# Cooperations in motivic homotopy theory

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**Abstract**

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A central task in motivic homotopy theory is to compute the stable motivic homotopy groups of spheres. These groups are organized into periodic layers, each of which being more tractable than the whole. A useful tool for computing the $v_1$-periodic layer is the kq-resolution, a spectral sequence which encodes descent from hermitian K-theory. This spectral sequence takes the form

$$\mathrm{E}_1^{s,f,w} = \pi^F_{s+f,w}(\mathrm{kq} \otimes \mathrm{kq}^{\otimes n}) \implies \pi^F_{s,w}(\mathbb{S}).$$

The primary goal of this thesis is to understand the kq-resolution. To this end, we compute the ring of cooperations for hermitian K-theory $\pi^F_{**}(\mathrm{kq} \otimes \mathrm{kq})$ over the real numbers and all finite fields of characteristic different from 2. As consequences, we determine (up to $v_1$-torsion) the $\mathrm{E}_1$-page of the kq resolution over these base fields, and we deduce a spectrum level splitting for the symplectic K-theory spectrum ksp over any base field of characteristic different from 2.

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# INTRODUCTION

## *1.1 Motivation*

The stable motivic homotopy groups of spheres are among the most important invariants in stable motivic homotopy theory. For $F$ a field, the bigraded ring $\pi^F_{**}(\mathbb{S})$ has many connections with both topological and arithmetic information regarding $F$. For example, it is a theorem of Morel [Mor04] that there is an isomorphism

$$\pi^F_{-n,-n}(\mathbb{S}) \cong \mathrm{K}^{\mathrm{MW}}_n(F),$$

where $\mathrm{K}^{\mathrm{MW}}_i(F)$ is the $i^{th}$ Milnor-Witt K-theory of $F$. As another example, take $F$ to be algebraically closed, and let $\mathrm{e} = \mathrm{char}(F)$ for $F$ of positive characteristic and $\mathrm{e} = 1$ for $F$ of characteristic 0. Then there is an isomorphism after inverting the characteristic [Lev14; WØ17]

$$\pi^F_{n,0}(\mathbb{S})[\mathrm{e}^{-1}] \cong \pi_n(\mathbb{S})[\mathrm{e}^{-1}],$$

where the right hand side denotes the classical stable homotopy groups of spheres.

An extremely useful tool for computing stable motivic homotopy groups is the motivic Adams spectral sequence. For any motivic ring spectrum E, there is an E-based motivic Adams spectral sequence computing $\pi^F_{**}(\mathbb{S})$ up to some completion. Perhaps the most well studied is the $\mathrm{H}\mathbb{F}_2$-based motivic Adams spectral sequence, where $\mathrm{H}\mathbb{F}_2$ is the motivic ring spectrum representing motivic cohomology with mod-$p$ coefficients. We will refer to this spectral sequence simply as the $\mathbf{mASS}^F(\mathbb{S})$, which has signature

$$\mathrm{E}_2^{s,f,w} = \mathrm{Ext}^{s,f,w}_{\mathcal{A}_2^\vee}(\mathbb{M}_2^F, \mathbb{M}_2^F) \implies \pi^F_{s,w}(\mathbb{S}).$$

Here $\mathcal{A}^\vee_F = \pi^F_{**}(\mathrm{H}\mathbb{F}_2 \otimes \mathrm{H}\mathbb{F}_2)$ denotes the mod-2 dual Steenrod algebra, $\mathbb{M}_2^F$ denotes the mod-2 homology of a point, and Ext is taken in the category of $\mathcal{A}^\vee_F$-comodules. This spectral sequence has been well studied over algebraically closed fields [DI10; HKO11a], the real numbers [BI22], and, following the computation of the dual Steenrod algebra $\mathcal{A}_2^\vee$ in positive characteristic [HKØ17], over finite fields [WØ17]. As we will heavily use the $\mathbf{mASS}^F(\mathbb{S})$ for the rest of this thesis, henceforth we will implicitly be working with 2-complete motivic spectra.

Another perspective which one can take to gain insight into $\pi_{**}^F(\mathbb{S})$ is by organizing its elements into periodic families. This is the heart of the chromatic approach to motivic homotopy theory. Instead of using the $\mathbf{mASS}^F(\mathbb{S})$ to compute the motivic stable stems one stem degree at a time, one can use different E-based motivic Adams spectral sequences to localize attention to an infinite periodic family in $\pi_{**}^F(\mathbb{S})$. One downside to this philosophy is that often these spectral sequences are less computable at the onset, and some amount of genuine work must be put into computing the $\mathrm{E}_1$-page. These spectral sequences take the form

$$\mathrm{E}_1^{s,f,w} = \pi_{s+f,w}^F(\mathrm{E} \otimes \overline{\mathrm{E}}^{\otimes f}) \implies \pi_{s,w}^F(\mathbb{S}),$$

where $\overline{\mathrm{E}}$ is the cofiber of the unit map $\mathbb{S} \to \mathrm{E}$. It is often easier to compute $\pi_{**}^F(\mathrm{E} \otimes \mathrm{E})$, known as the homotopy ring of cooperations, and then bootstrap up to the $\mathrm{E}_1$-page.

This is common practice in classical stable homotopy theory (see [Mah81; Gon98; BBBCX20]). Let bo denote the connective cover of the real topological K-theory spectrum KO. This spectrum is well-enough behaved that one may begin to study its associated Adams spectral sequence. Mahowald and others have studied the bo-**ASS**, known as the bo-resolution, at the prime 2 extensively [Mah81; LM87; BBBCX20]; their analysis of this spectral sequence leads to, among many things, the $v_1$-torsion free component of $\pi_*(\mathbb{S})$. Key to Mahowald's analysis is a spectrum level splitting of $\mathrm{bo} \otimes \mathrm{bo}$ as the following wedge sum (up to 2-completion):

$$\mathrm{bo} \otimes \mathrm{bo} \simeq \bigoplus_{k \geq 0} \Sigma^{4k} \mathrm{bo} \otimes \mathrm{H}\mathbb{Z}_k^{cl}. \tag{1.1.1}$$

The spectra $\mathrm{H}\mathbb{Z}_k^{cl}$ are finite complexes known as the integral Brown–Gitler spectra [BG73; GJM86]. This additive splitting allows one to understand $\mathrm{bo} \otimes \mathrm{bo}$ as a sort of "graded module" over bo where each summand is only slightly more complicated than bo itself.

In motivic homotopy theory, the generalized Adams spectral sequence has not been as thoroughly studied. For instance, the study of $v_1$-periodicity via these techniques is in its nascent stage (although there is work on the subject via the slice spectral sequence due to [BIK24; KQ24]). Let kq denote the very effective cover of the Hermitian K-theory spectrum KQ [ARØ20] (see Chapter 3 for details on the very effective slice filtration). In [CQ21], Culver–Quigley analyze the kq-**mASS** in $\mathbb{C}$-motivic homotopy theory. This spectral sequence is called the kq-resolution and has signature

$$\mathrm{E}_1^{s,f,w} = \pi^{\mathbb{C}}_{s+f,w}(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes f}) \implies \pi^{\mathbb{C}}_{s,w}(\mathbb{S}).$$

Their analysis of this spectral sequence leads to, among many things, the $v_1$-torsion free component of $\pi^{\mathbb{C}}_{**}(\mathbb{S})$. Toward computing the $\mathrm{E}_1$-page, Culver–Quigley compute the ring of cooperations $\pi^{\mathbb{C}}_{**}(\mathrm{kq} \otimes \mathrm{kq})$ by the $\mathbf{mASS}_2^{\mathbb{C}}(\mathrm{kq} \otimes \mathrm{kq})$, which takes the form

$$\mathrm{E}_2^{s,f,w} = \mathrm{Ext}^{s,f,w}_{\mathcal{A}^\vee_{\mathbb{C}}}\left(\mathbb{M}_2^{\mathbb{C}}, \mathrm{H}_{**}(\mathrm{kq} \otimes \mathrm{kq})\right) \implies \pi^{\mathbb{C}}_{**}(\mathrm{kq} \otimes \mathrm{kq}).$$

The techniques employed by Culver–Quigley are more algebraic in nature than those of Mahowald, stemming from the fact that to date, there is no construction of a motivic analogue of integral Brown–Gitler spectra. Thus, a motivic spectrum level splitting akin to (1.1.1) is currently unattainable.

## *1.2 Main results*

This dissertation, which combines the results of the papers [Mor26] and [Mor25], begins the study of the kq-resolution over a richer class of base fields. As a step in this direction, we compute the homotopy rings of cooperations $\pi^F_{**}(\mathrm{kq} \otimes \mathrm{kq})$ as a module over $\pi^F_{**}(\mathrm{kq})$ for $F = \mathbb{R}$ and $F = \mathbb{F}_q$ for $\mathrm{char}(\mathbb{F}_q) \neq 2$. As mentioned previously, henceforth all spectra are implicitly completed at the prime 2.

**Theorem A** (Theorem 7.4.4, Theorem 8.3.5)**.** The $\mathbf{mASS}^F(\mathrm{kq} \otimes \mathrm{kq})$ has signature

$$\mathrm{E}_2^{s,f,w} = \bigoplus_{k \geq 0} \Sigma^{4k,2k} \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee_F}(\mathbb{M}_2, B_0^F(k)) \implies \pi^F_{s,w}(\mathrm{kq} \otimes \mathrm{kq}),$$

where $B_0^F(k)$ denotes the $k^{th}$ integral motivic Brown–Gitler comodule. For $F = \mathbb{R}$ or $\mathbb{F}_q$ where $\mathrm{char}(\mathbb{F}_q) \neq 2$, we describe the $\mathrm{E}_\infty$-page, modulo $v_1$-torsion, as a module over $\pi_{**}^F(\mathrm{kq})$.

The reader familiar with Mahowald's analysis of the bo-resolution [Mah81] may be inclined to believe that this result may be proven in a similar fashion. This is not the case. Mahowald's arguments make great use of integral Brown–Gitler spectra, and as previously remarked, there is currently no known construction of motivic integral Brown–Gitler spectra except in very particular cases (see Example 2.3.3, Remark 2.3.1), and so another method of analysis must be employed. As is the case over $\mathbb{C}$, we generally follow the strategy employed by [BOSS19], in which they compute the classical ring of cooperations for tmf using algebraic methods. We outline this strategy below.

(1) We first introduce the integral motivic Brown–Gitler comodules $B_0^F(k)$ and show that there are isomorphisms of $\mathcal{A}(1)_F^\vee$-comodules:

$$\mathrm{H}_{**}(\mathrm{kq}) \cong (\mathcal{A} \,/\!/\, \mathcal{A}(1))_F^\vee \cong \bigoplus_{k \geq 0} \Sigma^{4k,2k} B_0^F(k).$$

(2) Next, we produce short exact sequences relating the integral Brown–Gitler comodules which allow for induction.

(3) Last, we make base-case computations in Ext using algebraic Atiyah–Hirzebruch spectral sequences (**aAHSS**) and induct to complete the proof of the theorem.

Steps (1) and (2) are direct consequences of [CQ21, Section 3]. Step (3) is the most difficult and occupies most of this dissertation. This is because the algebra $\mathrm{Ext}^{***}_{\mathcal{A}(1)_F^\vee}(\mathbb{M}_2^F, \mathbb{M}_2^F)$ has more elaborate structure than its classical and $\mathbb{C}$-motivic counterparts. Indeed, when presenting the charts for our spectral sequence computations, we find it best to follow the strategy of Guillou–Hill–Isaksen–Ravenel [GHIR20] and organize each page by coweight $cw = stem - weight$. Powers of the element $\tau \in \mathbb{M}_2^F$ give a coweight periodicity allowing us to organize our data into multiple sets of charts. These computation are manageable with careful bookkeeping.

Over $\mathbb{F}_q$, contrary to the classical, $\mathbb{C}$-motivic, and the $\mathbb{R}$-motivic incarnations of this question, this spectral sequence does not collapse on the $\mathrm{E}_2$-page. Since $B_0^{\mathbb{F}_q}(0) \cong \mathbb{M}_2^{\mathbb{F}_q}$, our description of the $\mathrm{E}_2$-page identifies the $k = 0$-summand as the $\mathrm{E}_2$-page of the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$, which also does not collapse at the $\mathrm{E}_2$-page. Curiously, we make the following observation.

**Theorem B** (Theorem 7.5.2, Theorem 8.4.2)**.** The differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq} \otimes \mathrm{kq})$ are determined by the $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{F}_q}, \mathbb{M}_2^{\mathbb{F}_q})$-module structure of the $E_2$-page and the differentials of the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$.

The differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$ are themselves lifted from the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z})$ along the natural quotient map $\mathrm{kq} \to \mathrm{H}\mathbb{Z}$. The above theorem implies that relative to the algebra $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{F}_q}, \mathbb{M}_2^{\mathbb{F}_q})$, the only interesting part of the ring of cooperations for Hermitian K-theory is determined by the integral motivic cohomology of $\mathbb{F}_q$.

Through our inductive process, we obtain the following result, where ksp denotes the very effective cover of $\Sigma^{4,2}\mathrm{KQ}$ (see Chapter 3):

**Theorem C** (Corollary 6.2.4)**.** The $\mathbf{mASS}^F(\mathrm{ksp})$ has signature:

$$\mathrm{E}_2^{s,f,w} = \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee_F}(\mathbb{M}_2^F, B_0^F(1)) \implies \pi^F_{s,w}(\mathrm{ksp}).$$

We describe the $\mathrm{E}_\infty$-page, modulo $v_1$-torsion, as a module over $\pi^F_{**}(\mathrm{kq})$.

For $F = \mathbb{F}_q$, this verifies and gives an alternative proof of Friedlander's seminal computation. This is an easy consequence of a stronger result which we show in a more general setting.

**Theorem D** (Theorem 3.0.1)**.** Let $F$ be any field with 2 invertible. Then there is an equivalence of motivic spectra

$$\mathrm{ksp} \simeq \mathrm{H}\mathbb{Z}_1^F \otimes \mathrm{kq},$$

where $\mathrm{H}\mathbb{Z}_1^F$ denotes the first integral $F$-motivic Brown–Gitler spectrum (see Example 2.3.3).

We prove this result by analyzing the very effective slice tower for KQ similar to how one would analyze the Whitehead tower for the classical spectrum KO.

As an application of our computations, we describe the $\mathrm{E}_1$-page of the kq-resolution.

**Theorem E** (Proposition 9.0.2, Theorem 9.0.3)**.** The $\mathbf{mASS}^F(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n})$ has signature

$$\mathrm{E}_2 = \bigoplus_{K \in \mathcal{K}_n} \Sigma^{4|K|,2|K|} \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^F, B_0^F(K)) \implies \pi^F_{s,w}(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n}),$$

where $\mathcal{K}_n = \{K = (k_1, \dots, k_n) : k_j \geq 1 \text{ for all } j\}$, $|K| = \sum_{j=1}^n k_j$, and $B_0^F(K) = \bigotimes_{j=1}^n B_0^F(k_j)$. For $n = 0$, the differentials are given by the differentials in the $\mathbf{mASS}^F(\mathrm{kq})$. For $n > 0$, the differentials are determined by the underlying module structure over the 0-line.

### *1.2.1 Notation*

# Chapter 2

# BACKGROUND

In this section we work over an arbitrary field $F$ of characteristic different from 2 such that $\mathrm{vcd}_2(F) < \infty$. For example, $\mathbb{C}, \mathbb{R}$, and $\mathbb{F}_q$ for $\mathrm{char}(\mathbb{F}_q) \neq 2$ all have this property. We recall the kq-resolution then review the dual motivic Steenrod algebra and motivic Brown–Gitler comodules.

## *2.1 The* kq*-resolution*

It was shown in [Hor05; CHN25] that there is an $(8,4)$-periodic motivic ring spectrum $\mathrm{KQ} \in \mathrm{SH}(F)$ representing *Hermitian K-theory*, in that $\pi^F_{s,w}(\mathrm{KQ}) = \mathrm{K}^{\mathrm{h}}_{s-2w}(F)$. In particular, $\pi^F_{8n,4n}(\mathrm{KQ}) = \mathrm{GW}(F)$, the Grothendieck–Witt ring of quadratic forms over $F$, and the unit map $\mathbb{S} \to \mathrm{KQ}$ induces Morel's isomorphism [RSØ19]

$$\pi^F_{-n,-n}(\mathbb{S}) \cong \mathrm{K}^{\mathrm{MW}}_n(F).$$

Let kq denote the very effective cover of KQ, which we will call the *very effective Hermitian K-theory spectrum* (see (3) for more details on the very effective filtration). There is a canonical Adams tower associated to the unit map $\mathbb{S} \to \mathrm{kq}$:

$$\begin{tikzcd}
\mathbb{S} \arrow[d] & \Sigma^{-1,0}\overline{\mathrm{kq}} \arrow[l] \arrow[d] & \Sigma^{-2,0}\overline{\mathrm{kq}} \otimes \overline{\mathrm{kq}} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\mathrm{kq} & \Sigma^{-1,0}\overline{\mathrm{kq}} \otimes \mathrm{kq} & \Sigma^{-2,0}\overline{\mathrm{kq}} \otimes \overline{\mathrm{kq}} \otimes \mathrm{kq} &
\end{tikzcd}$$

Applying $\pi^F_{**}(-)$ yields the motivic Adams spectral sequence kq-$\mathbf{mASS}(\mathbb{S})$ known as the kq-*resolution*. Its properties were first studied by Culver–Quigley [CQ21, Section 2]. We remark that we are working in the 2-complete category.

**Theorem 2.1.1** ([CQ21, Theorem 2.1])**.** *The* kq*-resolution is a strongly convergent spectral sequence of the form*

$$\mathrm{E}_1^{s,f,w} = \pi^F_{s+f,w}(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes f}) \implies \pi^F_{s,w}\mathbb{S}^\wedge_2.$$

*The $d_r$-differentials have the form*

$$d_r : \mathrm{E}_r^{s,f,w} \to \mathrm{E}_r^{s-1,f+r,w}.$$

**Remark 2.1.2.** We can also look at the kq-resolution in the integral, uncompleted setting. The unit map $\mathbb{S} \to \mathrm{kq}$ gives an isomorphism $\Pi^F_0(\mathrm{kq}) \cong \Pi^F_0(\mathbb{S})$, where $\Pi^F_n(X) := \bigoplus_{k\in\mathbb{Z}} \pi^F_{n+k,k}X$ denotes the $k$-th coweight stem (also known as the $k$-th Milnor–Witt stem). This implies that $\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n}$ is $n$-connected in Morel's homotopy $t$-structure. In fact, this is enough to show that $\mathbb{S}^\wedge_{\mathrm{kq}} \simeq \mathbb{S}$.

However, we will compute this $\mathrm{E}_1$-page by a motivic Adams spectral sequence based on the Eilenberg–Mac Lane spectrum $\mathrm{H}\mathbb{F}_2$. This spectral sequence computes the homotopy groups of the $(2,\eta)$-completion of $\mathrm{kq} \otimes \mathrm{kq}$. When $F$ is a field of finite virtual 2-cohomological dimension, then this coincides with the 2-completion [HKO11a, Theorem 1]. This force us to work in the 2-complete category (see also [CQ21, Section 2.2]).

Let bo denote the connective cover of the classical real K-theory spectrum KO. The kq-resolution is the motivic analogue of the bo-resolution in the following way. The $\mathrm{E}_1$-page of the bo-resolution is of the form

$$\mathrm{E}_1^{s,f} = \pi_{s+f}(\mathrm{bo} \otimes \overline{\mathrm{bo}}^{\otimes f}) \implies \pi_*\mathbb{S}^\wedge_2,$$

where we may identify $\mathbb{S}^\wedge_{\mathrm{bo}} \simeq \mathbb{S}^\wedge_2$ in the 2-complete setting [Bou79]. There is a *complex Betti realization functor*

$$\mathrm{Be}^{\mathbb{C}} : \mathrm{SH}(\mathbb{C}) \to \mathrm{Sp},$$

induced by the analytic topology on a scheme $X \in \mathrm{Sm}_{\mathbb{C}}$ after taking $\mathbb{C}$-points. If $F$ admits an embedding into $\mathbb{C}$, then there is complex realization $\mathrm{Be}^{\mathbb{C}} : \mathrm{SH}(F) \to \mathrm{Sp}$ defined in the same way. In fact, work of Bachmann–Østvær [BØ22] allows for one to employ complex

Betti realization when working over finite fields of odd characteristic. In particular, [BØ22, Proposition 4] gives an equivalence of categories (after 2-completion)

$$\mathrm{SH}(\mathbb{F}_p)^\wedge_2 \simeq \mathrm{SH}(\mathbb{Z}_p)^\wedge_2.$$

Choosing a complex embedding $\mathbb{Z}_p \hookrightarrow \mathbb{C}$ thus gives a complex Betti realization functor.

The following result lets us compare between kq and bo:

**Theorem 2.1.3** ([ARØ20, Lemma 2.13])**.** *There is an equivalence of spectra*

$$\mathrm{Be}^{\mathbb{C}}(\mathrm{kq}) \simeq \mathrm{bo}.$$

Betti realization is symmetric monoidal and exact [HO16], giving an immediate corollary:

**Corollary 2.1.4** ([CQ21, Cor 2.7])**.** *Betti realization sends the* kq*-resolution to the* bo*-resolution. In particular, there is a multiplicative map of spectral sequences:*

$$\mathrm{E}_r^{s,f,*} \to \mathrm{E}_r^{s,f}.$$

**Remark 2.1.5.** Note that the spectrum kq is not flat in the sense of Adams. Thus the $\mathrm{E}_2$-page does not identify with the Hopf algebroid cohomology $\mathrm{Ext}^{***}_{\pi^F_{**}(\mathrm{kq}\otimes\mathrm{kq})}\left(\pi^F_{**}\mathrm{kq}, \pi^F_{**}\mathrm{kq}\right)$, and so we must analyze the $\mathrm{E}_1$-page of the kq-resolution.

**Remark 2.1.6.** There is a complex Betti realization functor in the case of $F = \mathbb{R}$ to which we may apply Theorem 2.1.3. It is important to note that there is an obvious *real Betti realization*

$$\mathrm{Be}^{\mathbb{R}} : \mathrm{SH}(\mathbb{R}) \to \mathrm{Sp}$$

which is induced by the real analytic topology on a scheme $X \in \mathrm{Sm}_{\mathbb{R}}$ after taking $\mathbb{R}$-points. These functors behave very differently. For example, consider $\mathrm{kq} \in \mathrm{SH}(\mathbb{R})$. It was shown in [Ban25, Proposition 4.1] that $\pi_*(\mathrm{Be}^{\mathbb{R}}(\mathrm{kq})) \cong \mathbb{Z}[x]$, where $x \in \pi_4(\mathrm{Be}^{\mathbb{R}}(\mathrm{kq}))$. In particular, $\mathrm{Be}^{\mathbb{C}}(\mathrm{kq}) \not\simeq \mathrm{Be}^{\mathbb{R}}(\mathrm{kq})$.

### 2.2 The dual motivic Steenrod algebra

Let $\mathbb{M}_2^F$ denote the *mod-2 motivic homology of a point*. There is an isomorphism due to Voevodsky [Voe03a]:

$$\mathbb{M}_2^F = (\mathrm{K}_*^{\mathrm{M}}(F)/2)[\tau],$$

where $\mathrm{K}_*^{\mathrm{M}}(F)$ denotes the Milnor K-theory of $F$, $|\mathrm{K}_n^{\mathrm{M}}(F)| = (-n,-n)$, and $|\tau| = (0,-1)$. Recall the dual motivic Steenrod algebra:

**Theorem 2.2.1** ([Voe03b, Section 12])**.** *The dual motivic Steenrod algebra $\mathcal{A}_F^\vee = \pi_{**}^F(\mathrm{H}\mathbb{F}_2 \otimes \mathrm{H}\mathbb{F}_2)$ is given by*

$$\mathcal{A}_F^\vee = \mathbb{M}_2^F[\overline{\xi}_1, \overline{\xi}_2, \ldots, \overline{\tau}_0, \overline{\tau}_1, \ldots]/(\overline{\tau}_i^2 = \rho\overline{\tau}_{i+1} + \rho\overline{\tau}_0\overline{\xi}_{i+1} + \tau\overline{\xi}_{i+1}),$$

*where $|\overline{\xi}_i| = (2^{i+1}-2, 2^i-1)$ and $|\overline{\tau}_i| = (2^{i+1}-1, 2^i-1)$. There are structure maps*

$$\eta_L, \eta_R : \mathbb{M}_2^F \to \mathcal{A}_F^\vee, \quad \Delta : \mathcal{A}_F^\vee \to \mathcal{A}_F^\vee \otimes_{\mathbb{M}_2^F} \mathcal{A}_F^\vee$$

*which are determined by the following formulae:*

$$\begin{aligned}
&\eta_L(\rho) = \rho && \eta_L(\tau) = \tau;\\
&\eta_R(\rho) = \rho && \eta_R(\tau) = \tau + \overline{\tau}_0\rho;\\
&\Delta(\overline{\xi}_k) = \textstyle\sum_{i+j=k} \overline{\xi}_i \otimes \overline{\xi}_j^{2^i}, && \Delta(\overline{\tau}_i) = \textstyle\sum_{i+j=k} \overline{\tau}_i \otimes \overline{\xi}_j^{2^i} + 1 \otimes \overline{\tau}_k.
\end{aligned}$$

We will also encounter various subalgebras and quotient algebras of $\mathcal{A}_F^\vee$.

**Definition 2.2.2.** For $n \geq 0$, let $\mathcal{A}(n)_F^\vee$ be the quotient algebra

$$\mathcal{A}(n)_F^\vee \cong \mathcal{A}_F^\vee/(\overline{\xi}_1^{2^n}, \overline{\xi}_2^{2^{n-1}}, \ldots, \overline{\xi}_n^2, \overline{\xi}_{n+1}, \overline{\xi}_{n+2,}, \ldots, \overline{\tau}_{n+1}, \overline{\tau}_{n+2}, \ldots).$$

There is a sub-Hopf algebra of the motivic Steenrod algebra

$$\mathcal{A}(n)_F = \langle \mathrm{Sq}^1, \mathrm{Sq}^2, \ldots, \mathrm{Sq}^{2^n} \rangle;$$

let $(\mathcal{A} /\!/ \mathcal{A}(n))_F^\vee$ be the subalgebra of $\mathcal{A}_F^\vee$ given by

$$(\mathcal{A} /\!/ \mathcal{A}(n))_F^\vee = \mathbb{M}_2^F[\overline{\xi}_1^{2^n}, \overline{\xi}_2^{2^{n-1}}, \ldots, \overline{\tau}_{n+1}, \ldots]/(\overline{\tau}_i^2 = \rho\overline{\tau}_{i+1} + \rho\overline{\overline{\tau}_0}\overline{\xi}_{i+1} + \tau\overline{\xi}_{i+1}).$$

**Remark 2.2.3.** There is a canonical class $\rho \in \mathrm{K}_1^{\mathrm{M}}(F)/2$ representing $-1$, giving rise to a class in $\mathbb{M}_2^F$. If $-1$ is a square in $F$, then this class is trivial in motivic homology and $\mathcal{A}_F^\vee$ is an honest Hopf algebra. If $-1$ is not a square, then there is a nontrivial action of the motivic Steenrod algebra:

$$\mathrm{Sq}^1(\tau) = \rho.$$

In particular, upon dualizing we see that $\mathbb{M}_2^F$ is not central in $\mathcal{A}_F^\vee$, implying that $(\mathbb{M}_2^F, \mathcal{A}_F^\vee)$ is a Hopf algebroid rather than a Hopf algebra. See [Rav86, Appendix A] for more details on Hopf algebroids.

As in the classical case, one can express the comodule algebra $(\mathcal{A} /\!/ \mathcal{A}(n))_F^\vee$ as a cotensor product:

$$(\mathcal{A} /\!/ \mathcal{A}(n))_F^\vee = \mathcal{A}_F^\vee \square_{\mathcal{A}(n)_F^\vee} \mathbb{M}_2^F.$$

These $\mathcal{A}_F^\vee$-comodule algebras come up naturally in our work as the motivic homology of certain motivic spectra. For instance, we have that $\mathrm{H}_{**}(\mathrm{H}\mathbb{Z}) \cong (\mathcal{A} /\!/ \mathcal{A}(0))_F^\vee$. We also have the following:

**Theorem 2.2.4** ([ARØ20, Remark 2.14])**.** *There is an isomorphism of $\mathcal{A}_F^\vee$-comodule algebras*

$$\mathrm{H}_{**}(\mathrm{kq}) \cong (\mathcal{A} /\!/ \mathcal{A}(1))_F^\vee = \mathbb{M}_2^F[\overline{\xi}_1^2, \overline{\xi}_2, \ldots, \overline{\tau}_2, \overline{\tau}_3, \ldots]/(\overline{\tau}_i^2 = \rho\overline{\tau}_{i+1} + \rho\overline{\tau}_0\overline{\xi}_{i+1} + \tau\overline{\xi}_{i+1}).$$

We investigate now the $\mathcal{A}_F^\vee$-comodule structure on the homology $\mathrm{H}_{**}(\mathrm{kq})$. As $(\mathcal{A} /\!/ \mathcal{A}(1))_F^\vee \subseteq \mathcal{A}_F^\vee$ is a subcoalgebra, the coaction comes from the restriction of the comultiplication on $\mathcal{A}_F^\vee$:

$$\psi : \mathrm{H}_{**}(\mathrm{kq}) \to \mathcal{A}_F^\vee \otimes_{\mathbb{M}_2^F} \mathrm{H}_{**}(\mathrm{kq}).$$

Later, we will need to understand the $\mathcal{A}(1)_F^\vee$-comodule structure on the homology $\mathrm{H}_{**}(\mathrm{kq})$. Observe that by definition, $\mathcal{A}(1)_F^\vee$ takes the form

$$\mathcal{A}(1)_F^\vee = \mathbb{M}_2^F[\overline{\xi}_1, \overline{\tau}_0, \overline{\tau}_1]/(\overline{\tau}_0^2 = \rho\overline{\tau}_1 + \rho\overline{\tau}_0\overline{\xi}_1 + \tau\overline{\xi}_1, \overline{\tau}_1^2, \overline{\xi}_1^2).$$

Since $\mathcal{A}(1)_F^\vee$ is a quotient of $\mathcal{A}_F^\vee$, we can deduce that the coaction is given by projection of the usual coaction of $\mathcal{A}_F^\vee$ on $\mathrm{H}_{**}(\mathrm{kq})$:

$$\psi : \mathrm{H}_{**}(\mathrm{kq}) \to \mathcal{A}_F^\vee \otimes_{\mathbb{M}_2^F} \mathrm{H}_{**}(\mathrm{kq}) \to \mathcal{A}(1)_F^\vee \otimes_{\mathbb{M}_2^F} \mathrm{H}_{**}(\mathrm{kq}).$$

Moreover, since kq is a ring spectrum, its homology is a comodule algebra. Thus it suffices to determine the coaction on the generators. The following is immediate from this discussion.

**Proposition 2.2.5** ([CQ21, Corollary 3.6])**.** *The* $\mathcal{A}(1)^\vee_F$*-coaction on* $\mathrm{H}_{**}(\mathrm{kq})$ *is determined by the following formulae:*

$$\begin{aligned} &\psi(\overline{\xi}_1^2) = 1 \otimes \overline{\xi}_1^2; && \\ &\psi(\overline{\xi}_i) = 1 \otimes \overline{\xi}_i + \overline{\xi}_1 \otimes \overline{\xi}_{i-1}^2 && i \geq 2; \\ &\psi(\overline{\tau}_i) = 1 \otimes \overline{\tau}_i + \overline{\tau}_0 \otimes \overline{\xi}_i + \overline{\tau}_1 \otimes \overline{\xi}_{i-1}^2 && i \geq 2. \end{aligned}$$

The following result will be useful in the sequel.

**Proposition 2.2.6.** *There is a Künneth isomorphism of* $\mathcal{A}^\vee_F$*-comodule algebras*

$$\mathrm{H}_{**}(\mathrm{kq} \otimes \mathrm{kq}) \cong \mathrm{H}_{**}(\mathrm{kq}) \otimes_{\mathbb{M}_2^F} \mathrm{H}_{**}(\mathrm{kq}).$$

*Proof.* It was shown by Voevodsky [Voe03b] that the motivic Steenrod algebra $\mathcal{A}_F$ is free as a module over the cohomology of a point. One basis for this algebra is the motivic Milnor basis [Kyl17]. Since both $\mathrm{Sq}^1$ and $\mathrm{Sq}^2$ are elements of this basis, the quotient $\mathcal{A} /\!/ \mathcal{A}(1)_F \cong \mathcal{A}_F \otimes_{\mathcal{A}(1)_F} \mathbb{M}_2^F$ is also free over the cohomology of a point. Consider the following Künneth spectral sequence [DI05]:

$$\mathrm{E}_2 = \mathrm{Tor}^{\mathbb{M}_2^F}(\mathrm{H}_{**}(\mathrm{kq}), \mathrm{H}_{**}(\mathrm{kq})) \implies \mathrm{H}_{**}(\mathrm{kq} \otimes \mathrm{kq}).$$

Since $\mathrm{H}_{**}(\mathrm{kq}) \cong (\mathcal{A} /\!/ \mathcal{A}(1))^\vee_F$ is the $\mathbb{M}_2^F$-linear dual of a free module, it is also free. Thus the higher Tor vanishes and the spectral sequence collapses with no extensions, giving the result. □

### *2.3 Motivic Brown–Gitler comodules*

We next introduce motivic Brown–Gitler comodules. We define the *Mahowald weight filtration* on $\mathcal{A}^\vee_F$ by setting

$$wt(\overline{\xi}_i) = wt(\overline{\tau}_i) = 2^i, \quad wt(\tau) = wt(\rho) = 0$$

and letting $wt(xy) = wt(x) + wt(y)$. This naturally extends to the subalgebras $(\mathcal{A} /\!/ \mathcal{A}(n))^\vee_F$. The $F$*-motivic Brown–Gitler comodule* $B^F_n(k)$ is the $A(n)^\vee_F$-comodule

$$B^F_n(k) = \langle x \in (\mathcal{A} /\!/ \mathcal{A}(n))^\vee_F : wt(x) \leq 2^{n+1}k \rangle.$$

We will specialize to the following two cases throughout our work.

**Definition 2.3.1.** (compare with [CQ21, Def. 3.10]) The $k$*-th integral motivic Brown–Gitler comodule* is

$$B^F_0(k) \coloneqq \langle x \in (\mathcal{A} /\!/ \mathcal{A}(0))^\vee_F : wt(x) \leq 2k \rangle.$$

The $k$*-th* kq *motivic Brown–Gitler comodule* is

$$B^F_1(k) \coloneqq \langle x \in (\mathcal{A} /\!/ \mathcal{A}(1))^\vee_F : wt(x) \leq 4k \rangle.$$

We remark that in [CQ21], different notation is used for the Brown–Gitler comodules.

**Example 2.3.2.** The motivic Brown–Gitler comodule $B^F_0(0)$ is given by

$$B^F_0(0) = \langle x \in (\mathcal{A} /\!/ \mathcal{A}(0))^\vee_F : wt(x) \leq 0 \rangle \cong \mathbb{M}^F_2,$$

with trivial $\mathcal{A}(1)^\vee_F$-comodule structure.

**Example 2.3.3.** The motivic Brown–Gitler comodule $B^F_0(1)$ is given by

$$B^F_0(1) = \langle x \in (\mathcal{A} /\!/ \mathcal{A}(0))^\vee_F : wt(x) \leq 2 \rangle = \mathbb{M}^F_2\{1, \overline{\xi}_1, \overline{\tau}_1\},$$

where the $\mathcal{A}(1)^\vee_F$-comodule structure is given by

$$\psi(\overline{\xi}_1) = 1 \otimes \overline{\xi}_1 + \overline{\xi}_1 \otimes 1, \quad \psi(\overline{\tau}_1) = 1 \otimes \overline{\tau}_1 + \overline{\tau}_0 \otimes \overline{\xi}_1.$$

In other words, $B^F_0(1)$ is presented as the algebraic cell complex in (2.1).

In fact, we can realize $B^F_0(1)$ as the mod-2 motivic cohomology of a spectrum [CQ21, Example 3.12]. There is a geometric classifying space [MV99, Section 4] $B\mu_2$ of the affine

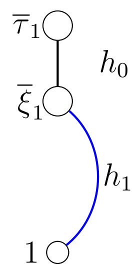


Figure 2.1: The motivic Brown–Gitler comodule $B_0^F(1)$

group scheme $\mu_2$ of second roots of unity. Take $L_2$ to be the simplicial 4-skeleton of $B\mu_2$. The inclusion of the $(3,2)$ cell of $L_2$ gives a map

$$S^{3,2} \hookrightarrow L_2.$$

Let $X$ be the cofiber; then $\mathrm{H}\mathbb{Z}_1^F := \Sigma^{4,2} F(X, \mathbb{S})$ is a finite complex realizing $B_0^F(1)$ in homology.

**Remark 2.3.1.** In classical topology there exist integral and bo Brown–Gitler spectra [Shi83; GJM86]. These are finite spectra $\mathrm{H}\mathbb{Z}_k$ and $\mathrm{bo}_k$, respectively, with the property that

$$\mathrm{H}_*(\mathrm{H}\mathbb{Z}_k) \cong B_0^{cl}(k), \quad \mathrm{H}_*(\mathrm{bo}_k) \cong B_1^{cl}(k).$$

However, while $C_2$-equivariant analogues exist for integral Brown–Gitler spectra [**LPT24**], there are currently no constructions of any motivic Brown–Gitler spectra. One possible construction could come from the motivic lambda algebra of Balderamma–Culver–Quigley [**BCQ23**] and using the classical work of Goerss–Jones–Mahowald [GJM86] as inspiration. An alternative construction, at least over $\mathbb{C}$, is in using the theory of filtered spectra in [GIKR22]. We will explore the construction of motivic Brown-Gitler spectra in future work.

The $\mathcal{A}(1)_F^\vee$-coaction on $(\mathcal{A} \,/\!/\, \mathcal{A}(1))_F^\vee$ preserves the Mahowald weight, as can be seen by our formulae in Proposition 2.2.5. This gives the following:

**Proposition 2.3.2.** *The motivic integral Brown–Gitler comodules $B_0^F(k)$ are $\mathcal{A}(1)_F^\vee$-subcomodules of $(\mathcal{A}\,/\!/\,\mathcal{A}(0))_F^\vee$.*

A key property of the integral motivic Brown–Gitler comodules is that they give rise to a splitting of the homology $\mathrm{H}_{**}(\mathrm{kq})$. We refer the reader to [CQ21] for a proof.

**Theorem 2.3.3** ([CQ21, Theorem 3.20])**.** *There is an isomorphism of $\mathcal{A}(1)_F^\vee$-comodules*

$$(\mathcal{A}\,/\!/\,\mathcal{A}(1))_F^\vee \cong \bigoplus_{k\geq 0} \Sigma^{4k,2k} B_0^F(k).$$

The following result will also be useful. Recall that $\overline{\mathrm{kq}} = \mathrm{cofib}(\mathbb{S} \to \mathrm{kq})$.

**Proposition 2.3.4.** *There is an isomorphism of $\mathcal{A}(1)_F^\vee$-comodules:*

$$\mathrm{H}_{**}(\overline{\mathrm{kq}}) \cong \bigoplus_{k\geq 1} \Sigma^{4k,2k} B_0^F(k).$$

*Proof.* The defining cofiber sequence

$$\mathbb{S} \to \mathrm{kq} \to \overline{\mathrm{kq}}$$

gives a long exact sequence in homology of the form

$$\cdots \to \mathrm{H}_{**}(\mathbb{S}) \to \mathrm{H}_{**}(\mathrm{kq}) \to \mathrm{H}_{**}(\overline{\mathrm{kq}}) \to \mathrm{H}_{*,*-1}(\mathbb{S}) \to \cdots .$$

We have isomorphisms $\mathrm{H}_{**}(\mathbb{S}) \cong \mathbb{M}_2^F \cong B_0^F(0)$ and $\mathrm{H}_{**}(\mathrm{kq}) \cong \bigoplus_{k\geq 0} \Sigma^{4k,2k} B_0^F(k)$. The unit map induces an inclusion of the bottom summand $B_0^F(0) \hookrightarrow \bigoplus_{k\geq 0} B_0^F(k)$, so by the long exact sequence, $\mathrm{H}_{**}(\overline{\mathrm{kq}})$ is trivial in those degrees. The long exact sequence also gives an isomorphism between $\mathrm{H}_{**}(\mathrm{kq})$ and $\mathrm{H}_{**}(\overline{\mathrm{kq}})$ outside of these degrees since $\mathrm{H}_{**}(\mathbb{S})$ vanishes, giving the result. □

We close this section by recalling useful short exact sequences of $\mathcal{A}(1)_F^\vee$-comodules relating motivic Brown–Gitler comodules. We refer the reader to [CQ21] for a proof.

**Proposition 2.3.5** ([CQ21, Lemma 3.21])**.** *There are short exact sequences of $\mathcal{A}(1)^\vee_F$-comodules:*

$$0 \to \Sigma^{4k,2k} B_0^F(k) \to B_0^F(2k) \to B_1^F(k-1) \otimes_{\mathbb{M}_2^F} (\mathcal{A}(1) \mathbin{/\!/} \mathcal{A}(0))^\vee_F \to 0,$$

$$0 \to \Sigma^{4k,2k} B_0^F(k) \otimes_{\mathbb{M}_2^F} B_0^F(1) \to B_0^F(2k+1) \to B_1^F(k-1) \otimes_{\mathbb{M}_2^F} (\mathcal{A}(1) \mathbin{/\!/} \mathcal{A}(0))^\vee_F \to 0.$$

**Remark 2.3.6.** The short exact sequences in Proposition 2.3.5 are motivic analogues of classical short exact sequences of Brown–Gitler comodules. These arise by applying mod-2 homology to cofiber sequences relating integral and bo Brown–Gitler spectra [Pea14]. One would imagine the same to be true for the conjectural motivic Brown–Gitler spectra.

# Chapter 3
# A SPLITTING OF SYMPLECTIC K-THEORY

In this chapter, we record a motivic analogue of a classical splitting result. Let bsp denote the connective cover of $\Sigma^4\mathrm{KO}$. Then there is an equivalence of spectra (up to 2-completion)[MM76, Theorem 1.5]:

$$\mathrm{bsp} \simeq \mathrm{bo} \otimes \mathrm{H}\mathbb{Z}_1^{\mathrm{cl}},$$

where $\mathrm{H}\mathbb{Z}_1^{\mathrm{cl}}$ denotes the first classical integral Brown–Gitler spectrum.

Recall the integral motivic Brown–Gitler spectrum $\mathrm{H}\mathbb{Z}_1^F$ constructed in (2.3.3). We prove the following:

**Theorem 3.0.1.** *There is an equivalence of motivic spectra*

$$\mathrm{ksp} \simeq \mathrm{kq} \otimes \mathrm{H}\mathbb{Z}_1^F.$$

## 3.1 Very effective slice filtration

The very effective slice filtration was introduced by Spitzweck–Østvær in [SØ12, Section 5] and further studied by Bachmann in [Bac17]. We follow the treatments given in [Ban25, Section 1] and [BH21b, Section 13].

**Definition 3.1.1** (Very effective spectra)**.** The category of *very effective motivic spectra*, denoted $\mathrm{SH}(F)^{\mathrm{veff}}$, is the full subcategory of $\mathrm{SH}(F)$ generated under colimits by $\Sigma^{n,0}\Sigma^\infty_+ S$, where $S \in \mathrm{Sm}_F$ and $n \geq 0$. The category of *very $n$-effective motivic spectra* for $n \in \mathbb{Z}$, denoted $\mathrm{SH}(F)^{\mathrm{veff}}(n)$, has objects $\Sigma^{2n,n}\mathrm{X}$ for $\mathrm{X} \in \mathrm{SH}(F)^{\mathrm{veff}}$.

These categories assemble into colimit-preserving inclusions

$$\cdots \subset \mathrm{SH}(F)^{\mathrm{veff}}(n+1) \subset \mathrm{SH}(F)^{\mathrm{veff}}(n) \subset \mathrm{SH}(F)^{\mathrm{veff}}(n-1) \subset \cdots \subset \mathrm{SH}(F).$$

Each inclusion functor $\tilde{\mathrm{i}}_n : \mathrm{SH}(F)^{\mathrm{veff}}(n) \to \mathrm{SH}(F)$ admit a right adjoint called the *n-th very effective cover*, denoted $\tilde{\mathrm{f}}_n$:

$$\tilde{\mathrm{i}}_n : \mathrm{SH}(F)^{\mathrm{veff}}(n) \rightleftarrows \mathrm{SH}(F) : \tilde{\mathrm{f}}_n.$$

If $\mathrm{X} \in \mathrm{SH}(F)$ is any motivic spectrum, the very effective covers give a functorial filtration

$$\cdots \to \tilde{\mathrm{f}}_{n+1}\mathrm{X} \to \tilde{\mathrm{f}}_n\mathrm{X} \to \tilde{\mathrm{f}}_{n-1}\mathrm{X} \to \cdots \to \mathrm{X}$$

which we will call the *very effective slice tower* for X.

**Definition 3.1.2** (Very effective cover)**.** For X a motivic spectrum, its *very effective cover* is defined to be $\tilde{\mathrm{f}}_0\mathrm{X}$.

**Example 3.1.3.** In this notation, we have that $\mathrm{kq} = \tilde{\mathrm{f}}_0\mathrm{KQ}$.

The $n^{th}$ *very effective slice* of a motivic spectrum X, denoted $\tilde{\mathrm{s}}_n\mathrm{X}$, is defined by the cofiber sequence

$$\tilde{\mathrm{f}}_{n+1}\mathrm{X} \to \tilde{\mathrm{f}}_n\mathrm{X} \to \tilde{\mathrm{s}}_n\mathrm{X}.$$

By construction, the very effective cover and slice functors behave well with respect to $\mathbb{P}^1$-suspension.

**Lemma 3.1.1** ([Bac17, Lemma 8])**.** *For any* $\mathrm{X} \in \mathrm{SH}(F)$, *we have*

$$\tilde{\mathrm{f}}_{n+1}(\Sigma^{2,1}\mathrm{X}) \simeq \Sigma^{2,1}\tilde{\mathrm{f}}_n\mathrm{X}, \quad \tilde{\mathrm{s}}_{n+1}(\Sigma^{2,1}\mathrm{X}) \simeq \Sigma^{2,1}\tilde{\mathrm{s}}_n\mathrm{X}.$$

**Remark 3.1.2.** The very effective slice filtration acts as a motivic stand-in for the Whitehead tower in classical stable homotopy. More precisely, let A and Y be classical spectra, and let $\mathrm{P}_n\mathrm{Y}$ be defined by the cofiber sequence

$$\tau_{\geq n+1}\mathrm{Y} \to \tau_{\geq n}\mathrm{Y} \to \mathrm{P}_n\mathrm{Y}$$

One way to obtain the Atiyah–Hirzebruch spectral sequence

$$\mathrm{E}_2 = \mathrm{A}_*(\mathrm{P}_n\mathrm{Y}) \implies \mathrm{A}_*(\mathrm{Y})$$

is by smashing the Whitehead tower for Y with A and applying homotopy groups. Now, let E and X be motivic spectra. Then there is a (generalized) very effective slice spectral sequence [CHR24]

$$\mathrm{E}_2 = \mathrm{E}_{**}(\tilde{\mathrm{s}}_n\mathrm{X}) \implies \mathrm{E}_{**}(\mathrm{X})$$

obtained by smashing the very effective slice tower for $X$ with E and applying homotopy groups. Over $F = \mathbb{C}$, Betti realization sends the very effective slice spectral sequence to an Atiyah–Hirzebruch spectral sequence.

### *3.2 A splitting of symplectic K-theory*

**Definition 3.2.1.** The very effective symplectic K-theory spectrum, denoted ksp, is the very effective cover

$$\mathrm{ksp} := \tilde{\mathrm{f}}_0(\Sigma^{4,2}\mathrm{KQ}).$$

Note that by (3.1.1), we have that $\Sigma^{4,2}\mathrm{ksp} \simeq \tilde{\mathrm{f}}_2\mathrm{KQ}$.

**Proposition 3.2.1.** *There is an isomorphism of $\mathcal{A}_F^\vee$-comodules:*

$$\mathrm{H}_{**}(\mathrm{ksp}) \cong (\mathcal{A} \,/\!/\, \mathcal{A}\langle \mathrm{Sq}^1, \mathrm{Sq}^5\rangle)_F^\vee$$

*Proof.* The idea is to mimic the classical proof, replacing the Whitehead tower with the very effective slice tower for KQ. This tower gives a cofiber sequence

$$\tilde{\mathrm{f}}_3\mathrm{KQ} \to \tilde{\mathrm{f}}_2\mathrm{KQ} \to \tilde{\mathrm{s}}_2\mathrm{KQ}. \tag{3.2.2}$$

By definition, we have $\tilde{\mathrm{f}}_2\mathrm{KQ} \simeq \Sigma^{4,2}\mathrm{ksp}$. Bachmann's identification of the very effective slices of KQ [Bac17, Theorem 16] shows that

$$\tilde{\mathrm{s}}_2\mathrm{KQ} \simeq \Sigma^{4,2}\mathrm{H}\mathbb{Z}, \quad \tilde{\mathrm{s}}_3\mathrm{KQ} \simeq 0.$$

By Bott periodicity for KQ and (3.1.1), we have

$$\tilde{\mathrm{f}}_4\mathrm{KQ} \simeq \tilde{\mathrm{f}}_4(\Sigma^{8,4}\mathrm{KQ}) \simeq \Sigma^{8,4}\tilde{\mathrm{f}}_0\mathrm{KQ} = \Sigma^{8,4}\mathrm{kq}.$$

Since $\tilde{\mathrm{s}}_3\mathrm{KQ} \simeq 0$, the defining cofiber sequence gives an equivalence $\tilde{\mathrm{f}}_4\mathrm{KQ} \simeq \tilde{\mathrm{f}}_3\mathrm{KQ} \simeq \Sigma^{8,4}\mathrm{kq}$. This allows us to rewrite (3.2.2) as

$$\Sigma^{8,4}\mathrm{kq} \to \Sigma^{4,2}\mathrm{ksp} \to \Sigma^{4,2}\mathrm{H}\mathbb{Z}. \tag{3.2.3}$$

Using the long exact sequence in homology and the isomorphisms from (2.2.4)

$$\mathrm{H}_{**}(\mathrm{H}\mathbb{Z}) \cong (\mathcal{A} \,/\!/\, \mathcal{A}(0))^\vee_F, \quad \mathrm{H}_{**}(\mathrm{kq}) \cong (\mathcal{A} \,/\!/\, \mathcal{A}(1))^\vee_F,$$

and comparing with the short exact sequence of $\mathcal{A}^\vee_F$-comodules:

$$0 \to \Sigma^{4,2}(\mathcal{A} \,/\!/\, \mathcal{A}\langle \mathrm{Sq}^1, \mathrm{Sq}^5\rangle)^\vee_F \to \Sigma^{4,2}(\mathcal{A} \,/\!/\, \mathcal{A}(0))^\vee_F \to \Sigma^{8,4}(\mathcal{A} \,/\!/\, \mathcal{A}(1))^\vee_F \to 0$$

gives the result. $\square$

*Proof of* (3.0.1). As kq is a ring spectrum, there is a multiplication map

$$\mu : \mathrm{kq} \otimes \mathrm{kq} \to \mathrm{kq}.$$

In homology, this map is the usual multiplication in $\mathcal{A}^\vee_F$, where we identify the left side using (2.2.6). This induces the kq-module structure on ksp, realized as a map

$$\mu' : \mathrm{ksp} \otimes \mathrm{kq} \to \mathrm{ksp}.$$

By (3.2.1), both $\mathrm{Sq}^2$ and $\mathrm{Sq}^1\mathrm{Sq}^2$ act nontrivially on the degree $(0,0)$ generator of $\mathrm{H}_{**}(\mathrm{ksp})$, hence there is an inclusion

$$\iota : \mathrm{H}\mathbb{Z}^F_1 \to \mathrm{ksp}$$

inducing the obvious map in homology. One can see now that the composition

$$\mathrm{H}\mathbb{Z}^F_1 \otimes \mathrm{kq} \xrightarrow{\iota\otimes 1} \mathrm{ksp} \otimes \mathrm{kq} \xrightarrow{\mu'} \mathrm{ksp}$$

is an isomorphism in homology. Since we are working in the 2-complete category, this gives an equivalence, finishing the proof. $\square$

As we will see, this spectrum level splitting allows us to calculate $\pi^F_{**}\mathrm{ksp}$ in a particularly nice way.

**Remark 3.2.2.** There is another way to relate the motivic spectra ksp and kq that is relevant to our work. In [BH21a], Bachmann-Hopkins construct Adams operations $\psi^q$ on kq. In [BIK24; KQ24], the homotopy groups of the spectrum L were computed over $F \in \{\mathbb{R}, \mathbb{F}_q, \mathbb{Q}_q, \mathbb{Q}, \overline{F}\}$, where $\mathrm{char}(\mathbb{F}_q) \neq 2$ and $\overline{F}$ reresents any algebraically closed field, and where L sits in a cofiber sequence

$$\mathrm{L} \to \mathrm{kq} \xrightarrow{\psi^3 - 1} \mathrm{kq}.$$

The spectrum $L$ is very closely related to $v_1$-periodicity. However, Bachmann-Hopkins go one step further and show that the map $\psi^3 - 1$ factors through the second very effective cover of kq. At least 2-locally, this defines a motivic image of J spectrum $\mathrm{j_O}$, which sits in a cofiber sequence

$$\mathrm{j_O} \to \mathrm{kq} \xrightarrow{\psi^3 - 1} \Sigma^{4,2}\mathrm{ksp}.$$

Culver–Quigley showed that the $v_1$-periodic of $\pi^{\mathbb{C}}_{**}(\mathbb{S})$ is isomorphic to $\pi^{\mathbb{C}}_{**}\mathrm{j_O}$ [CQ21, Theorem 7.13]. In future work, we will show that the homotopy groups $\pi^F_{**}(\mathrm{j_O})$ are isomorphic to the 0 and 1-lines of the kq-resolution over arbitrary base fields.

# Chapter 4

# THE INDUCTIVE PROCESS TO CALCULATE $\pi_{**}(\mathbf{kq} \otimes \mathbf{kq})$

In this section, we consider the motivic Adams spectral sequence converging to the homotopy ring of cooperations $\pi^F_{**}(\mathrm{kq} \otimes \mathrm{kq})$. We show that the $\mathrm{E}_2$-page splits using motivic Brown–Gitler comodules, then outline an inductive process to compute the $\mathrm{E}_2$-page by a series of algebraic Atiyah–Hirzebruch spectral sequences.

## 4.1 *The motivic Adams spectral sequence for* $\pi_{**}(\mathbf{kq} \otimes \mathbf{kq})$

There is an $\mathrm{H}\mathbb{F}_2$-based motivic Adams spectral sequence which computes the homotopy ring of cooperations (the $\mathbf{mASS}^F_2(\mathrm{kq} \otimes \mathrm{kq})$). Since $\mathrm{H}\mathbb{F}_2$ is flat in the sense of Adams, this takes the form

$$\mathrm{E}_2^{s,f,w} = \mathrm{Ext}^{s,f,w}_{\mathcal{A}^\vee_F}(\mathrm{H}_{**}(\mathrm{kq} \otimes \mathrm{kq})) \implies \pi^F_{s,w}(\mathrm{kq} \otimes \mathrm{kq}), \quad d_r : \mathrm{E}_r^{s,f,w} \to \mathrm{E}_r^{s-1,f+r,w}.$$

Here and henceforth, we use the abbreviation

$$\mathrm{Ext}^{***}_{\mathcal{A}^\vee_F}(M) := \mathrm{Ext}^{***}_{\mathcal{A}^\vee_F}(\mathbb{M}^F_2, M).$$

We can rewrite the $\mathrm{E}_2$-page of this spectral sequence in a convenient way.

**Theorem 4.1.1.** *The* $\mathrm{E}_2$*-page of the* $\mathbf{mASS}^F_2(\mathrm{kq} \otimes \mathrm{kq})$ *is given by*

$$\mathrm{E}_2^{s,f,w} = \bigoplus_{k \geq 0} \Sigma^{4k,2k} \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(B_0(k)).$$

*Proof.* By (2.2.4) and (2.2.6), we have an isomorphism of $\mathcal{A}^\vee_F$-comodule algebras

$$\mathrm{H}_{**}(\mathrm{kq} \otimes \mathrm{kq}) \cong (\mathcal{A} \,/\!/\, \mathcal{A}(1))^\vee_F \otimes_{\mathbb{M}^F_2} (\mathcal{A} \,/\!/\, \mathcal{A}(1))^\vee_F.$$

Combined with a standard change of rings argument [Rav86, Appendix A], we can rewrite the $\mathrm{E}_2$-page of the $\mathbf{mASS}_2^F(\mathrm{kq}\otimes\mathrm{kq})$ as

$$\mathrm{E}_2^{s,f,w} = \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee_F}((\mathcal{A}\,/\!/\,\mathcal{A}(1))^\vee_F).$$

Finally, using (7.1.1) and the fact that $\mathbb{M}_2^F$ is compact as an $\mathcal{A}(1)^\vee_F$-comodule, we can rewrite this as

$$\mathrm{E}_2^{s,f,w} = \bigoplus_{k\geq 0}\Sigma^{4k,2k}\mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee_F}(B_0(k)),$$

proving the theorem. □

Thus, to compute the $\mathrm{E}_2$-page of the $\mathbf{mASS}_2^F(\mathrm{kq}\otimes\mathrm{kq})$, we must compute the trigraded groups $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_F}(B_0(k))$ for $k\geq 0$. Note that as the integral Brown–Gitler comodules are subcomodules of $(\mathcal{A}\,/\!/\,\mathcal{A}(0))^\vee_F$, we understand their structure as $\mathcal{A}(1)^\vee$-comodules.

**Remark 4.1.2.** Up until this point, all computation has been a precise motivic analogue of the classical work of Mahowald [Mah81]. However, Mahowald's computation of $\pi_*(\mathrm{bo}\otimes\mathrm{bo})$ heavily relies on the fact that the topological integral Brown–Gitler spectra $\mathrm{H}\mathbb{Z}_k^{\mathrm{top}}$ give rise to a topological splitting:

$$\mathrm{bo}\otimes\mathrm{bo}\simeq\bigoplus_{k\geq 0}\Sigma^{4k}\mathrm{bo}\otimes\mathrm{H}\mathbb{Z}_k^{\mathrm{top}}.$$

As we have noted, there are currently no known motivic analogues of integral Brown–Gitler spectra except in very special cases, and so we must use different methods.

## *4.2 The algebraic Atiyah–Hirzebruch spectral sequence*

We now wish to compute the groups $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0(k))$ appearing in the decomposition of the $\mathrm{E}_2$-page from (4.1.1). To do this, one can use a series of algebraic Atiyah–Hirzebruch spectral sequences (**aAHSS**) along with the short exact sequences of Brown–Gitler comodules from (2.3.5). This is originally inspired by the work of Behrens–Ormsby–Stapleton–Stojanoska on bo-cooperations [BOSS19, Section 3]. We explain the method here for clarity:

1. Compute $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0(1))$ (modulo $v_1$-torsion) using an **aAHSS**:

$$\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2) \otimes \mathbb{M}_2\{[1], [\overline{\xi}_1], [\overline{\tau}_1]\} \implies \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0(1));$$

2. Compute $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0^F(k))$ (modulo $v_1$-torsion) by induction and (2.3.5).

**Remark 4.2.1.** Since we are only concerned with using the kq-resolution to compute the $v_1$-periodic component of $\pi_{**}(\mathbb{S})$, passing to $v_1$-torsion free Ext groups loses no information. Additionally, we will see that there are multiple $v_1$-torsion classes arising in these spectral sequences, and ignoring these classes drastically simplifies our calculations.

We start with the case $k = 0$. We have that $B_0(0) \cong \mathbb{M}_2$, and so the first summand of the $\mathbf{mASS}_2(\mathrm{kq} \otimes \mathrm{kq})$ is given by $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2)$. Notice that since $\mathrm{H}_{**}(\mathrm{kq}) \cong (\mathcal{A} /\!/ \mathcal{A}(1))^\vee$ by (2.2.4), a standard change of rings argument shows that the $\mathrm{E}_2$-page of the $\mathbf{mASS}^F(\mathrm{kq})$ is given by

$$\mathrm{E}_2^{s,f,w} = \mathrm{Ext}^{s,f,w}_{\mathcal{A}^\vee}(\mathrm{H}_{**}(\mathrm{kq})) \cong \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2) \implies \pi^F_{s,w}(\mathrm{kq}).$$

There are explicit formulas for the $\mathrm{E}_2$-page in $\mathbb{C}$-motivic [IS11, Theorem 6.6], $\mathbb{R}$-motivic [Hil11, Theorem 4.8], [GHIR20, Theorem 6.2], and $\mathbb{F}_q$-motivic homotopy [Kyl15, Theorems 4.4.2, 4.4.3]. We can use $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2)$ as input for an algebraic Atiyah–Hirzebruch spectral sequence $\mathbf{aAHSS}(B_0^F(1))$ computing the next summand, $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0(1))$, which we now outline.

Filter $B_0(1) = \mathbb{M}_2\{1, \overline{\xi}_1, \overline{\tau}_1\}$ by topological degree. Let $\mathrm{F}_i B_0(1)$ denote the subspace spanned by generators of degree $\leq i$. This gives a finite filtration of the form

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{F}_0 B_0(1) \arrow[r] \arrow[d] & \mathrm{F}_1 B_0(1) \arrow[r] \arrow[d] & \mathrm{F}_2 B_0(1) \arrow[r] \arrow[d] & \mathrm{F}_3 B_0(1) = B_0(1) \arrow[d] \\ & \mathbb{M}_2\{[1]\} & 0 & \mathbb{M}_2\{[\overline{\xi}_1]\} & \mathbb{M}_2\{[\overline{\tau}_1]\} \end{tikzcd}$$

We refer to this filtration as the cellular filtration. The $\mathbf{aAHSS}(B_0^F(1))$ comes from the exact couple arising from applying the functor $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(-)$ to this filtration. This spectral sequence has signature

$$\mathrm{E}_1^{s,f,w,a} = \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2) \otimes \mathbb{M}_2\{[1], [\overline{\xi}_1], [\overline{\tau}_1]\} \implies \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(B_0(1)).$$

The $\mathrm{E}_1$-page of this spectral sequence consists of 3 copies of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2)$ in Atiyah–Hirzebruch filtration degrees 0, 2, and 3, which we index by $a$. Each Atiyah–Hirzebruch filtration piece is suitably shifted by the degree of the cell it is attached to. To make this precise, we can rewrite the $\mathrm{E}_1$-page as

$$\mathrm{E}_1 \cong \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2) \oplus \Sigma^{2,1}\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2) \oplus \Sigma^{3,1}\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2).$$

We give a heuristic for this spectral sequence in Figure 4.1.

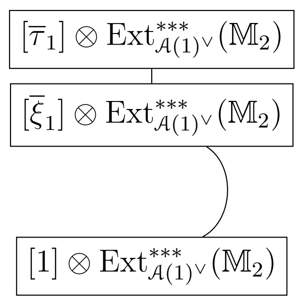


Figure 4.1: A heuristic for the $\mathbf{aAHSS}(B_0(1))$.

If $x \in \mathrm{Ext}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2)$ has degree $(s, f, w, a)$, then $|\Sigma^{p,q}x| = (s+p, f, w+q, a)$. We will let $\alpha[k]$ denote the copy of the class $\alpha \in \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2)$ in Atiyah–Hirzebruch filtration $k$. With this notation, differentials in the $\mathbf{aAHSS}(B_0(1))$ are of the form

$$d_r : \mathrm{E}_r^{s,f,w,a} \to \mathrm{E}_r^{s-1,f+1,w,a-r}.$$

Thus the $d_r$-differential lowers Atiyah–Hirzebruch filtration by $r$. In particular, the cell structure of $B_0(1)$ ensures that each differential may only be nonzero on one Atiyah–Hirzebruch filtration piece.

The differentials in this spectral sequence are computed by the cobar complex: to determine the differential on a class $\alpha[k]$, we first lift to a representative in the cobar complex for $\mathrm{F}_k B_0(1)$, compute the cobar differential on this representative, then project down to the appropriate filtration. In other words, the differentials witness the attaching maps in the algebraic cell complex for $B_0(1)$. The $d_1$-differential witnesses the $h_0$-attaching map between the cells $[\overline{\tau}_1]$ and $[\overline{\xi}_1]$, so is nonzero only on Atiyah–Hirzebruch filtration 3 and lands in Atiyah–Hirzebruch filtration 2. This gives the following formula:

**Proposition 4.2.2.** *In the* **aAHSS**$(B_0(1))$, *the* $d_1$*-differential is determined by*

$$d_1(\alpha[3]) = h_0\alpha[2].$$

Linearity over $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2)$ combined with the above formula gives the behavior of $d_1$. The $d_2$-differential witness the $h_1$-attaching map between the cells $[\overline{\xi}_1]$ and $[1]$, so is nonzero only on Atiyah–Hirzebruch filtration 2 and lands in Atiyah–Hirzebruch filtration 0. This gives the following formula:

**Proposition 4.2.3.** *In the* **aAHSS**$(B_0(1))$, *the* $d_2$*-differential is determined by*

$$d_2(\alpha[2]) = h_1\alpha[0].$$

Linearity combined with the above formula gives the behavior of $d_2$. By inspection, the only possible nonzero $d_3$-differential is between Atiyah–Hirzebruch filtrations 3 and 0, and will be given by the Massey product

$$d_3(\alpha[3]) = \langle \alpha, h_0, h_1 \rangle [0].$$

We will see that this differential is dependent on the base field $F$, as is evident from the formula. By inspection of the cellular filtration on $B_0(1)$, we see the following.

**Proposition 4.2.4.** *In the* **aAHSS**$(B_0^F(1))$, *we have that* $E_4 = E_\infty$.

The process of solving extension problems on the $\mathrm{E}_\infty$-page is more involved, so we postpone the discussion until we specialize to particular fields $F$.

**Remark 4.2.5.** Though not tantamount to computing the ring of cooperations, it still provides insight to compute $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0(1)^{\otimes i})$. Classically, Mahowald is able to express this group in terms of Adams covers of bo and bsp. We will see that a slightly more complicated story is true in the motivic case.

Now, we discuss the second step of our method. While one could analyze the algebraic Atiyah–Hirzebruch spectral sequence arising from the cellular filtration on $B_0(k)$, this spectral sequence becomes drastically more complicated due to the increasing complexity in cell structure. We will proceed in a different manner. After finding a formula for $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0(1))$, one can induct along the short exact sequences of (2.3.5) to compute $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0(k))$. If $k$ is even, then there is a short exact sequence

$$0 \to \Sigma^{2k,k} B_0(\tfrac{k}{2}) \to B_0(k) \to B_1(\tfrac{k}{2}-1) \otimes_{\mathbb{M}_2} (\mathcal{A}(1) \mathbin{/\!/} \mathcal{A}(0))^\vee \to 0.$$

Applying $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(-)$, we see that modulo $v_1$-torsion the connecting homomorphism is trivial. This reduces the computation of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0(k))$ to the kernel and cokernel in these short exact sequences in Ext. The kernels are taken care of by the inductive hypothesis, and the cokernels are a simple computation involving a change of rings isomorphism. This finishes the computation when $k$ is even.

When $k$ is odd, there is a short exact sequence

$$0 \to \Sigma^{2(k-1),k-1} B_0(\tfrac{k-1}{2}) \otimes_{\mathbb{M}_2} B_0(1) \to B_0(k) \to B_1(\tfrac{k-1}{2}-1) \otimes_{\mathbb{M}_2} (\mathcal{A}(1) \mathbin{/\!/} \mathcal{A}(0))^\vee \to 0.$$

As before, the long exact sequence induced by $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(-)$ splits into short exact sequences. The cokernels are again a simple computation, but the kernels require a more delicate analysis. Indeed, to compute $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}\big(B_0(\tfrac{k-1}{2}) \otimes_{\mathbb{M}_2} B_0(1)\big)$, we use another **aAHSS**. This spectral sequence comes from applying $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}\big(B_0(\tfrac{k-1}{2}) \otimes_{\mathbb{M}_2} -\big)$ to the cellular filtration on $B_0(1)$. The $\mathrm{E}_1$-page is given by

$$\mathrm{E}_1 = \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0(\tfrac{k-1}{2})) \otimes_{\mathbb{M}_2} \mathbb{M}_2\{[1], [\overline{\xi}_1], [\overline{\tau}_1]\}.$$

The induction hypothesis handles the left side of the tensor product, and a careful analysis of the spectral sequence gives us the Ext group in question. This finishes the computation when $k$ is odd.

By (4.1.1), assembling these results gives the $\mathrm{E}_2$-page of the **mASS**$(\mathrm{kq} \otimes \mathrm{kq})$, modulo $v_1$-torsion. We can now hope to analyze this spectral sequence by comparing with the classical case and base change. In [CQ21], this was shown to collapse on the $E_2$-page for all algebraically closed fields of characteristic 0, such as $\mathbb{C}$. We will show that this is also true over $\mathbb{R}$, but the situation is slightly more complicated over $\mathbb{F}_q$.

# Chapter 5

# COMPUTATIONS IN SH($\mathbb{C}$)

In this section, we review Culver–Quigley's [CQ21] computation of $\pi^{\mathbb{C}}_{**}(\mathrm{kq}\otimes\mathrm{kq})$. This serves both as an example of the methods described in Chapter 4 and as a place to recall results which will be useful in the sequel via base change. There are a few places where our arguments differ from the ones given in [CQ21]; we will explicitly indicate when this is the case.

## *5.1 Background*

Let $\mathbb{M}_2^{\mathbb{C}} = \mathbb{F}_2[\tau]$ denote the mod-2 motivic homology of a point [Voe03a, Corollary 6.10], where $|\tau| = (0,-1)$. Recall that as $\rho = 0$ in $\mathbb{M}_2^{\mathbb{C}}$, the dual Steenrod algebra is an honest Hopf algebra (see Remark 2.2.3). For the reader's convenience, we review its structure.

**Theorem 5.1.1** ([Voe03b, Theorem 12.6])**.** *The dual motivic Steenrod algebra $\mathcal{A}^\vee_{\mathbb{C}} = \pi^{\mathbb{C}}_{**}(\mathrm{H}\mathbb{F}_2\otimes\mathrm{H}\mathbb{F}_2)$ is the commutative Hopf algebra given by*

$$\mathcal{A}^\vee_{\mathbb{C}} = \mathbb{M}_2^{\mathbb{C}}[\overline{\xi}_1, \overline{\xi}_2, \dots, \overline{\tau}_0, \overline{\tau}_1, \dots]/(\overline{\tau}_i^2 = \tau\overline{\xi}_{i+1}),$$

*where $|\overline{\xi}_i| = (2^{i+1}-2, 2^i-1)$ and $|\overline{\tau}_i| = (2^{i+1}-1, 2^i-1)$. The unit map is given by inclusion, and the coproduct is determined by the formulae:*

$$\Delta(\overline{\xi}_k) = \sum_{i+j=k} \overline{\xi}_i \otimes \overline{\xi}_j^{2^i}, \quad \Delta(\overline{\tau}_i) = \sum_{i+j=k} \overline{\tau}_i \otimes \overline{\xi}_j^{2^i} + 1\otimes\overline{\tau}_k.$$

**Remark 5.1.2.** There is a well understood relationship between $\mathcal{A}^\vee_{\mathbb{C}}$ and the classical dual Steenrod algebra $\mathcal{A}^\vee$. Complex Betti realization gives a map $\mathrm{Be}^{\mathbb{C}} : \mathcal{A}^\vee_{\mathbb{C}} \to \mathcal{A}^\vee$ determined by

$$\mathrm{Be}^{\mathbb{C}}(\tau) = 1, \quad \mathrm{Be}^{\mathbb{C}}(\overline{\xi}_i) = \overline{\xi}_i^2, \quad \mathrm{Be}^{\mathbb{C}}(\overline{\tau}_i) = \overline{\xi}_i$$

This extends to an isomorphism in Ext [DI10]:

$$\mathrm{Ext}^{***}_{\mathcal{A}^\vee_{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}) \otimes_{\mathbb{M}_2^{\mathbb{C}}} \mathbb{M}_2^{\mathbb{C}}[\tau^{-1}] \cong \mathrm{Ext}^{***}_{\mathcal{A}^\vee}(\mathbb{F}_2) \otimes \mathbb{F}_2[\tau^{\pm 1}].$$

This observation is at the heart of the theory of synthetic spectra [Pst23], and will allow us to determine motivic information from classical information.

In addition to an understanding of the homology $\mathrm{H}_{**}(\mathrm{kq}) \cong (\mathcal{A} \mathbin{/\!/} \mathcal{A}(1))^\vee_{\mathbb{C}}$, we also have a computation of the homotopy groups $\pi^{\mathbb{C}}_{**}(\mathrm{kq})$ [IS11].

**Theorem 5.1.3** ([IS11, Theorem 4.10])**.** *The* $\mathbf{mASS}^{\mathbb{C}}(\mathrm{kq})$ *has* $\mathrm{E}_2$*-page*

$$\mathrm{E}_2^{s,f,w} = \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}) = \frac{\mathbb{M}_2^{\mathbb{C}}[h_0, h_1, a, b]}{(h_0h_1, \tau h_1^3, h_1 a, a^2 = h_0^2 b)},$$

*where* $|h_0| = (0,1,0)$*,* $|h_1| = (1,1,1)$*,* $|a| = (4,3,2)$ *and* $|b| = (8,4,4)$*. Moreover, the spectral sequence collapses at* $\mathrm{E}_2$ *with no extensions, hence there is an isomorphism*

$$\pi^{\mathbb{C}}_{**}(\mathrm{kq}) \cong \frac{\mathbb{Z}_2[\tau, h_1, a, b]}{(2h_1, \tau h_1^3, h_1 a, a^2 = 4b)},$$

*where the stem and filtration of the generators match those above.*

Following our discussion in Chapter 4, the first step towards computing the kq-resolution is computing the ring of cooperations $\pi^{\mathbb{C}}_{**}(\mathrm{kq} \otimes \mathrm{kq})$. Recall from Theorem 4.1.1 that the $\mathbf{mASS}^{\mathbb{C}}(\mathrm{kq} \otimes \mathrm{kq})$ has $\mathrm{E}_2$-page given by

$$\mathrm{E}_2^{s,f,w} = \bigoplus_{k \geq 0} \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(\Sigma^{4k,2k} B_0^{\mathbb{C}}(k)),$$

where $B_0^{\mathbb{C}}(k)$ are the $\mathbb{C}$-motivic integral motivic Brown–Gitler comodules. Therefore, we must compute the trigraded groups $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(k))$ for $k \geq 0$. Since $B_0^{\mathbb{C}}(0) = \mathbb{M}_2^{\mathbb{C}}$, the first summand is computed by Theorem 5.1.3. We move then to compute $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(1))$ by the algebraic Atiyah–Hirzebruch spectral sequence outlined in Section 4.2.

## 5.2 $\mathbf{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(1)^{\otimes i})$

Recall that the $\mathbf{aAHSS}(B_0^{\mathbb{C}}(1))$ takes the form

$$E_1^{s,f,w,a} = \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}) \otimes_{\mathbb{M}_2^{\mathbb{C}}} \mathbb{M}_2^{\mathbb{C}}\{[1],[\overline{\xi}_1],[\overline{\tau}_1]\} \implies \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(1)),$$

with differentials $d_r : E_r^{s,f,w,a} \to E_r^{s-1,f+1,w,a-r}$. We will let $\alpha[i]$ denote the copy of a class $\alpha \in \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}})$ in Atiyah–Hirzebruch filtration $i$. We have discussed in Section 4.2 the differentials for this spectral sequence. Namely, the $d_1$-differential is only nonzero on Atiyah–Hirzebruch filtration 3, where

$$d_1(\alpha[3]) = h_0\alpha[2],$$

and the $d_2$-differential is only nonzero on Atiyah–Hirzebruch filtration 2, where

$$d_2(\alpha[2]) = h_1\alpha[0].$$

For degree reasons we have that $d_3(\alpha) = 0$ for all remaining classes $\alpha[3] \in \mathrm{E}_3^{s,f,w,3}$. Thus the spectral sequence collapses on $\mathrm{E}_3 = \mathrm{E}_\infty$.

We depict the $\mathrm{E}_1$-page in Figure 5.1, the $\mathrm{E}_2$-page in Figure 5.2, and the $\mathrm{E}_3 = \mathrm{E}_\infty$-page in Figure 5.3. Charts are written in $(s,f)$-grading, with motivic weight and Atiyah–Hirzebruch filtration suppressed. A black $\bullet$ represents $\mathbb{M}_2^{\mathbb{C}}$. A hollow $\circ$ represents $\mathbb{F}_2$. A black vertical line represents multiplication by $h_0$. A black diagonal line of slope 1 represents multiplication by $h_1$. Differentials are blue, linear with respect to the underlying $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}})$-module structure, and preserve motivic weight. A dashed blue line represents a differential where either source or target is $\tau$-torsion.

The hidden extensions on the $\mathrm{E}_\infty$-page can be resolved using complex Betti realization. Recall from Section 2.1 that this is a functor $\mathrm{Be}^{\mathbb{C}} : \mathrm{SH}(\mathbb{C}) \to \mathrm{Sp}$. For $\mathrm{X} \in \mathrm{SH}(\mathbb{C})$, this gives a map in homotopy

$$\pi^{\mathbb{C}}_{s,w}\mathrm{X} \to \pi_s\mathrm{Be}^{\mathbb{C}}\mathrm{X}.$$

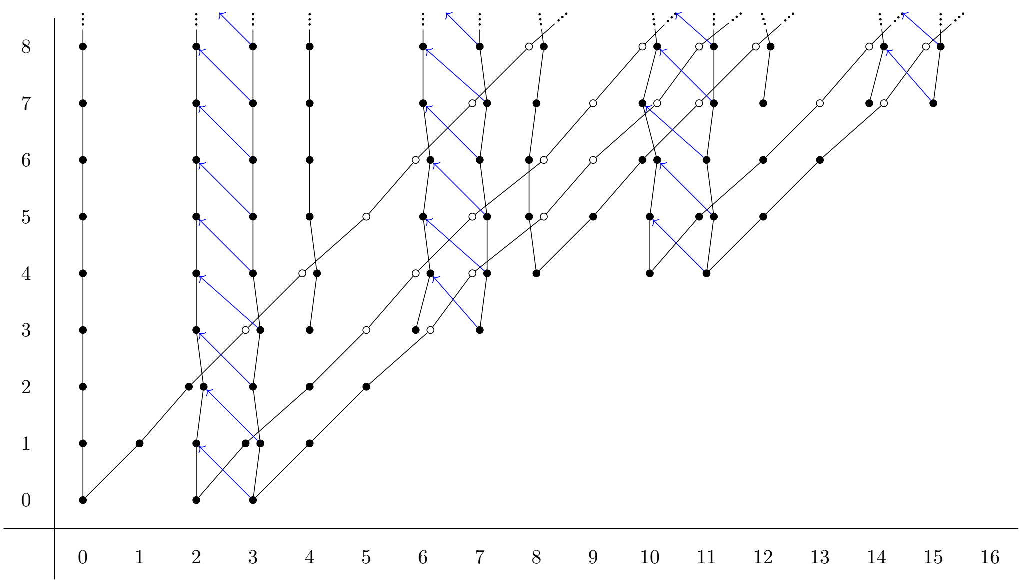


Figure 5.1: The $E_1$-page of the $\mathbf{aAHSS}(B_0^{\mathbb{C}}(1))$

In particular, there is a map of dual Steenrod algebras $\mathcal{A}_{\mathbb{C}}^{\vee} \to \mathcal{A}^{\vee}$ sending $\overline{\xi}_i \mapsto \overline{\xi}_i^2$ and $\overline{\tau}_i \mapsto \overline{\xi}_i$. Additionally, there is an isomorphism of $\mathcal{A}(1)_{\mathbb{C}}^{\vee}$-comodules

$$(B_0^{\mathbb{C}}(1))[\tau^{-1}] \cong B_0(1)[\tau^{\pm 1}],$$

where $B_0(1)$ is the classical Brown–Gitler comodule. This induces a map of Ext groups

$$\mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(B_0^{\mathbb{C}}(k)) \to \mathrm{Ext}^{s,f}_{\mathcal{A}(1)^{\vee}}(B_0(k)),$$

and allows one to lift hidden extensions from the classical case in [Mah81] to the $\mathbb{C}$-motivic case.

The following is a motivic analogue of a classical result.

**Corollary 5.2.1** ([CQ21, Corollary 3.31]). *The* $\mathbf{mASS}^{\mathbb{C}}(\mathrm{ksp})$ *takes the form*

$$\mathrm{E}_2^{s,f,w} = \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)_{\mathbb{C}}^{\vee}}(\mathbb{M}_2^{\mathbb{C}}, B_0^{\mathbb{C}}(1)) \implies \pi_{**}^{\mathbb{C}}(\mathrm{ksp}).$$

*Moreover, this spectral sequence collapes on the* $\mathrm{E}_2$*-page.*

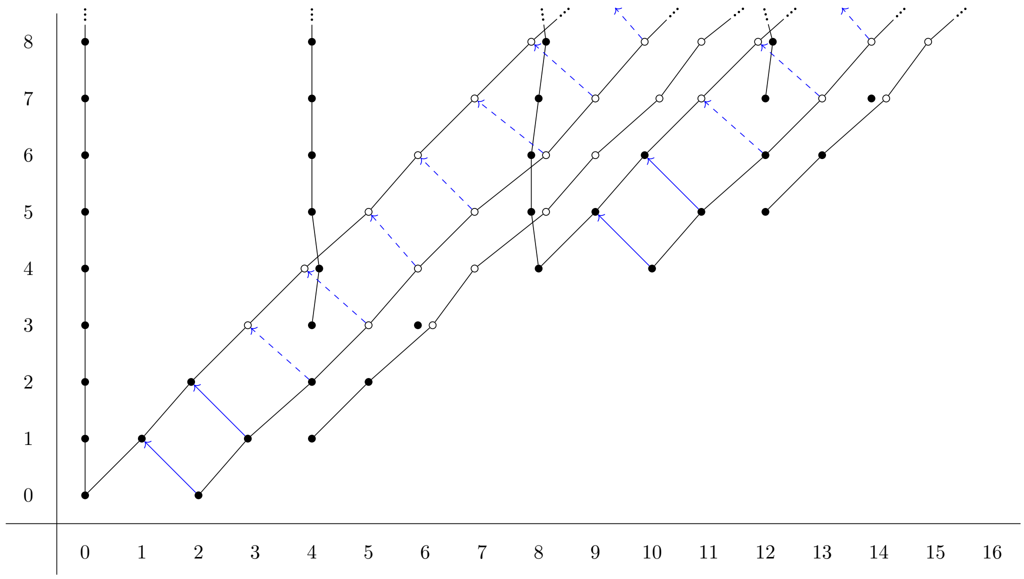


Figure 5.2: The $E_2$-page of the $\mathbf{aAHSS}(B_0^{\mathbb{C}}(1))$

*Proof.* By Theorem 3.0.1, there is an equivalence of motivic spectra

$$\mathrm{ksp} \cong \mathrm{H}\mathbb{Z}_1^{\mathbb{C}} \otimes \mathrm{kq},$$

where $\mathrm{H}\mathbb{Z}_1^{\mathbb{C}}$ is the motivic spectrum constructed in Section 2.3. The Künneth spectral sequence for mod-2 homology collapses for $\mathrm{H}\mathbb{Z}_1^{\mathbb{C}} \otimes \mathrm{kq}$, giving an isomorphism

$$\mathrm{H}_{**}(\mathrm{ksp}) \cong B_0^{\mathbb{C}}(1) \otimes_{\mathbb{M}_2^{\mathbb{C}}} (\mathcal{A} \,/\!/\, \mathcal{A}(1))_{\mathbb{C}}^{\vee}.$$

After a change of rings isomorphism, the $E_2$-page for the $\mathbf{mASS}^{\mathbb{C}}(\mathrm{ksp})$ takes the form

$$E_2 = \mathrm{Ext}^{***}_{\mathcal{A}(1)^{\vee}_{\mathbb{C}}}(B_0^{\mathbb{C}}(1)) \implies \pi^{\mathbb{C}}_{**}(\mathrm{ksp}).$$

This Ext group was computed by the $\mathbf{aAHSS}(B_0^{\mathbb{C}}(1))$ and is depicted in Figure 5.3. There are no differentials for degree reasons, so the spectral sequence collapses and gives the result. $\square$

Classically, one can express the groups $\mathrm{Ext}^{**}_{\mathcal{A}(1)^{\vee}}(B_0(1)^{\otimes i})$ in terms of Adams covers of bo and bsp. In motivic homotopy theory, this is not quite the case. Over $\mathbb{C}$, Culver–Quigley

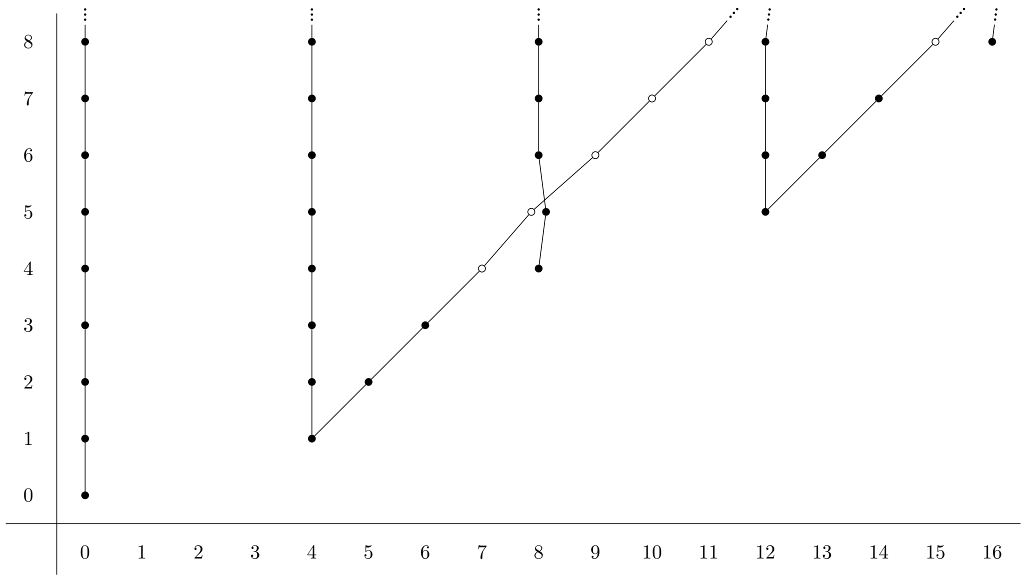


Figure 5.3: The $E_\infty$-page of the $\mathbf{aAHSS}(B_0^{\mathbb{C}}(1))$ with hidden extensions.

[CQ21] show that one cannot express the groups $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(1)^{\otimes i})$ in terms of Adams covers of kq and ksp. Instead, there is a periodic family of groups which will determine $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(1))$ composed of Adams covers of bo, bsp, kq and ksp.

In what follows, we will use the notation defined in Section 1.2.1. For $i \geq 0$, let $Z_i^{\mathbb{C}}$ be the trigraded group defined as follows:

- When $i \equiv 0\,(4)$, let

$$Z_i^{\mathbb{C}} = \bigoplus_{j=0}^{i/2-1} \Sigma^{4j,2j}\mathbb{M}_2^{\mathbb{C}}[h_0] \oplus \Sigma^{2i,i}\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}})$$

- When $i \equiv 1\,(4)$, let

$$Z_i^{\mathbb{C}} = \bigoplus_{j=0}^{(i-1)/2-1} \Sigma^{4j,2j}\mathbb{M}_2^{\mathbb{C}}[h_0] \oplus \Sigma^{2i-1,i-1}\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(1));$$

- When $i \equiv 2\,(4)$, let

$$Z_i^{\mathbb{C}} = \bigoplus_{j=0}^{i/2-1} \Sigma^{4j,2j}\mathbb{M}_2^{\mathbb{C}}[h_0] \oplus \Sigma^{2i-2,i}\mathbb{M}_2^{\mathbb{C}} \oplus \Sigma^{2i,i}\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(1))\langle 1\rangle;$$

- When $i \equiv 3\,(4)$, let

$$Z_i^{\mathbb{C}} = \bigoplus_{j=0}^{(i-1)/2} \Sigma^{4j,2j}\mathbb{M}_2^{\mathbb{C}}[h_0] \oplus \Sigma^{2i-1,i}\mathbb{M}_2^{\mathbb{C}}[h_1]/(h_1^2) \oplus \Sigma^{2i+2,i+1}\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(1))\langle 2\rangle.$$

Recall that we are only interested in information up to $v_1$-torsion (see Remark 4.2.1).

**Proposition 5.2.2** ([CQ21, Lemma 3.36])**.** *There is an isomorphism of $\mathcal{A}(1)^\vee_{\mathbb{C}}$-comodules and $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}})$-modules:*

$$\frac{\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(1)^{\otimes i})}{v_1\text{-torsion}} \cong Z_i^{\mathbb{C}}.$$

*Proof.* By applying the functor $\mathrm{Ext}^{***}_{\mathcal{A}^\vee_{\mathbb{C}}}\left(B_0^{\mathbb{C}}(1)^{\otimes i} \otimes_{\mathbb{M}_2^{\mathbb{C}}} -\right)$ to the cellular filtration of $B_0^{\mathbb{C}}(1)$, we may inductively compute these Ext groups by an **aAHSS**, where the base case was computed in Corollary 5.2.1. This spectral sequence takes the form

$$E_1 = \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(1)^{\otimes i}) \otimes_{\mathbb{M}_2^{\mathbb{C}}} \mathbb{M}_2^{\mathbb{C}}\{1, \overline{\xi}_1, \overline{\tau}_1\} \implies \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(1)^{\otimes i+1}).$$

The differentials take the same form as those in $\mathbf{aAHSS}(B_0^{\mathbb{C}}(1))$, namely we have $d_1(\alpha[3]) = h_0\alpha[2]$ and $d_2(\alpha[2]) = h_1\alpha[0]$, where $\alpha \in \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(1)^{\otimes i})$. For degree reasons, there is no $d_3$-differential. Hidden extensions may be obtained by comparison with the classical case, using the algebra map

$$\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(1)^{\otimes i}) \to \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0(1)^{\otimes i}).$$

This resolves all hidden extensions and concludes the proof. $\square$

### 5.3 $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(k))$ and the ring of cooperations

Next, we compute $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(k))$ for all $k \geq 1$.

**Theorem 5.3.1** ([CQ21, Theorem 3.38]). *There is an isomorphism of $\mathcal{A}(1)^\vee_{\mathbb{C}}$-comodules and $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}})$-modules:*

$$\frac{\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(k))}{v_1\text{-torsion}} \cong \Sigma^{4k-4,2k-2} Z^{\mathbb{C}}_{\alpha(k)} \oplus \bigoplus_{j=0}^{4k-8} \Sigma^{4j,2j}\mathbb{M}_2^{\mathbb{C}}[h_0],$$

*where $\alpha(k)$ is the number of 1's in the dyadic expansion of $k$.*

*Proof.* For notational convenience, all Ext groups in this proof will be implicitly computed modulo $v_1$-torsion. We will induce on $k$. The case of $k = 1$ was shown in Corollary 5.2.1. Now, suppose the theorem is true for all $i < k$.

Suppose that $k$ is even. Proposition 2.3.5 gives a short exact sequence of $\mathcal{A}(1)^\vee_{\mathbb{C}}$-comodules

$$0 \to \Sigma^{2k,k} B_0^{\mathbb{C}}(\tfrac{k}{2}) \to B_0^{\mathbb{C}}(k) \to B_1^{\mathbb{C}}(\tfrac{k}{2}-1) \otimes_{\mathbb{M}_2^{\mathbb{C}}} (\mathcal{A}(1) \,/\!/\, \mathcal{A}(0))^\vee_{\mathbb{C}} \to 0. \tag{5.3.1}$$

Applying the functor $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(-)$ gives a long exact sequence of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}})$-modules. Moreover, after killing $v_1$-torsion the connecting homomorphism is trivial, giving short exact sequences whose middle term is $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(k))$. Therefore, this Ext group decomposes into the Ext groups of the kernel and cokernel of the original short exact sequence (5.3.1). The kernel is handled by the inductive hypothesis: we have

$$\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(\Sigma^{2k,k} B_0^{\mathbb{C}}(\tfrac{k}{2})) \cong \Sigma^{2k,k}\left(\Sigma^{2k-4,k-2} Z^{\mathbb{C}}_{\alpha(k/2)} \oplus \bigoplus_{j=0}^{2k-8} \Sigma^{4j,2j}\mathbb{M}_2^{\mathbb{C}}[h_0]\right).$$

We can use a change of rings isomorphism to rewrite the cokernel as

$$\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}\left(B_1^{\mathbb{C}}(\tfrac{k}{2}-1) \otimes_{\mathbb{M}_2^{\mathbb{C}}} (\mathcal{A}(1) \,/\!/\, \mathcal{A}(0))^\vee_{\mathbb{C}}\right) \cong \mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee_{\mathbb{C}}}(B_1^{\mathbb{C}}(\tfrac{k}{2}-1)).$$

Since $\mathcal{A}(0)^\vee_{\mathbb{C}} \cong \mathbb{M}_2^{\mathbb{C}}[\overline{\tau}_0]/(\overline{\tau}_0^2)$ is exterior, the Ext groups in question are polynomial. Modulo $v_1$-torsion, we are only left with $h_0$-towers:

$$\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee_{\mathbb{C}}}(B_1^{\mathbb{C}}(\tfrac{k}{2}-1)) \cong \bigoplus_{j=0}^{k/2-1} \Sigma^{4j,2j}\mathbb{M}_2^{\mathbb{C}}[h_0].$$

Putting these pieces together with some reindexing on the $h_0$-towers and using the fact that $\alpha(k/2) = \alpha(k)$ gives the result.

Suppose now that $k$ is odd. Again, Proposition 2.3.5 gives a short exact sequence of $\mathcal{A}(1)^{\vee}_{\mathbb{C}}$-comodules

$$0 \to \Sigma^{2(k-1),k-1} B_0^{\mathbb{C}}(\tfrac{k-1}{2}) \otimes_{\mathbb{M}_2^{\mathbb{C}}} B_0^{\mathbb{C}}(1) \to B_0^{\mathbb{C}}(k) \to B_1^{\mathbb{C}}(\tfrac{k-1}{2} - 1) \otimes_{\mathbb{M}_2^{\mathbb{C}}} (\mathcal{A}(1) \,/\!/\, \mathcal{A}(0))^{\vee}_{\mathbb{C}} \to 0. \tag{5.3.2}$$

Applying the functor $\mathrm{Ext}^{***}_{\mathcal{A}(1)^{\vee}_{\mathbb{C}}}(-)$ gives a long exact sequence of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^{\vee}_{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}})$-modules, and as in the even case, the connecting homomorphism is trivial modulo $v_1$-torsion. Thus, the Ext group we are interested in decomposes into the Ext groups of the kernel and cokernel of 5.3.2. In this case, the cokernel is simpler to compute. A change of rings isomorphism gives

$$\mathrm{Ext}^{***}_{\mathcal{A}(1)^{\vee}_{\mathbb{C}}}\left(B_1^{\mathbb{C}}(\tfrac{k-1}{2} - 1) \otimes_{\mathbb{M}_2^{\mathbb{C}}} (\mathcal{A}(1) \,/\!/\, \mathcal{A}(0))^{\vee}_{\mathbb{C}}\right) \cong \mathrm{Ext}^{***}_{\mathcal{A}(0)^{\vee}_{\mathbb{C}}}(B_1^{\mathbb{C}}(\tfrac{k-1}{2} - 1)).$$

By the same argument as before, modulo $v_1$-torsion this Ext group consists of purely $h_0$-towers:

$$\mathrm{Ext}^{***}_{\mathcal{A}(0)^{\vee}_{\mathbb{C}}}(B_1^{\mathbb{C}}(\tfrac{k-1}{2} - 1)) \cong \bigoplus_{j=0}^{(k-1)/2-1} \Sigma^{4j,2j}\mathbb{M}_2^{\mathbb{C}}[h_0].$$

The Ext group of the kernel may be computed using an algebraic Atiyah–Hirzebruch spectral sequence $\mathbf{aAHSS}\left(B_0^{\mathbb{C}}(\tfrac{k-1}{2}) \otimes_{\mathbb{M}_2^{\mathbb{C}}} B_0^{\mathbb{C}}(1)\right)$. This has signature

$$E_1 = \mathrm{Ext}^{***}_{\mathcal{A}(1)^{\vee}_{\mathbb{C}}}(B_0^{\mathbb{C}}(\tfrac{k-1}{2})) \otimes_{\mathbb{M}_2^{\mathbb{C}}} \mathbb{M}_2^{\mathbb{C}}\{1, \overline{\xi}_1, \overline{\tau}_1\} \implies \mathrm{Ext}^{***}_{\mathcal{A}(1)^{\vee}_{\mathbb{C}}}\left(B_0^{\mathbb{C}}(\tfrac{k-1}{2}) \otimes_{\mathbb{M}_2^{\mathbb{C}}} B_0^{\mathbb{C}}(1)\right).$$

By induction, we may rewrite $\mathrm{Ext}^{***}_{\mathcal{A}(1)^{\vee}_{\mathbb{C}}}(B_0^{\mathbb{C}}(\tfrac{k-1}{2}))$ as

$$\mathrm{Ext}^{***}_{\mathcal{A}(1)^{\vee}_{\mathbb{C}}}(B_0^{\mathbb{C}}(\tfrac{k-1}{2})) \cong \Sigma^{2k-6,k-3} Z^{\mathbb{C}}_{\alpha(k)-1} \oplus \bigoplus_{j=0}^{2k-10} \Sigma^{4j,2j}\mathbb{M}_2^{\mathbb{C}}[h_0],$$

using that $\alpha(\tfrac{k-1}{2}) = \alpha(k) - 1$. This splitting leads to a splitting of the $E_1$-page of the algebraic Atiyah–Hirzebruch spectral sequence, and we may analyze each summand individually. On the left-hand summand, Proposition 5.2.2 implies that result of the spectral sequence is isomorphic to $\Sigma^{2k-2,k-1} Z^{\mathbb{C}}_{\alpha(k)}$. On the right-hand summand, the spectral sequence collapses on the $E_1$-page, giving back the original summand modulo $v_1$-torsion. Thus, after reindexing the $h_0$-towers, the Ext groups of the kernel and cokernel and right-hand side of the Ext groups assemble to give the result.

□

**Remark 5.3.2.** The above proof is where our argument differs most from the one given in [CQ21]. We explain these differences here. Classically, Mahowald [Mah81] showed that the groups $\mathrm{Ext}^{**}_{\mathcal{A}(1)^\vee}(B_0(k))$ can be expressed solely in terms of bo and bsp. To be precise, for a spectrum X, let $\mathrm{X}^{\langle n\rangle}$ denote the $n$-th Adams cover of $X$, so that

$$\mathrm{X} \leftarrow \mathrm{X}^{\langle 1\rangle} \leftarrow \mathrm{X}^{\langle 2\rangle} \leftarrow \mathrm{X}^{\langle 3\rangle} \leftarrow \cdots$$

is a minimal Adams resolution of X. Then there are isomorphisms:

$$\frac{\mathrm{Ext}^{**}_{\mathcal{A}(1)^\vee}(B_0(k))}{v_1\text{-torsion}} \cong \begin{cases} \mathrm{Ext}^{**}_{\mathcal{A}(1)^\vee}(\mathrm{H}_*(\mathrm{bo}^{\langle 2k-\alpha(k)\rangle})) & k \equiv 0\,(2), \\ \mathrm{Ext}^{**}_{\mathcal{A}(1)^\vee}(\mathrm{H}_*(\mathrm{bsp}^{\langle 2k-\alpha(k)-1\rangle})) & k \equiv 1\,(2). \end{cases}$$

Each of the above groups is describable by a direct sum of $h_0$-towers followed by some shift of $\mathrm{Ext}^{**}_{\mathcal{A}(1)^\vee}(\mathbb{F}_2)$. In particular, if $x \in \mathrm{Ext}^{s,f}_{\mathcal{A}(1)^\vee}(B_0(k))$ is in stem $s < 4(2k-\alpha(k))$, then $x$ does not support $h_1$-multiplication.

The results of Culver–Quigley assert that over $\mathbb{C}$ there is an isomorphism

$$\frac{\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(k))}{v_1\text{-torsion}} \cong Z^{\mathbb{C}}_{2k-\alpha(k)}.$$

However, this is not quite correct. The group $Z^{\mathbb{C}}_{2k-\alpha(k)}$ has classes in small stems which support $h_1$-multiplication, while the decomposition given by the short exact sequences of Proposition 2.3.5 do not allow for this. For example, for $k=2$ we have $2k-\alpha(k) = 4-1 = 3$, hence Culver–Quigley claim that $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(2))$ is isomorphic to

$$Z_3^{\mathbb{C}} = \mathbb{M}_2^{\mathbb{C}}[h_0] \oplus \Sigma^{4,2}\mathbb{M}_2^{\mathbb{C}}[h_0] \oplus \Sigma^{5,3}\mathbb{M}_2^{\mathbb{C}}[h_1]/(h_1^2) \oplus \Sigma^{8,4}\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(1))\langle 2\rangle.$$

We see that there is a class in stem 5 which supports $h_1$-multiplication. Alternatively, using the short exact sequence of Proposition 2.3.5 we see that the group $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(2))$ is isomorphic to the following the direct sum:

$$\Sigma^{8,4}\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(1)) \oplus \mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee_{\mathbb{C}}}(B_1^{\mathbb{C}}(0)) \cong \Sigma^{8,4}Z_1^{\mathbb{C}} \oplus \mathbb{M}_2^{\mathbb{C}}[h_0],$$

which agrees with the description in Proposition 5.2.2. This decomposition does not admit any $h_1$-multiplication in stem 5; in fact, there are no classes in stem 5. This type of difference extends to all values of $k \geq 2$, showing that the formula given in [CQ21, Theorem3.38] is incorrect. However, this observation is actually in line with the rest of the work done in Culver–Quigley and doesn't affect any end results. More precisely:

(1) It was shown in [CQ21, Lemma 3.38] that the groups $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(1)^{\otimes i})$ are not expressible in terms of Adams covers of kq and ksp. The purpose of the groups $Z_i^{\mathbb{C}}$ is exactly to account for this failure, mixing in an appropriate trigraded version of $\mathrm{Ext}^{**}_{\mathcal{A}(1)^\vee}(\mathrm{H}_*(\mathrm{bo}^{\langle i \rangle}))$ and $\mathrm{Ext}^{**}_{\mathcal{A}(1)^\vee}(\mathrm{H}_*(\mathrm{bsp}^{\langle i \rangle}))$. The difference being observed in the computation of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(k))$ can be accounted for by noting that this group *also* cannot be expressed using just $Z^{\mathbb{C}}_{2k-\alpha(k)}$. Rather, we must express these groups using shifts of $Z^{\mathbb{C}}_{\alpha(k)}$ and an appropriate trigraded version of the $h_0$-towers found in the classical groups $\mathrm{Ext}^{**}_{\mathcal{A}(1)^\vee}(\mathbb{F}_2)$; this last term is precisely what is given by $\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee_{\mathbb{C}}}(B_1^{\mathbb{C}}(\frac{k-1}{2}))$ in the case that $k \equiv 0\,(2)$, and by $\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee_{\mathbb{C}}}(B_1^{\mathbb{C}}(\frac{k-1}{2}-1))$ in the case that $k \equiv 1\,(2)$.

(2) Apart from this discrepancy, the contents of [CQ21] are unaffected. In particular, we will see that the $\mathbf{mASS}^{\mathbb{C}}(\mathrm{kq} \otimes \mathrm{kq})$ collapses on the $E_2$-page by using the same arguments as employed by Culver–Quigley.

The following is now immediate from Theorem 4.1.1.

**Proposition 5.3.3** ([CQ21, Proposition 3.41])**.** *The $E_2$-page of the $\mathbf{mASS}^{\mathbb{C}}(\mathrm{kq} \otimes \mathrm{kq})$ is given, modulo $v_1$-torsion, by*

$$E_2 = \bigoplus_{k \geq 0} \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(\Sigma^{4k,2k} B_0^{\mathbb{C}}(k)) \cong \bigoplus_{k \geq 0} \left( \Sigma^{4k,2i} Z^{\mathbb{C}}_{\alpha(k)} \oplus \bigoplus_{j=0}^{4k-8} \Sigma^{4j,2j} \mathbb{M}_2^{\mathbb{C}}[h_0] \right).$$

Recall from [Mah81, Theorem 2.9] that the $\mathbf{ASS}(\mathrm{bo} \otimes \mathrm{bo})$ collapses on $E_2$.

**Theorem 5.3.4** ([CQ21, Cor 3.43])**.** *The $\mathbf{mASS}^{\mathbb{C}}(\mathrm{kq} \otimes \mathrm{kq})$ collapses on the $E_2$-page.*

*Proof.* Betti realization sends the $\mathbf{mASS}^{\mathbb{C}}(\mathrm{kq} \otimes \mathrm{kq})$ to the $\mathbf{ASS}(\mathrm{bo} \otimes \mathrm{bo})$. At the level of $E_2$-pages, this is a map

$$\mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(\mathrm{H}_{**}(\mathrm{kq} \otimes \mathrm{kq})) \to \mathrm{Ext}^{s,f}_{\mathcal{A}(1)^\vee}(\mathrm{H}_*(\mathrm{bo} \otimes \mathrm{bo})),$$

obtained by inverting $\tau$ on the $E_2$-page of the $\mathbf{mASS}^{\mathbb{C}}(\mathrm{kq} \otimes \mathrm{kq})$ and setting $\tau = 1$. Since the $\mathbf{ASS}(\mathrm{bo} \otimes \mathrm{bo})$ collapses on $E_2$, we must have that there are no motivic differentials where both source and target are $\tau$-torsion free. However, our description of the $E_2$-page of the $\mathbf{mASS}^{\mathbb{C}}(\mathrm{kq} \otimes \mathrm{kq})$ shows that any $\tau$-torsion class must be $h_1$-torsion free, hence is some $h_1$-tower. For degree reasons, there can be no such differentials between these towers, concluding the proof. $\square$

# Chapter 6

# COMPUTATIONS IN SH($\mathbb{R}$)

In this chapter, we compute the homotopy ring of cooperations $\pi^{\mathbb{R}}_{**}(\mathrm{kq} \otimes \mathrm{kq})$. In Section 6.1, we recall some relevant $\mathbb{R}$-motivic algebra. In Section 6.2, we compute the group $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B^{\mathbb{R}}_0(1)^{\otimes i})$ by a series of algebraic Atiyah–Hirzebruch spectral sequences. Motivated by the presentation of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}^{\mathbb{R}}_2)$ given in [GHIR20], we organize these computations by coweight modulo 4. Consequently, our results are best displayed in multiple charts which should be read simultaneously. Throughout, we use the notation of Section 6.1.

## 6.1 Algebraic preliminaries and comparisons

In this section, we recall the particular details of the $\mathbb{R}$-motivic dual Steenrod algebra $\mathcal{A}^\vee_{\mathbb{R}}$ and introduce notation for modules to be used throughout. Then, we give a brief overview of the relationship between $\mathbb{C}$-motivic and $\mathbb{R}$-motivic homotopy and compare the respective kq-resolutions.

### 6.1.1 *The dual motivic Steenrod algebra and* $\mathrm{Ext}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}^{\mathbb{R}}_2)$

Let $\mathbb{M}^{\mathbb{R}}_2 = \mathbb{F}_2[\rho, \tau]$ denote the mod-2 motivic homology of a point [Voe03a, Corollary 6.10], where $|\tau| = (0, -1)$ and $|\rho| = (-1, -1)$. Recall from Theorem 2.2.1 that the dual motivic Steenrod algebra $\mathcal{A}^\vee_{\mathbb{R}}$ takes the form

$$\mathcal{A}^\vee_{\mathbb{R}} = \mathbb{M}^{\mathbb{R}}_2[\overline{\xi}_1, \overline{\xi}_2, \ldots, \overline{\tau}_0, \overline{\tau}_1, \ldots]/(\overline{\tau}_i^2 = \rho\overline{\tau}_{i+1} + \rho\overline{\tau}_0\overline{\xi}_{i+1} + \tau\overline{\xi}_{i+1}).$$

Theorem 4.1.1 asserts that the $\mathbf{mASS}(\mathrm{kq} \otimes \mathrm{kq})$ has $E_2$-page given by

$$E_2^{s,f,w} = \bigoplus_{k \geq 0} \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\Sigma^{4k,2k} B^{\mathbb{R}}_0(k)),$$

where $B_0^{\mathbb{R}}(k)$ are the $\mathbb{R}$-motivic integral motivic Brown–Gitler comodules. Therefore, to compute this $E_2$-page we must compute the trigraded groups $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(k))$ for $k \geq 0$. Since $B_0^{\mathbb{R}}(0) = \mathbb{M}_2^{\mathbb{R}}$, the first summand is isomorphic to the $E_2$-page of the **mASS**(kq). This was computed by Hill in [Hil11, Theorem 5.6]. We use a presentation adopted by Guillou–Hill–Isaksen–Ravenel [GHIR20, Theorem 6.2].

**Theorem 6.1.1** ([Hil11; GHIR20] )**.** *The* **mASS**(kq) *collapses on the $E_2$-page, which is given by*

$$\mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}}) = \mathbb{F}_2[\tau^4, \rho, h_0, h_1, a, b, \tau^2 h_0, \tau h_1, b, \tau^2 a]/I.$$

*The degrees of the generators are listed in Table 6.1 and the ideal of relations $I$ is detailed in Table 6.2.*

We organize Table 6.1 and Table 6.2 by coweight modulo 4.

| Generator | $(s, f, w)$ | Coweight |
|---|---|---|
| $\rho$ | $(-1, 0, -1)$ | 0 |
| $h_0$ | $(0, 1, 0)$ | 0 |
| $h_1$ | $(1, 1, 1)$ | 0 |
| $\tau^2 a$ | $(4, 3, 0)$ | 4 |
| $\tau^4$ | $(0, 0, -4)$ | 4 |
| $b$ | $(8, 4, 4)$ | 4 |
| $\tau h_1$ | $(1, 1, 0)$ | 1 |
| $\tau^2 h_0$ | $(0, 1, -2)$ | 2 |
| $a$ | $(4, 3, 2)$ | 2 |

Table 6.1: Generators for $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$

| Relation | $(s, f, w)$ | Coweight |
|---|---|---|
| $\rho h_0$ | $(-1, 1, -1)$ | 0 |
| $h_0 h_1$ | $(1, 2, 1)$ | 0 |
| $(\tau^2 h_0)^2 + \tau^4 h_0^2$ | $(0, 2, -4)$ | 4 |
| $\tau^4 h_1^3 + \rho \cdot \tau^2 a$ | $(3, 3, -1)$ | 4 |
| $\tau^2 h_0 \cdot a + h_0 \cdot \tau^2 a$ | $(4, 4, 0)$ | 4 |
| $h_1 \cdot \tau^2 a + \rho^3 b$ | $(5, 4, 1)$ | 4 |
| $a^2 + h_0^2 b$ | $(8, 6, 4)$ | 4 |
| $(\tau^2 a)^2 + \tau^4 h_0^2 b + \rho^2 \tau^4 h_1^2 b$ | $(8, 6, 0)$ | 8 |
| $\rho^2 \cdot \tau h_1$ | $(-1, 1, -2)$ | 1 |
| $h_0 \cdot \tau h_1 + \rho h_1 \cdot \tau h_1$ | $(1, 2, 0)$ | 1 |
| $h_1^2 \cdot \tau h_1$ | $(3, 3, 2)$ | 1 |
| $\tau h_1 \cdot \tau^2 a$ | $(5, 4, 0)$ | 5 |
| $\rho \cdot \tau^2 h_0$ | $(-1, 1, -3)$ | 2 |
| $\rho^3 \cdot a$ | $(1, 3, -1)$ | 2 |
| $\tau^2 h_0 \cdot h_1 + \rho(\tau h_1)^2$ | $(1, 2, -1)$ | 2 |
| $h_1(\tau h_1)^2 + \rho a$ | $(3, 3, 1)$ | 2 |
| $h_1 a$ | $(5, 4, 3)$ | 2 |
| $\tau^2 h_0 \cdot \tau^2 a + \tau^4 h_0 a$ | $(4, 4, -2)$ | 6 |
| $a \cdot \tau^2 a + \tau^2 h_0 \cdot h_0 b$ | $(8, 6, 2)$ | 6 |
| $\tau^2 h_0 \cdot \tau h_1$ | $(1, 2, -2)$ | 3 |
| $(\tau h_1)^3$ | $(3, 3, 0)$ | 3 |
| $\tau h_1 \cdot a$ | $(5, 4, 2)$ | 3 |

Table 6.2: Relations for $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_\mathbb{R}}(\mathbb{M}_2^\mathbb{R})$

The data comprising $\mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})$ is too complicated to display in one chart. To present this material more clearly, we give individual charts for each coweight modulo 4. Since the class $\tau^4$ has coweight 4, there is a coweight periodicity of order 4 as no elements in $\mathrm{Ext}^{s,f,w}_{\mathcal{A}^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})$ are $\tau^4$-torsion. Notice that there are no generators in coweight 3 modulo 4, and the relations ensure that there are never any classes in this coweight. Charts are written in $(s, f)$-grading, with motivic weight suppressed. We also suppress infinite $\rho$-towers, which would extend to the left in the charts, for readability. Figure 6.1 depicts the coweight 0 piece, Figure 6.2 depicts the coweight 1 piece, and Figure 6.3 depicts the coweight 2 piece.

A black ■ represents $\mathbb{F}_2[\tau^4][\rho, h_1]$. A black ● represents $\mathbb{F}_2[\tau^4]$. A vertical black line represents multiplication by $h_0$. A horizontal black line represents multiplication by $\rho$. A diagonal black line represents multiplication by $h_1$. A dashed horizontal line indicates that $\rho$-multiplication hits $\tau^4$ times a generator. For example, in Figure 6.1 there is a dashed line indicating $\rho \cdot \tau^2 a = \tau^4 h_1^3$.

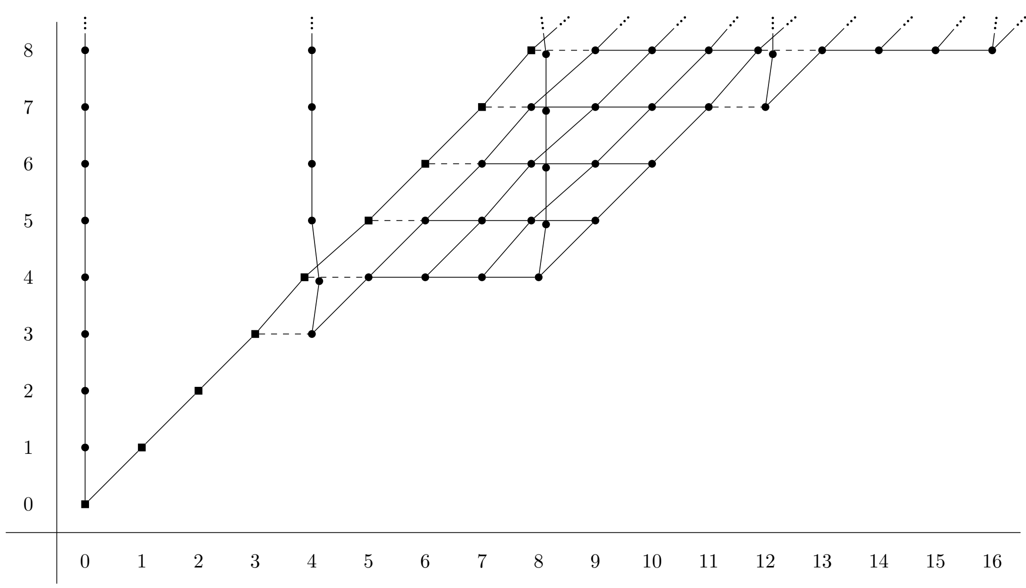


Figure 6.1: $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$ in $cw \equiv 0\,(4)$

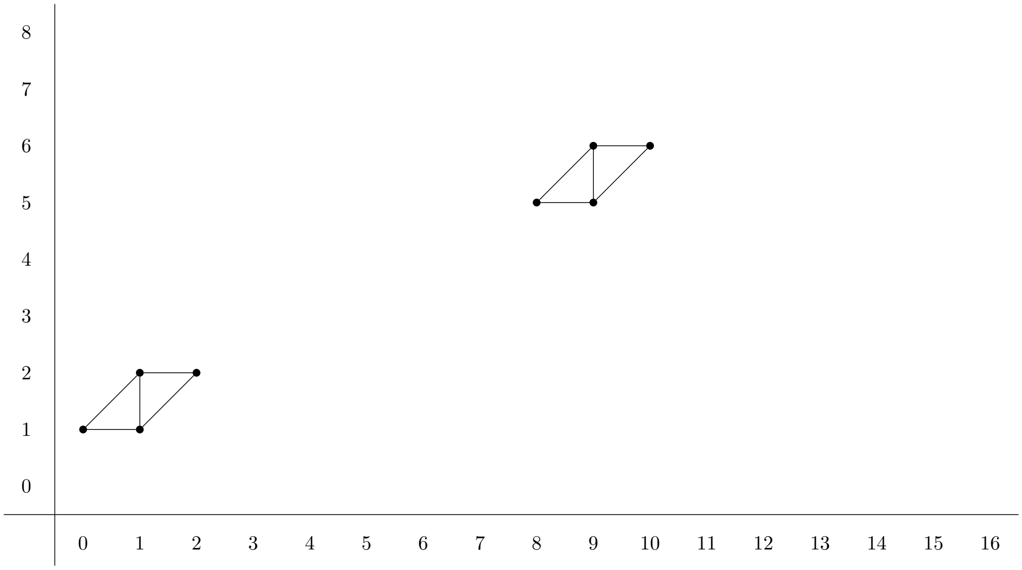


Figure 6.2: $\mathrm{Ext}^{***}_{\mathcal{A}(1)^{\vee}_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$ in $cw \equiv 1\,(4)$

### 6.1.2 *Some modules over* $\mathrm{Ext}^{***}_{\mathcal{A}(1)^{\vee}_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$

We now introduce notation for some modules over $\mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^{\vee}_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$ which will appear throughout. The reader is encouraged to consult Chapter A to follow along with these definitions.

**Definition 6.1.1.** Define the following $\mathrm{Ext}^{***}_{\mathcal{A}(1)^{\vee}_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$-modules:

- Let $\mathcal{D}$ denote the entire coweight $cw \equiv 1\,(4)$ piece of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^{\vee}_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$, called a *big diamond* (see Fig. 6.2);

- Let $\mathcal{S}$ denote the entire coweight $cw \equiv 2\,(4)$ piece of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^{\vee}_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$, called a *big staircase* (see Fig. 6.3);

- Let $P = \mathbb{F}_2[\tau^4][\rho]$, called a *$\rho$-tower*, with the obvious $\mathrm{Ext}^{***}_{\mathcal{A}(1)^{\vee}_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$-module structure (see Fig. A.1);

- Let $H = \mathbb{F}_2[\tau^4][h_0]$, called an *$h_0$-tower*, with the obvious $\mathrm{Ext}^{***}_{\mathcal{A}(1)^{\vee}_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$-module structure

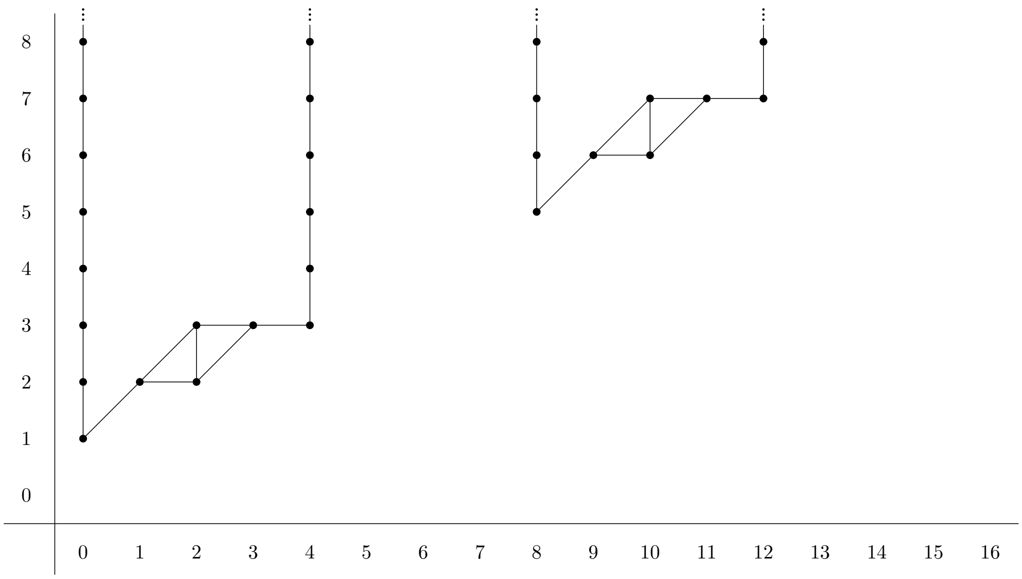


Figure 6.3: $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$ in $cw \equiv 2\,(4)$

(see Fig. A.2);

- Let $PH = \mathbb{F}_2[\tau^4][\rho, h_0]/(\rho h_0)$, called a $(\rho, h_0)$-*tower*, with the obvious $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$-module structure (see Fig. A.3);

- Let $D = \mathbb{F}_2[\tau^4][\rho, h_0, h_1]/(\rho^2, h_0^2, h_1^2, \rho h_0, h_0h_1, \rho h_1 = h_0)$, called a *diamond*, with the obvious $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$-module structure (see Fig. A.4);

- Let $S = \mathbb{F}_2[\tau^4][h_0, h_1, t, a]/(h_1^3, t^2, a^2, h_0h_1, h_1a, ta, h_0^2t, h_1^2t)$, called a *staircase*, with the $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$-module structure given by the relations (see Fig. A.5)

$$\rho \cdot t = h_1, \quad h_1 \cdot t = \rho \cdot a, \quad h_0 \cdot t = h_1^2 = \rho^2 \cdot a, \quad \rho^2 \cdot t = 0, \quad \rho^3 \cdot a = 0;$$

- Let $T = \mathbb{F}_2[\tau^4][\rho]/(\rho^2)$, called a *segment*, with the obvious $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$-module structure (see Fig. A.6);

- Let $J = \mathbb{F}_2[\tau^4][\rho, h_0]/(\rho^3, \rho h_0)$, called a *J-tower*, with the obvious $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_\mathbb{R}}(\mathbb{M}_2^\mathbb{R})$-module structure (see Fig. A.7);

- Let $JD = \mathbb{F}_2[\tau^4][\rho, h_0, t]/(\rho^3, t^2, \rho^2 t, \rho h_0, h_0^2 t)$, called a *JD-tower*, with the $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_\mathbb{R}}(\mathbb{M}_2^\mathbb{R})$-module structure given by the relations (see Fig. A.8)

$$\rho = h_1 t, \quad h_1 \cdot \rho t = h_0 t = \rho^2.$$

**Remark 6.1.2.** As should be indicated by the figures in Chapter A, the naming of these modules is derived from their general shapes when depicted in charts.

We construct now a more intricate family of modules. This module will track the behavior of the coweight $cw \equiv 0\,(4)$ piece of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_\mathbb{R}}(\mathbb{M}_2^\mathbb{R}, \mathbb{M}_2^\mathbb{R})$ throughout the process of the coming spectral sequences.

**Definition 6.1.2.** In this construction, we will attach generators to extend the coweight $cw \equiv 0\,(4)$ piece of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_\mathbb{R}}(\mathbb{M}_2^\mathbb{R})$. We continue until we have attached a generator in Adams filtration 0, at which point we stop. For $n \geq 0$, let $\mathcal{F}_{s,f,w}$ be the $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_\mathbb{R}}(\mathbb{M}_2^\mathbb{R})$-module concentrated in coweight $cw \equiv s - w\,(4)$, called a *big flag*, described by the following process:

(0) There is a copy of the coweight $cw \equiv 0\,(4)$ piece of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_\mathbb{R}}(\mathbb{M}_2^\mathbb{R})$ (see Fig. 6.1) with generator $x_0$ in degree $(s, f, w)$ called a *flag*;

$(4n+1)$ There is a $PH$ generated by $x_{4n+1}$ in degree $(s - 4 - 8n, f - 1 - 4n, w - 4 - 8n)$ subject to the relations

$$h_1 x_{4n+1} = \rho^3 x_{4n} \text{ and } b x_{4n+1} = \tau^2 a x_{4n}.$$

$(4n+2)$ There is a $P$ generated by $x_{4n+2}$ in degree $(s - 6 + 8n, f - 2 - 4n, w - 6 - 8n)$ subject to to the relations

$$h_1 x_{4n+2} = \rho x_{4n+1} \text{ and } b x_{4n+2} = h_1^2 x_{4n}.$$

$(4n+3)$ There is a $P$ generated by $x_{4n+3}$ in degree $(s-7-8n, f-3-4n, w-7-8n)$ subject to the relations

$$h_1 x_{4n+3} = x_{4n+2} \text{ and } b x_{4n+3} = h_1 x_{4n}.$$

$(4n+4)$ There is a $PH$ generated by $x_{4n+4}$ in degree $(s-8(n+1), f-4(n+1), w-8(n+1))$ subject to the relations

$$h_1 x_{4n+4} = x_{4n+3} \text{ and } b x_{4n+4} = x_{4n}.$$

In short, the submodule $\mathcal{F}_{s,f,w}$ looks like the coweight $cw \equiv 0\,(4)$ portion of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_\mathbb{R}}(\mathbb{M}_2^\mathbb{R})$ with generator shifted from $(0,0,0)$ to $(s,f,w)$, and then a complicated "tail" pattern coming from the generator of the flag which decreases in all degrees until Adams filtration 0 is reached. Importantly, at each step we are attaching a new generator by an $h_1$, and since $h_1$ has coweight 0 as an element of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_\mathbb{R}}(\mathbb{M}_2^\mathbb{R})$, this ensures that each stage of this process is concentrated in the same coweight modulo 4. As an example, we depict the big flag $\mathcal{F}_{12,3,6}$ in Fig. 6.4.

**Remark 6.1.3.** The big flag $\mathcal{F}_{0,0,0}$ is isomorphic to the coweight $cw \equiv 0\,(4)$ piece of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_\mathbb{R}}(\mathbb{M}_2^\mathbb{R})$ (see Fig. 6.1).

### *6.1.3 Comparison with $\mathbb{C}$-motivic*

There is a well-studied relationship between $\mathrm{SH}(\mathbb{C})$ and $\mathrm{SH}(\mathbb{R})$ (see, for example, [BI22, Section 2]). In [Hil11], Hill developed the $\rho$-Bockstein spectral sequence, an algebraic spectral sequence which allows for the computation of $\mathbb{R}$-motivic information from $\mathbb{C}$-motivic information. The observation that $\mathbb{M}_2^\mathbb{C} = \mathbb{M}_2^\mathbb{R}/\rho$ lifts to the dual Steenrod algebra, giving $\mathcal{A}^\vee_\mathbb{C} = \mathcal{A}^\vee_\mathbb{R}/\rho$. Filtering the cobar complex which computes $\mathrm{Ext}^{***}_{\mathcal{A}^\vee_\mathbb{R}}(\mathbb{M}_2^\mathbb{R})$ by powers of $\rho$, we obtain the $\rho$-Bockstein spectral sequence

$$\mathrm{E}_1 = \mathrm{Ext}^{***}_{\mathcal{A}^\vee_\mathbb{C}}(\mathbb{M}_2^\mathbb{C})[\rho] \implies \mathrm{Ext}^{***}_{\mathcal{A}^\vee_\mathbb{R}}(\mathbb{M}_2^\mathbb{R}).$$

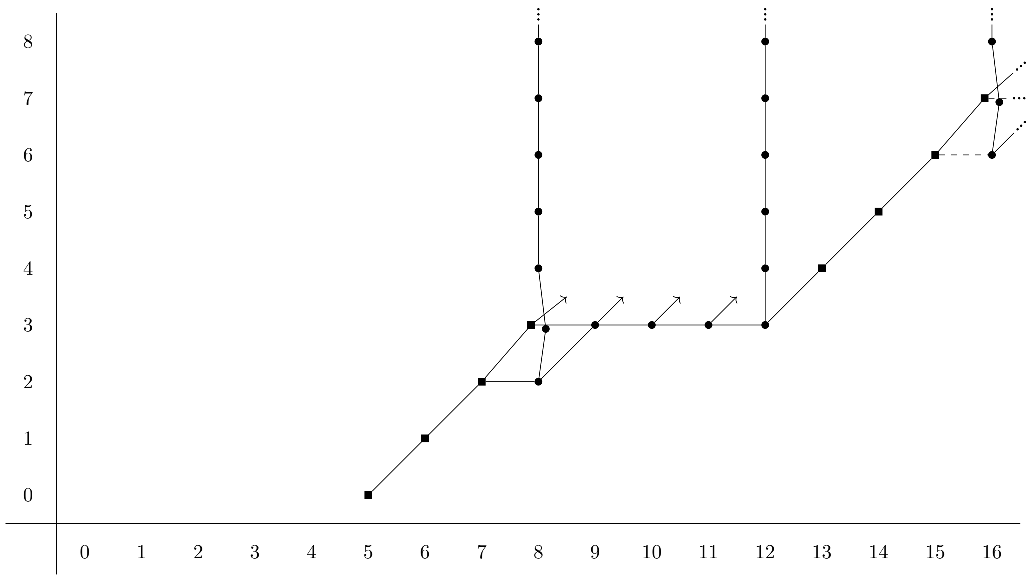


Figure 6.4: The $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$-module $\mathcal{F}_{12,3,6}$.

Note that this spectral sequence converges to the $E_2$-page of the $\mathbb{R}$-motivic **mASS**($\mathbb{S}$). One can also filter the cobar complex which computes $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$ by powers of $\rho$, giving a $\rho$-Bockstein spectral sequence

$$E_1 = \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}})[\rho] \implies \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}).$$

Note that this spectral sequence converges to the $E_2$-page of the **mASS**(kq).

There is also an extension of scalars functor $-\otimes\mathbb{C} : \mathrm{SH}(\mathbb{R}) \to \mathrm{SH}(\mathbb{C})$. For $X \in \mathrm{SH}(\mathbb{R})$, we denote its image under extensions of scalars by $X^{\mathbb{C}}$. This functor has the property $\rho\otimes\mathbb{C} = 0$ (modulo 2) and Eilenberg-Mac Lane spectra are sent to Eilenberg-Mac Lane spectra. In particular, this gives maps

$$\mathbb{M}_2^{\mathbb{R}} \to \mathbb{M}_2^{\mathbb{C}} \cong \mathbb{M}_2^{\mathbb{R}}/\rho, \quad \mathcal{A}^\vee_{\mathbb{R}} \to \mathcal{A}^\vee_{\mathbb{C}} \cong \mathcal{A}^\vee_{\mathbb{R}}/\rho.$$

Extension of scalars induces highly structured maps of Adams spectral sequences, which will allow us to lift $\mathbb{C}$-motivic extensions to $\mathrm{SH}(\mathbb{R})$.

In the 2-complete setting, one may lift $\rho$ to the sphere, giving an element $\rho \in \pi^{\mathbb{R}}_{-1,-1}\mathbb{S}$. This gives a cofiber sequence

$$\mathbb{S}^{-1,-1} \xrightarrow{\rho} \mathbb{S} \to \mathbb{S}/\rho.$$

This cofiber sequence induces a long exact sequence in $\mathrm{Ext}_{\mathcal{A}^\vee_{\mathbb{R}}}$. Since $\rho \otimes \mathbb{C} = 0$, extension of scalars on this long exact sequence gives a split long exact sequence in $\mathrm{Ext}_{\mathcal{A}^\vee_{\mathbb{C}}}$. Moreover, we can identify $\mathrm{H}_{**}(\mathbb{S}/\rho) \cong \mathbb{M}_2^{\mathbb{R}}/\rho$ since $\mathbb{M}_2^{\mathbb{R}}$ is $\rho$-torsion free. This gives us the following diagram.

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Ext}^{***}_{\mathcal{A}^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}) \arrow[r, "\rho"] \arrow[d] & \mathrm{Ext}^{***}_{\mathcal{A}^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}) \arrow[r] \arrow[d] & \mathrm{Ext}^{***}_{\mathcal{A}^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}/\rho) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{Ext}^{***}_{\mathcal{A}^\vee_{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}) \arrow[r, "0"] & \mathrm{Ext}^{***}_{\mathcal{A}^\vee_{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}) \arrow[r] & \mathrm{Ext}^{***}_{\mathcal{A}^\vee_{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}) \oplus \mathrm{Ext}^{***}_{\mathcal{A}^\vee_{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}) \arrow[r] & \cdots
\end{tikzcd}$$

The splitting on the bottom row gives a lift $\mathrm{Ext}^{***}_{\mathcal{A}^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}/\rho) \to \mathrm{Ext}^{***}_{\mathcal{A}^\vee_{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}})$. In fact, this map is an isomorphism due to the following.

**Proposition 6.1.4.** *Let $X \in \mathrm{SH}(\mathbb{R})$ be any spectrum such that $\mathrm{H}_{**}(X)$ is $\rho$-torsion free. Then there is an isomorphism*

$$\mathrm{Ext}^{***}_{\mathcal{A}^\vee_{\mathbb{R}}}(\mathrm{H}_{**}(X/\rho)) \xrightarrow{\cong} \mathrm{Ext}^{***}_{\mathcal{A}^\vee_{\mathbb{C}}}(\mathrm{H}_{**}(X^{\mathbb{C}})).$$

*Proof.* First, since $\mathrm{H}_{**}(X)$ is $\rho$-torsion free, we know that $\mathrm{H}_{**}(X/\rho) \cong \mathrm{H}_{**}(X)/\rho$, where $X/\rho \simeq X \otimes \mathbb{S}/\rho$. This follows from the long exact sequence in homology associated to the cofiber sequence

$$X \xrightarrow{\rho} X \to X/\rho.$$

Note that in this case, we have an isomorphism $\mathrm{H}_{**}(X)/\rho \cong \mathrm{H}_{**}(X^{\mathbb{C}})$. The result now follows from an isomorphism of cobar complexes

$$C_{\mathcal{A}^\vee_{\mathbb{R}}}(\mathrm{H}_{**}(X/\rho)) \cong C_{\mathcal{A}^\vee_{\mathbb{R}}}(\mathrm{H}_{**}(X))/\rho \cong C_{\mathcal{A}^\vee_{\mathbb{C}}}(\mathrm{H}_{**}(X^{\mathbb{C}})).$$

The left isomorphism holds because the cobar complex $C_{\mathcal{A}^\vee_{\mathbb{R}}}(\mathrm{H}_{**}(X/\rho))$ is entirely $\rho$-torsion free. The right isomorphism holds since $\mathrm{H}_{**}(X)/\rho \cong \mathrm{H}_{**}(X^{\mathbb{C}})$ and $\mathbb{M}_2^{\mathbb{R}}/\rho \cong \mathbb{M}_2^{\mathbb{C}}$. $\square$

One immediate consequence of Proposition 6.1.4 is that $\mathrm{Ext}^{***}_{\mathcal{A}^\vee_{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}})$ is a module over $\mathrm{Ext}^{***}_{\mathcal{A}^\vee_{\mathbb{R}}}(\mathbb{M}_2)$. This module structure is simple to describe. By the $\rho$-Bockstein, we can represent classes in $\mathrm{Ext}^{***}_{\mathcal{A}^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$ in the form $\rho^k x$ for some $x \in \mathrm{Ext}^{***}_{\mathcal{A}^\vee_{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}})$. If $y \in \mathrm{Ext}^{***}_{\mathcal{A}^\vee_{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}})$, then $\rho^k x \cdot y$ is only nonzero when $k = 0$, and in this case we have $x \cdot y = xy$.

**Corollary 6.1.5.** *Let $X \in \mathrm{SH}(\mathbb{R})$ be any spectrum such $\mathrm{H}_{**}(X)$ is $\rho$-torsion free. Then there is an isomorphism*

$$\pi^{\mathbb{R}}_{**}(X/\rho) \cong \pi^{\mathbb{C}}_{**}(X^{\mathbb{C}}).$$

*Proof.* The map from Proposition 6.1.4 gives an isomorphism of $E_2$-pages of Adams spectral sequences. $\square$

Note that $\mathrm{kq}^{\mathbb{C}}$ is the spectrum in $\mathrm{SH}(\mathbb{C})$ representing very effective Hermitian K-theory over $\mathbb{C}$, which we will abusively also denote by kq. Since $\mathrm{H}_{**}(\mathrm{kq}) \cong (\mathcal{A}_{\mathbb{R}}//\mathcal{A}(1))^\vee_{\mathbb{R}}$ is $\rho$-torsion free, it follows from Corollary 6.1.5 that $\pi^{\mathbb{R}}_{**}(\mathrm{kq}/\rho) \cong \pi^{\mathbb{C}}_{**}(\mathrm{kq})$. The Künneth spectral sequence for $\mathrm{H}_{**}(\mathrm{kq} \otimes \mathrm{kq})$ collapses by Proposition 2.2.6, so we also have

$$\pi^{\mathbb{R}}_{**}((\mathrm{kq} \otimes \mathrm{kq})/\rho) \cong \pi^{\mathbb{C}}_{**}(\mathrm{kq} \otimes \mathrm{kq}).$$

Combining Corollary 6.1.5 and Proposition 2.3.4, we can generalize this idea.

**Proposition 6.1.6.** *The $\rho$-periodic $E_1$-page of the real* kq*-resolution is isomorphic to the $E_1$-page of the complex* kq*-resolution.*

We can also prove analogous statements using the Brown–Gitler comodules. First, note that we have a diagram in Ext coming from the long exact sequences induced by the cofiber sequence

$$\mathrm{kq} \xrightarrow{\rho} \mathrm{kq} \to \mathrm{kq}/\rho$$

and extension of scalars, combined with the usual change of rings isomorphism:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}) \arrow[r, "\rho"] \arrow[d] & \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}) \arrow[r] \arrow[d] & \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}/\rho) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}) \arrow[r, "0"] & \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}) \arrow[r] & \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}) \oplus \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}) \arrow[r] & \cdots
\end{tikzcd}$$

This gives an isomorphism in Ext by the same argument as before:

$$\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}/\rho) \cong \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}}).$$

By construction, the Brown–Gitler comodules $B_0^{\mathbb{R}}(k)$ are $\rho$-torsion free since they are subcomodules of the dual Steenrod algebra. This extends naturally to tensor factors leading us to the following result:

**Proposition 6.1.7.** *There are isomorphisms for all $k \geq 0$:*

$$\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(k)^{\otimes i}/\rho) \cong \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}(B_0^{\mathbb{C}}(k)^{\otimes i}).$$

These isomorphisms will be helpful for deducing hidden extensions on the $E_\infty$-page of the $\mathbf{aAHSS}(B_0^{\mathbb{R}}(k))$ by comparing with the computations of [CQ21] recalled in Chapter 5.

**Remark 6.1.8.** If $F$ is a field with a real embedding, then there is a $C_2$-equivariant Betti realization functor

$$\mathrm{Be}^{C_2} : \mathrm{SH}(F) \to \mathrm{Sp}^{C_2},$$

obtained from the Betti realization functor of Remark 2.1.6 by remembering the $C_2$-action given by complex conjugation. This functor has been well studied, see for example [HO16], and exhibits a close relationship between $\mathbb{R}$-motivic and $C_2$-equivariant homotopy theory. It was shown by [Kon23] that

$$\mathrm{Be}^{C_2}(\mathrm{kq}^\wedge_{2,\eta}) \simeq (\mathrm{ko}_{C_2})^\wedge_{2,\eta}.$$

Moreover, there is a decomposition of the homology of a point as $\mathbb{M}_2^{C_2} \cong \mathbb{M}_2^{\mathbb{R}} \oplus NC$, where $NC$ is the so-called "negative cone". This leads to a splitting of the $C_2$-equivariant dual Steenrod algebra as $\mathcal{A}^\vee_{C_2} \cong \mathcal{A}^\vee_{\mathbb{R}} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathbb{M}_2^{C_2}$, and hence an isomorphism of Ext groups:

$$\mathrm{Ext}^{***}_{\mathcal{A}^\vee_{C_2}} \cong \mathrm{Ext}^{***}_{\mathcal{A}^\vee_{\mathbb{R}}} \oplus \mathrm{Ext}^{***}_{NC}.$$

This splitting implies that the $E_2$-page of the $C_2$-equivariant Adams spectral sequence computing the ring of cooperation $\pi^{C_2}_\star(\mathrm{ko}_{C_2} \otimes \mathrm{ko}_{C_2})$ contains the $E_2$-page of the $\mathbf{mASS}(\mathrm{kq}\otimes\mathrm{kq})$ as a summand. This will be investigated further in joint work with Li, Petersen, and Tatum.

### 6.2 The algebraic Atiyah–Hirzebruch spectral sequence

In this section, we compute the group $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(1)^{\otimes i})$ by a series of algebraic Atiyah–Hirzebruch spectral sequences. Motivated by the presentation of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$ given in [GHIR20], we organize these computations by coweight modulo 4. Consequently, our results are best displayed in multiple charts which should be read simultaneously. Throughout, we use the notation of Section 6.1.

#### 6.2.1 $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(1))$

Now, we will compute $\mathrm{Ext}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(1))$ by the **aAHSS**$(B_0^{\mathbb{R}}(1))$. This spectral sequence takes the form

$$E_1^{s,f,w,a} = \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}) \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathbb{M}_2^{\mathbb{R}}\{[1],[\overline{\xi}_1],[\overline{\tau}_1]\} \implies \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(1)).$$

The $d_1$ and $d_2$ differentials in this spectral sequence are determined by our discussion in Section 4.2. In particular the $d_1$-differential is only nonzero on Atiyah–Hirzebruch filtration 3, where

$$d_1(\alpha[3]) = h_0\alpha[2],$$

and the $d_2$-differential is only nonzero on Atiyah–Hirzebruch filtration 3, where

$$d_2(\alpha[2]) = h_1\alpha[0].$$

Note that in terms of coweight, both of these differentials are determined by multiplication by an element in $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$ of coweight 0. Given the complicated nature of this spectral sequence, we only present charts where the differential is nonzero and only include the relevant Atiyah–Hirzebruch filtrations. For example, since the $d_1$-differential is trivial on Atiyah–Hirzebruch filtration 0, we omit this filtration from the charts for the $E_1$-page.

Fig. 6.5 depicts the $E_1$-page with coweight $cw \equiv 0\,(4)$ in Atiyah–Hirzebruch filtrations 2 and 3. Fig. 6.6 depicts the $E_1$-page with coweight $cw \equiv 1\,(4)$ in Atyiah-Hirzebruch filtrations 2 and 3. Fig. 6.7 depicts the $E_1$-page with coweight $cw \equiv 2\,(4)$ in Atiyah–Hirzebruch filtrations 2 and 3. Fig. 6.8 depicts the $E_2$-page with coweight $cw \equiv 0\,(4)$ in Atiyah–Hirzebruch filtrations 0 and 2. Fig. 6.9 depicts the $E_2$-page with coweight $cw \equiv 1\,(4)$ in Atiyah–Hirzebruch filtrations 0 and 2. Fig. 6.10 depicts the $E_2$-page with coweight $cw \equiv 2\,(4)$ in Atiyah–Hirzebruch filtrations 0 and 2. Differentials are blue, linear with respect to the underlying $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$-module structure, and preserve motivic weight.

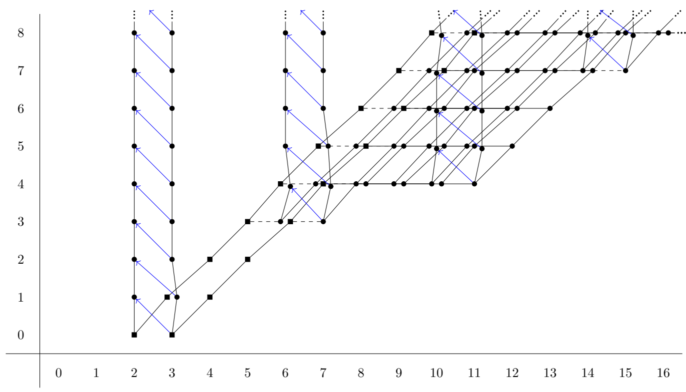


Figure 6.5: The $E_1$-page of the $\mathbf{aAHSS}(B_0^{\mathbb{R}}(1))$ with $cw \equiv 0\,(4)$ in Atiyah–Hirzebruch filtration 2 and 3.

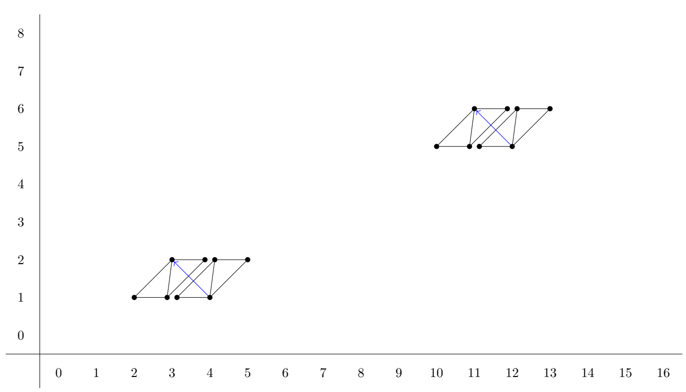


Figure 6.6: The $E_1$-page of the $\mathbf{aAHSS}(B_0^{\mathbb{R}}(1))$ with $cw \equiv 1\,(4)$ in Atiyah–Hirzebruch filtration 2 and 3.

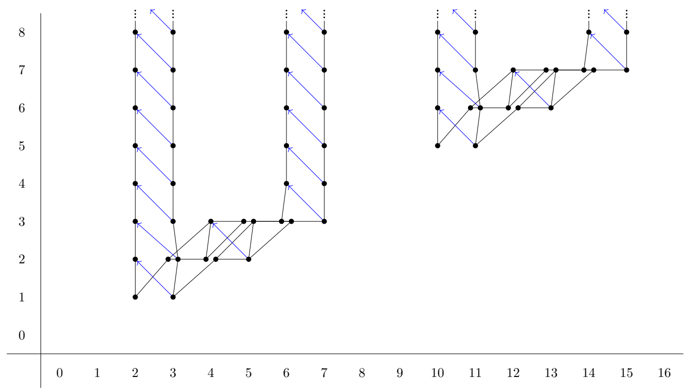


Figure 6.7: The $E_1$-page of the $\mathbf{aAHSS}(B_0^{\mathbb{R}}(1))$ with $cw \equiv 2\,(4)$ in Atiyah–Hirzebruch filtration 2 and 3.

As we remarked in Section 4.2, the $d_3$-differential in this spectral sequence is dependent on the field we are working over. While both classically and $\mathbb{C}$-motivically we saw that the $d_3$-differential, which is only potentially nonzero on Atiyah–Hirzebruch filtration 3, was actually trivial for degree reasons, we see that this is not the case here. By inspection, the only possible nonzero $d_3$ differential is between Atiyah–Hirzebruch filtrations 3 and 0, and will be given by the formula

$$d_3(\alpha[3]) = \langle \alpha, h_0, h_1 \rangle [0].$$

**Lemma 6.2.1.** *There is a $d_3$-differential in the* $\mathbf{aAHSS}(B_0^{\mathbb{R}}(1))$*:*

$$d_3(\rho[3]) = (\tau h_1)[0].$$

*Proof.* This differential is witnessed by the Massey product $\langle \rho, h_0, h_1 \rangle$. This Massey product is shown to be equal to $\tau h_1$ in [GHIR20, Thm 8.1] with zero indeterminacy. $\square$

In fact, since the differentials are linear over $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$ this determines all $d_3$-differentials. For example, we have

$$d_3(b\rho[3]) = b \cdot \tau h_1[0],$$

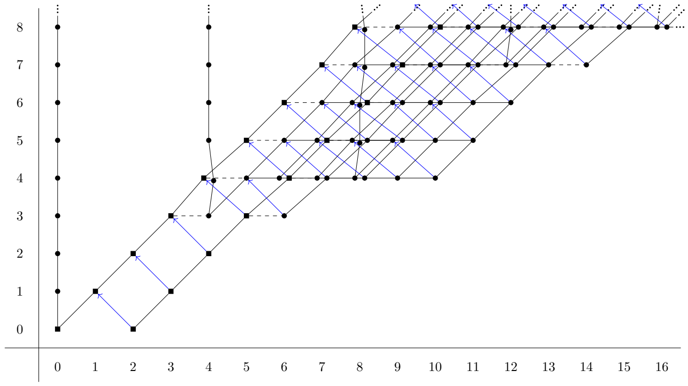


Figure 6.8: The $E_2$-page of the $\mathbf{aAHSS}(B_0^{\mathbb{R}}(1))$ with $cw \equiv 0\,(4)$ in Atiyah–Hirzebruch filtration 0 and 2.

and we have

$$d_3(\rho \cdot \tau h_1[3]) = (\tau h_1)^2[0].$$

It is important to note that since $\tau h_1$ has coweight 1 in $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$, and since each Atiyah–Hirzebruch filtration is zero in coweight 3, the $d_3$-differential is trivial on the coweight $cw \equiv 2\,(4)$ portion of Atiyah–Hirzebruch filtration 3. This coweight jump requires us to pay close attention to the $d_3$-differential. For example, in the coweight $cw \equiv 0\,(4)$ piece of Atiyah–Hirzebruch filtration 3, there is an infinite $\rho$-tower stemming from $\rho[3]$. However, while

$$d_3(\rho[3]) = \tau h_1[0]$$

and

$$d_3(\rho^2[3]) = \rho \cdot \tau h_1[0],$$

there are no other classes in the coweight $cw \equiv 1\,(4)$ piece of Atiyah–Hirzebruch filtration 0 for the rest of this $\rho$-tower to hit. In other words, for we have

$$d_3(\rho^n[3]) = 0[0],\ n \geq 2.$$

Fig. 6.11 depicts the $E_3$-page where the coweight of Atiyah–Hirzebruch filtration 3 is $cw \equiv 0\,(4)$ and the coweight of Atiyah–Hirzebruch filtration 0 is $cw \equiv 1\,(4)$. Fig. 6.12 depicts

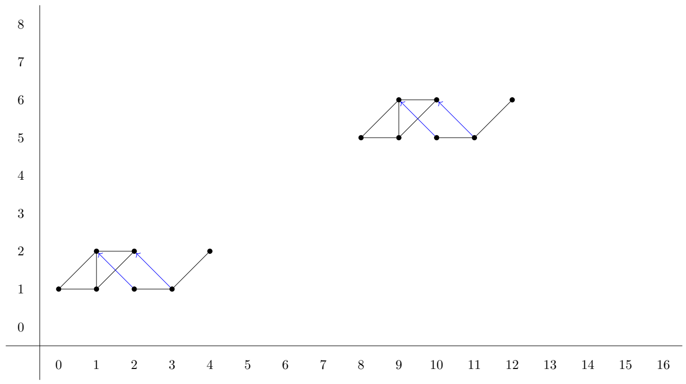


Figure 6.9: The $E_2$-page of the $\mathbf{aAHSS}(B_0^{\mathbb{R}}(1))$ with $cw \equiv 1\,(4)$ in Atiyah–Hirzebruch filtration 0 and 2.

the $E_3$-page where the coweight of Atiyah–Hirzebruch filtration 3 is $cw \equiv 1\,(4)$ and the coweight of Atiyah–Hirzebruch filtration 0 is $cw \equiv 2\,(4)$.

Recall from Proposition 4.2.4 that $E_4 = E_\infty$ for Atiyah–Hirzebruch filtration reasons. Hidden extensions can be realized using complex Betti realization, linearity over $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$, base change $\otimes\mathbb{C} : \mathrm{SH}(\mathbb{C}) \to \mathrm{SH}(\mathbb{R})$, and the isomorphism from Proposition 6.1.7:

$$\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(1)/\rho) \cong \mathrm{Ext}^{***}_{\mathcal{A}(1)_{\mathbb{C}}}(B_0^{\mathbb{C}}(1)).$$

The following result will be useful when depicting the $E_\infty$-page in charts.

**Lemma 6.2.2.** *The group* $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(1))$ *is zero in coweight* $cw \equiv 1\,(4)$*.*

*Proof.* In coweight $cw \equiv 1\,(4)$, the $E_\infty$-page of the $\mathbf{aAHSS}(B_0^{\mathbb{R}}(1))$ is isomorphic to the a direct sum of coweight $cw \equiv 1\,(4)$ in Atiyah–Hirzebruch filtration 0, coweight $cw \equiv 0\,(4)$ in Atiyah–Hirzebruch filtration 2, and coweight $cw \equiv 3\,(4)$ in Atiyah–Hirzebruch filtration 3 pieces. The differentials:

$$d_2(b^n \cdot \tau h_1[2]) = b^n h_1 \cdot \tau h_1[0], \quad d_2(b^n\rho \cdot \tau h_1[2]) = b^n \rho h_1 \cdot \tau h_1[0], \quad n \geq 0$$

and

$$d_3(b^n\rho[3]) = b^n \cdot \tau h_1[0], \quad d_3(b^n\rho^2[3]) = b^n\rho \cdot \tau h_1[0], \quad n \geq 0$$

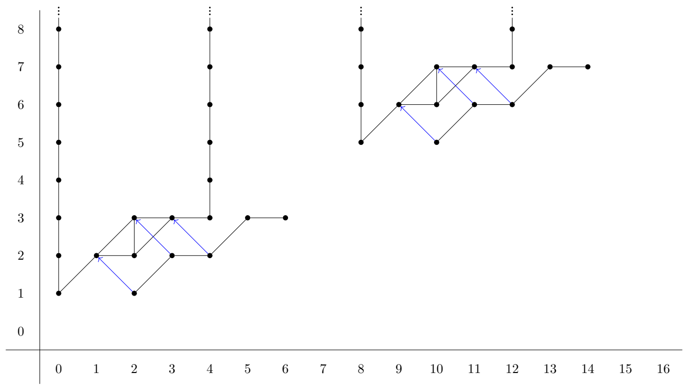


Figure 6.10: The $E_2$-page of the $\mathbf{aAHSS}(B_0^{\mathbb{R}}(1))$ with $cw \equiv 2\,(4)$ in Atiyah–Hirzebruch filtration 0 and 2.

ensure that Atiyah–Hirzebruch filtration 0 is trivial in coweight $cw \equiv 1\,(4)$. The differentials

$$d_1((\tau^2 a)^n b^m[3]) = (\tau^2 a)^n b^m h_0[2], \quad d_2((\tau^2 a)^n b^m[2]) = (\tau^2 a)^n b^m h_1[0], \quad n, m \geq 0$$

ensure that Atiyah–Hirzebruch filtration 2 is trivial in coweight $cw \equiv 0\,(4)$. Atiyah–Hirzebruch filtration 3 is trivial in coweight $cw \equiv 3\,(4)$ since $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$ is trivial in this coweight. $\square$

We organize the $E_\infty$-page by coweight modulo 4 in the same way that we depicted $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$. As indicated by Lemma 6.2.2, the different Atiyah–Hirzebruch filtration pieces contribute different pieces to each coweight of the $E_\infty$-page, suitably shifted by the degree of the cell in the filtration it is attached to. Since $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$ is concentrated in coweights $cw \equiv 0, 1,$ and $2\,(4)$, for each Atiyah–Hirzebruch filtration piece there is a coweight on the $E_\infty$-page which it does not contribute to. To make this clear, we describe each coweight piece of the $E_\infty$-page in terms of contributions from Atiyah–Hirzebruch filtration in Table 6.3.

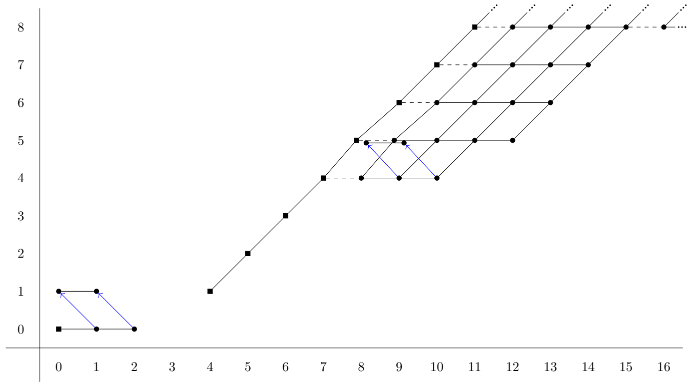


Figure 6.11: The $E_3$-page of the $\mathbf{aAHSS}(B_0^{\mathbb{R}}(1))$ with $cw \equiv 0\,(4)$ in Atiyah–Hirzebruch filtration 3 and $cw \equiv 1\,(4)$ in Atiyah–Hirzebruch filtration 0.

| $E_\infty$ coweight | Filtration 0 | Filtration 2 | Filtration 3 |
|---|---|---|---|
| 0 | 0 | 3 | 2 |
| 1 | 1 | 0 | 3 |
| 2 | 2 | 1 | 0 |
| 3 | 3 | 2 | 1 |

Table 6.3: Atiyah–Hirzebruch filtration coweight contributions in the $\mathbf{aAHSS}(B_0^{\mathbb{R}}(1))$.

With this in mind, we are prepared to give charts for the $E_\infty$-page. Fig. 6.13 depicts the $E_\infty$-page in coweight $cw \equiv 0\,(4)$. Fig. 6.14 depicts the $E_\infty$-page in coweight $cw \equiv 2\,(4)$. Fig. 6.15 depicts the $E_\infty$-page in coweight $cw \equiv 3\,(4)$. We use the same notation as previously used in this section, with some additions. A $\square$ represents $\mathbb{F}_2[\tau^4][\rho]$, where in contrast to a black $\blacksquare$, the $\rho$-tower does not support $h_1$-multiplication. Red lines indicate hidden extensions. A diagonal black $\nearrow$ indicates infinite $h_1$-multiplication on a $\rho$-tower which is compatible with an infinite $\rho$-tower off of a class which is $h_1$-divisible. For example, in coweight $cw \equiv 2\,(4)$, there is a class $x$ in degree $(3,1)$ which supports an infinite $h_1$-tower, and there is a class $y$ in

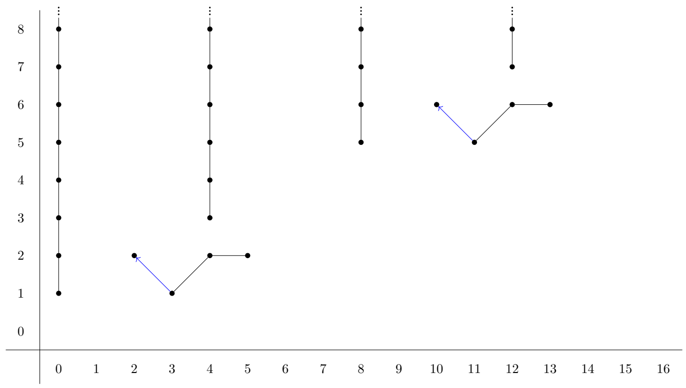


Figure 6.12: The $E_3$-page of the $\mathbf{aAHSS}(B_0^{\mathbb{R}}(1))$ with $cw \equiv 1\,(4)$ in Atiyah–Hirzebruch filtration 3 and $cw \equiv 2\,(4)$ in Atiyah–Hirzebruch filtration 0.

degree $(5, 2)$ which is part of an infinite $h_1$-tower and supports an infinite $\rho$-tower. Hidden in our charts, and as indicated by $\nearrow$, is the relation $h_1 x = \rho y$.

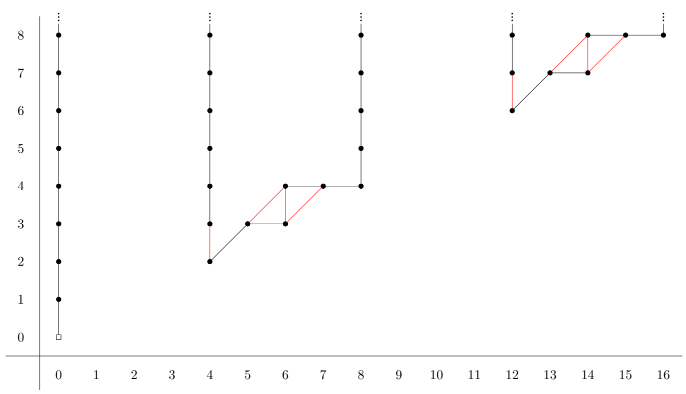


Figure 6.13: The $E_\infty$-page of the $\mathbf{aAHSS}(B_0^{\mathbb{R}}(1))$ in $cw \equiv 0\,(4)$.

**Remark 6.2.3.** The big flag $\mathcal{F}_{4,1,2}$ is isomorphic to the coweight $cw \equiv 2\,(4)$ piece of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(1))$ (see Fig. 6.14).

We note that most classes on the $E_\infty$-page are $v_1$-periodic. In fact, the only classes which

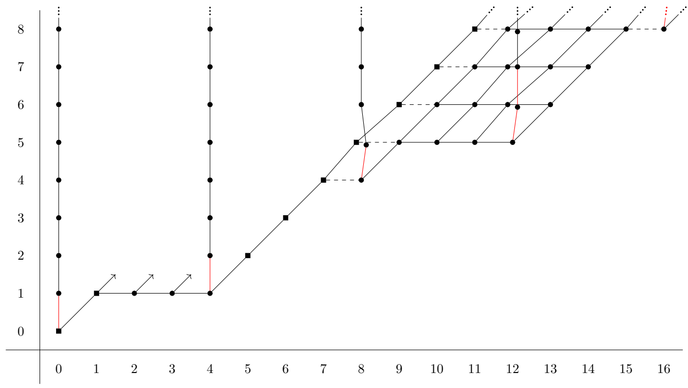


Figure 6.14: The $E_\infty$-page of the $\mathbf{aAHSS}(B_0^{\mathbb{R}}(1))$ in $cw \equiv 2\,(4)$.

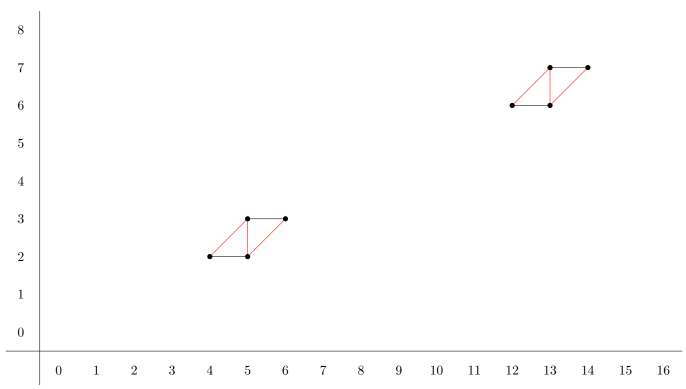


Figure 6.15: The $E_\infty$-page of the $\mathbf{aAHSS}(B_0^{\mathbb{R}}(1))$ in $cw \equiv 3\,(4)$.

are $v_1$-torsion are those in the $\rho$-tower in Adams filtration 0 of the coweight $cw \equiv 0\,(4)$ piece which are divisible by $\rho^3$.

We close this section with the following result.

**Corollary 6.2.4.** *There is an isomorphism*

$$\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(1)) \cong \pi^{\mathbb{R}}_{**}(\mathrm{ksp}).$$

*Proof.* The proof is identical to the one given in Corollary 5.2.1. $\square$

### 6.2.2 $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(1)^{\otimes 2})$ *and recursive submodules*

To illustrate the numerous $v_1$-torsion classes which arise in the process of computing $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right)$, which are ultimately to be ignored (see Remark 4.2.1), we explicitly compute $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(1)^{\otimes 2}\right)$. We form an algebraic Atiyah–Hirzebruch spectral sequence by taking the cellular filtration on $B_0^{\mathbb{R}}(1)$ from before and applying the functor $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(1) \otimes -\right)$. This has signature

$$E_1 = \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(1)) \otimes \mathbb{M}_2^{\mathbb{R}}\{1, \overline{\xi_1}, \overline{\tau_1}\} \implies \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(1)^{\otimes 2}).$$

The differentials are determined in an analogous manner as before, and hidden extensions can be realized by the same methods. Similar to the previous case, the behavior of the differentials ensures that the $E_\infty$-page is trivial in coweight $cw \equiv 1\,(4)$. We present the $E_\infty$ page, modulo $v_1$-torsion, below. Fig. 6.16 depicts the $E_\infty$-page in coweight $cw \equiv 0\,(4)$. Fig. 6.17 depicts the $E_\infty$-page in coweight $cw \equiv 1\,(4)$. Fig. 6.18 depicts the $E_\infty$-page in coweight $cw \equiv 2\,(4)$. Note that we do not present hidden extensions in red in these charts.

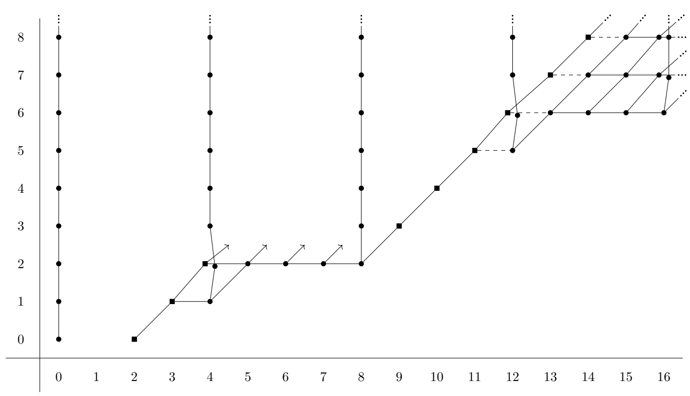


Figure 6.16: The $E_\infty$-page of the $\mathbf{aAHSS}\left(B_0^{\mathbb{R}}(1)^{\otimes 2}\right)$ in $cw \equiv 0\,(4)$, modulo $v_1$-torsion.

**Remark 6.2.5.** The big flag $\mathcal{F}_{8,2,4}$ is isomorphic to the summand of the coweight $cw \equiv 0\,(4)$ piece of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(1)^{\otimes 2})$ concentrated in stems $s \geq 2$ (see Fig. 6.16).

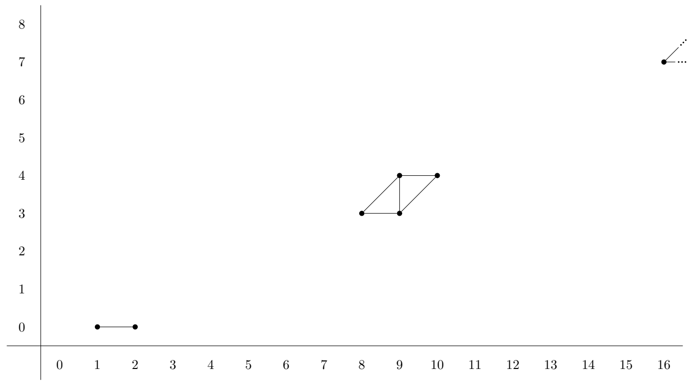

Figure 6.17: The $E_\infty$-page of the $\mathbf{aAHSS}\big(B_0^{\mathbb{R}}(1)^{\otimes 2}\big)$ in $cw \equiv 1\,(4)$, modulo $v_1$-torsion.

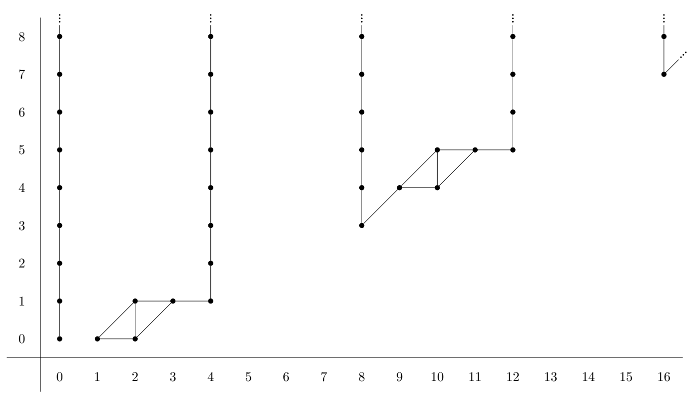

Figure 6.18: The $E_\infty$-page of the $\mathbf{aAHSS}\big(B_0^{\mathbb{R}}(1)^{\otimes 2}\big)$ in $cw \equiv 2\,(4)$, modulo $v_1$-torsion.

Before giving a general formula for $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right)$, we make a few simple observations. The group $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$, is only nonzero in coweights $cw \equiv 0, 1, 2\,(4)$. After running the $\mathbf{aAHSS}(B_0^{\mathbb{R}}(1))$, we see that there is a submodule

$$M_1 := \Sigma^{4,2}\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})\langle 1\rangle \subseteq \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(1)),$$

using the notation from Chapter 5. Note that if $x$ has coweight $cw \equiv t\,(4)$, then $\Sigma^{4,2}x\langle 1\rangle$ has coweight $cw \equiv t+2\,(4)$. In particular, $M_1$ is concentrated in coweights $cw \equiv 0, 2, 3\,(4)$. By observation, we see that in fact all classes in $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(1))$ are concentrated in these coweights. Similarly, after running the **aAHSS** $\left(B_0^{\mathbb{R}}(1)^{\otimes 2}\right)$ we see that there is a submodule

$$M_2 := \Sigma^{8,4}\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})\langle 2\rangle \subseteq \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(1)^{\otimes 2}\right).$$

It follows that $M_2$ is concentrated in coweights $cw \equiv 0, 1, 2\,(4)$, and by observation we see that in fact all classes in $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(1)^{\otimes 2}\right)$ are concentrated in these coweights. More generally, from the **aAHSS** $\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right)$ we see that the group $\mathrm{Ext}^{***}_{\mathcal{A}^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(1)^{\otimes i})$ contains a submodule

$$M_i = \Sigma^{4i,2i}\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})\langle i\rangle \subseteq \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right),$$

and that, modulo $v_1$-torsion, $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right)$ is concentrated in the same coweights as $M_n$ is. More precisely, we have the following result.

**Proposition 6.2.6.** *The group* $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right)/v_1\text{-torsion}$ *is trivial in coweight* $cw \equiv 1\,(4)$ *for* $n \equiv 1\,(2)$ *and in coweight* $cw \equiv 3\,(4)$ *for* $n \equiv 0\,(2)$.

We also have the following submodule of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(1)^{\otimes i})$.

**Proposition 6.2.7.** *Let* $i \geq 1$. *For all* $1 \leq j \leq i$, *the group* $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right)/v_1\text{-torsion}$ *contains as a submodule:*

$$\Sigma^{4i,2i}\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(1)^{\otimes j}\right)\langle i\rangle \subseteq \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right).$$

*Proof.* We have shown this to be true in the case of $i = 1$ in Corollary 6.2.4. The differentials in the **aAHSS** $\left(B_0^{\mathbb{R}}(1)^{\otimes 2}\right)$ on the submodule $M_1$ are isomorphic to the differentials in the **aAHSS**$(B_0^{\mathbb{R}}(1))$, hence the submodule $M_2$ of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(1)^{\otimes 2}\right)$ extends to give a submodule

$$\Sigma^{8,4}\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(1)\right)\langle 2\rangle \subseteq \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(1)^{\otimes 2}\right).$$

For arbitrary $i$, the submodule $M_{i-1} \subseteq \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(1)^{\otimes i-1}\right)$ extends to the submodule

$$\Sigma^{4(i-1),2(i-1)}\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(1)^{\otimes j}\right)\langle i-1\rangle$$

for each $1 \leq j \leq i-1$. The differentials in the **aAHSS** $\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right)$ are isomorphic to the differentials in the **aAHSS** $\left(B_0^{\mathbb{R}}(1)^{\otimes j}\right)$ when restricted to this submodule, hence the result of the spectral sequence gives the desired submodules in the target. $\square$

**Remark 6.2.8.** It is important that we work modulo $v_1$-torsion to make the above claim. For example, by computing the **aAHSS** $\left(B_0^{\mathbb{R}}(1)^{\otimes 3}\right)$ one can observe that the group $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(1)^{\otimes 3}\right)$ is nonzero in every coweight modulo 4. However, the only classes in coweight $cw \equiv 1\,(4)$ are $v_1$-torsion for degree reasons. Moreover, these $v_1$-torsion classes are permament cycles of the **aAHSS** $\left(B_0^{\mathbb{R}}(1)^{\otimes 4}\right)$, so their impact on this inductive process is negligible

### *6.2.3* $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right)$

To aid in our presentation of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right)$, we use the notation from Section 6.1.1 to construct a family of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$-modules which we organize by coweight modulo 4. As we will see, each module is concentrated in precisely 3 of these coweights, reflecting our observations from Proposition 6.2.6.

**Definition 6.2.1.** For each $i \geq 0$, let $Z_i^{\mathbb{R}}$ be the $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$-module given by the direct sum of the columns of the corresponding table.

For $i = 4k$, let $Z_i^{\mathbb{R}}$ be the direct sum of the columns of Table 6.4:

| $cw \equiv 0\,(4)$ | $cw \equiv 1\,(4)$ | $cw \equiv 2\,(4)$ |
|---|---|---|
| $\mathcal{F}_{4i,i,2i}$<br>$\oplus$<br>$\bigoplus_{j=1}^{2k} \Sigma^{4j-4,4\lfloor (j+1)/2\rfloor -4} H$ | $\Sigma^{2i,i}\mathcal{D}$ | $\Sigma^{2i,i}\mathcal{S}$<br>$\oplus$<br>$\bigoplus_{j=1}^{2k} \Sigma^{4j-4,4\lfloor j/2\rfloor -2} H$ |

Table 6.4: $Z_i^{\mathbb{R}}$ for $i = 4k$

For $i = 4k+1$, let $Z_i^{\mathbb{R}}$ be the direct sum of the columns of Table 6.5:

| $cw \equiv 0\,(4)$ | $cw \equiv 2\,(4)$ | $cw \equiv 3\,(4)$ |
|---|---|---|
| $\Sigma^{2i+2,i+1}\mathcal{S}\langle 1\rangle$<br>$\oplus$<br>$\bigoplus_{j=1}^{2k} \Sigma^{4j-4,4\lfloor (j+1)/2\rfloor -4}H$<br>$\oplus$<br>$\Sigma^{2i+2,i-1}J$ | $\mathcal{F}_{4i,i,2i}$<br>$\oplus$<br>$\bigoplus_{j=1}^{2k} \Sigma^{4j-4,4\lfloor j/2\rfloor -2}H$ | $\Sigma^{2i+2,i+1}\mathcal{D}\langle 1\rangle$ |

Table 6.5: $Z_i^{\mathbb{R}}$ for $i = 4k+1$

For $i = 4k+2$, let $Z_i^{\mathbb{R}}$ be the direct sum of the columns of Table 6.6:

| $cw \equiv 0\,(4)$ | $cw \equiv 1\,(4)$ | $cw \equiv 2\,(4)$ |
|---|---|---|
| $\mathcal{F}_{4i,i,2i}$<br>$\oplus$<br>$\bigoplus_{j=1}^{2k+1} \Sigma^{4j-4,4\lfloor (j+1)/2\rfloor -4}H$ | $\Sigma^{2i+4,i+2}\mathcal{D}\langle 2\rangle$<br>$\oplus$<br>$\Sigma^{2i-2,i-1}T$ | $\Sigma^{2i+4,i+2}\mathcal{S}\langle 2\rangle$<br>$\oplus$<br>$\bigoplus_{j=1}^{2k+1} \Sigma^{4j-4,4\lfloor j/2\rfloor -2}H$<br>$\oplus$<br>$\Sigma^{2i,i}JD\langle 1\rangle$ |

Table 6.6: $Z_i^{\mathbb{R}}$ for $i = 4k+2$

For $i = 4k+3$, let $Z_i^{\mathbb{R}}$ be the direct sum of the columns of Table 6.7:

| $cw \equiv 0\,(4)$ | $cw \equiv 2\,(4)$ | $cw \equiv 3\,(4)$ |
|---|---|---|
| $\Sigma^{2i-2,i-1}\mathcal{S}\langle -1\rangle$ <br> $\oplus$ <br> $\bigoplus_{j=1}^{2k+1}\Sigma^{4j-4,4\lfloor j/2\rfloor}H$ | $\mathcal{F}_{4i,i,2i}$ <br> $\oplus$ <br> $\bigoplus_{j=1}^{2k+2}\Sigma^{4j-4,4\lfloor (j+1)/2\rfloor -2}H$ | $\Sigma^{2i-2,i-1}\mathcal{D}\langle -1\rangle$ |

Table 6.7: $Z_i^{\mathbb{R}}$ for $i = 4k+3$

We are now prepared to state the $\mathbb{R}$-motivic analogue of Proposition 5.2.2.

**Theorem 6.2.9.** *There is an isomorphism of $\mathcal{A}(1)_{\mathbb{R}}^{\vee}$-comodules and $\mathrm{Ext}^{***}_{\mathcal{A}(1)^{\vee}_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$-modules:*

$$\frac{\mathrm{Ext}^{***}_{\mathcal{A}(1)^{\vee}_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right)}{v_1\text{-torsion}} \cong Z_i^{\mathbb{R}}$$

*Proof.* The case of $i = 1$ was computed in Corollary 6.2.4, and the case of $i = 2$ is presented in Fig. 6.16, Fig. 6.17, and Fig. 6.18. The general case is obtained by applying the functor $\mathrm{Ext}^{***}_{\mathcal{A}(1)^{\vee}_{\mathbb{R}}}(B_0^{\mathbb{R}}(1)^{\otimes i-1})$ to the cellular filtration of $B_0^{\mathbb{R}}(1)$ and computing the $\mathbf{aAHSS}(B_0^{\mathbb{R}}(1)^{\otimes i})$:

$$\mathrm{Ext}^{***}_{\mathcal{A}(1)^{\vee}_{\mathbb{R}}}(B_0^{\mathbb{R}}(1)^{\otimes i-1}) \otimes \mathbb{M}_2^{\mathbb{R}}\{[1],[\overline{\xi}_1],[\overline{\tau}_1]\} \implies \mathrm{Ext}^{***}_{\mathcal{A}(1)^{\vee}_{\mathbb{R}}}(B_0^{\mathbb{R}}(1)^{\otimes i}).$$

The differentials are determined by the attaching maps in the algebraic cell complex $B_0^{\mathbb{R}}(1)$ as before, hence Proposition 4.2.4 implies that $E_4 = E_\infty$. Hidden extensions may be obtained using complex Betti realization to the classical case, linearity over $\mathrm{Ext}^{***}_{\mathcal{A}(1)^{\vee}_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$, and the base change functor $\otimes\mathbb{C} : \mathrm{SH}(\mathbb{R}) \to \mathrm{SH}(\mathbb{C})$ combined with the isomorphism from Proposition 6.1.7:

$$\mathrm{Ext}_{\mathcal{A}(1)^{\vee}_{\mathbb{R}}}(B_0^{\mathbb{R}}(k)^{\otimes i}/\rho) \cong \mathrm{Ext}_{\mathcal{A}(1)^{\vee}_{\mathbb{C}}}(B_0^{\mathbb{C}}(k)^{\otimes i}).$$

The calculations are very similar for all $i$. We give the full argument in the case of $4k+1$ and leave the rest to the reader. As an example, we depict $Z_5^{\mathbb{R}}$ in Fig. 6.19, Fig. 6.20, and Fig. 6.21.

Let $i = 4k$, and consider the $\mathbf{aAHSS}(B_0^{\mathbb{R}}(1)^{\otimes 4k+1})$. By induction, the $E_1$-page is isomorphic to

$$Z_{4k}^{\mathbb{R}} \otimes \mathbb{M}_2^{\mathbb{R}}\{[1],[\overline{\xi}_1],[\overline{\tau}_1]\}.$$

Recall from Section 6.2.1 the contributions from each Atiyah–Hirzebruch filtration to each coweight of the $E_\infty$-page of the spectral sequence. In particular, since $Z^{\mathbb{R}}_{4k}$ is trivial in coweight $cw \equiv 3\,(4)$, the $E_\infty$-page consist of summands from only two Atiyah–Hirzebruch filtration pieces in coweights $cw \equiv 0, 1, 3\,(4)$. We proceed by analyzing the $E_\infty$-page of the **aAHSS** $\left(B^{\mathbb{R}}_0(1)^{\otimes 4k+1}\right)$ one coweight at a time.

$\boxed{\textbf{Coweight } cw \equiv 0\,(4)}$

We begin by analyzing the coweight $cw \equiv 0\,(4)$ portion of the spectral sequence. This consists of classes coming from Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 0\,(4)$ and from Atiyah–Hirzebruch filtration 3 in coweight $cw \equiv 2\,(4)$. There is a $d_1$-differential between the $h_0$-towers of Atiyah–Hirzebruch filtrations 2 and 3 in coweight $cw \equiv 2\,(4)$. Letting $x_j$ be a generator for $\Sigma^{4j-4,4\lfloor j/2\rfloor-2}H$, this differential is determined by

$$d_1(x_j[3]) = h_0 \cdot x_j[2].$$

There is also a $d_1$-differential between big staircases $\Sigma^{2i,i}\mathcal{S}$ in Atiyah–Hirzebruch filtrations 2 and 3 in coweight $cw \equiv 2\,(4)$. The classes in each staircase in Atiyah–Hirzebruch filtration 3 which do not support a $d_1$ differential are those which do not support an $h_0$-tower. These classes cannot support any further differentials for degree reasons, and so they survive to the $E_\infty$-page.

The $d_2$-differential between Atiyah–Hirzebruch filtrations 0 and 2 is given by the $h_1$-attaching map in $B^{\mathbb{R}}_0(1)$. In particular, since the $h_0$-towers from Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 0\,(4)$ do not support $h_1$ multiplication, each tower survives to $E_3$. None of these $h_0$-towers can be the target of a $d_3$ differentials for degree reasons, so they also survive to the $E_\infty$-page. The $d_2$-differential between the big flags $\mathcal{F}_{4i,i,2i}$ in Atiyah–Hirzebruch filtration 2 and 3 in coweight $cw \equiv 0\,(4)$ is determined by the structure given in Definition 6.1.2. In particular, the classes in the big flag in Atiyah–Hirzebruch filtration 0 which survive to $E_3$ are those classes which are not $h_1$-divisible. For $i = 4k$, these are precisely the $h_0$-towers in stems $s \equiv i+4\,(8)$, the $J$-towers in stems $s \equiv i\,(8)$ with Adams filtration $f \geq 1$, and the $(\rho, h_0)$-tower in Adams filtration $f=0$. For degree reasons, the $d_3$-differential is trivial on Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 0\,(4)$.

Thus, in coweight $cw \equiv 0$, the $E_\infty$-page is given by the $h_0$-towers from Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 0, (4)$ which do not come from the big flag:

$$\bigoplus_{k=0}^{2k} \Sigma^{4j-4, 4\lfloor (j+1)/2 \rfloor - 4} H;$$

the $h_0$-towers, $J$-towers and $(\rho, h_0)$-tower remaining from the big flag $\mathcal{F}_{4i,i,2i}$ in Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 0\,(4)$; and the remaining classes from the big staircase $\Sigma^{2i,i}\mathcal{S}$ in Atiyah–Hirzebruch filtration 3 in coweight $cw \equiv 2\,(4)$. Base change to $\mathbb{C}$ and linearity give hidden extensions between the $h_0$-towers and $J$-towers from Atiyah–Hirzebruch filtration 0 and the remaining classes from the big staircase in Atiyah–Hirzebruch filtration 3. To be precise, the classes in $\Sigma^{2i,i}\mathcal{S}[3]$ in smallest stem and Adams filtration which survive to $E_\infty$ are represented by

$$\Sigma^{2i,i}(h_1 \cdot (\tau^2 h_0))[3], \quad \Sigma^{2i,i}(h_1^2 \cdot (\tau^2 h_0))[3], \quad \Sigma^{2i,i}(\rho a)[3],$$

and their $\tau^4$-multiples. Let $x_{i-4}[0]$ denote the generator of $H \subseteq \mathcal{F}_{4i,i,2i}$ from Atiyah–Hirzebruch filtration 0 in lowest stem and Adams-filtration, and let $x_{i-5}[0]$ denote the generator of $J \subseteq \mathcal{F}_{4i,i,2i}$ from the same Atiyah–Hizebruch filtration. Base change to $\mathbb{C}$ gives hidden extensions

$$h_0 \cdot \Sigma^{2i,i}(h_1 \cdot (\tau^2 h_0))[3] = x_{i-4}[0], \quad h_1 \cdot \Sigma^{2i,i}(\rho a)[3] = \rho x_{i-5}[0].$$

Then, linearity over $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$ gives us that

$$h_1 \cdot \Sigma^{2i,i}(h_1^2 \cdot \tau^2 h_0)[3] = h_1 \cdot \Sigma^{2i,i}(\rho^2 a)[3] = \rho \cdot h_1 \cdot \Sigma^{2i,i}(\rho a)[3] = \rho^2 x_{i-5}[0]$$

and

$$h_0 \cdot \Sigma^{2i,i}(\rho a)[3] = h_0 \cdot \Sigma^{2i,i}(h_1(\tau h_1)^2)[3] = h_1 \cdot \Sigma^{2i,i}(h_0(\tau h_1)^2)[3] = h_1 \cdot \Sigma^{2i,i}(\rho^2 a)[3] = \rho^2 x_{i-5}[0].$$

Multiplication by $b$ gives the same extensions throughout the $E_\infty$-page. Altogether, this gives the summand

$$\Sigma^{2i+4,i+2}\mathcal{S}\langle 1\rangle.$$

For degree reasons, the elements in the $(\rho, h_0)$-tower from Atiyah–Hirzebruch filtration 0 which are divisible by $\rho^3$ must be $v_1$-torsion by inspection. Since our result is modulo this torsion, this gives the last summand

$$\Sigma^{2i-2,i-1}J.$$

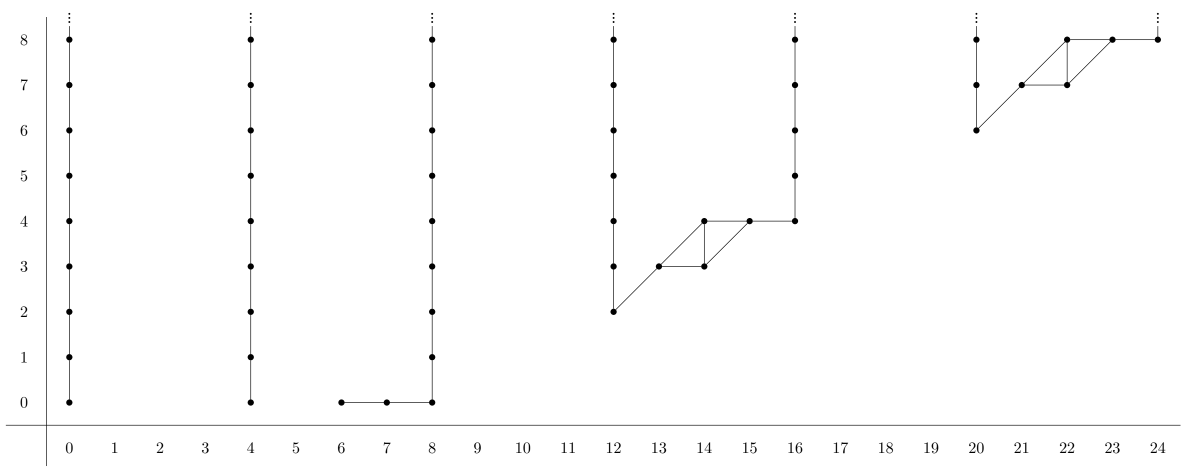


Figure 6.19: $Z_5^{\mathbb{R}}$ in $cw \equiv 0\,(4)$.

**Coweight** $cw \equiv 1\,(4)$

The coweight $cw \equiv 1\,(4)$ portion of the spectral sequence consists of classes coming from Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 1\,(4)$ and from Atiyah–Hirzebruch filtration 2 in coweight $cw \equiv 0\,(4)$. The $d_1$-differential between Atiyah–Hirzebruch filtrations 2 and 3 kills all of the classes from Atiyah–Hirzebruch filtration 2 in coweight $cw \equiv 0\,(4)$ which are divisible by $h_0$. The $d_2$-differential between Atiyah–Hirzebruch filtrations 0 and 2 kills all of the classes from Atiyah–Hirzebruch filtration 2 in coweight $cw \equiv 0\,(4)$ which support multiplication by

$h_1$. By observation, we see that every class remaining from the big flag $\mathcal{F}_{4i,i,2i}$ on the $E_2$-page is of this kind. Moreover, the filtration forces $E_3 = E_\infty$ on Atiyah–Hirzebruch filtration 2, so the only classes which survive are the bases of the $h_0$-towers

$$\bigoplus_{j=1}^{2k} \Sigma^{4j-4,4\lfloor (j+1)/2 \rfloor -4}\mathbb{F}_2[\tau^4].$$

The $d_2$-differential between Atiyah–Hirzebruch filtrations 0 and 2 kills all of the classes from Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 1\,(4)$ which are divisible by $h_1$. This can be recovered by appropriately shifting the analogous $d_2$-differential in the $\mathbf{aAHSS}(B_0^{\mathbb{R}}(1))$ (see Fig. 6.9). The $d_3$-differential between Atiyah–Hirzebruch filtrations 0 and 3 is determined by the formula $d_3(\rho[3]) = (\tau h_1)[0]$ established in Lemma 6.2.1. In particular, each remaining class from Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 1$ is the target of a differential. Similar to the $d_2$-differential just mentioned, this can be recovered by appropriately shifting the analogous $d_3$-differential in the $\mathbf{aAHSS}(B_0^{\mathbb{R}}(1))$ (see Fig. 6.11). Thus, there are no contributions to the $E_\infty$-page in coweight $cw \equiv 1\,(4)$ from Atiyah–Hirzebruch filtration 0. Notice that this forces the classes in the summand $\bigoplus_{j=0}^{2k} \Sigma^{4j-4,4\lfloor (j+1)/2 \rfloor -4}\mathbb{F}_2[\tau^4]$ to be $v_1$-torsion, hence the $E_\infty$-page is trivial in coweight $cw \equiv 1\,(4)$.

$\boxed{\textbf{Coweight } cw \equiv 2\,(4)}$

The coweight $cw \equiv 2\,(4)$ portion of the spectral sequence consists of classes coming from Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 2\,(4)$, from Atiyah–Hirzebruch filtration 2 in coweight $cw \equiv 1\,(4)$, and from Atiyah–Hirzebruch filtration 3 in coweight $cw \equiv 0\,(4)$. The $d_2$-differential between Atiyah–Hirzebruch filtrations 0 and 2 kills all of the classes from Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 2\,(4)$ which are divisible by $h_1$. Notice that while the behavior on the big staircase $\Sigma^{2i,i}\mathcal{S}$ can be recovered from the analogous differential in the $\mathbf{aAHSS}(B_0^{\mathbb{R}}(1))$ (see Fig. 6.10), the summand

$$\bigoplus_{j=1}^{2k} \Sigma^{4j-4,4\lfloor j/2 \rfloor -2} H$$

in Atiyah–Hirzebruch filtration 0 is entirely $h_1$-torsion, hence survives to $E_3$. The $d_3$-differential between Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 2\,(4)$ and Atiyah–Hirzebruch filtration 3 $cw \equiv 1\,(4)$ can be recovered from the $\mathbf{aAHSS}(B_0^{\mathbb{R}}(1))$ (see Fig. 6.12). Notice that there can be no differentials whose target are the $h_0$-towers $\bigoplus_{j=1}^{2k} \Sigma^{4j-4,4\lfloor j/2 \rfloor -2} H$ for degree reasons. This determines the contributions to the $E_\infty$-page in coweight $cw \equiv 2\,(4)$ coming from Atiyah–Hirzebruch filtration 0.

The $d_1$-differential between Atiyah–Hirzebruch filtrations 2 and 3 in coweight $cw \equiv 1\,(4)$ can be recovered from the $\mathbf{aAHSS}(B_0^{\mathbb{R}}(1))$ (see Fig. 6.6). Similarly, the $d_2$-differential between Atiyah–Hirzebruch filtrations 0 and 2 in coweight $cw \equiv 1\,(4)$ can be recovered from the $\mathbf{aAHSS}(B_0^{\mathbb{R}}(1))$ (See Fig. 6.9). For degree reasons, we have that $E_3 = E_\infty$ on Atiyah–Hirzebruch filtration 2, so this determines the contributions to the $E_\infty$-page in coweight $cw \equiv 2\,(4)$ from Atiyah–Hirzebruch filtration 2.

The $d_1$-differential between Atiyah–Hirzebruch filtrations 2 and 3 kills all classes from Atiyah–Hirzebruch filtration 3 in coweight $cw \equiv 0\,(4)$ which support $h_0$-multiplication. In particular, this kills all $h_0$-towers in the summand $\bigoplus_{j=1}^{2k} \Sigma^{4j-4,r\lfloor (j+1)/2 \rfloor -4} H$ and in the big flag $\mathcal{F}_{4i,i,2i}$. The $d_3$-differential between Atiyah–Hirzebruch filtrations 0 and 3 is determined by the formula $d_3(\rho[3]) = (\tau h_1)[0]$ established in Lemma 6.2.1. In particular, using the notation of Definition 6.1.2, we have

$$d_3(\rho \cdot x_{4k}[3]) = (\Sigma^{2i,i} \tau h_1)[0].$$

Linearity gives the rest of the differentials. Note that this is the differential which ensures that Atiyah–Hirzebruch filtration 0 is trivial in coweight $cw \equiv 1\,(4)$ on the $E_\infty$-page.

Thus, in coweight $cw \equiv 2\,(4)$, the $E_\infty$-page is given by the remaining $h_0$-towers from the big staircase $\Sigma^{2i,i}\mathcal{S}$ from Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 2\,(4)$; the $h_0$-towers which do not come from the big flag

$$\bigoplus_{j=1}^{2k} \Sigma^{4j-4,4\lfloor j/2 \rfloor -2} H$$

from Atiyah–Hirzebruch filtration 0 in coweight $cw \equiv 2\,(4)$; the remaining $\mathbb{F}_2[\tau^4]$'s from Atiyah–Hirzebruch filtration 2 in coweight $cw \equiv 2\,(4)$; and the remaining submodule of the big flag $\mathcal{F}_{4i,i,2i}$ from Atiyah–Hirzebruch filtration 3 in coweight $cw \equiv 0\,(4)$. Base change to $\mathbb{C}$ and linearity give the remaining hidden extensions, which we discuss now.

There is a hidden $h_0$-extension between each $\mathbb{F}_2[\tau^4]$ from Atiyah–Hirzebruch filtration 2 and the $h_0$-tower's concentrated in stems $s \equiv i + 4\,(8)$ from Atiyah–Hirzebruch filtration 0. Additionally, there is a hidden $h_0$-extension between each $\mathbb{F}_2[\tau^4]$ from Atiyah–Hirzebruch filtration 2 and the remaining submodule of the big flag $\mathcal{F}_{4i,i,2i}$ from Atiyah–Hirzebruch filtration 3. For example, there is an $h_0$-extension

$$h_0 \cdot h_1 x_{4k}[3] = \Sigma^{2i,i}(h_1 \cdot \tau h_1)[2].$$

Finally, there is a hidden $h_0$-extension between the $h_0$-towers concentrated in stems $s \equiv i\,(8)$ from Atiyah–Hirzebruch filtration 0 and the remaining submodule of the big flag $\mathcal{F}_{4i,i,2i}$ from Atiyah–Hirzebruch filtration 3. For example, there is an $h_0$-extension

$$h_0 \cdot \rho^3 x_{4k}[3] = \Sigma^{2i,i}(\tau^2 h_0)[0].$$

By inspection, we see that the hidden extensions assemble all of the remaining classes on the $E_\infty$-page into the big flag

$$\mathcal{F}_{4i+4,1+1,2i+2}.$$

$\boxed{\textbf{Coweight } cw \equiv 3\,(4)}$

Finally, the coweight $cw \equiv 3\,(4)$ portion of the spectral sequence consists of classes coming from Atiyah–Hirzebruch filtration 2 in coweight $cw \equiv 2\,(4)$ and from Atiyah–Hirzebruch filtration 3 in coweight $cw \equiv 1\,(4)$. The $d_1$-differential between Atiyah–Hirzebruch filtrations 2 and 3 kills all classes from Atiyah–Hirzebruch filtration 2 in coweight $cw \equiv 2\,(4)$ which are $h_0$-divisible. In particular, only the bases of the $h_0$-towers $\bigoplus_{j=1}^{2k} \Sigma^{4j-4,4\lfloor j/2 \rfloor -2} H$ survive, to the $E_2$-page:

$$\bigoplus_{j=1}^{2k} \Sigma^{4j-4,4\lfloor j/2 \rfloor -2} \mathbb{F}_2[\tau^4],$$

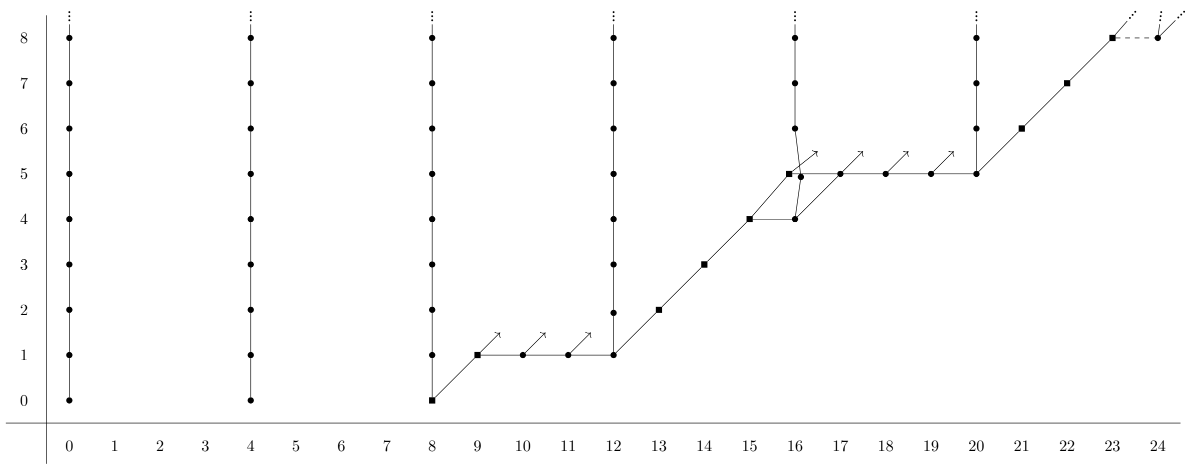

Figure 6.20: $Z_5^{\mathbb{R}}$ in $cw \equiv 2\,(4)$.

and only bases of the $h_0$-towers in the big staircase $\Sigma^{2i,i}\mathcal{S}$, as well as the classes that are not $h_0$-divisible, survive. The $d_2$-differential between Atiyah–Hirzebruch filtrations 0 and 2 kills all classes from Atiyah–Hirzebruch filtration 2 in coweight $cw \equiv 2\,(4)$ which support $h_1$-multiplication. The classes remaining from the $h_0$-towers that are not from the big staircase do not support $h_1$-multiplication, and neither do the two classes $b^n a$ and $b^n \rho a$ remaining from the staircases, so they survive to the $E_3$-page. For degree reasons, we have that $E_3 = E_\infty$ on Atiyah–Hirzebruch filtration 2, so this determines the contributions to the $E_\infty$-page in coweight $cw \equiv 3\,(4)$ from Atiyah–Hirzebruch filtration 2.

The $d_1$-differential between Atiyah–Hirzebruch filtrations 2 and 3 kills all classes from Atiyah–Hirzebruch filtration 3 in coweight $cw \equiv 1\,(4)$ which support $h_0$-multiplication. This can be recovered from the **aAHSS**$(B_0^{\mathbb{R}}(1))$. The $d_3$-differential between Atiyah–Hirzebruch filtrations 3 and 0 is only nonzero on the bottom left corner of the remnants of the diamond. This determines the contributions to the $E_\infty$-page in coweight $cw \equiv 3\,(4)$ from Atiyah–Hirzebruch filtration 3.

Thus, in coweight $cw \equiv 1\,(4)$, the $E_\infty$-page is given by the segments $T$ coming from Atiyah–Hirzebruch filtrations 2 and 3 and the bases of the $h_0$-towers from Atiyah–Hirzebruch filtration 2. These segments are connected by hidden extensions in the following way. Let $\Sigma^{2i,i}(h_1(\tau h_1))[3]$ and $\Sigma^{2i,i}\rho h_1(\tau h_1)[3]$ be the classes remaining from Atiyah–Hirzebruch filtration 3 in lowest Adams filtration, and let $\Sigma^{2i,i}a[2]$ and $\Sigma^{2i,i}\rho a[2]$ be the classes remaining from Atiyah–Hirzebruch filtration 2 in lowest Adams filtration. Base change to $\mathbb{C}$ gives a hidden extension

$$h_1 \cdot \Sigma^{2i,i}(h_1(\tau h_1))[3] = a[2].$$

Linearity over $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$ gives us that

$$h_1 \cdot \Sigma^{2i,i}(\rho h_1(\tau h_1))[3] = \rho a[2]$$

and

$$h_0 \cdot \Sigma^{2i,i}(h_1(\tau h_1))[3] = h_1 \cdot \Sigma^{2i,i}(h_0(\tau h_1))[3] = h_1 \cdot \Sigma^{2i,i}(\rho h_1(\tau h_1))[3] = \rho a[2].$$

Multiplication by $b$ gives the same extensions throughout the $E_\infty$-page. Altogether, this gives the summand

$$\Sigma^{2i+4,i+2}\mathcal{D}\langle 1\rangle.$$

For degree reasons, the bases of the $h_0$-towers from Atiyah–Hirzebruch filtration 2 are $v_1$-torsion.

Thus, we have shown that for $i = 4k+1$, $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right)/v_1$-torsion is given by the direct sum of columns of Table 6.5. The other congruence classes are proved in a similar fashion. $\square$

**Remark 6.2.10.** It is possible that one can recover Theorem 6.2.9 by using the $\rho$-Bockstein spectral sequence and [CQ21, Lemma 3.36]. This spectral sequence has signature

$$E_1 = \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{C}}}\left(B_0^{\mathbb{C}}(1)^{\otimes i}\right)[\rho] \implies \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(1)^{\otimes i}\right).$$

While we do not analyze this spectral sequence here, we remark that the extension problems we encounter throughout the algebraic Atiyah–Hirzebruch process are the same extension problems one encounters in the $\rho$-Bockstein spectral sequence.

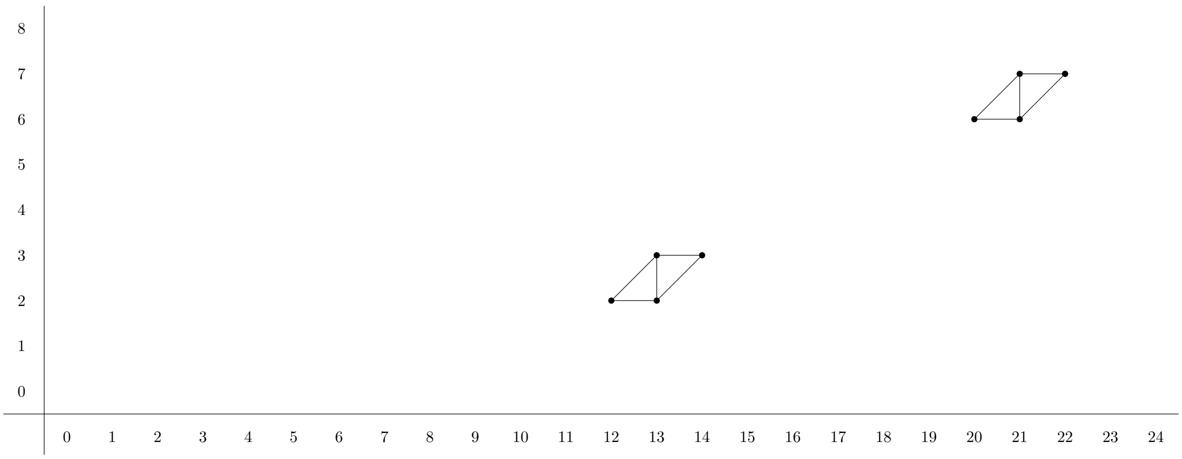


Figure 6.21: $Z_5^{\mathbb{R}}$ in $cw \equiv 3\,(4)$.

## *6.3 The ring of cooperations*

In this section, we continue the inductive procedure and compute the groups $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(k))$. Then, we assemble our results to compute the ring of cooperations $\pi^{\mathbb{R}}_{**}(\mathrm{kq} \otimes \mathrm{kq})$.

### *6.3.1* $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(k))$

The goal of this section is to compute $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(k))$. We begin with the following lemma.

**Lemma 6.3.1.** *There is an isomorphism of $\mathcal{A}(1)^\vee_{\mathbb{R}}$-comodules and $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$-modules:*

$$\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee_{\mathbb{R}}}(B_1^{\mathbb{R}}(k)) \cong \bigoplus_{j=0}^{k} \Sigma^{4j,2j}\mathbb{F}_2[\rho, \tau^2, h_0]/(\rho h_0).$$

*Proof.* Let $C_{\mathcal{A}(0)^\vee}(B_1^{\mathbb{R}}(k))$ denote the cobar complex computing $\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee_{\mathbb{R}}}(B_1^{\mathbb{R}}(k))$. This is a cosimplicial ring which takes the form

$$B_1^{\mathbb{R}}(k) \longrightarrow \mathcal{A}(0)^\vee_{\mathbb{R}} \otimes B_1^{\mathbb{R}}(k) \longrightarrow \mathcal{A}(0)^\vee_{\mathbb{R}} \otimes \mathcal{A}(0)^\vee_{\mathbb{R}} \otimes B_1^{\mathbb{R}}(k) \longrightarrow \cdots$$

after totalization. Since the coaction of $\mathcal{A}(0)^\vee_{\mathbb{R}}$ on $B_1^{\mathbb{R}}(k)$ fixes $\rho$, all of the maps in $C_{\mathcal{A}(0)^\vee_{\mathbb{R}}}(B_1^{\mathbb{R}}(k))$ are $\rho$-linear. This implies that the filtration by powers of $\rho$

$$C_{\mathcal{A}(0)^\vee_{\mathbb{R}}}(B_1^{\mathbb{R}}(k)) \supset \rho \cdot C_{\mathcal{A}(0)^\vee_{\mathbb{R}}}(B_1^{\mathbb{R}}(k)) \supset \rho^2 \cdot C_{\mathcal{A}(0)^\vee_{\mathbb{R}}}(B_1^{\mathbb{R}}(k)) \supset \cdots$$

is a filtration of chain complexes. Moreover, we have the following identifications

$$\mathbb{M}_2^{\mathbb{R}}/\rho \cong \mathbb{M}_2^{\mathbb{C}}, \quad \mathcal{A}(0)^\vee_{\mathbb{R}}/\rho = \mathcal{A}(0)^\vee_{\mathbb{C}}, \quad B_1^{\mathbb{R}}(k)/\rho = B_1^{\mathbb{C}}(k).$$

Thus, the filtration on $C_{\mathcal{A}(0)^\vee_{\mathbb{R}}}(B_1^{\mathbb{R}}(k))$ gives a $\rho$-Bockstein spectral sequence (see Section 6.1.3) of the form:

$$E_1 = \mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee_{\mathbb{C}}}(B_1^{\mathbb{C}}(k))[\rho] \implies \mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee_{\mathbb{R}}}(B_1^{\mathbb{R}}(k)).$$

The groups $\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee_{\mathbb{C}}}(B_1^{\mathbb{C}}(k))$ were calculated by Culver-Quigley [CQ21, Remark 3.25]. Modulo $v_1$-torsion, we have

$$\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee_{\mathbb{C}}}(B_1^{\mathbb{C}}(k)) \cong \bigoplus_{j=0}^{k} \Sigma^{4j,2j}\mathbb{M}_2^{\mathbb{C}}[h_0].$$

In the case of $k = 0$, we have that $B_1^{\mathbb{C}}(0) \cong \mathbb{M}_2^{\mathbb{C}}$, and this $\rho$-Bockstein spectral sequence was completely determined by Hill [Hil11, Theorem 3.1]. We can rewrite the $E_1$-page as

$$E_1 = \mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee_{\mathbb{C}}}(\mathbb{M}_2^{\mathbb{C}})[\rho] = \mathbb{M}_2^{\mathbb{C}}[h_0][\rho],$$

with a differential $d_1(\tau) = \rho h_0$. The Liebniz rule immediately determines the end result:

$$\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}) \cong \mathbb{F}_2[\rho, \tau^2, h_0]/(\rho h_0).$$

More generally, the $E_1$-page of the $\rho$-Bockstein takes the form

$$E_1 = \bigoplus_{j=0}^{k} \Sigma^{4j,2j}\mathbb{M}_2^{\mathbb{C}}[h_0][\rho],$$

and the differentials can be determined using the cobar complex. In particular, there are differentials

$$d_1(\Sigma^{4j,2j}\tau) = \Sigma^{4j,2j}(\rho h_0)$$

for each $0 \le j \le k$ on the classes $\tau$ in each $H$-tower in the decomposition of $\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee_{\mathbb{C}}}(B_1^{\mathbb{C}}(k))$. The Liebniz rule immediately determines the end result:

$$\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee_{\mathbb{R}}}(B_1^{\mathbb{R}}(k)) \cong \bigoplus_{j=0}^{k} \Sigma^{4j,2j}\mathbb{F}_2[\rho, \tau^2, h_0]/(\rho h_0),$$

concluding the proof. $\square$

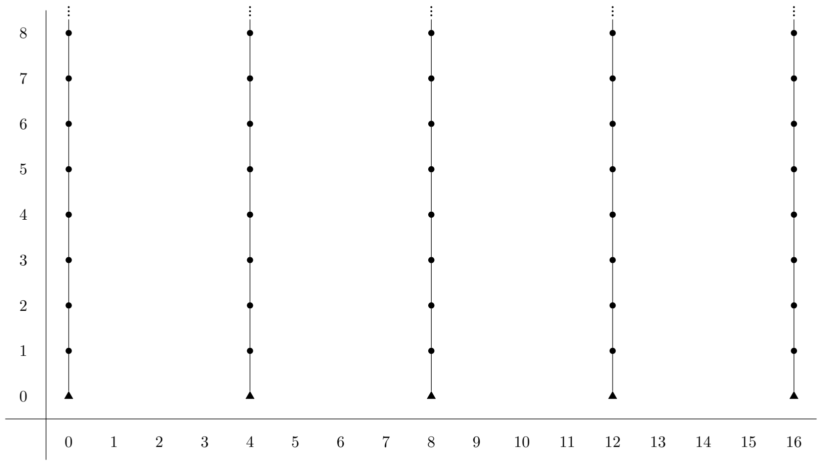


Figure 6.22: The group $\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee_{\mathbb{R}}}(B_1^{\mathbb{R}}(k))$. A black ▲ denotes $\mathbb{F}_2[\rho, \tau^2]$, and a black ● denotes $\mathbb{F}_2[\tau^2]$. A vertical black line represents multiplication by $h_0$.

We will now use the results of Theorem 6.2.9 and the short exact sequences of motivic Brown–Gitler comodules to culminate our algebraic Atiyah–Hirzebruch program. The following result should be compared with [Mah81, Prop 2.6], [BOSS19, Prop 3.3], and Theorem 5.3.1.

**Theorem 6.3.2.** *There is an isomorphism of $\mathcal{A}(1)^\vee_{\mathbb{R}}$-comodules and $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$-modules:*

$$\frac{\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(k))}{v_1\text{-torsion}} \cong \Sigma^{4k-4,2k-2} Z^{\mathbb{R}}_{\alpha(k)} \oplus \bigoplus_{j=0}^{4k-8} \left(\Sigma^{4j,2j} H \oplus \Sigma^{4j,2j-2} H\right),$$

*where $\alpha(k)$ is the number of 1's in the dyadic expansion of $k$. There are $\tau^2$-extensions between $\Sigma^{4j,2j}H$ and $\Sigma^{4j,2j-2}H$.*

*Proof.* We will induct on $k$ and use the short exact sequences of Proposition 2.3.5. For $k = 1$, this was shown in Corollary 6.2.4. We assume now the theorem is true for all $i < k$ and divide the proof into two cases: when $k$ is even, or when $k$ is odd.

Suppose that $k$ is even. Proposition 2.3.5 provides us with a short exact sequence of

$\mathcal{A}(1)^\vee_{\mathbb{R}}$-comodules:

$$0 \to \Sigma^{2k,k} B_0^{\mathbb{R}}(\tfrac{k}{2}) \to B_0^{\mathbb{R}}(k) \to B_1^{\mathbb{R}}(\tfrac{k}{2}-1) \otimes (\mathcal{A}(1)_{\mathbb{R}} \,/\!/\, \mathcal{A}(0))^\vee_{\mathbb{R}} \to 0. \tag{6.3.1}$$

Applying $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(-)$ gives a long exact sequence of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}})$-modules. Moreover, after killing $v_1$-torsion the connecting homomorphism is trivial, giving a short exact sequence whose middle term is $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(k))$. Therefore, the Ext group in question decomposes into the Ext groups of the kernel and cokernel of the short exact sequence (6.3.1). By the inductive hypothesis, we know that

$$\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(\Sigma^{2k,k} B_0^{\mathbb{R}}(\tfrac{k}{2})\right) \cong \Sigma^{2k,k}\left(\Sigma^{2k-4,k-2} Z^{\mathbb{R}}_{\alpha(k/2)} \oplus \bigoplus_{j=0}^{2k-8} (\Sigma^{4j,2j} H \oplus \Sigma^{4j,2j-2} H)\right).$$

A change of rings isomorphism gives an isomorphism for the right-hand side:

$$\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_1^{\mathbb{R}}(\tfrac{k}{2}-1) \otimes (\mathcal{A}(1) \,/\!/\, \mathcal{A}(0))^\vee_{\mathbb{R}}\right) \cong \mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee_{\mathbb{R}}}\left(B_1^{\mathbb{R}}(\tfrac{k}{2}-1)\right).$$

The group $\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee_{\mathbb{R}}}(B_1^{\mathbb{R}}(\frac{k}{2}-1))$ was calculated in Lemma 6.3.1. Thus the right-hand side and left-hand side assemble to give

$$\begin{aligned}\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(k)) \cong \Sigma^{4k-4,2k,2} Z^{\mathbb{R}}_{\alpha(k)} &\oplus \bigoplus_{j=0}^{2k-8} \left(\Sigma^{4j+2k,2j+2k} H \oplus \Sigma^{4j+4k,2j-2+2k} H\right)\\ &\oplus \bigoplus_{j=0}^{k/2-1} \Sigma^{4j,2j}\mathbb{F}_2[\rho,\tau^2,h_0](\rho h_0),\end{aligned}$$

using that $\alpha(\frac{k}{2}) = \alpha(k)$. Notice that for degree reasons, any $\rho$-divisible class coming from $\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee_{\mathbb{R}}}(B_1^{\mathbb{R}}(\frac{k}{2}-1))$ is $v_1$-torsion. Combining and reindexing the $H$-towers in the second and third summands gives the result.

Suppose now that $k$ is odd. Proposition 2.3.5 provides us with a different short exact sequence of $\mathcal{A}(1)^\vee_{\mathbb{R}}$-comodules:

$$0 \to \Sigma^{2(k-1),k-1} B_0^{\mathbb{R}}(\tfrac{k-1}{2}) \otimes B_0^{\mathbb{R}}(1) \to B_0^{\mathbb{R}}(k) \to B_1^{\mathbb{R}}(\tfrac{k-1}{2}-1) \otimes (\mathcal{A}(1) \,/\!/\, \mathcal{A}(0))^\vee_{\mathbb{R}} \to 0. \tag{6.3.2}$$

Again, modulo $v_1$-torsion the connecting homomorphism in the long exact sequence obtained by applying $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(-)$ is trivial, giving a short exact sequence whose middle term is

$\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(k))$. We proceed by analyzing the Ext groups of the kernel and cokernel of the short exact sequence (6.3.2). A change of rings isomorphism gives the right-hand side:

$$\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_1^{\mathbb{R}}(\tfrac{k-1}{2}-1)\otimes(\mathcal{A}(1)\,/\!/\,\mathcal{A}(0))^\vee_{\mathbb{R}}\right)\cong\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee_{\mathbb{R}}}\left(B_1^{\mathbb{R}}(\tfrac{k-1}{2}-1)\right).$$

The group $\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee_{\mathbb{R}}}\left(B_1^{\mathbb{R}}(\tfrac{k-1}{2}-1)\right)$ was calculated in Lemma 6.3.1. To calculate the left-hand side, we can use the **aAHSS** $\left(B_0^{\mathbb{R}}(\tfrac{k-1}{2})\otimes B_0^{\mathbb{R}}(1)\right)$. This spectral sequence is obtained by applying the functor $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(\tfrac{k-1}{2})\otimes -\right)$ to the cellular filtration for $B_0^{\mathbb{R}}(1)$ and has signature

$$E_1=\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(\tfrac{k-1}{2})\right)\otimes\mathbb{M}_2^{\mathbb{R}}\{[1],[\overline{\xi}_1],[\overline{\tau}_1]\}\implies\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(\tfrac{k-1}{2})\otimes B_0^{\mathbb{R}}(1)\right).$$

By the inductive hypothesis, we have an isomorphism modulo $v_1$-torsion

$$\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(\tfrac{k-1}{2})\right)\cong\Sigma^{2k-6,k-3}Z^{\mathbb{R}}_{\alpha(k)-1}\oplus\bigoplus_{j=0}^{2k-10}\left(\Sigma^{4j,2j}H\oplus\Sigma^{4j,2j-2}H\right),\tag{6.3.3}$$

using that $\alpha(\tfrac{k-1}{2})=\alpha(k)-1$. This splitting descends to a splitting of the spectral sequence, and we may analyze each summand of (6.3.3) individually. The left-hand summand is handled by Theorem 6.2.9, as it is isomorphic to the **aAHSS** $\left(\Sigma^{2k-6,k-3}B_0^{\mathbb{R}}(1)^{\otimes\alpha(k)}\right)$, so we have a summand of $\Sigma^{2k-2,k-1}Z^{\mathbb{R}}_{\alpha(k)}$ in $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(\tfrac{k-1}{2})\otimes B_0^{\mathbb{R}}(1)\right)$. The algebraic Atiyah–Hirzebruch spectral sequence on the right-hand summand collases on $E_2$. The remaining classes in Adams filtration 0 are $v_1$-torsion, and so the result is isomorphic to the original right-hand summand. Thus, we have an isomorphism:

$$\begin{aligned}&\Sigma^{2(k-1),k-1}\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}\left(B_0^{\mathbb{R}}(\tfrac{k-1}{2})\otimes B_0^{\mathbb{R}}(1)\right)\\ \cong{}&\Sigma^{2(k-1),k-1}\left(\Sigma^{2k-2,k-1}Z^{\mathbb{R}}_{\alpha(k)}\oplus\bigoplus_{j=0}^{2k-10}\left(\Sigma^{4j,2j}H\oplus\Sigma^{4j,2j-2}H\right)\right).\end{aligned}$$

Assembling the right-hand side and left-hand side gives

$$\begin{aligned}\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(B_0^{\mathbb{R}}(k))\cong{}&\Sigma^{4k-4,2k-2}Z^{\mathbb{R}}_{\alpha(k)}\oplus\bigoplus_{j=0}^{2k-10}\left(\Sigma^{4j+2(k-1),2j+k-1}H\oplus\Sigma^{4j+2(k-1),2j-2+k-1}H\right)\\ &\oplus\bigoplus_{j=0}^{(k-1)/2-1}\Sigma^{4j,2j}\mathbb{F}_2[\rho,\tau^2,h_0]/(\rho h_0).\end{aligned}$$

As before, for degree reasons, any $\rho$-divisible class coming from $\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee_{\mathbb{R}}}(B_1^{\mathbb{R}}\left(\frac{k-1}{2}-1\right))$ is $v_1$-torsion. Combining and reindexing the $H$-towers in the second and third summands gives the result. $\square$

**Remark 6.3.3.** The summand $\bigoplus_{j=0}^{4k-8}(\Sigma^{4j,2j}H \oplus \Sigma^{4j,2j-2}H)$ appearing in this decomposition is a ghost of the failure to express these Ext groups in terms of Adams covers of kq and ksp. It was shown in [Hil11] that the motivic Adams spectral sequence for $\mathrm{H}\mathbb{Z}$ collapses at the $E_2$-page and has signature

$$E_2 = \mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}) \cong \mathbb{F}_2[\rho, \tau^2, h_0]/(\rho h_0) \implies \pi^{\mathbb{R}}_{**}(\mathrm{H}\mathbb{Z}).$$

In the Adams covers for kq and ksp, one sees summands of $\pi^{\mathbb{R}}_{**}\mathrm{H}\mathbb{Z}$. In the context of our calculations, we only obtain summands of $\pi^{\mathbb{R}}_{**}(\mathrm{H}\mathbb{Z}/\rho)$.

### *6.3.2 The ring of cooperations*

In this subsection only, all results are implicitly computed modulo $v_1$-torsion. We assemble our results to compute the ring of cooperations $\pi^{\mathbb{R}}_{**}(\mathrm{kq}\otimes\mathrm{kq})$. First, our computation in Theorem 6.3.2 gives the following:

**Corollary 6.3.4.** *The $E_2$-page of the* **mASS**(kq $\otimes$ kq) *is given by*

$$E_2 = \bigoplus_{k\geq 0}\mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee_{\mathbb{R}}}(\Sigma^{4k,2k}B_0^{\mathbb{R}}(k)) \cong \bigoplus_{k\geq 0}\left(\Sigma^{4k-4,2k-2}Z^{\mathbb{R}}_{\alpha(k)} \oplus \bigoplus_{j=0}^{4k-8}\left(\Sigma^{4j,2j}H\oplus\Sigma^{4j,2j-2}H\right)\right).$$

*Proof.* The left hand equality, which is true even in the presence of $v_1$-torsion, is the content of Theorem 4.1.1. The right hand isomorphism follows from Theorem 6.3.2. $\square$

We now arrive at our main result.

**Theorem 6.3.5.** *The* **mASS**(kq $\otimes$ kq) *collapses on the $E_2$-page.*

*Proof.* There is a base change functor

$$-\otimes\mathbb{C} : \mathrm{SH}(\mathbb{R}) \to \mathrm{SH}(\mathbb{C})$$

which induces a highly structured morphism with the $\mathbb{C}$-motivic Adams spectral sequence for $\pi^{\mathbb{C}}_{**}(\mathrm{kq} \otimes \mathrm{kq})$. In particular, [CQ21, Corollary 3.43] implies that there can be no differentials in the **mASS**$(\mathrm{kq} \otimes \mathrm{kq})$ with both the source and target being $\tau$-free. However, by Corollary 6.3.4 we see that every class on the $E_2$-page is $\tau^4$-periodic. Hence there can be no differentials and the spectral sequence collapses.. $\square$

**Remark 6.3.6.** Determining the differentials in the kq-resolution is much more difficult. By complex Betti realization, we can deduce much of the behavior of the complex kq-resolution from the bo-resolution (see Corollary 2.1.4). However, there is information in the real kq-resolution which is both $\rho$-torsion and $v_1$-periodic that is not detected by these methods. We plan to analyze the real kq-resolution in future work using a combination of these methods and $C_2$-equivariant homotopy theory.

# Chapter 7

# COMPUTATIONS IN SH($\mathbb{F}_q$) WITH A TRIVIAL BOCKSTEIN ACTION

## 7.1 Background

Throughout, let $\mathbb{F}_q$ be a finite field with $\mathrm{char}(\mathbb{F}_q) \neq 2$. In this section, we review motivic cohomology and Hermitian K-theory, the dual Steenrod algebra, and Brown–Gitler comodules in the context of $\mathrm{SH}(\mathbb{F}_q)$. Then, we describe the $\mathrm{H}\mathbb{F}_2$ and kq-based motivic Adams spectral sequences and outline our plan to computing the ring of cooperations.

### 7.1.1 Motivic cohomology and Hermitian K-theory

Spitzweck defined the *integral motivic cohomology spectrum*, denoted $\mathrm{H}\mathbb{Z} \in \mathrm{SH}(\mathbb{Z})$, representing motivic cohomology with integral coefficients [Spi18]. The unique map $f : \mathbb{Z} \to \mathbb{F}_q$ induces a pullback $f^* : \mathrm{SH}(\mathbb{F}_q) \to \mathrm{SH}(\mathbb{Z})$; we abusively denote $\mathrm{H}\mathbb{Z} := f^*(\mathrm{H}\mathbb{Z}) \in \mathrm{SH}(\mathbb{F}_q)$ the integral motivic cohomology spectrum over $\mathbb{F}_q$. This represents motivic cohomology in the sense that if $X \in \mathrm{Sm}_{\mathbb{F}_q}$, then

$$[X, \Sigma^{s,w}\mathrm{H}\mathbb{Z}] = \mathrm{H}^{s,w}(X; \mathbb{Z}),$$

where we let $X$ also denote the motivic suspension spectrum of $X$. Of particular interest to us is when $X = \mathrm{Spec}(\mathbb{F}_q)$. As we will be computing with the 2-primary motivic Adams spectral sequence, we are only concerned with the 2-completion $\mathrm{H}^{p,q}(\mathbb{F}_q; \mathbb{Z})^\wedge_2$. These groups were calculated by Soulé [Sou79]:

$$\mathrm{H}^{-s,-w}(\mathbb{F}_q; \mathbb{Z})^\wedge_2 = \begin{cases} \mathbb{Z}_2 & s = w = 0 \\ \mathbb{Z}/(q^{-w} - 1)_2 & s = -1, w \leq 1 \\ 0 & \text{else.} \end{cases} \tag{7.1.1}$$

Notice that $\mathrm{H}^{-s,-w}(\mathbb{F}_q;\mathbb{Z}) = \pi^{\mathbb{F}_q}_{s,w}(\mathrm{H}\mathbb{Z})$, so this also calculates the homotopy groups of the integral motivic cohomology spectrum.

There is also a *mod-2 motivic cohomology spectrum*, denoted $\mathrm{H}\mathbb{F}_2$. By [Voe03a], we have that

$$\mathrm{H}^{-s,-w}(\mathbb{F}_q;\mathbb{Z}/2) = \pi^{\mathbb{F}_q}_{s,w}(\mathrm{H}\mathbb{F}_2) := \mathbb{M}_2^{\mathbb{F}_q} = (\mathrm{K}^{\mathrm{M}}_*(\mathbb{F}_q)/2)[\tau],$$

where $\mathrm{K}^{\mathrm{M}}_*(\mathbb{F}_q)$ denotes the *Milnor K-theory* of $\mathbb{F}_q$, and where $|\tau| = (0,-1)$ and $|\mathrm{K}^{\mathrm{M}}_n(\mathbb{F}_q)| = (-n,-n)$. Recall that Milnor K-theory $\mathrm{K}^{\mathrm{M}}_*(\mathbb{F}_q)$ is defined as the graded tensor algebra over $\mathbb{Z}$ generated by the symbols $[a]$ in degree one for $a \in \mathbb{F}_q^\times$, subject to the relations $[a]\otimes[1-a] = 0$ and $[a]+[b] = [ab]$. It is a routine exercise to show that $\mathrm{K}^{\mathrm{M}}_n(\mathbb{F}_q)$ vanishes in degrees $n \geq 2$. In particular, since every element of $\mathbb{F}_q^\times$ is either a square or a non square, this implies that

$$(\mathrm{K}^{\mathrm{M}}_0(\mathbb{F}_q)/2) = (\mathrm{K}^{\mathrm{M}}_1(\mathbb{F}_q)/2) = \mathbb{Z}/2.$$

To differentiate between the cases of $q \equiv 1\,(4)$ and $q \equiv 3\,(4)$, we let $u$ be any generator of $\mathrm{K}^{\mathrm{M}}_1(\mathbb{F}_q)$ for $q \equiv 1\,(4)$, and let $\rho = [-1]$ be a generator of $\mathrm{K}^{\mathrm{M}}_1(\mathbb{F}_q)$ for $q \equiv 3\,(4)$. Notice that if $q \equiv 1\,(4)$, then $-1$ has a square root in $\mathbb{F}_q$, hence $[-1] = 0 \in \mathrm{K}^{\mathrm{M}}_1(\mathbb{F}_q)$, so that $u$ is not represented by $-1$. Then we have identifications:

$$\mathbb{M}_2^{\mathbb{F}_q} = \begin{cases} \mathbb{F}_2[u,\tau]/(u^2) & q \equiv 1\,(4) \\ \mathbb{F}_2[\rho,\tau]/(\rho^2) & q \equiv 3\,(4), \end{cases}$$

where $|\tau| = (0,-1)$ and $|u| = |\rho| = (-1,-1)$.

There is a *Hermitian K-theory* spectrum, denoted KQ [Hor05; CHN25] which is the main character of our work. The homotopy groups $\pi^{\mathbb{F}_q}_{**}(\mathrm{KQ})$ intertwine the classical stable homotopy of the orthogonal, unitary, and symplectic K-groups of $\mathbb{F}_q$. We will only be interested in the orthogonal K-groups $\mathrm{KO}_*(\mathbb{F}_q) := \pi_*(\mathrm{BO}(\mathbb{F}_q)^+)$ and the symplectic K-groups $\mathrm{KSp}_*(\mathbb{F}_q) := \pi_*(\mathrm{BSp}(\mathbb{F}_q)^+)$; for more details, see [Hor05]. These are realized by Hermitian K-theory in the following way:

$$\pi^{\mathbb{F}_q}_{s,w}(\mathrm{KQ}) = \begin{cases} \mathrm{KO}_{s-2w}(\mathbb{F}_q) & w \equiv 0\,(4); \\ \mathrm{KSp}_{s-2w}(\mathbb{F}_q) & w \equiv 2\,(4). \end{cases}$$

The Hermitian K-theory spectrum is an $\mathbb{E}_\infty$ motivic ring spectrum [CHN25]. We let kq denote the very effective cover of KQ [ARØ20]. The ring structure on KQ lifts to kq, and the unit map $\mathbb{S} \to \mathrm{kq}$ induces Morel's isomorphism [RSØ19]

$$\pi^{\mathbb{F}_q}_{-n,-n}(\mathbb{S}) \cong \mathrm{K}^{\mathrm{MW}}_n(\mathbb{F}_q).$$

In particular, $\pi^{\mathbb{F}_q}_{0,0}(\mathrm{kq}) \cong \mathrm{GW}(\mathbb{F}_q)$, the Grothendieck–Witt ring. We will not need all of the details of the very effective slice filtration (see [Ban25, Section 1] for a wonderful recollection). For the purposes of our work, one should treat the very effective cover as an analogue of the connective cover. Indeed, for $s \geq 0$ there is an isomorphism $\pi^{\mathbb{F}_q}_{s,w}(\mathrm{KQ}) \cong \pi^{\mathbb{F}_q}_{s,w}(\mathrm{kq})$. However, we will see that the homotopy groups of kq do not vanish for $s < 0$ (an interpretation of these homotopy groups in negative stem degrees is given in Remark 7.3.2 and Remark 8.2.2).

There are cofiber sequences of spectra:

$$\Sigma^{1,1}\mathrm{kq} \xrightarrow{\eta} \mathrm{kq} \to \mathrm{kgl},$$

$$\Sigma^{2,1}\mathrm{kgl} \xrightarrow{\beta} \mathrm{kgl} \to \mathrm{H}\mathbb{Z},$$

where kgl denotes the effective algebraic K-theory spectrum, $\eta$ is the motivic Hopf map, and $\beta$ is the Bott periodicity class [ARØ20]. Combining the last morphism of each cofiber sequence gives a map

$$\mathrm{kq} \to \mathrm{H}\mathbb{Z} \tag{7.1.2}$$

which will feature heavily in our arguments.

Related to Hermitian K-theory is the *very effective symplectic K-theory* spectrum ksp, which is defined as the very effective cover of $\Sigma^{4,2}\mathrm{KQ}$. The homotopy groups of ksp are a shifted version of the homotopy groups of kq, which we investigate in more detail in later sections. We note that there is a cofiber sequence

$$\Sigma^{4,2}\mathrm{kq} \to \mathrm{ksp} \to \mathrm{H}\mathbb{Z} \tag{7.1.3}$$

coming from the very effective slice filtration on $\Sigma^{4,2}\mathrm{KQ}$. For more details on ksp and the very effective slice tower for KQ, see [Mor26, Section 3].

The Hermitian K-theory of finite fields was calculated by Friedlander [Fri76, Theorem 1.7]. We display the relevant results in Table 7.1.

| $n$ modulo 8 | $\mathrm{KO}_n(\mathbb{F}_q)$ | $\mathrm{KSp}_n(\mathbb{F}_q)$ |
|---|---|---|
| 0 | $\mathbb{Z}/2$ | 0 |
| 1 | $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ | 0 |
| 2 | $\mathbb{Z}/2$ | 0 |
| 3 | $\mathbb{Z}/(q^{(n+1)/2} - 1)$ | $\mathbb{Z}/(q^{(n+1)/2} - 1)$ |
| 4 | 0 | $\mathbb{Z}/2$ |
| 5 | 0 | $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ |
| 6 | 0 | $\mathbb{Z}/2$ |
| 7 | $\mathbb{Z}/(q^{(n+1)/2} - 1)$ | $\mathbb{Z}/(q^{(n+1)/2} - 1)$ |

Table 7.1: Friedlander's calculation of $\mathrm{KO}_n(\mathbb{F}_q)$ and $\mathrm{KSp}_n(\mathbb{F}_q)$, where $\mathrm{char}(\mathbb{F}_q) \neq 2$ and $n > 0$.

### *7.1.2 The dual Steenrod algebra and Brown–Gitler comodules*

The *dual Steenrod algebra* $\mathcal{A}^\vee := \pi_{**}^{\mathbb{F}_q}(\mathrm{H}\mathbb{F}_2 \otimes \mathrm{H}\mathbb{F}_2)$ was calculated in positive characteristic by Hoyois–Kelley–Østvær [HKØ17, Proposition 4.3]:

$$\mathcal{A}^\vee = \mathbb{M}_2^{\mathbb{F}_q}[\overline{\tau}_0, \overline{\tau}_1, \ldots, \overline{\xi}_1, \overline{\xi}_2, \ldots]/T,$$

where $|\overline{\tau}_i| = (2^{i+1} - 1, 2^i - 1)$ and $|\overline{\xi}_i| = (2^{i+1} - 2, 2^i - 1)$. The ideal $T$ of relations is dependent on the base field $\mathbb{F}_q$. In the cases we are concerned with, we have that

$$T = \begin{cases} (\overline{\tau}_i^2 = \tau\overline{\xi}_{i+1}) & q \equiv 1\,(4) \\ (\overline{\tau}_i^2 = \tau\overline{\xi}_{i+1} + \rho\overline{\tau}_{i+1} + \rho\overline{\tau}_0\overline{\xi}_{i+1}) & q \equiv 3\,(4). \end{cases}$$

Note that the pair $(\mathbb{M}_2^{\mathbb{F}_q}, \mathcal{A}^\vee)$ is a Hopf algebroid rather than a Hopf algebra [Rav86, Appendix A].

For $n \geq 0$, let $\mathcal{A}(n)^\vee$ be the quotient algebra

$$\mathcal{A}(n)^\vee \cong \mathcal{A}^\vee/(\overline{\xi}_1^{2^n}, \overline{\xi}_2^{2^{n-1}}, \ldots, \overline{\xi}_n^2, \overline{\xi}_{n+1}, \overline{\xi}_{n+2,}, \ldots, \overline{\tau}_{n+1}, \overline{\tau}_{n+2}, \ldots).$$

There is a sub-Hopf algebra of the motivic Steenrod algebra

$$\mathcal{A}(n) = \langle \mathrm{Sq}^1, \mathrm{Sq}^2, \ldots, \mathrm{Sq}^{2^n} \rangle;$$

let $(\mathcal{A} \mathbin{/\!/} \mathcal{A}(n))^\vee$ be the subalgebra of $\mathcal{A}^\vee$ given by

$$(\mathcal{A} \mathbin{/\!/} \mathcal{A}(n))^\vee = \mathbb{M}_2^{\mathbb{F}_q}[\overline{\xi}_1^{2^n}, \overline{\xi}_2^{2^{n-1}}, \ldots, \overline{\tau}_{n+1}, \ldots]/(\overline{\tau}_i^2 = \rho\overline{\tau}_{i+1} + \rho\overline{\tau_0}\overline{\xi}_{i+1} + \tau\overline{\xi}_{i+1}).$$

These subalgebras naturally arise via motivic homology: one can show that there are $\mathcal{A}^\vee$-comodule algebra isomorphisms

$$\mathrm{H}_{**}(\mathrm{H}\mathbb{Z}) \cong (\mathcal{A} \mathbin{/\!/} \mathcal{A}(0))^\vee, \quad \mathrm{H}_{**}(\mathrm{kq}) \cong (\mathcal{A} \mathbin{/\!/} \mathcal{A}(1))^\vee$$

by using the long exact sequence in homology associated to the cofiber sequences [ARØ20]

$$\mathrm{H}\mathbb{Z} \xrightarrow{\times 2} \mathrm{H}\mathbb{Z} \to \mathrm{H}\mathbb{F}_2, \quad \Sigma^{1,1}\mathrm{kq} \xrightarrow{\eta} \mathrm{kq} \to \mathrm{kgl}.$$

We next introduce motivic Brown–Gitler comodules. We define the *Mahowald weight filtration* on $\mathcal{A}^\vee$ by setting

$$\mathrm{wt}(\overline{\xi}_i) = \mathrm{wt}(\overline{\tau}_i) = 2^i, \quad \mathrm{wt}(\tau) = \mathrm{wt}(\rho) = \mathrm{wt}(u) = 0$$

and letting $\mathrm{wt}(xy) = \mathrm{wt}(x) + \mathrm{wt}(y)$. This naturally extends to the subalgebras $(\mathcal{A} \mathbin{/\!/} \mathcal{A}(n))^\vee$. The *motivic Brown–Gitler comodule* $B_n^{\mathbb{F}_q}(k)$ is the $\mathcal{A}(n)^\vee$-comodule

$$B_n^{\mathbb{F}_q}(k) = \langle x \in (\mathcal{A} \mathbin{/\!/} \mathcal{A}(n))^\vee : \mathrm{wt}(x) \leq 2^{n+1}k \rangle.$$

A key property of the Brown–Gitler comodules that we will use is the following.

**Theorem 7.1.1** ([CQ21, Theorem 3.20])**.** *There is an isomorphism of $\mathcal{A}(1)^\vee$-comodules*

$$(\mathcal{A} \mathbin{/\!/} \mathcal{A}(1))^\vee \cong \bigoplus_{k \geq 0} \Sigma^{4k,2k} B_0^{\mathbb{F}_q}(k).$$

The $0^{th}$ Brown–Gitler comodule is given by $B_0^{\mathbb{F}_q}(0) \cong \mathbb{M}_2^{\mathbb{F}_q}$, and the first Brown–Gitler comodule is given by $B_0^{\mathbb{F}_q}(1) \cong \mathbb{M}_2^{\mathbb{F}_q}\{1, \overline{\xi}_1, \overline{\tau}_1\}$. It was shown in [CQ21, Example 3.12] that there is a spectrum $\mathrm{H}\mathbb{Z}_1^{\mathbb{F}_q}$ such that

$$\mathrm{H}_{**}(\mathrm{H}\mathbb{Z}_1^{\mathbb{F}_q}) \cong B_0^{\mathbb{F}_q}(1)$$

as $\mathcal{A}^\vee$-comodules. Additionally, there is an equivalence of spectra up to 2-completion [Mor26, Proposition 3.7]:

$$\mathrm{ksp} \simeq \mathrm{kq} \otimes \mathrm{H}\mathbb{Z}_1^{\mathbb{F}_q}. \tag{7.1.4}$$

In general, the question of finding a motivic spectrum realizing a Brown–Gitler comodule is not simple to answer; see [Mor26, Remark 2.17].

There are short exact sequences of $\mathcal{A}(1)^\vee$-comodules relating motivic Brown–Gitler comodules. We refer the reader to [CQ21] for a proof.

**Proposition 7.1.2** ([CQ21, Lemma 3.21])**.** *There are short exact sequences of $\mathcal{A}(1)^\vee$-comodules:*

$$0 \to \Sigma^{4k,2k} B_0^{\mathbb{F}_q}(k) \to B_0^{\mathbb{F}_q}(2k) \to B_1^{\mathbb{F}_q}(k-1) \otimes (\mathcal{A}(1) /\!/ \mathcal{A}(0))^\vee \to 0,$$

$$0 \to \Sigma^{4k,2k} B_0^{\mathbb{F}_q}(k) \otimes B_0^{\mathbb{F}_q}(1) \to B_0^{\mathbb{F}_q}(2k+1) \to B_1^{\mathbb{F}_q}(k-1) \otimes (\mathcal{A}(1) /\!/ \mathcal{A}(0))^\vee \to 0.$$

An interesting feature in the mod-2 motivic cohomology of finite fields is that, while $\mathbb{M}_2^{\mathbb{F}_q}$ is consistent across all fields of characteristic different from 2, the *Bockstein homomorphism*, which is the connecting homomorphism associated to the coefficient sequence

$$0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0,$$

acts differently on $\mathrm{H}^{s,w}(\mathbb{F}_q; \mathbb{Z}/2)$ depending on whether $q \equiv 1\,(4)$ or $q \equiv 3\,(4)$. Notice that this Bockstein is represented by $\mathrm{Sq}^1$ in the motivic Steenrod algebra [Voe03a]. We can phrase the action of the Bockstein in terms of the action of the motivic Steenrod algebra on $\mathbb{M}_2^{\mathbb{F}_q}$.

**Proposition 7.1.3** ([WØ17])**.** *The action of* $\mathrm{Sq}^1$ *on* $\mathbb{M}_2^{\mathbb{F}_q}$ *is given by* $\mathrm{Sq}^1(\tau) = \rho$*. In particular, the action is trivial if and only if* $q \equiv 1\,(4)$*.*

This implies that $\mathbb{M}_2^{\mathbb{F}_q}$ is central in $\mathcal{A}^\vee$ when $q \equiv 1\,(4)$, meaning that $\mathcal{A}^\vee$ is a Hopf algebra over $\mathbb{M}_2^{\mathbb{F}_q}$. We will often refer to the case of $q \equiv 1\,(4)$ as the *trivial Bockstein action* and the case of $q \equiv 3\,(4)$ as the *nontrivial Bockstein action*. This subtle difference makes a noticeable impact on computation and is the reason we separate our work into distinct sections.

### 7.1.3 Motivic Adams spectral sequence

Let $\mathrm{E} \in \mathrm{SH}(\mathbb{F}_q)$ be a motivic ring spectrum. There is a canonical Adams tower associated to the unit map $\mathbb{S} \to \mathrm{E}$ taking the form

$$\begin{tikzcd}
\mathbb{S} \arrow[d] & \Sigma^{-1,0}\overline{\mathrm{E}} \arrow[l] \arrow[d] & \Sigma^{-2,0}\overline{\mathrm{E}} \otimes \overline{\mathrm{E}} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\mathrm{E} & \Sigma^{-1,0}\overline{\mathrm{E}} \otimes \mathrm{E} & \Sigma^{-2,0}\overline{\mathrm{E}} \otimes \overline{\mathrm{E}} \otimes \mathrm{E} &
\end{tikzcd}$$

where $\overline{\mathrm{E}}$ is the cofiber of the unit map $\mathbb{S} \to \mathrm{E}$. For any motivic spectrum $X$, we can apply the functor $\pi_{**}^{\mathbb{F}_q}(X \otimes -)$. This yields the *E-based motivic Adams spectral sequence for* $X$. By construction, the $E_1$-page of this spectral sequence has signature

$$E_1 = \pi_{s+f,w}^{\mathbb{F}_q}(\mathrm{E} \otimes \overline{\mathrm{E}}^{\otimes f} \otimes X) \implies \pi_{s,f}^{\mathbb{F}_q}(X_{\mathrm{E}}), \quad d_r : E_r^{s,f,w} \to E_r^{s-1,f+r,w}$$

where $X_{\mathrm{E}}$ denotes the $E$-nilpotent completion of $X$ [Bou79]. Convergence is in general not immediate, but we will only be interested in particular cases that are well-understood.

For $\mathrm{E} = \mathrm{kq}$, the resulting motivic Adams spectral sequence is called the *kq-resolution*. For a general $X$, this takes the form

$$E_1 = \pi_{s+f,w}^{\mathbb{F}_q}(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes f} \otimes X) \implies \pi_{s,f}^{\mathbb{F}_q}(X).$$

Convergence of the kq-resolution was proven for $X = \mathbb{S}$ by Culver–Quigley in [CQ21, Theorem 2.1] over any base field of characteristic different than 2. They use the $\mathbb{C}$-motivic kq-resolution for the sphere to compute the $v_1$-periodic stable stems. We computed the $E_1$-page of the kq-resolution in $\mathbb{R}$-motivic homotopy theory in [Mor26]. The primary aim of this paper is to compute the $E_1$-page of the kq-resolution in $\mathbb{F}_q$-motivic homotopy theory.

For $\mathrm{E} = \mathrm{H}\mathbb{F}_2$, the resulting motivic Adams spectral sequence will be referred to as the $\mathbf{mASS}^{\mathbb{F}_q}(X)$. In this case, since $\mathcal{A}^\vee = \pi_{**}^{\mathbb{F}_q}(\mathrm{H}\mathbb{F}_2 \otimes \mathrm{H}\mathbb{F}_2)$ is flat as a module over $\mathbb{M}_2^{\mathbb{F}_q}$, we are able to pass directly to the $E_2$-page. The $\mathbf{mASS}^{\mathbb{F}_q}(X)$ takes the form

$$E_2 = \mathrm{Ext}_{\mathcal{A}^\vee}^{s,f,w}(\mathrm{H}_{**}(X)) \implies \pi_{s,f}^{\mathbb{F}_q}(X).$$

Convergence for this motivic Adams spectral sequence was first studied by [HKO11a; DI10]. We will study the ring of cooperations for kq by computing the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq} \otimes \mathrm{kq})$.

**Remark 7.1.4.** Another way to extract $v_1$-periodicity is by studying the $\mathrm{BPGL}\langle 1\rangle$-motivic Adams spectral sequence, where $\mathrm{BPGL}\langle 1\rangle$ is the first truncated motivic Brown–Peterson spectrum [HK01]. The study of this spectral sequence was initiated in [MPT25], where we computed the ring of cooperations at all primes and over the fields $\mathbb{C}, \mathbb{R}$, and $\mathbb{F}_q$. The truncated motivic Brown–Peterson cooperations actually offer a spectrum level decomposition, which is absent in the case of kq (see Remark 7.4.7). However, we expect that at the prime 2, $v_1$-periodicity will be more easily accessible by means of the kq-resolution. It will be interesting to draw comparisons between these two parallel approaches.

In this section, we compute the ring of cooperations $\pi_{**}^{\mathbb{F}_q}(\mathrm{kq} \otimes \mathrm{kq})$ in the case where there is a trivial Bockstein action on $\mathbb{F}_q$, that is, when $q \equiv 1\,(4)$. We first describe the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z})$, then show that the differentials lift to the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$, completely determining the behavior of the spectral sequence. After calculating the structure of the $E_2$-page of the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq}\otimes\mathrm{kq})$ as a module over the $E_2$-page of the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$, we are able to lift differentials again, determining the spectral sequence and the ring of cooperations. To avoid overloading notation, we will often omit notation referring to the base field, with the assumption that all computations in this chapter are made in $\mathrm{SH}(\mathbb{F}_q)$.

## *7.2 Integral homology*

We begin by computing the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z})$. This spectral sequence has signature

$$\mathrm{E}_2 = \mathrm{Ext}_{\mathcal{A}^\vee}^{s,f,w}(\mathrm{H}_{**}(\mathrm{H}\mathbb{Z})) \implies \pi_{s,w}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z}).$$

Recall that there is an isomorphism of $\mathcal{A}^\vee$-comodule algebras $\mathrm{H}_{**}(\mathrm{H}\mathbb{Z}) \cong (\mathcal{A} \mathbin{/\!/} \mathcal{A}(0))^\vee$. This allows us to rewrite the $E_2$-page of using a change of rings isomorphism as

$$\mathrm{Ext}^{s,f,w}_{\mathcal{A}^\vee}((\mathcal{A} \mathbin{/\!/} \mathcal{A}(0))^\vee) \cong \mathrm{Ext}^{s,f,w}_{\mathcal{A}(0)^\vee}(\mathbb{M}_2^{\mathbb{F}_q}).$$

This Ext group was calculated by Kylling to be [Kyl15, Theorem 4.1.2]

$$\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee}(\mathbb{M}_2^{\mathbb{F}_q}) \cong \mathbb{F}_2[u, \tau, h_0]/(u^2),$$

where $|h_0| = (1, 0, 0)$. We depict this $\mathrm{E}_2$-page in Figure 7.1. Note that all charts are depicted in $(s, f)$-grading, and that motivic weight is suppressed.

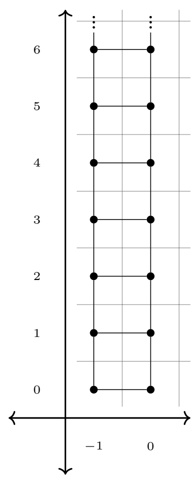


Figure 7.1: $\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee}(\mathbb{M}_2^{\mathbb{F}_q})$ for $q \equiv 1\,(4)$. A $\bullet$ denotes $\mathbb{F}_2[\tau]$. A vertical line represents multiplication by $h_0$, and a horizontal line represents multiplication by $u$.

We now determine all of the differentials in the spectral sequence.

**Lemma 7.2.1** ([Kyl15, Lemma 4.2.1])**.** *Let $q \equiv 1\,(4)$. The differentials in the* $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z})$ *are determined by*

$$d_{\nu_2(q-1)+\nu_2(i)}(\tau^i) = u\tau^{i-1}h_0^{\nu_2(q-1)+\nu_2(i)}.$$

*Proof.* Since $\pi^{\mathbb{F}_q}_{0,0}(\mathrm{H}\mathbb{Z}) \cong \mathbb{Z}_2$, any element in $\mathrm{E}_2^{0,f,0}$ must be a permanent cycle for all $f \geq 0$. This implies that 1 and all powers of $h_0$ are permanent cycles. Thus, if a class $x$ supports a differential, then $x$ must be divisible by $\tau$. Suppose that $y$ is not divisible by $\tau$ and that $x = \tau^k y$ for some $k \geq 1$. Then by the Liebniz rule, for any $r \geq 0$ we have

$$d_r(x) = d_r(y)\tau^k + y d_r(\tau^k) = y d_r(\tau^k).$$

Thus all differentials are uniquely determined by their value on $\tau$. Moreover, $\pi^{\mathbb{F}_q}_{0,-n}(\mathrm{H}\mathbb{Z}) = 0$ for any $n \geq 1$, so there can no $\tau$-divisibility in stem 0 on the $\mathrm{E}_\infty$-page. The particular values for the differentials are then a simple consequences of Equation (7.1.1) combined with the fact that Adams differentials preserve motivic weight. □

To illustrate the behavior of the differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z})$, we give an example.

**Example 7.2.1.** For example, take $q = 5$. We have that $\pi^{\mathbb{F}_5}_{-1,-1}(\mathrm{H}\mathbb{Z}) \cong \mathbb{Z}/4$, which implies that the class $uh_0^2$ is the class in stem $s = -1$ and weight $w = -1$ of lowest Adams filtration which must be the target of a differential. Since $\tau$ must die, this forces the differential

$$d_2(\tau) = uh_0^2.$$

The Liebniz rule then determines all $d_2$-differentials on odd powers of $\tau$. Similarly, we have $\pi^{\mathbb{F}_5}_{-1,-2}(\mathrm{H}\mathbb{Z}) \cong \mathbb{Z}/8$, so we must have that

$$d_3(\tau^2) = u\tau h_0^3,$$

and the Liebniz rule determines all $d_3$-differentials on $\tau^{2n}$ for $n$ odd.

We depict the $\mathrm{E}_\infty$-page in the case of $q = 5$ in Figure 7.2. An empty $\circ$ denotes $\mathbb{F}_2$. A class labeled with $\tau^n$ indicates the first nonzero power of $\tau$ which exists in that particular bidegree $(s, f)$. An empty class with $k$ circles around it indicates $\tau^{2^k}$-periodicity. For example, in bidegree $(-1, 2)$ we have $\mathbb{F}_2\{\rho\tau h_0^2\}[\tau^2]$, and in bidegree $(-1, 3)$ we have $\mathbb{F}_2\{\rho\tau^2 h_0^3\}[\tau^4]$.

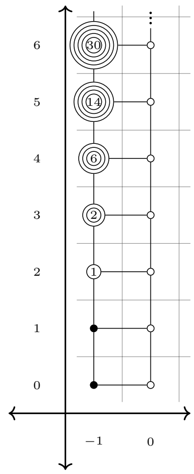


Figure 7.2: The $E_\infty$-page of the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z})$ for $q = 5$. A $\bullet$ denotes $\mathbb{F}_2[\tau]$. An empty $\circ$ denotes $\mathbb{F}_2$. An empty class with $k$ circles around it indicates $\tau^{2^k}$-periodicity. A vertical line represents multiplication by $h_0$, and a horizontal line represents multiplication by $u$. See Example 7.2.1 for more details.

### 7.3 Hermitian K-theory

We now compute the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$. This spectral sequence has signature

$$\mathrm{E}_2 = \mathrm{Ext}^{s,f,w}_{\mathcal{A}^\vee}(\mathrm{H}_{**}(\mathrm{kq})) \implies \pi^{\mathbb{F}_q}_{s,w}(\mathrm{kq}).$$

Recall that there is an isomorphism of $\mathcal{A}^\vee$-comodule algebras $\mathrm{H}_{**}(\mathrm{kq}) \cong (\mathcal{A} \,/\!/\, \mathcal{A}(1))^\vee$. This allows us to rewrite the $\mathrm{E}_2$-page using a change of rings isomorphism as

$$\mathrm{Ext}^{s,f,w}_{\mathcal{A}^\vee}((\mathcal{A} \,/\!/\, \mathcal{A}(1))^\vee) \cong \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{F}_q}).$$

This Ext group was calculated by Kylling to be [Kyl15, Theorem 4.1.2]:

$$\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{F}_q}) = \frac{\mathbb{F}_2[u, \tau, h_0, h_1, a, b]}{(u^2, h_0h_1, \tau h_1^3, h_1a, a^2 = h_0^2 b)}, \tag{7.3.1}$$

where $|h_0| = (1,0,0)$, $|h_1| = (1,1,1)$, $|a| = (4,3,2)$, and $|b| = (8,4,4)$. We note that this bares similarity to the $\mathbb{C}$-motivic analogue [Hil11]:

$$\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{C}}) = \frac{\mathbb{F}_2[\tau, h_0, h_1, a, b]}{(h_0h_1, \tau h_1^3, h_1 a, a^2 = h_0^2 b)},$$

with generators in the same tridegree as in Equation (7.3.1). Indeed, there is an abstract isomorphism

$$\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{F}_q}) \cong \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{C}}) \otimes \mathbb{M}_2^{\mathbb{F}_q}. \tag{7.3.2}$$

We depict this $E_2$-page in Figure 7.3. Be aware that the notation is slightly different than that of Figure 7.1 in that we suppress $u$-multiplication. In particular, the pair of $h_0$-towers in Figure 7.1 appears as a single $h_0$-tower in Figure 7.3.

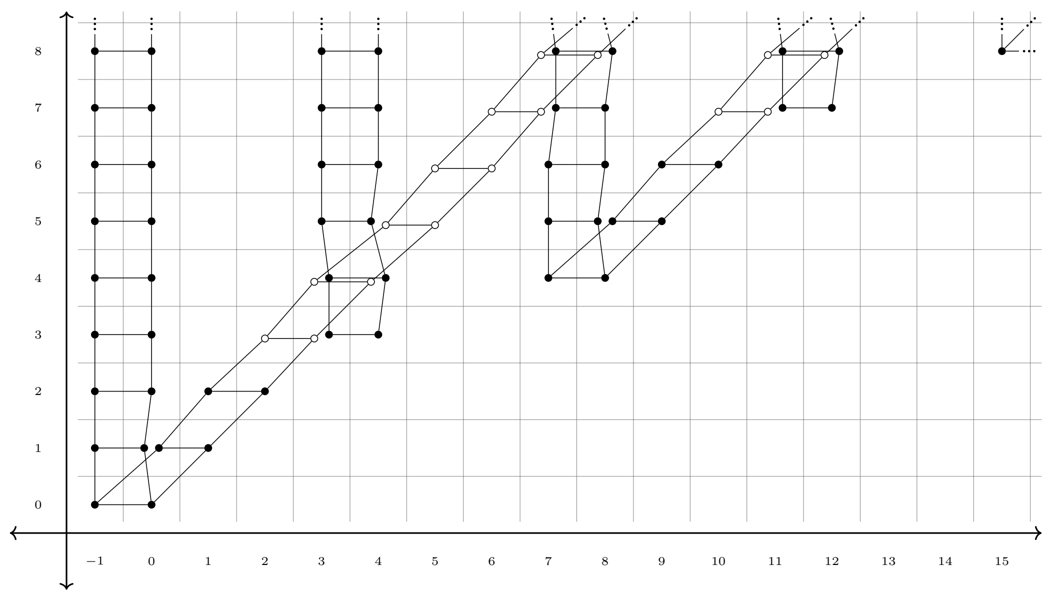


Figure 7.3: $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{F}_q})$ for $q \equiv 1\,(4)$. A $\blacksquare$ denotes $\mathbb{F}_2[u,\tau]/(u^2)$, and a $\square$ denotes $\mathbb{F}_2[u]/(u^2)$. A vertical line represents $h_0$-multiplication, and a diagonal line represents $h_1$-multiplication.

We will now show that we can lift the differentials from the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z})$ to the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$; see also [Kyl15, Section 4.2].

**Proposition 7.3.1.** *Let $q \equiv 1\,(4)$. The differentials in the* $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$ *are determined by the differentials in the* $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z})$.

*Proof.* We first determine the possible Adams differentials on the generators of the $E_2$-page. We see that $u$ cannot support a differential, as Adams differentials decrease stem degree by 1 and there are no classes in stem -2. Further, Adams differentials preserve weight. Since there are no classes in weight 0 in stem -1, as they are all divisible by $u$, we cannot have any differentials on 1 or any power of $h_0$. If $h_1$ were to support a differential, then degree considerations force it to be of the form $d_r(h_1) = \tau h_0^{r+1}$. However, this would imply that $d_r(h_1 h_0) = d_r(h_1)h_0 + h_1 d_r(h_0) = \tau h_0^{r+2}$, which is impossible since $h_1 h_0 = 0$. Thus $h_1$ does not support a differential. Finally, the only chance for $b$ to support a differential is $d_3(b) = h_1^7$, but since the weight of $b$ is 4 and the weight of $h_1^7$ is 7, this cannot happen. So, we see that if a class $x$ supports a differential, it must be $\tau$-divisible.

Recall the map $\mathrm{kq} \to \mathrm{H}\mathbb{Z}$ of Equation (7.1.2). This induces a map of spectral sequences $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq}) \to \mathbf{mASS}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z})$, which is realized on $E_2$-pages as the evident quotient map

$$\mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{F}_q}) = \frac{\mathbb{F}_2[u,\tau,h_0,h_1,a,b]}{(u^2, h_0h_1, \tau h_1^3, h_1 a, a^2 = h_0^2 b)} \to \mathbb{F}_2[u,\tau,h_0]/(u^2) = \mathrm{Ext}^{s,f,w}_{\mathcal{A}(0)^\vee}(\mathbb{M}_2^{\mathbb{F}_q}),$$

where $h_1, a, b \mapsto 0$. In particular, this implies that the differentials on powers of $\tau$ determined by Lemma 7.2.1 lift to the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$. Since $u, h_0, h_1, a$, and $b$ are permanent cycles, the Liebniz rule determines all differentials in the spectral sequence, finishing the proof. $\square$

Running the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$, we see that

$$\pi^{\mathbb{F}_q}_{s,w}(\mathrm{kq}) \cong \begin{cases} \mathrm{KO}_{s-2w}(\mathbb{F}_q) & w \equiv 0\,(4) \\ \mathrm{KSp}_{s-2w}(\mathbb{F}_q) & w \equiv 2\ (4) \end{cases}$$

for $s \geq 0$, agreeing with Friedlander's calculations given in Table 7.1.

**Remark 7.3.2.** By lifting differentials from the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z})$, we are able to more clearly identify the values of $\pi^{\mathbb{F}_q}_{s,w}(\mathrm{kq})$ for $s \leq 0$. The quotient map $\mathrm{kq} \to \mathrm{H}\mathbb{Z}$ induces an isomorphism of motivic Adams spectral sequence $E_2$-pages in stems $s \leq 0$, hence the differentials in the

$\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$ are isomorphic to the differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z})$ in this range. This gives an isomorphism $\pi^{\mathbb{F}_q}_{s,w}(\mathrm{kq}) \cong \pi^{\mathbb{F}_q}_{s,w}(\mathrm{H}\mathbb{Z})$ for $s \leq 0$. More generally, there we can express the homotopy of kq in terms of $\mathrm{H}\mathbb{Z}$:

$$\pi^F_{**}(\mathrm{kq}) \cong (\pi^F_{**}(\mathrm{H}\mathbb{Z}))[\eta, \tau^n\eta, \tau^n\eta^2, \alpha, \beta : n \geq 0]/(2\eta, \eta\alpha, \tau^n\eta\alpha, \tau^n\eta^2\alpha, \alpha^2 - 4\beta),$$

where $|\eta| = (1,1)$, $|\tau^n\eta| = (1, 1-n)$, $|\alpha| = (4,2)$, and $|\beta| = (8,4)$. Note that the additional generators $\tau^n\eta, \tau^n\eta^2$ stem from the fact that $\tau$ dies in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z})$ while $\tau\eta$ survives in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$.

## *7.4 Brown–Gitler comodules*

We begin the inductive process of computing the $E_2$-page of the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq} \otimes \mathrm{kq})$. To start, we will compute $\mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(1))$ by the $\mathbf{aAHSS}^{\mathbb{F}_q}(B_0^{\mathbb{F}_q}(1))$. This spectral sequence takes the form

$$E_1^{s,f,w,a} = \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)}(\mathbb{M}_2^{\mathbb{F}_q}) \otimes \mathbb{M}_2^{\mathbb{F}_q}\{[1], [\overline{\xi}_1], [\overline{\tau}_1]\} \implies \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)}(B_0^{\mathbb{F}_q}(1)).$$

We have discussed the differentials in this spectral sequence in Chapter 4. In particular, for $\alpha \in \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{F}_q})$, we have that

$$d_1(\alpha[3]) = h_0\alpha[2],$$

and

$$d_2(\alpha[2]) = h_1\alpha[0].$$

There is a potential third differential between Atiyah-Hirzebruch filtrations 3 and 0, but we see that for degree reasons this is not possible. Since the differentials are module maps over $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{F}_q})$, this determines the spectral sequence. Hidden extensions lift in the same way as in the $\mathbb{C}$-motivic case; see [CQ21, Section 3.4]. We depict the $E_\infty$-page in Figure 7.4.

In [Mor26, Theorem 3.7], we show that there is an equivalence of spectra

$$\mathrm{ksp} \simeq \mathrm{H}\mathbb{Z}_1^{\mathbb{F}_q} \otimes \mathrm{kq},$$

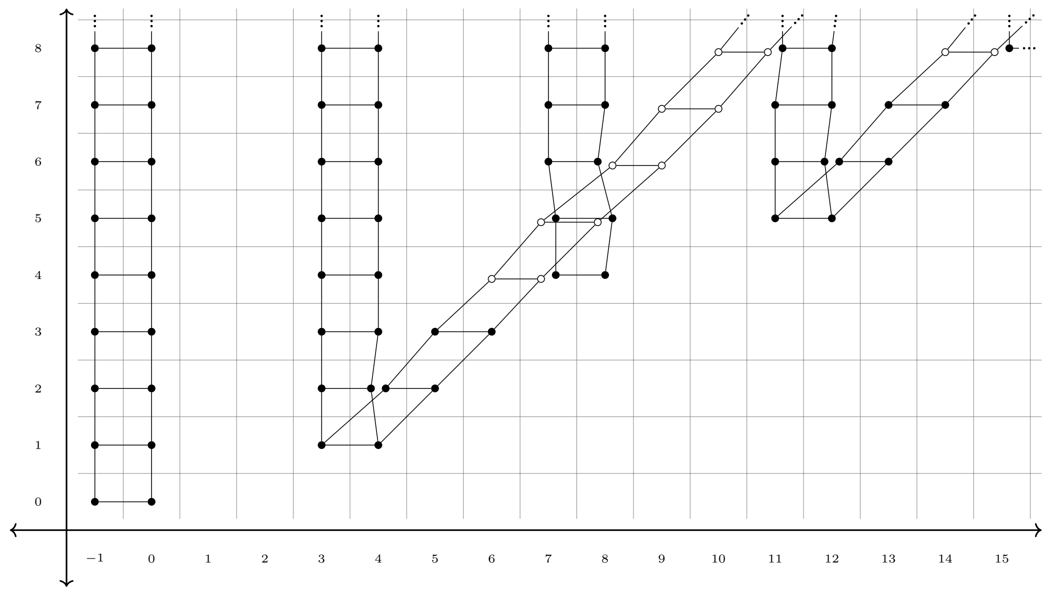


Figure 7.4: The $E_\infty$-page of the $\mathbf{aAHSS}(B_0^{\mathbb{F}_q}(1))$ for $q \equiv 1\,(4)$.

where ksp is the very effective symplectic K-theory spectrum and $\mathrm{H}\mathbb{Z}_1^{\mathbb{F}_q}$ is the spectrum realizing $B_0^{\mathbb{F}_q}(1)$ described in Section 7.1.2. The Künneth spectral sequence computing $\mathrm{H}_{**}(\mathrm{ksp})$ collapses, and so the change of rings isomorphism implies that the $E_2$-page of the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{ksp})$ is isomorphic to $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(1))$. However, there are differentials in this spectral sequence.

**Lemma 7.4.1.** *Let $q \equiv 1\,(4)$. In the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{ksp})$, the differentials are determined by the differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$ and the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z})$.*

*Proof.* Recall the cofiber sequence of Equation (7.1.3)

$$\Sigma^{4,2}\mathrm{kq} \to \mathrm{ksp} \to \mathrm{H}\mathbb{Z}.$$

By applying $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{H}_{**}(-))$ to this cofiber sequence, we get a long exact sequence in Ext groups. Notice that these Ext groups are the $E_2$-pages for the respective motivic Adams spectral sequences. For degree reasons, the connecting homomorphism

$$\delta : \mathrm{Ext}^{s,f,w}_{\mathcal{A}^\vee}(\mathrm{H}_{**}(\mathrm{H}\mathbb{Z})) \to \mathrm{Ext}^{s-1,f+1,w}_{\mathcal{A}^\vee}(\mathrm{H}_{**}(\Sigma^{4,2}\mathrm{kq}))$$

is trivial. Thus, after a few change of rings isomorphisms, we have a decomposition of the $E_2$-page of the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{ksp})$:

$$\mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(1)) \cong \Sigma^{4,2}\mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{F}_q})\langle 1\rangle \oplus \mathrm{Ext}^{s,f,w}_{\mathcal{A}(0)^\vee}(\mathbb{M}_2^{\mathbb{F}_q}),$$

where $\langle - \rangle$ denotes a shift in the Adams filtration degree. The projection $\mathrm{ksp} \to \mathrm{H}\mathbb{Z}$ induces evident quotient map on $E_2$-pages which lift the differentials from Lemma 7.2.1. The inclusion $\Sigma^{4,2}\mathrm{kq} \to \mathrm{ksp}$ induces the evident inclusion on $E_2$-page, lifting the differentials from Proposition 7.3.1 to the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{ksp})$ to this summand. For degree reasons there are no other differentials possible, finishing the proof. □

By running the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{ksp})$ and comparing with Table 7.1, we recover the expected isomorphism for $s \geq 4$

$$\pi^{\mathbb{F}_q}_{s,w}(\mathrm{ksp}) \cong \begin{cases} \mathrm{KSp}_{s-2w}(\mathbb{F}_q) & w \equiv 0\,(4); \\ \mathrm{KO}_{s-2w}(\mathbb{F}_q) & w \equiv 2\ (4). \end{cases}$$

We also obtain an isomorphism $\pi^{\mathbb{F}_q}_{s,w}(\mathrm{ksp}) \cong \pi^{\mathbb{F}_q}_{s,w}(\mathrm{kq})$ for $s \leq 0$.

The following is immediate from our description.

**Lemma 7.4.2.** *Let $q \equiv 1\,(4)$. There is an isomorphism*

$$\mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(1)) \cong \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{C}}(1)) \otimes \mathbb{M}_2^{\mathbb{F}_q}.$$

**Proposition 7.4.3.** *Let $q \equiv 1\,(4)$. For all $i \geq 0$, there is an isomorphism*

$$\frac{\mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(1)^{\otimes i})}{v_1\text{-torsion}} \cong \frac{\mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{C}}(1)^{\otimes i})}{v_1\text{-torsion}} \otimes \mathbb{M}_2^{\mathbb{F}_q}.$$

*Proof.* The case $i = 0$ was shown in Equation (7.3.2), and the case $i = 1$ was shown in Lemma 7.4.2. Assume the result holds for all $n \leq i$. We may calculate the Ext group in question by another algebraic Atiyah–Hirzebruch spectral sequence. For the rest of this proof, we implicitly compute modulo $v_1$-torsion. By applying the functor $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(1)^{\otimes i} \otimes -)$ to the cellular filtration of $B_0^{\mathbb{F}_q}(1)$, we obtain a spectral sequence with signature

$$E_1 = \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(1)^{\otimes i}) \otimes \mathbb{M}_2^{\mathbb{F}_q}\{[1], [\overline{\xi}_1], [\overline{\tau}_1]\} \implies \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(1)^{\otimes i+1}).$$

By induction, we can rewrite the $E_1$-page as

$$\left(\mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{C}}(1)^{\otimes i}) \otimes \mathbb{M}_2^{\mathbb{C}}\{1, [\overline{\xi}_1], [\overline{\tau}_1]\}\right) \otimes \mathbb{M}_2^{\mathbb{F}_q}.$$

As our spectral sequence is induced by the same filtration as the $\mathbf{aAHSS}(B_0^{\mathbb{F}_q}(1))$, the differentials are determined by the same formulae. For degree reasons, we see that there are no $d_3$-differentials. Hidden extensions follow as in the classical case. Since the differentials are the same as in the $\mathbb{C}$-motivic case by [CQ21, Lemma 3.36], we see that the abutment is isomorphic to

$$\mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{C}}(1)^{\otimes i+1}) \otimes \mathbb{M}_2^{\mathbb{F}_q},$$

which is what we wanted to show. □

**Theorem 7.4.4.** *Let $q \equiv 1\,(4)$. For all $k \geq 0$, there is an isomorphism*

$$\frac{\mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(k))}{v_1\text{-torsion}} \cong \frac{\mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{C}}(k))}{v_1\text{-torsion}} \otimes \mathbb{M}_2^{\mathbb{F}_q}.$$

*Proof.* We use the short exact sequences of $\mathcal{A}(1)^\vee$-comodules described in Proposition 7.1.2. For convenience, all Ext groups are implictly computed modulo $v_1$-torsion in this proof. The case of $k = 1$ was shown in Lemma 8.1.1. Now, suppose that the theorem is true for all $i < k$.

Suppose that $k$ is even. There is a short exact sequence of Brown–Gitler comodules

$$0 \to \Sigma^{2k,k} B_0^{\mathbb{F}_q}(\tfrac{k}{2}) \to B_0^{\mathbb{F}_q}(k) \to B_1^{\mathbb{F}_q}(\tfrac{k}{2} - 1) \otimes (\mathcal{A}(1) \mathbin{/\mkern-6mu/} \mathcal{A}(0))^\vee \to 0.$$

Applying the functor $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(-)$ gives a long exact sequence of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{F}_q})$-modules. We can use a change of rings isomorphism to rewrite the cokernel in Ext:

$$\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_1^{\mathbb{F}_q}(\tfrac{k}{2} - 1) \otimes (\mathcal{A}(1) \mathbin{/\mkern-6mu/} \mathcal{A}(0))^\vee) \cong \mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee}(B_1^{\mathbb{F}_q}(\tfrac{k}{2} - 1)).$$

As we are working mod $v_1$-torsion, the connecting homomorphism must be trivial. Thus, the Ext group in question decomposes into the Ext group of the kernel and the Ext group of the cokernel. The Ext group of the kernel, $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\Sigma^{2k,k} B_0^{\mathbb{F}_q}(\tfrac{k}{2}))$, is handled by the induction hypothesis. A simple computation in shows that the Ext group of the cokernel consists solely of a direct sum of $\mathbb{F}_2[u,\tau,h_0]/(u^2)$ indexed along the generators of $B_1^{\mathbb{F}_q}(\frac{k}{2}-1)$ as an $\mathcal{A}(0)^\vee$-comodule. In other words, since these summands are in the same stem as their $\mathbb{C}$-motivic counterpart [CQ21, Theorem 3.38], we have an isomorphism

$$\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee}(B_1^{\mathbb{F}_q}(\tfrac{k}{2}-1)) \cong \mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee}(B_1^{\mathbb{C}}(\tfrac{k}{2}-1)) \otimes \mathbb{M}_2^{\mathbb{F}_q}.$$

Altogether, we have

$$\begin{aligned}\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(k)) &\cong \big(\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\Sigma^{2k,k}B_0^{\mathbb{C}}(\tfrac{k}{2})) \oplus \mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee}(B_1^{\mathbb{C}}(\tfrac{k}{2}-1))\big) \otimes \mathbb{M}_2^{\mathbb{F}_q} \\ &\cong \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{C}}(k)) \otimes \mathbb{M}_2^{\mathbb{F}_q}\end{aligned}$$

where the last equality is due to [Mor26, Theorem 5.6].

Suppose that $k$ is odd. There is a short exact sequence of Brown–Gitler comodules

$$0 \to \Sigma^{2(k-1),k-1}B_0^{\mathbb{F}_q}(\tfrac{k-1}{2}) \otimes B_0^{\mathbb{F}_q}(1) \to B_0^{\mathbb{F}_q}(k) \to B_1^{\mathbb{F}_q}(\tfrac{k-1}{2}-1) \otimes (\mathcal{A}(1) \,/\!/\, \mathcal{A}(0))^\vee \to 0.$$

After applying $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(-)$ and using a change of rings isomorphism on the cokernel, the same argument as given above implies that the connecting homomorphism in the long exact sequence of Ext groups is trivial modulo $v_1$-torsion. Thus, we only need to understand the Ext group of the kernel and the Ext group of the cokernel. The Ext group of the cokernel follows from the same style of computation as in the previous case, giving an isomorphism

$$\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee}(B_1^{\mathbb{F}_q}(\tfrac{k-1}{2}-1)) \cong \mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee}(B_1^{\mathbb{C}}(\tfrac{k-1}{2}-1)) \otimes \mathbb{M}_2^{\mathbb{F}_q}.$$

The Ext group of the kernel may be calculated by another algebraic Atiyah–Hirzebruch spectral sequence. By applying the functor $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(\frac{k-1}{2}) \otimes -)$ to the cellular filtration on $B_0^{\mathbb{F}_q}(1)$, we obtain a spectral sequence with signature

$$E_1 = \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(\tfrac{k-1}{2})) \otimes \mathbb{M}_2^{\mathbb{F}_q}\{[1],[\overline{\xi}_1],[\overline{\tau}_1]\} \implies \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(\tfrac{k-1}{2}) \otimes B_0^{\mathbb{F}_q}(1)).$$

By the induction hypothesis, we have that

$$\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(\tfrac{k-1}{2})) \cong \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{C}}(\tfrac{k-1}{2})) \otimes \mathbb{M}_2^{\mathbb{F}_q}.$$

The differentials are induced by the cellular filtration on $B_0^{\mathbb{F}_q}(1)$, so by the same argument as given in Proposition 7.4.3, the differentials are the same as the $\mathbb{C}$-motivic case determined in [CQ21, Theorem 3.38] and there is no room for a $d_3$-differential. Thus the abutment is isomorphic to

$$\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{C}}(\tfrac{k-1}{2}) \otimes B_0^{\mathbb{C}}(1)) \otimes \mathbb{M}_2^{\mathbb{F}_q}.$$

Therefore, we have

$$\begin{aligned}\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(k)) &\cong \left(\mathrm{Ext}^{***}_{\mathcal{A}^\vee}(\Sigma^{2(k-1),(k-1)}B_0^{\mathbb{C}}(\tfrac{k}{2}-1)\otimes B_0^{\mathbb{C}}(1)) \oplus \mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee}(B_1(\tfrac{k-1}{2}-1))\right)\otimes \mathbb{M}_2^{\mathbb{F}_q}\\ &\cong \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{C}}(k))\otimes \mathbb{M}_2^{\mathbb{F}_q}\end{aligned}$$

where the last equivalence is again due to [Mor26, Theorem 5.6]. □

We can be more descriptive, but we must introduce some notation. For $M$ an $(s, f, w)$-trigraded module (such as an Ext group over the dual Steenrod algebra), let $\tau_{\geq n}M$ denote the truncation in the $s$-degree, so that $\tau_{\geq n}M = M$ for $s \geq n$ and $\tau_{\geq n}M = 0$ for $s < n$. Let $\mathrm{E} \in \mathrm{SH}(\mathbb{F}_q)$ be any motivic spectrum. Whenever E has a factor of $(\mathcal{A} /\!/ \mathcal{A}(1))^\vee$ in its mod-2 homology, we will use the notation $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{E})$ for the $E_2$-page of the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{E})$. Let $\mathrm{E}^{\langle m\rangle}$ denote the $m^{th}$ Adams cover of E, which is the $m^{th}$ term in a minimal $\mathrm{H}\mathbb{F}_2$-Adams resolution for E. Recall that a minimal Adams resolution is a diagram of the form [Rav86, Definition 2.1.3]

$$\begin{tikzcd}
\mathrm{E}^{\langle 0\rangle} := \mathrm{E} \arrow[d] & \mathrm{E}^{\langle 1\rangle} \arrow[l] \arrow[d] & \mathrm{E}^{\langle 2\rangle} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\mathrm{K}_0 \arrow[ur, dashed] & \mathrm{K}_1 \arrow[ur, dashed] & \mathrm{K}_2 \arrow[ur, dashed] &
\end{tikzcd}$$

and applying homotopy groups to this diagram gives the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{E})$. Here $\mathrm{K}_n := \bigoplus_B \mathrm{H}\mathbb{F}_2$ for a basis $B$ of the homology group $\mathrm{H}_{**}(\mathrm{E}^{\langle n\rangle})$, and $\mathrm{E}^{\langle n\rangle}$ is the fiber of the map $\mathrm{E}^{\langle n-1\rangle} \to \mathrm{K}_{n-1}$, defined inductively by taking $\mathrm{E}^{\langle 0\rangle} := \mathrm{E}$. The group $\mathrm{Ext}^{***}_{\mathcal{A}^\vee}(\mathrm{H}_{**}(\mathrm{E}^{\langle n\rangle}))$ can be obtained by described as

$$\mathrm{Ext}^{s,f,w}_{\mathcal{A}^\vee}(\mathrm{H}_{**}(\mathrm{E}^{\langle n\rangle})) \cong \begin{cases} \mathrm{Ext}^{s,f+n,w}_{\mathcal{A}^\vee}(\mathrm{H}_{**}(\mathrm{E})) & f \geq 0 \\ 0 & f < 0. \end{cases}$$

One may interpret this Ext group in charts by taking the charts for $\mathrm{Ext}^{***}_{\mathcal{A}^\vee}(\mathrm{H}_{**}(\mathrm{E}))$, relabeling the filtration $f = n$-line as $f = 0$, and then omitting anything below this now filtration 0 line. In other words, one slides the $s$-axis up to filtration $f = n$ and truncates all information in lower filtration.

Notice that $\tau_{\geq n}(\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{kq}^{\langle m\rangle}))$ and $\tau_{\geq n}(\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{ksp}^{\langle m\rangle}))$ have isolated non-nilpotent $h_1$-towers which are not connected to an $h_0$-tower. In these cases, we will use the modified truncation $\tau^{h_1}_{\geq n}(-)$ to indicate taking the truncation and then omitting out these isolated $h_1$-towers. As an example, we depict $\tau_{\geq 4}(\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{kq}^{\langle 2\rangle}))$ in Figure 7.5 and $\tau^{h_1}_{\geq 4}(\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{kq}^{\langle 2\rangle}))$ in Figure 7.6.

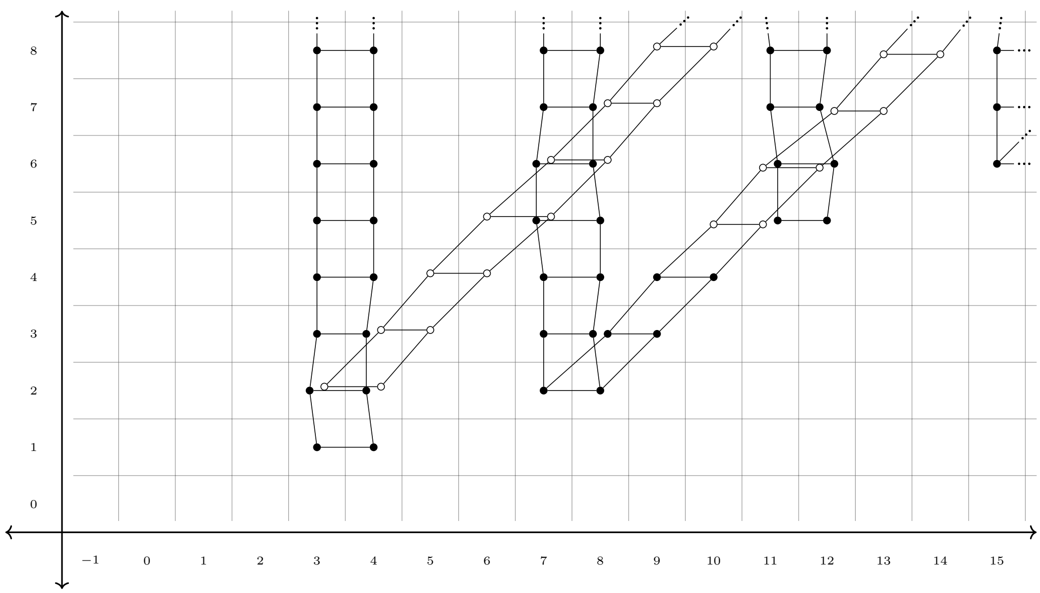


Figure 7.5: $\tau_{\geq 4}(\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{kq}^{\langle 2\rangle}))$, with an isolated $h_1$-tower.

**Definition 7.4.1.** For $i \geq 0$, let $Z_i^{\mathbb{C}}$ be the $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{C}})$-module defined as follows.

For $i \equiv 0\,(4)$, let

$$Z_i^{\mathbb{C}} = \tau^{h_1}_{\geq 2i}\left(\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{kq}^{\langle i\rangle})\right) \oplus \bigoplus_{j=0}^{i/2-1} \Sigma^{4j,2j}\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee}(\mathbb{M}_2^{\mathbb{C}}).$$

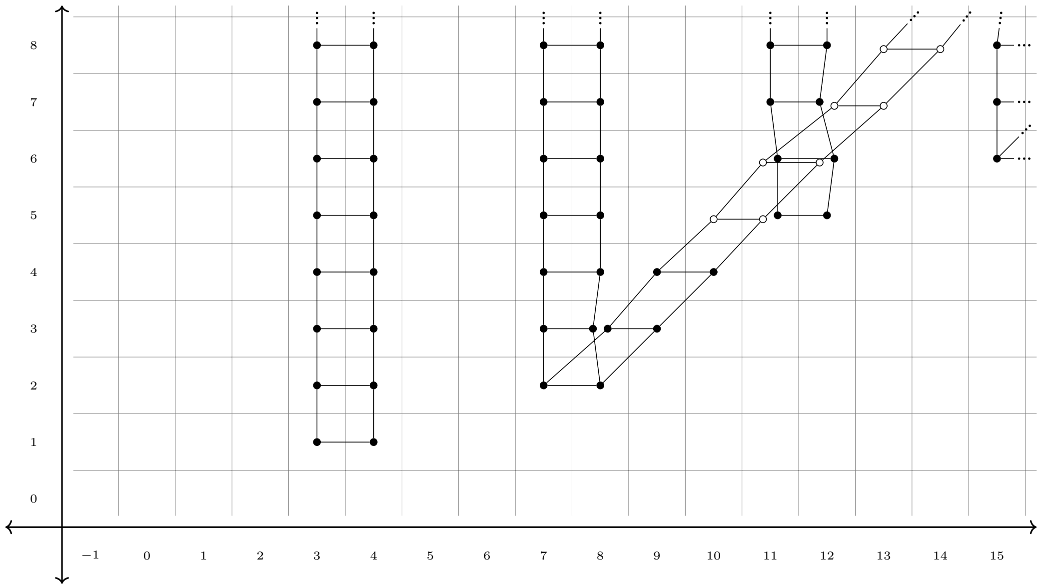


Figure 7.6: $\tau^{h_1}_{\geq 4}(\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{kq}^{\langle 2\rangle}))$, with no isolated $h_1$-towers.

For $i \equiv 1\,(4)$, let

$$Z^{\mathbb{C}}_i = \tau^{h_1}_{\geq 2i-2}\left(\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{ksp}^{\langle i-1\rangle})\right) \oplus \bigoplus_{j=0}^{(i-1)/2-1} \Sigma^{4j,2j}\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee}(\mathbb{M}^{\mathbb{C}}_2).$$

For $i \equiv 2\,(4)$, let

$$Z^{\mathbb{C}}_i = \tau^{h_1}_{\geq 2i}\left(\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{kq}^{\langle i\rangle})\right) \oplus \bigoplus_{j=0}^{i/2-1} \Sigma^{4j,2j}\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee}(\mathbb{M}^{\mathbb{C}}_2) \oplus \Sigma^{2i-2,i}\mathbb{F}_2[\tau].$$

For $i \equiv 3\,(4)$, let

$$Z^{\mathbb{C}}_i = \tau^{h_1}_{\geq 2i-2}\left(\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{ksp}^{\langle i-1\rangle})\right) \oplus \bigoplus_{j=0}^{(i-1)/2-1} \Sigma^{4j,2j}\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee}(\mathbb{M}^{\mathbb{C}}_2) \oplus \Sigma^{2i-3,i-2}\mathbb{F}_2[\tau,h_1]/(h_1^2).$$

**Corollary 7.4.5.** *Let $q \equiv 1\,(4)$. For all $k \geq 0$, there is an isomorphism:*

$$\frac{\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(k))}{v_1\text{-torsion}} \cong \left(\Sigma^{4k-4,2k-2}Z^{\mathbb{C}}_{\alpha(k)} \oplus \bigoplus_{j=0}^{k-2}\mathbb{M}^{\mathbb{C}}_2[h_0]\right) \otimes_{\mathbb{M}^{\mathbb{C}}_2} \mathbb{M}^{\mathbb{F}_q}_2,$$

*where $\alpha(k)$ is the number of 1's in the dyadic expansion of $k$, and the right hand summand is taken to be empty in the case of $k = 1$.*

*Proof.* By [Mor26, Theorem 5.6], there is an isomorphism

$$\frac{\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{C}}(k))}{v_1\text{-torsion}} \cong \Sigma^{4k-4,2k-2} Z^{\mathbb{C}}_{\alpha(k)} \oplus \bigoplus_{j=0}^{k-2} \mathbb{M}_2^{\mathbb{C}}[h_0].$$

The result then follows by Theorem 7.4.4 □

**Remark 7.4.6.** The notation in Definition 7.4.1, although equivalent, differs from the notation in [Mor26, Section 4]. We hope that this presentation in terms of Adams covers is more well suited for generalization.

**Remark 7.4.7.** As is true over $\mathbb{C}$ and $\mathbb{R}$, it is in this computation that we deviate from the classical story. Mahowald is able to express the groups $\mathrm{Ext}^{**}_{\mathcal{A}(1)^\vee}(B_0(1)^{\otimes i})$ solely in terms of Adams covers of bo and bsp [LM87]. The analogous statement in motivic homotopy is not true. If one were to express $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(1)^{\otimes i})$ in terms of Adams covers of kq or ksp, then one would see summands of non-nilpotent $h_1$-towers appearing in the formulae for $Z_i^{\mathbb{C}}$, reflecting that $\eta$ is non-nilpotent in $\pi_{**}^{\mathbb{F}_q}(\mathrm{kq}^{\langle n\rangle})$ and $\pi_{**}^{\mathbb{F}_q}(\mathrm{ksp}^{\langle n\rangle})$. However, these formulas do not allow for this, and indeed one can see through the **aAHSS** that such $h_1$-towers cannot be obtained. This is the reason behind the sophisticated truncation functors we introduced in the formulae given for $Z_i^{\mathbb{C}}$.

### *7.5 The ring of cooperations*

We now assemble our findings to compute $\pi_{**}^{\mathbb{F}_q}(\mathrm{kq}\otimes\mathrm{kq})$. First, we describe the $E_2$-page of the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq}\otimes\mathrm{kq})$.

**Corollary 7.5.1.** *Let $q \equiv 1\,(4)$. The $E_2$-page of the* $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq}\otimes\mathrm{kq})$*, modulo $v_1$-torsion, is given by*

$$E_2 = \bigoplus_{k\geq 0} \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\Sigma^{4k,2k} B_0^{\mathbb{F}_q}(k)) \cong \bigoplus_{k\geq 0} \Sigma^{4k,2k}\left(\Sigma^{4k-4,2k-2} Z^{\mathbb{C}}_{\alpha(k)} \oplus \bigoplus_{j=0}^{4k-8} \Sigma^{4j,2j}\mathbb{M}_2^{\mathbb{C}}[h_0]\right) \otimes \mathbb{M}_2^{\mathbb{F}_q}.$$

*Proof.* The first equivalence follows from **??**, and the second follows from Corollary 7.4.5. □

Notice that this description of the $E_2$-page is as a module over the $E_2$-page of the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$, that is, as a module over the algebra $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{F}_q})$. We leverage this to determine the differentials in our spectral sequence.

**Theorem 7.5.2.** *The differentials in the* $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq}\otimes\mathrm{kq})$ *are determined by the differentials in the* $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$.

*Proof.* The inclusion map $\mathrm{kq} \to \mathrm{kq} \otimes \mathrm{kq}$ induces a map of $E_2$-pages which pushes the differentials of Proposition 7.3.1 to the $k = 0$ summand of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{kq}\otimes\mathrm{kq})$. The $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{kq})$-module structure of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{kq}\otimes\mathrm{kq})$ and degree considerations lift all of the other differentials. To be precise, each summand of the $E_2$-page contains a submodule which is isomorphic to either $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{kq})$ or $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{ksp})$. Let $x$ be a generator for this submodule. Then $d_r(x) = 0$ for degree reasons for all $r$ as all classes in the prior stem have motivic weight at least 1 lower than $x$, as they are classes of the form $u\tau^n h_0^m$ for some $m, n \geq 0$, and Adams differentials preserve weight. Letting 1 denote the generator for $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{kq})$ of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{ksp})$, this is just the same as why $d_r(1) = 0$ for all $r$ in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$ or the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{ksp})$. The rest of the differentials on this submodule are now determined by the Liebniz rule and the underlying differentials in the $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{kq})$, following the proof of Proposition 7.3.1.

If $y$ is a generator for any submodule of the form $\mathbb{F}_2[u, \tau, h_0]/(u^2)$, then we must have that $d_r(y) = 0$ for all $r$ as all classes in the prior stem have weight at least 1 lower than $y$. Thus, the Liebniz rule implies that all the differentials will be given by the differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z})$. Finally there are no differentials on any ssummand of the form $\mathbb{F}_2[\tau]$ or $\mathbb{F}_2[\tau, h_1]/(h_1)^2$ for the same reason there is no differential on $h_1$ in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$, following the proof in Proposition 7.3.1. □

Figure 7.7 depicts a chart for the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq}\otimes\mathrm{kq})$ in the style of [BOSS19, Figure 3.2]. Different colors denote different summands of the $E_2$-page. For example, the $k = 0$ summand is given by $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{kq})$. This is depicted in black, and starts in bidegree $(0, 0)$. The $k = 1$ summand is given by $\Sigma^{4,2}\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{ksp})$. This is depicted in blue, and starts in bidegree $(4, 0)$. Since no classes which are divisible by $h_1$ support a differential, one can interpret these charts as either the $E_2$-page or the $E_\infty$-page. For the $E_2$-page, let a $\bullet$ of any color with a vertical arrow denote $\mathbb{F}_2[\tau, u, h_0]/(u^2)$. For the $E_\infty$-page, let a $\bullet$ of any color with a vertical arrow denote the $E_\infty$-page of the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z})$ shifted to that tridegree. A $\blacksquare$ denotes $\mathbb{F}_2[\tau, u]/(u^2)$, and a $\square$ denotes $\mathbb{F}_2[u]/(u^2)$.

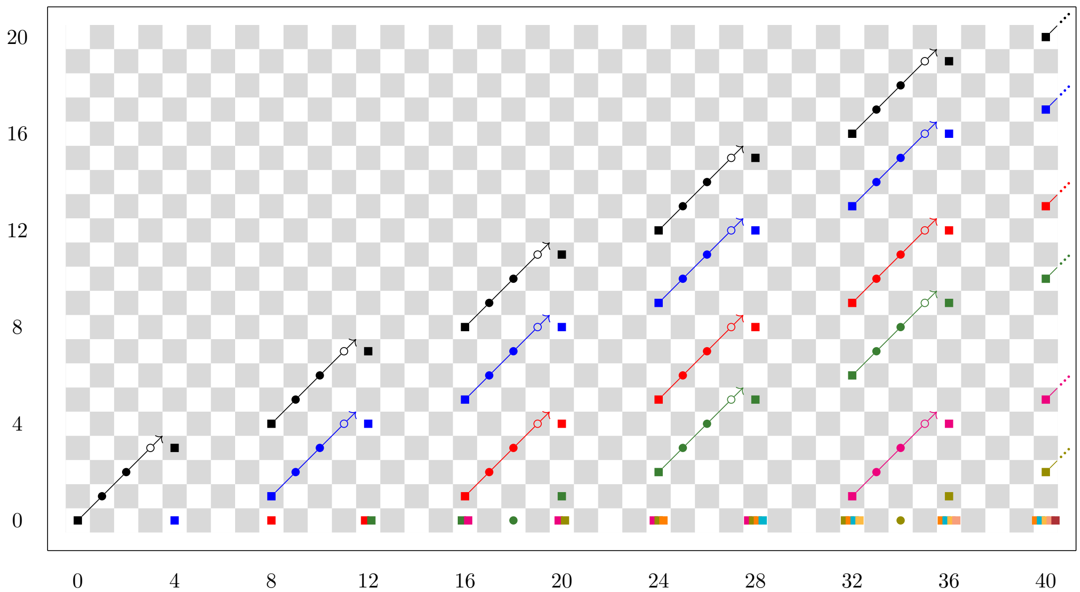


Figure 7.7: The $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq} \otimes \mathrm{kq})$ for $q \equiv 1\,(4)$.

# Chapter 8

# COMPUTATIONS IN SH($\mathbb{F}_q$) FOR A NONTRIVIAL BOCKSTEIN ACTION

In this section, we compute the ring of cooperations $\pi_{**}^{\mathbb{F}_q}(\mathrm{kq} \otimes \mathrm{kq})$ in the case where there is a nontrivial Bockstein action, that is, for $q \equiv 3\,(4)$. We follow the same process as was done in the previous section, and often refer to the previous section when we encounter identical proofs, but the Bockstein action makes the particular computations in Ext more complicated in this case. In particular, when computing over $\mathcal{A}(1)^\vee$, we display our data using multiple charts organized by coweight.

## *8.1 Integral homology*

We begin by computing the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z})$. This spectral sequence has signature

$$E_2 = \mathrm{Ext}_{\mathcal{A}^\vee}^{s,f,w}(\mathrm{H}_{**}(\mathrm{H}\mathbb{Z})) \implies \pi_{s,w}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z}).$$

Recall that there is an isomorphism of $\mathcal{A}^\vee$-comodules $\mathrm{H}_{**}(\mathrm{H}\mathbb{Z}) \cong (\mathcal{A} \,/\!/\, \mathcal{A}(0))^\vee$. This allows us to rewrite the $E_2$-page using a change of rings isomorphism as

$$\mathrm{Ext}_{\mathcal{A}^\vee}^{s,f,w}((\mathcal{A} \,/\!/\, \mathcal{A}(0))^\vee) \cong \mathrm{Ext}_{\mathcal{A}(0)^\vee}^{s,f,w}(\mathbb{M}_2^{\mathbb{F}_q}).$$

This Ext group was calculated by Kylling to be [Kyl15, Theorem 4.1.3]

$$\mathrm{Ext}_{\mathcal{A}(0)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q}) \cong \mathbb{F}_2[\rho, \tau^2, h_0, \rho\tau]/(\rho^2, \rho h_0, \rho(\rho\tau), (\rho\tau)^2), \tag{8.1.1}$$

where $|h_0| = (1, 0, 0)$ and $\rho\tau$ is indecomposable. We depict this $E_2$-page in Figure 8.1. Note that, although at first glance this chart looks nearly identical to Figure 8.1, it is much sparser in the motivic weight degree. For example, in stem 0 there are no classes of odd motivic weight, and in stem -1 the only classes of odd motivic weight are those of the form $\rho\tau^{2n}$. In particular, neither of the $h_0$ towers contain classes of odd motivic weight. To highlight the difference between these two sets of charts, we do not depict multiplication by $\rho\tau$.

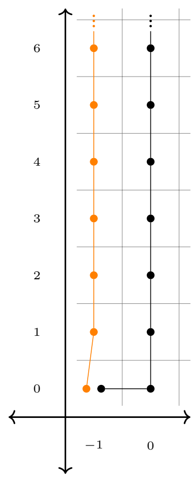


Figure 8.1: $\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee}(\mathbb{M}_2^{\mathbb{F}_q})$ for $q \equiv 3\,(4)$. A $\bullet$ denotes $\mathbb{F}_2[\tau^2]$, and the color orange denotes $\rho\tau$-divisibility. A vertical line represents multiplication by $h_0$, and a horizontal line represents multiplication by $\rho$.

We now determine all of the differentials in the spectral sequence.

**Lemma 8.1.1** ([Kyl15, Lemma 4.2.2])**.** *Let $q \equiv 3\,(4)$. The differentials in the* $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z})$ *are determined by*

$$d_{\nu_2(q^2-1)+i}(\tau^{2^i}) = \rho\tau^{2^i-1}h_0^{\nu_2(q^2-1)+i}.$$

*Proof.* The same proof as was given in Lemma 7.2.1 works by comparing with Equation (7.1.1), noting that since $\tau = 0$ on the $E_2$-page, the first class in stem 0 which must support a differential is $\tau^2$. $\square$

We depict the $E_\infty$-page in the case of $q = 3$ in Figure 8.2, which can be deduced using the same techniques as in Example 7.2.1. An empty $\circ$ denotes $\mathbb{F}_2$. A class labeled with $\tau^n$ indicates the first nonzero power of $\tau$ which exists in that particular bidegree. An empty class with $k$ circles around it indicates $\tau^{2^k}$-periodicity.

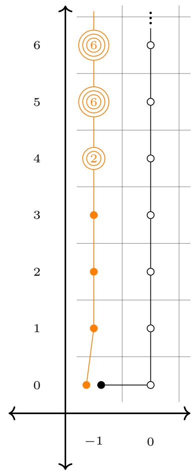


Figure 8.2: The $\mathrm{E}_\infty$-page of the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z})$ for $q \equiv 3\,(4)$. A $\bullet$ denotes $\mathbb{F}_2[\tau^2]$. An empty $\circ$ denotes $\mathbb{F}_2$. A class labeled with $\tau^n$ indicates the first nonzero power of $\tau$ which exists in that particular bidegree. An empty class with $k$ circles around it indicates $\tau^{2^k}$-periodicity. A vertical line represents multiplication by $h_0$, and a horizontal line represents multiplication by $\rho$.

## ***8.2 Hermitian K-theory***

We now compute the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$. This spectral sequence has signature

$$\mathrm{E}_2 = \mathrm{Ext}^{s,f,w}_{\mathcal{A}^\vee}(\mathrm{H}_{**}(\mathrm{kq})) \implies \pi^{\mathbb{F}_q}_{s,w}(\mathrm{kq}).$$

Recall that there is an isomorphism of $\mathcal{A}^\vee$-comodule algebras $\mathrm{H}_{**}(\mathrm{kq}) \cong (\mathcal{A} \,/\!/\, \mathcal{A}(1))^\vee$. This allows us to rewrite the $\mathrm{E}_2$-page using a change of rings isomorphism as

$$\mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}((\mathcal{A} \,/\!/\, \mathcal{A}(1))^\vee) \cong \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{F}_q}).$$

This Ext group was calculated by Kylling to be [Kyl15, Theorem 4.1.5]:

$$\mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{F}_3}) \cong \mathbb{F}_2[\rho, \tau^2, h_0, h_1, a, b, \tau h_1, \rho\tau]/I. \tag{8.2.1}$$

The degrees and coweights of the generators are listed in Table 8.1, where coweight refers to the difference between stem and weight. The ideal of relations $I$ is described in Table 8.2.

| Generator | $(s, f, w)$ | Coweight |
|---|---|---|
| $\rho$ | $(-1, 0, -1)$ | 0 |
| $h_0$ | $(0, 1, 0)$ | 0 |
| $h_1$ | $(1, 1, 1)$ | 0 |
| $b$ | $(8, 4, 4)$ | 4 |
| $\rho\tau$ | $(-1, 0, -2)$ | 1 |
| $\tau h_1$ | $(1, 1, 0)$ | 1 |
| $\tau^2$ | $(0, 0, -2)$ | 2 |
| $a$ | $(4, 3, 2)$ | 2 |

Table 8.1: Generators for $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_{\mathbb{F}_3}}(\mathbb{M}_2^{\mathbb{F}_q})$ for $q \equiv 3\,(4)$.

To clearly present the data of Equation (8.2.1), we give individual charts for each coweight modulo 2. Note that the element $\tau^2$ gives a coweight periodicity operator of degree 2. Figure 8.3 depicts the coweight $cw \equiv 0\,(2)$ piece. Figure 8.4 depicts the coweight $cw \equiv 1\,(2)$ piece. The notation agrees with the notation used earlier in this section.

As in the case of a trivial Bockstein action, we can lift the differentials from the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z})$ to completely determine the behavior of this spectral sequence. The proof follows from the same argument given in Proposition 7.3.1

**Proposition 8.2.1.** *Let $q \equiv 3\,(4)$. The differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$ are determined by the differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z})$.*

As in the previous section, by running the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$ we see that

$$\pi^{\mathbb{F}_q}_{s,w}(\mathrm{kq}) \cong \begin{cases} \mathrm{KO}_{s-2w}(\mathbb{F}_q) & w \equiv 0\,(4) \\ \mathrm{KSp}_{s-2w}(\mathbb{F}_q) & w \equiv 2\ (4) \end{cases}$$

for $s \geq 0$, agreeing with Friedlander's calculations given in Table 7.1.

| Relation | $(s, f, w)$ | Coweight |
|---|---|---|
| $\rho h_0$ | $(-1, 1, -1)$ | 0 |
| $h_0 h_1$ | $(1, 2, 1)$ | 0 |
| $\rho^2$ | $(-2, 0, -2)$ | 0 |
| $a^2 + h_0^2 b$ | $(8, 6, 4)$ | 4 |
| $h_0 \cdot \tau h_1 + \rho h_1 \cdot \tau h_1$ | $(1, 2, 0)$ | 1 |
| $h_1^2 \cdot \tau h_1$ | $(3, 3, 2)$ | 1 |
| $\rho\tau \cdot h_1^3$ | $(2, 3, 1)$ | 1 |
| $\rho \cdot \tau h_1 + h_1 \cdot \rho\tau$ | $(0, 1, -)$ | 1 |
| $\rho \cdot \rho\tau$ | $(-2, 0, -3)$ | 1 |
| $\tau^2 h_1^3 + \rho a$ | $(3, 3, 1)$ | 2 |
| $h_1 a$ | $(5, 4, 3)$ | 2 |
| $(\tau h_1)^2 = \tau^2 h_1^2$ | $(2, 2, 0)$ | 2 |
| $(\rho\tau)^2$ | $(-2, 0, -4)$ | 2 |
| $\tau h_1 \cdot \rho\tau + \rho\tau^2 h_1$ | $(0, 1, -2)$ | 2 |
| $\tau h_1 \cdot a$ | $(5, 4, 2)$ | 3 |

Table 8.2: Relations for $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{F}_3})$ for $q \equiv 3\,(4)$.

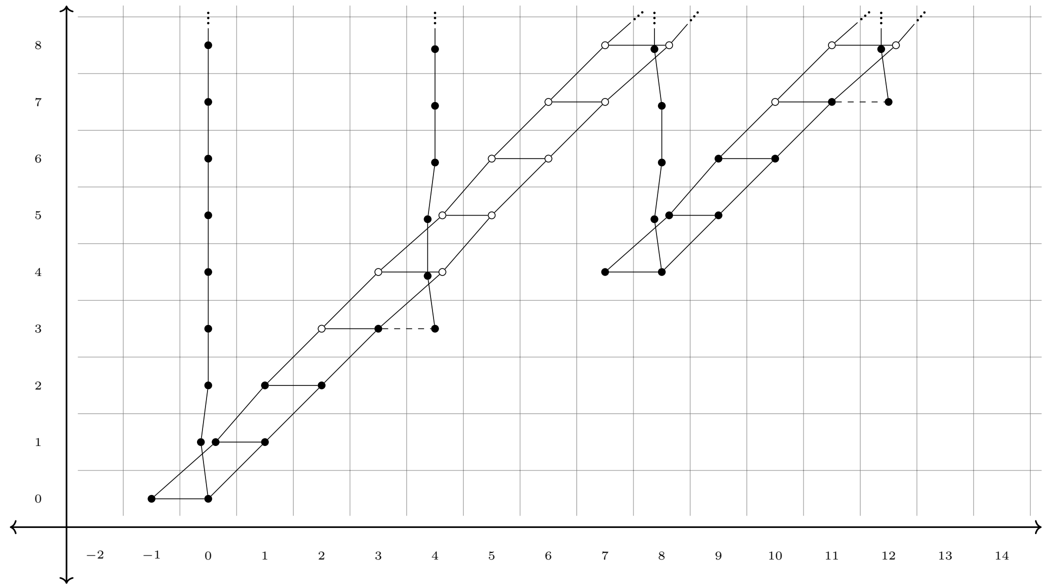


Figure 8.3: $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{F}_q})$ in coweight $cw \equiv 0\,(2)$ for $q \equiv 3\,(4)$.A $\bullet$ denotes $\mathbb{F}_2[\tau^2]$. An empty $\circ$ denotes $\mathbb{F}_2$. A vertical line represents multiplication by $h_0$, a horizontal line represents multiplication by $\rho$, and a diagonal line represents multiplication by $h_1$. A dashed line indicates that multiplication by $\rho$ hits a $\tau^2$-multiple of the indicated generator. For example, we have $\rho a = \tau^2 h_1^3$, which is indicated by the dashed line from $(4,3)$ to $(3,3)$.

**Remark 8.2.2.** The same argument given in Remark 7.3.2 shows that there is an isomorphism $\pi^{\mathbb{F}_q}_{s,w}(\mathrm{kq}) \cong \pi^{\mathbb{F}_q}_{s,w}(\mathrm{H}\mathbb{Z})$ for $s \leq 0$. More generally, we can express the homotopy of kq in terms of $\mathrm{H}\mathbb{Z}$:

$$\pi^{\mathbb{F}_q}_{**}(\mathrm{kq}) \cong (\pi^{\mathbb{F}_q}_{**}(\mathrm{H}\mathbb{Z}))[\eta, \tau^n\eta, \tau^n\eta^2, \alpha, \beta : n \geq 0]/(2\eta, \eta\alpha, \tau^n\eta\alpha, \tau^n\eta^2\alpha, \alpha^2 - 4\beta).$$

Notice that since $\tau h_1$ survives the spectral sequence, the homotopy of kq will have classes of the form $\tau^n\eta$ for all $n \geq 0$, even though $\tau = 0$ on the $E_2$-page of the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$.

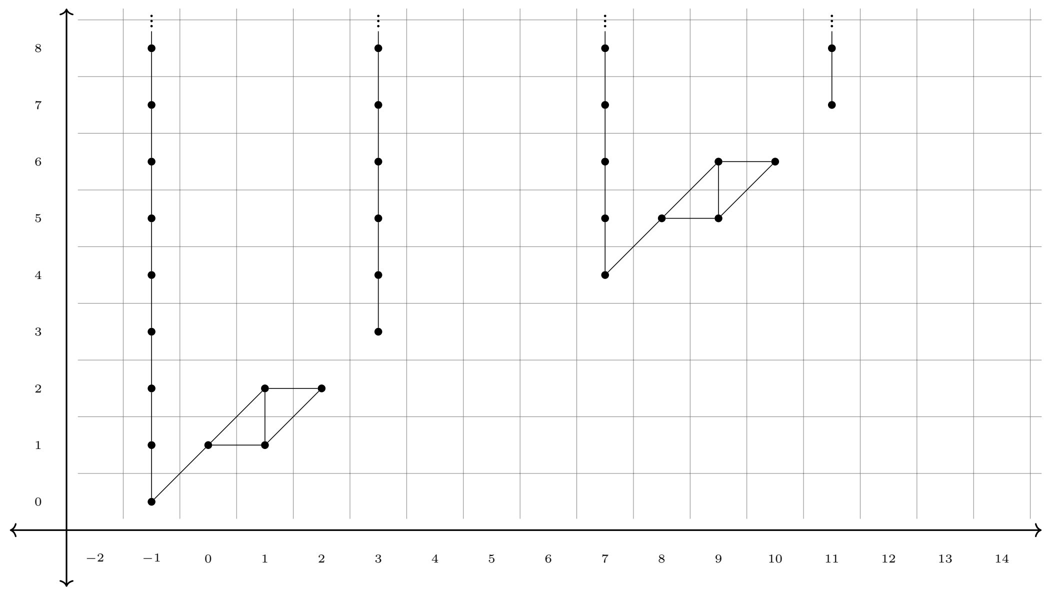


Figure 8.4: $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{F}_q})$ in coweights $cw \equiv 1\,(2)$ for $q \equiv 3\,(4)$. An empty $\circ$ denotes $\mathbb{F}_2$. A vertical line represents multiplication by $h_0$, a horizontal line represents multiplication by $\rho$, and a diagonal line represents multiplication by $h_1$.

### *8.3 Brown–Gitler comodules*

We begin the inductive process of computing the $E_2$-page of the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq} \otimes \mathrm{kq})$. To start, we will compute $\mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(1))$ by the $\mathbf{aAHSS}^{\mathbb{F}_q}(B_0^{\mathbb{F}_q}(1))$. This spectral sequence takes the form

$$E_1^{s,f,w,a} = \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)}(\mathbb{M}_2^{\mathbb{F}_q}) \otimes \mathbb{M}_2^{\mathbb{F}_q}\{[1], [\overline{\xi}_1], [\overline{\tau}_1]\} \implies \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)}(B_0^{\mathbb{F}_q}(1)).$$

We have discussed the differentials in this spectral sequence in Chapter 4. In particular, for $\alpha \in \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{F}_q})$, we have that

$$d_1(\alpha[3]) = h_0\alpha[2],$$

and

$$d_2(\alpha[2]) = h_1\alpha[0].$$

There is a potential third differential between Atiyah-Hirzebruch filtrations 3 and 0, which we saw did not exist for degree reasons in the case of a trivial Bockstein action on $\mathbb{F}_q$. However, this is not the case here.

**Lemma 8.3.1.** *Let $q \equiv 3\,(4)$. There is a $d_3$-differential in the* $\mathbf{aAHSS}(B_0^{\mathbb{F}_q}(1))$ :

$$d_3(\rho[3]) = (\tau h_1)[0].$$

*Proof.* This differential is given by the Massey product $\langle \rho, h_0, h_1\rangle[0]$. A simple computation in the cobar complex shows that this is equal to $\tau h_1$ with no indeterminacy. $\square$

Since the differentials in the $\mathbf{aAHSS}(B_0^{\mathbb{F}_q}(1))$ are module maps over $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{F}_q})$, this determines the spectral sequence. Hidden extensions lift as in the previous section. Note that the $d_1$ and $d_2$-differentials are given by multiplication with an element of coweight 0, while the $d_3$-differential is given by multiplication with an element of coweight 1.

We now provide charts for the $E_\infty$-page. Figure 8.5 depicts the coweight $cw \equiv 0\,(2)$ piece. Figure 8.6 depicts the coweight $cw \equiv 1\,(2)$ piece.

The equivalence of motivic spectra $\mathrm{ksp} \simeq \mathrm{H}\mathbb{Z}_1^{\mathbb{F}_q} \otimes \mathrm{kq}$ implies that the $E_2$-page of the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{ksp})$ is given by $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(1))$. Our computations with the $\mathbf{aAHSS}^{\mathbb{F}_q}(B_0^{\mathbb{F}_q}(1))$ show that

$$\mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(1)) \cong \Sigma^{4,2}\mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{F}_q})\langle 1\rangle \oplus \mathrm{Ext}^{s,f,w}_{\mathcal{A}(0)^\vee}(\mathbb{M}_2^{\mathbb{F}_q}).$$

The same argument as in Lemma 7.4.1 gives the following lemma.

**Lemma 8.3.2.** *Let $q \equiv 3\,(4)$. In the* $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{ksp})$*, the differentials are determined by the differentials in the* $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$ *and the* $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z})$.

By running the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{ksp})$ and comparing with Table 7.1, we recover the expected isomorphism for $s \geq 4$:

$$\pi^{\mathbb{F}_q}_{s,w}(\mathrm{ksp}) \cong \begin{cases} \mathrm{KSp}_{s-2w}(\mathbb{F}_q) & w \equiv 0\,(4); \\ \mathrm{KO}_{s-2w}(\mathbb{F}_q) & w \equiv 2\ (4). \end{cases}$$

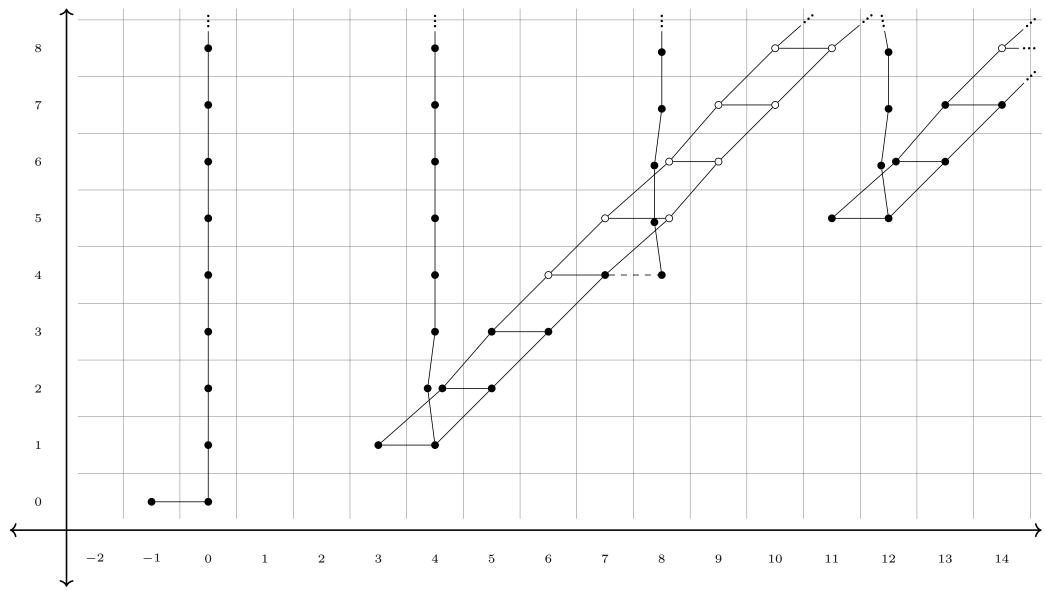

Figure 8.5: The $E_\infty$-page of the $\mathbf{aAHSS}(B_0^{\mathbb{F}_q}(1))$ in coweight $cw \equiv 0\,(2)$ for $q \equiv 3\,(4)$.

We also obtain an isomorphism $\pi_{s,w}^{\mathbb{F}_q}(\mathrm{ksp}) \cong \pi_{s,w}^{\mathbb{F}_q}(\mathrm{kq})$ for $s \leq 0$.

We now compute the group $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(B_0^{\mathbb{F}_q}(1)^{\otimes i})$. To do this, we introduce a family of trigraded groups and use the notation described in the paragraphs preceding Definition 7.4.1.

**Definition 8.3.1.** For $i \geq 0$, let $Z_i^{\mathbb{F}_q}$ be the $\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q})$-module defined as follows.

For $i \equiv 0\,(4)$, let

$$Z_i^{\mathbb{F}_q} = \tau_{\geq 2i}^{h_1}\left(\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathrm{kq}^{\langle i\rangle})\right) \oplus \bigoplus_{j=0}^{i/2-1} \Sigma^{4j,2j}\mathrm{Ext}_{\mathcal{A}(0)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q}).$$

For $i \equiv 1\,(4)$, let

$$Z_i^{\mathbb{F}_q} = \tau_{\geq 2i-2}^{h_1}\left(\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathrm{ksp}^{\langle i-1\rangle})\right) \oplus \bigoplus_{j=0}^{(i-1)/2-1} \Sigma^{4j,2j}\mathrm{Ext}_{\mathcal{A}(0)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q}).$$

For $i \equiv 2\,(4)$, let

$$Z_i^{\mathbb{F}_q} = \tau_{\geq 2i}^{h_1}\left(\mathrm{Ext}_{\mathcal{A}(1)^\vee}^{***}(\mathrm{kq}^{\langle i\rangle})\right) \oplus \bigoplus_{j=0}^{i/2-1} \Sigma^{4j,2j}\mathrm{Ext}_{\mathcal{A}(0)^\vee}^{***}(\mathbb{M}_2^{\mathbb{F}_q}) \oplus \Sigma^{2i-2,i}\mathbb{F}_2[\tau^2].$$

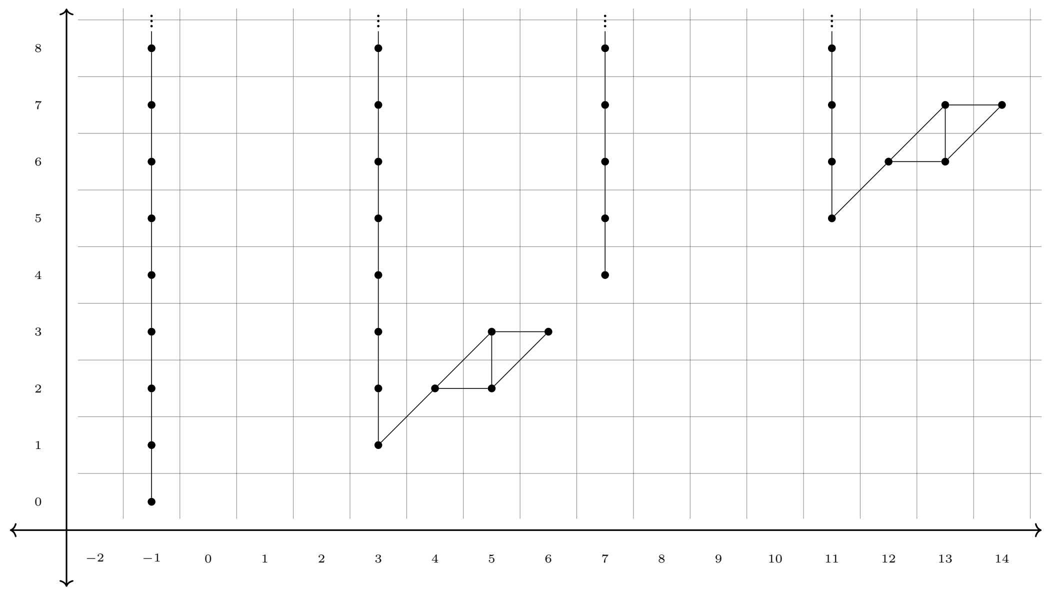


Figure 8.6: The $E_\infty$-page of the $\mathbf{aAHSS}(B_0^{\mathbb{F}_q}(1))$ in coweight $cw \equiv 1\,(2)$ for $q \equiv 3\,(4)$.

For $i \equiv 3\,(4)$, let

$$Z_i^{\mathbb{F}_q} = \tau^{h_1}_{\geq 2i-2}\left(\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{ksp}^{\langle i-1\rangle})\right) \oplus \bigoplus_{j=0}^{(i-1)/2-1} \Sigma^{4j,2j}\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee}(\mathbb{M}_2^{\mathbb{F}_q}) \oplus \Sigma^{2i-3,i-2}\mathbb{F}_2[\tau^2, h_1]/(h_1^2).$$

**Proposition 8.3.3.** *Let $q \equiv 3\,(4)$. For all $i \geq 0$, there is an isomorphism*

$$\frac{\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(1)^{\otimes i})}{v_1\text{-torsion}} \cong Z_i^{\mathbb{F}_q}.$$

*Proof.* The proof is similar to the proof of Proposition 7.4.3. The case of $i = 0$ was shown in Equation (8.2.1), and the case of $i = 1$ was shown in Lemma 8.3.2. In the general case, consider the algebraic Atiyah-Hirzebruch spectral sequence

$$E_1 = \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(1)^{\otimes i}) \otimes \mathbb{M}_2^{\mathbb{F}_q}\{[1], [\overline{\xi}_1], [\overline{\tau}_1]\} \implies \mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(1)^{\otimes i+1}).$$

The differentials are determined by the cellular filtration on $B_0^{\mathbb{F}_q}(1)$ in the exact same way as before, and hidden extensions follow as in the classical case. The result follows. $\square$

Before computing the groups $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(k))$, we make a preliminary calculation.

**Lemma 8.3.4.** *Let $q \equiv 3\,(4)$. There is an isomorphism*

$$\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee}(B_1^{\mathbb{F}_q}(k)) \cong \bigoplus_{j=0}^{k} \Sigma^{4j,2j}\mathbb{F}_2[\rho, \tau^2, h_0, \rho\tau]/(\rho^2, \rho h_0, \rho(\rho\tau), (\rho\tau)^2).$$

*Proof.* There is an equivalence of $\mathcal{A}(0)^\vee$-comodules

$$B_1^{\mathbb{F}_q}(k) \cong \bigoplus_{j=0}^{k} \Sigma^{4j,2j}\mathbb{M}_2^{\mathbb{F}_q}.$$

The result now follows from Equation (8.1.1). Alternatively, one may filter the cobar complex $C_{\mathcal{A}(0)^\vee}(B_1^{\mathbb{F}_q}(k))$ by powers of $\rho$ to yield a $\rho$-Bockstein spectral sequence computing the Ext group in question. Since $\rho^2 = 0 \in \mathbb{M}_2^{\mathbb{F}_q}$, the only differential is $d_1(\tau) = \rho h_0$, giving the desired structure. □

**Theorem 8.3.5.** *Let $q \equiv 3\,(4)$. There is an isomorphism*

$$\frac{\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(k))}{v_1\text{-torsion}} \cong \Sigma^{4k-4,2k-2}Z^{\mathbb{F}_q}_{\alpha(k)} \oplus \bigoplus_{j=0}^{k-2} \Sigma^{4j,2j}\mathbb{F}_2[\tau^2, h_0, \rho\tau]/((\rho\tau)^2),$$

*where $\alpha(k)$ is the number of 1's in the dyadic expansion of $k$, and the righthand summand is taken to be empty in the case of $k = 1$.*

*Proof.* The same argument as given in Theorem 7.4.4 works here as well. To be clear, one uses the short exact sequences of Proposition 7.1.2 to induce long exact sequences in Ext groups and induction on $k$, where the case of $k = 1$ was performed in Lemma 8.3.2. The result follows by noticing that, since we are working modulo $v_1$-torsion, the connecting homomorphism is trivial, and the class $\rho$ coming from $\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee}(B_1^{\mathbb{F}_q}(k))$ is $v_1$-torsion for degree reasons. □

### *8.4 The ring of cooperations*

We now assemble our findings to compute $\pi^{\mathbb{F}_q}_{**}(\mathrm{kq} \otimes \mathrm{kq})$. First, we describe the $E_2$-page of the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq} \otimes \mathrm{kq})$.

**Corollary 8.4.1.** *Let $q \equiv 3\,(4)$. The $E_2$-page of the* $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq} \otimes \mathrm{kq})$*, modulo $v_1$-torsion, is given by*

$$E_2 = \bigoplus_{k\geq 0} \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\Sigma^{4k,2k} B_0^{\mathbb{F}_q}(k)) \cong \bigoplus_{k\geq 0} \Sigma^{4k,2k}\left(\Sigma^{4k-4,2k-2} Z^{\mathbb{F}_q}_{\alpha(k)} \oplus \bigoplus_{j=0}^{k-2} \Sigma^{4j,2j}\mathbb{F}_2[\tau^2, h_0, \rho\tau]/((\rho\tau)^2)\right).$$

*Proof.* The first equivalence follows from Theorem 4.1.1, and the second follows from Theorem 8.3.5. □

**Theorem 8.4.2.** *The differentials in the* $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq}\otimes\mathrm{kq})$ *are determined by the differentials in the* $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq})$*.*

*Proof.* The same proof as was given in Theorem 7.5.2 works here. On each summand of the $E_2$-page, there is a submodule given by an Adams cover of kq or ksp. The differentials on this submodule are determined by the module structure over $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{F}_q})$. On each of the $\mathbb{F}_2[\tau^2, h_0, \rho\tau]/((\rho\tau)^2)$, the differentials are determined by the formulas for the differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z})$ in the same way we determined the differentials on the $\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee}(\mathbb{M}_2^{\mathbb{F}_q})$-summand of the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{ksp})$. □

Figure 8.7 depicts a chart for the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq} \otimes \mathrm{kq})$. A $\bullet$ indicates an $h_0$-tower. Note that an $h_0$-tower supports $\rho\tau$-multiplication. A $\square$ indicates a class which supports $\rho$ and $\rho\tau$-multiplication. A filled isosceles triangle indicates a class which supports $\rho, \rho\tau$, and $\tau h_1$-multiplication. Any shaded class is polynomial on $\tau^2$. As in Figure 7.7, one may interpret this chart as either the $E_2$-page of the $E_\infty$-page of the spectral sequence since no classes which are divisible by $h_1$ support a differential.

**Remark 8.4.3.** We expect that the formula given for $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0(k))$ generalizes well, since the short exact sequence of Brown–Gitler comodules Proposition 7.1.2 holds over any field with 2 invertible. We also expect that one can produce a spectrum level splitting of $\mathrm{kq} \otimes \mathrm{kq}$. Such a splitting would need to incorporate a mix of $h_1$-truncations of Adams covers of kq and ksp, and would need to account for the lack of non-nilpotent $h_1$-towers in low stem degrees.

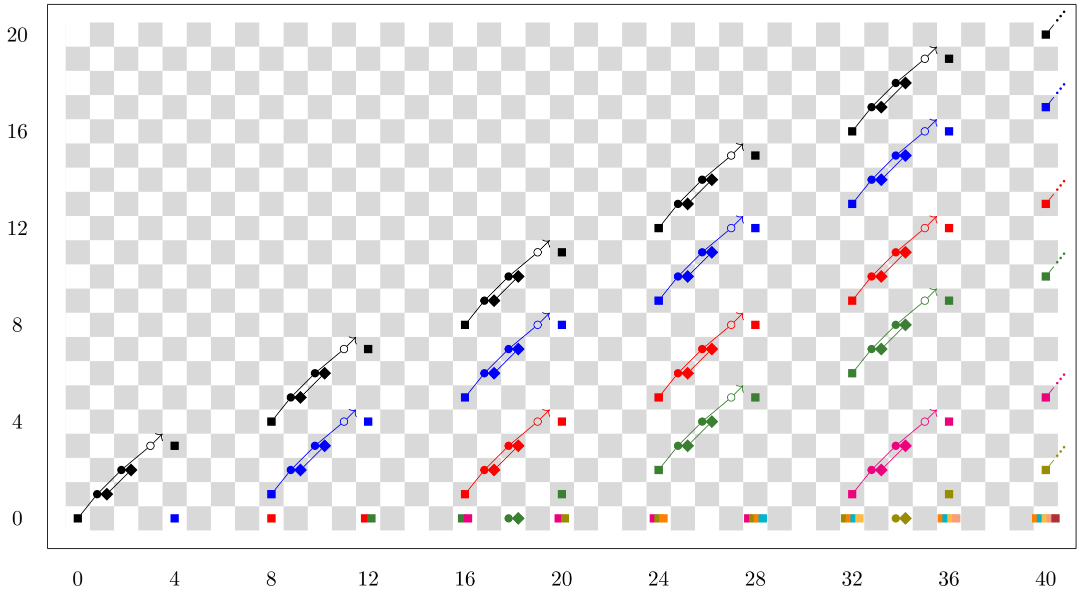


Figure 8.7: The $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq} \otimes \mathrm{kq})$ for $q \equiv 3\,(4)$.

**Remark 8.4.4.** In future work, and indeed one of our initial reasons for investigating the ring of cooperations over $\mathbb{F}_q$, we will use the computation of $\pi^F_{**}(\mathrm{kq} \otimes \mathrm{kq})$ for $F = \mathbb{C}, \mathbb{R}, \mathbb{F}_3$ to describe $\pi^{\mathbb{Z}[1/2]}_{**}(\mathrm{kq} \otimes \mathrm{kq})$ by the methods developed in [BØ22].

# Chapter 9

## APPLICATION TO THE KQ-RESOLUTION

We conclude by examining the $E_1$-page of the kq-resolution. In this chapter, results hold over the base fields $F \in \{\mathbb{C}, \mathbb{R}, \mathbb{F}_q\}$ for $\mathrm{char}(\mathbb{F}_q) \neq 2$.

Recall that the kq-resolution for the sphere has signature

$$\mathrm{E}_1 = \pi_{s+f,w}(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes f}) \implies \pi_{s,w}\mathbb{S}.$$

We can determine each filtration $(f = n)$-line of the $\mathrm{E}_1$-page by an extension of the techniques used for the computation of the homotopy ring of cooperations $\pi_{**}(\mathrm{kq} \otimes \mathrm{kq})$. For each $n \geq 0$, there is a motivic Adams spectral sequence of the form

$$\mathrm{E}_2 = \mathrm{Ext}^{s,f,w}_{\mathcal{A}^\vee}(\mathrm{H}_{**}(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n})) \implies \pi_{s,w}(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n}).$$

We begin by decomposing the $\mathrm{E}_2$-page.

**Lemma 9.0.1.** *There is a Künneth isomorphism*

$$\mathrm{H}_{**}(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n}) \cong \mathrm{H}_{**}(\mathrm{kq}) \otimes \mathrm{H}_{**}(\overline{\mathrm{kq}})^{\otimes n}.$$

*Proof.* We induct on $n$. The Künneth spectral sequence takes the form

$$\mathrm{E}_2 = \mathrm{Tor}^{\mathbb{M}_2}(\mathrm{H}_{**}(\mathrm{kq}), \mathrm{H}_{**}(\overline{\mathrm{kq}})) \implies \mathrm{H}_{**}(\mathrm{kq} \otimes \overline{\mathrm{kq}}).$$

Since $\mathrm{H}_{**}(\mathrm{kq}) = (\mathcal{A} \,/\!/\, \mathcal{A}(1))^\vee$ is the $\mathbb{M}_2$-linear dual of a free $\mathbb{M}_2$-module, it is also free. This implies that the spectral sequence collapes. Now, suppose the result is true for all $i < n$. We again have a Künneth spectral sequence which takes the form

$$\mathrm{E}_2 = \mathrm{Tor}^{\mathbb{M}_2}(\mathrm{H}_{**}(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n-1}), \mathrm{H}_{**}(\overline{\mathrm{kq}})) \implies \mathrm{H}_{**}(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n}).$$

By induction, we have a Künneth isomorphism on the left hand factor. Now, note that $\mathrm{H}_{**}(\overline{\mathrm{kq}})$ is also free over $\mathbb{M}_2$, which one can see by the long exact sequence in homology associated to the cofiber sequence

$$\mathbb{S} \to \mathrm{kq} \to \overline{\mathrm{kq}}.$$

Thus the spectral sequence collapses, giving the result. □

Note that there is an isomorphism of $\mathcal{A}(1)^\vee$-comodules [Mor26, Proposition 2.20]

$$\mathrm{H}_{**}(\overline{\mathrm{kq}}) \cong \bigoplus_{k\geq 1} \Sigma^{4k,2k} B_0(k). \tag{9.0.1}$$

We can now deduce the $\mathrm{E}_2$-page.

**Proposition 9.0.2.** *The* $\mathrm{E}_2$*-page of the* $\mathbf{mASS}(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n})$ *may be rewritten as*

$$\mathrm{E}_2^{s,f,w} \cong \bigoplus_{K\in \mathcal{K}_n} \Sigma^{4|K|,2|K|}\mathrm{Ext}^{s,f,w}_{\mathcal{A}(1)^\vee}(B_0(K)),$$

*where* $\mathcal{K}_n = \{K = (k_1, \dots, k_n) : k_j \geq 1 \text{ for all } j\}$, $|K| = \sum_{j=1}^n k_j$*, and* $B_0(K) = \bigotimes_{j=1}^n B_0(k_j)$.

*Proof.* By Lemma 9.0.1 and the change of rings isomorphism, we may rewrite the $\mathrm{E}_2$-page as

$$\mathrm{Ext}^{***}_{\mathcal{A}^\vee}(\mathrm{H}_{**}(\mathrm{kq}) \otimes \mathrm{H}_{**}(\overline{\mathrm{kq}})^{\otimes n}) \cong \mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{H}_{**}(\overline{\mathrm{kq}})^{\otimes n}).$$

The identification of (9.0.1) allows us to rewrite the first factor of $\mathrm{H}_{**}(\overline{\mathrm{kq}})$, leaving us with

$$\bigoplus_{k_1\geq 1} \Sigma^{4k_1,2k_1}\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}\left(B_0(k_1) \otimes \mathrm{H}_{**}(\mathrm{kq})^{\otimes n-1}\right).$$

Rewriting the next factor of $\mathrm{H}_{**}(\overline{\mathrm{kq}})$ gives us

$$\bigoplus_{k_1\geq 1} \Sigma^{4k_1,2k_1}\left(\bigoplus_{k_2\geq 1} \Sigma^{4k_2,2k_2}\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0(k_1) \otimes B_0(k_2) \otimes \mathrm{H}_{**}(\overline{\mathrm{kq}})^{\otimes n-2})\right),$$

which we may rewrite as

$$\bigoplus_{k_1,k_2\geq 1} \Sigma^{4(k_1+k_2),2(k_1+k_2)}\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}\left(B_0(k_1) \otimes B_0(k_2) \otimes \mathrm{H}_{**}(\overline{\mathrm{kq}})^{\otimes n-2}\right).$$

The result follows by extending these techniques and rewriting all factors of $\mathrm{H}_{**}(\overline{\mathrm{kq}})$ in terms of Brown–Gitler comodules. □

One may easily alter the charts given in Figure 7.7 and Figure 8.7 to illustrate the $\mathbf{mASS}(\mathrm{kq} \otimes \overline{\mathrm{kq}})$, i.e. the 1-line of the kq-resolution, by omitting the black $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2)$ summand originating in stem $s = 0$. For higher Adams filtration, one may compute $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0(K))$ by an analysis akin to the one we give in this paper, using a combination of the short exact sequences of Brown–Gitler comodules and algebraic Atiyah–Hirzebruch spectral sequences. In particular, while the Ext group is substantially larger, we can still decompose it, modulo $v_1$-torsion, into:

- shifts of $h_1$-truncations $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{kq}^{\langle n\rangle})$ and $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathrm{ksp}^{\langle n\rangle})$,
- $\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee}(\mathrm{H}\mathbb{Z})$-summands, and
- summands of the form $(\mathrm{Ext}^{***}_{\mathcal{A}(0)^\vee}(\mathrm{H}\mathbb{Z}))[h_1]/(h_0, h_1^2)$.

Note that the 0-line of the kq-resolution is $\pi_{**}(\mathrm{kq})$, and so we have described the $\mathbf{mASS}(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n})$ as a module over the $\mathbf{mASS}(\mathrm{kq})$.

**Theorem 9.0.3.** *The differentials in the* $\mathbf{mASS}(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n})$ *are determined by the differentials in the* $\mathbf{mASS}(\mathrm{kq})$.

*Proof.* In the same way that we were able to determine differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq} \otimes \mathrm{kq})$ by using the structure of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(k))$ as a module over $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{F}_q})$, the map

$$\mathrm{kq} \to \mathrm{kq} \otimes \mathrm{kq}^{\otimes n} \to \mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n}$$

allows us to determine the differentials in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{kq} \otimes \overline{\mathrm{kq}}^{\otimes n})$ by using the structure of $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(B_0^{\mathbb{F}_q}(K))$ as a module over $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee}(\mathbb{M}_2^{\mathbb{F}_q})$. The result follows. □

This gives an additive description of the $n$-line $E_1$-page of the kq-resolution, modulo $v_1$-torsion, in terms of $h_1$-truncated Adams covers of kq and ksp, summands of $\mathrm{H}\mathbb{Z}$, and stunted $\eta$-towers. Moreover, we have described the module structure of the $n$-line over the 0-line. We expect these observations to be useful in future work on the kq-resolution.

**Remark 9.0.4.** We will use these results in our forthcoming study of the kq-resolution over $\mathbb{R}$ and $\mathbb{F}_q$. In another direction, it would be interesting to investigate the the ring of cooperations for kq in $\mathrm{SH}(\mathbb{Q}_p)$ .

**Remark 9.0.5.** An interesting phenomenon encountered in this paper is that any differential in a motivic Adams spectral sequence we have considered has been determined by a differential in the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{H}\mathbb{Z})$. Similar observations were made by the author and, Petersen, and Tatum [MPT25] for the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{BPGL}\langle 1\rangle)$ and the $\mathbf{mASS}^{\mathbb{F}_q}(\mathrm{BPGL}\langle 1\rangle \otimes \mathrm{BPGL}\langle 1\rangle)$, by Ormsby [Orm11] for the $\mathbf{mASS}^{\mathbb{Q}_p}(\mathrm{BPGL}\langle n\rangle)$, and by Ormsby–Østvær [OØ13] for the $\mathbf{mASS}^{\mathbb{Q}}(\mathrm{BPGL}\langle n\rangle)$.

Say that a 2-complete motivic spectrum E is fp if its homology $\mathrm{H}_{**}(\mathrm{E})$ is finitely-presented as an $\mathcal{A}^\vee$-comodule. This is a motivic analogue of the notion of fp-spectra introduced by Mahowald–Rezk [MR99]. Examples of motivic fp-spectra include $\mathrm{H}\mathbb{F}_2, \mathrm{H}\mathbb{Z}, \mathrm{BPGL}\langle n\rangle$, kq, and mmf, the motivic modular forms spectrum (which is currently only defined over $\mathbb{C}$ [GIKR22]). Our observations motivate the following question.

**Question 9.0.6.** *Let* $\mathrm{E} \in \mathrm{SH}(F)$ *be an fp-spectrum, where* $F$ *is any field. To what extent are the differentials in the* $\mathbf{mASS}^F(\mathrm{E})$ *determined by the differentials in the* $\mathbf{mASS}^F(\mathrm{H}\mathbb{Z})$*?*

A similar question was studied by Ormsby–Østvær in the context of the slice spectral sequence [OØ14], where they determined that over fields of low cohomological dimension, there is an isomorphism

$$\pi^F_{**}(\mathrm{BPGL}\langle n\rangle) \cong (\pi^F_{**}(\mathrm{H}\mathbb{Z}))[v_1, \dots, v_n].$$

In particular, this implies that the differentials in the $\mathbf{mASS}^F(\mathrm{BPGL}\langle n\rangle)$ are completely determined by $v_n$-linearity and the differentials in the $\mathbf{mASS}^F(\mathrm{H}\mathbb{Z})$.

Notice that if an fp-spectrum E is additionally a ring spectrum, then an affirmative answer to the above question implies that there is a large degree of control over the E-based motivic Adams spectral sequence. As a particular topic of investigation, we give the following.

**Conjecture 9.0.7.** *Over any field $F$ of low cohomological dimension, the $E_1$-page of the* $\mathrm{BPGL}\langle 2\rangle$*-motivic Adams spectral sequence can be computed by appropriate lifts of the differentials in the* $\mathbf{mASS}^F(\mathrm{H}\mathbb{Z})$*. In particular, for $F = \mathbb{C}$ or $\mathbb{R}$, the* $\mathbf{mASS}^F(\mathrm{BPGL}\langle 2\rangle^{\otimes n})$ *collapses on the $E_2$-page.*

As a final note, let $G$ be a finite cyclic subgroup of the Morava stabilizer subgroup $\mathbb{S}_h$, $H \subseteq G$ any subgroup, and $\mathrm{BP}^{((G))}\langle m\rangle$ the $G$-equivariant truncated Brown–Peterson spectra. In recent work, Carrick–Hill [CH25] showed that the spectra $\mathrm{BP}^{((G))}\langle m\rangle^H$ are fp spectra for all $H \subseteq G$ and all $m \geq 0$. We expect an equivariant-motivic analogue, in the sense of [HKO11b], to be true. We will investigate these ideas in future work.

# Appendix A
# CHARTS

In this appendix, we record charts depicting some of the $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_\mathbb{R}}(\mathbb{M}_2^\mathbb{R})$-modules described in Section 6.1. We recall their definition below.

**Definition A.0.1.** Define the following $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_\mathbb{R}}(\mathbb{M}_2^\mathbb{R})$-modules:

- Let $P = \mathbb{F}_2[\tau^4][\rho]$, called a *$\rho$-tower*, with the obvious $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_\mathbb{R}}(\mathbb{M}_2^\mathbb{R})$-module structure (see Fig. A.1);
- Let $H = \mathbb{F}_2[\tau^4][h_0]$, called an *$h_0$-tower*, with the obvious $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_\mathbb{R}}(\mathbb{M}_2^\mathbb{R})$-module structure (see Fig. A.2);
- Let $PH = \mathbb{F}_2[\tau^4][\rho, h_0]/(\rho h_0)$, called a *$(\rho, h_0)$-tower*, with the obvious $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_\mathbb{R}}(\mathbb{M}_2^\mathbb{R})$-module structure (see Fig. A.3);
- Let $D = \mathbb{F}_2[\tau^4][\rho, h_0, h_1]/(\rho^2, h_0^2, h_1^2, \rho h_0, h_0h_1, \rho h_1 = h_0)$, called a *diamond*, with the obvious $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_\mathbb{R}}(\mathbb{M}_2^\mathbb{R})$-module structure (see Fig. A.4);
- Let $S = \mathbb{F}_2[\tau^4][h_0, h_1, t, a]/(h_1^3, t^2, a^2, h_0h_1, h_1a, ta, h_0^2t, h_1^2t)$, called a *staircase*, with the $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_\mathbb{R}}(\mathbb{M}_2^\mathbb{R})$-module structure given by the relations (see Fig. A.5)
$$\rho \cdot t = h_1, \quad h_1 \cdot t = \rho \cdot a, \quad h_0 \cdot t = h_1^2 = \rho^2 \cdot a, \quad \rho^2 \cdot t = 0, \quad \rho^3 \cdot a = 0;$$
- Let $T = \mathbb{F}_2[\tau^4][\rho]/(\rho^2)$, called a *segment*, with the obvious $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_\mathbb{R}}(\mathbb{M}_2^\mathbb{R})$-module structure (see Fig. A.6);
- Let $J = \mathbb{F}_2[\tau^4][\rho, h_0]/(\rho^3, \rho h_0)$, called a *J-tower*, with the obvious $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_\mathbb{R}}(\mathbb{M}_2^\mathbb{R})$-module structure (see Fig. A.7);

- Let $JD = \mathbb{F}_2[\tau^4][\rho, h_0, t]/(\rho^3, t^2, \rho^2 t, \rho h_0, h_0^2 t)$, called a *$JD$-tower*, with the $\mathrm{Ext}^{***}_{\mathcal{A}(1)^\vee_\mathbb{R}}(\mathbb{M}_2^\mathbb{R})$-module structure given by the relations (see Fig. A.8)

$$\rho = h_1 t, \quad h_1 \cdot \rho t = h_0 t = \rho^2.$$

A black ■ represents $\mathbb{F}_2[\tau^4][\rho]$. A black • represents $\mathbb{F}_2[\tau^4]$. A vertical black line represents multiplication by $h_0$. A horizontal black line represents multiplication by $\rho$. A black line of slope 1 represents multiplication by $h_1$.

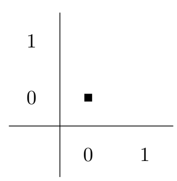


Figure A.1: The module $P$.

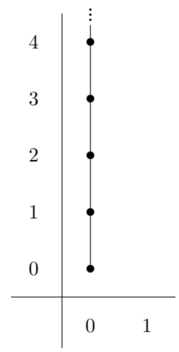


Figure A.2: The module $H$.

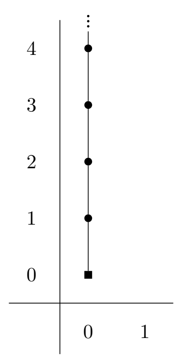


Figure A.3: The module $PH$.

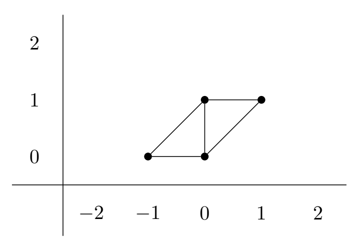


Figure A.4: The module $D$.

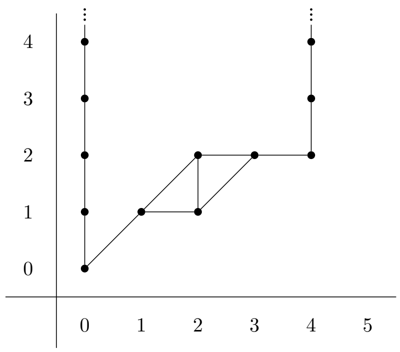


Figure A.5: The module $S$.

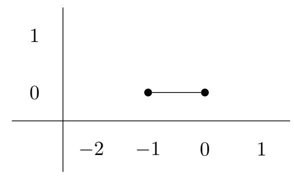


Figure A.6: The module $T$.

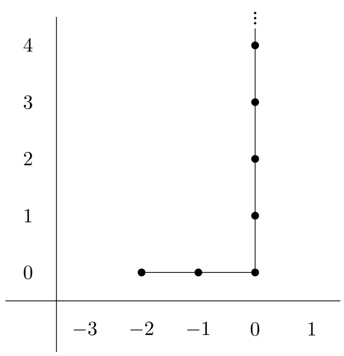


Figure A.7: The module $J$.

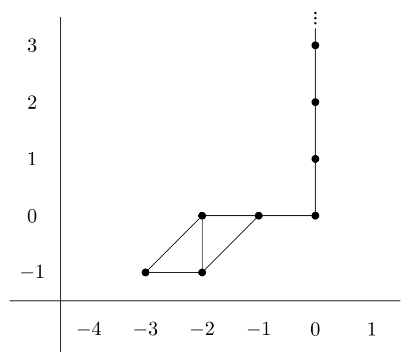


Figure A.8: The module $JD$.

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