Quadrature Domain Limit Shape for Activated Random Walk in One Dimension

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Abstract

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We study one-dimensional activated random walk (ARW) with sleep rate $\lambda > 0$, started from one active particle at each site of a bounded open set $A \subset \mathbb{R}$. For such initial configurations we prove that, after rescaling, the sleeping particles concentrate on an open set $A^* \subset \mathbb{R}$ determined by a quadrature inequality for concave test functions, with asymptotic particle density $\rho_c(\lambda)$ on A^*. This confirms the quadrature inequality Conjecture 6 of Levine and Silvestri for one-dimensional ARW.

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1 Activated Random Walk Background

1.1 History

The study of Activated Random Walk (ARW) is important in the context of what is called Self-Organized Criticality (SOC), which is present in complex natural systems where energy is built up slowly and is released intermittently. More specifically, SOC describes systems attracted to a critical point, without the need for fine-tuning parameters or for outside influence. Systems such as wildfires or landslides are said to exemplify SOC, as they are dissipative systems which eventually lead themselves to a "stable" state, generally by spreading wide enough until activity is ceased.

Mathematicians have been searching for a universal model of SOC since it was originally studied in [BTW87] by Bak, Tang, and Wiesenfeld in 1987. The search for the universal model of SOC has led to intensive research on the Abelian Sandpile model and other Stochastic Sandpile models. In the Abelian sandpile, we typically look at an undirected graph such as $\mathbb{Z}^d$. We have particles on the vertices of such a graph, and when a vertex has more particles than it does neighbors, it "topples" by sending one of its particles to each and every one of its neighbors. This often causes other vertices to have too many particles and thus cause them to topple as well, exemplifying something of an avalanche.

While the Abelian Sandpile has been studied extensively, at this point in time, the ARW appears to be the best candidate for the universal model of SOC. Whereas Brownian motion may model the diffusion of particles, it is argued that ARW is a suitable model for diffusion of energy. This is because when an ARW system is large enough, the resulting behavior does not seem to depend on the initial configuration, nor the underlying graph. The Abelian Sandpile, on the other hand, has been discovered to depend too much on the initial configuration, particularly in the context of the density conjecture [RSZ19]. Another way to put this is that ARW has much better "mixing properties" than the Abelian Sandpile. This and other discoveries have led researchers to pursue the ARW as the universal model of SOC. Unfortunately, the Abelian Sandpile has some algebraic structure which the ARW does not have, so new tools have needed to be developed in order to study and prove properties of ARW. Many of the conjectures surrounding the universality of the ARW are related to the universality of its critical density, and are surveyed in [LS23]. These will be discussed in this paper alongside some current work in progress related to this critical density.

1.2 ARW Construction

Activated Random Walk (ARW) is a particle system on an undirected graph, typically $\mathbb{Z}^d$, where particles move almost independently, with the exception of the following sleep conditions: Each particle is either "active" or "sleeping" at all times. Sleeping particles do not move, and only become active when encountered by an active particle. Active particles perform a simple random walk, and fall asleep at rate $\lambda > 0$ (the "sleep rate") if alone at a site. We allow any number of active particles to occupy the same site, but only one (or zero) sleeping particle may occupy a site at one time. We say a configuration in ARW is "stable" or has "fixated" if all particles are asleep, or, equivalently, if each site in our underlying graph is visited by a particle only finitely many times. We discuss more formal definitions and notation later in this section.

1.3 Critical density and the Density Conjecture

One of the most important results about ARW comes from Theorem 2 in [RS12], where it was shown that for any sleep rate $\lambda > 0$ there exists a well-defined critical density $\zeta_c(\lambda)$. This critical density lies in $[\frac{\lambda}{1+\lambda}, 1]$, and any ARW on $\mathbb{Z}$ with density $\zeta < \zeta_c$ fixates almost surely, and any ARW on $\mathbb{Z}$ with density $\zeta < \zeta_c$ will almost surely never fixate. The density ζ_c represents an ordinary phase transition between a continually active system and one where all present particles eventually sleep. The existence of ζ_c is a remarkable result because it means that we don't need to know where all of the particles in our system are, just the density of their placement, to know whether or not the system will fixate.

Much of the research surrounding ARW has to do with this critical density. Recently in [HJJ24], it is proven that the critical density on a finite line segment is also equal to the critical densities on both the line and on the n-cycle C_n. It is the proof for the n-cycle which has inspired our proof for the probability of stabilization when waking up one particle in a stable supercritical configuration, which is discussed later in this paper.

It is conjectured that this critical density, sometimes referred to as the *threshold density*, is equal to some other densities of note. The stationary density ζ_s is relevant to the ARW wired Markov chain, which is a driven-dissipative system [LS23]. The prediction that $\zeta_c = \zeta_s$ is known as the density conjecture and, while believed to be true, has yet to be proven for the Activated Random Walk. It has already been shown false in general for the sandpile model.

These densities are also conjectured coincide with a third density ζ_a, known as the aggregate density [LS23]. Consider an ARW on $\mathbb{Z}^d$ with sleep rate λ with initial configuration $n\delta_0$ (n particles at the origin and none anywhere else.) As the ARW takes its course, the particles will spread out in a roughly spherical fashion and the resulting stable configuration is known as the ARW aggregate. This is described formally in terms of the aggregate density ζ_a. Showing that ζ_c, ζ_s, and ζ_a coincide is still an open problem.

To prove Theorem 2 in [RS12], the existence of ζ_c, the authors utilize the Least Action Principle, the Abelian Property, and Monotonicity. These are some of the most important properties of the ARW, and are used frequently in our proofs, sometimes almost implicitly. These properties are stated in detail in Section 1.7, after we present the formal definitions for ARW.

1.4 Methods, definitions, and notation

Due to the construction of the Activated Random Walk, many of its properties have been challenging to prove using standard probabilistic techniques. One reason is that we typically consider ARW in a system where particles are conserved, though this can also sometimes actually be an advantageous tool for us to use. Another is that, unlike its brother the Abelian sandpile model, the ARW does not exemplify an algebraic structure. Despite these challenges, progress has been made on proving properties of ARW over the past 15 years or so, thanks to the development of tools with which to tackle such problems.

Among the most useful techniques is the site-wise representation. We may intuitively consider it to be the particles in ARW making the "decisions" to move left, right, or sleep, with their respective probabilities. However, it is also valid to consider the sites themselves to be making such decisions, as the particles themselves are interchangeable. The abelian

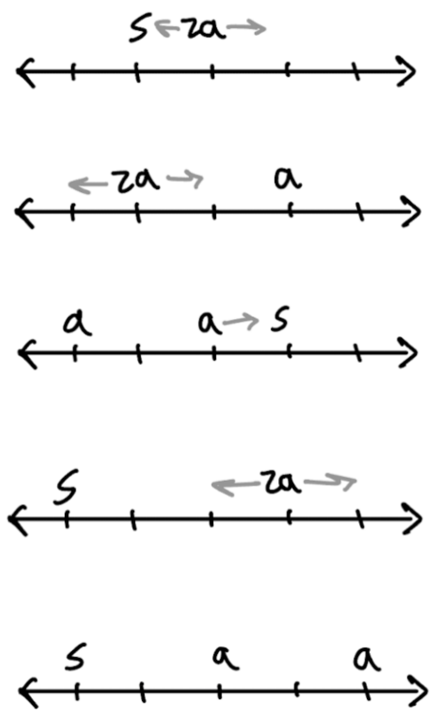

Figure 1: An example of Activated Random Walk (ARW) with three particles on a segment of $\mathbb{Z}$. Sleeping particles are represented by s and active particles are represented by a. We see that active particles move left or right with equal probability, and that sleeping particles are woken up when another particle lands at the same site. Active particles which are alone fall asleep at rate λ.

property of ARW enables us to take strong advantage of this representation. We often consider a stack of toppling instructions left, right, and sleep at each site, generated at random with the correct probabilities. By the abelian property, the order of instructions used does not matter, so we can choose to "topple", or follow instructions from, any site in any order, and still obtain the same resulting configuration. We are able to topple sites in any order which is convenient to our proofs. Sometimes it is also useful to refer to the "odometer," which counts the number of topplings that has happened at each site. This creates another useful tool to control our system.

Though not often as convenient, we may also consider the particle-wise construction. As discussed in [Rol20], this construction enables us to use other methods as the mass transport principle, coupling, resampling, and ergodicity. While we often favor the site-wise representation, some properties of ARW, such as mass conservation, at present have only been justified with the particle-wise construction.

In this section, we detail some of the formal definitions and notation for the processes, conjectures, and theorems discussed above. While there are various ways to formally describe the Activated Random Walk, here we follow the notation introduced in [RSZ19].

1.5 Activated Random Walk

Existing literature describes the ARW as a continuous-time stochastic process, starting with active particles placed on an undirected graph, typically $\mathbb{Z}^d$, with distribution ν. Each active particle will perform an independent random walk on the graph with a translation invariant jump distribution $p(x, y) = p(y - x)$. We typically take this as a uniform probability that the particle will jump to any of its neighbors, if it does choose to jump. If a particle is left alone at a site, it will fall asleep at rate $\lambda > 0$. When a particle is asleep it does not move, but if another particle visits the site where the sleeping particle is located, it will be woken up and reactivated immediately.

We use $\eta_t \in (N_s)^{\mathbb{Z}^d}$ to denote the state of the ARW at a time $t \geq 0$, where $N_s = N_0 \cup \{s\}$ and $N_0 = \{0, 1, 2, ...\}$, in order to indicate the number of active particles (or the existence of one sleeping particle) at any site in $\mathbb{Z}^d$. Here, the s represents a sleeping particle, and an integer assigned to a site represents the number of active particles at that site.

To formally describe the way sleeping particles and active particles interact, with active particles waking up sleeping ones, we make the following definitions: Define $|\cdot|$ to count particles, and $[\![\cdot]\!]$ to count active particles. Then we let $|s| = 1$, $[\![s]\!] = 0$, and $|n| = [\![n]\!] = n$ for $n \in N_0$. Further let $s + 1 = 2$, to show that if a site with a sleeping particle has one active particle added to it, we will thus have two active particles (and no sleeping particles) at that site. We also define

$$n \cdot s = \begin{cases} n, & n \geq 2, \\ s, & n = 1, \\ \text{undefined}, & n = 0. \end{cases}$$

The above three cases describe what happen when a particular site encounters a "sleep" instruction s. If a site with multiple particles encounters a sleep instruction, those particles will remain active. If a site with one particle encounters a sleep instruction, that particle will fall asleep. If there are no particles, the action of the sleep instruction on that site is undefined.

Using $\eta_t \in (N_s)^{\mathbb{Z}^d}$ to denote the state of the ARW at a time $t \geq 0$, we can describe how the process evolves as follows. Each site x will have a Poisson clock that rings at rate $(1+\lambda)[\![\eta(x)]\!]$. This clock is to represent the action of a sleep instruction happening randomly to a site at rate λ. When the clock rings, the state of the system goes through the transition $\eta \to \tau_{xs}\eta$, to represent the falling asleep of a particle at x with probability $\frac{\lambda}{1+\lambda}$, otherwise $\eta \to \tau_{xy}\eta$, to represent the movement of a particle from x to a neighbor site y, with probability $p(y-x)\frac{1}{1+\lambda}$. These transitions are given by

$$\tau_{xy}\eta(z) = \begin{cases} \eta(x) - 1, & z = x, \\ \eta(y) + 1, & z = y, \\ \eta(z), & \text{otherwise,} \end{cases}$$

and

$$\tau_{xs}\eta(z) = \begin{cases} \eta(x) \cdot s, & z = x, \\ \eta(z), & \text{otherwise.} \end{cases}$$

With all of this notating our process, we say that $(\eta_t)_t \geq 0$ fixates if $\eta_t(x)$ is eventually constant in t for each fixed $x \in \mathbb{Z}^d$.

1.6 Site-wise representation

As discussed above, the site-wise representation is one of the most useful tools that we have to attack problems around the Activated Random Walk. When we define the process in terms of the sites rather than the particles, we introduce a few new terms, again following the terminology and notation used in [RSZ19]. Let η denote a configuration in $(N_s)^{\mathbb{Z}^d}$, where a site x is considered unstable in the configuration if it has one or more active particles, i.e. $\eta(x) \geq 1$. If $\eta(x) < 1$, then the site x is considered stable.

- **Toppling** When we apply an operator τ_{xy} or τ_{xs} to a configuration η, we call this the act of *toppling* site x. Topplings can further be described as legal or acceptable (or not legal or not acceptable). A *legal* toppling is a toppling of an unstable site. If a site x contains one sleeping particle, $\eta(x) = s$, then toppling x, while not legal, would be considered *acceptable*. It is not acceptable to topple a site where $\eta(x) = 0$. Any legal toppling is also acceptable. In practice, being able to consider topplings of sites containing a sleeping particle is a useful property, and we formally define the outcomes as $s - 1 = 0$ and $s \cdot s = s$.

- **Odometer** Let the field $h \in (N_0)^{\mathbb{Z}^d}$ be the *odometer*, which counts the number of topplings that occur at each site.

To formally define a toppling, we define a fixed field of instructions $\mathcal{I} = (\tau^{x,j})_{x\in\mathbb{Z}^d,\, j\in\mathbb{N}}$, where for each site x and odometer value j, $\tau^{x,j}$ is equal to either τ_{xs} or τ_{xy} for some y. Then the toppling operation at x is defined by $\Phi_x(\eta, h) = (\tau^{x,h(x)+1}\eta,\ h + \delta_x)$. This is a fancy way of saying that when a toppling operation happens at a site, we apply a toppling operator to the configuration and the odometer h at said site will increase by 1.

Of course, we will typically be composing several toppling operators, so we define $\Phi_\alpha = \Phi_{x_k} \circ \Phi_{x_{k-1}} \circ \cdots \circ \Phi_{x_1}$ from a finite sequence of sites $\alpha = (x_1, \ldots, x_k)$. We say that α is a legal (or acceptable) sequence of topplings for a configuration and odometer (η, h) if, for every $1 \leq j \leq k$, the operator Φ_{x_j} is legal (acceptable) for $\Phi_{x_1,\ldots,x_{j-1}}(\eta, h)$.

It is also convenient to refer to a subset $V \subset \mathbb{Z}^d$ of sites as stable when every site $x \in V$ is stable. We say that α stabilizes in V if α is acceptable for (η, h) and $\Phi_\alpha(\eta, h)$ is stable in V. Finally, for a sequence of sites/topplings α, we define m_α to count how many times a given site x occurs in α, namely

$$m_\alpha(x) = \sum_\ell \mathbf{1}_{\{x_\ell = x\}}, \qquad x \in \mathbb{Z}^d.$$

We write $m_\alpha \leq m_\beta$ if $m_\alpha(x) \leq m_\beta(x)$ for all x, and use the same notation for two configurations $\eta \leq \tilde{\eta}$. This notation is necessary for stating the important properties and lemmas below.

1.7 Important lemmas

The following lemmas are essential tools for proving properties of ARW, and are so integral to our understanding of ARW that we sometimes use them implicitly. These lemmas are presented as they are stated in [RSZ19]. We will not prove these here, but the proofs can be found in [RS12], and many rely on the below properties:

Property 1.1. *If α is a legal sequence for η, then $\Phi_\alpha(\eta, h)$ depends on α only through m_α.*

Property 1 follows generally from the "abelianess of arithmetics" [RS12], and tells us that the resulting configuration $\Phi_\alpha(\eta, h)$ depends not on the order or exact locations of the sites in α themselves, but on the number of firings at each site in α.

Property 1.2. *$\Phi_\alpha \eta(x)$ is non-increasing in $m_\alpha(x)$ and non-decreasing in $m_\alpha(z)$ for $z \neq x$.*

This holds since toppling site x cannot increase $\eta(x)$ and toppling sites other than x cannot decrease $\eta(x)$.

Property 1.3. *If x is unstable in η and $\eta'(x) \geq \eta(x)$, then x is unstable in η'.*

This property is almost trivial, and holds because an unstable site cannot become stable by gaining more particles.

Property 1.4. *If moreover $\eta' \geq \eta$, then $\Phi_x \eta' \geq \Phi_x \eta$.*

Like Property 1.3, Property 1.4 is almost trivial, and is true because toppling the same site in two different configurations will alter both configurations the same way, thus keeping their relationship the same. The below lemmas are consequences of these properties:

Lemma 1.5 (Local abelianness)**.** *If α and β are acceptable sequences of topplings for the configuration (η, h), such that $m_\alpha = m_\beta$, then $\Phi_\alpha(\eta, h) = \Phi_\beta(\eta, h)$.*

This lemma says that if we have two acceptable sequences of topplings that are perhaps in different orders, we still end up with the same ending configuration and odometer, no matter which sequence we use. This essentially allows us to consider the toppling of a configuration in whatever order we like, which gives us a lot of flexibility!

For the next lemma, we introduce another piece of notation. For $V \subseteq \mathbb{Z}^d$, let

$$m_{V,\eta,h}(x) = \sup\{m_\beta(x) : \beta \subseteq V \text{ legal for } (\eta, h)\},$$

for $x \in \mathbb{Z}^d$. Further, notice that

$$m_{V,\eta,h}(x) = \sup\{m_{V',\eta,h}(x) : V' \subseteq V \text{ finite}\},$$

and let $m_{\eta,h} = m_{\mathbb{Z}^d,\eta,h}$.

Lemma 1.6 (Least Action Principle)**.** *If α is an acceptable sequence of topplings that stabilizes (η, h) in V, then $m_{V,\eta,h} \le m_\alpha$.*

We sometimes also phrase this in terms of two legal sequences, as in [RS12]: If α and β are legal sequences of topplings for η such that β is contained in V and α stabilizes η in V, then $m_\beta \le m_\alpha$. However, α need not be legal, only acceptable. The Least Action Principle thus tells us that if we have a sequence of topplings which stabilizes a starting configuration, then any other legal sequences on the same configuration will not topple any site any more often than the stabilizing sequence. In other words, if configuration is stabilized, then there are no more legal topplings we can do.

Lemma 1.7 (Global abelianness)**.** *If α and β are both legal toppling sequences for (η, h) that are contained in V and stabilize (η, h) in V, then $m_\alpha = m_\beta = m_{V,\eta,h}$. In particular, $\Phi_\alpha(\eta, h) = \Phi_\beta(\eta, h)$.*

This follows immediately from the Least Action Principle, by noting that $m_\alpha \le m_\beta \le m_\alpha$.

Lemma 1.8 (Monotonicity)**.** *If $V \subseteq V'$ and $\eta \le \eta'$, then $m_{V,\eta,h} \le m_{V',\eta',h}$.*

We know that a specific site x is stable if $\eta(x) < 1$, and we similarly say that the entire configuration η is *stabilizable* starting from odometer h if $m_{\eta,h}(x) < \infty$ for every $x \in \mathbb{Z}^d$. If $m_{\eta,h}(x) = \infty$ for every x, then the configuration η is *explosive*.

While the above lemmas and properties hold for a fixed instruction field $\mathcal{I}$, practically, we wish to consider a random field $\mathcal{I}$, as to create a realistic model for ARW as a stochastic process. To that end, $\mathcal{I}$ is distributed as follows: for each site $x \in \mathbb{Z}^d$ and odometer value $j \in \mathbb{N}$, let $\tau^{x,j}$ be τ_{xy} with probability $\frac{p(y-x)}{1+\lambda}$ and τ_{xs} with probability $\frac{\lambda}{1+\lambda}$. Further, let ν be a translation-invariant ergodic distribution with a finite density $\nu(\eta(0))$, which is independent of $\mathcal{I}$. Then we get the following lemma, which tells us that the probability of stabilizing from any starting configuration is always 0 or 1:

Lemma 1.9 (Zero-one law)**.** *For the ARW started from the initial distribution ν, the following events have the same probability, and this probability is either* 0 *or* 1*:*

$$\mathbb{P}^\nu\big(\textit{fixation of } (\eta_t)_{t\geq 0}\big) = \mathbb{P}^\nu\big(\eta_0 \textit{ is stabilizable}\big) = \mathbb{P}^\nu\big(m_{\eta_0}(0) < \infty\big) \in \{0,1\}.$$

This lemma also exemplifies our tendency to equate "stabilizable" to "a particle is active at the origin only finitely often." Finally, we have the mass-transport principle, which is an essential tool for proving the critical density theorem. Here, we present it in terms of a general graph $G = (V, E)$, though we usually consider this graph to be $\mathbb{Z}^d$.

Lemma 1.10 (Mass-transport principle)**.** *Consider the graph $G = (V, E)$, and consider the sequence of stationary random variables $\{h_v\}_{v\in V}$, for which $h_v(w) \geq 0$ and $\sum_w \mathbb{E}h_v(w) < \infty$ for all $w \in V$. Then*

$$\sum_v \mathbb{E}h_v(w) = \sum_w \mathbb{E}h_v(w).$$

The mass-transport principle tells us that, on average, the mass exiting a site x will be equal to the mass entering site x. The "mass" in question may be any non-negative function h_v. In the case of proving the density theorem, this function is usually taken to be an indicator function on a specific event.

2 Quadrature Inequality for Activated Random Walk

2.1 Background: Quadrature Domains

The term *quadrature* traditionally refers to methods for computing areas and integrals; in modern potential theory, a *quadrature domain* is a domain where integrals of certain functions can be computed from finitely many point evaluations.

The connection between quadrature domains and discrete stochastic processes was explored by Levine and Peres [LP10], who proved that the scaling limit of *internal diffusion-limited aggregation* (internal DLA, or IDLA) converges to a quadrature domain. In internal DLA, random walkers are released sequentially from the origin and freeze upon first reaching an unoccupied site. The limiting aggregate satisfies a quadrature inequality for superharmonic functions, characterizing the domain where particles accumulate. Surprisingly, the discrete randomness of individual particle trajectories disappears in the continuum limit, leaving only a deterministic domain characterized by a simple inequality.

More recently, Cairns introduced an abstract "smash sum" operation on bounded open sets and proved that, under natural assumptions (monotonicity, symmetries, associativity, and conservation of Lebesgue measure), this operation is uniquely determined and coincides with the continuous Diaconis–Fulton smash sum [Cai22]. In her framework, the smash sum is characterized as the unique bulky quadrature domain for a given weight, providing a dimension-independent continuum description of aggregation models such as IDLA. In this paper we show that one-dimensional activated random walk fits into the same picture, with weight determined by the Activated random walk's (ARW) critical density.

The key insight connecting ARW to quadrature domains is the following: if uniform density spreads from a compact set A to fill a larger domain A^* at a lower density $\zeta_a < 1$ (which we will later identify with the critical density $\rho_c = \rho_c(\lambda)$), then superharmonic functions provide natural test functions to characterize this spreading. The quadrature inequality captures the physical principle that the integral of any superharmonic function over the initial configuration bounds its integral over the final, spread-out configuration:

$$\int_A u(x)\,dx \;\geq\; \zeta_a \int_{A^*} u(x)\,dx$$

for all superharmonic u. This characterization provides a continuum description of ARW's scaling limit, analogous to how internal DLA's limit is described by quadrature domains.

This framework motivates the characterization of ARW's scaling limit through the following quadrature inequality theorem.

Very informally, one can think of the quadrature inequality as describing a "trash compactor" scene from Star Wars: A New Hope (1977). The ARW system behaves as if invisible walls are trying to squeeze the mass that starts in A into a more spread out region A^* (squirting up and out the sides of the compactor)containing the same mass, much like the compactor walls close in on the characters, even though in the movie they escape at the last second. Clip here: https://www.youtube.com/watch?v=6u3QInIMVME

2.2 Main result

In this paper, we focus on activated random walks in one dimension (on $\mathbb{Z}$). Our main result will be expressed in terms of the aggregate density ζ_a for activated random walk, as defined below:

Definition 2.1 (Conjecture 1 from [LS23] for $d = 1$)**.** *Let A_n denote the random set of sites visited by at least one walker during ARW started from n particles at the origin in $\mathbb{Z}$.*

Then $\zeta_a = \zeta_a(\mathbb{Z}, \lambda)$ is the positive constant such that for any $\epsilon > 0$, with probability tending to 1 as $n \to \infty$, the random set A_n contains all sites of $\mathbb{Z}$ that belong to the origin-centered interval with length $(1-\epsilon)n/\zeta_a$, and A_n is contained in the origin-centered interval of length $(1+\epsilon)/\zeta_a$.

Recall that if $U \subset \mathbb{R}^d$ is open, a function $u : U \to \mathbb{R}$ is called *superharmonic* if it satisfies the mean–value inequality

$$u(x) \;\ge\; \frac{1}{|B_r(x)|} \int_{B_r(x)} u(y)\, dy$$

for every ball $B_r(x) \subset U$. In our one-dimensional setting this is equivalent (for C^2 functions) to $u'' \le 0$, so superharmonic functions are exactly concave functions on intervals, and we will use this concavity picture throughout.

We will additionally utilize the following notation to represent the configuration of sleeping particles:

Definition 2.2. *For $\epsilon > 0$, an ARW configuration on $\epsilon\mathbb{Z}$ is a function $\eta : \epsilon\mathbb{Z} \to \mathbb{N} \cup \{\mathfrak{s}\}$, where $\eta(x) = k$ means there are k active particles at x and $\eta(x) = \mathfrak{s}$ means there is one sleeping particle at x. Let $\mathcal{S}(\eta)$ denote the random configuration obtained by running ARW with sleep rate $\lambda > 0$ started from η until the system is stable.*

In particular, for a bounded open set $A \subset \mathbb{R}$ and mesh $\epsilon > 0$, we write $\mathcal{S}(1_{A\cap\epsilon\mathbb{Z}})$ for the stabilized configuration started from one active particle at each site of $A \cap \epsilon\mathbb{Z}$.

Each realization of $\mathcal{S}(1_{A\cap\epsilon\mathbb{Z}})$ is a discrete set of sleeping particle locations in $\mathbb{R}$. We identify this set with the counting measure

$$\sum_{x\in\mathcal{S}(1_{A\cap\epsilon\mathbb{Z}})} \delta_x,$$

so we view $\mathcal{S}(1_{A\cap\epsilon\mathbb{Z}})$ as a random counting measure on $\mathbb{R}$.

Note also that the notation we use is very subtly different from the notation used in [LS23] to represent the same object, where the intersection is not in the subscript: $\mathcal{S}(1_A \cap \epsilon\mathbb{Z})$.

Throughout, $|\cdot|$ denotes Lebesgue measure on $\mathbb{R}$; for an interval $I = (a, b)$ we write $|I| = b - a$. We can now state our main result.

Theorem 2.3 (Quadrature inequality, Conjecture 6 from [LS23] with updated notation)**.** *Let $A \subset \mathbb{R}$ be a bounded open set whose boundary has Lebesgue measure 0. Let $\epsilon > 0$, and start with one active particle at each point of $A\cap\epsilon\mathbb{Z}$. Run ARW with sleep rate $\lambda > 0$. Then, as $\epsilon \to 0$,*

$$\epsilon\,\mathcal{S}(1_{A\cap\epsilon\mathbb{Z}}) \overset{*}{\rightharpoonup} \zeta_a 1_{A^*},$$

where A^* is the unique open subset of $\mathbb{R}$ satisfying

$$\int_A u dx \geq \zeta_a \int_{A^*} u dx$$

for all integrable superharmonic functions u on A^*.

Theorem 2.3 should be read as a characterization (and hence a definition) of the limiting domain A^*. In the rest of the paper, we regard A^* as the continuum limit shape associated with A in Theorem 2.3.

Figure 2: Original configuration with density 1 on the pixels (top left) and sleeping configurations after stabilization (read left-right, top-bottom) with sleep rates $\lambda = 4, 1$, and $1/2$. As we lower the sleep rate, the resulting configuration of sleeping particles is even more spread out (lower density), but still roughly in the same shape. Inspired by Figure 3 in [LS23], and using code from [Son25].

Levine and Silvestri define the aggregate density $\zeta_a(\lambda)$ via the visited set A_n, requiring $|A_n| \sim n/\zeta_a(\lambda)$ as $n \to \infty$ (Conjecture 1 of [LS23]). In dimension one, Hoffman–Johnson–Junge showed that the minimal interval $[L_n, R_n]$ containing A_n has length

$$R_n - L_n + 1 \;\sim\; \frac{n}{\rho_c(\lambda)}$$

with exponentially high probability as $n \to \infty$ [HJJ24]. Since $|A_n|$ and $R_n - L_n + 1$ differ by at most a constant, these two descriptions of the scaling limit must agree, giving:

Corollary 2.4. *For one-dimensional activated random walk, $\zeta_a(\lambda) = \rho_c(\lambda)$.*

Notation for critical density. For the remainder of the paper we therefore write ρ_c instead of ζ_a and refer to ρ_c as the critical density.

Notation for size. Throughout the paper, we use the symbol $|\cdot|$ in two related senses: for a subset $E \subset \mathbb{R}$ we write $|E|$ for its Lebesgue measure, and for a finite subset $F \subset \mathbb{Z}$ we write $|F|$ for its cardinality. Additionally, for $\epsilon > 0$ and $A_\epsilon := \{z \in \mathbb{Z} : z\epsilon \in A\}$, the quantity $|A_\epsilon|$ denotes the number of such lattice points, while $|A|$ denotes the Lebesgue measure of the set $A \subset \mathbb{R}$. When necessary for emphasis we may write $\mu(E)$ for the Lebesgue measure of E.

2.3 Geometric description of A^* in 1D, three ways

In one dimension, the structure of A^* is relatively concrete. When A consists of finitely many disjoint intervals, Levine and Peres showed that for IDLA the scaling limit is a *smash sum of balls* (which in one dimension are simply intervals) [LP10]. In the ARW setting we describe the same limit shape by first defining an equivalent set A^{comb} with a combinatorial origin.

Geometrically, A^{comb} is meant to describe what happens when the initial unit-density mass on A is allowed to spread out until the density drops to ρ_c. For each component A_i we first expand it to an interval A_i^* with the same center and total mass but density ρ_c. Whenever two expanded intervals overlap, we merge them into a single interval whose length is the sum of their lengths and whose center is their center of mass, reflecting the fact that mass from the two components has mixed. The final union A^{comb} is therefore the one-dimensional shape obtained by expanding each component to density ρ_c, and letting overlapping pieces merge together.

With this picture in mind, we now state the definition of A^{comb} for finite unions of intervals.

Definition 2.5 (A^{comb} for finite unions of intervals)**.** *Let $A_1, A_2, ..., A_n \subset \mathbb{R}$ be n disjoint open intervals with centers $c_1 < c_2 < ... < c_n$, and let $A = \bigcup_{i=1}^n A_i$. Define A_i^* for a single interval A_i as the interval with the same center as A_i and length $|A_i^*| = |A_i|/\rho_c$ for $i = 1, ..., n$.*

We define A^{comb} using the following iterative algorithm:

Initialization: *Let* $\mathcal{C}_0 = \{A_1^*, A_2^*, ..., A_n^*\}$*, where each* A_i^* *is defined as above. Specifically,* A_i^* *is an open interval* (a_i, b_i) *with center* $c_i = \frac{a_i+b_i}{2}$*, which coincides with the center of* A_i*, and length* $\ell_i = b_i - a_i$*, which is longer than the length of* A_i *by a factor of* $\frac{1}{\rho_c}$*.*

Iterate: *At each step, examine all unordered pairs* (I, J) *of intervals present in* $\mathcal{C}_k$*. If any overlap, select one such pair. Merge them into a single interval, replace those two by this single interval, and repeat the process with the new set* $\mathcal{C}_{k+1}$*. Continue until no overlapping pairs remain.*

More explicitly, for steps $k = 0, 1, 2, \ldots$*, define* $\mathcal{C}_{k+1}$ *from* $\mathcal{C}_k$ *as follows:*

1. *If there exist a pair of intervals* $I = (a, b)$ *and* $J = (a', b') \in \mathcal{C}_k$ *such that* $I \cap J \neq \emptyset$ *(i.e.,* $a < b'$ *and* $a' < b$*, and the intervals overlap), then:*
   - *Let* $\ell = |I| + |J| = (b - a) + (b' - a')$*.*
   - *Let* $c = \frac{|I|\cdot\frac{a+b}{2}+|J|\cdot\frac{a'+b'}{2}}{|I|+|J|} = \frac{(b-a)\cdot\frac{a+b}{2}+(b'-a')\cdot\frac{a'+b'}{2}}{(b-a)+(b'-a')}$*.*
   - *Replace* I *and* J *in* $\mathcal{C}_k$ *with the single interval* $(c - \ell/2,\ c + \ell/2)$ *to form* $\mathcal{C}_{k+1}$*.*
2. *If no such pair exists, terminate. Let* $\mathcal{C}_N$ *be the final collection of intervals.*

Define the resulting open set A^{comb} *for* A *as* $A^{\text{comb}} = \bigcup_{I \in \mathcal{C}_N} I$*.*

Notice that if the algorithm terminates with a single interval in $\mathcal{C}_N$ (i.e., $|\mathcal{C}_N| = 1$), then $A^{\text{comb}} = \mathcal{C}_N$ is the unique interval satisfying:

$$|A^{\text{comb}}| = \frac{\sum_{i=1}^n |A_i|}{\rho_c} = \frac{|A|}{\rho_c}$$

and

$$c_{A^{\text{comb}}} = \frac{\sum_{i=1}^n c_i |A_i|}{\sum_{i=1}^n |A_i|},$$

where c_i is the center of A_i.

Note also that if we start with just a single bounded open interval A, then A^{comb} is the unique open interval with length equal to $|A|/\rho_c$ and with center of mass equal to that of A.

We record the following basic property of the construction.

Remark 2.6. *At each step the number of intervals in $\mathcal{C}_k$ decreases by one whenever we merge a pair, and it never increases; since we start with finitely many intervals, the process must terminate.*

The map $A \mapsto A^{\mathrm{comb}}$ need not be injective; indeed, different starting sets A can have the same A^{comb}. The following example illustrates this phenomenon.

Example 2.7. *Suppose $A_1 = (0,3)$ and $A_2 = (-1,0) \cup (1,2) \cup (3,4)$. Then $|A_1| = |A_2| = 3$, and both have center of mass equal to 1.5. If the critical density is small, such as $\rho_c(\lambda) = 0.1$, then*

$$A_1^{\mathrm{comb}} = A_2^{\mathrm{comb}} = (-13.5, 16.5).$$

The above iterative process to construct A^{comb} in Definition 2.5 is abelian and will always result in the same collection $\mathcal{C}_N$ for a fixed starting set A, due to the abelian property of Activated Random Walk. In Section 5.4 we will provide proofs of these properties, which do not rely on Activated Random Walk dynamics at all.

The main result of this subsection is Proposition 2.10, that our explicit combinatorial construction coincides with the continuum smash sum. Motivated by Sakai's and Cairns's work, we fix a notation for the continuum smash sum of a general bounded open set A.

Informally, the smash sum of two bounded open sets $A, B \subset \mathbb{R}^d$ is the canonical open set that replaces $A \cup B$ in integrals of superharmonic functions. More generally, given a bounded nonnegative weight w with compact support, a (bulky) quadrature domain for w is an open set Q such that

$$\int_Q u(x)\,dx = \int_{\mathbb{R}^d} u(x)w(x)\,dx$$

for every integrable superharmonic function u on Q. By work of Sakai and Cairns, such a domain exists and is unique up to Lebesgue null sets, and for $w = 1_A + 1_B$ it is denoted $A \oplus B$ and called the (continuous Diaconis–Fulton) smash sum of A and B.

By Sakai's uniqueness theorem for quadrature domains, there is at most one bulky quadrature domain associated to a given weight w (up to null sets). See Sakai [Sak84], Sections 6–7; see also Cairns [Cai22] for the axiomatic formulation via the continuous Diaconis–Fulton smash sum.

Definition 2.8 (Smash-domain A^{sm})**.** *Let $A \subset \mathbb{R}$ be a bounded open set (possibly a countable union of intervals) and set $w := \rho_c 1_A$. By Sakai's existence and uniqueness theorem for quadrature domains [Sak84], together with Cairns's characterization of the continuous Diaconis–Fulton smash sum [Cai22], there exists a unique bulky quadrature domain for the weight w. We denote this domain by A^{sm} and refer to it as the smash-domain associated with A.*

Equivalently, A^{sm} is the unique open set (up to null sets) such that

$$\int_{A^{\mathrm{sm}}} u(x)\, dx = \rho_c \int_A u(x)\, dx$$

for all integrable superharmonic u on A^{sm}. Particularly, any open set B satisfying the quadrature inequality $\int_A u\, dx \geq \rho_c \int_B u\, dx$ for all such u must coincide with A^{sm}.

Remark 2.9. *In higher dimensions, Levine and Peres [LP10] construct the continuum smash sum for bounded open sets $A, B \subset \mathbb{R}^d$ via the obstacle problem with density $\rho = \mathbf{1}_A + \mathbf{1}_B$. In that setting, Proposition 2.12 and Corollary 2.13 of [LP10] show that the noncoincidence set D is a quadrature domain for ρ, satisfies $\lambda(\partial D) = 0$, and has volume $\lambda(D) = \int \rho = \lambda(A) + \lambda(B)$, where λ denotes Lebesgue measure. Then, the boundary of this smash sum has zero Lebesgue measure.*

In our setting, for a bounded open set $A \subset \mathbb{R}$, we take $\rho = \rho_c \mathbf{1}_A$, and A^{sm} is defined as the unique bulky quadrature domain associated with ρ in the sense of Sakai and Cairns; see Definition 2.8. Thus A^{sm} is exactly the continuum smash domain corresponding to $\rho_c \mathbf{1}_A$ in the framework of [LP10]. By the general theory of quadrature domains, we may choose A^{sm} so that its boundary has Lebesgue measure zero, and

$$\lambda(A^{\mathrm{sm}}) \;=\; \frac{1}{\rho_c} \int \rho_c \mathbf{1}_A\, dx \;=\; \frac{\lambda(A)}{\rho_c}.$$

For finite unions of intervals this agrees with our explicit combinatorial construction A_{comb} (Proposition 2.10), and for general bounded open A we obtain A^{sm} as the limit of the finite-union smash domains A_n^{sm} in the symmetric-difference metric, using the continuity of the smash sum as in Section 6 of [LP10] together with our approximation by finite unions (Section 8).

Proposition 2.10 (A^{comb} and A^{sm} agree with A^*)**.** *Let $A \subset \mathbb{R}$ be a finite union of disjoint open intervals. Then A^{comb} from Definition 2.5 agrees with A^{sm} (Sakai/Cairns characterization of the continuous Diaconis–Fulton smash sum) up to sets of measure zero, and hence with the set A^* appearing in Theorem 2.3.*

Proof. This follows from Theorem 5.19 together with the uniqueness of quadrature domains due to Sakai [Sak84] (see also [Cai22]). The detailed proof is given after Theorem 5.19 in Section 5.4. □

When A is a finite union of disjoint intervals, Proposition 2.10 shows that this abstract A^{sm} coincides (up to sets of measure zero) with the combinatorial construction of Definition 2.5.

2.4 Equivalent statement, without weak *

To demystify the statement of the conjecture and eliminate weak star convergence, we prove that the conjecture can be reframed using particle counts.

For the rest of this paper, let $|\cdot|$ denote the length of an interval. Specifically, for any interval $I = (a, b)$ (or $[a, b]$, etc.), we write $|I| = b - a$ for its length.

When translating results from the discrete lattice $\epsilon\mathbb{Z}$ to the continuum $\mathbb{R}$, it is necessary to rescale particle counts appropriately. For any fixed region $U \subset \mathbb{R}$, the number of lattice sites in U grows like $|U|/\epsilon$ as $\epsilon \to 0$, so the total number of particles in U diverges, unless rescaled.

To obtain a meaningful continuous density in the limit, the discrete particle count is multiplied by ϵ. This reflects that Riemann sums over $\epsilon\mathbb{Z}$ approximate Lebesgue integrals:

$$\epsilon \sum_{x \in U \cap \epsilon\mathbb{Z}} f(x) \;\approx\; \int_U f(x)\, dx.$$

Thus, all statements about limiting particle densities or measure convergence include this ϵ scaling to ensure the discrete sums correspond with continuous integrals.

Proposition 2.11. *Let $A \subset \mathbb{R}$ be a bounded open set, and let A^* be the quadrature domain associated with $w = \rho_c 1_A$ as in Definition 2.8.*

Let $U \subset \mathbb{R}$ be an open set, and $\mathcal{N}_\epsilon(U)$ be the random variable counting the number of particles in $U \cap \mathcal{S}(1_{A\cap\epsilon\mathbb{Z}})$ after stabilization:

$$\mathcal{N}_\epsilon(U) = \int_{\mathbb{R}} 1_U d\mathcal{S}(1_{A\cap\epsilon\mathbb{Z}}).$$

Then, as $\epsilon \to 0$,

$$\epsilon\, \mathcal{S}(1_{A\cap\epsilon\mathbb{Z}}) \xrightarrow{*} \rho_c 1_{A^*} \tag{1}$$

if and only if for every open set $U \subset \mathbb{R}$,

$$\lim_{\epsilon\to 0} \epsilon\, \mathbb{E}\Big[\mathcal{N}_\epsilon(U)\Big] = \rho_c\, |U \cap A^*|. \tag{2}$$

Note that $\mathcal{N}_\epsilon(U)$ is a random variable because the measure inside the integral, which is the final configuration of sleeping particles, is random.

Proof. Let A be a bounded open subset of $\mathbb{R}$.
($\Rightarrow$) Assume that $\epsilon\, \mathcal{S}(1_{A\cap\epsilon\mathbb{Z}}) \xrightharpoonup{*} \rho_c 1_{A^*}$ as $\epsilon \to 0$, and let $U \subset \mathbb{R}$ be an open set. Then by definition of weak star convergence, for any bounded continuous function f,

$$\lim_{\epsilon\to 0} \int_{\mathbb{R}} f d(\epsilon\, \mathcal{S}(1_{A\cap\epsilon\mathbb{Z}})) = \int_{\mathbb{R}} f \rho_c 1_{A^*} dx.$$

We can approximate the indicator function on our open set U with a monotone sequence of bounded, continuous functions as follows: Take the distance function $d(x, U^c) = \inf\{|x-y| : y \in U^c\}$, and let

$$f_n(x) \equiv \min\{1, n \cdot d(x, U^c)\}.$$

We have that $f_n : \mathbb{R} \to [0,1]$ is continuous for all x, since the distance function is itself continuous. Since distance is bounded below by 0, and we take the minimum with 1, we also have $0 \le f_n \le 1$. It is monotone increasing because, for any x, $n \cdot d(x, U^c) \le (n+1) \cdot d(x, U^c)$, thus $f_n(x) \le f_{n+1}(x)$, and the minimum function preserves this ordering.

We can also show that f_n converges pointwise to the indicator on U: Suppose $x \in U$.

Then there exists some $\epsilon > 0$ such that $B_\epsilon(x) \subseteq U$, which means $d(x, U^c) \geq \epsilon$. Then

$$f_n(x) = \min\{1, n \cdot d(x, U^c)\} = \min\{1, n \cdot \epsilon\}.$$

For large enough n, specifically $n \geq \frac{1}{\epsilon}$, we thus have $f_n(x) = 1$. If we instead let $x \in U^c$, then $d(x, U^c) = 0$. So for all n, $f_n(x) = \min\{1, n \cdot 0\} = 0$. Thus, our sequence f_n converges pointwise to 1_U.

Since f_n is bounded and continuous for each n, by weak star convergence we have

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} f_n \, d(\epsilon\, \mathcal{S}(1_{A \cap \epsilon\mathbb{Z}})) = \int_{\mathbb{R}} f_n \, \rho_c 1_{A^*} \, dx \tag{3}$$

for each fixed n. Taking the limit as n goes to ∞ on the right-hand side and applying the monotone convergence theorem [Fol99, Theorem 2.14] gives us

$$\lim_{n \to \infty} \int_{\mathbb{R}} f_n \, \rho_c 1_{A^*} \, dx = \int_{\mathbb{R}} \lim_{n \to \infty} f_n \, \rho_c 1_{A^*} \, dx = \int_{\mathbb{R}} 1_U \, \rho_c 1_{A^*} \, dx = \rho_c |U \cap A^*|.$$

Recall that $0 \leq f_n \leq 1$ and $f_n \uparrow 1_U$ pointwise. In addition to weak star convergence in (3), we have

$$0 \leq \int_{\mathbb{R}} f_n \, d(\epsilon\, \mathcal{S}(1_{A \cap \epsilon\mathbb{Z}})) \leq \epsilon\, \mathcal{S}(1_{A \cap \epsilon\mathbb{Z}})(\mathbb{R}),$$

and the total mass $\epsilon\, \mathcal{S}(1_{A \cap \epsilon\mathbb{Z}})(\mathbb{R})$ is uniformly bounded in ϵ (it is approximately equal to $|A|$). Thus, the family of integrals $\left\{\int_{\mathbb{R}} f_n \, d(\epsilon\, \mathcal{S}(1_{A \cap \epsilon\mathbb{Z}}))\right\}_{n,\epsilon}$ is uniformly bounded.

Fix $\epsilon > 0$. Since $f_n \uparrow 1_U$ and the integrand is nonnegative, the monotone convergence theorem yields

$$\lim_{n \to \infty} \int_{\mathbb{R}} f_n \, d(\epsilon\, \mathcal{S}(1_{A \cap \epsilon\mathbb{Z}})) = \int_{\mathbb{R}} 1_U \, d(\epsilon\, \mathcal{S}(1_{A \cap \epsilon\mathbb{Z}})).$$

By the dominated convergence theorem in ϵ, using the uniform bound above, we may switch the order of the limits in n and ϵ:

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} 1_U \, d(\epsilon\, \mathcal{S}(1_{A \cap \epsilon\mathbb{Z}})) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \int_{\mathbb{R}} f_n \, d(\epsilon\, \mathcal{S}(1_{A \cap \epsilon\mathbb{Z}})) = \lim_{n \to \infty} \lim_{\epsilon \to 0} \int_{\mathbb{R}} f_n \, d(\epsilon\, \mathcal{S}(1_{A \cap \epsilon\mathbb{Z}})).$$

Putting everything together,

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{\mathbb{R}} 1_U\, d(\epsilon\,\mathcal{S}(1_{A\cap\epsilon\mathbb{Z}})) &= \lim_{n\to\infty}\left(\lim_{\epsilon\to 0}\int_{\mathbb{R}} f_n\, d(\epsilon\,\mathcal{S}(1_{A\cap\epsilon\mathbb{Z}}))\right)\\ &= \lim_{n\to\infty}\int_{\mathbb{R}} f_n\,\rho_c 1_{A^*}\,dx\\ &= \int_{\mathbb{R}} 1_U\,\rho_c 1_{A^*}\,dx\\ &= \rho_c|U\cap A^*|.\end{aligned}$$

Since the integrand is nonnegative, Fubini's Theorem [Fol99, Theorem 2.37] allows us to swap the expectation with the integral below:

$$\begin{aligned}\epsilon\,\mathbb{E}\left[\mathcal{N}_\epsilon(U)\right] &= \mathbb{E}\left[\int_{\mathbb{R}} 1_U d(\epsilon\,\mathcal{S}(1_{A\cap\epsilon\mathbb{Z}}))\right]\\ &= \int_{\mathbb{R}} 1_U d\mu_\epsilon,\end{aligned}$$

where $\mu_\epsilon = \mathbb{E}\left[\epsilon\,\mathcal{S}(1_{A\cap\epsilon\mathbb{Z}})\right]$. We thus have

$$\lim_{\epsilon\to 0}\epsilon\,\mathbb{E}[\mathcal{N}_\epsilon(U)] = \int_{\mathbb{R}} 1_U d\mu_\epsilon = \rho_c|U\cap A^*|,$$

which was our desired equation (2).

($\Leftarrow$) Assume that for any open set $U\subset\mathbb{R}$,

$$\lim_{\epsilon\to 0}\epsilon\,\mathbb{E}\Big[\mathcal{N}_\epsilon(U)\Big] = \rho_c\,|U\cap A^*|. \tag{4}$$

Define a measure μ on $\mathbb{R}$ as $\mu(U) := \rho_c|U\cap A^*|$. Since $\mathcal{S}(1_A\cap\epsilon\mathbb{Z})$ is a random measure, we have again from Fubini's Theorem that

$$\epsilon\mathbb{E}\left[\mathcal{N}_\epsilon(U)\right] = \mathbb{E}\left[\int_{\mathbb{R}} 1_U d\left(\epsilon\mathcal{S}(1_{A\cap\epsilon\mathbb{Z}})\right)\right] = \int_{\mathbb{R}} 1_U d\mu_\epsilon,$$

where again $\mu_\epsilon := \mathbb{E}\left[\epsilon S(1_{A\cap\epsilon\mathbb{Z}})\right]$. Combining this with (4), for all open $U\subset\mathbb{R}$, we have

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}} 1_U d\mu_\epsilon = \rho_c|U\cap A^*|.$$

Note that since μ_ϵ is the expected counting measure after stabilization, $\mu_\epsilon(U)$ is equivalent to the expected number of sleeping particles at lattice sites in U. Then $\mu_\epsilon(U) = \int_{\mathbb{R}} 1_U d\mu_\epsilon$, and

$$\lim_{\epsilon \to 0} \mu_\epsilon(U) = \rho_c |U \cap A^*| = \mu(U) \tag{5}$$

for all U.

We wish to apply the Portmanteau Theorem, however, μ and μ_ϵ are not probability measures. We fix this normalizing by the total number of particles $|A|$.

Later, we show that when A is a single interval, the limiting measure is $\rho_c\, 1_{A^*}$ (Lemmas 4.5, 4.9, and 5.2), so by conservation of measure in Cor 7.3, $|A^*| = |A|/\rho_c$. We invoke that fact here to bound the size of A^*, not as a definition. Then $\mu(\mathbb{R}) = \rho_c|A^*| = |A|$, and

$$\begin{aligned}
\epsilon \cdot \mu_\epsilon(\mathbb{R}) &= \mathbb{E}\left[\mathcal{S}(1_{A \cap \epsilon\mathbb{Z}})(\mathbb{R})\right] \\
&= \epsilon \cdot \mathbb{E}[\text{total \# particles}] \\
&\approx \epsilon \cdot \frac{|A|}{\epsilon} \\
&= |A|.
\end{aligned}$$

Then, define two normalized probability measures on $\mathbb{R}$ by

$$P_\epsilon(U) := \frac{\epsilon \mu_\epsilon(U)}{|A|}$$

and

$$P(U) := \frac{\mu(U)}{|A|} = \frac{\rho_c |U \cap A^*|}{|A|}.$$

By (5), we have for any open $U \subset \mathbb{R}$,

$$
\begin{aligned}
\lim_{\epsilon\to 0} P_\epsilon(U) &= \lim_{\epsilon\to 0} \frac{\epsilon\mu_\epsilon(U)}{|A|} \\
&= \frac{\rho_c |U\cap A^*|}{|A|} \\
&= P(U).
\end{aligned}
$$

This is condition number (iv) from the Portmanteau Theorem [Bil99, Theorem 2.1], which requires $\liminf_n P_n G \geq PG$, where G is open, and P_n and P are probability measures.

By the equivalence of conditions (i) and (iv) in the Portmanteau Theorem, thus have that $P_\epsilon \Rightarrow P$ weakly.

Since weak convergence is preserved under scaling by constants, we also have that $\mu_\epsilon \Rightarrow \mu$ as measures. Then for any bounded continuous function f,

$$
\lim_{\epsilon\to 0} \int_{\mathbb{R}} f d\mu_\epsilon = \int_{\mathbb{R}} f d\mu = \int_{\mathbb{R}} f \rho_c 1_{A^*} dx.
$$

Recall that

$$
\mathbb{E}\left[\int_{\mathbb{R}} f d\left(\epsilon \mathcal{S}(1_{A\cap\epsilon\mathbb{Z}})\right)\right] = \int_{\mathbb{R}} f d\mu_\epsilon
$$

by linearity of expectation. Then

$$
\lim_{\epsilon\to 0} \mathbb{E}\left[\int_{\mathbb{R}} f d\left(\epsilon \mathcal{S}(1_{A\cap\epsilon\mathbb{Z}})\right)\right] = \lim_{\epsilon\to 0} \int_{\mathbb{R}} f d\mu_\epsilon = \int_{\mathbb{R}} f \rho_c 1_{A^*} dx,
$$

again where f is any bounded continuous function on $\mathbb{R}$. This is sufficient for showing weak* convergence because the limiting measure is deterministic. Then $\epsilon\, \mathcal{S}(1_{A\cap\epsilon\mathbb{Z}}) \xrightarrow{*} \rho_c 1_{A^*}$. □

Note that this reformulation implies the uniqueness of A^* for a given bounded open set $A \subset \mathbb{R}$.

Remark 2.12 (Uniqueness of A^*)**.** *Given a bounded open $A \subset \mathbb{R}$, if for any open set $U \subset \mathbb{R}$, we get*

$$
\lim_{\epsilon\to 0} \epsilon \mathbb{E}\left[\#\text{Particles}\left(U \cap \mathcal{S}(1_{A\cap\epsilon\mathbb{Z}})\right)\right] = \rho_c |U \cap A_1^*|
$$

and

$$\lim_{\epsilon \to 0} \epsilon \mathbb{E}\left[\#\text{Particles}\left(U \cap \mathcal{S}(1_{A \cap \epsilon\mathbb{Z}})\right)\right] = \rho_c |U \cap A_2^*|$$

for two subsets A_1^* and A_2^*, then for any open set $U \subset \mathbb{R}$,

$$\rho_c |U \cap A_1^*| = \rho_c |U \cap A_2^*|.$$

For any open $U \subset \mathbb{R}$, $|U \cap A_1^*| = |U \cap A_2^*|$, and thus A_1^* and A_2^* only differ by a set of Lebesgue measure zero. Then

$$|U \cap A_1^*| = |U \cap A_2^*| = |U \cap A_1^* \cap A_2^*|,$$

which implies that $A_1^* \cap A_2^*$ also satisfies the condition.

We also note that because the critical density $\rho_c = \rho_c(\lambda)$ is dependent on the sleep rate $\lambda > 0$, different sleep rates result in different densities of final configuration, as seen in Figure 2.

3 Not too many particles get left on any interval

For the duration of this document, we utilize activated random walk on $\mathbb{Z}$ with a fixed sleep rate $\lambda > 0$.

One big key to our argument is the property that we always have a low probability of "too many" particles getting left on any particular interval after stabilization. Since no subsection of the final configuration can be too dense, we will conclude that all but a measure zero amount of the sleeping particles land in the region that we call A^*. We see this more formally below.

We make heavy use of the following recent theorem.

Theorem 3.1 (Theorem 8.4 of [HJJ24])**.** *Let n be a positive integer. Consider activated random walk on $\mathbb{Z}$ with sleep rate $\lambda > 0$. Let σ be an initial configuration with no sleeping particles on the interval $[a, a+n-1]$ for some integer a. Let X_n be the number of particles left sleeping in the stabilization of σ on $[a, a+n-1]$. For any $\delta > 0$,*

$$\mathbb{P}_\sigma[X_n \geq (\rho_c + \delta)n] \leq Ce^{-cn},$$

where C and c are positive constants depending only on λ and δ.

This theorem says that, as an interval grows larger, the probability that the density of particles remaining in the interval after stabilization is greater than the critical density is decreasing exponentially by the size of (or number of sites in) the interval.

We utilize the following corollary of this theorem:

Corollary 3.2 (Corollary 3.3 of [BHS24], intervals I_a and $I_{b,\gamma}$ slightly modified)**.** *Let n be a positive integer. Consider activated random walk on $\mathbb{Z}$ with sleep rate $\lambda > 0$. Let σ be an initial configuration with at least one active particle on the interval $I_a = [a, a+n]$ for some integer a.*

Let b be an integer and $\gamma \in (0, 1]$ such that $I_{b,\gamma} = [b, b+n\gamma] \cap \mathbb{Z} \subseteq I_a$. Let U be a stable odometer on $I_{b,\gamma}$ that satisfies $U(b) = u_0$ and $U(x) > 0$ for every site $x \in I_{b,\gamma}$ containing a particle in σ. Define $Y_{I_{b,\gamma}}(u_0)$ to be the number of sleeping particles remaining in $I_{b,\gamma}$ after *applying U. Then, for any $\rho > \rho_c(\lambda)$, there exist constants $C, c > 0$ such that*

$$\mathbb{P}\big(Y_{I_{b,\gamma}}(u_0) \geq \rho n\gamma\big) \leq Ce^{-cn\gamma}$$

for all n sufficiently large.

In [BHS24], it is noted that this corollary can be proven using equation (81) from the proof of Theorem 8.4 from [HJJ24] along with the union bound, but here we provide a more explicit proof with the eventual goal of generalizing this result. To do so, we utilize the following upper bound on the odometer from [LS21]:

Lemma 3.3 (Lemma 4 from [LS21])**.** *Consider ARW on $\mathbb{Z}$ and sleep rate λ, starting with n active particles at the origin, and none elsewhere. For a legal sequence of firings, define the odometer function $u_n : \mathbb{Z} \to \mathbb{N}$ as*

$$u_n(x) = \sum_{j=1}^{k} \mathbb{1}(x_j = x)$$

to count the number of instructions used at site x. Then

$$\mathbb{P}\left(\sum_{x\in\mathbb{Z}} u_n(x) > n^4\right) \leq n^{-2}$$

for n large enough.

Since Lemma 3.3 is reliant on a starting configuration of n particles at the origin, rather than n particles on an interval of length n with one at each site, we show that this lemma can also apply to that case.

Corollary 3.4. *Consider an interval $[a, a+n] \subset \mathbb{Z}$ for some integer a, where one active particle starts at each site. Let $u_*(x)$ denote the odometer value at site $x \in \mathbb{Z}$ after stabilization. Then for any site $x \in \mathbb{Z}$,*

$$\mathbb{P}\left(u_*(x) > n^4\right) \leq n^{-2},$$

for n large enough.

Proof. Consider ARW on $\mathbb{Z}$, initially with n active particles at the origin. By the abelian property, we can move the particles in any order we deem convenient. Thus, we release the particles one at a time, and freeze when the reaches an unoccupied site (according to internal diffusion-limited aggregation, or IDLA.)

After completing IDLA, we have our n particles spread uniformly on some length n interval on $\mathbb{Z}$. Call this interval $I_n \subset \mathbb{Z}$. While this interval may not be perfectly centered at the origin, by translation invariance of ARW, anything that can be said about the odometer of I_n will also apply to any interval $[a, a+n]$ of the same length.

Let u_* be the odometer stabilizing the system starting with one active particle at each site in I_n. By construction, the total value of this odometer is less than or equal to the total value of the odometer stabilizing the n active particles starting from the origin, i.e. u_n from Lemma 3.3.

Note that the odometer value at any site cannot exceed the total odometer value summed over every site. Then for any site $x \in \mathbb{Z}$ (regardless of whether x is in I_n),

$$\mathbb{P}\left(u_*(x) > n^4\right) \leq \mathbb{P}\left(u_n(x) > n^4\right) \leq \mathbb{P}\left(\sum_x u_n(x) > n^4\right) \leq n^{-2},$$

for n large enough, where the last inequality comes directly from Lemma 3.3. By the above reasoning, the above odometer bound thus applies to any site in $[a, a+n]$, proving the claim. $\square$

We now prove Corollary 3.2.

Proof of Corollary 3.2. Let $\lambda > 0$, and consider ARW on the integer interval $I_a := [a,\, a+n]$ (so $|I_a| = n$), with an initial configuration σ having only active particles on I_a, and no particles outside I_a. We assume without loss of generality that σ places only 0 or 1 particle at each site; if not, we may topple sites until there are no remaining sites with more than one particle.

Let J be any subinterval of I_a of the form $J = [b, b+m]$ of length $m \leq n$. We may suppose $J = I_{b,\gamma} = [b, b+n\gamma] \cap \mathbb{Z}$ for some b and $\gamma \in (0,1]$ with $\lceil n\gamma \rceil = m$ as to match the corollary statement. Let U be a stable odometer on $I_{b,\gamma}$ that satisfies the corollary's conditions and $Y_{I_{b,\gamma}}(u_0) = Y_J$ be the number of sleeping particles in $J = I_{b,\gamma}$ after applying U.

Let Y_J denote the number of sleeping particles remaining in J after stabilization of σ on I_a. Given a density $\rho > \rho_c(\lambda)$, we aim to show:

$$\lim_{n\to 0} \mathbb{P}_\sigma \left(Y_J \geq \rho m\right) = 0.$$

We can use previous results to generalize a bound onto $[a+1,\, a+n-1]$. As in [HJJ24], we define the event $\text{Stable}_{n,\rho}(\sigma, u_0, f_0)$ as the event that there exists an odometer u on $[0, n]$ stable on $[1, n-1]$ for initial configuration σ with $u(0) = u_0$ and $\mathcal{R}_0(u) - \mathcal{L}_1(u) = f_0$, and

$$\sum_{v=1}^{n} \mathbb{1}\{\text{Instr}_v(u(v)) = \text{sleep}\} \geq \rho n,$$

with $(\text{Instr}_v)_{v\in[0,n]}$ the instructions for activated random walk with sleep rate $\lambda > 0$.

Equation (81) proved in [HJJ24] (see Theorem 8.4, section 8.2) says there exist constants C and c so that

$$\mathbb{P}(\text{Stable}_{n,\rho}(\sigma, u_0, f_0)) \leq Ce^{-cn}$$

for any initial configuration σ with all particles active in $[1, n-1]$, and for all odometer parameter choices of nonnegative u_0 and integer f_0.

By translation invariance of the ARW model and the instructions, the same proof and constants apply verbatim if we work on any interval $[a+1,\, a+n-1]$ (of the same size $n-1$) as a subinterval of $[a, a+n]$ for some $a \in \mathbb{Z}$. Thus, for any such interval,

$$\mathbb{P}(\text{Stable}_{n,\rho}(\sigma, u_a, f_a) \text{ on } [a+1,\, a+n-1]) \leq Ce^{-cn},$$

where $u(a) = u_a$ and $\mathcal{R}_a(u) - \mathcal{L}_{a+1}(u) = f_a$.

Now we consider the probability that, after stabilization on $I_a = [a,\, a+n]$, some subinterval $J = [b,\, b+m] \subset I_a$ contains at least ρm sleeping particles.

Suppose that $Y_J \geq \rho\gamma n = \rho m$. Let u_* be the odometer stabilizing I_a. Then u_* is stable on I_a and is thus stable on $[b, b+m] \subset I_a$. Then u_* leaves at least $\rho m - 1$ particles on

$[b+1, b+m]$. So $\text{Stable}_{m,\rho'}(\sigma, u_b, f_b)$ occurs for the interval $[b+1, b+m-1]$ at density $\rho' = \rho - \frac{1}{m}$ and for some u_b and f_b where $u_b \coloneqq u_*(b)$.

Note that for $m \geq m_0$, where m_0 is a constant depending only on λ and ρ, we have $\rho' > \rho_c(\lambda)$. Since 0 or 1 particles start at each site, the net flow at b is bounded by $-(a+n-b) \leq f_b \leq b-a+1$.

All of this means that if the event $Y_J \geq \rho m$ occurs, then either the event

$$\bigcup_{\substack{0 \leq u_b \leq n^4 \\ -(a+n-b) \leq f_b \leq b-a+1}} \text{Stable}_{m,\rho'}(\sigma, u_b, f_b)$$

also occurs, or we have a large odometer value $u_*(b) > n^4$. Then we can use the union bound to get our desired exponential bound on the probability of this event happening on a smaller odometer:

$$\begin{aligned}
\mathbb{P}\left(Y_J \geq \rho m | u_b \leq n^4\right) &= \mathbb{P}\left(\bigcup_{\substack{0 \leq u_b \leq n^4 \\ -(a+n-b) \leq f_b \leq b-a+1}} \text{Stable}_{m,\rho'}(\sigma, u_b, f_b)\right) \\
&\leq \sum_{\substack{0 \leq u_b \leq n^4 \\ -(a+n-b) \leq f_b \leq b-a+1}} \mathbb{P}\left(\text{Stable}_{m,\rho'}(\sigma, u_b, f_b)\right) \\
&\leq \sum_{\substack{0 \leq u_b \leq n^4 \\ -(a+n-b) \leq f_b \leq b-a+1}} Ce^{-cm} < n^5 \cdot Ce^{-cm}.
\end{aligned}$$

Last, we utilize the fact that the left odometer u_b cannot be too large. By Corollary 3.4, the value of the odometer u_* of any site in $I_a = [a, a+n]$ is bounded in probability by

$$\mathbb{P}\left(u_*(x) > n^4\right) \leq n^{-2}.$$

Therefore, as $J = I_{b,\gamma} \subset I_a$ and $b \in [a, a+n]$, we have that the above bound applies to $u_b \coloneqq u_*(b)$. Together with the bound from step 2, we get the following bound on the probability of the density of particles in J being too high:

$$\begin{aligned}
\mathbb{P}(Y_J \geq \rho m) &= \mathbb{P}\left(Y_J \geq \rho m | u_b \leq n^4\right) \mathbb{P}\left(u_b \leq n^4\right) + \mathbb{P}\left(Y_J \geq \rho m | u_b > n^4\right) \mathbb{P}\left(u_b > n^4\right) \\
&\leq \mathbb{P}\left(Y_J \geq \rho m | u_b \leq n^4\right) + \mathbb{P}\left(u_b > n^4\right) \\
&\leq n^5 \cdot Ce^{-cm} + n^{-2} \\
&= Cn^5 e^{-cn\gamma} + n^{-2}.
\end{aligned}$$

Since $m = \gamma n$ with $\gamma > 0$, there exist constants $C', c' > 0$ such that, for all n large enough,

$$\mathbb{P}(Y_J \geq \rho m) \leq C' e^{-c'n},$$

and hence we get $\mathbb{P}(Y_J \geq \rho m) \leq C' e^{-c'm}$. □

The previous corollary tells us that, after stabilization, we are unlikely to find a subinterval of our initial starting interval where we have a particle density greater than ρ_c. In the below corollary, we extend this result to intervals disjoint from the starting interval:

Corollary 3.5. *Let n be a positive integer. Consider activated random walk on $\mathbb{Z}$ with sleep rate $\lambda > 0$. Let σ be an initial configuration with at least one active particle on the interval $I := [a, a+n-1] \subset \mathbb{Z}$ for some integer a, and no particles outside of I.*

Let d be an integer, and $\gamma > 0$, and let $I_{d,\gamma} := [d, d+n\gamma-1] \cap \mathbb{Z}$ satisfy $I_{d,\gamma} \cap I = \emptyset$ (i.e., $I_{d,\gamma}$ is disjoint from I).

Define $Y_{I_{d,\gamma}}$ to be the number of sleeping particles remaining in $I_{d,\gamma}$ after stabilization. Then for any $\rho > \rho_c(\lambda)$,

$$\mathbb{P}\left[Y_{I_{d,\gamma}} \geq \rho n\gamma\right] \leq Ce^{-cn}$$

for some constants C and c depending only on λ and ρ.

This corollary is similar to Corollary 3.3, but it generalizes Theorem 8.4 even further to be able to express the unlikelihood of a large number of particles stabilizing within any interval, even an interval disjoint from where the active particles initialize.

Proof. Let $\lambda > 0$ be fixed. Consider ARW on $I = [a, a+n-1]$, with initial configuration σ

having only active particles in I and no particles outside I. Let $I_{d,\gamma} = [d, d + n\gamma - 1] \cap \mathbb{Z}$ be disjoint from I.

Define a new interval $\tilde{I}$ such that $I, I_{d,\gamma} \subset \tilde{I}$. Then by equating $\tilde{I}$ to I_a and $I_{d,\gamma}$ to $I_{b,\gamma}$ in Corollary 3.2 using the abelian property , this result is directly proven by Corollary 3.2. $\square$

4 Theorem 2.3: Single-interval case

In this section, we assume that $A \subset \mathbb{R}$ is a single bounded open interval. Our goal is to understand the corresponding smash domain A^* in this simplified setting.

After some light preliminary results, we first show that no smaller interval than A^* can contain almost all of the sleeping particles, and then prove that any interval strictly larger than A^* contains all but a negligible fraction of the particles. The general case, where A is a finite or countable union of intervals, will be treated in later sections.

Note that for a single interval A, Lemma 5.2 will show that the interval with the same center as A and length $|A|/\rho_c$ (equivalent to A^{comb} in Definition 2.5) satisfies the quadrature inequality. By uniqueness of smash sums, the smash-domain A^* for this A is therefore exactly this interval. Further, when A is one interval, A^* is the unique open interval with the same center as A and length $|A|/\rho_c$.

We also prove the uniform density statement $\epsilon\,\mathcal{S}(1_{A\cap\epsilon\mathbb{Z}}) \overset{*}{\rightharpoonup} \zeta_a 1_{A^*}$ as $\epsilon \to 0$ for this single-interval case.

4.1 A^* gets (almost) all the particles

Our goal is to show that no smaller interval than A^* gets all the particles with high probability, i.e., that A^* is the minimal interval containing (almost) all the particles. We begin with an important definition.

Definition 4.1. *For any open set $A \subset \mathbb{R}$, define*

$$A_\epsilon = \{z \in \mathbb{Z} : z\epsilon \in A\} \subset \mathbb{Z}.$$

The set A_ϵ is the discrete shadow of A, recording which lattice sites belong to A when we are in a space with mesh size ϵ. This definition is important as we wish to run out activated random walk on $\mathbb{Z}$ rather than $\mathbb{R}$, so we are able to use $A_\epsilon \subset \mathbb{Z}$ as our starting configuration.

We think of A_ϵ as a sampling of A on the lattice $\epsilon\mathbb{Z}$, recorded as integers rather than real points. We see in the following that the subset and intersection relationships between sets are maintained through this sampling.

Corollary 4.2. *Let $A, B \subset \mathbb{R}$ be open sets. If $B \subset A$, then $B_\epsilon \subset A_\epsilon$.*

Proof. Let $x \in B_\epsilon$. Then $x\epsilon \in B$ by definition. Since $B \subset A$, $x\epsilon \in A$. Then $x \in A_\epsilon$ by definition. Hence, $B_\epsilon \subset A_\epsilon$. □

Corollary 4.3. *Let $A, B \subset \mathbb{R}$ be open sets. If $A \cap B = \emptyset$, then $A_\epsilon \cap B_\epsilon = \emptyset$.*

Proof. We prove the contrapositive of the statement. Suppose that $A_\epsilon \cap B_\epsilon \neq \emptyset$. Let $x \in A_\epsilon \cap B_\epsilon$. Then $x\epsilon \in A$ and $x\epsilon \in B$, so $x\epsilon \in A \cap B$, and thus $A \cap B \neq \emptyset$. □

We also utilize the below approximation for the number of sites in A_ϵ, which frequently is equivalent to the number of particles in our system.

Recall that $|A_\epsilon|$ denotes the number of lattice points $z \in \mathbb{Z}$ with $z\epsilon \in A$, whereas $|A|$ is the Lebesgue measure of A. With this, we note that as $\epsilon \to 0$, the points $z \in A_\epsilon$ form a finer and finer sampling of A, and $|A_\epsilon| \approx \frac{|A|}{\epsilon}$.

Corollary 4.4 (Lattice approximation)**.** *Let $A \subset \mathbb{R}$ be a bounded open set that is a (possibly countable) union of disjoint open intervals, and whose boundary (∂A) has Lebesgue measure zero. Then*

$$\epsilon\,|A_\epsilon| \longrightarrow |A| \qquad \text{as } \epsilon \to 0,$$

or equivalently,

$$|A_\epsilon| = \frac{|A|}{\epsilon} + o\!\left(\frac{1}{\epsilon}\right) \qquad \text{as } \epsilon \to 0.$$

Proof. We can write A as a disjoint union of open intervals

$$A = \bigcup_{j=1}^{\infty} I_j, \qquad I_j = (a_j, b_j),$$

with $|A| = \sum_{j=1}^{\infty} |I_j| < \infty$ (bounded). For an interval $I = (a, b)$, define

$$M_\epsilon(I) := \#\{z \in \mathbb{Z} : z\epsilon \in I\},$$

so that $A_\epsilon = \{z \in \mathbb{Z} : z\epsilon \in A\}$ (Definition 4.1) satisfies

$$|A_\epsilon| = \sum_{j=1}^{\infty} M_\epsilon(I_j).$$

We first prove the claim for a single interval. For each interval $I = (a, b)$,

$$M_\epsilon(I) = \#\{z \in \mathbb{Z} : a < z\epsilon < b\} = \begin{cases} \lfloor \frac{b}{\epsilon} \rfloor - \lceil \frac{a}{\epsilon} \rceil + 1, & \lfloor \frac{b}{\epsilon} \rfloor \text{ and } \lceil \frac{a}{\epsilon} \rceil \text{ are not both integers} \\ \lfloor \frac{b}{\epsilon} \rfloor - \lceil \frac{a}{\epsilon} \rceil - 1, & \text{otherwise,} \end{cases}$$

and therefore:

$$\left| \frac{b-a}{\epsilon} - M_\epsilon(I) \right| \leq 2. \tag{6}$$

In particular,

$$M_\epsilon(I) = \frac{|I|}{\epsilon} + O(1) \qquad \text{as } \epsilon \to 0. \tag{7}$$

Next, we extend this to finite unions of intervals. Fix $n \in \mathbb{N}$ and set

$$A^n := \bigcup_{j=1}^{n} I_j.$$

For $\epsilon > 0$ small enough, the sets $\{z\epsilon \in I_j\}$, $1 \leq j \leq n$, are disjoint subsets of $\mathbb{Z}$, so

$$|A^n_\epsilon| = \sum_{j=1}^{n} M_\epsilon(I_j).$$

Using (7) for each j gives

$$|A^n_\epsilon| = \sum_{j=1}^{n} \frac{|I_j|}{\epsilon} + O_n(1) = \frac{|A^n|}{\epsilon} + O_n(1) \qquad \text{as } \epsilon \to 0,$$

where the error term depends on n but not on ϵ. Thus

$$\epsilon\, |A^n_\epsilon| \longrightarrow |A^n| \qquad \text{as } \epsilon \to 0$$

for each fixed n.

We have thus proved the estimate for each finite union $A^n = \bigcup_{j=1}^{n} I_j$. For a general (possibly countable) union $A = \bigcup_{j=1}^{\infty} I_j$, we approximate A from below by the increasing sequence (A^n) and use $|A \setminus A^n| \to 0$, and we apply the same finite–union argument to the open set $\overline{A}^c$ written as a (possibly countable) disjoint union of intervals.

Since A is bounded and $\bigcup_{j=1}^{\infty} I_j$ is a disjoint union, we have

$$|A \setminus A^n| = \sum_{j>n} |I_j| \xrightarrow[n\to\infty]{} 0.$$

Fix $\delta > 0$. Choose n large enough so that

$$|A \setminus A^n| < \delta. \tag{8}$$

For this fixed n, we already know that

$$\epsilon\,|A^n_\epsilon| \longrightarrow |A^n| \qquad \text{as } \epsilon \to 0.$$

Hence there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$,

$$\epsilon\,|A^n_\epsilon| \;\geq\; |A^n| - \delta. \tag{9}$$

Since $A^n_\epsilon \subset A_\epsilon$ and $|A^n| \geq |A| - \delta$ by (8), we obtain

$$\epsilon\,|A_\epsilon| \;\geq\; \epsilon\,|A^n_\epsilon| \;\geq\; |A| - 2\delta$$

for all sufficiently small ϵ. This shows that we cannot have too few lattice points in A.

For the complementary upper bound, we apply the same argument to the open set $(\overline{A})^c$. Since $|\partial A| = 0$, we have $|\overline{A}\,\Delta A| = 0$, so $\overline{A}^c$ is a bounded open set whose boundary also has Lebesgue measure zero. Writing $\overline{A}^c$ as a disjoint union of intervals and repeating the finite–union argument yields

$$\epsilon\,|(\overline{A}^c)_\epsilon| \longrightarrow |\overline{A}^c| \qquad \text{as } \epsilon \to 0.$$

Then for small ϵ,

$$\epsilon\,|(\overline{A}^c)_\epsilon| \;\geq\; |\overline{A}^c| - \delta.$$

Using $|\overline{A}^c| = |\mathbb{R} \setminus \overline{A}|$ and $|\overline{A}\,\Delta A| = 0$, this is equivalent to

$$\epsilon\,|A_\epsilon| \ \leq \ |A| + 2\delta$$

for all sufficiently small ϵ. Combining this with the lower bound above, we find that for every $\delta > 0$ there exists $\epsilon_0 > 0$ such that

$$|A| - 2\delta \ \leq \ \epsilon\,|A_\epsilon| \ \leq \ |A| + 2\delta$$

whenever $0 < \epsilon < \epsilon_0$. Since $\delta > 0$ was arbitrary, we conclude that

$$\epsilon\,|A_\epsilon| \longrightarrow |A| \qquad \text{as } \epsilon \to 0,$$

as claimed. $\square$

We now translate these discrete approximations into a first structural property of A^*. Intuitively, expanding A to density ρ_c should produce the smallest interval that can capture almost all of the sleeping particles. The next lemma makes this precise in the single-interval case.

Lemma 4.5 (No smaller interval captures almost all particles)**.** *Let $A \subset \mathbb{R}$ be a bounded open interval. Define A^* as the unique open set with the same center as A and size $|A^*| = |A|/\rho_c$. Let $B \subset \mathbb{R}$ be an open interval such that $A^* \setminus B$ contains an interval.*

Fix a value $0 < \delta < 1 - \frac{|A^* \cap B|}{|A|}\rho_c$. Let $\epsilon > 0$, and start with one active particle at each point in A_ϵ. Run ARW with sleep rate λ. Then,

$$\lim_{\epsilon \to 0} \mathbb{P}\left(\#\text{particles in } B_\epsilon \text{ after stabilization } > (1-\delta)\frac{|A|}{\epsilon}\right) = 0.$$

Proof. Note that, for any $0 < \delta_1, \delta_2 < 1$ satisfying $(1-\delta_1)\frac{|A|}{\epsilon} < (1-\delta_2)\frac{|A|}{\epsilon}$, we must have $\delta_1 > \delta_2$ (and vice versa). Then for $\delta_1 > \delta_2$, we get that

$$\mathbb{P}\left(\#\text{particles in } B_\epsilon > (1-\delta_1)\frac{|A|}{\epsilon}\right) > \mathbb{P}\left(\#\text{particles in } B_\epsilon > (1-\delta_2)\frac{|A|}{\epsilon}\right).$$

Thus, the lower bound on the number of particles, $(1-\delta)\frac{|A|}{\epsilon}$, observed in the event

$$\left(\#\text{particles in } B_\epsilon \text{ after stabilization } > (1-\delta)\frac{|A|}{\epsilon}\right)$$

grows larger as δ approaches 0, so we focus on showing the result for δ near $1 - \frac{|A^* \cap B|}{|A|}\rho_c$, which automatically shows the result for all smaller δ by monotonicity of probability.

Case 1: B is strictly contained in A. If B is strictly contained in A, then since $A \subseteq A^*$, $|A^* \cap B| = |B|$. Then the event with the largest allowed value of $\delta < 1 - \frac{|A^* \cap B|}{|A|}\rho_c$ is

$$\left(\#\text{particles in } B_\epsilon \text{ after stabilization } > \frac{|A^* \cap B|}{\epsilon}\rho_c = \frac{|B|}{\epsilon}\rho_c\right),$$

which is the event that the number of particles in B_ϵ exceeds the critical density. This is because the number of sites in B_ϵ is approximately $\frac{|B|}{\epsilon}$ as ϵ approaches zero by Corollary 4.4. We claim that by Corollary 3.2, the probability that B_ϵ contains more than the critical density of particles is decaying in the length of B_ϵ (i.e., by $1/\epsilon$).

Consider the initial configuration σ with one active particle at each site of A_ϵ. Let $|A_\epsilon| = n$ and $|B_\epsilon| = n\gamma$ for some $\gamma \in (0,1)$. We consider stabilization of the ARW process on A_ϵ, noting that the odometer must be greater than zero at every site in B_ϵ due to one active particle starting at each site (since $B_\epsilon \subset A_\epsilon$ by Corollary 4.2).

Corollary 3.2 gives us a probabilistic bound on the number of particles we can expect to find sleeping in a subset of the region where we had the original configuration of particles. Specifically, for any $\rho > \rho_c$, there exist constants $C, c > 0$ (depending on λ and ρ) such that

$$\mathbb{P}_\sigma\left[Y_{B_\epsilon}(u_0) \geq \rho|B_\epsilon|\right] \leq Ce^{-c|B_\epsilon|},$$

where $Y_{B_\epsilon}(u_0)$ is the number of sleeping particles remaining in B_ϵ after stabilization, for any initial leftmost odometer value u_0. This works because B_ϵ is contained in the original region of particles A_ϵ.

Therefore, the probability that the number of sleeping particles in B_ϵ after stabilization exceeds the critical density $\rho_c|B_\epsilon|$ decays exponentially in $|B_\epsilon|$ (and thus in $1/\epsilon$).

Case 2: B is disjoint from A. If $B \cap A = \emptyset$, then no particles start in B_ϵ. Any particles found in B_ϵ after stabilization must have traveled there from A_ϵ during the ARW process.

We apply Corollary 3.5 to bound the number of particles in B_ϵ. The conditions of Corollary 3.5 are satisfied because:

- $B_\epsilon \cap A_\epsilon = \emptyset$ by Corollary 4.3, since $B \cap A = \emptyset$,
- The initial configuration has at least one active particle at each site of A_ϵ, with no particles outside A_ϵ,
- B_ϵ is disjoint from the source region A_ϵ.

Let $n = |A_\epsilon| \approx \frac{|A|}{\epsilon}$ denote the number of lattice sites in A_ϵ (equivalently, the total number of particles in the system). By Corollary 3.5, for any $\rho > \rho_c$, there exist constants $C, c > 0$ (depending on λ and ρ) such that

$$\mathbb{P}_\sigma \left[Y_{B_\epsilon} \geq \rho n \right] \leq C e^{-cn},$$

where Y_{B_ϵ} is the number of sleeping particles in B_ϵ after stabilization. Since $n \approx \frac{|A|}{\epsilon}$, this gives

$$\mathbb{P}_\sigma \left[\text{particles in } B_\epsilon \geq \rho \frac{|A|}{\epsilon} \right] \leq C e^{-c|A|/\epsilon} \to 0$$

as $\epsilon \to 0$.

Taking $\rho = \rho_c + \delta'$ for small $\delta' > 0$, and noting that $(1-\delta)\frac{|A|}{\epsilon} > \rho_c \frac{|A|}{\epsilon}$ by our constraint on δ, we conclude that the probability that B_ϵ contains more than $(1-\delta)\frac{|A|}{\epsilon}$ particles vanishes exponentially as $\epsilon \to 0$.

Case 3: B and A overlap but $B \not\subset A$. If A and B overlap but B is not strictly contained in A, we can decompose B as $B = (B \cap A) \cup (B \setminus A)$, a disjoint union.

For the intersection $(B \cap A)_\epsilon$, by Case 1 and Corollary 3.2, the probability that the number of sleeping particles after stabilization exceeds the critical density decays exponentially in the length of the interval.

For the difference $(B \setminus A)_\epsilon$, since this region contains no initial particles, by Case 2, the probability that it contains a non-negligible number of sleeping particles after stabilization vanishes as $\epsilon \to 0$.

Thus, with high probability, the total number of particles in B_ϵ after stabilization cannot exceed the critical density on $(B \cap A)_\epsilon$ plus a negligible amount from $(B \setminus A)_\epsilon$, as desired.

More precisely, by the union bound applied to the events from Cases 1 and 2, we have the following.

From Case 1 applied to $(B \cap A)_\epsilon \subset A_\epsilon$, there exist constants $C_1, c_1 > 0$ such that

$$\mathbb{P}\left[\text{particles in } (B \cap A)_\epsilon > \rho_c \frac{|B \cap A|}{\epsilon}\right] \leq C_1 e^{-c_1|B\cap A|/\epsilon}.$$

From Case 2 applied to $(B \setminus A)_\epsilon$, which is disjoint from A_ϵ, there exist constants $C_2, c_2 > 0$ such that

$$\mathbb{P}\left[\text{particles in } (B \setminus A)_\epsilon \geq \delta' \frac{|A|}{\epsilon}\right] \leq C_2 e^{-c_2|A|/\epsilon}$$

for any $\delta' > 0$.

By the union bound, with probability at least $1 - C_1 e^{-c_1|B\cap A|/\epsilon} - C_2 e^{-c_2|A|/\epsilon}$ (which approaches 1 as $\epsilon \to 0$), we have

$$\text{particles in } B_\epsilon \leq \rho_c \frac{|B \cap A|}{\epsilon} + o(1/\epsilon).$$

To complete the argument, we must show this quantity is less than $(1-\delta)\frac{|A|}{\epsilon}$. Since $B \cap A \subset B \cap A^*$, we have $|B \cap A| \leq |B \cap A^*|$. Our constraint on δ states that $\delta < 1 - \frac{|A^* \cap B|}{|B|}\rho_c$, which rearranges to

$$\rho_c |A^* \cap B| < (1-\delta)|A|.$$

Therefore,

$$\rho_c \frac{|B \cap A|}{\epsilon} \leq \rho_c \frac{|B \cap A^*|}{\epsilon} < (1-\delta)\frac{|A|}{\epsilon}.$$

Adding the negligible $o(1/\epsilon)$ term does not affect the inequality for sufficiently small ϵ. Thus, with high probability, the number of particles in B_ϵ after stabilization is strictly less than $(1-\delta)\frac{|A|}{\epsilon}$, as desired. $\square$

4.2 Any extension of A^* does contain all the particles

In this section, we will show that any interval larger than A^* will, with high probability, contain all the particles in our system, which will enable us to conclude that A^* is, up to a negligible fraction of particles, the minimal interval containing the collection of sleeping particles.

We utilize the following results for the point-source model by Hoffman, Johnson, and Junge:

Proposition 4.6 (Proposition 8.7 from [HJJ24], with variable names changed slightly)**.** *Let $[L_n, R_n]$ be the smallest interval containing all sites ever visited by a particle in the point-source model on $\mathbb{Z}$ with n particles. For any $\eta > 0$,*

$$\mathbb{P}\left[[L_n, R_n] \subseteq \left[-\frac{n}{2(\rho_c - \eta)}, \frac{n}{2(\rho_c - \eta)}\right]\right] \geq 1 - Ce^{-cn}$$

for constants $c, C > 0$ depending only on λ and η.

Proposition 4.6 controls how far the visited set can spread: it shows that, with high probability, all visited sites lie in a centered interval whose length is arbitrarily close to the optimal value n/ρ_c. The next result complements this by controlling the *mass* inside such an almost-optimal interval, showing that only a small fraction of particles can lie outside it.

Proposition 4.7 (Proposition 8.8 from [HJJ24], with variable names changed slightly)**.** *Let $[\tilde{L}_n, \tilde{R}_n]$ be the smallest interval containing all sleeping particles after stabilization in the point-source model on $\mathbb{Z}$ with n particles. For any $\eta > 0$,*

$$\mathbb{P}\left[[\tilde{L}_n, \tilde{R}_n] \supseteq \left[-\frac{n}{2(\rho_c + \eta)}, \frac{n}{2(\rho_c + \eta)}\right]\right] \geq 1 - Ce^{-cn}$$

for constants $c, C > 0$ depending only on λ and η.

Proposition 4.8 (HJJ24, Prop. 8.8, reformulated)**.** *Let n particles start at the origin (point-source model). For any $\eta > 0$ there exist constants $c, C > 0$ and a centered interval $I_n \subset \mathbb{Z}$ of length $\frac{n}{\rho_c - \eta}$ such that*

$$\mathbb{P}\Big(\#\{\textit{particles outside } I_n\} > \eta n\Big) \leq Ce^{-cn}.$$

Although we are in a multi-source setting rather than a point-source one, we can again utilize the results from point-source by utilizing IDLA and the abelian property, just like in Corollary 3.4. We see that in the below proof.

The previous results show that for point-source initial data, almost all particles remain inside an interval whose length is close to $|A|/\rho_c$. We now transfer this control to our multi-source configuration on A_ϵ. The next lemma states that, when A is a single interval, only a vanishing fraction of particles can escape A^*.

Lemma 4.9 (Few particles leave A^*)**.** *Let $A \subset \mathbb{R}$ be a bounded open interval. Again, define A^* as the unique open set with the same center as A, and with size $|A^*| = |A|/\rho_c$. Let $B \subset \mathbb{R}$ be an open interval such that $A^* \subsetneq B$ and both components of $B \setminus A^*$ have positive length (i.e. B strictly extends A^* on the left and on the right).*

Fix $\delta > 0$. Let $\epsilon > 0$, and start with one active particle at each point in A_ϵ. Run ARW with sleep rate $\lambda > 0$. Then,

$$\lim_{\epsilon \to 0} \mathbb{P}\left(\#\text{particles in } B_\epsilon^c \text{ after stabilization } > \delta \frac{|A|}{\epsilon}\right) = 0.$$

Proof. Fix $\delta > 0$, and without loss of generality, assume A is a bounded open interval centered at the origin. Let $\epsilon > 0$, and put one active particle at each site of A_ϵ. The total number of particles in our system is $n = |A_\epsilon| = \frac{|A|}{\epsilon} + O(1)$.

As in Corollary 3.4, we first couple to a point-source configuration with n particles at a single site (say, the origin). We run IDLA until these particles occupy an interval $J_n \subset \mathbb{Z}$ of length n, and then run ARW starting from one active particle at each site of J_n. By the abelian property and translation invariance, the ARW odometer for the initial configuration on A_ϵ is dominated by the odometer for the point-source/IDLA configuration on J_n. In particular, any upper bound on the number of particles outside a given interval (such as B_ϵ) in the point–source model with n particles also holds for the original configuration on A_ϵ.

For ARW started from n particles at the origin, let A_n be the set of all visited sites. Again, define $L_n := \min A_n$ and $R_n := \max A_n$, so $A_n = [L_n, R_n]$ and that the length of the visited interval is

$$|A_n| = R_n - L_n + 1 \approx R_n - L_n.$$

By Proposition 4.6, for any $\eta > 0$,

$$\mathbb{P}\left[[L_n, R_n] \subseteq \left[-\frac{n}{2(\rho_c - \eta)}, \frac{n}{2(\rho_c - \eta)}\right]\right] \geq 1 - Ce^{-cn},$$

where constants $c, C > 0$ depend only on η and λ. Let E_n be the intersection of the events given by Propositions 4.6 and 4.8 with parameter η. More explicitly, let E_n be the event that both

1. $[L_n, R_n] \subset [-\frac{n}{2(\rho_c-\eta)}, \frac{n}{2(\rho_c-\eta)}]$ (Prop. 4.6),
2. and there exists a centered interval $I_n \subset \mathbb{Z}$ of length $\frac{n}{\rho_c-\eta}$ such that $\#\{\text{particles outside } I_n\} \leq \eta n$ (Prop. 4.8).

Then by Propositions 4.6 and 4.8 and the union bound, there exist constants $c', C' > 0$ (depending only on λ and η) such that $\mathbb{P}(E_n) \geq 1 - C'e^{-c'n}$.

Next, we choose an $\eta > 0$ such that $\eta < \frac{\delta}{2}$, and the confining interval $I_n = \left[-\frac{n}{2(\rho_c-\eta)}, \frac{n}{2(\rho_c-\eta)}\right]$ for A_n lies strictly within B (and contains A^*). We achieve this by defining $m := |B| - |A^*| > 0$, and then choose η small enough so that

$$\epsilon|A_n| \leq \epsilon\frac{n}{\rho_c - \eta} = \epsilon\frac{|A|/\epsilon}{\rho_c - \eta} = \frac{|A|}{\rho_c - \eta} < |A^*| + \frac{m}{2} < B.$$

For each fixed $\eta > 0$, Proposition 4.8 yields a centered interval $I_n \subset \mathbb{Z}$ of length

$$|I_n| = \frac{n}{\rho_c - \eta},$$

so after rescaling by ϵ, its length is approximately

$$\epsilon|I_n| \approx \frac{|A|}{\rho_c - \eta}.$$

The function

$$f(\eta) := \frac{|A|}{\rho_c - \eta}$$

is continuous for $\eta \in (0, \rho_c)$ and satisfies $f(0) = |A^*|$. Since $|B| > |A^*| = f(0)$, by continuity there exists $\eta_0 > 0$ such that

$$f(\eta_0) = \frac{|A|}{\rho_c - \eta_0} < |B|.$$

Now, set

$$\eta := \min\{\eta_0,\ \delta/2\}.$$

Then $\eta < \delta/2$ and

$$\frac{|A|}{\rho_c - \eta} \le \frac{|A|}{\rho_c - \eta_0} = f(\eta_0) < |B|.$$

Since A (and thus A^*) is centered at the origin and $A^* \subset B$ with $|B| > |A^*|$, the origin lies in the interior of B. Write $B = (b_-, b_+)$ and $A^* = (-a, a)$, so $|A^*| = 2a$ and $|B| = b_+ - b_-$. Because $A^* \subset B$ and $|B| > |A^*|$, we have $b_- < -a$ and $b_+ > a$, hence the distance from 0 to the boundary of B is

$$d := \min\{-b_-,\ b_+\} > a = \frac{|A^*|}{2}.$$

By our choice of η, for all sufficiently small ϵ, the length of ϵI_n satisfies

$$\frac{1}{2}|\epsilon I_n| = \frac{\epsilon}{2}\frac{n}{\rho_c - \eta} \le \frac{|A|}{2(\rho_c - \eta)} < d.$$

Therefore the interval ϵI_n is strictly contained in B, i.e. $\epsilon I_n \subset B$.

Intuitively, we know that $\frac{n}{\rho_c}$ is the expected length for containing all the particles, so expanding this slightly larger to $\frac{n}{\rho - \eta}$ means that only a fraction η of the total mass can lie outside, otherwise the average density would exceed ρ_c by "too much." After rescaling by ϵ, our choice of η ensures that $\epsilon I_n \subset B$ for all small ϵ, hence $I_n \subset B_\epsilon$. Therefore, on the event E_n,

$$\#\{\text{particles in } B_\epsilon^c\} \le \eta n = \eta\frac{|A|}{\epsilon} < \delta\frac{|A|}{\epsilon}.$$

Thus

$$\#\{\text{particles in } B_\epsilon^c \text{ after stabilization}\} > \delta\frac{|A|}{\epsilon}$$

cannot happen on E_n. Therefore,

$$\begin{aligned}\mathbb{P}\Big(\#\{\text{particles in } B_\epsilon^c \text{ after stabilization}\} > \delta\frac{|A|}{\epsilon}\Big) &\leq \mathbb{P}\left(E_n^c\right)\\ &\leq 1-(1-C'e^{-c'n})\\ &= C'e^{-c'n}\end{aligned}$$

for constants c', C' only depending on λ, which is fixed throughout this paper, and η, which is fixed based on δ. Hence,

$$\lim_{\epsilon\to 0}\mathbb{P}\left(\#\text{particles in } B_\epsilon^c \text{ after stabilization } > \delta\frac{|A|}{\epsilon}\right) = 0,$$

as desired. □

4.3 Uniform density inside A^*

In this subsection we complete the proof of Theorem 2.3 in the special case where $A \subset \mathbb{R}$ is a single bounded open interval. Recall that in this setting A^* is the unique open interval with the same center as A and length $|A^*| = |A|/\rho_c$ (Definition 2.5).

We have already shown that no smaller interval can contain almost all of the particles (Lemma 4.5) and that any interval strictly larger than A^* contains (almost) all the particles (Lemma 4.9). Specifically, A^* is the minimal interval capturing almost all of the mass after stabilization. In order to deduce the weak* convergence

$$\epsilon\, S\big(1_{A\cap\epsilon\mathbb{Z}}\big) \overset{*}{\rightharpoonup} \rho_c 1_{A^*}, \qquad \epsilon \to 0,$$

we must furthermore show that the particle density inside A^* converges to ρ_c uniformly on subintervals

Throughout this subsection we fix a bounded open interval $A \subset \mathbb{R}$ and write A^* for the corresponding interval from Definition 2.5. For an interval $J \subset \mathbb{R}$ and mesh $\epsilon > 0$ we write $N_\epsilon(J)$ for the number of sleeping particles in J after stabilization of ARW started from one active particle at each site of A_ϵ.

Lemma 4.10 (Uniform upper density bound in the one-interval case)**.** *Let $A \subset \mathbb{R}$ be a bounded open interval and fix $\eta > 0$. There exist constants $C, c > 0$ (depending only on λ and η) such that for every interval $J \subset \mathbb{R}$ and all sufficiently small $\epsilon > 0$,*

$$\mathbb{P}\left(N_\epsilon(J) \;\geq\; (\rho_c + \eta)\,\frac{|J|}{\epsilon}\right) \;\leq\; C\,e^{-c\,|J|/\epsilon}.$$

Consequently,

$$\limsup_{\epsilon \to 0} \epsilon\,\mathbb{E}\big[N_\epsilon(J)\big] \;\leq\; \rho_c\,|J| \quad \textit{for every interval } J \subset \mathbb{R}.$$

Proof. Fix $\eta > 0$ and an interval $J \subset \mathbb{R}$. We decompose J as

$$J = (J \cap A) \cup (J \setminus A),$$

and treat three cases similar to the proof of Lemma 4.5.

Case 1: $J \subset A$. Here $J_\epsilon \subset A_\epsilon$ by Corollary 4.2. By Corollary 3.2, for any $\rho > \rho_c$, there exist $C, c > 0$ such that

$$\mathbb{P}\big(N_\epsilon(J) \geq \rho\,|J_\epsilon|\big) \;\leq\; Ce^{-c|J_\epsilon|}.$$

Using $|J_\epsilon| = |J|/\epsilon + O(1)$ from Corollary 4.4, we obtain the desired exponential bound with $\rho = \rho_c + \eta$.

Case 2: $J \cap A = \emptyset$. Then $J_\epsilon \cap A_\epsilon = \emptyset$ by Corollary 4.3. Since particles start in A_ϵ and not J_ϵ, any sleeping particle in J_ϵ must have traveled there from A_ϵ. By Corollary 3.5, for any $\rho > \rho_c$ there exist $C, c > 0$ such that

$$\mathbb{P}\big(N_\epsilon(J) \geq \rho\,|A_\epsilon|\big) \;\leq\; Ce^{-c|A_\epsilon|}.$$

Since $|A_\epsilon| = |A|/\epsilon + O(1)$, this probability decays like $e^{-c'|A|/\epsilon}$, which is $e^{-c''|J|/\epsilon}$ once J is fixed, because $|J|$ is a constant independent of ϵ.

Case 3: J intersects A but neither is contained in the other. Write $J = (J \cap A) \cup$

$(J \setminus A)$ as a disjoint union, and note that

$$N_\epsilon(J) \;=\; N_\epsilon(J \cap A) + N_\epsilon(J \setminus A).$$

If $N_\epsilon(J) \geq (\rho_c + \eta)\,|J|/\epsilon$, then either

$$N_\epsilon(J \cap A) \;\geq\; (\rho_c + \eta/2)\,\frac{|J \cap A|}{\epsilon}, \quad \text{or} \quad N_\epsilon(J \setminus A) \;\geq\; (\eta/2)\,\frac{|J \setminus A|}{\epsilon}.$$

We can bound the first event by Case 1 (applied to the interval $J \cap A$), and the second by Case 2 (applied to $J \setminus A$, which is disjoint from A). Using the union bound gives the desired exponential estimate in $|J|/\epsilon$.

The bound on the expectation follows from the tail estimate by the standard decomposition

$$\mathbb{E}[N_\epsilon(J)] = \mathbb{E}\big[N_\epsilon(J)\,\mathbf{1}_{\{N_\epsilon(J)<(\rho_c+\eta)|J|/\epsilon\}}\big] + \mathbb{E}\big[N_\epsilon(J)\,\mathbf{1}_{\{N_\epsilon(J)\geq(\rho_c+\eta)|J|/\epsilon\}}\big],$$

and the fact that $N_\epsilon(J) \leq |A_\epsilon|$ is of order $1/\epsilon$ while the tail probability decays like $e^{-c|J|/\epsilon}$. Letting $\epsilon \to 0$ and then $\eta \to 0$ yields

$$\limsup_{\epsilon \to 0} \epsilon\, \mathbb{E}\big[N_\epsilon(J)\big] \;\leq\; \rho_c |J|.$$

□

The next lemma rules out the possibility that some subinterval of A^* carries density strictly below ρ_c. The argument is by contradiction: underfilling U would force overfilling of its complement $A^* \setminus U$, contradicting the upper bound of Lemma 4.10.

Lemma 4.11 (Uniform lower density inside A^*, one-interval case)**.** *Let $A \subset \mathbb{R}$ be a bounded open interval and let A^* be as above. If $U \subset A^*$ is an open interval, then*

$$\liminf_{\epsilon \to 0} \epsilon\, \mathbb{E}\big[N_\epsilon(U)\big] \;\geq\; \rho_c\,|U|.$$

Proof. Suppose for contradiction that there exist an open interval $U \subset A^*$, a $\delta > 0$, and a sequence $\epsilon_k \downarrow 0$ such that

$$\epsilon_k \, \mathbb{E}\big[N_{\epsilon_k}(U)\big] \;\le\; (\rho_c - \delta)\,|U| \qquad \text{for all } k. \tag{10}$$

We first recall that by conservation of particles and Corollary 4.4,

$$\lim_{\epsilon\to 0} \epsilon \, \mathbb{E}\big[N_\epsilon(\mathbb{R})\big] = |A|. \tag{11}$$

Additionally,

$$|A^*| = \frac{|A|}{\rho_c}. \tag{12}$$

Next we use the uniform upper density bound (Lemma 4.10). Applied to the interval U, we obtain

$$\limsup_{\epsilon\to 0} \epsilon \, \mathbb{E}\big[N_\epsilon(U)\big] \;\le\; \rho_c\,|U|. \tag{13}$$

Applied to the complement $A^* \setminus U$, we obtain

$$\limsup_{\epsilon\to 0} \epsilon \, \mathbb{E}\big[N_\epsilon(A^* \setminus U)\big] \;\le\; \rho_c\,|A^* \setminus U|. \tag{14}$$

Since U and $A^* \setminus U$ are disjoint and $U \cup (A^* \setminus U) = A^*$, we have

$$N_\epsilon(A^*) = N_\epsilon(U) + N_\epsilon(A^* \setminus U),$$

so by additivity of expectation,

$$\epsilon \, \mathbb{E}\big[N_\epsilon(A^*)\big] = \epsilon \, \mathbb{E}\big[N_\epsilon(U)\big] + \epsilon \, \mathbb{E}\big[N_\epsilon(A^* \setminus U)\big].$$

Taking $\limsup$ as $\epsilon \to 0$ and using (13) and (14), we obtain

$$\limsup_{\epsilon\to 0} \epsilon \, \mathbb{E}\big[N_\epsilon(A^*)\big] \;\le\; \rho_c\,|U| + \rho_c\,|A^* \setminus U| = \rho_c\,|A^*|. \tag{15}$$

On the other hand, Lemma 4.9 (few particles leave A^*) implies that for any fixed $\eta' > 0$,

$$\mathbb{P}\Big(N_\epsilon((A^*)^c) > \eta' \tfrac{|A|}{\epsilon}\Big) \longrightarrow 0,$$

hence

$$\epsilon\, \mathbb{E}\big[N_\epsilon((A^*)^c)\big] \longrightarrow 0.$$

Combining this with (11) yields

$$\lim_{\epsilon \to 0} \epsilon\, \mathbb{E}\big[N_\epsilon(A^*)\big] = |A|. \tag{16}$$

Now (16) together with (15) gives

$$|A| \;\leq\; \rho_c\, |A^*|.$$

Using (12), the right-hand side is exactly $|A|$, so we in fact have equality throughout and

$$\lim_{\epsilon \to 0} \epsilon\, \mathbb{E}\big[N_\epsilon(A^*)\big] = \rho_c\, |A^*|.$$

Additionally, the limsup bounds (13) and (14) must be sharp on U and $A^* \setminus U$. This forces

$$\lim_{\epsilon \to 0} \epsilon\, \mathbb{E}\big[N_\epsilon(U)\big] = \rho_c\, |U|.$$

But this contradicts our assumption (10), which asserts that along the sequence ϵ_k the quantity $\epsilon_k\, \mathbb{E}[N_{\epsilon_k}(U)]$ stays at most $(\rho_c - \delta)|U|$. Therefore (10) cannot hold, and we conclude that

$$\liminf_{\epsilon \to 0} \epsilon\, \mathbb{E}\big[N_\epsilon(U)\big] \;\geq\; \rho_c\, |U|,$$

as claimed. □

Combining the upper and lower bounds we obtain the exact density on subintervals of A^* in the one-interval case.

Lemma 4.12 (Exact density on intervals for a single A)**.** *Let $A \subset \mathbb{R}$ be a bounded open* *interval with associated A^* as above. Then for every open interval $J \subset \mathbb{R}$,*

$$\lim_{\epsilon \to 0} \epsilon \, \mathbb{E}\big[N_\epsilon(J)\big] \; = \; \rho_c \, |J \cap A^*|.$$

Proof. If $J \subset A^*$, the upper bound

$$\limsup_{\epsilon \to 0} \epsilon \, \mathbb{E}[N_\epsilon(J)] \; \leq \; \rho_c |J|$$

is given by Lemma 4.10, and the lower bound

$$\liminf_{\epsilon \to 0} \epsilon \, \mathbb{E}[N_\epsilon(J)] \; \geq \; \rho_c |J|$$

follows from Lemma 4.11. Thus

$$\lim_{\epsilon \to 0} \epsilon \, \mathbb{E}[N_\epsilon(J)] = \rho_c |J|.$$

If $J \cap A^* = \emptyset$, then J is at a positive distance from A^*. In that case, Lemma 4.9 with B any interval strictly containing A^* implies that the expected number of particles in J is negligible on the ϵ^{-1} scale, so

$$\lim_{\epsilon \to 0} \epsilon \, \mathbb{E}[N_\epsilon(J)] = 0 = \rho_c |J \cap A^*|.$$

For a general interval $J \subset \mathbb{R}$, we write $J = (J \cap A^*) \cup (J \setminus A^*)$ as a disjoint union and use additivity of N_ϵ and of Lebesgue measure to conclude that

$$\lim_{\epsilon \to 0} \epsilon \, \mathbb{E}\big[N_\epsilon(J)\big] = \rho_c \, |J \cap A^*|.$$

$\square$

We can now state and prove the one-interval version of Theorem 2.3.

Theorem 4.13 (Theorem 2.3 for a single interval)**.** *Let $A \subset \mathbb{R}$ be a bounded open interval and let A^* be the unique open interval with the same center as A and length $|A^*| = |A|/\rho_c$.*

Then, as $\epsilon \to 0$,

$$\epsilon\, S\big(1_{A\cap\epsilon\mathbb{Z}}\big) \xrightarrow{*} \rho_c\, 1_{A^*}.$$

Proof. By Proposition 2.11, it suffices to check that for every open set $U \subset \mathbb{R}$,

$$\lim_{\epsilon\to 0} \epsilon\, \mathbb{E}\big[N_\epsilon(U)\big] = \rho_c\, |U \cap A^*|.$$

Since every open $U \subset \mathbb{R}$ can be written as a countable union of disjoint open intervals, and both expectation and Lebesgue measure are countably additive, the claim follows from Lemma 4.12 applied to each interval component of U. □

In the next section, we will show that A^* from the single-interval case (and A^{comb} from every other case) satisfies the quadrature inequality from Theorem 2.3 (Lemma 5.2 and Definition 5.3).

5 A^* satisfies the Quadrature Inequality

Now, we wish to show that our construction of A^{comb} as assumed in Definition 2.5 correlates with the formal definition of A^* given in Theorem 2.3, that is, for a fixed starting set A, A^{comb} is the unique open subset of $\mathbb{R}$ that satisfies

$$\int_A u dx \geq \rho_c \int_{A^{\text{comb}}} u dx$$

for all integrable superharmonic functions u on A^{comb}.

In this section, we make frequent use of this variant of Jensen's Inequality:

Proposition 5.1 (Proposition 1 of [PPP11])**.** *Let A and B be bounded closed intervals from $\mathbb{R}$ such that $A \subset B$ and $0 < \mu(A) < \mu(B)$. If one of three equalities*

$$\frac{1}{\mu(A)} \int_A x d\mu(x) = \frac{1}{\mu(B)} \int x d\mu(x) = \frac{1}{\mu(B \setminus A)} \int_{B \setminus A} x d\mu(x)$$

is valid, then series of inequalities

$$\frac{1}{\mu(A)} \int_A f(x) d\mu(x) \leq \frac{1}{\mu(B)} \int f(x) d\mu(x) \leq \frac{1}{\mu(B \setminus A)} \int_{B \setminus A} f(x) d\mu(x)$$

hold for every convex function $f : B \to \mathbb{R}$.

We specifically utilize Proposition 5.1 for concave functions f, for which the inequalities are reversed as in Remark 1 from [PPP11]. We apply the Jensen inequality variants of [PPP11] to bounded open intervals, where the results hold identically even though we do not have closed intervals since Lebesgue measure assigns zero measure to boundary points.

Specifically, for any bounded open interval $I = (a, b)$, we apply Proposition 26 to its closure $\overline{I} = [a, b]$. Since the endpoints have Lebesgue measure zero, we have

$$\int_I f = \int_{\overline{I}} f \quad \text{and} \quad |I| = |\overline{I}|,$$

so the Jensen-type inequalities from Proposition 26 carry over unchanged to open intervals.

5.1 A is one interval

In this section, A^* is defined purely via the quadrature inequality; we will later show that this set indeed describes the macroscopic support of the ARW sleeping configuration and carries uniform density ρ_c.

Lemma 5.2. *Let $A \subset \mathbb{R}$ be a bounded open interval. Then an interval $B \subset \mathbb{R}$ has the same center of mass as A and length given by $|B| = |A|/\rho_c$ if and only if*

$$\int_A u dx \geq \rho_c \int_B u dx$$

for all integrable superharmonic (concave) functions u on B.

Proof. ($\Rightarrow$) Suppose that B has the same center of mass as A and $|B| = |A|/\rho_c$. This is equivalent to $\rho_c = |A|/|B|$. Then the inequality

$$\int_A u dx \geq \rho_c \int_B u dx$$

is equivalent to

$$\frac{1}{|A|}\int_A u dx \geq \frac{1}{|B|}\int_B u dx$$

for integrable superharmonic (concave) u on B. Because the centers of A and B are equal, we have the equality

$$\frac{1}{|A|}\int_A x dx = \frac{1}{|B|}\int_B x dx.$$

Thus, the desired inequality holds by Proposition 5.1. Note also that by construction, since $\rho_c \leq 1$, we must have $A \subseteq B$, so any integrable function on B is also integrable on A, and the left integral always exists.

($\Leftarrow$) Assume that $\int_A u dx \geq \rho_c \int_B u dx$ for all integrable concave functions u on B. We will show that the two sets have the same center and that their lengths satisfy the desired relationship.

Note first that constant functions are both integrable and concave on intervals in $\mathbb{R}$, including sets A and B. Then if $u \equiv 1$, we get that

$$\int_A u dx \geq \rho_c \int_B u dx,$$

implies

$$|A| \geq \rho_c |B|.$$

Similarly, if $u \equiv -1$, then the inequality gives $-|A| \geq -\rho_c|B|$. These two results are equivalent to $\frac{|A|}{|B|} \geq \rho_c$ and $\frac{|A|}{|B|} \leq \rho_c$ respectively, thus we must have $\frac{|A|}{|B|} = \rho_c$.

Last, we show that the centers of mass of A and B must coincide. Recall that the center of a measurable set $X \subset \mathbb{R}^n$ is defined as $\frac{1}{|X|}\int_X x dx$. Note also that a linear function $u(x) = x$ is concave. With this u, the integral inequality $\int_A u dx \geq \rho_c \int_B u dx$ would imply

$$\frac{1}{|A|}\int_A x dx \geq \frac{1}{|B|}\int_B x dx.$$

If $u(x) = -x$, it would still be concave, and the integral inequality would yield

$$\frac{1}{|A|}\int_A -x dx \geq \frac{1}{|B|}\int_B -x dx$$

$$\frac{1}{|A|}\int_A x dx \leq \frac{1}{|B|}\int_B x dx,$$

Then by definition, the centers of A and B must be equal. □

With this result, we can formally identify A^{comb} as A^* for the case where the bounded open set $A \subset \mathbb{R}$ is simply a single open interval. Note that this is consistent with Definition 2.5 presented earlier.

Definition 5.3. *Let $A \subset \mathbb{R}$ be a bounded open interval. Then the interval $A^* \subset \mathbb{R}$ is the unique open interval with the same center of mass as A and length given by $|A^*| = |A|/\rho_c$.*

5.2 A is two intervals

To explore A^* further and in a more general sense, we also expand on Definition 4.1, the definition of an ϵ-lattice approximation of A, for when A is a union of open intervals:

Corollary 5.4. *If $A = \bigcup_{i\in I} A_i$ is a (finite or countable) union of open intervals $A_i \subset \mathbb{R}$, then*

$$A_\epsilon = \bigcup_{i\in I} (A_i)_\epsilon$$

where $(A_i)_\epsilon := \{z \in \mathbb{Z} : z\epsilon \in A_i\}$.

Proof. We prove this by proving both subset directions. Fix ϵ. Suppose that $x \in A_\epsilon$. Then by definition, $x\epsilon \in A$, so $x\epsilon \in A_i$ for some i. Then $x \in A_{i,\epsilon}$ for such i as well, and thus $x \in \bigcup_{i\in I} A_{i,\epsilon}$.

Now suppose that $x \in \bigcup_{i\in I} A_{i,\epsilon}$. Then $x \in A_{i,\epsilon}$ for some i, and thus $x\epsilon \in A_i$ for such i. Then $x\epsilon \in A$ by construction of A, and thus $x \in A_\epsilon$, showing set equality. $\square$

For the rest of this section, if A is a single bounded open interval we denote by A^* the interval from Definition 5.3 (same center, length $|A|/\rho_c$). For $A = A_1 \cup A_2$ with A_i intervals, the notation A_i^* always refers to this single–interval construction applied to A_i, whereas A^* will denote the starred set for the whole union A, defined later.

We wish to rewrite Lemma 4.5 in a slightly more general sense, where rather than A being a single open interval, we have an A that is a union of multiple open intervals. We begin by considering A as the union of just two open intervals A_1 and A_2. Under some condition where the distance between A_1 and A_2 is sufficient, we expect that the A^* we seek as in 4.5 will be simply the union of A_1^* and A_2^*, whereas if the intervals A_1 and A_2 are two close together, the resulting A^* will just be one interval.

Geometrically, whether A_1^* and A_2^* remain disjoint depends on how large the gap between A_1 and A_2 is compared to their total mass: if the gap is too small, the expanded intervals overlap and merge into a single component of A^*. On the ARW side, the relevant quantity is the initial particle density in a bridge interval $\mathcal{I}$ (defined below), the smallest interval containing A_1^* and A_2^*. Lemma 5.5 makes this connection precise by showing that A_1^* and A_2^* remain disjoint if and only if the initial density of particles in I_ϵ is below the critical density ρ_c.

Lemma 5.5. *Let $A_1, A_2 \subset \mathbb{R}$ be two disjoint open intervals so that $A_1 = (a_1, b_1)$ with center c_1, $A_2 = (a_2, b_2)$ with center c_2, and $A = A_1 \cup A_2$. Define A_1^* (and similarly A_2^*) as the*

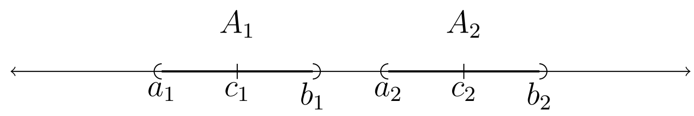

Figure 3: Two open intervals A_1 and A_2 on the real line $\mathbb{R}$.

interval with the same center as A_1 and length $|A_1^*| = |A_1|/\rho_c$, as in Definition 5.3. Further, define the interval

$$\mathcal{I} = \left(c_1 - \frac{|A_1|}{2\rho_c}, c_2 + \frac{|A_2|}{2\rho_c}\right)$$

which encapsulates A_1^* and A_2^*.

Consider an initial configuration for ARW where we start with one particle at each site in A_ϵ. Then A_1^* and A_2^* don't touch if and only if the density of particles in $\mathcal{I}_\epsilon$ (before stabilization) is less than ρ_c.

Proof. We start by establishing the conditions under which A_1^* and A_2^* touch. By definition, the leftmost point of A_2^* is $c_2 - \frac{|A_2|}{2\rho_c}$ and the rightmost point of A_1^* is $c_1 + \frac{|A_1|}{2\rho_c}$. Then if these two intervals overlap, we have

$$c_1 + \frac{|A_1|}{2\rho_c} \geq c_2 - \frac{|A_2|}{2\rho_c},$$

which is equivalent to

$$c_2 - c_1 \leq \frac{|A_1| + |A_2|}{2\rho_c}.$$

Then when A_1^* and A_2^* don't touch, we have

$$c_2 - c_2 > \frac{|A_1| + |A_2|}{2\rho_c}.$$

Now we can calculate the density of particles in $\mathcal{I}$. We know that the total number of particles will be equal to $|A_{1,\epsilon}| + |A_{2,\epsilon}| = \frac{|A_1|}{\epsilon} + \frac{|A_2|}{\epsilon}$ in the limit as $\epsilon \to 0$, as the initial configuration begins with one particle at each site of $A_{1,\epsilon}$ and $A_{2,\epsilon}$.

Using Corollary 4.4, we have $|A_{i,\epsilon}| = |A_i|/\epsilon + O(1)$ for $i = 1, 2$ and $|\mathcal{I}_\epsilon| = |\mathcal{I}|/\epsilon + O(1)$ as $\epsilon \to 0$. Hence

$$\lim_{\epsilon \to 0} \text{Density in } \mathcal{I}_\epsilon = \lim_{\epsilon \to 0} \frac{|A_{1,\epsilon}| + |A_{2,\epsilon}|}{|\mathcal{I}_\epsilon|} = \frac{|A_1| + |A_2|}{|\mathcal{I}|}.$$

Under the condition

$$c_2 - c_1 > \frac{|A_1| + |A_2|}{2\rho_c},$$

we have

$$|\mathcal{I}| = c_2 - c_1 + \frac{|A_1| + |A_2|}{2\rho_c} > \frac{|A_1| + |A_2|}{2\rho_c} + \frac{|A_1| + |A_2|}{2\rho_c} = \frac{|A_1| + |A_2|}{\rho_c},$$

and therefore

$$\lim_{\epsilon \to 0} \text{Density in } \mathcal{I}_\epsilon = \frac{|A_1| + |A_2|}{|\mathcal{I}|} < \rho_c.$$

The reverse direction of the above arguments also hold (the chain of inequalities can be reversed), and we conclude that A_1^* and A_2^* don't touch if and only if the density of particles in $\mathcal{I}$ is less than ρ_c. □

With the above result quantifying the distance between A_1 and A_2 (and their "starred" regions respectively), we can define the region A^* in the case where A is the union of two open intervals:

Definition 5.6. *Let $A_1, A_2 \subset \mathbb{R}$ be two disjoint open intervals so that $A_1 = (a_1, b_1)$ with center c_1, $A_2 = (a_2, b_2)$ with center c_2, and $A = A_1 \cup A_2$. Define A_1^* (and similarly A_2^*) as the interval with the same center as A_1 and length $|A_1^*| = |A_1|/\rho_c$.*

If A_1^* and A_2^* do not touch, satisfying the constraints of Lemma 5.5, then $A^* = A_1^* \cup A_2^*$, where A_1^* and A_2^* are each defined as in the one-interval case in Definition 5.3.

l If A_1^* and A_2^* touch, then A^* will be the unique open interval with its center of mass equal to the center of mass of A and length $|A^*| = \frac{|A|}{\rho_c}$. Specifically, the center of mass of A (and A^*) is

$$\frac{c_1|A_1| + c_2|A_2|}{|A_1| + |A_2|}.$$

Now, we want to prove that Definition 5.6 satisfies the quadrature inequality constraints for A^* from Theorem 2.3, similar to Lemma 5.2. We begin with the case that A_1^* and A_2^* do not overlap.

Lemma 5.7. *Let $A = A_1 \cup A_2 \subset \mathbb{R}$ be a bounded open set as defined in Definition 5.6. Then, if an open set B is defined as $B = A_1^* \cup A_2^* \subset \mathbb{R}$, where A_1^* and A_2^* are disjoint and defined* *as in Definition 5.3, we must have*

$$\int_A u dx \geq \rho_c \int_B u dx$$

for all integrable superharmonic (concave) functions u on B.

Proof. We first note that, by additivity of integrals, we have

$$\int_A u dx = \int_{A_1} u dx + \int_{A_2} u dx,$$

and

$$\int_B u dx = \int_{A_1^*} u dx + \int_{A_2^*} u dx$$

for any integrable function u on B (which is also integrable on A since $A \subset B$.)

Similar to Lemma 5.2, we have by Propositon 5.1 that

$$\frac{1}{|A_1|} \int_{A_1} x dx = \frac{1}{|A_1^*|} \int_{A_1^*} x dx$$

implies

$$\frac{1}{|A_1|} \int_{A_1} u dx \geq \frac{1}{|A_1^*|} \int_{A_1^*} u dx$$

$$\int_{A_1} u dx \geq \rho_c \int_{A_1^*} u dx$$

for superharmonic u. Using a similar inequality that we obtain for integrals on A_2 and A_2^*, we have

$$\begin{aligned}
\int_A u dx &= \int_{A_1} u dx + \int_{A_2} u dx \\
&\geq \rho_c \left(\int_{A_1^*} u dx + \int_{A_2^*} u dx \right) \\
&= \rho_c \int_B u dx,
\end{aligned}$$

as desired. □

Lemma 5.8. *If $A = A_1 \cup A_2 \subset \mathbb{R}$, then the center of mass of $A_1^* \cup A_2^*$ is equal to the center* *of mass of A.*

Note that this result holds whether or not A_1^* and A_2^* have any overlap.

Proof. Let $A = A_1 \cup A_2 \subset \mathbb{R}$, where A_1 and A_2 are open intervals in $\mathbb{R}$, and denote the center of mass of a set $X \subset \mathbb{R}$ as X^{CM}. Suppose interval $A_1 \subset \mathbb{R}$ has center (and center of mass) c_1, and interval $A_2 \subset \mathbb{R}$ has center c_2. Then the center of mass of A is

$$A^{\mathrm{CM}} = \frac{c_1|A_1| + c_2|A_2|}{|A_1| + |A_2|}.$$

Since $|A_1^*| = |A_1|/\rho_c$ (and similarly for A_2^*), and A_i^* has the same center of mass as its corresponding A_i by Definition 5.3, we get the center of mass of $A_1^* \cup A_2^*$ to be

$$\begin{aligned}(A_1^* \cup A_2^*)^{\mathrm{CM}} &= \frac{c_1|A_1^*| + c_2|A_2^*|}{|A_1^*| + |A_2^*|} \\ &= \frac{c_1\frac{|A_1|}{\rho_c} + c_2\frac{|A_2|}{\rho_c}}{\frac{|A_1|}{\rho_c} + \frac{|A_2|}{\rho_c}} \\ &= \frac{c_1|A_1| + c_2|A_2|}{|A_1| + |A_2|} = A^{\mathrm{CM}}.\end{aligned}$$

Thus, the center of mass of A and $A_1^* \cup A_2^*$ are equal, as desired. □

Below we prove another variant of Jensen's Inequality that will prove useful for showing A^* satisfies the quadrature inequality.

Proposition 5.9. *Let $I, J, K \subset \mathbb{R}$ be open intervals such that J has the same barycenter (center of mass) as $I \cup K$, and*

$$|J| = |I| + |K|.$$

Further, suppose that $I \cap K \subset J$. If $u : \mathbb{R} \to \mathbb{R}$ is a concave function that is integrable on J and $I \cup K$, then

$$\int_{I\cup K} u(x)\,dx + \int_{I\cap K} u(x)\,dx \geq \int_J u(x)\,dx.$$

Proof. Let $I, J, K \subset \mathbb{R}$ be open intervals such that $|J| = |I| + |K| = |I \cup K| + |I \cap K|$ and $I \cap K \subset J$. Let $u : \mathbb{R} \to \mathbb{R}$ be an integrable concave function on J. Define two probability measures ν and λ on $\mathbb{R}$ by

$$d\nu(x) = \frac{1}{|J|}\big(1_{I\cup K}(x) + 1_{I\cap K}(x)\big)\,dx, \qquad d\lambda(x) = \frac{1}{|J|}1_J(x)\,dx.$$

Both measures have total mass 1 and, since $|I \cup K| + |I \cap K| = |J|$, they share the same barycenter (center of mass):

$$\int_{\mathbb{R}} x\,d\nu(x) = \int_{\mathbb{R}} x\,d\lambda(x).$$

By Proposition 5.1, for any concave function u and probability measures with equal barycenter and with $I \cap K \subset J$, we have

$$\int_{\mathbb{R}} u(x)\,d\nu(x) \geq \int_{\mathbb{R}} u(x)\,d\lambda(x).$$

Substituting ν and λ yields

$$\frac{1}{|J|}\left(\int_{I\cup K} u(x)\,dx + \int_{I\cap K} u(x)\,dx\right) \geq \frac{1}{|J|}\int_J u(x)\,dx.$$

Multiplying both sides by $|J|$ gives

$$\int_{I\cup K} u(x)\,dx + \int_{I\cap K} u(x)\,dx \geq \int_J u(x)\,dx.$$

□

Geometrically, the measure ν places additional weight on $I \cap K$, so it is more concentrated than the uniform measure λ on J. Since u is concave, concentrating measure increases the integral average, yielding the stated inequality.

The above result formalizes the intuition that, for concave functions, concentrating mass into a single interval cannot increase the integral as long as the total length and barycenter are preserved. Among all configurations obtained by distributing the same total length around a fixed barycenter, the integral over a single interval is no larger than the integral over a more "spread-out" configuration such as $I \cup K$ with extra mass on $I \cap K$.

The following proposition applies this result to the two-interval configuration $A_1^* \cup A_2^*$

and to the single interval $B = (A_1 \cup A_2)^*$:

Proposition 5.10. *Let $A = A_1 \cup A_2 \subset \mathbb{R}$ be a bounded open set as defined in Definition 5.6. If a single open interval $B = (A_1 \cup A_2)^* \subset \mathbb{R}$ has the same center of mass as A and length given by $|B| = |A|/\rho_c$, then we must have that $A_1^* \cup A_2^* \subset B$.*

Proof. Suppose A_1 and A_2 are open intervals in $\mathbb{R}$ with centers c_1 and c_2, respectively. Then by definition,

$$A_1^* \cup A_2^* = (x, y) = \left(c_1 - \frac{|A_1|}{2\rho_c}, c_2 + \frac{|A_2|}{2\rho_c} \right)$$

and

$$B = (b_1, b_2) = \left(c - \frac{|A|}{2\rho_c}, c + \frac{|A|}{2\rho_c} \right),$$

where c is the center of mass of the original system A

$$c = A^{\text{CM}} = \frac{c_1|A_1| + c_2|A_2|}{|A_1| + |A_2|},$$

and $|A|$ is the total length $|A| = |A_1| + |A_2|$.

We first show that the left endpoint of B is less than the left endpoint of $A_1^* \cup A_2^*$, i.e. $b_1 < x$. Note that

$$b_1 = c - \frac{|A|}{2\rho_c} = \frac{c_1|A_1|}{|A_1| + |A_2|} + \frac{c_2|A_2|}{|A_1| + |A_2|} - \frac{|A_1|}{2\rho_c} - \frac{|A_2|}{2\rho_c},$$

and

$$\frac{|A_1||A_2|}{2\rho_c} + \frac{|A_2|^2}{2\rho_c} > 0.$$

Then we have

$$
\begin{aligned}
\frac{|A_1||A_2|}{2\rho_c} + \frac{|A_2|^2}{2\rho_c} &> 0 \\
c_1|A_1| + c_2|A_2| + \frac{|A_1||A_2|}{2\rho_c} + \frac{|A_2|^2}{2\rho_c} &> c_1|A_1| + c_2|A_2| \\
(|A_1| + |A_2|)\left(c_1 + \frac{|A_2|}{2\rho_c}\right) &> c_1|A_1| + c_2|A_2| \\
c_1 &> \frac{c_1|A_1|}{|A_1|+|A_2|} + \frac{c_2|A_2|}{|A_1|+|A_2|} - \frac{|A_2|}{2\rho_c} \\
c_1 - \frac{|A_1|}{2\rho_c} &> \frac{c_1|A_1|}{|A_1|+|A_2|} + \frac{c_2|A_2|}{|A_1|+|A_2|} - \frac{|A_1|}{2\rho_c} - \frac{|A_2|}{2\rho_c} \\
c_1 - \frac{|A_1|}{2\rho_c} &> b_1,
\end{aligned}
$$

so the left endpoint of B is less than the left endpoint of $A_1^* \cup A_2^*$, as desired. Similarly, the right endpoint of B is

$$b_2 = c + \frac{|A|}{2\rho_c} = \frac{c_1|A_1|}{|A_1|+|A_2|} + \frac{c_2|A_2|}{|A_1|+|A_2|} + \frac{|A_1|}{2\rho_c} + \frac{|A_2|}{2\rho_c},$$

and the right endpoint of $A_1^* \cup A_2^*$ is $c_2 + \frac{|A_2|}{2\rho_c}$. Now, to show that $b_2 > c_2 + \frac{|A_2|}{2\rho_c}$:

$$
\begin{aligned}
b_2 &= \frac{c_1|A_1|}{|A_1|+|A_2|} + \frac{c_2|A_2|}{|A_1|+|A_2|} + \frac{|A_1|}{2\rho_c} + \frac{|A_2|}{2\rho_c} \\
&= \left(\frac{c_1|A_1|}{|A_1|+|A_2|} + \frac{|A_2|}{2\rho_c}\right) + \left(\frac{c_2|A_2|}{|A_1|+|A_2|} + \frac{|A_1|}{2\rho_c}\right)
\end{aligned}
$$

Then

$$
\begin{aligned}
b_2 - \left(c_2 + \frac{|A_2|}{2\rho_c}\right) &= \left(\frac{c_1|A_1|}{|A_1|+|A_2|} + \frac{c_2|A_2|}{|A_1|+|A_2|} + \frac{|A_1|}{2\rho_c} + \frac{|A_2|}{2\rho_c}\right) - \left(c_2 + \frac{|A_2|}{2\rho_c}\right) \\
&= \frac{c_1|A_1|}{|A_1|+|A_2|} + \frac{c_2|A_2|}{|A_1|+|A_2|} - c_2 + \frac{|A_1|}{2\rho_c} \\
&= \frac{c_1|A_1| - c_2|A_1|}{|A_1|+|A_2|} + \frac{|A_1|}{2\rho_c} \\
&= \frac{(c_1 - c_2)|A_1|}{|A_1|+|A_2|} + \frac{|A_1|}{2\rho_c}.
\end{aligned}
$$

Since $c_2 > c_1$, $(c_1 - c_2) < 0$, but $\frac{|A_1|}{2\rho_c} > 0$ and $|A_1| + |A_2| > 0$, so the sum will be positive. Thus, $b_2 > c_2 + \frac{|A_2|}{2\rho_c}$, and $A_1^* \cup A_2^* \subset B$. □

Now, we can prove that our A^* for the two-intervalled A as in Definition 5.6 satisfies the quadrature inequality.

Lemma 5.11. *Let $A = A_1 \cup A_2 \subset \mathbb{R}$ be a bounded open set as defined in Definition 5.6. If a single open interval $B = (A_1 \cup A_2)^* \subset \mathbb{R}$ has the same center of mass as A and length given by $|B| = |A|/\rho_c$, then we must have*

$$\int_A u dx \geq \rho_c \int_B u dx$$

for all integrable superharmonic (concave) functions u on B.

Proof. We note that Proposition 5.1 does not automatically apply since A is not a single interval. However, we can apply Proposition 1 to A_1 and A_2 individually; For concave u on B, u to A_1 and A_2 and apply Lemma 5.2. Then

$$\int_{A_1} u dx \geq \rho_c \int_{A_1^*} u dx,$$

for all concave u on B, and we get a similar relationship for A_2 and A_2^*.

Note that any function that is integrable on B will also be integrable on $A_1^* \cup A_2^*$, since $A_1^* \cup A_2^* \subset B$ by Proposition 5.10. Then for all such u, we have

$$\begin{aligned}
\int_A u dx &= \int_{A_1} u dx + \int_{A_2} u dx \\
&\geq \rho_c \left(\int_{A_1^*} u dx + \int_{A_2^*} u dx \right) \\
&= \rho_c \left(\int_{A_1^* \cup A_2^*} u dx + \int_{A_1^* \cap A_2^*} u dx \right),
\end{aligned}$$

where the last equality comes from the inclusion-exclusion principle, and the fact that A_1^* and A_2^* are intervals that have nonempty overlap, so $A_1^* \cap A_2^*$ is nonempty. Note that by construction, the lengths of these intervals coincide:

$$|B| = \frac{|A|}{\rho_c} = \frac{|A_1| + |A_2|}{\rho_c} = |A_1^*| + |A_2^*| = |A_1^* \cup A_2^*| + |A_1^* \cap A_2^*|.$$

We also know that B has the same center of mass as A, and thus the same center of mass as $A_1^* \cup A_2^*$ by Lemma 5.8. Applying Proposition 5.9 with $J = B$, $I = A_1^*$, and $K = A_2^*$, we get

$$\begin{aligned}\int_A u dx &\geq \rho_c \left(\int_{A_1^* \cup A_2^*} u dx + \int_{A_1^* \cap A_2^*} u dx \right) \\ &\geq \rho_c \int_B u dx,\end{aligned}$$

for integrable superharmonic functions u on B, as desired. □

Lemma 5.12. *Let $A = A_1 \cup A_2 \subset \mathbb{R}$ be a bounded open set, with A_1 and A_2 being open intervals and A_1^* and A_2^* as defined in Definition 5.6.*

Suppose that $B \subset \mathbb{R}$ is an open set so that

$$\int_A u dx \geq \rho_c \int_B u dx$$

for all integrable superharmonic (concave) functions u on B. Then we must have that either $B = A_1^* \cup A_2^*$, or that B is the single interval with the same center as A and $|B| = |A|/\rho_c$.

Proof. We can split this up into two cases based on whether or not the starred intervals for A_1 and A_2 touch.

When they don't touch: If they do not touch, then we claim $B = A_1^* \cup A_2^*$ is the only set that satisfies the inequality. So for contradiction, suppose that we have $B \neq A_1^* \cup A_2^*$.

There are a few ways that B can be "wrong." First, suppose that B is too long, and $|B| > |A|/\rho_c$. Then for superharmonic $u \equiv 1$ on B, we'd have $\int_A u dx = |A| < \rho_c |B| = \rho_c \int_B u dx$.

Next, suppose that B is too short, and $|B| < |A|/\rho_c$. Similar to the last case, we can pick the concave function $u \equiv -1$ on B and get $\int_A u dx = -|A| < -\rho_c |B| = \rho_c \int_B u dx$.

Last, suppose that B has the exact right length, $|B| = |A|/\rho_c$. Then whether B is one, two, or a million intervals, I do not care. Let $u(x) = -|x - m|$, where m is the center of mass of B (and A). This function is strictly concave and maximized at $x = m$. In any case, we will always have $\int_A udx \geq \rho_c \int_B udx$.

When they do touch: By repeating the work from the case above, we get that if B does not have the same length as A, then we get a contradiction.

So we suppose that B is not the single interval I^* with center m and length $|A|/\rho_c$, but B has the same total measure. Then B must be a union of two or more separated intervals of total length $|A|/\rho_c$.

Let $u(x) = -|x - m|$, where m is the center of mass of B (and A). Then the integral $\int_B u(x)\,dx$ is maximized when B is as concentrated as possible around m, i.e., when B is a single interval centered at m. If B is a union of disjoint intervals, some of its measure must be further from m, so $\int_B u(x)\,dx < \int_{I^*} u(x)\,dx$.

Similarly, since $A \subseteq I^*$, we have $\int_A u(x)\,dx \leq \int_{I^*} u(x)\,dx$.

Thus, for this choice of u,

$$\int_A u(x)\,dx < \rho_c \int_B u(x)\,dx,$$

contradicting the assumed inequality.

Therefore, the only possible B for which the inequality holds for all concave u is I^*, the single interval centered at m and with length $|A|/\rho_c$. □

Corollary 5.13. *For $A = A_1 \cup A_2$, A^* as defined in Definition 5.6 is the unique open set satisfying*

$$\int_A udx \geq \rho_c \int_{A^*} udx$$

for all integrable superharmonic (concave) functions u on A^*.

Proof. Lemmas 5.7, 5.11, and 5.12 prove this. □

5.3 Examples and concavity tool.

We extend our definition of A^* to the case where A is the union of greater than two intervals, and then for when A is the union of countably many intervals. Since every open set in $\mathbb{R}$

can be expressed as a union of open intervals, this will cover all our cases for starting sets $A \subset \mathbb{R}$. We again utilize the result of Lemma 5.5 to quantify when the A_i^* intervals touch by using the density of some bridging intervals $\mathcal{I}$ (again, recall that for any i, A_i^* is the interval with the same center of mass as A_i and length $|A_i^*| = |A_i|/\rho_c$).

Defining A^* for the union of three intervals follows the same general ideas as the definition for two intervals, but it did prove a little more tricky than originally expected. It is possible for ρ_c, A_1, A_2 and A_3 to be so that A_2^* and A_3^* do not touch, but $(A_1 \cup A_2)^*$ (as in Definition 5.6) does touch A_3^*. Similarly, if A_1^* and A_2^* have a gap between them, it is possible that A_1^* and $(A_2 \cup A_3)^*$ do not. So rather than just looking at adjacent pairs of A_1, A_2 and A_3, we need to be careful to consider how our newly constructed starred intervals interact with each other.

We give a rough example to demonstrate this sad detail:

Example 5.14. *Define A as the union of the three open intervals $A_1 = (0, 100)$, $A_2 = (110, 210)$, and $A_3 = (400, 500)$. Let $\rho = 2/5$. Then $A_1^* = (-75, 175)$, $A_2^* = (35, 285)$ and $A_3^* = (325, 575)$. In this case, A_2^* and A_3^* do not touch. However, since A_1^* and A_2^* do touch, we have $(A_1 \cup A_2)^* = (-145, 355)$, which in turn has nonempty intersection with A_3^*.*

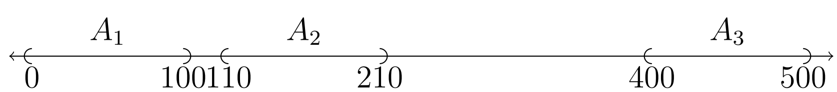

Figure 4: Example 5.14 of a starting configuration A, where A is a union of three intervals, and A_2^* does not intersect A_3^*, but $(A_1 \cap A_2)^*$ does intersect A_3^*.

The general idea will be to first construct A_i^* for $i = 1, 2, 3$, then check for intersection of these starred intervals. For any intervals that overlap, a new starred interval for its union will be constructed, then we will go back and check if there are any other new overlapping intervals as a result.

In order to show our general definitions of A^* satisfy the quadrature inequality, we utilize yet another consequence of Jensen's Inequality which expands on Proposition 5.9.

Proposition 5.15. *Let $A_1^*, A_2^*, \ldots, A_n^* \subset \mathbb{R}$ be open sets, and let $D \subset \mathbb{R}$ be an open set such* *that*

$$|D| = \sum_{i=1}^{n} |A_i^*|.$$

Suppose that the barycenter (center of mass) of D coincides with that of $A_1^* \cup \cdots \cup A_n^*$, i.e.

$$\frac{1}{|D|}\int_D x\,dx = \frac{1}{|A_1^*| + \cdots + |A_n^*|}\sum_{i=1}^{n}\int_{A_i^*} x\,dx.$$

Let $u : \mathbb{R} \to \mathbb{R}$ be a concave function integrable on all these sets. Then,

$$\sum_{i=1}^{n}\int_{A_i^*} u(x)\,dx \geq \int_D u(x)\,dx.$$

Proof. Define probability measures on $\mathbb{R}$ by

$$d\nu(x) = \frac{1}{|D|}\sum_{i=1}^{n} 1_{A_i^*}(x)\,dx, \qquad d\lambda(x) = \frac{1}{|D|}1_D(x)\,dx.$$

Then $\nu(\mathbb{R}) = \lambda(\mathbb{R}) = 1$, and by the barycenter assumption in the statement of the proposition we have

$$\int x\,d\nu(x) = \frac{1}{|D|}\sum_{i=1}^{n}\int_{A_i^*} x\,dx = \frac{1}{|D|}\int_D x\,dx = \int x\,d\lambda(x).$$

Set $\mu = \nu - \lambda$. Then μ is a signed measure on (a, b) (containing all the sets in question) satisfying

$$\int_a^b d\mu = \nu(\mathbb{R}) - \lambda(\mathbb{R}) = 0, \qquad \int_a^b x\,d\mu(x) = 0.$$

Moreover, the construction of ν as a mixture of the uniform measures on the intervals A_i^* and of λ as the uniform measure on their convex hull D implies that the cumulative integrals of μ are nonnegative:

$$\int_a^t \mu(a, x]\,dx \;\geq\; 0 \qquad \text{for all } t \in [a, b].$$

Therefore the hypotheses of Marshall–Olkin–Arnold/Karamata [MOA11, Prop. 16.B.4.a] are satisfied, and that proposition yields

$$\int \phi\,d\mu \;=\; \int \phi\,d\nu - \int \phi\,d\lambda \;\geq\; 0$$

for every convex function ϕ on (a, b). Taking $\phi = -u$, which is convex when u is concave, we obtain

$$\int u(x)\, d\nu(x) \;\geq\; \int u(x)\, d\lambda(x)$$

for all concave u. But because our measures integrate to

$$\int u(x)\, d\nu(x) = \frac{1}{|D|} \sum_{i=1}^{n} \int_{A_i^*} u(x)\, dx$$

and

$$\int u(x)\, d\lambda(x) = \frac{1}{|D|} \int_D u(x)\, dx,$$

we thus have that

$$\frac{1}{|D|} \sum_{i=1}^{n} \int_{A_i^*} u(x)\, dx \geq \frac{1}{|D|} \int_D u(x)\, dx,$$

which, upon multiplying both sides by $|D|$, yields

$$\sum_{i=1}^{n} \int_{A_i^*} u(x)\, dx \geq \int_D u(x)\, dx.$$

This completes the proof. $\square$

5.4 General construction of A^* (A^{comb})

Recall that when $A \subset \mathbb{R}$ is a finite union of disjoint open intervals, Definition 2.5 constructs A^{comb} by first expanding each component of A by the factor $1/\rho_c$ around its center of mass and then repeatedly merging any overlapping or touching intervals according to the smash-sum algorithm. When A is an arbitrary bounded open set, Definition 2.8 instead defines A^{sm} as the unique bulky quadrature domain associated with the weight $w = \rho_c \mathbf{1}_A$. By Proposition 2.10, these two constructions agree when A is a finite union of disjoint intervals (up to sets of measure zero), so in what follows we will freely use either viewpoint when analyzing A^*.

The next sections show that the smash construction from Definition 2.5 indeed yields the unique quadrature domain from Theorem 2.3 by analyzing the cases of one interval, two intervals, and then finitely many and countably many intervals.

We first show that the merge operation is associative and order-independent:

Proposition 5.16 (Length and center of mass preserved on a merge component)**.** *Let $I_1, \dots, I_n$ be n (possibly overlapping) open intervals on $\mathbb{R}$, with lengths*

$$\ell_i = |I_i| \quad \text{and centers} \quad c_i \in \mathbb{R}, \qquad i = 1, \dots, n.$$

Fix a nonempty index set $S \subset \{1, \dots, n\}$ such that the intervals $\{I_i : i \in S\}$ form a single merged interval at the end of the merging procedure in Definition 2.5.

Run the merging procedure of Definition 2.5, and suppose all intervals $\{I_i : i \in S\}$ are eventually replaced by a single interval G_S. Then

$$|G_S| \;=\; \sum_{i \in S} \ell_i, \qquad c(G_S) \;=\; \frac{\sum_{i \in S} \ell_i c_i}{\sum_{i \in S} \ell_i}, \tag{17}$$

and these quantities are independent of the order in which overlapping pairs inside S are merged.

Proof. First note the two–interval case. If I and J are merged into M as in Definition 2.5, then

$$|M| = |I| + |J|, \qquad |M|\, c(M) = |I|\, c(I) + |J|\, c(J),$$

so merging preserves total length and the first moment.

Now fix $S \subset \{1, \dots, n\}$ as in the statement and track only intervals built from $\{I_i : i \in S\}$. Any intermediate interval J obtained from these is produced by iterating the two–interval merge, hence there is a nonempty $T \subseteq S$ such that

$$|J| = \sum_{i \in T} \ell_i, \qquad |J|\, c(J) = \sum_{i \in T} \ell_i c_i. \tag{18}$$

We prove (17) by induction on $|S|$. If $|S| = 1$ there is nothing to show. For $|S| \geq 2$, pick $j \in S$ and let $S' = S \setminus \{j\}$. By induction, whenever the intervals from S' have merged to a single interval $G_{S'}$, it satisfies (18) with $T = S'$. When $G_{S'}$ is next merged with I_j, the two–interval rule gives a new interval H with

$$|H| = \sum_{i \in S} \ell_i, \qquad |H|\, c(H) = \sum_{i \in S} \ell_i c_i,$$

so H satisfies (18) with $T = S$. Any further merges involving only indices from S keep $T = S$ in (18), so the final interval G_S also satisfies (17).

Since the right-hand sides of (17) depend only on the original data $\{\ell_i, c_i : i \in S\}$, they are independent of the order of merges within S. $\square$

To formally prove that any merge order will result in the same ending configuration, we define the concept of a valid merging sequence.

Definition 5.17 (Valid merging sequence)**.** *Let $\mathcal{C}_0$ be a finite collection of open intervals on $\mathbb{R}$. A sequence of collections $\mathcal{C}_0, \mathcal{C}_1, \ldots, \mathcal{C}_N$ is called **valid** if, for each $k = 0, \ldots, N-1$:*

- *There exist intervals $I, J \in \mathcal{C}_k$, with $I \cap J \neq \emptyset$.*
- *$\mathcal{C}_{k+1}$ is formed from $\mathcal{C}_k$ by merging I and J into a single interval (by your weighted center/length formula), removing I and J, and adding the merged interval.*
- *The process repeats until there is no pair of overlapping intervals in $\mathcal{C}_N$.*

Each possible way of choosing overlapping pairs at each step yields a different valid sequence, but all satisfy the following:

- Every merge happens only between intervals that overlap at that step.
- The process ends (is stable) when all intervals are disjoint (no overlap).

With this notion of validity in hand, we can now show that the outcome of the merging procedure does not depend on the chosen valid sequence.

Theorem 5.18. *Let $\mathcal{C}_0$ be any finite collection of open intervals on $\mathbb{R}$, and let A^{comb} be constructed by merging overlapping intervals according to Definition 2.5, using any valid sequence of merges in the sense of Definition 5.17. Then:*

1. The final collection $\mathcal{C}_N$ is independent of the valid merge sequence.

2. Each final interval corresponds to the union of all intervals in a "merge component," and its location and length depend only on the initial data, not the merge path.

Proof. Let C_0 be the initial collection and let $\sim$ be the equivalence relation on C_0 generated by overlap: $I \sim J$ if there exists a chain $I = J_0, J_1, \dots, J_m = J$ with $J_r \cap J_{r+1} \neq \varnothing$ for all r. Call the equivalence classes $S_1, \dots, S_k$ the merge components.

By definition of a valid sequence, each merge always occurs between overlapping intervals, so it never mixes two different components. Within a given component S_j, the algorithm keeps merging overlapping intervals until only one interval from S_j remains. Denote this final interval by G_{S_j}.

By Proposition 5.16, for each S_j the interval G_{S_j} has length

$$|G_{S_j}| = \sum_{I \in S_j} |I|, \qquad c(G_{S_j}) = \frac{\sum_{I \in S_j} |I| \, c(I)}{\sum_{I \in S_j} |I|},$$

and these quantities are independent of the order in which overlapping pairs inside S_j are merged. Hence the outcome on each component is independent of the chosen valid sequence.

The final collection C_N obtained from any valid sequence therefore consists exactly of the intervals $\{G_{S_1}, \dots, G_{S_k}\}$, with the same location and lengths. Additionally, C_N and the set $A^{\mathrm{comb}} = \bigcup_j G_{S_j}$ depend only on C_0 and not on the valid merge sequence, which proves both claims of the theorem. $\square$

With the merging construction now shown to be well defined, we turn to its quadrature property.

Theorem 5.19 (Smash-domain satisfies the quadrature inequality)**.** *Let $A \subseteq \mathbb{R}$ be a bounded open set that is a finite union of disjoint open intervals, and let A^{comb} be constructed via Definition 2.5. Then A^{comb} satisfies the quadrature inequality:*

$$\int_A u \, dx \geq \rho_c \int_{A^{\mathrm{comb}}} u \, dx$$

for all integrable superharmonic concave functions u on A^{comb}.

For a general bounded open set A, the smash-domain A^{sm} defined in Definition 2.8 (via Sakai's and Cairns's quadrature-domain theory) already satisfies the same quadrature inequality by construction. In the special case when A is a finite union of disjoint open intervals, Theorem 5.19 shows that the combinatorial smash construction of Definition 2.5 produces a set A^{comb} that coincides, up to null sets, with this quadrature domain (see also Proposition 2.10).

Proof. We prove this claim by induction on the structure of the smash sum in Definition 2.5.

Let $A = A_1 \cup \cdots \cup A_n$ be a union of n disjoint open intervals, and let A^{comb} be constructed via Definition 2.5. We consider the final state of the algorithm after all merging is complete.

Base cases:

- $n = 1$: Proven in Lemma 5.2.
- $n = 2$: Proven in Lemma 5.11.

Inductive step: Assume the result holds for all unions of fewer than n intervals. We examine the output of the smash sum algorithm in Definition 2.5:

Case 1: The algorithm terminates with $|C_N| = k \geq 2$ disjoint intervals in the final collection. That is, $A^{\mathrm{comb}} = I_1 \cup \cdots \cup I_k$ where the I_j are disjoint.

Then we can write $A = B_1 \cup \cdots \cup B_k$, where each B_j is a union of some of the original A_i's that merged to form I_j. Since each B_j has fewer than n intervals, by the inductive hypothesis:

$$\int_{B_j} u\,dx \geq \rho_c \int_{I_j} u\,dx.$$

Since the I_j are disjoint and u is integrable, we have:

$$\begin{aligned}
\int_A u\,dx &= \sum_{j=1}^{k} \int_{B_j} u\,dx \\
&\geq \sum_{j=1}^{k} \rho_c \int_{I_j} u\,dx \\
&= \rho_c \int_{A^*} u\,dx.
\end{aligned}$$

Case 2: The algorithm terminates with $|C_N| = 1$, i.e., all intervals merge into a single interval A^{comb}.

By Proposition 5.16, we have $|A^{\mathrm{comb}}| = |A|/\rho_c$, and A^{comb} has the same center of mass as A. Then the result follows directly from the same argument as Lemma 5.2, using Proposition 5.1. $\square$

We can now prove Proposition 2.10 in more detail, concluding that A^{comb} is indeed equivalent to the smash sum from [Cai22]. The purpose of this proposition is to show that, when A is a finite union of intervals, the explicit combinatorial smash construction of Definition 2.5 coincides with the abstract smash-domain A* coming from quadrature-domain theory and from Theorem 2.3.

Proof of Proposition 2.10. Let $A \subset \mathbb{R}$ be a finite union of disjoint open intervals and let A^{comb} denote the set produced by Definition 2.5 (the one-dimensional smash construction). Set

$$w := \rho_c \, 1_A.$$

By Theorem 5.19, for every integrable superharmonic (equivalently, concave) function u on A^{comb} we have the quadrature inequality

$$\int_A u \, dx \;\geq\; \rho_c \int_{A^{\mathrm{comb}}} u \, dx. \tag{19}$$

Equivalently,

$$\int_{A^{\mathrm{comb}}} u \, dx \;\leq\; \int_{\mathbb{R}} u(x) \, w(x) \, dx$$

for all such u, so A^{comb} is a quadrature domain for the weight w.

On the other hand, by Sakai's existence and uniqueness theorem for quadrature domains [Sak84], as presented in [Cai22], there exists a unique bulky quadrature domain A^{sm} for the same weight w; this is precisely the continuous Diaconis–Fulton smash sum of the components of A. Further, A^{sm} also satisfies (19), and any two quadrature domains for w must agree up sets of measure zero. Therefore A^{comb} and A^{sm} are equal up to sets of measure zero.

Finally, recall that in Theorem 2.3 we denote by A^* the unique open set satisfying the quadrature inequality (19) for all integrable superharmonic functions u; by Sakai's theorem this set is exactly the quadrature domain for $w = \rho_c 1_A$, hence coincides with A^{sm} up to Lebesgue null sets. Combining these identifications, we obtain

$$A^{\mathrm{comb}} = A^{\mathrm{sm}} = A^* \quad \text{up to sets of measure zero,}$$

which proves the proposition. $\square$

By Sakai's uniqueness theorem for quadrature domains [Sak84] and Cairns's axiomatic characterization of the smash sum [Cai22], we know that for each weight w there is a unique bulky quadrature domain, characterized by the quadrature identity for all integrable superharmonic test functions. In higher dimensions we appeal to Sakai's potential-theoretic argument for this uniqueness, but in one dimension it is also possible to write a self-contained proof using only concavity and our Jensen-type inequalities.

Corollary 5.20. *For a bounded open set $A \subset \mathbb{R}$ consisting of finitely many disjoint intervals, the set A^* in Theorem 2 is given by the smash sum construction in Definition 2.5.*

Proof. By Theorem 5.19, the smash sum satisfies the quadrature inequality. By Remark 2.12, A^* is uniquely determined by this inequality up to measure zero. Therefore, the smash sum construction produces the unique A^* from Theorem Theorem 2.3. $\square$

For the rest of the paper, when A is a finite union of intervals we have $A^{\mathrm{comb}} = A^{\mathrm{sm}} = A^*$ up to sets of measure zero, and for general bounded open A we use the quadrature-domain definition $A^* = A^{\mathrm{sm}}$.

6 A^* gets almost all the particles: multiple intervals

In this section, we extend Lemma 4.5 from the single–interval case to general bounded open sets $A \subset \mathbb{R}$. Our goal is to prove the first part of Theorem 2.3: after stabilization, almost all particles lie inside A^* and no subinterval of A^* can sustain particle density significantly above the critical value ρ_c.

We first derive a uniform upper bound on the particle density in any interval J after stabilization, regardless of whether J lies inside or outside the starting region A (Lemma 6.1). This rules out local concentrations of particles above a density of ρ_c. We then combine this with global containment estimates (Lemmas 6.3 and 6.4), which show that with high probability the vast majority of particles end up inside A^* and that any open set strictly smaller than A^* cannot contain almost all of them. Together, these results establish that A^* is the unique domain capturing almost all of the mass at density at most ρ_c.

Lemma 6.1 (Uniform density bound)**.** *Let $A \subset \mathbb{R}$ be a bounded open set which is a finite union of open intervals, and let $\epsilon > 0$. Start with one active particle at each site of A_ϵ, and run ARW with sleep rate $\lambda > 0$ until stabilization.*

Then for any interval $J \subset \mathbb{R}$ and any $\eta > 0$, there exist constants $C, c > 0$ (depending only on λ and η) such that

$$\mathbb{P}\left(\#\textit{particles in } J_\epsilon \geq (\rho_c + \eta)\frac{|J|}{\epsilon}\right) \leq Ce^{-c|J|/\epsilon}.$$

Proof. Fix $\eta > 0$. We decompose $J = (J \cap A) \cup (J \setminus A)$ as a disjoint union and apply existing bounds to each piece. Note that this decomposition is possible for any interval J, regardless of whether it intersects A or not.

Case 1: $J \subseteq A$**.** By Corollary 3.2, since $J_\epsilon \subseteq A_\epsilon$ and the odometer is positive at every site in J_ϵ, for any $\rho > \rho_c$ there exist constants $C, c > 0$ such that

$$\mathbb{P}\left(\#\text{particles in } J_\epsilon \geq \rho|J_\epsilon|\right) \leq Ce^{-c|J_\epsilon|}.$$

Since $|J_\epsilon| \approx |J|/\epsilon$ by Corollary 4.4, the result follows.

Case 2: $J \cap A = \emptyset$**.** If the intersection of J and A is empty, then so is the intersection of J_ϵ and A_ϵ by Corollary 4.3. Since J_ϵ is disjoint from A_ϵ, particles can only reach J_ϵ by traveling from A_ϵ during the ARW process.

By Corollary 3.5, for any $\rho > \rho_c$, there exist constants $C, c > 0$ (depending only on λ and ρ) such that

$$\mathbb{P}\left(\#\text{particles in } J_\epsilon \geq \rho\frac{|A|}{\epsilon}\right) \leq Ce^{-c|A|/\epsilon}.$$

We consider two subcases depending on the relative sizes of J and A.

Subcase 2a: $|J| \geq \frac{\rho_c}{\rho_c+\eta}|A|$. Choose

$$\rho = (\rho_c + \eta)\frac{|J|}{|A|}.$$

Since $|J| \geq \frac{\rho_c}{\rho_c+\eta}|A|$, we have $\rho \geq \rho_c$, so Corollary 3.5 applies. This gives

$$\mathbb{P}\left(\#\text{particles in } J_\epsilon \geq (\rho_c + \eta)\frac{|J|}{\epsilon}\right) \leq Ce^{-c|A|/\epsilon}.$$

Note that this gives an exponential bound in terms of $|A|$, but we wish to have this bound decay in $|J|$ instead. Since $|J| \geq \frac{\rho_c}{\rho_c+\eta}|A|$, we have $|J| \geq c_0|A|$ for $c_0 = \frac{\rho_c}{\rho_c+\eta} > 0$. Therefore, $|J|/\epsilon \geq c_0|A|/\epsilon$, which implies $|A|/\epsilon \geq c_0|J|/\epsilon$. Hence,

$$\mathbb{P}\left(\#\text{particles in } J_\epsilon \geq (\rho_c + \eta)\frac{|J|}{\epsilon}\right) \leq Ce^{-c|A|/\epsilon} \leq Ce^{-cc_0|J|/\epsilon} = Ce^{-c'|J|/\epsilon}$$

for $c' = cc_0 > 0$ (depending on λ and η).

Subcase 2b: $|J| < \frac{\rho_c}{\rho_c+\eta}|A|$. Choose any $\rho > \rho_c$, say $\rho = \rho_c + \eta/2$. Then by Corollary 3.5,

$$\mathbb{P}\left(\#\text{particles in } J_\epsilon \geq (\rho_c + \eta/2)\frac{|A|}{\epsilon}\right) \leq Ce^{-c|A|/\epsilon}.$$

Since $|J| < \frac{\rho_c}{\rho_c+\eta}|A|$, we have

$$(\rho_c + \eta)\frac{|J|}{\epsilon} < (\rho_c + \eta) \cdot \frac{\rho_c}{\rho_c + \eta}\frac{|A|}{\epsilon} = \rho_c\frac{|A|}{\epsilon} < (\rho_c + \eta/2)\frac{|A|}{\epsilon}.$$

Therefore, the event $\{\#\text{particles in } J_\epsilon \geq (\rho_c+\eta)\frac{|J|}{\epsilon}\}$ is contained in the event $\{\#\text{particles in } J_\epsilon \geq$

$(\rho_c + \eta/2)\frac{|A|}{\epsilon}\}$, so

$$\mathbb{P}\left(\#\text{particles in } J_\epsilon \geq (\rho_c + \eta)\frac{|J|}{\epsilon}\right) \leq Ce^{-c|A|/\epsilon}.$$

In both subcases, we obtain an exponential decay in $1/\epsilon$. In Subcase 2a, the decay rate is explicitly in terms of $|J|/\epsilon$. In Subcase 2b, where $|J|$ is small relative to $|A|$, the decay is in terms of $|A|/\epsilon$, but since $|A|$ is a fixed constant (independent of ϵ), this still provides the required exponential decay. For applications in Case 3 below, only finitely many intervals J need to be considered, so the union bound over these intervals still yields exponential decay as $\epsilon \to 0$.

Case 3: J intersects A but neither contains the other. Write $J = (J \cap A) \cup (J \setminus A)$ as a disjoint union. Then, we have

$$\#\text{particles in } J_\epsilon = \#\text{particles in } (J \cap A)_\epsilon + \#\text{particles in } (J \setminus A)_\epsilon.$$

If the total number of particles in J_ϵ exceeds $(\rho_c+\eta)\frac{|J|}{\epsilon}$, then either the density in $(J \cap A)_\epsilon$ exceeds $\rho_c + \eta/2$, or a significant number of particles lie in $(J \setminus A)_\epsilon$ (which is outside A). By the union bound,

$$\begin{aligned}
&\mathbb{P}\left(\#\text{particles in } J_\epsilon \geq (\rho_c + \eta)\frac{|J|}{\epsilon}\right) \\
&\quad \leq \mathbb{P}\left(\#\text{particles in } (J \cap A)_\epsilon \geq (\rho_c + \eta/2)\frac{|J \cap A|}{\epsilon}\right) \\
&\qquad + \mathbb{P}\left(\#\text{particles in } (J \setminus A)_\epsilon \geq \frac{\eta}{2}\frac{|J \setminus A|}{\epsilon}\right).
\end{aligned}$$

Applying Case 1 to $(J \cap A)_\epsilon \subseteq A_\epsilon$, we obtain

$$\mathbb{P}\left(\#\text{particles in } (J \cap A)_\epsilon \geq (\rho_c + \eta/2)\frac{|J \cap A|}{\epsilon}\right) \leq C_1 e^{-c_1|J \cap A|/\epsilon}.$$

Applying Case 2 to $(J \setminus A)_\epsilon$ (which is disjoint from A_ϵ by Corollary 4.3), we obtain

$$\mathbb{P}\left(\#\text{particles in } (J \setminus A)_\epsilon \geq \frac{\eta}{2}\frac{|J \setminus A|}{\epsilon}\right) \leq C_2 e^{-c_2|J \setminus A|/\epsilon}.$$

Note that both $|J \cap A|$ and $|J \setminus A|$ are positive (since neither is empty, otherwise we would be in Case 1 or Case 2) and are determined by the geometry of J and A in $\mathbb{R}$, independent of ϵ. Thus, $\min\{|J \cap A|, |J \setminus A|\} \geq \delta$ for some constant $\delta > 0$ (depending on J and A). Therefore,

$$\mathbb{P}\left(\#\text{particles in } J_\epsilon \geq (\rho_c + \eta)\frac{|J|}{\epsilon}\right) \leq C_1 e^{-c_1|J\cap A|/\epsilon} + C_2 e^{-c_2|J\setminus A|/\epsilon}$$
$$\leq Ce^{-c\min\{|J\cap A|,|J\setminus A|\}/\epsilon},$$

which decays exponentially in $1/\epsilon$, where $C = C_1 + C_2$ and $c = \min\{c_1, c_2\}$ (depending on λ and η). □

In the following corollary we convert the above tail bound to an expectation bound, which will be helpful later:

Corollary 6.2 (No interval can get too many particles)**.** *Let $A \subset \mathbb{R}$ be a bounded open set that can be represented as a finite union of open intervals. For any interval $J \subseteq \mathbb{R}$,*

$$\limsup_{\epsilon\to 0} \epsilon \cdot \mathbb{E}[N_\epsilon(J)] \leq \rho_c|J|.$$

Proof. Fix $\eta > 0$. By Lemma 6.1, for any interval J,

$$\mathbb{P}\left(N_\epsilon(J) \geq (\rho_c + \eta)\frac{|J|}{\epsilon}\right) \leq Ce^{-c|J|/\epsilon}.$$

We can write

$$\mathbb{E}[N_\epsilon(J)] = \mathbb{E}\left[N_\epsilon(J) \cdot \mathbf{1}_{N_\epsilon(J)<(\rho_c+\eta)|J|/\epsilon}\right] + \mathbb{E}\left[N_\epsilon(J) \cdot \mathbf{1}_{N_\epsilon(J)\geq(\rho_c+\eta)|J|/\epsilon}\right].$$

For the first term:

$$\mathbb{E}\left[N_\epsilon(J) \cdot \mathbf{1}_{N_\epsilon(J)<(\rho_c+\eta)|J|/\epsilon}\right] \leq (\rho_c + \eta)\frac{|J|}{\epsilon}.$$

For the second term, note that $N_\epsilon(J) \leq |A|/\epsilon$ (total number of particles), so:

$$\mathbb{E}\left[N_\epsilon(J) \cdot \mathbf{1}_{N_\epsilon(J)\geq(\rho_c+\eta)|J|/\epsilon}\right] \leq \frac{|A|}{\epsilon} \cdot Ce^{-c|J|/\epsilon} = \frac{C|A|}{\epsilon}e^{-c|J|/\epsilon} \to 0$$

as $\epsilon \to 0$.

Therefore,

$$\limsup_{\epsilon \to 0} \epsilon \cdot \mathbb{E}[N_\epsilon(J)] \le (\rho_c + \eta)|J|.$$

Since $\eta > 0$ was arbitrary, taking $\eta \to 0$ gives

$$\limsup_{\epsilon \to 0} \epsilon \cdot \mathbb{E}[N_\epsilon(J)] \le \rho_c |J|.$$

□

Lemma 6.1 and Corollary 6.2 control the *local* density of particles in any interval, preventing concentration above the critical threshold ρ_c. To prove our main result of this section (Lemma 6.4), we also need a *global* containment result: almost all particles must lie within the target region A^*. The next lemma expands on Lemma 4.9 and establishes this complementary bound, showing that only a negligible fraction of particles will escape A^*.

Lemma 6.3 (Almost all particles lie in A^*)**.** *Let $A \subset \mathbb{R}$ be a bounded open set that can be represented as a finite union of open intervals. Define A^* as in Definition 2.5. Let $B \subset \mathbb{R}$ be an open set such that $A^* \subseteq B$.*

Fix $\eta > 0$. Let $\epsilon > 0$, and start with one active particle at each point in A_ϵ. Run ARW with sleep rate $\lambda > 0$. Then,

$$\lim_{\epsilon \to 0} \mathbb{P}\left(\#\textit{particles in } B_\epsilon^c \textit{ after stabilization} > \eta \frac{|A|}{\epsilon}\right) = 0.$$

Proof. The proof is identical to that of Lemma 4.9 (the case where A is a single interval), since the IDLA coupling argument applies equally to any bounded open set A with the same total mass $|A|$. □

Having established both a uniform density bound (Lemma 6.1) and a global containment result (Lemma 6.3), we can now prove that intervals strictly smaller than A^* cannot contain all the particles. The strategy is straightforward: if B is too small (i.e., $A^* \setminus B$ contains an interval), then by Lemma 6.3, most particles lie in A^*, but by Lemma 6.1, the portion of B intersecting A^* cannot hold all these particles at density exceeding ρ_c. This gap forces some particles to remain in $A^* \setminus B$, proving that B is insufficient, and that A^* is thus THE set containing almost all of the particles in the system after stabilization and satisfying the quadrature conjecture.

Unlike Lemma 4.5, we supply a proof of the following result that applies to any starting bounded set $A \subset \mathbb{R}$, regardless of how many intervals construct A.

Lemma 6.4 (Sets smaller than A^* do not contain all particles)**.** *Let $A \subset \mathbb{R}$ be a bounded open set that can be represented as a finite union of open intervals. Define A^* as in Definition 2.5. Let $B \subset \mathbb{R}$ be an open set such that $A^* \setminus B$ contains an interval.*

Fix a value $0 < \delta < 1 - \frac{|A^* \cap B|}{|A|}\rho_c$. Let $\epsilon > 0$, and start with one active particle at each point in A_ϵ. Run ARW with sleep rate λ. Then,

$$\lim_{\epsilon \to 0} \mathbb{P}\left(\#\textit{particles in } B_\epsilon \textit{ after stabilization} > (1-\delta)\frac{|A|}{\epsilon}\right) = 0.$$

Proof. We decompose B into two disjoint pieces:

$$B = (B \cap A^*) \cup (B \setminus A^*).$$

The total number of particles in the system is $n \approx |A|/\epsilon$. We bound the number of particles in each piece separately.

Step 1: Bound on $B \cap A^*$. Since $B \cap A^*$ is an open subset of $\mathbb{R}$, we can write it as a countable union of disjoint intervals $\{J_i\}_{i \in I}$, for some countable index set I. By Lemma 6.1, for any $\eta > 0$, there exist constants $C, c > 0$ (depending only on λ and η) such that each interval $J_i \subseteq B \cap A^*$ satisfies

$$\mathbb{P}\left(\#\text{particles in } (J_i)_\epsilon \geq (\rho_c + \eta)\frac{|J_i|}{\epsilon}\right) \leq Ce^{-c|J_i|/\epsilon}.$$

Fix $\delta > 0$. Since $\sum_{i \in I} |J_i| = |B \cap A^*|$, we can choose a finite subset $I_\delta \subset I$ such that

$$\sum_{i \in I_\delta} |J_i| \geq |B \cap A^*| - \delta.$$

By the union bound over I_δ,

$$\mathbb{P}\Big(\exists i \in I_\delta : \#\text{particles in } (J_i)_\epsilon \geq (\rho_c + \eta)\tfrac{|J_i|}{\epsilon}\Big) \leq |I_\delta|\, C \exp\Big(-c \min_{i \in I_\delta} |J_i|/\epsilon\Big).$$

On the complement of this event (no subinterval has too any particles) we have

$$\#\text{particles in } (B \cap A^*)_\epsilon \leq (\rho_c + \eta)\frac{1}{\epsilon}\Big(\sum_{i \in I_\delta} |J_i| + \delta\Big) \leq (\rho_c + \eta)\frac{|B \cap A^*| + \delta}{\epsilon}.$$

Here the term δ accounts for the contribution of the complement $\bigcup_{i \notin I_\delta} J_i$: by construction,

$$\sum_{i \notin I_\delta} |J_i| = |B \cap A^*| - \sum_{i \in I_\delta} |J_i| \leq \delta,$$

so even if we pessimistically assume density $(\rho_c + \eta)$ on this leftover set, it can contribute at most $(\rho_c + \eta)\delta/\epsilon$ additional particles. Absorbing δ into the constants, we obtain constants $\tilde{C}, \tilde{c} > 0$ such that

$$\mathbb{P}\Big(\#\text{particles in } (B \cap A^*)_\epsilon \geq (\rho_c + \eta)\frac{|B \cap A^*|}{\epsilon}\Big) \leq \tilde{C} e^{-\tilde{c}|B \cap A^*|/\epsilon},$$

for sufficiently small δ, where $\tilde{C} = C|I|$ and $\tilde{c} = c \cdot \min_i(|J_i|/|B \cap A^*|) > 0$ (depending on λ, η and the geometry of $B \cap A^*$).

Step 2: Bound on $B \setminus A^*$. Since $B \setminus A^* \subseteq (A^*)^c$, we have

$$\#\text{particles in } (B \setminus A^*)_\epsilon \leq \#\text{particles in } ((A^*)^c)_\epsilon.$$

To bound the right-hand side, we apply Lemma 6.3 with the choice $B = A^*$ (noting that $A^* \subseteq A^*$ trivially). For any $\eta' > 0$, Lemma 6.3 gives

$$\lim_{\epsilon \to 0} \mathbb{P}\left(\#\text{particles in } ((A^*)^c)_\epsilon > \eta' \frac{|A|}{\epsilon}\right) = 0.$$

Therefore,

$$\lim_{\epsilon \to 0} \mathbb{P}\left(\#\text{particles in } (B \setminus A^*)_\epsilon \geq \eta' \frac{|A|}{\epsilon}\right) = 0.$$

Step 3: Combine via union bound. Choose $\eta, \eta' > 0$ small enough that

$$(\rho_c + \eta)|B \cap A^*| + \eta'|A| < (1 - \delta)|A|.$$

By the constraint $\delta < 1 - \frac{|A^* \cap B|}{|A|}\rho_c$ in the lemma statement, we have

$$\rho_c|A^* \cap B| < (1 - \delta)|A|.$$

Define the gap

$$g := (1 - \delta)|A| - \rho_c|A^* \cap B| > 0.$$

Then, we want to choose η and η' such that

$$\eta|A^* \cap B| + \eta'|A| < g.$$

For instance, we can take $\eta = \frac{g}{3|A^* \cap B|}$ and $\eta' = \frac{g}{3|A|}$, which gives $\eta|A^* \cap B| + \eta'|A| = \frac{2g}{3} < g$.

With these choices,

$$(\rho_c + \eta)|A^* \cap B| + \eta'|A| = \rho_c|A^* \cap B| + \eta|A^* \cap B| + \eta'|A| < \rho_c|A^* \cap B| + g = (1 - \delta)|A|.$$

By the union bound applied to the events in Steps 1 and 2, with probability at least

$$1 - \tilde{C}e^{-\tilde{c}|B \cap A^*|/\epsilon} - C'e^{-c'|A|/\epsilon},$$

we have *both*

- #particles in $(B \cap A^*)_\epsilon \leq (\rho_c + \eta)\frac{|B \cap A^*|}{\epsilon}$ (by Step 1 with our choice of $\eta > 0$), and
- #particles in $(B \setminus A^*)_\epsilon \leq \eta'\frac{|A|}{\epsilon}$ (by Step 2 with our choice of $\eta' > 0$).

When both events hold, the total number of particles in B_ϵ satisfies

$$\begin{aligned}
\#\text{particles in } B_\epsilon &= \#\text{particles in } (B\cap A^*)_\epsilon + \#\text{particles in } (B\setminus A^*)_\epsilon \\
&\le (\rho_c+\eta)\frac{|B\cap A^*|}{\epsilon} + \eta'\frac{|A|}{\epsilon} \\
&= \frac{1}{\epsilon}\left[(\rho_c+\eta)|B\cap A^*| + \eta'|A|\right] \\
&< \frac{1}{\epsilon}(1-\delta)|A| \quad \text{(by our choice of } \eta \text{ and } \eta' \text{ above)} \\
&= (1-\delta)\frac{|A|}{\epsilon}.
\end{aligned}$$

Therefore, with probability at least $1-\tilde{C}e^{-\tilde{c}|B\cap A^*|/\epsilon} - C'e^{-c'|A|/\epsilon}$,

$$\#\text{particles in } B_\epsilon < (1-\delta)\frac{|A|}{\epsilon}.$$

Since both exponential terms $\tilde{C}e^{-\tilde{c}|B\cap A^*|/\epsilon}\to 0$ and $C'e^{-c'|A|/\epsilon}\to 0$ as $\epsilon\to 0$, we conclude

$$\lim_{\epsilon\to 0}\mathbb{P}\left(\#\text{particles in } B_\epsilon > (1-\delta)\frac{|A|}{\epsilon}\right) = 0.$$

$\square$

Remark 6.5. *Lemma 6.4 applies equally whether A is a single interval, a finite union of intervals, or even a countable union (as long as A is bounded). The proof relies only on the total mass $|A|$, the definition of A^* via the smash sum construction, and the uniform bounds from Lemmas 6.1 and 6.3, which do not depend on the internal structure of A.*

7 Particles distribute uniformly in A^*

In the previous sections, we established two key results: Lemma 6.1 shows that no subinterval can have particle density exceeding ρ_c after stabilization, and Lemma 6.3 shows that almost all the sleeping particles land within A^* (as we have defined it in Definition 2.5). However, these results alone do not guarantee that particles fill A^* uniformly with density ρ_c, as will be required for our main result.

7.1 Why do we need uniformity?

To see why additional work is needed, consider a scenario where $A^* = [0, 10]$ with $\rho_c = 0.5$. Suppose we have a total of $|A| = 5$ units of mass (corresponding to 5 particles in the initial configuration). After spreading, we expect all 5 units to fill $A^* = [0, 10]$ at density $\rho_c = 0.5$, giving $|A^*| = 5/0.5 = 10$ as required.

However, our current results would permit an alternative configuration where all 5 units of mass concentrate in the subinterval $[0, 5]$ at density 1.0, or even worse, concentrate in $[0, 5]$ at density 0.5, with 2.5 units of mass total, leaving $(5, 10]$ completely empty and the remaining 2.5 units somehow lost or distributed outside A^*.

Let us check whether this bad configuration violates our current lemmas:
Does any subinterval exceed density $\rho_c = 0.5$? **(Lemma 6.1)**

- In $[0, 5]$: density is 0.5 (if only 2.5 units are there) ≤ 0.5.
- In $(5, 10]$: density is $0 \leq 0.5$.

Thus, Lemma 6.1 is satisfied.

Are almost all particles in $A^* = [0, 10]$? **(Lemma 6.3)**

- If 2.5 units are in $[0, 5] \subset A^*$ and 2.5 units are outside A^*, then only half the particles are in A^*.
- However, if we allow a small error δ, this might still technically satisfy "almost all," depending on how we quantify it.

Thus, Lemma 6.3 is not clearly violated.

Although the configuration satisfies Lemma 6.1 and 6.3, this configuration is not what we are looking for in order to satisfy our theorem. According to Theorem 2.3, the limiting measure should be $\rho_c 1_{A^*}$, meaning:

$$\text{mass in } [0,5] = \rho_c \cdot |[0,5]| = 0.5 \cdot 5 = 2.5, \text{ and}$$

$$\text{mass in } (5,10] = \rho_c \cdot |(5,10]| = 0.5 \cdot 5 = 2.5.$$

The correct configuration has density $\rho_c = 0.5$ throughout A^*, not just in part of it. The interval $(5,10]$ should contain approximately 2.5 units of mass, not zero.

The issue is that we have only established an upper bound on particle density, as well as a global containment result. To prove weak$*$ convergence via Proposition 2.11, we need to show that for any open set $U \subseteq \mathbb{R}$,

$$\lim_{\epsilon \to 0} \mathbb{E}[\text{particles in } U \cap A^*_\epsilon] = \rho_c |U \cap A^*|.$$

For $U \subseteq A^*$, we particularly need

$$\lim_{\epsilon \to 0} \mathbb{E}[\text{particles in } U] = \rho_c |U|.$$

Our current lemmas provide the upper bound $\limsup_{\epsilon \to 0} \mathbb{E}[\text{particles in } U] \leq \rho_c |U|$, but we still need a matching lower bound to show that particles cannot simply avoid any portion of A^*. This section establishes this lower bound, completing the proof that particles distribute uniformly throughout A^* with density ρ_c.

7.2 Proving uniformity within A^*

Having established that particle density cannot exceed ρ_c anywhere (Corollary 6.2), we now prove the complementary lower bound: particles cannot avoid any region $U \subseteq A^*$. The proof uses a simple mass conservation argument. If particles are sparse in U, then by conservation of total mass, they must concentrate in the complement $A^* \setminus U$. But this would create density exceeding ρ_c, contradicting Corollary 6.2. Therefore, every subinterval $U \subseteq A^*$ must contain at least $\rho_c|U|$ units of mass.

Lemma 7.1 (Uniform lower density in A^*)**.** *Let $A \subset \mathbb{R}$ be a bounded open set that can be represented as a finite union of open intervals, and let A^* be constructed via Definition 2.8. Let $U \subseteq A^*$ be an open interval. For each $\epsilon > 0$, let $N_\epsilon(U)$ denote the number of sleeping particles in U after running ARW from A_ϵ. Then*

$$\liminf_{\epsilon \to 0} \epsilon \cdot \mathbb{E}[N_\epsilon(U)] \geq \rho_c|U|.$$

Proof. Suppose, for contradiction, that there exist an open interval $U \subset A^*$, a $\delta > 0$, and a sequence $\epsilon_k \downarrow 0$ such that

$$\epsilon_k \cdot \mathbb{E}[N_{\epsilon_k}(U)] \;\leq\; (\rho_c - \delta)\,|U| \qquad \text{for all } k. \tag{20}$$

By Lemma 6.3, almost all particles lie in A^*. Mass conservation therefore gives

$$\epsilon \cdot \mathbb{E}[N_\epsilon(A^*)] \;\longrightarrow\; |A| \quad \text{as } \epsilon \to 0.$$

Since

$$\epsilon \cdot \mathbb{E}[N_\epsilon(A^*)] = \epsilon \cdot \mathbb{E}[N_\epsilon(\mathbb{R})] - \epsilon \cdot \mathbb{E}[N_\epsilon((A^*)^c)]$$

and $\epsilon \cdot \mathbb{E}[N_\epsilon((A^*)^c)] \to 0$ by Lemma 6.3, we obtain

$$\epsilon \cdot \mathbb{E}[N_\epsilon(A^*)] \longrightarrow |A| \quad \text{as } \epsilon \to 0. \tag{21}$$

Write $A^* = U \cup (A^* \setminus U)$ as a disjoint union. For each ϵ,

$$\epsilon \cdot \mathbb{E}[N_\epsilon(A^*)] = \epsilon \cdot \mathbb{E}[N_\epsilon(U)] + \epsilon \cdot \mathbb{E}[N_\epsilon(A^* \setminus U)].$$

Evaluating this identity along the sequence ϵ_k and using (20), we obtain, for every k,

$$\epsilon_k \cdot \mathbb{E}[N_{\epsilon_k}(A^* \setminus U)] = \epsilon_k \cdot \mathbb{E}[N_{\epsilon_k}(A^*)] - \epsilon_k \cdot \mathbb{E}[N_{\epsilon_k}(U)]$$

$$\geq \rho_c|A^*| - (\rho_c - \delta)\,|U| - o(1) = \rho_c|A^* \setminus U| + \delta|U| - o(1),$$

where $o(1) \to 0$ as $k \to \infty$ by (21). Letting $k \to \infty$ and taking $\limsup$ on both sides yields

$$\limsup_{k\to\infty} \epsilon_k \cdot \mathbb{E}[N_{\epsilon_k}(A^* \setminus U)] \;\geq\; \rho_c|A^* \setminus U| + \delta|U|. \tag{22}$$

On the other hand, Corollary 6.2 applied to the interval $A^* \setminus U$ states that

$$\limsup_{\epsilon\to 0} \epsilon \cdot \mathbb{E}[N_\epsilon(A^* \setminus U)] \;\leq\; \rho_c|A^* \setminus U|.$$

The same bound holds along the subsequence ϵ_k:

$$\limsup_{k\to\infty} \epsilon_k \cdot \mathbb{E}[N_{\epsilon_k}(A^* \setminus U)] \;\leq\; \rho_c|A^* \setminus U|. \tag{23}$$

Combining (22) and (23) gives

$$\rho_c|A^* \setminus U| + \delta|U| \;\leq\; \rho_c|A^* \setminus U|,$$

which is impossible since $\delta > 0$ and $|U| > 0$. This contradiction shows that the assumption (20) was false, and therefore

$$\liminf_{\epsilon\to 0} \epsilon \cdot \mathbb{E}[N_\epsilon(U)] \;\geq\; \rho_c|U|$$

for every open interval $U \subset A^*$. □

Later, we will want the following extension of this lemma that says if we extend any section of A^* by a tiny bit, we still get a uniform lower density of particles within this extended range. This will be Corollary 8.4, and we will use this in Proposition 8.7.

The previous lemma shows that every open interval contained in A^* carries asymptotic density at least ρ_c. Together with the uniform upper bound from Corollary 6.2, this forces the limiting density inside A^* to be exactly ρ_c on all scales. We can now identify the limiting mass of particles in an arbitrary open set U by intersecting it with A^*!

Lemma 7.2. *Let $A \subset \mathbb{R}$ be a bounded open set that can be represented as a finite union of open intervals. For any open set $U \subseteq \mathbb{R}$,*

$$\lim_{\epsilon \to 0} \mathbb{E}[\text{particles in } U \cap A^*] = \rho_c |U \cap A^*|.$$

Proof. For any open interval $V \subseteq A^*$, Corollary 6.2 gives

$$\limsup_{\epsilon \to 0} \epsilon \cdot \mathbb{E}[N_\epsilon(V)] \leq \rho_c |V|,$$

and Lemma 7.1 gives

$$\liminf_{\epsilon \to 0} \epsilon \cdot \mathbb{E}[N_\epsilon(V)] \geq \rho_c |V|.$$

Therefore,

$$\lim_{\epsilon \to 0} \epsilon \cdot \mathbb{E}[N_\epsilon(V)] = \rho_c |V|.$$

For a general open set $U \subseteq \mathbb{R}$, the result follows by applying the above to $U \cap A^*$ and noting that $U \cap A^*$ can be written as a countable union of disjoint open intervals. By countable additivity of expectation and Lebesgue measure, the result extends to $U \cap A^*$. $\square$

Lemma 7.2 shows that, in the scaling limit, the stabilized ARW configuration has deterministic density ρ_c on the smash-domain A^* and zero density outside A^*. The total macroscopic mass is concentrated on A^* with constant density ρ_c. We record the resulting relation between $|A|$ and $|A^*|$ as a corollary.

Corollary 7.3 (Conservation of measure for A^*)**.** *Let $A \subset \mathbb{R}$ be a bounded open set that can be represented as a finite union of open intervals, and let A^* be the smash-domain associated with A as in Definition 2.8. Then*

$$|A^*| = \frac{|A|}{\rho_c}.$$

Proof. For each $\epsilon > 0$, start ARW from one active particle at each site of A_ϵ. By Corollary 4.4, the initial number of particles is

$$|A_\epsilon| = \frac{|A|}{\epsilon} + O(1),$$

and the ARW dynamics conserve particle number, so

$$\lim_{\epsilon \to 0} \epsilon \, \mathbb{E}\big[\text{total particles}\big] = |A|.$$

On the other hand, by Lemma 7.2 applied with $U = \mathbb{R}$ we have

$$\lim_{\epsilon \to 0} \epsilon \, \mathbb{E}\big[\text{particles in } A^*\big] = \rho_c \, |A^*|.$$

The total number of particles equals the number of particles in A^* plus the number of particles in $(A^*)^c$. Therefore,

$$|A| = \rho_c \, |A^*| + \lim_{\epsilon \to 0} \epsilon \, \mathbb{E}\big[\text{particles in } (A^*)^c\big].$$

By Lemma 7.2 with $U = (A^*)^c$, the latter limit is 0, so we obtain $|A| = \rho_c \, |A^*|$, proving the claim. $\square$

8 Approximation by a finite union of intervals

Until this point, we have basically worked under the assumption that our starting set $A \subset \mathbb{R}$ is at most a finite union of open intervals, and we have essentially ignored the requirement that A has a boundary with Lebesgue measure zero. However, it is also possible for A to be a countable union, and thus the assumption that the boundary has measure zero is no longer trivially satisfied.

In this section, we show that even if A is a *countable* union of open intervals, it suffices to prove Theorem 2.3 when the initial set A is a finite union of intervals. We then extend the result to general bounded open sets A with boundary having Lebesgue measure zero by approximation.

8.1 Gaps in A^{sm}

For any finite index set $F \subset \mathbb{N}$, define $A_F := \bigcup_{j \in F} I_j$ with corresponding smash-domain A_F^{sm}.

Lemma 8.1 (Existence of small gaps in A^{sm})**.** *Let $A = \bigcup_{j=1}^{\infty} I_j \subset \mathbb{R}$ be a bounded open set which is a union of disjoint intervals, and assume A's boundary has Lebesgue measure 0. Let A^{sm} be the associated smash-domain via Definition 2.8.*

Fix a finite index set F such that the complement has small total length

$$\sum_{j \notin F} |I_j| \leq \epsilon_F.$$

Let $B^{(1)}, \dots, B^{(m)}$ be the connected components of A_F^{sm} (and hence each $B^{(k)}$ is the block obtained by smashing together the intervals $(I_i)_{i \in F^{(k)}}$ for some partition $F = \bigcup_{k=1}^{m} F^{(k)}$). Then there exist open intervals

$$G_1^{(k)},\ G_2^{(k)} \subset \mathbb{R}, \qquad k = 1, \dots, m,$$

lying respectively to the left and right of $B^{(k)}$, such that:

1. *Each G has small intersection with A^{sm}:*

$$|G_1^{(k)} \cap A^{\mathrm{sm}}| < |G_1^{(k)}|, \qquad |G_2^{(k)} \cap A^{\mathrm{sm}}| < |G_2^{(k)}|, \quad \textit{for all } k = 1, \ldots, m;$$

2. *The total extra length added by all collars is controlled by the tail:*

$$\sum_{k=1}^{m} \Big(|(\inf G_1^{(k)}, \sup G_2^{(k)})| - |B^{(k)}| \Big) \ \leq\ C\, \epsilon_F,$$

for some constant $C > 0$.

Proof. Write $A = \bigcup_{j=1}^{\infty} I_j$ and fix a finite index set $F \subset \mathbb{N}$. Let $A_F := \bigcup_{j \in F} I_j$ and let A_F^{sm} be its smash-domain as in Definition 2.8. Assume $\sum_{j \notin F} |I_j| \leq \epsilon_F$.

Let $B^{(1)}, \ldots, B^{(m)}$ be the connected components of A_F^{sm}. We construct barriers for each $B^{(k)}$, and then control the total added length.

Fix $k \in \{1, \ldots, m\}$ and write $B^{(k)} = (a_k, b_k)$. Since A^{sm} is open, bounded, and has boundary of Lebesgue measure zero (Remark 2.9), almost every point is a Lebesgue density point of either A^{sm} or its complement $(A^{\mathrm{sm}})^c$. Then in the half-lines $(-\infty, a_k)$ and (b_k, ∞) we can choose points $x_1^{(k)} < a_k$ and $x_2^{(k)} > b_k$ that lie in $(A^{\mathrm{sm}})^c$ and are Lebesgue density 0 points of A^{sm}.

By the definition of density 0, for each $i = 1, 2$ there exists $\eta_i^{(k)} > 0$ such that for all $0 < r < \eta_i^{(k)}$,

$$\frac{|(x_i^{(k)} - r, x_i^{(k)} + r) \cap A^{\mathrm{sm}}|}{|(x_i^{(k)} - r, x_i^{(k)} + r)|} < 1.$$

Choose radii $0 < r_1^{(k)} < \eta_1^{(k)}$ and $0 < r_2^{(k)} < \eta_2^{(k)}$ small enough so that $x_1^{(k)} + r_1^{(k)} < a_k$ and $x_2^{(k)} - r_2^{(k)} > b_k$. Define the barriers

$$G_1^{(k)} := (x_1^{(k)} - r_1^{(k)}, x_1^{(k)} + r_1^{(k)}), \qquad G_2^{(k)} := (x_2^{(k)} - r_2^{(k)}, x_2^{(k)} + r_2^{(k)}).$$

Then $G_1^{(k)}$ and $G_2^{(k)}$ lie strictly to the left and right of $B^{(k)}$, respectively, and from the density condition we obtain

$$|G_1^{(k)} \cap A^{\mathrm{sm}}| < |G_1^{(k)}| \qquad \text{and} \qquad |G_2^{(k)} \cap A^{\mathrm{sm}}| < |G_2^{(k)}|.$$

Next, let

$$J^{(k)} := (\inf G_1^{(k)}, \sup G_2^{(k)})$$

denote the enlarged interval containing $B^{(k)}$ and its two barriers.

By shrinking the radii $r_1^{(k)}, r_2^{(k)}$ if necessary (for each fixed finite F), we may ensure that

$$\sum_{k=1}^{m} \big(|J^{(k)}| - |B^{(k)}|\big)$$

is as small as we wish when we only consider the finite smash-domain A_F^{sm} and ignore the complement/tail $\bigcup_{j \notin F} I_j$. Next, add the remaining intervals I_j with $j \notin F$, whose total length is at most ϵ_F. In one dimension, adding intervals of total length ϵ_F can enlarge the union of the smashed components by at most $C\epsilon_F$ in Lebesgue measure, for some constant $C > 0$ depending only on the smash operation.

Combining these two observations and choosing the radii so that the contribution from A_F^{sm} itself is negligible, we obtain

$$\sum_{k=1}^{m} \big(|J^{(k)}| - |B^{(k)}|\big) \le C\,\epsilon_F.$$

This is item (2) of the lemma, with $J^{(k)} = (\inf G_1^{(k)}, \sup G_2^{(k)})$. □

Corollary 8.2. *In the setting of Lemma 8.1, for each $G \in \{G_1^{(k)}, G_2^{(k)}\}$ we in fact can choose G such that*

$$G \cap A^{\mathrm{sm}} = \emptyset \quad \textit{and} \quad |G \cap A^{\mathrm{sm}}| = 0.$$

Proof. Fix $G \in \{G_1^{(k)}, G_2^{(k)}\}$. By Lemma 8.1 we have

$$|G \cap A^{\mathrm{sm}}| < |G|. \tag{24}$$

The closure $\overline{A^{\mathrm{sm}}}$ is closed and its boundary

$$\partial A^{\mathrm{sm}} := \overline{A^{\mathrm{sm}}} \setminus A^{\mathrm{sm}}$$

has Lebesgue measure zero by assumption. Let

$$U := \mathbb{R} \setminus \overline{A^{\mathrm{sm}}}.$$

Then U is open and thus a union of disjoint open intervals.

Decompose G into the three parts

$$G = \big(G \cap A^{\mathrm{sm}}\big) \cup \big(G \cap \partial A^{\mathrm{sm}}\big) \cup \big(G \cap U\big),$$

a disjoint union. Taking Lebesgue measures gives

$$|G| = |G \cap A^{\mathrm{sm}}| + |G \cap \partial A^{\mathrm{sm}}| + |G \cap U|.$$

Since ∂A^{sm} has measure zero, $|G \cap \partial A^{\mathrm{sm}}| = 0$, so

$$|G| = |G \cap A^{\mathrm{sm}}| + |G \cap U|. \tag{25}$$

Using (24) and 25, we obtain

$$|G \cap U| = |G| - |G \cap A^{\mathrm{sm}}| > 0$$

Since $G \cap U$ has positive measure, G is an interval and U is open and thus a union of intervals, $G \cap U$ contains an interval. If G satisfies the conditions of Lemma 8.1, then so does any subinterval of G. □

The above result shows that the barriers G between intervals in A_n^{sm} can be constructed "nicely." That is, G can always be constructed such that G does not add much area to the intervals which it separates. Next, we'll show that the barriers G are actually barriers that prevent the crossing of particles between intervals in A_n^{sm}.

Lemma 8.3 (Barriers prevent crossing between blocks of A^{sm})**.** *Consider the bounded open set $A = \bigcup_{j=1}^{\infty} I_j$ which is a union of disjoint open intervals.*

Let B be a connected component of A_F^{sm} for some index set F, and let G_1, G_2 be the left *and right barriers associated to B given by Lemma 8.1. Let G denote either G_1 or G_2.*

Suppose we have ARW with sleep rate λ started from one active particle at each point of $A \cap \epsilon\mathbb{Z}$. Then

$$\lim_{\epsilon \to 0} \mathbb{P}_\epsilon\Big(\textit{some particle visits both } (-\infty, \inf G) \textit{ and } (\sup G, \infty)\Big) = 0.$$

Note that particles enter G, but we claim that no particle enters G from one side, and exits out the other.

We model this proof after Proposition 8.7 in [HJJ24]. Accordingly, we adopt the following notation, parallel to Sections 4 and 8 of [HJJ24].

Let Instr denote the sitewise instruction stacks for ARW on $\mathbb{Z}$, and fix an interval $[a, b] \subset \mathbb{Z}$ containing the barrier G. For any (extended) odometer u on $[a, b]$, we write

$$\mathcal{L}_v(u) := \#\{\text{left instructions used at site } v\}, \qquad \mathcal{R}_v(u) := \#\{\text{right instructions used at site } v\},$$

and define the net flow across edge $(v, v+1)$ by

$$f_v := \mathcal{R}_v(u) - \mathcal{L}_{v+1}(u).$$

As in [HJJ24, Section 4], we fix a reference value $u_0 \in \mathbb{Z}$ for the odometer at the left endpoint and a net flow $f_0 \in \mathbb{Z}$ across the first edge, and consider the set

$$\mathcal{O}_n(\mathsf{Instr}, \sigma, u_0, f_0) := \big\{u \text{ extended odometer on } [0, n] : \\ u(0) = u_0,\ f_0 = \mathcal{R}_0(u) - \mathcal{L}_1(u),\ u \text{ stable on } [1, n-1]\big\},$$

where σ is the initial configuration on $[0, n]$. The minimal element of $\mathcal{O}_n(\mathsf{Instr}, \sigma, u_0, f_0)$ is denoted by $\mathfrak{m}$. In the layer-percolation process, we index infected cells by pairs $(r, s) \in \mathbb{N}^2$, where r is the column and s is the row. See [HJJ24, Section 3.2] for the full layer percolation definition.

Proof. As stated above, we model our proof after Proposition 8.7 in [HJJ24].

Fix a connected component B of A_F^{sm} and its right barrier G_2, with $\ell := \inf G_2$ and

$r := \sup G_2$; the left barrier case is symmetric. Without loss of generality, we also assume that B is centered at the origin.

By Lemma 8.1, for some $\delta > 0$ we have

$$\frac{|(\ell, r) \cap A^{\mathrm{sm}}|}{r - \ell} \leq \rho_c - \delta. \tag{26}$$

Let I_B be the interval obtained by taking B together with its collars, say $I_B := (\inf G_1, \sup G_2)$. Then by Lemma 8.1 we also have

$$\frac{|I_B \cap A^{\mathrm{sm}}|}{|I_B|} \leq \rho_c - \delta. \tag{27}$$

For each $\epsilon > 0$ we start ARW with one active particle at each site of $A \cap \epsilon\mathbb{Z}$, and we let

$$N(\epsilon) := \#\big(I_B \cap A \cap \epsilon\mathbb{Z}\big)$$

be the total number of particles initially placed inside the block B together with its collars. Since $A \subset A^{\mathrm{sm}}$ and by (27), for every $\eta > 0$ there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$,

$$N(\epsilon) \leq (\rho_c - \delta + \eta)\,\frac{|I_B|}{\epsilon}. \tag{28}$$

As in [HJJ24, Proposition 8.7], we now choose an integer $n = n(\epsilon)$ and construct a stable odometer on the interval $[-(n+1), n+1]$. We say that an event holds with overwhelming probability (w.o.p.) if its probability is bounded by $Ce^{-cN(\epsilon)}$ for some constants $c, C > 0$ depending only on λ and δ.

We define n to be the largest integer such that

$$\mathsf{Instr}_{n+1}(1) = \mathsf{Instr}_{-(n+1)}(1) = \mathsf{sleep} \quad \text{and} \quad n \leq \frac{N(\epsilon)}{2(\rho_c - \delta/2)}. \tag{29}$$

Recall that the first instructions $\mathsf{Instr}_v(1)$ are i.i.d. and equal to sleep with probability $\lambda/(1+\lambda)$. Then as in [HJJ24, (85)], there exists a constant $C < \infty$ such that, w.o.p. as $\epsilon \to 0$,

$$\frac{N(\epsilon)}{2(\rho_c - \delta/4)} \leq n.$$

Then for all sufficiently small ϵ,

$$\rho := \frac{N(\epsilon) - 2}{2n} \le \rho_c - \delta/4,$$

w.o.p., so the initial density in $[-n-1, n+1]$ is strictly subcritical. Then can apply Lemma 6.1 of [HJJ24]. Moreover, by (28) we have $n(\epsilon) \sim |I_B|/\epsilon$ as $\epsilon \to 0$, so for all sufficiently small ϵ the interval $[-(n+1), n+1]$ contains I_B and hence $[\ell, r]$.

We now follow the odometer construction of [HJJ24, Section 8.1]. Shift the configuration and instructions from $[-n, n]$ to $[0, 2n]$ by setting $\mathsf{Instr}'_v := \mathsf{Instr}_{v-n}$ and $\sigma'(v) := \sigma(v-n)$ for $v \in [0, 2n]$. Let $f_0 = -1$, and let u_0 be the index of the first left instruction after index 0 in Instr'_0. Let $\mathcal{O}_{2n}(\mathsf{Instr}', \sigma', u_0, f_0)$ be the set of extended odometers on $[0, 2n]$ as in [HJJ24, Section 4.1], and let $\mathfrak{m}$ be its minimal element.

By [HJJ24, Proposition 5.8], the value $\mathcal{R}_{2n}(\mathfrak{m})$ is concentrated around $2\rho n^2$. Let $s := N(\epsilon) - 2$ and define

$$r := f_0 + \sum_{v=1}^{2n} |\sigma'(v)| - \mathcal{R}_{2n}(\mathfrak{m}) - s. \tag{30}$$

Exactly as in [HJJ24, (86)], r lies in an interval of length $O(n^2)$ around $2\rho n^2$, and since $\rho \le \rho_c - \delta/4$ we may apply Lemma 6.1 of [HJJ24]. Thus, w.o.p. there is an infection path in the layer-percolation configuration corresponding to $(\mathsf{Instr}', \sigma')$ that ends at $(r, s)_{2n}$.

By the correspondence [HJJ24, Proposition 4.6], this infection path yields an extended odometer u' on $[0, 2n]$ with

$$u' \in \mathcal{O}_{2n}(\mathsf{Instr}', \sigma', u_0, f_0), \qquad u' \text{ stable on } [1, 2n-1], \qquad \mathcal{R}_{2n}(u') = 1.$$

We use Corollary 8.2 to justify that particles do not start near 0 and $2n$, which we claim are in the Gs. As in the proof of [HJJ24, Proposition 8.7], we have $u'(0) > 0$ and $u'(2n) > 0$, and the net flows $f_v := \mathcal{R}_v(u') - \mathcal{L}_{v+1}(u')$ satisfy $f_v \le -1$ for $v \le -1$ and $f_v \ge 1$ for $v \ge 0$. By [HJJ24, Lemma 8.3], this implies

$$u'(v) \ge 0 \quad \text{for all } v \in [0, 2n].$$

Shifting back from $[0, 2n]$ to $[-n, n]$, define $\tilde{u}(v) := u'(v+n)$ for $v \in [-n, n]$. Then $\tilde{u}$ is nonnegative and stable on $[-n+1, n-1]$, and by [HJJ24, Lemma 2.3] the stabilization using Instr' sends at most one particle to $-n-1$ and at most one particle to $2n+1$. Translating back, in our original coordinates the stabilization sends at most one particle to $-n-1$ and at most one particle to $n+1$; in particular, all but at most two particles stay inside $[-n, n]$, which contains I_B and hence $[\ell, r]$ for small ϵ.

We claim that, with overwhelming probability, there exists a site $x_G \in G$ such that $\tilde{u}(x_G) = 0$. Suppose, towards a contradiction, that $\tilde{u}(x) \geq 1$ for every $x \in G$. Then every site in G topples at least once during the stabilization on $[-n, n]$.

Recall that $G \subset (\ell, r)$ and, by (26), the initial density of particles in (ℓ, r) is at most $\rho_c - \delta$. By the same subcritical estimate (Lemma 6.1 of [HJJ24]) that we used above for $[-n-1, n+1]$, the stabilization with instruction stacks Instr and this initial configuration leaves, with overwhelming probability, strictly fewer than $(\rho_c - \delta/4)|G|$ sleeping particles in G.

On the other hand, if every site in G topples at least once (i.e. $\tilde{u}(x) \geq 1$ for all $x \in G$), then, by the same layer–percolation construction and point–source comparison used in [HJJ24, Sections 4 and 8], the number of sleeping particles left in G must be within $o(|G|)$ of the critical density ρ_c. For all sufficiently small ϵ, with overwhelming probability the final density in G is at least $\rho_c - \delta/8$.

These two statements are incompatible for small ϵ. Thus the event that $\tilde{u}(x) \geq 1$ for all $x \in G$ has probability decaying like $e^{-c|G|}$, and therefore with overwhelming probability there exists $x_G \in G$ with

$$\tilde{u}(x_G) = 0. \tag{31}$$

By the least action principle for ARW (Proposition 3 in [LS21]), the true stabilizing odometer on $[-n, n]$ is the pointwise minimal nonnegative odometer that stabilizes the configuration with the given instruction stacks. The odometer $\tilde{u}$ constructed above is nonnegative and stabilizes $[-n, n]$, and it satisfies $\tilde{u}(x_G) = 0$. Hence the true odometer u also satisfies

$$u(x_G) \leq \tilde{u}(x_G) = 0,$$

so in fact $u(x_G) = 0$. No instruction at x_G is ever used during stabilization on $[-n, n]$.

Any particle that moves from a site strictly to the left of G to a site strictly to the right of G must cross one of the edges $(x_G - \epsilon, x_G)$ or $(x_G, x_G + \epsilon)$. Such a crossing would require firing x_G at least once, which is impossible if $u(x_G) = 0$. The same reasoning rules out paths from right to left across G and paths from B to the complement of I_B on the other side of G.

Thus, on the high–probability event that (31) holds, no particle visits both $(-\infty, \ell)$ and (r, ∞). Since the above construction applies to each barrier G_2 and the left–barrier case is symmetric, we obtain

$$\lim_{\epsilon \to 0} \mathbb{P}_\epsilon\Big(\text{some particle visits both } (-\infty, \inf G) \text{ and } (\sup G, \infty)\Big) = 0,$$

which completes the proof. □

Combining Lemma 8.1 with Lemma 8.3, , we see that for any two connected components of A^{sm} there is a separating gap G with these two properties, so that adding a small amount of mass from $A \setminus A_F$ cannot cause distinct components to merge in the scaling limit.

Corollary 8.4. *For any interval $B^{(k)}$ in A^*, let $G_1^{(k)}$ and $G_2^{(k)}$ be its two bounding gaps to the left and right of $B^{(k)}$. Let $V = (\inf G_1^{(k)}, \sup G_2^{(k)})$. Then for any $U \subseteq V$ and $\delta > 0$,*

$$\liminf_{\epsilon \to 0} \epsilon \cdot \mathbb{E}[N_\epsilon(U)] \geq \rho_c(|B^{(k)}| - |V \setminus U|).$$

Proof. By Lemma 8.3, we have that for $\delta > 0$ and as ϵ approaches 0,

$$\#\text{particles that stabilize in } V \geq \frac{|B^{(k)}|\rho_c}{\epsilon}(1 - \delta)$$

with high probability, and by Corollary 6.2,

$$\#\text{ particles that stabilize outside } U \text{ but in } V \leq \frac{|V \setminus U|\rho_c}{\epsilon}(1 + \delta)$$

with high probability as $\epsilon \to 0$. Subtracting these two, we get

$$
\begin{aligned}
\epsilon \cdot (\#\text{ of particles that stabilize in } U) &\geq \epsilon\left(\frac{|B^{(k)}|\rho_c}{\epsilon}(1-\delta) - \frac{|V \setminus U|\rho_c}{\epsilon}(1+\delta)\right) \\
&\geq \rho_c(|B^{(k)}| - |V \setminus U|)(1-\delta')
\end{aligned}
$$

where $\delta' > 0$ is arbitrary. Then

$$
\epsilon \cdot (\#\text{ of particles that stabilize in } U) \geq \rho_c(|B^{(k)} - |V \setminus U|).
$$

Since this happens with high probability, this implies

$$
\liminf_{\epsilon \to 0} \epsilon \cdot \mathbb{E}[N_\epsilon(U)] \geq \rho_c(|B^{(k)}| - |V \setminus U|).
$$

□

8.2 Convergence of smash domains

We now record a basic continuity property of smash domains, obtained by applying the obstacle problem convergence results of Levine and Peres [LP10]. Roughly speaking, if the initial densities $\rho_c \mathbf{1}_{A_n}$ converge to $\rho_c \mathbf{1}_A$ in L^1, then the corresponding smash domains converge in measure.

Lemma 8.5 (Convergence of smash domains)**.** *Let $A \subset \mathbb{R}$ be a bounded open set (possibly a countable union of open sets), and choose an increasing sequence of bounded open sets $(A_n)_{n\geq 1}$ such that each A_n is a finite union of disjoint open intervals, $A_n \subset A_{n+1} \subset ... \subset A$,*

$$
\bigcup_{n \geq 1} A_n = A, \qquad \text{and} \qquad |A \setminus A_n| \longrightarrow 0 \quad \text{as } n \to \infty.
$$

For each n, let A_n^{sm} denote the smash-domain associated with the density $\rho_c \mathbf{1}_{A_n}$ in the sense of Definition 2.8, and let A^{sm} be the smash-domain associated with $\rho_c \mathbf{1}_A$. Then

$$
\left| A_n^{\mathrm{sm}} \triangle A^{\mathrm{sm}} \right| \longrightarrow 0 \quad \text{as } n \to \infty.
$$

Proof. Since $A_n \uparrow A$ and $|A \setminus A_n| \to 0$, we have $\rho_c \mathbf{1}_{A_n} \to \rho_c \mathbf{1}_A$ in $L^1(\mathbb{R})$. For each n, let D_n and D be the noncoincidence sets of the obstacle problem in [LP10] for the densities $\rho_c \mathbf{1}_{A_n}$ and $\rho_c \mathbf{1}_A$, respectively. By Proposition 2.12 and Lemma 2.16 of [LP10], D_n converges to D in measure: $|D_n \triangle D| \to 0$ as $n \to \infty$.

By Definition 2.8 and Remark 2.9, these noncoincidence sets are precisely the smash domains: $D_n = A_n^{\mathrm{sm}}$ and $D = A^{\mathrm{sm}}$. Hence

$$\left|A_n^{\mathrm{sm}} \triangle A^{\mathrm{sm}}\right| = |D_n \triangle D| \longrightarrow 0,$$

as claimed. □

The preceding lemma shows that the limit shape depends continuously (in the symmetric difference metric) on the initial set A. For the proof of Theorem 2.3, we will use this continuity together with a simple decomposition of a general bounded open set into a "large" finite-interval part, and a small remainder.

Corollary 8.6 (Approximation by finite unions)**.** *Let $A \subset \mathbb{R}$ be a bounded open set whose boundary has Lebesgue measure zero. Then there exists an increasing sequence of bounded open sets $(A_n)_{n\geq 1}$ such that:*

1. *Each A_n is a finite union of disjoint open intervals.*
2. *$A_n \subset A_{n+1} \subset ... \subset A$ and $\bigcup_{n\geq 1} A_n = A$.*
3. *$|A \setminus A_n| \to 0$ as $n \to \infty$.*

Moreover, for these sets we have

$$|A_n^{\mathrm{sm}} \triangle A^{\mathrm{sm}}| \longrightarrow 0 \quad \textit{and} \quad |(A \setminus A_n)^{\mathrm{sm}}| = \tfrac{1}{\rho_c} |A \setminus A_n| \longrightarrow 0 \quad \textit{as } n \to \infty.$$

Proof. Since A is a bounded open subset of $\mathbb{R}$ with $|\partial A| = 0$, its connected components form a countable collection of open intervals. Taking partial unions gives an increasing sequence (A_n) with the stated properties.

Applying Lemma 8.5 to this sequence yields $|A_n^{\mathrm{sm}} \triangle A^{\mathrm{sm}}| \to 0$. The identity $|(A \setminus A_n)^{\mathrm{sm}}| = \frac{1}{\rho_c} |A \setminus A_n|$ follows from Corollary 7.3. □

8.3 Proving Theorem 2.3 for the countable case is easy, actually

With the above results, we can extend the results about A^* from the finite union case to the countable interval case. We will show that in the below proposition:

Proposition 8.7. *Let $A = \bigcup_{j=1}^{\infty} I_j \subset \mathbb{R}$ be a bounded open set with $|\partial A| = 0$. To prove Theorem 2.3 it suffices to establish the desired convergence*

$$\epsilon\, S\big(1_{A_n \cap \epsilon\mathbb{Z}}\big) \xrightarrow{*} \rho_c 1_{A_n^*} \quad \text{as } \epsilon \to 0,$$

for any A_n which is a finite union of intervals that converges to A.

Proof. Fix such a finite union $A_n = A_F$ for a finite index set F and write the connected components of $(A_n)^{\mathrm{sm}}$ as $B^{(1)}, \dots, B^{(k)}$. By Corollary 8.2, we can choose Gs bounding each B that are "nice." In particular,

$$G \cap A^{\mathrm{sm}} = \emptyset, \quad |G \cap A^{\mathrm{sm}}| = 0,$$

and

$$\sum_{k=1}^{m} \Big(|(\inf G_1^{(k)}, \sup G_2^{(k)})| - |B^{(k)}| \Big) \ \leq\ C\, \epsilon_F,$$

where $C > 0$ is a constant and the complement of A with the approximation A_F is bounded by

$$\sum_{j \notin F} |I_j| \leq \epsilon_F.$$

Let $V^{(k)} := (\inf G_1^{(k)}, \sup G_2^{(k)})$. By Lemma 8.3, for each fixed n and k,

$$\mathbb{P}_\epsilon \Big(\text{a particle starting in } B^{(k)} \text{ leaves } V^{(k)} \Big) \to 0. \tag{32}$$

Fix any $U \subseteq B^{(k)}$. By Corollary 8.4,

$$\liminf_{\epsilon \to 0} \epsilon \cdot \mathbb{E}[N_\epsilon(U)] \geq \rho_c(|B^{(k)}| - |V^{(k)} \setminus U|). \tag{33}$$

We can simplify the right hand side of (33) as follows:

$$\begin{aligned} |B^{(k)}| - |V^{(k)} \setminus U| &= |B^{(k)} \setminus U| + |U| - (|V^{(k)} \setminus B^{(k)}| + |B^{(k)} \setminus U|) \\ &= |U| - |V^{(k)} \setminus B^{(k)}| \\ &\geq |U| - C\epsilon_F \end{aligned}$$

This gives us

$$\liminf_{\epsilon \to 0} \epsilon \cdot \mathbb{E}[N_\epsilon(U)] \geq \rho_c(|U| - C\epsilon_F). \tag{34}$$

We can take ϵ_F to be arbitrarily small by taking n large, which gets rid of the $C\epsilon_F$ term in (34). Then any interval $U \subseteq B^{(k)}$ never gets "too few" particles.

Also, by Corollary 6.2,

$$\limsup_{\epsilon \to 0} \epsilon \cdot \mathbb{E}[N_\epsilon(U)] \leq \rho_c|U|, \tag{35}$$

so U does not get "too many" particles with high probability.

Next, take U not in $V^{(k)}$ for A_n for some large n. We want to show that U does not get "many" particles. By Lemma 8.3, with high probability the only particles U can get will have started outside of $V^{(k)}$. By (2) of Lemma 8.1, at most ϵ_F/ϵ particles will stabilize in U with high probability, so

$$\limsup_{\epsilon \to 0} \epsilon \cdot \mathbb{E}[N_\epsilon(U)] \leq \epsilon_F. \tag{36}$$

Proposition 2.11, (34), (35), and (36) imply

$$\epsilon\, \mathcal{S}(1_{A \cap \epsilon\mathbb{Z}}) \xrightarrow{*} \rho_c 1_{A^*}, \tag{37}$$

which is the desired result of Theorem 2.3 in terms of the whole A rather than A_n! □

Proof of Theorem 2.3. By Corollary 8.6, we can approximate any bounded open set $A \subset \mathbb{R}$ whose boundary has Lebesgue measure zero by an increasing sequence of bounded open sets A_n such that by Lemma 8.5, the smash domains A_n^{sm} converge in measure to A^{sm}. By Definition 2.8 and Proposition 2.10 we have $A^{\mathrm{sm}} = A^*$ up to Lebesgue null sets. So by Proposition 8.7, it suffices to verify the weak-star convergence in Theorem 2.3 under the additional assumption that A is a finite union of intervals. In the case where A is a finite union of intervals, the statement of Theorem 2.3 follows from Lemma 7.2 together with Proposition 2.11. $\square$

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