depend on time.*

Students are also asked if the wave function proposed in the above statement satisfies the Schrödinger equation for the harmonic oscillator.

The questions above are interrelated. Next, students are asked if this state can be associated with a single energy, and to describe how they would find the time dependence of the state.

Generally students, working as a group, are able to correctly conclude that the proposed wave function is not a stationary state for the harmonic oscillator. Occasionally, however, the teaching assistants need to encourage students to try the proposed solution in the time-dependent Schrödinger equation. Some students are so convinced that it does satisfy the time-dependent Schrödinger equation that they claim verification is not necessary.

Since the initial state does not satisfy the energy eigenvalue (time-independent Schrödinger) equation, it cannot be associated with a single energy. Thus the time-dependence proposed by the student in the tutorial is not correct. To find its time dependence, it is necessary to express it as a linear combination of the stationary states of the harmonic oscillator, $\Psi(x,0) = \sum c_n \psi_n(x)$. In that form, it is simple to put in the correct complex exponential time-dependent factor in each term of the linear combination, $\Psi(x,t) = \sum c_n \psi_n(x) e^{-iE_n t/\hbar}$.

At this point, most groups are able to conclude that the state is not associated with a single energy, but many have trouble describing the procedure necessary to find the time dependence. Usually after some discussion and possible teaching assistant intervention, students conclude that they would need to express the function as a linear combination of the stationary states of the harmonic oscillator. Then, they could put in the appropriate time dependent factors.

Energy measurements

The third section of the tutorial begins by describing a system with a harmonic oscillator potential in a state such that only only the first two energy eigenvalues are possible values resulting from an energy measurement, and that these are equally likely. Students are asked whether this information is sufficient to completely specify the system. If not, they are asked to to write and sketch at least two states that satisfy this condition. They are then given a specific superposition that satisfies this condition. They are asked if this is an eigenstate, as well as to describe the time evolution of the state.

The state described is not completely specified by the information given because there could be any relative phase between the two states that comprise the superposition. No, state that matches the description, however, is an energy eigenstate. So, the probability distribution associated with any state fitting the description must evolve in time.

Most groups proceed through this section of the tutorial without much teaching assistant intervention. It is in many ways a review of what they learned in the previous tutorial, but requires students to generalize their results to the harmonic oscillator.

Next students are told to assume that the initial state of the system is given by $\Psi(x,0) = i\sqrt{\frac{1}{3}}\psi_o - \sqrt{\frac{2}{3}}\psi_1$. They are asked for the possible results of an energy measurement at time $t = 0$ as well as the probability the ground state energy is measured.

Here there is a probability of $1/3$ of measuring E_o and $2/3$ of measuring E_1.

Generally, most groups are able to answer this question with relative ease.

Students are then asked if the probability of measuring the ground state depends on the time the measurement is made, and to make sure their answers are consistent with the time evolution of the wave function.

As time evolves, the relative phase between the states comprising the superposition changes, but the probability of measuring the ground state energy stays the same.

This question is generally more difficult than the last, but most groups eventually conclude that the probability of measuring the ground state energy does not change with time.

Next students are asked to consider the system after an energy measurement is made. They are asked for the wave function immediately after such a measurement, whether the mathematical action of the Hamiltonian on the initial state produces this wave function, and for the results of a subsequent energy measurement. In answering, students must recognize that the measurement collapses the wave function to an eigenstate consistent with the measured eigenvalue, and the the system will remain in this state since it is a stationary state.

Generalization

In the final section of the tutorial, students are asked questions that help them generalize their results. They are asked to consider a general quantum mechanical state. For that state they must decide if the possible values for an energy measurement or their relative probabilities will change with time.

At this point, most groups conclude that neither the possible values nor the probabilities will change with time.

Students are then asked if the possible values for position measurements on this system or their relative probabilities will change as time evolves.

This question is more difficult, and related issues are explored more fully in the next tutorial.

Finally, students are asked to describe the general procedure of finding $\Psi(x,t)$ given $\Psi(x,0)$.

The wave function must be expanded in terms of the stationary states of the potential under consideration. Then the appropriate exponential time factors must be inserted into the superposition.

8.2.2 Measurement tutorial: The third tutorial in the new sequence

The new measurement tutorial consists of five sections. The first section focuses on eigenvalue equations for position momentum and energy. The second and third sections consider eigenvalues and eigenvectors for position and momentum respectively. The fourth section concerns position, momentum and energy measurements as well as the wave function before and after these measurements. The fifth section deals with compatible and incompatible observables.

Eigenvalue equations

In this first section, students are asked to write the operators associated with measurements of position, momentum, energy. They are also asked to write the eigenvalue equations associated with these operators. In completing both tasks, students are asked to write their answers in two ways: (1) in terms of symbols for the operators and (2) in terms of x and derivatives with respect to x. Students must also identify which symbols are constant and which depend upon x. In addition, students consider which of their answers depend on the specific system under consideration.

Groups are typically able to complete these exercises. Some students struggle to identify which symbols depend on x. This is particularly difficult in the case of position measurements. This particular case is discussed further in the next section.

Eigenvalues and eigenvectors of position

This section leads students to consider the eigenvectors and eigenvalues for position. It encourages students to consider the eigenvalue equation for position as $(x - \alpha)\phi(x) = 0$. They are asked describe the eigenfunctions in words, and whether the eigenvalues are continuous or discrete. Finally they are asked if the delta function is consistent with their description, and if the eigenfunctions of position are also eigenfunctions of energy.

Students often begin this exercise having memorized that the eigenfunctions of position are delta functions. Many have not thought about why this should be true, and these exercises provide an opportunity for such reflection. The last question also confronts the incorrect idea that the eigenfunctions for all observables are identical.

Eigenvalues and eigenvectors of momentum

This section follows a similar pattern as the last, but focuses on momentum operators. It further cements the idea that eigenfunctions for different observables are different.

Measurements of position, momentum, and energy

In this section students compare position, momentum, and energy measurements side by side. They are asked to describe the state of a system after each of these three possible measurements. They must decide whether the action of the corresponding operator mathematically produces the wave function after the measurement. They are also asked if the system evolves in time after each of the three measurements. In addition, they consider a second measurement of each type and are asked if it would yield the same or a different value as the first measurement. They also must decide if it matters when this second measurement occurs.

Generally, this section takes students some time to work through. The section makes it clear that energy measurements are special in that energy eigenstates are stationary states. The side-by-side format forces students to consider such differences.

Compatible and incompatible observables

The last section of the tutorial explores differences in sequences of measurements. It defines incompatible and compatible observables in the following way. If the order in which two observable quantities are measured could affect the outcome of the measurement, then it is said that they are incompatible observables. If the order does not matter, then they are compatible. The tutorial asks students to consider specific sequences of position, momentum, and energy measurements in exploring this idea.

8.3 Summary

The tutorial sequence described is intended to help students with many of the steps necessary to determine the time dependence of a general state. The effectiveness of the tutorial sequence in addressing specific student difficulties inherent in this process is described in the next chapter.

8.4 Notes to Chapter 8

[1] See Section 5.6 for a discussion of related student difficulties

Chapter 9

Assessment of the Tutorial Sequence on Time Dependence and Measurement

The effectiveness of the tutorial sequence described in Chapters 6, and 8 has been assessed in a number of different ways. We have been able to compare the performance of students who worked through the tutorial sequence to published student performance at similar institutions which do not use tutorials and to published performance of introductory graduate students. In addition we have been able to compare student performance at the University of Washington with and without tutorials. Finally, student performance has been compared before and after working through the tutorials. In all cases students were found to perform better after working through the tutorial sequence. In this chapter, these assessments will be described in the context in which they occurred.

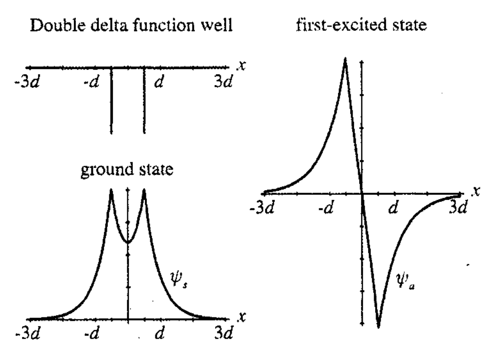
9.1 Comparison of Introductory Quantum Mechanics Classes With and Without Tutorial Instruction

In Autumn quarter of 2006, 44 students in Physics 324 were asked the question shown in Figure 9.1 on their second exam. This question focused on student ability to discriminate between stationary states from superpositions of stationary states. The second exam occurred after students had worked through the tutorial sequence described in Chapters 6, and 8 with the exception of the tutorial focusing on degeneracy in the two-dimensional harmonic oscillator. The question Figure 9.1 is very similar to the question asked at The Ohio State University in 2004 and described in Chapter 5. Both questions gave students a superposition of energy eigenstates and asked if the superposition was a stationary state.

Consider a system with the potential shown at right consisting of two delta function wells separated by a distance, d, centered about the origin. The symmetric ground state wave function, ψ_s, and the anti-symmetric first excited state wave function, ψ_a, for this system are also shown at right. Assume that the energy of the anti-symmetric first excited state is four times the energy of the symmetric ground state, $E_a = 4E_s$.

The system is initially in a state given by

$$\Psi(x, t=0) = \frac{1}{\sqrt{2}}[\psi_s + i\psi_a]$$



Is this state a stationary state? Explain how you know for full credit.

Figure 9.1 A question asked of 44 Physics 324 students on their second exam in Autumn quarter of 2006.

Table 9.1 compares the results of these two questions. With tutorial instruction the fraction of students answering correctly for the correct reasons is larger. In addition, both the incorrect answers and the correct answers with incorrect reasoning are less prevalent among students with tutorial instruction.

Table 9.1 Comparison of student performance with and without relevant tutorial instruction on superpositions of stationary states.

Response	without tutorials N=35	with tutorials N=44	
correct with correct reasoning	35% (12)	75%	(33)
correct incorrect reasoning	25% (8)	10%	(4)
incorrect	45% (15)	15%	(7)

9.2 Comparison of Student Performance at Multiple Levels with Varying Degrees of Tutorial Instruction

A survey conducted by Chandralekha Singh at the University of Pittsburgh[1] provides another measure of the effectiveness of the tutorial sequence. This 50 minute survey was given to 202 graduate students from seven different universities at the beginning of their graduate-level quantum mechanics course. It consisted of a number of questions three of which we focus on here. In the original survey, the written questions were followed up with interviews. The researchers found that students had common difficulties with quantum mechanics no matter where they were enrolled.

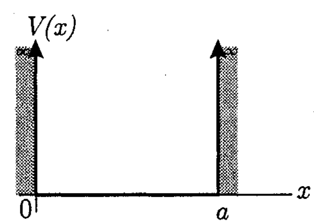
We asked some of these same survey questions of students enrolled in the Physics 324 class at the University of Washington some of whom had worked through tutorials on time dependence and others who had not. In addition we asked some of the survey questions of students in the Physics 315 class at the University of Washington, which does not use tutorials. Comparisons among the populations that answered questions from the survey provide some measure of the overall effectiveness of the entire tutorial sequence. Additional comparisons were made possible by similarities between questions on the survey and questions that have previously appeared on examinations at the University of Washington. This set of similar questions allowed us to compare student performance without tutorial instruction, with some tutorial instruction, and with an entire sequence of tutorials. The questions are discussed as appropriate in the following sections.

9.2.1 Assessment of student understanding of the time dependence of a superposition of stationary states

As we have seen in Chapters 4, 5, and 7, student understanding of time dependence in quantum mechanics is often incomplete. Facility in manipulating equations, while important, does not necessarily reflect an understanding of the physical implications of those manipulations. Moreover, errors in mathematical manipulation can reflect underlying conceptual errors.

The survey by Singh contains a question, shown in Figure 9.2, that deals explicitly with time dependence and requires formal mathematical manipulation. This question as well as very similar questions have been asked of quantum mechanics students at the University of Washington both before and after tutorial instruction.

The wave function of an electron in a one-dimensional infinite square well of width a at time $t = 0$ is given by $\Psi(x,0) = \sqrt{2/7}\phi_1(x) + \sqrt{5/7}\phi_2(x)$ where $\phi_1(x)$ and $\phi_2(x)$ are the ground state and first excited stationary states of the system.



($\phi_n(x) = \sqrt{2/a}\sin(n\pi x/a)$. $E_n = n^2\pi^2\hbar^2/(2ma^2)$ where $n = 1,2,3\ldots$)

Write down the wave function $\Psi(x,t)$ at time t in terms of $\phi_1(x)$ and $\phi_2(x)$.

Figure 9.2 A standard question concerning time dependence.

This question presents students with a quantum mechanical state at an initial time, $\Psi(x,0) = \sqrt{2/7}\phi_1(x) + \sqrt{5/7}\phi_2(x)$. The students are asked to determine the state at a subsequent time, t.

A solution to this problem involves several steps. Students must recognize that the Schrödinger equation, which governs the time evolution of this system, can be separated into a part that depends only on time and a part that depends only on space. They must recall that products of the solutions to these individual parts are solutions to the entire equation. They must also recognize that sums of such solutions are again solutions. Finally, they need to remember that they were given a sum of two solutions to the spatial part. Then, they can construct a solution to the entire equation that matches the initial condition by multiplying each spatial solution by the appropriate solution to the time dependent part of the original equation. Thus, they find $\Psi(x,t) = \sqrt{2/7}\phi_1(x)e^{-E_1 t/\hbar} + \sqrt{5/7}\phi_2(x)e^{-E_2 t/\hbar}$.

The reasoning underlying a solution such as this is quite involved, and this sort of problem lends itself to a memorized response. For this question we found that, most students included very little explanation for their response. This supports the idea that most who gave the correct answer arrived at it without going through all the logic outlined above. The fact that this problem is answerable without going through the requisite reasoning makes it a poor choice as a research question to get insights into student thinking. The frequency with which this question has been asked, however, makes it possible to compare student performance across classes and institutions.

At the University of Washington several similar versions of this question have been asked of different populations. Versions were asked on final examinations in Physics 315[2] in the winter quarters of both 2004 and 2006. This course is lecture based and does not use tutorials. In the fall of 2002 and 2005 a version of this question was given to students in Physics 324. In 2002 the students in Physics 324 had worked through quantum mechanics tutorials, but none of those tutorials explicitly addressed time dependence. In 2005 the students had worked through a sequence of tutorials, some of which explicitly addressed time dependence.

The performance of students from the graduate student survey by Singh as well as the Physics 324 class at the University of Washington is tabulated in Table 9.2. Notice, the Physics 324 students perform somewhat better than the published survey results after some tutorial instruction. This instruction is described in Chapter 12, but is not directly related to time dependence. There is, however, significant improvement after students worked through the tutorial sequence described in Chapters 6 and 8 that emphasize time dependence. The most common error in all populations is to use a single time-dependent phase factor (*e.g.*, $\Psi(x,t) = [\sqrt{2/7}\phi_1(x) + \sqrt{5/7}\phi_2(x)]e^{-iEt/\hbar}$). Another common error among students who did not have tutorial instruction was to conclude that the state does not depend on time.

Table 9.2 Performance of students in three distinct populations on a question concerning time dependence of a superposition of energy eigenstates.

Context	Correct	Common phase	No time dependence
Graduate student survey (N = 202)	43%	31%	9%
Physics 324 with some tutorials (N = 79)	67% (53)	15% (12)	4% (3)
Physics 324 tutorial sequence (N = 56)	91% (51)	4% (2)	0% (0)

9.2.2 Questions on measurement and expectation value

Other questions asked on the survey by Singh were not directly related to time dependence. Two of those questions used the same context as the question in Figure 9.2, but asked about measurement and expectation value. Those two questions are shown in Figure 9.3 and Figure 9.4. We asked both of these questions on the final exams of both Physics 315 and Physics 324.

The first of the two questions (Figure 9.3) asks the students to determine the possible values of energy that could be measured for the system in its initial state and the probabilities of measuring those energies. The solution to involves the recognition that the only values obtainable from a measurement of an observable are equal to the eigenvalues of the mathematical operator corresponding to that observable. Furthermore, the probability of obtaining a particular value for an observable is obtained by squaring the magnitude of the coefficient of the eigenfunction having that eigenvalue in an expansion of the wave function in terms of the eigenfunctions of the operator considered.

To answer this specific problem the student must recognize that the wave function is already expanded in terms of the energy eigenfunctions. Thus, it is possible to associate the square magnitudes of the coefficients with the probability of obtaining the corresponding energy. In this case, the only values possible are E_1 and E_2 with probabilities $2/7$ and $5/7$ respectively.

$$\Psi(x,0) = \sqrt{2/7}\phi_1(x) + \sqrt{5/7}\phi_2(x)$$

Suppose you measure the energy of an electron at time $t = 0$. What are the possible values of the energy and the probability of measuring each?

Figure 9.3 Survey question concerning measurement. The initial wave function is repeated for the convenience of the reader.

The second question asks for the expectation value of the energy. Knowing the possible values of the energy and their probabilities, the weighted average can be calculated. The probability of a particular value is multiplied by that value and all such quantities are added together. This results in $(2E_1 + 5E_2)/7$.

$$\Psi(x,0) = \sqrt{2/7}\phi_1(x) + \sqrt{5/7}\phi_2(x)$$

Calculate the expectation value of the energy in the state $\Psi(x,t))$ above.

Figure 9.4 Survey question concerning expectation value. The initial wave function is repeated for the convenience of the reader.

Both questions were given to students in on the final examinations in Physics 315 in the Winter quarter of 2006 and in Physics 324 during the Autumn quarter of 2005 at the University of Washington. Student performance is tabulated in Table 9.3. The difference between the Physics 315 students and the graduate students in the Singh survey is not significant[3], but the difference between the them and the Physics 324 students as well as the difference between the Physics 315 and 324 students is highly significant. While we have no evidence that these populations are the same before instruction, the performance of the 324 students is encouraging.

Table 9.3 Student responses to survey questions concerning measurement and expectation value

Context	**Correct possible values for energy and their probabilities**	**Correct expectation value**
Physics 315 [no tutorials] (N = 31)	61% (19)	29% (9)
Graduate student survey (N = 202)	67%	39%
Physics 324 [tutorial sequence] (N = 56)	96% (54)	77% (43)

Common errors reported in the survey of graduate students for the problem in Figure 9.3 included giving the expectation value as the only possible value and indicating that any energy eigenvalue for the infinite square well is possible. In the Physics 315 class only one of the 11 students with incorrect responses gave the expectation value as a possible result of an energy measurement, two of them gave incorrect responses indicated that any of the eigenvalues were possible, and four students gave the coefficients or the wave functions as the possible energies. Of the only two out of 56 students who made errors in the Physics 324 course, one made an error that stemmed from an incorrect explicit calculation of the energy eigenvalues. The other wrote a wave function, but did not otherwise answer the question.

In calculating the expectation value many of the graduate students who took the survey began with the wave function and the Hamiltonian rather than the energies and their probabilities, but made errors in the ensuing calculations. Some failed to integrate, writing $< E >= \Psi^* \hat{H} \Psi$. Others, about 17%, wrote the formal expression $< \Psi | E | \Psi >$, but were unable to proceed. In the Physics 315 class, six students wrote only formal expressions, but most of the errors were in the form of incorrect calculations of various types. Four of the Physics 324 students wrote only wrote the formal expression. Two attempted to calculate the expectation value of the position. The seven remaining made errors in calculations or did not answer.

9.2.3 Comparison of overall survey results

A more discriminating measure of student understanding is to compare the number of students from different populations that answered all three survey questions discussed correctly. The researchers who conducted the graduate survey did not report the number of students who answered all three questions correctly. It must, however, be less than 39%, the worst result on an individual question. That information along with results for the Physics 315 and 324 classes is tabulated in Table 9.4. The differences between the classes are statistically highly significant.

Table 9.4 Percentages of students who answered the first three and all four questions correctly.

Context	**Correct on first 3**
Physics 315 [no tutorials] (N = 31)	26% (8)
Physics 324 [tutorial sequence] (N = 56)	75% (42)
Graduate student survey (N = 202)	39% max.

Comparing our data with that of the graduate student survey has produced several interesting results. For *Tutorials in introductory physics* used in the first year calculus and algebra based courses, the Physics Education Group often uses the performance of 1^{st} year graduate students as an upper bound for student performance after tutorial instruction. It is clear that graduate student performance will not serve as a convenient upper bound on the capabilities of upper-division undergraduates. This might have been expected as many undergraduate physics majors go on to graduate study a short time after taking an undergraduate quantum mechanics course. Nevertheless, our results show undergraduate students with tutorial preparation performing significantly better than the average beginning graduate student. We interpret this as a strong indication that our tutorial sequence is having a positive effect.

9.3 Comparison of Student Performance before and after Tutorial Instruction

In addition to comparisons of student performance with and without tutorial instruction, questions covering similar topics have been administered both before and after tutorial instruction. In what follows "pretests" refer to questions administered before tutorial instruction without regard to lecture instruction. "Post-tests" refer to questions administered after tutorial instruction.

9.3.1 Assessment of student understanding of the time dependence of a general state

A unique situation occurred in Autumn quarter of 2006. The course instructor put the same question on the first examination and on the final examination. The first exam occurred immediately before students began working on the tutorial sequence described in Chapters 6 and 8. So, this first administration of this question can act as a pretest of the tutorial sequence. The second administration of the question then acts as a post-test.

A particle of mass, m, is placed in an 1d infinite square well potential: $V(x) = \infty$ for $x < 0$ and $x > L$, and $V(x) = 0$ for $0 \le x \le L$. At time $t = 0$ the particle is prepared in an initial state given by:

$$\Psi(x,0) = \begin{cases} 0 & \text{for } x < \frac{L}{2}, \\ \sqrt{\frac{2}{L}} & \text{for } \frac{L}{2} \le x < L, \\ 0 & \text{for } x > L. \end{cases}$$

Will $< x >$ change with time? Explain.

The energy measured and found to be E_2, the second energy eigenstate. After the measurement that yielded energy E_2 has been performed, the system is left alone for a long time. Qualitatively, how does the probability density of the particle evolve during that time?

Figure 9.5 A question asked twice of students in Physics 324 in the Autumn quarter of 2006.

The relevant problems are shown in Figure 9.5. Students are told that a particle in the infinite square well is initially in a state given by a step function in the right half of the well. The first question asks them if the expectation value for position would change with time. The correct answer is to recognize that the wave function is a superposition of many stationary states. Therefore, $< x >$ changes with time. As shown in Table 9.5 student performance improved from 60% correct with no incorrect reasoning to 85%.

Table 9.5 Comparison of student performance before and after tutorial instruction.

Response	pretest N=46	post-test N=43
correct with no incorrect reasoning	60% (27)	85% (36)
incorrect	40% (19)	15%(7)

The second problem on the exam question shown in Figure 9.5 asked students to describe the time evolution of the probability distribution for an isolated system after an energy measurement has been made. The correct answer is to recognize that the measurement collapses the wave function to an energy eigenstate with a time-independent probability distribution.

Table 9.6 Comparison of student performance before and after tutorial instruction.

Response	pretest N=46	post-test N=43
correct with no incorrect correct reasoning	35% (17)	90% (38)
"diffusion" based incorrect reasoning	35% (17)	5%(2)
"revival" based incorrect reasoning	10% (4)	5%(3)
"decay" based incorrect reasoning	5% (2)	0%(0)
other incorrect reasoning	15% (6)	0%(0)

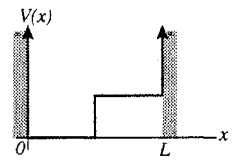
Student performance is shown in Table 9.6. As shown student performance improves from the 35% level to the 90% level. Additionally this question elicits many of the naïve mechanisms for time dependence discussed in Chapter 4. These methods of reasoning dominate student thinking before the tutorial sequence, but are greatly reduced after tutorial instruction.

9.3.2 Assessment of student recognition that stationary states depend on the potential

In Autumn quarter of 2006, 43 students were asked a question on their final exam that examined similar ideas as a question they answered before tutorial instruction on the relevant topics. This question is shown in Figure 9.6.

In this question students are shown a potential well with two distinct levels. They are also given a mathematical representation of the initial wave function for the system. It should be noted that the given wave function would be an energy eigenfunction for a system with an infinite square well potential. In this case, however, this function is clearly not an energy eigenfunction. Students are then asked if either the wave function or the probability density change with time. We have found that this set of questions are more illustrative than either one alone. Students should respond that both the wave

Consider the one-dimensional potential shown at right with a step between 0 and L and infinite potential outside. A system with this potential is initially prepared in a state given by: $\Psi(x,t=0) = \sqrt{\frac{2}{L}}\sin(2\pi x / L)$ between 0 and L and zero elsewhere.



1. Does the wave function, $\Psi(x,t)$, change as time passes? Explain.

2. Does the probability density, $\rho(x,t)$, change as time passes? Explain.

Figure 9.6 A question asked of 43 Physics 324 students on their final exam in Autumn quarter of 2006.

function and the associated probability density change with time since the state is not an energy eigenstate.

Table 9.7 Comparison of student performance before and after relevant tutorial instruction on energy eigenfunctions.

Response	**Pretest N=93**	**Post-test N=43**
correct with correct reasoning	15% (15)	50% (21)
common phase; eigenstate	55% (49)	40% (18)
other	30% (29)	10% (4)

Student performance on this question in 2006 is compared with a similar question that was asked before tutorial instruction in 2005 and 2006 in Table 9.7. On the post-test half of the 43 students were able to answer both questions correctly with correct reasoning. Again, the most common incorrect response was that the given state was an eigenstate or that it was associated with a single energy. Thus, the probability density would not change. The prevalence of this error decreased when compared to the pretest. It is also important to note that other errors were also reduced by a factor of three. Many of these other errors can be viewed as more serious than the common phase error. They often involve reasoning in a manner more closely associated with classical mechanics. This often takes the form of a focus on different regions of the well and positing different energies for different regions. Other errors are associated with the naïve mechanisms for time dependence that we termed “decay”, “diffusion”, or “revival”. Comparatively we view the common phase error as a possible step in the right direction from these more serious types of errors.

While performance at the 50% level on such a seemingly simple question on a final exam is not ideal, it attests to the difficulty of such questions for introductory quantum mechanics students. Student improvement on this measure is also indicative of the effectiveness of the tutorial sequence.

9.4 | Summary

The results described show that students that have tutorial instruction: perform better after that instruction than before, perform better than comparable students with only traditional instruction, and outperform first year graduate students. This is an indication that the tutorials are having the desired effect. We believe, however, that they could be better. While students who worked through the tutorial sequence outperformed those who had not, their performance on simple questions indicated that many students still have serious difficulties with the material. We believe that a continual cycle of research, curriculum development, and instruction is the key to further improvement.[4]

9.5 Notes to Chapter 9

1 See (Singh, 2005)

2 See Chapter 2 for a description of the courses involved.

3 Here statistical significance is taken to mean that there is less than a 5% chance that a class outperformed another by the measured amount or more under the assumption that the two classes are equivalent. Highly significant is taken to mean that this chance is less than 1%.

To compute these probabilities, make the assumption that the two populations are equivalent. Under that assumption selecting students for each class is arbitrary. Calculate the probability that a class is selected with the measured value or greater number of correct responses given the number of students in each class and the total number of correct responses.

4 The implementation of regular tutorial homework has the potential to improve performance as students in advanced courses spend a significant amount of time and effort to complete homework assignments.

Part II

Investigation of Student Ability to Relate Classical and Quantum Mechanics

Chapter 10

Introduction to Part II: Prior Research on Student Ability to Relate Classical and Quantum Mechanics

The long history of classical mechanics suggests that it was not a simple task to develop the language, concepts, and ideas necessary to transform real-world observations into a concise description of a system. In quantum mechanics, the description of a system's state bears little resemblance to the classical case. In classical mechanics, position and its rate of change are both needed to specify a state. In contrast, the quantum mechanical model involves a wave function that completely describes the system. It is not necessary to specify the time rate of change of the wave function as it is determined by the wave function itself as a solution to the Schrödinger equation. Furthermore, the wave function is not directly associated with any element of physical reality. This stands in stark contrast to elements in the theory of classical mechanics. Einstein, Podolsky, and Rosen expressed publicly their abhorrence of non-physical elements, such as the wave function, in their famous EPR paper.[1]

Given this situation, it is perhaps not surprising that students have difficulty in relating their understanding of classical mechanics to quantum mechanics. This part of the dissertation describes an investigation of student difficulties with this relationship and how the results were used in the development of curricula to address these difficulties.

In addition to this introduction containing relevant prior research, this part of the dissertation consists of three chapters. The first chapter describes our efforts to identify student difficulties associated with the relationship between classical and quantum mechanics. The second chapter discusses curriculum intended to address those difficulties and the assessment of that curriculum. The third chapter examines student understanding of some necessary requirements for quantum states to reproduce the predictions of classical mechanics.

10.1 Previous Research and Curriculum Development on Relating Classical and Quantum Mechanics

Several prior efforts have been made to examine student understanding of stationary quantum states and their relationship to classical mechanics. Some related curriculum development was guided by this research. In this section, a simple example from prior work is used to illustrate the depth of student difficulties. This is followed by a discussion of prior research and curriculum development.

10.1.1 Example of student difficulty in relating classical and quantum mechanics

An interview task designed by Brad Ambrose, a former graduate student at the University of Washington, illustrates student inability to relate classical and quantum mechanics even in simple contexts. The interview task was as follows:

> The interview volunteers were asked to draw qualitatively correct sketches of the wave functions for several energy eigenstates of an infinite square well potential and to explain their reasoning.[2]

Almost all of the interview subjects were able to draw the correct wave functions. Some of their explanations, however, indicated that they were using the corresponding classical case of a particle in a box incorrectly to reason about the quantum mechanical case. As an example, a student used the following reasoning to justify a correct sketch of the ground state.

> You can think of [the electron] as trying to get out and constantly running into a wall, but it can't go through, so it bounces off the wall and comes back . . . Since it's constantly going back and forth, it spends more time in the center because it crosses the center twice for every one time that it hits a wall.[3]

Clearly this student's justification is not correct. Classical mechanics predicts that a particle would spend an equal amount of time in any two regions of the well with equal width. Responses like the one above indicate that students have trouble describing physical situations in terms of probabilities, and that this difficulty may affect their ability to understand certain aspects of quantum mechanics. In particular, these students did not recognize that the ground state is very different from what one would predict classically Highly excited states are a better match to the classical case.[4]

10.1.2 Prior research on student understanding of probability and statistics

From the time that Max Born, in his 1926 paper, postulated a statistical interpretation for the squared magnitude of the wave function; probability has been ubiquitous in quantum mechanics. A vast literature exists outside of physics education research which describes the difficulties students have with probabilistic and statistical concepts[5]. A few of these are particularly relevant for quantum mechanics. Many quantum mechanics books, however, spend only a few pages, if any, addressing these known difficulties. Perhaps this is due to an assumption that these difficulties are not prevalent among the population that usually takes a quantum mechanics course. A few studies[6] have indicated that assumptions about students who take quantum mechanics possessing a robust understanding of concepts in probability and statistics may be unwarranted. The presence of such difficulties may be exacerbated by the fact that very few students take a course in statistics before they take quantum mechanics.[7]

Given the importance of probability and statistics in quantum mechanics and the vast literature concerning student difficulties in probability and statistics, there are many opportunities to investigate how student misconceptions in probability and statistics influence their ability to develop a functional understanding of quantum mechanics. An example from our research was discussed in Chapter 4 Section 4.2 where we found that quantum mechanics students are about as likely to exhibit the "gambler's fallacy" as introductory psychological statistics students and exhibit the same types of incorrect reasoning with nearly the same frequency.

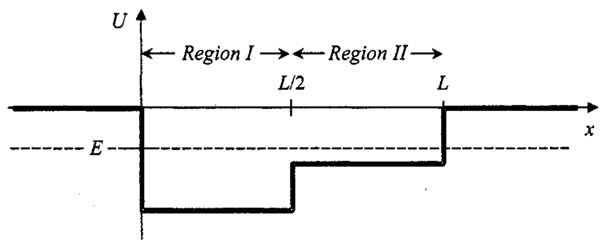
Although exploring more documented difficulties arising from probability and statistics in the context of quantum mechanics is a possible direction for future research, most of our work has been guided first by physical contexts.

10.1.3 Prior research on student ability to relate classical and quantum mechanics

Prior work at the University of Washington[8] as well as research conducted at the University of Maryland[9] provide insights into student ability to relate classical mechanics to quantum mechanics. To appreciate this work it is helpful to consider an example that is similar to questions asked of students in studies at the University of Washington as well studies at the University of Maryland. An example of such a question is shown in Figure 10.1.

In this question, students were show a potential well with two distinct levels. Students were asked to draw a qualitatively correct wave function for a highly excited energy eigenstate of the system.

An electron is in a bound state of the potential well shown at right. Consider the case in which the electron occupies a *highly excited* energy eigenstate of the well as shown. (*i.e.*, The energy of the electron is much greater than the ground state energy, and the wave function $\psi_n(x)$ of the electron has several nodes between $x = 0$ and $x = L$.)



The regions $0 < x < L/2$ and $L/2 < x < L$ are labeled as regions I and II, respectively.

Draw a qualitatively correct graph of the wave function.

Figure 10.1 A question asked of Physics 324 students in Autumn quarter of 2002 and 2003 as well as in the Summer quarter of 2003.

Given the potential well, students can take several distinct steps to come to a correct description of a highly excited energy eigenstate. First, they might consider the form of the wave function in different regions. They could use the Schrödinger equation to recognize that the energy eigenfunctions are exponential where the potential exceeds the total energy and sinusoidal elsewhere. In addition, they need to recognize that the difference between the total energy and potential determines the relative curvature of an energy eigenfunction. This information allows a student to make the conclusion that the wave number must be greater on the left side of the well shown in Figure 10.1.

Reasoning about the amplitude requires two additional steps. Students must recognize that the theory of quantum mechanics and classical theories must correspond for highly excited states. This allows them to conclude that the time averaged probability density for the quantum system must be indistinguishable from the time averaged probability density as predicted by a classical analysis. The next step is to determine the time averaged probability density as predicted by classical mechanics and use this to draw a wave function with qualitatively correct amplitude variations. In the case shown in Figure 10.1 a classical analysis would predict that a particle would spend more time on the right than on the left. Thus, the time averaged probability would be greater on the right, and hence the amplitude of the quantum mechanical wave function must be larger on the right.

Researchers conducting an investigation in the mid 1990s at the University of Maryland found that introductory students as well as students of quantum mechanics had a number of difficulties in approaching such problems. They found that students could not predict the most likely location of a particle given its wave function. They also found that students often thought that different eigenstates corresponded to different positions in the well. In addition, they found many students incorrectly employing classical reasoning to explain quantum phenomena.

Prior research at the University of Washington also employed questions similar to that shown in Figure 10.1 to probe student understanding. That work examined student's ability to relate potentials to probability densities for stationary states. The following student difficulties were all found in the context of stationary quantum states.[10]

- Mistaken belief that the probability density is larger in regions of lower potential.
- Failure to recognize that the amplitude of the wave function is constant in regions of constant potential.
- Mistaken belief that reasoning from classical mechanics can be used to account for the ground state wave function.
- Failure to recognize that the classical result for probability density is not valid for small values of momentum.

- Failure to express correctly the limit in which classical mechanics should be abandoned for quantum mechanics.

Both of these prior research efforts found evidence that students struggle to use classical mechanics to predict a time-averaged probability distribution for a physical system. Both studies also found that students have difficulties using these distributions to reason about stationary states of highly excited quantum mechanical systems. Curriculum was developed to help students with these two issues.

10.1.4 Previous Curriculum Development

At the outset of our research, we had access to two tutorials had been developed to address some of the student difficulties identified by research in the prior studies at the University of Washington and at the University of Maryland. The tutorial developed at the University of Maryland attempts to improve student ability to analyze macroscopic systems in terms of probability. The separate tutorial, developed at the University of Washington, guides students to relate this analysis to an analysis of similar quantum mechanical systems.

The tutorial developed at the University of Maryland presents students with classical systems modeled by a particle in a potential well. By imagining a set of photographs of the system taken at random times, students are led to conclude that the time averaged probability distribution would be largest where the particle spends the most time. Thus the probability distribution is greatest where the potential is greatest.[11]

A second tutorial has been developed at the University of Washington. The original version of that tutorial was created by Brad Ambrose, a former graduate student in the Physics Education Group.[12] A major goal of that tutorial is to help students reason about highly excited energy eigenstates.

They do this by considering the Schrödinger equation as well as the time averaged probability distribution for a corresponding classical system.

The tutorial consists of three parts. The first two parts lead students to recognize that the wave number and the form of the wave function depend upon the difference between the total and potential energies in the region under consideration. The second part also helps students relate an increase in energy to an increase in wave number. The final part of the tutorial leads students through the solution of the problem shown in Figure 10.1.

Since Autumn of 2002, we have been using and modifying these two tutorials. The modified tutorials are the first two tutorials in our current tutorial sequence for Physics 324. The research that led to the modifications is discussed in Chapter 11. The modified tutorials as well as their assessment is discussed in Chapter 12.

10.2 Notes to Chapter 10

[1] See (Einstein et al., 1935)

[2] See (Ambrose, 1999) for a full description.

[3] See (Ambrose, 1999) for a full description.

[4] Even highly excited energy eigenstates do match up with the predictions of classical mechanics. A superposition of stationary states is required to produce a time-dependent probability density.

[5] See (Sedlmeir, 1999) and (Shulte, 1981) as well as (Shaughnessy, 1992)

[6] See (Bao, 2002).

[7] A 1988 study found that only 2% of college bound high school students take a course in statistics. See (Shaughnessy, 1992) for references to this and other studies. In addition mathematics courses on probability and statistics are infrequently required of physics majors.

[8] See (Ambrose, 1999) for a detailed description of prior research on student understanding of quantum mechanics at the University of Washington.

[9] See (Bao, 2002) and (Bao, 1999) for descriptions of the work on student understanding of probability at the University of Maryland.

[10] See (Ambrose, 1999).

[11] In the original tutorial students have access to a video of an oscillating system. Students use VideoPoint software and Microsoft Excel to simulate this process. The original tutorial is described in (Bao, 2002).

[12] This tutorial is indebted to the presentation in French and Taylor's introduction to quantum mechanics. See (French, 1978).

Chapter 11

Identifying Student Difficulties in Relating Classical and Quantum Mechanics

An important goal of a quantum mechanics course can be to motivate the need for a quantum mechanical model. This requires students to be able recognize how the predictions of quantum mechanics and classical mechanics differ and the limiting circumstances under which the two predict the same behavior. Doing so, however, requires that students possess an understanding of classical mechanics in probabilistic terms and an understanding of limiting cases of quantum mechanics. This chapter focuses on an investigation of student understanding of these basic prerequisites.

The focus of the first section of this chapter is on student understanding of probability distributions in classical situations and for stationary quantum mechanical states. The second section extends the discussion to states that depend on time.

11.1 Student Difficulties Primarily Associated with Stationary States

Prior research on student understanding of the relationship between classical and quantum mechanics has almost solely focused on classical probabilities and stationary states. We also began our work by investigating these topics. That work which extends the research of others is described below.

11.1.1 Student ability to determine the time-averaged probability density of a physical system

To probe the ability of students to determine the time average probability density of physical systems, we have asked a number of questions before any tutorial instruction. These concern physical systems that are well described by classical mechanics. An example is shown in Figure 11.1 in which a ball is in a potential well composed, mainly, of two distinct levels. This context is very similar to that used by researchers at the University of Maryland in their development of the tutorial on classical probability.[1] Students were asked to consider photographs of this system taken at random times, and asked about the probability that a photograph will show the ball in various regions. Specifically, they were asked whether the probability of finding the ball pictured on the upper level was *greater than, less than,* or *equal to* the probability of finding the ball pictured on the lower level. This construct effectively forces students to work with the time averaged probability distribution for the ball's location in the well. Between 2002 and 2006, 186 students taking Physics 324 were presented with a potential like the one shown in Figure 11.1. Students were given this question after some lecture instruction on probability in quantum mechanics, but before any tutorial instruction. About 55% of those students answered correctly that the probability of finding the ball on the upper level was greater than the probability of finding the ball on the lower level. The two most common incorrect responses were that the ball is most likely to be found where the potential is the lowest or that it is equally likely to find the ball anywhere. Those who claimed that the ball would most likely be found where the potential was lowest often argued that this was where the ball "wanted" to be due to the forces on it. This same difficulty had been identified during previous research at the University of Washington[2].

A ball rolls back and forth in a track with very steep sides. Joined by a steep ramp, two levels of equal length form the base of the track. A large number of photographs of the system are taken at random times. You may assume that the time spent on the steep portions is negligible. Assume there is no friction or energy loss in the system, and that the ball rolls smoothly, without bouncing, forever.

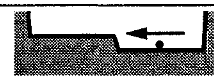


Figure 11.1 A question to probe student ability to use probability to analyze a system well described by classical mechanics.

We had expected that students would reason based on the fraction of time that the ball spent on each level. This, however, was not always the case even for students who answered correctly. The written responses to this question as well as verbal responses during instruction indicated that many students were reasoning based on the velocity alone or on vague notions about the height or length of the levels. The equal lengths allowed some students to answer without detailed explanations.

On the basis of this result, we made a change to the question so that the lengths were no longer equal and the heights were specified. Thus, students needed to calculate the probabilities of a ball being found on each level. Surprisingly, only five of the 100 students asked this question could correctly calculate these probabilities. Based on these results, we modified the tutorial originally developed at the University of Maryland to incorporate activities similar to those in this version of the pretest. We have also asked similar questions of students using other physical contexts. For example, a harmonic oscillator was used between 2002 and 2006. During this time, about 100 students were asked to draw the probability distribution for the location, based on random photographs, of an object executing harmonic motion. About 40% of those students were able to draw graphs that showed the likelihood of object being found at the extremes of the motion to be greater than in the center. About 45%, however, drew incorrect graphs showing the likelihood of finding the object at the center to be the greatest. Explanations for this incorrect response included the idea that the object is most likely to be found in the location where the potential is a minimum. Other students suggested that the middle is most likely location for the object because it must pass through that region twice for every one time it reaches the extremes. This response is similar to the interview response discussed earlier in Section 10.1.1 of Chapter 10. About 5% of students thought it was equally likely to find the object anywhere.

These results indicate that analyzing familiar classical systems in probabilistic terms is far from trivial for students when they come into an undergraduate quantum mechanics course. The results are part of our reason for using a tutorial on classical probability. A tutorial on this topic is necessary if students are to be able to compare the predictions of classical and quantum mechanics.

11.1.2 Research on student understanding of probability distributions

During the period from 2002 until 2006, we continuously used and modified the tutorials discussed in the previous chapter. As we did so, we found that certain difficulties not previously reported by others arose. Below we discuss some of the questions we used to probe student thinking and get a rough sense of the prevalence of incorrect ideas. Sometimes questions were asked before students worked through tutorials and sometimes afterwards. Often we found that specific difficulties persisted following instruction using early versions of the tutorials. This research guided some modifications to the tutorials that are described in the next chapter. As many of these difficulties were not related to the main focus of our research, however, the corresponding modifications have not always been explicitly assessed.

Difficulty in distinguishing probability from probability density

During the fall of 2002, we used a slightly modified version of the tutorial on classical probability developed at the University of Maryland. We observed during instruction that many students did not come to a complete understanding of probability density. In particular, some seemed to have difficulty with how the concept of probability differs from probability density (a probability per unit length in one dimension). Some seemed to think that there was a non-zero probability that a continuous variable could take on a particular value. For example, some would claim that the probability of finding a particle exactly at a particular location is non-zero.

On the basis of the results, we designed a pretest question that was given between 2004 and 2006 to 167 students in Physics 324, who had covered probability distributions in lecture but who had not yet had tutorial instruction. Students were shown a figure similar to the one in Figure 11.1. They were asked if they could determine the probability of finding the object in a random photograph of the system, exactly at a particular point or if they needed more information to do so. Only 35% correctly identified the probability of finding the object exactly at a specified point as zero or tending toward zero. Others gave specific numbers, sometimes based on the probability density, or suggested various pieces of additional information, such as the dimensions of the system, that they thought were needed to answer.

We have asked also asked a similar question an exam in Physics 324 after students had worked through an early version of the classical probability tutorial. This question was given to 82 students in the Autumn quarter of 2002. In this question, students were given a bell-shaped probability distribution for an object's position. They were asked for the probability of finding the particle exactly at a marked point. The results indicated that 60%, of the students were able to make a distinction between probability and probability density. Even some those who answered correctly, however, did not reason completely correctly. They knew that the probability was a probability density multiplied by a length, but were unclear about what length was appropriate. Some multiplied the density by the position coordinate rather than a length around a particular point.

Together, this pair of questions suggests that while the tutorial on classical probability may be somewhat effective at helping students interpret familiar physical systems in probabilistic terms, there remain about one-third of students who do not distinguish probability from probability density even after all instruction on this topic.

Difficulties in interpreting expectation values given a probability distribution

In 2005 and 2006 we asked 114 students in Physics 324 questions about the expectation values for the position and speed of the system shown in Figure 11.1. These questions were asked before tutorial instruction but after lecture instruction on probability and expectation values. The figure shows a ball rolling in a potential well with two equal length levels. In addition to the information shown in the figure, students were told that the velocity on the bottom level was twice that on the top. They were asked whether the average position was *to the right of, to the left of,* or *exactly at* the center of the system. Additionally, they were asked whether the average speed of the ball was *greater than, less than,* or *equal to* one and a half times the speed on the upper level.

To answer correctly concerning the position, students needed to recognize that the ball spends more time on the upper level. The upper level is on the left, hence, the expectation value is to the left of the center. About half of the students were able to answer correctly. About 30% suggested it would be to the right of the center, often reasoning that the ball "wanted" to be in a region of lower potential. About 20% answered that the average was exactly in the center because that was the center of the system.

To answer correctly concerning velocity, one must again recognize that the ball spends more time on the upper level. Hence, it spends more time at the lower speed, and the average speed would be less than an equally weighted average of the speed on the upper level and the speed on the lower level. Only one-third of the students, however, identified the expectation value of the speed as being less than one-and-one-half times the speed on the upper level. Around 25% of the students suggested that the average speed was greater than 1.5 times the speed on the upper level. Some of these students seemed to believe that the ball would spend more time on the lower level, but their reasoning was generally incomplete. About 20% suggested that the average speed was 1.5 times the speed on the upper level. These students were typically using an unweighed average of the speeds on the two levels.

The results suggest that fewer than half of the students in a typical undergraduate quantum mechanics course have a functional understanding of expectation value for simple systems for which they do not have a ready algorithm to calculate such quantities.

Failure to account for normalization for a probability distribution.

Another question was asked of the same 114 students described above after lecture instruction but before they had tutorial instruction. This question was intended to probe whether or not students thought a change in one part of the system could affect the probability of finding an object in a different part of the system. To do so, students were told to imagine that the lower level of the ramp shown in Figure 11.1 was extended, but that the speed of the ball on each level remains the same. They were asked if the probability of finding the ball on the upper level in a photograph taken at a random time would be *greater than, less than,* or *equal to* the probability of finding the ball on the upper level in a photograph taken at a random time when the two levels were of equal length.

To answer correctly, students must recognize that while the time spent on the upper level is the same in each case, the total time for an oscillation is larger with a larger lower level. Hence, the fraction of the total time spent on the upper level is smaller for the system with the extended lower level. Thus, the probability of finding a ball there will also be smaller.

Fewer than 45% of the students recognized that if the lower ramp in Figure 11.1 were lengthened that the probability of finding a ball pictured on the upper level would decrease. The most common incorrect answer was that the probabilities were equal with and without the extension to the ramp. Some of these students reasoned that the probability on the upper ramp does not change because neither the potential nor the velocity of the ball changes on the upper ramp. These students may think that changing one part of the system cannot affect the probability density in another part. They seem to be ignoring the requirement that the probability of finding the ball somewhere in the system must be one.

Failure to reason qualitatively about expectation values

In Autumn quarter of 2006, 46 students were asked a question that required them to write an expression for the expectation value of position given an initial quantum mechanical state. The question is shown in Figure 11.2. The initial wave function resembles a step within the infinite square well.

To answer, students must recognize that the expectation value for position, $\langle x \rangle$, is simply the position averaged over the probability distribution. Since the probability density is constant in the right half of the well and zero in the

A particle of mass, m, is placed in a on dimensional infinite square well potential: $V(x) = \infty$ for $x < 0$ and $x > L$, and $V(x) = 0$ for $0 \leq x \leq L$. At time $t = 0$ the particle is prepared in an initial state given by:

$$\Psi(x,0) = \begin{cases} 0 & \text{for } x < L/2 \\ \sqrt{2/L} & \text{for } L/2 \leq x < L \\ 0 & \text{for } x \geq L. \end{cases}$$

What is the expectation value of the particle's position, $\langle x \rangle$, at $t = 0$?

Figure 11.2 A question concerning the expectation value for position asked in Autumn of 2006.

left half, the expectation value is in the center of the right half of the well. Since the well has a length of L, $\langle x \rangle = 3L/4$.

This question was given to 46 students on the first exam of Autumn quarter in 2006 after they had completed relevant tutorial and lecture instruction. About 75% answered correctly. Only, 5% of these students answered by simply looking at the wave function and using qualitative arguments. Most gave an answer based on the evaluation of an integral.

About 15% of students who were asked this question answered incorrectly by making mathematical errors in formally solving this problem using an integral. It is noteworthy that none of these students discussed the reasonableness of their answers on conceptual grounds. Another 10% of the students assumed a sinusoidal form for the wave function contrary to the given initial wave form.

These results seem encouraging in light of the pre-instructional performance of students discussed in the previous subsection. Other, ostensibly similar, questions, however, appeared more difficult for students.

Failure to apply symmetry arguments in finding expectation values

On the final exam, a question similar to the one above was asked of 43 of the same students. There was one critical difference between the two questions. In this case the potential was a step and the wave function was a sinusoid. Figure 11.3 shows the question. In this case the probability density is symmetric. Thus, the expectation value will be in the center of the well.

> Consider a one-dimensional potential well with a step inside the well and infinite potential outside.
>
> $$V(x) = \begin{cases} \infty & \text{for } x < 0 \\ 0 & \text{for } 0 \le x < L/2 \\ V_o & \text{for } L/2 \le x < L \\ \infty & \text{for } x \ge L. \end{cases}$$
>
> A system with this potential is initially prepared in a state given by: $\Psi(x,t) = \sqrt{\frac{2}{L}}\sin(2\pi x/L)$ between 0 and L and zero elsewhere.
>
> At $t = 0$, is the expectation value of the position, $\langle x \rangle$, *greater than*, *less than*, or *equal to* $L/2$? Explain.

Figure 11.3 A question concerning the expectation value for position asked on the final exam in Physics 324 in Autumn of 2006.

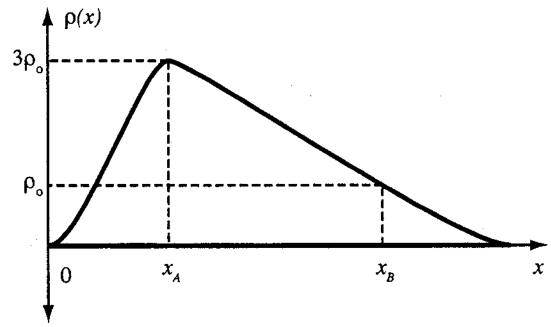
Only half of the students who took the final exam were able to answer this question correctly. About 30% of the students answered that the expectation value would be to the right of the center of the well. Most used the same reasoning. They argued that the particle would be moving more slowly where the potential is higher, therefore the probability density would be greater on the right side of the well. Thus, the expectation value would be to the right of center. In some ways this reasoning is encouraging. Students appear to be able to use the reasoning they learned concerning probability distributions predicted by classical mechanics. They are, however, using this reasoning in the wrong context. Such reasoning is appropriate for highly excited energy eigenstates of quantum mechanical systems not for arbitrary states of those systems. Moreover, they are not using the wave function that they are given.

Difficulties in distinguishing expected value, most likely value, and median

Some students have difficulty in distinguishing the three related concepts of expectation value, most likely value, and median value even after lecture and tutorial instruction. After all relevant instruction in 2002, 82 students in Physics 324 were asked the question shown in Figure 11.4 to probe these ideas. Students were shown an asymmetric probability distribution for the position of a particle and were asked to identify a position that divides the probability of finding the object into equal halves. Then they were asked if the expectation value for position is *greater than, less than,* or *equal to* the value they indicated for the previous question.

About 95% of those students could correctly identify a position that divides the probability of finding the object into equal halves. This position corresponds to the median value, and most students found it by ensuring the areas under the curve on either side of their selected position were equal (See Figure 11.5).

The normalized probability distribution for an object of mass, m, with total energy $19U_o$ is given at right. The probability density is zero outside the region shown.



In answering the questions below, treat the problem classically, not quantum mechanically.

A. [3 pts.] What is the probability that the object is located exactly at the point $x = x_A$?

B. [3 pts.] What is $\rho(x_A)$, the probability density exactly at the point $x = x_A$? Discuss how your answers in parts A and B are related.

C. [3 pts.] Clearly mark on the graph the approximate location, x_m, where the probability of finding the object to the left of the location is equal to the probability of finding the object to the right (*i.e.*, the probability is 50% on either side). Explain.

D. [3 pts.] From the graph of probability density, determine whether the expectation value of the position, <x>, is *greater than* or *less than* x_m, the value that you found in part C. Explain.

Figure 11.4 A question about a bell shaped probability distribution asked of 82 Physics 324 students in 2002 on their first exam.

Only 15%, however, could determine that the expectation value of the position would be to the right of their marked median position. This question is much more difficult than the previous one. Mathematically the question is similar to one involving an object with a mass density rather than probability density. The point at which the object would balance corresponds to the expectation

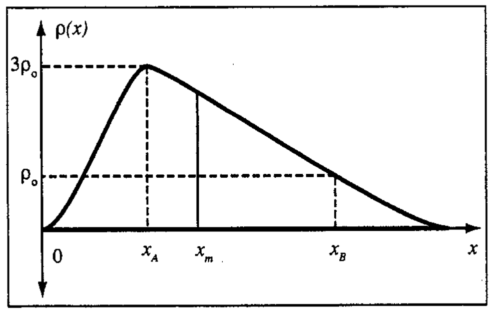


Figure 11.5 Students were expected to draw the line labeled x_m dividing the probability distribution into two equal areas.

value. Clearly, that point is to the right of the marked median position.

Most of the students, 60%, incorrectly suggested that the expectation value would be to the left of the marked position because the most likely position (the peak of the graph) is to the left of the median. About one-fifth said it would be at the same location as the median. The fact that some students have these difficulties is, perhaps, not surprising as students in the introductory course have similar difficulties with mass densities[3]. The fact that so many had these difficulties, however, was a more surprising finding.

Failure to recognize that the expectation values are time-independent for stationary states

We have also asked other questions on examinations that probe student understanding of expectation values. One such question is shown in Figure 11.6. It was asked of 58 students in Physics 324 during Autumn quarter of 2005 on their first exam after lecture and tutorial instruction concerning probability. This question gave students a physical context of a dropped ball making elastic collisions with the floor. They were asked several preliminary questions (not discussed here) in which they were required to sketch the classical potential and then to sketch a highly excited energy eigenstate of a quantum mechanical system in an analogous potential.

Students were then asked to sketch the expectation value as a function of time for both low energy and high energy eigenstates of the analogous quantum mechanical potential. The expectation value should be independent of time for energy eigenstates. Only 40% of students, however, recognized this fact and drew flat lines for the graphs. Fewer yet answered the question completely correctly. About 45% of the students drew graphs that were sinusoidal. Many reasoned that the graph should resemble the wave function or that the graph should oscillate because the corresponding classical situation concerns bouncing.

Part III. (15 pts) Consider a ball dropped from a height, h, making elastic collisions with the floor and bouncing back to the same height repeatedly. (Assume the collisions with the floor take place over a negligible distance and time.)

3. (4 pts) On the axes below sketch and clearly label qualitatively correct graphs of the expectation value of the position as a function of time for both a highly excited energy eigenstate, and for a low energy eigenstate for this well. Label these $<y>_{\text{high}}$ and $<y>_{\text{low}}$. Explain.

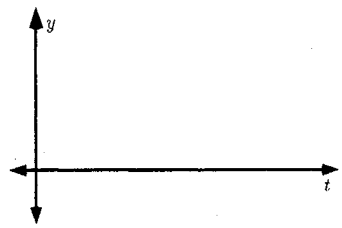


Figure 11.6 A question asked of 58 students during their first exam in Physics 324 during Autumn quarter of 2005.

Summary

It is clear from the results that even after all instruction in the first quarter of quantum mechanics many students lack a functional understanding of probability distributions. In particular, they have difficulties in relating probability distributions to various quantities that are derivable from them such as expectation values, medians, or most likely values. Clearly there is a need for tutorial instruction. Many of these difficulties drove and are continuing to drive modifications to the tutorial on classical probability. Since these difficulties are secondary to the main focus of this project, however, they have not been explicitly assessed after the tutorial modifications.

11.1.3 Research on student understanding of the correspondence principle.

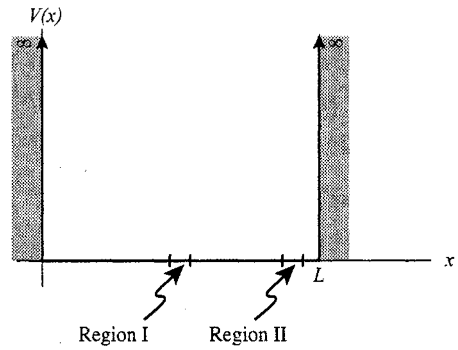
One step in solving the problem shown in Figure 10.1 of Chapter 10 (drawing quantum mechanical highly excited energy eigenstates given a potential) is the recognition that quantum mechanics should produce results that are indistinguishable from classical mechanics for highly excited states. This correspondence allows one to predict the amplitude variations of a highly excited energy eigenfunction if one knows the classical probability distribution. The prior research shows that students have difficulty with both the classical and the quantum mechanical aspects of this relationship. We were interested in probing student understanding of the correspondence itself. Investigating student understanding of this aspect, however, proved to be rather complicated, in part because student understanding of each component was not robust.

Three questions are discussed below that illustrate student reasoning.

Failure to distinguish between predictions based on classical and quantum mechanics

One of the questions we used was given in Autumn of 2006 to 40 students. They were asked the question shown in Figure 11.7 after they had worked through a version of the classical probability tutorial but before they had worked through a version of the relating classical and quantum mechanics tutorial. The question requires students to distinguish between predictions based on classical and quantum mechanics. Students were shown an infinite square well potential with two small equal with regions marked on the horizontal axis. One region was in the center of the well and the other on the right side. They were told that the system was analyzed and a prediction was made that the particle would be more likely to be found in the region in the center than at the edge. They were then asked if this analysis could be done correctly using (1) classical mechanics and (2) quantum mechanics. They were also told that a different analysis predicted that the probability of finding a particle in each region would be equal. Then they were asked if this second analysis could have been done correctly using (1) classical mechanics and (2) quantum mechanics.

A particle with non-zero total energy is confined to an infinite square potential well. The graph of that potential well is shown at right. Two regions of equal width are marked on the graph. Region I is located exactly in the center of the well. Region II is to the right of region I as shown.



1. This system is analyzed, and using that analysis it is predicted that the time-averaged probability of finding the particle in region I will be *greater than* the time-averaged probability of finding the particle in region II.
   a. Could this analysis have been done correctly using classical mechanics? Explain.
   b. Could this analysis have been done correctly using quantum mechanics? Explain.
2. This system is reanalyzed, and using that analysis it is predicted that the time-averaged probability of finding the particle in region I will be *equal to* the time-averaged probability of finding the particle in region II.
   a. Could this analysis have been done correctly using classical mechanics? Explain.
   b. Could this analysis have been done correctly using quantum mechanics? Explain.

Figure 11.7 A question asked of 40 students in Autumn of 2006 that requires students to distinguish between predictions based on classical and quantum mechanics.

We expected students to reason that a classical description results in a flat time-averaged probability distribution for the infinite square well. A quantum mechanical description of a low energy eigenstate gives a higher probability of a particle being found near the center of the infinite square well. For a highly excited energy eigenstate, on the other hand, the probability of finding a particle in a region near the center is approximately equal to the probability of finding it in a region near the edge.

Only about 15% of the students answered correctly and reasoned in the manner described above. Another 45%, who answered correctly, reasoned correctly about classical mechanical predictions but gave incomplete reasoning for the quantum mechanical case. They did not describe the nature of the quantum states that account for these predictions. About 20% answered incorrectly and stated that quantum mechanics could only account for a peaked probability distribution but not a flat distribution.

Tendency to associate quantum mechanical stationary states with classical descriptions of motion

As discussed at the beginning of this chapter, prior research has demonstrated that students frequently claim that the quantum mechanical ground state best approximates predictions of the behavior of a system given by classical mechanics. We have asked a question to probe the prevalence of this idea.

A question on this issue was asked in Autumn quarter of 2004 and 2005 in Physics 324 after lecture instruction but before tutorial instruction on relating classical to quantum mechanics. The question directly asks if the ground state is the quantum mechanical state that best approximates the classical behavior of a particle in an infinite square well. Of the 79 students who answered this questions, 40% claimed that the ground state best approximates the behavior predicted by classical mechanics. After an early version of the tutorial on relating classical and quantum mechanics used in 2002. About 25% of students answered in this way. Thus, students ability to reason about the correspondence between quantum and classical mechanics seems to improve slightly after tutorial instruction.

Failure to distinguish between the quantum and classical regimes

Almost 80 students in Physics 324 during 2002 were asked a question that required them to differentiate between classical and quantum mechanics. About 55 of them had worked through the tutorial on relating classical to quantum mechanics (The others did not attend this particular tutorial). The same question was also asked of 36 students at Arizona State University after they had taken a modern physics course. The question is shown in Figure 11.8. Students were told that a system with the potential well shown on the left side of the figure is described by the wave function shown on the right. They were asked if the wave function on the right side of the figure represents the system in a classical or a quantum mechanical regime. To answer students should reason that according to classical mechanics, a particle spends the most time where the potential is greatest. Therefore, a classical analysis would predict the probability density to be greatest in those regions. The wave function on the right side of the figure, however, has the greatest probability density where the potential is the smallest. Thus, it cannot represent a classical description of the system.

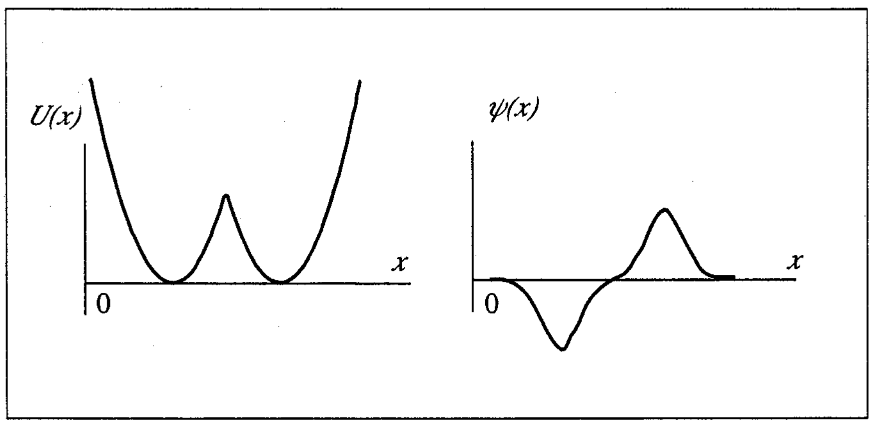


Figure 11.8 A question asked of 78 students in Physics 324 in Autumn of 2002 after they had worked through two tutorials and 36 students who had taken modern physics at Arizona State University without tutorials. It requires students to distinguish between predictions based on classical and quantum mechanics.

In analyzing this question we were able to separate students into those who attended the tutorial on relating classical to quantum mechanics and those who did not. About 75% of the students who attended tutorial, answered the question correctly. Only 40% of the students who did not attend tutorial answered correctly. Although we have not made a comparison of the level of these students in the class, the results are encouraging. More work, however, clearly needs to be done.

Summary

In order for students to relate classical and quantum mechanics, they must understand many underlying ideas. They must be able to associate a classical system with a probability distribution. They must also be able to interpret probability distributions in terms of the probability of finding a particle in a given region. Finally, they must be able to associate the magnitude of a quantum mechanical wave function with the probability distribution. This subsection has described addition efforts to probe student understanding in this area and document some of the difficulties they encounter.

11.1.4 Research on student understanding of the role of mass in quantum mechanics

The tutorials described at the beginning of this chapter lead students to draw qualitatively correct wave functions for systems highly excited energy eigenstates. A typical problem was shown in Figure 10.1. To extend this work, we decided to probe student understanding of mass in quantum mechanics. In particular, we investigated whether students could build on their reasoning for the high energy case to understand a large mass case.

Students encounter statements about quantum mechanics long before their their first undergraduate quantum mechanics course. In many cases they are told, as early as high school, that quantum mechanics applies when things are very small. We were interested in whether they would be able to discuss the sense in which such a statement is true after having completed a quantum mechanics course.

We have asked a series of questions that extend previous results concerning highly excited energy eigenstates by requiring students to differentiate between the high energy and large mass limits. We wanted to know if students could recognize situations in which the two cases give different results. A question that requires such recognition is shown in Figure 11.9. This question requires similar reasoning as the question shown in Figure 10.1, which asks students to sketch quantum mechanical wave functions for a highly excited energy eigenstate. In this case, however, students are asked to consider a system with a very massive particle having the same energy as the original particle in the eigenfunction shown. They are asked to sketch the wave function for both the high energy case and the large mass case.

A particle of mass, m, is bound in the potential well shown at right. The wavefunction for the ground state is also shown. It has energy $2E_s$.

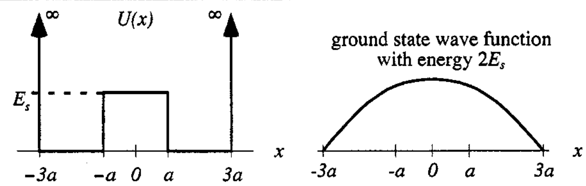


Figure 11.9 A context used to elicit difficulties associated with the role of mass in quantum mechanics.

To come to a correct answer, students must recognize that the time-independent Schrödinger equation, $\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E-V)\psi$, applies. In the high energy case, the potential nearly becomes negligible in comparison to the total energy. Thus the wave function is almost indistinguishable from a highly excited energy eigenstate for the infinite square well. In the large mass case, the Schrödinger equation predicts that as mass increases keeping the total energy constant, so does the relative curvature of the wave function. Thus, the relative curvature of the wave function must increase at every point. The only way for this to happen is if the new wave function has many nodes. Therefore, the eigenfunction with eigenvalue closest to the constant energy is no longer the ground state. In the large mass case, the ground state energy will be much lower that it is for a less massive particle. This information can be used to draw wave function. Note that since the energy remains fixed, the difference in potential between the middle and sides of the well will not be negligible for the large mass case. Hence, the high energy and large mass cases will be different.

"At very highly excited energy states ($E>>E_s$) the energy would be large enough that the raised potential part between -a and a would not affect the wave function much at all."

"... However, in the highly massive case the energy stayed the same and had a value where the raised potential region would affect the wave function."

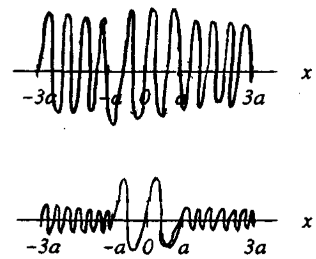


Figure 11.10 Student responses regarded as correct for a highly excited state as well as a system with a very large mass given the potential shown in Figure 11.9

A correct sketch for both the high energy and large mass cases is shown in figure 11.10. Notice that this student correctly drew a wave function for the high energy case that is almost indistinguishable from the high energy limit of the infinite square well. For the large mass case, the student correctly drew a wave function that clearly shows the effect of the potential step in the center of the well.

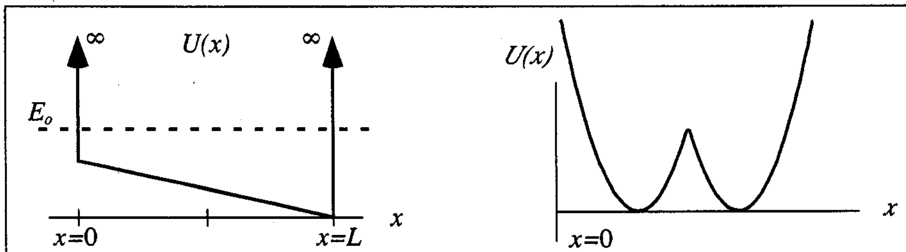


Figure 11.11 Questions designed to probe student understanding of the role played by mass in quantum mechanics.

In addition to the question illustrated in Figure 11.9, other similar questions were designed to probe student ability to differentiate between the high energy and large mass limits of quantum mechanical systems. The potentials associated with those other questions are shown in Figure 11.11. Note that they are both similar to the previous problem in that the high energy and large mass limits will be distinct.

About 90 Physics 324 students were asked these types of questions, concerning a high energy and a large mass limit, in Autumn of 2002 and 2003 on their first or second exam. Fewer than 5% responded correctly with correct reasoning even after tutorial instruction. Student difficulties seemed to be relatively independent of the shape of the potential used. Although few students answered such questions correctly, several patterns emerged. Typical explanations are shown in Figure 11.12. Nearly half of the students, 45%, indicated that mass would have no effect. Often they made statements like the first student's explanation shown in Figure 11.12. About 20% of students drew a function consistent with the claim that increasing mass decreases velocity and thus increases the amplitude. Often they seemed to be using the classical relationship between mass velocity and kinetic energy, $KE = \frac{1}{2}mv^2$, and keeping KE constant. Thus, they believed that increasing the mass decreases the velocity. If the particle moved slower everywhere, they thought the likelihood of finding it would increase at every point. About 15% of students, on the other hand, drew a function consistent with claim that increasing the mass increases the kinetic energy decreasing the amplitude. These students also seemed to be using the relationship $KE = \frac{1}{2}mv^2$, but this time keeping the velocity constant and associating an increased kinetic energy with a decreased probability of finding a particle at every point. Regardless of their answers 40% of the students drew graphs for large mass and high energy that were indistinguishable. Preliminary interviews indicate that student difficulties with classical limits may be intertwined with their understanding of relativistic limits.

> *The mass would not matter if the energy was the same.*
>
> *It would decrease the velocity of the particle everywhere. So, amplitudes would be greater.*
>
> *Greater KE [implies] lower prob[ability] so lower amplitude to wave.*

Figure 11.12 Examples of incorrect student reasoning concerning the large mass limit.

11.1.5 Summary

Research on student ability to relate stationary quantum mechanical states to corresponding classical systems has been discussed in this section along with student understanding of some of the necessary pre-requisites. We found that students have a number of difficulties with probability distributions, the correspondence principle, and the role of mass in quantum mechanics.

As we began investigating student understanding of time dependence in quantum mechanics, the subject of the first part of this dissertation, we found that student's ideas about how classical and quantum mechanics are related could influence their understanding of time dependence. This is the subject of the next section.

11.2 Student Difficulties Primarily Associated with Time-Dependent States

Previous research on relating classical and quantum mechanics has focused entirely on stationary states. As we began to investigate student understanding of time dependence in quantum mechanics (the subject of Part I), it became apparent that student difficulties with the time dependence of stationary states and superpositions of stationary states affected their ability to reason about the relationship between classical and quantum mechanics. In particular, students would frequently associate macroscopic motion with a stationary quantum state. Upon further investigation, we found that only a few students associated time-dependent probability distributions with macroscopic motion. They tended to focus only on a time-averaged probability distribution. This section will focus on these issues.

Our investigation of student understanding of the incompatibility between stationary quantum states and macroscopic motion focused on the following two aspects: (1) student understanding that classical motion is associated with time-dependent probability distributions and (2) student understanding that quantum mechanical energy eigenstates are *not* associated with time-dependent probability distributions. In Chapter 13 we examine student understanding of the superpositions of stationary states necessary to approximate the time-dependent probability distributions predicted by classical mechanics.

11.2.1 Failure to associate time dependent probability densities with macroscopic motion

As part of our investigation, we were interested in determining whether students could translate what they knew of about motion in classical mechanics into the probabilistic language employed by quantum mechanics. In particular, we wanted to know if students would associate a time-dependent probability distribution with a moving macroscopic object. To probe their thinking, we designed a number of questions. These were refined as time progressed. Four questions used at various stages of development are discussed below along with a brief description of results.

In the summer of 2003, students in Physics 324 (N=9) were asked, "Does the probability density for a classical harmonic oscillator change with time?" To answer, students should recognize that classical mechanics predicts that the position of an object whose motion is constrained by a simple harmonic oscillator potential. Therefore, the probability density must change with time. Only one student of the nine correctly answered that it did change. Most claimed that the probability density would not change with time.

Based on the results from the question above, we decided to probe student thinking in greater detail. We designed a question in which students needed to respond to correct and incorrect statements about a situation similar to the one described in the previous question. We based the correct and incorrect student statements on student responses to the previous question. Thus, in Autumn of 2003, students in Physics 324 (N=40) were asked the question shown in Figure 11.13. This question is similar to the one discussed above except students must respond to two student statements. These statements consisted of both correct and incorrect answers with explanations.

In this case, about 35% of the students correctly indicated that the probability density would depend on time. Many of the rest seemed to be thinking of a time-averaged probability density. This tendency to associate macroscopic motion with time-independent probability distributions may be related to common treatments of scattering and will be discussed further in Chapter 14.

Two other students are discussing the probability density as a function of time for a classical particle in an infinite square well. Do you agree with any parts of either student's statement? Explain.

Student 1: *"Classically you know exactly where the particle is at any instant in time. So, my graph shows a spike of area one that will move back and forth as time progresses. My graph, however, only shows the probability density at a particular time t_i."*

Student 2: *"I don't agree. Classically at any time the particle is equally likely to be found anywhere. So, I have drawn a straight line for the probability density at a time t_i."*

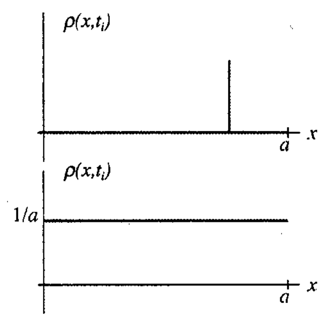
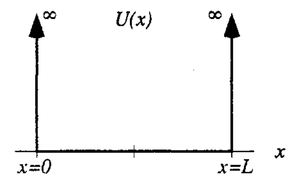


Figure 11.13 A question asked of Physics 324 students in Autumn quarter of 2003 concerning probability densities predicted by classical mechanics.

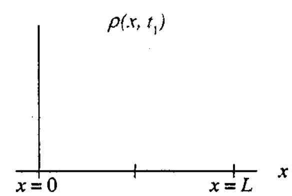
In Autumn of 2005, students in Physics 324 (N=53) were asked a question to probe similar issues. That question is shown in Figure 11.14. It has three parts. Students are first asked to describe the motion of a point particle in an infinite square well potential using a classical mechanical model. A correct description is a particle bouncing back and forth with a constant speed. Next they are told that the position and velocity are known at an initial time, and asked to sketch a reasonable graph of the probability density for the position of the particle at a specific later time. Any spike was treated as a reasonable graph of the probability distribution at a later time. Finally students are asked to draw the graph of the probability density for the position of the particle in a randomly taken photograph. In this case randomly taking a photograph is equivalent to taking the time-average of the probability distribution. Hence, a flat line would represent the probability distribution.

A classical point particle with non-zero total energy is confined to an infinite square potential well. The graph of that potential well is shown at right.



1. Describe the motion of the particle.

2. Suppose you know the position and velocity of the particle at $t = 0$. Sketch a reasonable graph of the probability density for the position of the particle at a **specific** later time, t_1 *(i.e.,* $\rho(x, t_1)$*).* Explain.



3. Imagine that you were going to take a photograph of the particle at some **random** time. Sketch a graph of the probability density for the position of the particle *(i.e.,* $\rho(x)$*).* Explain.

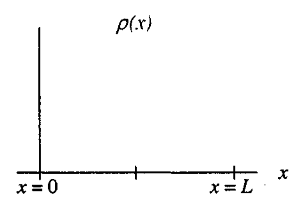


Figure 11.14 A question asked of Physics 324 students in Autumn quarter of 2005 concerning probability densities predicted by classical mechanics.

The first question yielded results that were difficult to categorize. Almost all of the students stated the ball would bounce back and forth. Some correctly added that it would do so with constant speed. Others, incorrectly, stated that it would be like the harmonic oscillator or that it would move sinusoidally.

About 30% of the students correctly answered the second two parts of the question, independent of reasoning. Only half drew a spiked probability distribution for the second question. Most drew flat or Gaussian-like curves. About one quarter of the students drew the same distribution for both cases.

Are any of the quantum mechanical states below a good approximation to the behavior of a classical particle in an infinite square well as represented by both of the graphs above? Explain your reasoning.

- the ground state
- a highly excited energy eigenstate
- a superposition of many eigenstates
- other (please describe)
- No quantum mechanical state can reproduce the behavior illustrated by the two graphs above.

Figure 11.15 The continuation question asked of Physics 324 students in Autumn quarter of 2005 concerning probability densities predicted by classical mechanics.

The question shown in Figure 11.14 was followed by another question illustrated in Figure 11.15. In this final question, students are asked what quantum mechanical state best represents the behavior of a classical particle in an infinite square well. Interestingly, more than 40% of the students suggested that the ground state would best represent the particle as modeled by classical mechanics.[4] This is significant because the ground state is, in a sense, the least classical of quantum mechanical states. The ground state corresponds to a time-independent probability distribution spread throughout the well. A macroscopic object like a billiard ball bouncing between two hard bumpers could not be spread evenly between them. Obviously, many students do not see the contradiction. They are not making a connection between what they study in the quantum mechanics course and macroscopic phenomena. The results from the four questions above suggest that students have difficulty thinking in probabilistic terms about systems that are well described by classical mechanics. In addition, instruction that leads students to think about time-averaged probability distributions for such systems does not necessarily prepare students to consider the more general time-dependent case.

11.2.2 Belief that stationary states account for macroscopic motion

As discussed in subsection 5.4, many students seem to believe that the expectation values for all quantities in quantum mechanics are time independent. Strictly speaking, such a belief would imply that quantum mechanics can't account for motion. Since many students give answers consistent with all valid states being stationary, the question arises as to whether they believe quantum mechanics can account for any motion. We were interested in student ideas concerning this question. To probe student thinking, we administered several questions and re-examined results for questions that had previously been administered.

A partial answer to how students regard the time dependence of stationary states is provided by the naïve mechanisms for time dependence discussed in Chapter 4. In that chapter, we gave evidence that students often assume that stationary states "decay" or "diffuse". We have found that students often use these mechanisms to associate a time dependence with systems that is not predicted by the formalism of quantum mechanics.

Additional insight into student thinking about the time dependence of stationary states is provided by results from the following question suggesting that students associate macroscopic motion with stationary quantum states. In Autumn of 2003 and 2004, Physics 324 students (N=71) were asked a question based on the wave function shown Figure 11.16.

Students were told that this wave function represents a highly excited energy eigenstate in the infinite square well and asked whether it could represent a pebble bouncing back and forth in an essentially infinite square well of macroscopic size.[5] To answer, students should recognize that this is a stationary state with a time independent probability distribution. So, it cannot represent a pebble bouncing back and forth in an infinite square well.

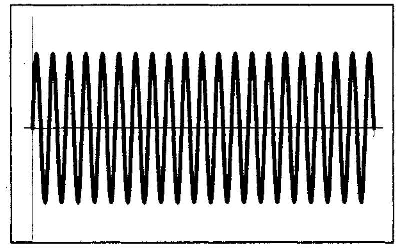


Figure 11.16 A highly excited energy eigenstate shown to 71 students taking Physics 324 in the Autumn of 2003 and 2004 as part of a question concerning classical time-dependence.

In 2003, about one-third of the 40 students who answered this question incorrectly said that this state could represent classical motion. They often said that the classical motion would be represented by a flat line probability distribution which is a correct description of the time-averaged probability distribution.

We also administered this question in Autumn of 2004. In that class 16% of the 31 students stated that the highly excited energy eigenstate could represent a particle moving back and forth.[6]

11.2.3 Commentary

The results from this question and the preceding question suggest that a relatively large number of students believe that quantum mechanics can reproduce the familiar results from classical mechanics even if it is limited to stationary states. These students fail to recognize the need for a quantum mechanical formalism involving superpositions of stationary states that produce time-dependent results. The results appear to persist despite lecture instruction. The tutorials on classical probability and relating classical and quantum mechanics do not adequately address these problems.

11.3 Summary

We have found that students have numerous difficulties relating classical and quantum mechanics. Difficulties understanding classical systems, and probability distributions make it difficult for students to reason about the correspondence between classical time-averaged probability distributions and the probability distributions associated with highly excited energy eigenstates of quantum mechanical systems. In addition, students often fail to recognize that classical mechanics predicts time-dependent probability distributions, and they often assume that stationary quantum states are associated with classical motion.

The next chapter describes the evolution and assessment of curriculum designed to address these and other difficulties.

11.4 Notes to Chapter 11

1 See (Bao, 2002).

2 See (Ambrose, 1999).

3 See (Ortiz et al., 2005) for a description of similar difficulties in the context of mass density. In particular, consider the balanced baseball bat.

4 We have seen similar results described in Section 11.1.3.2. Others have also seen these types of responses as discussed in Section 10.1.1 of Chapter 10.

5 Note Brad Ambrose at the University of Washington used this same context to probe student ideas about time-averaged probability distributions for highly excited quantum mechanical states.

6 A possible reason for the discrepancy in results is that a previous question on the 2004 test indicated explicitly that a pebble bouncing back and fourth would have a most likely location that changes with time. This extra information, may have accounted for the discrepancy in the results.

Chapter 12

Evolution of Curriculum to Address Student Difficulties Relating Classical and Quantum Mechanics

As discussed in the previous chapter, at the onset of this investigation we were using two slightly modified tutorials that addressed student difficulties in relating classical to quantum mechanics. One had been developed by Lei Bao and the Physics Education Research Group at the University of Maryland. The other was developed by Brad Ambrose and the Physics Education Group at the University of Washington and originally designed to be used in an interactive lecture setting. Both of these were extensively modified and reorganized in the course of our investigation. Some changes were based on results presented in the first part of this dissertation, which revealed extensive student difficulties with time dependence. Some changes were based on the results presented in the previous chapter.

A description of the two tutorials that guide students in relating classical and quantum mechanics is given below. It precedes a discussion of preliminary results assessing the effectiveness of this sequence of tutorials.

12.1 Revised Curriculum

In this section, the modified version of the tutorial entitled *Classical Probability* is briefly described. This is followed by evidence that this tutorial alone is insufficient to enable students to draw highly excited energy eigenstates given a potential well. Finally, the current version of the tutorial entitled *Relating Classical and Quantum Mechanics* is described.

12.1.1 Tutorial: Classical Probability

We have extensively modified the tutorial on classical probability that was originally developed at the University of Maryland.[1]. The original version of that tutorial, modified to fit our instructional format, was discussed in Chapter 10. The current version of the tutorial is described below and is included in Appendix A.

Both the original and the modified tutorial present students with classical systems of a particle in a potential well. By imagining a set of photographs of the system taken at random times, students are led to construct a probability density. They reason that the time-averaged probability distribution would be largest where the particle spends the most time. Thus the probability distribution is greatest where the potential is greatest.

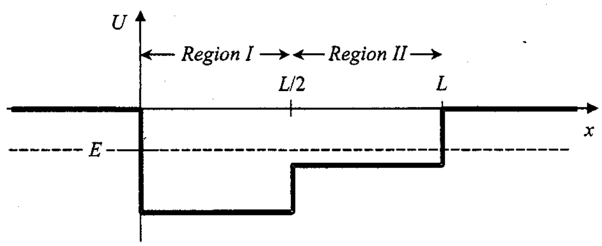
Based on the student difficulties related to probability densities and classical time dependence that we described in the previous chapter, we changed the tutorial in several ways. In addition to the goals of the original tutorial we wanted to address: a failure to distinguish probability from probability density, a difficulty in using probability distributions to find averages, and a failure to recognize that classical systems can be described by both a time-independent as well as a time-dependent probability distribution.

In the process of working through the current tutorial students are now led through an improved sequence concerning the probability of finding a ball in a potential well with two levels of unequal length. They are led to differentiate between probability and probability density and to use the probability density to find averages. Students now also compare the time-averaged probability distribution for the classical description of the system to the probability density as a function of time as described by classical mechanics. The latter, they should conclude, is a spike moving back and forth in the well.

12.1.2 One tutorial is not sufficient

The question arises as to whether standard instruction on quantum mechanics together with a single tutorial on classical probability is sufficient to give students the ability to use the relationship between classical and quantum mechanics to predict the shape of highly excited energy eigenstates. To help answer this question, we have asked the question shown in Figure 10.1 of Chapter 10 after students had completed the tutorial on classical probability as well as relevant lecture instruction, but before they had received any specialized tutorial instruction on relating classical and quantum mechanics. The figure is reproduced Figure 12.1 for convenience.

Fewer than 5% of the 117 students in Physics 324 during 2002 and 2003 who were asked this question could draw a qualitatively correct graph. An acceptable graph consisted of a sinusoidal graph in both regions with the graph in region I having smaller amplitude and smaller wavelength than the graph in region II. The most common error given by about 60% of the students was to say that the amplitude is greatest on the left. Many of these students justified this statement with a claim that the energy is greatest there, An electron is in a bound state of the potential well shown at right. Consider the case in which the electron occupies a *highly excited* energy eigenstate of the well as shown. (*i.e.*, The energy of the electron is much greater than the ground state energy, and the wave function $\psi_n(x)$ of the electron has several nodes between $x = 0$ and $x = L$.)



The regions $0 < x < L/2$ and $L/2 < x < L$ are labeled as regions I and II, respectively.

Draw a qualitatively correct graph of the wave function.

Figure 12.1 A question asked of Physics 324 students in Autumn quarter of 2002 and 2003 as well as in the Summer quarter of 2003. (repeated for the convenience of the reader)

perhaps focusing on the kinetic energy of the corresponding classical system. Recall that the researchers at the University of Maryland also found students mis-applying concepts appropriate in classical mechanics to quantum mechanics.

Some students (45%) claimed that the wave number in each region was equal. Many justified this claim by writing that the energy is constant. Often they wrote $E = \hbar f$. Seemingly, the students are conflating a constant frequency with a constant wavelength, perhaps in analogy with a traveling wave.

Earlier work at the University of Washington found similar results on this question after traditional instruction (3 correct from a class of 30). These results suggest that the classical probability tutorial alone is not sufficient, and that the tutorial explicitly helping students relate classical and quantum mechanics is needed. The evolution of that tutorial based on our research is discussed below.

12.1.3 Tutorial: Relating Classical and Quantum Mechanics

As discussed previously, the tutorial we developed on relating classical and quantum mechanics has evolved from an original version created by Brad Ambrose, a former graduate student in the Physics Education Group at the University of Washington. This tutorial is also indebted to the presentation in French and Taylor's introduction to quantum mechanics.[2]

The major goal of the tutorial is to help students be able to relate high energy eigenstates to classical systems. That is achieved by helping students to predict the form of a highly excited energy eigenstate. They are led to develop this skill by reasoning on the basis of the Schrödinger equation as well as the time averaged probability distribution for a corresponding classical system.

The tutorial consists of four parts concerning: (1) the shape of the wave function based on solutions to the Schrödinger equation, (2) the infinite square well, (3) a comparison between classical and quantum systems, and (4) an application. These sections, along with observations from informal interactions between students and instructors during the tutorial sessions, are discussed below. The parts of the tutorial that differ from the original version are emphasized below. While modifications have been made throughout the tutorial, the third part is entirely new and based on research described in the previous chapter and the first part of this dissertation.

The Schrödinger equation and the shape of the wave function

The tutorial begins by asking students to consider the mathematical form of the wave function for a system in three regions: (1) where the total energy is greater than the potential, (2) where the total energy is less than the potential energy, and (3) where the total energy is equal to the potential energy.

Students are told to base their answers on the time-independent Schrödinger equation, $\frac{d^2\psi(x)}{dx^2} = -\frac{2m}{\hbar^2}[E - U(x)]\psi(x)$.

After some group discussion, most students are able to recognize that the solutions will be determined by the sign of the quantity, $[E - U(x)]$. They recognize that solutions will be sinusoidal in regions where the total energy exceeds the potential, exponential in regions where the potential exceeds the total energy, and linear in regions where the potential and total energy are equal.

The next section first reminds students that sinusoidal wave functions can be expressed as $\psi(x) = A\cos(kx) + B\sin(kx)$. (By the time they study this tutorial, these ideas have already been covered in lecture.) Then students are given a graph of a sinusoid, and asked to determine the wave number as well as to interpret its meaning. The graph shows three half-cycles of a sinusoid in a length of L.

Students often produce an overly complicated mathematical solution while struggling to recall a memorized relationship. In such cases, teaching assistants often ask students to consider the units for the wave number, and whether they could read the quantity directly off the graph without calculation. Usually students then recognize that they can interpret the wave number as the number of radians in every meter. They can tell that the given graph goes through 3π radians in a length L. Thus the wave number is $3\pi/L$. The purpose of this exercise is to make certain that students connect a higher wave number with more oscillations in each meter. While after standard instruction students often have a sense for the meaning of wave length, we have found that most students have no intuition for wave number.

The last question in this part asks students to give the relationship between the wave number and quantities in the time-independent Schrödinger equation. Most students correctly apply the time-independent Schrödinger equation to the expression $\psi(x) = A\cos(kx) + B\sin(kx)$ and find the relationship $k^2 = \frac{2m}{\hbar^2}[E - U(x)]$.

The infinite square well

In the second part of the tutorial, students are given the specific example of the infinite square well. This system is used because of its simplicity. It also ensures that the tutorial is flexible enough that it can be used early in the course when the square well is the only quantum mechanical system with which students are familiar. Students are asked to sketch the wave functions for the energy eigenstates associated with n=1, n=4, and n=8. They then are asked to comment on what happens to the wave number as the energy increases, as well as to write expressions for the wave functions and their probability densities as functions of time.

Most students require a fair amount of time to draw the wave functions, but are able to do this on their own. Often they draw them incorrectly with amplitudes that decrease with increasing wave number. When teaching assistants ask if this is deliberate, most students recognize that the amplitudes should be the same. Usually students have no trouble associating increasing wave number with increasing energy. Often, however, they leave out the time dependent phase when writing a mathematical expression for the wave function. Teaching assistants emphasize this omission in discussions with students in anticipation of later instruction on time dependence.

At this point students have used the time-independent Schrödinger equation to reason about the shape of energy eigenstates for the infinite square well. In the next part, they consider highly excited systems and their classical counterparts.

Classical and quantum systems compared

The third part of the tutorial was added based on research described earlier that indicates students have trouble reasoning about time-dependent probability distributions. In this part of the tutorial, students are given three different systems and for each are asked to draw a graph of the probability density at a particular time and a graph of the time averaged probability density. The first system is a macroscopic particle with an infinite square well potential. The probability density at a particular time for such a system is just a spike at the location of the particle. The time averaged probability density is simply a flat line indicating that the particle is equally likely to be found anywhere if sampled at a random time. The second system is the ground state of a quantum mechanical infinite square well. The probability density for this system is the same for all times. It is zero at the edges and peaked in the center. The third system is a highly excited energy eigenstate for a quantum mechanical infinite square well. The probability density is again the same for all times. The probability density, however, oscillates rapidly in space between 0 and $2/L$.

Typically students take a significant amount of time to draw the graphs associated with the three systems. The graph for the probability density of the macroscopic system at a particular time is often difficult for them. It is our experience that most have never been asked to think about such a system at an instant in time in terms of a probability density.

After making their graphs, students are told that energy eigenstates that they have been considering are often called stationary states. They are asked why this is an appropriate name.

Students often answer quickly that these states do not change. Teaching assistants must sometimes ask students explicitly if they are describing the wave function or the probability density. Students then recognize that the wave functions change, but the probability densities do not. At times an analogy to stationary orbits is discussed.

Next, students are told to imagine that they are given three large ensembles of identical systems. Each ensemble corresponds to one of the systems for which they drew the probability densities: a macroscopic system, a quantum mechanical system in the ground state, and a quantum mechanical system in a highly excited energy eigenstate. Students are told to imagine that they have access to a large number of detectors to measure position. Then they are asked, "If, for each system, you made a measurement of position at the same time t_1, could you decide to which case your ensemble corresponds?" They are also asked, "If instead, you made measurements of position at random times, could you decide to which case your ensemble corresponds?"

Generally students recognize that all three systems have different probability densities at a particular time. They could tell them apart based on the measurements. For the second question, most students agree that they could tell the difference between the ensemble with ground states and the other two ensembles. Students often disagree, however, about whether they could tell the time averaged probability density for the ensemble of macroscopic systems from that for the ensemble of systems in a highly excited energy eigenstate. The next question in the tutorial provides guidance on this question by asking if their answers would change if the resolution of the available detectors were changed.

Generally after discussion among group members, students agree that at for a high enough energy the highly excited energy eigenstate and the macroscopic system would be indistinguishable by random measurements of position. The are led to recognize that this observation forms the basis for a method of finding the wave functions for highly excited energy eigenstates. This method is elicited from the students in the next part of the tutorial.

Before this part ends students are confronted with a student dialogue to help them consider the correspondence between the classical and quantum models more carefully. We have observed many students struggling because the classical time-averaged probability density at a given point is half of the square of the amplitude of the corresponding highly excited quantum mechanical energy eigenstate. The dialogue requires students to resolve this paradox by recognizing that the average of the sinusoid that represents the time averaged probability density of the highly excited energy eigenstate is only half of the square of the amplitude of that state's wave function.

Using classical mechanics to reason about highly excited energy eigenstates

In this part of the tutorial, students combine ideas from previous parts of the tutorial to sketch a highly excited energy eigenstate for the two-level potential well shown in Figure 12.2. This context is identical to that of the question in Figure 10.1 of Chapter 10.

Students are first asked to treat the problem from a classical standpoint. They must recognize that a particle would spend less time on the deeper side of the potential well. Thus, the particle's time averaged probability density would be less on that side.

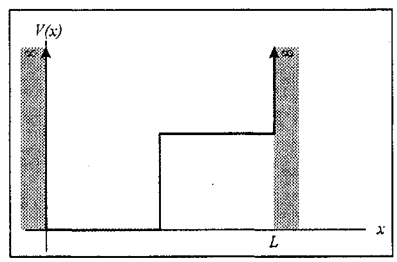


Figure 12.2 A two-level potential well that students were asked to consider as part of a tutorial on relating classical and quantum mechanics.

Students then consider the same potential from a quantum mechanical perspective. They are led to reason that the wave number would be greatest on the side with the deeper potential because the difference between the total energy and the potential is greatest on that side.

Finally, students sketch a graph of a highly excited energy eigenstate for the well. They reason that the wave function has a large wave number and a small amplitude where the potential is deepest.

Commentary on Tutorial: Relating Classical and Quantum Mechanics

Note that this tutorial leaves many questions open about the relationship between classical and quantum mechanics. The next tutorial that focuses on this topic, *Uncertainty: Classical and Quantum*, typically occurs much later in the course after several tutorials focusing time-dependence. It is still in a preliminary stage of development, but does attempt to address some of these open questions. It is discussed in Chapter 13. The tutorials already discussed have, however, been assessed with respect to their effect. The results are presented in the next section.

12.2 Assessment of Modified Tutorials

Some assessment of the tutorials has taken place after the first tutorial, *Classical Probability*, and some after the pair of tutorials. Both sets of post-tests are discussed below.

12.2.1 Assessment of the modified *Classical Probability* tutorial

Some of the questions originally shown in Figure 11.7 and discussed as a pre-test to the *Relating Classical and Quantum Mechanics* tutorial can serve as a post-test for the *Classical Probability* tutorial.

In 2006 after students had worked through the modified tutorial, *Classical Probability*, they were shown the question in Figure 11.7 (The potential well is reproduced in Figure 12.3 for convenience). The context was an infinite square well with small, equal width regions marked in the center and near the edge of the well as shown.

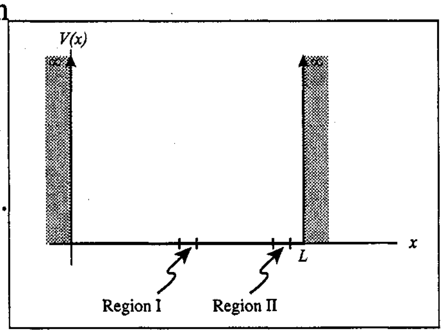


Figure 12.3 An infinite square well potential with marked regions of equal width.

Student performance overall was discouraging, however, it is possible to isolate the questions concerning classical probability. One question asked whether classical mechanics would be consistent with a time averaged probability in which finding a particle in the central region is more likely than in the region near the edge. The other question asked if classical mechanics would be consistent with a time averaged probability in which finding a particle in each region is equal. About 80% of the students answered both of these questions correctly. They said that only the second case is possible since the time-averaged probability density is constant across the well. Fewer than 10% reasoned in manner consistent with the probability density being peaked in the center. Thus the tutorial seemed to have addressed the incorrect idea that the particle spends more time in the center of a classical infinite square well.

12.2.2 Assessment of both tutorials

After tutorial instruction using the sequence of two tutorials described, students have been asked many questions on their examinations to assess the effectiveness of the instruction. There were many versions. Some asked students to draw a wave function given a potential. Others required the converse. A few are discussed below.

A number of these questions required students to draw a highly excited energy eigenstate for a given potential. Figure 12.4 shows one such question.[3] The question is somewhat more difficult than the pretest question asked before tutorial instruction shown in Figure 12.1 and discussed in Section 12.1.2. The post-test question requires students to draw a wave function with a continuous variation in amplitude and wave number. A total of 133 students were asked to draw a highly excited energy eigenstate in a "V-shaped" potential between 2004 and 2006. About 60% of those students were able to do so correctly. On the corresponding pretest question, fewer than 5% answered correctly. These results are tabulated in Table 12.1.

Consider a particle in a highly excited energy eigenstate of the potential given at right.

$$U(x) = \begin{cases} \infty, x < 0 \\ +\beta x, x > 0 \end{cases}$$

On the graph of *U(x)* indicate the classical turning point(s).

In the space at right, sketch a qualitatively correct graph of the wave function. Use the same horizontal scale used in the graph of *U(x)*. Explain the reasoning you used to draw your graph.

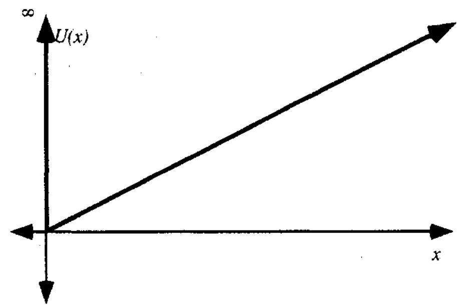


Figure 12.4 A question asked of 51 students on their first exam in Autumn quarter of 2004 requiring them to reason from a potential to a highly excited energy eigenstate.

Another example of a post-test is shown in Figure 12.5. In 2004 and 2006, 97 students were asked to draw a potential given the graph of a highly excited energy eigenstate. About 80% of the students were able to answer correctly.

The real part of a highly excited energy eigenstate for a quantum mechanical system is shown at right.

In the space provided, sketch a potential and a total energy that are consistent with this wave-function. Use the same horizontal scale, and clearly show relative lengths on your vertical scale.

Clearly label your graphs, and explain the reasoning you used to draw them.

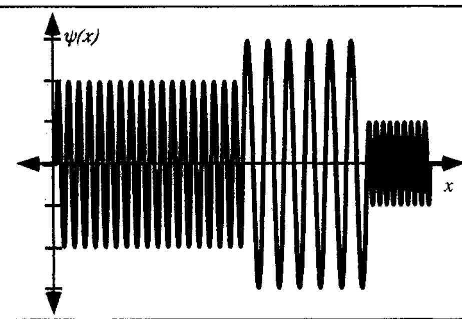
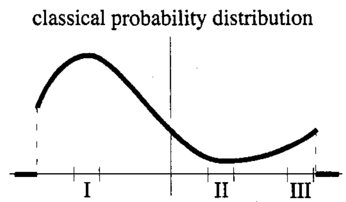


Figure 12.5 A question asked of 51 students on their first exam in Autumn quarter of 2004 requiring them to reason from a highly excited energy eigenstate to a potential.

Another version of a post-test question was given in the Summer quarter of 2003 to 11 students. The question is shown in Figure 12.6. The students were given the classical time-averaged probability distribution for a system and asked to rank the potential for different regions in that system. They were also asked to rank the relative number of nodes contained in different regions for the wave function of the corresponding quantum mechanical system. All of the students were able to rank the potentials correctly, and most, 9 of 11, were able to rank the number of nodes correctly.

[10 pts.] The classical time averaged probability distribution for a particle moving in an unknown potential well is shown at right. Consider quantum mechanically a particle in a highly excited energy eigenstate of the same well (*i.e.*, a particle in the well with an energy much greater than that of the ground state).



1. Rank the potential in the regions marked I, II, and III from largest to smallest. (*Note:* Each region has the same width.) Explain.

2. Rank the number of nodes in the wave function within the regions marked from largest to smallest. Explain.

Figure 12.6 A question concerning relating classical and quantum mechanics asked of 11 Physics 324 students in the Summer quarter of 2003.

12.3 | Commentary

Clearly, as shown in Table 12.1, students are performing much better after specialized instruction. We find these results encouraging. After the tutorials, students seem to be better able to use the relationship between quantum and classical mechanics to reason about highly excited energy eigenstates.

Table 12.1 Student performance before and after tutorials on questions requiring them to relate the shape of highly excited energy eigenstates to potentials.

population	**wave function given potential**	**potential given wave function**
Pretest Au 2002 – Au 2003 (N=117)	<5%	–
Post-test Au 2004–Au 2006 (N=133)	60%	–
Post-test Au 2004, Au 2006 (N=97)	–	80%

Another goal of the tutorials, however, might be to enable students to placing limits on quantum systems that are intended to reproduce classical motion. This is a more complex task. The next chapter discusses additional research and curriculum development related to this issue.

12.4 | Notes to Chapter 12

1 See (Bao, 2002) for a description of the curriculum to address student difficulties with classical probability developed at the University of Maryland.

2 See (French, 1978).

3 Two versions of this question have been given. One version, given to 97 students in 2004 and 2006, is shown in Figure 12.4. The other version, given in 2005, differed in that students had to come up with the potential. They were told to draw the potential for a physical situation described by a perfectly elastic ball bouncing on a hard floor. Of the 58 students who answered the question that required them to determine the potential, 36 did so correctly. So, these 36 had the same potential to use to draw a highly excited energy eigenstate as did the 97 others who were given this potential directly. About 60% of both groups (59/97 and 21/36) could correctly draw a highly excited energy eigenstate. Thus, we have combined these results.

Chapter 13

Revisiting the Relationship between Classical and Quantum Mechanics

As discussed in the previous two chapters, students have difficulties in reasoning about highly excited energy eigenstates of quantum mechanical systems on the basis of the analogous systems described by classical mechanics. While it is valid to reason about highly excited quantum mechanical states using the results from a classical analysis of an analogous problem, the logic is far from simple. Students have difficulties with the steps involved. Often they do not understand how a classical time-averaged probability density can be used to infer the shape of the probability density associated with a quantum mechanical highly excited energy eigenstate.

Quantum mechanics and classical mechanics both purport to describe physical systems. Some systems are well described by classical mechanics. If quantum mechanics is to describe faithfully those same systems, it must yield predictions that are indistinguishable from those predicted by classical mechanics. Systems that are well described by classical mechanics have high energies on a quantum scale. Classical mechanics puts no restrictions on the precision with which positions or velocities are known. A macroscopic simple harmonic oscillator, for example, could be represented by a spiked probability density moving back and forth harmonically as predicted by classical mechanics. A highly excited energy eigenstate, however, cannot reproduce this motion because it has a time-independent probability distribution. A sum of such states, however, can reproduce a time-dependent probability distribution. The time average of such a superposition must correspond to the time-average of the classical probability distribution as discussed in Chapter 10. This chain of reasoning is long and involved. It is, perhaps, not surprising that students have difficulties with many of these ideas.

We recognize that the nature of a true quantum mechanical description of a "classical process" is still up for debate. A general understanding of the measurement problem is not settled among experts.[1] While there is occasional debate about restructuring the presentation of measurement in the typical quantum mechanics course,[2] it has not been our goal to change how this topic is taught in a standard course. Rather we focus on examining on student ability to recognize that a single highly-excited energy eigenstate does not correspond to a time-dependent "classical" system, and that to give a more complete quantum mechanical account of the motion of a "classical object", a time dependent quantum mechanical probability distribution is required.

This chapter focuses on student recognition that superpositions of stationary quantum mechanical states are necessary to predict time-dependent probability distributions. It starts with a description of the tutorial, which serves to illustrate how the concept of uncertainty fits into this topic. This is followed by a description of the specific difficulties that the tutorial is intended to address.

13.1 Tutorial on Uncertainty: Classical and Quantum

The tutorial entitled "Uncertainty: Classical and Quantum" consists of four parts. The first part considers a classical description of a macroscopic system. The second, considers uncertainties in both quantum mechanics and classical mechanics. The third part considers some requirements for a quantum mechanical description of a macroscopic system. The fourth asks students to re-consider the two-level system discussed in the earlier tutorial "Relating Classical Mechanics to Quantum Mechanics" (See Chapter 12).

13.1.1 Classical description

The tutorial begins by asking students to consider a macroscopic physical system. A description of this system is shown in Figure 13.1. Students are asked to draw a graph and write an equation for the potential of this system.

A frictionless parabolic ramp of height 4.5 m and width 2 m is shown at right. A heavy iron block of mass 1 kg will be set in motion on the ramp.

In this tutorial we will investigate some requirements for both the classical and the quantum mechanical descriptions of the motion.

Our quantum mechanical description will describe the experiment using the idealized Hamiltonian $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$.

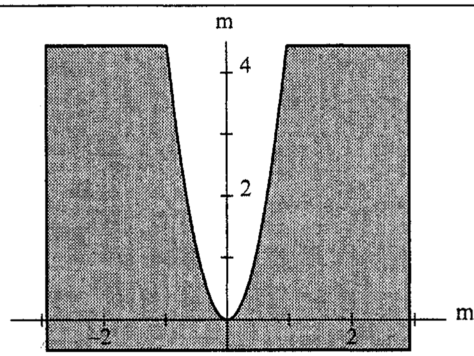


Figure 13.1 A description of a macroscopic physical system used as part of tutorial instruction.

Writing a mathematical description proves to be somewhat difficult for students as they must consider the dimensions of the parabolic ramp and the gravitational potential. Eventually most conclude that the approximate gravitational potential energy in Joules is given by $45x^2$ where x is in meters.

Students are asked if $40J$ is a reasonable estimate for the total energy of such a system, and if $0.5J$ is a reasonable estimate for the standard deviation in repeated measurements of the total energy–provided measurements are made with simple equipment. After some calculation, groups usually agree that these values are within reason. Albeit, most groups suggest that a smaller standard deviation would be possible.

Having thought about these issues in the context of classical mechanics, students are in a position to think about similar issues in the context of quantum mechanics.

13.1.2 Classical and quantum uncertainties

In the second part of the tutorial, students are led to consider a quantum mechanical description of the same system. They are asked to consider the ground state of such a system, and to find the uncertainty in energy for such a state. In the process, they are asked to respond to a student dialog about the situation. That dialog is shown in Figure 13.2. The first student in the dialog expresses the idea that ΔE is the separation between energy levels. The second student expresses the idea that ΔE represents a change in energy. This question usually prompts group discussion during the tutorial, but generally groups come to the correct conclusion that ΔE for the ground state is zero. It is often also an opportunity for teaching assistants to probe students about their understanding of the energy-time uncertainty relationship.

Next students are asked to consider a superposition of the ground state and first excited state, $(\psi_o + \psi_1)\sqrt{2}$. They are asked to draw a histogram illustrating the possible results of energy measurements on a system in this state. They must also calculate the quantum mechanical uncertainty in the energy for this state.

> **Student: 1** *"The energy of the first excited state is $3\hbar\omega/2$ and the next higher or lower state is $\hbar\omega$ above or below that. So, $\Delta E = \hbar\omega$."*
> **Student: 2** *"No, $\Delta E = 0$. The only way to have a non-zero ΔE is for the energy of the system to change. This happens, for instance, when something decays"*

Figure 13.2 Student dialog concerning the uncertainty in energy of the ground state.

Students generally use the relation $\Delta E = \sqrt{\langle E^2\rangle - \langle E\rangle^2}$. Oftentimes, however, students make the errors in calculating the expectation values. Very few students recognize that to calculate the expectation values they need only multiply the possible values by their respective probabilities and sum the results. Teaching assistants often prompt students to consider this alternate way of determining the uncertainty.

Finally students are asked whether they expect the quantum mechanical uncertainty for the macroscopic system initially described to be *greater than, less than,* or *equal to* the uncertainty of $0.5J$ found from the experiment described in the first part of the tutorial. At this point the instructors make sure that students understand that if quantum mechanics is to predict the behavior of the macroscopic system it must not have a larger uncertainty than the experimental uncertainty.

13.1.3 Quantum mechanical probability distribution

The goal of the third part of the tutorial is to help students recognize some of the necessary conditions for a quantum mechanical state to reproduce a probability distribution that approximates that of the "classical" system described in the first part. It begins by asking students to write a general wave function for a harmonic oscillator with appropriate time-dependence. Most students are able to write $\sum c_n \psi_n e^{-iE_n t/\hbar}$.

Next they are told to suppose that such a state is to describe a macroscopic system with energy $40J$ and $\Delta E = 0.5J$.[3] They are then asked to draw a histogram that illustrates the possible results of energy measurements and their relative probabilities. They must then mark the approximate expectation value for the energy on their graph. Most groups are able to draw a graph spiked around $40J$.

The students are asked for the quantum number of an energy eigenstate with energy approximately equal to the expectation value for the energy of the system. Students use their original equation for the potential to determine the very large quantum number for this eigenstate. Then they are asked to provide an order of magnitude estimate for the number of states that appreciably contribute to the linear combination that forms the wave function. They are also asked whether this linear combination can form more than one state, and whether those states or their probability densities depend on time.

Most students are able to use the measured $0.5J$ of uncertainty to determine that a very large number of state could contribute appreciably to the linear combination that makes up the state describing the physical system. They also generally are able to conclude that these states could be combined in many different ways. (*e.g.,* There are many possible values for c_n in $\sum c_n \psi_n e^{-iE_n t/\hbar}$.) Thus, they also recognize that the resulting wave functions, as well as their probability distributions, depend on time.

Finally, the students are asked how the time-averaged probability distribution for the state above compares to the time averaged probability distribution for a single state. This question generally takes students some time to answer. In answering they must recognizing that by time averaging the probability density, all the cross terms go to zero. Thus, they are left with a sum of probability densities each associated with highly excited energy eigenstates and having similar envelopes. The result is a time-averaged probability distribution that is experimentally indistinguishable from that of the corresponding "classical" system.

13.1.4 Relating classical and quantum mechanics revisited

In the final part of the tutorial, students are told to re-consider the two-level system shown in Figure 12.2 and reproduced here as Figure 13.3 for convenience. Students are asked to, once again, explain how they would find a highly excited energy eigenstate of the potential well shown.

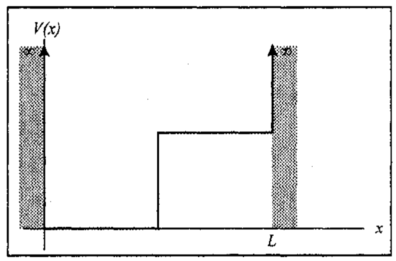


Figure 13.3 A two-level potential well that students were asked to consider as part of a tutorial on relating classical and quantum mechanics. (reproduced for the convenience of the reader)

To answer, students must recognize why a highly excited energy eigenstate has an envelope that resembles the time averaged classical probability distribution. They must surmise that a quantum state that predicted the same behavior as classical mechanics must involve a large number of highly excited energy eigenstates. The time averaged probability density of such a state is made up of a sum of the probability densities of many highly excited energy eigenstates all having similar envelopes. So, the envelope of a single highly excited energy eigenstate resembles the classical time averaged probability distribution.

13.1.5 Commentary on tutorial

The sequence of steps described completes the argument that began in the tutorial on relating classical to quantum mechanics. Note, the tutorial does not suggest a particular state that represents a macroscopic system in all respects. Rather it only puts some bounds on such a state.

13.2 Investigation of Student Understanding of Uncertainty in Classical and Quantum Mechanics.

The tutorial described above was motivated by results from various questions we have asked students to probe their reasoning about the ability of quantum mechanics to describe macroscopic motion. Some of these questions have been in the context of uncertainty. Examples are given below. These questions were asked before students had worked through the tutorial described above, but after they had encountered these ideas in lecture.

In the Autumn quarter of 2005 and 2006, we asked 84 students for the value of ΔE for the first excited state of a quantum mechanical harmonic oscillator. To answer, students should recognize that for an energy eigenstate the energy is known exactly. Thus, $\Delta E = 0$.

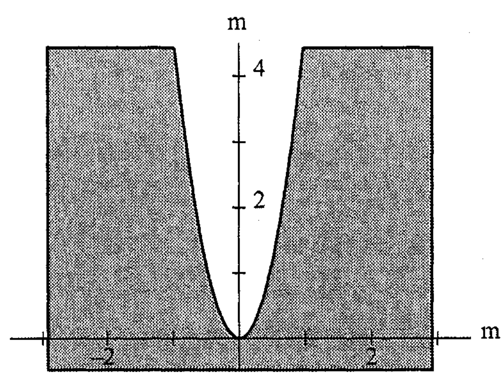
Although this was the eighth week of a ten week course, only 45% (38) of students could answer this question correctly. Many, 25% (21), attempted calculations with expectation values such as $\Delta E = \sqrt{\langle E^2 \rangle - \langle E \rangle^2}$ but could not carry them through. Others, 20% (15) tried to use an uncertainty principle without success. Most of their work was mathematical in nature, and it was difficult to identify their reasons for employing different mathematical approaches. Most, however, failed to recognize that the fact that a measurement on an energy eigenstate yields only one value of the energy. This implies $\Delta E = 0$.

Another question asked in Autumn 2006 provided additional insight into student thinking. That question is shown in Figure 13.4. It is similar to the context used in the tutorial. It asks students to consider an iron block oscillating on a smooth parabolic ramp. They then must consider two student statements concerning the applicability of quantum mechanics. The statements suggest that the uncertainty principle implies that quantum mechanics cannot apply to the physical situation described. The first student argues based on an assumption that the energy is very well known classically. The student concludes that the system could not change with time due to the energy time uncertainty principle. This assumption is not correct because many quantum state of this system would have an energy within the range of energies given by the uncertainty in a real physical measurement. Thus, ΔE is non-zero, and the system can change in a finite amount of time without violating the energy-time uncertainty principle.

The second student reasons that classically the position and momentum are known very well. So, quantum mechanics can't describe this physical system. This argument is also incorrect. Measurements of these quantities made on a real physical system of this size have uncertainties much larger than those required by the uncertainty principle.

A frictionless parabolic ramp of height 4.5 m and width 2 m is shown at right. A heavy iron block of mass 1 kg will be set in motion on the ramp.

Consider the following student dialog concerning the applicability of quantum mechanics to this situation.



Student 1: *"Quantum mechanics can't describe a classical situation like this because of the energy-time uncertainty principle. In the classical case we know the energy very well. This implies that the system hardly changes at all with time. This is not the case with the real macroscopic harmonic oscillator that we have here."*

Student 2: *"I agree. In fact, it also violates the position and energy uncertainty relationship. Classically we know both the position and energy very well at any given time. In quantum mechanics since the operators for position and energy don't commute, we can't know both of them well at the same time."*

With which student or students, if any, do you agree? Explain.

Figure 13.4 A question asked of Physics 324 students in Autumn quarter of 2006 concerning uncertainty in classical and quantum mechanics.

Student reasoning associated with the above question is shown in Figure 13.5. Many of the incorrect responses fell into two categories. Some students, as exemplified by the first four responses shown in Figure 13.5, believe that the "classical " system has a definite energy. Some of those, like the second student, see no inconsistency between this statement and the fact that the system changes with time. Others like the third and fourth students explicitly state that there is no time dependence for this system after they reason on the basis of $\Delta E \Delta t \geq \hbar/2$. This is reminiscent of the student belief that stationary states account for macroscopic motion discussed in Section 11.2.2 of Chapter 11.

Another category of students typified by the final two responses in Figure 13.5, is characterized by a separation between quantum and classical mechanics. According to many of these students quantum mechanics cannot be applied to macroscopic systems.

Don't agree with either. [In] Quantum Mechanics we have a constant energy.

[Student 1 is] wrong: The position of the block in the system moves around, but the energy stays the same in both classical and quantum.

Student 1 is right that we know the energy very well, which implies that the system doesn't change with time. This can be true for the real oscillator.

The system does NOT change with time. (No friction) We know E very well, so $\Delta E \Rightarrow 0$ *and* $\Delta t \Rightarrow\uparrow \infty$. *Just as we would expect.*

The system is not a quantum system. I agree with student 1 and 2.

. . . This is a classical situation (mass on ramp). There is no uncertainty

Figure 13.5 Examples of student reasoning to the question shown in Figure 13.4.

13.3 Summary

The preliminary results discussed in this chapter indicate that many students have failed to develop a complete understanding of the relationship between classical and quantum mechanics during a typical quantum mechanics course. Our preliminary research indicates that this topic is quite challenging for students. The tutorial we have developed on this topic is preliminary and at this point only address a few of the difficulties students have with this topic. Additional investigations in this area are needed. We believe that this work could give insights into how students view uncertainties in measurement that could transcend the context of the quantum mechanics course.

13.4 Notes to Chapter 13

[1] See (Zurek, 1981),(Zurek, 2003), (Schlosshauer, 2004), and (Schlosshauer, 2006).

[2] See (Griffiths et al., 2007) for letters encouraging a restructuring of the quantum mechanics course in response to a review of research into student understanding of quantum mechanics contained in (Singh et al., 2006).

[3] Students are to treat the extended object as a point particle. In practice, this issue does not arise with students during instruction.

Part III

Additional Topics

Chapter 14

Student Difficulties Associated with Probability Current

An aspect our investigation drew on previous research by others on tunneling and one-dimensional scattering. We quickly found that students had difficulty with aspects of these topics that seemed to be related to the standard presentation in terms of stationary states. In this way, we were led to consider student understanding of probability current. In the process, we found that difficulties with standard questions and instruction impinged on student ability to understand both scattering and time dependence in quantum mechanics.

14.1 Results from Prior Research

A few studies have probed student understanding of one-dimensional quantum tunneling. Some has been conducted at the University of Washington[1], and some elsewhere. A few studies have examined student difficulties[2], and a few attempts have been made to address those difficulties[3]. Still other work has focused on the creation of standardized instruments to measure student understanding[4]. There have also been attempts to use this research as a stepping stone to improve student understanding of technical applications of quantum tunneling through the use of computer aided instruction[5].

Some common difficulties that have been identified include the following: a tendency to believe that the energy of a particle is less after tunneling through a potential barrier, the belief that reflection cannot occur at a potential barrier, and a miss-interpretation of graphs of potential energy. Table 14.1 shows that these common difficulties have been found by different researchers at different institutions. The asterisks indicate that a difficulty was observed at a particular institution.

Table 14.1 Common difficulties identified by researchers at different institutions: UW–University of Washington, UMD–University of Maryland, UME–University of Maine, UCO–University of Colorado.

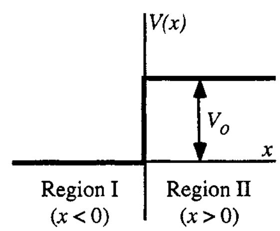
difficulty	UW	UMD/UME	UCO
Energy is lost in tunneling	*	*	*
reflection doesn't occur	*	*	
potential is misinterpreted	*	*	*

14.2 Preliminary Results Based on Student Responses to Typical Scattering Questions

Our early results corroborated those of other studies by showing the presence of the difficulties above among students at the University of Washington. An example of a question we used in this study is shown in Figure 14.1. It was asked of our students in Autumn quarter of 2002. It is very similar to questions used by other researchers. In it students are told mono-energetic electrons are incident on a potential step from the left. The students were asked whether all of the particles would end up to the right of the step, and to draw wave functions for situations where particles were incident both from the left with energy above that of the step. Other versions asked students to consider electrons incident from the right and energies less than that of the step.

Monoenergetic electrons (*i.e.*, electrons that all have the same energy) travel through a region in which the potential energy $V(x)$ varies with x as follows:

$$V(x) = \begin{cases} 0, & x < 0 \\ V_o, & x > 0 \end{cases}$$ (See figure at right.)



Let E represent the total energy of each electron, and $\psi(x)$, the wave function associated with the electrons (a complex function, in general).

A. Suppose the electrons are incident ***from left to right*** (*i.e.*, in the $+x$ direction) with $\boldsymbol{E > V_o}$.

1. Do all of the electrons end up in Region II? Explain.
2. On the axes provided, draw the shape of both the real and imaginary parts of $\psi(x)$. Clearly indicate how, if at all, the wave function in $x < 0$ is qualitatively different from the wave function in $x > 0$ (*e.g.*, in which region are the nodes in the wave function closer together?). If the wave function is equal to zero anywhere, show that explicitly. Briefly explain.

Figure 14.1 A question asked of Physics 324 students in Autumn quarter of 2002 after traditional instruction on scattering.

To answer correctly, students need to recognize that there is a probability that the electron's position could be measured to the left of the step. Thus, not all of the electrons end up to the right of the step. To draw the wave function, they could use the reasoning discussed in Chapter 10 to conclude that the wave function would have greater relative curvature in regions where the difference between the total energy and the potential energy is greatest, and that its form would be sinusoidal where the total energy exceeds the potential and exponential where the potential exceeds the total energy.

About one quarter of the 57 students who answered the first question incorrectly thought that none of the particles would be reflected when incident from the left. A subset of 26 of those students were also asked the same question but with particles incident from the right. About half of them answered incorrectly that none of the particles would be reflected. As expected, fewer students think reflection occurs when a particle is incident on a step from a region with higher potential to one with lower potential. This corroborates the findings of others.

We also observed a small number of students responding in ways that Wittmann and Morgan[6] found consistent with the misconception that energy is lost in tunneling. A potential step, however, is not an ideal context to elicit this type of response.

In addition to the errors discussed above, we were able to observe some responses that have not been detailed in the literature. An initial version of the question only required students to sketch the real part of the wave function. In a subsequent version, however, we decided to ask 31 students to draw the imaginary part as well. Very few could do so correctly. The most common errors are tabulated in Table 14.2

Table 14.2 Common student difficulties in sketching the imaginary part of the wave function.

difficulty	$E > V$	$E < V$
No $\Im m\psi$	30% (9/31)	15% (5/31)
No $\Im m\psi$ inside step	15% (4/31)	5% (2/31)
No $\Im m\psi$ outside step	5% (1/31)	10% (3/31)
No $\Re e\psi$ inside step	–	15% (4/31)

About 45% (14) suggested that there were areas where there was no imaginary part of the wave function in the case that the total energy was above the step. Of those, 65% (9) said there was no imaginary part anywhere. Another

30% (4) said that there was no imaginary part inside the step. About 5% (1) said the wave function was only imaginary in the step.

When the total energy was greater than the potential, about 30% (10) stated that there were areas where there was no imaginary part of the wave function. Half of those suggested that there was no imaginary part of the wave function at all. About 20% (2) of those drew an imaginary part only outside of the step. Another 30% (3) drew an imaginary part only inside of the step. In addition 15% of the 31 students suggested that the wave function had no real part inside the step when the step height exceeded the total energy.

Clearly many students are not able to correctly predict the shape of the wave function in different regions. They also appear be thinking about the real and imaginary parts of the wave function in a non standard way.

14.3 | Probability Current

In addition to the problems discussed above, during instruction we found many students struggling with ideas associated with probability. When faced with problems such as that shown in Figure 14.1, students were often perplexed because the amplitude of what they were calling the transmitted wave was greater than that of the incident or reflected waves. This seemed counter-intuitive to them. In addition, they had become accustomed to reasoning about the amplitude of the wave function in bound state problems by considering how long an analogous classical particle would spend in the region under consideration. This approach was also problematic because, for problems such as the one shown in Figure 14.1, the extent of the system is infinite and the wave functions are not normalizable. To approach this problem in a manner consistent with what is done in the rest of the course we attempted to consider student understanding of probability current.

An analysis of the problem shown in Figure 14.1 in terms of probability current is illustrative. A probability current can be thought of as a term with units of velocity multiplied by a term with units of probability density. Thus, the same probability current could be obtained from a small velocity and a large probability density or a large velocity and a small probability density. Since probability is conserved, the probability current incident on the left side of the step minus the probability current reflected from the step must be equal to the probability current transmitted. The velocity of a particle in an analogous classical system would be greater on the left than on the right since the potential is smaller there. Thus even though the amplitude of the transmitted wave function is larger than that of the incident wave function, the transmitted probability current can still be less than the incident probability current as intuition would suggest.[7]

The research we have don on this topic is much less complete than that described in Parts I and II of this dissertation. Below, we discuss results from questions designed to probe what we thought might be issues that caused difficulties for students about current density and probability density followed by a preliminary version of a tutorial to address these issues.

14.4 Identifying and Addressing Difficulties with Current Density.

From our work with in-service teachers we know that the concept of density is not trivial. So, we began our investigation if student understanding of probability current by asking questions about current density and probability density.

In the summer of 2003, we asked a small number of Physics 324 students (N=9) the question shown Figure 14.2. In this question students were asked to compare the speeds of cars on two highways given a comparison of how many cars per minute travel on each highway. It is not possible to answer, because there is no information given about the density of cars on the highways. The information given could be consistent with any speed ranking. To answer the question correctly the students had to recognize and explain their inability to answer. Only 45% of the students could do this. Another 45% thought some additional information might be required, but could not explain themselves completely. No clearly discernible pattern was observed in these responses. About 10% indicated that the speed on the first highway must be faster. These results indicate that current densities even in familiar contexts are not completely understood by students at this level.

> A greater number of cars per minute pass a signpost on highway 1 than on highway 2. How do the speeds of cars on highway 1 and highway 2 compare? Explain.

Figure 14.2 A question asked of nine physics 324 students in the summer of 2003.

As part of a tutorial on probability current students worked through the exercise shown in Figure 14.3 which focuses on ideas similar to those involved in the question shown in Figure 14.2. They were asked questions that led them to relate the particle current at each end of a one-dimensional region to the change in the number of particles in that region. In the process they must recognize that the current involves a velocity and a density.

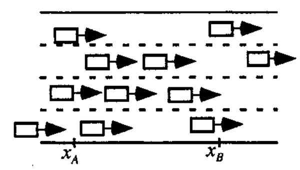
In practice, this section proved quite difficult. Students took up to 25 minutes to reason their way though this short section working together in small groups. Many students had significant difficulties with a density that varied from place to place.

14.5 Identifying and Addressing Difficulties with Probability Current

The next step in the tutorial led students to consider probability currents as opposed to currents of physical objects. Students were led to derive an expression for a quantum mechanical probability current and apply it to bound states. Students in the summer of 2003 were later shown the context illustrated in Figure 14.4. It involves a steady state situation in which a beam

I. Classical current density

Cars travel at different speeds on the roadway illustrated at right. Their speeds are given by the function $v(x,t)$. The linear density of cars on the road is given by $\lambda(x,t)$.



A. What is the number of cars in a length of roadway that extends from x_A to x_B in terms of $\lambda(x,t)$?

B. How many cars per second pass point x_A in terms of the variables given? point x_B?

C. Write an expression relating the time rate of change of the quantity you found in question A to your answers to question B.

D. The quantity $v(x,t)\lambda(x,t)$ is sometimes called a current. Discuss the reasonableness of this name.

Figure 14.3 A portion of a probability current tutorial that students in Physics 324 worked through in Summer quarter of 2003.

of electrons are incident from the left on a thin potential barrier with energy less than the barrier height. Two regions of equal width are marked on the x-axis to the left and right of the barrier.

Students were asked about the probability current at the left end of region I and at the left end of region II. Naturally, if probability is not to increase or decrease between these marks, the probability current must be the same at each location. This question was asked after some standard lecture instruction on probability current. The students had also worked through the portion of the probability current tutorial described above. More than half of the students, however, thought there would be a greater probability current on the left. Some of these students seemed to be comparing the incident rather than the total probability current to the transmitted probability current. Others based their answer on a belief that energy was lost in tunneling.

Consider the potential barrier shown at right. Regions of equal lengths are marked along the x-axis.

A. A mono-energetic beam of electrons with energy, E_o, has been incident upon the potential barrier shown from the left for a long time.

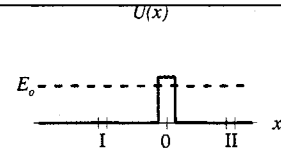


Figure 14.4 The context of a question about tunneling asked of students in Physics 324 during Summer quarter of 2003.

The students continued their tutorial instruction in the next class meeting by being led to analyze reflection and transmission at a step using probability current. From conservation of probability and the continuity of the wave function, they find the reflection and transmission coefficients.

After this instruction, the class (N=11) was asked the question shown in Figure 14.4 on their second exam. In this question students had to understand the relationship between probability current and a change in probability. They could estimate the number of squares to the left of $a/4$ in each drawing, convert these numbers to probabilities, find the difference, and divide by the elapsed time to find the probability current at $a/4$.

This question proved to be difficult. Most students did not answer, or simply wrote down the expression for quantum mechanical probability current. Some students, 35%, had a reasonably correct approach to the problem. Several of these, however, claimed the current was negative or did not provide full explanation.

14.6 | Shortcomings of Traditional Presentations

During the research and curriculum development discussed above, we listened to students and attempted to better understand their thoughts about time-dependence[8]. We quickly recognized that there were not only some serious shortcomings in student understanding, but also some subtle issues

C. [9 pts.] The probability density for a particle in an infinite square well is shown at $t = 0$ s and at $t = 0.3$ s.

1. Is the particle in an eigenstate? Explain.

2. Estimate the probability current at $x = a/4$ and $t = 0.15$ s. Explain.

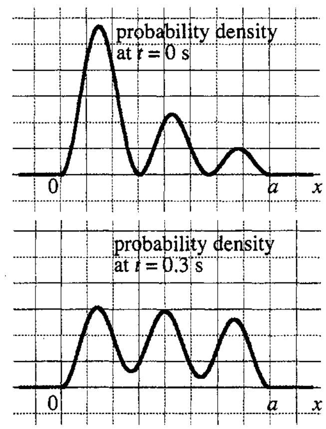


Figure 14.5 A question asked of Physics 324 students in Summer quarter of 2003 on their second exam. This question probes student understanding of probability current in a graphical manner.

in the questions and presentations typically used in scattering. These issues may have been a factor in student understanding of time dependence in certain cases. Foremost among these issues is the fact that these questions often describe the motion of a particle yet ask students to reason about an energy eigenstate. Moreover, the particle is often said to have a definite energy, which reinforces the idea that a single energy eigenstate is involved. The problem, of course, is that the probability distribution for a single energy eigenstate is static. Thus, no motion is possible. In Section 11.2.2 we showed that students often associate a highly excited energy eigenstate with a physical particle bouncing back and fourth in a potential well. Many students incorrectly think that a physical particle moving back and forth in a well has a time-independent probability distribution. The description of scattering problems discussed above can serve to reinforce these incorrect notions.

Furthermore, a barrier is usually assumed to be the only potential in the problem. The physical space is unbounded. Thus, while ratios of probabilities make sense, probability densities for a stationary state do not. The authors of some other curricular materials[9] go even further. They ask students draw the wave function for a macroscopic charged bead on long segmented wire of varying potential. They expect students to draw an energy eigenstate. It is clear that an energy eigenstate with its spatial probability distribution spread out all along the wire could in no way represent the moving macroscopic bead. The association of an energy eigenstate with a moving particle seriously conflicts with a consistent approach to time dependence in quantum mechanics. A single moving particle does not have the same probability distribution or wave function as the plane wave solution desired.

It is often suggested that using a beam of particles solves the problem above. It is true that the desired plane wave solution has a time independent probability distribution, and the probability of finding an electron in any small region would also be nearly time independent. The wave functions, however, are in no way identical. The single particle wave-function is a one dimensional function, whereas the beam has a multi-particle wave function which has as many dimensions as particles in the beam.

We are fully aware that there are many useful semi-classical approximations that can be made in treating this topic. These ideas could be introduced inductively from the phenomena as was done historically, or they could be presented deductively as approximations from the theory. In either case we feel that if semi-classical terms are used without an explicit acknowledgment, it can be very difficult for students to leave the course with a consistent picture of quantum mechanics. For instance, a statement that a particle has a local De-Broglie wavelength may make sense semi-classically, but leads directly to the possibility of writing the momentum as a function of position violating the uncertainty principle.

These considerations are not intended to suggest that the time-independent approach to scattering is useless. When using this approach, however, it must be emphasized that the plane wave solutions are in no way physically realizable[10], just as plane waves in electromagnetism are not physically realizable. Students should recognize these states are very useful mathematical tools which can be used to build up physical states, but in no way represent the particles or beams themselves.

14.7 Conclusion

Our investigation reveled significant difficulties. We found, however, that developing an approach that was consistent with the approach we had taken for time dependence was difficult to incorporate into a standard course. The topics involved were not directly related to topics typically emphasized in an introductory quantum mechanics course. While we believe such a treatment is necessary for a coherent development of scattering and tunneling type problems, it was not our goal to develop a new model course, but rather to supplement the typical course. We hope, however, that this preliminary investigation will help instructors and authors think critically about the coherence of the course from a students perspective with regard to scattering.

14.8 | Notes to Chapter 14

1 See the dissertation of Brad Ambrose (Ambrose, 1999) for a detailed description.

2 See the dissertations of both Brad Ambrose and Lei Bao(Ambrose, 1999, Bao, 1999) as well as the more recent work conducted at the University of Maine(Wittmann et al., 2005 and Morgan, 2006, Morgan et al., 2004, Wittmann, 2004), and in Europe (Domert et al., 2005).

3 See the tutorials described in Brad Ambrose's dissertation (Ambrose, 1999), and a description of part of the *New Model Course in Quantum Mechanics* in (Wittmann et al., 2005) .

4 See (Cataloglu, 2002) and (McKagan, 2006)

5 See a description of the visual quantum mechanics project in (Zollman et al., 2002)

6 See (Wittmann et al., 2005).

7 See (French, 1978) for more details on the use of probability current in scattering and tunneling.

8 See Part I for a description of or research on student understanding of time dependence in quantum mechanics.

9 See (Wittmann et al., 2005) for an example.

10 See (Taylor, 1972) for a both time-dependent and time-independent approaches to scattering. The following is a relevant quote from that source.

> In this book we shall use the improper vectors of Dirac. However, it cannot be overemphasized that *only the proper vectors (the vectors in $\mathcal{H}$) represent physically realizable states.* Improper vectors, such as $|\mathbf{p}>$, do not represent physical states and have significance only as objects in terms of which the proper vectors can be expanded. This distinction is especially important in scattering theory where several results that must obviously be true for a physical state vector are nonetheless false for improper vectors. For example, the central result of scattering theory is that any vector representing the evolution of a collision process behaves just like a free-particle state vector long before and long after the collision takes place. This result is not true when applied to the improper scattering eigenstates.

Chapter 15

Student Difficulties Associated with Angular Momentum in Quantum Mechanics

We have found that students consistently report angular momentum to be one of the most challenging topics in introductory quantum mechanics. This is, perhaps, not surprising as an understanding of angular momentum in quantum mechanics is built on a understanding of quantum mechanics in one and multiple dimensions. Due to the multitude of ideas necessary to understand angular momentum, our research has only touched on the subject. We have collected some data concerning student understanding of spin systems, of the measurement of angular momentum, and of the addition of angular momentum. In addition, we have developed but not systematically assessed curriculum addressing the addition of angular momentum.

Below, we describe questions we have used to probe student understanding.

In those cases in which post-tests were preceded by students having worked through a preliminary version of a tutorial on that topic, we briefly describe the tutorial.

15.1 | Classical Spin

The question shown in Figure 15.1 was administered to students in Physics 325 during Winter quarter of 2003 (N=49). Students were asked to describe the classical motion of a spinning charged ball in a magnetic field. About one-third of the 49 students correctly described the ball as precessing, but only 20% were able to give even a partially correct justification. Most students, 45%, wrote that the axis of rotation would tend to line up with the field, and 10% suggested a translational motion.

Positive charge is placed uniformly on the surface of a rubber ball. The ball is rotating with an angular velocity ω and is placed within a region of space in which there exists a uniform magnetic field. Describe the subsequent motion of the ball. Explain your answer using ideas from classical physics.

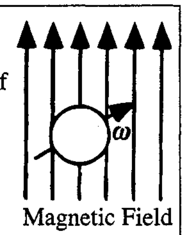


Figure 15.1 A question concerning classical angular momentum asked in the winter quarter of 2003 of 49 students in Physics 325.

This suggests that students do not have a strong foundation in angular momentum in the context of classical mechanics. Thus, curriculum supplementing traditional instruction on quantum mechanics cannot rely on students having such a background.

15.2 | Spinors

In the winter quarter of 2003, students in Physics 325 (N=49) were asked a series of questions to probe their understanding of spin one-half systems. The question shown in Figure 15.2, gave students a spin state for a spin system in the S_z basis, $X = \frac{3}{5}S_{z+} + \frac{4}{5}S_{z-}$. They were asked whether it is possible to determine the probability of obtaining $\hbar/2$ from measurements of S_x, S_y, and S_z. To answer correctly a student needs to recognize that the state has been completely specified, and it is possible to determine the probability of measuring the spin along an arbitrary axis. About 55% answered correctly that all three probabilities could be found. Only 30%, however, did so with complete and correct reasoning. Most incorrect responses suggested that the spinor in the S_z basis contained only information about measurements in the z direction. Other students supported their incorrect answers with the statement that S_x, S_y, and S_z don't commute.

A spin 1/2 system is in a spin state characterized by the following spinor in the S_z basis.

$\chi = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$. Using this spinor given in the S_z basis is it possible to determine all three probabilities below? (You don't need to calculate any values.)

the probability of obtaining $\hbar/2$ from a measurement of S_z

the probability of obtaining $\hbar/2$ from a measurement of S_x

the probability of obtaining $\hbar/2$ from a measurement of S_y

Explain why or why not.

Figure 15.2 A question concerning a spin one-half system asked of 49 students in Physics 325 during Winter quarter of 2003.

Another question that we asked in Winter 2003 is shown in Figure 15.3. It requires students to write down a spinor consistent with the information that the likelihood of measuring spin up in the z direction is 0.64. They were then asked if this information uniquely determines the state. Only 35% of students were able to write down a consistent spinor (*e.g.*, $\mathrm{X} = .8S_{z+} + .6iS_{z-}$). More than half of the students explicitly stated incorrectly that the information given was sufficient to characterize the state of the system, and only 30% correctly concluded that the information given was not sufficient.

For a certain spin $1/2$ system, it is known that the probability of obtaining $\hbar/2$ from a measurement of the spin in the z direction is 0.64.

Write down a spinor in the S_z basis that is consistent with this information. Explain.

Is this information sufficient to specify completely the spin state? Explain.

Figure 15.3 A question concerning a spin one-half system asked of 49 students in Physics 325 during Winter quarter of 2003.

15.3 Measurement of Angular Momentum

We have also considered student understanding of more general systems with angular momentum. In the Winter quarters of 2005–2007, we asked a pair of questions about the measurement of angular momentum to about 90 students in Physics 325 before any tutorial instruction on angular momentum. The first in this pair of questions is shown in Figure 15.4. Students were first told that the orbital angular momentum squared for a system was measured to be $12\hbar^2$. They were then shown several wave functions and asked to choose wave functions that could have represented the state of the system before the measurement.

To answer, one must recognize that only states that have a component with orbital quantum number of $l = 3$ could yield the required value of $12\hbar^2$ for a measurement of the orbital angular momentum squared. The first three states given have terms with $l = 3$. So, they are possibilities.

Only 30% of students selected only these first three states. No clear pattern of incorrect responses emerged although 15% of students picked all four states.

Suppose the orbital angular momentum squared, L^2, is measured for an electron in a spherically symmetric potential and is found to be $12\hbar^2$.

A. Which of the following represent possible wave functions for the electron just before the measurement is made? Here $f(r)$ and $g(r)$ are normalized functions of the radial coordinate r only. The $Y_l^m(\theta,\varphi)$ are the normalized spherical harmonics and the χ_i are the eigenspinors of S_z.

1. $\Psi(r,\theta,\varphi) = f(r)[(\sqrt{1/6})Y_3^0(\theta,\varphi)\,\chi_+ + (\sqrt{2/6})Y_3^1(\theta,\varphi)\,\chi_-] + g(r)(\sqrt{3/6})Y_2^0(\theta,\varphi)\,\chi_+$
2. $\Psi(r,\theta,\varphi) = f(r)[(\sqrt{1/3})Y_3^0(\theta,\varphi)\,\chi_+ + (\sqrt{2/3})Y_3^1(\theta,\varphi)\,\chi_-]$
3. $\Psi(r,\theta,\varphi) = f(r)Y_3^0(\theta,\varphi)\,\chi_+$
4. $\Psi(r,\theta,\varphi) = g(r)\,Y_2^0(\theta,\varphi)\,\chi_+$

Explain your choice(s).

Figure 15.4 The first of a pair of questions asked of Physics 325 students in 2005 and 2006 concerning the measurement of angular momentum.

The next question asked students to indicate which equations could represent the state after the measurement. This question is shown in Figure 15.5. Students needed to recognize that the final state must contain only terms with $l = 3$. Thus, only the middle two options are possible final states.

Table 15.1 Results of questions concerning the measurement of angular momentum.

	percent correct				
Question	**Winter 2003 N=21**	**Winter 2005 N=25**	**Winter 2006 N=35**	**Winter 2007 N=30**	**Total**
ψ before measurement	-	40% (10)	35% (12)	25% (7)	30% (29/90)
ψ after measurement	-	40% (10)	20% (7)	30% (9)	30% (26/90)

Fewer than 30% of the students answered the second question correctly. Many of the students, 40%, suggested that the wave function must collapse to a single state after the measurement by selecting wave function 3, 4, or both. The results the first two questions are shown in Table 15.1.

Which of the following represent possible wave functions for the electron just after the measurement is made? Here $f(r)$ and $g(r)$ are normalized functions of the radial coordinate r only. The $Y_l^m(\theta,\varphi)$ are the normalized spherical harmonics and the χ_i are the eigenspinors of S_z.

1. $\Psi(r,\theta,\varphi) = f(r)[(\sqrt{1/6})Y_3^0(\theta,\varphi)\chi_+ + (\sqrt{2/6})Y_3^1(\theta,\varphi)\chi_-] + g(r)(\sqrt{3/6})Y_2^0(\theta,\varphi)\chi_+$

2. $\Psi(r,\theta,\varphi) = f(r)[(\sqrt{1/3})Y_3^0(\theta,\varphi)\chi_+ + (\sqrt{2/3})Y_3^1(\theta,\varphi)\chi_-]$

3. $\Psi(r,\theta,\varphi) = f(r)Y_3^0(\theta,\varphi)\chi_+$

4. $\Psi(r,\theta,\varphi) = g(r)\,Y_2^0(\theta,\varphi)\chi_+$

Explain your choice(s).

Figure 15.5 The second of a pair of questions asked of Physics 325 students in 2005 and 2006 concerning the measurement of angular momentum.

15.4 Addition of Angular Momentum

Several questions about the addition of angular momentum were asked of 58 Physics 325 students in the Winter quarter of 2003 and 2006. The first question shown in Figure 15.6 gives students a wave function and asks them to give the possible values for the z component of the orbital angular momentum and their probabilities. It is possible to read off the possibilities 0 and $\hbar$ directly from the m values of the spherical harmonic functions, $Y_l^m(\theta,\phi)$. Examining the coefficients the probabilities can be determined to be 2/3 and 1/3 respectively. Only 35% of the students, however, could write down the correct possibilities, and their probabilities. Some errors included using the coefficients or the l values to determine the possible values for the z component of the angular momentum.

For the next three questions, assume that the electron has the wave function $\Psi(r,\theta,\varphi) = f(r)[(\sqrt{1/6})Y_3^0(\theta,\varphi)\chi_+ + (\sqrt{2/6})Y_3^1(\theta,\varphi)\chi_-] + g(r)(\sqrt{3/6})Y_2^0(\theta,\varphi)\chi_+$.

C. Suppose you measured the z-component of the orbital angular momentum, L_z. What values might you get, and what is the probability of each? Explain.

Figure 15.6 The first of three questions asked of 58 Physics 325 students in the Winter of 2003 and 2006.

The next two questions are shown in Figure 15.7. The second question asked students to determine the possible values for a measurement of total angular momentum squared. The third question asked for the z component of the total angular momentum. Probabilities were not required.

Fewer than 10% of the students determined the correct possible values for the total angular momentum squared. Only about 10% of students could determine that $1/2$ was the correct quantum number for the z component of the total angular momentum. The incorrect answers did not fit into readily recognizable patterns, and a high percentage of students wrote unproductive equations or did not write anything down.

Let $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$ be the total angular momentum of the electron.

Suppose you measured the total angular momentum squared, J^2. What values might you get? Explain.

Suppose you measured the z-component of the total angular momentum, J_z. What values might you get? Explain.

Figure 15.7 The second two of three questions asked of 58 Physics 325 students in the Winter of 2003 and 2006.

A similar set of questions were asked of 31 students on their first exam Winter quarter of 2004 in Physics 325. These students did not have any specialized instruction on angular momentum, but had completed all of their homework assignments on this topic. Two-thirds could determine the possible values for the z component of the orbital angular momentum and their probabilities. About 35% could determine the possible values for the total angular momentum squared, and 60% could determine the possible values for the z component of total angular momentum. While this performance is better than that of the prior group of students, it is certainly not ideal.

In addition to poor performance on written questions, our observations during instruction indicated that students often incorrectly apply the vector representation often found in modern physics texts for the addition of angular momentum. Typically this representation has a vector sweeping out a cone representing one angular momentum added to another such vector representing another angular momentum (See Figure 15.8).

We have developed a tutorial worksheet which helps students to interpret this representation. It leads them to conclude that this representation is good for determining the z-component of the angular momentum, and could be used heuristically, but is often misleading when determining other quantities.

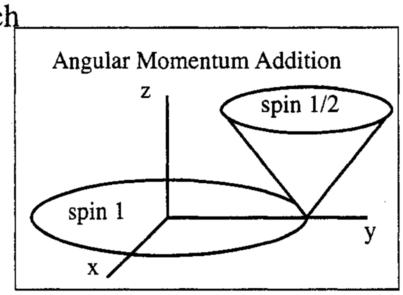


Figure 15.8 A common graphical representation of angular momentum addition. In this case a spin 1 system with no z-component of angular momentum is added to a spin "up" spin 1/2 system.

In 2005, 37 students who had worked through this tutorial completed a question on their first exam similar to those described above. About 80% of these students could determine the possibilities for the z component of the orbital angular momentum and their probabilities. Another 35% could determine the possible values for the total angular momentum squared, and 60% could determine the possible values for the z component of total angular momentum. These results are shown in Table 15.2.

Table 15.2 Results of questions concerning angular momentum and its addition before, without, and after specialized tutorial instruction.

	percent correct		
Question	**before tutorial instruction Wi 2003, 2006, 2007 N=86**	**without tutorial instruction Wi 2004 N=31**	**after tutorial instruction Wi 2005 N=37**
poss. and prob. values for L_z	35%	70%	80%
possible values for J^2	10%	35%	35%
possible values for J_z	10%	60%	60%

While student response to this instruction has been encouraging, student performance on exam questions requiring the addition of angular momentum is not significantly impacted by the tutorials we have developed. Due to the complexities of angular momentum in quantum mechanics, we hypothesize that a sequence of tutorials spanning several weeks would be necessary to see real improvement in student understanding.

Chapter 16

Difficulties Associated with Identical Particles

In working with students in the Physics 324 and 325 courses, we have been able to observe students as they have worked through the portion of the syllabus on identical particles in quantum mechanics. We have found a number of student difficulties associated with this topic. Some seem to be related to difficulties with topics in probability. For example, difficulties in understanding joint probabilities as well as the meaning of independent and dependent events. Lack of experience with probability seems to obscure, for some students, the differences between a quantum and a classical description of reality. Other difficulties are related to symmetry or to the time-dependence of quantum states. All of these are discussed below, after a brief review of relevant research from mathematics.

16.1 Previous Research

There has been considerable research in mathematics on student learning of probability and statistics. Research on student understanding of joint probabilities has shown that students have significant difficulties with conjunctive probabilities and conditional probabilities.[1] For example, students often fail to recognize that the probability of two events both occurring must be less than that of either event alone. This is termed the "conjunction fallacy". Conditional probabilities refer to the probability that an event will occur given that another has occurred. These joint probability topics are important to the study of identical particles in quantum mechanics because the probability distributions obtained are joint probabilities. In physics, however, aside from anecdotal reports from experienced instructors,[2], there has been very little reported research on student understanding of identical particles in quantum mechanics.

16.2 Difficulties Related to Joint Probabilities

We designed a series of questions to probe student understanding of joint probabilities[3]. These were given to 80 Physics 325 students in the Winter quarters of 2004–2007. The questions are shown in Figures 16.1 and 16.2. Both are in the context of weighted dice.

The first question (shown in Figure 16.1) requires that students recognize that the probability of one of two mutually exclusive events occurring is simply the sum of the probabilities of each event occurring separately. About 80% of the 80 students answered this question correctly. Common incorrect answers included assuming the die was equally weighted, contrary to the text in the question, and multiplying the probabilities rather than adding them. Observations of students during in-class instruction, however, suggests that many of the students who answered incorrectly could have answered correctly if the die were not weighted. It appeared that most students could reason about simple probability problems, but often did not have a procedure for handling more complex situations. This observation is important because students are expected to handle more complex situations with less familiar notation in their study of identical particles. Our observations are consistent with reported results in the mathematics education literature that indicate that students can more readily solve probability problems posed in a concrete form, often in terms of frequencies, than problems posed more abstractly in terms of probabilities.[4]

> Suppose that you are given a red die weighted such that the probability of rolling an "i" on the red die is given by $P_r(i)$. (*i.e.*, the probability that rolling the red die would result in a 1 is $P_r(1)$.) *Note:* Do not assume $P_r(i) = 1/6$.
>
> Write an expression for the probability that the roll would result in a "1" *or* a "2". Explain.

Figure 16.1 The first part of a question asked of Physics 325 students in the Winter quarters of 2004–2007 that probes their understanding of joint probabilities in the context of dice.

The second question (shown in Figure 16.2) consisted of two parts. In the first part, students are told they have two weighted dice and asked about the probability a specific outcome from a roll of both dice.

> You are now given a blue die that is weighted differently such that the probability of rolling a "j" on the blue die is given by $P_b(j)$.
>
> Write an expression for the probability that rolling both dice would result in a blue "1" and a red "6." Explain.
>
> Write an expression for the probability of rolling a "1" and a "6," regardless of color. Explain.

Figure 16.2 The second part of a question asked of Physics 325 students in the Winter quarters of 2004–2007 that probes their understanding of joint probabilities in the context of dice.

To answer, students must recognize that the probability of two independent events both occurring is the product of the probabilities that each occur separately. On this question, again, about 80% of students answered correctly. Common incorrect responses included assuming the dice were not weighted and adding rather than multiplying probabilities.

Next the students were asked to find the probability of rolling a one and a six regardless of color. In this case they had to combine what they did in the previous two questions. They needed to multiply to find the probability of rolling a red one and a blue six, and they needed to again multiply to find the probability of rolling a blue one and a red six. Since these two possibilities are mutually exclusive, they must add to find the probability that one or the other happens. Only about half of the students from 2004 and 2005 answered this question correctly. The results from 2006 and 2007 are somewhat better with between 65% and 70% answering correctly. Two common incorrect responses are shown in Figure 16.3. The first response in that figure shows a student adding all the probabilities as if they were each mutually exclusive alternatives. In the second response the student adds probabilities and then multiplies. This may be due to an incorrect application of a memorized rule that adding is associated with "or" and multiplying is associated with "and."

$$P_r(1) + P_r(6) + P_b(1) + P_b(6)$$

$$(P_r(1) + P_b(1)) \times (P_r(6) + P_b(6)) = P_r(1)P_r(6) + P_b(1)P_b(6) + P_r(1)P_b(6) + P_b(1)P_r(6)$$

Figure 16.3 Examples of incorrect student responses to a question about the probability of rolling a "1" and a "6" regardless of color.

Student performance on the first three questions (Figures 16.1 and 16.2) taken together gives an indication of their understanding of the probability necessary for a study of identical particles. Only half of the 116 Physics 325 students who answered these three questions were able to do so correctly. This is noteworthy as instructors often assume that these ideas are well understood by students at this level. These ideas also influence student ability to interpret expressions concerning identical particles in quantum mechanics.

Have you made any assumptions in answering the two previous questions? Explain.

Figure 16.4 The last question asked of Physics 325 students in the Winter quarters of 2004–2007 that probes their understanding of joint probabilities in the context of dice.

Finally students were asked to describe any assumptions that they made in answering the questions (See Figure 16.2). While other assumptions may also have been made, we were interested to see if student mentioned that the dice were independent, non-interacting, or uncoupled. 20% of the 46 students who answered this question in 2004 and 2005 mentioned independent, non-interaction, or uncoupled. In 2006 and 2007, 60% of the 70 students did so. The difference between these results may be due, in part, to the presence of an additional question described below.

Table 16.1 Results of questions concerning probability from Winter quarters of 2004–2007.

	percent correct				
question	**Winter 2004 N=26**	**Winter 2005 N=20**	**Winter 2006 N=34**	**Winter 2007 N=36**	**Total**
P(red 1 or red 2)	90% (23)	65% (13)	80% (27)	85% (31)	80% (94/116)
P(blue 1 and red 6)	90% (23)	75% (15)	90% (30)	75% (27)	80% (95/116)
P(1 and 6)	55% (14)	55% (11)	70% (23)	65% (24)	60% (72/116)
Correct on first three	50% (13)	55% (11)	40% (14)	55% (20)	50% (58/116)
Assumptions	20% (5)	25% (5)	60% (20)	60% (22)	45% (52/116)
possibility of non-independent events	-	-	55% (19)	40% (15)	50% (34/70)

In the Winter quarters of 2006 and 2007 an additional question was asked of students. It is shown in Figure 16.4. It required students to recognize that if events are not independent the probability of both of them occurring need not be the product of the individual probabilities. Only 50% of the 70 Physics 325 students who answered this question did so correctly. In addition, the presence of this question may account for the discrepancy in the number of students mentioning independence in their answer to the question about their assumptions.

Consider two arbitrary events A and B. The probability of event A occurring is 1/5, and the probability of event B occurring is 1/6. Could the probability of both of them occurring be 1/15? Explain.

Figure 16.5 The last part of a question asked of Physics 325 students in Winter quarter of 2006 that probes their understanding of joint probabilities.

16.3 | Difficulties Related to Wave Function Symmetry

In the winter quarter of 2003 and 2006 after traditional instruction but before specialized instruction we asked students in Physics 325 a series of questions that required them to reason about the symmetries of wave functions. These questions are shown in Figure 16.6.

Students were expected to base their reasoning on the overall symmetry of the wave function and on the fact that electrons are Fermions. Thus, the overall wave function should be anti-symmetric. They should have concluded that the third and fifth wave function are the only two that are valid. Fewer than one-quarter of the 68 students who answered this question were able to do so correctly. Many incorrect responses were hard to categorize, but some focused on the symmetries of the spatial or spin part alone. Others assumed that a function had to be either symmetric or anti-symmetric when in general some functions are neither symmetric nor anti-symmetric.

Consider a crude model for helium (Z = 2) in which the atomic electrons do not interact with each other. Let $\psi_{nlm}(\bar{r})$ represent normalized single-electron energy eigenstates.

1. For each wave function listed below, determine whether or not that wave function is a valid wave function for the electrons in helium. **Explain your reasoning in each case.**

   a. $A_1\ \{\psi_{100}(\bar{r}_1)\psi_{210}(\bar{r}_2)\}\ |\uparrow\uparrow\rangle$

   b. $A_2\ \{\psi_{100}(\bar{r}_1)\psi_{210}(\bar{r}_2)\}\ |\uparrow\downarrow\rangle$

   c. $A_3\ \{\psi_{100}(\bar{r}_1)\psi_{100}(\bar{r}_2)\}\ \{|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle\}$

   d. $A_4\ \{\psi_{100}(\bar{r}_1)\psi_{210}(\bar{r}_2)-\psi_{210}(\bar{r}_1)\psi_{100}(\bar{r}_2)\}\ \{|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle\}$

   e. $A_5\ \{\psi_{100}(\bar{r}_1)\psi_{210}(\bar{r}_2)-\psi_{210}(\bar{r}_1)\psi_{100}(\bar{r}_2)\}\ \{|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle\}$

Figure 16.6 A question asked before specialized instruction of students in Physics 325 Winter quarter of 2003 and 2006.

16.4 Difficulties Related to Two Particle Probability Densities

In Winter quarter of 2003 students in Physics 325 were asked the question shown in Figure 16.7 after some standard instruction on identical particles but before tutorial instruction. Students were shown several plots of probability densities, and asked which could represent a system of distinguishable particles, identical Bosons, or identical Fermions. They were also asked, if possible, to comment on the spin state of the system.

Students could have reasoned that for identical particles the probability density would have to be symmetric under interchange of co-ordinates. In addition, they needed to reason that in a system with an antisymmetric spatial wave function the particles would be unlikely to be found at the same place since the wave function must be zero at that point. Thus, the first two graphs could only be distinguishable particles. The second two graphs could be identical Fermions, identical Bosons or distinguishable particles. The third graph corresponds to a symmetric spatial wave function. So, it could be a

Bosonic system in a symmetric spin state or a Fermionic system in a anti-symmetric spin state. Conversely the fourth system could be Bosonic in an anti-symmetric spin state or Fermionic in a symmetric spin state.

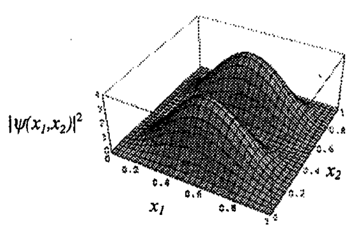
A very small number of students were able to answer the entire question correctly. It is, therefore, more instructive to consider the number of students who at least recognized that only the second two graphs could correspond to identical particles. Only seven or 20% of the 36 students were able to do this.

Consider the following plots of $|\psi(x_1,x_2)|^2$ for a two particle system in a one dimensional infinite square well of unit width.

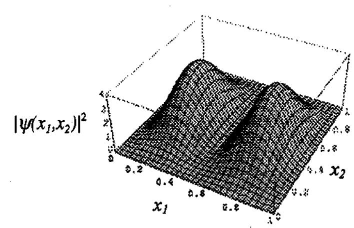
a. Determine whether each plot could represent non-identical particles. If so, what can be concluded about their spin state?

b. Determine whether each plot could represent identical bosons. If so, what can be concluded about their spin state?

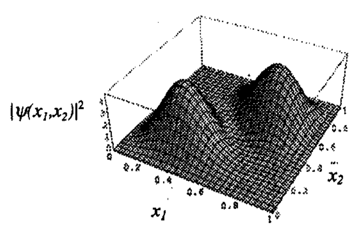
c. Determine whether each plot could represent identical fermions. If so, what can be concluded about their spin state?



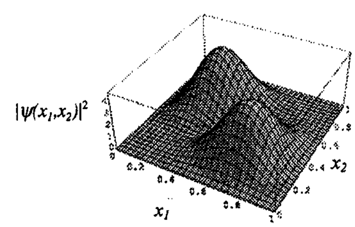
Case i.



Case ii.



Case iii.



Case iv.

Figure 16.7 A question probing students understanding of probability densities in the context of identical particles. This question was administered to 36 students in Physics 325 during Winter quarter of 2003 before specialized tutorial instruction.

Since this question proved quite difficult, we tried reducing the complexity in 2007. We used the same graphs shown in Figure 16.7, but asked only which plots could represent the probability density for a system of two identical Fermions in a stationary state with symmetric spin state. The question is shown in Figure 16.8.

> Consider the following plots of $|\psi(x_1, x_2)|^2$ for a two particle system in a one dimensional infinite square well of unit width. Which, if any, of the plots below could be for a system of two identical Fermions in a stationary state with a symmetric spin state? Explain your reasoning for each case.

Figure 16.8 A question probing students understanding of probability densities in the context of identical particles. This question was administered to 28 students in Physics 325 during Winter quarter of 2007 before specialized tutorial instruction.

As explained above, a correct answer is that only the last graph could represent Fermions in a stationary state with symmetric spin. The students who responded this question found it quite difficult. Only two of the 28 students, 5%, answered correctly regardless of explanation. Many used symmetry arguments incorrectly in their answers. For example, 25% of students eliminated the last two graphs on the grounds that they were symmetric. These students seemed to assume that a symmetric probability distribution implies a symmetric wave function.

16.5 Addressing Difficulties with Identical Particles

A tutorial on identical particles in quantum mechanics has been in use and under development at the University of Washington since Winter quarter of 2003. The first page of the most recent version of this tutorial focuses on classical joint probability in the context of dice and includes questions similar to those shown in Figure 16.1 and Figure 16.2. The questions are posed twice. The first time the dice are not weighted, and students can easily reason with numbers. In the second case the questions are identical to those described in Figure 16.1. In this setting, however, students are able to discuss their reasoning within their small groups. This discussion usually helps students generalize their numerical results. It also provides some students with their first opportunity to critically examine why probabilities combine the way they do.

Subsequently students are asked about two-particle wave functions in quantum mechanics. They are told that $\Psi(x_1, x_2)$ represents the wave function for a system of two non-interacting distinguishable particles a and b, and that the first set of coordinates refer to particle a and the second to particle b. They are also told that $\psi_a(x)$ and $\psi_b(x)$ represent the wave functions of the individual particles. Then they are asked to interpret the quantity $|\Psi(x_1, x_2)|^2 dx_1 dx_2$, and relate it to $\psi_a(x)$ and $\psi_b(x)$. Students must interpret $|\Psi(x_1, x_2)|^2 dx_1 dx_2$ as the probability that particle a is in a small region dx_1 around x_1 and particle b is in a small region dx_2 around x_2. They must then recall that $|\psi_a(x)|^2 dx$ is the probability that particle a is in a region dx surrounding x, and similarly for particle b. Then they can use what they discussed in terms of classical probability to write $|\Psi(x_1, x_2)|^2 dx_1 dx_2 = |\psi_a(x_1)|^2 |\psi_b(x_2)|^2 dx_1 dx_2$. Finally they conclude that a wave function which is consistent with this expression is given by $\Psi(x_1, x_2) = \psi_a(x_1)\psi_b(x_2)$.

They are then asked to identify the error made by a student writing $\Psi(x_1, x_2) = \psi_a(x_1) + \psi_b(x_2)$. Finally they must write an expression that gives the probability that a particle is found in a region dx_1 and another is found in a region dx_2 without regard to which particle is in which region. Students are usually able to reach agreement on $|\psi_a(x_1)|^2 |\psi_b(x_2)|^2 dx_1 dx_2 + |\psi_a(x_2)|^2 |\psi_b(x_1)|^2 dx_1 dx_2$. It sometimes takes some prompting for groups to recognize that this is not the same as using the wave function $\psi_a(x_1)\psi_b(x_2) + \psi_a(x_2)\psi_b(x_1)$.

Next students are led to consider classical interacting distinguishable systems. This is done in the context of animals in cages. Students are given the frequency with which two lions sleep in various parts of their cages. They are told the two lions are put in the same cage and asked to calculate the probabilities for all possible sleeping arrangements under the assumption that each lion's behavior is independent of the others. They are then shown another set of probabilities, told that it is experimental data and asked if they believe it is valid. Most students recognize that the data would be possible if the lions do not behave independently.

Next students consider non-interacting indistinguishable particles. They first must determine that their wave function for distinguishable particles is invalid for indistinguishable particles. Then they are told that the wave function for identical particles must be symmetric under exchange for Bosons and anti-symmetric for Fermions. They then write the corresponding wave functions and calculate the corresponding probability distributions. This last step is critical because students routinely assume that the cross terms go away upon multiplication even though no integration is involved. They are asked about the normalization to bring this contrast to the forefront. They are then asked to consider several student statements about identical particles. The point of this exercise is for students to recognize that even though the particles have no interaction term in their Hamiltonian that they still behave as if they were not statistically independent. They are also asked for $\psi_a(x)$ and $\psi_b(x)$ such that $|\Psi(x_1, x_2)|^2 = |\psi_a(x_1)|^2|\psi_b(x_2)|^2 dx_1 dx_2 + |\psi_a(x_2)|^2|\psi_b(x_1)|^2 dx_1 dx_2$ is a good approximation for identical particles. Most groups resolve this by drawing two wave functions that are well localized in different places.

Finally students are led through an example of a system with two particles in an infinite square well. Through this example students are able to produce all the salient features of the graphs in Figure 16.7 using pencil and paper.

16.6 | Assessing the Effectiveness of Instruction

On the first exam in the winter quarter of 2004 and 2005, we were able to ask some of the same symmetry questions that were asked before specialized tutorial instruction in 2003 and 2006. Unfortunately, only two questions were common among all of the classes. They were the third and fifth questions shown in Figure 16.6. Before tutorial instruction 35% of students could answer both of these questions correctly. After tutorial instruction about 65% of the students could answer these questions correctly. We are not satisfied that these two questions are ideal probes of the tutorials effectiveness. Nevertheless, these results are encouraging.

In 2007 we were able to ask a question on the first exam in Physics 325 (N=38). The question is similar to the questions shown in Figure 16.7 and Figure 16.8. It is shown in Figure 16.9. Students were asked to consider two identical bosons in a system with a one-dimensional double delta-function potential. They were asked to consider an energy eigenstate associated with a particular energy and an anti-symmetric spin state. They were shown the one-dimensional energy eigenstates, and asked to rank $|\Psi(x_1, x_2)|^2$ for several points marked on a graph for this state.

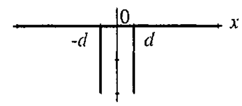
To answer correctly, students need to recognize that two-bosons in an energy eigenstate with an anti-symmetric spin state must have an anti-symmetric spatial state so that their total state is symmetric. They are told that this state has an energy that is the sum of the two one-dimensional single particle energy eigenfunctions shown. Thus, the state is given by $\frac{1}{\sqrt{2}}(\psi_A(x_1)\psi_S(x_2) - \psi_A(x_2)\psi_S(x_1))$, where the wave functions used are the one dimensional wave functions for the anti-symmetric and symmetric single particle states. Clearly this wave function is zero along the line $x_1 = x_2$. The probability density is also symmetric about that line. Thus the ranking is $A = E > B = C = D$. This is a stationary state. So, the ranking is unchanged as time progresses.

About 25% of the 38 students gave a correct ranking at the initial time, and 15% at a later time. While this performance is not ideal, it is an improvement when compared to the pretest where fewer than 5% of students answered this type of question correctly.

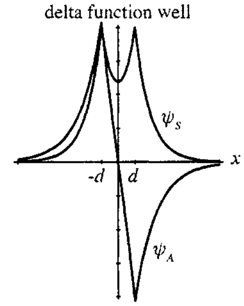
Part III. (13 pts) Bosons in a double delta function well

Consider two identical spin one bosons in the one-dimensional double delta function potential well shown at right. The delta functions are separated by a distance $2d$ as shown. Mathematically, the potential is described by: $V(x) = -\alpha\delta(x+d) - \alpha\delta(x-d)$. The two bound states for a single boson, one symmetric and the other anti-symmetric, are also shown at right. The energy of each state is E_S and E_A respectively where $E_A > E_S$.

Double delta function well

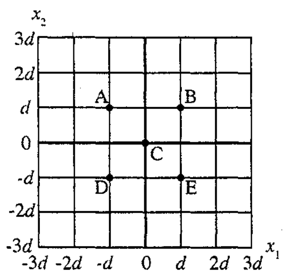


Energy eigenstates for double delta function well



Consider an energy eigenstate for the two-particle system with energy $E_S + E_A$, and a spin state given by: $\frac{1}{\sqrt{2}}\{|1,0> - |0,1>\}$ in the basis of the z-components of the individual angular momenta.

1. (9 pts) For the above state, rank the values of $|\Psi(x_1, x_2)|^2$ at the points A, B, C, D, and E shown on the graph below. Explain your ranking.



2. (4 pts) For the above state, rank the values of $|\Psi(x_1, x_2)|^2$ at the points A, B, C, D, and E shown on the previous graph after a time $t = \frac{2\pi\hbar}{E_A}$ has elapsed. Explain your ranking.

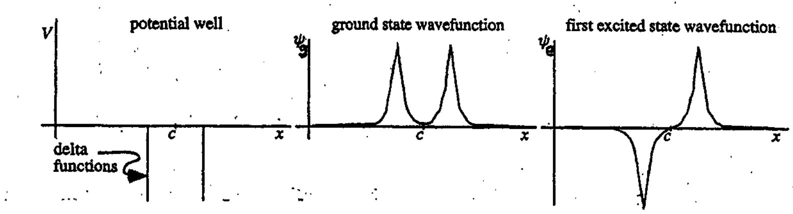
Figure 16.9 A question probing students understanding of probability densities in the context of identical particles. This question was administered to 38 students in Physics 325 during Winter quarter of 2007 after specialized tutorial instruction on their final exam.

16.7 | Related Difficulties with Time Dependence

As we were developing and using the tutorial on identical particles we identified some additional difficulties with time dependence. Figure 16.10 shows a question that requires students to differentiate between mixed states and superpositions. To answer, students needed to recognize that after finding the particle to the left of point c the system could only be in a superposition of stationary states.

We have found that about 40% of students believe that the ground state or the excited state are possible states of the system after the measurement. This distinction between mixed states and superpositions may be especially subtle after students have recently considered multi-particle systems.

A particle is placed in the double delta function potential shown below. The two delta functions are close enough together to allow tunneling of the particle between the wells. Point *c* lies halfway between the delta functions. (Recall that the attractive delta function potential has only one bound state, leading to the ground and excited state wave functions shown below.)



A. (5 pts) If the position of particle is measured to be to the left of point *c*, then at a later time, which of the following statements are true? Explain.

a.) The particle may be in the ground state.
b.) The particle may be in the excited state.
c.) The particle will be in a superposition of the ground and excited state.
d.) The expectation value of the particle's position will not change with time.

Figure 16.10 A question asked of 33 Physics 325 student during Winter quarter of 2004 which focuses on the difference between mixed states and superpositions.

The study of identical particles seems to introduce new difficulties as well as revive old ones. Figure 16.11 shows a question concerning a two particle system in a stationary state. Students are given this state and asked if the probability of finding the particles in a particular region changes with time. They are also asked if one, two, or multiple values could result from an energy measurement on this system. To answer, students must recognize that the probability density is constant and it is only possible to measure a single energy since the system is in an energy eigenstate. Most students, 60%, answered both of these questions. The most common error, made by 20% of students was to state that two values for the energy are possible. Upon examining the responses, it was clear they were thinking of the energies of the single particle ground state and that of the single particle excited state.

B. (10 pts) Now two non-interacting identical fermions are placed in the double delta function potential shown above. At $t = 0$ the two-particle wave function has a spatial part given by: $[\psi_g(x_1)\psi_e(x_2) - \psi_g(x_2)\psi_e(x_1)]/\sqrt{2}$, and a total spin of 1.

1. (5 pts) As time elapses, will the probability of finding both particles to the left of point c *change or remain the same*? Explain.

2. (5 pts) At $t = 0$ a measurement of the total energy of the system is made. Which of the following is true? Explain.

a.) Only one value is possible for the energy of the system.
b.) There are two possible values for the energy of the system.
c.) There are many possible values for the energy of the system.

Figure 16.11 A question on identical particles asked of 33 students in Physics 325 during Winter quarter of 2004.

On a similar free response question in Winter quarter of 2005 students were only asked for the possible values of an energy measurement. 70% of the 37 students answered correctly, but the same common incorrect response was prevalent. 20% of the students answered that it was possible to measure one of the two energies associated with the single particle states.

16.8 Conclusions

Our investigation of student understanding of identical particles in quantum mechanics has led us to several conclusions. Students have difficulty with joint probabilities even in classical contexts especially when problems are posed abstractly. Students have difficulties with wave function symmetries under particle interchange. Some confuse requirements on wave function symmetries with requirements on probability distributions. Difficulties with time dependence also re-emerge in this new context. We have begun to develop and asses curriculum to address some of these student difficulties, but that assessment is still at a preliminary stage.

16.9 | Notes to Chapter 16

[1] See (Sedlmeir, 1999) for a review of research concerning student understanding of statistics.

[2] See (Styer, 1996).

[3] Some students in 2003 were also asked some of these questions. The additional warning not to assume $P_r(i) = 1/6$, however, was not included. Many students made this assumption, and thus answered incorrectly. It was not clear if the students could have answered correctly without making this assumption, so the question was revised to include the warning.

[4] See (Sedlmeir, 1999).

Chapter 17

Student Understanding of Perturbation Theory in Quantum Mechanics

Perturbation theory is a standard part of most introductory quantum mechanics courses. This is likely due to the dearth of exactly soluble problems in quantum mechanics. We have found that instruction on this topic allows us to study student learning of this topic but also affords us an opportunity to study some special aspects of student's ideas about superpositions and their time-dependence. In time dependent perturbation theory, for example, students must consider the state of a perturbed quantum mechanical system as a superposition of the stationary states of a different system, namely the unperturbed system. Thus, students are exposed to a superposition of states which are themselves not stationary for the current potential. This allows us to probe student understanding of such superpositions and their time-dependence.

17.1 Student Difficulties with Time-Independent Perturbation Theory

During the Winter quarters of 2003-2007 131 students in Physics 325 were asked a series of questions concerning time-independent perturbation theory. Those students did have some standard lecture instruction on this topic, but had not yet worked through tutorials concerning time-independent perturbation theory. Figure 17.1 shows that series of questions. The questions involve an infinite square well potential. Students are asked for the sign of the first order correction to the energy of both the ground and first excited states given a perturbation. In the first case, the perturbation is a delta function on the right side of the well. In the second case, it is a step up on the left side of the well and down on the right side of the well.

Students were given that $E_n^1 = \langle\psi_n|H'|\psi_n\rangle$. They need to recognize that this implies that first order correction to the energy of a particular stationary state is given by the weighted average of the perturbation using the probability density for that state. An application of this idea leads one to conclude that the correction to both the ground state and the first excited state is positive when the perturbation is a delta function in the right half of the well. When the perturbation is $+V_o$ in the left half of the well and $-V_o$ in the right, the correction to both the ground state and the first excited state is zero.

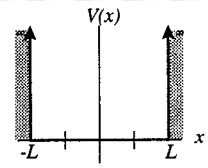
Only 20% of students answered correctly. Some of the others seemed to believe the energy correction could be found by averaging the perturbation weighted by the wave function rather than by the probability density. About 5% used this type of reasoning for throughout all parts of the problem. Another 15%, used this reasoning only for the questions concerning the second perturbation.

In the second problem, 10% of students reasoned about the two halves of the well separately. Many concluded that the energy would increase in half the well and decrease in the other half half. They did not seem to be troubled by the requirement that an energy eigenstate be associated with a single energy.

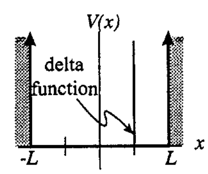
As part of their response about 5% of the students explicitly mentioned that the wave function would be constrained to only part of the well by the perturbation. One student said, "Because the potential change creates a boundary that the wave function will have to propagate through or be reflected by" to explain their contention that the wave function is zero to the right of the delta function. Other students revealed similar beliefs concerning the second perturbation by drawing a a sinusoid that only existed on the right side of the well where the potential was the lowest.

Consider a particle in an infinite square "well" of width $2L$ (see figure at right).

Two possible perturbations to the original potential are described below. For what follows, recall $E^1_n = \langle \psi_n | H' | \psi_n \rangle$.



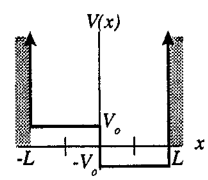
A. Suppose the original well is perturbed by a delta function potential as shown at right. The perturbation to the Hamiltonian is given by: $H'(x) = +\alpha\,\delta(x - L/2)$.



1. Is the first-order correction to the energy, $E^1_{n=1}$, of the ground state *positive*, *negative*, or *zero?* Explain.

2. Is the first-order correction to the energy, $E^1_{n=2}$, of the first excited state *positive*, *negative*, or *zero?* Explain.

B. Consider the well shown at right. Write an expression for the perturbation, H', to the original well that results in this well.



For this perturbation:

1. Is the first-order correction to the energy, $E^1_{n=1}$, of the ground state *positive*, *negative*, or *zero?* Explain.

2. Is the first-order correction to the energy, $E^1_{n=2}$, of the first excited state *positive*, *negative*, or *zero?* Explain.

Figure 17.1 A question concerning time-independent perturbation theory asked of 107 students in Physics 325 between 2003 and 2006.

Experience working with students in tutorial sections indicates that the belief that the first order correction can be found using $\int V(x)\psi(x)dx$ rather than $\int V(x)\psi^*(x)\psi(x)dx$ is more common than the results from the previous question illustrates. This experience suggests that many students have this idea, but are not able to apply it consistently.

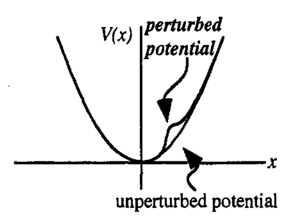
A preliminary version of a tutorial worksheet had been designed to address some of these difficulties, and to provide students with a mechanism for understanding the effect of applying small perturbations to quantum mechanical systems.

Students who completed the tutorial worksheet in 2003 (N=50) were asked the question shown in Figure 17.2. The question involves a small perturbation to the harmonic oscillator potential consisting of a "bump" on the right hand side. About 75% were able to reason correctly that the first order energy correction to both the ground and first excited states would be positive. Only 10% of students choose positive for the correction to the ground state and negative for the first excited state indicative of using the wave function rather than the probability distribution in the calculation.

For all parts of this problem, consider a particle in a one dimensional harmonic oscillator (*i.e.*, $V(x) = \frac{1}{2}m\omega^2x^2$).

(a) Suppose that the oscillator were perturbed as shown. Would each of the following quantities be *positive, negative,* or *zero*? Explain.

i. the first-order correction to the energy of the ground state



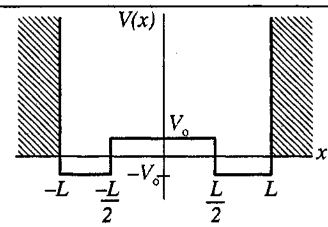
ii. the first-order correction to the energy of the first excited state

Figure 17.2 An examination question asked of 50 students in Physics 325 during Winter quarter of 2003 concerning time-independent perturbation theory.

In 2004 and 2006 a similar question, shown in Figure 17.3, was asked on the second exam of Physics 325 (N=70). The questions asked were the same, but the potential was designed to probe two additional difficulties. During instruction it was found that some students were assuming the first order correction to the energy could be obtained from an un-weighted average of the perturbation. In the case of the potential shown in Figure 17.3 a student believing this would conclude that both corrections are zero. Classroom experience also indicated that some students assumed incorrectly that the integral of an even function is always positive. In this case a student reasoning this way would predict that the correction is positive in both cases.

The correct answer to the examination question requires students to average the perturbation weighted by the appropriate probability distribution. Since the distribution for the ground state weights the center more, the correction is positive. The first excited state weights each quarter of the well the same. So, the correction is zero. About 60% of students were able to answer correctly. About 20% made an error consistent with taking an un-weighted average of the perturbation. Another 10% of student responses were consistent with the idea that an even function always has a positive integral.

[6 pts] Treat the potential well shown as a small perturbation to the infinite square well.



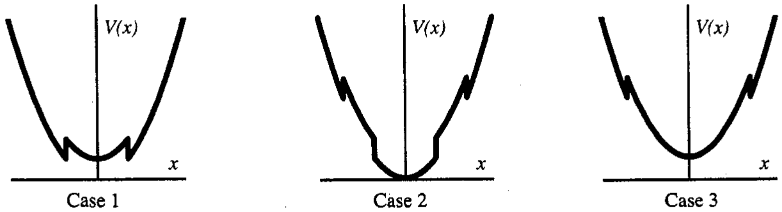
1. Would the first order correction to the energy of the ground state be *positive, negative,* or, *zero?* Explain.

2. Would the first order correction to the energy of the first excited state be *positive, negative,* or, *zero?* Explain.

Figure 17.3 An examination question asked of 70 students in Physics 325 during Winter quarter of 2004 and 2006 concerning time-independent perturbation theory.

A different question was asked on the second examination during the Winter quarters of 2005 and 2007 (N=75). It is shown in Figure 17.4. This question required students to determine the sign of the first order correction to the ground state energy for three different perturbations. Students were also asked to rank the magnitudes of these corrections for three different perturbations. The perturbations were all chosen to be the same height. The perturbation in case three was the longest. The perturbations in cases one and two were the same total length. In case one, the perturbation was centered about the origin; whereas in case two, the perturbation is located far from the origin.

Consider the perturbation to the harmonic oscillator shown in each case below.



$$H_1'(x) = \begin{cases} 0 & x < -1, \\ \epsilon & -1 \leq x \leq 1, \\ 0 & x > 1. \end{cases} \quad H_2'(x) = \begin{cases} 0 & x < -2, \\ \epsilon & -2 \leq x \leq -1, \\ 0 & -1 < x < 1, \\ \epsilon & 1 \leq x \leq 2, \\ 0 & x > 2. \end{cases} \quad H_3'(x) = \begin{cases} 0 & x < -2, \\ \epsilon & -2 \leq x \leq 2, \\ 0 & x > 2. \end{cases}$$

For each case, is the first-order correction to the ground state energy *positive*, *negative*, or *zero?* Explain. (Assume ϵ is positive.)

Rank the absolute value of the first-order correction to the ground state energy in each case from greatest to least. Explain.

Figure 17.4 An examination question asked of 75 students in Physics 325 during the Winter quarters of 2005 and 2007 concerning time-independent perturbation theory.

To answer the first part correctly, students need to recognize that all of the perturbations are positive. Thus, all of the energy corrections are positive. To answer the second part, students need to recognize that the ground state has a probability density which is largest near the origin. Thus, the energy correction corresponding to case 3 exceeds that of case one which exceeds that of case two.

About 75% of the students could determine the sign of the correction for all three perturbations, and 55% of students were able to correctly answer both parts of this question. About 10% of students answered in a manner consistent with the correction being the unweighed average of the perturbation. Another 15% focused only on the center of the well stating that case two will not be perturbed while one and three will be.

17.2 Student Difficulties with Time Dependent Perturbation Theory

We have also had the opportunity to examine student understanding of time dependent perturbation theory. This study was largely motivated by our investigation of student difficulties concerning time dependence in quantum mechanics.

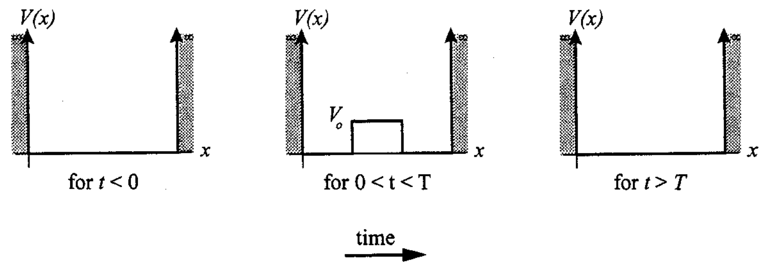
Between 2005 and 2007 we asked the questions shown in Figure 17.5 and Figure 17.6 to 71 students in Physics 325. Students were told that an infinite square well system, initially in the ground state, is perturbed for a finite time as shown. Students were asked whether energy measurements made before, during, or after the perturbation could yield any of the energy eigenvalues of the unperturbed system, or only certain energy eigenvalues of the unperturbed system. They were also asked if a countable or uncountably infinite number of values were possible for such an energy measurement. In addition they were asked if the probability density for the system would change with time for periods before, during, and after the perturbation.

Students needed to recognize that before the perturbation is turned on any measurement of energy will give the ground state energy of the infinite square well. The probability density is constant in time. While the perturbation is on, the system no longer has the same energy eigenvalues. The new wave function can be written as a superposition of the energy eigenstates for the new potential, and thus the probability density changes with time. After the perturbation is turned off the system is, in general, left in a superposition of the stationary states of the infinite square well. So, it is possible to measure multiple values for the energy and the probability density changes with time.

About 20% of students could answer the energy questions correctly. About 35% of students answered the questions about the probability density correctly. Only about 5% of the students answered both parts correctly. The

A time dependent perturbation to the infinite square well potential is illustrated below. (*Note*: $V_o << E_1^o$) At a time $t_- << 0$ the total energy of the system is measured and found to be the ground state energy of the unperturbed well. In what follows let the symbols E_i^o represent the energy eigenvalues for the unperturbed well.

A time dependent perturbation to the infinite square well



Consider a single additional energy measurement made at one of three possible times. In each case describe as specifically as possible what values could possibly result from the measurement. (*e.g.* Are any of the E_i^o possible values? Are only certain of the E_i^o possible? Are there a countable or unaccountably infinite number of possible values?)

Case 1: The measurement is made at a time $t_- < t < 0$. Explain.

Case 2: The measurement is made at a time $0 < t < T$. Explain.

Case 3: The measurement is made at a time $t > T$. Explain.

Figure 17.5 A question concerning energy measurements in time-dependent perturbation theory asked of 71 students in Physics 325 during the Winter quarters of 2005–2007.

Assume no energy measurements have been made after time t_-. Does the spatial probability distribution change with time during each of the following time intervals.

Case 1: between $t = t_-$ and $t = 0$ Explain.

Case 2: between $t = 0$ and $t = T$ Explain.

Case 3: after $t = T$ Explain.

Figure 17.6 A question concerning spatial probability distributions in time-dependent perturbation theory asked of 71 students in Physics 325 during the Winter quarters of 2005–2007.

results of the questions concerning probability density may overestimate student understanding since 40% of those students who gave the correct answer explicitly said that the probability density decayed to that of the ground state or slowly revived the original state. This is how they accounted for the probability density changing after the perturbation was turned off. Two of the most common incorrect responses were to claim that it was possible to measure eigenvalues corresponding to the square well while the perturbation was on, and to claim that the probability density would not change after the perturbation was turned off. These errors were each made by at least 40% of the students.

A tutorial was developed that attempts to address the issues discussed above and others. After tutorial instruction in Winter of 2006, 40 Physics 325 students answered the question shown in Figure 17.7 as part of their final exam. The question asks asks the students to rank the probabilities of measuring the ground state energy of a system before a perturbation is applied, during the perturbation, after the perturbation, and a long time after the perturbation. Initially the particle is in the first excited state.

While the perturbation is on, it is not possible to measure the energy eigenvalues of the unperturbed well. After the perturbation, the probability of measuring the ground state is non-zero since the matrix element for the perturbation connecting the ground state and first excited state is nonzero. This can be inferred from symmetry as the matrix element is an integral of an anti-symmetric wave function multiplied by an anti-symmetric potential multiplied by a symmetric wave function. Once the perturbation is turned off, the probability of measuring the ground state remains constant no matter how long you wait. Only ten percent of students could rank the probabilities completely correctly. About 60% of the students were able to conclude that the probability of obtaining the ground state energy is the same just after the perturbation is turned off and a long time later. Clearly additional research is needed concerning this topic along with additional curriculum development.

Consider a one dimension harmonic oscillator. A long time before $t = 0$, the energy of the oscillator is measured and found to be that of the first excited state, E_1. Immediately after time $t = 0$, a small perturbation is applied in the form $H' = \varepsilon x$, where ε is a small parameter with the correct units. Just before a time equal to one-fourth of the period of the oscillator (*i.e.*, $t = \pi/2\omega$), the perturbation is turned off. In what follows, E_i are the eigenstates of the unperturbed harmonic oscillator.

$$H'(t) = \begin{cases} 0; & t \leq 0 \\ \varepsilon x; & 0 < t < \pi/2\omega \\ 0; & t \geq \pi/2\omega \end{cases}$$

[5 pts.] Rank the probabilities that a measurement of the energy will yield E_o at the following times: $t = 0$, $t = \pi/4\omega$, $t = \pi/2\omega$, and a time long after the perturbation is finished. Use the following symbols: $P_o(0)$, $P_o(\pi/4\omega)$, $P_o(\pi/2\omega)$, $P_o(\infty)$. If any probability is zero or one, state so explicitly. Use only first-order time-dependent perturbation theory. Explain.

Figure 17.7 A question on time-dependent perturbation theory asked of students in Physics 325 on their final exam during Winter quarter of 2006.

17.3 | Summary

This chapter has illustrated student difficulties concerning both time-independent and time-dependent perturbation theory. In addition, it has shown how student difficulties with time-dependence influence student reasoning about these more advanced topics. The prevalence of student difficulties suggest the need for further research and curriculum development.

Another area of student understanding that we have begun to explore concerns degenerate perturbation theory. We have developed preliminary curriculum in this area, and made observations of students working with this curriculum. Student difficulties in this area, however, remain largely unexplored.

Chapter
18

Student Difficulties Associated with the Formalism Used in Quantum Mechanics

Most instructors would probably agree that envisioning functions as vectors in an infinite dimensional Hilbert space is one of the more challenging aspects of learning quantum mechanics. On the other hand, it is often felt that an initial exposure to the Dirac "bra-ket" notation at the undergraduate level is helpful even if the students do not gain a greater understanding until graduate school. Our results confirm that, in fact, students have significant difficulties with "Dirac" notation. Many are even unable to transcribe what they have done previously with wave functions, $\Psi(x, t)$, into this new notation.

In this Chapter wee illustrate some problems that indicate the extent of student problems with both Dirac notation, and Fourier analysis.

18.1 Difficulties Associated with Dirac "bra-ket" Notation

The question shown in Figure 18.1 was asked of 72 students who took Physics 324 in the Summer of 2003 as well as in Autumn of 2003 and 2004. Students are shown a probability density and asked to find $\angle x_1|\psi_o\rangle$, $\angle x_4|\psi_o\rangle$, $\angle x'|\psi_o\rangle$ where x_1 and x_4 are marked on the graph and $x' = x_1 + x_4$ is a location where the probability density is zero. The question was asked well after students had been introduced to Dirac notation in class.

Consider a particle in the ground state, ψ_o, of an *unspecified* potential well. The probability distribution for the ground state is shown at right. Assume the ground state was chosen to be real at $t = 0$. Answer the following question for a particle in the state ψ_o at time $t = 0$.

Find $<x_1|\psi_o>$, $<x_4|\psi_o>$, and $<x'|\psi_o>$ where $x' = x_1 + x_4$. Explain.

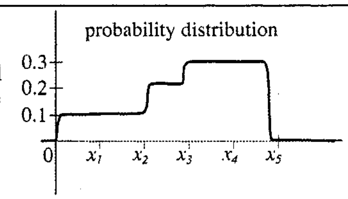


Figure 18.1 A question asked of Physics 324 students before specialized instruction in 2003 and 2004 that requires a basic understanding of Dirac notation.

To answer correctly students needed to make the identification $\psi(x') = \langle x'|\psi\rangle$. Hence, they can read from the graph that $\langle x_1|\psi\rangle = \sqrt{.1}$, $\langle x_4|\psi\rangle = \sqrt{.3}$, and $\langle x'|\psi\rangle = 0$. Even excluding errors made by forgetting the square root, only four of the 72 students answered this question correctly. Moreover, 65% of students claimed that the third quantity was the sum of the first two. They reasoned as if $\langle(x_1 + x_4)|\psi\rangle = \langle x_1|\psi\rangle + \langle x_4|\psi\rangle$. In addition, many students did not treat these quantities as numbers. Answers such as $\langle x_1|\psi\rangle = .1\psi$ or $\langle x_1|\psi\rangle = .1x$ were also common.

Other results indicate that even after all instruction, some of these types of difficulties remain. Figure 18.2 shows a question asked of students in the Summer and Autumn quarters of 2003 on their final exam. These students had some specialized tutorial instruction intended to help them understand functions as infinite dimensional vectors. The question showed students a graph of the real part of the wave function and asked them to give an approximate value for $\langle x_1|\psi\rangle$ where x_1 was a marked location on the graph. To answer students needed to interpret the Dirac notation, $\langle x_1|\psi\rangle$, as representing the value of the wave function at a particular point, $\psi(x_1)$.

About 75% of the students who took this exam question answered correctly. As on the question above, a common error was multiplying or dividing by the spatial coordinate x. Although this result is better than the previous question, there remain 25% of the students who could not answer this very basic question.

[4 pts.] A wavefunction in an infinite square well of width a is shown at right. The wavefunction is entirely real at this instant in time.

1. Give an approximate value for $\langle x_l|\psi\rangle$. Explain.

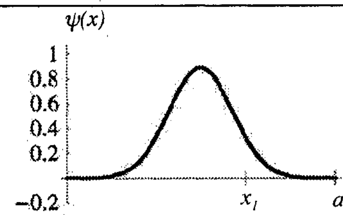


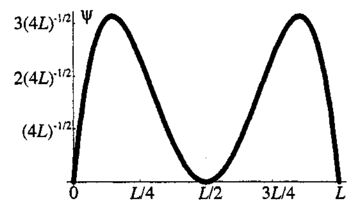
Figure 18.2 A question asked of Physics 324 students in Summer and Autumn of 2003 that requires a basic understanding of Dirac notation.

Figure 18.3 shows a similar, but slightly more involved, question asked of students on their final exam in Autumn of 2004. Again these students had worked through specialized tutorial instruction on viewing functions as vectors. In this question students were again given the graph of a wave function. They were asked for $\langle x_o|\psi\rangle$, $\langle\psi|10\hat{x}|\psi\rangle$, and $\langle x'|\psi\rangle$ where x_o was a given point $x' = 10x_o$ and $\hat{x}$ is the position operator.

Only about 55% of the students answered all three parts of this question correctly. About 10% of students thought that $\langle 10x_o|\psi\rangle = 10\langle x_o|\psi\rangle$, and another 10% reasoned as if $\langle\psi|\hat{x}|\psi\rangle$ represents the value of the wave function at the point x.

While student improvement on these tasks after tutorial instruction is encouraging, these results indicate that Dirac notation remains difficult for many students even after this instruction. More work needs to be done to help students reach a basic level of familiarity and facility with this notation.

A particle is confined to a one-dimensional infinite square well of width L. At the instant shown at right the normalized wavefunction is entirely real. In what follows, let $x_o = 0.8L$, $x' = 10x_o$, and let $\hat{x}$ be the x operator.



1. Give an approximate value for the following quantities. If any do not have finite values, state so explicitly. Explain your answers.
   (a) $< x_o|\psi >$

   (b) $< \psi|10\hat{x}|\psi >$

   (c) $< x'|\psi >$

Figure 18.3 A question asked of Physics 324 students in Autumn of 2004 that requires a basic understanding of Dirac notation.

18.2 | Difficulties Associated with Fourier Analysis

We have asked a number of questions to probe student understanding of Fourier analysis as it is used in a typical quantum mechanics course. Below we discuss several questions and the difficulties those questions elicited.

In 2003 and 2004, Physics 324 students were asked the question shown in Figure 18.4. It required them to given an interpretation of the quantity $|a(p)|^2$ in the expression $\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} a(p)e^{ipx/\hbar}dp$.

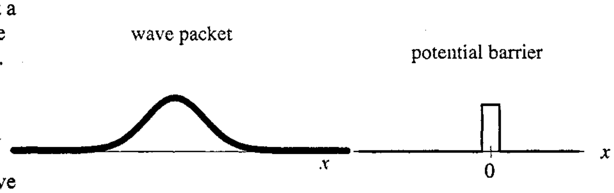
A wave function can be written as $\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} a(p)e^{ipx/\hbar}dp$. Give a physical interpretation for $|a(p)|^2$. Explain.

Figure 18.4 A question asked of Physics 324 students in 2003 and 2004 which required them to recognize a momentum space wave function.

Only 40% of the 72 students who answered this question could identify $|a(p)|^2dp$ as the probability of measuring a value for the momentum in the range dp about p. About 15% thought that $|a(p)|^2$ is the spatial probability distribution, but many other responses were also given. Some said it was the "power" or "energy" spectrum.

In the summer of 2003 another question was given in Physics 324. About 9 students were asked to consider several student statements about a wave packet moving toward a potential barrier. These students had already studied the Fourier decomposition of one-dimensional wave functions. The question is shown in Figure 18.5. The statements have drawn from comments we had heard from students in previous classes.

The wave packet shown at right at a particular time is incident from the left on the potential barrier shown.



1. Do you agree or disagree with each of the following statements concerning the wave packet shown? Briefly explain.

Student 1: "The wave packet is a solution to the Schrödinger equation, and is therefore associated with a particular energy."

Student 2: "The wave packet solves Schrödinger 's equation, but is associated with a very large but finite number of energies."

Student 3: "The wave packet solves Schrödinger 's equation, but is associated with an infinite number of energies all of which are multiples of some smallest energy."

Student 4: "The wave packet solves Schrödinger 's equation, but is associated with an infinite number of energies with all possible values contributing not just multiples of some smallest energy."

Student 5: "The wave packet does not solve Schrödinger 's equation. It is not a plane wave."

Figure 18.5 A question asked of 9 students in the Summer quarter of 2003 requiring them to reason about a wave packet.

About half (5) of those students correctly agreed with the fourth student that the wave packet solves the Schrödinger equation and is associated with an infinite number of energies that are not multiples of a smallest energy. The most common incorrect response was that the wave packet solves the Schrödinger equation, and is associated with an infinite number of energies that are multiples of a smallest energy. Many students justified this response by stating that energy must be quantized.

Another question asked students to draw the probability distribution for the momentum of the wave packet at the instant shown. A correct answer requires students to recognize that many momentum eigenstates must be added together to make the wave packet shown. They must also recognize that these must be centered about a positive momentum value if the packet is to move toward the right. Only about one-third of the students correctly drew a distribution centered on a positive momentum value. Another one-third drew distributions centered on zero momentum. Most students did not provide sufficient explanations to illustrate their reasoning, but it is clear that students have difficulties with even basic requirements for Fourier decompositions of wave packets.

We have also asked some questions that more directly probe student understanding of Fourier analysis. In the Spring of 2003 we asked a mechanical engineering class, ME374, the question shown in Figure 18.6 and in Figure 18.7. In Figure 18.6 the students were shown several signals. They were asked if each signal could be made of a linear combination of sinusoids. If they thought the signal could be made up of multiple sinusoids, they were asked if there was a lowest frequency sinusoid.

To answer students needed to recognize that the signals with finite duration (signals I, III, and V) could be created from an infinite number of periodic sinusoids with no smallest frequency. The periodic signals (signals II, IV, and VI) could also be created from linear combination of periodic sinusoids, but in the case there would be a smallest frequency.

None of the 30 students who answered this question did so completely correctly. One third of the students thought that the finite signals could not be made from linear combinations of periodic sinusoidal signals. The students who did correctly say that signal III and V could be made of sinusoids (about 15%) incorrectly said that all the contributions would be multiples of some lowest frequency. In particular, many stated that signal III could be made from a single sinusoid and signal V could be made from three.

A. Six signals are shown below. The periodic signals II, IV, and VI extend from the remote past to the remote future. For each signal:

1. Can the signal be made from a linear combination of sinusoidal signals? If so, would a sinusoid of a single frequency be sufficient, or would sinusoids of two or more frequencies be necessary? Explain.
2. For those signals that can be made from a linear combination of two or more sinusoidal signals:

   Are all of the frequencies present in the linear combination multiples of some lowest frequency? If so, determine the lowest frequency for that signal. Explain. If not, explain why not.

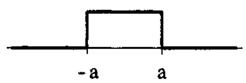
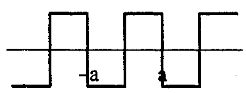
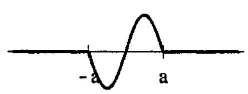
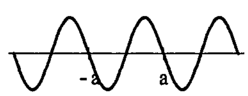
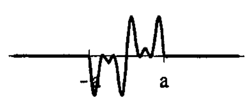
	1. *one* or *more*? Explain.	2. If two or more frequencies: Is there a lowest frequency? *yes, no?* Explain.
I.  -a a time		
II.  -a a time		
III.  -a a time		
IV.  -a a time		
V.  -a a time		
VI.  -a a time		

Figure 18.6 A question about Fourier analysis asked in a mechanical engineering class.

The question shown in Figure 18.7 is about a linear system. It was motivated by some responses discussed in Part I that students in quantum mechanics had given to questions which required them to reason about the possible results of energy measurements made on isolated systems. In those questions, many students reasoned that a superposition of stationary states would decay to a single stationary state after a long time. The question shown in Figure 18.7 describes a mass, spring, and damper system with an input signal given by a sum of three sinusoidal signals with different frequencies. The students are asked how many frequencies are found in the response a long time later, and if they can determine those frequencies.

> Consider a system made from ideal masses, springs, and dampers. The system is driven with a signal given by $F(t) = 3\sin(2\pi t/3) + 5\sin(2\pi t/4) + 3\sin(2\pi t/5)$. After a long time has elapsed:
>
> How many frequencies will be found in the response? Explain.
>
> With the given information is it possible to determine which frequencies will be found in the response? If so, what are they? Explain. If not, explain why not.

Figure 18.7 A question about linear systems asked in a mechanical engineering class.

To answer correctly, students must recognize that in a linear system the same frequencies would be present in the response as were in the input signal.

The students had studied such systems in their class, and solved problems that required Fourier analysis. Only 55% of the students who took this quiz, however, correctly concluded that three input frequencies would be present in the long term response. Only 35% said that the same same three frequencies present in the input would be present in the response. One-third of the students thought that the response would only have one frequency. Some of those students thought that after a long time the other frequencies would decay away. This response is very similar to the "decay" response prevalent among quantum mechanics students who believe any wave function will decay to a stationary state. Others suggested the response should decay to a steady state. This may be similar to students describing wave functions diffusing to a flat line. This similarity to the way students describe the long term time dependence of wave functions deserves further study.

Overall, our results suggest that many students do not seem to have an understanding of Fourier analysis, nor do many seem to understand the usefulness of Fourier analysis for linear systems.

18.3 Summary

In this chapter, we have presented evidence that students have significant difficulties with Dirac notation. These difficulties arise even in converting notation such as $\psi(x)$ into $\langle x|\psi\rangle$. During instruction we have noted much more confusion among students about the relationship between functions and vectors in an infinite dimensional Hilbert space. In addition, this chapter has discussed student difficulties with Fourier analysis, and has shown that some difficulties we see in quantum mechanics courses have close parallels to difficulties in contexts outside of quantum mechanics. There are many ways in which this work could be expanded and used to inform instruction in a variety of courses.

Chapter 19

Conclusion

Research conducted by the Physics Education Group during the course of the past 35 years has demonstrated the effectiveness of an iterative cycle of research, curriculum development, and assessment in improving instruction in physics and physical science at a variety of levels. For the most part, however, this research has focused on foundational topics. This dissertation describes an extension of those research techniques to an upper-division physics course primarily taken by physics majors. The research described in this dissertation has spanned the breadth of concepts typically covered in an undergraduate quantum mechanics course and has led to the design of preliminary versions of 50-minute tutorials on some important topics. These have been implemented and tested over a period of several years in weekly tutorial sections associated with the Physics 324-325 quantum mechanics sequence at the University of Washington. We have found that research-based curriculum development, in which assessment plays an integral part, can be effective at this level of instruction both for expanding the research base on student understanding and for improving student learning. Moreover, we have found that students at this level have been very receptive to this method of instruction, often asking that it be incorporated into other upper-division courses in addition to quantum mechanics.

Part I of this dissertation describes ways in which students reason about time-dependence in quantum mechanics before instruction as well as specific difficulties that many students have in applying the model based on the Schrödinger equation that they learn in a typical quantum mechanics course. For example, many students reason as if they believe that all quantum mechanical states are stationary, or that all operators have the same eigenfunctions. Part I also illustrates the design of tutorials to address these and other difficulties. Encouraging results concerning the assessment of that curriculum's effectiveness are reported.

Part II of the dissertation begins by describing research that led to modifications that we made to existing curriculum intended to help students relate classical to quantum mechanics. We describe how our investigation of student understanding of time-dependence led to our identification of some common incorrect over-generalizations that students often made after working through initial versions of the tutorials. For example, students often incorrectly associate stationary states with macroscopic systems having time-dependent probability distributions. In the process of attempting to help students come to a better understanding of the relationship between classical and quantum mechanics, we also identified some difficulties that students have with the ideas of uncertainty, approximations, and the limitations of models in quantum mechanics. All of these insights guided modifications to the tutorials.

Finally, Part III of the dissertation describes research on student understanding of certain more advanced topics in quantum mechanics. These include probability current, angular momentum, identical particles, and perturbation theory. Many of the difficulties that we identified relate to probabilities, symmetries, measurements, approximation, and mathematics. Most of these difficulties, however, were shown to have roots in more basic ideas concerning probability or time-dependence.

Insights gained as well as specific results obtained during this investigation transcend the quantum mechanics course. For example, we have seen strong similarities between the difficulties that quantum mechanics students have with solutions to the Schrödinger equation and difficulties that mechanical engineering students have with mechanical systems governed by differential equations. Such difficulties may be common among students studying a variety of differential equations. If so, it may be possible to use similar strategies in addressing them. The prevalence of many difficulties concerning fundamental ideas among quantum mechanics students may have implications for the teaching of modern physics topics courses prior to quantum mechanics and for the proposals to teach these topics at ever earlier levels. In addition, this work has increased our understanding of the way in which students approach uncertainties, approximations, and the limitations of models. These ideas are important, not only for future physicists or scientists, but for the general population and their teachers. Thus, just as this work has built on the work of others, it is our hope that our efforts can form the basis of further improvement in the understanding of student learning and contribute to a cumulative improvement in instruction.

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Appendix A Current Tutorials

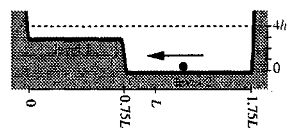
This appendix contains tutorials which are currently being used as part of the Physics 324–325 sequence at the University of Washington. The order of the tutorials is the order in which they are typically used.

Page 1 of Tutorial: Classical Probability

CLASSICAL PROBABILITY Name ____________

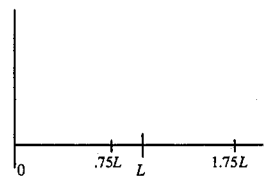
Balls on tracks

A ball rolls back and forth in a track with very steep sides. Two levels of unequal length joined by a steep ramp form the base of the track. A large number of photographs of the system are taken at random times. You may assume that the time spent on the steep portions is negligible. Assume there is no friction or energy loss in the system, and that the ball rolls smoothly, without bouncing, forever. Level 1 has a length of 3/4*L*, and level 2 has a length of *L*. Assume the ball was dropped from a height of 4*h*.



A. *Probability*

1. Sketch the gravitational potential energy corresponding to this situation between $x = 0$ and $x = 1.75L$.



2. Is the speed of the ball on level 1 *greater than, less than,* or *equal to* its speed on level 2? Explain.

   For the ball, is the amount of time that it spends on level 1 *greater than, less than,* or *equal to* the amount of time it spends on level 2? Explain.

3. Suppose a single photograph were taken at a random time. On the basis of your results above, is the probability of the photograph showing the ball on level 1 *greater than, less than,* or *equal to* that of the photograph showing the ball on level 2? Explain.

4. Determine the probability of finding the ball on each level. Explain.

✓ Check your results with a tutorial instructor.

Page 2 of Tutorial: Classical Probability

B. *Probability density*

Imagine splitting level 2 into two unequal segments: segment "2A" from $x = .75L$ to $x = L$, and segment "2B" from $x = L$ to $x = 1.75L$.

1. Find the probability that out of anywhere in the system the ball is found:

   a. along segment 2A

   b. along segment 2B

   Explain your reasoning.

Consider the following ratio for each segment: The probability of finding the ball along that segment divided by the length of that segment.

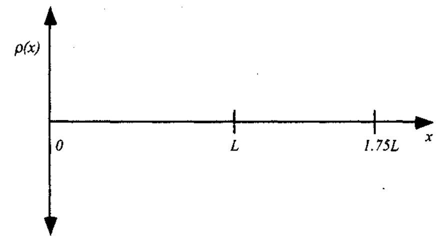
2. Is the above ratio larger for segment 2A, larger for segment 2B, or is it the same for both segments? Explain.

The ratio defined above is called *probability density*.

3. Compare and contrast *probability density* with other densities that you have encountered in physics. What are the units of probability density in this case?

Page 3 of Tutorial: Classical Probability

4. In the space at right, carefully draw a graph of *probability density, ρ(x),* versus *position* from $x = 0$ to $x = 1.75L$. Label relevant values on the vertical axis.



5. What feature of the graph represents the probability of finding the ball in an arbitrarily chosen interval between $x = x_1$ and $x = x_2$?

6. What is the probability of finding the ball *exactly* at $x = L$? Explain.

7. What answer would you expect for the probability of finding the ball anywhere between $x = 0$ and $x = 1.75L$? Show that your graph of probability density gives you the answer you expect.

8. Suppose you were given an arbitrary probability density function $\rho(x)$ *(i.e., one that does not have a shape as simple as the one above).* Write a mathematical expression for the probability of finding the ball between $x = x_1$ and $x = x_2$.

Page 4 of Tutorial: Classical Probability

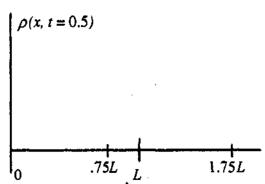
C. *Averages*

Suppose a large number of photographs of the experiment on the first page are taken at random times, and the position of the ball in every photograph is measured. Write an expression that shows how the probability density, $\rho(x)$, can be used to calculate the average of these positions.

Write an expression that shows how the probability density, $\rho(x)$, can be used to calculate the average speed.

D. *Probability distributions as a function of time*

Suppose you know the position and velocity of the ball at $t = 0$. Sketch a reasonable probability distribution for the location of the ball at $t = 0.5$ s, and describe the behavior of this probability distribution as a function of time (*i.e.* describe $\rho(x,t)$). Explain.



How, if at all, is $\rho(x,t)$ related to the probability distribution that you drew in section B4 above?

✓ Check your results with a tutorial instructor.

Page 1 of Tutorial: Relating Classical Mechanics to Quantum Mechanics

RELATING CLASSICAL MECHANICS TO QUANTUM MECHANICS

I. The Schrödinger equation and the shape of the wave function

The Schrödinger equation in one dimension is given by $i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V\Psi$. Its solutions, in general, can be written as $\Psi(x,t) = \sum c_n\psi_n(x)e^{-iE_nt/\hbar}$. In this tutorial we consider only solutions in the form $\Psi(x,t) = \psi_n(x)e^{-iE_nt/\hbar}$, where the functions $\psi_n(x)$ satisfy the time independent Schrödinger equation, $\frac{d^2\psi(x)}{dx^2} = -\frac{2m}{\hbar^2}\left[E - U(x)\right]\psi(x)$.

A. What can be said about the form of the function, $\psi(x)$, if the total energy in a given region is:

1. greater than the potential energy in that region?

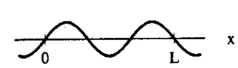
2. less than the potential energy in that region?

3. equal to the potential energy in that region?

B. In those region(s) in which the function $\psi(x)$ is sinusoidal, it may be expressed mathematically in the form, $\psi(x) = A\cos kx + B\sin kx$, where k is a constant.

1. In describing periodic waves, k is often called the *wave number*. Give an interpretation of the wave number. (*Hint:* How is the wave number related to the *wavelength* λ of a sinusoidal wave?)

2. Find the wave number, k, in terms of L for the following wave.



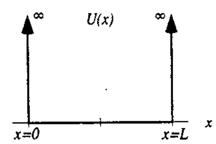
3. How is the wave number related to quantities that appear in the Schrödinger equation?

Page 2 of Tutorial: Relating Classical Mechanics to Quantum Mechanics

QM 2 *Relating classical mechanics to quantum mechanics*

II. The infinite square well potential

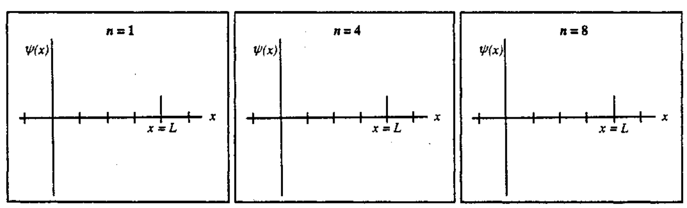
Consider a particle with total energy E in an infinite square well potential of width L. (As shown in the graph, the potential energy rises abruptly from zero to a very large value, essentially infinite, at $x = 0$ and at $x = L$.)



The allowed energy states correspond to the functions, $\psi_n(x)$, and energy eigenvalues, E_n, below:

$$\psi_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right); \quad E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

A. Sketch the function, $\psi_n(x)$, that corresponds to a particle in each of the energy states below. Label the vertical axis with the appropriate values for each case.



B. What happens to the *wave number* of the function as the energy increases?

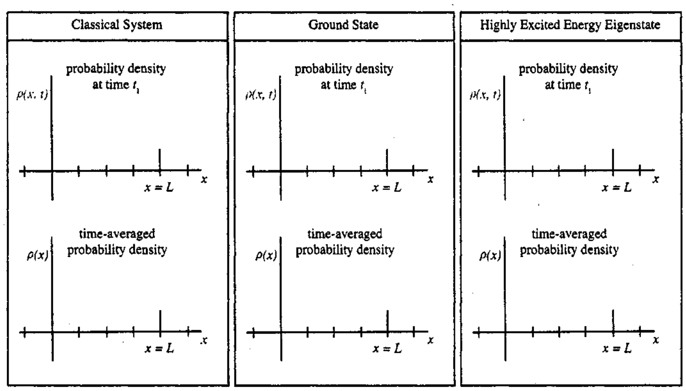
C. Write the full wave functions, $\Psi(x,t)$, for each of the states above.

D. Find the probability density as a function of position and time, $\rho(x,t)$, for each of the states above.

Page 3 of Tutorial: Relating Classical Mechanics to Quantum Mechanics

III. Classical and quantum systems compared

A. In this section we consider the quantum mechanical infinite square well, as well as its classical analog having the same potential. For each of the systems below complete the graphs of (1) probability density at some arbitrary time, t_1, and (2) the time-averaged probability density. Explain your reasoning.



B. The special states that we have been considering (*i.e.*, those that can be written as $\Psi(x,t) = \psi_n(x)e^{-iE_nt/\hbar}$, rather than $\Psi(x,t) = \sum c_n\psi_n(x)e^{-iE_nt/\hbar}$) are often given the name *stationary states*. Why do you think this is appropriate?

C. Suppose that you have been given a large ensemble of identically prepared systems all corresponding to a single one of the three possible cases above. Imagine that you also have a large number of detectors that you could use to measure position.

1. If, for each system, you made a measurement of position at the same time t_1, could you decide to which case your ensemble corresponds? Explain.

2. If, instead, you made measurements of position at random times, could decide to which case your ensemble corresponds? Explain.

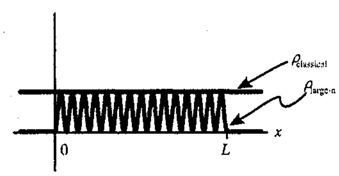
Page 4 of Tutorial: Relating Classical Mechanics to Quantum Mechanics

3. Would your answers to parts 1 and 2 change if the resolution of your detectors changed (*i.e.*, was made better or worse)?

Quantum mechanics was developed in order to account for phenomena that classical mechanics could not. If it is to be regarded as a complete theory, it must also be able to account for systems that classical mechanics can describe. This idea underlies the *correspondence principle.* A consequence of the correspondence principle is that the time average probability density for a highly excited system should be experimentally indistinguishable from that of the analogous classical system.

D. Consider the following discussion between two physics students:

Student 1: "For a classical particle in a box of length L, the time-averaged probability density is uniform along the length of the box and is equal to (1/L). I have drawn a time-averaged quantum mechanical probability distribution corresponding to large n. As you can see this probability distribution approaches the classical one just as the correspondence principle says it should.



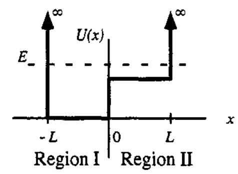
Student 2: "No, the area under the quantum mechanical distribution that you have drawn is half that of the classical probability distribution. That violates the correspondence principle.

Which student, if either, is correct? Identify any errors made by either student.

Page 5 of Tutorial: Relating Classical Mechanics to Quantum Mechanics

IV. Using classical mechanics to reason about highly excited energy eigenstates

A particle of energy E is in a bound state of the idealized potential well shown at right.



A. First, treat this problem from a purely ***classical*** standpoint.

1. Give an example of a real physical situation that corresponds to this potential well.

2. In which region of the well would the particle have greater kinetic energy? Explain.

3. In which region of the well would the particle spend more time? Explain.

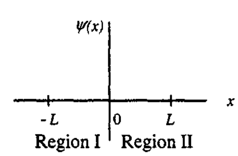
B. Now treat the problem from a quantum mechanical point of view. Consider specifically the case in which the particle occupies a *highly excited* energy eigenstate of the well for which the function $\psi_n(x)$ has many nodes (*i.e.*, zero points) between $x = -L$ and $x = L$.

1. What general shape do you expect for the function $\psi_n(x)$ in each region of the well? Base your answers on part IA.

2. How would you expect the *amplitude* of the function $\psi_n(x)$ to compare in the two regions of the potential well? Base your answers on part IV A and the consequences of the correspondence principle. Explain.

3. How would you expect the *wave number* of the function $\psi_n(x)$ to compare in the two regions of the potential well? Explain.

4. Sketch a graph of this highly excited eigenstate.



Page 1 of Tutorial: Time Dependence in Quantum Mechanics

TIME DEPENDENCE IN QUANTUM MECHANICS

I. Eigenstates of the infinite square well

Consider two identical quantum mechanical particles in separate, identical infinite square well potentials, $V(x)$, defined at right. At time $t = 0$ the first particle is in the ground state and the second particle is in the first excited state.

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 < x < a \\ \infty & x > a \end{cases}$$

A. Sketch and label the wave function for each particle at $t = 0$. Can you use a single two–dimensional sketch for each wave function? Explain.

Do the ground state and first excited state wave functions change with time? If so, how?

Are wave functions in your sketches solutions of the time-dependent Schrödinger equation? (*i.e.* Do they satisfy $i\hbar\frac{\partial\Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x)\Psi(x,t)$?)

Page 2 of Tutorial: Time Dependence in Quantum Mechanics

B. Sketch the probability density for each particle at t = 0. Can you use a single two-dimensional sketch for each particle's probability density? Explain.

Do the probability densities for the ground and first excited states change with time? If so, how?

C. A short time later the graph of the second particle's wave function looks like it did at t = 0.

1. What is the minimum amount of time, *T*, required for this to happen?

2. Sketch the first particle's wave function at this new time, *T*.

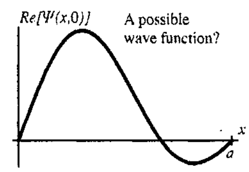
3. Sketch both wave functions at time $t = 2T$.

D. For a state with a time independent probability density, what feature of the graph of the wave function for that state reflects the idea that the probability density is constant in time?

Page 3 of Tutorial: Time Dependence in Quantum Mechanics

II. Linear superpositions of eigenstates

Consider the graph shown at right at time $t = 0$. (Assume $\text{Im}[\psi] = 0$.)



A. Could this graph be a possible wave function for the particle in the infinite square well? If so, is it an eigenstate, or could it be made from a sum of eigenstates? Explain.

Do you agree or disagree with the following students? Explain.

"It is a sum of eigenstates which makes it an eigenstate."

"The wave function above is a solution of the Schrödinger equation. So, there must be a definite energy associated with it."

B. Does the wave function change as time progresses? Explain.

Do you agree or disagree with the following students?

"At times the wave function is like the ground state at other times it is like the first excited state. So, sometimes it has energy E_1 and at other times E_2."

"No the c_ns are always the same. So, the probabilities of measuring E_1 or E_2 don't change even though it looks like ψ_2 at times when $e^{-iE_1t/\hbar} = 0$, and like ψ_1 at times when $e^{-iE_2t/\hbar} = 0$."

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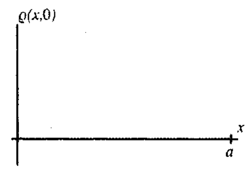
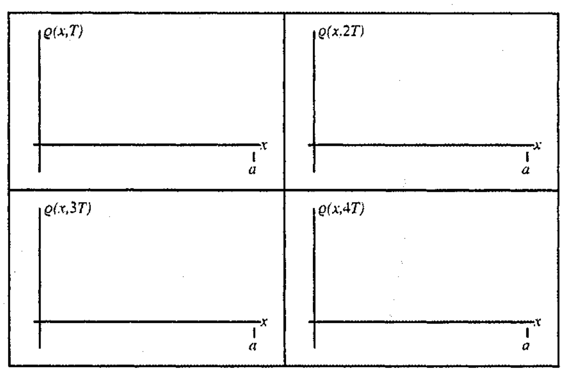
C. Draw the probability density associated with the wave function above at $t = 0$.



D. Ask an instructor for the functional form of the wave function above. Does the probability density change with time? If so, draw a sequence of graphs that illustrate the probability density at times: T, $2T$, $3T$, and $4T$.



E. Obtain a handout showing the time evolution of the probability density. Resolve any inconsistencies with your answers above.

Page 5 of Tutorial: Time Dependence in Quantum Mechanics

F. What will happen to the state after a very long time has elapsed? Explain.

G. What must be true of a quantum mechanical state if the probability density associated with it is to change with time?

H. What aspect or aspects of a classical particle's motion in an infinite square well are represented by a quantum mechanical highly excited energy eigenstate?

What must be true of a quantum mechanical state if it is to account for the time dependence of a classical particle's motion?

Page 1 of Tutorial: Energy Measurements

ENERGY MEASUREMENTS Name ____________

I. The nature of superpositions

A. Consider the following student's statement about a quantum mechanical infinite square well where ψ_1 and ψ_2 are the ground state ($n = 1$) and first excited state ($n = 2$), and each satisifies the time-independent Schrödinger equation. (*i.e.,* $H\psi_1 = E_1\psi_1$, and $H\psi_2 = E_2\psi_2$.)

"The wave function given by $\Psi(x,0) = \sqrt{\frac{1}{2}}\psi_1 + \sqrt{\frac{1}{2}}\psi_2$ *represents a lack of knowledge about the state of the system. The system is definitely in either the ground state or the first excited state. The wave function simply tells you that the probability is 1/2 that the system is really in the ground state and 1/2 that it is really in the first excited state."*

Do you agree or disagree with this statement?

B. Suppose you have the following two ensembles of systems in a quantum mechanical infinite square well at time $t = 0$:

(1) half of the systems are in state ψ_1 and half of the systems are in state ψ_2

(2) all of the systems are in state $(\psi_1 + \psi_2)/\sqrt{2}$.

At $t = 0$ would the probability of finding a particle on the left half of the well in a randomly selected system from ensemble (1) be *greater than, less than*, or *equal to* the probability of finding a particle on the left half of the well in a randomly selected system from ensemble (2)? Would your answer change as time evolves? Explain.

C. Check that your answers to parts A and B are consistent. Explain.

Page 2 of Tutorial: Energy Measurements

II. Eigenstates of the square well in the harmonic oscillator potential.

Consider a quantum mechanical harmonic oscillator. This system is prepared so that at $t = 0$ it is in a state described by a wave function identical to that of the ground state of the infinite square well. That is, $\Psi(x,0) = \sqrt{1/L}\cos(\pi x/2L)$ between $-L$ and L.

A. Does the probability density associated with this state depend on time? If so, describe its time dependence. If not, explain why not.

B. Consider the following student's statement.

"This state does depend on time. The full wave function is given by $\Psi(x,t) = \sqrt{1/L}\cos(\pi x/2L)e^{-iE_1 t/\hbar}$. The probability density, however, does not depend on time since the phase cancels out upon squaring."

Do you agree or disagree? Explain.

Does this student's wave function satisfy the Schrödinger equation for the harmonic oscillator? Show and explain your work.

C. Is the initial state of the harmonic oscillator associated with a single energy? Explain. (*Hint:* Consider the time-independent Schrödinger equation?)

Is $\Psi(x, 0)$ a valid state in the harmonic oscillator? If so, describe how you would write the full time dependent wave function $\Psi(x, t)$.

✓ **Discuss your answers with an instructor before continuing.**

Page 3 of Tutorial: Energy Measurements

III. Energy Eigenstates

Again consider a quantum mechanical harmonic oscillator.

A. Assume that energy measurements on this oscillator in a particular state could only yield E_o or E_1 with equal probability. Write and sketch $\Psi(x, 0)$. If it is not possible to completely specify $\Psi(x, 0)$ with the given information, write and sketch at least two possible $\Psi(x, 0)$ consistent with the given information.

B. Now assume the quantum mechanical harmonic oscillator is prepared so that its initial state is given by:

$$\Psi_i = \Psi(x,0) = i\sqrt{\frac{1}{3}}\psi_0 - \sqrt{\frac{2}{3}}\psi_1$$

where ψ_0 and ψ_1 are the ground state ($n = 0$) and first excited state ($n = 1$), and each satisifies the time-independent Schrödinger equation. (*i.e.,* $H\psi_o = E_o\psi_o$, and $H\psi_1 = E_1\psi_1$.)

1. Sketch a graph of each term in the linear combination that constitutes $\Psi(x,0)$. Describe how a graph of $\Psi(x, t)$ would change with time.

2. Is the Ψ_i an energy eigenstate? Justify your answer.

3. Suppose that at time $t = 0$ you measured the energy of the system.

   What value or values could a measurement of the energy yield?

   What is the probability that your energy measurement would yield the value E_o?

Page 4 of Tutorial: Energy Measurements

Is the probability above dependent on the time at which the measurement is made? Explain.

Explain how your answers about energy measurements are consistent with the time dependence of $\Psi(x, t)$ that you discussed above.

4. On the basis of your results above, can "the energy of the system" in this state be represented by a single number or is this quantity not well-defined?

C. Suppose that an energy measurement made on the system in its initial state Ψ_i described above yields the first excited state energy, E_1.

1. Write down an expression for the state of the system immediately after the energy measurement is made.

2. Do you agree that acting on the initial wave function with the Hamiltonian corresponds to measuring the energy? Explain.

3. Imagine that a few minutes later you measured the energy of the system again. (Assume that, in the time interval between the first and second measurements, the system remained isolated from the environment.) On the basis of your results above, what would you expect the results of your energy measurement to be? Explain.

✓ **Discuss your answers with an instructor before continuing.**

Page 5 of Tutorial: Energy Measurements

IV. Summary

Assume $\Psi(x, 0)$ is a general quantum mechanical wave function describing the initial state of an isolated system with a time-independent potential.

A. Would the possible values for energy measurements on this system change as time evolves? Explain.

Would the relative probabilities of measuring these values change as time evolves? Explain.

B. Would the possible values for position measurements on this system change as time evolves? Explain.

Would the relative probabilities of measuring these values change as time evolves? Explain.

C. Describe how you would find the state of the system, $\Psi(x, t)$, as a function of position and time given the functional form of $\Psi(x, 0)$.

Page 1 of Tutorial: Position, Momentum, and Energy Measurements

POSITION, MOMENTUM, AND ENERGY MEASUREMENTS

I. Eigenvalue equations

The generalized statistical interpretation of quantum mechanics postulates that measurements of an observable of a system can yield only eigenvalues of the hermitian operator corresponding to that observable. In this tutorial, we examine some consequences of this interpretation.

A. For each of the observables below, write the corresponding operator in terms of x (and derivatives with respect to x). Explain. If in any case more information is needed, describe the required information.

Position	Momentum	Energy

B. What equations must you solve to determine the possible results of the following measurements? In each case, write two equations: one in terms of the symbols for the formal operators; the other in terms of x (and derivatives with respect to x). In each case indicate explicitly any symbols that are not functions of x.

A position measurement	A momentum measurement	An energy measurement

C. Which of the equations above depends upon the system under consideration? Explain.

II. Eigenvalues and eigenvectors of position.

Consider the eigenvalue equation for position that you wrote in part IB. Suppose α were a possible eigenvalue.

A. Rewrite the equation such that one side is equal to zero.

B. Describe the eigenfunction corresponding to the eigenvalue α in words.

C. Are the eigenvalues discrete or continuous?

Page 2 of Tutorial: Position, Momentum, and Energy Measurements

D. Describe how $\delta(x - \alpha)$ is consistent with your description of the function above.

E. Is this position eigenfunction also an eigenfunction of energy? If you need additional information to answer, describe that information.

III. Eigenvalues and eigenvectors of momentum.

A. Use your equation in IB to determine a mathematical expression for the eigenfunctions of momentum.

B. Are the eigenvalues corresponding to these eigenfunctions discrete or continuous? Explain.

C. Is this momentum eigenfunction also an eigenfunction of energy or position? If you need additional information to answer, describe that information.

Page 3 of Tutorial: Position, Momentum, and Energy Measurements

IV. Measurements of position, momentum, and energy

A system is initially in an arbitrary superposition of energy eigenstates. At $t = 0$ a measurement of the system is made.

A. Describe the state of the system immediately after each of the following measurements. Explain

A position measurement yielding $x = \alpha$	A momentum measurement yielding $p = \lambda$	An energy measurement yielding $E = E_i$

B. In each case state whether the mathematical action of the operator on the wave function determines the state of the system after the measurements. Explain.

position	momentum	energy

C. Does the probability density for a system change as time evolves if a measurement at $t = 0$ yields: Explain.

a position, $x = \alpha$	A momentum, $p = \lambda$	an energy, $E = E_i$

D. Imagine that immediately after the first measurement, a second measurement *of the same* type is made. What would you expect for the results of this second measurement compared to the results of the first measurement? Explain.

Position	Momentum	Energy

E. Now suppose the second measurement is not made *at the same time as* the first, but rather is made a few minutes later. What would you expect for the results of this second measurement compared to the results of the first measurement? Explain.

Position	Momentum	Energy

Page 4 of Tutorial: Position, Momentum, and Energy Measurements

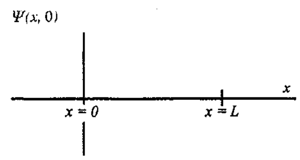
QM 4 ***Position, momentum, and energy measurements***

V. Compatible and incompatible observables

A. At time $t = 0$, an electron in an infinite square well is in the third excited state ($n = 4$). Let E_4 represent the energy of this state.

1. On the axes provided at right, sketch the wave function for the electron at $t = 0$. Assume the wave function is real at $t = 0$.

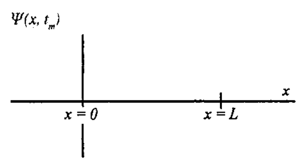
2. On your graph of the wave function at $t = 0$, clearly label the locations where the electron would be (i) *most* likely and (ii) *least* likely to be found at that time. Explain.



wave function at $t = 0$

Suppose that at a later time you measured the *position* of the electron and found it to be located exactly at $x_o = L/8$.

3. On the axes at right, draw the resulting shape of the wave function immediately after the measurement. (Assume the wave function is real.) Explain.



wave function immediately after measurement

4. Is the wave function that you have drawn an energy eigenstate or a superposition of many energy eigenstates? Explain how you can tell from the shape of the wave function.

B. Suppose that a few minutes later you measured the *energy* of the electron.

1. On the basis of your results in part A, would you expect only *one* possible result for your energy measurement (*e.g.*, E_4), or *many* possible results? Explain.

2. Is the probability of finding a particular result for your energy measurement (*e.g.*, E_4) dependent on time? Explain.

Would your answer to the above question change if you waited an extremely long time?

Page 5 of Tutorial: Position, Momentum, and Energy Measurements

If the order in which two observable quantities, *A* and *B*, are measured could affect the outcome of the measurements, then it is said that *A* and *B* are *incompatible observables.* If the order in which *A* and *B* are measured does not matter, then *A* and *B* are *compatible observables.*

C. Do you agree or disagree with the following statement?

"Some observables, like position, are not even compatible with themselves. If you make a measurement of position, wait a little time, and then make another measurement of position, the two measurements may not give the same value. So they are incompatible."

D. On the basis of your results above, are *position* and *energy* compatible or incompatible observables? Explain.

E. Are *position* and *momentum* compatible or incompatible observables? Explain.

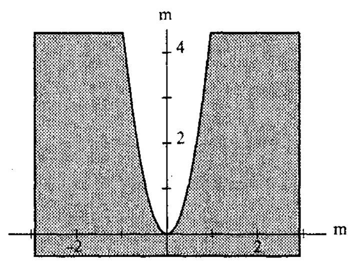
F. Are *momentum* and *energy* compatible or incompatible observables? Explain.

G. Do any of your answers above depend upon the nature of the particular system under consideration? Explain.

Page 1 of Tutorial: Uncertainty: Classical and Quantum

UNCERTAINTY: CLASSICAL AND QUANTUM Name ____________

A frictionless parabolic ramp of height 4.5 m and width 2 m is shown at right. A heavy iron block of mass 1 kg will be set in motion on the ramp.

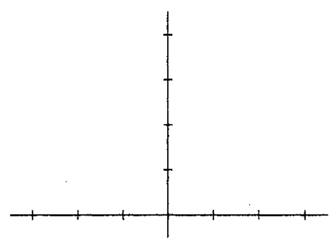


In this tutorial we will investigate some requirements for both the classical and the quantum mechanical descriptions of the motion.

Our quantum mechanical description will describe the experiment using the idealized Hamiltonian $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$.

I. Classical description

A. Draw the potential for the above system on the graph at right. Explain.



What mathematical function of x describes this potential? Explain.

B. What determines the total energy E_{CM} classically? Could 40 J be a possible value for E_{CM}?

C. Would you expect predictions about the motion of this system based on classical mechanics to agree with experimental values? Briefly explain.

D. Suppose an experiment is performed in which the total energy of the block is measured. That experiment is repeated many times and a standard deviation ΔE is calculated. Could 0.5 J be a reasonable estimate for ΔE assuming that measurements are made with simple equipment such as meter sticks and spring scales?

Page 2 of Tutorial: Uncertainty: Classical and Quantum

II. ΔE and ΔE_{QM}

Consider the quantum mechanical description of a harmonic oscillator with Hamiltonian $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$.

A. Suppose this system is in the first excited state. Determine the uncertainty (standard deviation) in the energy, ΔE_{QM}. Explain.

Consider the following student dialogue concerning this question.

Student 1: *"The energy of the first excited state is $3\hbar\omega/2$ and the next higher or lower state is $\hbar\omega$ above or below that. So, $\Delta E = \hbar\omega$."*

Student 2: *"No, $\Delta E = 0$. The only way to have a non-zero ΔE is for the energy of the system to change. This happens, for instance, when something decays."*

With which student or students, if any, do you agree? Explain.

B. Now consider a system in a superposition of the ground state ψ_o and the first excited state ψ_1 given by: $(\psi_o + \psi_1)/\sqrt{2}$.

Draw a histogram showing the possible results of energy measurements on this system and their probabilities.

Calculate the uncertainty in the energy, ΔE_{QM}, for the system in this state. Explain.

Check that your answer is consistent with your histogram.

Page 3 of Tutorial: Uncertainty: Classical and Quantum

C. In a quantum mechanical description of the macroscopic system described in part I, do you expect ΔE_{QM} to be *greater than, less than,* or *equal to* the ΔE of 0.5 J found from the experiment described in part ID? Base your response on the idea that quantum mechanics should be able to predict the results of any experiment. Explain.

✓ **Discuss your answers with an instructor before continuing.**

III. A quantum mechanical description and its probability density

An arbitrary state of the harmonic oscillator, $\Psi(x,t)$, can be written as a linear combination of the energy eigenstates of the harmonic oscillator.

A. Write such a general sum with appropriate time dependence.

B. Suppose the wave function above is to describe the system in part I with its $E = 40\ J$ and $\Delta E = 0.5\ J$.

1. Sketch a histogram that illustrates the possible results of energy measurements on this system and their relative probabilities.

2. Mark the approximate value of $\langle E \rangle$ on your graph. What is the approximate quantum number of the energy eigenstate with energy nearest $\langle E \rangle$?

Page 4 of Tutorial: Uncertainty: Classical and Quantum

3. Approximately what is the maximum number of eigenstates that contribute appreciably to the linear combination that forms $\Psi(x,t)$? (An order of magnitude estimate is all that is required.)

4. Can a linear combination of these states form more than one wave function?

5. Write $\rho(x,t)$ for the wave function that you wrote in part A. Does it depend on time?

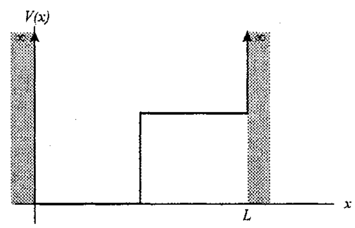
   Find the time average of $\rho(x,t)$. Does it depend on time?

   How does the time average of $\rho(x, t)$ compare to the time-averaged probability density for a single highly excited energy eigenstate?

Page 5 of Tutorial: Uncertainty: Classical and Quantum

IV. Relating classical to quantum mechanics revisited

In the tutorial *Relating classical to quantum mechanics* you used classical mechanics as a guide in drawing a highly excited energy eigenstate for the system that has the potential shown at right. How do your results above help inform the argument you used to reason about the amplitude of the wave function in each region?



Page 1 of Tutorial: Degenerate States

DEGENERATE STATES

A two–dimensional isotropic harmonic oscillator is described by the Hamiltonian
$H=-\frac{\hbar^2}{2M}\nabla^2+\frac{1}{2}M\omega^2r^2$.

Consider solutions in the form $\Psi(\vec{r},t)=\psi(\vec{r})f(t)$. Use the Schrödinger equation, $i\hbar\frac{\partial\Psi}{\partial t}=H\Psi$, to find *f(t)*. Does your answer depend upon the co-ordinate system used?

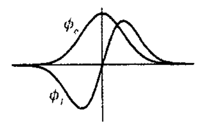
I. Cartesian co-ordinates

The first two energy eigenfunctions for the one–dimensional harmonic oscillator are:
$\left(\frac{M\omega}{\pi\hbar}\right)^{\frac{1}{4}}e^{\frac{-M\omega x^2}{2\hbar}}$ and $\left(\frac{4M^3\omega^3}{\pi\hbar^3}\right)^{\frac{1}{4}}xe^{\frac{-M\omega x^2}{2\hbar}}$. The eigenvalues corresponding to these eigenfunctions are $\frac{1}{2}\hbar\omega$ and $\frac{3}{2}\hbar\omega$.

A. Use the above information to determine the energy eigenfunctions and eigenvalues for the ground state and the first excited state of the two dimensional harmonic oscillator in Cartesian co-ordinates. (*Hint:* Look for solutions in the form $\psi(\vec{r})=X(x)Y(y)$.) Are these states unique?

Page 2 of Tutorial: Degenerate States

QM 2 ***Degenerate States***



B. Consider the functions: $\varphi_o(x) = \left(\frac{M\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{M\omega x^2}{2\hbar}}$ and

$\varphi_1(x) = \left(\frac{4M^3\omega^3}{\pi\hbar^3}\right)^{\frac{1}{4}} x e^{-\frac{M\omega x^2}{2\hbar}}$.

Also consider the following four possible initial states for the two-dimensional harmonic oscillator:

(a) $\psi_a = \varphi_o(x)\varphi_o(y)$, (b) $\psi_b = \varphi_1(x)\varphi_o(y)$, (c) $\psi_c = (\varphi_1(x)\varphi_o(y) + \varphi_o(x)\varphi_1(y))/\sqrt{2}$, and

(d) $\psi_d = (\varphi_o(x)\varphi_o(y) + \varphi_o(x)\varphi_1(y))/\sqrt{2}$.

1. Describe the first two states in words, or with a diagram. Indicate any symmetries.

2. For each of the four wave functions, answer the following three questions:

   Is this state associated with a single energy? If so, which energy? Explain.

   Does this state change as time evolves? If so, describe how it changes. If not, why not?

   Does the probability density associated with this state change as time evolves? If so, describe how it changes. If not, why not?

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3. Can a superposition of stationary states be a stationary state? If so, under what conditions?

II. Polar co-ordinates

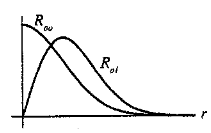
Again consider a two–dimensional harmonic oscillator described by the Hamiltonian

$$H = -\frac{\hbar^2}{2M}\nabla^2 + \frac{1}{2}M\omega^2 r^2 .$$

Recall $\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$.

A. Consider solutions to the time independent Schrödinger equation in the form $\psi(\vec{r}) = R(r)\Theta(\theta)$ Use the time-independent Schrödinger equation, $H\psi = E\psi$, to find $\Theta(\theta)$. Is there only one solution for a given energy?

B. Consider the functions: $R_{oo}(r) = Ne^{\frac{-M\omega r^2}{2\hbar}}$, $R_{o1}(r) = N'r e^{\frac{-M\omega r^2}{2\hbar}}$, and $\Theta_1(\theta) = e^{i\theta}$.



Also consider the following two possible initial states for the two-dimensional harmonic oscillator:

(e) $\psi_e = R_{oo}(r)$, and (f) $\psi_f = R_{o1}(r)\Theta_1(\theta)$.

1. Describe both states in words, or with a diagram. Indicate any symmetries.

Page 4 of Tutorial: Degenerate States

2. For both of the wave functions, answer the following four questions:

   Can this state be written in terms of the states φ_n from part I? If so, write it in that manner.

   Is this state associated with a single energy? If so, which energy? Explain.

   Does this state change as time evolves? If so, describe how it changes. If not, why not?

   Does the probability density associated with this state change as time evolves? If so, describe how it changes. If not, why not?

3. Does a change of coordinates from Cartesian to polar affect the stationary states for a system? If so, how?

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III. Cartesian and polar

1. Suppose a two–dimensional harmonic oscillator were initially in the state $N\left(R_{01}(r)\Theta_1(\theta)+\varphi_o(x)\varphi_1(y)\right)$.

   Is this state associated with a single energy? If so, which energy? Explain.

   Does this state change as time evolves? Explain.

   Does the probability density associated with this state change as time evolves? Explain.

Page 1 of Tutorial: Addition of Angular Momentum

ADDITION OF ANGULAR MOMENTUM

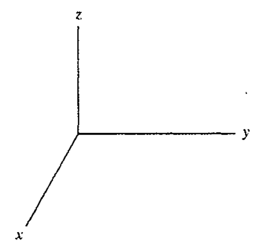
I. Angular momentum in quantum mechanics

An electron is known to be in a spin angular momentum state given by $\chi = |1/2, +1/2\rangle$ in the basis formed by the eigenvectors of S_z.

A. What is the value of the spin angular momentum squared, S^2, for this electron? Explain.

B. What is the z-component of the spin angular momentum S_z for this electron? Explain.

C. On the diagram at right, draw a vector that could represent the angular momentum of a *classical* particle with an angular momentum equal to the spin of this electron. Label the lengths of the known components and the length of the vector itself.



D. Suppose a measurement of the x-component of the spin angular momentum, S_x, is made on the electron described above. What are the possible values you could obtain and what are their probabilities? Explain.

E. Suppose a measurement of the y-component of the spin angular momentum, S_y, is made on the electron described above. What are the possible values you could obtain and what are their probabilities? Explain.

F. Would you say that the x-component of the spin angular momentum, S_x, is well defined? Would you say that the y-component of the spin angular momentum, S_y, is well defined? Explain.

Page 2 of Tutorial: Addition of Angular Momentum

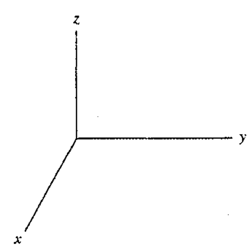
G. What are the two ways in which the results in parts D and E are inconsistent with the picture drawn in part C for a classical particle? Explain.

The angular momentum of a quantum mechanical system is often represented as the infinite set of vectors lying along a cone of height l_z with side of length $\sqrt{l(l+1)}\hbar$. (*e.g.*, figure 4.9 on page 165 of *Introduction to Quantum Mechanics 2ed* by David Griffiths.)

H. How is this representation inconsistent with your answers to D and E? Explain.

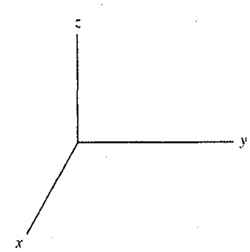
In spite of these inconsistencies, this representation can be a useful tool in thinking about quantum mechanical angular momentum.

I. On the graph at right, show how the spin angular momentum of the electron in the state $\chi = |1/2, +1/2\rangle$ would be represented in this representation.



This same electron is known to be in the orbital angular momentum state Y_2^0.

J. On the graph at right, show how the orbital angular momentum of the electron would be represented in the representation described above.



Page 3 of Tutorial: Addition of Angular Momentum

II. Addition of angular momentum

The total angular momentum $\boldsymbol{J}$ of an electron is defined to be the sum of the orbital angular momentum and spin angular momentum $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$.

A. Write down an expression in Dirac notation for the angular momentum state of the electron from part I. Use the basis formed by the eigenvectors of L_z and S_z.

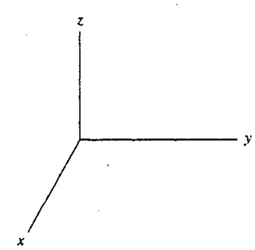
B. Using the definition of $\boldsymbol{J}$, determine the relationship between the ith-component of the total angular momentum J_i and the ith-components of the orbital and spin angular momenta L_i and S_i, respectively. Explain.

C. Based on your answer in part B, what is the z-component of the total angular momentum J_z for this electron? Is this quantity well defined? Explain.

D. Based on your answer in part B, what is the x-component of the total angular momentum J_x for this electron? Is this quantity well defined? Explain.

E. Based on your answer in part B, what is the y-component of the total angular momentum J_y for this electron? Is this quantity well defined? Explain.

F. On the graph at right, show how the total angular momentum of the electron would be represented in the representation described in part I. Is it possible to draw just one cone consistent with the known values of l and s?



Page 4 of Tutorial: Addition of Angular Momentum

QM 4 ***Addition of angular momentum***

Since all three components of the angular momentum vector for a quantum mechanical system cannot be simultaneously determined, we have seen that the magnitude of the sum of two angular momentum vectors cannot be determined. In fact, the quantum number j associated with the total angular momentum vector $\mathbf{J} = \mathbf{L} + \mathbf{S}$ can take values from $|l - s|$ to $l + s$ in integer steps. Here, l and s are the quantum numbers associated with the orbital and spin angular momenta, respectively.

Additionally, we have found that the z-component of the total angular momentum is equal to the sum of the z-components of the orbital and spin angular momenta, $J_z = L_z + S_z$. This relation, written in terms of the z-component quantum numbers, is $j_z = l_z + s_z$. Consider the electron with spin state $\chi = |1/2, +1/2\rangle$ and orbital angular momentum state Y_2^0.

1. What are the possible values of j_z for this electron? Explain.

2. What are the possible values of j for this electron? Explain.

3. Write down an expression in Dirac notation for the total angular momentum state of the electron in the basis formed by the eigenvectors of J^2 and J_z leaving the coefficients undetermined.

4. Find the coefficients in your expression above. You may find the Clebsch-Gordon table on page 188 of *Introduction to Quantum Mechanics 2ed* by David Griffiths helpful.

5. Suppose a measurement of the total angular momentum squared, J^2, is made on the electron. What are the possible values you could obtain and what are their probabilities? Explain.

Page 1 of Tutorial: Identical Particles

IDENTICAL PARTICLES

I. Combining probabilities

A. Suppose that you have a red six-sided die. Assume that the likelihood of a side landing up after a roll is equal for all sides.

1. What is the probability that rolling the die would result in a "1"? What is the probability that the roll would result in a "1" *or* a "2"?

2. Suppose you had a second die identical to the first but blue. What is the probability that rolling both dice would result in a blue "1" and a red "6"?

   What is the probability of rolling a "1" and a "6," regardless of color?

B. Suppose that you are now given two other dice weighted such that the probability of rolling an "i" on the blue die is given by $P_b(i)$, and the probability of rolling a "j" on the red die is given by $P_r(j)$.

1. What is the probability that rolling the red die would result in a "1"? What is the probability that the roll would result in a "1" *or* a "2"?

2. What is the probability that rolling both dice would result in a blue "1" and a red "6"?

   What is the probability of rolling a "1" and a "6," regardless of color?

Page 2 of Tutorial: Identical Particles

II. Two-particle wave functions

Consider the wave function, $\Psi(x_1, x_2)$, for a system of two non-interacting particles, *a* and *b*. Let the first argument of $\Psi(x_1, x_2)$ represent the set of space coordinates that refer to particle *a*, and let the second argument represent the same set of coordinates for particle *b*. Let $\psi_a(x)$ and $\psi_b(x)$ represent the wave functions of the individual particles.

A. Non-interacting, *distinguishable* particles

1. Interpret the quantity $|\Psi(x_1,x_2)|^2 dx_1 dx_2$ in your own words.

2. For the case in which the particles are ***distinguishable***, write an expression for $\Psi(x_1,x_2)$ in terms of ψ_a and ψ_b. Explain how you decided to combine ψ_a and ψ_b in the way you did. (*Hint:* $\Psi(x_1, x_2)$ represents a wave function rather than a probability density. You may want to consider how you can apply the reasoning that you used in section I to the present situation?)

3. Consider the following *incorrect* statement:

   "The wave function, $\Psi(x_1, x_2)$, for particles a and b refers to the probability of detecting particle a at x_1 and particle b at x_2. The 'and' means that we can add, that is, use superposition, to write $\Psi(x_1, x_2)$ as the linear combination, $(1/\sqrt{2})$ $\{\psi_a(x_1) + \psi_b(x_2)\}$."
   Identify which parts of this statement are correct and which parts are incorrect. Explain.

4. Write an expression that gives the probability that a particle is found in a region dx_1 and another is found in a region dx_2 without regard to which particle is in which region.

Page 3 of Tutorial: Identical Particles

III. Interacting distinguishable systems

A zookeeper divides two identical long thin lion cages into four sections each. Each night the zookeeper records in which section of each cage a lion sleeps. This information is used to determine a probability function. The results are as follows.

	Lion A	Lion B
section	Probability	Probability
1	.1	.5
2	.3	.1
3	.4	.1
4	.2	.3

Later both lions are transferred to the same cage.

A. Make a table that shows the probabilities for each possible sleeping arrangement for the two lions in the four-section cage. (Assume each lion's behavior is independent of the other's.)

Section for lion A				
4				
3				
2				
1				
Section for lion B	1	2	3	4

B. The zookeeper presents the following table of probabilities as the actual results of an experiment.

Section for lion A				
4	.01	.02	.03	.15
3	.05	.04	.13	.09
2	.09	.12	.03	.06
1	.14	.01	.01	.02
Section for lion B	1	2	3	4

1. Are the results shown in the table above possible?

2. What conclusions can be drawn from the above table? Explain.

Page 4 of Tutorial: Identical Particles

IV. Non-interacting, *indistinguishable* particles

Now consider the case in which particles *a* and *b* are ***indistinguishable.*** The wave function, $\Psi(x_1, x_2)$, for a system of two indistinguishable particles must have the property that exchanging x_1 and x_2 gives the *same* wave function to within an overall sign.

1. Is your expression for $\Psi(x_1, x_2)$ (in terms of ψ_a and ψ_b) from part II question A2 valid in this case? Explain.

A wave function representing a system of indistinguishable particles that remains unchanged after exchanging any two sets of coordinates is said to be *symmetric* under exchange. The particles represented by such a wave function are called *bosons*. A wave function is instead said to be *anti-symmetric* under exchange if exchanging any two sets of coordinates causes the wave function to change sign (but otherwise remain unchanged). Particles represented by such wave functions are called *fermions*.

2. Write an expression for $\Psi(x_1, x_2)$ in terms of ψ_a and ψ_b that would be valid to describe a system of two bosons. Explain..

3. In the space below, write an expression for $\Psi(x_1, x_2)$ in terms of ψ_a and ψ_b that would be valid to describe a system of two fermions. Explain.

Page 5 of Tutorial: Identical Particles

4. Use your results to write down and simplify (as much as possible) an expression for $|\Psi(x_1, x_2)|^2$ for each of these two cases. Can you give a physical interpretation to each term in your expressions? (How do your expressions compare to what you would get for the case of two *distinguishable* particles?)

   $|\Psi(x_1, x_2)|^2$ for bosons:

   $|\Psi(x_1, x_2)|^2$ for fermions:

5. Are your expressions for $\Psi(x_1,x_2)$ normalized? If not, normalize them. (Assume ψ_a and ψ_b are orthonormal single particle wave functions.)

Page 6 of Tutorial: Identical Particles

6. After completing question 4, four students make the following comments.

Student 1: "Identical particles should work like the dice did when we ignored the color. The total probability should be the probability of particle **a** being found near $\mathbf{x}_1$ multiplied by the probability of particle **b** being found near $\mathbf{x}_2$ plus the probability of particle **b** being found near $\mathbf{x}_1$ multiplied by the probability of particle **a** being found near $\mathbf{x}_2$."

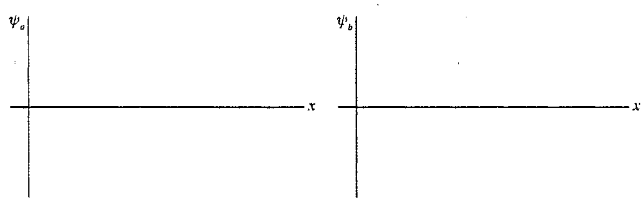
Student 2: "I disagree, here we are not dealing with independent events. So, the probability is more complicated than a simple sum of two products."

Student 3: "Yes, if the overlap between the two functions at $\mathbf{x}_1$ and $\mathbf{x}_2$ is large, the probability of finding particles in those locations is always enhanced to be greater than a simple sum of two products."

Student 4: "No, the probabilities must be independent because the particles are non-interacting."

With which students do you agree, or do you agree with none of them? Explain your reasoning.

B. In certain circumstances $|\psi(x_1,x_2)|^2$ for indistinguishable particles can be approximated by a simple sum of two products as student 1 suggests. Sketch a possible ψ_a and ψ_b for such a circumstance. Explain how your sketch is consistent with your answer to question 4 above.



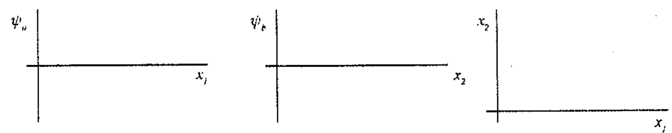
Page 7 of Tutorial: Identical Particles

V. Two particles in an infinite square well

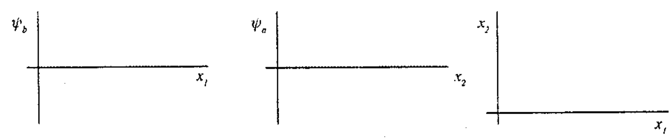
A. Consider two particles in a one dimensional infinite square well of unit width. Let ψ_a be the ground state of this well, and let ψ_b be the first excited state. Assume that the combined spin state is symmetric.

1. Consider the case of two distinguishable particles.

   a. Sketch $\psi_a(x_1)$, and $\psi_b(x_2)$ below. Also, mark "+" signs on the x_2 versus x_1 axes to indicate high points in the graph of the spatial part of the two particle wave function, and use "–" signs to indicate low points.



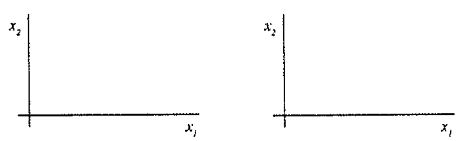
   b. Sketch $\psi_b(x_1)$, and $\psi_a(x_2)$ below. Also, mark "+" signs on the x_2 versus x_1 axes to indicate a high points in the graph of the spatial part of the two particle wave function, and use "–" signs to indicate low points.



   c. Mark any lines of symmetry on the above representations of the spatial part of the two-particle wave function.

2. Now consider the case of two indistinguishable bosons.

   a. Mark "+" signs on the x_2 versus x_1 axes to indicate high points in the graph of the spatial part of the two particle wave function, and use "-" signs to indicate low points. Do the same for a graph of the spatial part of the probability distribution.

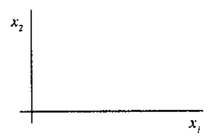


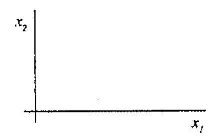
   b. Mark any lines of symmetry on the above plots.

Page 8 of Tutorial: Identical Particles

3. Finally consider the case of two indistinguishable fermions.

   a. Mark "+" signs on the x_2 versus x_1 axes to indicate high points in the graph of the spatial part of the two particle wave function, and use "–" signs to indicate low points. Do the same for a graph of the spatial part of the probability distribution.





   b. Mark any lines of symmetry on the above plots.

4. From your plots, are bosons or fermions with the same spin state more likely to be found in the same region of space? Explain.

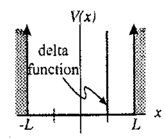
VI. Time dependence

Which, if any, of your plots from part V would change as time progresses? Explain.

Page 1 of Tutorial: First Order Time Independent Perturbation Theory

FIRST ORDER TIME INDEPENDENT PERTURBATION THEORY

I. Perturbations to the energy

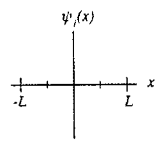


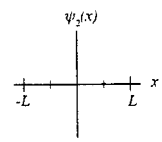
Consider an infinite square "well" of width $2L$, shown at right. Suppose that the "well" is perturbed by a delta-function potential:

$$H'(x) = +\alpha \frac{\hbar^2}{mL^2}\delta\left(x - \tfrac{L}{2}\right),$$

where α is a small dimensionless constant.

A. In the spaces at right, draw the wave functions for the (unperturbed) ground state ($n = 1$) and first excited state ($n = 2$). Draw each graph using the same vertical scale.





B. Write the normalized equation for each of these wave functions.

As presented in lecture, an important result from time-independent perturbation theory is that the first-order correction to the n^{th} energy eigenvalue is given by $E_n^1 = \langle \psi_n^0 | H' | \psi_n^0 \rangle$, where H' is the Hamiltonian operator that describes the perturbation.

C. Write an integral expression for E_n^1 for the perturbation given above. (Do not bother to write out $\psi_n^0(x)$ explicitly in terms of x.)

D. Would this perturbation produce a first order correction to the energy that is *greater than, less than*, or *equal to* that produced by a delta function perturbation at the origin: *(i.e.,* $\hat{H}(x) = \alpha\delta(x)$.) Explain your reasoning.

1. for the ground state?

2. for the first excited state?

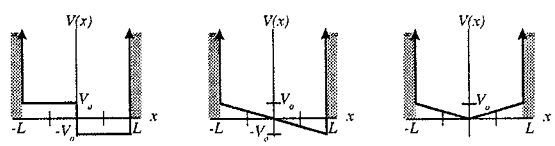
Page 2 of Tutorial: First Order Time Independent Perturbation Theory

E. Are there energy eigenstates for which the first-order correction to the energy caused by the perturbation $H'(x) = +\alpha \frac{\hbar^2}{mL^2}\delta\left(x - \frac{L}{2}\right)$ is *zero?* If so, what is the lowest energy eigenstate for which this is true? If not, explain why not.

F. For the unperturbed square "well" given above, the ground state energy is $E_1^0 = \pi^2\hbar^2/(8mL^2)$ and the ground state wave function is $\psi_1{}^0(x) = \sqrt{1/L}\cos(\pi x/2L)$.

Determine the ground state energy, *correct to first order,* when the delta function perturbation, $H'(x) = +\alpha\,\delta\left(x - \frac{L}{2}\right)$, is applied. Show all work.

G. Consider the three potential wells shown below.



1. For each well write an expression for the perturbation, H', necessary to change the original infinite square well into the well shown.

2. For each well, determine whether the first-order correction to the **ground state energy** is *zero* or *nonzero.* Explain your reasoning in each case.

3. For each well, determine whether the first-order correction to the energy of the **first excited state** is *zero* or *nonzero.* Explain your reasoning in each case.

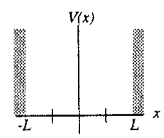
4. For which of the wells above, if any, is the first order correction to the energy equal to zero for all energy levels? Explain.

Page 3 of Tutorial: First Order Time Independent Perturbation Theory

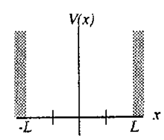
5. If $E^1_n = 0$ for all n, does that imply that the energy of each of the perturbed eigenfunctions is equal to that of the corresponding unperturbed eigenfunctions? Explain. (Hint: Consider the full Schrödinger equation and its solutions for both the perturbed and unperturbed case.)

H. Sketch potential wells that result from perturbations of the infinite square well such that the following will be true. Explain your sketches.

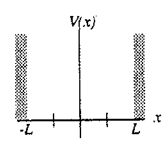
1. The first order correction to the energy is positive for all energy levels?



2. The first order correction to the energy is negative for all energy levels?



3. The first order correction to the energy is zero for all energy levels? (Assume the potential is not equal to zero everywhere.)

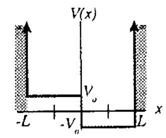


Page 4 of Tutorial: First Order Time Independent Perturbation Theory

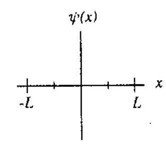
QM 4 ***First order time-independent perturbation theory***

II. Perturbations to the wave functions

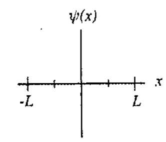
A. A potential well is shown at right.



1. Sketch the wave function for a particle in an highly-excited energy eigenstates of this well. Explain.



2. Sketch the wave function for a particle in the ground state of this well. Explain.



As presented in lecture, an important result from time-independent perturbation theory is that the first-order correction to the n^{th} wave function is given by

$\psi_n^1 = \sum_{m\neq n} \frac{\langle \psi_m^0 | H' | \psi_n^0 \rangle}{E_n^0 - E_m^0} \psi_m^0$, where H' is the Hamiltonian operator that describes the perturbation.

B. Determine the signs of the coefficients of the first few terms in the expansion of ψ_1^1. Are they consistent with your graph from part A2? Explain.

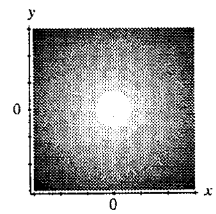
Determine the signs of the coefficients of the first few terms in the expansion of ψ_2^1. Are they consistent with your graph from part A1? Explain.

C. Compare your graphs to those provided by an instructor.

Page 1 of Tutorial: Degenerate Time Independent Perturbation Theory

DEGENERATE TIME-INDEPENDENT PERTURBATION THEORY

I. Two–dimensional isotropic harmonic oscillator



A two–dimensional isotropic harmonic oscillator is described by the Hamiltonian $H = -\frac{\hbar^2}{2M}\nabla^2 + \frac{1}{2}M\omega^2 r^2$. A contour plot of the potential is shown at right. The first two energy eigenfunctions for the one–dimensional harmonic oscillator are:

$\varphi_o(x) = A_o e^{-\alpha x^2}$ and $\varphi_1(x) = A_1 x e^{-\alpha x^2}$ where

$A_o = \left(\frac{M\omega}{\pi\hbar}\right)^{\frac{1}{4}}$, $A_1 = \left(\frac{4M^3\omega^3}{\pi\hbar^3}\right)^{\frac{1}{4}}$, and $\alpha = \frac{M\omega}{2\hbar}$.

The eigenvalues corresponding to these eigenfunctions are $\frac{1}{2}\hbar\omega$ and $\frac{3}{2}\hbar\omega$ respectively.

A. Use the above information to determine the energy eigenfunctions and eigenvalues for the ground state and the first excited state of the two dimensional harmonic oscillator in Cartesian co-ordinates. (*Hint:* Look for solutions in the form $\psi(\vec{r}) = X(x)Y(y)$.) Describe any degeneracies.

1. Ground state

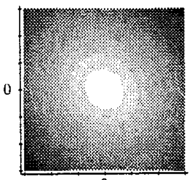
2. First excited state

Page 2 of Tutorial: Degenerate Time Independent Perturbation Theory

QM 2 *Degenerate time-independent perturbation theory*

II. An anisotropy

A small anisotropy is added to the potential as illustrated by the contour plot at right. The new Hamiltonian is now $H = -\frac{\hbar^2}{2M}\nabla^2 + \frac{1}{2}M\omega^2 r^2 + \gamma\frac{1}{2}M\omega^2(x+y)^2$ where γ is small and positive.



A. The full Hamilton can be written as the Hamiltonian for the isotropic oscillator plus a perturbation. $H = H^o + H'$. What is H'?

B. Describe any symmetries in the following functions

1. The original potential of the isotropic oscillator

2. The perturbation

3. The full potential of the anisotropic oscillator

C. Show that the first order correction to the energy of the ground state is $\gamma\hbar\omega/2$. The following integrals may be useful:

$$\int_{-\infty}^{\infty} |\varphi_o(x)|^2 dx = \int_{-\infty}^{\infty} |\varphi_1(x)|^2 dx = 1$$

$$\int_{-\infty}^{\infty} x^2 |\varphi_o(x)|^2 dx = \frac{\hbar}{2M\omega}$$

$$\int_{-\infty}^{\infty} x^2 |\varphi_1(x)|^2 dx = \frac{3\hbar}{2M\omega}$$

$$\int_{-\infty}^{\infty} x\varphi_o(x)\varphi_1(x) dx = \sqrt{\frac{\hbar}{2M\omega}}$$

Page 3 of Tutorial: Degenerate Time Independent Perturbation Theory

D. Find the first order correction to the first excited state. The following steps may be helpful.

1. Calculate the matrix elements of the perturbation in the basis formed by the degenerate first excited state eigenfunctions. (*Hint:* Your matrix should have two unique values $\gamma\hbar\omega$, and $\gamma\hbar\omega/2$.)

2. Find the eigenvalues and eigenvectors of the above matrix.

E. Write the *"good"* first excited state wave functions in terms of x and y.

Page 4 of Tutorial: Degenerate Time Independent Perturbation Theory

III. Symmetry

A. Find a hermitian operator, A, that commutes with H^0 and H'. Describe the action of this operator.

B. Show that the linear combinations you found in the previous part are simultaneous eigenstates of H^0 and A.

Page 5 of Tutorial: Degenerate Time Independent Perturbation Theory

IV. An exact solution

Make a change of variables so that you can rewrite the Hamiltonian for the anisotropic oscillator as a sum of two separate harmonic oscillator Hamiltonians.

A. What is the energy of the ground state.

B. What is the energy of the first and second excited state.

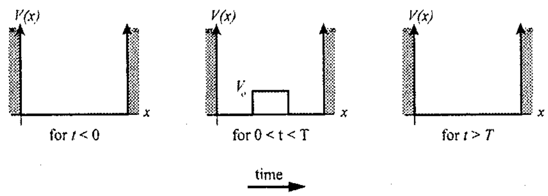
C. Compare your answers to those you found using degenerate perturbation theory. *(Hint:* $\sqrt{1+2\gamma} \approx 1+\gamma+\cdots$*)*

Page 1 of Tutorial: Time Dependent Perturbation Theory

TIME DEPENDENT PERTURBATION THEORY

A time dependent perturbation to the infinite square well potential is illustrated below. (*Note*: $V_o << E_1^o$) At a time $t_- << 0$ the total energy of the system is measured and found to be the ground state energy of the unperturbed well. In what follows let the symbols E_i^o represent the energy eigenvalues for the unperturbed well.

A time dependent perturbation to the infinite square well



I. Before the perturbation

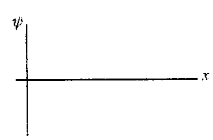
A. Describe the time dependence, if any, of the wave–function for $t_- < t < 0$.

B. Describe the time dependence, if any, of the spatial probability distribution for $t_- < t < 0$.

II. During the perturbation

A. Consider a highly excited energy eigenstate for this system as it appears during the perturbation.

1. Sketch a graph of the wave function for this highly excited energy eigenstate.



2. Is the function that you drew an energy eigenfunction of the unperturbed well? Explain.

Page 2 of Tutorial: Time Dependent Perturbation Theory

3. Can you represent this wave–function at a particular time as a linear combination of the eigenstates of the unperturbed square well? Explain.

4. Do the absolute squares of the coefficients in this expansion represent the probability that a measurement of the energy, made while the perturbation is on, will yield E_i^o? Explain.

B. A general wave–function

1. As you found with the highly excited case, in general a wave–function for a time $0 < t < T$ can be represented by a linear combination of the eigenstates of the unperturbed well.
   (*e.g.*, $\psi(x) = \sum_n c_n \varphi_n(x)$.) What is $\psi(x,t)$?

   Why does $\psi(x,t) \neq \sum_n c_n \varphi_n(x) e^{-iE_n t/\hbar}$?

2. Even though φ_n are not the eigenfunctions for the well at times between 0 and T, we can write $\psi(x,t) = \sum_n c_n(t) \varphi_n(x) e^{-iE_n t/\hbar}$. Why?

Page 3 of Tutorial: Time Dependent Perturbation Theory

The time–dependent Schrödinger equation is given by $H\psi = i\hbar \dfrac{\partial \psi}{\partial t}$.

3. Write the Hamiltonian between $t = 0$ and $t = T$ as the sum of the unperturbed Hamiltonian and the perturbation.

4. Use the time–dependent Schrödinger equation and the expansion for $\psi(x,t)$ to find $\dfrac{dc_m(t)}{dt} = -\dfrac{i}{\hbar}\sum_n H'_{mn} c_n(t) e^{i\omega_{mn}t}$. Here $H'_{mn} = \langle \varphi_m | H' | \varphi_n \rangle$ and $\omega_{mn} = E_m - E_n / \hbar$.

5. Use the fact that the system was measured to be in the ground state at time t_- to find an expression for the coefficients other than c_0 in the expansion at a time T, $c_m(T)$, correct to first order.

Page 4 of Tutorial: Time Dependent Perturbation Theory

6. Are any of the coefficients equal to zero at time T to first order? Explain.

III. After the perturbation

A. Immediately after time T, can you represent this wave–function as a linear combination of the eigenstates of the unperturbed square well? Explain.

B. Immediately after time T, do the absolute squares of the coefficients in this expansion represent the probability of a measurement of the energy yielding E_i^o ? Explain.

C. Describe the time dependence, if any, of the wave–function for $t > T$.

D. Describe the time dependence, if any, of the probability distribution for $t > T$.

Appendix B Miscellaneous Tutorials

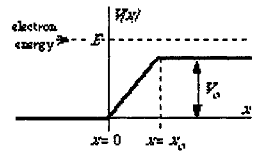
This appendix contains tutorials which are not currently being used as part of the Physics 324–325 sequence at the University of Washington but have been used in the past. The order of the tutorials is the order in which they were typically used.

Page 1 of Tutorial: Reflection and Transmission of Matter

REFLECTION AND TRANSMISSION OF MATTER Name ____________

I. Behavior of classical particles in a region of varying potential energy

Monoenergetic electrons travel in the $+x$ direction and enter a region in which the potential energy, $V(x)$, increases linearly with x. (See figure at right.) Suppose that the energy, E, of each electron is larger than, V_o.

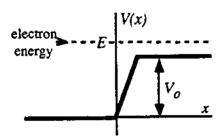


Answer the following questions by treating the electrons as ***classical*** particles.

A. Describe the motion of the electrons as they move through the region shown in the graph.

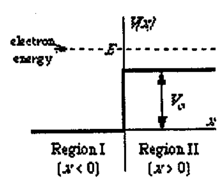
B. Use the graph of $V(x)$ versus x to determine the magnitude and direction of the net force on an electron at all locations shown in the graph. Explain how you used the graph.

C. Suppose that the potential energy increases from zero to V_o over a *smaller* interval than before, as shown at right. How would this change affect the magnitude of the maximum force on the electron and the net work done? Explain.



D. What physical situation would yield a potential energy that varies as shown above? (Hint: The particles have charge.)

E. Suppose that the potential energy, $V(x)$, varied with x as shown in the graph at right. As before, the electron energy, E, is larger than V_o.



$$V(x) = \begin{cases} 0, & x < 0 \\ V_o, & x > 0 \end{cases}$$

1. Find the:
   - magnitude of the net force.
   - direction of the net force.
   - work done by the net force.

   Explain.

2. Would all of the electrons reach Region II?

Page 2 of Tutorial: Reflection and Transmission of Matter

II. Behavior of classical waves at a boundary

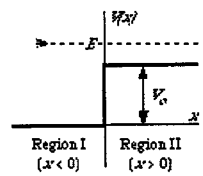
Consider a classical wave incident on a boundary from the left. Assume waves travel at a higher speed in the medium to the left of the boundary than they do in the medium to the right.

A. Will there be a reflected wave?

B. Would there be a reflected wave if the original wave were incident from the right?

III. Behavior of quantum particles at a boundary

Consider a particle in an eigenstate of the potential shown with energy E. Treat it as a ***quantum*** particle.



A. Imagine the particle is incident from the left.

1. Write down the general form of the spatial wave function in Region I. (Hint: This is the solution to the time independent Schrödinger equation in region I.)

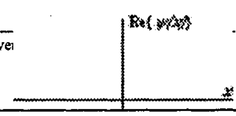
   Using the fact that the full wave function, $\Psi(x, t)$, may be written as $\Psi(x, t) = \psi(x) \exp(-iEt/\hbar)$, show that you can associate an incident wave, $\psi_{inc}(x)$, with part of your wave function and a reflected wave, $\psi_{refl}(x)$, with another part. Explain.

   Would you expect there to be a reflected wave if you were to treat the particle classically?

2. Write down the general form of the spatial wave function in Region II.

   Can you eliminate one term in your general solution based on physical grounds? Explain.

   Show that you can associate a transmitted wave, $\psi_{tran}(x)$, with your remaining term.



Page 3 of Tutorial: Reflection and Transmission of Matter

3. On the graph at right, sketch the real part of the incident, reflected, and transmitted wavefunctions associated with the particle. Be sure the relative sizes of the de Broglie wavelength are indicated in your sketch.

4. The wave function must satisfy certain conditions at the boundary between Regions I and II: $\psi(x)$ and $d\psi(x)/dx$ must be continuous at $x = 0$.

   Apply the above boundary conditions to your expressions for $\psi_{inc}(x)$ and $\psi_{refl}(x)$, and $\psi_{tran}(x)$. If the amplitude of $\psi_{inc}(x)$ is treated as given information, do the boundary conditions provide enough information for you to solve the problem completely?

5. Find the amplitudes of the reflected and transmitted wavefunctions in terms of the amplitude of the incident wavefunction. Assume the amplitude of the incident wavefunction is real. Are your results consistent with the graph you drew above?

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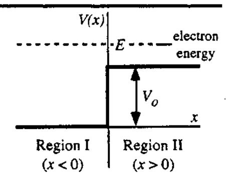
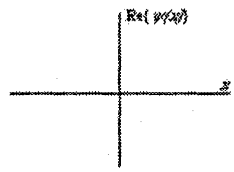
Tutorial: Reflection and transmission of matter

B. Now assume the electron is incident from the right.



1. On the graph below, sketch the real part of the incident, reflected, and transmitted wavefunctions associated with the particle. Be sure the relative sizes of the de Broglie wavelength are indicated in your sketch.



2. Write down the general form of the wave function in Regions I and II. Interpret each term.

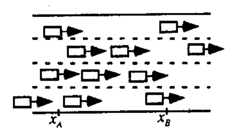
3. Find the amplitudes of the reflected and transmitted wavefunctions in terms of the amplitude of the incident wavefunction. Assume the amplitude of the incident wavefunction is real. Are your results consistent with the graph you drew above?

Page 1 of Tutorial: Probability Current

PROBABILITY CURRENT

I. Classical current density

Cars travel at different speeds on the roadway illustrated at right. Their speeds are given by the function $v(x,t)$. The linear density of cars on the road is given by $\lambda(x,t)$.



A. What is the number of cars in a length of roadway that extends from x_A to x_B in terms of $\lambda(x,t)$?

B. How many cars per second pass point x_A in terms of the variables given? point x_B?

C. Write an expression relating the time rate of change of the quantity you found in question A to your answers to question B.

D. The quantity $v(x,t)\lambda(x,t)$ is sometimes called a current. Discuss the reasonableness of this name.

II. Probability current in quantum mechanics.

In part I above you found a current for the number of cars on a road. Now we wish to determine a probability current for a state in quantum mechanics.

A. Your expression for the current in part I involved the velocity as a function of position. Is it possible write v as a function of x in quantum mechanics? (*Hint:* Consider the uncertainty principle.)

B. Given a wave function $\psi(x)$ for a quantum mechanical particle, write an expression for the probability of finding that particle in the region between x_A and x_B.

Page 2 of Tutorial: Probability Current

C. What is the time rate of change of this probability?

D. Use the product rule to write this as the sum of two terms.

E. Use Schrödinger's equation, $i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi$, to replace the time derivatives in your expression above with spatial derivatives. (*Hint:* Also use the complex conjugate of Schrödinger's equation.)

F. Write the result as an integral of a derivative and integrate.

G. Compare the expression you have written above to the expression you found in part I. In particular consider the units.

H. What quantity would it be reasonable to call probability current. (*Hint:* Consider the sign carefully.)

Page 3 of Tutorial: Probability Current

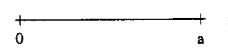
III. Applications to bound states

A. Consider a quantum mechanical particle in an infinite square well potential, *V(x)*, defined at right.

$$V(x)=\begin{cases}\infty & x<0\\ 0 & 0<x<a\\ \infty & x>a\end{cases}$$

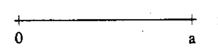
1. Sketch the ground state wave function for this particle. Does this state change with time? If so, how?

ground state wave function



2. Sketch the probability density for this particle in the ground state. Does this probability density change with time? If so, how?

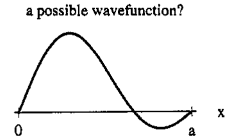
ground state probability density



3. Verify your previous answer by using the expression for probability current that you found in the previous section.

B. Consider the graph shown at right at time $t=0$.

a possible wavefunction?



1. Could this graph be a possible wavefunction for the particle in the infinite square well?

Page 4 of Tutorial: Probability Current

2. Is the above graph an eigenstate?

   Do you agree or disagree with the following student? Explain.

   "It is a sum of eigenstates which makes it an eigenstate."

3. Can the wavefunction above be associated with a single value for the energy?

   Do you agree or disagree with the following student?

   "The wavefunction above is a solution to the Schrödinger equation. So, there must have a definite energy associated with it."

4. Does the wavefunction change as time progresses?

5. Draw the probability density associated with this graph.

ground state probability density

6. Does the probability density change with time?

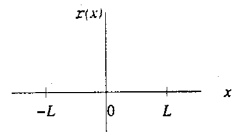
   Use probability current density to support your answer. An instructor can provide you with the mathematical form of the graph shown.

7. Obtain a handout showing the time evolution of the probability density. Resolve any inconsistencies.

Page 5 of Tutorial: Probability Current

IV. Applications to the harmonic oscillator.

A. Sketch the probability density for a classical harmonic oscillator at an instant in time. Explain your sketch.



B. Does the probability density for a classical harmonic oscillator change with time? If so, describe how it changes. If not, explain why not.

C. Does the probability density of a particle in the ground state of a quantum mechanical harmonic oscillator change with time? If so, describe how it changes. If not, explain why not.

D. Does the probability density of a particle in a highly excited energy eigenstate of the quantum mechanical harmonic oscillator change with time? If so, describe how it changes. If not, explain why not.

E. Use ideas from parts II and III to explain the apparent inconsistency between the correspondence principal and questions B and D above.

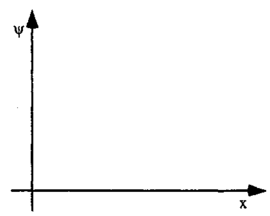
Page 1 of Tutorial: Functions as Vectors

TREATING FUNCTIONS AS VECTORS

I. Discrete wave functions

A. Consider a particle in the ground state of a quantum mechanical infinite square well of width L. Assume the wave function is completely real at $t = 0$.

1. Sketch a graph of the wave function for this state.

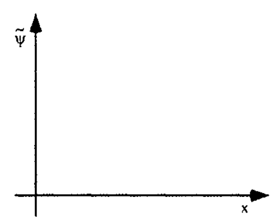


2. Discretize the graph you drew above by dividing the well into three regions of equal width and approximating the value of the wave function in each region by its value at the center.

   a. Write the values for your discretized data in the following table. The symbol $\tilde{\psi}$ will stand for the discretized wave function.

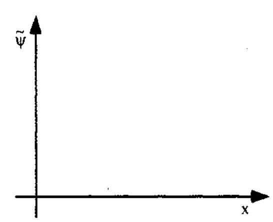
x	$\tilde{\psi}$
x_1	
x_2	
x_3	

   b. Make a plot of this discretized data.



B. Make a similar plot and table for the first excited state. Plot the full wave function on the same axes.

x	$\tilde{\psi}$
x_1	
x_2	
x_3	



Page 2 of Tutorial: Functions as Vectors

II. Orthogonality

A. What mathematical condition corresponds to the statement that two eigenfunctions are orthogonal?

The discretized data that you have calculated are intended to approximate the ground state and first excited state wave functions.

What mathematical condition on these numbers corresponds to the statement that the two eigenstates are orthogonal? Check to see if this condition is satisfied.

B. Tabulate data, similar to that which you found in part I, for the $n = 3$ and 4 states.

	$n = 1$	$n = 2$	$n = 3$	$n = 4$
	$\tilde{\psi}_1$	$\tilde{\psi}_2$	$\tilde{\psi}_3$	$\tilde{\psi}_4$
x_1				
x_2				
x_3				

1. Does the orthogonality condition that you found in question II A hold for all your discretized eigenfunctions? If not, where does it fail?

2. How could you make discrete data more accurately represent the eigenfunctions?

3. How many regions would you need to ensure that all of the eigenstates of the infinite square well were orthogonal? Explain.

Page 3 of Tutorial: Functions as Vectors

4. If you represent the discretized data that you tabulated above as a set of vectors, what vector operation enforces orthogonality? Does this same condition apply if you discretized your wave functions using a larger number of regions?

III. Normalization

A. What condition on this discretized data corresponds to the statement that the eigenstates are normalized? (*Hint:* Make sure your answer agrees with the corresponding condition using full wave functions.)

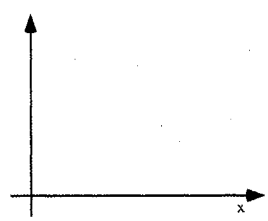
B. Is your data for the first three states in the case of three regions "normalized"? If not, "normalize" it.

IV. Basis vectors

A. In sections I and II you implicitly used a set of three basis functions, $e_i(x)$, to write down the components of your discretized eigenfunctions.

1. Describe this set using the graph at right to sketch an example.

2. Are your basis functions orthonormal according to the condition you found in question III A above? If not, make them orthonormal.



Page 4 of Tutorial: Functions as Vectors

3. Write the discretized first excited state, $\tilde{\psi}_2$, as a sum of your basis functions.

4. Do the squares of the coefficients of the $e_i(x)$ in the expansion above have a physical interpretation? Explain.

B. As the number of regions, n, becomes very large, what happens to the number of the basis functions, $e_i(x)$, and their graphs?

In the limit that n becomes large, Dirac notation denotes the probability of finding a particle in a region Δx_i surrounding x_i by $|\langle x_i|\psi\rangle|^2 \Delta x_i$. As n becomes infinite, Dirac notation represents these probabilities with the symbol, $|\langle x|\psi\rangle|^2 dx$.

1. How can you interpret $|\langle x|\psi\rangle|^2$ above? Explain.

2. Explain how are these new symbols are relate to the coefficients in your expansion in question A 4.

3. Discuss the reasonableness of associating each of the coefficients in an expansion of a wave function with a single value of the position, x.

Page 5 of Tutorial: Functions as Vectors

Dirac notation would represent the *components* of your re-defined finite dimensional vectors as $<s_i|\tilde{\psi}_n>$. Here the $|s_i>$ form a basis.

C. Write $\tilde{\psi}_1$ as an explicit linear combination of the $|s_i>$

D. On the graph at right draw a graphical representation of the second basis vector.

E. Give an interpretation of $<s_2|\tilde{\psi}>$ if $\tilde{\psi}$ is an arbitrary discretized wave function.

Page 6 of Tutorial: Functions as Vectors

F. Use Dirac notation to write $\tilde{\psi}$ as an explicit linear combination of the $|s_i\rangle$.

G. Consider the limit as you make your discretized data a better and better approximation to the real wave function.

1. What happens to the number of basis vectors? Could you label them with a single value of the position? Explain.

2. What happens to the graph of each basis vector?

3. How must you modify your answer to part D above?

4. How would your represent $\psi(x_i)$ in Dirac notation?

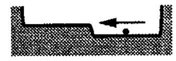
Appendix C Pretests

This appendix consists of pretest questions which have been discussed in the dissertation. It includes pretests that are part of the current sequence and those that have been used in the past. Where two very similar pretest questions were discussed in the dissertation, only one representative example has been included in this appendix. The order of the pretest roughly corresponds to the order of topics in a typical introductory quantum mechanics course. Multiple pretests on a given topic are grouped together.

Page 1 of Pretest: Classical Probability

PRETEST 1: CLASSICAL PROBABILITY

Name ____________________



A ball rolls back and forth in a track with very steep sides. Joined by a steep ramp, two levels of equal length form the base of the track. A large number of photographs of the system are taken at random times. You may assume that the time spent on the steep portions is negligible. Assume there is no friction or energy loss in the system, and that the ball rolls smoothly, without bouncing, forever.

1. In a randomly selected photograph is the probability of the ball being on the higher level *greater than, less than*, or *equal to* the probability of the ball being on the lower level? Explain.

2. Sketch a graph of the ball's time averaged probability density versus horizontal position. Explain.

3. Is the average position of the ball *to the right of, to the left of*, or *exactly at* the center of the system? Explain.

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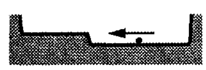
Page 2 of Pretest: Classical Probability

Now assume that the system is adjusted such that the speed of the ball on the lower level is twice that on the higher level.

4. Is the average speed of the ball *greater than, less than,* or *equal to* one and a half times the speed on the upper level? Explain.

5. What is the probability of finding a ball pictured exactly at the center of the lower level? Explain. If it is not possible to answer this question with the information given, describe the additional information that is needed.

The length of the lower level alone is increased, but the speed of the ball on that level remains twice the speed of the ball on the upper level.

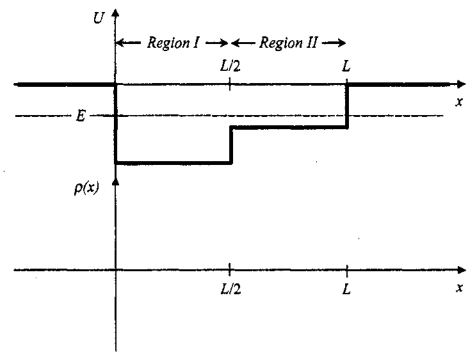


6. Is the probability of finding the ball on the upper level in a photograph taken at random *greater than, less than,* or *equal to* the probability of finding the ball on the upper level in a photograph taken at random when the two levels were of equal length? Explain.

Page 1 of Pretest: Relating Classical Mechanics to Quantum Mechanics

Pretest Name ________________

A. A classical particle with total energy E is bound in the potential well shown. In the space below, draw a qualitatively correct sketch of the probability density for the particle. Explain.

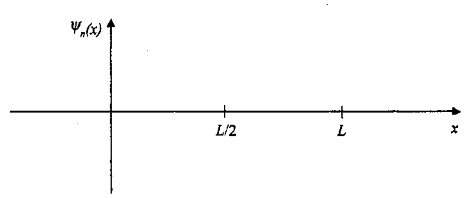


B. Now suppose that the graph of $U(x)$ from part A represents a quantum mechanical system consisting of an electron in a potential well. Assume that the energy, E, marked on the graph corresponds to a *highly excited* energy eigenstate. (*i.e.*, The energy of the electron is much greater than the ground state energy, and the wave function $\psi_n(x)$ of the electron has many nodes between $x = 0$ and $x = L$.)

1. Would the amplitude of the wave function in region I be *greater than, less than,* or *equal to* the amplitude of the wave function in region II? Explain.

2. Would the spacing between the nodes of the wave function in Region I be *greater than, less than*, or *equal to* the spacing between the nodes in region II? Explain.

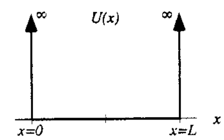
3. In the space at right, draw a qualitatively correct graph of the wave function. Explain.



Page 1 of Pretest: Relating Classical Mechanics to Quantum Mechanics

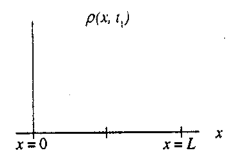
Pretest Name ________________

A. A classical point particle with non-zero total energy is confined to an infinite square potential well. The graph of that potential well is shown at right.

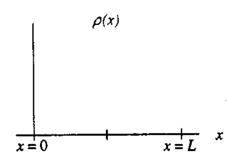


1. Describe the motion of the particle.

2. Suppose you know the position and velocity of the particle at $t = 0$. Sketch a reasonable graph of the probability density for the position of the particle at a **specific** later time, t_1 *(i.e.,* $\rho(x, t_1)$*).* Explain.



3. Imagine that you were going to take a photograph of the particle at some **random** time. Sketch a graph of the probability density for the position of the particle *(i.e.,* $\rho(x)$*).* Explain.



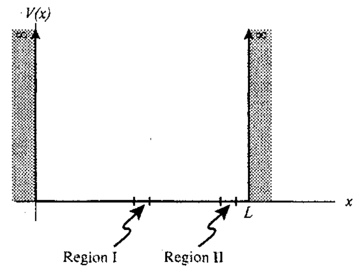
B. Are any of the quantum mechanical states below a good approximation to the behavior of a classical particle in an infinite square well as represented by both of the graphs above? Explain your reasoning.

- the ground state
- a highly excited energy eigenstate
- a superposition of many eigenstates
- other (please describe)
- No quantum mechanical state can reproduce the behavior illustrated by the two graphs above.

Page 1 of Pretest: Relating Classical Mechanics to Quantum Mechanics

Pretest Name ________________

A. A particle with non-zero total energy is confined to an infinite square potential well. The graph of that potential well is shown at right. Two regions of equal width are marked on the graph. Region I is located exactly in the center of the well. Region II is to the right of region I as shown.



1. This system is analyzed, and using that analysis it is predicted that the time-averaged probability of finding the particle in region I will be *greater than* the time-averaged probability of finding the particle in region II.

   a. Could this analysis have been done correctly using classical mechanics? Explain.

   b. Could this analysis have been done correctly using quantum mechanics? Explain.

2. This system is reanalyzed, and using that analysis it is predicted that the time-averaged probability of finding the particle in region I will be *equal to* the time-averaged probability of finding the particle in region II.

   a. Could this analysis have been done correctly using classical mechanics? Explain.

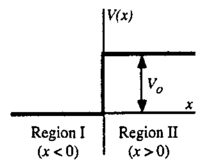
   b. Could this analysis have been done correctly using quantum mechanics? Explain.

Page 1 of Pretest: Reflection and Transmission of Matter

Pretest Name ____________________

Monoenergetic electrons (*i.e.*, electrons that all have the same energy) travel through a region in which the potential energy $V(x)$ varies with x as follows:

$$V(x)=\begin{cases}0, & x<0\\ V_o, & x>0\end{cases}$$ (See figure at right.)



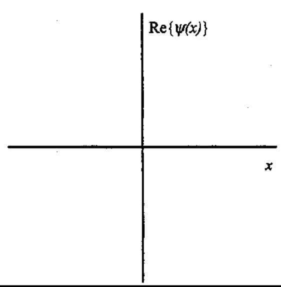
Let E represent the total energy of each electron, and $\psi(x)$, the wave function associated with the electrons (a complex function, in general).

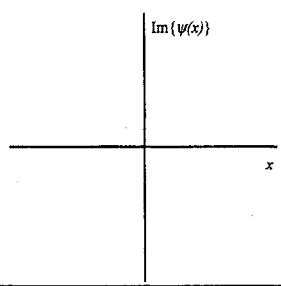
A. Suppose the electrons are incident ***from left to right*** (*i.e.*, in the $+x$ direction) with $E > V_o$.

1. Is the speed of the electrons in Region I *greater than, less than,* or *equal to* the speed of the electrons in Region II? Explain.

2. Do all of the electrons end up in Region II? Explain.

3. On the axes provided, draw the shape of both the real and imaginary parts of $\psi(x)$. Clearly indicate how, if at all, the wave function in $x < 0$ is qualitatively different from the wave function in $x > 0$ (*e.g.*, in which region are the nodes in the wave function closer together?). If the wave function is equal to zero anywhere, show that explicitly. Briefly explain.

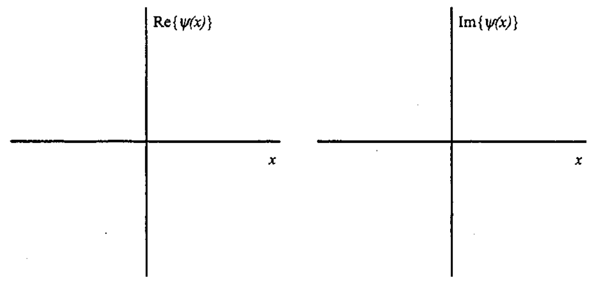




Page 2 of Pretest: Reflection and Transmission of Matter

B. Suppose the electrons are incident ***from left to right*** (*i.e.*, in the $+x$ direction) with $E < V_o$.

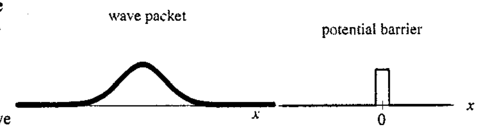
On the axes provided, draw the shape of both the real and imaginary parts of $\psi(x)$. Clearly indicate how, if at all, the wave function is qualitatively different in the two regions. If the wave function is equal to zero anywhere, show that explicitly. Briefly explain.



Page 1 of Pretest: Reflection and Transmission of Matter

PRETEST: REFLECTION AND TRANSMISSION Name ___________________

A. The wave packet shown at right at a particular time is incident from the left on the potential barrier shown.



1. Do you agree or disagree with each of the following statements concerning the wave packet shown? Briefly explain.

Student 1: "The wave packet is a solution to the Schrödinger equation, and is therefore associated with a particular energy."

Student 2: "The wave packet solves Schrödinger 's equation, but is associated with a very large but finite number of energies."

Student 3: "The wave packet solves Schrödinger 's equation, but is associated with an infinite number of energies all of which are multiples of some smallest energy."

Student 4: "The wave packet solves Schrödinger 's equation, but is associated with an infinite number of energies with all possible values contributing not just multiples of some smallest energy."

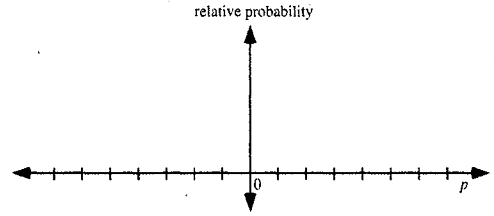
Student 5: "The wave packet does not solve Schrödinger 's equation. It is not a plane wave."

(continued on next page)

Page 2 of Pretest: Reflection and Transmission of Matter

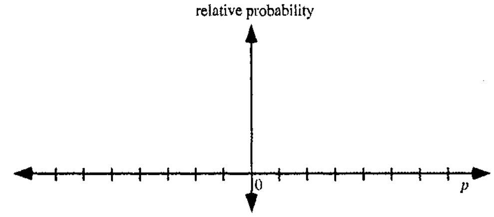
B. Many systems are prepared identically to the one described above. Suppose a measurement of momentum was made on each of the identically prepared systems.

Draw a graph that illustrates the relative probability of obtaining different values of momentum. If only one value of momentum is possible, state that explicitly. Explain.



C. Another set of identical systems are prepared as described in part A. A long time later (*i.e.* well after the centers of the wave packets have reached the potentials) a measurement of momentum is made on each system.

Draw a graph that illustrates the relative probability of obtaining different values of momentum. If only one value of momentum is possible, state that explicitly. Explain.



Page 1 of Pretest: Probability Current

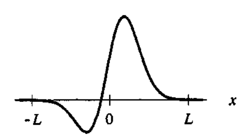
PRETEST: PROBABILITY CURRENT Name ________________

A. Does the probability density for a classical harmonic oscillator change with time? If so, describe how it changes. If not, explain why not.

B. Does the probability density of a particle in the ground state of a quantum mechanical harmonic oscillator change with time? If so, describe how it changes. If not, explain why not.

C. Does the probability density of a particle in a highly excited energy eigenstate of the quantum mechanical harmonic oscillator change with time? If so, describe how it changes. If not, explain why not.

D. Could the graph at right represent a possible wavefunction for a particle in a quantum mechanical harmonic oscillator? If so, is it associated with a single energy? Explain.



Page 2 of Pretest: Probability Current

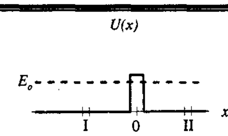
E. In quantum mechanics is it possible to write the velocity of a particle as a function of its position? Briefly explain.

F. A greater number of cars per minute pass signposts on highway 1 than on highway 2. Are the cars on highway 1 traveling at a speed *greater than*, *less than*, or *equal to* the speed of cars on highway 2? Explain.

Page 1 of Pretest: Probability Current

PRETEST: PROBABILITY CURRENT CONTINUED Name ________________

Consider the potential barrier shown at right. Regions of equal lengths are marked along the x-axis.



A. A mono-energetic beam of electrons with energy, E_o, has been incident upon the potential barrier shown from the left for a long time.

1. Is the probability of finding an electron in region I *zero* or *non-zero*? region II? Explain.

2. Is the probability current at the left end of region I *greater than, less than*, or *equal to* zero? at the left end of region II? Explain.

B. The mono-energetic beam of electrons with energy, E_o, is replaced by a mono-energetic beam of electrons with a much higher energy.

1. Is the probability of finding an electron in region I *greater than, less than*, or *equal to* the probability of finding an electron in region II? Explain.

2. Is the probability current at the left end of region I *greater than, less than*, or *equal to* the probability current at the left end of region II? Explain.

Page 2 of Pretest: Probability Current

Pretest

C. The beam is now replaced with a mono-energetic beam of particles with mass much greater than that of electrons, but with energy E_o.

1. Is the probability of finding an electron in region I *greater than, less than*, or *equal to* the probability of finding an electron in region II? Explain.

2. Is the probability current at the left end of region I *greater than, less than*, or *equal to* the probability current at the left end of region II? Explain.

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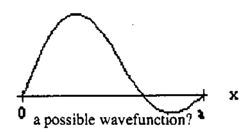
Page 1 of Pretest: Time Dependence in Quantum Mechanics

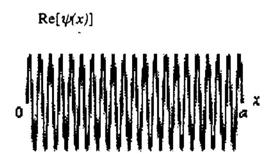
Pretest Name ____________________

A. A particle is in a quantum mechanical infinite square well. Could the graph at right represent a possible wavefunction for the particle at some instant in time? If so, is it associated with a single energy? Explain.



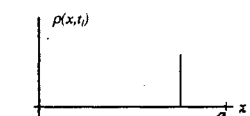
B. A student is discussing the highly excited energy eigenstate of the infinite square well shown at right. Do you agree or disagree with the student's statement? Explain.

Student: *"This state describes a particle bouncing back and forth between the two edges of the well."*

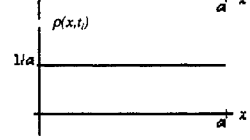


C. Two other students are discussing the probability density as a function of time for a classical particle in an infinite square well. Do you agree with any parts of either student's statement? Explain.

Student 1: *"Classically you know exactly where the particle is at any instant in time. So, my graph shows a spike of area one that will move back and forth as time progresses. My graph, however, only shows the probability density at a particular time t_i."*



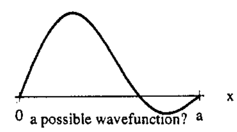
Student 2: *"I don't agree. Classically at any time the particle is equally likely to be found anywhere. So, I have drawn a straight line for the probability density at a time t_i."*



Page 1 of Pretest: Time Dependence in Quantum Mechanics

Pretest Name ___________________

A. A particle is in a quantum mechanical infinite square well. Could the graph at right represent a possible wavefunction for the particle at some instant in time? Explain.



Answer either question one or question two.

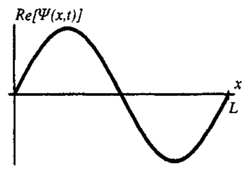
1. If the graph above does represent a possible wavefunction, is it associated with a single energy? Explain. If possible, sketch a graph that could not represent a possible wavefunction.

2. If the graph above does not represent a possible wavefunction, explain why not and sketch a graph that could represent a possible wavefunction. Is the graph that you have drawn associated with a single energy? Explain.

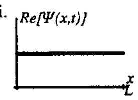
Page 1 of Pretest: Time Dependence in Quantum Mechanics

PRETEST: TIME DEPENDENCE **Name:**_______________

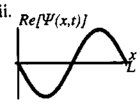
1. Consider an isolated system with an infinite square well potential. At time $t = 0$ the system is in the first excited state of the infinite square well and is entirely real as shown at right.



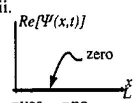
A. Which, if any, of the following graphs could represent the real part of the wave function for the isolated system at some later time?

i.


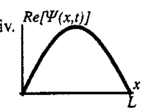
▫yes ▫no

ii.


▫yes ▫no

iii.


▫yes ▫no

iv.


▫yes ▫no

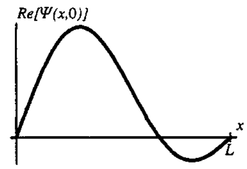
Briefly explain:

B. Which, if any, of the above graphs could represent the real part of the wave function for the isolated system a *very* long time later?

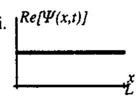
Page 2 of Pretest: Time Dependence in Quantum Mechanics

Pretest: Time dependence

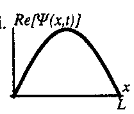
2. Consider another isolated system with an infinite square well potential. At time $t = 0$, the state of this system can be represented by the wave function shown at right. At $t = 0$, this function can be written as a linear combination of only the ground state (with energy E_1) and the first excited state (with energy E_2) of the infinite square well and is entirely real.



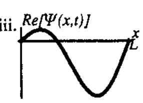
A. Which, if any, of the following graphs could represent the real part of the wave function for the isolated system at some later time?

i.


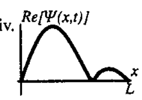
▫yes ▫no

ii.


▫yes ▫no

iii.


▫yes ▫no

iv.


▫yes ▫no

Explain:

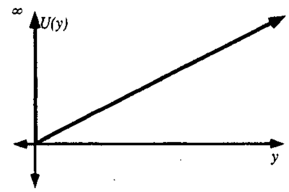
B. For each graph that you marked "yes" above, suppose the energy were measured at the instant corresponding to the graph shown. What value or values are possible results of this energy measurement?

C. Which, if any, of the above graphs could represent the real part of the wave function for the isolated system a *very* long time later? Explain.

Page 1 of Pretest: Energy Measurements

PRETEST: ENERGY MEASUREMENTS Name ________________

A. Consider a quantum mechanical system with a potential similar to that of a ball making elastic collisions with a surface. Such a potential is shown at right. This system is prepared so that at $t = 0$ it is in a state described by a wave function identical to that of the ground state of the infinite square well. That is, $\Psi(y,0) = \sqrt{2/L}\sin(\pi y/L)$.



1. Does the wave function associated with this state depend on time? If so, describe how you would write down its time dependence. If not, explain why not.

2. Does the probability density associated with this state depend on time? If so, describe its time dependence. If not, explain why not.

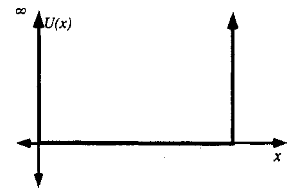
(continued on reverse)

Page 2 of Pretest: Energy Measurements

B. A particle in an infinite square well (shown at right) is prepared so that its initial state, $\Psi(x, 0)$ is:

$$\Psi(x,0) = 0.6\ \psi_1 + 0.8i\ \psi_2\ ,$$

where ψ_1 and ψ_2 are the two lowest energy eigenstates of the infinite square well, with $E_1 = \varepsilon$ and $E_2 = 4\varepsilon$.



1. Suppose you measured the energy of this particle at a time, $t = 0$.

   Could this energy measurement result in the values in the table below? For those values that are possible results, give the probability of their occurrence. If more information is needed, state so explicitly.

value	A possible result?	Probability
$0.36E_1 + 0.64E_2$		
E_1		
$0.5E_1 + 0.5E_2$		

   Explain your reasoning.

2. Which, if any, of your answers would change if the measurement of the energy described above were, instead, made at a time later than $t = 0$? Explain.

value	Probability changes?
$0.36E_1 + 0.64E_2$	
E_1	
$0.5E_1 + 0.5E_2$	

3. Suppose that the first measurement had been made at time $t = 0$. Then a few minutes later, at time t_1, you measured the energy of the particle again. (Assume that, during the time interval $0 < t < t_1$, the particle remains isolated from the environment.)

   Which of the following statements would best describe the result of your second energy measurement? **Circle one and explain your reasoning.**

   i. The second energy measurement would *definitely* give the same result as the first.

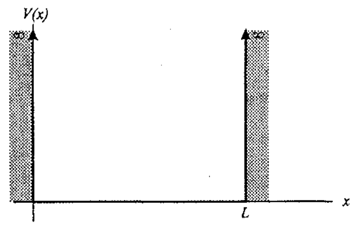
   ii. The second energy measurement could *possibly* (but not necessarily) give the same result as the first.

   iii. The second energy measurement would *definitely not* give the same result as the first.

Page 1 of Pretest: Position, Momentum, and Energy Measurements

PRETEST: POSITION, MOMENTUM AND ENERGY MEASUREMENTS

A particle is in the infinite square well potential shown at right.



At time $t = 0$, the energy of the system is measured and found to be E_2, the first excited state energy.

A. Immediately after the measurement, is the system in an eigenstate of position? Explain.

B. At time $t_1 > 0$ you measure the *position* of the particle and find that it is located at $x = 0.3L$.

Suppose a few minutes later, at a time t_2 you measured the *energy* of the particle. (Assume that, during the time interval $t_1 < t < t_2$, the particle remains isolated from the environment.)

Which of the following statements would best describe the result of your energy measurement? **Circle one and explain your reasoning.**

i. The value that you measure would *definitely* be the ground state energy.
ii. The value that you measure would *definitely* be the first excited state energy.
iii. The value that you measure could *possibly* (but not necessarily) be the first excited state energy.
iv. The value that you measure would *definitely not* be the first excited state energy.

Page 2 of Pretest: Position, Momentum, and Energy Measurements

QM 2 *Pretest: Position, momentum and energy measurements*

Suppose that instead of measuring the energy at time t_2, you measure the energy a *very* long time later. (Assume that, during the time interval between position and energy measurements, the particle remains isolated from the environment.)

Which of the following statements would best describe the result of your energy measurement? **Circle one and explain your reasoning.**

i. The value that you measure would *definitely* be the ground state energy.
ii. The value that you measure would *definitely* be the first excited state energy.
iii. The value that you measure could *possibly* (but not necessarily) be the first excited state energy.
iv. The value that you measure would *definitely not* be the first excited state energy.

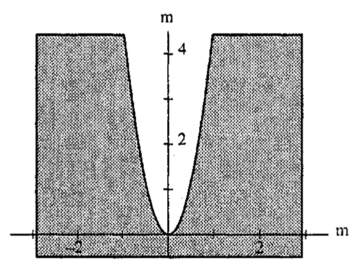
Page 1 of Pretest: Uncertainty: Classical and Quantum

UNCERTAINTY: CLASSICAL AND QUANTUM Name ____________

A. Consider the quantum mechanical description of a harmonic oscillator with Hamiltonian $H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$.

Suppose this system is in the first excited state. Calculate the uncertainty in the energy, ΔE_{QM}, for the system in this state. Explain.

B. A frictionless parabolic ramp of height 4.5 m and width 2 m is shown at right. A heavy iron block of mass 1 kg will be set in motion on the ramp.



Consider the following student dialog concerning the applicability of quantum mechanics to this situation.

Student 1: *"Quantum mechanics can't describe a classical situation like this because of the energy-time uncertainty principle. In the classical case we know the energy very well. This implies that the system hardly changes at all with time. This is not the case with the real macroscopic harmonic oscillator that we have here."*

Student 2: *"I agree. In fact, it also violates the position and energy uncertainty relationship. Classically we know both the position and energy very well at any given time. In quantum mechanics since the operators for position and energy don't commute, we can't know both of them well at the same time."*

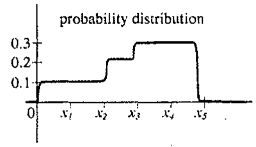
With which student or students, if any, do you agree? Explain.

If you do not agree with one or both of the students, explain how $\Delta E \Delta t \geq \frac{\hbar}{2}$ and/or $\Delta x \Delta E \geq \left|\frac{1}{2i}\langle[\hat{x},\hat{H}]\rangle\right|$ apply in this situation.

Page 1 of Pretest: Functions as Vectors

PRETEST: FUNCTIONS AS VECTORS Name ____________________

A. Consider a particle in the ground state, ψ_o, of an *unspecified* potential well. The probability distribution for the ground state is shown at right. Assume the ground state was chosen to be real at $t = 0$. Answer the following question for a particle in the state ψ_o at time $t = 0$.



Find $\langle x_1|\psi_o\rangle$, $\langle x_4|\psi_o\rangle$, and $\langle x'|\psi_o\rangle$ where $x' = x_1 + x_4$. Explain.

B. A wave function can be written as $\psi(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{+\infty} a(p)e^{ipx/\hbar}dp$. Give a physical interpretation for $|a(p)|^2$. Explain.

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Page 2 of Pretest: Functions as Vectors

C. Find $\langle E_o|\psi_o\rangle$, $\langle E_1|\psi_o\rangle$, and $\langle E'|\psi_o\rangle$ where E_o is the energy of the ground state, E_1 is the energy of the first excited state and $E' = E_o + E_1$. (Assume that there is an eigenstate of this well with energy E'.) Explain.

Page 1 of Pretest: Functions as Vectors

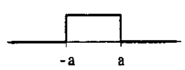
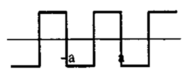
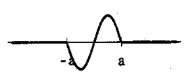
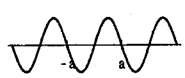
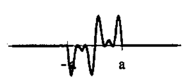
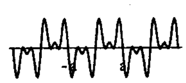
☐ I have never studied Fourier series. ***☐ I have studied Fourier series in __________***
☐ I have never studied Fourier transforms. ***☐ I have studied Fourier transforms in ________***

Pretest Name ________________

A. Six signals are shown below. The periodic signals II, IV, and VI extend from the remote past to the remote future. For each signal:

1. Can the signal be made from a linear combination of sinusoidal signals? If so, would a sinusoid of a single frequency be sufficient, or would sinusoids of two or more frequencies be necessary? Explain.
2. For those signals that can be made from a linear combination of two or more sinusoidal signals:

   Are all of the frequencies present in the linear combination multiples of some lowest frequency? If so, determine the lowest frequency for that signal. Explain. If not, explain why not.

	1. *one* or *more*? Explain.	2. If two or more frequencies: Is there a lowest frequency? *yes, no?* Explain.
I.  time		
II.  time		
III.  time		
IV.  time		
V.  time		
VI.  time		

Page 2 of Pretest: Functions as Vectors

B. Consider a system made from ideal masses, springs, and dampers. The system is driven with a signal given by $F(t)=3\sin(2\pi t/3)+5\sin(2\pi t/4)+3\sin(2\pi t/5)$. After a long time has elapsed:

1. How many frequencies will be found in the response? Explain.

2. With the given information is it possible to determine which frequency or frequencies will be found in the response? If so, what are they? Explain. If not, explain why not.

Page 1 of Pretest: Degenerate States

PRETEST: DEGENERATE STATES Name ____________

Consider a three–dimensional harmonic oscillator described by the Hamiltonian $H=-\frac{\hbar^2}{2M}\nabla^2+\frac{1}{2}M\omega^2r^2$. In Cartesian co–ordinates the energy eigenfuncitions are given by $\psi(x,y,z)=N\varphi_{n_x}(x)\varphi_{n_y}(y)\varphi_{n_z}(z)$ with eigenvalues $E_{n_xn_yn_z}=\hbar\omega\left(\frac{3}{2}+n_x+n_y+n_z\right)$. In spherical co–ordinates the energy eigenfunctions are given by $\psi(r,\theta,\phi)=NR_{nl}(r)Y_l^m(\theta,\phi)$ with eigenvalues $E_{nl}=\hbar\omega\left(\frac{3}{2}+2n+l\right)$.

For each state below answer the following questions:
1) Is this an allowed state?
2) Does the wave function change as time evolves?
3) Does the probability density associated with the state change as time evolves?

In each case assume N is the proper normalization.

A. $\Psi(\vec{r},t=0)=N\left(\varphi_1(x)\varphi_0(y)\varphi_0(z)+\varphi_0(x)\varphi_0(y)\varphi_1(z)\right)$

1. Is this an allowed state? Explain.

2. Does the state change as time evolves?

3. Does the probability density change as time evolves? Explain.

B. $\Psi(\vec{r},t=0)=N\left(\varphi_1(x)\varphi_0(y)\varphi_0(z)+\varphi_1(x)\varphi_0(y)\varphi_1(z)\right)$

1. Is this an allowed state? Explain.

2. Does the state change as time evolves?

3. Does the probability density change as time evolves? Explain.

Page 2 of Pretest: Degenerate States

C. $\Psi(\vec{r},t=0)=N\left(\varphi_1(x)\varphi_1(y)\varphi_0(z)+R_{11}(r)Y_1^1(\theta,\phi)\right)$

1. Is this an allowed state? Explain.

2. Does the state change as time evolves?

3. Does the probability density change as time evolves? Explain.

D. $\Psi(\vec{r},t=0)=N\left(\varphi_2(x)\varphi_1(y)\varphi_0(z)+R_{11}(r)Y_1^1(\theta,\phi)\right)$

1. Is this an allowed state? Explain.

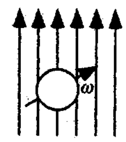
2. Does the state change as time evolves?

3. Does the probability density change as time evolves? Explain.

Page 1 of Pretest: Spinors

PRETEST Name:__________________

1. Positive charge is placed uniformly on the surface of a rubber ball. The ball is rotating with an angular velocity ω and is placed within a region of space in which there exists a uniform magnetic field. Describe the subsequent motion of the ball. Explain your answer using ideas from classical physics.



2. A spin 1/2 system is in a spin state characterized by the following spinor in the S_z basis. $\chi = \frac{1}{5}\begin{pmatrix}3\\4\end{pmatrix}$. Using this spinor given in the S_z basis is it possible to determine all three probabilities below? (You don't need to calculate any values.)

   the probability of obtaining $\hbar/2$ from a measurement of S_z

   the probability of obtaining $\hbar/2$ from a measurement of S_x

   the probability of obtaining $\hbar/2$ from a measurement of S_y

   Explain why or why not.

3. For a certain spin 1/2 system, it is known that the probability of obtaining $\hbar/2$ from a measurement of the spin in the z direction is 0.64.

   a. Write down a spinor in the S_z basis that is consistent with this information. Explain.

   b. Is this information sufficient to specify completely the spin state? Explain.

Page 1 of Pretest: Addition of Angular Momentum

PRETEST: ADDITION OF ANGULAR MOMENTUM Name ____________

Suppose the orbital angular momentum squared, L^2, is measured for an electron in a spherically symmetric potential and is found to be $12\hbar^2$.

A. Which of the following represent possible wave functions for the electron just before the measurement is made? Here $f(r)$ and $g(r)$ are normalized functions of the radial coordinate r only. The $Y_l^m(\theta,\varphi)$ are the normalized spherical harmonics and the χ_i are the eigenspinors of S_z.

1. $\Psi(r,\theta,\varphi) = f(r)[(\sqrt{1/6})Y_3^0(\theta,\varphi)\,\chi_+ + (\sqrt{2/6})Y_3^1(\theta,\varphi)\,\chi_-] + g(r)(\sqrt{3/6})Y_2^0(\theta,\varphi)\,\chi_+$
2. $\Psi(r,\theta,\varphi) = f(r)[(\sqrt{1/3})Y_3^0(\theta,\varphi)\,\chi_+ + (\sqrt{2/3})Y_3^1(\theta,\varphi)\,\chi_-]$
3. $\Psi(r,\theta,\varphi) = f(r)Y_3^0(\theta,\varphi)\,\chi_+$
4. $\Psi(r,\theta,\varphi) = g(r)\,Y_2^0(\theta,\varphi)\,\chi_+$

Explain your choice(s).

B. Which of the following represent possible wave functions for the electron just after the measurement is made? Here $f(r)$ and $g(r)$ are normalized functions of the radial coordinate r only. The $Y_l^m(\theta,\varphi)$ are the normalized spherical harmonics and the χ_i are the eigenspinors of S_z.

1. $\Psi(r,\theta,\varphi) = f(r)[(\sqrt{1/6})Y_3^0(\theta,\varphi)\,\chi_+ + (\sqrt{2/6})Y_3^1(\theta,\varphi)\,\chi_-] + g(r)(\sqrt{3/6})Y_2^0(\theta,\varphi)\,\chi_+$
2. $\Psi(r,\theta,\varphi) = f(r)[(\sqrt{1/3})Y_3^0(\theta,\varphi)\,\chi_+ + (\sqrt{2/3})Y_3^1(\theta,\varphi)\,\chi_-]$
3. $\Psi(r,\theta,\varphi) = f(r)Y_3^0(\theta,\varphi)\,\chi_+$
4. $\Psi(r,\theta,\varphi) = g(r)\,Y_2^0(\theta,\varphi)\,\chi_+$

Explain your choice(s).

Page 2 of Pretest: Addition of Angular Momentum

QM 2 *Pretest: Addition of angular momentum*

For the next three questions, assume that the electron has the wave function
$\Psi(r,\theta,\varphi) = f(r)[(\sqrt{1/6})Y_3^0(\theta,\varphi)\,\chi_+ + (\sqrt{2/6})Y_3^1(\theta,\varphi)\,\chi_-] + g(r)(\sqrt{3/6})Y_2^0(\theta,\varphi)\,\chi_+$.

C. Suppose you measured the z-component of the orbital angular momentum, L_z. What values might you get, and what is the probability of each? Explain.

Let $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$ be the total angular momentum of the electron.

D. Suppose you measured the total angular momentum squared, J^2. What values might you get? Explain.

E. Suppose you measured the z-component of the total angular momentum, J_z. What values might you get? Explain.

Page 1 of Pretest: Identical Particles

PRETEST: IDENTICAL PARTICLES Name ____________

A. Suppose that you are given a red die weighted such that the probability of rolling an "i" on the red die is given by $P_r(i)$. (*i.e.*, the probability that rolling the red die would result in a 1 is $P_r(1)$.) *Note:* Do not assume $P_r(i)=1/6$.

1. Write an expression for the probability that the roll would result in a "1" *or* a "2." Explain.

B. You are now given a blue die that is weighted differently such that the probability of rolling a "j" on the blue die is given by $P_b(j)$.

1. Write an expression for the probability that rolling both dice would result in a blue "1" and a red "6." Explain.

2. Write an expression for the probability of rolling a "1" and a "6," regardless of color. Explain.

3. Have you made any assumptions in answering parts 1. and 2. above? Explain.

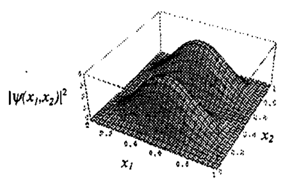
Page 2 of Pretest: Identical Particles

C. Consider two arbitrary events A and B. The probability of event A occurring is 1/5 and the probability of event B occurring is 1/6. Could the probability of both of them occurring be 1/15? Explain.

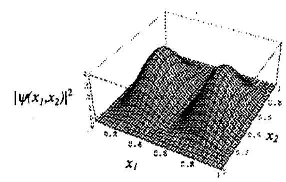
Page 1 of Pretest: Identical Particles

PRETEST: IDENTICAL PARTICLES Name ____________

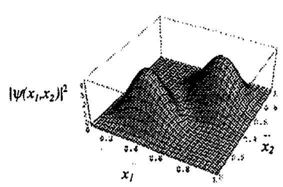
A. Consider the following plots of $|\psi(x_1,x_2)|^2$ for a two particle system in a one dimensional infinite square well of unit width. Which, if any, of the plots below could be for a system of two identical fermions in a stationary state with a symmetric spin state. Explain your reasoning for each case.



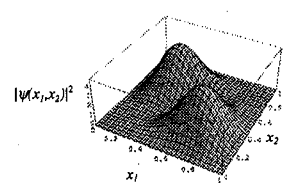
Case i.



Case ii.



Case iii.



Case iv.

Page 2 of Pretest: Identical Particles

B. Consider a crude model for helium (Z = 2) in which the atomic electrons do not interact with each other. Let $\psi_{nlm}(\vec{r})$ represent normalized single-electron energy eigenstates.

1. For each wave function listed below, determine whether or not that wave function is a valid wave function for the electrons in helium. **Explain your reasoning in each case.**

   a. $A_1\ \{\psi_{100}(\vec{r}_1)\psi_{210}(\vec{r}_2)\}\ |\uparrow\uparrow\rangle$

   b. $A_2\ \{\psi_{100}(\vec{r}_1)\psi_{210}(\vec{r}_2)\}\ |\uparrow\downarrow\rangle$

   c. $A_3\ \{\psi_{100}(\vec{r}_1)\psi_{100}(\vec{r}_2)\}\ \{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\}$

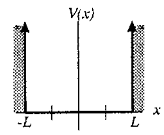
   d. $A_4\ \{\psi_{100}(\vec{r}_1)\psi_{210}(\vec{r}_2) - \psi_{210}(\vec{r}_1)\psi_{100}(\vec{r}_2)\}\ \{|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\}$

   e. $A_5\ \{\psi_{100}(\vec{r}_1)\psi_{210}(\vec{r}_2) - \psi_{210}(\vec{r}_1)\psi_{100}(\vec{r}_2)\}\ \{|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\}$

Page 1 of Pretest: Time Independent Perturbation Theory

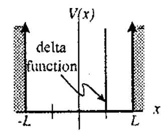
PRETEST Name ____________________

Consider a particle in an infinite square "well" of width $2L$ (see figure at right).



Two possible perturbations to the original potential are described below. For what follows, recall $E^1_n = \langle \psi_n | H' | \psi_n \rangle$.

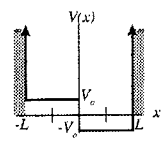
A. Suppose the original well is perturbed by a delta function potential as shown at right. The perturbation to the Hamiltonian is given by: $H'(x) = +\alpha\,\delta(x - L/2)$.



1. Is the first-order correction to the energy, $E^1_{n=1}$, of the ground state *positive, negative,* or *zero?* Explain.

2. Is the first-order correction to the energy, $E^1_{n=2}$, of the first excited state *positive, negative,* or *zero?* Explain.

B. Consider the well shown at right. Write an expression for the perturbation, H', to the original well that results in this well.



For this perturbation:

1. Is the first-order correction to the energy, $E^1_{n=1}$, of the ground state *positive, negative,* or *zero?* Explain.

2. Is the first-order correction to the energy, $E^1_{n=2}$, of the first excited state *positive, negative,* or *zero?* Explain.

Page 1 of Pretest: Degenerate Time Independent Perturbation Theory

QM 2 **PRETEST**

A two–dimensional isotropic harmonic oscillator is described by the Hamiltonian $H = -\frac{\hbar^2}{2M}\nabla^2 + \frac{1}{2}M\omega^2 r^2$. The first two energy eigenfunctions for the one–dimensional harmonic oscillator are:

$\varphi_o(x) = A_o e^{-\alpha x^2}$ and $\varphi_1(x) = A_1 x e^{-\alpha x^2}$ where $A_o = \left(\frac{M\omega}{\pi\hbar}\right)^{\frac{1}{4}}$, $A_1 = \left(\frac{4M^3\omega^3}{\pi\hbar^3}\right)^{\frac{1}{4}}$, and $\alpha = \frac{M\omega}{2\hbar}$.

The eigenvalues corresponding to these eigenfunctions are $\frac{1}{2}\hbar\omega$ and $\frac{3}{2}\hbar\omega$ respectively.

A perturbation

A small anisotropy is added to the potential. The new Hamiltonian is now $H = -\frac{\hbar^2}{2M}\nabla^2 + \frac{1}{2}M\omega^2 r^2 + \gamma\frac{1}{2}M\omega^2(x+y)^2$ where γ is small and positive.

The full Hamilton can be written as the Hamiltonian for the isotropic oscillator plus a perturbation. $H = H^o + H'$.

A. Is the first order correction to the ground state energy *positive, negative,* or, *zero*? Explain.

B. Which, if any, of the following wave functions are valid first excited states for the *unperturbed* oscillator? Explain

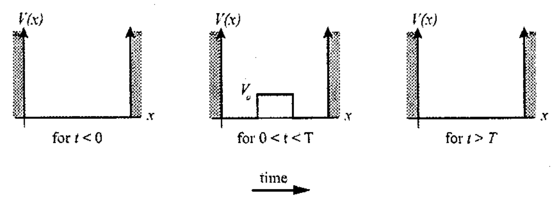
1. $\varphi_o(x)\varphi_1(y)$
2. $\varphi_o(y)\varphi_1(x)$
3. $(\varphi_o(x)\varphi_1(y) + \varphi_o(y)\varphi_1(x))/\sqrt{2}$
4. $(\varphi_o(x)\varphi_1(y) - \varphi_o(y)\varphi_1(x))/\sqrt{2}$

Page 1 of Pretest: Time Dependent Perturbation Theory

PRETEST: TIME DEPENDENT PERTURBATION THEORY Name ____________

A time dependent perturbation to the infinite square well potential is illustrated below. (*Note*: $V_o << E_i^o$) At a time $t_- << 0$ the total energy of the system is measured and found to be the ground state energy of the unperturbed well. In what follows let the symbols E_i^o represent the energy eigenvalues for the unperturbed well.

A time dependent perturbation to the infinite square well



Consider a single additional energy measurement made at one of three possible times. In each case describe as specifically as possible what values could possibly result from the measurement. (*e.g.* Are any of the E_i^o possible values? Are only certain of the E_i^o possible? Are there a countable or an uncountably infinite number of possible values?)

Case 1: The measurement is made at a time $t_- < t < 0$. Explain.

Case 2: The measurement is made at a time $0 < t < T$. Explain.

Case 3: The measurement is made at a time $t > T$. Explain.

Page 2 of Pretest: Time Dependent Perturbation Theory

QM 2 *Pretest: Time dependent perturbation theory*

Assume no energy measurements have been made after time t_-. Does the spatial probability distribution change with time during each of the following time intervals.

Case 1: between $t = t_-$ and $t = 0$ Explain.

Case 2: between $t = 0$ and $t = T$ Explain.

Case 3: after $t = T$ Explain.

Appendix D Exam Questions

This appendix consists of exam questions which have been discussed in the dissertation. Where two very similar exam questions were discussed in the dissertation, only one representative example has been included in this appendix. That example is labeled according to the date that it was given. The order of the exam questions roughly corresponds to the order in which they were first discussed in the dissertation.

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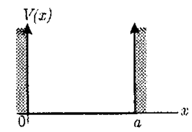
Name ______________________________ *Student ID* ______________ *Score*______

last   first

Part IV. (14 pts) An infinite square well



The wave function of an electron in a one-dimensional infinite square well of width a at time $t = 0$ is given by $\Psi(x,0) = \sqrt{2/7}\phi_1(x) + \sqrt{5/7}\phi_2(x)$ where $\phi_1(x)$ and $\phi_2(x)$ are the ground state and first excited stationary states of the system. ($\phi_n(x) = \sqrt{2/a}\sin(n\pi x/a)$. $E_n = n^2\pi^2\hbar^2/(2ma^2)$ where $n = 1, 2, 3...$)

Answer the following questions about this system:

1. **(3 pts)** Write down the wave function $\Psi(x,t)$ at time t in terms of $\phi_1(x)$ and $\phi_2(x)$.

2. **(3 pts)** Suppose you measure the energy of an electron at time $t = 0$. What are the possible values of the energy and the probability of measuring each?

3. **(3 pts)** Calculate the expectation value of the energy in the state $\Psi(x,t)$ above.

4. **(5 pts)** Which of the following wave functions are allowed for an electron in a one-dimensional infinite square well of width a with boundaries at $x = 0$ and $x = a$: $A\sin^3(\pi x/a)$, $A\left[\sqrt{2/5}\sin(\pi x/a) + \sqrt{3/5}\sin(2\pi x/a)\right]$, and $Ae^{-((x-a/2)/a)^2}$? In each of the three cases, A is a suitable normalization constant. You must provide a clear explanation for each case.

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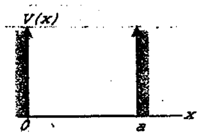
4. [15 pts.] The wave function of an electron in a one-dimensional infinite square well of width a at time t = 0 is given by:



$$\Psi(x,0) = \sqrt{2/7}\,\phi_1(x) + \sqrt{5/7}\,\phi_2(x)$$

where $\phi_1(x)$ and $\phi_2(x)$ are the ground state and first excited stationary states of the system. ($\phi_n(x) = \sqrt{2/a}\sin(n\pi x/a)$. $E_n = n^2\pi^2\hbar^2/(2ma^2)$ where $n = 1,2,3\ldots$)

Answer the following questions about this system:

(a) [4 pts] Write down the wave function Ψ(x,t) at time t in terms of $\phi_1(x)$ and $\phi_2(x)$.

(b) [4 pts] Suppose you measure the energy of an electron at time t = 0. What are the possible values of the energy and the probability of measuring each?

(c) [4 pts] Calculate the expectation value of the energy in the state Ψ(x,t) above.

(d) [3 pts] Which of the following wave functions are allowed for an electron in a one-dimensional infinite square well of width a with boundaries at x = 0 and x = a?

(i) $A\sin^3(\pi x/a)$

(ii) $A\left[\sqrt{2/5}\sin(\pi x/a) + \sqrt{3/5}\sin(2\pi x/a)\right]$

(iii) $Ae^{-((x-a/2)/a)^2}$

In each of the three cases, A is a suitable normalization constant. You must provide a clear explanation for each case.

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2. (*15 pts*) The system is a particle in a one-dimensional potential well. At $t = 0$ the wave function is

$$\psi(x) = \frac{1}{2}u_0(x) + \frac{\sqrt{3}}{2}u_1(x)$$

where u_0 and u_1 are the wave functions for the two lowest energy states with energies E_0 and E_1. u_0 and u_1 are real and orthonormal.

a) (*4 pts*) If you measure the energy of this system what is the probability of finding:

E_0 ____ $\frac{1}{4}(E_0 + 3E_1)$ ____ E_1 ____ $\frac{1}{2}(E_0 + E_1)$ ____

b) (*3 pts*) Write $\Psi(x,t)$ in terms of u_0 and u_1.

c) (*8 pts*) In each case indicate (*without calculations*) whether the quantity varies with time for the state $\Psi(x,t)$.

	Yes	No
$\langle x \rangle$		
$\langle H \rangle$		
$\Psi^*(x,t)\Psi(x,t)$		
$\int \Psi^*(x,t)\Psi(x,t)\,dx$		

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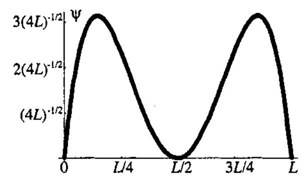
A particle is confined to a one-dimensional infinite square well of width L. At the instant shown at right the normalized wavefunction is entirely real. In what follows, let $x_o = 0.8L$, $x' = 10x_o$, and let $\hat{x}$ be the x operator.



1. Give an approximate value for the following quantities. If any do not have finite values, state so explicitly. Explain your answers.
   (a) $< x_o|\psi >$
   (b) $< \psi|10\hat{x}|\psi >$
   (c) $< x'|\psi >$
2. Now assume a measurement of position has been made, and the particle has been found near x_o with small uncertainty. Describe whether the each of the quantities from the previous question are now (*i.e.*, immediately after the measurement) greater than they were before the measurement, less than they were before the measurement, or equal to their values before the measurement. Explain your answers and state explicitly if it is not possible to decide.
   (a) $< x_o|\psi >$
   (b) $< \psi|10\hat{x}|\psi >$
   (c) $< x'|\psi >$
3. Assume a very long time has now elapsed since the measurement. For each of the following quantities state whether you expect their current values to be different from their values before the measurement. Explain your answers and state explicitly if it is not possible to decide.
   (a) $< x_o|\psi >$
   (b) $< \psi|10\hat{x}|\psi >$
   (c) $< x'|\psi >$

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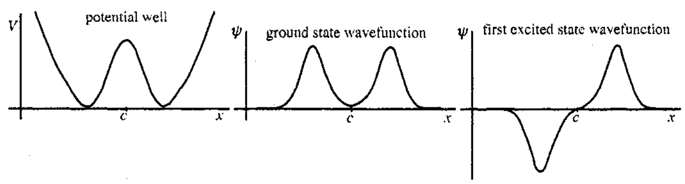
Name ___________________________

Part IV.

The potential for a nitrogen atom in an ammonia molecule (NH_3) is shown below. The ground state and the first excited state wavefunctions are also shown.



Consider a large ensemble of identical systems s_1, s_2, s_3, s_4, *etc.*, each in the **ground state** of this potential. Assume the position of the nitrogen atom in the first system has been measured and found to be to the left of $x = c$.

A. Suppose you now measure with small uncertainty the position x_2, x_3, x_4, *etc.* of the nitrogen atom in each of the next nine systems. Would you expect the average of the values for **all ten** systems (*i.e.*, the average of $x_1—x_{10}$) to be *greater than, less than*, or *equal to* $x = c$? Explain.

B. Would you expect the average of the values for the position of the nitrogen atom in **only** systems $s_2—s_{10}$ to be *greater than, less than*, or *equal to* $x = c$? Explain.

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March 15, 2004

Name ______________________________

Final Exam *100 points*

Numerical answers should be left in terms of π, $\sqrt{2}$, $\hbar$, etc

1. (*20 pts*) The system is a particle in a one-dimensional potential well. At $t = 0$ the wave function is

$$\psi(x) = \frac{1}{2}u_0(x) + \frac{\sqrt{3}}{2}u_1(x)$$

where u_0 and u_1 are the wave functions for the two lowest energy states with energies E_0 and E_1. u_0 and u_1 are real and orthonormal.

a) (*5 pts*) If you were to measure the energy of such a system what is the probability of finding

$\frac{1}{4}(E_0 + 3E_1)$ __________

E_1 __________

$\frac{1}{2}(E_0 + E_1)$ __________

b) (*5 pts*) Write $\Psi(x,t)$ in terms of u_0 and u_1.

c) (*5 pts*) Evaluate $\langle H \rangle$ for the state $\psi(x)$.

d) (*5 pts*) In each case indicate (*without calculations*) whether the quantity varies with time.

$\langle H \rangle$ Yes No

$\Psi^*(x,t)\Psi(x,t)$ Yes No

$\int \Psi^*(x,t)\Psi(x,t)\,dx$ Yes No

Physics 324 Exam 1 Autumn 2005

Physics 324 **First Midterm Exam** **Autumn 2005**

NAME (print legibly)

This is an open book exam. In addition you are permitted to use one sheet of notes and a calculator. Show all of your work to receive full credit. The exam is worth 50 pts (1 pt/min).

Part I. (20 pts) A particle of mass, m, is placed in an infinite square well potential: $V(x) = \infty$ for $x < 0$ and $x > L$, and $V(x) = 0$ for $0 \le x \le L$. The particle is prepared into an initial state (at time $t = 0$) given by:

$$\Psi(x,0) = Nx \quad \text{for} \quad 0 \le x \le L$$

In what follows you might find it useful to know that

$$\int x\sin(nx) = \frac{1}{n^2}\sin(nx) - \frac{x}{n}\cos nx.$$

1. (4 pts) Normalize $\Psi(x,0)$ to find N.

2. (3 pts) What is the expectation value of the particle's position, $\langle x\rangle$, at time $t = 0$?

3. (3 pts) Will $\langle x\rangle$ change with time? Explain.

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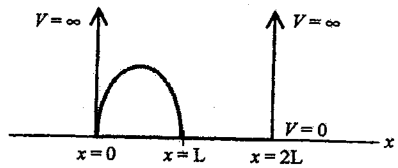
Part II. (30 points) NAME: ________________

A particle is in an infinite square well of width 2L with potential $V(x) = 0$ for $0 \le x \le 2L$, and infinite elsewhere. At time $t = 0$, it is described by this initial wave function:

$$\psi(x,0) = Ax(L-x) \quad 0 \le x \le L$$
$$= 0 \quad L \le x \le 2L$$
$$= +\infty \quad \text{elsewhere}$$

The figure shows this initial wave function superimposed on the potential well.



1. (6 points) Normalize $\psi(x,0)$ to find the value of A in terms of L.

It should be possible to expand this function in the eigenstates of the infinite square well, $\varphi_n = [1/L]^{1/2} \sin\{n\pi x/(2L)\}$, as follows: $\psi(x,0) = \sum c_n\, [1/L]^{1/2} \sin\{n\pi x/(2L)\}$, where the sum runs from $n = 1$ to $n = \infty$.

2. (4 pts) Use Fourier's trick to write an integral expression for the coefficients c_n.

3. (12 pts) Evaluate the c_n by performing the integration with the help of the integrals listed on the front page of this exam. Hint: as the <u>last</u> step, separate into n = odd and n = even solutions.

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Part II (continued) NAME: ____________________

4. (3 pts) Estimate the value of $< x >$ at time $t = 0$ using the figure on page 3.

5. (5 pts) Would you expect the value of $< x >$ to change with time? Explain your reasoning.

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Part II. (15 pts) A particle of mass m is moving in a harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2x^2$. Recall that for the harmonic oscillator we can write the position and the momentum operator in terms of the ladder operators:

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-), \quad p = i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-), \quad \text{with} \quad a_+\psi_n = \sqrt{n+1}\psi_{n+1}, \quad a_-\psi_n = \sqrt{n}\psi_{n-1}$$

where

$$H\psi_n = E_n\psi_n, \quad E_n = \hbar\omega(n+\frac{1}{2}), \quad n = 0, 1, 2, \ldots \quad \text{and} \quad \int dx\, \psi_n^*\psi_m = \delta_{nm}.$$

At $t = 0$ the system in prepared in an intial state

$$\Psi(x, 0) = \frac{1}{\sqrt{2}}(\psi_1(x) + \psi_2(x))$$

1. (6 pts) What is the expectation value of the particle's position, $\langle x \rangle$, at time $t = 0$? (*Hint:* Use the ladder operators!)

2. (5 pts) What is the expectation value of the particle's momentum, $\langle p \rangle$, at time $t = 0$? (*Hint:* Use the ladder operators!)

3. (4 pts) Are $\langle x \rangle$ and $\langle p \rangle$ evolving with time? Explain.

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3. (*10 pts*) (*no partial credit*)
Alice and Bob are members of a security agency and want to quickly trap an eavesdropper, Eve. Alice decides to send only photons designating a "1" bit using a 45° Polaroid sheet. Bob, as usual, uses, at random, Horizontal and −45° Polaroid sheets.

a) (*5 pts*) What fraction of the photons does Bob detect?

fraction= $0 \quad \frac{3}{16} \quad \frac{1}{2} \quad \frac{1}{4} \quad \frac{3}{8} \quad \frac{5}{8}$.

b) (*5 pts*) The eavesdropper, Eve, always uses a Horizontal sheet.
What fraction of the photons sent by Alice does Bob detect?

fraction= $0 \quad \frac{3}{16} \quad \frac{1}{2} \quad \frac{1}{4} \quad \frac{3}{8} \quad \frac{5}{8}$.

4. (*10 pts*) A beam of spin $\frac{1}{2}$ particles is a **mixture** with 2/3 of the particles having spin up and 1/3 spin down.
Another beam of particles is described by

$$\chi_0 = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$$

Are the two beams identical? If so, give a convincing argument.
If not, propose an experiment with Stern-Gerlach device with one blocked exit to distinguish between them.

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Problem 6. (30 pts) A particle is in a 1-dimensional harmonic oscillator potential, $V(x) = m\omega^2 x^2/2$. At time $t = 0$, the particle is described by the following wavefunction:

$$\Psi(x,0) = \frac{(1+i)}{\sqrt{10}}\Phi_0(x) + \frac{(2-2i)}{\sqrt{10}}\Phi_3(x)$$

where $\Phi_n(x)$ are the spatial energy eigenfunctions that satisfy the time independent Schrodinger equation: $\hat{H}\Phi_n(x) = E_n\Phi_n(x)$.

A.)

i.) (3 pts) Are $\Phi_n(x)$ also eigenstates of momentum? Explain. (Hint, momentum eigenfunctions have the form $e^{\pm ipx/\hbar}$.

ii.) (3 pts) Write an expression for $\Psi(x,t)$, the wavefunction at a later time, t.

iii.) (3 pts) If the energy of the particle were measured, what is the probablity that the value E_0 would be measured?

iv.) (4 pts) What is the expectation value for the energy at time $t = 0$?

v.) (5 pts) Find the probability density, $\rho(x,t)$, for finding the particle at position x at time t. Express your answer in terms of the spatial wavefunctions $\Phi_n(x)$.

Physics 324 Final Exam Autumn 2002 Page 2

Problem 6, continued from the previous page

B.) At some time, $t' > 0$, the energy of the particle is measured to be E_3.

i.) (4 pts) Write down the normalized wavefunction for the particle, $\Psi(x,t)$, for times $t > t'$ (*i.e.* after the measurement) in terms of the spatial wavefunctions, $\Phi_n(x)$.

ii.) (4 pts) After the measurement, is the probability density, $\rho(x,t)$, time-dependent? Explain.

C.) At an even later time, $t'' > t'$, the **momentum** of the particle is measured.

i.) (4 pts) After this momentum measurement is made, can the energy of the particle be represented by a single number (*i.e.* is the particle in an energy eigenstate)? Explain.

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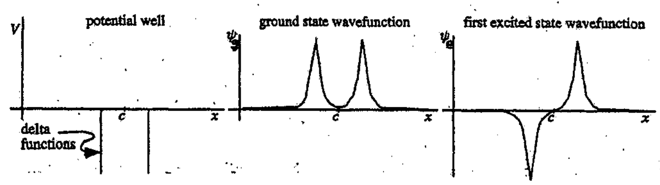
PART III: (15 pts)

A particle is placed in the double delta function potential shown below. The two delta functions are close enough together to allow tunneling of the particle between the wells. Point c lies halfway between the delta functions. (Recall that the attractive delta function potential has only one bound state, leading to the ground and excited state wave functions shown below.)



A. (5 pts) If the position of particle is measured to be to the left of point c, then at a later time, which of the following statements are true? Explain.

a.) The particle may be in the ground state.
b.) The particle may be in the excited state.
c.) The particle will be in a superposition of the ground and excited state.
d.) The expectation value of the particle's position will not change with time.

B. (10 pts) Now two non-interacting identical fermions are placed in the double delta function potential shown above. At $t = 0$ the two-particle wave function has a spatial part given by: $[\psi_g(x_1)\psi_e(x_2) - \psi_g(x_2)\psi_e(x_1)]/\sqrt{2}$, and a total spin of 1.

~~1. (5 pts) As time elapses, will the probability of finding both particles to the left of point c change or remain~~ *the same?* Explain.

2. (5 pts) At $t = 0$ a measurement of the total energy of the system is made. Which of the following is true? Explain.

a.) Only one value is possible for the energy of the system.
b.) There are two possible values for the energy of the system.
c.) There are many possible values for the energy of the system.

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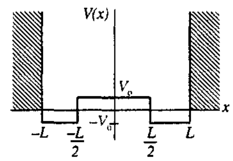
Name ______________________________

Part III. [16 pts.]

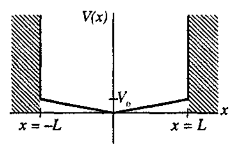
A. [6 pts] Treat the potential well shown as a small perturbation to the infinite square well.



1. Would the first order correction to the energy of the ground state be *positive, negative,* or, *zero?* Explain.

2. Would the first order correction to the energy of the first excited state be *positive, negative,* or, *zero?* Explain.

B. [7 pts] A particle is in the potential well shown at right. This well is only slightly perturbed from the infinite square well. At time $t = 0$ s the particle's wave function is given by



$$\psi(x) = \sqrt{1/L}\,\sin\left(\frac{\pi x}{L}\right).$$

1. If a measurement of the particle's energy were made at time $t = 0$ s, which of the following would be true?
   - There is one particular value of the energy that could be measured. It is equal to E_o, the ground state energy of the unperturbed well.
   - There is one particular value of the energy that could be measured. It is greater than E_o, the ground state energy of the unperturbed well.
   - There is one particular value of the energy that could be measured. It is less than E_o, the ground state energy of the unperturbed well.
   - There is not just one particular value of the energy that might be measured. Instead there are more than one possible values that might be obtained from a measurement of energy.

   Breifly explain.

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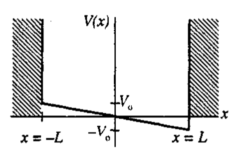
2. Suppose instead that the energy of the particle in Part B1 had been measured at a time $t > 0$ s. Which of the following is true?

   - The number of possible values that might be obtained for a measurement of the energy is greater than it was in part B1.
   - The number of possible values that might be obtained for a measurement of the energy is less than it was in part B1.
   - The same value or values for the energy are possible as in part B1 and they have the same relative probabilities.
   - The same value or values for the energy are possible as in part B1 but they have different relative probabilities.

   Explain your choice.

C. [3 pts] Which of your answers in part B1, if any, would change if the perturbed well were instead shaped as shown at right? Explain.



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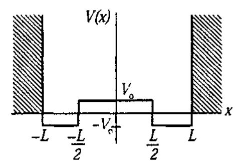
Name ______________________________

Part III. [15 pts.]

A. [6 pts] Treat the potential well shown as a small perturbation to the infinite square well.

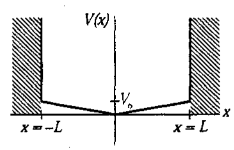


1. Would the first order correction to the energy of the ground state be *positive, negative,* or, *zero?* Explain.

2. Would the first order correction to the energy of the first excited state be *positive, negative,* or, *zero?* Explain.

B. [6 pts] A particle is in the potential well shown at right. This well is only slightly perturbed from the infinite square well. At time $t = 0$ s the particle's wave function is given by

$$\psi(x) = \sqrt{\frac{1}{L}} \cos(\pi x / 2L)$$



1. If a measurement of the particle's energy were made at time $t = 0$ s, which of the following would be true?
   - There is one particular value of the energy that could be measured. It is equal to E_o, the ground state energy of the unperturbed well.
   - There is one particular value of the energy that could be measured. It is greater than E_o, the ground state energy of the unperturbed well.
   - There is one particular value of the energy that could be measured. It is less than E_o, the ground state energy of the unperturbed well.
   - There is not just one particular value of the energy that might be measured. Instead, there are many possible values that might be obtained from a measurement of energy.

   Briefly explain.

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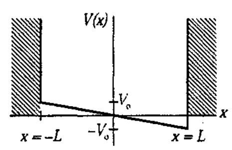
2. Suppose, instead, that the energy of the particle in Part B1 had been measured at a time $t > 0$ s. Which of the following is true? **Circle all that apply.**

   - Just after $t = 0$, the number of possible values that might be obtained from a measurement of the energy is greater than it was in part B1.
   - Just after $t = 0$, the number of possible values that might be obtained from a measurement of the energy is less than it was in part B1.
   - Just after $t = 0$, the same value or values for the energy are possible as in part B1 and they have the same relative probabilities.
   - Just after $t = 0$, the same value or values for the energy are possible as in part B1 but they have different relative probabilities.
   - Only one value for the energy is possible regardless of the time at which the measurement is made.
   - Multiple values for the energy are possible regardless of the time at which the energy measurement is made.
   - A long time after $t = 0$, only one value for the energy is possible.
   - A long time after $t = 0$, multiple values for the energy are possible.

   Explain your choice(s).

C. [3 pts] Which of your answers in part B1, if any, would change if the perturbed well were instead shaped as shown at right? Explain.



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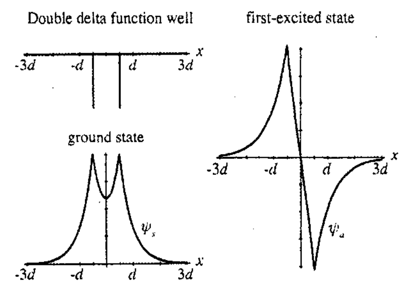
Part III. [22 Pts.]

Consider a system with the potential shown at right consisting of two delta function wells separated by a distance, *d*, centered about the origin. The symmetric ground state wave function, ψ_s, and the anti-symmetric first excited state wave function, ψ_a, for this system are also shown at right. Assume that the energy of the anti-symmetric first excited state is four times the energy of the symmetric ground state, $E_a = 4E_s$.



The system is initially in a state given by

$$\Psi(x, t=0) = \frac{1}{\sqrt{2}}[\psi_s + i\psi_a]$$

1. [5 pts.] Is the initial probability of finding the particle to the left of the origin *greater than, less than,* or *equal to* the probability of finding the particle to the right of the origin? Explain.

2. [3 pts.] Is this state a stationary state? Explain how you know for full credit.

3. [4 pts.] Is there a time when the wave function is equal to: (i) the ground state wave function, ψ_s? (ii) the first excited state wave function, ψ_a? If so, give a time when this is true? Give your answers in terms of the variables given or describe a limiting case. Explain.

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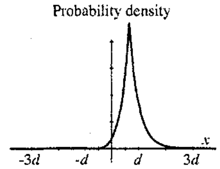
4. [3 pts.] Is there a time when the probability distribution is given by a flat line? If so, give a time when this is true? Give your answer in terms of the variables given or describe a limiting case. Explain.

5. [7 pts.] Is there a time when the probability distribution looks like the graph at right? If so, give a time when this is true? Give your answer in terms of the variables given or describe a limiting case. Explain. (*Hint:* Be mindful of the initial state given above.)



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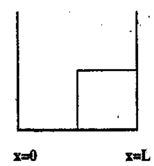
Physics 324 **First Midterm Exam** **Autumn 2006**

NAME (print legibly)

This is an open book exam. In addition you are permitted to use one sheet of notes and a calculator. Show all of your work to receive full credit. The exam is worth 50 pts (1 pt/min).

Part I. (20 pts) A particle of mass, m, is placed in an infinite square well potential: $V(x) = \infty$ for $x < 0$ and $x > L$, and $V(x) = 0$ for $0 \le x \le L$. At time $t = 0$ the particle is prepared in an initial state given by:

$$\Psi(x,0) = \begin{cases} 0 & \text{for } x < \frac{L}{2} \\ \sqrt{\frac{2}{L}} & \text{for } \frac{L}{2} \le x \end{cases}$$



1. (4 pts) What is the expectation value of the particle's position, $\langle x \rangle$, at time $t = 0$?

2. (4 pts) Will $\langle x \rangle$ change with time? Explain.

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Part I. (continued)

3. (4 pts) If the energy is measured and is found to be $\mathbf{E_2}$, the second energy eigenstate, and then the position of the particle is measured, at what position(s) is the particle most likely to be found ? Explain.

4. (4 pts) After the measurement that yielded energy E_2 has been performed, the system is left alone for a long time. Qualitatively, how does the probability density of the particle evolve during that time?

5. (4 pts) If the potential energy of the bottom of the square well was V_0, rather then 0, what would be the energy of the ground state of the particle in this new well.

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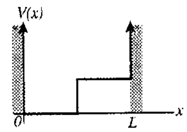
Part IV. (26 pts)



Consider the one-dimensional potential shown at right with a step between 0 and L and infinite potential outside. A system with this potential is initially prepared in a state given by: $\Psi(x,t=0)=\sqrt{\frac{2}{L}}\sin(2\pi x/L)$ between 0 and L and zero elsewhere.

1. (6 pts.) At t = 0, is the expectation value of the position, $\langle x\rangle$, *greater than, less than,* or *equal to L/2*? Explain.

2. (4 pts.) Does the wave function, $\Psi(x,t)$, change as time passes? Explain.

3. (4 pts.) Does the probability density, $\rho(x,t)$, change as time passes? Explain.

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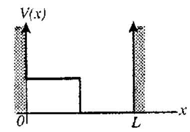
Now consider a system with an identical initial wave function, $\Psi(x, t=0)$; but the potential shown at right.



4. (6 pts.) At $t = 0$, is the probability of measuring the position to be $L/4$ in the system with the new potential *greater than, less than,* or *equal to* the probability of measuring the position to be $L/4$ in the system with the previous potential? Explain.

5. (6 pts.) At $t = 0$, is the probability of measuring the momentum to be to the left with magnitude $2\pi/\hbar$ in the system with the new potential *greater than, less than,* or *equal to* the probability of measuring the momentum to be to the left with magnitude $2\pi/\hbar$ in the system with the previous potential? Explain.

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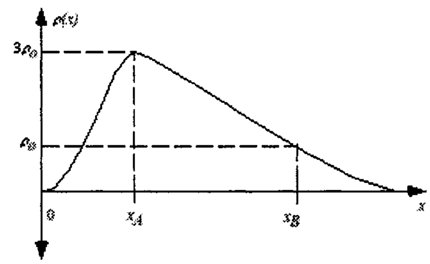
Part III. [15 pts.]

The normalized probability distribution for an object of mass, m, with total energy $19U_o$ is given at right. The probability density is zero outside the region shown.



In answering the questions below, treat the problem classically, not quantum mechanically.

A. [3 pts.] What is the probability that the object is located exactly at the point $x = x_A$?

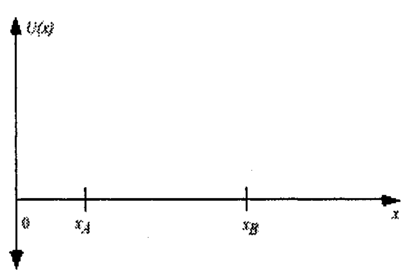
B. [3 pts.] What is $\rho(x_A)$, the probability density exactly at the point $x = x_A$? Discuss how your answers in parts A and B are related.

C. [3 pts.] Clearly mark on the graph the approximate location, x_m, where the probability of finding the object to the left of the location is equal to the probability of finding the object to the right (*i.e.*, the probability is 50% on either side). Explain.

D. [3 pts.] From the graph of probability density, determine whether the expectation value of the position, $\langle x \rangle$, is *greater than* or *less than* x_m, the value that you found in part C. Explain.

E. [3 pts.] Draw a qualitatively correct graph of potential energy, $U(x)$, versus x.

Let $U(x_B) = U_o$. Label the values of $U(x_A)$ and $U(x_B)$ (in terms of U_0) on your graph.



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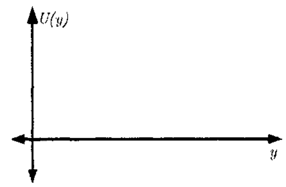
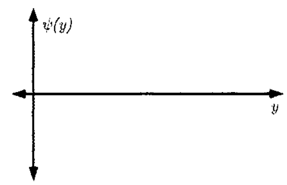
Part III. (15 pts) Consider a ball dropped from a height, h, making elastic collisions with the floor and bouncing back to the same height repeatedly. (Assume the collisions with the floor take place over a negligible distance and time.)

1. (4 pts) Sketch a graph of the potential energy, $U(y)$, on the axes at right for this situation where y is the vertical position above the floor. Indicate the total energy on your sketch as well as the classical turning point(s).



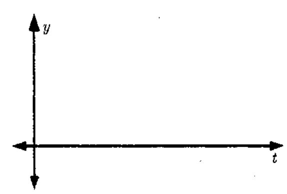
2. (7 pts) Now consider the quantum mechanical analogue of this system. Sketch a highly excited energy eigenstate for this potential well. Describe, explain, and justify in words any variations that your graph exhibits in:



amplitude

wave number

3. (4 pts) On the axes below sketch and clearly label qualitatively correct graphs of the expectation value of the position as a function of time for both a highly excited energy eigenstate, and for a low energy eigenstate for this well. Label these $<y>_{high}$ and $<y>_{low}$. Explain.

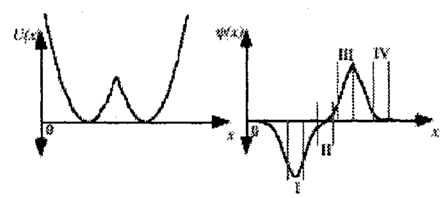


Arizona State University Spring 2003

Name ____________________

A. A particle is in an energy eigenstate of the potential well shown below. The corresponding wave function, $\psi(x)$, is given at far right.

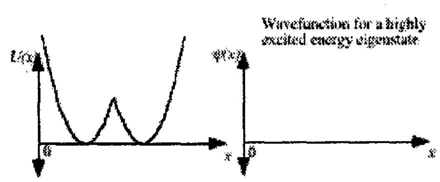
1. Does this wave function represent the particle in the classical or the quantum mechanical regime? Explain.



2. Four regions (I–IV) of equal width are marked on the graph. Rank the regions in order of decreasing probability of finding the particle in that region. State explicitly if any of these probabilities are zero. Explain your ranking.

B. Now consider a highly excited energy eigenstate of this same potential well.

1. Draw the wave function for the particle. Briefly describe the important features of your graph, and explain why you drew them as you did.



2. On the graph of $U(x)$ in part B, consider two small regions of equal width both within the classically allowed region. Region I is near the classical turning point on the right side of the potential, and region II is near the minimum on the right side of the potential. In which region is the particle more likely to be found? Explain.

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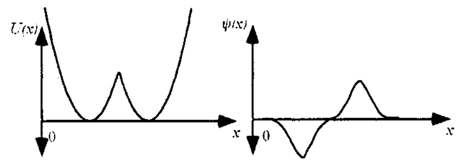
Name ___________________

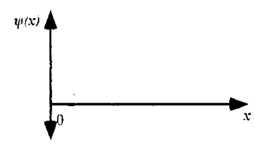
Part III. [15 pts.]

A. Consider the potential well shown. A wave function, $\psi(x)$, for a particle in this well is shown at far right.

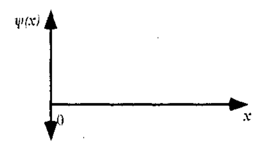


1. Does this wave function represent the particle in the classical or the quantum mechanical regime? Explain.

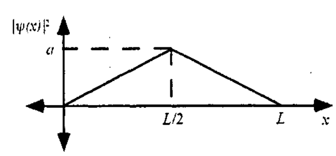
2. Draw the wave function for a particle in a highly excited energy eigenstate of this well. Describe the important features of your graph, and explain why you drew them as you did.



3. Suppose the particle in part 1 were replaced by a particle with greater mass but the same total energy. (Assume the particle is in a bound state.) How, if at all, would the wave function shown in part 1 change? In either case, sketch a possible wave function of the particle with greater mass.



B. A probability density, $|\psi(x)|^2$, is shown at right. Here a and L are real numbers not under your control.



Either:

explain why $\psi(x)$ is normalizable, and normalize it by constructing a new wave function, $\varphi(x)$, from $\psi(x)$ where $\varphi(x)$ is normalized;

or

explain why $\psi(x)$ is not normalizable.

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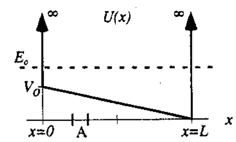
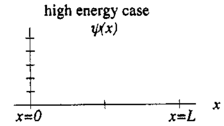
Part III. [30 pts.]

Consider a particle of mass m bound in the potential well shown at right. The energy of the ground state, E_o, is marked on the graph.

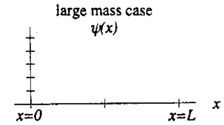


1. Write down the Schrödinger equation for this potential. (Note: You need not solve it here or in what follows.)

2. Sketch the wave function for a very high energy eigenstate of this particle (i.e., an eigenstate with energy much much greater than E_o). Describe the important features of your graph, and explain why you drew them as you did.



3. The particle of mass m is now replaced by a particle with a much larger mass. The new particle is in an energy eigenstate that happens to have the same energy, E_o, as the original particle's ground state. Sketch the wavefunction for this particle. Describe the important features of your graph, and explain why you drew them as you did.



4. Do the wavefunctions you have drawn above look identical? Explain why they are similar or why they are different.

5. Is the probability of finding the particle in question 2 within the region marked A *greater than, less than*, or *equal to* the probability of finding the particle in question 3 within the region marked A? Explain.

Physics 324 Exam 1 Autumn 2004

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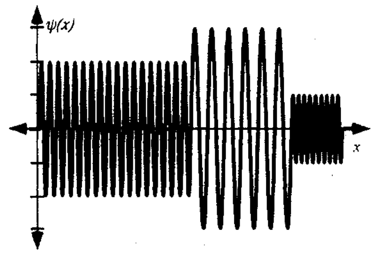
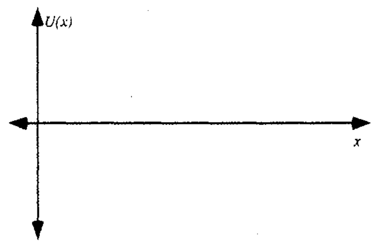
last first

A. The real part of a highly excited energy eigenstate for a quantum mechanical system is shown at right.

In the space provided, sketch a potential and a total energy that are consistent with this wave-function. Use the same horizontal scale, and clearly show relative lengths on your vertical scale.

Clearly label your graphs, and explain the reasoning you used to draw them.

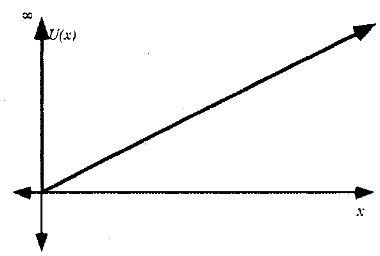




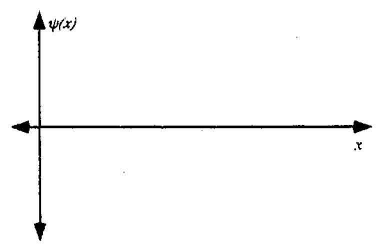
B. Consider a particle in a highly excited energy eigenstate of the potential given at right.

$$U(x) = \begin{cases} \infty, x < 0 \\ +\beta x, x > 0 \end{cases}$$

On the graph of *U(x)* indicate the classical turning point(s).



In the space at right, sketch a qualitatively correct graph of the wave function. Use the same horizontal scale used in the graph of *U(x)*. Explain the reasoning you used to draw your graph.



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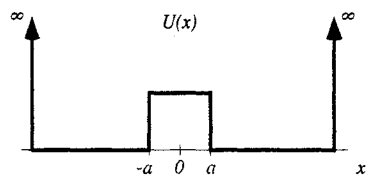
Name ______________________

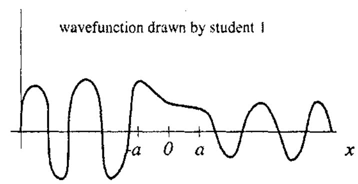
Part III. [30 pts.]

A. [10 pts.] A potential well with infinite sides is shown at right. Two students have drawn wavefunctions for a particle bound in this well. (see below)



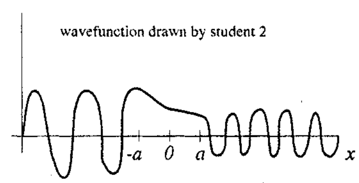
1. Could the sketch drawn by student 1 represent a valid wavefunction in this well?

   If so, explain your reasoning and state whether the probability density associated with this wavefunction would change with time. If not, explain why not.



2. Could the sketch drawn by student 2 represent a valid wavefunction in this well?

   If so, explain your reasoning and state whether the probability density associated with this wavefunction would change with time. If not, explain why not.



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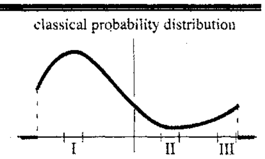
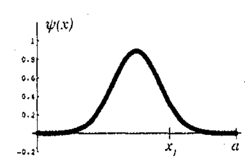
B. [10 pts.] The classical time averaged probability distribution for a particle moving in an unknown potential well is shown at right. Consider quantum mechanically a particle in a highly excited energy eigenstate of the same well (*i.e.*, a particle in the well with an energy much greater than that of the ground state).



1. Rank the potential in the regions marked I, II, and III from largest to smallest. (*Note:* Each region has the same width.) Explain.

2. Rank the number of nodes in the wave function within the regions marked from largest to smallest. Explain.

C. [10 pts.] A wavefunction in an infinite square well of width a is shown at right. The wavefunction is entirely real at this instant in time.



1. Give an approximate value for $\langle x_i|\psi\rangle$. Explain.

2. Would this value change with time? Explain.

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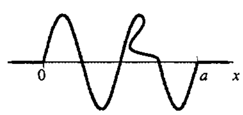
Name ______________________

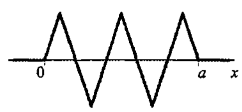
Part III. [25 pts.]

A. [12 pts.] For each graph shown below answer the following two questions. Could the graph possibly represent the real part of a wave function for a particle in an infinite square well of width *a*? If so, would a measurement of the particle's energy yield only one possible value, or one of many possible values? Explain briefly.

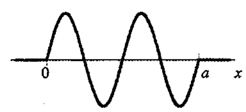


Possible or impossible? One or many?

Possible or impossible? One or many?



Possible or impossible One or many?



Possible or impossible? One or many?

B. [4 pts.] Can the wave function for an eigenstate of a particle in the infinite square well at time t = 0 be: purely real? purely imaginary? complex with non-zero real and imaginary parts? Explain.

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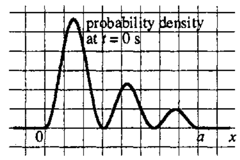
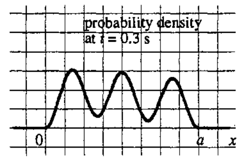
C. [9 pts.] The probability density for a particle in an infinite square well is shown at $t = 0$ s and at $t = 0.3$ s.

1. Is the particle in an eigenstate? Explain.

2. Estimate the probability current at $x = a/4$ and $t = 0.15$ s. Explain.





Physics 325 Exam 1 Winter 2005

Name ______________________________ *Student ID* __________ *Score* ______
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A. An electron has the following wavefunction
$\Psi(r,\theta,\varphi) = f(r)[(\sqrt{1/6})Y_3^0(\theta,\varphi)\,\chi_+ + (\sqrt{2/6})Y_3^1(\theta,\varphi)\,\chi_-] + g(r)(\sqrt{3/6})Y_2^0(\theta,\varphi)\,\chi_+$.

Here $f(r)$ and $g(r)$ are normalized functions of the radial coordinate r only. The $Y_l^m(\theta,\varphi)$ are the normalized spherical harmonics and the χ_i are the eigenspinors of S_z

1. Suppose you measured the z-component of the orbital angular momentum (L_z). What values might you get, and what is the probability of each? Explain.

Let **J** = **L** + **S** be the total angular momentum of the electron.

2. Suppose you measured the total angular momentum squared (J^2). What values might you get? Explain.

3. Suppose you measured the z-component of the total angular momentum (J_z). What values might you get? Explain.

B. A beam of electrons with spin up in the z-direction enters a Stern-Gerlach apparatus oriented in the x-direction. The electrons that emerge with spin $+\hbar/2$ in the x-direction have their spin in the x-direction measured again a long time after the initial measurement. What values for the spin are possible? If a range of values is possible, specify that range. Explain.

C. An electron with spin up in the z-direction enters a Stern-Gerlach apparatus oriented in the y-z plane at a 45 degree angle to the z-axis. The apparatus measures the spin along that axis. What values for the spin are possible? If a range of values is possible, specify that range. Explain.

Physics 325 Exam 1 Winter 2007

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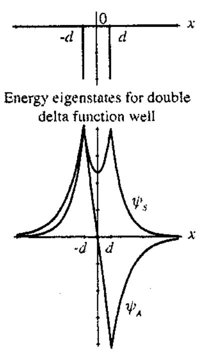
Part III. (13 pts) Bosons in a double delta function well

Consider two identical spin one bosons in the one-dimensional double delta function potential well shown at right. The delta functions are separated by a distance $2d$ as shown. Mathematically, the potential is described by: $V(x) = -\alpha\delta(x+d) - \alpha\delta(x-d)$. The two bound states for a single boson, one symmetric and the other anti-symmetric, are also shown at right. The energy of each state is E_S and E_A respectively where $E_A > E_S$.

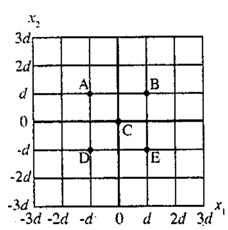
Double delta function well

Energy eigenstates for double delta function well



Consider an energy eigenstate for the two-particle system with energy $E_S + E_A$, and a spin state given by: $\frac{1}{\sqrt{2}}\{|1,0> - |0,1>\}$ in the basis of the z-components of the individual angular momenta.

1. (9 pts) For the above state, rank the values of $|\Psi(x_1,x_2)|^2$ at the points A, B, C, D, and E shown on the graph below. Explain your ranking.



2. (4 pts) For the above state, rank the values of $|\Psi(x_1,x_2)|^2$ at the points A, B, C, D, and E shown on the previous graph after a time $t = \frac{2\pi\hbar}{E_A}$ has elapsed. Explain your ranking.

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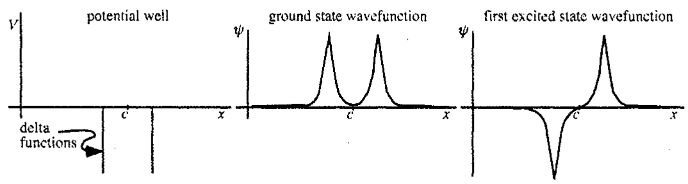
Name ____________________________

Part IV.

A particle is placed in the double delta function potential shown below. Point c lies halfway between the delta functions.



Consider an ensemble of many identical systems s_1, s_2, s_3, s_4, and etc. each in the **ground state** of this potential.

A. If you were to measure the position of the particle in each of the systems in the ensemble, in approximately what fraction, P_L, of the systems would you expect to find the particle to the left of point *c?* Explain.

B. Suppose instead that you measured the position of the particle in only the first three systems, and found the particle to the left of point *c* in all three cases.

   Is the probability of finding the particle to the left of point *c* in the fourth system *greater than, less than*, or *equal to* P_L? Explain.

C. Now assume that measurements of position have been made on all of the systems in the ensemble. Consider the subset of the ensemble consisting only of those systems in which the particle was found to the left of point *c*.

   If you wait a long time after making those measurements, which of the following would be true about the systems in this subset:

   (a) Some of those systems would be in the ground state and some in the first excited state.
   (b) All of those systems would be in the ground state.
   (c) All of those systems would be in the first excited state.
   (d) Other

   Explain.

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D. Two non-interacting fermions are placed now placed in the potential well above. At $t = 0$, the two-particle wave function has spatial part equal to $\frac{1}{\sqrt{2}}(\psi_g(x_1)\psi_e(x_2) - \psi_g(x_2)\psi_e(x_1))$ and total spin of 1.

(ψ_g and ψ_e are the ground and first excited state wavefunctions shown above.)

1. As time elapses will the probability of finding both particles to the left of point *c* *change* or *remain the same?* Explain.

2. At t = 0, a measurement of the energy of the system is made. Which of the following is true?

   - Only one value is possible for the energy of the system.
   - There are two possible values for the energy of the system.
   - There are many possible values for the energy of the system.

   Explain.

Physics 325 Exam 2 Winter 2003

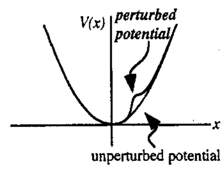
Name:______________________________ **Total:**______

1. **XX pts. total**

   For all parts of this problem, consider a particle in a one dimensional harmonic oscillator (*i.e.*, $V(x) = \frac{1}{2}m\omega^2x^2$).

   (a) Suppose that the oscillator were perturbed as shown. Would each of the following quantities be *positive*, *negative*, or *zero*? Explain.

   i. the first-order correction to the energy of the ground state



   ii. the first-order correction to the energy of the first excited state

   (b) Suppose instead that the perturbation were a delta function centered at the origin (*i.e.*, $H'(x) = \beta\delta(x)$).

   i. Calculate the ground state energy *correct to first order*. Show your work. (*Hint:* $\psi_o(x) = (\frac{m\omega}{\pi\hbar})^{\frac{1}{4}}e^{-\frac{m\omega}{2\hbar}x^2}$.)

   ii. What is the lowest energy state that has a first order energy correction equal to zero? Explain.

   iii. Would the second order correction to the energy of this same state be *positive*, *negative*, or *zero*? Explain.

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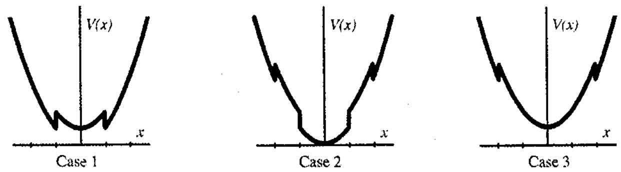
Part III. (15 pts) Time independent perturbation theory
Consider the perturbation to the harmonic oscillator shown in each case below.



$$H_1'(x) = \begin{cases} 0 & x < -1 \\ \epsilon & -1 \le x \le 1 \\ 0 & x > 1 \end{cases} \quad H_2'(x) = \begin{cases} 0 & x < -2 \\ \epsilon & -2 \le x \le -1 \\ 0 & -1 < x < 1 \\ \epsilon & 1 \le x \le 2 \\ 0 & x > 2 \end{cases} \quad H_3'(x) = \begin{cases} 0 & x < -2 \\ \epsilon & -2 \le x \le 2 \\ 0 & x > 2 \end{cases}$$

1. (6 pts) For each case, is the first-order correction to the ground state energy *positive, negative,* or *zero?* Explain. (Assume ϵ is positive.)

   Case 1:

   Case 2:

   Case 3:

2. (3 pts) Rank the absolute value of the first-order correction to the ground state energy in each case from greatest to least. Explain.

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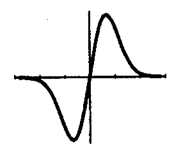
last first

Consider two systems: (1) an unperturbed harmonic oscillator, and (2) an oscillator with the same potential as case 2 above. In each case assume the initial state of the system is given by the first excited state of the unperturbed harmonic oscillator, $\Psi(x, t=0) = (\frac{4M^3\omega^3}{\pi\hbar^3})^{1/4} x e^{-\frac{M\omega}{2\hbar}x^2}$.



$\Psi(x,0) = (\frac{4M^3\omega^3}{\pi\hbar^3})^{1/4} x e^{-\frac{M\omega}{2\hbar}x^2}$

3. (3 pts) Compare and contrast the time-evolution of the probability density associated with the two systems above.

4. (3 pts) Compare and contrast the time-evolution of the possible values for energy measurements made on the two systems above.

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Name ______________________________ *Student ID* __________ *Score* ______
last   first

V. [20 pts.]

Consider a one dimension harmonic oscillator. A long time before $t = 0$, the energy of the oscillator is measured and found to be that of the first excited state, E_1. Immediately after time $t = 0$, a small perturbation is applied in the form $H' = \varepsilon x$, where ε is a small parameter with the correct units. Just before a time equal to one-fourth of the period of the oscillator (*i.e.*, $t = \pi/2\omega$), the perturbation is turned off. In what follows, E_i are the eigenstates of the unperturbed harmonic oscillator.

$$H'(t) = \begin{cases} 0; & t \le 0 \\ \varepsilon x; & 0 < t < \pi/2\omega \\ 0; & t \ge \pi/2\omega \end{cases}$$

A. [5 pts.] Rank the probabilities that a measurement of the energy will yield E_o at the following times: $t = 0$, $t = \pi/4\omega$, $t = \pi/2\omega$, and a time long after the perturbation is finished. Use the following symbols: $P_o(0)$, $P_o(\pi/4\omega)$, $P_o(\pi/2\omega)$, $P_o(\infty)$. If any probability is zero or one, state so explicitly. Use only first-order time-dependent perturbation theory. Explain.

B. [5 pts.] Rank the probabilities that a measurement of the energy will yield E_2 at the following times: $t = 0$, $t = \pi/4\omega$, $t = \pi/2\omega$, and a time long after the perturbation is finished. Use the following symbols: $P_2(0)$, $P_2(\pi/4\omega)$, $P_2(\pi/2\omega)$, $P_2(\infty)$. If any probability is zero or one, state so explicitly. Use only first-order time-dependent perturbation theory. Explain.

C. [5 pts.] Rank the probabilities that a measurement of the energy will yield E_3 at the following times: $t = 0$, $t = \pi/4\omega$, $t = \pi/2\omega$, and a time long after the perturbation is finished. Use the following symbols: $P_3(0)$, $P_3(\pi/4\omega)$, $P_3(\pi/2\omega)$, $P_3(\infty)$. If any probability is zero or one, state so explicitly. Use only first-order time-dependent perturbation theory. Explain.

D. [5 pts.] Consider the system described above and an identically prepared system to which no time-dependent perturbation is applied. After waiting a very long time is the probability of finding the system with the perturbation in the ground state, $P_o(\infty)_{pert}$ *greater than, less than,* or *equal to* the probability of finding the system without the perturbation in the ground state, $P_o(\infty)_{no\text{-}pert}$. If either probability is zero or one, state so explicitly. Explain.

Vita

Andrew Dale Crouse was born on the sixteenth of October 1972 in New Orleans, Louisiana. Adopted shortly thereafter by Dale and Jeanette Crouse, he was raised in Belleville, Illinois, and graduated from high school there. In 1994, he received his undergraduate degree in Engineering Physics from The University of Illinois at Urbana–Champaign. He then joined the Peace Corps where he taught physics and mathematics in Sigatoka, Fiji. Upon his return to the United States, he began graduate study in physics at Brandeis University in Waltham, Massachusetts, where he earned an M.A. in 1998. Afterwards, he again taught mathematics and physics internationally in Caracas, Venezuela, before returning to school to earn an M.S. in Physics at the University of Washington in 2003. In the summer of 2004, he married Nicole Fedio. They had their first child, Ava, in 2006.