Applications of the Cyclic Sieving Phenomenon to Words, Branching Rules, and Tableaux

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Abstract

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Since Reiner-Stanton-White [RSW04] defined the cyclic sieving phenomenon associated to a finite cyclic group action and a polynomial, many compelling examples of cyclic sieving have been found. In this thesis, we focus on what we can accomplish using these known cyclic sieving phenomena as tools. We seek to show that the cyclic sieving phenomenon is more than just a coincidence.

There is a natural notion of *refinement* for many CSP's. We formulate and prove a refinement of a famous CSP on words of fixed content by also fixing the *cyclic descent type*. The argument presented is completely different from Reiner-Stanton-White's representation-theoretic approach. Instead, we build up the words of fixed content and cyclic descent type through a branching algorithm and use it to derive major index generating functions. We also prove a refinement of cyclic sieving on shifted subset sums by extending some ideas of Wagon-Wilf [WW94].

We show that the cyclic sieving phenomenon of Reiner–Stanton–White together with necklace generating functions arising from work of Klyachko offer a remarkably unified and direct approach to a series of results due to Kraśkiewicz–Weyman, Stembridge, and Schocker related to the so-called higher Lie modules and branching rules for inclusions $C_a \wr S_b \hookrightarrow S_{ab}$. Our arguments are bijective except for the use of the cyclic sieving phenomenon. Extending

this approach gives monomial expansions for certain graded Frobenius series arising from a generalization of Thrall's problem.

Following a conjecture of Dennis White, Brendon Rhoades proved an interesting cyclic sieving phenomenon for the action of promotion on rectangular standard Young tableaux [Rho10a, Theorem 1.3], which gives an effective enumeration formula for the fixed point sizes of the powers on promotion of rectangular Young tableaux. However, currently unknown are the actual fixed point sets counted by Rhoades's formula. In this thesis, we construct all of the sufficiently large rectangular tableaux fixed by powers of promotion. To do so, we introduce tableau stabilization and prove some of its interesting properties. We also present open problems related to tableau stabilization.

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GLOSSARY

: $\alpha \vDash n$: α is a composition of n

: $a_{\lambda,r} = \#\{Q \in SYT(\lambda) : \mathrm{maj}(Q) \equiv_n r\}$

: CDes: the cyclic descent set on words

: cdes: the number of cyclic descents

: χ^r: the representation of C_n given by $\chi^r(\sigma_n) = \omega_n^r$.

: cont: the content statistic on words or tableaux

: $c_{\lambda,\mu}^{\nu}$: the Littlewood-Richardson coefficient.

: C_n: the cyclic group of order n, often embedded in S_n as generated by $\sigma_n = (1\,2\,\ldots\,n)$

: Des: the descent set on words or tableaux

: des: the number of descents

: $e_n(\mathbf{x}) = \sum_{i_1<\cdots<i_n} x_{i_1} \ldots x_{i_n}$, the elementary symmetric function of degree n

: ϵ: the sign representation of S_n

: freq(w): the frequency of w

: $F_{n,r}$: the set of words of length n with frequency r

: flex: the flex statistic on words: $\mathrm{flex}(w) = \mathrm{freq}(w) \cdot \mathrm{lex}(w)$

: $h_n(\mathbf{x}) = \sum_{i_1\leq\cdots\leq i_n} x_{i_1} \ldots x_{i_n}$, the homogeneous symmetric function of degree n

: $\mathcal{L}_\lambda$: the Lie representation indexed by λ

: $\mathcal{L}^{r,1}_{(a^b)} = (\chi^r_a \wr 1)\uparrow^{S_{ab}}_{C_a \wr S_b}$, the generalized Schosker representations

: $\mathcal{L}^{r,\epsilon}_{(a^b)} = (\chi^r_a \wr \epsilon)\uparrow^{S_{ab}}_{C_a \wr S_b}$

: $\lambda \vdash n$: λ is a partition of n

: λ/μ: the skew shape obtained by removing the boxes in μ from λ

: lex: the lex statistic on words

: $M_{n,r}$: the set of words of length n with major index congruent to r modulo n

: maj: the major index statistic on words or tableaux

: $\mathrm{NFD}_{n,r}$: the set of necklaces of length n words with frequency dividing r

: N_n: the set of necklaces of length n words

: $[n]_q = 1 + q + \cdots + q^{n-1}$

: $[n]_q! = [n]_q[n-1]_q \ldots [1]_q$

: $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$

: $\left(\!\binom{n}{k}\!\right)_q = \binom{n+k-1}{k}_q$

: $\binom{n}{\alpha_1,\ldots,\alpha_m}_q = \frac{[n]_q!}{[\alpha_1]_q \ldots [\alpha_m]_q}$

: P^a_b: the set of all tuples of b paritions with total size a

: $P(w)$: the insertion tableau of w

: $\mathrm{p} : \mathrm{SYT}(\lambda/\mu) \to \mathrm{SYT}(\lambda/\mu)$: Schützenberger's promotion operator

: period(w): the period of w

: $p_n(\mathbf{x}) = x_1^n + x_2^n + \ldots$, the power symmetric function of degree n

: $Q(w)$: the recording tableau of w

: Rect: the rectification operator

: Rect*: the anti-rectification operator

: RSK: the Robinson=Schensted-Knuth correspondence

: $\binom{S}{k}$: the set of k-element subsets of S

: $\left(\!\binom{S}{k}\!\right)$: the set of k-element multi-subsets of S

: sh: the shape operator on tableau

: $s_\lambda(\mathbf{x}) = \mathrm{SSYT}^{\mathrm{cont}}(\mathbf{x})$: the Schur function indexed by λ

: S_n: the symmetric group of order n

: σ_n: generator for C_n, often $\sigma_n = (1\,2\,\ldots\,n) \in S_n$

: $S^{\underline{\lambda}}$: the irreductible representation of $C_a \wr S_b$ indexed by $\underline{\lambda} \in P_b^a$

: SYT(λ): the set of standard Young tableaux of shape λ

: SSYT(λ): the set of semistandard Young tableaux of shape λ

: SSYT($\lambda;\alpha$): the set of semistandard Young tableaux of shape λ and content α

: stab: the stabilization statistic on standard skew tableaux with weakly decreasing row vectors

: stab*: the anti-stabilization statistic on standard skew tableaux with weakly increasing row vectors

: $T \leftarrow w$: the row insertion of w into T

: W_n: the set of words of length n.

: W_α: the set of words of content α.

: $W_{\alpha,\delta}$: the set of words of content α and cyclic descent type δ

: $W^{\mathrm{stat}}(q) = \sum_{w \in W} q^{\mathrm{stat}(w)}$

: W^g: the set of elements of W fixed by g

: ω: the involution on symmetric functions interchanging s_λ and $s_{\lambda'}$

: ω_n: a fixed primitive n-th root of unity.

: $\mathbf{x} = (x_1, x_2 \dots)$

: $f[g]$: the plethysm of two symmetric functions f, g

: $\equiv_n$: congruent modulo n

: $\wr$: the wreath product on groups and representations

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DEDICATION

Chapter 1
INTRODUCTION

Since Reiner, Stanton and White defined the cyclic sieving phenomenon in 2004 [RSW04], the cyclic sieving phenomenon has been of increasing interest to mathematics researchers. The cyclic sieving phenomenon (CSP) is a relationship between a cyclic action on a set W and a polynomial $f(q)$. Typically, $f(q)$ is a well-known q-analog and/or a statistic generating function on the set W, making the CSP seem like a series of elegant coincidences. My goal in this thesis is to convince you that the cyclic sieving phenomenon is more than a series of elegant coincidences.

Definition 1.1.1. For any set W and map $g : W \to W$, define

$$W^g \coloneqq \{w \in W : g(w) = w\}.$$

Definition 1.1.2. Suppose C_n is a cyclic group of order n generated by σ_n, W is a finite set on which C_n acts, and $f(q) \in \mathbb{N}[q]$. We say the triple $(W, C_n, f(q))$ exhibits the *cyclic sieving phenomenon (CSP)* if for all $k \in \mathbb{Z}$,

$$\#W^{\sigma_n^k} = f(\omega_n^k), \tag{1.1}$$

where ω_n is a fixed primitive n-th root of unity.

The CSP has two more equivalent definitions. We say $(W, C_n, f(q))$ exhibits the CSP if and only if

$$f(q) \equiv \sum_{\text{Orbits } \mathcal{O} \subset W} \sum_{j=0}^{|\mathcal{O}|-1} q^{j \cdot \frac{n}{|\mathcal{O}|}} \mod(q^n - 1) \tag{1.2}$$

where the sum is over C_n orbits $\mathcal{O}$ of W, or if and only if

$$\chi^W(\sigma_n^k) = f(\omega_n^k) \quad \text{for all } k \in \mathbb{Z}, \tag{1.3}$$

where χ^W is the character of $\mathbb{C}[W]$ as a C_n-representation. A condition like (1.2) makes sense since fixing a polynomial's evaluations at the n-the roots of unity determines that polynomial uniquely modulo $q^n - 1$. We refer the interested reader to [RSW04, Proposition 2.1] for the proof of the equivalence of (1.1) and (1.2). Equation (1.2) means every C_n-action on a finite set W gives rise to a CSP $(W, C_n, f(q))$, where $f(q)$ is the right hand side of (1.2). Equation (1.3) is equivalent to the CSP since

$$\chi^W(\sigma_n^k) = \#\{w \in W : \sigma_n^k \cdot w = w\}$$

by the definition of character. See [Sag11] for a nice survey and introduction to cyclic sieving.

In many instances of cyclic sieving, and nearly all of those considered here, $f(q)$ is the generating function for some statistic on W. Given a statistic $\mathrm{stat}\colon W \to \mathbb{N}$, let

$$W^{\mathrm{stat}}(q) := \sum_{w \in W} q^{\mathrm{stat}\, w} \in \mathbb{N}[q]. \tag{1.4}$$

Example 1.1.3. Here are some examples of triples that exhibit the cyclic sieving phenomenon:

1. q-binomials for subsets or multisubsets: $\left(\binom{[n]}{k}, \langle\sigma_n\rangle, \binom{n}{k}_q\right)$ and $\left(\left(\!\binom{[n]}{k}\!\right), \langle\sigma_n\rangle, \left(\!\binom{n}{k}\!\right)_q\right)$, where $\sigma_n \cdot \{a_1, \ldots a_k\} = \{a_1 + 1 \pmod n, \ldots, a_k + 1 \pmod n\}$ [RSW04, Theorem 1.1(b)].

2. Major index generating functions for words: $\left(\mathrm{W}_\alpha, \langle\sigma\rangle, \mathrm{W}_\alpha^{\mathrm{maj}}(q)\right)$ where W_α is the set of words of content α, and $\sigma \cdot w_1 w_2 \ldots w_n = w_2 \ldots w_n w_1$ [RSW04, Proposition 4.4].

3. Promotion on rectangular standard Young tableaux:

$$\left(\mathrm{SYT}(a^b), \langle \mathrm{p} \rangle, \frac{[ab]_q!}{\prod_{c \in (a^b)} [h_c]_q}\right)$$

where $\mathrm{SYT}(a^b)$ is the set of $a \times b$ rectangular Standard Young tableaux, and $\mathrm{p} : \mathrm{SYT}(a^b) \to \mathrm{SYT}(a^b)$ is Schützenberger's promotion operator [Rho10a, Theorem 1.3]. Here, $\frac{[ab]_q!}{\prod_{c \in (a^b)} [h_c]_q}$ is the q-analog of the hook length formula for the shape (a^b).

The cyclic sieving phenomenon is an active area of current research and will be a powerful mathematical tool going forward. Example 1.1.3(2) is equivalent to a special case of more general cyclic sieving phenomenon for finite Coxeter groups [RSW04, Theorem 1.6]. More cyclic sieving phenomena have been found related to increasing tableaux [Pec14], triangulations of an $n+2$-gon [RSW04], non-crossing matchings and webs [PPR09], and reduced words for the longest element in the hyperoctahedral group [PS10]. In [AA19], Alexandersson and Amini characterize when a statistic generating function $W^{\mathrm{stat}}(q)$ on W has an associated cyclic action of C_n so that $(W, C_n, W^{\mathrm{stat}}(q))$ exhibits the CSP. They also realize the set of cyclic actions associated to a statistic under cyclic sieving can be viewed as the lattice points of a cone. See [Sag11] for a survey of cyclic sieving phenomena. Cyclic sieving truly is a phenomenon.

These CSPs are not just coincidences that we prove and then are done with them. They becomes tools themselves! What can we do with this cyclic sieving relationship between cyclic actions and polynomials? We can find refinements of the cyclic sieving relationship in Chapter 3, which we do for Example 1.1.3(2). We can explore its consequences for branching rules in representation theory in Chapter 4, which we again do for Example 1.1.3(2). We can find the fixed points that are counted by CSPs Chapter 4, which we do for Example 1.1.3(3). We believe Chapter 4 demonstrates that cyclic sieving is a powerful tool in representation theory.

In Chapter 2, we review the necessary background. Such background includes notation for combinatorial concepts, words and statistics on words, Young tableaux, and representation theory. Background that is essential for only a single chapter is saved for that chapter.

In Chapter 3, we realize that the $(\mathrm{W}_\alpha, C_n, W_\alpha^{\mathrm{maj}}(q))$ CSP, Example 1.1.3(2), can be partitioned into smaller CSPs $(\mathrm{W}_{\alpha,\delta}, C_n, \mathrm{W}_{\alpha,\delta}^{\mathrm{maj}}(q))$ for all compositions α, δ [AS18b]. We

call such a partition a *refinement* of a CSP since the same statistic is used and the C_n-action is restricted. In this case, we refine by fixing what we call the cyclic descent type of a word. The cyclic descent type tracks the number of cyclic descents that are added as all copies of each letter are added in increasing order where they fit, see Definition 3.3.1. Each $(\mathrm{W}_{\alpha,\delta}, C_n, \mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q))$ CSP can be viewed as a product of subset and multisubsets CSPs, Example 1.1.3(1), up to some modular adjustments. This follows from a tree-branching algorithm to construct the words in $\mathrm{W}_{\alpha,\delta}$.

Our refinement makes Example 1.1.3(2) less coincidental since we break it into smaller pieces that are built out of CSPs of the form in Example 1.1.3(1). In addition, this gives a new proof of Example 1.1.3(2) that uses Example 1.1.3(1). While Example 1.1.3(2) has been proven via representation theory, a similar representation-theoretic proof of our refinement in unknown and would be very interesting.

In Chapter 4, we explore the representation-theoretic consequences of $(\mathrm{W}_{\alpha}, C_n, \mathrm{W}^{\mathrm{maj}}_{\alpha}(q))$ exhibiting the CSP, Example 1.1.3(2) [AS18a]. The usual paradigm is to use representation theory to prove cyclic sieving results. In Chapter 4, we do the reverse - we use this cyclic sieving result to prove results in representation theory. We can reinterpret $(\mathrm{W}_{\alpha}, C_n, \mathrm{W}^{\mathrm{maj}}_{\alpha}(q))$ exhibiting the CSP for all contents α as the following symmetric function identity between content generating functions:

$$\mathrm{NFD}^{\mathrm{cont}}_{n,r}(\mathbf{x}) = \mathrm{M}^{cont}_{n,r}(\mathbf{x}) \tag{1.5}$$

where

$$\mathrm{NFD}_{n,r} = \{\text{necklaces } N \text{ of length n words } : \mathrm{freq}(N) \mid r\},$$
$$\mathrm{M}_{n,r} = \{\text{words } w \text{ of length n } : \mathrm{maj}(w) \equiv_n r\}.$$

One of the major tools in symmetric group representation theory is the Frobenius characteristic map, which assigns each S_n-representation V to a symmetric function $\mathrm{ch}(V)$. Expanding

$\mathrm{ch}(V)$ in terms of the Schur functions then gives the irreducible decomposition of V itself. Equation (1.5) has helped us find the Schur expansion of various induced representations of the symmetric group.

Let $C_n = \langle \sigma_n \rangle$ denote the cyclic group generated by the long cycle $\sigma_n = (1\,2\,\ldots\,n) \in S_n$. Let χ^r denote the representation of C_n given by $\chi^r(\sigma_n) = \omega_n^r$. Kraśkiewicz–Weyman [KW01] first gave the Schur expansion of $\chi^r{\uparrow}_{C_n}^{S_n}$. The induced representation $\chi^1{\uparrow}_{C_n}^{S_n}$ is of particular interest since it is isomorphic to the multilinear S_n representation on the free Lie algebra, [Kly74, Prop. 1].

Theorem 1.1.4. [KW01] We have

$$\mathrm{ch}\left(\chi^r{\uparrow}_{C_n}^{S_n}\right) = \sum_{\lambda \vdash n} \#\{Q \in \mathrm{SYT}(\lambda) : \mathrm{maj}(Q) \equiv_n r\} s_\lambda(\mathbf{x})$$

We give a new proof of Theorem 1.1.4 using (1.5) that is much simpler than Kraśkiewicz and Weyman's. Our proof also generalizes Klyachko's work, [Kly74] to prove that

$$\mathrm{ch}\,\chi^r{\uparrow}_{C_n}^{S_n} = \mathrm{NFD}_{n,r}^{\mathrm{cont}}(\mathbf{x}).$$

Another important tool in our proof is the Robinson-Schensted-Knuth correspondence, which gives the Schur expansion of $\mathrm{M}_{n,r}^{\mathrm{cont}}(\mathbf{x})$ directly. One lesson from our approach is that viewing the representations $\{\chi^r{\uparrow}_{C_n}^{S_n}\}$ as part of a family rather than in isolation enabled us to use the cyclic sieving phenomenon.

We generalize our proof of Theorem 1.1.4 in two directions. First, we consider inducing 1-dimensional representations of arbitrary cyclic subgroups of S_n, reproving a result of Stembridge, [Ste89]. The Schur expansions obtained are governed by generalized major index statistics that arise very naturally from the combinatorics of orbits and cyclic sieving. Secondly, we consider inducing 1-dimensional representations of the wreath product $C_a \wr S_b$ up to S_{ab}, generalizing a result of Schocker, [Sch03]. As opposed to Schocker's approach, in our approach, our formulas arise naturally from the underlying combinatorics using Möbius

inversion, Burnside's lemma, and a sign-reversing involution. Also, of particular interest is the representation $\chi^{1,1}\uparrow^{S_{ab}}_{C_a \wr S_b}$, which is isomorphic to Lie representations arising in Thrall's Problem.

Unfortunately, our generalization to Schocker's formula involves a computationally complex sum, subtractions and divisions. Finally, we explore an approach to obtaining a better formula without these issues. We give a plethystic formula for the Frobenius characteristic of the irreducible representations of $C_a \wr S_b$ induced up to S_{ab}. We reduce the problem of finding the Schur expansion of these characteristics to finding a combinatorial statistic that interpolates between the major index and the shape under the Robinson-Schensted correspondence. We believe progress on this difficult problem is more likely if we consider the whole family of irreducible representations induced from $C_a \wr S_b$ up to S_{ab}.

Chapter 5 introduces tableaux stabilization in order to count the fixed points counted by the CSP in Example 1.1.3(3). After Dennis White conjectured Example 1.1.3(3) in 2007, Brendon Rhoades proved the CSP in Example 1.1.3(3) with a heavy reliance on representation theory in 2010 [Rho10a, Theorem 1.3]. He identifies the action of promotion with the action of the long cycle, σ_{ab}, on the Specht module indexed by (a^b), up to a sign, [Rho10a, Proposition 3.5]. Using (1.1), we can reinterpret his result as follows, Corollary 1.1.5. Kevin Purbhoo also gave an alternate proof of Corollary 1.1.5 using the Wronksi map [Pur13, Theorem 1.5].

Corollary 1.1.5. [Rho10a, Corollary 9.1] For $a, b \in \mathbb{Z}_{\geq 1}$, and $d \mid ab$,

$$\#\,\mathrm{SYT}(a^b)^{\mathrm{p}^d} = \#\text{ standard } \left(\frac{ab}{d}\right)\text{−ribbon tableaux of shape } (a^b).$$

What is noticeably missing from both of these proofs is a description of what tableaux are fixed by a given power of promotion! We know the size of the set $\mathrm{SYT}(a^b)^{\mathrm{p}^d}$, but what are its elements? In Chapter 5, we describe the elements of in $\mathrm{SYT}(a^b)^{\mathrm{p}^d}$ for sufficiently large a and $d = \frac{ab}{2}$ [Ahl19]. However, knowing Corollary 1.1.5 was instrumental in our ability to actually find these fixed points since it motivates our construction.

Our construction relies on a new notion we call tableau stabilization [Ahl19]. Suppose S is a standard skew tableau whose row sizes decrease from top to bottom. Consider attaching $k-1$ shifted copies of S to the right of itself so the result, $S^{(k)}$, is a standard tableau. When rectifying $S^{(k)}$, eventually one of the shifted copies of S experiences only horizontal slides, and then we say S *stabilizes* at that point. We derive bounds on stabilization that help us construct $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$ for $a \geq 2b-1$. Along the way, we derive a formula for the shape of the stabilized tableau when the initial tableau has constant row sizes. We also investigate stabilization as a permutation statistic and present open problems.

Chapter 2

BACKGROUND

2.1 Combinatorial Background

2.1.1 Basic Notation

Let $\mathbb{Z}, \mathbb{Z}_{\geq 0}, \mathbb{Z}_{\geq 1}$ denote the set of integers, nonnegative integers, and positive integers, respectively. For $n \in \mathbb{Z}_{\geq 1}$, let $[n] \coloneqq \{1, \ldots, n\}$, and for $a, b \in Z$, let $[a, b] = \{i \in Z : a \leq i \leq b\}$. Let $\#S$ denote the cardinality of a set S, and

$$\binom{S}{k} := \{\text{all } k\text{-element subsets of } S\}, \tag{2.1}$$

$$\left(\!\binom{S}{k}\!\right) := \{\text{all } k\text{-element multisubsets of } S\}. \tag{2.2}$$

For any set X, let $X[q]$ denote the set of polynomials in q with coefficents in X. We use the following standard q-analogues:

$$[n]_q := 1 + q + \cdots + q^{n-1} = \frac{q^n - 1}{q - 1},$$

$$[n]_q! := [n]_q[n-1]_q \cdots [1]_q,$$

$$\binom{n}{\alpha_1, \ldots, \alpha_m}_q := \frac{[n]_q!}{[\alpha_1]_q! \cdots [\alpha_m]_q!} \in \mathbb{Z}_{\geq 0}[q],$$

$$\left(\!\binom{n}{k}\!\right)_q := \binom{n+k-1}{k}_q := \binom{n+k-1}{k, n-1}_q.$$

The cyclic group $C_n = \langle \sigma_n \rangle$ of order n acts on $\binom{[n]}{k}$ and $\left(\!\binom{[n]}{k}\!\right)$ by rotating the elements

within $[n]$:

$$\sigma_n \cdot \{a_1, \dots a_k\} = \{a_1 + 1 \pmod n, \dots, a_k + 1 \pmod n\}. \tag{2.3}$$

For example, $\sigma_4 \cdot \{0, 0, 0, 2, 2, 3\} = \{0, 1, 1, 1, 3, 3\}$. Under this action, the q-analogs $\binom{n}{k}_q$ and $\left(\!\binom{n}{k}\!\right)_q$ give rise to the CSPs in Example 1.1.3(1).

Theorem 2.1.1. [RSW04, Thm. 1.1]. The triples

$$\left(\binom{[0,n-1]}{k}, C_n, \binom{n}{k}_q\right) \qquad \text{and} \qquad \left(\left(\!\binom{[0,n-1]}{k}\!\right), C_n, \left(\!\binom{n}{k}\!\right)_q\right)$$

exhibit the CSP.

Example 2.1.2. The triple $(\binom{[0,5]}{2}, C_6, \binom{6}{2}_q)$ exhibits the CSP. Omitting braces on the subsets, the orbits are

$$\{01, 12, 23, 34, 45, 05\}, \{02, 13, 24, 35, 04, 15\}, \{03, 14, 25\}.$$

Then, let

$$f(q) \coloneqq \binom{6}{2}_q = \frac{[6]_q[5]_q}{[2]_q} = 1 + q + 2q^2 + 2q^3 + 3q^4 + 2q^5 + 2q^6 + q^7 + q^8.$$

This means

$$f(\omega_6) = f(\omega_6^2) = f(\omega_6^4) = f(\omega_6^5) = 0,\ f(\omega_6^3) = 3,\ f(\omega_6^6) = 15,$$

which agrees with (1.1). Moreover,

$$f(q) \equiv 2(q^0 + q^1 + \cdots + q^5) + (q^0 + q^2 + q^4) \pmod{q^6 - 1},$$

which agrees with (1.2).

In addition $\left(\left(\!\binom{[0,5]}{2}\!\right), C_6, \left(\!\binom{6}{2}\!\right)_q\right)$ exhibits the CSP. The orbits are

$$\{00, 11, 22, 33, 44, 55\}, \{01, 12, 23, 34, 45, 50\}, \{02, 13, 24, 35, 04, 15\}, \{03, 14, 25\}.$$

Then, let

$$h(q) \coloneqq \left(\!\binom{6}{2}\!\right)_q = \binom{7}{2}_q = 1 + q + 2q^2 + 2q^3 + 3q^4 + 3q^5 + 3q^6 + 2q^7 + 2q^8 + q^9 + q^{10}.$$

This means

$$h(\omega_6) = h(\omega_6^2) = h(\omega_6^4) = h(\omega_6^5) = 0,\ h(\omega_6^3) = 3,\ h(\omega_6^6) = 21,$$

which agrees with (1.1). Moreover,

$$h(q) \equiv 3(q^0 + q^1 + \cdots + q^5) + (q^0 + q^2 + q^4) \pmod{q^6 - 1},$$

which agrees with (1.2).

Not only are examples of cyclic sieving phenomena becoming more and more plentiful, but cyclic sieving is well-behaved with respect to subgroups and products. Under cyclic sieving, extending cyclic actions to sets or multisets of elements corresponds to a nice plethysm of the polynomials. Theorem 2.1.3 describes these properties of cyclic sieving more precisely.

Theorem 2.1.3. Suppose C_n is a cyclic group of order n acting on W and X. If $(W, C_n, f(q))$ and $(X, C_n, g(q))$ exhibit the CSP, then so do

1. $(W, C_d, f(q))$ where $C_d \subseteq C_n$ is a subgroup of C_n,
2. $(W, C_d, f(q^{-1}))$.
3. $(W \times X, C_n, f_1(q)g(q))$.
4. $\left(\binom{W}{k}, C_n, e_k[f(q)]\right)$ where e_k is the elementary symmetric functions of degree k, and $[\,]$ is the plethysm operation.

5. $\left(\left(\!\binom{W}{k}\!\right), C_n, h_k[f(q)]\right)$ where h_k is the homogeneous symmetric function of degree k, and $[\,]$ is the plethysm operation.

Facts 1, 2, follow from (1.1), noting (1.1) still makes sense for $f(q) \in \mathbb{Z}[q, q^{-1}]$. See [BER11, Propositions 7,8] for proofs of facts 3, 4, 5.

In most triples $(W, C_n, f(q))$ that have been found to exhibit the CSP, $f(q)$ is a statistic generating function on W for some well-known statistic. A statistic on a set W is simply a map $\mathrm{stat} : W \to X$, where usually, $X = \mathbb{Z}_{\geq 0}$. We have adopted the following notation for statistic generating functions. Given $\mathrm{stat}\colon W \to \mathbb{Z}_{\geq 0}$, we write the corresponding generating function as

$$W^{\mathrm{stat}}(q) \coloneqq \sum_{w \in W} q^{\mathrm{stat}(w)}.$$

This notation emphasizes the set and the statistic without dummy variables and $\sum$'s getting in the way.

We use natural multivariable analogues of this notation as well. For example, letting $\mathbf{x} = (x_1, x_2, \dots)$, $\mathrm{stat}_1 : W \to \{(\alpha_1, \alpha_2, \dots,) \in \mathbb{Z}_{\geq 0}^{\infty} : \sum_j \alpha_j < \infty\}$ and $\mathrm{stat}_2 : W \to \mathbb{Z}_{\geq 0}$,

$$W^{\mathrm{stat}_1, \mathrm{stat}_2}(\mathbf{x}; q) \coloneqq \sum_{w \in W} \mathbf{x}^{\mathrm{stat}_1(w)} q^{\mathrm{stat}_2(w)} \in \mathbb{Z}_{\geq 0}[[x_1, x_2, \dots]][q]$$

where $\mathbf{x}^{(\alpha_1, \dots, \alpha_m)} \coloneqq x_1^{\alpha_1} \cdots x_m^{\alpha_m}$.

While the polynomials $\binom{n}{k}_q$ and $\left(\!\binom{n}{k}\!\right)_q$ in Theorem 2.1.1 are not expressed in the form of statistic generating functions, they can be quite nicely expressed in such a form (see [Mac95, Example I.2.2] or [Sta99, Proposition 7.8.3]):

$$\binom{n}{k}_q = q^{-\binom{k}{2}} \binom{[0, n-1]}{k}^{\mathrm{sum}}(q), \tag{2.4}$$

$$\left(\!\binom{n}{k}\!\right)_q = \left(\!\binom{[0, n-1]}{k}\!\right)^{\mathrm{sum}}(q). \tag{2.5}$$

Here the sum statistic denotes the sum of the elements of a subset or submultiset of $\mathbb{Z}$. These generating functions will appear in Chapter 3.

2.1.2 Words

A *word* is a sequence of positive integers. Let W_n denote the set of all words of length n. For any sequence w, let w_j denote the j-th element of w, so that for $w \in \mathrm{W}_n$, $w = w_1 \ldots, w_n$. A word $w_1 w_2 \ldots w_n$ is *increasing* if $w_1 \leq w_2 \leq \cdots \leq w_n$ and *decreasing* if $w_1 \geq w_2 \geq \cdots \geq w_n$. A *subsequence* of a sequence $w_1 w_2 \ldots w_n$ is a sequence of the form $w_{i_1} w_{i_2} \ldots w_{i_k}$ for some $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.

A *composition* α of n, denoted $\alpha \vDash n$, is any sequence $(\alpha_1, \ldots, \alpha_m)$ of non-negative integers satisfying $\alpha_1 + \cdots + \alpha_m = n$. The content of a word $w = w_1 w_2 \ldots w_n$, denoted $\mathrm{cont}(w)$, is the sequence whose j-th element is the number of j's that appear in w. Note that for $w \in \mathrm{W}_n, \mathrm{cont}(w) \vDash n$. Write

$$\mathrm{W}_\alpha \coloneqq \{\text{words } w : \mathrm{cont}(w) = \alpha\} \tag{2.6}$$

for the set of words with content α. The words of content $(1, 1, \ldots, 1)$ with n 1's are the permutations of length n. These words can be viewed as bijections $w : [n] \to [n]$ by $i \mapsto w_i$ and then they form the symmetric group S_n under composition. It is difficult to understate the importance of the symmetric group in mathematics research and in this thesis.

The symmetric group S_n acts on W_n by permuting the letters according to

$$\tau \cdot w_1 w_2 \cdots w_n \coloneqq w_{\tau^{-1}(1)} w_{\tau^{-1}(2)} \cdots w_{\tau^{-1}(n)} \tag{2.7}$$

for all $\tau \in S_n$. In particular, letting $\sigma_n \coloneqq (1\ 2\ \cdots\ n) \in S_n$ and $C_n \coloneqq \langle \sigma_n \rangle$, the cyclic group C_n acts on W_n by rotation according to

$$\sigma_n \cdot w_1 w_2 \cdots w_n \coloneqq w_n w_1 \cdots w_{n-1}. \tag{2.8}$$

We employ the following statistics on words. The *descent set* of $w = w_1 \dots w_n \in \mathrm{W}_n$ tracks the positions where the values decrease:

$$\mathrm{Des}(w) := \{1 \leq i < n : w_i > w_{i+1}\}, \quad \mathrm{des}(w) = \#\,\mathrm{Des}(w).$$

The *cyclic descent set* of $w = w_1 \dots w_n \in \mathrm{W}_n$ tracks the cyclic positions where the values decrease:

$$\mathrm{CDes}(w) := \{1 \leq i \leq n : w_i > w_{i+1 \,(\mathrm{mod}\, n)}\}, \quad \mathrm{cdes}(w) = \#\,\mathrm{CDes}(w).$$

The *major index* of w sums the descent positions:

$$\mathrm{maj}(w) := \sum_{i \in \mathrm{Des}(w)} i.$$

The inversion statistic $\mathrm{inv} : \mathrm{W}_n \to \mathbb{Z}_{\geq 0}$ tracks the number of pairs whose order is reversed:

$$\mathrm{inv}(w) = \#\{(i,j) : i < j, w_i > w_j\}.$$

MacMahon, [Mac], proved that for all $\alpha \vDash n$,

$$\mathrm{W}_\alpha^{\mathrm{maj}}(q) = \binom{n}{\alpha}_q = \mathrm{W}_\alpha^{\mathrm{inv}}(q). \tag{2.9}$$

This shows inv and maj have the same distribution on W_α. Despite this and the inversion statistic coinciding with length in the symmetric group as a Coxeter group, the major index statistic will be far more useful in this thesis. First, the CSP $(\mathrm{W}_\alpha, C_n, \mathrm{W}_\alpha^{\mathrm{maj}}(q))$ refines to $(\mathrm{W}_{\alpha,\delta}, C_n, W_{\alpha,\delta}^{\mathrm{maj}}(q))$ by fixing the cyclic descent type, while $(\mathrm{W}_\alpha, C_n, \mathrm{W}_\alpha^{\mathrm{inv}}(q))$ does not refine to $(\mathrm{W}_{\alpha,\delta}, C_n, \mathrm{W}_{\alpha,\delta}^{\mathrm{inv}}(q))$. Secondly, major index is also a statistic on standard Young tableaux that behaves well under the Robinson-Schensted-Knuth correspondence.

Example 2.1.4. For example, $(2,0,2,0,4) \vDash 8$, $w = 15531553 \in \mathrm{W}_{(2,0,2,0,4)} \subseteq W_8$, and

$$\sigma_8 \cdot 15531553 = 31553155.T$$

The descents of w are marked with periods below:

$$\mathrm{Des}(w) = \mathrm{Des}(155.3.155.3) = \{3,4,6\}, \quad \mathrm{des}(w) = 3.$$

The cyclic descents of w are marked with periode below:

$$\mathrm{CDes}(w) = \mathrm{CDes}(155.3.155.3.) = \{3,4,6,7\}, \quad \mathrm{cdes}(w) = 4.$$

Therefore,

$$\mathrm{maj}(w) = \sum_{i \in \mathrm{Des}(w)} i = 3 + 4 + 6 = 13.$$

Also, $\mathrm{inv}(w) = 9$.

The set of all words in $\mathbb{P}$ is a monoid under concatenation. A word is *primitive* if it is not a power of a smaller word. Any non-empty word w may be written uniquely as $w = v^f$ for $f \geq 1$ with v primitive. We call $|v|$ the *period* of w, written $\mathrm{period}(w)$, and f the *frequency* of w, written $\mathrm{freq}(w)$. An orbit of W_n under rotation is a *necklace*, usually denoted $[w]$. We have $\mathrm{period}(w) = \#[w]$ and $\mathrm{freq}(w) \cdot \mathrm{period}(w) = |w|$. Let $\mathrm{lex}(w)$ denote the index at which w appears when lexicographically ordering the necklace $[w]$, and

$$\mathrm{flex}(w) = \mathrm{freq}(w) \cdot \mathrm{lex}(w).$$

Content, primitivity, period, frequency, and cdes are all constant on necklaces. For $n \geq 1$

and $\alpha \vdash n$, we write

$$\mathrm{N}_n \coloneqq \{\text{necklaces of length } n \text{ words}\},$$
$$\mathrm{N}_\alpha \coloneqq \{\text{necklaces of content } \alpha\}.$$

Example 2.1.5. The necklace of $w = 15531553 = (1553)^2$ is

$$[w] = \{15531553, 55315531, 53155315, 31553155\} \in \mathrm{N}_{(2,0,2,0,4)} \subseteq \mathrm{N}_8,$$

The necklace $[w]$ is not primitive and has period 4, frequency 2, and cdes 4. Lexicographically ordering $[w]$ gives

$$[w] = \{15531553, 31553155, 53155315, 55315531\},$$

which means

$$\begin{aligned} \mathrm{lex}(15531553) &= 1, \quad \mathrm{flex}(15531553) = 2, \\ \mathrm{lex}(31553155) &= 2, \quad \mathrm{flex}(31553155) = 4, \\ \mathrm{lex}(53155315) &= 3, \quad \mathrm{flex}(53155315) = 6, \\ \mathrm{lex}(55315531) &= 4, \quad \mathrm{flex}(55315531) = 8. \end{aligned}$$

Theorem 2.1.6. [RSW04, Proposition 4.4]. Let $\alpha \vDash n$. The triple $(\mathrm{W}_\alpha, C_n, \mathrm{W}_\alpha^{\mathrm{maj}}(q))$ exhibits the CSP.

Example 2.1.7. $(\mathrm{W}_{(3,3)}, C_6, \mathrm{W}_{(3,3)}^{\mathrm{maj}}(q))$ exhibits the CSP. The necklaces are:

- {111222, 211122, 221112, 222111, 122211, 112221},
- {112122, 211212, 221121, 122112, 212211, 12122 },
- {112212, 211221, 121122, 212112, 221211, 122121},

- $\{121212, 212121\}$.

Then,

$$f(q) \coloneqq W^{\mathrm{maj}}_{(3,3)}(q) = \binom{6}{3}_q = 1 + q + 2q^2 + 3q^3 + 3q^4 + 3q^5 + 3q^6 + 2q^7 + q^8 + q^9.$$

This means

$$f(\omega_6) = f(\omega_6^3) = f(\omega_6^5) = 0,\ f(\omega_6^2) = f(\omega_6^4) = 2,\ f(\omega_6^6) = 20,$$

which agrees with (1.1). Moreover,

$$f(q) \equiv 3(q^0 + q^1 + \cdots + q^5) + (q^0 + q^3) \pmod{q^6 - 1},$$

which agrees with (1.2).

2.1.3 Tableaux

A partition λ of n, denoted $\lambda \vdash n$, is a composition whose parts are weakly decreasing. The *Young diagram* of a partition λ is the upper-left justified collection of cells with λ_i entries in the ith row starting from the top. A *semistandard Young tableau* of shape λ is a filling of the Young diagram of λ with entries from $\mathbb{Z}_{\geq 1}$ which weakly increases along rows and strictly increases along columns. The set of semistandard Young tableaux of shape λ is denoted $\mathrm{SSYT}(\lambda)$. The *content* of $P \in \mathrm{SSYT}(\lambda)$, denoted $\mathrm{cont}(P)$, is the composition whose j-th entry is the number of j's in P. For a composition $\alpha \vDash n$, let $\mathrm{SSYT}(\lambda; \alpha)$ denote the set of semistandard Young tableaux of shape λ with content α. The set of *standard Young tableaux* of shape λ, denoted $\mathrm{SYT}(\lambda)$, is $\mathrm{SSYT}(\lambda; (1, \ldots, 1))$. The *descent set* of a standard tableau $Q \in \mathrm{SYT}(\lambda)$ is

$$\mathrm{Des}(Q) = \{i \in [n-1] : i+1 \text{ lies in a lower row of } Q \text{ than } i\}.$$

Similarly to words, let $\mathrm{maj}(Q) = \sum_{i \in \mathrm{Des}(Q)} i$.

Example 2.1.8. For example, $(5,5,3,2) \vdash 15$, and

$$P = \begin{array}{|c|c|c|c|c|}\hline 1&1&3&3&3\\\hline 2&3&4&4&4\\\hline 3&4&5\\\cline{1-3} 5&5\\\cline{1-2}\end{array} \in \mathrm{SSYT}((5,5,3,2),(2,1,5,4,3)).$$

is a semistandard tableau, drawn in English notation. Also,

$$Q = \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2} 6\\\cline{1-1}\end{array} \in \mathrm{SYT}(3,2,1)$$

is a standard tableau with $\mathrm{Des}(Q) = \{2,5\}$ and $\mathrm{maj}(Q) = 7$. The filling

$$F = \begin{array}{|c|c|c|c|}\hline 1&\mathbf{4}&5&8\\\hline 2&\mathbf{3}\\\cline{1-2} 6&7\\\cline{1-2}\end{array}$$

is not a semistandard tableau because its second column is not increasing.

When Young tableaux arise, the algorithms of row insertion and jeu-de-taquin often appear as well. Row insertion gives rise to the Robinson-Schensted-Knuth correspondence, which will critical role in this thesis. Jeu-de-taquin leads to rectification, which will play a critical role in Chapter 5. We delay introducing jeu de taquin until we need it in Chapter 5.

Definition 2.1.9. Given a tableau T and $x \in \mathbb{Z}_{\geq 1}$, we *row insert* x into T by inserting x at the end of the first row if x is weakly larger than every cell in the first row. Otherwise, x bumps down the smallest element y larger than x in the first row of T, and y is recursively row inserted into the rest of T. Let $T \leftarrow x$ be the result of row inserting x into T, and let

$$T \leftarrow w_1 w_2 \dots w_n \coloneqq (\dots((T \leftarrow w_1) \leftarrow w_2)\dots) \leftarrow w_n.$$

The Robinson–Schensted–Knuth correspondence (RSK) is a bijection

$$\begin{aligned} \mathrm{RSK}\colon \mathrm{W}_n &\to \bigsqcup_{\lambda \vdash n} \mathrm{SSYT}(\lambda) \times \mathrm{SYT}(\lambda), \\ w &\mapsto (P(w), Q(w)). \end{aligned}$$

where $P(w) = \varnothing \leftarrow w$ is called the insertion tableau, and $Q(w)$, the recording tableau, records the order in which the cells were created. Example 2.1.10 should help illustrate this algorithm.

Example 2.1.10. Here is an example of a single row insertion, with the changed cells, called the bumping chain, highlighted in yellow:

$$\begin{array}{|c|c|c|c|c|}\hline 1&3&5&7&8\\\hline 2&9&10&11&14\\\hline 4&12&15\\\cline{1-3} 13\\\cline{1-1}\end{array} \leftarrow 6 = \begin{array}{|c|c|c|c|c|}\hline 1&3&5&\cellcolor{yellow}6&8\\\hline 2&\cellcolor{yellow}7&10&11&14\\\hline 4&\cellcolor{yellow}9&15\\\cline{1-3} \cellcolor{yellow}12\\\cline{1-1} \cellcolor{yellow}13\\\cline{1-1}\end{array}.$$

If $w = 474153$, then

$$P(w) = \varnothing \leftarrow 474153 = \begin{array}{|c|}\hline 4\\\hline\end{array} \leftarrow 74153 = \begin{array}{|c|c|}\hline 4&7\\\hline\end{array} \leftarrow 4153 = \begin{array}{|c|c|}\hline 4&4\\\hline 7\\\cline{1-1}\end{array} \leftarrow 153$$
$$= \begin{array}{|c|c|}\hline 1&4\\\hline 4\\\cline{1-1} 7\\\cline{1-1}\end{array} \leftarrow 53 = \begin{array}{|c|c|c|}\hline 1&4&5\\\hline 4\\\cline{1-1} 7\\\cline{1-1}\end{array} \leftarrow 3 = \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 4&4\\\cline{1-2} 7\\\cline{1-1}\end{array}.$$

Then, $Q(w)$ records the order in which the cells were created, so

$$Q(w) = \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&6\\\cline{1-2} 4\\\cline{1-1}\end{array}.$$

Two well-known properties of the RSK correspondence are

$$\mathrm{cont}(w) = \mathrm{cont}(P(w)), \qquad \mathrm{Des}(w) = \mathrm{Des}(Q(w)). \tag{2.10}$$

The first property in (2.10) is clear from the definition. The second property in (2.10) is originally due to Schützenberger [Sch63, Remarque 2]. See [Sta99, Lemma 7.23.1] for a proof of (2.10) in the decisive permutation case and [Sta99, p.404] for further historical remarks. See [Sag01, Chapter 3] for more details on RSK.

2.1.4 Symmetric Functions

To do the representation theory of the symmetric group in this thesis, we will need some knowledge about the ring of symmetric functions. This is the ring Λ of all power series in $\mathbb{C}[x_1, x_2, \dots,]$ of bounded degree that are invariant under permuting the variables. For convenience, let

$$\mathbf{x} = (x_1, x_2, \dots).$$

One easy way to construct a symmetric function is to start with a single monomial and add all other monomials obtained by permuting the variables. This gives the monomial symmetric functions - for each partition $\lambda = (\lambda_1, \dots, \lambda_k)$, let

$$m_\lambda(\mathbf{x}) \coloneqq \sum_{i_1,\dots,i_k \text{ distinct}} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \dots x_{i_m}^{\lambda_k}$$

The set $\{m_\lambda(\mathbf{x})\}_{\lambda \in \mathrm{Par}}$ forms a basis for the ring of symmetric functions Λ. One particularly simple case of the monomials symmetric functions is the *power-sum symmetric functions*

$$p_n(\mathbf{x}) \coloneqq m_{(n)}(\mathbf{x}) = x_1^n + x_2^n + \cdots$$

Since Λ is a ring,

$$p_{(\lambda_1,\dots,\lambda_k)}(\mathbf{x}) \coloneqq p_{\lambda_1}(\mathbf{x}) \cdots p_{\lambda_k}(\mathbf{x})$$

is symmetric for each partition $\lambda = (\lambda_1, \ldots, \lambda_k)$. Moreover, $\{p_\lambda(\mathbf{x})\}_{\lambda \in \mathrm{Par}}$ is a basis for Λ as well, proven by Newton in the 1600s [Mac95, (2.11), (2.11')].

The most important symmetric functions for our purposes are the Schur functions, also indexed by partitions. For a partition λ, the *Schur function* s_λ is the content generating function on semistandard tableaux of shape λ,

$$s_\lambda(\mathbf{x}) \coloneqq \mathrm{SSYT}(\lambda)^{\mathrm{cont}}(\mathbf{x}) = \sum_{P \in \mathrm{SSYT}(\lambda)} \mathbf{x}^{\mathrm{cont}(P)}. \tag{2.11}$$

Unlike the monomial symmetric functions, it is not obvious the Schur functions are symmetric - see [Sag01, Proposition 4.4.2]. Like the mononials, $\{m_\lambda(\mathbf{x})\}_{\lambda \in \mathrm{Par}}$ is a basis for Λ. Two important instances of Schur functions are the complete homogeneous symmetric functions

$$h_n(\mathbf{x}) \coloneqq s_{(n)}(\mathbf{x}) = \sum_{i_1 \leq \cdots \leq i_n} x_{i_1} \cdots x_{i_n} \tag{2.12}$$

and the elementary symmetric functions

$$e_n(\mathbf{x}) \coloneqq s_{(1^n)}(\mathbf{x}) = \sum_{i_1 < \cdots < i_n} x_{i_1} \cdots x_{i_n}. \tag{2.13}$$

2.2 Representation Theory

A complex *representation* of a finite group G is a complex vector space V together with an action of G on V where each $g \in G$ acts linearly on V. A complex representation V of G is *irreducible* if it is not the direct sum of two smaller representations of G. Over $\mathbb{C}$, V is irreducible if it has no nontrivial subspaces that are invariant under G. Two representations V, W or G are equivalent, if there exists a linear bijection $\varphi : V \to W$ so that

$$g \cdot \varphi(v) = \varphi(g \cdot v), \qquad \text{for all } g \in G, v \in V.$$

The *character* of a representation V is the map $\chi : G \to \mathbb{C}$ given by $\chi(g) = \mathrm{Tr}_V(g)$, the trace of the action of g on V as a linear map. Amazingly, two representations are equivalent if and ony if they have the same character.

The complex irreducible inequivalent representations of S_n are canonically indexed by partitions $\lambda \vdash n$ and are called Specht modules, written S^λ. The actual construction of these modules is quite complex and we will not deal with it here. The interested reader may consult [Sag01] or [Ful97]. The *Frobenius characteristic map* ch is defined by $\mathrm{ch}\, S^\lambda := s_\lambda(\mathbf{x})$ and is extended additively to all S_n-representations. Since Schur functions are $\mathbb{Z}$-linearly independent, computing the irreducible decomposition of an S_n-module M corresponds to computing the Schur expansion of $\mathrm{ch}\, M$.

Example 2.2.1. For example, $(4,3,1,1) \vdash 9$, and $S^{(4,3,1,1)}$ is a representation of S_9 with character $\chi^{(4,3,1,1)}$. Its Frobenius characteristic is

$$\mathrm{ch}\, S^{(4,3,1,1)} = s_{(4,3,1,1)}(\mathbf{x}) = \mathrm{SSYT}(4,3,1,1)^{\mathrm{cont}}(\mathbf{x}).$$

Many proofs of CSPs use representation theory. This is not only because of (1.3), but also because we can harness the power of representation theory to count fix points, and many representation theory formulas involve roots of unity. We present a proof of Theorem 2.1.6 using representation theory. Our proof also illustrates many of the tools we will use in this thesis: the representation theory of the symmetric group, Young tableaux, and the Robinson-Schensted-Knuth correspondence.

Proof of Theorem 2.1.6. Fix $\alpha \vDash n$. The complex vector space over words of content α, $\mathbb{C}[\mathrm{W}_\alpha]$, is an S_n-representation under the action in (2.7) extended linearly. The decomposition of $\mathbb{C}[\mathrm{W}_\alpha]$ as an S_n-representation is well-known. For each $\lambda \vdash n$, the multiplicity of the irreducible representation χ^λ in $\mathbb{C}[\mathrm{W}_\alpha]$ is the Kostka number $\#\,\mathrm{SSYT}(\lambda;\alpha)$. This means the

character of $\mathbb{C}[\mathrm{W}_\alpha]$ is given by

$$\chi^{\mathrm{W}_\alpha}(\tau) = \sum_{\lambda \vdash n} \#\,\mathrm{SSYT}(\lambda;\alpha)\chi^\lambda(\tau) \qquad \text{for all } \tau \in S_n. \tag{2.14}$$

Next, we can similarly decompose $\mathrm{W}_\alpha^{\mathrm{maj}}(q)$ with the Robinson-Schensted-Knuth correspondence.

$$\begin{aligned} \mathrm{W}_\alpha^{\mathrm{maj}}(q) &= \sum_{\lambda \vdash n} \sum_{P \in \mathrm{SSYT}(\lambda;\alpha)} \sum_{Q \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(Q)} \qquad \text{by RSK and (2.10)} \\ &= \sum_{\lambda \vdash n} \#\,\mathrm{SSYT}(\lambda;\alpha)\ \mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) \end{aligned} \tag{2.15}$$

The major index generating function on standard Young tableaux and the corresponding irreducible S_n characters are related by

$$\mathrm{SYT}^{\mathrm{maj}}(\omega_n^k) = \chi^\lambda(\sigma_n^k) \tag{2.16}$$

where ω_n is a primitive n-th root of unity [Spr74, Proposition 4.5]. The cyclic group $C_n = \langle \sigma_n \rangle$ embeds in S_n where σ_n is the long cycle $(1\,2\,\ldots\,n)$. Then,

$$\begin{aligned} \#\,\mathrm{W}_\alpha^{\sigma_n^k} &= \chi^{\mathrm{W}_\alpha}(\sigma_n^k) \qquad \text{by definition of character} \\ &= \sum_{\lambda \vdash n} \#\,\mathrm{SSYT}(\lambda;\alpha)\chi^\lambda(\sigma_n^k) \qquad \text{by (2.14)} \\ &= \sum_{\lambda \vdash n} \#\,\mathrm{SSYT}(\lambda;\alpha)\,\mathrm{SYT}(\lambda)^{\mathrm{maj}}(\omega_n^k) \qquad \text{by (2.16)} \\ &= \mathrm{W}_\alpha^{\mathrm{maj}}(\omega_n^k) \qquad \text{by (2.15).} \end{aligned}$$

Hence, $(\mathrm{W}_\alpha, C_n, \mathrm{W}_\alpha^{\mathrm{maj}}(q))$ exhibits the CSP by (1.1).

□

Chapter 3

REFINED CYCLIC SIEVING ON WORDS FOR THE MAJOR INDEX STATISTIC

3.1 Introduction

Our main result in this chapter is a refinement of the CSP in Theorem 2.1.6, namely $(\mathrm{W}_\alpha, C_n, \mathrm{W}_\alpha^{\mathrm{maj}}(q))$. Recall W_α denotes the set of words of content alpha, and maj denotes the major index statistic. Reiner, Stanton, and White deduced Theorem 2.1.6 from the following more general result about Coxeter systems.

Theorem 3.1.1. [RSW04, Theorem 1.6]. Let (W, S) be a finite Coxeter system and $J \subseteq S$. Let W_J be the corresponding parabolic subgroup, W^J the set of minimal length representatives for left cosets $X := W/W_J$, and $X^\ell(q) := \sum_{w \in W^J} q^{\ell(w)}$. Let C be a cyclic subgroup of W generated by a Springer regular element. Then $(X, C, X^\ell(q))$ exhibits the cyclic sieving phenomenon.

Theorem 2.1.6 follows from Theorem 3.1.1 when $W = S_n$ by identifying W/W_J with words of fixed content α, where α is the composition recording the lengths of consecutive subsequences of J, and C is generated by an n-cycle. One must also use the classical result of MacMahon that maj is equidistributed with the inversion statistic on words, from which it follows that $\mathrm{W}_\alpha^{\mathrm{maj}}(q) = X^\ell(q)$ [Mac, §1].

Definition 3.1.2. A *refinement* of a CSP triple $(W, C_n, W^{\mathrm{stat}}(q))$ is a triple $(V, C_n, V^{\mathrm{stat}}(q))$ where $V \subset W$ has the restricted C_n-action and $(V, C_n, V^{\mathrm{stat}}(q))$ still exhibits the CSP.

If $(V, C_n, V^{\mathrm{stat}}(q))$ refines $(W, C_n, W^{\mathrm{stat}}(q))$, then so does $(U, C_n, U^{\mathrm{stat}}(q))$ where $U := W - V$. Thus, a CSP refinement partitions W into smaller CSPs with the same statistic. If W is an orbit, its only refinements are W and $\varnothing$. In Section 3.7, we define a statistic on words, *flex*, which is universal in the sense that it refines to all C_n-orbits. Such universal statistics are essentially equivalent to the choice of a total ordering for each orbit $\mathcal{O}$ of W.

We partition words of fixed content into fixed *cyclic descent type* (CDT). One computes $\mathrm{CDT}(w)$ by building up w by adding all 1's, 2's, ..., and counting the number of *cyclic descents* introduced at each step. For precise details, see Definition 3.3.1 and Example 3.3.2.

We write the set of words with fixed content and cyclic descent type as

$$\mathrm{W}_{\alpha,\delta} := \{\text{words } w : \mathrm{cont}(w) = \alpha, \mathrm{CDT}(w) = \delta\}. \tag{3.1}$$

Recall the *cyclic descent set* of w is $\mathrm{CDes}(w) := \{1 \leq i \leq n : w_i > w_{i+1 (\mathrm{mod}\, n)}\}$, and we write $\mathrm{cdes}(w) := \#\,\mathrm{CDes}(w)$ for the number of cyclic descents. Any position $1 \leq i \leq n$ that is not a cyclic descent is a *cyclic weak ascent*. Cyclic descents were introduced by Cellini in an algebraic context; see [Cel98]. Since then, cyclic descents have been used by Lam and Postnikov in studying alcoved polytopes [LP18] and by Petersen in studying P-partitions [Pet05].

Theorem 3.1.3. Let $\alpha \vDash n$ and δ be any composition. The triple

$$(\mathrm{W}_{\alpha,\delta}, C_n, \mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q))$$

refines the CSP triple $(\mathrm{W}_\alpha, C_n, \mathrm{W}^{\mathrm{maj}}_{\alpha}(q))$.

Indeed, we derive an explicit product formula for $\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q)$ mod $(q^n - 1)$ involving q-binomial coefficients, Theorem 3.4.19. The formula results in a q-identity similar to the Vandermonde convolution identity; see Corollary 3.4.20. The argument involves constructing $\mathrm{W}_{\alpha,\delta}$ algorithmically by recursively building a certain tree.

The two-letter case of Theorem 3.1.3 can be rephrased as follows. Fix $n \in \mathbb{Z}_{\geq 1}$ and $k, b \in \mathbb{Z}_{\geq 0}$. Let $S_{k,b}$ denote the set of subsets Δ of $\mathbb{Z}/n$ of size k where $\#\{i \in \Delta : i+1 \notin \Delta\} = b$. Define the statistic $\mathrm{mbs}\colon S_{k,b} \to \mathbb{N}$ by identifying $\mathbb{Z}/n$ with $\{1, \ldots, n\}$ and setting $\mathrm{mbs}(\Delta) := \sum_{i \in \Delta : i+1 \notin \Delta} i$, which sums the maximum of the cyclic blocks of Δ.

Corollary 3.1.4. The triple

$$(S_{k,b}, C_n, S^{\mathrm{mbs}}_{k,b}(q))$$

exhibits the CSP.

Example 3.1.5. When $n = 5, k = 3, b = 2$,

$$S_{k,b} = \{\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 1\}, \{4, 5, 2\}, \{5, 1, 3\}\},$$

which have mbs statistic $6, 8, 5, 7, 4$, respectively, so $S^{\text{mbs}}_{k,b}(q) = q^4 + q^5 + q^6 + q^7 + q^8$. We then have $S^{\text{mbs}}_{k,b}(\omega_5) = 0, S^{\text{mbs}}_{k,b}(1) = 5$, in agreement with (1.1).

Theorem 8.3 in [RSW04] and hence Theorem 2.1.6 builds on a representation-theoretic result due to Springer [Spr74, Proposition 4.5]. Our argument is highly combinatorial, but it is not entirely bijective. Finding an explicit bijection would be quite interesting. See Section 3.7 for more details.

A key building block of our proof of Theorem 3.1.3 involves cyclic sieving on multisubsets and subsets, which was also first stated in [RSW04]. We describe refinements of these results as well,Theorem 3.6.4 and Theorem 3.6.11, restricting to certain gcd requirements in the subset case. We present a completely different inductive proof of our subset refinement in the spirit of our proof of Theorem 3.1.3. Both our proof of Theorem 3.6.11 and Theorem 3.1.3 use an extension lemma, Lemma 3.2.3, which allows us to extend CSPs from smaller cyclic groups to larger ones.

The rest of this chapter is organized as follows. In Section 3.2, we introduce the concept of modular periodicity and prove our extension lemma, Lemma 3.2.3. In Section 3.3, we define cyclic descent type. In Section 3.4, we decompose words with fixed content and cyclic descent type and prove a product formula for $\mathrm{W}^{\text{maj}}_{\alpha,\delta}(q)$ modulo $q^n - 1$, Theorem 3.4.19. Section 3.5 uses the results of Section 3.4 to prove our main result, Theorem 3.1.3. Section 3.6 refines cyclic sieving on multisubsets and subsets with respect to shifted sum statistics. In Section 3.7, we introduce the flex statistic and use it to reinterpret Theorem 3.1.3.

3.2 Modular Periodicity and an Extension Lemma

We now introduce the concept of *modular periodicity* and use it to give an extension lemma, Lemma 3.2.3, which allows us to extend CSP's from certain subgroups to larger groups. We

will verify the hypotheses of Lemma 3.2.3 in the subsequent sections to deduce Theorem 3.1.3.

Definition 3.2.1. We say a statistic $\mathrm{stat}\colon W \to \mathbb{Z}$ has *period a modulo b on W* if for all $i \in \mathbb{Z}$,

$$\#\{w \in W : \mathrm{stat}(w) \equiv_b i\} = \#\{w \in W : \mathrm{stat}(w) \equiv_b i + a\}.$$

Similarly, we say a Laurent polynomial $f(q) \in \mathbb{C}[q, q^{-1}]$ has *period a modulo b* if

$$q^a f(q) \equiv f(q) \pmod{q^b - 1},$$

or equivalently if $(q^b - 1) \mid (q^a - 1)f(q)$.

For example, $1 + 5q + q^2 + 5q^3 + q^4 + 5q^5$ has period 2 modulo 6. Note that stat has period a modulo b on W if and only if $W^{\mathrm{stat}}(q)$ has period a modulo b. The following basic properties of modular periodicity will be useful throughout this chapter.

Lemma 3.2.2. Let $f(q) \in \mathbb{C}[q, q^{-1}]$ and $a, b, c \in \mathbb{Z}$.

(i) If $f(q)$ has period a modulo c and period b modulo c, then $f(q)$ has period $ua + vb$ modulo c for any $u, v \in \mathbb{Z}$. In particular, $f(q)$ has period $\gcd(a, b)$ modulo c.

(ii) If $f(q)$ has period a modulo b and period b modulo c, then $f(q)$ has period a modulo c.

(iii) If $f(q)$ has period a modulo c and $b \mid c$, then $f(q)$ has period a modulo b.

(iv) If $f(q)$ has period a modulo b, then so does $f(q)h(q)$ for any Laurent polynomial $h(q)$.

(v) If $f(q)$ has period a modulo b and $a \mid b$, then

$$f(q) \equiv \frac{a}{b}\left(\frac{q^b - 1}{q^a - 1}\right) f(q) \pmod{q^b - 1}.$$

Proof. (i), (iii), (iv), and (v) are straightforward. For (ii), suppose

$$(q^b - 1) \mid (q^a - 1)f(q), \qquad (q^c - 1) \mid (q^b - 1)f(q).$$

Write $q^c - 1 = \prod_{k=1}^{c}(q - \mathrm{W}_c^k)$. If $q - \omega_c^k$ does not divide $f(q)$, then it must divide $q^b - 1$ and hence $q^a - 1$. It follows that

$$(q^c - 1) \mid (q^a - 1)f(q).$$

□

Lemma 3.2.3. Suppose $C_n = \langle \sigma_n \rangle$ acts on W. Let $g \mid n$ and $C_g := \langle \sigma_n^{n/g} \rangle \subset C_n$. If

(i) $(W, C_g, f(q))$ exhibits the CSP,

(ii) $f(q)$ has period g modulo n, and

(iii) for all C_n-orbits $\mathcal{O} \subset W$, we have $\frac{n}{|\mathcal{O}|} \mid g$,

then $(W, C_n, f(q))$ exhibits the CSP.

Proof. Let

$$F(q) := \sum_{C_n\text{-orbits } \mathcal{O} \subset W} \frac{q^n - 1}{q^{n/|\mathcal{O}|} - 1}.$$

By (1.2), $(W, C_n, F(q))$ exhibits the CSP, so $(W, C_g, F(q))$ also exhibits the CSP by Theorem 2.1.3. Thus, by (1.2) and condition (i),

$$f(q) = F(q) + p(q)(q^g - 1) \tag{3.2}$$

for some $p(q) \in \mathbb{C}[q]$. Each summand of $F(q)$ has period g modulo n since

$$(q^n - 1) \mid (q^g - 1)\frac{q^n - 1}{q^{n/|\mathcal{O}|} - 1},$$

by condition (iii). Putting this together with condition (ii), $f(q)$ and $F(q)$ have period g

modulo n. Using Lemma 3.2.2(v) twice along with (3.2) now gives

$$\begin{aligned} f(q) &\equiv \frac{g}{n}\frac{q^n-1}{q^g-1}f(q) \\ &= \frac{g}{n}\frac{q^n-1}{q^g-1}(F(q)+p(q)(q^g-1)) \\ &\equiv \frac{g}{n}\frac{q^n-1}{q^g-1}F(q) \\ &\equiv F(q) \qquad \mod (q^n-1). \end{aligned}$$

□

3.3 Cyclic Descent Type

In this section, we introduce the *cyclic descent type* of a word. We also verify hypothesis (iii) of Lemma 3.2.3 for $\mathrm{W}_{\alpha,\delta}$ for a particular g; see Lemma 3.3.3.

Let $w^{(i)}$ denote the subsequence of w with all letters larger than i removed. We have a "filtration"

$$\varnothing \preceq w^{(1)} \preceq w^{(2)} \preceq \cdots \preceq w^{(m-1)} \preceq w^{(m)} = w,$$

where $u \preceq v$ means that u is a subsequence of v. We think of this filtration as building up w by recursively adding all of the copies of the next largest letter "where they fit." The *cyclic descent type* of a word w, denoted $\mathrm{CDT}(w)$, is the sequence which tracks the number of new cyclic descents at each stage of the filtration. Precisely, we have the following.

Definition 3.3.1. The *cyclic descent type* (CDT) of a word w is the weak composition of $\mathrm{cdes}(w)$ given by

$$\mathrm{CDT}(w) := (\mathrm{cdes}(w^{(1)}), \mathrm{cdes}(w^{(2)}) - \mathrm{cdes}(w^{(1)}), \ldots, \mathrm{cdes}(w^{(m)}) - \mathrm{cdes}(w^{(m-1)})). \tag{3.3}$$

Note that CDT is constant on necklaces since rotating w rotates each $w^{(i)}$ and cdes is constant under rotations. Furthermore, $\mathrm{cdes}(w^{(1)}) = 0$ always, so $\mathrm{CDT}(w)$ always begins with 0.

Example 3.3.2. Suppose $w = 143124114223$, so

$$\begin{aligned} w^{(1)} &= 1111 & \operatorname{cdes}(w^{(1)}) &= 0, \\ w^{(2)} &= 112.1122. & \operatorname{cdes}(w^{(2)}) &= 2, \\ w^{(3)} &= 13.12.11223. & \operatorname{cdes}(w^{(3)}) &= 3, \\ w^{(4)} &= 14.3.124.114.223. & \operatorname{cdes}(w^{(4)}) &= 5. \end{aligned}$$

Hence, $\operatorname{CDT}(143124114223) = (0, 2-0, 3-2, 5-3) = (0, 2, 1, 2)$.

Recall from (3.1) that

$$\mathrm{W}_{\alpha,\delta} := \{w \in \mathrm{W}_n : \operatorname{cont}(w) = \alpha, \operatorname{CDT}(w) = \delta\}.$$

We could define $\mathrm{W}_{\alpha,\delta}$ more "symmetrically" by replacing cont with "cyclic weak ascent type," which would be the point-wise difference of cont and CDT. However, content is ubiquitous in the literature, so we use it.

Lemma 3.3.3. If $\alpha = (\alpha_1, \ldots, \alpha_m)$, $\delta = (\delta_1, \ldots, \delta_m)$, $N \subset \mathrm{W}_{\alpha,\delta}$ is a necklace, and $g := gcd(\alpha_1, \ldots, \alpha_m, \delta_1, \ldots \delta_m)$, then $\frac{n}{|N|} \mid g$.

Proof. We can write $N = [w]$ with $w = u^{\frac{n}{|N|}}$ since $\operatorname{freq}(w) = \frac{n}{|N|}$. Hence, using pointwise multiplication,

$$\operatorname{cont}(w) = \frac{n}{|N|} \cdot \operatorname{cont}(u), \qquad \operatorname{CDT}(w) = \frac{n}{|N|} \cdot \operatorname{CDT}(u).$$

In particular, $\frac{n}{|N|}$ divides $\alpha_1, \ldots, \alpha_m, \delta_1, \ldots \delta_m$, so $\frac{n}{|N|} \mid g$.

□

3.4 Runs and Falls

In this section, we give a method to algorithmically construct $\mathrm{W}_{\alpha,\delta}$ and use it to prove a product formula for $\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q)$, Theorem 3.4.19. We conclude the section by using this formula

to verify hypothesis (ii) of Lemma 3.2.3 for $\mathrm{W}_{\alpha,\delta}$; see Proposition 3.4.21.

3.4.1 A Tree Decomposition for $\mathrm{W}_{\alpha,\delta}$

We now describe a way to create words with a fixed content and CDT in terms of insertions into *runs* and *falls*. This procedure is organized into a tree, Definition 3.4.11, whose edges are labeled with sets and multisets. Lemma 3.4.8 describes changes in the major index upon traversing an edge of this tree.

Definition 3.4.1. Write $w = w_1 \cdots w_n \in \mathrm{W}_n$. A *fall* in w is a maximal set of distinct consecutive indices $i, i+1, \ldots, j-1, j$ such that $w_i > w_{i+1} > \cdots > w_j$, where we take indices modulo n. A *run* in a non-constant word w is a maximal set of distinct consecutive indices $i, i+1, \ldots, j$ such that $w_i \leq w_{i+1} \leq \cdots \leq w_j$, where we take indices modulo n. The constant word $w = \ell^n$ by convention has no runs, and it has n falls.

Note that each letter in w is part of a unique fall and a unique run, except when $w = \ell^n$ is constant. Index falls from 0 from left to right starting at the fall containing the first letter of w, and do the same with runs. It is easy to see that w has $n - \mathrm{cdes}(w)$ falls and $\mathrm{cdes}(w)$ runs, since they are separated by cyclic weak ascents and cyclic descents, respectively.

Definition 3.4.2. We write

$$F(w) := [0, |w| - \mathrm{cdes}(w) - 1] \qquad \text{and} \qquad R(w) := [0, \mathrm{cdes}(w) - 1]$$

for the indices of the falls and runs of w, respectively.

Example 3.4.3. Let $w = 26534611 = 2\dot{6}53\dot{4}6\dot{1}1\dot{} = 26.5.346.11 \in \mathrm{W}_8$, where upper dots indicate cyclic weak ascents and lower dots indicate cyclic descents. Since $\mathrm{cdes}(w) = 3$, we have $F(w) = [0, 4]$ and $R(w) = [0, 2]$. The 5 falls of w are $2, 653, 4, 61, 1$, with respective indices $0, 1, 2, 3, 4$. The 3 runs of w are $1126, 5, 346$, with respective indices $0, 1, 2$.

Definition 3.4.4. Fix a letter ℓ and pick a subset F of the falls $F(w)$. Assume ℓ does not appear in any of the falls in F. We *insert ℓ into falls F* by successively inserting ℓ into each fall $w_i > w_{i+1} > \cdots > w_j$ in F so that $w_i \cdots \ell \cdots w_j$ is still decreasing.

Similarly, we may fix a letter ℓ and pick a *multisubset* R of $R(w)$ (this time ℓ may already appear in a run in R). We *insert ℓ into runs R* by successively inserting ℓ into each run $w_i \leq w_{i+1} \leq \cdots \leq w_j$ in R so that $w_i \cdots \ell \cdots w_j$ is still weakly increasing.

When inserting ℓ into a run already containing ℓ, the resulting word is independent of precisely which of the possible positions is used. This is the reason we insert into runs and falls instead of positions.

Note that there is a slight ambiguity in our description of insertion into falls and runs, since it may be possible to insert either at the beginning or at the end of w while still satisfying the relevant inequalities. Given the choice, we always insert at the beginning of w.

Example 3.4.5. Let $w = 2\dot{6}5\dot{3}\dot{4}6\dot{1}\dot{1}$. Insert 7 into falls of w with indices 0 and 3 to successively obtain $\underline{7}2\dot{6}5\dot{3}\dot{4}6\dot{1}\dot{1}$ and then $w' := 72\dot{6}5\dot{3}\dot{4}\underline{7}6\dot{1}\dot{1}$. Note that $w' = 7.26.5.347.6.11$ has two more runs (or cyclic descents) than w. Now insert 7 into the runs of w' with multiset of indices $\{0, 2, 3, 3\}$ to successively obtain $\underline{7}7.26.5.347.6.11$, $77.26.5\underline{7}.347.6.11$, $77.26.57.34\underline{7}7.6.11$, and $w'' := 77.26.57.34\underline{7}77.6.11$.

Let

$$\widetilde{\mathrm{W}}_n = \{w \in \mathrm{W}_n : w \text{ ends in a } 1\}, \tag{3.4}$$

$$\widetilde{\mathrm{W}}_{\alpha,\delta} = \{w \in \mathrm{W}_{\alpha,\delta} : w \text{ ends in a } 1\}. \tag{3.5}$$

We restrict to $\widetilde{\mathrm{W}}_n$ and $\widetilde{\mathrm{W}}_{\alpha,\delta}$ since the major index generating function is easier to find and extends to $\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q) \pmod{q^n - 1}$.

Definition 3.4.6. Fix $w \in \widetilde{\mathrm{W}}_n$, a letter ℓ not in w, and

$$F \subset F(w) = [0, |w| - \mathrm{cdes}(w) - 1] \qquad \text{and} \qquad R \underset{\text{mult.}}{\subset} [0, \mathrm{cdes}(w) + |F| - 1]$$

where $\underset{\text{mult.}}{\subset}$ denotes a multisubset. Let w' be obtained by inserting ℓ into falls F of w. Note that $[0, \operatorname{cdes}(w)+|F|-1] = R(w')$ indexes the runs of w'. Now let w'' be obtained by inserting ℓ into runs R of w'. We say w'' is obtained by *inserting the triple* (ℓ, F, R) into w. Observe that $\operatorname{cdes}(w'') = \operatorname{cdes}(w') = \operatorname{cdes}(w) + |F|$ and $w'' \in \widetilde{\mathrm{W}}_{n+|F|+|R|}$.

First we define the cyclic descent type $\delta = \mathrm{CDT}(w)$ of any $w \in \mathrm{W}_\alpha$. Then we give a product formula for $\mathrm{W}_{\alpha,\delta}^{\mathrm{maj}}(q)$ modulo $q^n - 1$, Theorem 3.4.19. The $q = 1$ specialization gives a formula for $\#\,\mathrm{W}_{\alpha,\delta}$, Proposition 3.4.16. Along the way, we describe how to build words in $\mathrm{W}_{\alpha,\delta}$ by walking along a tree whose edges are labeled by sets and multisets. We describe a fixed point lemma arising from the tree which will play a key role in Section 3.5. We also introduce the notion of modular periodicity in order to transfer certain results between different cyclic groups or moduli.

We next describe the effect of inserting a single letter on maj. We restrict to $\widetilde{\mathrm{W}}_n$ so we preserve a cyclic weak ascent at the end and never add a letter to the end. The fact that the increments in major index from inserting a new letter into all possible positions form a permutation was first observed by Gupta [Gup78]. Lemma 3.4.7 tells us exactly the increment in major index based on which run or fall the newly inserted letter fits into.

Lemma 3.4.7. Suppose $w' \in \widetilde{\mathrm{W}}_{n+1}$ is obtained by adding a letter ℓ to $w \in \widetilde{\mathrm{W}}_n$ in any position. Then w' is obtained by inserting ℓ into some run or fall of w, and

$$\operatorname{maj}(w') - \operatorname{maj}(w) = \begin{cases} \operatorname{cdes}(w) - r & \text{if } \ell \text{ is inserted into run } r \text{ of } w \\ \operatorname{cdes}(w) + 1 + f & \text{if } \ell \text{ is inserted into fall } f \text{ of } w. \end{cases} \tag{3.6}$$

Proof. If $\operatorname{cdes}(w') = \operatorname{cdes}(w)$, then ℓ is inserted into some run of w, and otherwise $\operatorname{cdes}(w') = \operatorname{cdes}(w) + 1$ and ℓ is inserted into some fall of w. Inserting ℓ into run r of w will increment the position of $\operatorname{cdes}(w) - r$ descents by 1 each, so

$$\operatorname{maj}(w') - \operatorname{maj}(w) = \operatorname{cdes}(w) - r.$$

Let $\mathrm{comaj}(w) := 1 + 2 + \cdots + (|w| - 1) - \mathrm{maj}(w)$, which is the sum of $i \in [|w| - 1]$ where $w_i \leq w_{i+1}$. Inserting ℓ into fall f of w will increment the position of $(|w| - 1) - \mathrm{cdes}(w) - f$ weak ascents by 1 each, so

$$\mathrm{comaj}(w') - \mathrm{comaj}(w) = (|w| - 1) - \mathrm{cdes}(w) - f,$$

from which it follows that

$$\mathrm{maj}(w') - \mathrm{maj}(w) = \mathrm{cdes}(w) + 1 + f.$$

□

Lemma 3.4.8. Suppose w'' is obtained by inserting the triple (ℓ, F, R) into $w \in \widetilde{\mathrm{W}}_n$. Then

$$\mathrm{maj}(w'') - \mathrm{maj}(w) = \binom{|F|+1}{2} + (\mathrm{cdes}(w))(|F| + |R|) + |F||R| + \sum_{f \in F} f - \sum_{r \in R} r. \tag{3.7}$$

Proof. Let w' be obtained by inserting ℓ into falls F of w. It suffices to show

$$\mathrm{maj}(w') - \mathrm{maj}(w) = \binom{|F|+1}{2} + (\mathrm{cdes}(w))|F| + \sum_{f \in F} f \tag{3.8}$$

and

$$\mathrm{maj}(w'') - \mathrm{maj}(w') = (\mathrm{cdes}(w'))|R| - \sum_{r \in R} r \tag{3.9}$$

since $\mathrm{cdes}(w') = \mathrm{cdes}(w'') = \mathrm{cdes}(w) + |F|$. Both (3.8) and (3.9) follow from iterating Lemma 3.4.7 and recalling cdes is incremented by 1 each time we insert into a fall. □

Notation 3.4.9. For the rest of this section, fix a strong composition $\alpha = (\alpha_1, \ldots, \alpha_m)$ of $n \geq 1$ and $\delta = (\delta_1, \ldots, \delta_m) \vDash k$ with $\delta_1 = 0$. We emphasize that α and δ have the same

number, m, of parts. For $\ell = 1, \ldots, m$, let

$$n_\ell := \alpha_1 + \cdots + \alpha_\ell, \tag{3.10}$$

$$k_\ell := \delta_1 + \cdots + \delta_\ell. \tag{3.11}$$

For $w \in \mathrm{W}_{\alpha,\delta}$, we have the defining conditions $|w^{(\ell)}| = n_\ell$ and $\mathrm{cdes}(w^{(\ell)}) = k_\ell$. Furthermore, let

$$\mathrm{S}_\ell := \binom{[0, n_{\ell-1} - k_{\ell-1} - 1]}{\delta_\ell}, \qquad \mathrm{M}_\ell := \left(\!\!\binom{[0, k_\ell - 1]}{\alpha_\ell - \delta_\ell}\!\!\right)$$

and

$$g := \gcd(\alpha_1, \alpha_2, \ldots, \alpha_m, \delta_1, \delta_2, \ldots, \delta_m).$$

If $w \in \widetilde{\mathrm{W}}_{\alpha,\delta}$, then the set S_ℓ consists of all subsets of the falls $F(w^{(\ell-1)})$ which, when ℓ is inserted into those falls of $w^{(\ell-1)}$, result in a word w' with k_ℓ cyclic descents. The multiset M_ℓ similarly consists of all choices of runs $R(w')$ which, when ℓ is inserted into those runs, result in a word with length n_ℓ.

Remark 3.4.10. We restrict to strong compositions α for notational simplicity, though the results in this section may easily be generalized to arbitrary weak compositions by "flattening" weak compositions to strong ones by removing zeros.

Definition 3.4.11. Construct a rooted, vertex-labeled and edge-labeled tree $T_{\alpha,\delta}$ recursively as follows. Begin with a tree $T^{(1)}$ containing only a root labeled by the word 1^{α_1}. For $\ell = 2, \ldots, m$, to obtain $T^{(\ell)}$, do the following. For each leaf w of $T^{(\ell-1)}$ and for each triple (ℓ, F, R) with

$$F \in \mathrm{S}_\ell \qquad \text{and} \qquad R \in \mathrm{M}_\ell,$$

add an edge labeled by (F, R) to $T^{(\ell-1)}$ from w to w'' where w'' is obtained by inserting (ℓ, F, R) into w. Define $T_{\alpha,\delta} := T^{(m)}$.

Example 3.4.12. Let $\alpha = (4, 2, 3)$ and $\delta = (0, 2, 1)$. Figure 3.1 is the subgraph of $T_{\alpha,\delta}$ consisting of paths from the root to leaves that are rotations of 112113323.

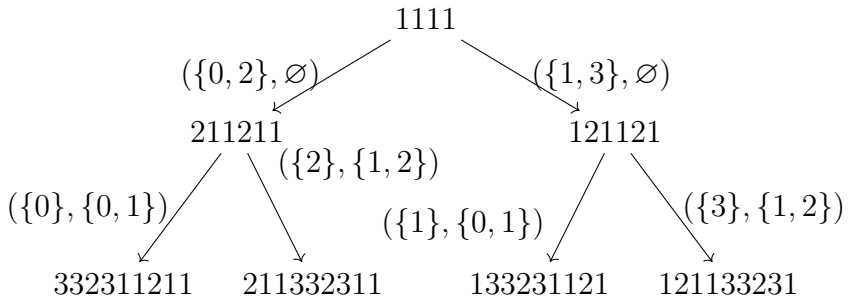


Figure 3.1: Subgraph of tree $T_{\alpha,\delta}$ with $\alpha = (4, 2, 3)$, $\delta = (0, 2, 1)$.

For this full $T_{\alpha,\delta}$, the root has $\binom{4}{2} = 6$ children since 1111 has 4 falls. Each child of the root itself has $\binom{4}{1}\left(\!\binom{3}{2}\!\right) = 24$ children. Hence, $T_{\alpha,\delta}$ has 144 leaves. Notice that the cyclic rotations of 311211332 appearing as leaves in Figure 3.1 are precisely those ending in 1. It will shortly become apparent that in this example, $\#\,\mathrm{W}_{\alpha,\delta} = \frac{9}{4} \cdot 144 = 324$.

Lemma 3.4.13. The vertices of $T_{\alpha,\delta}$ which are $\ell < m$ edges away from the root are precisely the elements of $\{w^{(\ell+1)} : w \in \widetilde{\mathrm{W}}_{\alpha,\delta}\}$, each occurring once. In particular, the leaves of $T_{\alpha,\delta}$ are precisely the elements of $\widetilde{\mathrm{W}}_{\alpha,\delta}$, each occurring once.

Proof. By definition of S_ℓ and M_ℓ, any leaf of $T^{(\ell)}$ has content $(\alpha_1, \ldots, \alpha_\ell)$, cyclic descent type $(\delta_1, \ldots, \delta_\ell)$, and ends in a 1, so is in $\{w^{(\ell)} : w \in \widetilde{\mathrm{W}}_{\alpha,\delta}\}$. Conversely, given any $w \in \widetilde{\mathrm{W}}_{\alpha,\delta}$, the word $w^{(\ell)}$ is obtained by inserting a unique triple (ℓ, F, R) into $w^{(\ell-1)}$ by repeated applications of Lemma 3.4.7. □

Definition 3.4.14. By Lemma 3.4.13, the tree $T_{\alpha,\delta}$ encodes a bijection

$$\Phi \colon \widetilde{\mathrm{W}}_{\alpha,\delta} \xrightarrow{\sim} \prod_{\ell=2}^{m} \mathrm{S}_\ell \times \mathrm{M}_\ell$$

given by reading the edge labels from the root to w. We suppress the dependence of Φ on α and δ from the notation since they can be computed from the input w.

Lemma 3.4.15. For any $w \in \mathrm{W}_{\alpha,\delta}$,

$$\#[w] = \frac{n}{\alpha_1} \cdot \#\left([w] \cap \widetilde{\mathrm{W}}_{\alpha,\delta}\right). \tag{3.12}$$

Consequently,

$$\#\,\mathrm{W}_{\alpha,\delta} = \frac{n}{\alpha_1} \#\widetilde{\mathrm{W}}_{\alpha,\delta}. \tag{3.13}$$

Proof. Each $w \in \mathrm{W}_{\alpha,\delta}$ has $\mathrm{period}(w) = n/\operatorname{freq}(w)$ distinct cyclic rotations, of which $\alpha_1/\operatorname{freq}(w)$ end in 1. □

Proposition 3.4.16. Using Notation 3.4.9, we have

$$\#\,\mathrm{W}_{\alpha,\delta} = \frac{n}{\alpha_1} \prod_{\ell=2}^{m} \binom{n_{\ell-1} - k_{\ell-1}}{\delta_\ell} \left(\!\!\binom{k_\ell}{\alpha_\ell - \delta_\ell}\!\!\right). \tag{3.14}$$

In particular, $\mathrm{W}_{\alpha,\delta} \neq \varnothing$ if and only if

$$\begin{aligned} 0 \leq \delta_\ell \leq \alpha_\ell \qquad & \text{for all } 1 \leq \ell \leq m, \text{and} \\ \delta_1 + \cdots + \delta_{\ell+1} \leq \alpha_1 + \cdots + \alpha_\ell \qquad & \text{for all } 1 \leq \ell < m. \end{aligned} \tag{3.15}$$

Proof. The product in (3.14) is $\#\prod_{\ell=2}^{m} \mathrm{S}_\ell \times \mathrm{M}_\ell$, which is $\#\widetilde{\mathrm{W}}_{\alpha,\delta}$ by the bijection Φ. Now (3.14) follows from (3.13), and (3.15) follows from (3.14). □

3.4.2 Major Index Generating Functions

We next use the bijection Φ and Lemma 3.4.8 to give a product formula for $\widetilde{\mathrm{W}}^{\mathrm{maj}}_{\alpha,\delta}(q)$, Theorem 3.4.17. We then use modular periodicity to obtain an analogous expression for $\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q)$ modulo $q^n - 1$, Theorem 3.4.19.

Theorem 3.4.17. Using Notation 3.4.9, we have

$$\widetilde{\mathrm{W}}^{\mathrm{maj}}_{\alpha,\delta}(q) = \prod_{\ell=2}^{m} q^{k_\ell \alpha_\ell} \binom{n_{\ell-1}-k_{\ell-1}}{\delta_\ell}_q \left(\!\!\binom{k_\ell}{\alpha_\ell-\delta_\ell}\!\!\right)_{q^{-1}} \tag{3.16}$$

$$= q^{\eta(\alpha,\delta)} \prod_{\ell=2}^{m} \binom{n_{\ell-1}-k_{\ell-1}}{\delta_\ell}_q \left(\!\!\binom{k_\ell}{\alpha_\ell-\delta_\ell}\!\!\right)_{q} \tag{3.17}$$

where

$$\eta(\alpha,\delta) := n - \alpha_1 + \binom{k}{2} + \sum_{\ell=2}^{m} \binom{\delta_\ell}{2}.$$

Proof. Combining Φ with Lemma 3.4.8 shows that

$$\widetilde{\mathrm{W}}^{\mathrm{maj}}_{\alpha,\delta}(q) = \prod_{\ell=2}^{m} \sum_{\substack{F\in \mathrm{S}_\ell \\ R\in \mathrm{M}_\ell}} q^{\epsilon(\ell,F,R)}, \tag{3.18}$$

where

$$\epsilon(\ell,F,R) := \binom{\delta_\ell+1}{2} + k_{\ell-1}\alpha_\ell + \delta_\ell(\alpha_\ell-\delta_\ell) + \mathrm{sum}(F) - \mathrm{sum}(R).$$

Noting that

$$\binom{\delta_\ell+1}{2} + k_{\ell-1}\alpha_\ell + \delta_\ell(\alpha_\ell-\delta_\ell) = k_\ell\alpha_\ell - \binom{\delta_\ell}{2},$$

simplifying (3.18) gives

$$\widetilde{\mathrm{W}}^{\mathrm{maj}}_{\alpha,\delta}(q) = \prod_{\ell=2}^{m} q^{k_\ell\alpha_\ell - \binom{\delta_\ell}{2}} \, \mathrm{S}^{\mathrm{sum}}_\ell(q) \, \mathrm{M}^{\mathrm{sum}}_\ell(q^{-1}). \tag{3.19}$$

Equation (3.16) now follows from (2.4), (2.5), and the definition of S_ℓ and M_ℓ. As for (3.17), consider the reversal bijection $r\colon \mathrm{M}_\ell \to \mathrm{M}_\ell$ induced by

$$x \mapsto k_\ell - 1 - x$$

on $[0, k_\ell - 1]$. This bijection satisfies $\mathrm{sum}(r(A)) = (k_\ell - 1)(\alpha_\ell - \delta_\ell) - \mathrm{sum}(A)$, so

$$\mathrm{M}_\ell^{\mathrm{sum}}(q^{-1}) = q^{-(k_\ell-1)(\alpha_\ell-\delta_\ell)}\,\mathrm{M}_\ell^{\mathrm{sum}}(q). \tag{3.20}$$

Plugging (3.20) into (3.19) and noting that

$$\begin{aligned}
\sum_{\ell=2}^{m}\left(k_\ell\alpha_\ell - \binom{\delta_\ell}{2} - (k_\ell-1)(\alpha_\ell-\delta_\ell)\right) &= \sum_{\ell=2}^{m}\left(\alpha_\ell - \frac{\delta_\ell}{2} - \frac{\delta_\ell^2}{2} + k_\ell\delta_\ell\right) \\
&= n - \alpha_1 - \frac{k}{2} + \sum_{\ell=2}^{m}\left(-\frac{\delta_\ell^2}{2} + \sum_{j=2}^{\ell}\delta_j\delta_\ell\right) \\
&= n - \alpha_1 - \frac{k}{2} + \frac{1}{2}\sum_{\ell=2}^{m}\sum_{j=2}^{m}\delta_j\delta_\ell \\
&= n - \alpha_1 - \frac{k}{2} + \frac{k^2}{2}
\end{aligned}$$

gives

$$\widetilde{\mathrm{W}}_{\alpha,\delta}^{\mathrm{maj}}(q) = q^{n-\alpha_1+\binom{k}{2}}\prod_{\ell=2}^{m}\mathrm{S}_\ell^{\mathrm{sum}}(q)\,\mathrm{M}_\ell^{\mathrm{sum}}(q).$$

Using (2.4) and (2.5) now yields (3.17). □

Lemma 3.4.18. Let $\alpha \vDash n$, $\delta \vDash k$. The statistic maj has period k modulo n on $\mathrm{W}_{\alpha,\delta}$. Moreover, maj is constant modulo $d := \gcd(n,k)$ on necklaces in $\mathrm{W}_{\alpha,\delta}$, and

$$\mathrm{W}_{\alpha,\delta}^{\mathrm{maj}}(q) \equiv \frac{n}{\alpha_1}\widetilde{\mathrm{W}}_{\alpha,\delta}^{\mathrm{maj}}(q) \qquad (\mathrm{mod}\ q^d - 1). \tag{3.21}$$

Proof. Since cyclically rotating $w \in \mathrm{W}_{\alpha,\delta}$ increments each cyclic descent by 1 modulo n, we have

$$\mathrm{maj}(\sigma_n \cdot w) \equiv_n \mathrm{maj}(w) + k. \tag{3.22}$$

In particular, maj has period k modulo n on necklaces in $\mathrm{W}_{\alpha,\delta}$. Furthermore, maj is constant on necklaces in $\mathrm{W}_{\alpha,\delta}$ modulo d. Now (3.21) follows from (3.12). □

Theorem 3.4.19. Using Notation 3.4.9, let $d := \gcd(n,k)$. Then, modulo q^n-1,

$$\begin{aligned}\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q) &\equiv \frac{d}{\alpha_1}\left(\frac{q^n-1}{q^d-1}\right)\prod_{\ell=2}^{m} q^{k_\ell\alpha_\ell}\binom{n_{\ell-1}-k_{\ell-1}}{\delta_\ell}_q \left(\!\!\binom{k_\ell}{\alpha_\ell-\delta_\ell}\!\!\right)_{q^{-1}} \\ &\equiv \frac{d}{\alpha_1}\left(\frac{q^n-1}{q^d-1}\right) q^{\binom{k}{2}+\sum_{\ell=2}^{m}\binom{\delta_\ell}{2}-\alpha_1}\prod_{\ell=2}^{m}\binom{n_{\ell-1}-k_{\ell-1}}{\delta_\ell}_q \left(\!\!\binom{k_\ell}{\alpha_\ell-\delta_\ell}\!\!\right)_q .\end{aligned} \tag{3.23}$$

Proof. By Lemma 3.4.18, maj has period k modulo n on $\mathrm{W}_{\alpha,\delta}$. Hence by Lemma 3.2.2(i), maj has period d modulo n on $\mathrm{W}_{\alpha,\delta}$. Using Lemma 3.2.2(v) and (3.21) gives

$$\begin{aligned}\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q) &\equiv \frac{d}{n}\left(\frac{q^n-1}{q^d-1}\right)\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q) \\ &\equiv \frac{d}{n}\left(\frac{q^n-1}{q^d-1}\right)\left(\frac{n}{\alpha_1}\widetilde{\mathrm{W}}^{\mathrm{maj}}_{\alpha,\delta}(q)+p(q)(q^d-1)\right) \\ &\equiv \frac{d}{\alpha_1}\left(\frac{q^n-1}{q^d-1}\right)\widetilde{\mathrm{W}}^{\mathrm{maj}}_{\alpha,\delta} \qquad (\mathrm{mod}\, q^n-1),\end{aligned}$$

where $p(q)\in\mathbb{C}[q]$. Theorem 3.4.19 now follows from Theorem 3.4.17. □

Corollary 3.4.20. Using Notation 3.4.9, let $d := \gcd(n,k)$. Then, modulo q^n-1,

$$\mathrm{W}^{\mathrm{maj}}_{\alpha}(q) = \binom{n}{\alpha}_q \equiv \sum_{\delta}\frac{d}{\alpha_1}\left(\frac{q^n-1}{q^d-1}\right)\prod_{\ell=2}^{m} q^{k_\ell\alpha_\ell}\binom{n_{\ell-1}-k_{\ell-1}}{\delta_\ell}_q \left(\!\!\binom{k_\ell}{\alpha_\ell-\delta_\ell}\!\!\right)_{q^{-1}} \tag{3.24}$$

where the sum is over weak compositions δ of k satisfying (3.15). In particular,

$$\#\,\mathrm{W}_\alpha = \binom{n}{\alpha} = \sum_{\delta}\frac{n}{\alpha_1}\prod_{\ell=2}^{m}\binom{n_{\ell-1}-k_{\ell-1}}{\delta_\ell}\left(\!\!\binom{k_\ell}{\alpha_\ell-\delta_\ell}\!\!\right). \tag{3.25}$$

Note that the two-letter case of (3.25) is a special case of the classical Vandermonde convolution identity [Sta12, Ex. 1.1.17].

3.4.3 Verifying Hypothesis (ii) of Lemma 3.2.3 for $\mathrm{W}_{\alpha,\delta}$

Proposition 3.4.21. Using Notation 3.4.9, $\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q)$ has period g modulo n.

Proof. Let $d = \gcd(n,k)$. By Theorem 3.4.19,

$$\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q) \equiv \frac{d}{\alpha_1}\left(\frac{q^n-1}{q^d-1}\right)\prod_{\ell=2}^{m} q^{k_\ell\alpha_\ell}\binom{n_{\ell-1}-k_{\ell-1}}{\delta_\ell}_q \left(\!\!\binom{k_\ell}{\alpha_\ell-\delta_\ell}\!\!\right)_{q^{-1}}$$

modulo q^n-1. The action of rotation on elements of $\mathrm{S}_\ell = \binom{[0,n_{\ell-1}-k_{\ell-1}]}{\delta_\ell}$ increases their sum by δ_ℓ modulo $n_{\ell-1}-k_{\ell-1}$. Thus by (2.4), $\binom{n_{\ell-1}-k_{\ell-1}}{\delta_\ell}_q$ has period δ_ℓ modulo $n_{\ell-1}-k_{\ell-1}$. Similarly by (2.5), $\left(\!\!\binom{k_\ell}{\alpha_\ell-\delta_\ell}\!\!\right)_{q^{-1}}$ has period $\alpha_\ell-\delta_\ell$ modulo k_ℓ. For $\ell = 2,\ldots,m$, by Lemma 3.2.2(iv) we then have

$$\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q) \text{ has period } \delta_\ell \text{ modulo } n_{\ell-1}-k_{\ell-1}, \text{ and} \tag{3.26}$$

$$\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q) \text{ has period } \alpha_\ell-\delta_\ell \text{ modulo } k_\ell. \tag{3.27}$$

We show $\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q)$ has period α_ℓ and δ_ℓ modulo n by downward induction on ℓ, for $m \geq \ell \geq 2$. Note that the base case $\ell = m$ is accounted for by our argument as well.

Suppose $\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q)$ has period α_j and δ_j modulo n for all $j > \ell$. By Lemma 3.4.18, $\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q)$ has period k modulo n. By Lemma 3.2.2(i), $\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q)$ thus has period

$$k_\ell = k - (\delta_m + \cdots + \delta_{\ell+1})$$

modulo n. Since $\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q)$ has period $\alpha_\ell-\delta_\ell$ modulo k_ℓ, $\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q)$ has period $\alpha_\ell-\delta_\ell$ modulo n by Lemma 3.2.2(ii).

As noted, $\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q)$ has period δ_ℓ modulo $n_{\ell-1}-k_{\ell-1}$. By Lemma 3.2.2(i), $\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q)$ also has period

$$n_{\ell-1}-k_{\ell-1} = n - (\alpha_m + \cdots + \alpha_{\ell+1}) - k + (\delta_m + \cdots + \delta_{\ell+1}) - (\alpha_\ell - \delta_\ell)$$

modulo n. Hence, as $\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q)$ has period δ_ℓ modulo $n_{\ell-1}-k_{\ell-1}$, $\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q)$ has period δ_ℓ modulo n by Lemma 3.2.2(ii). By another application of Lemma 3.2.2(i), $\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q)$ has period α_ℓ modulo n as well, completing the induction.

Indeed, $\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q)$ has period $\delta_1 = 0$ modulo n trivially, and $\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q)$ has period $\alpha_1 = n - (\alpha_m + \cdots + \alpha_2)$ modulo n by Lemma 3.2.2(i). Putting everything together, $\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q)$ has periods $\alpha_1, \ldots, \alpha_m, \delta_1, \ldots, \delta_m$ modulo n, so by one more application of Lemma 3.2.2(i), $\mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q)$ has period g modulo n. □

3.5 Refining the CSP to fixed content and Cyclic Descent Type

In this section, we verify the final hypothesis (i) of Lemma 3.2.3 for $\mathrm{W}_{\alpha,\delta}$ and deduce Theorem 3.1.3. Throughout this section we continue to follow Notation 3.4.9. We recall in particular that

$$\mathrm{S}_\ell := \binom{[0, n_{\ell-1} - k_{\ell-1} - 1]}{\delta_\ell}, \qquad \mathrm{M}_\ell := \left(\!\!\binom{[0, k_\ell - 1]}{\alpha_\ell - \delta_\ell}\!\!\right)$$

and

$$g := \gcd(\alpha_1, \ldots, \alpha_m, \delta_1, \ldots, \delta_m).$$

3.5.1 A Fixed Point Lemma

To prove our main result, Theorem 3.1.3, one approach would be to find a C_n-equivariant isomorphism between a known CSP triple and $(\mathrm{W}_{\alpha,\delta}, C_n, \mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q))$. Such a triple is hinted at by (3.16) and the bijection Φ using products of CSP's coming from Theorem 2.1.1, though the approach encounters immediate difficulties. For instance, $\widetilde{\mathrm{W}}_{\alpha,\delta}$ is not generally closed under the C_n-action. In this section, we instead give a fixed point lemma, Lemma 3.5.5, which is intuitively a weakened version of the equivariant isomorphism approach.

Definition 3.5.1. Since $g \mid n_{\ell-1} - k_{\ell-1}$, $g \mid k_\ell$, and $g \mid n$, C_g acts on each of S_ℓ, M_ℓ, and $\mathrm{W}_{\alpha,\delta}$ by restricting the actions of $C_{n_{\ell-1}-k_{\ell-1}}$, C_{k_ℓ}, and C_n to their unique subgroups of size g. For instance, the action of C_g on $\mathrm{W}_{\alpha,\delta}$ is generated by rotation by n/g.

We further let C_g act diagonally on the products $\mathrm{S}_\ell \times \mathrm{M}_\ell$ and $\prod_{\ell=2}^m \mathrm{S}_\ell \times \mathrm{M}_\ell$. We emphasize that despite having C_g-actions on $\mathrm{W}_{\alpha,\delta}$ and $\prod_{\ell=2}^m \mathrm{S}_\ell \times \mathrm{M}_\ell$, the bijection $\Phi\colon \widetilde{\mathrm{W}}_{\alpha,\delta} \xrightarrow{\sim} \prod_{\ell=2}^m \mathrm{S}_\ell \times \mathrm{M}_\ell$ is not in general equivariant since $\widetilde{\mathrm{W}}_{\alpha,\delta}$ is not closed under the C_g action on $\mathrm{W}_{\alpha,\delta}$.

Definition 3.5.2. Given a multisubset of some set $[0, a]$, we may encode it as a multiplicity word $w_0 w_1 \dots w_a$ where w_i is the multiplicity of i. In particular, we may consider the bijection $\Phi\colon \widetilde{\mathrm{W}}_{\alpha,\delta} \xrightarrow{\sim} \prod_{\ell=2}^m \mathrm{S}_\ell \times \mathrm{M}_\ell$ as mapping words to sequences of pairs of certain words.

Example 3.5.3. Consider the leaf $w = 211332311$ in Figure 3.1 from Example 3.4.12. Reading edge labels gives $\Phi(w) = ((\{0,2\}, \varnothing), (\{2\}, \{1,2\}))$. Recalling that S_2 consists of subsets of $[0, 4-1]$, M_2 consists of multisubsets of $\varnothing$, S_3 consists of subsets of $[0, 4-1]$, and M_3 consists of multisubsets of $[0, 3-1]$, the corresponding sequence of words is $((1010, \epsilon), (0010, 011))$, where ϵ denotes the empty word. Table 3.1 summarizes several similar translations.

w	$\Phi(w)$	sequence of pairs of words
211332311	$((\{0,2\}, \varnothing), (\{2\}, \{1,2\}))$	$((1010, \epsilon), (0010, 011))$
121133231	$((\{1,3\}, \varnothing), (\{3\}, \{1,2\}))$	$((0101, \epsilon), (0001, 011))$
$(211332311)^2$	$((\{0,2,4,6\}, \varnothing), (\{2,6\}, \{1,2,4,5\}))$	$((1010^2, \epsilon), (0010^2, 011^2))$
2221123311	$((\{0,2\}, \{0,0\}), (\varnothing, \{1,1\}))$	$((1010, 20), (000000, 02))$

Table 3.1: Examples of words, corresponding sequences of edge labels in $T_{\alpha,\delta}$, and corresponding sequences of words. Note that the second word is a cyclic rotation of the first.

Lemma 3.5.4. Suppose $w = u^k$ for some word u. If $\Phi(u) = ((x_2, y_2), \dots, (x_m, y_m))$ encoded as multiplicity words as in Definition 3.5.2, then $\Phi(w) = ((x_2^k, y_2^k), \dots, (x_m^k, y_m^k))$.

Proof. The insertion triples needed to build w are the sequences of k shifted copies of the insertion triples needed to build u. □

Lemma 3.5.5. An element $\tau \in C_g$ fixes $w \in \widetilde{\mathrm{W}}_{\alpha,\delta}$ if and only if τ fixes $\Phi(w)$.

Proof. For $\tau \in C_n$, let $o(\tau)$ denote the order of τ. It is easy to see that $\tau \in C_n$ fixes $w \in \mathrm{W}_n$ if and only if there is some word u such that $w = u^{o(\tau)}$.

Suppose $\tau \in C_g$ fixes w, so that $w = u^{o(\tau)}$. By Lemma 3.5.4,

$$\Phi(w) = ((x_2^{o(\tau)}, y_2^{o(\tau)}), \ldots, (x_m^{o(\tau)}, y_m^{o(\tau)})).$$

Each of the words $x_i^{o(\tau)}$ and $y_i^{o(\tau)}$ is fixed by τ, so $\Phi(w)$ is fixed by τ. The reverse implication follows analogously using the fact that Φ is a bijection. □

3.5.2 Verifying Hypothesis (i) of Lemma 3.2.3 for $\mathrm{W}_{\alpha,\delta}$

Theorem 3.5.6. Using Notation 3.4.9, $(\mathrm{W}_{\alpha,\delta}, C_g, \mathrm{W}^{\mathrm{maj}}_{\alpha,\delta}(q))$ exhibits the CSP.

Proof. We use the notation and actions in Definition 3.5.1. Recall that

$$\mathrm{S}_\ell := \binom{[0, n_{\ell-1} - k_{\ell-1} - 1]}{\delta_\ell}, \qquad \mathrm{M}_\ell := \left(\!\!\binom{[0, k_\ell - 1]}{\alpha_\ell - \delta_\ell}\!\!\right).$$

From Theorem 2.1.1, for each $2 \le \ell \le m$, $\left(\mathrm{S}_\ell, C_g, \binom{n_\ell - k_\ell}{\delta_\ell}_q\right)$ and $\left(\mathrm{M}_\ell, C_g, \left(\!\!\binom{k_\ell}{\alpha_\ell - \delta_\ell}\!\!\right)_{q^{-1}}\right)$ exhibit the CSP. Taking products,

$$\left(\prod_{\ell=2}^{m} S_\ell \times \mathrm{M}_\ell, C_g, \prod_{\ell=2}^{m} \binom{n_\ell - k_\ell}{\delta_\ell}_q \left(\!\!\binom{k_\ell}{\alpha_\ell - \delta_\ell}\!\!\right)_{q^{-1}}\right)$$

exhibits the CSP. Comparing this to Theorem 3.4.17, we have

$$\widetilde{\mathrm{W}}^{\mathrm{maj}}_{\alpha,\delta} \equiv \prod_{\ell=2}^{m} \binom{n_\ell - k_\ell}{\delta_\ell}_q \left(\!\!\binom{k_\ell}{\alpha_\ell - \delta_\ell}\!\!\right)_{q^{-1}}$$

modulo $q^g - 1$, as $\sum_{\ell=2}^{m} k_\ell \alpha_\ell \equiv_g 0$ because $g \mid \alpha_\ell$ for all ℓ. Thus,

$$\left(\prod_{\ell=2}^{m} \mathrm{S}_\ell \times \mathrm{M}_\ell, C_g, \widetilde{\mathrm{W}}^{\mathrm{maj}}_{\alpha,\delta}(q)\right) \tag{3.28}$$

exhibits the CSP.

By Lemma 3.4.15, for any $w \in \mathrm{W}_{\alpha,\delta}$,

$$\#[w] = \frac{n}{\alpha_1} \cdot \#\left([w] \cap \widetilde{\mathrm{W}}_{\alpha,\delta}\right).$$

Since $[w]$ is an orbit under C_n, an element $\tau \in C_n$ fixes w if and only if τ fixes $[w]$ pointwise. Thus, for any $\tau \in C_n$,

$$\#\,\mathrm{W}_{\alpha,\delta}^{\tau} = \frac{n}{\alpha_1} \cdot \#\widetilde{\mathrm{W}}_{\alpha,\delta}^{\tau}. \tag{3.29}$$

Combining (3.29) and Lemma 3.5.5 now shows that for any $\tau \in C_g$,

$$\#\,\mathrm{W}_{\alpha,\delta}^{\tau} = \frac{n}{\alpha_1} \cdot \#\left(\prod_{\ell=2}^{m} \mathrm{S}_\ell \times \mathrm{M}_\ell\right)^{\tau}. \tag{3.30}$$

Hence, by (3.30), the CSP in (3.28), and (1.1),

$$\left(\mathrm{W}_{\alpha,\delta}, C_g, \frac{n}{\alpha_1}\widetilde{\mathrm{W}}_{\alpha,\delta}^{\mathrm{maj}}(q)\right)$$

exhibits the CSP. By (3.21), $\frac{n}{\alpha_1}\widetilde{\mathrm{W}}_{\alpha,\delta}(q) \equiv \mathrm{W}_{\alpha,\delta}^{\mathrm{maj}}(q)$ modulo $q^d - 1$, hence also modulo $q^g - 1$ since $g \mid d$, completing the proof. □

We have now finished the verification of the conditions in Lemma 3.2.3 for $\mathrm{W}_{\alpha,\delta}$. Condition (i) is Theorem 3.5.6, Condition (ii) is Proposition 3.4.21, and Condition (iii) is Lemma 3.3.3. This completes the proof of Theorem 3.1.3.

3.6 Refinements of Binomial CSP's

A key step in the proof of Theorem 3.5.6 was Theorem 2.1.1 due to Reiner-Stanton-White, which says that the triples

$$\left(\binom{[0,n-1]}{k}, C_n, \binom{n}{k}_q\right) \qquad \text{and} \qquad \left(\left(\!\!\binom{[0,n-1]}{k}\!\!\right), C_n, \left(\!\!\binom{n}{k}\!\!\right)_q\right)$$

exhibit the CSP. Indeed, [RSW04] contains two proofs, one via representation theory [RSW04, §3] and another by direct calculation [RSW04, §4]. In this section, we give two refinements of related CSP's involving an action of C_d on sets of subsets (Theorem 3.6.11) and multisubsets (Theorem 3.6.4) for all $d \mid n$, using shifted sum statistics. Our proof of the subset refinement, Theorem 3.6.11, does not use Theorem 2.1.1, so it can be used as an alternative proof of the subset case of Theorem 2.1.1. Our method is inspired by the rotation of subintervals used by Wagon and Wilf in [WW94, §3].

3.6.1 Cyclic Actions and Notation

We define two different cyclic actions of the cyclic group of order d on $[0, n-1]$ and induce these actions to $\binom{[0,n-1]}{k}$ and $\left(\!\binom{[0,n-1]}{k}\!\right)$. We also fix notation for the rest of the section.

Notation 3.6.1. Fix $n \in \mathbb{Z}_{\geq 1}, k \in \mathbb{Z}_{\geq 0}$, and $d \mid n$. Let

$$\mathrm{S} = \binom{[0, n-1]}{k}, \qquad \mathrm{M} = \left(\!\binom{[0, n-1]}{k}\!\right)$$

For all $j \in [1, \frac{n}{d}]$, let

$$I_d^j := [(j-1)d, jd-1],$$

which we call a *d-interval*. For any composition $\alpha = (\alpha_1, \ldots, \alpha_{n/d}) \vDash k$ with n/d parts, let

$$\mathrm{S}_\alpha := \{A \in S : \#(A \cap I_d^j) = \alpha_j \text{ for all } j\}, \tag{3.31}$$

$$\mathrm{M}_\alpha := \{A \in M : \#(A \cap I_d^j) = \alpha_j \text{ for all } j\}, \tag{3.32}$$

where the intersection in (3.32) preserves the multiplicity of A. We also fix cyclic groups C_d, C_d' of order d whose actions are described below.

Let C_d act on $[0, n-1]$ by simultaneous rotation of d-intervals, which is generated by the

permutation

$$\sigma_d := (0\ 1\ \ldots\ (d-1))\ldots((n-d)\ (n-d+1)\ \ldots\ (n-1)) \tag{3.33}$$

in cycle notation. On the other hand, C_n has a unique subgroup C'_d of order d which also acts on $[0, n-1]$ and is generated by the permutation

$$\sigma_n^{n/d} = \left(0\ \left(\frac{n}{d}\right)\ \ldots\left(n-\frac{n}{d}\right)\right)\ldots\left(\left(\frac{n}{d}-1\right)\ \left(\frac{2n}{d}-1\right)\ \ldots(n-1)\right). \tag{3.34}$$

Induce these actions of C_d and C'_d up to S and M by

$$g\cdot\{a_1,\ldots,a_k\} := \{g\cdot a_1,\ldots,g\cdot a_k\}.$$

Notice that the action of C_d restricts to S_α and M_α for any $\alpha = (\alpha_1,\ldots,\alpha_{n/d}) \vDash k$.

Let (G, X) be a pair where G is a group acting on a set X. A morphism of group actions $(G, X) \to (G', X')$ is a pair (ϕ, ψ) where $\phi\colon G \to G'$ is a group homorphism and $\psi\colon X \to X'$ is a map of sets which satisfy

$$\psi(g\cdot x) = \phi(g)\cdot\psi(x) \text{ for all } g\in G, x\in X.$$

Remark 3.6.2. The actions of C_d and C'_d on $[0, n-1]$ are isomorphic since σ_d and $\sigma_n^{n/d}$ have the same cycle type. This isomorphism explicitly arises from $\phi\colon \sigma_d \mapsto \sigma_n^{n/d}$ with $\psi\colon 0\mapsto 0$, $1\mapsto \frac{n}{d}$, etc. Thus the actions of C_d and C'_d on S and M are isomorphic as well.

Recall the sum statistic sums the elements of a set of multiset. We also use the following shifted sum statistic. For $A \in S$, let

$$\mathrm{sum}'(A) := \sum_{a\in A} a - \sum_{i=0}^{k-1} i = \mathrm{sum}(A) - \binom{k}{2}. \tag{3.35}$$

Recall from (2.4) and (2.5) that

$$\mathrm{S}^{\mathrm{sum}'}(q) = \binom{n}{k}_q, \qquad \mathrm{M}^{\mathrm{sum}}(q) = \left(\!\!\binom{n}{k}\!\!\right)_q. \tag{3.36}$$

Using (3.36), we may restate Theorem 2.1.1 as saying that $(\mathrm{S}, C_n, \mathrm{S}^{\mathrm{sum}'}(q))$ and $(\mathrm{M}, C_n, \mathrm{M}^{\mathrm{sum}}(q))$ exhibit the CSP. Moreover, under the restricted action of $C'_d \subset C_n$ on M and S, $(\mathrm{S}, C'_d, \mathrm{S}^{\mathrm{sum}'}(q))$ and $(\mathrm{M}, C'_d, \mathrm{M}^{\mathrm{sum}}(q))$ exhibit the CSP by Theorem 2.1.3. By Remark 3.6.2,

$$(\mathrm{S}, C_d, \mathrm{S}^{\mathrm{sum}'}(q)) \qquad \text{and} \qquad (\mathrm{M}, C_d, \mathrm{M}^{\mathrm{sum}}(q)) \tag{3.37}$$

also exhibit the CSP.

Example 3.6.3. Let $n = 8$, $k = 4$, and $d = 4$. Abbreviating $\{0, 4, 5, 6\}$ as 0456, etc., gives

$$\begin{aligned}\mathrm{S}_{(1,3)} = \{&0456, 0457, 0467, 0567, 1456, 1457, 1467, 1567,\\ &2456, 2457, 2467, 2567, 3456, 3457, 3467, 3567\}.\end{aligned}$$

Here, C_4 acts on $[0, 8-1]$ by the permutation $(0123)(4567)$, and C'_4 acts by $(0246)(1357)$. $\mathrm{M}_{(1,3)}$ contains $\mathrm{S}_{(1,3)}$ in addition to, for instance, 0444.

3.6.2 A Multisubset Refinement

We next prove a refinement of the CSP triple $(\mathrm{M}, C_d, \mathrm{M}^{\mathrm{sum}}(q))$ in (3.37) by fixing sizes of intersections with the d-intervals.

Theorem 3.6.4. Recall Notation 3.6.1, and fix a composition $\alpha = (\alpha_1, \ldots, \alpha_{n/d}) \vDash k$. Then, $(\mathrm{M}_\alpha, C_d, \mathrm{M}^{\mathrm{sum}}_\alpha(q))$ refines the CSP triple $(\mathrm{M}, C_d, \mathrm{M}^{\mathrm{sum}}(q))$.

Proof. Separating the d-intervals into different multisubsets gives

$$\mathrm{M}_\alpha \cong \left(\!\!\binom{[0, d-1]}{\alpha_1}\!\!\right) \times \cdots \times \left(\!\!\binom{[0, d-1]}{\alpha_{n/d}}\!\!\right), \tag{3.38}$$

which preserves the natural C_d-action and sum statistic modulo d. Since

$$\left(\left(\!\!\binom{[0,d-1]}{\alpha_j}\!\!\right), C_d, \left(\!\!\binom{[0,d-1]}{\alpha_j}\!\!\right)^{\mathrm{sum}} (q) \right)$$

exhibits the CSP for all j, the result follows from Theorem 2.1.3. □

The following analogous result holds for subsets.

Proposition 3.6.5. Recall Notation 3.6.1, and fix a composition $\alpha = (\alpha_1, \ldots, \alpha_{n/d}) \vDash k$. Then $(\mathrm{S}_\alpha, C_d, \mathrm{S}_\alpha^{\mathrm{sum}^*}(q))$ exhibits the CSP, where

$$\mathrm{sum}^*(A) := \mathrm{sum}(A) - \sum_{j=1}^{n/d} \binom{\alpha_j}{2}. \tag{3.39}$$

Proof. Separating the d-intervals into different subsets gives

$$\mathrm{S}_\alpha \cong \binom{[0,d-1]}{\alpha_1} \times \cdots \times \binom{[0,d-1]}{\alpha_{n/d}}, \tag{3.40}$$

which preserves the C_d-action and sum statistic modulo d. Since

$$\left(\binom{[0,d-1]}{\alpha_j}, C_d, \binom{[0,d-1]}{\alpha_j}^{\mathrm{sum} - \binom{\alpha_j}{2}} (q) \right)$$

exhibits the CSP for all j, $(\mathrm{S}_\alpha, C_d, \mathrm{S}_\alpha^{\mathrm{sum}^*}(q))$ exhibits the CSP by Theorem 2.1.3.

□

Remark 3.6.6. Since we must shift the sum statistic by different amounts depending on α, Proposition 3.6.5 is not a CSP refinement, in contrast to Theorem 3.6.4.

3.6.3 A Subset Refinement

We next prove an honest refinement of the CSP triple $(\mathrm{S}, C_d, \mathrm{S}^{\mathrm{sum}'}(q))$ in (3.37). To do so, we restrict to certain subsets of S for each divisibility chain ending in n. Our proof

again inductively extends CSP's up from cyclic subgroups of C_d using Lemma 3.2.3. In this subsection we first define our restricted subsets and give some examples. We then present a series of lemmata verifying the conditions of Lemma 3.2.3 before proving our refinement, Theorem 3.6.11.

Definition 3.6.7. Suppose $e \mid d \mid n$. Let

$$\mathrm{G}_{d,e} := \{A \in \mathrm{S} : \gcd(d, \#(A \cap I_d^1), \#(A \cap I_d^2), \ldots, \#(A \cap I_d^{n/d})) = e\}. \tag{3.41}$$

We have $\mathrm{G}_{n,\gcd(n,k)} = \mathrm{S}$ and $\mathrm{G}_{n,e} = \varnothing$ for all other e.

Remark 3.6.8. Note that $\mathrm{G}_{d,e}$ decomposes as the disjoint union

$$\mathrm{G}_{d,e} = \coprod \mathrm{S}_\alpha, \tag{3.42}$$

ranging over all $\alpha = (\alpha_1, \ldots, \alpha_{n/d}) \vDash k$ satisfying

$$\gcd(d, \alpha_1, \ldots, \alpha_{n/d}) = e.$$

Example 3.6.9. If $n = 4, k = 2$, then abbreviating $\{0, 2\}$ as 02, etc., gives

$$\mathrm{G}_{1,1} = \{01, 02, 03, 12, 13, 23\} = S,$$
$$\mathrm{G}_{2,1} = \{02, 03, 12, 13\}, \qquad \mathrm{G}_{2,2} = \{01, 23\},$$
$$\mathrm{G}_{4,1} = \varnothing, \qquad \mathrm{G}_{4,2} = \{01, 02, 03, 12, 13, 23\} = S, \qquad \mathrm{G}_{4,4} = \varnothing.$$

Consequently, $\mathrm{G}_{4,2} \cap \mathrm{G}_{2,1} = \{02, 03, 12, 13\}$ and $\mathrm{G}_{4,2} \cap \mathrm{G}_{2,2} = \{01, 23\}$.

Definition 3.6.10. Suppose D is a totally ordered chain in the divisibility lattice ending with $\gcd(n, k) \mid n$, i.e. $D = d_p \mid d_{p-1} \mid \cdots \mid d_0 \mid n$ where $d_0 := \gcd(n, k)$. Write

$$\mathrm{G}_D := \mathrm{G}_{n,d_0} \cap \mathrm{G}_{d_0,d_1} \cap \cdots \cap \mathrm{G}_{d_{p-1},d_p} \subset \S.$$

We may now state our subset refinement. The proof is postponed to the end of this subsection.

Theorem 3.6.11. Using Notation 3.6.1, let D be a totally ordered chain in the divisibility lattice ending with $\gcd(n,k) \mid n$ and starting with $e \mid d$. Then, $(\mathrm{G}_D, C_d, \mathrm{G}_D^{\mathrm{sum}'}(q))$ refines the CSP triple $(S, C_d, S^{\mathrm{sum}'}(q))$.

Example 3.6.12. If $n = 4$, $k = 2$, and $D = 1 \mid 2 \mid 4$, then $G = \mathrm{G}_{4,2} \cap \mathrm{G}_{2,1}$ has C_2 orbits $\{02, 13\}$ and $\{03, 12\}$. Moreover,

$$G^{\mathrm{sum}'}(q) = q^1 + 2q^2 + q^3 \equiv 2(q^0 + q^1) \pmod{q^2 - 1},$$

so $(G, C_2, G^{\mathrm{sum}'}(q))$ exhibits the CSP by (1.2).

In fact, the subset case of Theorem 2.1.1 is the special case $D = \gcd(n,k) \mid n$ of Theorem 3.6.11, so the proof below of Theorem 3.6.11 yields an alternative proof of the subset case of Theorem 2.1.1.

Corollary 3.6.13. $(\mathrm{S}, C_n, \mathrm{S}^{\mathrm{sum}'}(q))$ exhibits the CSP.

Lemma 3.6.14. Let D be a totally ordered chain in the divisibility lattice ending with $\gcd(n,k) \mid n$ and beginning with $e \mid d$. Suppose C'_e is the unique subgroup of C_d of order e.

(i) $\mathrm{G}_D = \coprod \mathrm{S}_\alpha$, where the disjoint union is over some set of α satisfying $\alpha = (\alpha_1, \ldots, \alpha_{n/d}) \vDash k$ and $\gcd(d, \alpha_1, \ldots, \alpha_{n/d}) = e$.

(ii) G_D is closed under the C_d and C_e-actions on S.

(iii) The C'_e and C_e-actions on G_D are isomorphic.

(iv) For any C_d-orbit $\mathcal{O}$ of G_D, we have $\frac{d}{|\mathcal{O}|} \mid e$.

(v) The sum' statistic has period e modulo d on G_D.

Proof. For (i), by (3.42) we have $\mathrm{G}_{d,e} = \coprod \mathrm{S}_\alpha$ where α ranges over all compositions satisfying the constraints in (i). Similarly, if $c \mid b$, then $\mathrm{G}_{b,c} = \coprod \mathrm{S}_\beta$ where in particular $\beta = (\beta_1, \ldots, \beta_{n/b}) \vDash k$. Now if $d \mid c$, we may break up each b-interval into b/d d-intervals. It is then easy to see that

$$\mathrm{S}_\alpha \cap \mathrm{S}_\beta = \varnothing \text{ or } \mathrm{S}_\alpha \,. \tag{3.43}$$

Now (i) follows inductively.

For (ii), by (i) it suffices to show that each S_α is closed under the C_d and C_e-actions. Since σ_d rotates d-intervals, it preserves the size of each d-interval, so σ_d indeed maps S_α to itself. The same argument applies with σ_e in place of σ_d.

For (iii), by (i), it suffices to show the C_e and C'_e-actions on S_α are isomorphic. Recalling (3.40), we have

$$\mathrm{S}_\alpha \cong \binom{[0, d-1]}{\alpha_1} \times \cdots \times \binom{[0, d-1]}{\alpha_{n/d}}.$$

By Remark 3.6.2, the actions of C_e and C'_e on $\binom{[0,d-1]}{\alpha_j}$ are isomorphic for each j, so their actions on S_α are isomorphic as well.

For (iv), pick $A \in \mathcal{O}$ with $A \in \mathrm{S}_\alpha$ for α as in (i). Let $A_j := A \cap I_d^j$, which has α_j elements. Viewing A_j as a multiplicity word w_j as in Definition 3.5.2, we see that A_j has $d - \alpha_j$ zeros and α_j ones. For all j, w_j is some word repeated $\frac{d}{|\mathcal{O}|}$ times. Using the two-letter case of Lemma 3.3.3, we have $\frac{d}{|\mathcal{O}|} \mid \mathrm{freq}(w_j) \mid \alpha_j$. Thus $\frac{d}{|\mathcal{O}|} \mid \gcd(d, \alpha_1, \ldots, \alpha_{n/d}) = e$.

For (v), it suffices to show that sum$'$ has period e modulo d on S_α for α as in (i). By the gcd condition, there exist $c_1, \ldots, c_{n/d} \in \mathbb{Z}$ such that

$$c_1\alpha_1 + \cdots + c_{n/d}\alpha_{n/d} \equiv e \pmod{d}.$$

For some particular $A \in \mathrm{S}_\alpha$, consider cyclically rotating the elements of $A \cap I_d^j$ forward by c_j

in I_d^j for all j. The result is a bijection $\phi\colon \mathrm{S}_\alpha \to \mathrm{S}_\alpha$ such that for all $A \in \mathrm{S}_\alpha$,

$$\mathrm{sum}'(\phi(A)) \equiv \mathrm{sum}'(A) + e \pmod{d}.$$

Hence sum′ indeed has period e modulo d on S_α. □

Example 3.6.15. Let $n = 12, k = 8$, and $D = 1 \mid 2 \mid 4 \mid 12$. Then

$$\mathrm{G}_D = \mathrm{G}_{12,4} \cap \mathrm{G}_{4,2} \cap \mathrm{G}_{2,1} = \mathrm{G}_{4,2} \cap \mathrm{G}_{2,1}\,.$$

We have $\mathrm{G}_{2,1} = \coprod \mathrm{S}_\alpha$ where $\alpha = (\alpha_1, \ldots, \alpha_6) \vDash 8$ and $\gcd(2, \alpha_1, \ldots, \alpha_6) = 1$. Similarly $\mathrm{G}_{4,2} = \coprod \mathrm{S}_\beta$ where $\beta = (\beta_1, \beta_2, \beta_3) \vDash 8$ and $\gcd(4, \beta_1, \beta_2, \beta_3) = 2$. In fact,

$$\varnothing \subsetneq \mathrm{G}_D \subsetneq \mathrm{G}_{2,1}$$

since, for instance, $\mathrm{S}_\alpha \subset \mathrm{G}_D$ when $\alpha = (4, 0, 1, 1, 1, 1)$ while $\mathrm{S}_\alpha \subset \mathrm{G}_{2,1} - \mathrm{G}_D$ when $\alpha = (2, 0, 2, 1, 2, 1)$.

Lemma 3.6.16. Let $d \mid n$. The C_d action on $\mathrm{G}_{d,d}$ is trivial and $(\mathrm{G}_{d,d}, C_d, \mathrm{G}_{d,d}^{\mathrm{sum}'}(q))$ exhibits the CSP.

Proof. All subsets in $\mathrm{G}_{d,d}$ have each d-interval either full or empty, so C_d fixes every $A \in \mathrm{G}_{d,d}$. By (1.2), $(\mathrm{G}_{d,d}, C_d, \mathrm{G}_{d,d}^{\mathrm{sum}'}(q))$ thus exhibits the CSP if and only if $\mathrm{G}_{d,d}^{\mathrm{sum}'}(q) \equiv \#\,\mathrm{G}_{d,d} \bmod (q^d - 1)$. If $\mathrm{G}_{d,d} = \varnothing$ the result is trivial, so take $\mathrm{G}_{d,d} \neq \varnothing$. For any $A \in \mathrm{G}_{d,d}$, since each d-interval is full or empty, we have $d \mid k$ and

$$\mathrm{sum}'(A) \equiv \frac{k}{d}\binom{d}{2} - \binom{k}{2} \equiv \frac{k(d-k)}{2} \equiv 0 \pmod{d}. \tag{3.44}$$

□

We may now prove Theorem 3.6.11.

Proof of Theorem 3.6.11. We induct on d. If $d = 1$, then $(\mathrm{G}_D, C_1, \mathrm{G}_D^{\mathrm{sum}'}(q))$ exhibits the CSP trivially. For the induction step, we first claim that $(\mathrm{G}_D, C_e, \mathrm{G}_D^{\mathrm{sum}'}(q))$ exhibits the CSP. If $e = d$, then $\mathrm{G}_D \subset \mathrm{G}_{d,d}$, so by Lemma 3.6.16 the C_e action is trivial. It is easy to see that CSP's with trivial actions refine to arbitrary subsets, so $(\mathrm{G}_D, C_e, \mathrm{G}_D^{\mathrm{sum}'}(q))$ exhibits the CSP in this case. If $e < d$, by conditioning on the sizes of the intersections of the e-intervals, we can write

$$\mathrm{G}_D = \coprod_{f|e} \mathrm{G}_{f|D} \tag{3.45}$$

where $f \mid D$ denotes the chain with f prepended to D. Hence $(\mathrm{G}_{f|D}, C_e, \mathrm{G}_{f|D}^{\mathrm{sum}'}(q))$ exhibits the CSP by induction for each $f \mid e$, since $f \mid D$ begins with $f \mid e$. Thus $(\mathrm{G}_D, C_e, \mathrm{G}_D^{\mathrm{sum}'}(q))$ exhibits the CSP by (3.45), proving the claim.

In order to realize the $(\mathrm{G}_D, C_d, \mathrm{G}_D^{\mathrm{sum}'}(q))$ CSP triple from the $(\mathrm{G}_D, C_e, \mathrm{G}_D^{\mathrm{sum}'}(q))$ CSP triple, we verify the conditions of Lemma 3.2.3. From Lemma 3.6.14(ii), the restriction of the C_d-action on G_D to the subgroup $C_e' \subset C_d$ of size e is isomorphic to the C_e-action on G_D, giving Condition (i). Condition (ii) is Lemma 3.6.14(v), and Condition (iii) is Lemma 3.6.14(iv). Thus $(\mathrm{G}_D, C_d, \mathrm{G}_D^{\mathrm{sum}'}(q))$ exhibits the CSP by Lemma 3.2.3. □

3.7 The Flex Statistic

We conclude by formalizing the notion of *universal* sieving statistics and giving an example, *flex*, in the context of words. We end with an open problem.

Definition 3.7.1. Given a finite set W with a C_n-action, we say $\mathrm{stat}\colon W \to \mathbb{Z}_{\geq 0}$ is a *universal CSP statistic* for (W, C_n) if $(\mathcal{O}, C_n, \mathcal{O}^{\mathrm{stat}}(q))$ exhibits the CSP for all C_n-orbits $\mathcal{O}$ of W.

Recall $\mathrm{lex}(w)$ denotes the index at which w appears after lexicographically ordering the necklace $[w]$, and

$$\mathrm{flex}(w) := \mathrm{freq}(w)\,\mathrm{lex}(w).$$

See Example 2.1.5 for an example.

Lemma 3.7.2. The function flex is a universal CSP statistic for (W_n, C_n).

Proof. Let N be any necklace of length n words. Since $\mathrm{freq}(N) = \frac{n}{|N|}$, and $\mathrm{lex}(N) = \{1, \ldots, |N|\}$, we have

$$N^{\mathrm{flex}}(q) = \sum_{j=1}^{|N|} q^{j \cdot \frac{n}{|N|}} \equiv \sum_{j=0}^{|N|-1} q^{j \cdot \frac{n}{|N|}} = \frac{q^n - 1}{q^{n/|N|} - 1}, \tag{3.46}$$

moduloe $(q^n - 1)$, so $(N, C_n, N^{\mathrm{flex}}(q))$ exhibits the CSP by (1.2). □

Given a universal sieving statistic stat on some set W, stat takes on the values $\{0, n/d, \ldots, n - n/d\}$ modulo n on any orbit of size d. The converse holds as well. In this sense, up to shifting values by n, universal sieving statistics are equivalent to total orderings on each orbit $\mathcal{O}$ of W.

Standing in contrast to Lemma 3.7.2, $(N, C_n, N^{\mathrm{maj}}(q))$ does not exhibit the CSP when $N = [123123]$, so maj is not a universal CSP statistic on (W_n, C_n). However, maj trivially refines to the orbit $N = \{1^n\}$ for any n. Since refinement is not generally closed under intersections, it is not clear if there is any useful sense in which maj on words can be "maximally refined."

It follows from Lemma 3.7.2 and (1.2) that Theorem 3.1.3 is equivalent to the following.

Theorem 3.7.3. The statistics flex and maj are equidistributed modulo n on $\mathrm{W}_{\alpha,\delta}$.

Indeed, we were originally led to Theorem 3.1.3 through an exploration of the irreducible multiplicities of the so-called higher Lie modules (see e.g. [Sch03]), which uncovered the fact that flex and maj are equidistributed modulo n on W_α. Data exploration led us to conjecture this equidistribution refined to fixed cyclic descent type as in Theorem 3.7.3. These connections will be explained in a future publication. They also naturally suggest the problem of finding explicit bijections proving Theorem 3.7.3, which we leave as an open problem.

Open Problem 3.7.4. For $\alpha \vDash n$ and δ any weak composition, find a bijection $\varphi\colon \mathrm{W}_{\alpha,\delta} \to \mathrm{W}_{\alpha,\delta}$ satisfying

$$\mathrm{maj}(\varphi(w)) \equiv \mathrm{flex}(w) \pmod{n}. \tag{3.47}$$

Chapter 4

NECKLACES, BRANCHING RULES, AND THRALL'S PROBLEM

4.1 Introduction

The *Lie module* $\mathcal{L}_n$ is the nth degree component of the free Lie algebra over $\mathbb{C}$ with m generators, which is naturally a $\mathrm{GL}(\mathbb{C}^m)$-module. The Lie modules were famously studied by Thrall [Thr42] in the 1940's and have been extensively studied by Brandt [Bra44], Klyachko [Kly74], Kraśkiewicz–Weyman [KW01], Garsia [Gar90], Gessel–Reutenauer [GR93], Reutenauer [Reu93], Sundaram [Sun94], Schocker [Sch03], and many others. Thrall more generally introduced a certain $\mathrm{GL}(\mathbb{C}^m)$-decomposition $\oplus_{\lambda\in\mathrm{Par}}\mathcal{L}_\lambda$ of the tensor algebra of $\mathbb{C}^m$ arising from the Poincaré–Birkhoff–Witt Theorem, where $\mathcal{L}_{(n)} = \mathcal{L}_n$. The $\mathcal{L}_\lambda$ are sometimes called the *higher Lie modules*. Thrall's original paper considered the determination of the multiplicity of the irreducible V^μ in $\mathcal{L}_\lambda$, which is often referred to as *Thrall's problem*. This problem is still open 75 years later. See Section 4.2.4 and [Reu93] for more background on Thrall's problem and [Rei15] for a recent summary of related work. See Section 4.2 for missing definitions.

Kraśkiewicz–Weyman [KW01] gave a combinatorial solution to Thrall's problem when $\lambda = (n)$. In particular, they showed the multiplicity of V^μ in $\mathcal{L}_{(n)}$ is

$$\#\{T \in \mathrm{SYT}(\mu) : \mathrm{maj}(T) \equiv_n 1\},$$

i.e. the number of standard tableaux of shape μ with major index 1 modulo n. Their argument crucially hinges upon (2.16):

$$\mathrm{SYT}(\mu)^{\mathrm{maj}}(\omega_n^r) = \chi^\mu(\sigma_n^r)$$

where ω_n is a primitive nth complex root of unity, σ_n is an n-cycle in the symmetric group S_n, and χ^μ is the character of the S_n-irreducible indexed by a partition μ of n. The analysis in [KW01] is somewhat indirect. It involves results of Lusztig and Stanley on coinvariant algebras and an intricate though beautiful argument involving ℓ-decomposable partitions.

Equation (2.16) bears a striking resemblance to the cyclic sieving phenomenon, Defini-

tion 1.1.2. Indeed, Theorem 2.1.6 due to Reiner–Stanton–White is intimately related to (2.16), as we already saw in our proof of Theorem 2.1.6 at the end of Chapter 2.

Since the sets W_α are precisely the S_n-orbits for the natural S_n action on length n words, Theorem 2.1.6 may be thought of as a "universal sieving result" as follows. A very similar observation appeared in [BER11, Prop. 3.1].

Corollary 4.1.1. Let W be a finite set of length n words closed under the S_n-action. Then, the triple

$$\big(W, C_n, W^{\mathrm{maj}}(q)\big)$$

exhibits the CSP.

The flex statistic, see Example 2.1.5, was introduced in [AS18b]. It was designed to be "universal" for cyclic rather than symmetric actions on words in the following sense.

Lemma 4.1.2. [AS18b, Lemma 8.3] Let W be a finite set of length n words closed under the C_n-action, where C_n acts by cyclic rotations. Then, the triple

$$\big(W, C_n, W^{\mathrm{flex}}(q)\big)$$

exhibits the CSP.

A corollary of these universal sieving results is the following equidistribution result. We proved a more refined version in Theorem 3.7.3.

Theorem 4.1.3. [AS18b, Theorem 8.4] Let W_n denote the set of length n words, let maj_n denote the major index modulo n taking values in $\{1, \ldots, n\}$, and let cont denote the *content* of a word. We then have

$$\mathrm{W}_n^{\mathrm{cont},\mathrm{maj}_n}(\mathbf{x}; q) = \mathrm{W}_n^{\mathrm{cont},\mathrm{flex}}(\mathbf{x}; q).$$

In Section 4.3, we show that the following well-known result of Kraśkiewicz–Weyman is essentially a corollary of Theorem 4.1.3. Here χ^r is the linear representation of the cyclic group C_n given by $\chi^r(\sigma_n) = \omega_n^r$.

Theorem 4.1.4. [KW01](Theorem 1.1.4) We have

$$\operatorname{ch} \chi^r \uparrow_{C_n}^{S_n} = \sum_{\lambda \vdash n} a_{\lambda,r} s_\lambda(\mathbf{x})$$

where

$$a_{\lambda,r} \coloneqq \#\{Q \in \mathrm{SYT}(\lambda) : \mathrm{maj}(Q) \equiv_n r\}.$$

Klyachko [Kly74, Prop. 1] showed that the Lie modules $\mathcal{L}_n$ and the induced representations $\chi^1 \uparrow_{C_n}^{S_n}$ are Schur–Weyl duals. The $\lambda = (n)$ case of Thrall's problem thus follows from Theorem 1.1.4 when $r = 1$. More precisely, Klyachko expressed both the characteristic of $\chi^1 \uparrow_{C_n}^{S_n}$ and the character of $\mathcal{L}_n$ as content generating functions on primitive necklaces of length n words. We generalize this observation in Section 4.3 as follows, which also naturally motivates the flex statistic.

Theorem 4.1.5. Let $\mathrm{NFD}_{n,r}$ denote the set of necklaces of length n words with *frequency* dividing r, $\mathrm{F}_{n,r}$ denote the set of length n words with flex equal to r, and $\mathrm{M}_{n,r}(\mathbf{x})$ denote the set of length n words with maj_n equal to r. Then

$$\operatorname{ch} \chi^r \uparrow_{C_n}^{S_n} = \mathrm{NFD}_{n,r}^{\mathrm{cont}}(\mathbf{x}) = \mathrm{F}_{n,r}^{\mathrm{cont}}(\mathbf{x}) = \mathrm{M}_{n,r}^{\mathrm{cont}}(\mathbf{x}).$$

Our new proof of Kraśkiewicz–Weyman's result reduces the problem of finding a bijective proof of a well-known symmetry result following from Theorem 1.1.4 to finding a bijective proof of the above equidistribution result, Theorem 4.1.3; see Corollary 4.3.5. It also provides a thus far rare example of an instance of cyclic sieving being used to prove other results rather than vice-versa.

In Section 4.4, we give a new proof of a result of Stembridge [Ste89] which settled a conjecture of Stanley describing the irreducible multiplicities of induced representations $\chi^r \uparrow_{\langle\sigma\rangle}^{S_n}$ for arbitrary $\sigma \in S_n$. The corresponding generalized major index statistics arise very naturally from the combinatorics of orbits and cyclic sieving.

In Section 4.5, we prove and generalize a result of Schocker [Sch03] concerning the higher

Lie modules. Thrall's problem may be reduced to the $\lambda = (a^b)$ case by the Littlewood–Richardson rule. Bergeron–Bergeron–Garsia [BBG90] identified the Schur–Weyl dual of $\mathcal{L}_{(a^b)}$ as a certain induced module $\chi^{1,1}\uparrow_{C_a \wr S_b}^{S_{ab}}$ where $C_a \wr S_b$ is a wreath product; see Section 4.2.5 for details. Schocker gave a formula for the multiplicity of the irreducible V^μ in $\mathcal{L}_{(a^b)}$, though it involves many subtractions and divisions in general. We generalize Schocker's formula to all one-dimensional representations of $C_a \wr S_b$. In our approach, the subtractions and divisions in Schocker's formula arise naturally from the underlying combinatorics using Möbius inversion and Burnside's lemma.

The basic outline of each argument is the same: we obtain an orbit generating function from an explicit basis of a $\mathrm{GL}(V)$-module, we construct an appropriate necklace generating function, we use cyclic sieving to rewrite this generating function using words and descent statistics like the major index, and we finally apply RSK to get a Schur expansion. Transitioning from an orbit generating function to a necklace generating function where we can apply cyclic sieving involves various combinatorial techniques.

In Section 4.6, we discuss applying aspects of our approach to Thrall's problem in general. The arguments in the preceding sections strongly suggest attacking Thrall's problem by considering all branching rules for the inclusion $C_a \wr S_b \hookrightarrow S_{ab}$ rather than considering only one such rule. To that end, consider the irreducible representations $S^{\underline{\lambda}}$ of $C_a \wr S_b$, which are indexed by the set of a-tuples $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(a)})$ of partitions with $\sum_{r=1}^{a} |\lambda^{(r)}| = b$. We first give the following plethystic expression for the corresponding characteristic.

Theorem 4.1.6. For all integers $a, b \geq 1$, we have

$$\mathrm{ch}\, S^{\underline{\lambda}}\uparrow_{C_a \wr S_b}^{S_{ab}} = \prod_{r=1}^{a} s_{\lambda^{(r)}}[\mathrm{NFD}_{a,r}^{\mathrm{cont}}(\mathbf{x})].$$

We then identify the analogues of the flex and maj_n statistics in this context, which send words to such a-tuples of partitions. We consequently give the following monomial expansion of the corresponding graded Frobenius series. See Section 4.2.5 and Section 4.6 for details.

Theorem 4.1.7. Fix integers $a, b \geq 1$. We have

$$\sum_{\underline{\lambda}} \dim S^{\underline{\lambda}} \cdot \mathrm{ch}\left(S^{\underline{\lambda}}{\downarrow}^{S_{ab}}_{C_a \wr S_b}\right) q^{\underline{\lambda}} = \sum_{w \in \mathrm{W}_{ab}} \mathbf{x}^{\mathrm{cont}(w)} q^{\mathrm{flex}^b_a(w)} = \sum_{w \in \mathrm{W}_{ab}} \mathbf{x}^{\mathrm{cont}(w)} q^{\mathrm{maj}^b_a(w)}$$

where the sum is over all a-tuples $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(a)})$ of partitions with $\sum_{r=1}^{a} |\lambda^{(r)}| = b$ and the $q^{\underline{\lambda}}$ are independent indeterminates.

The rest of this chapter is organized as follows. In Section 4.2, we review combinatorial and representation-theoretic background. In particular, we summarize work related to Kraśkiewicz–Weyman's result, Theorem 1.1.4, in Section 4.2.3, and we discuss the current status of Thrall's problem in Section 4.2.4. In Section 4.3, we present our proof of Kraśkiewicz–Weyman's result, Theorem 1.1.4, using cyclic sieving. In Section 4.4, we give an analogous proof of Stembridge's result, Theorem 4.4.11. In Section 4.5, we give generalizations of Schocker's result, Theorem 4.5.11. In Section 4.6, we define the statistics flex^b_a and maj^b_a, prove Theorem 4.1.6 and Theorem 4.1.7, and discuss how the approach could be used to find the branching rules for $C_a \wr S_b \hookrightarrow S_{ab}$.

4.2 Background

4.2.1 RSK and symmetric functions

We will repeatedly use the RSK correspondence to transition from the monomial to the Schur basis in the ring of symmetric functions. These arguments all rely on the following result.

Lemma 4.2.1. For all $D \subset [n-1]$,

$$\{w \in \mathrm{W}_n : \mathrm{Des}(w) = D\} = \sum_{\lambda \vdash n} \#\{Q \in \mathrm{SYT}(\lambda) : \mathrm{Des}(Q) = D\} s_\lambda(\mathbf{x}).$$

Proof. From (2.10), we have $\mathrm{cont}(P(w)) = \mathrm{cont}(w)$ and $\mathrm{Des}(Q(w)) = \mathrm{Des}(w)$ under the RSK

correspondence. This means RSK restricts to a bijeciton

$$\mathrm{RSK} : \{w \in \mathrm{W}_n : \mathrm{Des}(w) = D\} \to \bigcup_{\lambda \vdash n} \mathrm{SSYT}(\lambda) \times \{Q \in \mathrm{SYT}(\lambda) : \mathrm{Des}(Q) = D\},$$

where the content of w is tracked in the first semistandard Young tableaux. Therefore,

$$\begin{aligned}\{w \in \mathrm{W}_n : \mathrm{Des}(w) = D\}^{\mathrm{cont}}(\mathbf{x}) &= \sum_{\lambda \vdash n} \sum_{\substack{Q \in \mathrm{SYT}(\lambda) \\ \mathrm{Des}(Q) = D}} \sum_{P \in \mathrm{SSYT}(\lambda)} x^{\mathrm{cont}(P)} \\ &= \sum_{\lambda \vdash n} \#\{Q \in \mathrm{SYT}(\lambda) : \mathrm{Des}(Q) = D\} s_\lambda(\mathbf{x})\end{aligned}$$

by definition of $s_\lambda(\mathbf{x})$, (2.11). □

4.2.2 Schur–Weyl Duality

We next summarize a few key points from the representation theory of $\mathrm{GL}(\mathbb{C}^m)$ and its relationship to the representation theory of S_n. See [Ful97] for more.

Let V be a complex vector space of dimension m. Endow $V^{\otimes n}$ with the diagonal left $\mathrm{GL}(V)$-action and the natural right S_n-action given by permutation of indexes. Given any S_n-module M, define a corresponding $\mathrm{GL}(V)$-module by

$$E(M) \coloneqq V^{\otimes n} \otimes_{\mathbb{C}S_n} M,$$

which we call the *Schur-Weyl dual* of M. The irreducible inequivalent polynomial representations of $\mathrm{GL}(V)$ are precisely the Schur Weyl duals of all S^λ where λ is a partition with at most $\dim(V)$ non-zero parts [Ful97, Thm. 8.2.2].

Let E be a finite-dimensional, polynomial representation of $\mathrm{GL}(V)$ and pick a basis $\{v_1, \ldots, v_m\}$ for V. The *Schur character* of E, denoted $\mathrm{ch}\, E$, is the trace of the action of $\mathrm{diag}(x_1, \ldots, x_m) \in \mathrm{GL}(V)$ on E, where the diagonal matrix is with respect to the basis $v_1, \ldots, v_m$. Polynomiality of E implies $\mathrm{ch}\, E \in \mathbb{C}[x_1, \ldots, x_m]$. Moreover, $\mathrm{ch}(E)$ is a symmetric

function of $x_1, \ldots, x_m$. In fact,

$$\operatorname{ch} V^\lambda = \operatorname{ch} E(S^\lambda) = s_\lambda(x_1, \ldots, x_m, 0, 0, \ldots).$$

Thus, for any S_n-module M, we have

$$\lim_{m\to\infty} \operatorname{ch} E(M) = \operatorname{ch} M.$$

In light of this, we often leave dependence on m or V implicit.

4.2.3 Kraśkiewicz–Weyman Symmetric Functions

The symmetric functions appearing in Theorem 4.1.3 have a wealth of important interpretations. Here we summarize some of these interpretations.

Definition 4.2.2. For $n \geq 1$, let

$$\mathrm{KW}_n(\mathbf{x}; q) \coloneqq \sum_{\substack{\lambda \vdash n \\ r \in [n]}} a_{\lambda,r} s_\lambda(\mathbf{x})\, q^r \tag{4.1}$$

where $a_{\lambda,r} \coloneqq \#\{Q \in \mathrm{SYT}(\lambda) : \mathrm{maj}(Q) \equiv_n r\}$. We call $\mathrm{KW}_n(\mathbf{x}; q)$ the nth *Kraśkiewicz–Weyman* symmetric function.

These symmetric functions are intimately related to the irreducible representations of certain cyclic groups.

Definition 4.2.3. Recall $\sigma_n \coloneqq (1\,2\,\cdots\,n) \in S_n$ and $C_n \coloneqq \langle \sigma_n \rangle \leq S_n$ be the cyclic group of order n it generates. Fixing any primitive nth root of unity ω_n, write the irreducible characters of C_n as $\chi^1, \ldots, \chi^n$ where

$$\chi^r(\sigma_n) \coloneqq \omega_n^r.$$

We sometimes write χ_n^r if we want to specify the cyclic group C_n as well.

Theorem 1.1.4 gives our first interpretation of $\mathrm{KW}_n(\mathbf{x}; q)$,

$$\mathrm{KW}_n(\mathbf{x}; q) = \sum_{r=1}^{n} \mathrm{ch}\, \chi^r \uparrow_{C_n}^{S_n} q^r. \tag{4.2}$$

Since the regular representation of C_n is $\oplus_{r=1}^{n} \chi^r$, when $q = 1$ the right-hand side of (4.2) is the Frobenius characteristic of the regular representation of S_n, denoted $\mathbb{C}S_n$. The right-hand side of (4.2) is hence similar to a graded Frobenius series for $\mathbb{C}S_n$ and tracks branching rules for the inclusion $C_n \hookrightarrow S_n$. By Theorem 4.1.5, we can also write this series as

$$\mathrm{KW}_n(\mathbf{x}; q) = \sum_{r=1}^{n} \mathrm{NFD}_{n,r}^{\mathrm{cont}}(\mathbf{x})\, q^r. \tag{4.3}$$

Now consider the action of σ_n on the S_n-irreducible S^λ. Since $\sigma_n^n = 1 \in S_n$, the action of σ_n on S^λ is diagonal with eigenvalues $\omega_n^{k_1}, \omega_n^{k_2}, \ldots$ where ω_n is a fixed primitive nth root of unity and $1 \leq k_i \leq n$ for each i. Let $P_\lambda(q) \coloneqq q^{k_1} + q^{k_2} + \cdots$ be the generating function of the *cyclic exponents* $k_1, k_2, \ldots$, which were studied extensively by Stembridge [Ste89]. Using the right-hand side of (4.2) and Frobenius reciprocity quickly gives the following.

Theorem 4.2.4 (See [Ste89, Prop. 1.2, Thm. 3.3])**.** The cyclic exponent generating function for S_n is given by

$$\mathrm{KW}_n(\mathbf{x}; q) = \sum_{\lambda \vdash n} P_\lambda(q) s_\lambda(\mathbf{x}). \tag{4.4}$$

Next, extend the regular representation $\mathbb{C}S_n$ to an $S_n \times C_n$-module by letting S_n act on the left and C_n act on the right. There is a straightforward notion of an $S_n \times C_n$-Frobenius characteristic map given by sending an irreducible $S^\lambda \otimes \chi^r$ to $s_\lambda(\mathbf{x}) q^r$ where q is an indeterminate satisfying $q^n = 1$. The following now follows easily using the right-hand side of (4.2).

Corollary 4.2.5. [KW01] The $S_n \times C_n$-Frobenius characteristic of the regular representation is

$$\mathrm{KW}_n(\mathbf{x}; q) = \mathrm{ch}_{S_n \times C_n}\, \mathbb{C}S_n. \tag{4.5}$$

It is well-known that the type A_{n-1} coinvariant algebra R_n is a graded S_n-module which is isomorphic as an *ungraded* S_n-module to $\mathbb{C}S_n$. We may give R_n an $S_n \times C_n$-module structure by letting C_n act on the kth degree component of R_n by $\sigma_n \cdot f \coloneqq \omega_n^k f$, where ω_n is a fixed primitive nth root of unity. Springer and, independently, Kraśkiewicz–Weyman showed that $\mathbb{C}S_n$ and R_n are isomorphic as $S_n \times C_n$-modules. Consequently, from the right-hand side of (4.5), we have the following.

Theorem 4.2.6 (Springer [Spr74, Prop. 4.5]; cf. [KW01, Thm. 1])**.** The $S_n \times C_n$-Frobenius characteristic of the coinvariant algebra R_n is

$$\mathrm{KW}_n(\mathbf{x}; q) = \mathrm{ch}_{S_n \times C_n} R_n. \tag{4.6}$$

The graded Frobenius characteristic of the coinvariant algebra is the modified Hall–Littlewood symmetric function $\widetilde{Q}_{(1^n)}(\mathbf{x}; q)$ [GP92, (I.8)]. Consequently, (4.6) gives

$$\mathrm{KW}_n(\mathbf{x}; q) \equiv \widetilde{Q}_{(1^n)}(\mathbf{x}; q) \qquad (\mathrm{mod}\ q^n - 1). \tag{4.7}$$

See also [Rho10b, §3] for a nice summary of this connection.

We may instead use the right-hand side of (4.1) as a starting point. From Lemma 4.2.1, it follows that

$$\mathrm{KW}_n(\mathbf{x}; q) = \mathrm{W}_n^{\mathrm{cont},\mathrm{maj}_n}(\mathbf{x}; q). \tag{4.8}$$

From Theorem 4.1.3 and (4.8), our final interpretation of $\mathrm{KW}_n(\mathbf{x}; q)$ in this subsection is

$$\mathrm{KW}_n(\mathbf{x}; q) = \mathrm{W}_n^{\mathrm{cont},\mathrm{flex}}(\mathbf{x}; q). \tag{4.9}$$

4.2.4 Thrall's Problem

We next define the Lie modules $\mathcal{L}_\lambda$ and summarize the status of Thrall's problem. See [Reu93] for more details.

The *tensor algebra* of V is $T(V) \coloneqq \oplus_{n=0}^{\infty} V^{\otimes n}$, which is naturally a graded $\mathrm{GL}(V)$-

representation. Let $\mathcal{L}(V)$ be the Lie subalgebra of $T(V)$ generated by V, called the *free Lie algebra* on V, so that $\mathcal{L}(V)$ is a graded $\mathrm{GL}(V)$-representation with graded components $\mathcal{L}_n(V) = V^{\otimes n} \cap \mathcal{L}(V)$ called *Lie modules*. The *universal enveloping algebra* $\mathfrak{U}(\mathcal{L}(V))$ is isomorphic to $T(V)$ itself. By the Poincaré–Birkhoff–Witt Theorem,

$$\mathfrak{U}(\mathcal{L}(V)) \cong \bigoplus_{\lambda=1^{m_1}2^{m_2}\cdots} \mathrm{Sym}^{m_1}(\mathcal{L}_1(V)) \otimes \mathrm{Sym}^{m_2}(\mathcal{L}_2(V)) \otimes \cdots$$

as graded $\mathrm{GL}(V)$-representations, where the sum is over all partitions and $\mathrm{Sym}^m(M)$ is the mth symmetric power of M [Reu93, Lemma 8.22]. The *higher Lie module* associated to $\lambda = 1^{m_1}2^{m_2}\cdots$ is defined to be

$$\mathcal{L}_\lambda(V) \coloneqq \mathrm{Sym}^{m_1}(\mathcal{L}_1(V)) \otimes \mathrm{Sym}^{m_2}(\mathcal{L}_2(V)) \otimes \cdots . \tag{4.10}$$

The Lie modules hence yield a $\mathrm{GL}(V)$-module decomposition $T(V) \cong \oplus_{\lambda\in\mathrm{Par}}\mathcal{L}_\lambda(V)$.

Thrall's problem is the determination of the multiplicity of V^μ in $\mathcal{L}_\lambda(V)$, for instance by counting explicit combinatorial objects. The well-known Littlewood–Richardson rule solves the analogous problem for $V^\mu \otimes V^\nu$. It follows from (4.10) and the Littlewood–Richardson rule that, for the purposes of Thrall's problem, we may restrict our attention to the case when $\lambda = (a^b)$ is a rectangle. Since

$$\mathcal{L}_{(a^b)}(V) = \mathrm{Sym}^b(\mathcal{L}_{(a)}(V)), \tag{4.11}$$

the single-row case is particularly fundamental.

Hall [MH59, Lemma 11.2.1] introduced what is now called the *Hall basis* for $\mathcal{L}_n(V)$, which, in the $m \to \infty$ limit, is in content-preserving bijection with primitive necklaces $\mathrm{NFD}_{n,1}$. For each primitive necklace, Hall associates a bracketing of its elements using what is now known as the Lyndon factorization [CFL58]. He gives an explicit, though computationally complex, algorithm to express any bracketing as a linear combination of the bracketings associated to primitive necklaces. Linear independence of these generators follows from a dimension count.

Klyachko consequently observed that the Schur character of $\mathcal{L}_n$ is the corresponding content generating function $\mathrm{NFD}_{n,1}^{\mathrm{cont}}(\mathbf{x})$. Taking symmetric powers, it follows that in the $m \to \infty$ limit, $\mathcal{L}_{(a^b)}(V)$ has a basis indexed by multisets of primitive necklaces and the Schur character is the following content generating function.

Lemma 4.2.7 (See [Kly74, Proposition 1])**.** We have, in the $m \to \infty$ limit,

$$\operatorname{ch} \mathcal{L}_{(a)} = \mathrm{NFD}_{a,1}^{\mathrm{cont}}(\mathbf{x}) \quad \text{and} \quad \operatorname{ch} \mathcal{L}_{(a^b)} = \left(\!\!\binom{\mathrm{NFD}_{a,1}}{b}\!\!\right)^{\mathrm{cont}}(\mathbf{x}).$$

One formulation of Thrall's problem is hence to find the Schur expansion of the expressions in Lemma 4.2.7.

While we will not have direct need of it, we would be remiss if we did not mention the following beautiful and important result of Gessel and Reutenauer [GR93, (2.1)]. The expansion of $\operatorname{ch} \mathcal{L}_\lambda$ in terms of Gessel's fundamental quasisymmetric functions is

$$\operatorname{ch} \mathcal{L}_\lambda = \sum_{\substack{\sigma \in S_n \\ \sigma \text{ has cycle type } \lambda}} F_{n,\mathrm{Des}(\sigma)}(\mathbf{x}), \tag{4.12}$$

where

$$F_{n,D}(\mathbf{x}) = \sum_{\substack{i_1 \le \cdots \le i_n \\ i_j < i_{j+1} \text{ if } j \in D}} x_{i_1} \dots x_{i_n}.$$

Gessel and Reutenauer gave an elegant bijective proof of (4.12) in [GR93] involving multisets of primitive necklaces as in Lemma 4.2.7. Another formulation of Thrall's problem is thus to convert the right-hand side of (4.12) to the Schur basis.

Klyachko [Kly74] was the first to observe the intimate connections between Lie modules and the linear representations χ^r in Definition 4.2.3. Klyachko proved the $r = 1$ case of Theorem 4.1.5, that $\operatorname{ch} \chi^1 \uparrow_{C_n}^{S_n} = \mathrm{NFD}_{n,1}^{\mathrm{cont}}(\mathbf{x})$. Combining Klyachko's result, Lemma 4.2.7, the $r = 1$ case of Theorem 4.1.5, and Kraśkiewicz–Weyman's result, Theorem 1.1.4, solves Thrall's problem when $\lambda = (n)$. Recall that if $\lambda \vdash n$, then $a_{\lambda,r} \coloneqq \#\{Q \in \mathrm{SYT}(\lambda) : \mathrm{maj}(Q) \equiv_n r\}$.

Corollary 4.2.8. For all $\lambda \vdash n \geq 1$, the multiplicity of V^λ in $\mathcal{L}_{(n)}$ is $a_{\lambda,1}$.

Since $\chi^r \uparrow_{C_n}^{S_n}$ depends up to isomorphism only on n and $\gcd(n,r)$, we also have the following well-known symmetry.

Corollary 4.2.9. For all $\lambda \vdash n \geq 1$, we have $a_{\lambda,r} = a_{\lambda,\gcd(n,r)}$.

Remark 4.2.10. A bijective proof of this symmetry is currently unknown.

Thrall's problem is an instance of a plethysm problem as we next describe. See [Sta99, Appendix 2] for more details. Given polynomial representations of general linear groups

$$\rho\colon \mathrm{GL}(V) \to \mathrm{GL}(W) \qquad \text{and} \qquad \tau\colon \mathrm{GL}(W) \to \mathrm{GL}(X)$$

where V, W, X are finite-dimensional complex vector spaces, the *plethysm* of their Schur characters is the Schur character of their composite:

$$(\mathrm{ch}\,\tau)[\mathrm{ch}\,\rho] \coloneqq \mathrm{ch}(\tau \circ \rho).$$

It is easy to see that $\mathrm{ch}\,\mathrm{Sym}^b(W) = h_b(x_1, \ldots, x_m)$ where $m = \dim(W)$. Consequently, (4.11) gives

$$\mathrm{ch}\,\mathcal{L}_{(a^b)} = h_b[\mathrm{ch}\,\mathcal{L}_a]. \tag{4.13}$$

Yet another formulation of Thrall's problem is thus to expand $h_b[\mathrm{ch}\,\mathcal{L}_a]$ in the Schur basis. Such plethysm problems are notoriously difficult. However, a combinatorial description for the Schur expansion of $(\mathrm{ch}\,\mathcal{L}_a)[h_\nu]$ in terms of the *charge* statistic was given by Lascoux–Leclerc–Thibon in [LLT94, Thm. 4.2] and [LLT97, Thm. III.3].

Remark 4.2.11. At present, Thrall's problem has only been solved in the following cases:

- when $\lambda = (n)$ has a single part (see Corollary 4.2.8);
- when $\lambda = (1^n)$, $\mathcal{L}_{(1^n)}$ is the trivial representation;

- when $\lambda = (2^b)$, $\operatorname{ch} \mathcal{L}_{(2^b)} = \sum s_\mu$ where the sum is over $\mu \vdash 2b$ with even column sizes (see [Mac95, Ex. I.8.6(b), p. 138]).

4.2.5 Wreath Products

The Schur–Weyl duals of the higher Lie modules $\mathcal{L}_\lambda$ have also been identified in terms of induced representations of certain wreath products. Here we summarize this connection as well as some related aspects of the representation theory of wreath products which will be used in Section 4.6. Our presentation largely mirrors [Ste89].

Definition 4.2.12. Given a group G, the *wreath product* of G with S_n, denoted $G \wr S_n$, is the semidirect product explicitly described as follows. $G \wr S_n$ is the set $G^n \times S_n$ with multiplication given by

$$(g_1, \ldots, g_n, \sigma) \cdot (h_1, \ldots, h_n, \tau) \coloneqq (g_1 h_{\sigma^{-1}(1)}, \ldots, g_n h_{\sigma^{-1}(n)}, \sigma\tau)$$

for all $g_1, \ldots g_n, h_1, \ldots, h_n \in G$ and $\sigma, \tau \in S_n$. Furthermore, given $\alpha \vDash n$, set $G \wr \prod_i S_{\alpha_i} \coloneqq \prod_i (G \wr S_{\alpha_i})$, which has a natural inclusion into $G \wr S_n$. Roughly speaking, $G \wr S_n$ can be considered as the group of $n \times n$ "pseudo-permutation" matrices with entries from G.

Now suppose U is a G-set and V is an S_n-set. There is a natural notion of $U \wr V$ as a $G \wr S_n$-set. Explicitly, let $U \wr V$ be the set $U^n \times V$ with $G \wr S_n$-action given by

$$(g_1, \ldots, g_n, \sigma) \cdot (u_1, \ldots, u_n, v) \coloneqq (g_1 \cdot u_{\sigma^{-1}(1)}, \ldots, g_n \cdot u_{\sigma^{-1}(n)}, \sigma \cdot v)$$

for all $g_1, \ldots, g_n \in G, \sigma \in S_n, u_1, \ldots, u_n \in U, v \in V$. There is an analogous notion if U is a G-module and V is an S_n-module, namely $U \wr V \coloneqq U^{\otimes n} \otimes V$ with $G \wr S_n$-action

$$(g_1, \ldots, g_b, \sigma) \cdot (u_1 \otimes \cdots \otimes u_b \otimes v) \coloneqq (g_1 \cdot u_{\sigma^{-1}(1)}) \otimes \cdots \otimes (g_b \cdot u_{\sigma^{-1}(b)}) \otimes (\sigma \cdot v)$$

extended $\mathbb{C}$-linearly.

Since S_a acts naturally and faithfully on $[a]$, $[a] \wr 1_b$ has a natural $S_a \wr S_b$-action, where

1_b denotes the trivial S_b-set. Identifying $[a] \wr 1_b$ with the set $[ab]$ and noting that the action remains faithful gives an inclusion $S_a \wr S_b \hookrightarrow S_{ab}$. Similarly we have an inclusion $C_a \wr S_b \hookrightarrow S_{ab}$. More concretely, $C_a \wr S_b$ acts faithfully on $[ab]$ by permuting the b size-a intervals in $[ab]$ amongst themselves and cyclically rotating each size-a interval independently.

Remark 4.2.13. The induction product of two symmetric group representations corresponds to the product of their Frobenius characteristics, so that if U is an S_a-module and V is an S_b-module, then [Sta99, Prop. 7.18.2],

$$\operatorname{ch}\left(U \otimes V \uparrow_{S_a \times S_b}^{S_{a+b}}\right) = (\operatorname{ch} U)(\operatorname{ch} V). \tag{4.14}$$

In Section 4.2.4, we considered the plethysm of Schur characters of general linear group representations. The corresponding operation for Frobenius characters of symmetric group representations is less well-known and involves wreath products as follows. Given two symmetric functions f and $g = m_1 + m_2 + \cdots$ where the m_i are all monomials, their *plethysm* is given by [Sta99, Def. A2.6]

$$f[g] \coloneqq f(m_1, m_2, \dots), \tag{4.15}$$

which is well-defined since f is symmetric. Then, if U is an S_a-module and V is an S_b-module, we have (see [Sta99, Thm. A2.8] or [Mac95, Appendix A, (6.2)])

$$\operatorname{ch}\left((U \wr V) \uparrow_{S_a \wr S_b}^{S_{ab}}\right) = \operatorname{ch}(V)[\operatorname{ch}(U)]. \tag{4.16}$$

When G is a finite group, Specht [Spe32] described the complex inequivalent irreducible representations of $G \wr S_n$ in terms of those for G and S_n, the conjugacy classes of G, and wreath products. In the case $C_a \wr S_b$, they are indexed by the following objects.

Theorem 4.2.14 ([Spe32]; see [Ste89, Thm. 4.1])**.** The complex inequivalent irreducible representations of $C_a \wr S_b$ are indexed by a-tuples $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(a)})$ of partitions with

$\sum_{r=1}^{a} |\lambda^{(r)}| = b$. In particular, they are given by

$$S^{\underline{\lambda}} \coloneqq \Big((\chi_a^1 \wr S^{\lambda^{(1)}}) \otimes \cdots \otimes (\chi_a^a \wr S^{\lambda^{(a)}}) \Big) \uparrow_{C_a \wr S_{\alpha(\underline{\lambda})}}^{C_a \wr S_b}, \tag{4.17}$$

where

$$\alpha(\underline{\lambda}) \coloneqq (|\lambda^{(1)}|, \ldots, |\lambda^{(a)}|) \vDash b,$$
$$S_{\alpha(\underline{\lambda})} \coloneqq S_{|\lambda^{(1)}|} \times \cdots \times S_{|\lambda^{(a)}|},$$

χ_a^r is as defined in Definition 4.2.3, and $C_a \wr S_{\alpha(\underline{\lambda})}$ is viewed naturally as a subgroup of $C_a \wr S_b$.

One consequence of Theorem 4.2.14 is

$$\dim(S^{\underline{\lambda}}) = \binom{b}{\alpha(\underline{\lambda})} \prod_{r=1}^{a} \#\,\mathrm{SYT}(\lambda^{(r)}). \tag{4.18}$$

Another consequence is an explicit description of the one-dimensional representations of $C_a \wr S_b$, which are as follows.

Definition 4.2.15. Fix integers $a, b \geq 1$. Let

$$\chi^{r,1} \coloneqq \chi_a^r \wr 1_b \qquad \text{and } \chi^{r,\epsilon} \coloneqq \chi_a^r \wr \epsilon_b$$

where $r = 1, \ldots, a$ and 1_b and ϵ_b are the trivial and sign representations of S_b, respectively. When $b = 1$, $\epsilon_b = 1_b$, in which case $\chi^{r,1} = \chi^{r,\epsilon} = \chi_a^r$. We sometimes write $\chi_{(a^b)}^{r,1}$ or $\chi_{(a^b)}^{r,\epsilon}$ if we want to specify the group $C_a \wr S_b$ as well.

Bergeron–Bergeron–Garsia [BBG90] extended Klyachko's observation by showing that the Schur–Weyl dual of $\mathcal{L}_{(a^b)}$ is $\chi^{1,1}\uparrow_{C_a \wr S_b}^{S_{ab}}$. We next give a different argument of this fact which is straightforward given the preceding background and which uses a lemma we will require later in Section 4.6.

Lemma 4.2.16. We have

$$\operatorname{ch} \chi^{1,1}{\uparrow}^{S_{ab}}_{C_a \wr S_b} = \left(\!\!\binom{\mathrm{NFD}_{a,1}}{b}\!\!\right)^{\mathrm{cont}}(\mathbf{x}).$$

Proof. By Lemma 4.2.19 below and the fact that $C_a \wr S_b \subseteq S_a \wr S_b \subseteq S_{ab}$, we have

$$\chi^{1,1}{\uparrow}^{S_{ab}}_{C_a \wr S_b} = (\chi^1_a \wr 1_b){\uparrow}^{S_{ab}}_{C_a \wr S_b} \cong (\chi^1_a{\uparrow}^{S_a}_{C_a} \wr 1_b){\uparrow}^{S_{ab}}_{S_a \wr S_b}. \tag{4.19}$$

By (4.16) and the $r = 1$ case of Theorem 4.1.5,

$$\operatorname{ch}(\chi^1_a{\uparrow}^{S_a}_{C_a} \wr 1_b){\uparrow}^{S_{ab}}_{S_a \wr S_b} = (\operatorname{ch} 1_b)[\operatorname{ch} \chi^1_a{\uparrow}^{S_a}_{C_a}] = h_b[\mathrm{NFD}^{\mathrm{cont}}_{a,1}(\mathbf{x})], \tag{4.20}$$

since $\operatorname{ch}(1_b) = h_b(\mathbf{x})$. Now, $h_b[\mathrm{NFD}^{\mathrm{cont}}_{a,1}(\mathbf{x})] = \left(\!\!\binom{\mathrm{NFD}_{a,1}}{b}\!\!\right)^{\mathrm{cont}}(\mathbf{x})$ from the definition of plethysm, (4.15), and the definition of h_b, (2.12). The result will be complete once we prove Lemma 4.2.19. □

Corollary 4.2.17 ([BBG90, §4.4]; see also [Reu93, Thm. 8.24])**.** The Schur–Weyl dual of $\mathcal{L}_{(a^b)}$ is $\chi^1_a{\uparrow}^{S_{ab}}_{C_a \wr S_b}$.

Proof. Combine Lemma 4.2.7 and Lemma 4.2.16. □

Indeed, the Schur–Weyl duals of general $\mathcal{L}_\lambda$ can be expressed very explicitly in terms of induced linear representations as follows. Suppose $\sigma \in S_n$ has cycle type λ. Write Z_λ for the centralizer of σ in S_n. When $\lambda = (a^b)$, it is straightforward to see that $Z_{(a^b)} \cong C_a \wr S_b$. Furthermore, when $\lambda = 1^{b_1}2^{b_2}\cdots k^{b_k}$ is written in exponential notation, we have $Z_\lambda \cong Z_{(1^{b_1})} \times Z_{(2^{b_2})} \times \cdots \times Z_{(k^{b_k})}$.

Corollary 4.2.18 (see [Reu93, Thm. 8.24])**.** Suppose $\lambda = 1^{b_1}2^{b_2}\cdots k^{b_k} \vdash n$. Let $\chi^{1,1}_\lambda$ denote the linear representation of $Z_\lambda \leq S_n$ given by the (outer) tensor product of the representations $\chi^{1,1}_{(i^{b_i})}$ of $C_i \wr S_{b_i}$ for $1 \leq i \leq k$. Then, the Schur–Weyl dual of $\mathcal{L}_\lambda$ is $\chi^{1,1}_\lambda{\uparrow}^{S_n}_{Z_\lambda}$.

Proof. Using in order (4.10), multiplicativity of Schur characters under tensor products, Corollary 4.2.17, (4.14), Lemma 4.2.20 and transitivity of induction, the fact that $Z_\lambda \cong \prod_{i=1}^k Z_{(i^{b_i})}$, and the definition of $\chi_\lambda^{1,1}$, we have

$$\begin{aligned}
\operatorname{ch} \mathcal{L}_\lambda &= \operatorname{ch}\left(\bigotimes_{i=1}^k \mathcal{L}_{(i^{b_i})}\right) = \prod_{i=1}^k \operatorname{ch} \mathcal{L}_{(i^{b_i})} = \prod_{i=1}^k \operatorname{ch} \chi_{(i^{b_i})}^{1,1} \uparrow_{Z_{(i^{b_i})}}^{S_{ib_i}} \\
&= \operatorname{ch}\left(\bigotimes_{i=1}^k \chi_{(i^{b_i})}^{1,1} \uparrow_{Z_{(i^{b_i})}}^{S_{ib_i}}\right) \uparrow_{S_{1b_1}\times S_{2b_2}\times\cdots}^{S_n} = \operatorname{ch}\left(\bigotimes_{i=1}^k \chi_{(i^{b_i})}^{1,1}\right) \uparrow_{Z_\lambda}^{S_n} \\
&= \operatorname{ch} \chi_\lambda^{1,1} \uparrow_{Z_\lambda}^{S_n}
\end{aligned}$$

The result will be complete once we prove Lemma 4.2.20. □

Lemma 4.2.19. Suppose that H is a subgroup of a group G, that U is an H-module, and that V is an S_n-module. Then

$$(U \wr V) \uparrow_{H\wr S_n}^{G\wr S_n} \cong \left(U \uparrow_H^G\right) \wr V$$

as $G \wr S_n$-modules.

Proof. As sets, we have

$$\begin{aligned}
(U \wr V) \uparrow_{H\wr S_n}^{G\wr S_n} &= \mathbb{C}(G \wr S_n) \otimes_{\mathbb{C}(H\wr S_n)} (U^{\otimes n} \otimes V), \\
\left(U \uparrow_H^G\right) \wr V &= \left(\mathbb{C}G \otimes_{\mathbb{C}H} U\right)^{\otimes n} \otimes V.
\end{aligned}$$

Define

$$\begin{aligned}
\phi\colon (U \wr V) \uparrow_{H\wr S_n}^{G\wr S_n} &\to \left(U \uparrow_H^G\right) \wr V, \\
\psi\colon \left(U \uparrow_H^G\right) \wr V &\to (U \wr V) \uparrow_{H\wr S_n}^{G\wr S_n}
\end{aligned}$$

by

$$\begin{aligned}
&\phi((g_1,\dots,g_n,\tau)\otimes(u_1\otimes\cdots\otimes u_n\otimes v))\\
&\qquad\coloneqq (g_1\otimes u_{\tau^{-1}(1)})\otimes\cdots\otimes(g_n\otimes u_{\tau^{-1}(n)})\otimes(\tau\cdot v),\\
&\psi((g_1\otimes x_1)\otimes\cdots\otimes(g_n\otimes x_n)\otimes y)\\
&\qquad\coloneqq (g_1,\dots,g_n,1)\otimes(x_1\otimes\cdots\otimes x_n\otimes y)
\end{aligned}$$

extended $\mathbb{C}$-linearly. It is straightforward to check directly that ϕ and ψ are well-defined, $G\wr S_n$-equivariant, and mutual inverses. Note that showing $\psi\circ\phi(x)=x$ requires using the relation

$$(g_1,\dots,g_n,\tau)\otimes z=(g_1,\dots,g_n,1)\otimes(\tau\cdot z)$$

in $\mathbb{C}(G\wr S_n)\otimes_{\mathbb{C}(H\wr S_n)}(U^{\otimes n}\otimes V)$ for $g_1,\dots,g_n\in G,\tau\in S_n,z\in U^{\otimes n}\otimes V$. □

Lemma 4.2.20. Suppose that $H_1,\dots,H_k$ are subgroups of groups $G_1,\dots,G_k$ and that U_i is an H_i-module for $1\le i\le k$. Then

$$(U_1\otimes\cdots\otimes U_k)\uparrow^{G_1\times\cdots\times G_k}_{H_1\times\cdots\times H_k}\cong U_1\uparrow^{G_1}_{H_1}\otimes\cdots\otimes U_k\uparrow^{G_k}_{H_k}$$

as $G_1\times\cdots\times G_k$-modules.

Proof. Having chosen bases for both sides, there is a natural $\mathbb{C}$-linear map between them. It is easy to check this is also $G_1\times\cdots\times G_k$-equivariant. The details are omitted. □

4.3 Cyclic Sieving and Kraśkiewicz–Weyman's Result

In this section, we first build on work of Klyachko to prove Theorem 4.1.5. We then recover Kraśkiewicz–Weyman's result, Theorem 1.1.4, and discuss some benefits of our approach.

Klyachko observed in [Kly74, Prop. 1] that $E(\chi^1\uparrow^{S_n}_{C_n})$, like $\mathcal{L}_{(n)}$, also has a basis indexed by primitive necklaces. Klyachko's argument may be readily generalized to $E(\chi^r\uparrow^{S_n}_{C_n})$ as

follows. Recall from the introduction that

$$\begin{aligned} \mathrm{NFD}_{n,r} &\coloneqq \{N \in \mathrm{N}_n : \mathrm{freq}(N) \mid r\}, \\ \mathrm{F}_{n,r} &\coloneqq \{w \in \mathrm{W}_n : \mathrm{flex}(w) = r\}, \\ \mathrm{M}_{n,r} &\coloneqq \{w \in \mathrm{W}_n : \mathrm{maj}_n(w) = r\}. \end{aligned}$$

In particular, $\mathrm{NFD}_{n,n} = \mathrm{N}_n$, and $\mathrm{NFD}_{n,1}$ is the set of primitive necklaces of length n.

Theorem 4.3.1. There is a basis for $E(\chi^r{\uparrow}_{C_n}^{S_n})$ indexed by necklaces of length n words with letters from $[m]$ and with frequency dividing r. Moreover,

$$\mathrm{ch}\, \chi^r{\uparrow}_{C_n}^{S_n} = \mathrm{NFD}_{n,r}^{\mathrm{cont}}(\mathbf{x}). \tag{4.21}$$

Proof. Suppose the underlying vector space V has basis $\{v_1, \ldots, v_m\}$. By a slight abuse of notation, we may view χ^r as the vector space $\mathbb{C}$ with the left C_n-action $\sigma_n \cdot 1 \coloneqq \omega_n^r$. Since $\chi^r{\uparrow}_{C_n}^{S_n} \coloneqq \mathbb{C}S_n \otimes_{\mathbb{C}C_n} \chi^r$, we have

$$E(\chi^r{\uparrow}_{C_n}^{S_n}) = V^{\otimes n} \otimes_{\mathbb{C}S_n} \mathbb{C}S_n \otimes_{\mathbb{C}C_n} \chi^r \cong V^{\otimes n} \otimes_{\mathbb{C}C_n} \chi^r \tag{4.22}$$

where C_n acts on $V^{\otimes n}$ on the right by "rotating" the components of simple tensors. A spanning set for $V^{\otimes n} \otimes_{\mathbb{C}C_n} \chi^r$ is given by all $v_{i_1} \otimes \cdots \otimes v_{i_n} \otimes 1$, which we abbreviate as $[i_1 \ \cdots \ i_n]$. Acting by σ_n^{-1} on χ^r on the left or on $V^{\otimes n}$ on the right gives the relation

$$[i_1 \ \cdots \ i_n] = \omega_n^r [i_2 \cdots i_n \, i_1]. \tag{4.23}$$

This relation shows that $[i_1 \cdots i_n]$ is well-defined on the level of necklaces, at least up to nonzero scalar multiplication, which explains our notation. If the word $i_1 \cdots i_n$ has frequency

f and period p, we then find

$$\begin{aligned}[i_1 \ \cdots \ i_n] &= \frac{1}{n}\sum_{j=0}^{n-1} \omega_n^{jr}[i_{j+1} \ \cdots \ i_n \ i_1 \ \cdots \ i_j] \\ &= \frac{1}{n}\sum_{k=0}^{p-1}\left(\sum_{\ell=0}^{f-1}\omega_n^{(\ell p+k)r}\right)[i_{k+1}\cdots i_n i_1 \cdots i_k] \\ &= \frac{1}{n}\left(\sum_{\ell=0}^{f-1}\omega_n^{\ell pr}\right)\sum_{k=0}^{p-1}\omega_n^{kr}[i_{k+1}\cdots i_n i_1\cdots i_k].\end{aligned}$$

Since ω_n^p is a primitive $n/p = f$-th root of unity, the factor $\sum_{\ell=0}^{f-1}\omega_n^{\ell pr}$ is nonzero if and only if $\omega_n^{pr} = 1$, so if and only if $f \mid r$. Picking representatives for necklaces with frequency dividing r thus gives a spanning set for $E(\chi^r{\uparrow}_{C_n}^{S_n})$, and it is easy to see it is in fact a basis. Diagonal matrices act on this basis via

$$\operatorname{diag}(x_1,\ldots,x_n)\cdot[i_1 \ \cdots \ i_n] = \mathbf{x}^{\operatorname{cont}(i_1\cdots i_n)}[i_1 \ \cdots \ i_n], \tag{4.24}$$

from which it follows that the Schur character is the content generating function of necklaces of length n words with letters from $[m]$ and with frequency dividing r. Letting $m \to \infty$, (4.21) follows. □

Lemma 4.3.2. We have

$$\mathrm{NFD}_{n,r}^{\mathrm{cont}}(\mathbf{x}) = \mathrm{F}_{n,r}^{\mathrm{cont}}(\mathbf{x}) = \mathrm{M}_{n,r}^{\mathrm{cont}}(\mathbf{x}).$$

Proof. Consider the map

$$\begin{aligned}\iota\colon \mathrm{F}_{n,r} &\to \mathrm{NFD}_{n,r} \\ \iota(w) &\coloneqq [w].\end{aligned}$$

Since $\operatorname{flex}(w) = \operatorname{freq}(w)\operatorname{lex}(w) = r$, we have $\operatorname{freq}(w) \mid r$, so $[w] \in \mathrm{NFD}_{n,r}$. Thus, ι is in fact a

map from $\mathrm{F}_{n,r}$ to $\mathrm{NFD}_{n,r}$. Since each necklace in $\mathrm{NFD}_{n,r}$ contains exactly one word with flex equal to r, ι is a content-preserving bijection. Therefore, $\mathrm{NFD}^{\mathrm{cont}}_{n,r}(\mathbf{x}) = \mathrm{F}^{\mathrm{cont}}_{n,r}(\mathbf{x})$.

Using Theorem 4.1.3, we have

$$\mathrm{W}^{\mathrm{cont,flex}}_{n}(\mathbf{x}; q) = \mathrm{W}^{\mathrm{cont,maj}_n}_{n}(\mathbf{x}; q),$$

which means $\mathrm{F}^{\mathrm{cont}}_{n,r}(\mathbf{x}) = \mathrm{M}^{\mathrm{cont}}_{n,r}(\mathbf{x})$. □

Remark 4.3.3. From Theorem 4.3.1 and Lemma 4.3.2, the Schur character of $\chi^r{\uparrow}^{S_n}_{C_n}$ may be described as a content generating function for certain necklaces or for certain words. This proves Theorem 4.1.5 from the introduction.

We may now present our remarkably direct proof of Kraśkiewicz–Weyman's result, Theorem 1.1.4, using cyclic sieving.

Proof (of Theorem 1.1.4). The argument in Theorem 4.3.1 exhibited an explicit basis of the Schur module $E(\chi^r{\uparrow}^{S_n}_{C_n})$, showing that

$$\sum_{r=1}^{n} \mathrm{ch}\, \chi^r{\uparrow}^{S_n}_{C_n}\, q^r = \sum_{r=1}^{n} \mathrm{NFD}^{\mathrm{cont}}_{n,r}(\mathbf{x})\, q^r.$$

From Lemma 4.3.2, the bijection $\iota\colon\ \mathrm{F}_{n,r} \xrightarrow{\sim} \mathrm{NFD}_{n,r}$ given by $w \mapsto [w]$ gives

$$\sum_{r=1}^{n} \mathrm{NFD}^{\mathrm{cont}}_{n,r}(\mathbf{x})\, q^r = \mathrm{W}^{\mathrm{cont,flex}}_{n}(\mathbf{x}; q).$$

Using universal cyclic sieving on words for S_n-orbits and C_n-orbits as described in the introduction, Theorem 4.1.3 now gives

$$\mathrm{W}^{\mathrm{cont,flex}}_{n}(\mathbf{x}; q) = \mathrm{W}^{\mathrm{cont,maj}_n}_{n}(\mathbf{x}; q).$$

Using the RSK algorithm, Lemma 4.2.1 gives

$$\mathrm{W}_n^{\mathrm{cont},\mathrm{maj}_n}(\mathbf{x};q) = \sum_{\substack{\lambda\vdash n \\ r\in[n]}} a_{\lambda,r} s_\lambda(\mathbf{x})\, q^r.$$

Combining all of these equalities and extracting the coefficient of q^r gives the result. □

Every step of the preceding proof uses an explicit bijection with the exception of the appeal to cyclic sieving through Theorem 4.1.3. This suggests the problem of finding a bijective proof of Theorem 4.1.3.

Open Problem 4.3.4. For each $n \geq 1$, find an explicit, content-preserving bijection

$$\phi\colon \mathrm{W}_n \to \mathrm{W}_n$$

such that $\mathrm{maj}_n(w) = \mathrm{flex}(\phi(w))$.

Corollary 4.3.5. A solution to Open Problem 4.3.4 would yield an explicit, bijective proof of the identity

$$\sum_{\lambda\vdash n} a_{\lambda,r} s_\lambda(\mathbf{x}) = \sum_{\lambda\vdash n} a_{\lambda,s} s_\lambda(\mathbf{x}) \tag{4.25}$$

for any $r, s \in \mathbb{Z}$ where $\gcd(n,r) = \gcd(n,s)$.

Proof. We have content-preserving bijections

$$\bigsqcup_{\lambda\vdash n} \mathrm{SSYT}(\lambda) \times \{Q \in \mathrm{SYT}(\lambda) : \mathrm{maj}(Q) \equiv_n r\} \overset{\mathrm{RSK}}{\to} \mathrm{M}_{n,r} \overset{\phi}{\to} \mathrm{F}_{n,r} \overset{\iota}{\to} \mathrm{NFD}_{n,r}\,.$$

Now note that

$$\mathrm{NFD}_{n,r} = \mathrm{NFD}_{n,\gcd(n,r)} = \mathrm{NFD}_{n,\gcd(n,s)} = \mathrm{NFD}_{n,s}\,.$$

We thus have an explicit, content-preserving bijection

$$\bigsqcup_{\lambda\vdash n} \mathrm{SSYT}(\lambda) \times \{Q \in \mathrm{SYT}(\lambda) : \mathrm{maj}(Q) \equiv_n r\}$$
$$\xrightarrow{\sim} \bigsqcup_{\lambda\vdash n} \mathrm{SSYT}(\lambda) \times \{Q \in \mathrm{SYT}(\lambda) : \mathrm{maj}(Q) \equiv_n s\}$$

from which (4.25) follows. □

Remark 4.3.6. The most difficult step in our proof of Theorem 1.1.4 is the universal S_n-cyclic sieving result, Corollary 4.1.1, or equivalently Theorem 2.1.6. The proof in [RSW04] of Theorem 2.1.6 perhaps unsurprisingly uses several of the interpretations of the Kraśkiewicz–Weyman symmetric functions from Section 4.2.3, in particular Theorem 4.2.6 involving $\mathrm{ch}_{S_n\times C_n} R_n$. However, both Kraśkiewicz–Weyman's and Springer's original proofs of Theorem 4.2.6 hinge upon (2.16). Indeed, Kraśkiewicz–Weyman showed explicitly in [KW01, Prop. 3] that $\mathrm{ch}_{S_n\times C_n} \mathbb{C}S_n = \mathrm{ch}_{S_n\times C_n} R_n$ is easily equivalent to (2.16). Springer's argument proving (2.16) uses a Molien-style formula, while Kraśkiewicz–Weyman's argument uses a recursion involving ℓ-cores and skew hooks.

One may thus ask about the relationship between (2.16) and the cyclic sieving result, Theorem 2.1.6. Using stable principal specializations, one can consider earlier approaches to have been "in the s-basis" and our approach to have been "in the h-basis" in the following sense. Let τ^λ be the S_n-character of $1\uparrow_{S_\lambda}^{S_n}$, which has $\mathrm{ch}(1\uparrow_{S_\lambda}^{S_n}) = h_\lambda$. We have

$$\chi^\lambda(\sigma_n^r) = \mathrm{SYT}(\lambda)^{\mathrm{maj}}(\omega_n^r) = (1-q)\cdots(1-q^n)s_\lambda(1,q,q^2,\ldots)|_{q=\omega_n^r},$$
$$\tau^\lambda(\sigma_n^r) = \mathrm{W}_\lambda^{\mathrm{maj}}(\omega_n^r) \qquad = (1-q)\cdots(1-q^n)h_\lambda(1,q,q^2,\ldots)|_{q=\omega_n^r}.$$

where the first equality is (2.16), the second is [Sta99, Prop. 7.19.11], the third is Theorem 2.1.6, and the fourth is [Sta99, Prop. 7.8.3] and [Mac13, Art. 6]. Our approach suggests that, as far as the Kraśkiewicz–Weyman Theorem is concerned, the h-basis arises more directly.

In [AS18b], the authors proved a refinement of Theorem 2.1.6. Since earlier approaches to

Theorem 2.1.6 involving representation theory could not readily be adapted to this refinement, the argument instead uses completely different and highly combinatorial techniques. Thus, the arguments in [AS18b] and the proof of Theorem 1.1.4 together give an essentially self-contained proof of Kraśkiewicz–Weyman's result.

4.4 Induced Representations of Arbitrary Cyclic Subgroups of S_n

We next generalize the discussion in Section 4.3 to branching rules for general inclusions $\langle\sigma\rangle \hookrightarrow S_n$, recovering a result of Stembridge, Theorem 4.4.11. Following the outline of the previous section, we express the relevant characters in turn as a certain orbit generating function, Theorem 4.4.2, a necklace generating function, Lemma 4.4.6, and a generating function on words, Lemma 4.4.7. Two variations on the major index, $\mathbf{maj}_\nu$ and maj_ν, arise quite naturally from our argument. The CSP Theorem 2.1.6 again plays a decisive role.

Throughout this section, let $\sigma \in S_n$, let C be the cyclic group generated by σ, and let $\ell \coloneqq \#C$ be the order of σ. Fixing a primitive ℓ-th root of unity ω_ℓ, let $\chi^r \colon C \to \mathbb{C}$ for $r = 1, \ldots, \ell$ be the linear C-module given by $\chi^r(\sigma) \coloneqq \omega_\ell^r$. We begin by updating our notation for this setting and generalizing Theorem 4.3.1.

Definition 4.4.1. In analogy with necklaces, suppose $\mathcal{O}$ is an orbit of W_n under the restricted C-action. The *period* of $\mathcal{O}$ is $\#\mathcal{O}$ and the *frequency* of $\mathcal{O}$, written $\mathrm{freq}(\mathcal{O})$, is the stabilizer-order of any element of $\mathcal{O}$, or equivalently $\mathrm{freq}(\mathcal{O}) = \frac{\ell}{\#\mathcal{O}}$. The set of orbits of words whose frequency divides r is

$$\mathrm{OFD}_{C,r} \coloneqq \{C\text{-orbits } \mathcal{O} \text{ of } \mathrm{W}_n : \mathrm{freq}(\mathcal{O}) \mid r\}.$$

Theorem 4.4.2. There is a basis for $E(\chi^r{\uparrow}_C^{S_n})$ indexed by C-orbits of length n words with letters from $[m]$ and with frequency dividing r. Moreover,

$$\mathrm{ch}\left(\chi^r{\uparrow}_C^{S_n}\right) = \mathrm{OFD}_{C,r}^{\mathrm{cont}}(\mathbf{x}). \tag{4.26}$$

Proof. The proof of Theorem 4.3.1 goes through verbatim with the C-action replacing the C_n-action. □

Our goal is broadly to replace $\mathrm{OFD}^{\mathrm{cont}}_{C,r}(\mathbf{x})$ with a necklace generating function, apply cyclic sieving to get a major index generating function on words, and then apply RSK to get a Schur expansion.

Notation 4.4.3. For the rest of the section, suppose that σ has disjoint cycle decomposition $\sigma = \sigma_1 \cdots \sigma_k$ with $\nu_i \coloneqq |\sigma_i|$. Consequently, $\ell = |\langle\sigma\rangle| = \mathrm{lcm}(\nu_1, \ldots, \nu_k)$. Further, write

$$C_\nu \coloneqq \{\sigma_1^{r_1} \cdots \sigma_k^{r_k} \in S_n : r_1, \ldots, r_k \in \mathbb{Z}\} \cong C_{\nu_1} \times \cdots \times C_{\nu_k}$$

where $C_{\nu_i} \coloneqq \langle\sigma_i\rangle \subset S_n$. Thus, we have $C \subset C_\nu \subset S_n$.

In Section 4.3, we considered the C_n-orbits of W_n, namely necklaces $N \in \mathrm{N}_n$. The frequency of N is the stabilizer-order of N, i.e. $\mathrm{freq}(N) = \#\,\mathrm{Stab}_{C_n}(N)$. We may group together C_n-orbits of W_n according to their stabilizer sizes by letting

$$\mathrm{NF}_{n,r} \coloneqq \{N \in \mathrm{N}_n : \mathrm{freq}(N) = r\} \tag{4.27}$$

be the set of necklaces of length n words with frequency r. Similarly, $\mathrm{NFD}_{n,r}$ consists of C_n-orbits of W_n whose stabilizer is contained in the common stabilizer of $\mathrm{NF}_{n,r}$.

Analogously, the C_ν-orbits of W_n can be identified with products of necklaces $N_1 \times \cdots \times N_k$ or equivalently with tuples $(N_1, \ldots, N_k)$ where $N_j \in \mathrm{N}_{\nu_j}$. Since

$$\mathrm{Stab}_{C_\nu}(N_1 \times \cdots \times N_k) = \prod_{j=1}^{k} \mathrm{Stab}_{C_{\nu_j}}(N_j),$$

we may group together C_ν-orbits of W_n according to their stabilizers as follows.

Definition 4.4.4. Given $\nu = (\nu_1, \ldots, \nu_k)$ and $\rho = (\rho_1, \ldots, \rho_k)$, let

$$\mathrm{NF}_{\nu,\rho} \coloneqq \mathrm{NF}_{\nu_1,\rho_1} \times \cdots \times \mathrm{NF}_{\nu_k,\rho_k},$$
$$\mathrm{NFD}_{\nu,\rho} \coloneqq \mathrm{NFD}_{\nu_1,\rho_1} \times \cdots \times \mathrm{NFD}_{\nu_k,\rho_k}.$$

The elements of $\mathrm{NF}_{\nu,\rho}$ all have the same stabilizer, and the elements of $\mathrm{NFD}_{\nu,\rho}$ are precisely those whose stabilizer is contained in the common stabilizer of elements of $\mathrm{NF}_{\nu,\rho}$. We write $\rho \mid \nu$ to mean that $\rho_i \mid \nu_i$ for all $i = 1, \ldots, r$. Note that $\mathrm{NF}_{\nu,\rho} \neq \varnothing$ if and only if $\rho \mid \nu$.

Given a group G acting on a set W and a subgroup H of G, each G-orbit of W is partitioned into H-orbits. Consequently, C_ν-orbits of W_n are unions of C-orbits, which we exploit as follows.

Lemma 4.4.5. Let $\mathcal{O}$ be a C-orbit of W_n. Let $N_1 \times \cdots \times N_k$ be the C_ν-orbit containing $\mathcal{O}$ and suppose $N_1 \times \cdots \times N_k \in \mathrm{NF}_{\nu,\rho}$. Then

$$\#\mathcal{O} = \operatorname{lcm}\left(\frac{\nu_1}{\rho_1}, \ldots, \frac{\nu_k}{\rho_k}\right),$$

which depends only on ν and ρ. In particular,

$$\mathcal{O} \in \mathrm{OFD}_{C,r} \qquad \text{if and only if} \qquad \ell \mid r \cdot \operatorname{lcm}\left(\frac{\nu_1}{\rho_1}, \ldots, \frac{\nu_k}{\rho_k}\right).$$

Proof. By assumption, $\mathrm{freq}(N_j) = \rho_j$ and $N_j \in \mathrm{N}_{\nu_j}$, so $\#N_j = \nu_j/\rho_j$. It follows that $\mathcal{O}$ is in bijection with the group generated by a permutation of cycle type $(\nu_1/\rho_1, \ldots, \nu_k/\rho_k)$, so that $\#\mathcal{O} = \operatorname{lcm}(\nu_1/\rho_1, \ldots, \nu_k/\rho_k)$. The second claim follows by noting that

$$\mathcal{O} \in \mathrm{OFD}_{C,r} \Leftrightarrow \mathrm{freq}(\mathcal{O}) \mid r \Leftrightarrow (\ell/\#\mathcal{O}) \mid r \Leftrightarrow \ell \mid r \cdot \#\mathcal{O}.$$

□

Lemma 4.4.6. We have

$$\mathrm{OFD}^{\mathrm{cont}}_{C,r}(\mathbf{x}) = \sum \frac{\prod_{j=1}^{k} \frac{\nu_j}{\rho_j}}{\mathrm{lcm}\left(\frac{\nu_1}{\rho_1}, \ldots, \frac{\nu_k}{\rho_k}\right)} \mathrm{NF}^{\mathrm{cont}}_{\nu,\rho}(\mathbf{x}),$$

where the sum is over all ρ such that $\rho \mid \nu$ and $\ell \mid r \cdot \mathrm{lcm}\left(\frac{\nu_1}{\rho_1}, \ldots, \frac{\nu_k}{\rho_k}\right)$.

Proof. Consider the map

$$\Omega : \mathrm{OFD}_{C,r} \to \bigsqcup \mathrm{NF}_{\nu,\rho}$$

sending $\mathcal{O} \in \mathrm{OFD}_{C,r}$ to the C_ν-orbit containing $\mathcal{O}$, where the union is over all ρ such that $\rho \mid \nu$ and $\ell \mid r \cdot \mathrm{lcm}\left(\frac{\nu_1}{\rho_1}, \ldots, \frac{\nu_k}{\rho_k}\right)$. By Lemma 4.4.5, Ω does in fact map $\mathrm{OFD}_{C,r}$ into this union, and Ω is surjective. Also, each C_ν-orbit contained in $\mathrm{NF}_{\nu,\rho}$ has size $\mathrm{lcm}\left(\frac{\nu_1}{\rho_1}, \ldots, \frac{\nu_k}{\rho_k}\right)$, and $\#N_1 \times \cdots \times N_k = \prod_{j=1}^{k} \frac{\nu_j}{\rho_j}$, so the fiber of each $N_1 \times \cdots \times N_k \in \mathrm{NF}_{\nu,\rho}$ has size

$$\#\Omega^{-1}(N_1 \times \ldots \times N_k) = \frac{\prod_{j=1}^{k} \frac{\nu_j}{\rho_j}}{\mathrm{lcm}\left(\frac{\nu_1}{\rho_1}, \ldots, \frac{\nu_k}{\rho_k}\right)}.$$

The result now follows from Ω being content-preserving. □

In Section 4.3, we used cyclic sieving to turn generating functions involving $\mathrm{NFD}^{\mathrm{cont}}_{n,r}(\mathbf{x})$ into Schur expansions. Thus our next goal is to turn the necklace generating function in Lemma 4.4.6 into an analogous generating function over $\mathrm{NFD}^{\mathrm{cont}}_{\nu,\rho}(\mathbf{x})$. To accomplish this, one could in principle use Möbius inversion on the lattice of stabilizers of C_ν-orbits to convert from $\mathrm{NF}^{\mathrm{cont}}_{\nu,\rho}(\mathbf{x})$ to $\mathrm{NFD}^{\mathrm{cont}}_{\nu,\rho}(\mathbf{x})$. However, the following argument is more direct.

Lemma 4.4.7. For $r = 1, \ldots, n$,

$$\mathrm{OFD}^{\mathrm{cont}}_{C,r}(\mathbf{x}) = \sum \mathrm{NFD}^{\mathrm{cont}}_{\nu,\tau}(\mathbf{x}),$$

where the sum is over all k-tuples of integers $\tau \in [\nu_1] \times \cdots \times [\nu_k]$ such that $\sum_{j=1}^{k} \frac{\ell}{\nu_j}\tau_j \equiv_\ell r$.

Proof. We have

$$\mathrm{NFD}^{\mathrm{cont}}_{\nu,\tau}(\mathbf{x}) = \sum_{\rho|\nu,\tau} \mathrm{NF}^{\mathrm{cont}}_{\nu,\rho}(\mathbf{x}),$$

where $\rho \mid \nu, \tau$ means $\rho_j \mid \nu_j$ and $\rho_j \mid \tau_j$ for all j. Consequently,

$$\sum_{\substack{\tau \in [\nu_1] \times \cdots \times [\nu_k] \\ \sum_{j=1}^k \frac{\ell}{\nu_j} \tau_j \equiv_\ell r}} \mathrm{NFD}^{\mathrm{cont}}_{\nu,\tau}(\mathbf{x}) = \sum_{\rho|\nu} c^r_{\nu,\rho} \, \mathrm{NF}^{\mathrm{cont}}_{\nu,\rho}(\mathbf{x})$$

where

$$c^r_{\nu,\rho} \coloneqq \# \left\{ \tau \in [\nu_1] \times \cdots \times [\nu_k] : \rho \mid \tau, \sum_{j=1}^k \frac{\ell}{\nu_j} \tau_j \equiv_\ell r \right\}.$$

Since $\rho_j \mid \nu_j$ and $\rho_j \mid \tau_j$, write $\gamma_j \coloneqq \frac{\nu_j}{\rho_j} \in \mathbb{Z}_{\geq 1}$ and $\delta_j \coloneqq \frac{\tau_j}{\rho_j} \in \mathbb{Z}_{\geq 1}$. Then,

$$\sum_{j=1}^k \frac{\ell}{\nu_j} \tau_j = \sum_{j=1}^k \frac{\ell}{\gamma_j} \delta_j,$$

so

$$c^r_{\nu,\rho} = \# \left\{ \delta \in [\gamma_1] \times \cdots \times [\gamma_k] : \sum_{j=1}^k \frac{\ell}{\gamma_j} \delta_j \equiv_\ell r \right\}.$$

Defining a group homomorphism

$$\begin{aligned} \phi \colon \prod_{i=1}^k \mathbb{Z}/\gamma_j &\to \mathbb{Z}/\ell \\ (\delta_1, \ldots, \delta_k) &\mapsto \sum_{j=1}^k \frac{\ell}{\gamma_j} \delta_j, \end{aligned}$$

we now have $c^r_{\nu,\rho} = \#\phi^{-1}(r)$. Since $\frac{\ell}{\gamma_1}\mathbb{Z} + \cdots + \frac{\ell}{\gamma_k}\mathbb{Z} = \frac{\ell}{\operatorname{lcm}(\gamma_1,\ldots,\gamma_k)}\mathbb{Z}$, it follows that

$$\begin{aligned} \operatorname{im}\phi &= \{r \in \mathbb{Z}/\ell : \ell \mid r \cdot \operatorname{lcm}(\gamma_1, \ldots, \gamma_k)\}, \\ \#\operatorname{im}\phi &= \operatorname{lcm}(\gamma_1, \ldots, \gamma_k). \end{aligned}$$

For $r \in \operatorname{im}\phi$, we then have

$$c^r_{\nu,\rho} = \#\phi^{-1}(r) = \#\ker\phi = \frac{\gamma_1 \cdots \gamma_k}{\operatorname{lcm}(\gamma_1, \ldots, \gamma_k)}.$$

The result follows from Lemma 4.4.6. □

Our next goal is to convert the necklace expansion in Lemma 4.4.7 into a Schur expansion. Recalling from Section 4.3 that $\mathrm{M}_{n,r} \coloneqq \{w \in \mathrm{W}_n : \mathrm{maj}_n(w) = r\}$, Lemma 4.3.2 tells us

$$\mathrm{NFD}^{\mathrm{cont}}_{\nu,\tau}(\mathbf{x}) = \prod_{j=1}^{k} \mathrm{NFD}^{\mathrm{cont}}_{\nu_j,\tau_j}(\mathbf{x}) = \prod_{j=1}^{k} \mathrm{M}^{\mathrm{cont}}_{\nu_j,\tau_j}(\mathbf{x}). \tag{4.28}$$

Interpreting the right-hand side of (4.28) in terms of words and comparing with the indexing set in Lemma 4.4.7 motivates the following variations on the major index.

Definition 4.4.8. Suppose $\nu \vDash n$, $\tau \in [\nu_1] \times \cdots \times [\nu_k]$, and $\ell = \operatorname{lcm}(\nu_1, \ldots, \nu_k)$. Let $\mathbf{maj}_\nu \colon \mathrm{W}_n \to [\nu_1] \times \cdots \times [\nu_k]$ be defined as follows. For $w \in \mathrm{W}_n$, write $w = w^1 \cdots w^k$ where each w^j is a word in W_{ν_j}. Set

$$\mathbf{maj}_\nu(w) \coloneqq (\mathrm{maj}_{\nu_1}(w^1), \ldots, \mathrm{maj}_{\nu_k}(w^k)).$$

Furthermore, let $\mathrm{maj}_\nu \colon \mathrm{W}_n \to [\ell]$ be defined by

$$\mathrm{maj}_\nu(w) \coloneqq \sum_{j=1}^{k} \frac{\ell}{\nu_j} \mathbf{maj}_\nu(w)_j \qquad (\mathrm{mod}\ \ell).$$

Consequently, we have $\mathrm{maj}_{(n)} = \mathrm{maj}_n$. Note that both $\mathbf{maj}_\nu$ and maj_ν are functions of

$\mathrm{Des}(w)$. We may thus define both $\mathbf{maj}_\nu$ and maj_ν on $Q \in \mathrm{SYT}(n)$ using only $\mathrm{Des}(Q)$ in the same way. Equivalently, we may set $\mathbf{maj}_\nu(Q) \coloneqq \mathbf{maj}_\nu(w)$ and $\mathrm{maj}_\nu(Q) \coloneqq \mathrm{maj}_\nu(w)$ for any w such that $Q = Q(w)$.

Example 4.4.9. Let $\nu = (5,3,3)$ and $w = 44121361631$, so that $\ell = 15$, $w_1 = 44121$, $w_2 = 361$, and $w_3 = 631$. We have

$$\mathbf{maj}_\nu(w) = (\mathrm{maj}_5(w_1), \mathrm{maj}_3(w_2), \mathrm{maj}_3(w_3)) = (1,2,3)$$

and, hence, $\mathrm{maj}_\nu(w) = \frac{15}{5} \cdot 1 + \frac{15}{3} \cdot 2 + \frac{15}{3} \cdot 3 = 13 \pmod{15}$.

Definition 4.4.10. Suppose $\nu \vDash n$, $\tau \in [\nu_1] \times \cdots \times [\nu_k]$. Let

$$\mathrm{M}_{\nu,\tau} \coloneqq \{w \in \mathrm{W}_n : \mathbf{maj}_\nu(w) = \tau\},$$

Theorem 4.4.11. [Ste89, Theorem 3.3] Let C be a cyclic subgroup of S_n generated by an element of cycle type $\nu = (\nu_1, \ldots, \nu_k)$, and let $\ell = \mathrm{lcm}(\nu_1, \ldots, \nu_k)$. We have

$$\sum_{r=1}^{\ell} \mathrm{ch}\left(\chi^r \uparrow_C^{S_n}\right) q^r = \mathrm{W}_n^{\mathrm{cont},\mathrm{maj}_\nu}(\mathbf{x}; q) = \sum_{\substack{\lambda \vdash n \\ r \in [\ell]}} a_{\lambda,r}^\nu s_\lambda(\mathbf{x}) q^r$$

where $a_{\lambda,r}^\nu \coloneqq \#\{Q \in \mathrm{SYT}(\lambda) : \mathrm{maj}_\nu(Q) = r\}$. In particular, the multiplicity of S^λ in $\chi^r \uparrow_C^{S_n}$ is $a_{\lambda,r}^\nu$.

Proof. From the definition of $\mathbf{maj}_\nu$ and Definition 4.4.4, we have

$$\mathrm{NFD}_{\nu,\tau}^{\mathrm{cont}}(\mathbf{x}) = \mathrm{M}_{\nu,\tau}^{\mathrm{cont}}(\mathbf{x}). \tag{4.29}$$

Using Theorem 4.4.2 and Lemma 4.4.7, we then have

$$\begin{aligned}
\sum_{r=1}^{\ell} \mathrm{ch}\left(\chi^r{\uparrow}_C^{S_n}\right) q^r &= \sum_{r=1}^{\ell} \sum_{\substack{\tau\in[\nu_1]\times\cdots\times[\nu_k] \\ \sum_{j=1}^{k} \frac{\ell}{\nu_j}\tau_j \equiv_\ell r}} \mathrm{NFD}_{\nu,\tau}^{\mathrm{cont}}(\mathbf{x})\; q^r \\
&= \sum_{r=1}^{\ell} \sum_{\substack{\tau\in[\nu_1]\times\cdots\times[\nu_k] \\ \sum_{j=1}^{k} \frac{\ell}{\nu_j}\tau_j \equiv_\ell r}} \mathrm{M}_{\nu,\tau}^{\mathrm{cont}}(\mathbf{x})\; q^r \\
&= \sum_{r=1}^{\ell} \{w \in \mathrm{W}_n : \mathrm{maj}_\nu(w) = r\}^{\mathrm{cont}}(\mathbf{x})\; q^r \\
&= \mathrm{W}_n^{\mathrm{cont},\mathrm{maj}_\nu}(\mathbf{x}; q).
\end{aligned}$$

Since $\mathrm{maj}_\nu(w)$ depends only on $\mathrm{Des}(w)$, we can apply the RSK bijection again through Lemma 4.2.1 to get

$$\mathrm{W}_n^{\mathrm{cont},\mathrm{maj}_\nu}(\mathbf{x}; q) = \sum_{\substack{\lambda\vdash n \\ r\in[\ell]}} a_{\lambda,r}^{\nu} s_\lambda(\mathbf{x}) q^r.$$

□

Remark 4.4.12. Stembridge showed the equality of the first and third terms in Theorem 4.4.11 using the skew analogue of (2.16) and branching rules along Young subgroups of S_n. By contrast, $\mathrm{W}_n^{\mathrm{cont},\mathrm{maj}_\nu}(\mathbf{x}; q)$ played a key role in our approach.

Since the isomorphism type of $\chi^r{\uparrow}_C^{S_n}$, or equivalently the Schur expansion of $\mathrm{OFD}_{C,r}^{\mathrm{cont}}(\mathbf{x})$, depends only on ν, the cycle type of a generator of C, and $\gcd(\ell, r)$, we have the following generalization of Corollary 4.2.9.

Corollary 4.4.13. For all $n \geq 1$ and $\lambda, \nu \vdash n$, we have $a_{\lambda,r}^{\nu} = a_{\lambda,\gcd(\ell,r)}^{\nu}$, where $\ell = \mathrm{lcm}(\nu_1, \nu_2, \ldots)$.

For use in the next section, we record the Schur expansion of $\mathrm{M}_{\nu,\tau}^{\mathrm{cont}}(\mathbf{x})$. The proof is analogous to the last step of the proof of Theorem 4.4.11 using Lemma 4.2.1.

Corollary 4.4.14. If $\nu, \tau \vDash n$, then

$$\mathrm{M}^{\mathrm{cont}}_{\nu,\tau}(\mathbf{x}) = \sum_{\lambda \vdash n} \mathbf{a}^{\nu}_{\lambda,\tau} s_{\lambda}(\mathbf{x})$$

where

$$\mathbf{a}^{\nu}_{\lambda,\tau} \coloneqq \#\{Q \in \mathrm{SYT}(\lambda) : \mathbf{maj}_{\nu}(Q) = \tau\}.$$

We also have a corresponding symmetry result. Contrast it with Corollary 4.2.9.

Corollary 4.4.15. Suppose $\nu = (\nu_1, \ldots, \nu_k)$ is the cycle type of some $\sigma \in S_n$, $\tau \in [\nu_1] \times \cdots \times [\nu_k]$, $\pi \in S_k$, and $\lambda \vdash n$. Then, $\mathbf{a}^{\nu}_{\lambda,\tau} = \mathbf{a}^{\pi\cdot\nu}_{\lambda,\pi\cdot\tau}$.

Proof. Since reordering does not affect contents, we have

$$\mathrm{NFD}^{\mathrm{cont}}_{\nu,\tau}(\mathbf{x}) = \mathrm{NFD}^{\mathrm{cont}}_{\pi\cdot\nu,\pi\cdot\tau}(\mathbf{x}).$$

Now apply Corollary 4.4.14 and equate coefficients of $s_{\lambda}(\mathbf{x})$. □

4.5 Inducing 1-dimensional Representations from $C_a \wr S_b$ to S_{ab}

We next apply the approach of Section 4.3 and Section 4.4 to prove a generalization of a formula due to Schocker [Sch03] for the Schur expansion of $\mathcal{L}_{(a^b)}$. In particular, we give Schur expansions of the characteristics of

$$\mathcal{L}^{r,1}_{(a^b)} \coloneqq \chi^{r,1}{\uparrow}^{S_{ab}}_{C_a \wr S_b} \qquad \text{and} \qquad \mathcal{L}^{r,\epsilon}_{(a^b)} \coloneqq \chi^{r,\epsilon}{\uparrow}^{S_{ab}}_{C_a \wr S_b}.$$

Note that $\mathrm{ch}\,\mathcal{L}_{(a^b)} = \mathrm{ch}\,\mathcal{L}^{1,1}_{(a^b)}$ by Corollary 4.2.17.

The argument in Corollary 4.2.17 and the fact that $\mathrm{ch}(\epsilon_b) = e_b(\mathbf{x})$ immediately yield the following more general result, which also follows from an appropriate modification of Theorem 4.3.1.

Lemma 4.5.1. We have

$$\mathcal{L}^{r,1}_{(a^b)} = \left(\!\!\binom{\mathrm{NFD}_{a,r}}{b}\!\!\right)^{\mathrm{cont}}(\mathbf{x}) \qquad \text{and} \qquad \mathcal{L}^{r,\epsilon}_{(a^b)} = \binom{\mathrm{NFD}_{a,r}}{b}^{\mathrm{cont}}(\mathbf{x}).$$

Our first goal is to manipulate the necklace generating functions in Lemma 4.5.1 in such a way that we may apply cyclic sieving. We use Burnside's lemma and a sign-reversing involution to unravel these multiset and subset generating functions, respectively.

Lemma 4.5.2. We have

$$\left(\!\!\binom{\mathrm{NFD}_{a,r}}{b}\!\!\right)^{\mathrm{cont}}(x_1, x_2, \ldots) = \sum_{\nu \vdash b} \frac{1}{z_\nu} \prod_{j=1}^{\ell(\nu)} \mathrm{NFD}^{\mathrm{cont}}_{a,r}(x_1^{\nu_j}, x_2^{\nu_j}, \ldots).$$

Proof. Multisets of b necklaces from $\mathrm{NFD}_{a,r}$ can be thought of as S_b-orbits of length-b tuples $(N_1, \ldots, N_b)$ of necklaces $N_i \in \mathrm{NFD}_{a,r}$ under the natural S_b-action. The tuples $(N_1, \ldots, N_b)$ fixed by an element $\sigma \in S_b$ are those tuples which are constant on blocks corresponding to cycles of σ. It follows that if σ has cycle type $\nu \vdash b$,

$$\{T \in \mathrm{NFD}^b_{a,r} : \sigma \cdot T = T\}^{\mathrm{cont}}(x_1, x_2, \ldots) = \prod_{j=1}^{\ell(\nu)} \mathrm{NFD}^{\mathrm{cont}}_{a,r}(x_1^{\nu_j}, x_2^{\nu_j}, \ldots). \tag{4.30}$$

By Burnside's lemma, we may count S_b-orbits of necklaces $(N_1, \ldots, N_b)$ of fixed content by averaging the number of σ-fixed tuples of fixed content over all $\sigma \in S_b$. The result follows by grouping together permutations of a given cycle type. □

Lemma 4.5.3. We have

$$\binom{\mathrm{NFD}_{a,r}}{b}^{\mathrm{cont}}(\mathbf{x}) = \sum_{\nu \vdash b} \frac{(-1)^{b-\ell(\nu)}}{z_\nu} \prod_{j=1}^{\ell(\nu)} \mathrm{NFD}^{\mathrm{cont}}_{a,r}(x_1^{\nu_j}, x_2^{\nu_j}, \ldots).$$

Proof. Multiplying both sides by $b!$, using (4.30) and the fact $\mathrm{sgn}(\sigma) = (-1)^{b-\ell(\nu)}$ for $\sigma \in S_b$

with cycle type ν, the result is equivalent to

$$\begin{aligned}\{(N_1,\dots,N_b)\in \mathrm{NFD}^b_{a,r} : (N_1,\dots,N_b) \text{ are distinct}\}^{\mathrm{cont}}(\mathbf{x})\\ = \sum_{\sigma\in S_b} \mathrm{sgn}(\sigma)\{T\in \mathrm{NFD}^b_{a,r} : \sigma\cdot T = T\}^{\mathrm{cont}}(\mathbf{x}).\end{aligned} \tag{4.31}$$

On the right-hand side of (4.31), each b-tuple $(N_1,\dots,N_b)$ is counted

$$\mathrm{wgt}(N_1,\dots,N_b) \coloneqq \sum_{\substack{\sigma\in S_b \\ \text{s.t. } \sigma\cdot(N_1,\dots,N_b)=(N_1,\dots,N_b)}} \mathrm{sgn}(\sigma)$$

times. If $N_1,\dots,N_b$ are distinct, then only $\sigma = \mathrm{id}$ contributes, so $\mathrm{wgt}(N_1,\dots,N_b) = 1$. If $N_1,\dots,N_b$ are not distinct, then without loss of generality, suppose $N_1 = N_2$. Then, modifying the cycle(s) containing 1 and 2 as in

$$(1\ \cdots)(2\ \cdots) \leftrightarrow (1\ \cdots\ 2\ \cdots)$$

gives a sign-reversing involution on $\{\sigma\in S_b : \sigma\cdot(N_1,\dots,N_b) = (N_1,\dots,N_b)\}$, meaning $\mathrm{wgt}(N_1,\dots,N_b) = 0$. This proves (4.31). □

Remark 4.5.4. Using standard properties of plethysm (see e.g. [Mac95, §I.8])) and the power-sum expansions of e_b and h_b (see [Sta99, (7.22)-(7.23)]), Lemma 4.5.2 and Lemma 4.5.3 are equivalent to

$$\mathrm{ch}\,\mathcal{L}^{r,1}_{(a^b)} = h_b[\mathrm{ch}\,\chi^r{\uparrow}^{S_a}_{C_a}] = \sum_{\nu\vdash b}\frac{1}{z_\nu}p_\nu[\mathrm{ch}\,\chi^r{\uparrow}^{S_a}_{C_a}], \tag{4.32}$$

$$\mathrm{ch}\,\mathcal{L}^{r,\epsilon}_{(a^b)} = e_b[\mathrm{ch}\,\chi^r{\uparrow}^{S_a}_{C_a}] = \sum_{\nu\vdash b}\frac{(-1)^{b-\ell(\nu)}}{z_\nu}p_\nu[\mathrm{ch}\,\chi^r{\uparrow}^{S_a}_{C_a}]. \tag{4.33}$$

Consequently, one may replace the combinatorial manipulations in Lemma 4.5.2 and Lemma 4.5.3 with symmetric function manipulations. In the next section, we will prove Theorem 4.1.6, which generalizes the first equalities in (4.32) and (4.33).

Remark 4.5.5. Let ω be the involution on the algebra of symmetric functions defined by $\omega(s_\lambda(\mathbf{x})) = s_{\lambda'}(\mathbf{x})$ where λ' is the *conjugate* of λ, obtained by reflecting λ through the line $y = -x$. One may show in a variety of ways that

$$\omega\left(\operatorname{ch}\chi^r{\uparrow}_{C_n}^{S_n}\right) = \operatorname{ch}\chi^s{\uparrow}_{C_n}^{S_n} \qquad \text{where} \qquad s = \binom{n}{2} - r. \tag{4.34}$$

For instance, we can prove (4.34) using Theorem 1.1.4 as follows. Since conjugation $Q \mapsto Q'$ satisfies $\operatorname{Des}(Q') = [n-1] \setminus \operatorname{Des}(Q)$, we have

$$\begin{aligned} a_{\lambda',r} &= \#\{Q \in \operatorname{SYT}(\lambda') : \operatorname{maj}(Q) \equiv_n r\} \\ &= \#\left\{Q' \in \operatorname{SYT}(\lambda) : \operatorname{maj}(Q') \equiv_n \binom{n}{2} - r\right\} = a_{\lambda,\binom{n}{2}-r}. \end{aligned} \tag{4.35}$$

Therefore, by Theorem 1.1.4, letting $s = \binom{n}{2} - r$,

$$\omega\left(\operatorname{ch}\chi^r{\uparrow}_{C_n}^{S_n}\right) = \omega\left(\sum_{\lambda\vdash n} a_{\lambda,r}s_\lambda(\mathbf{x})\right) = \sum_{\lambda\vdash n} a_{\lambda',r}s_\lambda(\mathbf{x}) = \sum_{\lambda\vdash n} a_{\lambda,s}s_\lambda(\mathbf{x}) = \chi^s{\uparrow}_{C_n}^{S_n}.$$

From the symmetry result Corollary 4.2.9, it follows that $\operatorname{ch}\chi^r{\uparrow}_{C_n}^{S_n}$ is fixed under ω when n is odd. When n is even, $\operatorname{ch}\chi^r{\uparrow}_{C_n}^{S_n}$ may or may not be fixed. For instance, when $r = 1$, we find

$$\omega\left(\operatorname{ch}\mathcal{L}_n\right) = \omega\left(\operatorname{ch}\chi^1{\uparrow}_{S_n}^{C_n}\right) = \begin{cases} \operatorname{ch}\mathcal{L}_n^{(2)} = \operatorname{ch}\chi^2{\uparrow}_{S_n}^{C_n} & \text{if } n/2 \text{ is odd,} \\ \operatorname{ch}\mathcal{L}_n = \operatorname{ch}\chi^1{\uparrow}_{S_n}^{C_n} & \text{otherwise.} \end{cases}$$

Here $\mathcal{L}_n^{(2)}$ is the deformation of $\mathcal{L}_n$ recently studied by Sundaram [Sun18]. Further standard properties of plethysm together with (4.32) and (4.33) give

$$\left.\begin{aligned} \omega\left(\operatorname{ch}\mathcal{L}_{(a^b)}^{r,1}\right) &= \operatorname{ch}\mathcal{L}_{(a^b)}^{r,\epsilon} \qquad \text{for } a \text{ odd, and} \\ \omega\left(\operatorname{ch}\mathcal{L}_{(a^b)}^{r,1}\right) &= \operatorname{ch}\mathcal{L}_{(a^b)}^{s,1} \\ \omega\left(\operatorname{ch}\mathcal{L}_{(a^b)}^{r,\epsilon}\right) &= \operatorname{ch}\mathcal{L}_{(a^b)}^{s,\epsilon} \end{aligned}\right\} \quad \text{for } a \text{ even, where } s = \binom{a}{2} - r.$$

Consequently, one may obtain the Schur expansion of $\operatorname{ch} \mathcal{L}^{r,\epsilon}_{(a^b)}$ from the Schur expansion of $\operatorname{ch} \mathcal{L}^{r,1}_{(a^b)}$ simply by applying the ω map if and only if a is odd. When a is even, these two cases are more fundamentally different.

Next, we convert $\mathrm{NFD}^{\mathrm{cont}}_{a,r}(x_1^{\nu_j}, x_2^{\nu_j}, \ldots)$ into a linear combination of $\mathrm{NF}^{\mathrm{cont}}_{k,s}(\mathbf{x})$'s and then apply Mobius inversion to convert to a linear combination of $\mathrm{NFD}^{\mathrm{cont}}_{k,s}(\mathbf{x})$'s. We will need the following variation on the number-theoretic Möbius function μ.

Definition 4.5.6. Suppose $d \mid e$ and $f \mid e$. Set

$$\mu_f(d,e) \coloneqq \sum_{\substack{g \\ \text{s.t. } \operatorname{lcm}(f,d) \mid g \mid e}} \mu\left(\frac{g}{f}\right).$$

This expression simplifies considerably as follows. Let $\operatorname{rad}(m)$ denote the squarefree positive integer with the same prime divisors as m.

Lemma 4.5.7. Suppose $d \mid e$ and $f \mid e$. Then

$$\mu_f(d,e) = \begin{cases} \mu\left(\frac{\operatorname{lcm}(f,d)}{f}\right) & \text{if } \operatorname{rad}\left(\frac{e}{f}\right) = \operatorname{rad}\left(\frac{\operatorname{lcm}(f,d)}{f}\right) = \frac{\operatorname{lcm}(f,d)}{f}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We see

$$\mu_f(d,e) = \sum_{\substack{g \\ \text{s.t. } \operatorname{lcm}(f,d) \mid g \mid e}} \mu\left(\frac{g}{f}\right) = \sum_{\substack{h \\ \text{s.t. } \frac{\operatorname{lcm}(f,d)}{f} \mid h \mid \frac{e}{f}}} \mu(h)$$

Since $\mu(h) \neq 0$ only when h is radical, $\mu(h) \neq 0$ only when $\frac{\operatorname{lcm}(f,d)}{f}$ is radical and $h \mid \operatorname{rad}(e/f)$. Restricting to this case, we can write $\operatorname{rad}(e/f) = k \operatorname{lcm}(f,d)/f$ for some integer k. Since k

and $\operatorname{lcm}(f,d)/f$ must be relatively prime, we have

$$\begin{aligned}
\mu_f(d,e) &= \sum_{\substack{h \\ \text{s.t. } \frac{\operatorname{lcm}(f,d)}{f} \mid h \mid \frac{k\cdot\operatorname{lcm}(f,d)}{f}}} \mu(h) \\
&= \sum_{s\mid k} \mu\left(\left(\frac{\operatorname{lcm}(f,d)}{f}\right) s\right) \\
&= \mu\left(\frac{\operatorname{lcm}(f,d)}{f}\right) \sum_{s\mid k} \mu(s) \\
&= \begin{cases} \mu\left(\frac{\operatorname{lcm}(f,d)}{f}\right) & \text{if } k = 1, \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

giving the result. □

Lemma 4.5.8. We have

$$\operatorname{NFD}^{\text{cont}}_{a,r}(x_1^k, x_2^k, \ldots) = \sum_{s\mid rk} \mu_s(k, rk) \operatorname{NFD}^{\text{cont}}_{ak,s}(x_1, x_2, \ldots).$$

Proof. The left-hand side is the content generating function for k-tuples of length a necklaces with frequency dividing r of the form $(N, \ldots, N)$, repeating the same necklace k times. By concatenation, we may equivalently view such tuples as length ak necklaces whose frequency f satisfies $k \mid f \mid rk$. Consequently,

$$\operatorname{NFD}^{\text{cont}}_{a,r}(x_1^k, x_2^k, \ldots) = \sum_{\substack{f \\ \text{s.t. } k\mid f\mid rk}} \operatorname{NF}^{\text{cont}}_{ak,f}(x_1, x_2, \ldots), \tag{4.36}$$

recalling $\operatorname{NF}_{n,f} \coloneqq \{N \in \operatorname{N}_n : \operatorname{freq}(N) = f\}$. Möbius inversion on the identity $\operatorname{NFD}^{\text{cont}}_{ak,f}(\mathbf{x}) = \sum_{s\mid f} \operatorname{NF}^{\text{cont}}_{ak,s}(\mathbf{x})$ gives

$$\operatorname{NF}^{\text{cont}}_{ak,f}(\mathbf{x}) = \sum_{s\mid f} \mu\left(\frac{f}{s}\right) \operatorname{NFD}^{\text{cont}}_{ak,s}(\mathbf{x}). \tag{4.37}$$

Thus, by (4.37), (4.36) becomes

$$\begin{aligned}\mathrm{NFD}^{\mathrm{cont}}_{a,r}(x_1^k, x_2^k, \ldots) &= \sum_{\substack{f \\ \text{s.t. } k|f|rk}} \sum_{s|f} \mu\left(\frac{f}{s}\right) \mathrm{NFD}^{\mathrm{cont}}_{ak,s}(x_1, x_2, \ldots) \\ &= \sum_{s|rk} \left(\sum_{\substack{f \\ \text{s.t. } \mathrm{lcm}(k,s)|f|rk}} \mu\left(\frac{f}{s}\right) \right) \mathrm{NFD}^{\mathrm{cont}}_{ak,s}(x_1, x_2, \ldots) \\ &= \sum_{s|rk} \mu_s(k, rk)\, \mathrm{NFD}^{\mathrm{cont}}_{ak,s}(x_1, x_2, \ldots).\end{aligned}$$

by Definition 4.5.6. □

Notation 4.5.9. Given a sequence $\nu = (\nu_1, \ldots, \nu_k) \in \mathbb{Z}^k_{\geq 1}$ and an integer $r \in \mathbb{Z}_{\geq 1}$, let

$$r * \nu \coloneqq (r\nu_1, \ldots, r\nu_k).$$

Given another sequence $\tau = (\tau_1, \ldots, \tau_k)$, recall that $\tau \mid \nu$ means $\tau_j \mid \nu_j$ for all j. Further recall

$$\mathrm{NFD}_{\nu,\tau} = \mathrm{NFD}_{\nu_j,\tau_j} \times \cdots \times \mathrm{NFD}_{\nu_k,\tau_k}$$

from Definition 4.4.4. Finally, extend $\mu_f(d, e)$ to sequences multiplicatively:

$$\mu_{(f_1,\ldots,f_k)}((d_1, \ldots, d_k), (e_1, \ldots, e_k)) \coloneqq \prod_{j=1}^{k} \mu_{f_j}(d_j, e_j).$$

Corollary 4.5.10. We have

$$\begin{aligned}\mathrm{ch}\, \mathcal{L}^{r,1}_{(a^b)} &= \sum_{\nu \vdash b} \frac{1}{z_\nu} \sum_{\tau | r*\nu} \mu_\tau(\nu, r * \nu)\, \mathrm{NFD}^{\mathrm{cont}}_{a*\nu,\tau}(\mathbf{x}), \\ \mathrm{ch}\, \mathcal{L}^{r,\epsilon}_{(a^b)} &= \sum_{\nu \vdash b} \frac{(-1)^{b-\ell(\nu)}}{z_\nu} \sum_{\tau | r*\nu} \mu_\tau(\nu, r * \nu)\, \mathrm{NFD}^{\mathrm{cont}}_{a*\nu,\tau}(\mathbf{x}).\end{aligned}$$

Proof. Combine Lemma 4.5.1, Lemma 4.5.2 or Lemma 4.5.3, and Lemma 4.5.8. □

We may now state and generalize Schocker's formula for $\operatorname{ch} \mathcal{L}_{(a^b)} = \operatorname{ch} \mathcal{L}^{1,1}_{(a^b)}$.

Theorem 4.5.11 (See [Sch03, Thm. 3.1])**.** For all $a, b \geq 1$ and $r = 1, \ldots, a$, we have

$$\operatorname{ch} \mathcal{L}^{r,1}_{(a^b)} = \sum_{\lambda \vdash ab} \left(\sum_{\nu \vdash b} \frac{1}{z_\nu} \sum_{\tau | r*\nu} \mu_\tau(\nu, r*\nu) \mathbf{a}^{a*\nu}_{\lambda,\tau} \right) s_\lambda(\mathbf{x}) \qquad \text{and}$$

$$\operatorname{ch} \mathcal{L}^{r,\epsilon}_{(a^b)} = \sum_{\lambda \vdash ab} \left(\sum_{\nu \vdash b} \frac{(-1)^{b-\ell(\nu)}}{z_\nu} \sum_{\tau | r*\nu} \mu_\tau(\nu, r*\nu) \mathbf{a}^{a*\nu}_{\lambda,\tau} \right) s_\lambda(\mathbf{x}),$$

where, recalling the definition of $\mathbf{maj}_{a*\nu}$ from Definition 4.4.8,

$$\mathbf{a}^{a*\nu}_{\lambda,\tau} \coloneqq \#\{Q \in \mathrm{SYT}(\lambda) : \mathbf{maj}_{a*\nu}(Q) = \tau\}.$$

Proof. Combine Corollary 4.4.14 and Corollary 4.5.10. □

Remark 4.5.12. Schocker's approach to [Sch03, Thm. 3.1] uses Jöllenbeck's non-commutative character theory and involved manipulations with Klyachko's idempotents and Ramanujan sums. Much of Schocker's argument generalizes immediately to all r. The argument presented above is comparatively self-contained and direct. Two perhaps mysterious aspects of the formula, the appearance of Möbius functions and the average over S_b, arose naturally from Burnside's lemma and a change of basis using Möbius inversion. Our argument uses explicit bijections at each step except for the appeal to Burnside's lemma and the use of Lemma 4.5.3.

4.6 Higher Lie Modules and Branching Rules

The argument in Section 4.3 solves Thrall's problem for $\lambda = (n)$ by considering all branching rules for $C_n \hookrightarrow S_n$ simultaneously and using cyclic sieving and RSK to convert from the monomial to the Schur basis. We now turn to analogous considerations for the higher Lie modules and more generally branching rules for $C_a \wr S_b \hookrightarrow S_{ab}$. We give an analogue of the flex statistic and the monomial basis expansion for such branching rules from Section 4.2.3. We then show how to convert from the monomial to the Schur basis assuming the existence

of a certain statistic on words we call mash which interpolates between maj_n and the shape under RSK.

We now recall and prove Theorem 4.1.6 from the introduction, after introducing some notation.

Definition 4.6.1. Fix integers $a, b \geq 1$. Define

$$\mathrm{P}_a^b \coloneqq \left\{ \underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(a)}) : \lambda^{(1)}, \ldots, \lambda^{(a)} \text{ are partitions }, \sum_{r=1}^{a} |\lambda^{(r)}| = b \right\},$$

which indexes the irreducible $C_a \wr S_b$-representations by Theorem 4.2.14.

Theorem 4.6.2. For all $a, b \geq 1$ and $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(a)}) \in \mathrm{P}_a^b$, we have

$$\mathrm{ch}\, S^{\underline{\lambda}} \uparrow_{C_a \wr S_b}^{S_{ab}} = \prod_{r=1}^{a} s_{\lambda^{(r)}}[\mathrm{NFD}_{a,r}^{\mathrm{cont}}(\mathbf{x})].$$

Proof of Theorem 4.1.6. We have

$$\begin{aligned}
S^{\underline{\lambda}} \uparrow_{C_a \wr S_b}^{S_{ab}} &= \left[\bigotimes_{r=1}^{a} (\chi_a^r \wr S^{\lambda^{(r)}}) \right] \uparrow_{C_a \wr S_{\alpha(\underline{\lambda})}}^{S_{ab}} \\
&\cong \left[\bigotimes_{r=1}^{a} (\chi_a^r \wr S^{\lambda^{(r)}}) \right] \uparrow_{C_a \wr S_{\alpha(\underline{\lambda})}}^{S_{a*\alpha(\underline{\lambda})}} \uparrow_{S_{a*\alpha(\underline{\lambda})}}^{S_{ab}} \\
&\cong \left[\bigotimes_{r=1}^{a} (\chi_a^r \wr S^{\lambda^{(r)}}) \uparrow_{C_a \wr S_{|\lambda^{(r)}|}}^{S_{a|\lambda^{(r)}|}} \right] \uparrow_{S_{a*\alpha(\underline{\lambda})}}^{S_{ab}} \\
&\cong \left[\bigotimes_{r=1}^{a} (\chi_a^r \wr S^{\lambda^{(r)}}) \uparrow_{C_a \wr S_{|\lambda^{(r)}|}}^{S_a \wr S_{|\lambda^{(r)}|}} \uparrow_{S_a \wr S_{|\lambda^{(r)}|}}^{S_{a|\lambda^{(r)}|}} \right] \uparrow_{S_{a*\alpha(\underline{\lambda})}}^{S_{ab}} \\
&\cong \left[\bigotimes_{r=1}^{a} (\chi_a^r \uparrow_{C_a}^{S_a} \wr S^{\lambda^{(r)}}) \uparrow_{S_a \wr S_{|\lambda^{(r)}|}}^{S_{a|\lambda^{(r)}|}} \right] \uparrow_{S_{a*\alpha(\underline{\lambda})}}^{S_{ab}},
\end{aligned}$$

where the first and third isomorphisms use transitivity of induction, the second isomorphism uses Lemma 4.2.20, and the fourth isomorphism uses Lemma 4.2.19. Consequently, using

(4.14), (4.16), and Theorem 4.3.1, we have

$$\begin{aligned}\operatorname{ch} S^{\underline{\lambda}}\uparrow_{C_a \wr S_b}^{S_{ab}} &= \prod_{r=1}^{a} \operatorname{ch}\left(\chi_a^r \uparrow_{C_a}^{S_a} \wr S^{\lambda^{(r)}}\right)\uparrow_{S_a \wr S_{|\lambda^{(r)}|}}^{S_{a|\lambda^{(r)}|}} \\ &= \prod_{r=1}^{a} (\operatorname{ch} S^{\lambda^{(r)}})[\operatorname{ch} \chi_a^r \uparrow_{C_a}^{S_a}] \\ &= \prod_{r=1}^{a} s_{\lambda^{(r)}}[\mathrm{NFD}_{a,r}^{\mathrm{cont}}(\mathbf{x})].\end{aligned}$$

□

Given a word w, let the shape of w, denoted $\mathrm{sh}(w)$, be the common shape of $P(w)$ and $Q(w)$ under RSK.

Definition 4.6.3. Fix $a, b \geq 1$. Construct statistics

$$\mathrm{flex}_a^b, \mathrm{maj}_a^b \colon \mathrm{W}_{ab} \to \mathrm{P}_a^b$$

as follows. Given $w \in \mathrm{W}_{ab}$, write $w = w^1 \cdots w^b$ where $w^j \in \mathrm{W}_a$. In this way, consider w as a word of size b whose letters are in W_a. For each $r \in [a]$, let $w^{(r)}$ denote the subword of w whose letters are those w^j such that $\mathrm{flex}(w^j) = r$. Totally order W_a lexicographically, so that RSK is well-defined for words with letters from W_a. Set

$$\mathrm{flex}_a^b(w) \coloneqq (\mathrm{sh}(w^{(1)}), \ldots, \mathrm{sh}(w^{(a)})).$$

Define maj_a^b in the same way but with flex replaced by maj_a. Consequently, $\mathrm{maj}_n^1(w)$ is the n-tuple of partitions whose only non-empty entry is a single cell at position $\mathrm{maj}_n(w)$.

Example 4.6.4. Let $w = 212023101241$ and suppose $a = 3$, $b = 4$. Write

$$w = (212)(023)(101)(241).$$

The parenthesized terms have flex statistics $2, 1, 2, 2$ and maj_3 statistics $1, 3, 1, 2$, respectively.

When computing $\mathrm{flex}_3^4(w)$, we then have $w^{(1)} = (023), w^{(2)} = (212)(101)(241), w^{(3)} = \varnothing$. Since $(101) <_{\mathrm{lex}} (212) <_{\mathrm{lex}} (241)$, $\mathrm{sh}(w^{(2)}) = \mathrm{sh}(213) = (2,1)$. Consequently,

$$\mathrm{flex}_3^4(212023101241) = ((1),(2,1),\varnothing).$$

When computing $\mathrm{maj}_3^4(w)$, we have $w^{(1)} = (212)(101), w^{(2)} = (241), w^{(3)} = (023)$. Since $(101) <_{\mathrm{lex}} (212)$, $\mathrm{sh}(w^{(1)}) = \mathrm{sh}(21) = (1,1)$. Hence

$$\mathrm{maj}_3^4(212023101241) = ((1,1),(1),(1)).$$

We now recall and prove Theorem 4.1.7 from the introduction.

Theorem 4.6.5. Fix $a, b \geq 1$. We have

$$\begin{aligned}\sum_{\underline{\lambda} \in \mathrm{P}_a^b} \dim S^{\underline{\lambda}} \cdot \mathrm{ch}\left(S^{\underline{\lambda}}{\uparrow}_{C_a \wr S_b}^{S_{ab}}\right) q^{\underline{\lambda}} &= \mathrm{W}_{ab}^{\mathrm{cont},\mathrm{flex}_a^b}(\mathbf{x};q) \\ &= \mathrm{W}_{ab}^{\mathrm{cont},\mathrm{maj}_a^b}(\mathbf{x};q)\end{aligned}$$

where the $S^{\underline{\lambda}}$ are irreducible representations of $C_a \wr S_b$ and the $q^{\underline{\lambda}}$ are independent indeterminates.

Proof of Theorem 4.1.7. Fix $\underline{\lambda} \in \mathrm{P}_a^b$. For the left-hand side, using Theorem 4.1.6 and (4.18),

$$\dim S^{\underline{\lambda}} \cdot \mathrm{ch}\left(S^{\underline{\lambda}}{\uparrow}_{C_a \wr S_b}^{S_{ab}}\right) = \binom{b}{\alpha(\underline{\lambda})} \prod_{r=1}^{a} \#\,\mathrm{SYT}(\lambda^{(r)}) \cdot s_{\lambda^{(r)}}[\mathrm{NFD}_{a,r}^{\mathrm{cont}}(\mathbf{x})]. \tag{4.38}$$

For the right-hand side, we have

$$\mathrm{W}_{ab}^{\mathrm{cont},\mathrm{flex}_a^b}(\mathbf{x};q)\Big|_{q^{\underline{\lambda}}} = \{w \in \mathrm{W}_{ab} : \mathrm{flex}_a^b(w) = \underline{\lambda}\}^{\mathrm{cont}}(\mathbf{x}).$$

Say $\alpha(\underline{\lambda}) = (\alpha_1, \ldots, \alpha_a)$. In order for $w \in \mathrm{W}_{ab}$ to have $\mathrm{flex}_a^b(w) = \underline{\lambda}$, we must have $\mathrm{sh}(w^{(r)}) = \lambda^{(r)}$ for each $r \in [a]$. Recalling $\mathrm{F}_{a,r} \coloneqq \{w \in \mathrm{W}_a : \mathrm{flex}(w) = r\}$, we may thus

choose each $w^{(r)} \in (\mathrm{F}_{a,r})^{\alpha_r}$ with $\mathrm{sh}(w^{(r)}) = \lambda^{(r)}$ independently and then shuffle them in $\binom{b}{\alpha(\underline{\lambda})}$ ways to form w. Consequently,

$$\begin{aligned}\{w \in \mathrm{W}_{ab} : \mathrm{flex}_a^b(w) = \underline{\lambda}\}^{\mathrm{cont}}(\mathbf{x}) \\ = \binom{b}{\alpha(\underline{\lambda})} \prod_{r=1}^{a} \{w^{(r)} \in (\mathrm{F}_{a,r})^{\alpha_r} : \mathrm{sh}(w^{(r)}) = \lambda^{(r)}\}^{\mathrm{cont}}(\mathbf{x}).\end{aligned} \tag{4.39}$$

The content generating function for words with a given shape $\mu \vdash n$ under RSK is given by

$$\{w \in \mathrm{W}_n : \mathrm{sh}(w) = \mu\}^{\mathrm{cont}}(\mathbf{x}) = \#\,\mathrm{SYT}(\mu)\, s_\mu(\mathbf{x}), \tag{4.40}$$

since the number of possible Q-tableaux is $\#\,\mathrm{SYT}(\mu)$ and the content generating function for P-tableaux is $s_\lambda(\mathbf{x})$. Changing the alphabet from $\mathbb{Z}_{\geq 1}$ to $\mathrm{F}_{a,r}$ and using Lemma 4.3.2 gives

$$\begin{aligned}\{w^{(r)} \in (\mathrm{F}_{a,r})^{\alpha_r} : \mathrm{sh}(w^r) = \lambda^{(r)}\}^{\mathrm{cont}}(\mathbf{x}) &= \#\,\mathrm{SYT}(\lambda^{(r)}) s_{\lambda^{(r)}}[\mathrm{F}_{a,r}^{\mathrm{cont}}(\mathbf{x})] \\ &= \#\,\mathrm{SYT}(\lambda^{(r)}) s_{\lambda^{(r)}}[\mathrm{NFD}_{a,r}^{\mathrm{cont}}(\mathbf{x})].\end{aligned} \tag{4.41}$$

The first equality in Theorem 4.1.7 now follows from combining (4.39) and (4.41) with (4.38). The second equality in Theorem 4.1.7 follows similarly. □

While Theorem 4.1.7 determines the monomial expansion of the graded Frobenius series tracking branching rules for $C_a \wr S_b \hookrightarrow S_{ab}$, we are ultimately interested in the corresponding Schur expansion. We next describe how the approach in the preceding sections might be used to find this Schur expansion. The key properties used in the proof of Theorem 1.1.4 converting from the monomial basis to the Schur basis were that maj_n is equidistributed with flex on each W_α and $\mathrm{maj}_n(w)$ depends only on $Q(w)$. In order to apply a similar argument for $\mathrm{ch}(S^{\underline{\lambda}}\!\uparrow_{C_a \wr S_b}^{S_{ab}})$, we need a statistic as follows.

Open Problem 4.6.6. Fix $a, b \geq 1$. Find a statistic

$$\mathrm{mash}_a^b \colon \mathrm{W}_{ab} \to \mathrm{P}_a^b$$

with the following properties.

(i) For all $\alpha \vDash ab$, maj_a^b (or equivalently flex_a^b) and mash_a^b are equidistributed on W_α.

(ii) If $v, w \in \mathrm{W}_{ab}$ satisfy $Q(v) = Q(w)$, then $\mathrm{mash}_a^b(v) = \mathrm{mash}_a^b(w)$.

Finding such a statistic mash_a^b would determine the Schur decomposition of $\mathrm{ch}(S^{\underline{\lambda}}\uparrow_{C_a \wr S_b}^{S_{ab}})$ as follows.

Corollary 4.6.7. Suppose mash_a^b satisfies Properties (i) and (ii) in Open Problem 4.6.6. Then

$$\mathrm{ch}(S^{\underline{\lambda}}\uparrow_{C_a \wr S_b}^{S_{ab}}) = \sum_{\nu \vdash ab} \frac{\#\{Q \in \mathrm{SYT}(\nu) : \mathrm{mash}_a^b(Q) = \underline{\lambda}\}}{\dim(S^{\underline{\lambda}})} s_{\nu(\mathbf{x})},$$

where $\mathrm{mash}_a^b(Q) \coloneqq \mathrm{mash}_a^b(w)$ for any $w \in \mathrm{W}_{ab}$ with $Q(w) = Q$.

Proof. We use, in order, Theorem 4.1.7, Property (i), RSK, and Property (ii) to compute

$$\begin{aligned}
\sum_{\underline{\lambda} \in \mathrm{P}_a^b} \dim(S^{\underline{\lambda}}) \,\mathrm{ch}(S^{\underline{\lambda}}\uparrow_{C_a \wr S_b}^{S_{ab}})) q^{\underline{\lambda}} &= \mathrm{W}_{ab}^{\mathrm{cont},\mathrm{maj}_a^b}(\mathbf{x}; q) \\
&= \sum_{\alpha \vDash ab} \mathrm{W}_\alpha^{\mathrm{maj}_a^b}(q)\, \mathbf{x}^\alpha \\
&= \sum_{\alpha \vDash ab} \mathrm{W}_\alpha^{\mathrm{mash}_a^b}(q)\, \mathbf{x}^\alpha \\
&= \mathrm{W}_{ab}^{\mathrm{cont},\mathrm{mash}_a^b}(\mathbf{x}; q) \\
&= \sum_{\nu \vdash ab} (\mathrm{SSYT}(\nu) \times \mathrm{SYT}(\nu))^{\mathrm{cont},\mathrm{mash}_a^b}(\mathbf{x}; q) \\
&= \sum_{\nu \vdash ab} \mathrm{SSYT}(\nu)^{\mathrm{cont}}(\mathbf{x})\, \mathrm{SYT}(\nu)^{\mathrm{mash}_a^b}(q) \\
&= \sum_{\nu \vdash ab} \mathrm{SYT}(\nu)^{\mathrm{mash}_a^b}(q) s_\nu(\mathbf{x}).
\end{aligned}$$

The result follows by equating coefficients of $q^{\underline{\lambda}}$. □

Remark 4.6.8. When $a = 1$ and $b = n$, we may replace $\underline{\lambda}$ with $\lambda \vdash n$. Under this identification, $\mathrm{maj}_1^n(w) = \mathrm{sh}(w)$, which clearly satisfies Properties (i) and (ii). When $a = n$

and $b = 1$, we may replace $\underline{\lambda}$ with an element $r \in [n]$. Under this identification, we may set $\mathrm{mash}_n^1(w) = \mathrm{maj}_n(w)$, which satisfies Properties (i) and (ii). In this sense mash_a^b interpolates between the major index maj_n and the shape under RSK, hence the name.

While maj_a^b trivially satisfies Property (i), it fails Property (ii) already when $a = b = 2$, as in the following example.

Example 4.6.9. Let $v = 2314$ and $w = 1423$. Then,

$$Q(v) = Q(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}$$

while

$$\mathrm{maj}_2^2(v) = (\varnothing, (1,1))$$
$$\mathrm{maj}_2^2(w) = (\varnothing, (2)).$$

Remark 4.6.10. When defining flex_a^b and maj_a^b, we somewhat arbitrarily chose the lexicographic order on W_a. Any other total order would work just as well. However, maj_a^b continues to fail Property (ii) using any other total order when $a = b = 2$ in Example 4.6.9 since either $14 < 23$ or $23 < 14$.

Chapter 5

TABLEAU STABILIZATION AND RECTANGULAR TABLEAUX FIXED BY PROMOTION POWERS

5.1 Introduction

In this chapter, we introduce tableau stabilization, a new phenomenon we found in order to construct sufficiently large rectangular tableaux that are fixed by promotion powers. Central to defining and investigating tableau stabilization are Schützenberger's jeu de taquin and the rectification operator, which are already well-established algorithms in the theory of Young tableaux. Tableau stabilization is the phenomenon that if we attach sufficiently many shifted copies of a skew tableau to its right and then rectify, some copy and all of those to its right only experience horizontal slides. We will investigate when the vertical slides stop, i.e. when the skew tableau stabilizes. We will also determine the shape of the stabilized tableau if the initial skew tableau had all equal row sizes. The equal row size case is used to construct sufficiently large rectangular tableaux that are fixed by promotion powers. We leave the reader with open problems on tableau stabilization, most notably extending our bound from equal row sizes and finding its distribution as a permutation statistic. See Section 5.2 for definitions and further background.

Definition 5.1.1. For any standard skew tableau S, let $S^{(k)}$ denote the result of attaching $(k-1)$ shifted copies of S to the right of S so that the result is a standard skew tableau. Let m denote the size of S and k be a positive integer. We say S *stabilizes* at k if the entries in $[(k-1)m+1, km]$ lie in the same rows in $\mathrm{Rect}(S^{(k)})$ and $S^{(k)}$. Let $\mathrm{stab}(S)$ denote the minimum value at which S stabilizes.

Example 5.1.2. Let Rect denote the rectification operator. Consider

$$S = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 5 & 6 \\ \hline \end{array}\!\!\begin{array}{|c|c|}\hline 2 & 4 \\ \hline \end{array} \qquad T = \begin{array}{|c|c|}\hline 1 & 6 \\ \hline \end{array}\!\!\begin{array}{|c|c|}\hline 2 & 5 \\ \hline \end{array}\!\!\begin{array}{|c|c|}\hline 3 & 4 \\ \hline \end{array}$$

$$S^{(3)} = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 3 & 7 & 9 & 13 & 15 \\ \hline 5 & 6 & 11 & 12 & 17 & 18 \\ \hline \end{array}\!\!\begin{array}{|c|c|c|c|}\hline 2 & 4 & 8 & 10 & 14 & 16 \\ \hline \end{array}, \qquad T^{(3)} = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 6 & 7 & 12 & 13 & 18 \\ \hline \end{array}\!\!\begin{array}{|c|c|c|c|c|c|}\hline 2 & 5 & 8 & 11 & 14 & 17 \\ \hline \end{array}\!\!\begin{array}{|c|c|c|c|c|c|}\hline 3 & 4 & 9 & 10 & 15 & 16 \\ \hline \end{array},$$

$$\mathrm{Rect}(S^{(3)}) = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 3 & 5 & 6 & 7 & 9 & 13 & 15 \\ \hline 2 & 4 & 11 & 12 & 17 & 18 \\ \hline 8 & 10 & 14 & 16 \\ \hline \end{array}, \qquad \mathrm{Rect}(T^{(3)}) = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 4 & 5 & 6 & 7 & 12 & 13 & 18 \\ \hline 2 & 8 & 10 & 11 & 14 & 17 \\ \hline 3 & 9 & 15 & 16 \\ \hline \end{array}.$$

Notice 2 does not lie in the same row in $S^{(3)}$ and $\mathrm{Rect}(S^{(3)})$, but $7, 8, \ldots, 12$ do. Hence, $\mathrm{stab}(S) = 2$. As $13, 14, \ldots, 18$ also stay in the say row in S, S stabilizes at 3 as well. Notice 10 does not lie in the same row in $T^{(3)}$ and $\mathrm{Rect}(T^{(3)})$, but $13, 14, \ldots, 18$ do. Hence, $\mathrm{stab}(T) = 3$.

Also consider

$$U = \begin{array}{|c|c|c|}\hline 4 & 5 & 6 \\ \hline \end{array}\!\!\begin{array}{|c|c|}\hline 3 & 7 \\ \hline \end{array}\!\!\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array},$$

$$U^{(3)} = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 4 & 5 & 6 & 11 & 12 & 13 & 18 & 19 & 20 \\ \hline \end{array}\!\!\begin{array}{|c|c|c|c|c|c|c|}\hline 3 & 7 & 10 & 14 & 17 & 21 \\ \hline \end{array}\!\!\begin{array}{|c|c|c|c|c|c|}\hline 1 & 2 & 8 & 9 & 15 & 16 \\ \hline \end{array},$$

$$\mathrm{Rect}(U^{(3)}) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 2 & 3 & 4 & 5 & 6 & 11 & 12 & 13 & 18 & 19 & 20 \\ \hline 7 & 9 & 10 & 14 & 17 & 21 \\ \hline 8 & 15 & 16 \\ \hline \end{array}.$$

Notice that 9 does not in the same row in $U^{(3)}$ and $\mathrm{Rect}(U^{(3)})$, but $15, 16, \ldots, 21$ do, so $\mathrm{stab}(U) = 3$.

Tableaux stabilization is defined on any skew tableaux whose row sizes weakly decrease from top to bottom. Otherwise, $S^{(k)}$ need not be a standard skew tableaux, see Remark 5.3.4. Two

of the most basic facts about tableaux stabilization are that once a skew tableaux stabilizes, it continues to stabilize, and any skew tableaux must stabilize eventually, Lemma 5.3.9. A more interesting property of stabilization is that it is constant on dual equivalence classes, Theorem 5.3.6.

The equal row size case is of particular interest to us because we use it to construct sufficiently large rectangular tableaux that are fixed by certain promotion powers. We also have better results in the equal row size case.

It is natural to ask what bounds there are for when a tableaux stabilizes. In Example 5.1.2, S, T, U all had 3 rows and stabilized at 3. Note S in Example 5.1.2 actually stablized earlier. Does every skew tableaux with b rows whose sizes weakly decrease from top to bottom stabilize at b? We have verfied that the answer is yes for all standard skew tableaux of size at most 7 and random searching on larger tableaux has failed to produce a counterexample. These computations were performed in CoCalc [Ste19].

Conjecture 5.1.3. Any standard skew tableau with b rows and decreasing row sizes stabilizes at b.

In the equal row size case, Conjecture 5.1.3 is true, Theorem 5.1.4. Theorem 5.1.4 will give us explicit bounds on the dimensions of the rectangular tableaux we construct to be fixed by various promotion powers. Although we have not proven Conjecture 5.1.3, we have deduced a weaker, but still linear, bound for skew tableaux with b rows of decreasing sizes, Theorem 5.1.5. This weaker bound shows Conjecture 5.1.3 is true when $b = 2$. The $b = 3$ case is open.

Theorem 5.1.4. Any standard skew tableau with b rows of the same size stabilizes at b.

Theorem 5.1.5. Any standard skew tableau with $b \geq 2$ rows and decreasing row lengths stabilizes at $2b - 2$.

We have a way to determine the shape of $\mathrm{Rect}(S^{(k)})$ if S has b rows of equal size r, and $k \geq b - 1$. This approach is instrumental in proving Theorem 5.1.4. Both Theorem 5.1.4 and

Theorem 5.1.6 are essential in our proof of Theorem 5.1.12. See Section 5.2 and Section 5.3 for missing definitions. Though they state it differently, Donguk and Rhee effectively proved the $r = 1$ case, [PR17].

Theorem 5.1.6. (r = 1, [PR17]) Suppose S is a standard skew tableau with b rows each of size r. Let $w_1, \ldots, w_b$ denote the entries in each row read from left to right, starting from the bottom. For $k \geq b-1$, $\mathrm{Rect}(S^{(k)})$ has shape $(\lambda_1, \ldots, \lambda_b)$, where

$$\lambda_j = kr + \sum_{i=1}^{b-j} c_i - \sum_{i=1}^{j-1} c_i \quad \text{for all } j = 1, \ldots, b, \tag{5.1}$$

and

$$c_i = (\text{the length of the first row of } P(w_i w_{i+1})) - r \quad \text{for all } i = 1, \ldots, b-1.$$

We defined tableaux stabilization in order to construct the sufficiently large rectangular tableaux fixed by powers of promotion. Dennis White found and Brendon Rhoades proved a very interesting cyclic sieving phenomenon, [RSW04], regarding the action of promotion on rectangular standard Young tableaux, Theorem 5.1.8 [Rho10a, Theorem 1.3]. This results shows that the number of rectangular standard Young tableaux of shape (a^b) fixed by d promotions equals the number of standard $\frac{ab}{d}$-ribbon tableaux of shape (a^b), Corollary 5.1.10 [Rho10a, Corollary 9.1], which is the same as the number of standard tableaux of the shape associated to the $\frac{ab}{d}$-quotient of (a^b) [DLT94], Corollary 5.1.10.

For $a, b \in \mathbb{Z}_{\geq 1}$, let

$$(a^b) \coloneqq \underbrace{(a, \ldots, a)}_{b \text{ times}},$$

and let $\mathrm{SYT}(a^b)$ denote the set of standard Young tableaux of shape (a^b).

Theorem 5.1.7. [Hai92, Theorem 4.4] Schützenberger's promotion operator $\mathrm{p} : \mathrm{SYT}(a^b) \to \mathrm{SYT}(a^b)$ has order ab.

By Theorem 5.1.7, the cyclic group $\langle \mathrm{p} \rangle$ generated by $\mathrm{p} : \mathrm{SYT}(a^b) \to \mathrm{SYT}(a^b)$ has order ab.

Following a conjecture Dennis White, Rhoades proved that using the q-analog of the hook length formula on shape (a^b) gives rise to a CSP for the action of $\langle \mathrm{p} \rangle$ [Rho10a, Theorem 1.3], —Example 1.1.3(3). We restate the result here for the reader's convenience and later reference. Let h_c denote the hook length of cell c in the Young diagram of (a^b).

Theorem 5.1.8. [Rho10a, Theorem 1.3] (Conjectured by White, 2007) For $a, b \in \mathbb{Z}_{\geq 1}$,

$$\left(\mathrm{SYT}(a^b), \langle \mathrm{p} \rangle, \frac{[ab]_q!}{\prod_{c \in (a^b)} [h_c]_q} \right)$$

exhibits the CSP.

The polynomial $\frac{[ab]_q!}{\prod_{c \in (a^b)} [h_c]_q}$ in Theorem 5.1.8 has notable connections to tableau statistics and representation theory [Rho10a]. First, the hook length formula [FRT54] tells us

$$\# \mathrm{SYT}(a^b) = \frac{(ab)!}{\prod_{c \in (a^b)} h_c},$$

making $\frac{[ab]_q!}{\prod_{c \in (a^b)} [h_c]_q}$ the q-analog of the hook length formula for the shape (a^b). Secondly, it is a q-shift of the major index generating function on Standard Young tableau of shape λ:

$$\frac{[ab]_q!}{\prod_{c \in (a^b)} [h_c]_q} = q^{-a\binom{b}{2}} \mathrm{SYT}(\lambda)^{\mathrm{maj}}(q).$$

Thirdly, $\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q)$ corresponds to the multiplicities of the Specht module S^λ in the degree components of the coinvariant algerba [Sta99, Corollary 7.21.5]. Namely, for $\lambda \vdash n$,

$$\mathrm{SYT}(\lambda)^{\mathrm{maj}}(q) = \sum_{k \geq 0} \langle (R_n)_k, S^\lambda \rangle q^k,$$

where $(R_n)_k$ is the kth degree component of the coinvariant algebra R_n of S_n.

Rhoades proves Theorem 5.1.8 by identifying the action of promotion with the action of the long cycle, σ_{ab}, on the Specht module indexed by (a^b), up to a sign, [Rho10a, Proposition 3.5]. He uses the Kazhdan–Lusztig construction of the irreducible S_n-representations,

which is governed by descents sets of tableaux and the top coefficients of the Kazhdan–Lusztig polynomials. Identifying permutations with their insertion tableaux, he shows that the symmetrized Kazhdan–Lusztig μ function on rectangular tableaux is invariant under simultaneous promotion. Moreover, he defines a cyclic descent set on rectangular tableaux which promotion cycles.

Because σ_{ab} acts by promotion up to a sign on the Specht module indexed by (a^b), we have, by definition of character,

$$\#\,\mathrm{SYT}(a^b)^{\mathrm{p}^d} = |\chi^{(a^b)}(\sigma_{ab}^d)|, \tag{5.2}$$

where χ^λ is the character of the Specht module indexed by λ. Corollary 1.1.5, which we restate as Corollary 5.1.9 here, then follows from (5.2), the Murnagnan–Nakayama Rule, and the fact that all r-ribbon tableaux of a given shape have the same height parity [JK81, 2.7.26].

Corollary 5.1.9. [Rho10a, Corollary 9.1] For $a, b \in \mathbb{Z}_{\geq 1}$, and $d \mid ab$,

$$\#\,\mathrm{SYT}(a^b)^{\mathrm{p}^d} = \#\text{ standard } \left(\frac{ab}{d}\right)\text{-ribbon tableaux of shape } (a^b).$$

Kevin Purbhoo also gave an alternate proof of Corollary 5.1.9 using the Wronksi map [Pur13, Theorem 1.5]. The Wronksi map takes a b-dimensional subspace X of polynomials with degree up to $a-b-1$ and outputs the determinant of the Jacobian of a basis for X, which is well-defined up to scalar multiplication. The generic fibers of the Wronski map are in bijection with rectangular standard Young tableaux of shape (a^b). If we restrict to points in the fiber of the Wronski map that are fixed by a certain C_d-action, Purbhoo shows that the generic number is both $\#\,\mathrm{SYT}(a^b)^{\mathrm{p}^d}$ and the number of $\frac{ab}{d}$-ribbon tableaux of shape (a^b), proving Corollary 5.1.9.

Now, r-ribbon tableaux of shape λ only exist when λ has empty r-core [DLT94]. Moreover, when λ has empty r-core, r-ribbon tableaux of shape λ bijectively correspond to standard

fillings of the r-quotient of λ, [DLT94] or [Wil16, Lemma 2.1]. With these two facts, we can rephrase Corollary 5.1.9 as follows.

Corollary 5.1.10. [Rho10a, Corollary 9.1] For $a, b \in \mathbb{Z}_{\geq 1}$ and $d \mid ab$,

$$\#\,\mathrm{SYT}(a^b)^{\mathrm{p}^d} = \begin{cases} \#\,\mathrm{SYT}(Q_{\frac{ab}{d}}(\lambda)), & \text{if } (a^b) \text{ has empty } \frac{ab}{d}\text{-core,} \\ 0, & \text{else} \end{cases} \tag{5.3}$$

where $Q_r(\lambda)$ is the r-quotient of λ combined anti-diagonally into a single a skew shape.

However, neither Rhoades's nor Purbhoo's proof describes which rectangular tableaux are fixed by d promotions. The problem of constructing these fixed points is still open. We make substantial progress on this problem by characterizing all of the sufficiently large tableaux fixed by a given power of promotion. By Theorem 5.1.7, $\mathrm{SYT}(a^b)^{\mathrm{p}^d}$ is only nonempty when $d \mid ab$. Furthermore, we show in Section 5.2 that all nonempty cases are of the form $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$ up to conjugation. Thus, it suffices to answer Question 5.1.11.

Question 5.1.11. For $a, b, r \in \mathbb{Z}_{\geq 1}$, which tableaux lie in $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$?

Some cases of Question 5.1.11 have already been answered. The $a = 1$ case is a trivial consequence of Theorem 5.1.7: $\mathrm{SYT}(r^b)^{\mathrm{p}^{br}} = \mathrm{SYT}(r^b)$. The $r = 1, a \geq b$ case was solved by Kevin Purbhoo and Donguk Rhee, [PR17] in 2017. Their construction uses a similar algorithm to our tableau stabilization. For each $w = w_1 \dots w_b \in S_b$, they put $w_1, \dots, w_b$ in cells placed anti-diagonally. Then they perform rectification while refilling the anti-diagonal cells with n plus whatever entry just left it. This algorithm agrees with stabilizing the anti-diagonal tableau and then restricting to cells left of this anti-diagonal. Finally, they perform the analogous algorithm with outer slides toward the southwest corner of (a^b) and attach the two results along the anti-diagonal to get a rectangular tableaux of shape (a^b) fixed by p^b.

The $a = 2$ case was solved by Dennis White, [Whi06], and independently by Donguk Rhee in his Master's thesis, [Rhe12] using the same construction. For the sake of completeness

and so there is a record of this construction in the literature, we present their construction in Section 5.7. Dennis White also shared some ideas with me for recursively constructing tableaux in $\mathrm{SYT}((2r)^b)^{\mathrm{p}^{br/k}}$ [Whi06], which inspired Theorem 5.7.7.

We answer Question 5.1.11 for all $b, r \in \mathbb{Z}_{\geq 1}$ and $a \geq 2b-1$, Theorem 5.1.12. In addition, the tableaux we construct in $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$ are in natural bijection with $\mathrm{SYT}(Q_a((ar)^b))$, which are in bijection with standard $\frac{ab}{d}$-ribbon tableaux of shape (a^b), as in Corollary 5.1.10, solving [Rho10a, Problem 9.4] for $a \geq 2b-1$ and $a = 2$. Describing the tableaux in $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$ for $a \in [3, 2b-2]$ remains open, see Open Problem 5.9.6.

Let Rect denote the rectification operator and Rect^* denote the anti-rectification operator, which slides a skew tableau until it is right-justified, see Definition 5.5.1. For partitions λ, μ, let $\lambda \cup \mu$ denote the result of combining λ and μ anti-diagonally into a single skew shape, see Example 5.2.15. By Corollary 5.1.10, the tableaux in $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$ are in bijection with $\mathrm{SYT}((r) \cup \cdots \cup (r))$, with b pieces. For any tableau $S \in \mathrm{SYT}((r) \cup \cdots \cup (r))$ and integer $a \geq 2b-1$, let $R_a(S)$ be the rectangular tableau formed by row-concatenating $\mathrm{Rect}(S^{(b-1)}), S^{(a-2b+2)} + (b-1)r$, and $\mathrm{Rect}^*(S^{(b-1)}) + (a-b+1)r$ together from left to right.

Theorem 5.1.12. For $b, r \in \mathbb{Z}_{\geq 1}$ and any integer $a \geq 2b-1$, the tableaux in $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$ are all constructed as follows:

$$\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}} = \left\{ R_a(S) : S \in \mathrm{SYT}(\underbrace{(r) \cup \cdots \cup (r)}_{b \text{ times}}) \right\}.$$

In addition,

$$\#\,\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}} = \binom{br}{r, r, \ldots, r}. \tag{5.4}$$

Moreover, since promotion commutes with itself, $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$ is closed under the promotion operator. Using the extension of promotion to skew shapes, we will show that promotion commutes with the R_a operator.

Corollary 5.1.13. For all $a \geq 2b-1$ and $S \in \mathrm{SYT}(\underbrace{(r) \cup \cdots \cup (r)}_{b \text{ times}})$,

$$\mathrm{p}(R_a(S)) = R_a(\mathrm{p}(S)).$$

The $a = 2$ case uses a similar construction, but it does not require the machinery of tableau stabilization. Using Corollary 5.1.10, the tableaux in $\mathrm{SYT}((2r)^b)^{\mathrm{p}^{br}}$ are in bijection with $\mathrm{SYT}((r^{\lceil \frac{b}{2} \rceil}) \cup (r^{\lfloor \frac{b}{2} \rfloor}))$. For $S \in \mathrm{SYT}((r^{\lceil \frac{b}{2} \rceil}) \cup (r^{\lfloor \frac{b}{2} \rfloor}))$. Let $R_2(S)$ is formed by attaching $\mathrm{Rect}(S)$ and $\mathrm{Rect}^*(S) + br$ together from left to right. We show promotion commutes with the R_2 operator as well.

Theorem 5.1.14. [Whi06] [Rhe12] For $b, r \in \mathbb{Z}_{\geq 1}$,

$$\mathrm{SYT}((2r)^b)^{\mathrm{p}^{br}} = \left\{ R_2(S) : S \in \mathrm{SYT}((r^{\lceil \frac{b}{2} \rceil}) \cup (r^{\lfloor \frac{b}{2} \rfloor})) \right\},$$

In addition,

$$\#\,\mathrm{SYT}((2r)^b)^{\mathrm{p}^{br}} = \binom{br}{\lfloor \frac{b}{2} \rfloor\, r} \#\,\mathrm{SYT}(r^{\lceil \frac{b}{2} \rceil}) \cdot \#\,\mathrm{SYT}(r^{\lfloor \frac{b}{2} \rfloor}).$$

Corollary 5.1.15. For all $S \in \mathrm{SYT}((r^{\lceil \frac{b}{2} \rceil}) \cup (r^{\lfloor \frac{b}{2} \rfloor}))$,

$$\mathrm{p}(R_2(S)) = R_2(\mathrm{p}(S)).$$

In Section 5.2, we review the necessary background. In Section 5.3, we introduce tableau stabilization and prove some of its general properties, notably Theorem 5.1.5. In Section 5.4, we restrict our attention to stabilization on tableau with equal row sizes, prove Theorem 5.1.4, and prove Theorem 5.1.6. In Section 5.5, we discuss anti-stabilization and prove it coincides with stabilization when both are defined. In Section 5.6, we employ tableau stabilization to explicitly construct $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$ for $a \geq 2b-1$, proving Theorem 5.1.12, and describe promotion on $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$ for $a \geq 2b-1$. In Section 5.7, we explicitly construct $\mathrm{SYT}((2r)^b)^{\mathrm{p}^{br}}$, proving Theorem 5.1.14, and describe the action of promotion on $\mathrm{SYT}((2r)^b)^{\mathrm{p}^{br}}$. In Section 5.8, we study stabilization as a permutation statistic. In Section 5.9, we present open problems.

5.2 Background

5.2.1 Tableaux

A *partition* $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a weakly decreasing sequence of positive integers. We say $\lambda \vdash n$ or $|\lambda| = n$ if $\lambda_1 + \cdots + \lambda_k = n$. We view λ as a left-justified diagram with $\lambda_1, \ldots, \lambda_k$ cells in its rows from top to bottom, using English notation. If λ, μ are partitions so that all of the cells in μ are in λ, the *skew shape* λ/μ is the set of cells in λ but not in μ.

A (semistandard) *tableau* is a filling of a partition with entries from $\mathbb{Z}_{\geq 1}$ so that the rows are weakly increasing and the columns are strictly increasing. A (semistandard) skew tableau is such a filling of a skew shape. For a skew tableau S, let $|S|$ denote the number of cells in S. A skew tableau S is *standard* if it uses $1, \ldots, |S|$ exactly once. For partitions λ, μ, let $\mathrm{SYT}(\lambda), \mathrm{SYT}(\lambda/\mu)$ denote the set of standard skew tableaux of shape $\lambda, \lambda/\mu$, respectively. We say a skew tableau has *straight shape* if its shape is a partition. For any skew tableau S, let $\mathrm{rv}(S)$, the *row vector* of S, denote the sequence of row sizes from top to bottom so that $\mathrm{rv}(S)_j$ denotes the number of cells in row j of S. The row vector of S coincides with the shape of S when S has straight shape. The *reading word* of S is the sequence of entries in S read left to right along the rows, from bottom to top. Let λ', S' denote the conjugates of λ, S, respectively, obtained by interchanging the rows and columns. For a standard skew tableau S, its descent set is

$$\mathrm{Des}(S) = \{i : (i+1) \text{ lies strictly below } i \text{ in } S\}.$$

Example 5.2.1.

$$S = \begin{array}{|c|c|c|c|c|} \multicolumn{2}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{3-5} \multicolumn{2}{c|}{} & 1 & 7 & \multicolumn{1}{c}{} \\ \cline{3-5} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 2 & 4 & 6 & 9 \\ \cline{2-5} \multicolumn{1}{c|}{3} & 5 & 8 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array}$$

is a standard skew tableau with skew shape $(7,5,5,3)/(5,5,1,0)$, $|S| = 9$, $\mathrm{rv}(S) = (2,0,4,3)$, $\mathrm{rv}(S)_3 = 4$, reading word 358246917, and $\mathrm{Des}(S) = \{1,2,4,7\}$.

5.2.2 Row Insertion and Jeu de Taquin

This paper uses row insertion, the Robinson–Schensted–Knuth correspondence (RSK), and jeu de taquin heavily. We will state the known properties we will use here, but we assume the reader is already familiar with row insertion, RSK, and jeu de taquin. We recommend [Sag01, Chapter 3] or [Ful97, Chapter 1] for background on these algorithms and their properties.

Lemma 5.2.2. [Ful97, §1.1] Let T be any tableau, and consider the sequential row insertions $(T \leftarrow x) \leftarrow y$.

(i) If $x \leq y$, the bumping chain of y is strictly right of the bumping chain of x.

(ii) If $x > y$, the bumping chain of y is weakly left of the bumping chain of x.

We will need the following generalization of Lemma 5.2.2(ii) that allows for intermediate insertions between x and y. Because insertions can only decrease the value of each cell, Lemma 5.2.2(ii) still holds if the intermediate bumping chains avoid the bumping chain of x.

Lemma 5.2.3. Let T be any tableau, and consider the sequential row insertions $(\ldots((T \leftarrow x) \leftarrow x_1) \cdots \leftarrow x_k) \leftarrow y$. If the bumping chains of $x_1, \ldots, x_k$ are disjoint from the bumping chain of x and $x > y$, then the bumping chain of y is weakly left of the bumping chain of x.

The insertion tableau was originally defined to study increasing subsequences of words by Schensted, who proved that the length of the first row of $P(w)$ is the maximum length of an increasing subsequence of w [Sch61]. Later, Greene generalized this result to describe $\mathrm{rv}(P(w))$ in terms of increasing subsequences of w [Gre74]. Since reversing a word w transposes the shape of $P(w)$, analogous results hold for decreasing subsequences and the conjugate shape.

Theorem 5.2.4. [Gre74] Suppose w is a word and let $\lambda = (\lambda_1, \ldots, \lambda_b) = \mathrm{rv}(P(w))$. Then, for any $k \leq b$, $\lambda_1 + \cdots + \lambda_k$ is the maximum length of a subsequence of w which is the disjoint union of k increasing subsequences. If $\lambda' = (\lambda'_1, \ldots, \lambda'_{\ell(\lambda)})$, then for any $k \leq \ell(\lambda)$, $\lambda'_1 + \cdots + \lambda'_k$ is the maximum length of a subsequence of w which is the disjoint union of k decreasing subsequences.

Definition 5.2.5. For any skew tableau S, an *inner slide* on S is the act of sliding into an outer corner of the inner shape of S and continuing slide into the hole so that increasing rows and columns are preserved until we reach an outer corner. Let $\mathrm{Rect}(S)$ denote the *rectification* of S, obtained by continually performing inner slides until straight shape is achieved. $\mathrm{Rect}(S)$ is well-defined in that it is independent of the order of the slides. SeeExample 5.2.6.

Example 5.2.6. Consider

$$S = \begin{array}{|c|c|c|c|}\hline \multicolumn{1}{c}{} & 2 & 4 & 9 \\ \cline{2-4} \multicolumn{1}{c}{} & 1 & 5 & 8 \\ \cline{2-4} 3 & 6 & 10 & 13 \\ \cline{1-4} 7 & 11 & 12 & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array}.$$

We continually perform inner slides to the yellow $*$ cell as follows. The green cells indicate which cells were just moved by the previous inner slide.

$$\begin{array}{|c|c|c|c|}\hline \multicolumn{1}{c}{} & * & 1 & 6 \\ \cline{2-4} \multicolumn{1}{c}{} & 3 & 4 & 9 \\ \cline{2-4} 2 & 7 & 8 & 11 \\ \cline{1-4} 5 & 10 & 12 & 13 \\ \cline{1-4} \end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline \multicolumn{1}{c}{} & 1 & 4 & 6 \\ \cline{2-4} * & 3 & 8 & 9 \\ \cline{1-4} 2 & 7 & 11 & 13 \\ \cline{1-4} 5 & 10 & 12 & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline * & 1 & 4 & 6 \\ \cline{1-4} 2 & 3 & 8 & 9 \\ \cline{1-4} 5 & 7 & 11 & 13 \\ \cline{1-4} 10 & 12 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array} \rightarrow \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 6 \\ \cline{1-4} 2 & 7 & 8 & 9 \\ \cline{1-4} 5 & 11 & 13 & \multicolumn{1}{c}{} \\ \cline{1-3} 10 & 12 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}.$$

Therefore,

$$\mathrm{Rect}(S) = \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 6 \\ \cline{1-4} 2 & 7 & 8 & 9 \\ \cline{1-4} 5 & 11 & 13 & \multicolumn{1}{c}{} \\ \cline{1-3} 10 & 12 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}.$$

Rectification has many important roles in algebraic combinatorics. It is related to the RSK correspondence as in Lemma 5.2.7. It is involved in one of the many ways to define the Littlewood–Richardson coefficients, Definition 5.2.8. Inner and outer slides are also essential in defining promotion, demotion, and evacuation, Definitions 5.2.9 and 5.2.11.

Lemma 5.2.7. [Ful97, §2.1, Corollary 2] For any skew tableau S,

$$\mathrm{Rect}(S) = P(w), \tag{5.5}$$

where w is the reading word of S.

Definition 5.2.8. [Ful97, Chapter 5.1] Given any skew shape λ/μ and $\nu \vdash |\lambda| - |\mu|$, the corresponding *Littlewood–Richardson coefficient* is

$$c^{\lambda}_{\mu,\nu} \coloneqq \#\{S \in \mathrm{SYT}(\lambda/\mu) : \mathrm{Rect}(S) = T_0\}$$

for any $T_0 \in \mathrm{SYT}(\nu)$. The number $c^{\lambda}_{\mu,\nu}$ is independent of the choice of $T_0 \in \mathrm{SYT}(\nu)$ [Ful97, Corollary 5.1.1].

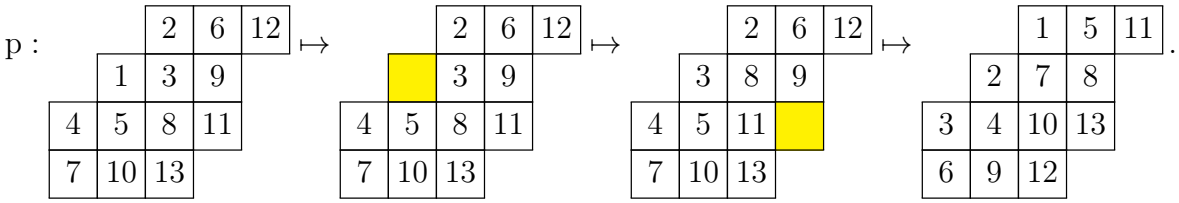
Schützenberger's promotion operator on standard tableaux, Definition 5.2.9, appears in the context of representation theory. Promotion also generalizes to linear extensions of finite posets, including skew tableaux.

Definition 5.2.9. For any skew shape λ/μ, *promotion* $\mathrm{p} : \mathrm{SYT}(\lambda/\mu) \to \mathrm{SYT}(\lambda/\mu)$ is given by erasing the label 1, sliding that cell until it hits an outer corner of λ/μ, filling that outer corner with $|\lambda| - |\mu| + 1$, and finally decrementing all of the entries by 1. *Demotion* is given by erasing the label n, sliding that cell until it hits an inner corner of λ, filling that inner corner with 0, and finally incrementing all of the entries by 1. Demotion and promotion are inverses when λ/μ is rectangular, but not in general. The names "promotion" and "demotion" for these operations are often interchanged, such as in [Rho10a]

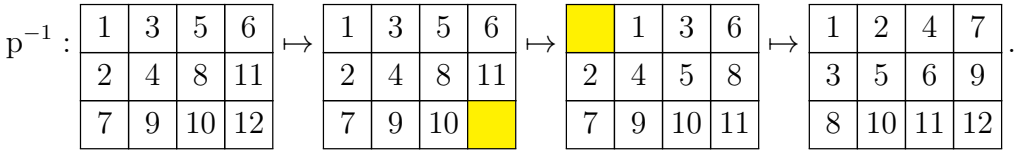
Example 5.2.10.

$$\mathrm{p}: \begin{array}{|c|c|c|c|}\hline 1&3&5&6\\\hline 2&4&8&11\\\hline 7&9&10&12\\\hline\end{array} \mapsto \begin{array}{|c|c|c|c|}\hline &3&5&6\\\hline 2&4&8&11\\\hline 7&9&10&12\\\hline\end{array} \mapsto \begin{array}{|c|c|c|c|}\hline 2&3&5&6\\\hline 4&8&10&11\\\hline 7&9&12&\\\hline\end{array} \mapsto \begin{array}{|c|c|c|c|}\hline 1&2&4&5\\\hline 3&7&9&10\\\hline 6&8&11&12\\\hline\end{array}.$$

For an example on a skew shape,



We will only use demotion on rectangular shapes, when it is the inverse of promotion.

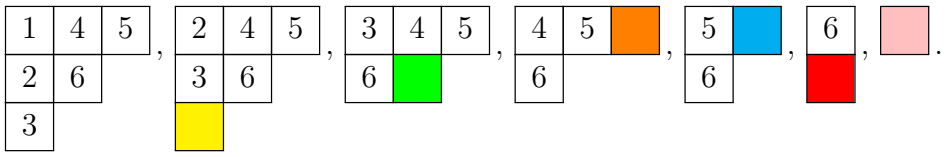


Definition 5.2.11. Suppose $\lambda \vdash n$. For $T \in \mathrm{SYT}(\lambda)$, the *evacuation* of T, $e(T) \in \mathrm{SYT}(\lambda)$ is the standard tableau that records the reverse order in which the cells are vacated as the smallest entry is repeatedly removed and the skew tableau rectified.

Example 5.2.12. Consider

$$T = \begin{array}{|c|c|c|}\hline 1 & 4 & 5 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} \end{array}.$$

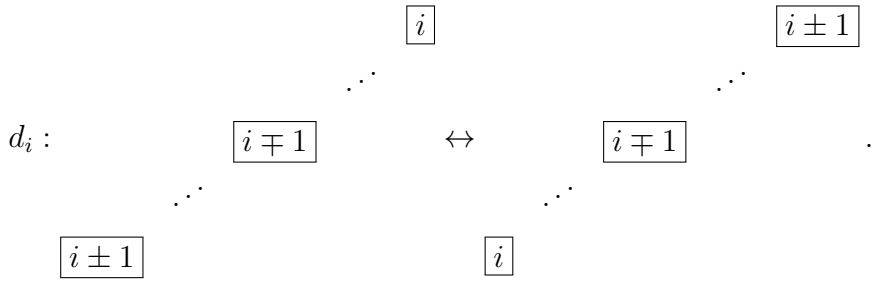
Repeatedly removing the smallest entry and rectifying, we get the sequence



Recording the reverse order in which the cells are vacated gives

$$e(T) = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}.$$

Definition 5.2.13. Two standard skew tableaux S, T are *dual equivalent* if they differ by a sequence *elementary dual equivalence moves*. The elementary dual equivalence move d_i swaps the cells of $i \pm 1$ and i if $i \mp 1$ appears between them in reading order:



Haiman studies dual equivalence in [Hai92]. He defines dual equivalence as the property that the same sequence of slides produces the same shape and then proves this property is characterized by Definition 5.2.13, [Hai92, Theorem 2.6]. Moreover, elementary dual equivalences moves commute with rectification [Hai92, Lemma 2.3]:

$$d_i(\mathrm{Rect}(S)) = \mathrm{Rect}(d_i(S)), \tag{5.6}$$

for all integers i and standard skew tableaux S. Haiman uses dual equivalence to prove Theorem 5.1.7. Two permutations are dual equivalent if they differ by moves of the form

$$\ldots (i \pm 1) \ldots (i \mp 1) \ldots i \cdots \leftrightarrow \ldots i \ldots (i \mp 1) \ldots (i \pm 1) \ldots,$$

for any integer i. By Definition 5.2.13, dual equivalence of standard tableaux of the same skew shape corresponds to dual equivalence of their reading words [Hai92, Lemma 2.11]. Dual equivalence classes of permutations are indexed by their common recording tableau: for all $v, w \in S_n$,

$$Q(v) = Q(w) \iff v \text{ and } w \text{ are dual equivalent.} \tag{5.7}$$

5.2.3 Quotients and Cores of Partitions

We review the definition of quotients and cores of partitions. We present a version due to James and Kerber [JK81, Section 2.7] by viewing the boundary path of a partition as a 0,1 sequence. Many other equivalent variations for the quotient and cores of partitions are known, such as [Mac95, I, Exercise 1.8]. In our case, it will prove convenient to view the r-quotient as a skew shape by combining its pieces anti-diagonally.

Definition 5.2.14. Consider the map

$$\varphi : \{\text{Partitions}\} \to \{\text{Infinite binary strings with initial 0's and terminal 1's}\}$$

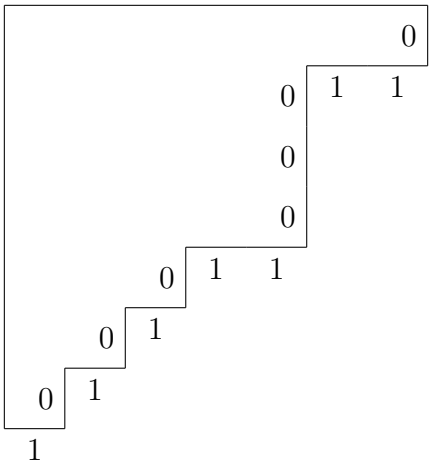
given by tracing the boundary path of the partition from southwest to northeast and writing 0 for each up step and 1 for each right step. The initial 0's represent the up steps along the negative y-axis, and the terminal 1's represent the right steps along the positive x-axis. We can thus view these sequences as finite binary strings where we can remove initial 0's and terminal 1's.

For any partition λ and $r \in \mathbb{Z}_{\geq 1}$, we form the *$r$-quotient*, $Q_r(\lambda)$, and *r-core*, $C_r(\lambda)$, of λ as follows. Let $w = \varphi(\lambda)$, and for each $j = 1, \ldots, r$, let $w^{(j)}$ denote the subsequence of w whose positions are congruent to j modulo r. Let $\lambda^{(1)} = \varphi^{-1}(w^{(1)}), \ldots, \lambda^{(r)} = \varphi^{-1}(w^{(r)})$. Then, $Q_r(\lambda)$, is the result of combining $\lambda^{(1)}, \ldots, \lambda^{(r)}$ into a single skew shape anti-diagonally, denoted $\lambda^{(1)} \cup \cdots \cup \lambda^{(r)}$, with $\lambda^{(1)}$ southwest of $\lambda^{(2)}$ southwest of $\lambda^{(3)}$, etc. Finally, let $\tilde{w}$ denote the binary sequence obtained after performing all swaps of the form

$$1x_1 \ldots x_{r-1}0 \to 0x_1 \ldots x_{r-1}1$$

on w until the 0's are are far left as possible. The order of swaps is irrelevant since we only swap two numbers whose indices are congruent modulo r. Then, set $C_r(\lambda) = \varphi^{-1}(\tilde{w})$.

Example 5.2.15. Suppose $\lambda = (7, 5, 5, 5, 3, 2, 1)$ and $r = 3$. Tracing the boundary path of λ and labeling vertical steps with 0's and horizontal steps with 1's,



Therefore,

$$w = \varphi(\lambda) = 10101011000110,$$

so

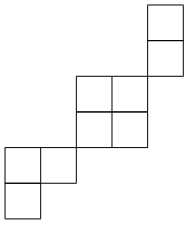
$$w^{(1)} = 10101 = 1010, \qquad w^{(2)} = 01100 = 1100, \qquad w^{(3)} = 1001 = 100,$$

which correspond to

$$\lambda^{(1)} = \varphi^{-1}(w^{(1)}) = (2,1), \qquad \lambda^{(2)} = \varphi^{-1}(w^{(2)}) = (2,2), \qquad \lambda^{(3)} = \varphi^{-1}(w^{(3)}) = (1,1).$$

Thus,

$$Q_3(\lambda) = (2,1) \cup (2,2) \cup (1,1) = \quad = (5,5,4,4,2,1)/(4,4,2,2).$$



Finally, performing all possible swaps of the form $1x_1x_20 \to 0x_1x_21$ on w gives

$$\tilde{w} = 00000010111111 = 10,$$

which corresponds to

$$C_3(\lambda) = \varphi^{-1}(\tilde{w}) = (1).$$

We will need to know what rectangles have empty r-core and what the r-quotient is in that case. Then, we can apply Corollary 5.1.10 to find $\#\,\mathrm{SYT}(a^b)^{\mathrm{p}^d}$, and we will know when we have constructed all of the tableaux in $\#\,\mathrm{SYT}(a^b)^{\mathrm{p}^d}$. These results were known by Dennis White and are implicit in [Whi06], unpublished work. We include their proofs here for completeness.

Lemma 5.2.16. Suppose $a, b, r \in \mathbb{Z}_{\geq 1}$ and $r \mid ab$. Then, (a^b) has empty r-core if and only if $r \mid a$ or $r \mid b$.

Proof. The binary word corresponding to the rectangle $\lambda = (a^b)$ is

$$\varphi(\lambda) = 1^a 0^b.$$

From Definition 5.2.14, λ has empty r-core if and only if performing all possible swaps of the form $1x_1 \dots x_{r-1}0 \to 0x_1 \dots x_{r-1}1$ to 1^a0^b results in 0^b1^a. Hence, (a^b) has empty r-core if and only if the positions of the zeros, namely $a+1, \dots, a+b$ are congruent to $1, 2, \dots, b$ modulo r, in some order. We say two multisets $\{\beta_1, \dots, \beta_m\}, \{\gamma_1, \dots, \gamma_m\}$ are congruent modulo r, denoted $\{\beta_1, \dots, \beta_m\} \equiv_r \{\gamma_1, \dots, \gamma_m\}$, if there exists a permutation $w \in S_b$ so that $\beta_{w_j} \equiv_r \gamma_j$ for all $j = 1, \dots, m$. Then, the r-core of (a^b) is empty if and only if

$$\{a+1, \dots, a+b\} \equiv_r \{1, \dots, b\}. \tag{5.8}$$

By the division algorithm, we can write $a = kr + i$ and $b = \ell r + j$ with $k, \ell \in \mathbb{Z}_{\geq 0}$ and $i, j \in [0, r-1]$. Then,

$$\begin{aligned} \{a+1, \dots, a+b\} \equiv_r \{i+1, \dots, i+b\} &\equiv_r \{0, \dots, r-1\}^\ell \cup \{i+1, \dots, i+j\} \\ \{1, \dots, b\} &\equiv_r \{0, \dots, r-1\}^\ell \cup \{1, \dots, j\}. \end{aligned} \tag{5.9}$$

Combining (5.8) and (5.9) gives

$$\{i+1,\dots,i+j\} \equiv_r \{1,\dots,j\}. \tag{5.10}$$

Since $i,j \in [0,r-1]$, (5.10) holds if and only if $i=0$ or $j=0$, or equivalently $r \mid a$ or $r \mid b$.

□

Remark 5.2.17. We can also characterize a partition λ having empty r-core using r-ribbons. The r-core of a partition λ is the smallest partition μ that an be obtained from λ by successively removing r-ribbons. From this perspective, it is clear that if $r \mid a$ or $r \mid b$, then the r-core of (a^b) is empty. However, it is less obvious that the r-core of (a^b) is empty only if $r \mid a$ or $r \mid b$.

Corollary 5.2.18. Suppose $a,b,d \in \mathbb{Z}_{\geq 1}$ and $d \mid ab$. If $\mathrm{SYT}(a^b)^{\mathrm{p}^d} \neq \varnothing$, then

$$\mathrm{SYT}(a^b)^{\mathrm{p}^d} = \mathrm{SYT}((kr)^b)^{\mathrm{p}^{br}}, \quad \text{or} \quad \mathrm{SYT}(a^b)^{\mathrm{p}^d} = \mathrm{SYT}(a^{kr})^{\mathrm{p}^{ar}}$$

for some $k,r \in \mathbb{Z}_{\geq 1}$.

Proof. By Corollary 5.1.10, if $\mathrm{SYT}(a^b)^{\mathrm{p}^d} \neq \varnothing$, then (a^b) has empty $\frac{ab}{d}$-core. By Lemma 5.2.16, this means

$$\frac{ab}{d} \mid a \ \text{ or } \ \frac{ab}{d} \mid b \implies b \mid d, \text{ or } a \mid d. \tag{5.11}$$

If $b \mid d$, let $r \coloneqq \frac{d}{b} \in \mathbb{Z}_{\geq 1}$, so $d = br$ and

$$d \mid ab \implies r \mid a \implies a = kr \text{ for some } k \in \mathbb{Z}_{\geq 1}.$$

Hence, $\mathrm{SYT}(a^b)^{\mathrm{p}^d} = \mathrm{SYT}((kr)^b)^{\mathrm{p}^{br}}$. Otherwise, $a \mid d$, so similarly, $\mathrm{SYT}(a^b)^{\mathrm{p}^d} = \mathrm{SYT}((a^{kr})^{\mathrm{p}^{ar}}$ for some $k,r \in \mathbb{Z}_{\geq 1}$. □

Thus, to $\mathrm{SYT}(a^b)^{\mathrm{p}^d}$ is empty unless it is of the form $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$ or $\mathrm{SYT}(b^{ar})^{\mathrm{p}^{br}}$ by Corollary 5.2.18. However, since promotion commutes with the conjugation map $T \mapsto T'$, we

have

$$\mathrm{SYT}(b^{ar})^{\mathrm{p}^{br}} = \{T' : T \in \mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}\}.$$

Therefore, it suffices to consider to describe the tableaux in $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$ to find $\mathrm{SYT}(a^b)^{\mathrm{p}^d}$ in general, as in Question 5.1.11.

Lemma 5.2.19. For $a, b \in \mathbb{Z}_{\geq 1}$, let $s = b \pmod a \in [0, a-1]$. Then, for any $r \in \mathbb{Z}_{\geq 1}$,

$$Q_a((ar)^b) = \underbrace{(r^{\lceil \frac{b}{a} \rceil}) \cup \cdots \cup (r^{\lceil \frac{b}{a} \rceil})}_{s \text{ times}} \cup \underbrace{(r^{\lfloor \frac{b}{a} \rfloor}) \cup \cdots \cup (r^{\lfloor \frac{b}{a} \rfloor})}_{a-s \text{ times}}$$

Proof. Following Definition 5.2.14, mapping $\nu \coloneqq (ar)^b$ to a binary sequence gives

$$\varphi(\nu) = 1^{ar}0^b.$$

Hence, $Q_a(\nu) = \nu^{(1)} \cup \cdots \cup \nu^{(a)}$ where

$$\begin{aligned} \nu^{(1)} = \cdots = \nu^{(s)} &= \varphi^{-1}(1^r\ 0^{\lceil \frac{b}{a} \rceil}) = (r^{\lceil \frac{b}{a} \rceil}), \\ \nu^{(s+1)} = \cdots = \nu^{(a)} &= \varphi^{-1}(1^r\ 0^{\lfloor \frac{b}{a} \rfloor}) = (r^{\lfloor \frac{b}{a} \rfloor}). \end{aligned}$$

□

5.2.4 Generalized Sums

Later in Section 5.4, it will be convenient to use sums $\sum_{j=m}^{n}$ where m need not be less than or equal to n. For a sequence $a_0, a_1, \ldots$, we generalize the notion $\sum_{j=m}^{n} a_j$ from $m \leq n$ to all $m, n \in \mathbb{Z}_{\geq 0}$ as follows, as in Section 2.6 of [GKP94].

Definition 5.2.20. For $m, n \in \mathbb{Z}_{\geq 0}$, define

$$\sum_{j=m}^{n} a_j \coloneqq \sum_{j=0}^{n} a_j - \sum_{j=0}^{m-1} a_j, \tag{5.12}$$

where $\sum_{j=0}^{-1} a_j = 0$. In particular, for $x > y$, we have

$$\sum_{j=m}^{n} a_j = \begin{cases} 0, & \text{if } m = n-1, \\ -\sum_{j=m-1}^{n+1} a_j, & \text{else.} \end{cases} \tag{5.13}$$

We make Definition 5.2.20 so that

$$\sum_{j=m}^{n} a_j + \sum_{j=n+1}^{p} a_j = \sum_{j=m}^{p} a_j \tag{5.14}$$

for all $m, n, p \in \mathbb{Z}_{\geq 0}$, which is an immediate consequence of (5.12). We introduce this notation so we can rewrite (5.1) more elegantly as

$$\lambda_j = kr + \sum_{i=j}^{b-j} c_i. \tag{5.15}$$

In addition, we will make use of the following lemma involving $\sum_{j=m}^{n} a_j$ with $m > n$ in Section 5.4.

Lemma 5.2.21. For any positive integers $b \geq t \geq 1$ and any sequence $a_1, a_2, \ldots,$

$$\sum_{j=1}^{t} \sum_{i=1}^{j+b-t-1} a_i = \sum_{j=1}^{t} \sum_{i=j}^{b-j} a_i.$$

Proof. We have

$$\begin{aligned} \sum_{j=1}^{t} \sum_{i=j}^{j+b-t-1} a_i &= \sum_{j=1}^{t} \sum_{i=j}^{b-j} a_i + \sum_{j=1}^{t} \sum_{i=b-j+1}^{b-t-1} a_i + \sum_{j=1}^{t} \sum_{i=b-t}^{b-t+j-1} a_i \qquad \text{by (5.14)} \\ &= \sum_{j=1}^{t} \sum_{i=j}^{b-j} a_i + \sum_{j=1}^{t} \left(\sum_{i=b-j+1}^{b-t-1} a_i + \sum_{i=b-t}^{b-j} a_i \right) \end{aligned}$$

by re-indexing the last sum by $j \mapsto t+1-j$. Then,

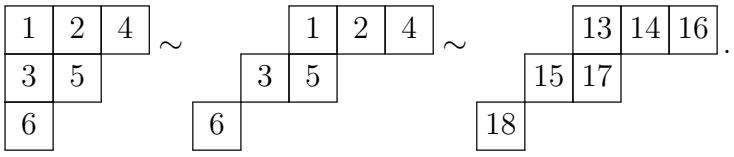
$$\sum_{j=1}^{t}\sum_{i=j}^{j+b-t-1} a_i = \sum_{j=1}^{t}\sum_{i=j}^{b-j} a_i + \sum_{j=1}^{t}\sum_{i=b-j+1}^{b-j} a_i \qquad \text{by (5.14)}$$
$$= \sum_{j=1}^{t}\sum_{i=j}^{b-j} a_i \qquad \text{by (5.13).}$$

□

5.3 Tableau Stabilization

In this section, we recall tableau stabilization from the introduction and explore its properties. After defining row shift equivalence and tableau stabilization, we show stabilization is invariant under dual equivalence. We show stabilization continues after its first occurrence and prove an upper bound for stabilization when it is defined.

Definition 5.3.1. For any skew tableau S, let $S+x$ denote the result of adding x to each cell of S. We say two skew tableaux S_1 and S_2 are *row shift equivalent*, denoted $\sim$, if there exists a constant x so S_1+x and S_2 differ only by horizontal slides. For example,



Row shift equivalence is a natural equivalence relation for our purposes since stabilization is unaffected by horizontal slides. Using Definition 5.3.1, we can rephrase the definition of tableaux stabilization, Definition 5.1.1, in various ways as follows.

Definition 5.3.2. For $c \le d$, let $S|_{[c,d]}$ be the restriction of S to the cells with entries in $[c,d]$. Suppose that S is skew tableaux with m entries and decreasing row vector. Recall that $S^{(k)}$ is the result of attaching $(k-1)$ shifted copies of S to the right of S so that the result is a standard skew tableau. This means $S^{(k)}\big|_{[(j-1)m+1,jm]} \sim S$ for all $j = 1,\dots,k$. We say S *stabilizes at* k if any of the following equivalent conditions hold:

(a) $\mathrm{Rect}(S^{(k)})\big|_{[(k-1)m+1,km]} \sim S$,

(b) $\mathrm{Rect}(S^{(k)})\big|_{[(k-1)m+1,km]}$ and S have the same number of cells in each row,

(c) $\mathrm{rv}(\mathrm{Rect}(S^{(k)})) - \mathrm{rv}(\mathrm{Rect}(S^{(k-1)})) = \mathrm{rv}(S)$.

Let $\mathrm{stab}(S)$ denote the minimum value at which S stabilizes. See Example 5.1.2 for examples.

It is easy to see that Definition 5.3.2(a), (b), (c) are all equivalent to Definition 5.1.1. First, (a) is just a rephrasing of Definition 5.1.1. Secondly, (a) clearly implies (b). Thirdly, (b) and (c) are equivalent because

$$\mathrm{rv}(\mathrm{Rect}(S^{(k)})\big|_{[(k-1)m+1,km]}) = \mathrm{rv}(\mathrm{Rect}(S^{(k)})) - \mathrm{rv}(\mathrm{Rect}(S^{(k-1)})).$$

Finally, if (c) holds, then since $\mathrm{Rect}(S^{(k)})\big|_{(k-1)m+1,km]}$ is obtained by sliding the cells in $S + (k-1)m$, the slides from $S + (k-1)m$ to $\mathrm{Rect}(S^{(k)})\big|_{(k-1)m+1,km]}$ can only be horizontal. Hence, $\mathrm{Rect}(S^{(k)})\big|_{(k-1)m+1,km]} \sim S$.

Remark 5.3.3. By definition of rectification, we have

$$\mathrm{Rect}(S^{(j)}) = \mathrm{Rect}(S^{(j+1)})\big|_{[1,jm]} \text{ for all } j \geq 1,$$

so

$$\mathrm{Rect}(S^{(j)})\big|_{[(i-1)m+1,im]} = \mathrm{Rect}(S^{(k)})\big|_{[(i-1)m+1,im]} \text{ for all } 1 \leq i \leq j \leq k.$$

Thus, to determine if S stabilizes at j for any $j \leq k$, it suffices to consider $\mathrm{Rect}(S^{(k)})$.

Remark 5.3.4. We only consider standard skew tableaux with decreasing row vectors so that $S^{(k)}$ is a standard skew tableau for all $k \in \mathbb{Z}_{\geq 1}$. If the row vector of S is not decreasing, then $S^{(k)}$ need not be a tableau. For example,

$$S = \begin{array}{|c|c|}\hline & 2 \\ \hline 1 & 3 \\ \hline \end{array} \implies S^{(2)} = \begin{array}{|c|c|c|c|}\hline & 2 & 5 & \\ \hline 1 & 3 & 4 & 6 \\ \hline \end{array},$$

which is not a skew tableau both because of its shape and the third column not being increasing.

Lemma 5.3.5. Suppose S, T are standard skew tableaux with $S \sim T$. Then, $\mathrm{Rect}(S^{(k)}) = \mathrm{Rect}(T^{(k)})$ for all $k \in \mathbb{Z}_{\geq 1}$ and $\mathrm{stab}(S) = \mathrm{stab}(T)$.

Proof. Consider any $k \in \mathbb{Z}_{\geq 1}$. By definition of row shift equivalence restricted to standard skew tableaux, $S^{(k)}$ and $T^{(k)}$ have the same reading word, so $\mathrm{Rect}(S^{(k)}) = \mathrm{Rect}(T^{(k)})$ by Lemma 5.2.7. Let $m = |S| = |T|$. Then, if S stabilizes at k,

$$\mathrm{Rect}(T^{(k)})\big|_{[(k-1)m,km]} = \mathrm{Rect}(S^{(k)})\big|_{[(k-1)m,km]} \sim S \sim T,$$

so T stabilizes at k as well. Similarly the converse holds, and $\mathrm{stab}(S) = \mathrm{stab}(T)$. □

Theorem 5.3.6. If S, T are dual equivalent standard skew tableaux, then $\mathrm{stab}(S) = \mathrm{stab}(T)$.

Proof. By Definition 5.2.13, we may assume $T = d_i(S)$ without loss of generality. Shifting the entries by a constant preserves this relationship, so $T + x = d_{i+x}(S + x)$ for all postive integers x. It follows by the definition of $S^{(k)}$ in Definition 5.1.1 that

$$T^{(k)} = d_i \circ d_{i+m} \circ \cdots \circ d_{i+(k-1)m}(S^{(k)}) \text{ for all } k \geq 1, \tag{5.16}$$

where $m = |S| = |T|$, which means $S^{(k)}, T^{(k)}$ are dual equivalent. By (5.6), (5.16) implies

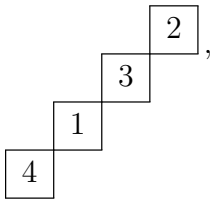
$$\mathrm{Rect}(T^{(k)}) = d_i \circ d_{i+m} \circ \cdots \circ d_{i+(k-1)m}(\mathrm{Rect}(S^{(k)})) \text{ for all } k \geq 1. \tag{5.17}$$

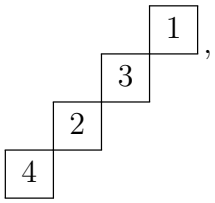
Since elementary dual equivalence moves preserve the row vector,

$$\mathrm{rv}(\mathrm{Rect}(T^{(k)})) = \mathrm{rv}(\mathrm{Rect}(S^{(k)})) \text{ for all } k \geq 1.$$

Hence, $\mathrm{stab}(S) = \mathrm{stab}(T)$ by Definition 5.3.2(c). □

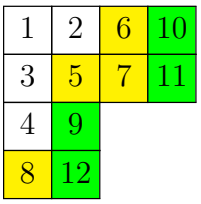
Example 5.3.7. Consider

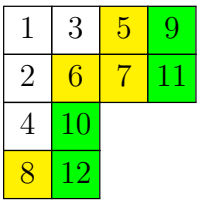
$S =$ 

, $T =$ 

,

which satisfy $T = d_2(S)$. Then,

$\text{Rect}(S^{(3)}) =$ 

, $\text{Rect}(T^{(3)}) =$ 

,

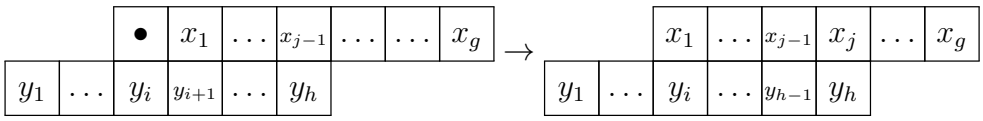
which satisfy $\text{Rect}(T^{(3)}) = d_2 \circ d_6 \circ d_{10}(\text{Rect}(S^{(3)}))$. Notice that d_2 swaps 1 and 2 in S and T but swaps 2 and 3 in $\text{Rect}(S^{(3)})$ and $\text{Rect}(T^{(3)})$. Here, $\text{stab}(S) = \text{stab}(T) = 3$, which is consistent with Theorem 5.3.6.

In order to show any standard skew tableau S with decreasing row vector eventually stabilizes and then continues to stabilize, consider $S^{(\infty)}$, obtained by attaching infinitely many shifted copies of S to the right of S so that the result uses each of $1, 2, \ldots$ exactly once. As the row vector of S is decreasing, the rows and columns of $S^{(\infty)}$ are increasing. We can rectify $S^{(\infty)}$ with inner slides just like any skew tableau using the following lemma.

Lemma 5.3.8. Suppose S is a standard skew tableau with decreasing row vector, and let $m = |S|$. When rectifying $S^{(\infty)}$, if all inner slides so far pass horizontally through the entries in $[(k-1)m+1, km]$ for some $k \in \mathbb{Z}_{\geq 1}$, then they pass horizontally through all entries greater than km.

Proof. We localize our attention to the entries in $[(k-1)m+1, km]$ and suppose all inner

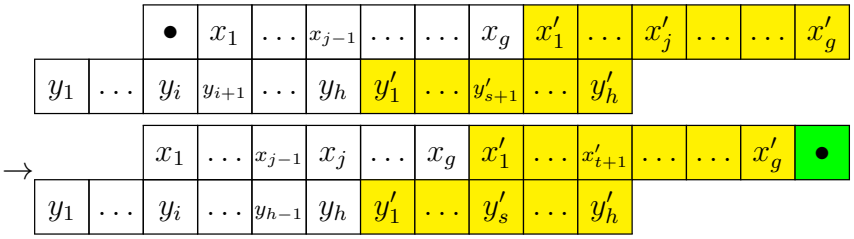
slides so far have proceed horizontally through $[(k-1)m+1, km]$. Consider one such slide:



where $g \geq h$ and $j = h - i + 1$. This means

$$x_\ell < y_{\ell+i-1} \qquad \text{for all } \ell = 1, \ldots, h-i+1.$$

Then, if we include the relevant entries in $[km+1, (k+1)m]$, we have



where $x'_\ell = x_\ell + m, y'_\ell = y_\ell + m, s = i + g - h \geq i, t = j - s \leq j$. This slide also proceeds horizontally through $[km+1, (k+1)m]$ because

$$x'_\ell < y'_{\ell+i-1} \leq y'_{\ell+s-1} \qquad \text{for all } \ell = 1, \ldots, h-s+1.$$

Also, any cell whose diagonally southwest neighbor is in a different shifted copy of S slides horizontally. It follows inductively this slide and all slides so far proceed horizontally through all entries greater than km.

□

Lemma 5.3.9. Suppose S is a standard skew tableau with decreasing row vector. Then, S must stabilize eventually, and if S stabilizes at k, then S stabilizes at k' for all $k' \geq k$.

Proof. By Remark 5.3.3, we have $\text{Rect}(S^{(j)}) \subset \text{Rect}(S^{(\infty)})$ for all j. Each inner slide rectifying $S^{(\infty)}$ can involve at most $(b-1)$ vertical moves as $S^{(\infty)}$ only has b rows. Since $S^{(\infty)}$ rectifies

in a finite number of inner slides, rectifying $S^{(\infty)}$ involves a finite number of vertical moves. Hence, for some k, entries greater than $(k-1)m$ experience no vertical moves, which means

$$\mathrm{Rect}(S^{(\infty)})\big|_{[(k-1)m+1,km]} \sim S,$$

Thus, S stabilizes at k. Since the entries in $[(k-1)m+1, km]$ have experienced no vertical moves, all entries greater than km have experienced no vertical moves by Lemma 5.3.8. Therefore, for all $k' \geq k$,

$$\mathrm{Rect}(S^{(\infty)})\big|_{[(k'-1)m+1,k'm]} \sim S,$$

and S stabilizes at k'.

□

While Lemma 5.3.9 shows that any standard skew tableau with decreasing row vector must stabilize eventually, the bounds its proof gives for stabilization are very weak. If a skew tableau S has b rows, each inner slide to rectify $S^{(\infty)}$ that starts in row i can have at most $b-i$ vertical slides. By Lemma 5.3.8, those vertical slides must happen before or within the first $b-i$ shifted copies of S that have yet to experience vertical slides. Thus, any skew tableaux of shape λ/μ with decreasing row vector stabilizes at $\sum_{j=1}^{b} \mu_j(b-j)$.

This naive bound depends on the inner shape, whereas the general bound we will prove only depend is linear in the number of rows, Theorem 5.1.5. Any standard tableau T of straight shape has $\mathrm{stab}(T) = 1$ trivially since $\mathrm{Rect}(T) = T$. Any nonempty standard skew tableau S with 1 row also has $\mathrm{stab}(S) = 1$ trivially. Theorem 5.1.5 gives a general bound, $2b-2$, for $b \geq 2$ rows.

Definition 5.3.10. For any word $u = u_1 u_2 \dots u_n$ and $m \in \mathbb{Z}_{\geq 1}$, let

$$u + m = (u_1 + m)(u_2 + m) \dots (u_n + m).$$

In addition, for $k \in \mathbb{Z}_{\geq 1}$, let

$$u^{(k,m)} = u(u+m)(u+2m)\dots(u+(k-1)m)$$

If m is implicit, let $u^{(k)} = u^{(k,m)}$. For example, if $u = 134, k = 2$, and $m = 5$, then

$$(134)^{(2)} = (134)^{(2,5)} = 134689.$$

Notation 5.3.11. Suppose S is a standard skew tableau with decreasing row vector $(r_b, r_{b-1}, \dots, r_2, r_1)$, so that its size is $m \coloneqq r_1 + \cdots + r_b$. Let $w_1, w_2, \dots, w_b$ denote the sequences of entries in S from left to right in rows $b, (b-1), \dots, 1$, respectively, so that the reading word of S is $w_1 w_2 \dots w_b$. For the rest of this paper, m will be implicit, so $u^{(k)} = u^{(k,m)}$ for any word u. Thus, the reading word of $S^{(k)}$ is $w_1^{(k)} w_2^{(k)} \dots w_b^{(k)}$. Hence, by Lemma 5.2.7,

$$\mathrm{Rect}(S^{(k)}) = P(w_1^{(k)} w_2^{(k)} \dots w_b^{(k)}). \tag{5.18}$$

For $j = 1, \dots, b$ and $k \geq 1$, let

$$T_j^{(k)}(S) \coloneqq P(w_1^{(k)} w_2^{(k)} \dots w_j^{(k)}). \tag{5.19}$$

so that $T_b^{(k)}(S) = \mathrm{Rect}(S^{(k)})$.

Our goal is to understand the tails of the rows in $T_b^{(k)}(S)$ for sufficiently large k, which will turn out to $k \geq 2b-2$, Lemma 5.3.15. We will do so using row insertion rather than jeu de taquin. In order to understand what happens to the tails of the rows under these row insertions, Lemma 5.3.13, we need to understand various comparisons between the elements we are inserting and the tails of each row Lemma 5.3.12.

Lemma 5.3.12. Suppose u, v are increasing words on $[m]$ with respective lengths $r \leq s$. Recall $P(uv)$ denotes the insertion tableau of the concatenated word uv, and let

$$c \coloneqq \mathrm{rv}(P(uv))_1 - s.$$

Then, for each $k \geq 1$,

$$v_t^{(k)} < u_{t+c}^{(k)} \text{ for all } t \in [rk - c]. \tag{5.20}$$

Proof. By Theorem 5.2.4, $\mathrm{rv}(P(uv))_1 = c + s$ is the maximum length of an increasing subsequence of uv. For any $j \in [r - c]$, the subsequence

$$u_1 u_2 \dots u_{j+c} v_j v_{j+1} \dots v_s$$

of uv has size $c + s + 1$, so it is not increasing. As u, v are increasing, this forces

$$v_j < u_{j+c} \text{ for all } j \in [r - c]. \tag{5.21}$$

Now, by Definition 5.3.10,

$$u_{ir+j}^{(k)} = u_j + (i-1)m, \qquad v_{is+j'}^{(k)} = v_{j'} + (i-1)m \tag{5.22}$$

for all $i \in [0, k-1], j \in [r], j' \in [s]$. If $i \in [0, k-1], j \in [r-c]$, then $j + c \leq r$, so

$$v_{is+j}^{(k)} = v_j + (i-1)m < u_{j+c} + (i-1)m = u_{ir+j+c} \leq u_{is+j+c} \tag{5.23}$$

using (5.21), $r \leq s$, and that u is increasing. On the other hand, if $i \in [0, k-2], j \in [r-c+1, s]$, then $v_j \leq m$ and $j + c \geq r + 1$, so

$$v_{is+j}^{(k)} = v_j + (i-1)m \leq im < u_{(i+1)r+1} = u_{ir+r+1} \leq u_{ir+j+c} \leq u_{is+j+c}, \tag{5.24}$$

also using $r \leq s$ and that u is increasing. Combining (5.23) and (5.24) gives Lemma 5.3.12. $\square$

Lemma 5.3.13. Suppose $i, m \in \mathbb{Z}_{\geq 1}$, u, v are increasing words on $[(i-1)m+1, im]$ with respective lengths $r \leq s$, and w, w' are increasing words on $[(i-1)m]$. Then, for all $k \geq 2$,

$$\boxed{wu^{(k)}} \leftarrow w'v^{(k)} = \begin{array}{|c|c|}\hline \text{x} & (v+m)^{(k-1)} \\ \hline \text{x'} & (u+m)^{(k-1)} \\ \hline \end{array}$$

where x, x' are increasing words on $[im]$, not necessarily of the same length.

Proof. First, consider the insertion $wu^{(k)} \leftarrow w'$. Since each element of w' is $\leq (i-1)m$ and all elements of $u^{(k)}$ are $\geq (i-1)m+1$, the elements of $u^{(k)}$ that w' bumps down must form a consecutive sequence w'' starting from the beginning of $u^{(k)}$. Note w'' may be empty. Letting $c \coloneqq \mathrm{rv}(P(uv)) - s$, $u \leftarrow v$ creates a tableau whose first row has size $c+s$, meaning v bumps down $r-c$ elements from u. Hence, in $wu^{(k)} \leftarrow w'v$, after w' bumps down w'' from $u^{(k)}$, v bumps down at most $r-c$ elements from u. Thus, as each element of v is $\leq im$, v also bumps down at least $c+s-r$ of the initial elements that are after both w'' and u. In particular, as $r \leq s$, the c initial elements of $(u+m)^{(k-1)}$ are bumped down in $wu^{(k)} \leftarrow w'v$.

Let $c' \geq c$ be the number of initial elements of $(u+m)^{(k-1)}$ that are bumped down in $wu^{(k)} \leftarrow w'v$. By Lemma 5.3.12 and u being increasing,

$$(v+m)_t^{(k-1)} < (u+m)_{t+c}^{(k-1)} \leq (u+m)_{t+c'}^{(k-1)} \text{ for all } t \in [r(k-1)-c'].$$

It follows inductively that the insertion of $(v+m)^{(k-1)}$ bumps down all of the remaining elements of $(u+m)^{(k-1)}$ in row 1. The result follows.

$\square$

Example 5.3.14. Suppose $u = 134, v = 256$, $k = 3$ and $m = 6$. Then, $\boxed{u^{(3,6)}} \leftarrow v^{(4,6)}$

$$= \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline 1&3&4&7&9&10&13&15&16\\\hline\end{array} \leftarrow 2568(11)(12)(14)(17)(18)$$

$$= \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 1&2&4&5&6&8&11&12&14&17&18\\\hline\end{array}$$
$$\begin{array}{|c|c|c|c|c|c|c|}\hline 3&7&9&10&13&15&16\\\hline\end{array}$$

$$= \begin{array}{|c|c|}\hline 12456 & (v+6)^{(2)}\\\hline 3 & (u+6)^{(2)}\\\hline\end{array}$$

Numbers bigger than 9 in a sequence are parenthesized to avoid confusion.

Lemma 5.3.15. For $j = 2, \ldots, b$, and $k \geq 2b-2$, $T_j^{(k)}(S)$ is of the form

x_1	$(w_j + (j-1)m)^{(k-j+1)}$
x_2	$(w_{j-1} + jm)^{(k-j)}$
$\vdots$	$\vdots$
x_i	$(w_{j+1-i} + (j-2+i)m)^{(k-j+2-i)}$
$\vdots$	$\vdots$
x_{j-1}	$(w_2 + (2j-3)m)^{(k-2j+3)}$
x_j	$(w_1 + (2j-3)m)^{(k-2j+3)}$

where each x_i is an increasing word on $[(j-2+i)m]$, not necessarily of the same length.

Proof. We proceed by induction on j. The base case $j = 1$ holds since $w_1^{(k)}$ is increasing. Inductively assuming the result holds for some fixed $j \leq b-1$, we have

$$T_{j+1}^{(k)}(S) = T_j^{(k)}(S) \leftarrow w_{j+1}^{(k)}. \tag{5.25}$$

By repeated applications of Lemma 5.3.13, the bumping process in $T_j^{(k)}(S) \leftarrow w_{j+1}^{(k)}$ bumps $(w_j + jm)^{(k-j)}$ from row 1 to row 2, which bumps $(w_{j-1} + (j+1)m)^{(k-j-1)}$ from row 2 to row 3, ..., which bumps $(w_{j+1-i} + (j-1+i)m)^{(k-j+1-i)}$ from row i to row $i+1$ for all $i = 1, \ldots, j$.

This means $(w_{j+2-i} + (j-2+i)m)^{(k-j+2-i)}$ bumps into row i for $i = 1, \ldots, j, j+1$. So, by Lemma 5.3.13, row i of $T^k_{j+1}(S)$ ends in $(w_{j+2-i} + (j-1+i)m)^{(k-j+1-i)}$ for $i = 1, \ldots, j$ and in $(w_1 + (2j-1)m)^{(k-2j+1)}$ in newly created row $j+1$. Smaller elements are also bumped down from each row, but they do not affect this pattern. This proves Lemma 5.3.15 for $j+1$ and completes our induction. □

Proof of Theorem 5.1.5. Suppose S is a standard skew tableau with decreasing row vector and continue Notation 5.3.11. Plugging $j = b$ and $k = 2b-2$ into Lemma 5.3.15 gives

$$\mathrm{Rect}(S^{(2b-2)}) = T_b^{(2b-2)}(S) = \begin{array}{|c|c|}\hline x_1 & (w_b + (b-1)m)^{(b-1)} \\ \hline x_2 & (w_{b-1} + bm)^{(b-2)} \\ \hline \vdots & \vdots \\ \hline x_i & (w_{b+1-i} + (b-2+i)m)^{(b-i)} \\ \hline \vdots & \vdots \\ \hline x_{b-1} & (w_2 + (2b-3)m)^{(1)} \\ \hline x_b & (w_1 + (2b-3)m)^{(1)} \\ \hline \end{array},$$

where each x_i is an increasing word on $[(b-2+i)m]$. Hence,

$$\mathrm{Rect}(S^{(2b-2)})\big|_{[(2b-3)m+1,(2b-2)m]} \sim S,$$

so S stabilizes at $2b-2$.

□

5.4 Stabilization on Skew Tableaux with Constant Row Vectors

In this section, we restrict our attention to skew tableaux with constant row vectors. We prove a formula, (5.1), for the shape of the stabilized tableau, proving Theorem 5.1.6. This lets us improve our upper bound for stabilization from $2b-2$ to b in the constant row vector case, proving Theorem 5.1.4.

Notation 5.4.1. Suppose S is a standard skew tableau with constant row vector (r^b), so that it has b rows of size r, and its size is $m \coloneqq br$. We then continue with Definition 5.3.10 and Notation 5.3.11 as before. Recall $w_1, w_2, \ldots, w_b$ are the reading words of the rows of S from bottom to top,

$$T_j^{(k)}(S) \coloneqq P(w_1^{(k)} w_2^{(k)} \ldots w_j^{(k)}) \quad \text{for} \quad j = 1, \ldots, b \ \text{ and } \ k \geq 1, \tag{5.26}$$

and $T_b^{(k)}(S) = \mathrm{Rect}(S^{(k)})$. In addition, let

$$c_i = c_i(S) \coloneqq \mathrm{rv}(P(w_i w_{i+1}))_1 - r, \ \text{for all } i = 1, \ldots, b-1. \tag{5.27}$$

See Example 5.4.3 for an example computation of the c_i's.

First, we derive a lower bound on the partial sums of $\mathrm{rv}(T_b^{(k)}(S))$ using Theorem 5.2.4. Secondly, we will show that equality always holds for this bound using Lemma 5.2.2 and Lemma 5.2.3, proving Theorem 5.1.6. Recall from (5.13) that for all $n, p \in \mathbb{Z}_{\geq 0}$ with $n > p$ that

$$\sum_{j=n}^{p} a_j = \begin{cases} 0, & \text{if } y = x - 1, \\ -\sum_{j=p-1}^{n+1} a_j, & \text{else.} \end{cases}$$

Lemma 5.4.2. Suppose S is standard skew tableau with row vector (r^b) and use Notation 5.4.1. If $k \geq b-1$ and $\mathrm{rv}(T_b^{(k)}(S)) = (\lambda_1, \ldots, \lambda_b)$, then, for $t = 1, \ldots, b$,

$$\sum_{j=1}^{t} \lambda_j \geq \sum_{j=1}^{t} \left(kr + \sum_{i=j}^{b-j} c_i \right). \tag{5.28}$$

Proof. Consider any positive integers $t \leq b$ and $k \geq b-1$. By Theorem 5.2.4, we will be done if we show there exists t disjoint increasing subsequences of $w(S,k) \coloneqq w_1^{(k)} \ldots w_b^{(k)}$ whose total length is $\sum_{j=1}^{t} \left(kr + \sum_{i=j}^{b-j} c_i \right)$. We will exhibit t such increasing subsequences of $w(S,k)$ explicitly. Our exhibition of these subsequences uses the following $b \times k$ matrix of

words:

$$M_k(S) = \begin{matrix} w_b & w_b + m & \dots & w_b + (k-1)m \\ \vdots & \vdots & \vdots & \vdots \\ w_2 & w_2 + m & \dots & w_2 + (k-1)m \\ w_1 & w_1 + m & \dots & w_1 + (k-1)m \end{matrix}$$

Notice concatenating these words from left to right and then top to bottom gives $w(S,k)$. In addition, we will use the lattice paths on the Cartesian plane in Figure 5.1.

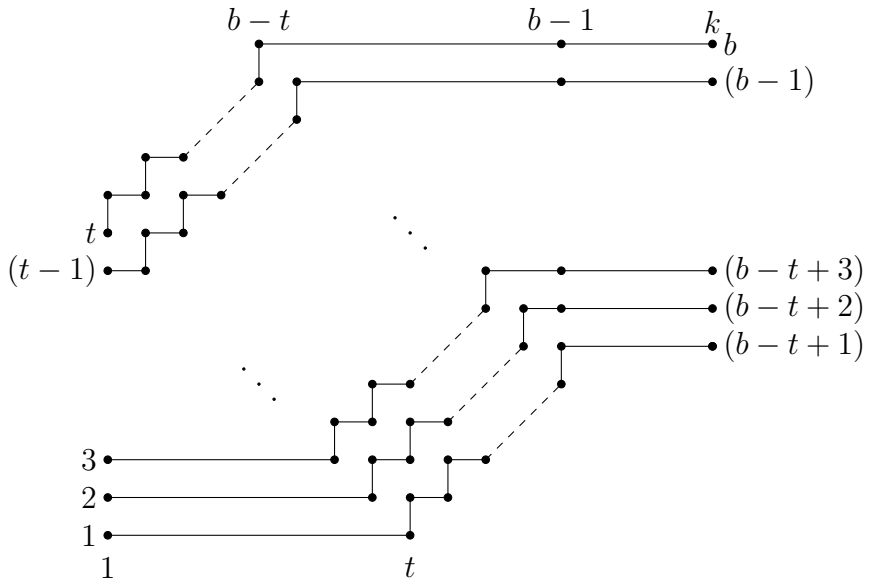


Figure 5.1: The lattice paths used to construct increasing subsequences of $w(S,k)$

Label the paths $L_1, \dots, L_t$ in Figure 5.1 from bottom to top. For $j = 1, \dots, t$, the path L_j proceeds as follows:

(a) Walk horizontally from $(j,1)$ to $(j, t+1-j)$, then

(b) Alternate between $(0,1)$ and $(1,0)$ steps, starting with $(0,1)$, until reaching $(b-j, b-t+j)$, then

(c) Walk horizontally to $(k, b-t+j)$.

Now, overlay the $b \times k$ grid containing these lattice paths on $M_k(S)$. We can then convert each lattice path L_j into a word y_j as follows. First concatenate the words in $M_k(S)$ that correspond to lattice points in L_j in order starting from $(1, j)$ and ending at $(k, b-t+j)$ to obtain x_j. Note x_j need not be increasing because it contains $(w_i + sm)(w_{i+1} + sm)$ on each column where there is a vertical step. To obtain an increasing subsequence associated to L_j, we will replace each $(w_i + sm)(w_{i+1} + sm)$ by a maximum length increasing subsequence of $(w_i + sm)(w_{i+1} + sm)$. By Theorem 5.2.4 and the definition of c_i, (5.27), there exists a maximum length increasing subsequence u_i of $w_i w_{i+1}$ with length $c_i + r$. Thus, replace each instance of $(w_i + sm)(w_{i+1} + sm)$ in x_j with $u_i + sm$ to obtain the increasing subsequence y_j. Hence,

$$\begin{aligned} y_j = w_j^{(t-j)}(u_j + (t-j)m)(u_{j+1} + (t+1-j)m) \dots (u_{b-t+j-1} + (b-j)m) \\ (w_{b-t+j} + (b+1-j)m)^{(k-b+j)}. \end{aligned}$$

Since the lattice paths $L_1, \dots, L_t$ are disjoint, $y_1, \dots, y_t$ are disjoint as well. For $j = 1, \dots, t$, as $\ell(w_i) = r$ and $\ell(u_i) = c_i + r$ for all i, we have

$$\ell(y_j) = (t-j)r + \left(\sum_{i=j}^{b-t+j-1} (c_i + r) \right) + (k-b+j)r = kr + \sum_{i=j}^{j+b-t-1} c_i.$$

Hence,

$$\sum_{j=1}^{t} \ell(y_j) = tkr + \sum_{j=1}^{t} \sum_{i=j}^{j+b-t-1} c_i = \sum_{j=1}^{t} \left(kr + \sum_{i=j}^{b-j} c_i \right)$$

by Lemma 5.2.21. Thus, $y_1, \dots, y_t$ are t disjoint increasing subsequences of $w(S, k)$ with total length $\sum_{j=1}^{t} \left(kr + \sum_{i=j}^{b-j} c_i \right)$, completing the proof. $\square$

Example 5.4.3. Suppose

$$S = \begin{array}{|c|c|c|c|c|c|c|c|c|}\hline \multicolumn{6}{c}{} & 7 & 9 & 10 \\ \cline{7-9} \multicolumn{4}{c}{} & 2 & 4 & 12 \\ \cline{5-7} \multicolumn{2}{c}{} & 6 & 8 & 11 \\ \cline{3-5} 1 & 3 & 5 \\ \cline{1-3}\end{array}.$$

Thus, $b = 4, r = 3, m = 12$, $w_1 = 135, w_2 = 68(11), w_3 = 24(12), w_4 = 79(10)$,

$$P(w_1w_2) = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 3 & 5 & 6 & 8 & 11 \\ \hline\end{array}, P(w_2w_3) = \begin{array}{|c|c|c|c|}\hline 2 & 4 & 11 & 12 \\ \hline 6 & 8 \\ \cline{1-2}\end{array},$$

$$P(w_3w_4) = \begin{array}{|c|c|c|c|c|}\hline 2 & 4 & 7 & 9 & 10 \\ \hline 12 \\ \cline{1-1}\end{array},$$

so $c_1 = 3, c_2 = 1, c_3 = 2$. Maximum length increasing subsequences of w_1w_2, w_2w_3 and w_3w_4 are

$$u_1 = 13568(11), \quad u_2 = 68(11)(12), \quad u_3 = 2479(10),$$

respectively. The matrix of words for this example is

$$M_3(S) = \begin{matrix} w_4 & w_4+12 & w_4+24 \\ w_3 & w_3+12 & w_3+24 \\ w_2 & w_2+12 & w_2+24 \\ w_1 & w_1+12 & w_1+24 \end{matrix} = \begin{matrix} 79(10) & (19)(21)(22) & (31)(33)(34) \\ 24(12) & (14)(16)(24) & (26)(28)(36) \\ 68(11) & (18)(20)(23) & (30)(32)(35) \\ 135 & (13)(15)(17) & (25)(27)(29) \end{matrix}.$$

The lattice paths from the proof of Lemma 5.4.2 for $b = 4, k = 3$ and $t = 1, 2, 3, 4$ are shown in Figure 5.2 from left to right, respectively.

Figure 5.2 corresponds to the following lists of disjoint increasing subsequences of $w_1^{(3)}w_2^{(3)}w_3^{(3)}w_4^{(3)}$:

1. $u_1(u_2 + 12)(u_3 + 24)$ with size 15,

2. $w_1(u_1 + 12)(u_2 + 24), u_2(u_3 + 12)(w_4 + 24)$ with total size 25,

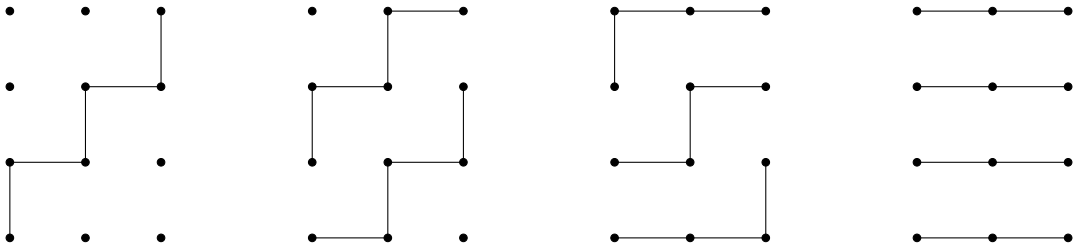


Figure 5.2: The lattice paths used to construct increasing subsequences of $w(S,k)$ for $b=4, k=3, t=1,2,3,4$

3. $w_1(w_1+12)(u_1+12), w_2(u_2+12)(w_3+24), u_3(w_4+12)(w_4+24)$ with total size 33,

4. $w_1^{(3)}, w_2^{(3)}, w_3^{(3)}, w_4^{(3)}$ with total size 36,

which are consistent with Lemma 5.4.2. Letting $(\lambda_1,\lambda_2,\lambda_3,\lambda_4) = \mathrm{rv}(T_4^{(3)}(S))$, by Lemma 5.4.2, we must have

$$\begin{aligned}
\lambda_1 &\geq 3r + \sum_{i=1}^{3} c_i = 15,\\
\lambda_1+\lambda_2 &\geq 15 + 3r + \sum_{i=2}^{2} c_i = 25,\\
\lambda_1+\lambda_2+\lambda_3 &\geq 25 + 3r + \sum_{i=3}^{1} c_i = 33,\\
\lambda_1+\lambda_2+\lambda_3+\lambda_4 &\geq 33 + 3r + \sum_{i=4}^{0} c_i = 36.
\end{aligned}$$

Not only do these inequalities hold, but they are all equalities. We have $\mathrm{rv}(T_4^{(3)}(S)) = (15,10,8,3)$, as shown below:

$T_4^{(3)}(S) = \mathrm{Rect}(S^{(3)}) =$

1	2	4	6	7	9	10	14	16	19	21	22	31	33	34
3	5	8	11	12	23	24	26	28	36					
13	15	17	18	20	30	32	35							
25	27	29												

.

Proof of Theorem 5.1.6. Suppose S is a standard skew tableau with row vector (r^b), and continue Notation 5.4.1. Fix $k \geq b-1$. Recall that

$$T_i^{(k)}(S) = P(w_1^{(k)} w_2^{(k)} \dots w_i^{(k)}) \quad \text{for} \quad i = 1, \dots, b.$$

We need to show

$$\mathrm{rv}(T_b^{(k)}(S))_j = kr + \sum_{i=j}^{b-j} c_i \quad \text{for all} \quad j = 1, \dots, b \ \text{ and } \ k \geq b-1. \tag{5.29}$$

We proceed by induction on b. If $b = 1$, $P(w_1^{(k)})$ is a tableau with 1 row of size kr, so Theorem 5.1.6 holds for $b = 1$. Inductively assume that Theorem 5.1.6 holds for all standard skew tableaux with less than b rows and all $k \geq b-2$.

Let $B_1, \dots, B_{kr}$ denote the bumping chains of the row insertions

$$T_{b-1}^{(k)}(S) = T_{b-2}^{(k)}(S) \leftarrow w_{b-1}^{(k)}.$$

Each letter of $w_{b-1}^{(k)}$ has its own bumping chain. Since $w_{b-1}^{(k)}$ is increasing, $B_1, \dots, B_{kr}$ proceed strictly left to right by Lemma 5.2.2. By the induction hypothesis, we have

$$\mathrm{rv}(T_{b-2}^{(k)}(S))_j = kr + \sum_{i=j}^{b-2-j} c_i, \ \text{for all } j = 1, \dots, b-2,$$
$$\mathrm{rv}(T_{b-1}^{(k)}(S))_j = kr + \sum_{i=j}^{b-1-j} c_i, \ \text{for all } j = 1, \dots, b-1.$$

Therefore, for $j = 1, \dots, b-2$,

$$\mathrm{rv}(T_{b-1}^{(k)}(S))_j - \mathrm{rv}(T_{b-2}^{(k)}(S))_j = c_{b-1-j},$$

which implies that c_{b-1-j} of the bumping chains among $B_1, \dots, B_{kr}$ end in row j. Furthermore, for $t = 1, \dots, b-2$, exactly $c_{b-1-t} + \dots + c_{b-2}$ of $B_1, \dots, B_{kr}$ end in or above row t.

Next, let $B'_1, \ldots, B'_{kr}$ denote the bumping chains of the row insertions

$$T_b^{(k)}(S) = T_{b-1}^{(k)}(S) \leftarrow w_b^{(k)}.$$

Since $w_b^{(k)}$ is increasing, $B'_1, \ldots, B'_{kr}$ again proceed strictly left to right by Lemma 5.2.2. We claim that B'_j is weakly left of $B_{j+c_{b-1}}$ for all $j = 1, \ldots, kr - c_{b-1}$. where

$$c_{b-1} = \mathrm{rv}(P(w_{b-1}w_b))_1 - r,$$

from (5.27). By Lemma 5.3.12,

$$(w_b^{(k)})_j < (w_{b-1}^{(k)})_{j+c_{b-1}} \text{ for all } j = 1, \ldots, kr - c_{b-1}. \tag{5.30}$$

Since $B_2, \ldots, B_{kr}$ are disjoint from B_1, and $(w_b^{(k)})_1 < (w_{b-1}^{(k)})_{1+c_{b-1}}$, Lemma 5.2.3 implies that B'_1 is weakly left of $B_{1+c_{b-1}}$. Inductively assuming that $B'_1, \ldots, B'_j$ are weakly left of $B_{1+c_{b-1}}, \ldots, B_{j+c_{b-1}}$, respectively, $B_{j+2+c_{b-1}}, \ldots, B_m, B'_1, \ldots B'_j$ are all disjoint from $B_{j+1+c_{b-1}}$ and $(w_b^{(k)})_{j+1} < (w_{b-1}^{(k)})_{j+c_{b-1}}$. Thus, by Lemma 5.2.3, B'_{j+1} is weakly left of $B_{j+1+c_{b-1}}$, proving our claim. Hence, B'_j must end in a strictly lower row than $B_{j+c_{b-1}}$ for all $j = 1, \ldots, kr - c_{b-1}$.

Combining this with exactly $c_{b-1-t} + \cdots + c_{b-2}$ of $B_1, \ldots, B_{kr}$ ending in or above row t, at most $c_{b-t} + \cdots + c_{b-2} + c_{b-1}$ of $B'_1, \ldots, B'_{kr}$ end in or above row t for $t = 1, \ldots, b-1$. Thus, using our original induction hypothesis,

$$\begin{aligned}
\sum_{j=1}^{t} \mathrm{rv}(T_b^{(k)}(S))_j &\leq \left(\sum_{j=1}^{t} \mathrm{rv}(T_{b-1}^{(k)}(S))_j \right) + c_{b-t} + \cdots + c_{b-1} \\
&= \sum_{j=1}^{t} \left(kr + \sum_{i=j}^{b-1-j} c_i \right) + \sum_{j=1}^{t} c_{b-j} \\
&= \sum_{j=1}^{t} \left(kr + \sum_{i=j}^{b-j} c_i \right).
\end{aligned} \tag{5.31}$$

for $t = 1, \ldots, b-1$. Combining (5.31) with Lemma 5.4.2,

$$\sum_{j=1}^{t} \mathrm{rv}(T_b^{(k)}(S))_j = \sum_{j=1}^{t} \left(kr + \sum_{i=j}^{b-j} c_i \right) \quad \text{for all} \quad t = 1, \ldots, b-1. \tag{5.32}$$

For the $t = b$ case,

$$\sum_{j=1}^{b} \left(kr + \sum_{i=j}^{b-j} c_i \right) = bkr + \sum_{j=1}^{b} \sum_{i=j}^{j-1} c_i = bkr. \tag{5.33}$$

by Lemma 5.2.21 with $t = b$. Also,

$$\sum_{j=1}^{b} \mathrm{rv}(T_b^k(S))_j = bkr$$

since $\ell(w_1^{(k)} w_2^{(k)} \ldots w_b^{(k)}) = bkr$. Therefore,

$$\sum_{j=1}^{b} \mathrm{rv}(T_b^{(k)}(S))_j = \sum_{j=1}^{b} \left(kr + \sum_{i=j}^{b-j} c_i \right). \tag{5.34}$$

Combining (5.32) and (5.34) yields

$$\sum_{j=1}^{t} \mathrm{rv}(T_b^{(k)}(S))_j = \sum_{j=1}^{t} \left(kr + \sum_{i=j}^{b-j} c_i \right) \text{ for all } t = 1, \ldots, b.$$

This shows that the partial sums of the two sequences $\{\mathrm{rv}(T_b^{(k)}(S))_j\}_{j=1}^{b}$ and $\left\{ kr + \sum_{i=j}^{b-j} c_i \right\}_{j=1}^{b}$ both agree, so the sequences agree as well, proving (5.29).

□

Proof of Theorem 5.1.4. Suppose S is a standard skew tableau with row vector (r^b). In order

to show S stabilizes at b, it suffices to show that

$$\mathrm{rv}(\mathrm{Rect}(S^{(b)}))_j - \mathrm{rv}(\mathrm{Rect}(S^{(b-1)}))_j = \mathrm{rv}(S)_j \quad \text{for all} \quad j = 1, \ldots, b. \tag{5.35}$$

by Definition 5.3.2(c). In fact, using Theorem 5.1.6,

$$\mathrm{rv}(\mathrm{Rect}(S^{(b)}))_j - \mathrm{rv}(\mathrm{Rect}(S^{(b-1)}))_j = br + \sum_{i=j}^{b-j} c_i - (b-1)r - \sum_{i=j}^{b-j} c_i = r = \mathrm{rv}(S)_j$$

for all $j = 1, \ldots, b$. □

Corollary 5.4.4. Suppose S is a standard skew tableau with b rows each of size r. Let $w_1, \ldots, w_b$ denote the entries in each row read from left to right, starting from the bottom. For $k \geq \mathrm{stab}(S) - 1$, $\mathrm{Rect}(S^{(k)})$ has shape $(\lambda_1, \ldots, \lambda_b)$, where

$$\lambda_j = kr + \sum_{i=1}^{b-j} c_i - \sum_{i=1}^{j-1} c_i \quad \text{for all } j = 1, \ldots, b,$$

where

$$c_i = (\text{the length of the first row of } P(w_i w_{i+1})) - r \quad \text{for all } i = 1, \ldots, b-1.$$

Proof. This is exactly Theorem 5.1.6 with the bound $k \geq b - 1$ changed to $k \geq \mathrm{stab}(S) - 1$. Let $m = |S|$ and suppose $k \in [\mathrm{stab}(S) - 1, b - 1]$. Because $k \geq \mathrm{stab}(S) - 1$,

$$\mathrm{Rect}(S^{(b-1)})\big|_{[(k-1)m+1,(b-1)m]} \sim S^{(b-1-k)}$$

by Lemma 5.3.9. Since $\mathrm{Rect}(S^{(k)}) \subseteq \mathrm{Rect}(S^{(b-1)})$ as well,

$$\begin{aligned}\mathrm{rv}(\mathrm{Rect}(S^{(k)}))_j &= \mathrm{rv}(\mathrm{Rect}(S^{(b-1)}))j - \mathrm{rv}(S^{(b-1-k)})j \\ &= (b-1)r + \sum_{i=1}^{b-j} c_i - \sum_{i=1}^{j-1} c_i - (b-1-k)r \\ &= kr + \sum_{i=1}^{b-j} c_i - \sum_{i=1}^{j-1} c_i.\end{aligned}$$

for all $j = 1, \ldots, b$. □

Example 5.4.5. Continuing with the same S as in Example 5.4.3,

$\mathrm{Rect}(S^{(4)}) =$

1	2	4	6	7	9	10	14	16	19	21	22	31	33	34	43	45	46
3	5	8	11	12	23	24	26	28	36	38	40	48					
13	15	17	18	20	30	32	35	42	44	47							
25	27	29	37	39	41												

,

so $\mathrm{Rect}(S^{(4)})\big|_{[25,36]} \sim S$, $\mathrm{Rect}(S^{(4)})\big|_{[13,24]} \not\sim S$, and $\mathrm{stab}(S) = 3$. If

$T =$

							5	10	11	
						8	9	12		
			2	4	6					
	1	3	7							

,

then

$\mathrm{Rect}(T^{(4)}) =$

1	2	4	5	8	9	10	11	17	22	23	29	34	35	41	46	47
3	6	12	14	16	18	20	21	24	32	33	36	44	45	48		
7	13	25	26	28	30	38	40	42								
15	19	27	31	37	39	43										

,

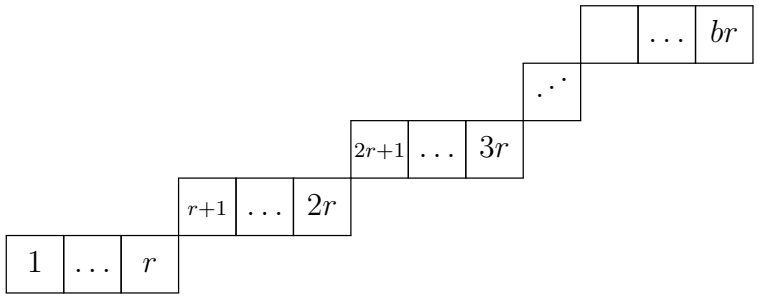
so $\mathrm{Rect}(T^{(4)})\big|_{[37,48]} \sim T$, $\mathrm{Rect}(T^{(4)})\big|_{[25,36]} \not\sim T$, and $\mathrm{stab}(S) = 4$.

As seen in Example 5.4.5, some, but not all, skew tableaux with b rows of equal size

stabilize before b. To prove the bound is tight in Theorem 5.1.4 is tight, it suffices to find a standard tableau $I_{b,r}$ with row vector (r^b) and $\mathrm{stab}(I_{b,r}) = b$ for each $b, r \in \mathbb{Z}_{\geq 1}$. We can form such an $I_{b,r}$ by making its reading word increasing, as in Example 5.4.6.

Example 5.4.6. Consider

$S =$



Then, we have $c_1 = c_2 = \cdots = c_{b-1} = r$, so in particular, setting $k = b - 1$ and $j = b$ in Theorem 5.1.6 gives

$$\mathrm{rv}(\mathrm{Rect}(I_{b,r}^{(b-1)}))_b = (b-1)r + \sum_{i=b}^{0} c_i = (b-1)r - \sum_{i=1}^{b-1} r = 0.$$

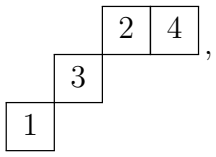
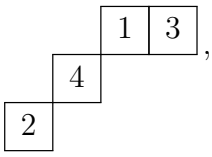
Hence, $\mathrm{Rect}(I_{b,r}^{(b-1)})$ consists of $b-1$ rows. Therefore, $\mathrm{Rect}(I_{b,r}^{(b-1)})\Big|_{[b(b-2)+1,b(b-1)]} \not\sim I_{b,r}$, so $I_{b,r}$ does not stabilize at $b-1$. But, by Theorem 5.1.4, $I_{b,r}$ stabilizes at b, so $\mathrm{stab}(I_{b,r}) = b$. Note that $I_{b,r}$ is not the only skew tableaux with b rows of size r and $\mathrm{stab}(I_{b,r}) = b$, as illustrated in Example 5.4.5.

Remark 5.4.7. We believe any standard tableau S with decreasing row vector stabilizes at b, Conjecture 5.1.3, though we are currently unable to prove it. We cannot directly generalize our proof of Theorem 5.1.4 to prove Conjecture 5.1.3 because Theorem 5.1.6, a key ingredient in proving Theorem 5.1.4, does not generalize to this setting. Let $w_1, \ldots, w_b$ denote the reading words of the rows of S, from bottom to top, $r_i = \ell(w_i)$ for all i, and

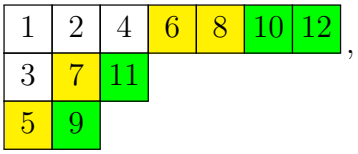
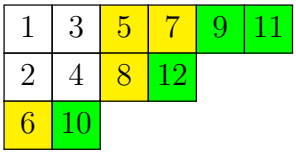
$$c_i \coloneqq \mathrm{rv}(P(w_i w_{i+1}))_1 - r_{i+1} \text{ for all } i = 1, \ldots, (b-1).$$

Unlike Theorem 5.1.6, in this more general setting, $\mathrm{rv}(\mathrm{Rect}(S^{(b)}))$ depends on more than just $b, r_1, \ldots, r_b, c_1, \ldots, c_{b-1}$, as demonstrated in Example 5.4.8. Hence, some other technique is required to prove Conjecture 5.1.3, if it is indeed true.

Example 5.4.8. Consider

$S =$  , $T =$  ,

which both have $b = 3, r_1 = 1, r_2 = 1, r_3 = 2, c_1 = 1, c_2 = 0$. Then,

$\mathrm{Rect}(S^{(3)}) =$  , $\mathrm{Rect}(T^{(3)}) =$  .

Thus, $\mathrm{rv}(\mathrm{Rect}(S^{(3)})) = (7, 3, 2) \neq (6, 4, 2) = \mathrm{rv}(\mathrm{Rect}(T^{(3)}))$.

5.5 Anti-Stabilization

In this section, we discuss anti-stabilization, which involves rectifying toward a southeast corner rather than the northwest corner. We show that the stabilized and anti-stabilized skew tableaux have reflected shapes and that the stabilization and anti-stabilization statistics coincide when both are defined.

Definition 5.5.1. For any skew tableau S with b rows, fix a so that $S \subset (a^b)$, and let $\mathrm{Rect}^*(S)$, the *anti-rectification* of S, be the skew tableau obtained by continually performing outer slides on S within (a^b) until the southeast corner of (a^b) is filled. $\mathrm{Rect}^*(S)$ is independent of a up to row shift equivalence. Also, for a standard skew tableau S of size m, let $S^\dagger$ denote the tableau obtained from S by rotating 180° and then flipping the entries by $x \mapsto m + 1 - x$, and let $(\lambda/\mu)^\dagger$ denote the skew shape obtained by rotating λ/μ by 180°.

Suppose S is a standard skew tableau of size m with increasing row vector. Then, define $S^{(*k)}$ to be the standard skew tableau obtained by attaching $k - 1$ shifted copies of S to

the left of $S + (k-1)m$ so that $S^{(*k)}\big|_{[(j-1)m+1,jm]} \sim S$ for all $j = 1, \ldots, k$. Then, we say S *anti-stabilizes* at k if $\mathrm{Rect}^*(S^{(*k)})\big|_{[1,m]} \sim S$. Let $\mathrm{stab}^*(S)$ denote the minimum value at which S anti-stabilizes.

Example 5.5.2. Consider

$S =$

		4
	2	5
1	3	

.

Then,

$S^{(*3)} =$

				4	9	14
	2	5	7	10	12	15
1	3	6	8	11	13	

,

$\mathrm{Rect}^*(S^{(*3)}) =$

					4	9
	2	5	7	10	12	14
1	3	6	8	11	13	15

,

$\mathrm{Rect}^*(S^{(*3)})^\dagger =$

1	3	5	8	10	13	15
2	4	6	9	11	14	
7	12					

.

Since $\mathrm{Rect}^*(S^{(*3)})\big|_{[11,15]} \not\sim S$, but $\mathrm{Rect}^*(S^{(*3)})\big|_{[6,10]} \sim S$, $\mathrm{stab}^*(S) = 2$.

Remark 5.5.3. Similarly to Remark 5.3.4, if S does not have increasing row vector, $S^{(*k)}$ need not be a standard skew tableau. Hence, the notions of stabilization and anti-stabilization only make sense simultaneously when the row vector is constant.

Anti-stabilization has many of the same properties as stabilization. If we apply the same reasoning using anti-rectification instead of rectification and $S^{(*k)}$ instead of $S^{(k)}$, we get the following analogues of Lemma 5.3.5, Lemma 5.3.9, Theorem 5.1.5, and Theorem 5.1.4 for anti-stabilization.

Corollary 5.5.4. Suppose S is a standard skew tableau with b rows and increasing row vector. Then,

(a) Anti-stabilization is well-defined up to row shift equivalence.

(b) S anti-stabilizes eventually, and if S anti-stabilizes at k, it anti-stabilizes at any $k' \geq k$.

(c) If $b \geq 2$, then S anti-stabilizes at $2b-2$.

(d) If S has constant row vector, then S anti-stabilizes at b.

If S is a standard skew tableau with constant row vector, then $S^{(k)}$ and $S^{(*k)}$ are both standard tableaux with the same reading word. Thus, by Lemma 5.2.7,

$$S^{(*k)} \sim S^{(k)},\ \mathrm{Rect}(S^{(*k)}) = \mathrm{Rect}(S^{(k)}),\ \text{ and }\ \mathrm{Rect}^*(S^{(*k)}) = \mathrm{Rect}^*(S^{(k)}). \tag{5.36}$$

Lemma 5.5.5. For any standard skew tableau S,

$$\mathrm{Rect}^*(S) = \mathrm{Rect}(S^\dagger)^\dagger = (e(\mathrm{Rect}(S)))^\dagger,$$

where e is the evacuation operator.

Proof. Let $w = w_1 \dots w_m$ be the reading word of S, so w is permutation of size m. Anti-rectifying S and rectifying $S^\dagger$ differ by a rotation of 180°, implying

$$\mathrm{Rect}^*(S) = \mathrm{Rect}(S^\dagger)^\dagger. \tag{5.37}$$

As † reverses the order of the cells in the reading word and reverses the entry values, the reading word of $S^\dagger$ is $w_0 w w_0$, where $w_0 = [m, m-1, \dots, 2, 1]$ is the reversal permutation of size m. Thus, by Lemma 5.2.7,

$$\mathrm{Rect}(S^\dagger)^\dagger = P(w_0 w w_0)^\dagger. \tag{5.38}$$

Moreover, $P(w_0ww_0) = e(P(w))$ by [Sag01, Theorem 3.9.4], so

$$P(w_0ww_0)^\dagger = e(P(w))^\dagger = (e(\mathrm{Rect}(S)))^\dagger \tag{5.39}$$

by Lemma 5.2.7. Combining (5.37), (5.38), and (5.39) gives Lemma 5.5.5.

□

Lemma 5.5.6. Suppose S is a standard skew tableau with row vector (r^b). Then, for all $k \geq 1$, and $j = 1, \ldots, b$,

$$\mathrm{rv}(\mathrm{Rect}^*(S^{(k)}))_j = \mathrm{rv}(\mathrm{Rect}(S^{(k)}))_{b+1-j}. \tag{5.40}$$

In addition,

$$\mathrm{stab}^*(S) = \mathrm{stab}(S). \tag{5.41}$$

Proof. For any $k \geq 1$, $S^{(k)} \sim S^{(*k)}$ from (5.36). Thus, by Lemma 5.5.5,

$$\mathrm{rv}(\mathrm{Rect}^*(S^{(k)}))_j = \mathrm{rv}(e(\mathrm{Rect}(S^{(k)}))^\dagger)_j = \mathrm{rv}(\mathrm{Rect}(S^{(k)})^\dagger)_j = \mathrm{rv}(\mathrm{Rect}(S^{(k)}))_{b+1-j}$$

for all $k \geq 1$ and $j = 1, \ldots, b$, proving (5.40). By Definition 5.3.2(c), S stabilizes at k if and only if

$$\mathrm{rv}(\mathrm{Rect}(S^{(k)}))_j - \mathrm{rv}(\mathrm{Rect}(S^{(k-1)}))_j = r \quad \text{for all } j = 1, \ldots, b. \tag{5.42}$$

Similarly, S anti-stabilizes at k if and only if

$$\mathrm{rv}(\mathrm{Rect}^*(S^{(k)}))_j - \mathrm{rv}(\mathrm{Rect}^*(S^{(k-1)}))_j = r \quad \text{for all } j = 1, \ldots, b. \tag{5.43}$$

By (5.40), (5.42) holds if and only if (5.43) holds. Thus, S stabilizes at k if and only if S anti-stabilizes at k, and (5.41) follows.

□

5.6 Sufficiently Large Tableaux Fixed by Powers of Promotion

In this section, we first give an alternative method for doing multiple promotions at once. Recall $\mathrm{p} : \mathrm{SYT}(a^b) \to \mathrm{SYT}(a^b)$ denotes Schützenberger's promotion operator, Definition 5.2.9. We then construct the sufficiently large rectangular standard tableaux fixed by promotion powers. In particular, we construct the tableaux in $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$ for $a \geq 2b-1$ and prove Theorem 5.1.12. Theorem 5.1.6 plays a key role in showing these tableaux are rectangular, tableau stabilization is central in showing these tableaux are fixed by p^{br}, and the bound in Theorem 5.1.4 lets us control the size of these rectangular tableaux. We also prove proving Corollary 5.1.13, which describes promotion's action on $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$ by .

Lemma 5.6.1. Suppose $a, b \in \mathbb{Z}_{\geq 1}$ and $n = ab$. For any tableau $T \in \mathrm{SYT}(a^b)$ and $k = 1, \ldots, n$, we have

$$\mathrm{p}^k(T)\big|_{[1,n-k]} = \mathrm{Rect}(T|_{[k+1,n]}) - k, \tag{5.44}$$

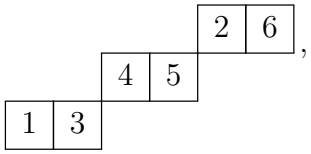
$$\mathrm{p}^k(T)\big|_{[n-k+1,n]} = \mathrm{Rect}^*(T|_{[1,k]}) + (n-k). \tag{5.45}$$

Proof. Equation (5.44) follows from Definition 5.2.9 and rectification being well-defined. By Theorem 5.1.7, $\mathrm{p}^n = \mathrm{id}$ on $\mathrm{SYT}(a^b)$, so $\mathrm{p}^k = (\mathrm{p}^{-1})^{n-k}$. Then, (5.45) follows from the fact that T is rectangular, Definition 5.2.9, and anti-rectification being well-defined.

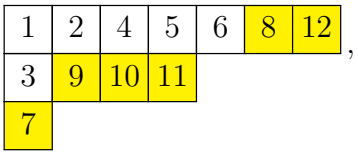
□

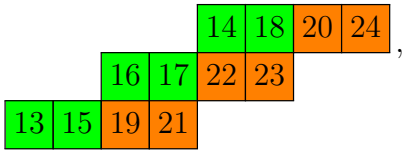
Definition 5.6.2. Suppose S is a standard skew tableau with row vector (r^b), and let $k \coloneqq \mathrm{stab}(S)$. For any $a \geq 2k-1$, let $R_a(S)$ denote the filling of rectangular shape formed by row-concatenating $\mathrm{Rect}(S^{(k-1)})$, $S^{(a-2k+2)} + (k-1)br$, and $\mathrm{Rect}^*(S^{(k-1)}) + (a-k+1)br$ together from left to right.

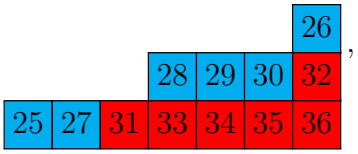
Example 5.6.3. Let

$S =$ 

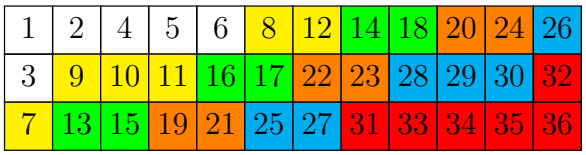
which has $k = \mathrm{stab}(S) = 3$, and let $a = 6 \geq 5$. Observe

$\mathrm{Rect}(S^{(2)}) =$ 

$S^{(2)} + 12 =$ 

$\mathrm{Rect}^*(S^{(2)}) + 24 =$ 

so

$R_6(S) =$ 

Theorem 5.6.4. Suppose S is a standard skew tableau with row vector (r^b). Then, for any integer $a \geq 2\,\mathrm{stab}(S) - 1$,

$$R_a(S) \in \mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}.$$

Proof. Let $k \coloneqq \mathrm{stab}(S)$, and let $R \coloneqq R_a(S)$. First, we check that R has rectangular shape $((ar)^b)$. By Corollary 5.4.4 and (5.40),

$$\mathrm{rv}(\mathrm{Rect}(S^{(k-1)}))_j = (k-1)r + \sum_{i=j}^{b-j} c_i, \tag{5.46}$$

$$\mathrm{rv}(\mathrm{Rect}^*(S^{(k-1)}))_j = \mathrm{rv}(\mathrm{Rect}(S^{(k-1)}))_{b+1-j} = (k-1)r + \sum_{i=b+1-j}^{j-1} c_i \tag{5.47}$$

for all $j = 1, \ldots, b$. Hence, for $j = 1, \ldots, b$,

$$\begin{aligned}
\mathrm{rv}(R)_j &= \mathrm{rv}(\mathrm{Rect}(S^{k-1}))_j + \mathrm{rv}(S^{(a-2k+2)})_j + \mathrm{rv}(\mathrm{Rect}^*(S_{k-1}))_j \\
&= (k-1)r + \sum_{i=j}^{b-j} c_i + (a-2k+2)r + (k-1)r + \sum_{i=b+1-j}^{j-1} c_i \\
&= ar + \sum_{i=j}^{j-1} c_i \\
&= ar,
\end{aligned}$$

which shows $\mathrm{rv}(R) = ((ar)^b)$. Since R is left-justified, it is a filling of shape $((ar)^b)$.

Secondly, we check that R is a standard tableau. We can break R into three pieces by restricting:

- $R_1 \coloneqq R|_{[1,kbr]} = \mathrm{Rect}(S^{(k)})$, because S stabilizes at k
- $R_2 \coloneqq R|_{[kbr+1,(a-k+1)br]} \sim S^{(a-2k+1)}$,
- $R_3 \coloneqq R|_{[(a-k+1)br+1,abr]} = \mathrm{Rect}^*(S^{(k-1)}) + (a-k+1)br$.

The fillings R_1 and R_3 are skew tableaux because they are the rectification and anti-rectification of skew tableaux, respectively. The filling R_2 is a skew tableaux since it is formed by row-concatenating shifted copies of $R_1|_{[(k-1)br+1,kbr]}$. Since R_1, R_2, R_3 use the entries $[1, kbr]$, $[kbr+1, (a-k+1)br]$, and $[(a-k+1)br+1, abr]$ respectively, R uses each of $1, 2, \ldots, abr$ exactly once. Because R_2 adjoins to the right of R_1 and R_3 adjoins to the right of R_2, the rows and columns of R are increasing. Hence, R is a standard tableau.

Thirdly, we check that R is fixed by br promotions. Since S stabilizes at k,

$$R|_{[1,jbr]} = \mathrm{Rect}(S^{(j)}) \quad \text{for all } j \in [1, a-k+1]. \tag{5.48}$$

By Lemma 5.5.6, S also anti-stabilizes at k, so similarly,

$$R|_{[jbr+1,abr]} = \mathrm{Rect}^*(S^{(a-j)}) + jbr \quad \text{for all } j \in [k-1, a]. \tag{5.49}$$

Note that both sides of (5.49) are empty when $j = a$. Then, for all $j \in [k-1, a]$,

$$\begin{aligned} \mathrm{p}^{jbr}(R)\big|_{[1,(a-j)br]} &= \mathrm{Rect}(R|_{[jbr+1,abr]}) - jbr \qquad \text{by (5.44)} \\ &= \mathrm{Rect}(\mathrm{Rect}^*(S^{(a-j)}) + jbr) - jbr \qquad \text{by (5.49)} \\ &= \mathrm{Rect}(S^{(a-j)}) \qquad \text{as rectification is well-defined} \\ &= R|_{[1,(a-j)br]} \qquad \text{by (5.48).} \end{aligned} \tag{5.50}$$

Similarly, for all $j \in [1, a-k+1]$,

$$\begin{aligned} \mathrm{p}^{jbr}(R)\big|_{[(a-j)br+1,abr]} &= \mathrm{Rect}^*(R|_{[1,jbr]}) + (a-j)br \qquad \text{by (5.45)} \\ &= \mathrm{Rect}^*(\mathrm{Rect}(S^{(j)})) + (a-j)br \qquad \text{by (5.48)} \\ &= \mathrm{Rect}^*(S^{(j)}) + (a-j)br \quad \text{as anti-rectification is well-defined} \\ &= R|_{[(a-j)br+1,abr]} \qquad \text{by (5.49).} \end{aligned} \tag{5.51}$$

Putting (5.50) and (5.51) together,

$$\mathrm{p}^{jbr}(R) = R \ \text{ for all } \ j \in [k-1, a-k+1].$$

In particular, because $a \geq 2k-1$, we have $k-1, k \in [k-1, a-k+1]$, so

$$\mathrm{p}^{br}(R) = \mathrm{p}^{kbr-(k-1)br}(R) = \mathrm{p}^{kbr}(\mathrm{p}^{-(k-1)br}(R)) = \mathrm{p}^{kbr}(R) = R.$$

□

Notation 5.6.5. Fix $b, r \in \mathbb{Z}_{\geq 1}$, and let β the block anti-diagonal skew tableaux

$$\beta = \underbrace{(r) \cup (r) \cup \cdots \cup (r)}_{b \text{ times}}.$$

Although Theorem 5.6.4 holds for any standard skew tableau S with row vector (r^b), any rectangular tableau of the form $R_a(S)$ for some standard skew tableaux S with row vector (r^b) is actually of the form $R_a(S')$ for some $S' \in \mathrm{SYT}(\beta)$. This is because any standard skew tableau S with row vector (r^b) is row shift equivalent to one of shape β by performing horizontal slides, and $\mathrm{Rect}(S^{(k)}) = \mathrm{Rect}((S')^{(k)})$ whenever $S \sim S'$, Lemma 5.3.5.

Corollary 5.6.6. For all $a \in \mathbb{Z}_{\geq 1}$,

$$\left\{ R_a(S) : S \in \mathrm{SYT}(\beta), \mathrm{stab}(S) \leq \frac{a+1}{2} \right\} \subseteq \mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}. \tag{5.52}$$

Proof. For fixed $S \in \mathrm{SYT}(\beta)$, $R_a(S)$ is defined for $a \geq 2\,\mathrm{stab}(S) - 1$. Thus, for fixed $a \in \mathbb{Z}_{\geq 1}$ and all $S \in \mathrm{SYT}(\beta)$ with $\mathrm{stab}(S) \leq \frac{a+1}{2}$, $R_a(S)$ is defined and lies in $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$ by Theorem 5.6.4.

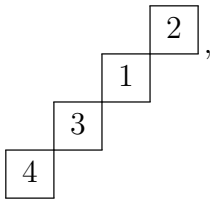
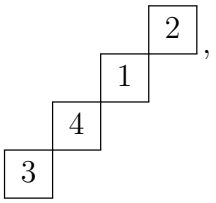
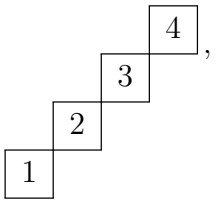
It remains to show the elements of $\{R_a(S) : S \in \mathrm{SYT}(\beta)\}$ are distinct. Suppose $R_a(S) = R_a(S')$ for $S, S' \in \mathrm{SYT}(\beta)$. Letting $k = \mathrm{stab}(S)$, $\mathrm{rv}(R_a(S)|_{[(k-1)br+1,kbr]}) = (r^b)$ and hence $\mathrm{rv}(R_a(S')|_{[(k-1)br+1,kbr]}) = (r^b)$ as well. Therefore, by Definition 5.6.2,

$$S \sim R_a(S)|_{[(k-1)br+1,kbr]} = R_a(S')|_{[(k-1)br+1,kbr]} \sim S',$$

forcing $S = S'$ because $S, S' \in \mathrm{SYT}(\beta)$.

□

Example 5.6.7. Consider

$S =$  , $T =$  , $U =$  ,

which have

$$\mathrm{stab}(S) = 2, \qquad \mathrm{stab}(T) = 3, \qquad \mathrm{stab}(U) = 4.$$

Thus, the smallest $R_a(S), R_{a'}(T), R_{a''}(U)$ we can construct with Definition 5.6.2 are

$$R_3(S) = \begin{array}{|c|c|c|}\hline 1&2&6\\\hline 3&5&10\\\hline 4&7&11\\\hline 8&9&12\\\hline\end{array} \in \mathrm{SYT}(3^4)^{\mathrm{p}^4},$$

$$R_5(T) = \begin{array}{|c|c|c|c|c|}\hline 1&2&5&6&10\\\hline 3&4&9&13&14\\\hline 7&8&12&17&18\\\hline 11&15&16&19&20\\\hline\end{array} \in \mathrm{SYT}(5^4)^{\mathrm{p}^4},$$

$$R_7(U) = \begin{array}{|c|c|c|c|c|c|c|}\hline 1&2&3&4&8&12&16\\\hline 5&6&7&11&15&19&20\\\hline 9&10&14&18&22&23&24\\\hline 13&17&21&25&26&27&28\\\hline\end{array} \in \mathrm{SYT}(7^4)^{\mathrm{p}^4}.$$

On the other hand, neither

$$V = \begin{array}{|c|c|c|}\hline 1&3&4\\\hline 2&5&8\\\hline 6&7&9\\\hline 10&11&12\\\hline\end{array} \in \mathrm{SYT}(3^4)^{\mathrm{p}^4}, \qquad \text{nor} \qquad W = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&4&8\\\hline 5&6&7&11&12\\\hline 9&10&14&15&16\\\hline 13&17&18&19&20\\\hline\end{array} \in \mathrm{SYT}(5^4)^{\mathrm{p}^4}$$

are of the form $R_a(S)$ for $S \in \mathrm{SYT}((1)\cup(1)\cup(1)\cup(1))$ because $V|_{[4(j-1)+1,4j]}$ and $W|_{[4(j-1)+1,4j]}$

never use all 4 rows for any value of j.

The containment in Corollary 5.6.6 can be strict in general, as illustrtated in Example 5.6.7. However, equality in Corollary 5.6.6 holds for $a \geq 2b-1$, when $\mathrm{stab}(S) \leq \frac{a+1}{2}$ holds for all $S \in \mathrm{SYT}(\beta)$ by Theorem 5.1.4. Thus, Corollary 5.6.6 specializes to

$$\{R_a(S) : S \in \mathrm{SYT}(\beta)\} \subseteq \mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}} \qquad \text{when } a \geq 2b-1. \tag{5.53}$$

The fact that equality holds in (5.53) is the content of Theorem 5.1.12.

Remark 5.6.8. In Theorem 5.1.12, $R_a(S)$ is defined by row concatenating shifted copies of $\mathrm{Rect}(S^{(b-1)})$, $S^{(a-2b+2)}$, and $\mathrm{Rect}^*(S^{(b-1)})$ instead of shifted copies of $\mathrm{Rect}(S^{(k-1)})$, $S^{(a-2k+2)}$, and $\mathrm{Rect}^*(S^{(k-1)})$ where $k = \mathrm{stab}(S)$ as in Definition 5.6.2. These 2 constructions agree by the definition of tableau stabilization, Definition 5.1.1, and because $\mathrm{stab}(S) \leq b$ for $S \in \mathrm{SYT}(\beta)$, Theorem 5.1.4.

Proof of Theorem 5.1.12. Fix a positive integer $a \geq 2b-1$. Equality in (5.53) will follow from both sides having the same size. By Lemma 5.2.19, we have the quotient

$$Q_e((er)^b) = \underbrace{(r) \cup (r) \cup \cdots \cup (r)}_{b \text{ times}} = \beta \text{ whenever } e \geq b. \tag{5.54}$$

Thus, $Q_a((ar)^b) = \beta$ because $a \geq 2b-1 \geq b$. By Corollary 5.1.10 and (5.54),

$$\#\,\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}} = \#\,\mathrm{SYT}(Q_a((ar)^b)) = \#\,\mathrm{SYT}\,(\beta) = \#\,\{R_a(S) : S \in \mathrm{SYT}(\beta)\}\,,$$

which shows both sides of (5.53) have the same size. Therefore,

$$\{R_a(S) : S \in \mathrm{SYT}(\beta)\} = \mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}.$$

Finally, we can choose $S \in \mathrm{SYT}(\beta)$ by choosing the sets of r elements that go in each of b

rows, which determines S uniquely since each row of S is increasing. Thus,

$$\#\,\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}} = \binom{br}{r,\ldots,r}.$$

□

Not only can we describe the elements of $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$ for $a \geq 2b-1$, but we can also describe the action of promotion on $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$, which is closed under promotion. In fact, using the definition of promotion for skew shapes - Definition 5.2.9, promotion commutes with the R_a operator:

$$\mathrm{p}(R_a(S)) = R_a(\mathrm{p}(S)) \text{ for all } S \in \mathrm{SYT}(\beta),$$

as in Corollary 5.1.13.

Proof of Corollary 5.1.13. For any $T \in \mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$, we have

$$\mathrm{p}^{br}(\mathrm{p}(T)) = \mathrm{p}(\mathrm{p}^{br}(T)) = \mathrm{p}(T), \quad \text{implying} \quad \mathrm{p}(T) \in \mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}.$$

Thus, $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$ is closed under promotion. So, fixing $S \in \mathrm{SYT}(\beta)$,

$$\mathrm{p}(R_a(S)) = R_a(S') \qquad \text{for some } S' \in \mathrm{SYT}(\beta)$$

by Theorem 5.1.12. It suffices to show $S' = \mathrm{p}(S)$. By the well-defined-ness of rectification, we get the same result whether we perform an inner slide at 1's cell before or after rectification, so

$$\mathrm{p}(S^{(b)})\big|_{[1,b^2r-1]} = \mathrm{Rect}(S^{(b)}\big|_{[2,b^2r]}) - 1. \tag{5.55}$$

Also, since $S \in \mathrm{SYT}(\beta)$, the inner slide starting at 1's cell in $S^{(b)}$ is horizontal, so

$$\mathrm{p}(S^{(b)}) = \mathrm{p}(S)^{(b)}. \tag{5.56}$$

Then,

$$\begin{aligned} R_a(S')|_{[1,b^2r-1]} &= \mathrm{p}(R_a(S))|_{[1,b^2r-1]} \\ &= \mathrm{Rect}(R_a(S)|_{[2,b^2r]}) - 1 \qquad \text{by Definition 5.2.9} \\ &= \mathrm{Rect}(S^{(b)}\big|_{[2,b^2r]}) - 1 \qquad \text{by Definition 5.6.2} \\ &= \mathrm{Rect}(\mathrm{p}(S^{(b)})\big|_{[1,b^2r-1]}) \qquad \text{by (5.55)} \\ &= \mathrm{Rect}(\mathrm{p}(S)^{(b)}\big|_{[1,b^2r-1]}) \qquad \text{by (5.56)} \\ &= R_a(p(S))|_{[1,b^2r-1]} \qquad \text{by Definition 5.6.2.} \end{aligned} \tag{5.57}$$

By Theorem 5.1.4, $R_a(S')|_{[(b-1)br+1,b^2r]}$ and $R_a(\mathrm{p}(S))|_{[(b-1)br+1,b^2r]}$ each have row vector (r^b), which together with (5.57), forces b^2r to be in the same cell in $R_a(S')$ and $R_a(\mathrm{p}(S))$. Combining this with (5.57) again,

$$R_a(S')|_{[1,b^2r]} = R_a(\mathrm{p}(S))|_{[1,b^2r]}.$$

In particular,

$$S' \sim R_a(S')|_{[(b-1)br+1,b^2r]} = R_a(\mathrm{p}(S))|_{[(b-1)br+1,b^2r]} \sim \mathrm{p}(S),$$

forcing $S' = \mathrm{p}(S)$ since $S', \mathrm{p}(S) \in \mathrm{SYT}(\beta)$. □

Corollary 5.6.9. Let $\lambda = ((b+1)r, br, \ldots, r)$ and $\mu = (br, (b-1)r, \ldots, 0)$. Then, for all integers $a \geq 2b-1$, $\nu \vdash br$, and $T_0 \in \mathrm{SYT}(\nu)$,

$$\#\{T \in \mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}} : T|_{[1,br]} = T_0\} = c^{\lambda}_{\mu,\nu},$$

the Littlewood-Richardson coefficient from Definition 5.2.8.

Proof. Note that $\beta = \lambda/\mu$. By Theorem 5.1.12 and $R_a(S)|_{[1,br]} = \mathrm{Rect}(S)$, we have

$$\begin{aligned}\#\{T \in \mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}} : T|_{[1,br]} = T_0\} &= \#\left\{R_a(S) : S \in \mathrm{SYT}(\beta),\, R_a(S)|_{[1,br]} = T_0\right\} \\ &= \#\left\{R_a(S) : S \in \mathrm{SYT}(\beta), \mathrm{Rect}(S) = T_0\right\} \\ &= \#\left\{S \in \mathrm{SYT}(\lambda/\mu) : \mathrm{Rect}(S) = T_0\right\} \\ &= c^{\lambda}_{\mu,\nu}.\end{aligned}$$

by Definition 5.2.8. □

5.7 Other Tableaux Fixed by Promotion Powers

In this section, we construct the tableaux in $\mathrm{SYT}((2r)^b)^{\mathrm{p}^{br}}$, proving Theorem 5.1.14, with a similar construction to Theorem 5.1.12. Then, we describe the action of promotion on $\mathrm{SYT}((2r)^b)^{\mathrm{p}^{br}}$ in Corollary 5.7.4. Such a construction and promotion action was already known by Dennis White [Whi06]. Next, we characterize the block diagonal skew tableaux fixed by a power of promotion in terms of tableaux of straight shape fixed by certain powers of promotion, Theorem 5.7.7, inspired by White [Whi06]. This has consequences for describing the tableaux in $\mathrm{SYT}((2r)^b)^{\mathrm{p}^{br/k}}$ in terms of rectangular tableaux fixed by smaller promotion powers, Corollary 5.7.9.

We next describe the tableaux in $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$ when $a = 2$. Using Lemma 5.2.19 and that $\lceil \frac{b}{2} \rceil = \lfloor \frac{b}{2} \rfloor$ when b is even, we have the quotient

$$\gamma = Q_2((2r)^b) = (r^{\lceil \frac{b}{2} \rceil}) \cup (r^{\lfloor \frac{b}{2} \rfloor}). \tag{5.58}$$

Therefore, by Corollary 5.1.10,

$$\#\,\mathrm{SYT}((2r)^b)^{\mathrm{p}^{br}} = \#\,\mathrm{SYT}\left((r^{\lceil \frac{b}{2} \rceil}) \cup (r^{\lfloor \frac{b}{2} \rfloor})\right) = \#\,\mathrm{SYT}(\gamma). \tag{5.59}$$

Definition 5.7.1. Fix $b, r \in \mathbb{Z}_{\geq 1}$ and let $\gamma = (r^{\lceil \frac{b}{2} \rceil}) \cup (r^{\lfloor \frac{b}{2} \rfloor})$. For $S \in \mathrm{SYT}(\gamma)$, let $R_2(S)$

be the filling formed by row-concatenating $\mathrm{Rect}(S)$ and $\mathrm{Rect}(S^*) + br$ together from left to right.

Example 5.7.2. Suppose $b = 4, r = 2$, and choose

$$S = \begin{array}{|c|c|}\hline 2 & 5 \\ \hline 6 & 8 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 & 7 \\ \hline \end{array} \in \mathrm{SYT}((2^2) \cup (2^2)).$$

Then,

$$\mathrm{Rect}(S) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 8 \\ \hline 3 & 6 \\ \cline{1-2} 4 & 7 \\ \cline{1-2} \end{array}, \qquad \mathrm{Rect}^*(S) = \begin{array}{|c|c|c|c|}\hline & & 2 & 5 \\ \hline & & 3 & 6 \\ \hline 1 & 4 & 7 & 8 \\ \hline \end{array},$$

$$R_2(S) = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 8 \\ \hline 3 & 6 & 10 & 13 \\ \hline 4 & 7 & 11 & 14 \\ \hline 9 & 12 & 15 & 16 \\ \hline \end{array}.$$

We use the name $R_2(S)$ because the construction is similar to the definition of $R_a(S)$ if we set $a = 2$. Note that we cannot just set $a = 2$ in Definition 5.6.2 because Definition 5.6.2 requires $a \geq 2\,\mathrm{stab}(S) - 1$. Similarly to Theorem 5.6.4, we will show $R_2(S)$ has shape $((2r)^b)$, is a standard tableau, and is fixed by p^{br}.

Lemma 5.7.3. The map

$$\mathrm{Rect} : \mathrm{SYT}(\gamma) \to \bigcup_{\substack{\rho=(\rho_1,\ldots,\rho_b) \vdash br, \\ \text{s.t. } \rho_j + \rho_{b+1-j} = 2r}} \mathrm{SYT}(\rho)$$

is bijective.

Proof. Say $\gamma = \lambda/\mu$ as a skew shape. By [Sta86, Lemma 3.3], we have the Littlewood-

Richardson coefficient

$$c^{\lambda}_{\mu,\rho} = \begin{cases} 1, & \text{if } \rho_j + \rho_{b+1-j} = 2r \text{ for all } j \in [1,b], \\ 0, & \text{else,} \end{cases} \tag{5.60}$$

for all $\rho = (\rho_1, \ldots, \rho_b) \vdash br$. Lemma 5.7.3 follows immediately from (5.60) and Definition 5.2.8.

□

Proof of Theorem 5.1.14. [Whi06] [Rhe12] Fix $S \in \mathrm{SYT}(\gamma)$. For all $j = 1, \ldots, b$ and

$$\begin{aligned} \mathrm{rv}(R_2(S)) &= \mathrm{rv}(\mathrm{Rect}(S))_j + \mathrm{rv}(\mathrm{Rect}^*(S))_j \\ &= \mathrm{rv}(\mathrm{Rect}(S))_j + \mathrm{rv}(e(\mathrm{Rect}(S))^{\dagger})_j \qquad \text{by Lemma 5.5.5} \\ &= \mathrm{rv}(\mathrm{Rect}(S))_j + \mathrm{rv}(\mathrm{Rect}(S))_{b+1-j} \\ &= 2r \qquad \text{by Lemma 5.7.3} \end{aligned} \quad .$$

Indeed, $R_2(S)$ is a filling of shape $((2r)^b)$. The fillings $R_2(S)|_{[1,br]} = \mathrm{Rect}(S)$ and $R_2(S)|_{[br+1,2br]} = \mathrm{Rect}^*(S) + br$ are skew tableaux using the entries $[1, br]$ and $[br+1, 2br]$, repsectively. Thus, $R_2(S)$, their row-concatenation from left to right, is a standard tableau. Next,

$$\begin{aligned} \mathrm{p}^{br}(R_2(S))\big|_{[1,br]} &= \mathrm{Rect}(\mathrm{Rect}^*(S) + br) - br \qquad \text{by (5.44)} \\ &= \mathrm{Rect}(S) \qquad \text{as rectification is well-defined} \\ &= R_2(S)|_{[1,br]} \qquad \text{by Definition 5.7.1 ,} \end{aligned} \tag{5.61}$$

and

$$\begin{aligned} \mathrm{p}^{br}(R_2(S))\big|_{[br+1,2br]} &= \mathrm{Rect}^*(\mathrm{Rect}(S)) + br \qquad \text{by (5.45)} \\ &= \mathrm{Rect}^*(S) + br \qquad \text{as anti-rectification is well-defined} \\ &= R_2(S)|_{[br+1,2br]} \qquad \text{by Definition 5.7.1 .} \end{aligned} \tag{5.62}$$

Putting (5.61) and (5.62) together gives $\mathrm{p}^{br}(R_2(S)) = R_2(S)$. Thus,

$$\{R_2(S) : S \in \mathrm{SYT}(\gamma)\} \subseteq \mathrm{SYT}((2r)^b)^{\mathrm{p}^{br}}. \tag{5.63}$$

The fact that $\{R_2(S) : S \in \mathrm{SYT}(\gamma)\}$ is a set and not a multiset is a consequence of Lemma 5.7.3. By (5.59), both sides of (5.63) have the same size, so

$$\mathrm{SYT}((2r)^b)^{\mathrm{p}^{br}} = \{R_2(S) : S \in \mathrm{SYT}(\gamma)\}.$$

Finally, we count these tableaux by first partitioning $\{1, 2, \ldots, br\}$ into two blocks of size $r \cdot \lceil \frac{b}{2} \rceil$ and $r \cdot \lfloor \frac{b}{2} \rfloor$. Then, we choose a filling of $(r^{\lceil \frac{b}{2} \rceil})$ with the first block and a filling of $(r^{\lfloor \frac{b}{2} \rfloor})$ with the second block so that both fillings have increasing rows and columns. This yields

$$\#\,\mathrm{SYT}((2r)^b)^{\mathrm{p}^{br}} = \binom{br}{\lfloor \frac{b}{2} \rfloor\, r} \#\,\mathrm{SYT}(r^{\lceil \frac{b}{2} \rceil}) \cdot \#\,\mathrm{SYT}(r^{\lfloor \frac{b}{2} \rfloor}).$$

□

We also describe the action of promotion on $\mathrm{SYT}((2r)^b)^{\mathrm{p}^{br}}$, which is closed under promotion. Like R_a in Corollary 5.1.13, the R_2 operator also commutes with promotion.

Corollary 5.7.4. [Whi06] [Rhe12] For any $S \in \mathrm{SYT}(\gamma)$,

$$\mathrm{p}(R_2(S)) = R_2(\mathrm{p}(S)).$$

Proof. Similar to Corollary 5.1.13, $\mathrm{SYT}((2r)^b)^{\mathrm{p}^{br}}$ is closed under promotion. Thus, fixing $S \in \mathrm{SYT}(\gamma)$,

$$p(R_2(S)) = p(R_2(S')) \qquad \text{for some } S' \in \mathrm{SYT}(\gamma).$$

Our reasoning from the proof of Corollary 5.1.13 still holds through (5.57), so

$$R_2(S')|_{[1,b^2r-1]} = R_2(p(S))|_{[1,br-1]}\,. \tag{5.64}$$

By Lemma 5.7.3, $R_2(S')|_{[1,br]}$ and $R_2(\mathrm{p}(S))|_{[(b-1)br+1,b^2r]}$ each have shapes $\rho = (\rho_1, \ldots, \rho_b)$ satisfying $\rho_j + \rho_{b+1-j} = 2r$, which together with (5.64), forces br to be in the same cell in $R_2(S')$ and $R_2(\mathrm{p}(S))$. Thus,

$$R_2(S')|_{[1,br]} = R_2(\mathrm{p}(S))|_{[1,br]}\,. \tag{5.65}$$

Hence,

$$\mathrm{Rect}(S') = R_2(S')|_{[1,br]} = R_2(\mathrm{p}(S))|_{[1,br]} = \mathrm{Rect}(S).$$

Finally, Lemma 5.7.3 forces $S' = \mathrm{p}(S)$. □

For any $k \mid br$,

$$\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br/k}} \subseteq \mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}.$$

By Corollary 5.1.13 and Corollary 5.7.4, we have, for $a = 2$ or $a \geq 2b-1$,

$$\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br/k}} = \{R_a(S) : S \in \mathrm{SYT}(Q_a((ar)^b), \mathrm{p}^{br/k}(S) = S\}. \tag{5.66}$$

For $a = 2$, we will describe $\{S \in \mathrm{SYT}(\gamma) : \mathrm{p}^{br/k}(S) = S\}$ in terms of rectangular tableaux fixed by smaller promotion powers.

Definition 5.7.5. For a standard tableau T of size n, $m \in \mathbb{Z}_{\geq 1}$, and $A = \{a_1, \ldots, a_k\} \subset [m]$ where $k \mid n$, let $I_{m,A}(T)$ denote the skew tableau obtained from T by replacing $ik + j$ by $im + a_j$ for all $i = 0, 1, \ldots, n/k - 1$ and $j = 1, \ldots, k$. For example, if

$$T = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 5 & 8 \\ \hline 3 & 6 & 9 & 11 \\ \hline 4 & 7 & 10 & 12 \\ \hline \end{array},$$

then

$$I_{6,\{2,4,5\}}(T) = \begin{array}{|c|c|c|c|}\hline 2 & 4 & 10 & 16 \\ \hline 5 & 11 & 17 & 22 \\ \hline 8 & 14 & 20 & 23 \\ \hline \end{array}.$$

Definition 5.7.6. On a tableau with entries $i_1 < \cdots < i_p$, let promotion act on the indices as it would on a standard tableau. For example, since

$$\mathrm{p}: \begin{array}{|c|c|c|c|}\hline 1&3&5&6\\\hline 2&4&8&11\\\hline 7&9&10&12\\\hline\end{array} \mapsto \begin{array}{|c|c|c|c|}\hline 1&2&4&5\\\hline 3&7&9&10\\\hline 6&8&11&12\\\hline\end{array},$$

for any $i_1 < \cdots < i_{12}$, we have

$$\mathrm{p}: \begin{array}{|c|c|c|c|}\hline i_1&i_3&i_5&i_6\\\hline i_2&i_4&i_8&i_{11}\\\hline i_7&i_9&i_{10}&i_{12}\\\hline\end{array} \mapsto \begin{array}{|c|c|c|c|}\hline i_1&i_2&i_4&i_5\\\hline i_3&i_7&i_9&i_{10}\\\hline i_6&i_8&i_{11}&i_{12}\\\hline\end{array}.$$

We realize there are other variations on how promotion acts on tableaux with distinct non-standard entries, such as in [Rho10a], but this will prove convenient for our purposes.

Theorem 5.7.7. (inspired by [Whi06]) For any partitions $\lambda^{(1)}, \ldots, \lambda^{(b)}$ with total size n, and $k \mid n$,

$$\begin{aligned}\mathrm{SYT}(\lambda^{(1)} \cup \cdots \cup \lambda^{(b)})^{\mathrm{p}^{n/k}} = \big\{ & I_{n/k,M_1}(T_1) \cup \cdots \cup I_{n/k,M_b}(T_b) : \\ & T_j \in \mathrm{SYT}(\lambda^{(j)})^{\mathrm{p}^{|\lambda^{(j)}|/k}}, (M_1, \ldots, M_b) \in \binom{[n/k]}{|\lambda^{(1)}|/k, \ldots, |\lambda^{(b)}|/k} \big\},\end{aligned} \tag{5.67}$$

if $k \mid \gcd(|\lambda^{(1)}|, \ldots, |\lambda^{(b)}|)$, where $\cup$ denotes anti-diagonal concatenation. Else,

$$\mathrm{SYT}(\lambda^{(1)} \cup \cdots \cup \lambda^{(b)})^{\mathrm{p}^{n/k}} = \varnothing.$$

Example 5.7.8. Suppose $r = 2, b = 5$. Pick $k = 2$,

$$T_1 = \begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4&6\\\hline\end{array} \in \mathrm{SYT}(3,3)^{\mathrm{p}^3}, \quad T_2 = \begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array} \in \mathrm{SYT}(2,2)^{\mathrm{p}^2},$$

$$(M_1, M_2) = (\{1,4,5\}, \{2,3\}) \in \binom{[5]}{3,2}.$$

Then,

$$I_{5,M_1}(T_1) = \begin{array}{|c|c|c|}\hline 1 & 4 & 9 \\ \hline 5 & 6 & 10 \\ \hline \end{array}, \quad I_{5,M_2}(T_2) = \begin{array}{|c|c|}\hline 2 & 7 \\ \hline 3 & 8 \\ \hline \end{array},$$

$$I_{5,M_1}(T_1) \cup I_{5,M_2}(T_2) = \begin{array}{|c|c|c|}\hline & & 2 & 7 \\ \cline{3-4} & & 3 & 8 \\ \cline{3-4} \hline 1 & 4 & 9 \\ \cline{1-3} 5 & 6 & 10 \\ \cline{1-3} \end{array} \in \mathrm{SYT}((3,3) \cup (2,2))^{\mathrm{p}^5}.$$

Proof. Suppose $\lambda^{(1)}, \ldots, \lambda^{(b)}$ are partitions with total size n and $k \mid \gcd(|\lambda^{(1)}|, \ldots, |\lambda^{(b)}|)$. Let

$$m = \frac{n}{k}, \qquad m_j = \frac{|\lambda^{(j)}|}{k} \quad \text{for all } j = 1, \ldots, b.$$

Consider $T_j \in \mathrm{SYT}(\lambda^{(j)})^{\mathrm{p}^{m_j}}$ for $j = 1, \ldots, b$ and $(M_1, \ldots, M_b) \in \binom{m}{m_1, \ldots, m_b}$. Let

$$S_j = I_{m,M_j}(T_j) \qquad \text{for all } j = 1, \ldots, b. \tag{5.68}$$

and

$$S = S_1 \cup \cdots \cup S_b.$$

Also, write

$$\mathrm{p}^m(S) = \mathrm{p}^m(S)_1 \cup \cdots \cup \mathrm{p}^m(S)_b. \tag{5.69}$$

For each j, the entries in S_j are $\cup_{i=1}^{k}(M_j + (i-1)m)$. Thus, since p^m performs a total of m decrements modulo n, the entries in $\mathrm{p}^m(S)_j$ are

$$\bigcup_{\ell=1}^{k-1}(M_j + (\ell-1)m) \bigcup (M_j + n - m) = \bigcup_{\ell=1}^{k}(M_j + (\ell-1)m).$$

Thus, the same set of entries appears in S_j and $\mathrm{p}^m(S_j)$. By this and the fact that m_j of the

m promotions on S slide through S_j,

$$\mathrm{p}^m(S)_j = \mathrm{p}^{m_j}(S_j). \tag{5.70}$$

Now,

$$\begin{aligned} \mathrm{p}^{m_j}(S_j) &= \mathrm{p}^{m_j}(I_{m,M_j}(T_j)) \qquad \text{by (5.68)} \\ &= I_{m,M_j}(\mathrm{p}^{m_j}(T_j)) \qquad \text{by Definition 5.7.5} \\ &= I_{m,M_j}(T_j) \qquad \text{because } T_j \in \mathrm{SYT}(\lambda^{(j)})^{\mathrm{p}^{m_j}} \\ &= S_j \qquad \text{by (5.68)} \end{aligned} \tag{5.71}$$

for all $j = 1, \ldots, b$. Combining (5.69), (5.70), and (5.71) yields $\mathrm{p}^m(S) = S$, so $\supseteq$ holds in (5.67).

On the other hand, suppose $S \in \mathrm{SYT}(\lambda^{(1)} \cup \cdots \cup \lambda^{(b)})^{\mathrm{p}^m}$. Let $S = S_1 \cup \cdots \cup S_b$, M_j denote the entries in S_j that lie in $[m]$, $m_j = \#M_j$. Since $\mathrm{p}^m(S) = S$, the set of entries in S_j must be fixed by m decrements modulo n. When applying p^m to S, we apply m_j inner slides and m decrements modulo n to S_j, so letting $\{i_1, \ldots, i_{|\lambda^{(j)}|}\}$ denote the set of entries of S_j in increasing order, we must have

$$i_{s+m_j} \equiv_n i_s + m \qquad \text{for all } s = 1, \ldots, |\lambda^{(j)}|. \tag{5.72}$$

Equation (5.72) means that shifting the indices by m_j modulo m corresponds to shifting the entries by m modulo n. This means $|\lambda^{(j)}| = k_j m_j$ for some $k_j \in \mathbb{Z}_{\geq 1}$, $M_j = \{i_1, \ldots, i_{m_j}\}$, and that the set of entries in S_j is

$$\bigcup_{\ell=1}^{k_j} (M_j + (\ell - 1)m).$$

As $S \in \mathrm{SYT}(\lambda^{(1)} \cup \cdots \cup \lambda^{(b)})$, S must use the entries $1, 2, \ldots, n$, which forces

$$k_1 = \cdots = k_b \coloneqq k.$$

Now we have

$$k \mid \gcd(|\lambda^{(1)}|, \dots, |\lambda^{(b)}|), \quad m = \frac{n}{k}, \quad m_j = \frac{|\lambda^{(j)}|}{k} \text{ for all } j = 1, \dots b.$$

Hence, $\mathrm{SYT}(\lambda^{(1)} \cup \cdots \cup \lambda^{(b)})^{\mathrm{p}^{n/k}} = \varnothing$ unless $k \mid \gcd(|\lambda^{(1)}|, \dots, |\lambda^{(b)}|)$.

Now, assume that $k \mid \gcd(|\lambda^{(1)}|, \dots, |\lambda^{(b)}|)$. Since the set of entries in S_j is $\bigcup_{\ell=1}^{k_j}(M_j + (\ell-1)m)$, we can write $S_j = I_{m,M_j}(T_j)$ for some $T_j \in \mathrm{SYT}(\lambda^{(j)})$. Then,

$$\begin{aligned}
&\mathrm{p}^m(S) = S \\
\implies &\mathrm{p}^{m_j}(S_j) = S_j \qquad \text{by (5.69) and (5.70)} \\
\implies &\mathrm{p}^{m_j}(I_{m,M_j}(T_j)) = I_{m,M_j}(T_j) \\
\implies &I_{m,M_j}(\mathrm{p}^{m_j}(T_j)) = I_{m,M_j}(T_j) \qquad \text{by Definition 5.7.5} \\
\implies &\mathrm{p}^{m_j}(T_j) = T_j
\end{aligned}$$

for all $j = 1, \dots, b$. This proves $\subseteq$ holds in (5.67) and proves Theorem 5.7.7.

□

Corollary 5.7.9. For any $r, b \in \mathbb{Z}_{\geq 1}$, and $k \mid br$,

$$\begin{aligned}
\mathrm{SYT}((2r)^b)^{\mathrm{p}^{br/k}} = \Big\{ & R_2(I_{br/k,M_1}(T_1) \cup I_{br/k,M_2}(T_2)) : T_1 \in \mathrm{SYT}\left(r^{\lceil \frac{b}{2} \rceil}\right)^{\mathrm{p}^{r\lfloor \frac{b}{2} \rfloor/k}}, \\
& T_2 \in \mathrm{SYT}\left(r^{\lceil \frac{b}{2} \rceil}\right)^{\mathrm{p}^{r\lfloor \frac{b}{2} \rfloor/k}}, (M_1, M_2) \in \binom{[br/k]}{r\lceil \frac{b}{2} \rceil /k, r\lfloor \frac{b}{2} \rfloor /k} \Big\}
\end{aligned}$$

if $k \mid \gcd(r^{\lfloor \frac{b}{2} \rfloor}, r^{\lceil \frac{b}{2} \rceil})$. Else,

$$\mathrm{SYT}((2r)^b)^{\mathrm{p}^{br/k}} = \varnothing.$$

Example 5.7.10. Continuing Example 5.7.8,

$$R_2 \quad \begin{array}{|c|c|c|}\hline 1&4&9\\\hline 5&6&10\\\hline\end{array} \quad \begin{array}{|c|c|}\hline 2&7\\\hline 3&8\\\hline\end{array} = \begin{array}{|c|c|c|c|c|}\hline 1&2&3&8&13\\\hline 4&6&7&12&18\\\hline 5&9&11&14&19\\\hline 10&15&16&17&20\\\hline\end{array} \in \mathrm{SYT}(4^5)^{\mathrm{p}^5}.$$

Proof. By Theorem 5.1.14 and Corollary 5.7.4,

$$\mathrm{SYT}((2r)^b)^{\mathrm{p}^{br/k}} = \left\{ R_2(S) : S \in \mathrm{SYT}(r^{\lfloor \frac{b}{2} \rfloor} \cup r^{\lceil \frac{b}{2} \rceil})^{\mathrm{p}^{br/k}} \right\} \tag{5.73}$$

Then, Corollary 5.7.9 follows from (5.73) and Theorem 5.7.7. □

Remark 5.7.11. One can generalize Corollary 5.7.9 to give a similar description of $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br/k}}$ for $a \geq 2b-1$. However, such a generalization would not give us anything new. In order for $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br/k}}$ to be nonempty, we must have $b \mid \frac{br}{k}$ or $ar \mid \frac{br}{k}$ by (5.11). Since $ar \geq (2b-1)r \geq \frac{br}{k}$, $ar \not| \frac{br}{k}$, so

$$b \mid \frac{br}{k} \implies k \mid r.$$

Note also that $ak \geq a \geq 2b-1$. Then, by Theorem 5.1.12,

$$\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br/k}} = \mathrm{SYT}\left(\left(ak \cdot \frac{r}{k}\right)^b\right)^{\mathrm{p}^{b \cdot \frac{r}{k}}} = \left\{ R_{ak}(\tilde{S}) : \tilde{S} \in \mathrm{SYT}(\underbrace{(r/k) \cup \cdots \cup (r/k)}_{b \text{ times}}) \right\}.$$

These tableaux were already constructed in Theorem 5.1.12.

5.8 Stabilization as a Permutation Statistic

We can identify each permutation $w \in S_n$ with the unique skew tableau of shape $\underbrace{(1) \cup (1) \cup \cdots \cup (1)}_{n \text{ times}}$ whose reading word is w. Under this identification, we translate the stabilization statistic to the symmetric group. We realize stabilization is invariant on dual equivalence classes and is bounded below by the number of ascents of w plus 1. We characterize the words with

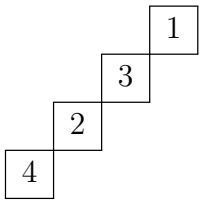
stabilization statistic 1 and 2 in terms of their recording tableau.

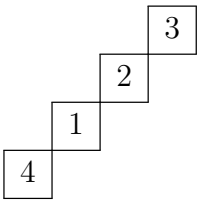
Definition 5.8.1. For $w \in S_n$, let $T(w)$ be the unique standard skew tableau of shape $\underbrace{(1) \cup (1) \cup \cdots \cup (1)}_{n \text{ times}}$ with reading word w, and define $\mathrm{stab}(w) \coloneqq \mathrm{stab}(T(w))$. For $w \in S_n$, let

$$\mathrm{asc}(w) = \#\{i : w_i < w_{i+1}\}$$

denote the number of ascents of w.

Example 5.8.2.

$\mathrm{stab}(4231) = \mathrm{stab}$  $= 3,$

$\mathrm{stab}(4123) =$  $= 3,$

$\mathrm{asc}(4231) = 1,$ $\mathrm{asc}(4123) = 2.$

Recall permutations $v, w \in S_n$ are equivalent if they differ by dual equivalence moves of the form

$$\ldots (i \pm 1) \ldots (i \mp 1) \ldots i \cdots \leftrightarrow \ldots i \ldots (i \mp 1) \ldots (i \pm 1) \ldots,$$

for any integer i. More importantly, recall from (5.7) that $v, w \in S_n$ are dual equivalent if and only if they have the same recording tableau, namely $Q(v) = Q(w)$.

Lemma 5.8.3. If $v, w \in S_n$ are dual equivalent, then $\mathrm{stab}(v) = \mathrm{stab}(w)$.

Proof. Since v, w are dual equivalent, $T(v)$ and $T(w)$ are dual equivalent [Hai92, Lemma 2.11]. It follows by Theorem 5.3.6 that

$$\mathrm{stab}(v) = \mathrm{stab}(T(v)) = \mathrm{stab}(T(w)) = \mathrm{stab}(w).$$

□

Lemma 5.8.4. For all $w \in S_n$, $\text{stab}(w) > \text{asc}(w)$.

Proof. Suppose $w = w_1 \dots w_n \in S_n$, and let $r = \text{stab}(w)$, so $T(w)$ stabilizes at r. In particular, this means $\text{Rect}(T(w)^{(r)})$ has n rows. The reading word of $T(w)^{(r)}$ is

$$w^{(r)} \coloneqq w_1 \, (w_1 + n) \, \dots \, (w_1 + (r-1)n) \, \dots \, w_n \, (w_n + n) \, \dots \, (w_n + (r-1)n), \tag{5.74}$$

so

$$\text{Rect}(T(w)^{(r)}) = P(w^{(r)}) \tag{5.75}$$

by Lemma 5.2.7. By Theorem 5.2.4, $w^{(r)}$ must have a decreasing subsequence of size n. Yet, at most one of $w_j, (w_j + n), \dots, (w_j + (r-1)n)$ can be in a decreasing subsequence of $w^{(r)}$. Thus, $w^{(r)}$ must have a decreasing subsequence of the form

$$(w_1 + (r-1)n) \dots (w_{i_1} + (r-1)n) \, (w_{i_1+1} + (r-2)n) \dots (w_{i_2} + (r-2)n) \dots (w_{i_{r-1}+1}) \dots (w_n)$$

for some $1 \leq i_1 \leq i_2 \cdots \leq i_{r-1} \leq n$. Since this subsequence is decreasing,

$$w_1 > \cdots > w_{i_1}, \quad w_{i_1+1} > \cdots > w_{i_2}, \quad \dots, \quad w_{i_{r-1}+1} > \cdots > w_n.$$

Therefore, w can only have ascents at possibly $i_1, \dots, i_{r-1}$, so w has at most $r-1$ ascents. Hence, $\text{stab}(w) > \text{asc}(w)$.

□

Lemma 5.8.3 and Lemma 5.8.4 give us some information about the distribution of stab on S_n, but what else can we say about this distribution? By Theorem 5.1.4, we know $1 \leq \text{stab}(w) \leq n$ for $w \in S_n$. In Figure 5.3, we give the distribution of stab on $S_1, \dots, S_8$. We have a formula for the number of permutations in S_n with stab 1 or stab 2. In fact, we characterize exactly which permutation has stab 1 and which permutations have stab 2 in terms of their recording tableau.

1
1 1
1 4 1
1 8 14 1
1 18 63 37 1
1 33 175 434 76 1
1 68 549 2345 1927 149 1
1 124 1787 7807 23760 6552 288 1

Figure 5.3: The distribution of stab on $S_1, \ldots, S_8$. The k-th entry from the left in row n is $\#\{w \in S_n : \mathrm{stab}(w) = k\}$.

Lemma 5.8.5. The permutation $w = n\,(n-1)\,\ldots\,2\,1$ is the only permutation with $\mathrm{stab}(w) = 1$.

Proof. First, if $w = n\,(n-1)\,\ldots\,2\,1$, then $\mathrm{Rect}(T(w)) = P(w)$ has a single column, so $\mathrm{stab}(w) = 1$. If $w \in S_n$ with $\mathrm{stab}(w) = 1$, then $P(w)$ consists of n rows, forcing $w = n\,(n-1)\,\ldots\,2\,1$.

□

Notation 5.8.6. Fix $n \in \mathbb{Z}_{\geq 1}$. For $k = 1, \ldots, n$, let

$$T_k = \begin{array}{|c|c|}\hline 1 & k{+}1 \\ \hline \vdots & \vdots \\ \hline k & 2k \\ \hline 2k{+}1 \\ \cline{1-1} \vdots \\ \cline{1-1} n \\ \cline{1-1}\end{array}, \qquad \text{for } k < \left\lfloor \frac{n}{2} \right\rfloor, \qquad T_k = \begin{array}{|c|c|}\hline 1 & k{+}1 \\ \hline \vdots & \vdots \\ \hline n{-}k & n \\ \hline n{-}k{+}1 \\ \cline{1-1} \vdots \\ \cline{1-1} k \\ \cline{1-1}\end{array}, \qquad \text{for } k \geq \left\lfloor \frac{n}{2} \right\rfloor.$$

This also means

$$T_k = P(k \ldots 1\, n \ldots (k+1)) = Q(k \ldots 1\, n \ldots (k+1)).$$

Theorem 5.8.7. For all $n \in \mathbb{Z}_{\geq 1}$,

$$\{w \in S_n : \mathrm{stab}(w) = 2\} = \{w \in S_n : Q(w) = T_k \text{ for some k}\}. \tag{5.76}$$

Consequently,

$$\#\{w \in S_n : \mathrm{stab}(w) = 2\} = \binom{n+1}{\lfloor \frac{n+1}{2} \rfloor} - 2. \tag{5.77}$$

See OEIS entry A201686.

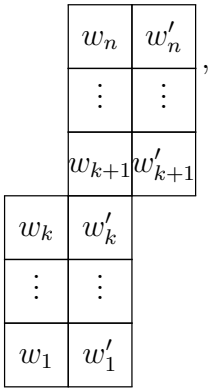
We break the proof of Theorem 5.8.7 into 3 steps. First, we use Lemma 5.8.4 to prove (5.76). Secondly, we calculate the number of standard tableaux of a given size with 1 or 2 columns. Thirdly, we use this result to prove (5.77).

Proof of (5.76). Suppose $w \in S_n$ has $\mathrm{stab}(w) = 2$. By Lemma 5.8.4, $\mathrm{asc}(w) \leq 1$, so we can write $w = w_1 \ldots w_k w_{k+1} \ldots w_n$ where

$$w_1 > \cdots > w_k, \quad w_{k+1} > \cdots > w_n$$

for some $k = 1, \ldots, n$. If $w_k > w_{k+1}$ as well, then $w = n\,(n-1)\,\ldots\,2\,1$, which has $\mathrm{stab}(w) = 1$, contradicting $\mathrm{stab}(w) = 2$. Thus, $w_k < w_{k+1}$. Since $\mathrm{stab}(w) = 2$, $\mathrm{Rect}(T(w)^{(2)})$ must have

$n+1, \ldots, 2n$ in distinct rows. Notice that we can perform inner slides to $T(w)^{(2)}$ to get



where $x' = x + n$ for all x. In order for $T(w)$ to stabilize at 2, $w'_1, \ldots, w'_n$ can only slide horizontally. Since $w_k < w_{k+1}$ and thus $w'_k < w'_{k+1}$, w'_k will slide vertically if the cell above it is vacated while it is in the second column. Since any entry above w'_k sliding into column 1 forces w'_k to slide up, w'_k must slide horizontally into column 1 before any of the entries above it do. Hence, $w_k, \ldots, w_1$ must all slide up either above w'_k or to the top before $w_n, \ldots, w_{k+1}$ experience any horizontal slides.

If $k \geq \left\lfloor \frac{n}{2} \right\rfloor$, then this means

$$P(w^{(2)}) = \mathrm{Rect}(T(w)^{(2)}) = \begin{array}{|c|c|c|}\hline w_k & w_n & w'_n \\ \hline \vdots & \vdots & \vdots \\ \hline w_{2k-n+1} & w_{k+1} & w'_{k+1} \\ \hline w_{2k-n} & w'_k \\ \cline{1-1} \vdots & \vdots \\ \cline{1-1} w_1 & w'_{n-k+1} \\ \cline{1-1} w'_{n-k} \\ \cline{1-1} \vdots \\ \cline{1-1} w'_1 \\ \cline{1-1} \end{array} \quad \Longrightarrow \quad P(w) = \begin{array}{|c|c|}\hline w_k & w_n \\ \hline \vdots & \vdots \\ \hline w_{2k-n+1} & w_{k+1} \\ \cline{1-1} w_{2k-n} \\ \cline{1-1} \vdots \\ \cline{1-1} w_1 \\ \cline{1-1} \end{array}.$$

Since $w = w_1 \dots w_k w_{k+1} \dots w_n$, this means during RSK, the whole first column of w was created first, followed by the second column. Thus,

$$Q(w) = \begin{array}{|c|c|}\hline 1 & k{+}1 \\ \hline \vdots & \vdots \\ \hline n{-}k & n \\ \cline{1-1} n{-}k{+}1 \\ \cline{1-1} \vdots \\ \cline{1-1} k \\ \cline{1-1} \end{array} = T_k.$$

If $k < \left\lfloor \frac{n}{2} \right\rfloor$, then instead we have

$$P(w^{(2)}) = \text{Rect}(T(w)^{(2)}) = \text{Rect}$$

	w_n	w'_n
	$\vdots$	$\vdots$
	w_{2k+1}	w'_{2k+1}
w_k	w_{2k}	w'_{2k}
$\vdots$	$\vdots$	$\vdots$
w_1	w_{k+1}	w'_{k+1}
w'_k		
$\vdots$		
w'_1		

In particular, based on the position of $w_1 \dots w_{2k}$, $\text{RSK}(w_1 \dots, w_{2k})$ began with a column of size k followed by a second column of size k. Thus,

$$P(w_1 \dots w_{2k}) = \begin{array}{|c|c|}\hline w_k & w_{2k} \\ \hline \vdots & \vdots \\ \hline w_1 & w_{k+1} \\ \hline \end{array}, \qquad Q(w_1 \dots w_{2k}) = \begin{array}{|c|c|}\hline 1 & k+1 \\ \hline \vdots & \vdots \\ \hline k & 2k \\ \hline \end{array}.$$

As w has a decreasing subsequence of size $n-k$, the first column of $Q(w)$ must have size at

least $n-k$. As $Q(w)$ contains $Q(w_1 \dots w_{2k})$, we must have

$$Q(w) = \begin{array}{|c|c|}\hline 1 & {\scriptstyle k+1} \\ \hline \vdots & \vdots \\ \hline k & 2k \\ \hline {\scriptstyle 2k+1} \\ \cline{1-1} \dots \\ \cline{1-1} n \\ \cline{1-1} \end{array} = T_k.$$

This shows that if $\mathrm{stab}(w) = 2$, then $Q(w) = T_k$ for some k, and hence

$$\{w \in S_n : \mathrm{stab}(w) = 2\} \subseteq \{w \in S_n : Q(w) = T_k \text{ for some k}\}. \tag{5.78}$$

In order to show the reverse containment of (5.78) and thus complete the proof of (5.76), suppose $Q(w) = T_k$ for some k. To show $\mathrm{stab}(w) = 2$, it suffices to verify $\mathrm{stab}(w) = 2$ for only 1 such word w in each dual equivalence class by Lemma 5.8.3. As the recording tableau characterizes dual equivalence classes and

$$Q(k \dots 1\, n \dots (k+1)) = T_k \quad \text{for all } k = 1, \dots n,$$

it suffices to check that

$$\mathrm{stab}(k \dots 1\, n \dots (k+1)) = 2 \quad \text{for all } k = 1, \dots n.$$

Let $w = k \dots 1\, n \dots (k{+}1)$ so that according to (5.74), $w^{(2)} = k\,(n{+}k) \dots 1\,(n{+}1)\,n\,2n \dots (k{+}$

1) $(n+k+1)$. Then,

$$P(w^{(2)}) = \begin{array}{|c|c|c|}\hline 1 & k{+}1 & n{+}k{+}1 \\ \hline \vdots & \vdots & \vdots \\ \hline k & 2k & n{+}2k \\ \hline 2k{+}1 & n{+}2k{+}1 \\ \cline{1-2} \vdots & \vdots \\ \cline{1-2} n & 2n \\ \cline{1-2} n{+}1 \\ \cline{1-1} \vdots \\ \cline{1-1} n{+}k \\ \cline{1-1} \end{array} \text{ if } k < \left\lfloor \frac{n}{2} \right\rfloor, \qquad P(w^{(2)}) = \begin{array}{|c|c|c|}\hline 1 & k{+}1 & n{+}k{+}1 \\ \hline \vdots & \vdots & \vdots \\ \hline n{-}k & n & 2n \\ \hline n{-}k{+}1 & n+1 \\ \cline{1-2} \vdots & \vdots \\ \cline{1-2} k & 2k \\ \cline{1-2} 2k{+}1 \\ \cline{1-1} \vdots \\ \cline{1-1} n{+}k \\ \cline{1-1} \end{array} \text{ if } k \geq \left\lfloor \frac{n}{2} \right\rfloor.$$

Either way, $\mathrm{stab}(w) = 2$ by (5.75), proving (5.76).

□

Lemma 5.8.8. The number of standard tableaux of size n with at most 2 columns is $\binom{n}{\lfloor n/2 \rfloor}$.

Proof. Because RSK: $S_n \to \cup_{\lambda \vdash n} \mathrm{SYT}(\lambda) \times \mathrm{SYT}(\lambda)$ is a bijection,

$$\#\{w \in S_n : Q(w) = T\} = \#\,\mathrm{SYT}(\mathrm{rv}(T)). \tag{5.79}$$

For all $k = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$, let

$$Q_k = \begin{array}{|c|c|}\hline 1 & \lfloor \frac{n}{2} \rfloor + 1 \\ \hline \vdots & \vdots \\ \hline k & \lfloor \frac{n}{2} \rfloor + k \\ \hline \vdots & \\ \hline \lfloor \frac{n}{2} \rfloor & \\ \hline \lfloor \frac{n}{2} \rfloor + k + 1 & \\ \hline \vdots & \\ \hline n & \\ \hline \end{array},$$

which has shape $(2^k, 1^{n-2k})$. Now, $\{Q_k : k = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ is the set of size n standard tableaux whose descent set is a subset of $\left\{ \lfloor \frac{n}{2} \rfloor \right\}$. Thus,

$$\begin{aligned}
\#\{\text{1 or 2 column standard tableaux of size } n\} &= \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \#\,\mathrm{SYT}(2^k, 1^{n-2k}) \\
&= \#\{w \in S_n : Q(w) = Q_k \text{ for some } k\} \qquad \text{by (5.79)} \\
&= \#\left\{ w \in S_n : \mathrm{Des}(Q(w)) \subset \left\{ \left\lfloor \frac{n}{2} \right\rfloor \right\} \right\} \\
&= \#\left\{ w \in S_n : \mathrm{Des}(w) \subset \left\{ \left\lfloor \frac{n}{2} \right\rfloor \right\} \right\} \qquad \text{by (??)} \\
&= \#\{w \in S_n : w_1 > \cdots > w_{\lfloor \frac{n}{2} \rfloor}, w_{\lfloor \frac{n}{2} \rfloor + 1} > \cdots > w_n\} \\
&= \binom{n}{\lfloor \frac{n}{2} \rfloor}
\end{aligned}$$

by choosing the set $\{w_1, \ldots, w_{\lfloor \frac{n}{2} \rfloor}\}$ out of $[n]$, which determines w uniquely.

□

Proof of (5.77). The set $\{T_k : k = 1, \ldots, n\}$ contains 2 tableaux of shape $(2^r, 1^{n-2r})$ for

$r = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$ and 1 of shape $\left(2^{n/2}\right)$ when n is even. Thus, by (5.76),

$$\begin{aligned} \#\{w \in S_n : \mathrm{stab}(w) = 2\} &= \#\{w \in S_n : Q(w) = T_k \text{ for some } k\} \\ &= 2 \sum_{r=1}^{\lfloor \frac{n-1}{2} \rfloor} \#\,\mathrm{SYT}(2^r, 1^{n-2r}) + \#\,\mathrm{SYT}\left(2^{n/2}\right) \end{aligned}$$

where $\#\,\mathrm{SYT}\left(2^{n/2}\right) = 0$ if is odd. Each 1 or 2-column tableau of size $n+1$ except

$$\begin{array}{|c|}\hline 1 \\ \hline \vdots \\ \hline n \\ \hline n{+}1 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 1 & n{+}1 \\ \cline{1-1} \vdots & \\ \cline{1-1} n & \\ \cline{1-1} \end{array}$$

can be created by adding $(n+1)$ to a standard tableaux of shape $(2^r, 1^{n-2r})$ for some $r = 1, \ldots, \lfloor \frac{n-1}{2} \rfloor$ in 2 ways or to a standard tableaux of shape $\left(2^{n/2}\right)$ in 1 way. Therefore,

$$\begin{aligned} \#\{w \in S_n : \mathrm{stab}(w) = 2\} &= 2 \sum_{r=1}^{\lfloor \frac{n-1}{2} \rfloor} \#\,\mathrm{SYT}(2^r, 1^{n-2r}) + \#\,\mathrm{SYT}\left(2^{n/2}\right) \\ &= \#\{1 \text{ or } 2 \text{ column standard tableaux of size } n+1\} - 2 \\ &= \binom{n+1}{\lfloor \frac{n+1}{2} \rfloor} - 2 \end{aligned}$$

by Lemma 5.8.8, proving (5.77).

□

5.9 Open Problems

We finish by discussing related open questions about tableau stabilization and promotion. While we have proven some of the important properties of tableau stabilization, much remains unknown. The most glaring open problem is Conjecture 5.1.3.

Conjecture 5.1.3: Any standard skew tableau with b rows and decreasing row lengths

stabilizes at b.

We have proven this bound is tight for skew tableaux with constant row vectors, Theorem 5.1.4 and Example 5.4.6. Due to Example 5.4.8, our approach to proving Theorem 5.1.4 does not readily generalize to proving Conjecture 5.1.3.

The distribution of the stabilization statistic remains to be explored as well.

Open Problem 5.9.1. What is the distribution of stab on tableaux of a fixed skew shape?

I expect Open Problem 5.9.1 to be especially difficult. The permutation case, i.e. shape $(1) \cup \cdots \cup (1)$, could be more tractable. The triangular array in Figure 5.3 describes stabilization's distribution on permutations in $S_1, \ldots, S_8$. The distribution is unimodal and log-concave for these small cases. While we know the leftmost entry is always 1, is the rightmost entry is always 1 as well? In addition, since stab is invariant on dual equivalence classes, $Q(w)$ determines $\mathrm{stab}(w)$.

Conjecture 5.9.2. The permutation $w = 1\,2\,\ldots\,(n-1)\,n$ is the only permutation with $\mathrm{stab}(w) = n$.

Open Problem 5.9.3. What is the distribution of stab on S_n?

Open Problem 5.9.4. Is the distribution of stab on S_n unimodal? Is it log-concave?

Open Problem 5.9.5. Characterize $\mathrm{stab}(w)$ directly in terms of $Q(w)$ for all $w \in S_n$.

We have made substantial progress on the problem of specifying the fixed points of the powers of promotion on rectangular tableaux, but some cases remain open. We constructed the tableaux in $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$ for $a \geq 2b-1$ in Section 5.6 and for $a = 2$ in Section 5.7. The $a = 1$ case is trivial, and the complete $r = 1$ case with $a \geq b$ was solved by Purbhoo and Rhee, [PR17]. To completely solve the problem of specifying the fixed points of the powers of promotion on rectangular tableaux, Open Problem 5.9.6 remains.

Open Problem 5.9.6. Fix $b, r \in \mathbb{Z}_{\geq 1}$. For each $a \in [3, 2b-2]$ ($a \in [3, b-1]$ for $r = 1$), describe the tableaux in $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$.

Generalizing Purbhoo and Rhee's construction for $r = 1, a \geq b$ to $r \geq 2, a \geq b$ is a nontrivial potential future task. Note that for $a \geq b$ or $a \geq 2b-1$, we are looking for the same number of tableaux. By Corollary 5.1.10 and (5.54), we still have

$$\#\,\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}} = \binom{br}{r, r, \ldots, r}, \qquad \text{for all } a \geq b.$$

Purbhoo and Rhee's proof relies on the fact that for all $T \in \mathrm{SYT}(b^b)^{\mathrm{p}^b}$, the entries in the anti-diagonal cells $(b,1), (b-1,2), \ldots, (1,b)$ of T form a permutation in S_b when taken modulo b. For example,

1	2	3	7	8
4	5	10	12	13
6	9	15	17	18
11	14	20	22	23
16	19	21	24	25

$\in \mathrm{SYT}(5^5)^{\mathrm{p}^5}$

has the anti-diagonal entries $[16, 14, 15, 12, 8] \equiv_5 [1, 4, 5, 2, 3] \in S_5$. One might hope that for any $T \in \mathrm{SYT}((b \cdot r)^b)^{\mathrm{p}^{br}}$, the cells $(b,1), \ldots, (b,r), (b-1, r+1), \ldots, (b-1, 2r), \ldots, (1, (b-1)r+1), \ldots (1, br)$ have entries that form a permutation in S_{br} when taken modulo br. But, considering such examples as

1	2	4	5	6	8
3	7	9	10	11	12

$\in \mathrm{SYT}(6^2)^{\mathrm{p}^6}$,

1	2	5	6	8	14
3	7	9	11	12	15
4	10	13	16	17	18

$\in \mathrm{SYT}(6^3)^{\mathrm{p}^6}$,

this is not the case, since $[3, 7, 9, 5, 6, 8] \equiv_6 [3, 1, 3, 5, 6, 2] \notin S_6$ and $[4, 10, 9, 11, 8, 14] \equiv_6 [4, 4, 3, 5, 2, 2] \notin S_6$. In the first case, there is not even a symmetric choice of 3 consecutive entries in row 1 and 3 consecutive entries in row 2 which reduces to a permutation modulo 6. We want a symmetric choice with respect to reflection so we can attach 2 symmetric shapes together to make a rectangle like we did for tableau stabilization.

The $a < b$ case is even more challenging since $\#\,\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}} = \#\,\mathrm{SYT}(Q_a((ar)^b))$ changes according to Lemma 5.2.19, using Corollary 5.1.10. Since now the pieces of the

quotient are rectangles, one might have to do some tableau stabilization-like procedure with rectangles to find $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$, and it is unclear how this would work without just reducing to tableau stabilization. When $a = 2$, just row-concatenating the rectification to the anti-rectification works, but this does not easily generalize to $a \in [3, 2b-1]$. Row-concatenating $\mathrm{Rect}(S^{\left(\lceil \frac{a}{2} \rceil\right)})$ and $\mathrm{Rect}^*(S^{\left(\lfloor \frac{a}{2} \rfloor\right)})$ will not produce a rectangular tableaux in general.

For example, consider $a = 3, r = 2, b = 6$, whence

$$\#\,\mathrm{SYT}(6^6)^{\mathrm{p}^{12}} = \#\,\mathrm{SYT}(Q_2(6^6)) = \#\,\mathrm{SYT}((2,2) \cup (2,2) \cup (2,2)).$$

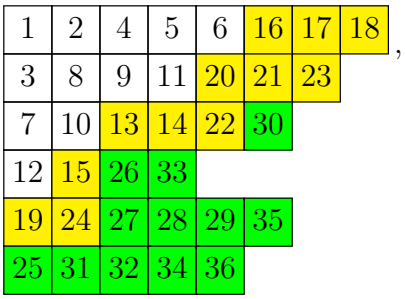
by Corollary 5.1.10 and Lemma 5.2.19. Choose

$$S = \begin{array}{|c|c|c|c|c|c|}\hline \multicolumn{4}{c}{} & 5 & 9 \\ \cline{5-6} \multicolumn{4}{c}{} & 6 & 11 \\ \cline{5-6} \multicolumn{2}{c}{} & 2 & 4 \\ \cline{3-4} \multicolumn{2}{c}{} & 8 & 10 \\ \cline{3-4} 1 & 3 \\ \cline{1-2} 7 & 12 \\ \cline{1-2} \end{array} \in \mathrm{SYT}((2,2) \cup (2,2) \cup (2,2)).$$

Then, we have

$$\mathrm{Rect}(S^{(2)}) = \begin{array}{|c|c|c|c|c|c|c|c|}\hline 1 & 2 & 4 & 5 & 6 & 16 & 17 & 18 \\ \hline 3 & 8 & 9 & 11 & 20 & 21 & 23 \\ \cline{1-7} 7 & 10 & 13 & 14 & 22 \\ \cline{1-5} 12 & 15 \\ \cline{1-2} 19 & 24 \\ \cline{1-2} \end{array}, \qquad \mathrm{Rect}^*(S) = \begin{array}{|c|c|c|c|c|}\hline \multicolumn{4}{c}{} & 6 \\ \cline{5-5} \multicolumn{3}{c}{} & 2 & 9 \\ \cline{4-5} \multicolumn{1}{c}{} & 3 & 4 & 5 & 11 \\ \cline{2-5} 1 & 7 & 8 & 10 & 12 \\ \cline{1-5} \end{array}.$$

Row-concatenating the properly shifted versions of these along 6 rows gives



which is not even a partitioned-shaped, let alone rectangular.

In Theorem 5.1.12 and Theorem 5.1.14, our construction of $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$ for $a \geq 2b-1$ and White and Rhee's construction for $a = 2$ included a natural bijection $R_a : \mathrm{SYT}(Q_a((ar)^b)) \to \mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$. Moreover, promotion commuted with R_a as in Corollary 5.1.13 and Corollary 5.7.4. One would hope these properties extended to a construction of $\mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}}$ for $a \in [3, 2b-2]$ as well.

Open Problem 5.9.7. Fix $b, r \in \mathbb{Z}_{\geq 1}$ and $a \in [3, 2b-2]$. Find a bijection

$$R_a : \mathrm{SYT}((ar)^b)^{\mathrm{p}^{br}} \to \mathrm{SYT}(Q_a((ar)^b))$$

satisfying $\mathrm{p}(R_a(S)) = R_a(\mathrm{p}(S))$ for all $S \in \mathrm{SYT}(Q_a((ar)^b))$, or show no such bijection exists.

BIBLIOGRAPHY

[AA19] P. Alexandersson and N. Amini. The cone of cyclic sieving phenomena. *Discrete Math.*, 342(6):1581–1601, 2019.

[Ahl19] C. Ahlbach. Tableau stabilization and rectangular tableaux fixed by promotion powers. *In Preparation*, 2019.

[AS18a] C. Ahlbach and J. P. Swanson. Cyclic sieving, necklaces, and branching rules related to Thrall's problem. *Electron. J. Combin.*, 25(4):Paper 4.42, 38, 2018.

[AS18b] C. Ahlbach and J.P. Swanson. Refined cyclic sieving on words for the major index statistic. *European J. Combin.*, 73:37–60, 2018.

[BBG90] F. Bergeron, N. Bergeron, and A. M. Garsia. Idempotents for the free Lie algebra and q-enumeration. In *Invariant theory and tableaux (Minneapolis, MN, 1988)*, volume 19 of *IMA Vol. Math. Appl.*, pages 166–190. Springer, New York, 1990.

[BER11] A. Berget, S.-P. Eu, and V. Reiner. Constructions for cyclic sieving phenomena. *SIAM J. Discrete Math.*, 25(3):1297–1314, 2011.

[Bra44] A. Brandt. The free Lie ring and Lie representations of the full linear group. *Trans. Amer. Math. Soc.*, 56:528–536, 1944.

[Cel98] P. Cellini. Cyclic Eulerian elements. *European J. Combin.*, 19(5):545–552, 1998.

[CFL58] K.-T. Chen, R. H. Fox, and R. C. Lyndon. Free differential calculus, IV. The quotient groups of the lower central series. *Ann. of Math. (2)*, 68:81–95, 1958.

[DLT94] J. Désarménien, B. Leclerc, and J.-Y. Thibon. Hall-Littlewood functions and Kostka-Foulkes polynomials in representation theory. *Sém. Lothar. Combin.*, 32:Art. B32c, 38, 1994.

[FRT54] J. S. Frame, G. de B. Robinson, and R. M. Thrall. The hook graphs of the symmetric groups. *Canadian J. Math.*, 6:316–324, 1954.

[Ful97] W. Fulton. *Young Tableaux*, volume 35 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 1997.

[Gar90] A. M. Garsia. Combinatorics of the free Lie algebra and the symmetric group. In *Analysis, et cetera*, pages 309–382. Academic Press, Boston, MA, 1990.

[GKP94] Graham, Knuth, and Patashnik. *Concrete Mathematics: A Foundation for Computer Science*. Addison–Wesley, 2nd edition, 1994.

[GP92] A. M. Garsia and C. Procesi. On certain graded S_n-modules and the q-Kostka polynomials. *Adv. Math.*, 94(1):82–138, 1992.

[GR93] I. Gessel and C. Reutenauer. Counting permutations with given cycle structure and descent set. *J. Combin. Theory Ser. A*, 64(2):189–215, 1993.

[Gre74] C. Greene. An extension of Schensted's theorem. *Advances in Math.*, 14:254–265, 1974.

[Gup78] H. Gupta. A new look at the permutations of the first n natural numbers. *Indian J. Pure Appl. Math.*, 9(6):600–631, 1978.

[Hai92] M. D. Haiman. Dual equivalence with applications, including a conjecture of Proctor. *Discrete Mathematics*, 99(1):79 – 113, 1992.

[JK81] G. James and A. Kerber. *The Representation Theory of the Symmetric Group*, volume 16 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass., 1981.

[Kly74] A. A. Klyachko. Lie elements in the tensor algebra. *Siberian Mathematical Journal*, 15(6):914–920, 1974.

[KW01] W. Kraśkiewicz and J. Weyman. Algebra of coinvariants and the action of a Coxeter element. *Bayreuth. Math. Schr.*, (63):265–284, 2001.

[LLT94] A. Lascoux, B. Leclerc, and J.-Y. Thibon. Green polynomials and Hall-Littlewood functions at roots of unity. *European J. Combin.*, 15(2):173–180, 1994.

[LLT97] A. Lascoux, B. Leclerc, and J-Y. Thibon. Ribbon tableaux, Hall-Littlewood functions, quantum affine algebras, and unipotent varieties. *J. Math. Phys.*, 38(2):1041–1068, 1997.

[LP18] Thomas Lam and Alexander Postnikov. Alcoved polytopes II. In *Lie groups, geometry, and representation theory*, volume 326 of *Progr. Math.*, pages 253–272. Birkhäuser/Springer, Cham, 2018.

[Mac] P. A. MacMahon. Two Applications of General Theorems in Combinatory Analysis: (1) To the Theory of Inversions of Permutations; (2) To the Ascertainment of the Numbers of Terms in the Development of a Determinant which has Amongst its Elements an Arbitrary Number of Zeros. *Proc. London Math. Soc.*, S2-15(1):314.

[Mac13] P. A. MacMahon. The indices of permutations and the derivation therefrom of functions of a single variable associated with the permutations of any assemblage of objects. *Amer. J. Math.*, 35(3):281–322, 1913.

[Mac95] I. G. Macdonald. *Symmetric Functions and Hall Polynomials*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, second edition, 1995.

[MH59] Jr. M. Hall. *The Theory of Groups*. The Macmillan Co., New York, N.Y., 1959.

[Pec14] O. Pechenik. Cyclic sieving of increasing tableaux and small Schröder paths. *J. Combin. Theory Ser. A*, 125:357–378, 2014.

[Pet05] K. Petersen. Cyclic descents and P-partitions. *J. Algebraic Combin.*, 22(3):343–375, 2005.

[PPR09] K. Petersen, P. Pylyavskyy, and B. Rhoades. Promotion and cyclic sieving via webs. *J. Algebraic Combin.*, 30(1):19–41, 2009.

[PR17] K. Purbhoo and D. Rhee. Minimal orbits of promotion. *Electron. J. Combin.*, 24(1):Paper 1.41, 17, 2017.

[PS10] K. Petersen and L. Serrano. Cyclic sieving for longest reduced words in the hyperoctahedral group. *Electron. J. Combin.*, 17(1):Research Paper 67, 12, 2010.

[Pur13] K. Purbhoo. Wronskians, cyclic group actions, and ribbon tableaux. *Trans. Amer. Math. Soc.*, 365(4):1977–2030, 2013.

[Rei15] V. Reiner. Thrall's problem and coarsenings. Banff Workshop on Positivity in Algebraic Combinatorics, 2015.

[Reu93] C. Reutenauer. *Free Lie Algebras*, volume 7 of *London Mathematical Society Monographs. New Series*. The Clarendon Press, Oxford University Press, New York, 1993. Oxford Science Publications.

[Rhe12] D. Rhee. *Cyclic sieving phemonon of promotion on rectangular tableaux,*. Master's thesis, University of Waterloo, 2012.

[Rho10a] B. Rhoades. Cyclic sieving, promotion, and representation theory. *J. Combin. Theory Ser. A*, 117(1):38–76, 2010.

[Rho10b] B. Rhoades. Hall-Littlewood polynomials and fixed point enumeration. *Discrete Math.*, 310(4):869–876, 2010.

[RSW04] V. Reiner, D. Stanton, and D. White. The cyclic sieving phenomenon. *J. Combin. Theory Ser. A*, 108(1):17–50, 2004.

[Sag01] B. E. Sagan. *The Symmetric Group*, volume 203 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 2001.

[Sag11] B. E. Sagan. The cyclic sieving phenomenon: a survey. In *Surveys in Combinatorics 2011*, volume 392 of *London Math. Soc. Lecture Note Ser.*, pages 183–233. Cambridge Univ. Press, Cambridge, 2011.

[Sch61] C. Schensted. Longest increasing and decreasing subsequences. *Canad. J. Math.*, 13:179–191, 1961.

[Sch63] M.-P. Schützenberger. Quelques remarques sur une construction de Schensted. *Math. Scand.*, 12:117–128, 1963.

[Sch03] M. Schocker. Multiplicities of higher Lie characters. *J. Aust. Math. Soc.*, 75(1):9–21, 2003.

[Spe32] W. Specht. *Eine Verallgemeinerung der symmetrischen Gruppe*. PhD thesis, Humboldt-Universität zu Berlin, null, 1932.

[Spr74] T. A. Springer. Regular elements of finite reflection groups. *Invent. Math.*, 25:159–198, 1974.

[Sta86] R.P. Stanley. Symmetries of plane partitions. *J. Combin. Theory Ser. A*, 43(1):103–113, 1986.

[Sta99] R. P. Stanley. *Enumerative Combinatorics. Volume 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

[Sta12] R. P. Stanley. *Enumerative Combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2012.

[Ste89] J. Stembridge. On the eigenvalues of representations of reflection groups and wreath products. *Pacific J. Math.*, 140(2):353–396, 1989.

[Ste19] W. Stein. Cocalc. Data Collection, 2018-2019.

[Sun94] S. Sundaram. The homology representations of the symmetric group on Cohen-Macaulay subposets of the partition lattice. *Adv. Math.*, 104(2):225–296, 1994.

[Sun18] S. Sundaram. Variations on the S_n-module Lie_n. 2018. arXiv:1803.09368.

[Thr42] R. M. Thrall. On symmetrized Kronecker powers and the structure of the free Lie ring. *Amer. J. Math.*, 64:371–388, 1942.

[Whi06] D. White. Csp on promotion. Manuscript, 2006.

[Wil16] M. Wildon. A generalized sxp rule proved by bijections and involutions. *Preprint, arxiv:1508.07030v2*, 2016.

[WW94] S. Wagon and H. Wilf. When are subset sums equidistributed modulo m? *Electron. J. Combin.*, 1:Research Paper 3, 1994.