

Factorization Homology for Embedded Submanifolds

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A thesis submitted in partial requirement for the degree of Master of Science.

University of Washington 2020

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Program authorized to offer degree:

Mathematics

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Abstract

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In this thesis I will explore the theory of factorization homology including prerequisite material required to understand the definitions and structures used in the theory. I will begin with a brief survey of some basic structures in infinity category theory. Using these tools I will explore the theory of stratified spaces and how it pertains to factorization homology. Once we have built the symmetric monoidal ∞ -categories $\mathcal{S}nglr$, $\mathcal{B}sc$, and $\mathcal{M}fld(\mathcal{B})$ we can then define factorization homology. From there we will explore factorization homology for embedded submanifolds. The thesis concludes with the construction of an ∞ -category of basics whose ∞ -category of manifolds has 3-manifolds with boundary with an embedded submanifold of dimension 1 on the boundary as objects. This is the beginning of PhD work I will be pursuing with Marcy Robertson at the University of Melbourne in Melbourne, Australia.

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1 Three Models for Infinity Categories

I will be making frequent use of three different models for $(\infty, 1)$ -categories, and well as the Quillen equivalences between the model structures on their respective categories. These three models are **quasicategories**, **categories enriched in Kan complexes**, and **categories enriched in topological spaces**.

1.1 Simplicial Sets

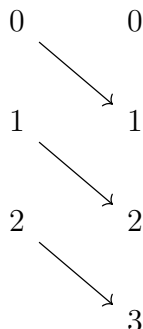
We will begin our examination of these with a brief review of simplicial sets. Simplicial sets may be somewhat familiar to those who have encountered simplicial complexes before. The key difference being the notion of degeneracy maps. In order to define simplicial sets we must first understand the simplex category.

Definition 1.1. Let Δ denote the **simplex category**, whose objects are finite ordinals, denoted $[n]$ and whose morphisms are order preserving maps.

This category has two distinguished classes of morphisms that generate the entire category:

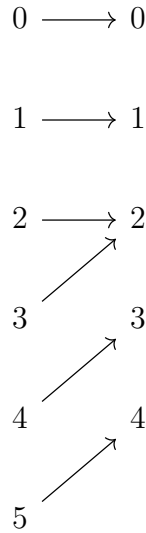
Definition 1.2. We call injective maps $[n-1] \rightarrow [n]$ face maps. We denote by $\partial_i^n : [n-1] \rightarrow [n]$ the specific face map that excludes i from it's image.

Example 1.3. A diagram depicting the map ∂_0^3 :

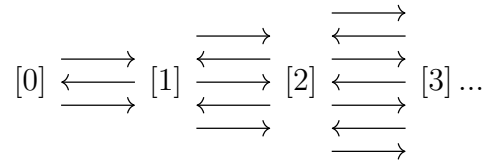


Definition 1.4. We denote $\sigma_i^n : [n+1] \rightarrow [n]$ the unique surjection such that $\sigma(i) = \sigma(i+1) = i$. We call such maps **degeneracy maps**.

Example 1.5. A diagram depicting the map σ_2^4 :



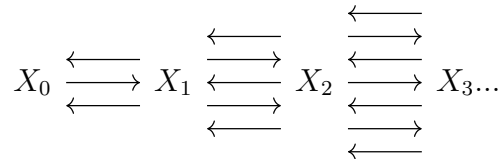
All morphisms in this category are generated by face and degeneracy maps. In this way we may depict the simplex category Δ in a diagram of this manner:



In this diagram the right arrows represent face maps and the left represent degeneracy maps. The rest of the arrows in Δ are compositions of these.

Definition 1.6. A presheaf $\Delta^{op} \rightarrow Set$ is called a simplicial set.

We call the images of face maps under the presheaf that is a simplicial set, face maps still. This is also true for the degeneracy maps. We also conflate the notation such that $\partial_i^n : X_n \rightarrow X_{n-1}$ represents the image of $\partial_i^n : [n-1] \rightarrow [n]$, same for the notation denoting degeneracy maps. The best way to visualize a simplicial set is as the diagram of the simplex category but with the arrows reversed. We let X_n denote the image $[n]$ under a given presheaf $X : \Delta^{op} \rightarrow Set$. Then the following diagram represents how one might visualize the simplicial set X :



Now the left arrows represent face maps and the right represent degeneracy maps.

We call the set, X_n , the set of n -simplices in X . We think of the elements of these sets as n -dimensional triangles. In this way X_0 can be thought of as a set of points, X_1 a set of lines, X_2 as set of honest two dimensional triangles, X_3 a set of triangular prisms and so forth. We may induce a natural ordering on the vertices, 0-simplices, of these triangles from the face maps. Let $\sigma \in X_1$, let $a = \partial_0^1(\sigma)$ and let $b = \partial_1^1(\sigma)$ then we may see σ as a directed line as such:

$$a \rightarrow b$$

It is easy enough to see from this that the face maps point a simplex to it's boundary components inducing an ordering on these from the labeling of the maps. This seems quite useful and may in fact be familiar to anyone who has worked with simplicial complexes. But what are these degeneracy maps doing? Since they are pointing from the set of n -simplices to the set of $(n + 1)$ -simplices we must be associating lower dimensional triangles to higher ones in some manner. So let's look at the most basic example and hopefully make a little sense of this. There is one degeneracy map from the set of 0-simplices, X_0 , to the set of 1-simplices, X_1 . It comes from the unique map $[1] \rightarrow [0]$ that sends $0, 1 \mapsto 0$.

Example 1.7. Let $\alpha \in X_0$. Then the image of α under the unique degeneracy map $\partial_0^0(\alpha)$ may be depicted as 1-simplex pointing α to itself:

$$\alpha \rightarrow \alpha$$

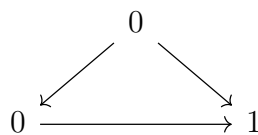
If we imagine 0-simplices as points in space then the image of a 0-simplex under the unique degeneracy map can be regarded as the constant path from the point to itself. This is not a rigorous examination, but simply a tool to help visualize what this degeneracy map is doing. One might think of a constant path as a rather lame path. In this manner we will think of degeneracy maps as pointing n -dimensional triangles to lame (degenerate) $(n + 1)$ -dimensional triangles.

Before we described degeneracy maps as doubling up on some index i . Let $\Xi \in X_n$ then the image $\sigma_i^n(\Xi)$ will see the i^{th} vertex doubled up to create a higher dimensional triangle.

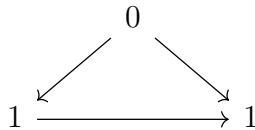
Example 1.8. Let $\Xi \in X_1$ be a 1-simplex in the simplicial set X . We will depict it as such:

$$0 \rightarrow 1$$

Then we may visual $\sigma_0^1(\Xi)$ as:



Similarly we may see $\sigma_1^1(\Xi)$ as:



There are distinguished simplicial sets that we use to explore the content of other simplicial sets. They are the representables, and we consider them to be the "standard simplices". This name is somewhat misleading as they are actually entire simplicial sets. The reason for this naming convention will become more apparent momentarily.

Definition 1.9. We use $\Delta[n]$ to denote the simplicial set represented by $[n] \in \Delta$. In this way we see the set of m -simplices of $\Delta[n]$ to be:

$$\Delta[n]_m := \Delta([m], [n])$$

We refer to this simplicial set as **the standard n -simplex**.

We can see this simplicial set as an n -dimensional triangle along with all the necessary information of the degenerate higher simplices lying about. In order to understand how this simplicial set can probe the content of another we must first define the notion of a map between simplicial sets.

Definition 1.10. Let X, Y be simplicial sets. A **simplicial set map** $X \rightarrow Y$ is simply defined to be a natural transformation from the functor $X : \Delta^{op} \rightarrow Set$ to the functor $Y : \Delta^{op} \rightarrow Set$.

So this definition is the correct notion but may be particularly illuminated at first glance. Let's look at the diagram corresponding to the content of this natural transformation.

$$\begin{array}{ccc}
 X_n & \xrightarrow{X(f)} & X_m \\
 \alpha_n \downarrow & & \downarrow \alpha_m \\
 Y_n & \xrightarrow{Y(f)} & Y_m
 \end{array}$$

What this shows is that a simplicial set map is a sequence of set maps α_i for all $i \in \mathbb{N}$ pointing the set of i -simplices in X to the set of i -simplices in Y in such a way that commutes with face and degeneracy maps of each. We use this notion of simplicial set map to define the category of simplicial sets.

Definition 1.11. We define the **category of simplicial sets**, $sSet$ to be the category whose objects are simplicial sets and whose morphisms are simplicial set maps.

Now that we have these definitions down we can properly explore the way in which standard simplices probe the content of simplicial sets. By way of the Yoneda lemma we see that given a simplicial set X we have $X_n \cong sSet(\Delta[n], X)$.

As a pre-sheaf category, the category of simplicial sets has the lovely property that limits and colimits are defined levelwise in set. As a goto example let's take a look at products in $sSet$.

Definition 1.12. Let X, Y be simplicial sets. We define the product simplicial set $X \times Y$ levelwise in set in the following manner:

$$(X \times Y)_n := X_n \times Y_n$$

Our face and degeneracy maps are defined as the product maps as well:

$$\partial^n((a, b)) := (\partial^n(a), \partial^n(b))$$

$$\sigma^n((a, b)) := (\sigma^n(a), \sigma^n(b))$$

The category of simplicial sets is a Cartesian closed category. We will frequently cite the closed nature of this category using the following definition:

Definition 1.13. Let X, Y be simplicial sets. We define the **mapping space** $sSet(X, Y)$ as the simplicial set defined:

$$sSet(X, Y)_n := sSet(X \times \Delta[n], Y)$$

The set of n -simplices is simply the set of simplicial maps from the product space $X \times \Delta[n]$ to Y .

We will use this closed structure in order to regard $sSet$ as a category enriched in simplicial sets, or more commonly refer to it as a simplicial category.

Definition 1.14. Let $X, Y \in sSet$ be simplicial sets the **join of simplicial sets**, $X \star Y$ is the simplicial set defined:

$$(X \star Y)_n = X_n \times Y_n \times \bigcup_{j+i=n-1} X_i \times Y_j$$

1.2 Horns, Fillers, Kan Complexes, and Quasicategories

In order to make sense of quasicategories we must first understand the notion of horns and fillers.

Definition 1.15. The i^{th} -horn of $\Delta[n]$, denoted $\Lambda^i[n]$ is a subsimplicial set of the standard n -simplex $\Delta[n]$ obtained by discarding the unique non-degenerate n -simplex as well as its i^{th} face. We call a horn outer if $i = 0, n$ and inner otherwise.

Definition 1.16. A simplicial set $X : \Delta^{op} \rightarrow \text{Set}$ is said to be a **Kan complex** if for all $n \geq 0$, $0 \leq i \leq n$ any map $\Lambda^i[n] \rightarrow X$ admits a map $\Delta[n] \rightarrow X$ such that the following diagram commutes:

$$\begin{array}{ccc} \Delta[n] & \longrightarrow & X \\ \uparrow & \nearrow & \\ \Lambda^i[n] & & \end{array}$$

This summed up as all horns in a Kan complex admit fillers. Keep in mind these fillers need not be unique.

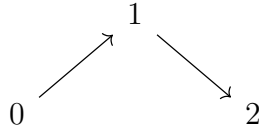
Definition 1.17. A simplicial set $X : \Delta^{op} \rightarrow \text{Set}$ is said to be a quasicategory if all inner horns, $\Lambda^i[n] \rightarrow X$ for $0 < i < n$ admit fillers.

I will frequently use the convention of referring to quasicategories as ∞ -categories. A natural question from here would be: in what way do quasicategories model $(\infty, 1)$ -categories? In order to make sense of this let us first define a functor $\mathcal{N} : \text{Cat} \rightarrow \text{sSet}$ called "the nerve" construction.

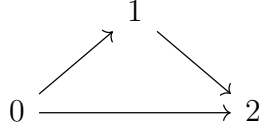
Definition 1.18. The nerve of a small category, C , is the image of C under the functor $\mathcal{N} : \text{Cat} \rightarrow \text{Set}$ defined by sending a category to the unique simplicial set $\mathcal{N}(C)$ whose set of 0-simplices is the set of objects of C and whose set of n -simplices is the set of all chains of n composable morphisms in C . The face maps ∂_0^1 and ∂_1^1 are defined by sending the 1-simplex to the object representing its source and target respectively. The outer face maps ∂_n^n and ∂_0^n are defined by forgetting the last or first morphism in the chain of composable morphisms respectively. The inner maps ∂_k^n are defined by replacing the k^{th} and $(k+1)^{\text{th}}$ morphism in the chain with its composition.

Remark 1.19. We will note without proof that the nerve of a category is a quasicategory with unique horn fillers.

In this manner we can make sense of how quasicategories are in fact category like. We let $X : \Delta^{op} \rightarrow Set$ be a quasicategory. Let's first imagine we have a 2-horn, $\eta : \Lambda_1^2 \rightarrow X$:



Now η has a not-necessarily-unique filler $\eta' : \Delta_n \rightarrow X$:



This 2-cell introduces a 1 simplex pointing $0 \rightarrow 2$ and a 2-simplex which we can think of as showing the composition relationship this “arrow” has to the other two. It is important to note that there may not be a unique filler and in this way we may have multiple arrows acting as the composite arrow. We can use a 3-cell argument to show that there is a contractible “space” of composite arrows. We can also use fillers of horns of 3-cells to show that associativity is held. I will leave this as exercise for anyone curious enough to work through it on their own.

Definition 1.20. Let $X \in sSet$ be a quasicategory. We define **the opposite quasicategory** X^{op} as such:

$$\begin{aligned} X_n^{op} &:= X_n \\ \partial_i : (X_n^{op} \rightarrow X_{n-1}^{op}) &= \partial_{n-i} : (X_n \rightarrow X_{n-1}) \\ \sigma_i : (X_n^{op} \rightarrow X_{n+1}^{op}) &= \sigma_{n-1} : (X_n \rightarrow X_{n+1}) \end{aligned}$$

Definition 1.21. Let $p : X \rightarrow Y$ be a map of simplicial sets we define the ∞ -**overcategory** $X_{/p}$ to be $(X_{/p})_n := sSet_p(\Delta[n] \star X, Y)$ where the subscript on the right means we are restricting to the set of simplicial set maps that when restricted to X are p . In this way we can define an ∞ -overcategory $X_{/x}$ where $x \in X_0$ by simply taking p to be the map $\Delta^0 \rightarrow X$ pointing out x . We could do the same for any n -simplex in X but I will be restricting my attention to ∞ -overcategories over a specific 0-simplex.

1.3 Simplicial and Topological Categories

Simplicial and Topological Categories

In this section I will introduce our two other models for $(\infty, 1)$ -categories, but let's make sense of this notation. An infinity category should be a category like structure that has a notion of n -arrows (or cells) for all $n > 0$. An $(\infty, 1)$ -category is then an infinity category such that all n -arrows are invertible for $n > 1$. To begin we need to understand the notion of enrichment.

Definition 1.22. Let \mathcal{V} be a monoidal category we define a small **category enriched in** \mathcal{V} , C to be:

- A set of objects $Obj(C)$
- For every ordered pair of objects $(a, b) \in Obj(C) \times Obj(C)$ we have a hom-object in \mathcal{V} , $C(a, b) \in \mathcal{V}$
- For each ordered triple $(a, b, c) \in Obj(C) \times Obj(C) \times Obj(C)$ we have a "composite morphism" $C(a, b) \rightarrow C(a, c)$ in \mathcal{V}
- For each object $a \in C$ we have a morphism $j_a : I \rightarrow C(a, a)$ where I is the monoidal unit in \mathcal{V} , this is acting as our identity morphism $a \rightarrow a$

Further we require that for all objects $a, b, c, d \in C$ the following diagrams commute in \mathcal{V} :

$$\begin{array}{ccc}
 (C(c, d) \otimes C(b, c)) \otimes C(a, b) & \xrightarrow{\alpha} & C(c, d) \otimes (C(b, c) \otimes C(a, b)) \\
 \circ \otimes Id_{C(a, b)} \downarrow & & \downarrow Id_{C(c, d)} \otimes \circ \\
 C(b, d) \otimes C(a, b) & \xrightarrow{\circ} & C(a, d) \xleftarrow{\circ} C(c, d) \otimes C(a, c)
 \end{array}$$

which shows composition is associative,

$$\begin{array}{ccccc}
 C(b, b) \otimes C(a, b) & \xrightarrow{\circ} & C(a, b) & \xleftarrow{\circ} & C(a, b) \otimes C(a, a) \\
 j_b \otimes Id_{C(a, b)} \uparrow & \nearrow i & & \nwarrow i & \uparrow Id_{C(a, b)} \otimes j_a \\
 I \otimes C(a, b) & & & & C(a, b) \otimes I
 \end{array}$$

which shows composition is unital.

Definition 1.23. A **topological category** is a category, C , enriched in topological spaces.

Now, the way one views a topological category as a model for $(\infty, 1)$ -categories is as follows. We let C be a topological category. Given two objects $a, b \in Obj(C)$ and two morphisms $f, g \in hom(a, b)$. A 2-arrow pointing $f \rightarrow g$ in C is induced by a path from f to g in the topological space $hom(a, b)$. This exhibits an inverse by the reversal of the path. An n -arrow is then induced inductively by describing an $(n - 2)$ -homotopy between points paths in $hom(a, b)$. For example a 3-arrow will simply be a homotopy between paths $f, g \in hom(a, b)$. A 4-arrow will be a homotopy of homotopies, a 2-homotopy, pointing one

homotopy from f to g to another and so forth. Since all paths are invertible in a topological space we do in fact get the structure we wish for.

Definition 1.24. A **simplicial category** is a category, C enriched in simplicial sets. In the sense that our hom-objects $hom(a, b)$ are simplicial sets.

We care specifically about those simplicial categories whose hom-simplicial-sets are Kan complexes, in that they admit fillers for *all* horns. In order to make sense of why these categories enriched in Kan complexes are models for $(\infty, 1)$ -categories we must first take a close look at Kan complexes.

Definition 1.25. We say a 1-simplex $f : a \rightarrow b$ has an inverse if there exists a 1-simplex $f^{-1} : b \rightarrow a$ and two 2-simplices of the form:

$$\begin{array}{ccc}
 & b & \\
 f \nearrow & & \searrow f^{-1} \\
 a & \xrightarrow{id_a} & a
 \end{array}
 \quad
 \begin{array}{ccc}
 & a & \\
 f^{-1} \nearrow & & \searrow f \\
 b & \xrightarrow{id_b} & b
 \end{array}$$

Where id_a and id_b represent the unique degenerate simplices pointing a and b to themselves respectively.

We let $sCat$ denote the category of simplicial categories. We let $TopCat$ denote the category of topological categories.

There exists a model structure on the category $sSet$ we call the Joyal model structure. In this model structure our fibrant objects are exactly those simplicial sets that are quasi-categories. Similarly there exists a model structure on $sCat$ in which the fibrant objects are *Kan*-enriched categories. Here I will describe the functor that induces a quillen equivalence on these categories. This is done so that I may be able to pass back and forth between these categories using the theories of each. I will not go deeply into the model category theory, but it is important to know that these categories carry the same homotopy theory. In order to make sense of this functor we must first define a series of simplicial categories that will induce the construction itself.

Definition 1.26. We let $S[n]$ denote the simplicial category that has objects the set $\{0, 1, \dots, n\}$. The hom-simplicial-sets are defined:

$$S[n](i, j) := \Delta[1]^{j-i-1}$$

This is the standard $(j - i - 1)$ -cube for $j \geq i$. If $i > j$ we simply define it as the empty simplicial set.

Definition 1.27. The **homotopy coherent nerve** is a functor $\mathcal{N} : sCat \rightarrow sSet$ that induces a quillen adjunction and equivalence. Given a simplicial category \mathcal{C} we define $\mathcal{N}(\mathcal{C})$ by the following characterization:

$$\mathcal{N}(\mathcal{C})_m := Hom_{sCat}(S[n], \mathcal{C})$$

2 Stratified Spaces and The Category Bsc

2.1 Stratified Spaces

Let's begin by defining how one might view a poset P as a topological space. Let $U \subseteq P$, we say U is open if and only if it is closed upward. This means for open U if $a \in U$ and $b \geq a$ then $b \in U$. One can verify this is a topology. Note that a set map $P \rightarrow P'$ between posets is a poset map if and only if it is continuous under this topology. From this we can invoke a nice functor:

$$Poset \rightarrow Top$$

This functor is full, and faithful. Making $Poset$ a full subcategory of Top . This functor also preserves limits.

Definition 2.1. Let P be a poset viewed as its image under the functor defined above. A **stratified space** is a topological space X equipped with a map of topological spaces $X \rightarrow P$. We call this map a **stratification**. We call X the underlying topological space and I will often conflate these referring to a stratified space, $X \rightarrow P$, simply by it's underlying space X .

Definition 2.2. Let $(X \rightarrow P)$ and $(X' \rightarrow P')$ be stratified spaces. A **map of stratified spaces** is commuting square of the following form:

$$\begin{array}{ccc} X & \longrightarrow & P \\ \downarrow & & \downarrow \\ X' & \longrightarrow & P' \end{array}$$

We will frequently conflate the map of underlying topological spaces with the map of stratified spaces.

Example 2.3. Let $(p : X \rightarrow P)$ be the stratified space $X = \mathbb{R}$ and $P = \{*\}$, giving our space the trivial stratification. Let $(q : Y \rightarrow Q)$ be the stratified space $Y = \mathbb{R}^2$ and $Q = \{0 < 1\}$ with the stratification that sends all points $(x, 0) \mapsto 0$ and all else to 1. To verify this is well

defined consider the open sets of Q , which are $U = \{0, 1\}$ and $V = \{1\}$. We have $q^{-1}(V) = \emptyset$ and $q^{-1}(U) = \mathbb{R}$ making this map continuous. If we let $f_{top} : X \rightarrow Y$ be a map defined by sending $x \mapsto (x, 0)$ and let $f_{pos} : P \rightarrow Q$ be a map sending $* \mapsto 0$. Then the following diagram commutes:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{f_{top}} & \mathbb{R}^2 \\ p \downarrow & & \downarrow q \\ \{*\} & \xrightarrow{f_{pos}} & \{0 < 1\} \end{array}$$

Thus if we can define a map of stratified spaces $f : X \rightarrow Y$ using this diagram.

We refer to the fiber X_p over $p \in P$ of a stratified space $(X \rightarrow P)$ as a stratum. Each point $x \in X$ belongs to a stratum, namely the fiber over the image of that point.

Definition 2.4. Let $(X \rightarrow P)$ and $(Y \rightarrow P')$ be stratified spaces. We say a map of stratified spaces $f : X \rightarrow Y$ is an **open embedding of stratified spaces** when it is an open embedding on the underlying topological spaces, and for each $p \in P$ the restriction map:

$$f|_p : X_p \rightarrow Y_{fp}$$

is an open embedding of topological spaces.

Definition 2.5. We define the category $StTop^{open}$ as the category with objects are given by stratified spaces whose underlying topological space is second countable and Hausdorff, and whose morphisms are given by open embeddings of stratified spaces.

The concept of a basic is essentially singularity type. We make sense of singularities through a notion of cone as defined here:

Definition 2.6. Let $(X \rightarrow P)$ be a stratified space. We define the **cone** $C(X \rightarrow P)$ as $(C(X) \rightarrow C(P))$, where $C(X)$ is the following pushout:

$$\begin{array}{ccc} \{0\} \times X & \longrightarrow & \mathbb{R}_{\geq 0} \times X \\ \downarrow & & \downarrow \\ * & \longrightarrow & C(X) \end{array}$$

and $C(P)$ is defined by adjoining a minimal element $*$. This is the easiest way to understand this space but we may also define it as a pushout in $Poset$:

$$\begin{array}{ccc} \{0\} \times P & \longrightarrow & [1] \times P \\ \downarrow & & \downarrow \\ * & \longrightarrow & C(P) \end{array}$$

The poset $[1]$ is simply the poset $\{0 < 1\}$. The cone of a stratified space inherits a stratified space structure from the the stratified space $(\mathbb{R}_{\geq 0} \rightarrow [1])$ given by $0 \mapsto 0$ and for all $x \geq 0$, $x \mapsto 1$.

Example 2.7. Let $\mathbb{S}^1 \rightarrow \{*\}$ be the circle with trivial stratification. As an example let's consider the cone over this stratified space:

$$\begin{array}{ccc} \{0\} \times \mathbb{S}^1 & \longrightarrow & \mathbb{R}_{\geq 0} \times \mathbb{S}^1 \\ \downarrow & & \downarrow \\ * & \longrightarrow & C(\mathbb{S}^1) \end{array}$$

One should think of pushouts in the category of topological spaces as gluings. In particular in this example we are gluing a single point to the set of points $\{(0, x) : x \in \mathbb{S}^1\}$. So we begin with an infinite cylinder with boundary, and then by gluing a single point to this boundary we are squeezing it to a single point. Thus we end up with a sort of infinite cone, which on the level of topological spaces is homeomorphic, but not diffeomorphic to the real plane. Now let's examine how this example works on the level of posets. Adjoining a minimal element to $\{*\}$ simply gives us a poset isomorphic to $\{0 < 1\}$. So our stratification is a map from the infinite cone $C(\mathbb{S}^1) \rightarrow \{0 < 1\}$. It inherits a stratification from the map $\mathbb{R}_{\geq 0} \rightarrow \{0 < 1\}$ that sends all non-zero elements to 1 and sends $0 \mapsto 0$. In this manner our stratification sends all points in the infinite cone to 1 except the cone point which gets sent to 0.

Using this information we can construct a category of **Topological** or C^0 stratified spaces. To begin we let $\mathcal{T} \subset StTop^{open}$ defined:

- \mathcal{T} contains the empty stratified space $(\emptyset \rightarrow \emptyset)$.
- If $(X \rightarrow P) \in \mathcal{T}$ and both X and P are compact spaces then $C(X \rightarrow P) \in \mathcal{T}$.
- If $(X \rightarrow P) \in \mathcal{T}$ then the stratified space $(X \times \mathbb{R} \rightarrow P)$ given by the projection map $X \times \mathbb{R} \rightarrow X \rightarrow P$ is in \mathcal{T} .
- If $(Y \rightarrow S) \rightarrow (X \rightarrow P)$ is an open embedding of stratified spaces and $(X \rightarrow P) \in \mathcal{T}$ then $(Y \rightarrow S) \in \mathcal{T}$.
- if $(X \rightarrow P)$ is a stratified space such that there exist stratified spaces $\{X_i \rightarrow P_i\}_{i \in \mathcal{I}} \in \mathcal{T}$ and maps $\{(X_i \rightarrow P_i) \rightarrow (X \rightarrow P)\}_{i \in \mathcal{I}}$ such that both $\{X_i \rightarrow X\}_{i \in \mathcal{I}}$ and $\{P_i \rightarrow P\}_{i \in \mathcal{I}}$ are open covers then $(X \rightarrow P) \in \mathcal{T}$.

Definition 2.8. We let $\mathcal{T}' \subset StTop^{open}$ be the smallest full subcategory containing \mathcal{T} we then define the category of C^0 stratified spaces $Snglr^{C^0}$ to be the subcategory of \mathcal{T}' such that every object's underlying space X is paracompact.

Definition 2.9. A C^0 **basic** is a C^0 stratified space of the form:

$$\mathbb{R}^n \times C(Z)$$

Here $n \geq 0$ and $\mathbb{R}^n = (\mathbb{R}^n \rightarrow *)$ has the trivial stratification, and Z is a compact C^0 stratified space.

In order to define the notions of **depth** and **dimension** of a stratified space we must first define the notion of **Lesbegue Covering Dimension**.

Definition 2.10. Let X be a topological space. The **Lesbegue covering dimension** of X is the smallest number n such that every open cover \mathcal{C} of X admits an open refinement \mathcal{C}' such that for every point $x \in X$, x is contained in at most $n + 1$ sets of \mathcal{C}' .

Definition 2.11. The local dimension $dim_x(X)$ of a nonempty C^0 stratified space $(X \rightarrow P)$ is defined as the covering dimension of X at x . The **dimension of a C^0 stratified space** is then defined as:

$$dim(X) := \sup_{x \in X} dim_x(X)$$

We define the dimension of $(\emptyset \rightarrow \emptyset)$ to be -1 .

Definition 2.12. We define the depth of a stratified space $(S : X \rightarrow P)$, denoted $depth_x(X)$ to be the difference between the local dimension of X at x and the local dimension of X_{Sx} at x . Recall that X_{Sx} denotes the stratum of X that x belongs to.

$$depth_x(X) := dim_x(X) - dim_x(X_{Sx})$$

We then define the **depth of a stratified space** to be:

$$depth(X) := \sup_{x \in X} depth_x(X)$$

We say that a stratified space $(X \rightarrow P)$ is of *pure dimension* n if the local dimension $dim_x(X) = n$ for all $x \in X$.

Definition 2.13. We define $Snglr_{k,n}^{C^0} \subset Snglr^{C^0}$ to be the full subcategory consisting of those stratified spaces of pure dimension n and depth exactly k . We let

$$Snglr_{\leq k,n}^{C^0} \subset Snglr_{\leq k,\leq n}^{C^0} \supset Snglr_{k,\leq n}^{C^0}$$

be defined in the obvious manner.

We let $Snglr_{\leq k,\infty}^{C^0}$ be the full subcategory of $Snglr^{C^0}$ with stratified spaces of any dimension and depth at most k .

2.2 Conical Smoothness

In order to define the category Bsc and $Snglr$ we need an appropriate notion of smoothness for C^0 stratified spaces. Ayala, Francis and Tanaka call this concept conical smoothness [2, Definition 2]. To begin exploring this we will first go over the notion of differentiability along \mathbb{R}^i . We begin with a stratified space $U = \mathbb{R}^i \times C(Z)$ where Z is some compact C^0 stratified space. We are going to need some notion of the tangent space at a point so we begin with the identifications:

$$T\mathbb{R}^i \times C(Z) \cong \mathbb{R}_{\vec{v}}^i \times \mathbb{R}^i \times C(Z) = \mathbb{R}_{\vec{v}}^i \times U$$

We want to think of $T\mathbb{R}^i \times C(Z)$ as a sort of tangent bundle for U . We will denote the points of U as tuples $(u, [s, z])$ with $(u, s, z) \in \mathbb{R} \times \mathbb{R}_{\geq 0} \times Z$ using the convention that $[s, z] = *$ when $Z = \emptyset$. In order to recover a notion of a vector space on this tangent structure we consider the homeomorphism:

$$\gamma : \mathbb{R}_{>0} \times T\mathbb{R}^i \times C(Z) \rightarrow \mathbb{R}_{>0} \times T\mathbb{R}^i \times C(Z)$$

defined:

$$(t, \vec{v}, u, [s, z]) \mapsto (t, t\vec{v} + u, u, [ts, z])$$

The idea here is that given a choice (u, t) we have a map $\gamma_{u,t} : \mathbb{R}_{\vec{v}}^i \times C(Z) \rightarrow \mathbb{R}_{\vec{v}}^i \times C(Z)$ giving us a notion of scaling and translating on a basic. Further we have the following identities corroborating our vector space intuition:

$$\begin{aligned} \gamma_{t_2, u_2} \circ \gamma_{t_1, u_1} &= \gamma_{t_2 t_1, u_2 + t_2 u_1} \\ (\gamma_{t, u})^{-1} &= \gamma_{\frac{1}{t}, -\frac{u}{t}} \end{aligned}$$

In order to define the notion of being conically smooth along \mathbb{R}^i we must first define a specific map $f_{\Delta} : \mathbb{R}_{>0} \times T\mathbb{R}^i \times C(Z) \rightarrow \mathbb{R}_{>0} \times T\mathbb{R}^i \times C(Z)$. To do so let's lay down some notation. Given two compact C^0 stratified spaces $(Z \rightarrow P)$ and $(Z' \rightarrow P')$ along with a continuous map of stratified spaces $f : \mathbb{R}^i \times C(Z) \rightarrow \mathbb{R}^{i'} \times C(Z')$ that on the level of posets $C(P) \rightarrow C(P')$ send the cone point to the cone point we denote $f|_{\mathbb{R}^i} : \mathbb{R}^i \times \{*\} \rightarrow \mathbb{R}^{i'} \times \{*\}$ to be the restriction of f to the cone point stratum. We then have a map:

$$f|_{\mathbb{R}^i} \times f : \mathbb{R}_{\vec{v}}^i \times U \rightarrow \mathbb{R}_{\vec{v}'}^{i'} \times U'$$

Here $U = \mathbb{R}^i \times C(Z)$ and $U' = \mathbb{R}^{i'} \times C(Z')$. We use this map for the following definition:

Definition 2.14. The map $f_{\Delta} : \mathbb{R}_{>0} \times T\mathbb{R}^i \times C(Z) \rightarrow \mathbb{R}_{>0} \times T\mathbb{R}^i \times C(Z)$ will be defined by using the identification $T\mathbb{R}^i \times C(Z) \cong \mathbb{R}_{\vec{v}}^i \times U$ as such:

$$f_{\Delta} := id_{\mathbb{R}_{>0}} \times f|_{\mathbb{R}^i} \times f$$

Equipped with this map we can now define what it means to be conically smooth along \mathbb{R}^i . We first let $(Z \rightarrow P)$ and $(Z' \rightarrow P')$ be compact C^0 stratified spaces along with a continuous map of stratified spaces $f : \mathbb{R}^i \times C(Z) \rightarrow \mathbb{R}^{i'} \times C(Z')$.

Definition 2.15. We say f is C^1 along \mathbb{R}^i if there is a continuous extension $\tilde{D}f$ making the following diagram commute:

$$\begin{array}{ccc} \mathbb{R}_{\geq 0} \times T\mathbb{R}^i \times C(Z) & \overset{\tilde{D}f}{\dashrightarrow} & \mathbb{R}_{\geq 0} \times \mathbb{R}^{i'} \times C(Z') \\ \uparrow & & \uparrow \\ \mathbb{R}_{> 0} \times T\mathbb{R}^i \times C(Z) & \xrightarrow{\gamma^{-1} \circ f_{\Delta} \circ \gamma} & \mathbb{R}_{> 0} \times T\mathbb{R}^{i'} \times C(Z') \end{array}$$

We call the restriction of $\tilde{D}f$ to $t = 0$, $Df : T\mathbb{R}^i \times C(Z) \rightarrow T\mathbb{R}^{i'} \times C(Z')$. Then we say that the map f is C^r along \mathbb{R}^i if $Df : T\mathbb{R}^i \cong \mathbb{R}^{2i} \times C(Z) \rightarrow \mathbb{R}^{2i'} \times C(Z')$ is C^{r-1} along \mathbb{R}^{2i} . Then we say f is **conically smooth along** \mathbb{R}^i if it is C^r for all $r > 0$.

2.3 The 1-Categories Bsc , $Snglr$, $Mfld$

The idea with conically smooth stratified spaces is similar to that of smooth manifolds. We would like to introduce new singularity types we call basics, such that every point in a stratified space has a neighborhood homeomorphic to a basic in conically smooth fashion. In order to define our category of conically smooth stratified spaces we must first define a full subcategory of basics. We will do so inductively by first defining the category of basics with negative depth. In order to begin this process we must first assume we have defined $Bsc_{\leq k+1, \leq \infty}$ in order to make sense of a conically smooth atlas.

Definition 2.16. Let $X \in Snglr_{\leq k+1, \leq \infty}^{C^0}$ be a C^0 stratified space. A **conically smooth atlas on X** is a collection

$$\tilde{\mathcal{A}} := \{U, \varphi : U \rightarrow X\}$$

with $U \in Bsc_{\leq k+1, \leq \infty}$ and φ an open embedding of stratified spaces.

We require that $\tilde{\mathcal{A}}$ is an open cover of X .

We require that given any $(U, \varphi), (V, \psi) \in \tilde{\mathcal{A}}$ and any point $x \in \varphi(U) \cap \psi(V)$ there is a diagram in $Bsc_{\leq k+1, \leq \infty}$ of the form $U \leftarrow W \rightarrow V$ such that the resulting diagram in $Snglr_{\leq k+1, \leq \infty}^{C^0}$ commutes:

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \psi \\ U & \xrightarrow{\varphi} & X \end{array}$$

When we define $Bsc_{\leq k+1, \leq \infty}$ we will make sense of the way in which this atlas is conically smooth.

In [2] they show that the relation, $\tilde{\mathcal{A}} \sim \tilde{\mathcal{A}}' \leftrightarrow \tilde{\mathcal{A}} \cup \tilde{\mathcal{A}}'$ is an atlas, is an equivalence relation. Further note that set of equivalent atlases on X , \mathcal{A} is ordered by inclusion of collections and thus $\bigcup_{\tilde{\mathcal{A}} \in \mathcal{A}} \tilde{\mathcal{A}}$ is the maximal element.

So to begin our induction we are simply going to declare that the empty manifold $\emptyset \in Snglr_{\leq -1, \leq \infty}^{C^0}$ has a unique empty maximal atlas, and that $Bsc_{\leq -1, \leq \infty} := \emptyset$ is the empty category. There are no basics of negative depth. With these things defined we can make sense of what the objects of our category of basics are.

Definition 2.17. We define the category $Bsc_{\leq k+1, \leq \infty}$ to be the category whose objects are pairs (U, \mathcal{A}_Z) where U is a C^0 basic of the form $\mathbb{R}^i \times C(Z)$ and $Z \in Snglr_{\leq k+1, \leq \infty}^{C^0}$ is a C^0 stratified space of at most depth $k + 1$ equipped with a maximal atlas \mathcal{A}_Z .

In this manner, we can begin building more basics simply from the empty manifold with it's empty atlas. The first of these will be $\mathbb{R}^i \times C(\emptyset) = \mathbb{R}^i$. Thus conically smooth atlases include coverings by Euclidean space. We can then continue to build new basics $\mathbb{R}^i \times C(Y)$ where Y is a compact C^0 stratified space that admits an atlas in the traditional sense. We continue this process to build all our basics and atlases.

A morphisms between objects $f : (U = \mathbb{R}^i \times C(Z), \mathcal{A}_Z) \rightarrow (V = \mathbb{R}^j \times C(Y), \mathcal{A}_Y) \in Bsc_{\leq k+1, \leq \infty}$ is an open embedding of C^0 stratified spaces that satisfies the following properties:

- If on the level of posets f sends the cone point of U to the cone point of V then f is conically smooth along \mathbb{R}^i , Df is injective and:

$$\mathcal{A}_{|_{f^{-1}(V \setminus \mathbb{R}^j)}} = (f|_{f^{-1}(V \setminus \mathbb{R}^j)})^* \mathcal{A}_{V \setminus \mathbb{R}^j}$$

- If on the level of posets f does not send the cone point of U to the cone point of V then we require it factors as open embeddings of C^0 stratified spaces in the following way:

$$f : U \xrightarrow{f_0} \mathbb{R}^j \times \mathbb{R}_{>0} \times Y \hookrightarrow V$$

Further, we require $\{(U, f_0)\}$ to an atlas the represents equivalence class $\mathcal{A}_{\mathbb{R}^j \times \mathbb{R}_{>0} \times Y}$.

Now in our definition for a conically smooth atlas we required that the diagram $U \leftarrow W \rightarrow V$ be in $Bsc_{\leq k+1, \leq \infty}$ in this manner we have required our "transition maps" to be, in a sense, conically smooth. Equipped with this definition we can define our category of basics.

Definition 2.18. The category Bsc is defined:

$$Bsc := \bigcup_{k \geq -1} Bsc_{\leq k, \leq \infty}$$

Definition 2.19. Similarly to Bsc we define the category of conically smooth stratified spaces, $Snglr$, to be the category whose objects are pairs (X, \mathcal{A}) , with X a C^0 stratified spaces and \mathcal{A} a choice of equivalence class of conically smooth atlas on X . Morphisms are stratified open embeddings $f : X \rightarrow Y$ such that $\mathcal{A}_X = f^* \mathcal{A}_Y$.

In order to define the notion of a map of conically smooth stratified spaces being conically smooth we must first declare that the empty map $\emptyset \rightarrow \emptyset$. We will use this to make sense of what it means for any other map of conically smooth stratified spaces to be, itself, conically smooth.

Definition 2.20. To start we will define what it means for a map of basics to be conically smooth. So let $f : X \rightarrow Y$ be a map of stratified spaces, and let $X = \mathbb{R}^i \times C(Z)$, $Y = \mathbb{R}^j \times C(W)$ be basics. We then say that f is **conically smooth** when:

- If onn the level of posets, f does not send the cone point of X to the cone point of Y , then f factors as it did in our definition of maps in Bsc , specifically:

$$f : U \xrightarrow{f_0} \mathbb{R}^j \times \mathbb{R}_{>0} \times Y \hookrightarrow \mathbb{R}^j \times C(W)$$

- If on the level of posets f does send the cone point of X to the cone point of Y then f is conically smooth along \mathbb{R}^i , and the restriction $f^{-1}(Y \setminus \mathbb{R}^j) \rightarrow Y \setminus \mathbb{R}^j$ is conically smooth.

Definition 2.21. We say a map of conically smooth stratified spaces $f : X \rightarrow Y$ is, itself, conically smooth if for each pair of charts $\varphi : U \hookrightarrow X$, $\psi : V \hookrightarrow Y$ with $f(\varphi(U)) \subset \psi(V)$, the composition:

$$\psi^{-1} f \varphi : U \rightarrow V$$

is conically smooth. Since this is a map of basics this is well defined.

Definition 2.22. We let $Srat$ denote the categories whose objects are pairs (X, \mathcal{A}) as in $Snglr$ but whose morphisms are now conically smooth maps that are not necessarily open embeddings.

From this we can see that $Snglr \subset Strat$ is a subcategory. In [2] they show that conically smooth maps properly compose, and that $Strat$ admits finite products.

3 Infinity Categories of Basics

3.1 The $(\infty, 1)$ -categories $Snglr$, $\mathcal{B}sc$ and $\mathcal{M}fld$

Factorization homology is defined using infinity categorical versions of the categories we have defined thus far. In order to define these we will first show that they carry a natural enrichment in Kan and then use the simplicial nerve functor to arrive at an honest quasicategory. To begin let's define a set which we will then use to define a set-valued pre-sheaf on $Strat$. Let X, Y, Z be objects in $Strat$.

Definition 3.1. $Strat_Z(X, Y) := \{f : (Z \times X) \rightarrow (Z \times Y) | pr_Z = pr_Z \circ f\}$.

Or the set of maps, f , that fit into the following commutative diagram:

$$\begin{array}{ccc} Z \times X & \xrightarrow{f} & Z \times Y \\ & \searrow pr_Z & \swarrow pr_Z \\ & Z & \end{array}$$

We use this definition to define an enrichment of $Strat$ of set valued presheafs on $Strat$.

Definition 3.2. We let $Strat(X, Y)$ denote the presheaf that sends a conically smooth stratified space Z to the set $Strat_Z(X, Y)$. We then define the set-valued presheaf enriched category $\mathcal{S}trat$ as the category whose objects are the objects of $Strat$ and the set-valued presheaf between conically smooth stratified spaces X and Y is as defined: $\mathcal{S}trat(X, Y)$.

We will use this presheaf enrichment as a means to construct a simplicial enrichment by first defining the following cosimplicial conically smooth stratified space.

Definition 3.3. We define the map $\Delta_e^\bullet : \Delta \rightarrow \mathcal{S}trat$ by:

$$[n] \mapsto \Delta_e^n := \{t : (0, \dots, n) \rightarrow \mathbb{R} | \sum_{i=0}^n t_i = 1\}$$

Definition 3.4. This allows us to construct a product preserving functor:

$$(-)_\Delta : Fun(\mathcal{Strat}^{op}, Set) \rightarrow sSet$$

Simply by restriction along Δ_\bullet .

In [2] they prove that the simplicial sets $\mathcal{Strat}(X, Y)_{|\Delta} := [n] \mapsto \mathcal{Strat}_{\Delta_e^n}(X, Y)$ and $\mathcal{Snglr}(X, Y)_{|\Delta}$, defined similarly, are Kan complexes. Thus we may use them to define a Kan enrichment we conflate notation referring to \mathcal{Strat} as both the presheaf and Kan enriched categories. Further we may throw the category \mathcal{Strat} under the simplicial nerve functor to get a quasicategory in $sSet$. We will often invoke the theory of quasicategories when working with this category.

3.2 Right Fibrations and The Unstraightening Functor

In order to define an infinity category of basics we must first understand the notion of a right fibration.

Definition 3.5. A map of simplicial sets $f : X \rightarrow Y$ is said to be a **right fibration** if it has the right lifting property with respect to right horn inclusions. Meaning every diagram of the form:

$$\begin{array}{ccc} \Lambda^i[n] & \longrightarrow & X \\ \downarrow & \dashrightarrow & \downarrow f \\ \Delta[n] & \longrightarrow & Y \end{array}$$

exhibits the dotted lift arrow for all $0 < i \leq n$.

Definition 3.6. Let $Kan \subset sSet$ be the full subcategory spanned by those simplicial sets that are Kan complexes. We view the mapping space $sSet(X, Y)$ as a simplicial set in order to regard Kan as a simplicial category. In [3] Lurie shows that for Kan complexes X, Y the space $sSet(X, Y)$ is a Kan complex. In this manner the image of Kan under the homotopy coherent nerve, $\mathcal{N}(Kan)$ is a quasicategory and we refer to it as **the infinity category of spaces** and denote it \mathcal{Spaces} .

Definition 3.7. We define the category of presheaves on a quasicategory C as:

$$PShv(C) := sSet(C^{op}, \mathcal{Spaces})$$

We let $Z \rightarrow C$ and $Z' \rightarrow C$ be two right fibrations. The simplicial set $Fun_C(Z, Z')$ has a maximal sub kan complex and in this way we can view the category of right fibrations, $RFib_C$, over C to be enriched in Kan . Invoking the homotopy coherent nerve we then regard it as a quasicategory.

Lurie shows in [3] that there is an equivalence of infinity categories:

$$PShv(C) \cong RFib_C$$

This is invoked by an $(\infty, 1)$ -category version of the Grothendieck construction that he refers to as the "unstraightening functor."

The unstraightening functor is a functor $Un : [C^{op}, sSet] \rightarrow sSet/C$. In order to define this functor we will first take a look at its right adjoint, the "straightening functor".

Definition 3.8. Let C, X be ∞ -categories. Let $\mathfrak{C}(-)$ denote the left adjoint to the homotopy coherent nerve. Let $p : X \rightarrow C$ be a map of simplicial sets. Then we consider the following pushout diagram in $sCat$:

$$\begin{array}{ccc} \mathfrak{C}(X) & \longrightarrow & \mathfrak{C}(X^\triangleright) \\ \downarrow & & \downarrow \\ \mathfrak{C}(C) & \longrightarrow & K_p \end{array} \tag{1}$$

Here X^\triangleright is the join of X with a single vertex $*$. Which we can think of as a cone construction in $sSet$. The top horizontal map is simply inclusion. The left vertical map is $\mathfrak{C}(p)$. As a pushout the objects of K_p are disjoint unions of the objects in $\mathfrak{C}(C)$ and $\mathfrak{C}(C^\triangleright)$ under the identification of images of X . The left adjoint of the homotopy coherent nerve sends 0-simplices to objects. In this manner we can regard the cone point $*$ of X^\triangleright as an object in K_p . We then define the **straightening functor** $St : sSet/C \rightarrow [C^{op}, sSet]$ as the functor that sends:

$$p \mapsto K_p(-, *)$$

Definition 3.9. The **unstraightening functor** $Un : [C^{op}, sSet] \rightarrow sSet/C$ is the right adjoint to St .

In [3] Lurie shows that this adjunction is a Quillen equivalence between the right fibration model structure on $sSet/C$ and the global projective model structure on simplicial presheaves and in particular induces an equivalence on the ∞ -categories these models present,

$$RFib_{/\mathcal{C}} \cong PShv(\mathcal{C}).$$

We would like to be able to describe structures on stratified spaces. In order to do this we must first understand the homotopy theoretical viewpoint of a G -structure on a smooth manifold. Normally given our group G we would want a group homomorphism $G \rightarrow O(n)$. So what we will do instead is look at the classifying spaces BG and $BO(n)$. In this manner we can describe a structure on n -manifold X as the following lift:

$$\begin{array}{ccc} & & BG \\ & \nearrow \text{dashed} & \downarrow \\ X & \xrightarrow{\tau_X} & BO(n) \end{array}$$

The map τ_X is classifying the tangent bundle on X . We use τ_X to define a functor from $Mfld_n \rightarrow Spaces_{/BO(n)}$ that sends a manifold X to τ_X . We want something like this for the stratified setting.

Definition 3.10. The **tangent classifier** is composite functor defined:

$$\tau : \mathcal{Snglr} \xrightarrow{\text{yoneda}} PShv(\mathcal{Snglr}) \xrightarrow{\iota} PShv(\mathcal{Bsc})$$

The map ι is a restriction map. Let's parse for a moment what is going on here. The Yoneda embedding in this context is sending a conically smooth stratified space X to the space of embeddings $\mathcal{Snglr}(-, X)$. The map ι is then restricting it to the presheaf $\mathcal{Snglr}_{|\mathcal{Bsc}}(-, X)$. So our functor τ sends a conically smooth stratified space X to the presheaf of embeddings of basics into X .

3.3 Infinity Categories of Basics

Definition 1 An ∞ -category of basics is a quasicategory X equipped with a right fibration $X \rightarrow \mathcal{Bsc}$.

Given this definition and the equivalence of ∞ -categories $PShv(\mathcal{Bsc}) \cong RFib_{\mathcal{Bsc}}$ we can describe the tangent classifier as assigning to each conically smooth stratified space an infinity category of basics. Later on we will define the notion of an ∞ -category of manifolds associated to an infinity category of basics. The idea is that given an ∞ -category of basics \mathcal{B} we can build an ∞ -category of stratified spaces with a notion of tangent bundle associated to \mathcal{B} . The tangent classifier assigns to each conically smooth stratified space X an ∞ -category of basics \mathcal{B} such that the associated ∞ -category of manifolds would include this space X as an object.

Definition 3.11. For X a conically smooth stratified space, the **enter path category** $Entr(X)$ is defined as the following ∞ -overcategory:

$$Entr(X) := \mathcal{B}sc_{/X}$$

The ∞ -category of basics over X .

Definition 3.12. We define the functor $\tau_X : Entr(X) \rightarrow \mathcal{B}sc$ as the forgetful functor. Equivalently is the image of $\tau(X)$ under the unstraightening functor. In this manner we think of $Entr(X)$ as the ∞ -category of basics assigned to X by the tangent classifier.

Definition 3.13. Fix an ∞ -category of basics $\mathcal{B} \rightarrow \mathcal{B}sc$ we then define the category of \mathcal{B} -manifolds as the following pullback square:

$$\begin{array}{ccc} \mathcal{M}fld(\mathcal{B}) & \longrightarrow & (RFib_{/\mathcal{B}sc})_{/\mathcal{B}} \\ \downarrow & & \downarrow \\ \mathcal{S}nglr & \xrightarrow{\tau} & RFib_{/\mathcal{B}sc} \end{array}$$

Here the right vertical map is simply forgetting the map into \mathcal{B} . The idea here is that the notion of tangent bundle that the tangent classifier assigns to a conically smooth stratified space factors through our ∞ -category of basics \mathcal{B} . The intuition is that the structure that local structure on our stratified space corroborates what \mathcal{B} describes. More concretely we view objects of $\mathcal{M}fld(\mathcal{B})$ to be pairs (X, g) where X is a conically smooth stratified space and g is a lift of the tangent classifier:

$$\begin{array}{ccc} & & \mathcal{B} \\ & \nearrow & \downarrow \\ Entr(X) & \xrightarrow{\tau_X} & \mathcal{B}sc \end{array}$$

From this we see that a \mathcal{B} -manifold is a conically smooth stratified space X such that τ_X factors through the ∞ -category of basics $\mathcal{B} \rightarrow \mathcal{B}sc$.

Definition 3.14. We define the topological category \mathcal{D}_n as:

- Objects are disjoint unions of copies of \mathbb{R}^n
- Hom-spaces are endowed with the Whitney C^∞ topology

We can then regard this as a quasicategory by first throwing our hom-spaces under the *Sing* functor and then taking its image under the homotopy coherent nerve. Equivalently this can be defined as the homotopy coherent nerve of $\mathcal{B}sc_{0,n}$. It's inclusion into $\mathcal{B}sc$ is a

right fibration, thus when supplied with this map we may regard \mathcal{D}_n as an infinity category of basics.

Example 3.15. A \mathcal{D}_n -manifold is simply a smooth n -manifold.

There is a specific class of infinity categories of basics that we will be using frequently. Those that come from certain subcategories of \mathcal{B}_{sc} , here we are regarding \mathcal{B}_{sc} as a Kan-enriched category.

Definition 3.16. Given a Kan-enriched category C we define a **sieve** on C to be a full subcategory $L \subset C$ such that if $c \in C$ and $d \in L$ with $hom_C(c, d)$ nonempty then necessarily $c \in L$.

So we wish to think of infinity categories of basics as a means of specifying singularity types. A sieve on $\mathcal{B} \hookrightarrow \mathcal{B}_{sc}$ can be thought of as such a list with the convenient property that any arbitrary singularity type of a neighborhood of a point in a member of this list is again a member of the list. There is a model independent notion of right fibration but here we can simply think of the image of this Kan-enriched category under the homotopy coherent nerve to avoid this.

Example 3.17. We define the ∞ -category of basics \mathcal{D}_n^∂ to be the full subcategory of \mathcal{B}_{sc} built from the objects $\{\mathbb{R}^n, \mathbb{R}^{n-1} \times C(*)\}$. Here \mathbb{R}^n is given the trivial stratification $\mathbb{R}^n \rightarrow \{*\}$. We build the stratification map of $\mathbb{R}^{n-1} \times C(*)$ from the definition of the product of stratified spaces and the cone over stratified spaces. More specifically the point $*$ is given the trivial stratification map. Such that the cone $C(*) \cong \mathbb{R}_{\geq 0}$ with the stratification map $C(*) \rightarrow \{0 < 1\}$ that sends the point associated to $0 \in \mathbb{R}_{\geq 0}$ to 0 and all else to 1. Since the product of a singleton with any set is isomorphic to that set we see the map stratification map for \mathbb{R}^{n-1} as a map to $\{0 < 1\}$ that sends the open upper half of \mathbb{R}^n to 1 and the boundary set of points of the form $(0, x_1, \dots, x_{n-1}) \mapsto 0$. Thus the ∞ -category of manifolds $\mathcal{Mfld}(\mathcal{D}_n^\partial)$ has as objects n -manifolds with boundary.

Example 3.18. We define the ∞ -category of basics \mathcal{D}_n^{fr} to be the image of $\mathcal{D}_n(-, \mathbb{R}^n)$ under the unstraightening functor.

$$\begin{array}{ccc} \mathcal{D}_n^{fr} & \longrightarrow & (PShv(\mathcal{D}_n))_{/\mathcal{D}_n(-, \mathbb{R}^n)} \\ \downarrow & & \downarrow \\ \mathcal{D}_n & \longrightarrow & PShv(\mathcal{D}_n) \end{array}$$

Let's take a moment to parse this. There is one vertex in \mathcal{D}_n , it is \mathbb{R}^n and we have that $Emb(\mathbb{R}^n, \mathbb{R}^n) \cong O(n)$. In this manner we can think of a vertex in \mathcal{D}_n^{fr} to be a copy of \mathbb{R}^n together with a framing of \mathbb{R}^n . In this manner a \mathcal{D}_n^{fr} -manifold is a framed n -manifold.

We wish to define a notion of manifold with an embedded submanifold. In order to do so we must first define the infinity category of basics that these manifolds arise from. To begin we will illuminate the following definition:

Definition 3.19. The category $D_{d\subset n}^{Kink}$ is a sieve on \mathcal{Bsc} and therefore an infinity category of basics when supplied with its inclusion into \mathcal{Bsc} . It has two objects $\{\mathbb{R}^n, \mathbb{R}^{n-d} \times C(\mathbb{S}^{n-d-1})\}$ we write the latter $\mathbb{R}^{d\subset n}$. Our morphism spaces are as such:

- $D_{d\subset n}^{Kink}(\mathbb{R}^n, \mathbb{R}^n) := Emb(\mathbb{R}^n, \mathbb{R}^n)$ the space of smooth open embeddings
- $D_{d\subset n}^{Kink}(\mathbb{R}^{d\subset n}, \mathbb{R}^n) := \emptyset$
- $D_{d\subset n}^{Kink}(\mathbb{R}^n, \mathbb{R}^{d\subset n}) := Emb(\mathbb{R}^n, \mathbb{R}^n \setminus \mathbb{R}^d)$
- $D_{d\subset n}^{Kink}(\mathbb{R}^{d\subset n}, \mathbb{R}^{d\subset n}) :=$ the set of $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that there exists an $\tilde{f} : \mathbb{R}^d \times \mathbb{R}_{\geq 0} \times \mathbb{S}^{n-d-1} \rightarrow \mathbb{R}^d \times \mathbb{R}_{\geq 0} \times \mathbb{S}^{n-d-1}$ filling this diagram:

$$\begin{array}{ccc}
\mathbb{R}^d \times \mathbb{R}_{\geq 0} \times \mathbb{S}^{n-d-1} & \xrightarrow{\tilde{f}} & \mathbb{R}^d \times \mathbb{R}_{\geq 0} \times \mathbb{S}^{n-d-1} \\
\text{Cyl} \downarrow & & \downarrow \text{Cyl} \\
\mathbb{R}^n & \xrightarrow{f} & \mathbb{R}^n
\end{array}$$

Definition 3.20. The category $D_{d\subset n}$ is defined as the following pullback:

$$\begin{array}{ccc}
D_{d\subset n} & \longrightarrow & D_n \\
\downarrow & & \downarrow \\
D_{d\subset n}^{Kink} & \longrightarrow & D_n^{PL}
\end{array}$$

Here D_n^{PL} is the monoid of piecewise linear self embeddings of \mathbb{R}^n .

Definition 3.21. We define $D_{d\subset n}^{fr}$ as the following pullback:

$$\begin{array}{ccc}
D_{d\subset n}^{fr} & \longrightarrow & D_n^{fr} \\
\downarrow & & \downarrow \\
D_{d\subset n} & \longrightarrow & D_n
\end{array}$$

3.4 Symmetric Monoidal Infinity-Categories and Disc Algebras

We will make use of Lurie’s notion of symmetric monoidal infinity categories in our definition of factorization homology. In order to make sense of this definition we must first delve back into the theory of fibrations of simplicial sets.

Definition 3.22. We say a map of simplicial sets $f : X \rightarrow Y$ is an inner fibration if it lifts against all inner horns. Meaning for all $0 < i < n$ any diagram of the form:

$$\begin{array}{ccc} \Lambda^i[n] & \longrightarrow & X \\ \downarrow & \nearrow \text{dotted} & \downarrow \\ \Delta[n] & \longrightarrow & Y \end{array}$$

exhibit the dotted lift arrow.

Definition 3.23. Given an inner fibration $p : X \rightarrow Y$ we say a 1-simplex $f : x \rightarrow x'$ in X is a p -**coCartesian edge** if any diagram of the form:

$$\begin{array}{ccc} \Delta^{\{0,1\}} & & \\ \downarrow & \searrow f & \\ \Lambda^0[n] & \longrightarrow & X \\ \downarrow & \nearrow \text{dotted} & \downarrow p \\ \Delta[n] & \longrightarrow & Y \end{array}$$

where the arrow f is pointing out the edge f in X , exhibits the dotted lift arrow.

Definition 3.24. We say that an inner fibration $p : X \rightarrow Y$ is a **coCartesian fibration** if for every edge $f : y \rightarrow y'$ there exists a lift \hat{y} and p -coCartesian edge $\hat{f} : \hat{y} \rightarrow z$ with $p(\hat{f}) = f$.

Definition 3.25. A **symmetric monoidal ∞ -category** is an ∞ -category \mathcal{C}^\otimes equipped with a coCartesian fibration $p : \mathcal{C}^\otimes \rightarrow \mathcal{N}(Fin_*)$ to the nerve of the category of finite pointed sets that satisfies the following property:

- There is an equivalence ∞ -categories $\mathcal{C}_{\langle n \rangle}^\otimes \cong (\mathcal{C}_{\langle 1 \rangle}^\otimes)^n$ induced by the maps $\rho_i : \langle n \rangle \rightarrow \langle 1 \rangle$ in Fin_* for $1 \leq i \leq n$ defined $\rho_i(j) = 1$ for $i = j$ and $\rho_i(j) = *$ otherwise.

Factorization homology will enjoy some nice properties with respect to symmetric monoidal structures. Before we can elucidate this we must first endow our categories \mathcal{Snglr} and \mathcal{Bsc} with symmetric monoidal structures. We will begin by viewing \mathcal{Snglr} as a Kan-enriched

category. From here we take disjoint unions as our symmetric monoidal product. This gives us a symmetric monoidal *Kan*-enriched category. To get a symmetric monoidal ∞ -category in the sense of Lurie we simply take it's image under the homotopy coherent nerve. We can do the same to endow $\mathcal{B}sc$, and $\mathcal{S}nglr$ with our desired product in this manner.

In [1] they show that $\mathcal{M}fld(\mathcal{B})$ for any ∞ -category of basics \mathcal{B} inherits a symmetric monoidal structure from $\mathcal{S}nglr$. They do so by extending its defining diagram to a diagram of symmetric monoidal ∞ -functors.

Definition 3.26. The **disc category on an ∞ -category of basics**, $\mathcal{B} \rightarrow \mathcal{B}sc$ is defined as the smallest symmetric monoidal subcategory containing $\mathcal{B} \subset \mathcal{B}sc$ and we will denote it $Disc(\mathcal{B})$.

We will be using the notation $Alg_{Disc(\mathcal{B})}(\mathcal{V}) := Fun^{\otimes}(Disc(\mathcal{B}), \mathcal{V})$ to refer to the category of symmetric monoidal functors. We will refer to it's objects and $Disc(\mathcal{B})$ -algebras.

4 Factorization Homology

4.1 The Definition of Factorization Homology

Factorization takes in as input some \mathcal{B} -manifold M as well as an algebraic input A from some symmetric monoidal ∞ -category \mathcal{V}^{\otimes} and produces an object $\int_M A$ of \mathcal{V}^{\otimes} . The idea is that this object can be viewed as an invariant of our manifold M thus providing a generalization of traditional homology theories. We can view this object as an invariant of A which leads to connections with Hochschild homology specifically when we choose $M = \mathcal{S}^1$. We are interested in developing a notion of factorization homology for sutured manifolds. To begin exploring this we will look at factorization homology for manifolds with embedded submanifolds. Specifically the factorization homology for $Disc(D_{dCn}^{fr})$ -manifolds.

Definition 4.1. Let \mathcal{V}^{\otimes} be a symmetric monoidal ∞ -category. We define the factorization homology functor as the left adjoint to the restriction functor $Fun^{\otimes}(\mathcal{M}fld(\mathcal{B}), \mathcal{V}) \rightarrow Alg_{Disc(\mathcal{B})}(\mathcal{V})$. We denote it as:

$$\int_{-} : Alg_{Disc(\mathcal{B})}(\mathcal{V}) \rightarrow Fun^{\otimes}(\mathcal{M}fld(\mathcal{B}), \mathcal{V})$$

This left adjoint need not exist, but will exist and correspond with the left adjoint of the underlying ∞ -categories ignoring the symmetric monoidal structure.

Equivalently we may look at factorization homology as a specific left kan extension shown here as the dotted arrow.

$$\begin{array}{ccc} \mathcal{D}isk(\mathcal{B}) & \longrightarrow & \mathcal{V}^\otimes \\ \downarrow & \nearrow \text{dotted} & \\ \mathcal{M}fld(\mathcal{B}) & & \end{array}$$

Let's explore this with the simplest example available. Choose $\mathcal{B} = \mathcal{D}^n$ which is the topological category of disjoint unions, including the empty one, of framed copies of \mathbb{R}^n . Mapping spaces are the space of open embeddings with the C^∞ Whitney topology. We view disjoint unions as the symmetric monoidal product and regard this category as a symmetric monoidal ∞ -category in this way. An algebra on this disk category is then a symmetric monoidal ∞ -functor:

$$A : \mathcal{D}isk(\mathcal{D}_n^{fr}) \rightarrow \mathcal{V}^\otimes$$

We called this functor A for a reason. First note that \mathbb{R}^n with any given framing is isomorphic to \mathbb{R}^n with any other framing. In this manner we will conflate isomorphism classes and consider A to be the image of \mathbb{R}^n under this functor. We will regard A as the algebraic input to factorization homology. So what does a $\mathcal{D}isk(\mathcal{D}_n^{fr})$ -algebra look like. Well, let's consider the specific case when $n = 1$.

The space of open embeddings $Emb(\mathbb{R} \sqcup \mathbb{R}, \mathbb{R})$ viewed up to isotopy amounts only to two choices. Fix an ordering on the connected components of $\mathbb{R} \sqcup \mathbb{R}$ then the embedding either respects this or it does not. We will let $j : \mathbb{R} \sqcup \mathbb{R} \rightarrow \mathbb{R}$ denote the order preserving map. Then we will regard $A(j) : A \otimes A \rightarrow A$ as the multiplication for our algebra A .

There is exactly one map $Emb(\emptyset, \mathbb{R})$. We will denote this map $u : \emptyset \rightarrow \mathbb{R}$. We will regard $A(u)$ as the algebraic unit in the algebra A . These do in fact satisfy unitality and associativity but I have left any proof of this out.

Let's now consider the swap map $\sigma : \mathbb{R} \sqcup \mathbb{R} \rightarrow \mathbb{R} \sqcup \mathbb{R}$. The fact that $j \circ \sigma$ is not isotopic to j shows us that our multiplication map is not necessarily commutative, in the sense that switching the order of our arguments does not give us the same map or even a map that is isotopic. We will think of this as saying that a $\mathcal{D}isk(\mathcal{D}_1^{fr})$ -algebra is not necessarily commutative up to homotopy. In this manner we can consider the algebra A to be E_1 or A_∞ .

If we attempt this same procedure for \mathbb{R}^2 we will see that any maps $\mathbb{R}^2 \sqcup \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are isotopic. In particular if we let $\sigma : \mathbb{R}^2 \sqcup \mathbb{R}^2 \rightarrow \mathbb{R}^2 \sqcup \mathbb{R}^2$ be the swap map then for any $j : \mathbb{R}^2 \sqcup \mathbb{R}^2 \rightarrow \mathbb{R}^2$ we have $j \cong j \circ \sigma$. There are non-homotopic homotopies exhibiting this though and thus our commutativity up to homotopy is not canonical. There is in fact a winding number obstruction preventing this. In this manner we can see $\mathcal{D}isk(\mathcal{D}_2^{fr})$ -algebras as being E_2 . In [1] they prove that:

$$Alg_{\mathcal{D}isk}(\mathcal{D}_n^{fr})(\mathcal{V}) \cong Alg_{E_n}(\mathcal{V})$$

Hopefully these illustrations give some intuition as to why this is the case.

Since we wish to explore the factorization homology for sutured manifolds, let's first take a look at the case of embedded 1-manifolds in a 3-manifold. We will be using the ∞ -category

of basics D_{3C1}^{fr} .

$$\int_- : \mathcal{A}lg_{\mathcal{D}isc(D_{3C1}^{fr})}(\mathcal{V}) \rightarrow Fun^\otimes(\mathcal{M}fld(D_{3C1}^{fr}), \mathcal{V})$$

Let's now make a bit more sense of this. What exactly is the datum of a $\mathcal{D}isc(D_{3C1}^{fr})$ -algebra. In **Proposition 4.1** of [1] Ayala, Francis, and Tanaka prove that for $\mathcal{V} = Mod_{\mathbb{k}}^\otimes$ that there is an equivalence:

$$\int_{S^1} A \cong HC_*(A)$$

In **Proposition 4.8** of [1] they show that the datum of a $\mathcal{D}isk(D_{dCn}^{fr})$ -algebra is a triple (A, B, α) where A is a $\mathcal{D}isc(D_n^{fr})$ -algebra, B is a $\mathcal{D}isc_d^{fr}$ -algebra as α is a map of $\mathcal{D}isc(D_{d+1}^{fr})$ -algebras:

$$\alpha : \int_{S^{n-d-1}} A \rightarrow HC_{D_d^{fr}}^*(B)$$

So in particular for $\mathcal{V} = Mod_{\mathbb{k}}^\otimes$ we have that a $\mathcal{D}isc_{3C1}^{fr}$ -algebra consists of the datum of a triple (A, B, α) . Here A is a $\mathcal{D}isc(D_3^{fr})$ -algebra, B is a $\mathcal{D}isc(D_1^{fr})$ algebra and α is a map of $\mathcal{D}isc(D_2^{fr})$ -algebras:

$$\alpha : HC_*(A) \rightarrow HC^*(B)$$

4.2 Sutured Manifolds and Future Work

Definition 4.2. A **sutured manifold** is a pair (M, γ) with M a compact 3-manifold with boundary and γ an embedded 1-manifold that exists entirely in the boundary such that γ splits the boundary into two components which we call $R_-(\gamma)$ and $R_+(\gamma)$. In order to make a factorization homology for these manifolds we need to first define an ∞ -category of basics such that it's corresponding ∞ -category of manifolds is a sutured manifold. Below is an outline for a potential definition of an ∞ -category of basics whose ∞ -category of manifolds is that of 3-manifolds with boundary together with an embedded 1-manifold on the boundary:

The first step in my PhD project is to successfully define a notion of factorization homology for sutured manifolds. In order to make sense of this notion we must first understand how we can view a 3-manifold with boundary equipped with a 1-manifold embedded on the boundary as a conically smooth stratified space.

Example 4.3. Let M be a 3-manifold with boundary and let L be an embedded 1-manifold on the boundary. Here I show how $L \subset M$ can be seen as a conically smooth stratified space of depth 2. We endow M with a stratification $M \rightarrow \{l < d < m\}$ defined by sending $(M \setminus \partial M) \rightarrow m$, $(\partial M \setminus L) \rightarrow d$ and $L \rightarrow l$. We then can see M as covered by stratified open embeddings of the form $\mathbb{R}^3 \rightarrow \{m\}$, $\mathbb{R}^2 \times C(*) \rightarrow \{d < m\}$ and $\mathbb{R}^1 \times \mathbb{R}^1 \times C(*) \rightarrow \{l < d < m\}$. Thus M can be seen as carrying an atlas of those stratified open embeddings $\{\varphi : \mathbb{R}^3 \rightarrow M\} \cup \{\psi : \mathbb{R}^2 \times C(*) \rightarrow M\} \cup \{\xi : \mathbb{R}^1 \times \mathbb{R}^1 \times C(*) \rightarrow M\}$ such that the maps

- $\varphi : \mathbb{R}^3 \rightarrow (M \setminus \partial M)$
- $\psi| : \mathbb{R}^2 \times \{0\} \rightarrow \partial M$
- $\psi| : \mathbb{R}^2 \times \mathbb{R}_{>0} \rightarrow (M \setminus \partial M)$
- $\xi| : \mathbb{R}^1 \times \{0\} \times \{0\} \rightarrow L$
- $\xi| : \mathbb{R}^1 \times \mathbb{R}_{>0} \times \{0\} \rightarrow (\partial M \setminus L)$
- $\xi| : \mathbb{R}^1 \times \mathbb{R}_{>0} \times \mathbb{R}_{>0} \rightarrow (M \setminus \partial M)$

carry the specified codomain and are smooth.

Definition 4.4. The ∞ -category of basics D_{SM}^{kink} is defined as the sieve on \mathcal{B}_{sc} with object set $\{\mathbb{R}^3, \mathbb{R}^2 \times C(*), \mathbb{R}^1 \times \mathbb{R}^1 \times C(*)\}$. We will denote the second object as \mathbb{R}_+^3 and the third object as $\mathbb{R}_{1 \subset \partial 3}$. We will follow the format of example 1 from this section. We define this as a topological category but regard it as a quasicategory simply by taking its image under the sing functor followed by the homotopy coherent nerve. Alright so our mapping spaces then are going to be defined as the following:

- $D_{SM}^{kink}(\mathbb{R}^3, \mathbb{R}^3) := Emb(\mathbb{R}^3, \mathbb{R}^3)$ the space of smooth open embeddings with the compact open topology
- $D_{SM}^{kink}(\mathbb{R}^3, \mathbb{R}_+^3) = D_{SM}^{kink}(\mathbb{R}^3, \mathbb{R}_{1 \subset \partial 3}) := Emb(\mathbb{R}^3, \mathbb{R}^2 \times \mathbb{R}_{\geq 0})$
- $D_{SM}^{kink}(\mathbb{R}_+^3, \mathbb{R}_+^3) := Emb(\mathbb{R}^2 \times \mathbb{R}_{\geq 0}, \mathbb{R}^2 \times \mathbb{R}_{\geq 0})$
- $D_{SM}^{kink}(\mathbb{R}_+^3, \mathbb{R}^3) = D_{SM}^{kink}(\mathbb{R}_{1 \subset \partial 3}, \mathbb{R}^3) = D_{SM}^{kink}(\mathbb{R}_{1 \subset \partial 3}, \mathbb{R}_+^3) := \emptyset$
- $D_{SM}^{kink}(\mathbb{R}_+^3, \mathbb{R}_{1 \subset \partial 3}) := Emb(\mathbb{R}^2 \times \mathbb{R}_{\geq 0}, (\mathbb{R}^2 \times \mathbb{R}_{\geq 0}) \setminus \mathbb{R}^1)$ which is the space of embeddings that miss $\{0\} \times \mathbb{R}^1 \times \mathbb{R}_{\geq 0}$
- $D_{SM}^{kink}(\mathbb{R}_{1 \subset \partial 3}, \mathbb{R}_{1 \subset \partial 3}) \subset Emb(\mathbb{R}^2 \times \mathbb{R}_{\geq 0}, \mathbb{R}^2 \times \mathbb{R}_{\geq 0})$ the subset of embeddings of $\mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ into itself that fit into this diagram:

$$\begin{array}{ccccc}
\mathbb{R}^1 & \longrightarrow & \mathbb{R}^2 \times \mathbb{R}_{\geq 0} & \longleftarrow & (\mathbb{R}^2 \times \mathbb{R}_{\geq 0}) \setminus \mathbb{R}^1 \\
f_1 \downarrow & & \downarrow f & & \downarrow f_1 \\
\mathbb{R}^1 & \longrightarrow & \mathbb{R}^2 \times \mathbb{R}_{\geq 0} & \longleftarrow & (\mathbb{R}^2 \times \mathbb{R}_{\geq 0}) \setminus \mathbb{R}^1
\end{array}$$

Where the right and left vertical maps are smooth and the middle is conically smooth.

A D_{SM}^{kink} -manifold is a piecewise linear 3-manifold with boundary together an embedded

submanifold of dimension 1 on the boundary. In order to generalize this to smooth manifolds we define the following ∞ -category of basics:

Definition 4.5. The ∞ -category of basics D_{SM} is defined as the following pullback:

$$\begin{array}{ccc} D_{SM} & \longrightarrow & D_n \\ \downarrow & & \downarrow \\ D_{SM}^{kink} & \longrightarrow & D_n^{PL} \end{array}$$

Where D_n^{PL} is the monoid of piecewise linear self embeddings of \mathbb{R}^n and the maps are injections.

Then a D_{SM} -manifold is a smooth 3-manifold with boundary together with an embedded submanifold of dimension 1 on its boundary. For the sake of adding more information with which to build invariants we may wish to frame these manifolds and we do so by defining the following ∞ -category of basics:

Definition 4.6. The ∞ -category of basics D_{SM}^{fr} is defined as the following pullback:

$$\begin{array}{ccc} D_{SM}^{fr} & \longrightarrow & D_n^{fr} \\ \downarrow & & \downarrow \\ D_{SM} & \longrightarrow & D_n \end{array}$$

A D_{SM}^{fr} -manifold is a D_{SM} -manifold with a framing of both the 3-manifold and submanifold of dimension 1 that are compatible.

References

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- [3] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.