Inverse Transport with Angularly Averaged Measurements

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This is to certify that I have examined this copy of a doctoral dissertation by

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and have found that it is complete and satisfactory in all respects,
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Abstract

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The inverse problem in radiative transfer is considered. The measurement setup involves controlling incoming radiation at the boundary of a convex domain in $\mathbb{R}^n$ and measuring the outgoing radiation in an attempt to reconstruct (actually, prove uniqueness and stability theorems about the reconstruction of) the total cross section and scattering coefficients. Varying degrees of radiation production/measurement accuracy lead to different results. Assuming the index of refraction is constant, if the incoming radiation is directionally dependent, and the outgoing radiation is averaged over angle, then the total cross section is determined. With an additional smallness on the scattering kernel, these measurements also determine the scattering kernel (assuming we know *a-priori* its angular dependence). These results are generalized to the case of a spatially varying, though possibly anisotropic, index of refraction. Finally, we consider the case where the incoming radiation is isotropic, and the measurements are averaged over angle. In this setup, the contribution to the measurement is an integral of the scattering kernel against a product of harmonic functions, plus an additional term that is small when absorption and scattering are small. The linearized problem is severely ill-posed. We construct a regularized inverse that allows for reconstruction of the low frequency content of the scattering kernel, up to quadratic error, from the nonlinear map. An iterative scheme is used to improve this error so that it is small when the high frequency content of the scattering kernel is small.

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Chapter 1

INTRODUCTION AND SUMMARY OF PREVIOUS RESULTS

1.1 Motivation for this Work and Historical Introduction

Around the time I arrived at the University of Washington, I decided that I wanted to work in some area that would help make renewable energy our primary source of power. After sending out lots and lots of emails, I realized that any technique that would improve weather measurements would make weather prediction better, and in turn that would help make the economics of wind and solar energy more attractive. After a few more emails, I decided that imaging of weather systems sounded like something that I could work into a nice inverse problem. Gunther told me that the transport equation was used as a model for wave propagation through the atmosphere. It is for that reason that I studied transport.

The reason for studying it in the context of medical imaging was my advisor's membership in the Focused Research Group (FRG) on "Inverse Problems in Radiative Transfer." This group consists of Guillaume Bal (Columbia), Arnold Kim (University of California, Merced), Steve McDowall (Western Washington), John Schotland (University of Pennsylvania), Plamen Stefanov (Purdue), and Gunther Uhlmann (UW). It was formed to study transport problems with angularly averaged measurements, and the intended application was medical imaging.

Discussions with FRG group members, in Banff Canada and Rio de Janerio Brasil, led to the work presented in chapter 2. This work has been published in *Inverse Problems* [24]. A natural extension of this work is to include a background metric that models a varying index of refraction, see chapter 3. That work has been submitted to *Communications in PDE* jointly with Stephen McDowall. Later, I spent a Spring in New York working with Guillaume Bal. He presented to me the problem of isotropic sources and averaged measurements. Working jointly with him and Francois Monard, the work that appears in chapter 4 was done. This work has been published in *Inverse Problems in Imaging*, [11].

1.2 Background, Physical Measurement Setup, and Motivation

Linear transport equations are used in many applications such as the propagation of the energy density of waves in heterogeneous media [8, 18, 38, 50], neutrons in nuclear reactors [32], and more recently, near-infra-red photons in tissues and its application in optical tomography, a medical imaging modality [5, 17, 34]. The equation we consider here (1.1) models the phase space propagation of energy density in one of the above mentioned forms. This model is valid only under certain assumptions. The validity of this model will not be considered here, though the interested reader may consult [10]. Here we give a heuristic explanation. As flux (of electromagnetic energy for example) passes through point x in direction v, it is being absorbed by the surrounding medium at a rate $\sigma_a(x)u(x,v)$, and is being scattered off into different directions v'' at a rate $u(x,v)k(x,v,v'')$. This gives another energy loss term $u(x,v)\int_{\mathbb{S}^{n-1}} k(x,v,v'')\,dv''$. Some flux passing through x in direction v' is scattered into direction v at a rate $u(x,v')k(x,v',v)$, giving a gain term $\int_{\mathbb{S}^{n-1}} u(x,v')k(x,v',v)\,dv'$. Assuming $\int_{\mathbb{S}^{n-1}} k(x,v,v'')\,dv''$ depends only on x (call it then σ_s), and that the system has reached a steady-state, we have

$$v \cdot \nabla_x u(x,v) = -\sigma_a(x)u(x,v) - \sigma_s(x)u(xv) + \int_{\mathbb{S}^{n-1}} u(x,v')k(x,v',v)\,dv'.$$

Defining the total cross section $\sigma := \sigma_a + \sigma_s$ we now have (1.1).

Inverse transport theory for (1.1) consists of reconstructing σ and k from various measurements. In this chapter we only consider measurements of the following type: A scientist controls the energy flux entering a bounded domain in $\mathbb{R}^n$, and measures the outgoing flux. Various restrictions on flux-production/measurement ability results in different reconstruction results.

While, as mentioned, inverse transport has a variety of applications, this work is motivated by optical tomography (OT) in medical imaging. While the exact form of OT measurements will change as new technology is introduced, it is clear that some form of directional averaging will be present for some time [5]. By this, we mean that it is unrealistic to expect a measurement apparatus to produce and detect flux as a function of both position and direction. Current measurements are closer to the form

$$\int_{\mathbb{S}^{n-1}} u(x,v)|\nu_x \cdot v|\, dv,$$

where $u(x,v)$ is the energy density at position x in direction v, and ν_x is the outward normal at x. In this case, we are measuring the outgoing current flux. The type of incoming flux one wishes to produce may also vary, see chapter 4.

The main contribution of this thesis is to consider two such averaged measurement setups. In the first case, incoming flux is directional, and outgoing flux is averaged, see chapter 2. In the second, both incoming and outgoing are non-directional, see chapter 4. The first case is also further explored with flux propogating on a Riemannian manifold in chapter 3. We will have to formulate the forward and inverse problem separately in all three cases. However, for the time being it is helpful to make the following general formulation: Let $X \subset \mathbb{R}^n$ be an open, bounded, and strictly convex domain with C^k boundary, for some $k \geq 1$. Define the incoming and outgoing bundles $\partial_\pm SX$ by $\partial_\pm SX := \{(x,v) \in \partial X \times \mathbb{S}^{n-1} : \pm\nu_x \cdot v > 0\}$. Let us denote by u the solution (if it exists) to the following boundary value problem for the stationary linear transport (Boltzmann) equation:

$$\begin{gathered} v \cdot \nabla_x u(x,v) + \sigma(x)u(x,v) = \int_{\mathbb{S}^{n-1}} k(x,v',v)u(x,v')dS' \\ u|_{\partial_- SX} = u_-. \end{gathered} \tag{1.1}$$

We then use some knowledge of $u_+ := u|_{\partial_+ SX}$ in an attempt to recover σ, k.

1.3 Previous Results

The problem of reconstructing σ, k is not new. Typically this inverse problem is done numerically or under strong assumptions such as homogeneity of the coefficients [3, 4, 7, 27, 46]. By "done numerically", we mean that σ, k are found as optimal parameters that minimize some "modeled vs. measured" penalty function.

Optimization based optical tomography has been used successfully, for example, in the identification of joints affected by rheumatoid arthritis [33]. It has also been used for imaging in small animals [14, 36]. Optical tomography is an attractive medical imaging alternative, especially due to its low cost and portability.

It should be pointed out that these results have been obtained using the *diffusion approximation* to transport. If scattering is high enough, then the transport equation may be approximated with the following diffusion equation

$$\nabla \cdot D(x)\nabla U(x) + \mu_a(x)U(x) = 0 \quad \text{in } X$$
$$U(x) + 3\epsilon L_3 v(x) \cdot D(x)\nabla U(x) = \Lambda(f)(x), \quad \text{on } \partial X.$$

The symmetric diffusion tensor D is related to σ and k, as is μ_a. The operator Λ maps the incoming angular distribution $f = f(x,v)$ to a real number, $\Lambda(f)(x)$. L_3 is the extrapolation length in this Robin-type boundary condition. The *transport mean free path* ϵ is a small parameter that measures the average distance photons travel before they are significantly deflected from their original path. In the limit $\epsilon \to 0$, the error between $U(x)$ and $\int u(x,v)\,dv$ is $O(\epsilon^2)$ [21]. The diffusion approximation is useful because the equation is solved in three spacial dimensions as opposed to three spacial and (at least) two angular dimensions, as is the case for transport. It should be pointed out that scattering of infrared light is very high in living tissue, and as a result the diffusion approximation is reasonable here. When compared to transport based tomography, diffusion based tomography is less accurate, although the differences are negligible when the noise levels are high [35]. Figure 1.1 shows values for an absorption coefficient in a three dimensional (simulated) body. Assuming the scattering coefficient is known, diffusion and transport based optical tomography is used to reconstruct this inclusion, see figure 1.2. One can see that the transport based reconstruction (on the left) is more accurate than the diffusion based reconstruction (center). Both of these figures are reprinted with permission from [35]. We also point out that neither reconstruction is particularly accurate. This lack of resolution is due to the fact that the incoming flux does not have angular dependence, and the outgoing flux is angularly averaged. The results in chapter 4 show that this inversion is severely ill-posed. Note that this (non-ideal) type of measurement is the type available in practice. We note here that when using transport-based inversion, one usually assumes that the transport kernel has the form $k(x, v', v) = f(x)g(v \cdot v')$, where the Henyey-Greenstein phase function

$$g(v \cdot v') = (1 - a^2)(1 + a^2 - 2av \cdot v')^{-3/2}$$

. Here $a \in [0, 1]$ is the anisotropy factor [52]. For more information about optical tomography, see http://www.optical-tomography.net.

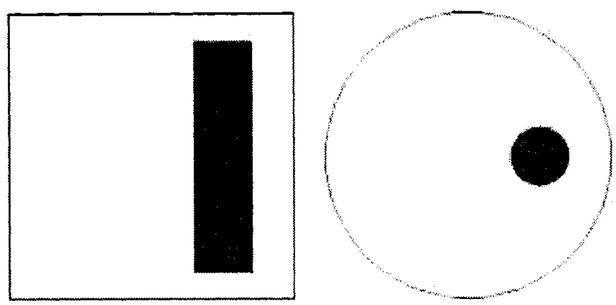

Figure 1.1: XZ (at y=0) and XY (at z=1) cross sections of the computational domain

Results on inverse transport through the atmosphere and underwater are plentiful. In these situations, it is more or less valid to assume the medium of interest has a one dimensional structure. In the aforementioned applications, the σ and k vary only in the vertical direction. Here (as in optical tomography) one often assumes that the scattering kernel has a form $k(x, v', v) = f(x)\Theta(v \cdot v')$, with Θ known. One is able then to obtain inversion formulas (as opposed to optimization based inversions) [28].

Inverse transport in the atmosphere is typically used to determine concentrations of molecules such as aerosols, Rayleigh molecules, ozone, and nitrogen dioxide [26]. Measurements are typically passive. In this case, measurements are made of reflected or transmitted sunlight, or transmitted radiation of some other form (e.g. the earth emits infrared radiation due to its nonzero temperature). Some measurements are active and measure backscattered laser and microwave radiation emitted from (typically ground based) sources [26].

A passive imaging technique was studied in [12]. Here, vertical profiles of ozone concentration was reconstructed using an optimization scheme. An inversion formula was also found, and shown to be severely ill-posed.

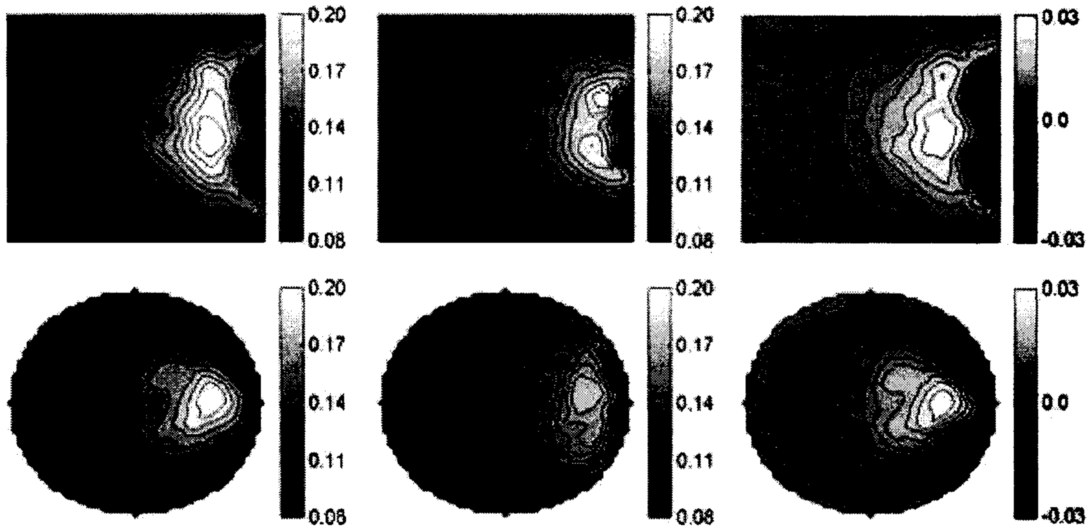

Figure 1.2: Cross sections of the reconstructed absorption coefficients in domain of small size. Top row: XZ cross section at y=0 for transport reconstruction (left), diffusion reconstruction (middle), and their difference (right). Bottom row: Corresponding XY cross sections at z=1. Reconstructions are done with noise-free data.

It is desirable to obtain results assuming only enough so that the forward problem is well-posed. The first main results along these lines were based on full phase-space measurements. What we mean by "full-phase space measurements" is the following. Particle densities depend on their position x and their direction v, which in this paper we assume is normalized to $|v| = 1$. Phase space measurements mean that $u(x, v)$ can be arbitrarily chosen and measured at the domain's boundary as a function of its phase-space variables (x, v) in $n + (n-1)$ dimensions for $n-$dimensional problems. This means having $4(n-1)$ dimensions of available data to reconstruct the optical parameters.

Given these measurements, and assumptions guaranteeing a unique solution to (1.1), it makes sense to consider the *albedo operator* defined as

$$\mathcal{A}u_-(x, v) := u_+(x, v) \tag{1.2}$$

given that $u_+ = u|_{\partial_+ SX}$ and u satisfies (1.1). It is then shown that knowledge of $\mathcal{A}$ uniquely determines σ and k under mild assumptions in dimensions three and higher [25, 40, 20] , and under a smallness assumption on k in dimension two [41]. We will now outline these results and the techniques used to obtain them.

First, under mild assumptions σ, k, such as $\sigma \in L^\infty$, $\|k\|_{L^\infty} < \text{diam}(X)\text{Vol}(\mathbb{S}^{n-1})$, the solution to (1.1) is decomposed as

$$u := (I-K)^{-1}Ju_- = Ju_- + KJu_- + (I-K)^{-1}K^2 Ju_-$$

where we have defined

$$\begin{aligned} Ju_-(y,v) &:= E(x_-(y,v),y)u_-(x_-(y,v),v), \\ Kf(x,v) &:= \int_0^{\tau_-(x,v)} E(x,x-tv) \int_{\mathbb{S}^{n-1}} k(x-tv) f(x-tv,v')\, dS(v')dt, \\ E(x,y) &:= \exp\left(- \int_0^{|y-x|} \sigma\Big(x + \frac{y-x}{|y-x|}s\Big) ds \right). \end{aligned}$$

The term Ju_- represents flux that enters the boundary, is attenuated, and exits without scattering. The term KJu_- enters, is attenuated along its path, and exits after scattering once.

This decomposition leads to a similar decomposition of $\mathcal{A}$ into

$$\mathcal{A} = \alpha_0 + \alpha_1 + \alpha_2$$

so that knowledge of α_0 is equivalent to knowledge of $\{(u_-, Ju_-) : u_- \in L^1(\partial_- SX)\}$. We also have α_1 corresponding to KJu_- and α_2 corresponding with $(I-K)^{-1}K^2Ju_-$ in a similar fashion.

One notices that knowledge of the function $E(x,y)$ for $x, y \in \partial X$ is equivalent to knowing the X-ray transform of σ. So to recover σ, one needs to separate α_0 from α_1, α_2. As shown in [20] this can be done in dimensions two and higher since α_0 is a delta distribution more singular than α_1 and α_2. Similarly, knowledge of α_1 would allow one to recover k. In dimensions three and higher, α_1 is a delta distribution, whereas α_2 is not, and a limiting argument shows that recover of α_1 is possible. In two dimensions, both kernels are locally integrable, so a different argument is needed. The argument used is of the following type: Let B be a Banach space, and suppose $f : B \to B$ and we can decompose $f(k) = Tk + g(k)$ where T is linear with a bounded inverse and g comes with the estimate

$\|g(k) - g(\tilde{k})\| \leq \|k - \tilde{k}\|(\|k\| + \|\tilde{k}\|)$. Continuing, suppose $f(k) = f(\tilde{k})$, then we have

$$\|k - \tilde{k}\| \leq C\|Tk - T\tilde{k}\| = C\|g(k) - g(\tilde{k})\| \leq C'\|k - \tilde{k}\|(\|k\| + \|\tilde{k}\|).$$

So, under the *a-priori* assumption $\|k\|, \|\tilde{k}\| < (2C')^{-1}$ we must then have $k = \tilde{k}$.

The model (1.1) says that in the absence of scattering, flux propagates in straight lines. Clearly this model cannot hold in medium with varying index of refraction. When the refractive index is non-constant, Fermat's principle says that light bends (in the absence of scattering, and in the short wavelength limit) in order to minimize the action

$$\int_A^B n(x(t), \dot{x}(t))\, dt.$$

Here n is the refractive index, and $x(t)$ is the position of the wavefront at time t. This model can be extrapolated to derive a transport equation valid when refractive index varies, see chapter 3.1. Assuming the metric is simple, and given bounds on curvature, it may be shown that the albedo operator determines σ in dimensions $n \geq 2$, and k in dimensions $n \geq 3$ [29], and k in dimension two with an additional smallness assumption on k [30].

Chapter 2

DIRECTIONAL SOURCES AND AVERAGED MEASUREMENTS, EUCLIDEAN CASE

2.1 Introduction

If the refractive index is constant, then, in the absence of scattering light travels in straight lines. The content of this chapter is inverse transport in this setup. The presence of angularly averaged measurements means that the Schwartz kernel corresponding to single-scattering is then seen to be an invertible weighted ray transform. This realization, which was pointed out to me by Plamen Stefanov after numerous discussions between members of the Focused Research Group on radiative transfer, forms the basis of this work.

We formulate this mathematically as follows. Let $X \subset \mathbb{R}^n, n \geq 2$ be an open bounded convex set with C^1 boundary ∂X Define $\partial_\pm SX := \{(x,v) \in \partial X \times \mathbb{S}^{n-1} : \pm\nu_x \cdot v > 0\}$, where ν_x is the outer normal to ∂X at $x \in \partial X$. For $(x,v) \in X \times \mathbb{S}^{n-1}$, let $\tau_\pm(x,v)$ be the distance from x to ∂X traveling in the direction of $\pm v$. We give $\partial_\pm SX$ the measure $d\xi(x,v) = |\nu_x \cdot v| d\mu(x) dS(v)$, where $d\mu, dS$ are the volume forms on $\partial X, \mathbb{S}^{n-1}$ respectively. We also define $\tau_\pm(x,v)$ as the "travel time to" to ∂X starting at point x, headed in direction $\pm v$. So $x \pm \tau_\pm(x,v)v \in \partial_\pm SX$. Let us denote by the solution (if it exists) to the following boundary value problem for the stationary linear transport (Boltzmann) equation:

$$v \cdot \nabla_x u(x,v) + \sigma(x) u(x,v) = \int_{\mathbb{S}^{n-1}} \Theta(x,v',v) u(x,v') dS'$$

$$u|_{\partial_- SX} = u_-.$$

Since σ and Θ are separated mathematically, we treat them as separate functions in this paper, but keep in mind the following caveat: In practice, assumptions made about σ imply assumptions about Θ and vice versa. Notice we assume σ depends only on position, and $v \in \mathbb{S}^{n-1}$. This slightly restrictive (but physically realistic) assumption on σ is necessary: If $\Theta = 0$, this problem reduces to the usual X-ray transform. One can then show that if

σ depends on the direction v (as opposed to $|v|$), no boundary measurement can uniquely determine σ (see the introduction of [20]). The "monochromatic" assumption $v \in \mathbb{S}^{n-1}$ greatly simplifies the problem, and is commonly used.

Solving the forward problem requires some assumptions such as

$$0 \le \sigma, \Theta \in L^\infty, \quad \int_{\mathbb{S}^{n-1}} \Theta(x, v', v)\, dS' \le \sigma(x)$$

This assumption is physically reasonable, since it means flux production is less than flux elimination. Alternatively, the forward problem is also well posed if Θ is small enough. Since the main results of this chapter (actually, in this whole paper) require small enough Θ, we stick with this assumption only, see [20]. We will also assume $\Theta, \sigma \in L^\infty$ are bounded functions.

In this chapter, we work with the following type of measurement of the outgoing flux:

Definition 2.1.1. Given nonvanishing $m \in C(\partial_+ SX)$ with $m(x,v)|\nu_x \cdot v|^{-1} \in L^\infty(\partial_+ SX)$, we define the *angularly averaged albedo operator*, $\mathcal{M} : L^1(\partial_- SX, d\xi) \to L^1(\partial X)$ (the mapping properties will be verified later) by

$$\mathcal{M}u_-(x) := \int_{\nu_x \cdot v > 0} u(x,v) m(x,v)\, dS, \tag{2.1}$$

where u satisfies (1).

$\mathcal{M}u_-(x)$ corresponds to a measurement at $x \in \partial X$ with incoming boundary condition u_-. If for example, $m(x,v) = \nu_x \cdot v$, then we are measuring the power/area flowing out through the point x. In general, m corresponds to some limited aperture of our measuring device. Our "full" measurement set is $\{(u_-, \mathcal{M}u_-) : u_- \in L^1(\partial_- SX)\}$. These are (a continuous version of) measurements available in practice, see [5].

Similar to [20], we decompose the Schwarz kernel of $\mathcal{M}$ into terms of varying singularity. The most singular term corresponds to the ballistic (no scattering) part of the solution. From this we are able to reconstruct σ. With additional assumptions, these measurements uniquely reconstruct the unknown part of Θ. To be more specific, we assume Θ has the special form $\Theta(x, v, v') = k(x) g(x, v, v')$ where $g \in C^\infty$ is a, positive, a-priori known phase-function, and the scattering kernel $k \in L^\infty$ is small and supported away from the boundary.

We also assume that σ, g are close to some real analytic coefficients in a manner to be made precise later. These assumptions kill the (weak) blow-up of the single scattering term in the decomposition, and allow us to view this within the framework established in [23]. In particular, the single scattering (linear in k) term is a weighted X-ray transform of k. This transform is injective. The smallness assumption made on k allows us then to prove injectivity of the nonlinear map $k \mapsto \mathcal{M}$. The assumption on supp(k) is reasonable, since in practice the measurement apparatus can always be placed some non-zero distance from the scattering body. The assumption $\Theta = k(x)g(x, v, v')$ is also of practical interest. For example, in atmospheric science it is common to assume a phase function $g(v, v') = 1+(v\cdot v')^2$ corresponding to Rayleigh scattering [?]. In the case of wave propagation in tissues, it is common to assume the Henyey-Greenstein phase function $g(v, v') = (1 - a^2)(1 + a^2 - 2av \cdot v')^{-3/2}$, where $a \in [0, 1]$ is the anisotropy factor [52]. The choice of phase function corresponds to an assumption on the underlying scattering mechanism at the measurement frequency/velocity. Note that these scattering phase functions make g real analytic.

2.2 Statement of the main results

As we will see, the "full" measurement set $\{(u_-, \mathcal{M}u_-) : u_- \in L^1(\partial_- SX)\}$ is well defined. However, we do not need nearly so much information to recover σ, k.

In all theorems below, (σ, k) are coefficients in (1.1) with $\Theta(x, v', v) = k(x)g(x, v', v)$, with g known. Also, $\mathcal{M}, m$ are as in definition (2.1.1). Given $\mathcal{H} \subset \partial_- SX$, we define the set of lines $\Gamma(\mathcal{H}) = \{x_0 + t_0 v_0 : (x_0, v_0) \in \mathcal{H}, t_0 \in [0, \tau_+(x_0, v_0)]\}$.

Theorem 2.2.1 (Uniqueness result for σ). *Assume, $0 \le \sigma \in L^\infty$, and $\|\Theta\|_{L^\infty} < (Vol(\mathbb{S}^{n-1}) diam(X))^{-1}$. Let $\mathcal{H}_\sigma \subset \partial_- SX$ be open. If the X-ray transform over all lines in $\Gamma(\mathcal{H}_\sigma)$ is injective, then σ is uniquely determined by $\{(u_-, \mathcal{M}u_-) : u_- \in L^1(\mathcal{H}_\sigma)\}$*

Fixing $D > 0$, and making the definition

$$\mathcal{K}_\varepsilon^D := \{k \in L^\infty(X) : \text{dist}(\text{supp}(k), \partial X) > D, \|k\|_{L^\infty} \le \varepsilon\}$$

, we have the following uniqueness result for k.

Theorem 2.2.2 (Uniqueness result for k). *Suppose Θ has the form $\Theta = k(x)g(x, v', v)$, with (g, m) both non-vanishing, and (g, m, σ) known real analytic functions. Let $\mathcal{H}_k \subset \partial_- SX$ be open, and assume that for every $(x, v) \in X \times \mathbb{S}^{n-1}$ there exists $\gamma \in \Gamma(\mathcal{H}_k)$ passing through x in direction v_0 perpendicular to v. Then there exits $\varepsilon > 0$ small enough so that for a.e. $x \in \partial X$, knowledge of $\{(u_-, \mathcal{M}u_-(x)) : u_- \in L^1(\mathcal{H}_k)\}$ uniquely determines k over the class of all $\tilde{k} \in \mathcal{K}_\varepsilon^D$.*

Moreover, ε may be chosen such that the result holds in some C^∞ neighborhood of g, and a C^2 neighborhood of (m, σ).

Remark. We will find a solution to (1.1) in terms of a Neumann series. Estimates derived from this series require $\varepsilon < (\text{diam}(X)\text{vol}(\mathbb{S}^{n-1})\|g\|_{L^\infty})^{-1}$ (proposition 2.5.3). ε also depends upon the coefficients m, and σ. As mentioned earlier, the Schwarz kernel term linear in k is a weighted X-ray transform. To "invert" the weighted transform, one uses the open mapping theorem. This results in a coefficient (and thus a smallness requirement for k), depending upon (g, m, σ), that we cannot calculate explicitly. We take $\varepsilon > 0$ small enough to meet both of these requirements. Since real analytic functions are dense, we have a uniqueness result for an open and dense set of (g, m, σ).

In every dimension, $\Gamma(\mathcal{H}_k)$ must include lines passing through every point of supp(k). If $n = 2$, then these lines must pass through every point in 1/2 of the possible directions (only one of the directions $+v$ and $-v$ is necessary). If $n \geq 3$, we may omit some directions. For example, we could choose directions confined to any great circle $C \subset \mathbb{S}^{n-1}$. See sections 4, 5 for details.

To prove theorems 2.2.1 and 2.2.2 we decompose the Schwarz kernel of the operator $\mathcal{M}$ into an infinite sum $\sum_{i=0}^{\infty} \alpha_i$. The individual terms $\alpha_i \in \mathcal{D}'(\partial X \times \partial_- SX)$, are each associated with i scattering events. α_0 is a singular distribution, so our measurements enable us to access it separately from the less singular part $\sum_{i=1}^{\infty} \alpha_i$. Since knowledge of $\{(u_-, \mathcal{M}u_-) : u_- \in L^1(\mathcal{H})\}$ is equivalent to knowledge of $\sum_{i=0}^{\infty} \alpha_i\big|_{\partial X \times \mathcal{H}}$, we work with the latter in proving our theorems.

We also prove a stability estimate for the inverse problem. Suppose that $(k, \tilde{k})$, $(\sigma, \tilde{\sigma})$ are two pairs of possibly different coefficients giving rise to averaged albedo operators $(\mathcal{M}, \tilde{\mathcal{M}})$, weight functions $(w, \tilde{w})$, and kernels $(\alpha_i, \tilde{\alpha}_i)$. Let $\Delta k = k - \tilde{k}$, $\Delta\alpha_i = \alpha_i - \tilde{\alpha}_i$ and so on.

For any fixed $x \in \partial X$, let $\mathcal{S} := \| \sum_{i=1}^{\infty} \Delta\alpha_i(x, \cdot, \cdot) \|_{H^1(\partial_- SX)}$. As we will see in the proof of lemma 2.6.2, $\sum_{i=2}^{\infty} \Delta\alpha_i(x, \cdot, \cdot) \in H^1$. Since for every i, $\alpha_i(x, \cdot, \cdot) \in C^2$, we have $\mathcal{S} < \infty$. Now if $u_- \in H^{-1}(\partial_- SX)$, to contribution to $\Delta\mathcal{M}u_-$ due to scattering is $|\sum_{i=1}^{\infty} \int_{\partial_- SX} \Delta\alpha_i(x, x_0, v_0) u_-(x_0, v_0)\, d\xi(x_0, v_0)| \leq \mathcal{S} \|u_-\|_{H^{-1}}$. We also define $a(x_0, v_0) := \int_0^{\tau_+(x_0,v_0)} \sigma(x_0 + t_0 v_0)\, dt_0$, and $\tilde{a}$ similarly. Our measurements directly give e^a. To recover σ, we take the logarithm, then invert an X-ray transform. We measure this difference $a - \tilde{a}$ as $\mathcal{A} := \|a - \tilde{a}\|_{H^1(\partial_- SX)}$. Estimates involving $\mathcal{A}$ are standard when dealing with X-ray transforms. They are not directly related to error in the measured data. They are however related to error coming from processing of the data a, or discretization effects.

Theorem 2.2.3 (Stability). *Suppose Θ has the form $\Theta = k(x)g(x, v', v)$, with (g, m) both non-vanishing, and (g, m, σ) known real analytic functions. Assume $k, \tilde{k} \in \kappa_\varepsilon^D$ for ε small enough so that theorem 2.1 holds with coefficients (g, m, σ). Also suppose $\|\sigma\|_{H^s}, \|\tilde{\sigma}\|_{H^s} \leq M < \infty$, for some $s > n/2 + 2$. Then there exits C depending on $(M, D, s, X, m, g, \varepsilon, \sigma)$ such that*

$$\|\Delta k\|_{L^2} \leq C(\mathcal{S} + \varepsilon\, \mathcal{A}^{1-(4+n)/(2s)} + \varepsilon\, \mathcal{A}^{2(1-(4+n)/(2s))})$$

and

$$\|\Delta\sigma\|_{L^\infty} \leq C\mathcal{A}^{1-(2+n)/(2s)},$$

with the inequality holding in a C^∞ neighborhood of g, and a C^2 neighborhood of (m, σ) (so long as $\|\sigma\|_{H^2} \leq M$).

Remark. The author knows of no other stability result dealing with angularly averaged measurements in transport. The closest result is known only in two dimensions, and with angularly dependent measurements, see [41].

The dependence of C on ε can be stated as $C \leq (1 - (\varepsilon\|g\|\mathrm{Vol}(\mathbb{S}^{n-1})\mathrm{diam}(X)))^{-1}$. The dependence of C on σ comes about for the same reason that ε depended on σ in theorem 2.2.2.

As our uniqueness result requires knowledge of $(\mathcal{M}u_-, u_-)$ only for $u_- \in L^1(\mathcal{H}_k)$, one expects that a stability estimate involving $\| \sum_{i=1}^{\infty} \Delta\alpha_i \|_{H^1(\mathcal{H}_k)}$ is possible. In proving theorem

2.2.3 we obtain something of this sort by using a cutoff, $\beta \in C^\infty(\partial_- SX)$. We only require $\beta \neq 0$ on $\mathcal{H}_k$. It should be noted that β is also needed for theorem 2.2.2. This results in a more complicated version of theorem 2.2.3 which depends on $\mathcal{S}_\beta := \|\beta \sum_{i=1}^\infty \alpha_i\|_{H^1(\partial_- SX)}$. The resultant estimate for k is then changed to

$$\|\Delta k\|_{L^1} \leq C_\beta(\mathcal{S}_\beta + \varepsilon \mathcal{A}^{1-(4+n)/2s} + \varepsilon \mathcal{A}^{2(1-(4+n)/2s)}). \tag{2.2}$$

Here C_β depends also on β. Now theorem 2.2.3 is a special case.

2.3 Decomposition of the averaged albedo operator

Here we show that $\mathcal{M}$ as a distribution on $\partial_- SX$ has a kernel with one singular term representing the solution in the absence of scattering (the ballistic solution), and a series of terms corresponding to integer numbers of scattering. This decomposition is essential since it reduces the problem to that where the input source is a delta distribution in both position and angle. This leads to a highly simplified way of viewing the problem. See figure 2.1. It will also allow us to use theorems from integral geometry to recover the scattering kernel k. First, we will need the following theorem from [20].

Lemma 2.3.1 (Change of Variables). *Assume that $f \in L^1(X \times \mathbb{S}^{n-1})$. Then*

$$\int_{X\times\mathbb{S}^{n-1}} f(x,v)dxdS = \int_{\partial_\mp SX}\int_0^{\tau_\pm(x,v)} f(x_0 \pm tv, v)\, dtd\xi.$$

Here we recall how the forward problem can be solved: First, u solves (1.1) if and only if for all t,

$$\begin{aligned} 0 &= e^{-\int_0^t \sigma(x-sv)ds}\left((v\cdot\nabla+\sigma)u(x-tv,v) - \int_{\mathbb{S}^{n-1}} \Theta(x-tv,v',v)u(x-tv,v')\,dS\right) \\ &= -\frac{d}{dt}e^{-\int_0^t \sigma(x-sv)ds}u(x-tv,v) - e^{-\int_0^t \sigma(x-sv)ds}\int_{\mathbb{S}^{n-1}} \Theta(x-tv,v',v)u(x-tv,v')\,dS. \end{aligned}$$

Next, integrate from 0 to $\tau_-(x,v)$ to obtain the following.

$$\begin{aligned} &(I-K)u = Ju_-, \text{ with} \\ &Ju_-(y,v) := E(y-\tau_-(y,v)v, y)u_-(y-\tau_-(y,v)v, v), \\ &Kf(x,v) := \int_0^{\tau_-(x,v)} E(x,x-tv)\int_{\mathbb{S}^{n-1}} \Theta(x-tv,v',v)f(x-tv,v')\,dS'dt. \\ &\quad E(a,b) := e^{-\int_0^{|b-a|}\sigma\left(a+\frac{b-a}{|b-a|}s\right)ds}. \end{aligned} \tag{2.3}$$

It can be shown that the solution to (2.3) defines a distributional solution to (1.1). Under our assumptions, the integral equation can be solved with a Neumann series in the space $L^1(X \times \mathbb{S}^{n-1}, \tau^{-1}dxdS)$, where $\tau(x,v) = \tau_-(x,v) + \tau_+(x,v)$. For details see proposition 2.3 in [20]. Since the solution solves (2.3) in an L^1 sense, $t \mapsto u(x+tv,v)$ need not be differentiable. If it is, then the solution we have found is a classical solution. If not, then it must be interpreted in the sense of distributions. If $t \mapsto u(x+tv,v)$ is not differentiable for any u in the function class associated with the solution to (2.3), then no classical solution exists. In all cases we do have the following, which is useful in proving the trace lemma 2.3.3.

Lemma 2.3.2 (Lipshitz Continuity Along Lines). *Suppose $u_- \in L^\infty(\partial_- SX)$ is a bounded function. Then for every $(x,v) \in X \times \mathbb{S}^{n-1}$, the map $t \to u(x-tv,v)$ is Lipshitz continuous with Lipshitz constant depending only on $\|\Theta\|_{L^\infty}, \|\sigma\|_{L^\infty}, \|u_-\|_{L^\infty}$, and X.*

Proof. First, for every bounded function $f \in L^\infty(X \times \mathbb{S}^{n-1})$, $(x,v) \in X \times \mathbb{S}^{n-1}$, the integral $|Kf(x,v)| \le \mathrm{Vol}(\mathbb{S}^{n-1})\mathrm{diam}(X)\|\Theta\|_{L^\infty}\|f\|_{L^\infty}$. Now proposition 2.3 in [20] shows that the solution $u(x,v) = \sum_{i=0}^\infty K^i J u_-(x,v)$, for $(x,v) \in X \times \mathbb{S}^{n-1}$. Therefore, using our assumption $\|\Theta\|_{L^\infty} \le (\mathrm{Vol}(\mathbb{S}^{n-1})\mathrm{diam}(X))^{-1}$, we have that the solution u is bounded inside $X \times \mathbb{S}^{n-1}$.

Next, for $l \in \mathbb{R}$ such that $x, x+lv \in X$ we consider

$$|u(x+lv,v) - u(x,v)| \tag{2.4}$$

The top line of (2.3) shows that for $(x,v) \in X \times \mathbb{S}^{n-1}$,

$$\begin{aligned} u(x,v) &= \int_0^{\tau_-(x,v)} E(x, x-tv) \int_{\mathbb{S}^{n-1}} \Theta(x-tv, v', v) u(x,v')\, dS' dt + E(x_0, x) u_-(x_0, v) \\ x_0 &:= x - \tau_-(x,v)v \end{aligned} \tag{2.5}$$

For simplicity, extend Θ to be zero outside of X then take the limit of the integral in t to be ∞. We can write $u(x+tv,v)$ in a similar manner, then break the integral over t into the intervals $[0,l]$ and $[l,\infty]$. This nets two integrals, the second of which is almost identical to the integral defining $u(x,v)$ (after a change of variables $t \mapsto t-l$). The result is that (2.4) is bounded by

$$\int_0^l E(x+lv, x+lv-tv) \int_{\mathbb{S}^{n-1}} \Theta(x+lv-tv, v', v) u(x+lv-tv, v')\, dS' dt$$
$$+ \int_0^\infty [E(x+lv, x-tv) - E(x, x-tv)] \int_{\mathbb{S}^{n-1}} \Theta(x-tv, v', v) u(x-tv, v')\, dS' dt$$
$$+ [E(x_0, x+lv) - E(x_0, x)]\, u_-(x_0, v)$$

Now use the inequality

$$|E(a,b) - E(a, b+lv)| \le l\|\sigma\|_{L^\infty}$$

to see that (2.4) is bounded by

$$l\big[\|\Theta\|\|u\|\mathrm{Vol}(\mathbb{S}^{n-1})\mathrm{diam}(X) + \|\sigma\|\mathrm{Vol}(\mathbb{S}^{n-1})\|\Theta\|\|u\| + \|\sigma\|\|u_-\|\big].$$

Where all norms are L^∞. □

As stated, since Θ is small enough we may solve the forward problem with an Neumann series. *A-priori*, this solves the forward problem only in $X \times \mathbb{S}^{n-1}$. In other words $u(x,v) = \sum_{i=0}^\infty K^i J u_-(x,v)$ for $(x,v) \in X \times \mathbb{S}^{n-1}$. As our measurements are on $\partial_+ SX$, we must restrict this result to $\partial_+ SX$. Call this restriction operator R. The result is the albedo operator $A : L^1(\partial_- SX) \to L^1(\partial_+ SX)$, defined by $Au_- = R\sum_{i=0}^\infty K^i J u_-$. In [20] proposition 2.3 they build upon [?, ?] and show that this restriction is well defined and continuous. We prove slightly more below.

Lemma 2.3.3. *For $(x,v) \in \partial_+ SX$, $u_- \in L^\infty(\partial_- SX)$ a bounded function, we have*

$$Au_-(x,v) = u|_{\partial_+ SX} = \sum_{i=0}^\infty K^i J u_-(x,v).$$

Where $K^i J u_-(x,v)$ is given by the formula (2.3), and the series converges in $L^1(\partial_- SX)$. Moreover, the series extends uniquely to a continuous operator : $L^1(\partial_- SX, d\xi) \to$ $L^1(\partial_+ SX, d\xi)$.

Remark. Given this, we write our solution at the boundary from now on as $\sum_{i=0}^\infty K^i J u_-$.

Proof. First suppose u_- is bounded. Let B denote the operator defined by the expression $Bu_-(x,v) := \sum_{i=0}^{\infty} K^i J u_-(x,v)$, for $(x,v) \in X \times \mathbb{S}^{n-1}$, or $(x,v) \in \partial_+ SX$, and $u_- \in L^1(\partial_- SX)$. The proof of proposition 2.3 in [20] shows that inside $X \times \mathbb{S}^{n-1}$, Bu_- gives the solution to (2.3).

Now extend X slightly to a new convex domain Y (extending Θ, σ to be zero outside of X) still small enough so that $\|\Theta\|_{L^\infty} < (\text{diam}(Y)\text{Vol}(\mathbb{S}^{n-1}))^{-1}$. We extend the incoming boundary data $u_-(x,v)$ back in the direction v to give us $\tilde{u}_- \in \partial_- SY$. We then have a solution to the transport problem given by $\tilde{u}(x,v) = \sum_{i=0}^{\infty} \tilde{K}^i \tilde{J} w_-(x,v)$, with the operators $\tilde{K}, \tilde{J}$ defined analogously to K, J. Since X is convex, once flux enters $Y \setminus X$ it will not return to X. Also, no scattering or flux extinction occurs in $Y \setminus X$. Therefore, the solution $\tilde{u}$ to the transport problem on our new domain, when restricted to X, should not differ from u. Indeed, consider the integral $\tilde{K} f(x,v)$, $(x,v) \in \bar{X} \times \mathbb{S}^{n-1}$. Since Θ is zero outside of X, we could take the line integral to be from 0 to $\tau_-(x,v)$, with $\tau_-(x,v)$ being either the distance to ∂X or to ∂Y. Also, our definition of $\tilde{u}_-$ ensures that $Ju_-(x,v) = \tilde{J}\tilde{u}_-(x,v)$. Therefore, for $(x,v) \in X \times \mathbb{S}^{n-1}$, $B(x,v) = \sum_{i=0}^{\infty} Ju_-(x,v) = \sum_{i=0}^{\infty} \tilde{K}^i \tilde{J} \tilde{w}_-(x,v)$. Now inside Y, $\tilde{u}$ will satisfy lemma 2.3.2 if u_- is bounded. Therefore Bu_- satisfies lemma 2.3.2 for $(x,v) \in \bar{X} \times \mathbb{S}^{n-1}$.

This means that $Bu_-(x,v)$, $(x,v) \in \partial_+ SX$ gives a function on $\partial_+ SX$ corresponding to the flux leaving ∂X. We now show that this function is a continuous restriction of $u = Bu_-$ to the boundary. This proof depends upon the boundedness of $\tau_-(x,v)^{-1} Bu_-(x,v)$ in $L^1(X \times \mathbb{S}^{n-1})$, and the absolute continuity of $t \mapsto Bu_-(x - tv, v)$. For convienence we define the following norm and corresponding space

$$\|f\|_{\mathcal{W}} := \|v \cdot \nabla f\|_{L^1(X \times \mathbb{S}^{n-1})} + \|\tau_-^{-1} f\|_{L^1(X \times \mathbb{S}^{n-1})}.$$

Now suppose g is absolutely continuous on $[0,a]$ for some $a > 0$. Then $g' \in L^1$ is defined a.e. (see [37]), and

$$g(0) = -\int_0^t g'(s)\, ds + g(t).$$

From this we obtain

$$|g(0)| = \frac{1}{a}\int_0^a |g(0)|\, ds \le \|g'\| + \frac{1}{a}\|g\|. \tag{2.6}$$

Since Lipshitz continuity implies absolute continuity (see [37] p.112), $t \mapsto Bu_-(x - tv, v)$ is absolutely continuous for $(x, v) \in \bar{X} \times \mathbb{S}^{n-1}$, this implies that for $x' \in \partial X$,

$$Bu_-(x', v) = -\int_0^t \partial_s Bu_-(x' - sv, v)\, ds + Bu_-(x' - tv, v).$$

Where the derivative $\partial_s Bu_-(x' - sv, v) = (v \cdot \nabla Bu_-)(x' - tv, v)$ exists a.e. Using (2.6) we see that

$$|Bu_-(x', v)| \leq \int_0^{\tau_-(x',v)} |(v \cdot \nabla Bu_-)(x' - tv, v)|\, dt + \frac{1}{\tau_-(x', v)} \int_0^{\tau_-(x',v)} |Bu_-(x' - tv, v)|\, dt.$$

Integrating over $\partial_- SX$ and using lemma 2.3.1 we finally obtain

$$\begin{aligned} \|Bu_-\|_{L^1(\partial_- SX, d\xi)} &\leq \|Bu_-\|_{\mathcal{W}} \\ &\leq C_{X,\sigma,\Theta} \|u_-\|_{L^1(\partial_- SX, d\xi)}. \end{aligned}$$

Where $C_{X,\sigma,\Theta}$ is a constant depending only on X, σ, Θ. The last inequality comes from the proof of proposition 2.3 in [20]. Therefore the expression Bu_-, thought of as a function in $L^1(\partial_+ SX, d\xi)$ gives a continuous restriction of Bu_-, thought of as a function in $\mathcal{W}$. The last inequality, and the fact that bounded functions are dense in $L^1(\partial_- SX, d\xi)$, shows that B extends to a continuous operator $\tilde{B} : L^1(\partial_- SX, d\xi) \to L^1(\partial_+ SX, d\xi)$. ☐

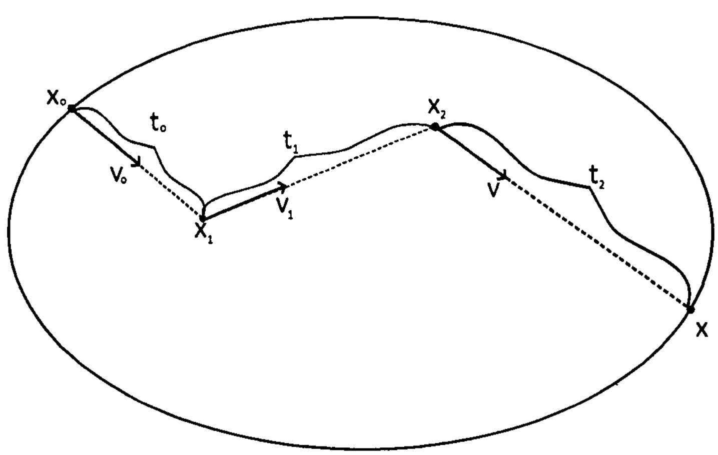

Figure 2.1: Double Scattering

We now write the decomposition of $\mathcal{M}$. Specifically, we decompose the linear functional $L^1(\partial_- SX) \ni u_- \mapsto \mathcal{M}u_-(x)$ for a fixed $x \in \partial X$ into a series of Schwartz kernels. The first term is a singular distribution that will be useful for recovering σ. The next term is a weighted line integral of k that will be shown to be injective. The other terms are quadradic in k, and can be shown not to destroy the injectivity. Later, we will need to work with a fixed measurement point $x \in \partial X$, and therefore prove pointwise convergence of the decomposition. See the paragraph after definition 2.5.3.

Theorem 2.3.1 (Kernel Decomposition). *Suppose* $\|\Theta\|_{L^\infty} < (diam(X)\, Vol(\mathbb{S}^{n-1}))^{-1}$. *Then* $\mathcal{M} : L^1(\partial_- SX, d\xi) \to L^1(\partial X)$ *continuously and for a.e.* $x \in \partial X$, *and is given by the integral*

$$\begin{aligned}\mathcal{M}u_-(x) &= \sum_{i=0}^{\infty} \int_{\nu_x \cdot v > 0} K^i J u_-(x,v) m(x,v)\, dS \\ &= \sum_{i=0}^{\infty} \int_{\partial_- SX} \alpha_i(x, x_0, v_0) u_-(x_0, v_0)\, d\xi(x_0, v_0),\end{aligned} \tag{2.7}$$

with

$$\begin{aligned}
\alpha_0(x,\cdot,\cdot) &\quad \textit{is a singular distribution,}\\
&\qquad \textit{supported on a manifold of dimension } n-1,\\
\alpha_1(x,x_0,v_0) &:= \int_0^{\tau_+(x_0,v_0)} \Theta(x_1,v_0,v)E(x_0,x_1,x)(t_1)^{1-n}m(x,v)\,dt_0,\\
\alpha_2(x,x_0,v_0) &:= \int_0^{\tau_+(x_0,v_0)} \int_X \Theta(x_1,v_0,v_1)\Theta(x_2,v_1,v)E(x_0,x_1,x_2,x)(t_1t_2)^{1-n}m(x,v)\,dx_2dt_0,\\
&\vdots,\\
\alpha_i(x,x_0,v_0) &:= \int_0^{\tau_+(x_0,v_0)} \int_X \cdots \int_X \Theta(x_1,v_0,v_1)\cdots\Theta(x_i,v_{i-1},v)E(x_0,\dots,x_i,x)\\
&\qquad \times (t_1\cdots t_i)^{1-n}m(x,v)\,dx_i\cdots dx_2dt_0
\end{aligned}$$

where,

$$\begin{aligned}
x_1 &:= x_0 + t_0v_0,\\
E(a,b) &:= e^{-\int_0^{|b-a|}\sigma\left(a+\frac{b-a}{|b-a|}s\right)ds},\\
E(a_1,\dots,a_m) &:= E(a_1,\dots,a_{m-1})E(a_{m-1},a_m),\\
t_m &:= |x_{m+1}-x_m|,\ (m\le i-1),\quad t_i := |x-x_i|,\\
v_m &:= (x_{m+1}-x_m)t_m^{-1},\quad v := (x-x_i)t_i^{-1}.
\end{aligned}$$

Proof. Note that lemma (2.3.3) shows the map $L^1(\partial_-SX,d\xi)\ni u_-\mapsto u|_{\partial_+SX}\in L^1(\partial_+SX,d\xi)$ is well defined and continuous. Since $m(x,v)|\nu_x\cdot v|^{-1}$ is bounded, the map $u_-\mapsto \mathcal{M}u_-\in L^1(\partial X)$ is continuous, i.e. $\|\mathcal{M}u_-\|_{L^1(\partial X)}\le C\|u_-\|_{L^1(\partial_-SX)}$. In particular, this holds if $u_-\in C^\infty(\partial_-SX)$. In this special case, using lemma 2.3.3, the solution decomposes as $u|_{\partial_+SX}=\sum_{i=0}^\infty K^iJu_-$. Since each term preserves non-negative functions, each represents a bounded map, i.e.,

$$\|K^iJu_-\|_{L^1(\partial_+SX)}\le C\|u_-\|_{L^1(\partial_-SX)}.$$

Consider now the function

$$\mathcal{M}^i u_-(x) := \int_{\nu_x \cdot v > 0} K^i J u_-(x,v) m(x,v)\, dS,$$

which obeys

$$\begin{aligned}\|\mathcal{M}^i u_-\|_{L^1(\partial X)} &\le \int_{\partial X} \int_{\nu_x \cdot v>0} |K^i J u_-(x,v)| m(x,v)\, dS d\mu(x)\\ &\le \sup\left\{\frac{m(x,v)}{|\nu_x \cdot v|}\right\} \|K^i J u_-\|_{L^1(\partial_+ SX, d\xi)}\\ &\le \sup\left\{\frac{m(x,v)}{|\nu_x \cdot v|}\right\} C\|u_-\|_{L^1(\partial_- SX)}\end{aligned}$$

Since $m(x,v)|\nu_x \cdot v|$ is bounded, so is $\mathcal{M}^i$. Now, for every i, $\mathcal{M}^i$ is bounded as a map from $C^\infty(\partial_- SX) \to \mathcal{D}'(\partial_+ SX)$. Indeed, if $u_- \to 0$ in C^∞, then in particular $u_- \to 0$ in L^1, and therefore $\int_{\partial X} \mathcal{M}^i u_- \phi d\mu(x) \to 0$ for any $\phi \in C^\infty(\partial X)$. Therefore, we may write

$$\int_{\nu_x \cdot v>0} K^i J u_-(x,v) m(x,v)\, dv = \int_{\partial_- SX} \alpha_i(x, x_0, v_0) u_-(x_0, v_0)\, d\xi(x_0, v_0), \;\; (i = 0,1,2\ldots), \tag{2.8}$$

for some set of Schwartz kernels α_i.

Momentarily we will show that each kernel is as stated in the theorem. Assuming this, and denoting by u the solution to (1.1) we have

$$\begin{aligned}\mathcal{M}u_-(x) &= \int_{\nu_x \cdot v>0} u(x,v) m(x,v)\, dS\\ &= \int_{\nu_x \cdot v>0} \sum_{i=0}^{\infty} K^i J u_-(x,v) m(x,v)\, dS\\ &= \sum_{i=0}^{\infty} \int_{\nu_x \cdot v>0} K^i J u_-(x,v) m(x,v)\, dS\\ &= \sum_{i=0}^{\infty} \int_{\partial_- SX} \alpha_i(x, x_0, v_0) u_-(x_0, v_0)\, d\xi.\end{aligned} \tag{2.9}$$

The third equality comes from the fact that $\sum_{i=0}^{\infty} K^i J u_- \in L^1(\partial_+ SX, d\xi)$, which combined with the boundedness of $m(x,v)|\nu_x \cdot v|^{-1}$ gives us $m \cdot \sum_{i=0}^{\infty} K^i J u_- \in L^1(\partial_+ SX, dSd\mu)$. Therefore, for *a.e.* $x \in \partial X$, we have $\sum_{i=0}^{\infty} K^i J u_-(x,\cdot) m(x,\cdot) \in L^1(\{v \in \mathbb{S}^{n-1} : \nu_x \cdot v > 0\})$. Indeed, if the integral $\int_{\nu_x \cdot v>0} \sum K^i J u_-(x,v) m(x,v) dS$ was infinite for a set of $x \in \partial X$ with positive measure, then $\|m \cdot \sum K^i J u_-\|_{L^1(\partial_+ SX)}$ would be too. Since K and J preserve non-negativity, we may use monotone convergence to switch the order of differentiation and integration. This will prove theorem 2.3.1.

We now show (2.8). For every fixed x, $\alpha_0(x,\cdot,\cdot)$ is a singular distribution. For example, let $F_x := \{(x_0,v_0) \in \partial_- SX : x_0 = x - \tau_-(x,v_0)v_0\}$, and $U_x \subset F_x$ be open. Now suppose that a sequence $\{u_j\} \subset C^\infty(\partial_- SX)$ converges pointwise to the characteristic function of the set U_x. Suppose further that for every j, $u_j(x_0,v_0) = 1$ for $(x_0,v_0) \in U_x$, and $0 \le u_j \le 1$, then

$$\begin{aligned}\int_{\partial_- SX} \alpha_0(x,x_0,v_0)u_j(x_0,v_0)\,d\xi &= \int_{\nu_x\cdot v>0} Ju_j(x,v)m(x,v)\,dS\\ &\ge \int_{\{v:(x-\tau_-(x,v)v,v)\in U_x\}} E(x-\tau_-(x,v)v,x)m(x,v)\,dS\\ &\ne 0\end{aligned}$$

for every j. Now $u_j \to 0$ in every $L^p(\partial_- SX)$ space, $p \in [1,\infty]$, so $\alpha_0 \notin L^p$ for every $p \in [1,\infty]$. Therefore the $n-1$ dimensional submanifold $F_x \subset \operatorname{supp}(\alpha_0(x,\cdot,\cdot))$. Similarly we can show that $\alpha_0(x,\cdot,\cdot) = 0$ when restricted to the compilment of F_x. Therefore the support of α_0 has dimension $n-1$. Heuristically, α_0 can be written

$$\alpha_0(x,x_0,v_0) := E(x_0,x)\delta_{\{x-\tau_-(x,v_0)v_0\}}(x_0)\frac{m(x,v_0)}{|\nu_{x_0}\cdot v_0|}.$$

When $i=1$,

$$\begin{aligned}&\int_{\nu_x\cdot v>0} KJu_-(x,v)m(x,v)\,dS\\ &= \int_{\nu_x\cdot v>0}\int_0^{\tau_-(x,v)} E(x,x-t_1v)\int_{\mathbb{S}^{n-1}} \Theta(x-t_1v,v_0,v)Ju_-(x-t_1v,v_0)\,dS_0dt_1m(x,v)dS\\ &= \int_{\nu_x\cdot v>0}\int_0^{\tau_-(x,v)} E(x,x-t_1v)\int_{\mathbb{S}^{n-1}} \Theta(x-t_1v,v_0,v)u_-(x-t_1v-\tau_-(x-t_1v,v_0)v_0,v_0)\\ &\times E(x-t_1v-\tau_-(x-t_1v,v_0)v_0,x_0)\,dS_0dt_1m(x,v)dS\\ &= \int_X\int_{\mathbb{S}^{n-1}} \Theta(x_1,v_0,v)u_-(x_1-\tau_-(x_1,v_0)v_0,v_0)E(x_1-\tau_-(x_1,v_0)v_0,x_1,x)t_1^{1-n}m(x,v)\,dS_0dx_1\\ &= \int_{\partial_- SX}\int_0^{\tau_+(x_0,v_0)} u_-(x_0,v_0)E(x_0,x_1,x)\Theta(x_0+t_0v_0,v_0,v)t_1^{1-n}m(x,v)\,dt_0d\xi.\\ &= \int_{\partial_- SX} \alpha_1(x,x_0,v_0)u_-(x_0,v_0)\,d\xi.\end{aligned} \tag{2.10}$$

Where we first changed variables $x_1 = x - t_1 v,\ dx_1 = t_1^{n-1} dt_1 dS$, then made the change $x_0 = x_1 - \tau_-(x_1, v_0)v_0$ and used lemma 2.3.1. This proves the case $i = 1$. Next, an inductive argument shows that when $i \geq 2$,

$$K^i h(x,v) = \int_0^{\tau_-(x,v)} \int_X \cdots \int_X \int_{\mathbb{S}^{n-1}} E(x, x - t_i v, x_{i-1}, x_{i-2}, \dots, x_1)\Theta(x - t_i v, v_{i-1}, v)$$
$$\times\, \Theta(x_{i-1}, v_{i-2}, v_{i-1}) \dots \Theta(x_1, v_0, v_1)(t_1, \dots, t_{i-1})^{1-n} h(x_1, v_0)\, dS_0 dx_1 \dots dx_{i-1} dt_i. \tag{2.11}$$

This formula is put into(2.7) with $h = Ju_-$, and handled in a manner similar to the case $i = 1$. I.e., we first change variables $x_i = x - t_i v,\ dx_i = t_i^{n-1} dt_i dS$, then make the change $x_0 = x_1 - \tau_-(x_1, v_0)v_0$ and use lemma 2.3.1. □

2.4 Recovering the absorption

We prove proposition 2.4.2, which shows that $\mathcal{M}$ determines the X-Ray transform of σ. This of course determines σ in the manner specified by theorem 2.2.1. The proof relies on the fact that α_0 is a singular distribution, and we may therefore separate it from the rest of the terms in our expansion (2.7) since they obey a certain boundedness property. See proposition 2.4.1.

We will need to work with the following moment of the Riesz potential, defined as an operator $T : L^q(X) \to L^q(X)$.

$$Tf(y) := \int_X \frac{f(y)}{|x-y|^{n-1}}\, dy. \tag{2.12}$$

For any $q \in [1, \infty]$, we have $\|Tf\|_{L^q} \leq Vol(\mathbb{S}^{n-1})\mathrm{diam}(X)\|f\|_{L^q}$. See proposition 5.1 in appendix A of [?]. Note that for every $0 < \varepsilon < n$, $y \in \mathbb{R}^n$, $x \mapsto |x-y|^{1-n} \in L^{(n-\varepsilon)/(n-1)}(X)$. Choose $0 < \varepsilon < 1$, and put $p = (n-\varepsilon)/(n-1)$, $q = (1 - 1/p)^{-1}$, and $C_p = sup_{y \in \partial X} \||y - \cdot|^{1-n}\|_{L^p(X)}$.

Under the assumptions of theorem 2.2.1, (as opposed to theorems 2.2.2, or 2.2.3) the operator $\sum_{i=1}^{\infty} \alpha_i : L^1(\partial_- SX, d\xi) \to L^1(\partial X)$ is not smoothing. The chance that k is nonzero at the boundary prevents this. More importantly, it prevents us from obtaining L^2 estimates on $\sum_{i=1}^{\infty} \alpha_i$. However, we still do have the following property.

Proposition 2.4.1. *Let (σ, Θ, m, X) satisfy the conditions of theorem 2.2.1, then for (p,q) as above, there exists $C > 0$ such that for a.e. $x \in \partial X$ (wherever theorem 2.3.1 holds),*

$$\left|\sum_{i=1}^{\infty} \int_{\partial_- SX} \alpha_i(x, x_0, v_0) f(x_0, v_0)\, d\xi\right| \le \|\Theta\|_{L^\infty} \|m\|_{L^\infty} C_p C \int_{\mathbb{S}^{n-1}} \|f(\cdot - \tau_-(\cdot, v_0)v_0, v_0)\|_{L^q(X)}\, dS$$
$$\le \|\Theta\|_{L^\infty} \|m\|_{L^\infty} C_p C \left[Vol(\mathbb{S}^{n-1})\right]^{1/p} diam(X) \|f\|_{L^q(\partial_- SX, d\xi)}.$$

Proof. The second inequality follows from the first, Holder's inequality, and lemma 2.3.1. We now show the first inequality when $i = 1$.

$$\begin{aligned}
&\left|\int_{\partial_- SX} \alpha_1(x, x_0, v_0) f(x_0, v_0)\, d\xi\right| \\
&\le \|\Theta\|_{L^\infty} \|m\|_{L^\infty} \int_{\partial_- SX} \int_0^{\tau_+(x_0, v_0)} \frac{|f(x_0, v_0)|}{|x - (x_0 + t_0 v_0)|^{n-1}}\, d\xi dt_0 \\
&\le \|\Theta\|_{L^\infty} \|m\|_{L^\infty} \int_{X \times \mathbb{S}^{n-1}} \frac{|f(x_1 - \tau_-(x_1, v_0))|}{|x - x_1|^{n-1}}\, dx_1 dS \\
&\le \|\Theta\|_{L^\infty} \|m\|_{L^\infty} C_p \int_{\mathbb{S}^{n-1}} \|f(\cdot - \tau_-(\cdot, v_0)v_0)\|_{L^q(X)}\, dS.
\end{aligned}$$

When $i = 2$,

$$\begin{aligned}
&\left|\int_{\partial_- SX} \alpha_2(x, x_0, v_0) f(x_0, v_0)\, d\xi\right| \\
&\le \|\Theta\|_{L^\infty}^2 \|m\|_{L^\infty} \int_{\partial_- SX} \int_0^{\tau_+(x_0, v_0)} f(x_0 + t_0 v_0) \\
&\times \int_{X_2} \frac{1}{|x_2 - (x_0 + t_0 v_0)|^{n-1}} \frac{1}{|x_2 - x|^{n-1}}\, dx_2 dt_0 d\xi \\
&= \|\Theta\|_{L^\infty}^2 \|m\|_{L^\infty} \int_{\mathbb{S}^{n-1}} \int_X f(x_1 - \tau_-(x_1, v_0)v_0, v_0) \int_X \frac{1}{|x_2 - x_1|^{n-1}} \frac{1}{|x_2 - x|^{n-1}}\, dx_2 dx_1 dS \\
&= \|\Theta\|_{L^\infty}^2 \|m\|_{L^\infty} \int_{\mathbb{S}^{n-1}} \int_X f(x_1 - \tau_-(x_1, v_0)v_0, v_0) \left(T \frac{1}{|\cdot - x|^{n-1}}\right)(x_1)\, dx_1 dS \\
&\le \|\Theta\|_{L^\infty}^2 \|m\|_{L^\infty} \|T\| C_p \int_{\mathbb{S}^{n-1}} \|f(\cdot - \tau_-(\cdot, v_0)v_0, v_0)\|_{L^q(X)}\, dS.
\end{aligned}$$

For $i \geq 2$,

$$\left|\int_{\partial_- SX} \alpha_i(x, x_0, v_0) f(x_0, v_0)\, d\xi\right|$$
$$\leq \|\Theta\|_{L^\infty}^i \|m\|_{L^\infty} \int_{\mathbb{S}^{n-1}} \int_X f(x_1 - t_0 v_0, v_0) T^{i-1}\left(\frac{1}{|\cdot - x|^{n-1}}\right)(x_1)\, dx_1 dS$$
$$\leq \|\Theta\|_{L^\infty} \|m\|_{L^\infty} \left[\|\Theta\|_{L^\infty}\|T\|\right]^{i-1} C_p \int_{\mathbb{S}^{n-1}} \|f(\cdot - \tau_-(\cdot, v_0)v_0, v_0)\|_{L^q(X)}\, dS.$$

Taking into account $\|T\| \leq \mathrm{Vol}(\mathbb{S}^{n-1})\mathrm{diam}(X)$, our smallness assumption on Θ ensures that the sum converges and has the appropriate bound. □

Proposition 2.4.2. *Suppose theorem 2.3.1 holds at fixed $(x, v) \in X \times \mathbb{S}^{n-1}$ (this happens a.e). Let (σ, Θ, m, X) satisfy the conditions of theorem 2.2.1. Choose $0 \leq \varphi \in C_c^\infty(\mathbb{R}), \varphi(0) = 1$. Put $\varphi^\eta(x_0) = \varphi\left(\frac{|x - \tau_-(x,v)v - x_0|}{\eta}\right)$. Let $0 \leq \delta_v^\eta \in C^\infty$ be an approximation of the delta distribution on $\mathbb{S}^{n-1}$ concentrated at v ($lim_{\eta\to 0}\delta_v^\eta = \delta_v$ in $\mathcal{D}'(\mathbb{S}^{n-1})$). Define $u_\eta \in L^1(\partial_- SX)$ by $u_\eta(x_0, v_0) = \varphi^\eta(x_0)\delta_v^\eta(v_0)m(x,v)^{-1}$. Then*

$$lim_{\eta\to 0}\mathcal{M}u_\eta(x) = E(x - \tau_-(x,v)v, x)$$

Proof. We have $\mathcal{M}u_\eta = \sum_{i=0}^\infty \int_{\partial_- SX} \alpha_i u_\eta$. One can verify that $\lim_{\eta\to 0}\int_{\partial_- SX} \alpha_0 u_\eta\, d\xi = \lim_{\eta\to 0}\int_{\nu_x \cdot v > 0} J u_\eta(x, v_0) m(x, v_0)\, dS = E(x - \tau_-(x,v)v, x)$. Lastly, as a result of proposition 2.4.1,

$$\left|\sum_{i=1}^\infty \int_{\partial_- SX} \alpha_i(x, x_0, v_0) u_\eta(x_0, v_0)\, d\xi\right|$$
$$\leq C \int_{\mathbb{S}^{n-1}} \delta_v^\eta(v_0) \|\varphi^\eta(\cdot - \tau_-(\cdot, v_0)v_0)\|_{L^q(X)}\, dS$$
$$= C \int_{\mathbb{S}^{n-1}} \delta_v^\eta(v_0) \left\{\int_X |\varphi^\eta(x - \tau_-(x, v_0)v_0)|^q\, dx\right\}^{1/q} dS$$
$$= \int_{\mathbb{S}^{n-1}} \delta_v^\eta(v_0) \left\{\int_{\partial X}\int_0^{\tau_+(x_0, v_0)} |\varphi^\eta(x_0)|^q\, |\nu_{x_0} \cdot v_0|\, dt_0 d\mu(x_0)\right\}^{1/q} dS$$
$$\leq \int_{\mathbb{S}^{n-1}} \delta_v^\eta(v_0)\mathrm{diam}(X)^{1/q}\|\varphi^\eta\|_{L^q(\partial X)} \to 0 \text{ as } \eta \to 0.$$

The last equality comes from a change of variables $x = x_0 + t_0 v_0$. The Jacobian for this is $|\nu_{x_0} \cdot v_0|$. □

2.5 *Recovering the scattering kernel*

In this section we prove that our measurements uniquely determine the scattering kernel. This is done by way of proposition 2.5.1, and 2.5.3. Together these show that if two a-priori different scattering kernels $k, \tilde{k}$ give the same measurements, we have the inequality $\|k - \tilde{k}\|_{L^2} \leq C\varepsilon\|k - \tilde{k}\|_{L^2}$. For small enough ε the kernels must then be equal. To do this, we only need to take measurements at one point. Fixing this measurement point $x \in \partial X$, we henceforth drop the explicit dependence of α_i on x, and abuse notation writing $\alpha_i(x, x_0, v_0) = \alpha_i(x_0, v_0)$. Recall also that we assume $\Theta(x, v, v') = k(x)g(x, v, v')$ for some known function g, and $\text{dist}(\text{supp}(k), \partial X) \geq D > 0$.

Our uniqueness proof will require knowledge of $\sum_{i=1}^{\infty} \alpha_i(x_0, v_0)$ for all $(x_0, v_0) \in \mathcal{H}_k$. We obtain this from our measurements in the following fashion. First, it is easy to show that for $i \geq 1$,

$$\|\alpha_i\|_{L^\infty} \leq \text{diam}(X) D^{1-n} \|m\|_{L^\infty} \|T\|^{i-1} (\|k\|_{L^\infty} \|g\|_{L^\infty})^i.$$

Therefore, for small enough $\|k\|$, $\|\sum_{i=1}^{\infty} \alpha_i\|_{L^\infty} < \infty$. Since we assume we know σ, we in turn know α_0, and therefore $\sum_{i=1}^{\infty} \alpha_i$.

Suppose $\beta \in C^\infty(\partial_- SX)$, with $\beta|_{\bar{\mathcal{H}}_k} \neq 0$. Then knowledge of $\beta \sum_{i=1}^{\infty} \alpha_i$ is equivalent to knowledge of $\sum_{i=1}^{\infty} \alpha_i\big|_{\mathcal{H}_k}$. We assume that this is our data. This formulation will allow us to obtain the necessary estimates.

2.5.1 SINGLE SCATTERING AS AN INTEGRAL TRANSFORM

In this section we use the main results of [23] to prove proposition 2.5.1.

Definition 2.5.1. Given $\beta \in C^\infty(\partial_- SX)$, $\eta : X \times \mathbb{S}^{n-1} \to \mathbb{R}$, we define the *weighted X-ray transform.*

$$I_{\eta,\beta} f(x_0, v_0) := \beta(x_0, v_0) \int_0^{\tau_+(x_0,v_0)} \eta(x_0 + t_0 v_0, v_0) f(x_0 + t_0 v_0)\, dt_0.$$

We also have the L^2 adjoint

$$I^*_{\eta,\beta} h(x) := \int_{\mathbb{S}^{n-1}} \eta(x, v) \beta(x - \tau_-(x, v)v, v) h(x - \tau_-(x, v)v, v)\, dS.$$

Remark. If $\beta \equiv 1$, then $I_{\eta,\beta} f$ gives us the (weighted) line integrals of f over all lines passing through X. In the general case, β restricts the number of lines we have access to.

Given appropriate assumptions, the single scattering term α_1 is in fact a weighted X-ray transform of the unknown term k. It is natural then to ask if this transform is injective. We will use the following results in integral geometry, developed in a much more general setting in [23], which builds upon work developed for the boundary rigidity problem of compact Riemannian manifolds, [42, 43, 44]. We have adapted the theorems and definitions to appear more simply in our specific case.

Definition 2.5.2. We say Γ is a *regular* family of lines if for any $(x, \zeta) \in X \times \mathbb{S}^{n-1}$, there exists $\gamma \in \Gamma$ through x normal to ζ.

Theorem 2.5.1. *Let $\beta \in C^\infty(\partial_- SX)$, $\mathcal{H} \subset \{(x_0, v_0) : \beta(x_0, v_0) \neq 0\} \subset \partial_- SX$ be open. Suppose $\Gamma(\mathcal{H})$ is a regular family of lines in X, and η is real analytic and non-vanishing in U, where $U \subset \bar{U} \subset X$. Then $I_{\eta,\beta}$ is injective on $\mathcal{D}'(U)$.*

This is a direct consequence of the fact (proven in [23]) that if $\text{supp}(f) \in U$, and $\int_\gamma f = 0$ for all $\gamma \in neigh(\gamma_0)$, then $N^*\gamma_0 \cap WF_A(f) = \emptyset$. $N^*\gamma_0$ is the conormal bundle of γ_0, and $WF_A f$ is the (real) analytic wavefront set of f. See [48] for a definition of the analytic wavefront set. Therefore f is real analytic in X and has compact support, hence $f \equiv 0$.

Note that injectivity of weighted transforms is a nontrivial matter. As a counterexample to injectivity Boman [15] gives a smooth positive weight function w such that $I_{w,1}$ is not injective when X is the closed unit disk.

We will also require the following estimate, valid whenever $I_{\eta,\beta}$ is injective, e.g. if $\Gamma(\mathcal{H})$ is an injective family of lines. Note that the hypothesis for β, η are weaker here:

Theorem 2.5.2. *Let U be as in theorem 2.5.1. Suppose $\eta \in C^\infty$, and $\beta \in C^\infty(\partial_- SX)$.*

(a) If $I_{\eta,\beta}$ is injective on $L^2(U)$, then we have $C > 0$ such that

$$\|f\|_{L^2(U)}/C \leq \|I^*_{\eta,\beta} I_{\eta,\beta} f\|_{H^1(X)} \leq C\|f\|_{L^2(U)}.$$

(b) There exists a C^2 neighborhood of (η, β) on which the above estimate remains true with a uniform constant C.

Part (a) is proved by first showing that the operator $N_{\eta,\beta} := I^*_{\eta,\beta} I_{\eta,\beta}$ is an elliptic classical ΨDO of order -1. Therefore there exists a classical ΨDO Q of order 1 such that $QN_{\eta,\beta} f = f + Kf$ with K having C^∞_c Schwarz kernel. This immediately leads to the estimate $\|f\|_{L^2(U)} \leq C\|N_{\eta,\beta} f\|_{H^1(X)} + C_s\|f\|_{H^s(X)}$ for every s. Using proposition 6.7 in appendix A of [47], we then have that $N_{\eta,\beta}$ has closed range. Since $I_{\eta,\beta}$ is injective on $L^2(U)$, $N_{\eta,\beta} : L^2(U) \to H^1(X)$ is injective. $N_{\eta,\beta}$ is therefore a bijection between $L^2(U)$ and its range. The open mapping theorem then gives us the estimate (a). Part (b) is a consequence of the C^2 continuity properties (w.r.t. (η, β)) of the operator $N_{\eta,\beta}$. In proposition 2.5.2 we prove something slightly different, which gives the same result.

Let us now apply these results to our problem. In our setup, we suppose $k, \tilde{k}$ are two a-priori different scattering coefficients in (1.1), each of the form $\Theta = k(x)g(x, v', v)$. We assume $k, \tilde{k} \in \mathcal{K}^D_\varepsilon$. These give rise to averaged albedo operators $\mathcal{M}$ and $\tilde{\mathcal{M}}$, and Schwarz kernels α_i and $\tilde{\alpha}_i$. Set $\Delta k = k - \tilde{k}$, and $\Delta\alpha_i = \alpha_i - \tilde{\alpha}_i$. We will view the decomposition of $\mathcal{M}$ as a series of weighted X-ray transforms. In particular, $\alpha_1 = I_{w,1}k$, with w defined below.

Definition 2.5.3. Let $\chi \in C^\infty_c(X)$ with $\chi \equiv 1$ on $\{y \in X : \text{dist}(y, \partial X) > D\}$. For our fixed $x \in \partial X$, and any $x_1 \in X, v_0 \in \mathbb{S}^{n-1}$, put $v = (x - x_1)|x - x_1|^{-1}$. We then define the weight function w by

$$w(x_1, v_0) := \frac{g(x_1, v_0, v)E(x_0, x_1, x)m(x, v)}{|x - x_1|^{n-1}}\chi(x_1).$$

Note that $E(x_0, x_1, x) = E(x_0, x_1)E(x_1, x)$. We can extend σ to be defined on $\mathbb{R}^n$ by extending it to be 0 outside of $\bar{X}$. Then $E(x_0, x_1) = exp\left\{\int_0^\infty \sigma(x_1 - sv_0)ds\right\}$. Also recall that x is fixed. Therefore w is a proper weight function, depending only on (x_1, v_0). If $m_0, g_0, \sigma_0|_X$ are real analytic (with m_0, g_0 non-vanishing), then the corresponding weight function w_0 is real analytic and non-vanishing on $\{x \in X : \text{dist}(x, \partial X) > D\}$. Theorem 5.1 gives us that $I_{w_0,\beta}$ is injective on $\mathcal{D}'(\{x \in X : \text{dist}(x, \partial X) > D\})$, and then using theorem 5.2 we have thereby proved

Lemma 2.5.1. *Suppose $\beta \in C^\infty(\partial_- SX)$, $\mathcal{H}_k \subset \{(x_0, v_0) : \beta(x_0, v_0) \neq 0\} \subset \partial_- SX$ is open with $\Gamma(\mathcal{H}_k)$ regular. Suppose further that fixed (m, g, σ) are real analytic. Then there exists*

$C > 0$ such that

$$\|\Delta k\|_{L^2(X)} \leq C\|I^*_{w,\beta} I_{w,\beta} \Delta k\|_{H^1(X)},$$

with the inequality holding in a C^2 neighborhood of (m, g, σ).

Proposition 2.5.1. *Suppose $\beta \in C^\infty(\partial_- SX)$, $\mathcal{H}_k \subset \{(x_0, v_0) : \beta(x_0, v_0) \neq 0\} \subset \partial_- SX$ is open with $\Gamma(\mathcal{H}_k)$ regular. Fix real analytic (m, g, σ) and suppose that $\{(u_-, \mathcal{M}u_-(x)) : u_- \in L^1(\mathcal{H}_k)\} = \{(u_-, \tilde{\mathcal{M}}u_-(x)) : u_- \in L^1(\mathcal{H}_k)\}$. Then there exits $C > 0$, independent of $k, \tilde{k} \in \mathcal{K}^D_\varepsilon$ such that*

$$\|\Delta k\|_{L^2(X)} \leq C\|I^*_{w,\beta} I_{w,\beta} \Delta k\|_{H^1(X)} = C\|I^*_{w,\beta} \beta \Delta\alpha_1\|_{H^1(X)} = C\left\|I^*_{w,\beta} \beta \sum_{i=2}^{\infty} \Delta\alpha_i\right\|_{H^1(X)},$$

with the inequality holding in a C^2 neighborhood of (m, g, σ).

Proof. The first inequality has already been proved. The first equality is a result of our definition of w and the relation $\beta I_{w,1} = I_{w,\beta}$. Since $\sigma = \tilde{\sigma}$, we have $\alpha_0 = \tilde{\alpha}_0$. The agreement of $\mathcal{M}, \tilde{\mathcal{M}}$ then gives us $\beta \sum_{i=1}^{\infty} \Delta\alpha_i = 0$, which gives the last equality. □

2.5.2 THE SMALL NON-LINEARITY

In this section we bound $\|I^*_{w,\beta} \beta \sum_{i=2}^{\infty} \Delta\alpha_i\|_{H^1(X)}$ from above by $\varepsilon C\|\Delta k\|_{L^2(X)}$. Combined with proposition 2.5.1 and the assumption that ε is small enough, this proves theorem 2.2.2. We will estimate the H^1 norm of every term separately by writing $\Delta\alpha_i$ as a sum of weighted integral transforms, see definition (2.5.1). We will then apply proposition 2.5.2, which is very similar to proposition 4 in [23]. However, to use that result here we would need to assume that η and β were both small.

Proposition 2.5.2. *There exists $C > 0$, depending only on X such that*

$$\|I^*_{\eta_1,\beta_1} I_{\eta_2,\beta_2} f\|_{H^1} \leq C\|\eta_1\|_{C^1}\|\beta_1\|_{C^1}\|\eta_2\|_{C^2}\|\beta_2\|_{C^1}\|f\|_{L^2}.$$

Where all norms are over the space X.

Proof. After a re-parametrization, we have

$$I^*_{\eta_1,\beta_1} I_{\eta_2,\beta_2} f(x) = \int_{\mathbb{S}^{n-1}} \int_{-\tau_-(x,v)}^{\tau_+(x,v)} A(x, r, v) f(x + rv)\, dr dS,$$

with

$$A(x,r,v) = \eta_1(x,v)\eta_2(x+rv,v)\beta_1(x-\tau_-(x,v)v,v)\beta_2(x-\tau_-(x,v)v,v). \tag{2.13}$$

If A is odd with respect to v, the integral vanishes. Therefore, we may change kernels to $A_{even}(x,r,v) = 1/2(A(x,r,v)+A(x,r,-v))$. Changing variables $y = x+rv$ we then have

$$I^*_{\eta_1,\beta_1} I_{\eta_2,\beta_2} f(x) = 2\int_X A_{even}\left(x, |y-x|, \frac{y-x}{|y-x|}\right) \frac{f(y)}{|y-x|^{n-1}}\, dy =: \mathcal{A}_{even} f(x).$$

Expanding A_{even} in a taylor series, we have

$$A_{even}(x,r,v) = A_{even,0}(x,v) + rA_{even,1}(x,r,v),$$

with $A_{even,0}, A_{even,1} \in C^1$. The resultant operator $\mathcal{A}_{even,1}$ has a singularity of order r^{n-2} in its kernel. We are therefore justified differentiating inside the integral and using the mapping properties of Riesz potentials (see 2.12) to get

$$\|\mathcal{A}_{even,1} f\|_{H^1} \le C\|A_{even,1}\|_{C^1}\|f\|_{L^2} \le C\|\eta_1\|_{C^1}\|\beta_1\|_{C^1}\|\eta_2\|_{C^2}\|\beta_2\|_{C^1}\|f\|_{L^2}.$$

We now consider the operator $\partial_j \mathcal{A}_{even,0}$. This type of operator is handled in [31], chapter 11, §11, theorem 11.1. See also chapter 9, §7. We sketch the result here.

Formally differentiating, we see that the operator $\partial_j \mathcal{A}_{even,0}$ should consist of an operator with a weak singularity r^{1-n}, and one with a strong r^{-n} one. The weak singularity is handled in a manner similar to above. As for the strong one, it can be shown that the kernel has mean value $= 0$ in the v variable. The result (formally at least) is a principle value distribution. The L^2 norm of the principle-value type operator is bounded by the L^∞ norm of its kernel. Being more careful, one can show that this formal differentiation is justified, up to an operator bounded in L^2 by $\|A_{even,0}\|_{L^\infty}$. Hence,

$$\|\mathcal{A}_{even,0} f\|_{H^1} \le C\|A_{even,0}\|_{C^1}\|f\|_{L^2} \le C\|\eta_1\|_{C^1}\|\beta_1\|_{C^1}\|\eta_2\|_{C^1}\|\beta_2\|_{C^1}\|f\|_{L^2}.$$

□

To write $\Delta\alpha_i$ as a sum of weighted integral transforms, we need to separate one instance of the scattering phase function g (appearing in $\Delta\alpha_i$) into a sum of terms that look like weighting functions. The following lemma will be useful.

Lemma 2.5.2. *Let $f \in C^\infty(K \times \mathbb{S}^{n-1})$ where $K \subset \mathbb{R}^n$ is compact. Then there exist functions φ_j and constant C, uniform in a C^∞ neighborhood of f, such that*

$$f(x,\theta) = \sum_{j=0}^{\infty} f_j(x)\varphi_j(\theta), \text{ with}$$

$$\|\varphi_j\|_{L^\infty} \le 1,$$

$$\|f_j\|_{C^2} \le \frac{C}{1+j^2}.$$

Proof. For fixed x, we expand f (or some derivative of f, call it $\partial_x^\alpha f$) in spherical harmonics. $f(x,\theta) = \sum_{l=0}^{\infty} f_j(x)\varphi_j(\theta)$, with f_j and its first two derivatives bounded by $C(x)(1+j^2)^{-1}$ (in fact rapidly decreasing). The coefficients $f_j(x)$ are given in terms of an explicit integral, and from this integral we see that $C(x)$ depends continuously on x. In fact,

$$C(x) \le \frac{c_{n,N}}{1+j^2}\|(\Delta)^N \partial_x^\alpha f(x,\cdot)\|_{L^2(\mathbb{S}^{n-1})}$$

where $c_{n,N}$ depends only on n and some large enough N. Since K is compact we have the desired result. □

Proposition 2.5.3. *Fix (m, g, σ), with $g \in C^\infty$, $m \in C^2(\partial_+SX)$, $\sigma \in C^2(X)$. Suppose $k, \tilde{k} \in \mathcal{K}_\varepsilon^D$, and $\beta \in C^\infty(\partial_-SX)$, and $\|k\|_{L^\infty} < [\|g\|_{L^\infty} diam(X)\, Vol(\mathbb{S}^{n-1})]^{-1}$. Then there exists $C > 0$ such that*

$$\|I^*_{w,\beta}\beta \sum_{i=2}^{\infty} \Delta\alpha_i\|_{H^1(X)} \le C\varepsilon\|\Delta k\|_{L^2(X)},$$

with the inequality holding in a C^∞ neighborhood of g, and a C^2 neighborhood of m and σ.

Proof. Applying the relation $ab - \tilde{a}\tilde{b} = (a-\tilde{a})b + \tilde{a}(b-\tilde{b})$ inductively we have

$$k(x_1)\cdots k(x_i) - \tilde{k}(x_1)\cdots\tilde{k}(x_i) = \sum_{l=1}^{i} \tilde{k}(x_1)\cdots\tilde{k}(x_{l-1})\Delta k(x_l)k(x_{l+1})\cdots k(x_i).$$

Therefore $\Delta\alpha_i$ is a sum of terms having the exact same form as α_i except for the replacement of some $k's$ with $\tilde{k}$, and a Δk in the l^{th} position. For example,

$$\Delta\alpha_2(x_0,v_0) = \int_0^{\tau_+(x_0,v_0)} E(x_0, x_0+t_0v_0)\int_X \left[\Delta k(x_0+t_0v_0)k(x_2) + \tilde{k}(x_0+t_0v_0)\Delta k(x_2)\right]$$
$$\times\, g(x_0+t_0v_0, v_0, v_1)g(x_2, v_1, v_2)E(x_0+t_0v_0, x_2, x)(t_1t_2)^{1-n}m(x,v)\, dx_2dt_0. \tag{2.14}$$

We will now write (2.14) in the form of a weighted X-ray transform. Since $\sigma = 0$ outside of X, we may define $\mathcal{E}(x_1, v_0) := e^{\int_0^\infty \sigma(x_1 - sv_0)ds} = E(x_0, x_1)$. Then, using lemma 2.5.2 we have (formally)

$$
\begin{aligned}
&\Delta\alpha_2(x_0, v_0) = \\
&\quad \sum_j \int_0^{\tau_+(x_0,v_0)} g_j(x_0 + t_0 v_0, v_0)\mathcal{E}(x_0 + t_0 v_0, v_0) \\
&\quad \times \int_X \left[\Delta k(x_0 + t_0 v_0)k(x_2) + \tilde{k}(x_0 + t_0 v_0)\Delta k(x_2)\right] \\
&\quad \times \varphi_j(v_1)g(x_2, v_1, v_2)E(x_0 + t_0 v_0, x_2, x)(t_1 t_2)^{1-n} m(x, v)\, dx_2 dt_0 \\
&\quad = \sum_j \int_0^{\tau_+(x_0,v_0)} g_j(x_0 + t_0 v_0, v_0)\mathcal{E}(x_0 + t_0 v_0, v_0)\left[\Psi_{2,1,j}(x_0 + t_0 v_0) + \Psi_{2,2,j}(x_0 + t_0 v_0)\right] dt_0 \\
&\quad = \sum_j I_{g_j \mathcal{E},1}\left[\Psi_{2,1,j} + \Psi_{2,2,j}\right](x_0, v_0).
\end{aligned}
\tag{2.15}
$$

The above expansion, and more to follow will be justified by the final result showing H^1 convergence. In the above equation, we have implicitly defined $\Psi_{2,1,j}$ and $\Psi_{2,2,j}$. We now define the general function $\Psi_{i,l,j}$, which corresponds to the i^{th} scattering kernel, Δk being integrated as a function of x_l, and the j^{th} term in the expansion of $g(x_1, v_0, v_1)$. For $j \in \mathbb{N}, 2 \le i, 1 \le l \le i$, we have

$$
\begin{aligned}
\Psi_{i,l,j}(x_1) := \int_X \cdots \int_X &\tilde{k}(x_1)\cdots\tilde{k}(x_{l-1})\Delta k(x_l)k(x_{l+1})\cdots k(x_i) \\
&\times \varphi_j(v_1)g(x_2, v_1, v_2)\cdots g(x_i, v_{i-1}, v_i)E(x_1, \ldots, x_i, x)(t_1\cdots t_i)^{1-n} m(x, v)\, dx_i \ldots dx_2.
\end{aligned}
\tag{2.16}
$$

Which leads to

$$
\Delta\alpha_i(x_0, v_0) = \sum_{l=1}^{i}\sum_{j=1}^{\infty} I_{g_j\mathcal{E}}\Psi_{i,l,j}(x_0, v_0). \tag{2.17}
$$

And using the relation $\beta I_{g_j\mathcal{E},1} = I_{g_j\mathcal{E},\beta}$,

$$
\begin{aligned}
\|I^*_{w,\beta}\beta\sum_{i=2}^{\infty}\Delta\alpha_i\|_{H^1(X)} &\le \sum_{i=2}^{\infty}\sum_{l=1}^{i}\sum_{j=1}^{\infty}\|I^*_{w,\beta}I_{g_j\mathcal{E},\beta}\Psi_{i,l,j}\|_{H^1(X)} \\
&\le \sum_{i=2}^{\infty}\sum_{l=1}^{i}\sum_{j=1}^{\infty}\frac{C}{1+j^2}\|\Psi_{i,l,j}\|_{L^2(X)}.
\end{aligned}
\tag{2.18}
$$

We now need an L^2 estimate for $\Psi_{i,l,j}$ in terms of $\|\Delta k\|_{L^2(X)}$. We first pull the L^∞ norms of (m,g) out of 2.16. Likewise, $\|E\|_{L^\infty} \leq 1$, and $t_i^{1-n} \leq \operatorname{dist}(\operatorname{supp}(k), \partial X)^{1-n} \leq D^{1-n}$. Substituting $t_k = |x_k - x_{k+1}|$ into (2.16), and recalling the operator $Tf(y) = \int_X f(x)(|x - y|^{1-n})\,dx$ see (2.12), we may then write

$$
\begin{aligned}
\|\Psi_{i,l,j}\|_{L^2(X)} &\leq \|m\|_{L^\infty} D^{1-n} \|g\|_{L^\infty}^i \varepsilon^{i-1} \\
&\quad \times \left\| \int_X \frac{1}{|\cdot - x_2|^{n-1}} \cdots \int_X \frac{1}{|x_{l-2} - x_{l-1}|^{n-1}} \int_X \frac{\Delta k(x_l)}{|x_{l-1} - x_l|^{n-1}} \right. \\
&\quad \left. \times \int_X \frac{1}{|x_l - x_{l+1}|^{n-1}} \cdots \int_X \frac{1}{|x_{i-1} - x_i|^{n-1}} dx_i \ldots dx_2 \right\|_{L^2(X)} \\
&= \|m\|_{L^\infty} D^{1-n} (\|g\|_{L^\infty}\varepsilon)^{i-1} \|T^{l-1}(\Delta k \cdot T^{i-l}1)\|_{L^2} \\
&\leq C\varepsilon(\|g\|_{L^\infty}\varepsilon)^{i-2} \|T^{l-1}(\Delta k \cdot T^{i-l}1)\|_{L^2} \\
&\leq C\varepsilon(\|g\|_{L^\infty}\varepsilon)^{i-2} \|T\|^{l-1} \|\Delta k \cdot T^{i-l}1\|_{L^2} \\
&\leq C\varepsilon(\|g\|_{L^\infty}\varepsilon)^{i-2} \|T\|^{l-1} \|\Delta k\|_{L^2} \|T^{i-l}1\|_{L^\infty} \\
&\leq C\varepsilon(\|g\|_{L^\infty}\varepsilon\|T\|)^{i-2} \|\Delta k\|_{L^2}.
\end{aligned}
\tag{2.19}
$$

Whence

$$
\begin{aligned}
\|I^*_{w,\beta}\beta \sum_{i=2}^\infty \Delta\alpha_i\|_{H^1(X)} &\leq \sum_{i=2}^\infty \sum_{l=1}^i \sum_{j=1}^\infty \frac{C}{1+j^2} \|\Psi_{i,l,j}\|_{L^2(X)} \\
&\leq \sum_{i=2}^\infty \sum_{l=1}^i \sum_{j=1}^\infty \frac{C}{1+j^2} \varepsilon(\|g\|_{L^\infty}\varepsilon\|T\|)^{i-2} \|\Delta k\|_{L^2(X)} \\
&\leq \varepsilon C \|\Delta k\|_{L^2(X)} \sum_{i=2}^\infty i \cdot (\|g\|_{L^\infty}\varepsilon\|T\|)^{i-2}.
\end{aligned}
$$

The series converging for small enough ε. □

Proof of theorem 2.2.2. First, fix real analytic (m, g, σ). Given the hypothesis of theorem 2.2.2 we are ensured that both propositions 2.5.1 and 2.5.3 hold. We therefore have a new constant $C > 0$, such that

$$\|\Delta k\|_{L^2(X)} \leq C\varepsilon \|\Delta k\|_{L^2(X)}$$

Moreover, the inequality holds, with the same C, in some C^∞ neighborhood of g, and a C^2 neighborhood of m and σ. For $\varepsilon < C^{-1}$ we must therefore have $\Delta k = 0$. □

Proof of theorem 2.2. First, fix real analytic (m, g, σ). Given the hypothesis of theorem 2.2 we are ensured that both propositions 5.1 and 5.3 hold. We therefore have a new constant $C > 0$, such that

$$\|\Delta k\|_{L^2(X)} \leq C\varepsilon \|\Delta k\|_{L^2(X)}$$

Moreover, the inequality holds, with the same C, in some C^∞ neighborhood of g, and a C^2 neighborhood of m and σ. For $\varepsilon < C^{-1}$ we must therefore have $\Delta k = 0$. □

2.6 The stability estimate

We first prove the following two lemmas:

Lemma 2.6.1. *Let $X \subset \mathbb{R}^n$ be a bounded domain with C^1 boundary. Let $j = 0$ or 1. Assume $\eta \in H^j(X \times \mathbb{S}^{n-1})$, and $\beta \in C^\infty(\partial_- SX)$. Then $I_{\eta,\beta} : H^j(X) \to H^j(\partial_- SX, d\xi)$ and $I^*_{\eta,\beta} : H^j(\partial_- SX, d\xi) \to H^j(X)$, both continuously.*

Proof. Since $I_{\eta,\beta} f = \beta I_{\eta,1} f$, and $I^*_{\eta,\beta} h = I^*_{\eta,1} \beta h$, it will suffice to prove the lemma for $\beta \equiv 1$. For $(x_0, v_0) \in \partial_- SX$, $t_0 \in [0, \mathrm{diam}(X)]$, put $F(x_0, v_0, t_0) = \eta(x_0 + t_0 v_0, v_0) f(x_0 + t_0 v_0))$. Then,

$$\begin{aligned}
\|I_{\eta,1} f\|_{L^2(\partial_- SX, d\xi)} &= \left\| \int_0^{\mathrm{diam}(X)} F(\cdot, \cdot, t_0), dt_0 \right\|_{L^2(\partial_- SX, d\xi)} \\
&\leq \int_0^{\mathrm{diam}(X)} \|F(\cdot, \cdot, t_0)\|_{L^2(\partial_- SX, d\xi)} \, dt_0 \\
&\leq \sqrt{\mathrm{diam}(X)} \sqrt{\int_0^{\mathrm{diam}(X)} \|F(\cdot, \cdot, t_0)\|^2_{L^2(\partial_- SX, d\xi)} \, dt_0}, \\
&= \sqrt{\mathrm{diam}(X)} \sqrt{\int_0^\infty \int_{\mathbb{S}^{n-1}} \int_{\partial X} |\eta(x_0 + t_0 v_0, v_0) f(x_0 + t_0 v_0)|^2 \, d\xi dt_0} \\
&= \sqrt{\mathrm{diam}(X)} \sqrt{\int_{X \times \mathbb{S}^{n-1}} |\eta(x, v_0) f(x)|^2 \, dx dS} \\
&\leq \sqrt{\mathrm{diam}(X)} \|\eta\|_{L^\infty(X \times \mathbb{S}^{n-1})} \sqrt{Vol(\mathbb{S}^{n-1})} \|f\|_{L^2(X)}, \quad .
\end{aligned}$$

Where we have used the continuous Minkowskii and Cauchy-Shwartz inequalities, as well as lemma 2.3.1. This proves the $H^0 = L^2$ boundedness of both $I_{\eta,1}$ and $I^*_{\eta,1}$, which must have the same norm.

The proof of the H^1 mapping properties of $I^*_{\eta,1}$ requires a bit more care. First let (U_k, φ_k) be a finite coordinate cover of ∂_-SX with cutoff functions χ_k. For $h : \partial_-SX \to \mathbb{R}$, the pullback (not adjoint) of $\chi_k h$, $(\varphi_k^{-1})^*\chi_k h : \mathbb{R}^{2n-2} \to \mathbb{R}$. We then define the Sobolev norm

$$\|h\|_{H^1(\partial_-SX,d\xi)} = \sum_k \|(\varphi_k^{-1})^*\chi_k h\|_{H^1(\mathbb{R}^{2n-2})}. \tag{2.20}$$

One may check that the resultant topology is invariant under a change of coordinates. To prove $I^*_{\eta,1} : H^1(\partial_-SX, d\xi) \to H^1(X)$, it will therefore suffice to consider $I^*_{\eta,1}h$ for h supported in one coordinate chart (U, φ), to which we give coordinate functions (y^i, u^m). Put $x_0(x,v) = \varphi(x - \tau_-(x,v)v, v)$. Our boundary regularity implies that $x_0(x,v) \in C^1(X \times \mathbb{S}^{n-1}, \partial_-SX)$. We then write

$$\begin{aligned} I^*_{\eta,1}h(x) &= \int_{\mathbb{S}^{n-1}} \eta(x,v)h(x - \tau_-(x,v)v, v)\, dS \\ &= \int_{\mathbb{S}^{n-1}} \eta(x,v)(\varphi^{-1})^*h(x_0(x,v))\, dS \end{aligned}$$

Then if $\partial_j := \partial/\partial x^j$, $\eta, h \in C^1$,

$$\begin{aligned} \partial_j I^*_{\eta,1}h(x) &= I^*_{\partial_j\eta,1}h(x) + \sum_i \int_{\mathbb{S}^{n-1}} \eta(x,v)\frac{\partial(\varphi^{-1})^*h}{\partial y^i}(x_0(x,v))\frac{\partial x_0^i}{\partial x^j}(x,v)\, dS \\ &= I^*_{\partial_j\eta,1}h(x) + \sum_i I^*_{\eta\partial_j x_0,1}\frac{\partial(\varphi^{-1})^*h}{\partial y^i}. \end{aligned}$$

From which it follows that

$$\begin{aligned} \|\partial_j I^*_{\eta,1}h\|_{L^2} &\le C\|\partial_j\eta\|_{L^\infty}\|h\|_{L^2} + C\|\eta\|_{L^\infty}\left\|\frac{\partial(\varphi^{-1})^*h}{\partial y^i}\frac{\partial x_0}{\partial x^j}\right\|_{L^2} \\ &\le C\|\eta\|_{C^1}\|h\|_{H^1}. \end{aligned}$$

This defines the bilinear map $(\eta, h) \mapsto \partial_j I^*_{\eta,1}h \in L^2(X)$ on a dense subset of $C^1(X \times \mathbb{S}^{n-1}) \times H^1(\partial_-SX, d\xi)$. We therefore have a unique continuous extension to the whole space, bounded by the same constant. Obtaining the H^1 norm is now easy. The proof for $I_{\eta,1}$ is done similarly. □

Lemma 2.6.2. *Define $\mathcal{S}_\beta := \|\beta\sum_{i=1}^\infty \alpha_i\|_{H^1(\partial_-SX,d\xi)}$. Suppose Θ has the form $\Theta = k(x)g(x,v',v)$, with (g,m) both non-vanishing, and (g,m,σ) known real analytic functions. Assume $k, \tilde{k} \in \kappa_\varepsilon^D$ for $\varepsilon < (diam(X)vol(\mathbb{S}^{n-1})\|g\|_{L^\infty})^{-1}$. Also suppose $\|\sigma\|_{H^s}, \|\tilde{\sigma}\|_{H^s} \le M <$*

∞, for some $s > n/2+2$. Then there exists $C > 0$, depending only on $(M, D, s, X, m, g, \varepsilon, \sigma)$ such that for small enough ε,

$$\|\Delta k\|_{L^2} \leq C(1+M)\mathcal{S}_\beta + C\varepsilon(1+M^2)(\|\Delta\sigma\|_{C^2} + \|\Delta\sigma\|_{C^2}^2),$$

with the estimate holding in a C^∞ neighborhood of g and a C^2 neighborhood of (m, σ) (so long as $\|\sigma\|_{H^2} \leq M$).

Proof. In this proof, C denotes different constants. One can check that the dependence of C is as stated in the lemma.

First write $\Delta\alpha_1 = I_{w,1}\Delta k + I_{\Delta w,1}k$. We then multiply by β, apply $I^*_{w,\beta}$ to both sides, and use lemma (2.5.1), which yields

$$\begin{aligned}\|\Delta k\|_{L^2} &\leq C\|I^*_{w,\beta}I_{w,\beta}\Delta k\|_{H^1} \\ &\leq C\|I^*_{w,\beta}I_{\Delta w,\beta}k\|_{H^1} + C\|I^*_{w,\beta}\beta\Delta\alpha_1\|_{H^1} \\ &= C\|I^*_{w,\beta}I_{\Delta w,\beta}k\|_{H^1} + C\left\|I^*_{w,\beta}\beta\sum_{i=1}^{\infty}\Delta\alpha_i - I^*_{w,\beta}\beta\sum_{i=2}^{\infty}\Delta\alpha_i\right\|_{H^1} \\ &\leq C\|I^*_{w,\beta}I_{\Delta w,\beta}k\|_{H^1} + C\left\|I^*_{w,\beta}\beta\sum_{i=1}^{\infty}\Delta\alpha_i\right\|_{H^1} + C\left\|I^*_{w,\beta}\beta\sum_{i=2}^{\infty}\Delta\alpha_i\right\|_{H^1}.\end{aligned} \tag{2.21}$$

The following three claims prove lemma 2.6.2.

Claim 1. $\|I^*_{w,\beta}I_{\Delta w,\beta}k\|_{H^1} \leq C(1+M^2)(\|\Delta\sigma\|_{C^2} + \|\Delta\sigma\|_{C^2}^2)\|k\|_{L^2}$

We prove this by using proposition 2.5.2 to obtain $\|I^*_{w,\beta}I_{\Delta w,\beta}k\|_{H^1} \leq C\|w\|_{C^2}\|\Delta w\|_{C^2}\|k\|_{L^2}$. Then, we note that $\|\sigma\|_{C^2} \leq C\|\sigma\|_{H^s} \leq CM$ due to the continuous embedding $H^{n/2+2} \subset C^2$, and our a-priori assumption on σ. Similarly $\|\tilde{\sigma}\|_{C^2} \leq CM$. So we have $C > 0$ such that $\|w\|_{C^1} \leq C(1+\|\sigma\|_{C^1}) \leq C(1+M)$. As for Δw, we use the inequality $|e^{s_1} - e^{s_2}| \leq |s_1 - s_2|$, valid for $s_1, s_2 \geq 0$ to obtain $\|\Delta w\|_{L^\infty} \leq C\|\Delta\sigma\|_{L^\infty}$. The derivatives are similar, and make use of the relation $ab - \tilde{a}\tilde{b} = (a-\tilde{a})b + \tilde{a}(b-\tilde{b})$. The result is $\|\Delta w\|_{C^2} \leq C(1+M+M\|\Delta\sigma\|_{L^\infty})\|\Delta\sigma\|_{C^2}$.

Claim 2. $\|I^*_{w,\beta}\beta\sum_{i=1}^{\infty}\Delta\alpha_i\|_{H^1} \leq C(1+M)\|\beta\sum_{i=1}^{\infty}\Delta\alpha_i\|_{H^1} = C(1+M)\mathcal{S}_\beta$

This is a consequence of lemma 2.6.1

Claim 3. $\|I^*_{w,\beta}\beta\sum_{i=2}^{\infty}\Delta\alpha_i\|_{H^1} \leq C\varepsilon\|\Delta k\|_{L^2} + C\varepsilon^2(1+M^2)(\|\Delta\sigma\|_{C^1} + \|\Delta\sigma\|_{C^1}^2)$.

This may be proved in a manner similar to proposition 2.5.3, except in this case we have both Δk and $\Delta\sigma$ to contend with. Not to worry, we will make repeated use of $ab - \tilde{a}\tilde{b} = (a - \tilde{a})b + \tilde{a}(b - \tilde{b})$ to get estimates involving Δk and $\Delta\sigma$. Consider $\Delta\alpha_i$. It looks like α_i, except it has the factor

$$\begin{aligned}
&k(x_1)\cdots k(x_i)E(x_0,\ldots,x) - \tilde{k}(x_1)\cdots\tilde{k}(x_i)\tilde{E}(x_0,\ldots,x)\\
&= \big[k(x_1)\cdots k(x_i) - \tilde{k}(x_1)\cdots\tilde{k}(x_i)\big]E(x_0,\ldots,x) + \tilde{k}(x_1)\cdots\tilde{k}(x_i)\big[E(x_0,\ldots,x) - \tilde{E}(x_0,\ldots,x)\big]\\
&= \big[k(x_1)\cdots k(x_i) - \tilde{k}(x_1)\cdots\tilde{k}(x_i)\big]E(x_0,x_1)E(x_1,\ldots,x)\\
&\quad + \tilde{k}(x_1)\cdots\tilde{k}(x_i)\big[E(x_0,x_1) - \tilde{E}(x_0,x_1)\big]E(x_1,\ldots,x)\\
&\quad + \tilde{k}(x_1)\cdots\tilde{k}(x_i)\tilde{E}(x_0,x_1)\big[E(x_1,\ldots,x) - \tilde{E}(x_1,\ldots,x)\big]
\end{aligned}$$

Where we have taken special care to separate the terms $E(x_0,x_1), \tilde{E}(x_0,x_1)$ since they will appear as $\mathcal{E}(x_1,v_0), \tilde{\mathcal{E}}(x_1,v_0)$ in the weight function (see the paragraph after (2.14)). Recall in the proof of theorem 2.2.2, we also encountered $\Delta\alpha_i$. At that point, we were assumed $\sigma = \tilde{\sigma}$. Here, we denote that special case by $\Delta|_{\sigma=\tilde{\sigma}}\alpha_i$. Therefore, expanding $g(x_1,v_0,v_1)$ using lemma 2.5.2

$$\Delta\alpha_i = \Delta|_{\sigma=\tilde{\sigma}}\alpha_i + \sum_{j=1}^{\infty} I_{g_j\Delta\mathcal{E},1}\Lambda_{i,j} + \sum_{j=1}^{\infty} I_{g_j\tilde{\mathcal{E}},1}\Delta\Lambda_{i,j}.$$

We define the Λ's below.

$$\begin{aligned}
\Lambda_{i,j}(x_1) &:= \int_X\cdots\int_X \tilde{k}(x_1)\cdots\tilde{k}(x_i)\varphi_j(v_1)g(x_2,v_1,v_2)\cdots g(x_i,v_{i-1},v_i)\\
&\quad\times E(x_1,\ldots,x)(t_1\cdots t_i)^{1-n}m(x,v)dx_i\cdots dx_2.\\
\Delta\Lambda_{i,j}(x_1) &:= \int_X\cdots\int_X \tilde{k}(x_1)\cdots\tilde{k}(x_i)\varphi_j(v_1)g(x_2,v_1,v_2)\cdots g(x_i,v_{i-1},v_i)\\
&\quad\times \big[E(x_1,\ldots,x) - \tilde{E}(x_1,\ldots,x)\big](t_1\cdots t_i)^{1-n}m(x,v)dx_i\cdots dx_2.
\end{aligned} \tag{2.22}$$

We will need L^2 estimates for $\Lambda_{i,j}, \Delta\Lambda_{i,j}$.

Claim 4. $\|\Lambda_{i,j}\|_{L^2} \leq C\varepsilon^2(\varepsilon\|g\|_{L^\infty}\|T\|)^{i-2}$.

This is proved in a manner similar to (2.19), except rather than pulling out a $\|\Delta k\|_{L^2}$, we pull out another ε.

Claim 5. $\|\Delta\Lambda_{i,j}\|_{L^2} \leq C\varepsilon^2(\varepsilon\|g\|_{L^\infty}\|T\|)^{i-2}i\|\Delta\sigma\|_{L^\infty}$.

This is proved in a manner similar to claim 4, except rather than pulling out $\|E(x_1,\ldots,x_i,x)\|_{L^\infty} \leq$

1, we first write $E(x_1, \ldots, x_i, x) = exp\{\int_\gamma \sigma\}$, $\tilde{E}(x_1, \ldots, x_i, x) = exp\{\int_\gamma \tilde{\sigma}\}$, where γ is the piecewise linear path connecting $(x_1, \ldots, x_i, x)$. Next, we use the inequality $|e^{-a} - e^{-b}| \leq |a - b|$, valid for $a, b \geq 0$. This yeilds $\sup|E(x_1, \ldots, x_i, x) - \tilde{E}(x_1, \ldots, x_i, x)| \leq \text{diam}(X) i \|\Delta\sigma\|_{L^\infty}$. The result now follows.

Claims 4, 5, propositions 2.5.2, 2.5.3, and lemma 2.5.2 now yield the following, which proves claim 3.

$$
\begin{aligned}
\|I^*_{w,\beta}\beta \sum_{i=2}^{\infty} \Delta\alpha_i\|_{H^1} &= \|I^*_{w,\beta^2} \sum_{i=2}^{\infty} \Delta\alpha_i\|_{H^1} \\
&\leq \sum_{i=2}^{\infty} \|I^*_{w,\beta^2} \Delta|_{\sigma=\tilde{\sigma}} \alpha_i\|_{H^1} + \sum_{i=2}^{\infty}\sum_{j=1}^{\infty} \|I^*_{w,\beta^2} I_{g_j \Delta\mathcal{E},1} \Lambda_{i,j}\|_{H^1} \\
&+ \sum_{i=2}^{\infty}\sum_{j=1}^{\infty} \|I^*_{w,\beta^2} I_{g_j \tilde{\mathcal{E}},1} \Delta\Lambda_i\|_{H^1} \\
&\leq C\varepsilon\|\Delta k\|_{L^2} + C\sum_{i=2}^{\infty}\sum_{j=1}^{\infty} \frac{\|w\|_{C^1}}{1+j^2} \left\{ \|\Delta\mathcal{E}\|_{C^2} \|\Lambda_{i,j}\|_{L^2} + \|\tilde{\mathcal{E}}\|_{C^2} \|\Delta\Lambda_{i,j}\|_{L^2} \right\} \\
&\leq C\varepsilon\|\Delta k\|_{L^2} + C\sum_{i=2}^{\infty} \left\{ (1+M^2)(\|\Delta\sigma\|_{C^2} + \|\Delta\sigma\|^2_{C^2}) \|\Lambda_{i,j}\|_{L^2} + M^2 \|\Delta\Lambda_{i,j}\|_{L^2} \right\} \\
&\leq C\varepsilon\|\Delta k\|_{L^2} + C(1+M^2)(\|\Delta\sigma\|_{C^2} + \|\Delta\sigma\|^2_{C^2})\varepsilon^2 + CM^2\varepsilon^2 \|\Delta\sigma\|_{L^\infty} \\
&\leq C\varepsilon\|\Delta k\|_{L^2} C(1+M^2)(\|\Delta\sigma\|_{C^2} + \|\Delta\sigma\|^2_{C^2})\varepsilon^2.
\end{aligned}
$$

Where we have estimated $\|\Delta\mathcal{E}\|_{C^2} \leq C(1+M)(\|\Delta\sigma\|_{C^2} + \|\Delta\sigma\|^2_{C^2})$ using the same technique as in claim 1.

□

We are now in a position to prove theorem 2.2.3.

Proof. Lemma 2.6.2 gives us

$$
\|\Delta k\|_{L^2} \leq C(\mathcal{S} + \varepsilon\|\Delta\sigma\|_{C^2} + \varepsilon\|\Delta\sigma\|^2_{C^2}). \tag{2.23}
$$

Letting $\|\cdot\|_s$ denote norm in H^s, the interpolation inequality

$$
\|f\|_{\lambda_1 s_1 + \lambda_2 s_2} \leq \|f\|^{\lambda_1}_{s_1} \|f\|^{\lambda_2}_{s_2}, \quad \lambda_1 + \lambda_2 = 1,\ \lambda_i \geq 0,
$$

along with the continuous embedding $H^{n/2+2} \subset C^2$ allow us to write

$$\|\Delta\sigma\|_{C^2} \leq C\|\Delta\sigma\|_{n/2+2} \leq C\|\Delta\sigma\|_s^{1/s+\mu}\|\Delta\sigma\|_{L^2}^{1-1/s-\mu}.$$

The above being valid for $(n+2)/(2s) \leq \mu < 1 - 1/s$. By a result of Mukhometov [?],

$$\|\Delta\sigma\|_{L^2} \leq C\mathcal{A}. \tag{2.24}$$

This, along with (2.23) gives us

$$\|\Delta k\|_{L^1} \leq C(\mathcal{S}_\beta + \varepsilon\mathcal{A}^{1-(4+n)/2s} + \varepsilon\mathcal{A}^{2(1-(4+n)/2s)}).$$

Setting $\beta \equiv 1$ gives us first statement in the theorem.

As for the second statement, we use (2.24) and the interpolation inequality to show that for $n/(2s) \leq \mu < 1 - 1/s$ we have

$$\|\Delta\sigma\|_{L^\infty} \leq C\|\Delta\sigma\|_{n/2} \leq C\|\Delta\sigma\|_s^{1/s+\mu}\|\Delta\sigma\|_{L^2}^{1-1/s-\mu} \leq C\|\Delta\sigma\|_s^{1/s+\mu}\mathcal{A}^{1-1/s-\mu}.$$

□

Chapter 3

DIRECTIONAL SOURCES AND AVERAGED MEASUREMENTS, RIEMANNIAN CASE

3.1 Introduction

When the index of refraction is nonconstant, the situation is more complicated. The case of light propagating in medium with variable index of refraction, but without scattering or absorption, gives rise to geometrical optics in the high frequency limit. These results imply light travels along geodesics, and leads us to the model (3.1), which may be shown to correspond with a rigorous transport model. Starting here, we extend the results of chapter 2.

Let $X \subset \mathbb{R}^n$ be an open bounded set with smooth boundary ∂X and let g be a smooth Riemannian metric on $\bar{X}$. We shall make geometric assumptions on the Riemannian manifold (X, g) in due course. If $u(x, v)$ represents the density of particles at position x with velocity v in the unit tangent sphere at x, $\Omega_x X$, then the stationary linear transport equation is

$$Tu(x,v) = -\mathcal{D}u(x,v) - \sigma(x,v)u(x,v) + \int_{\Omega_x X} k(x,v',v)u(x,v')\,dv' = 0. \tag{3.1}$$

The operator $\mathcal{D}$ is the derivative along the geodesic flow (see (3.2) below) which in the case of g being Euclidean is simply $\mathcal{D}u(x,v) = v \cdot \nabla_x u(x,v)$. The measure dv' is the volume form on $\Omega_x X$ induced from the Euclidean volume form on $T_x X$ determined by g at x; here $T_x X$ is the full tangent space to X at x.

The case of a Euclidean metric corresponds to transport in material with a constant index of refraction. Though this is easier to study, in practice it is always an approximation. Derivation of the transport equation in case of an inhomogeneous, yet still isotropic, refractive index was studied in [9]. Examination of this result shows that our model (3.1) with conformally Euclidean metric $g_{ij}(x) = c_0^{-2} n(x)^2 \delta_{ij}$ correctly describes energy density propagation in isotropic material. Here c_0 is the speed of light in a vacuum, and n is the refractive index. This should not be surprising when one considers Fermat's principle, which states that (in the absence of absorption or scattering) light traveling from point A to point B will take a path which is a stationary point of the action

$$I(s) := \int_A^B n(x_s(t), \dot{x}_s(t))\, dt.$$

It is a classical result that stationary points of this action are geodesics of the metric $g_{ij} = n^2\delta_{ij}$. As obvious as this may make the model (3.1) seem, quite a few incorrect models have seen publication! See [9] for further discussion. Motivated by this success with the conformally Euclidean metric, we allow (3.1) with an anisotropic metric g_{ij} to model transport with an anisotropic index of refraction n_{ij}. In other words, $g_{ij} = c_0^{-2}\sum_k n_{ik}n_{kj}$. Showing the equivalence between the results in [9] (specifically equation (17) and (3.1) in this chapter) with a conformally Euclidean metric is somewhat non-trivial. For that reason we sketch the procedure here: The transport equation (17) in [9] is written in Euclidean coordinates and describes the propagation of phase space energy density, which has units of Joules per unit of solid angle. This energy density is a function of position x and direction $\hat{v} := v(\sum(v^i)^2)^{-1/2}$ (note that in [9] this is called Ω). To compare these two models we need to write (3.1) in terms of $\tilde{u}(x,\hat{v}) := u(x, |v|\hat{v})$. This is done by substituting $\tilde{u}(x, \hat{v}(v))$ into (3.1) and then finding the Christoffel symbols in terms of the metric. If one chooses the metric to be $g_{ij}(x) = c_0^{-2}n(x)^2\delta_{ij}$ the expressions are seen to be equivalent.

Define the "incoming" and "outgoing" bundles

$$\Gamma_\pm = \{(x,v) \,:\, x \in \partial X, \text{ and } \pm\langle v, \nu_x\rangle > 0\}$$

where ν_x is the unit outer normal vector to the boundary ∂X at x and $\langle\cdot,\cdot\rangle$ is the inner product, each with respect to g at x.

Given an incoming flux of particles u_- defined on Γ_-, let u be the solution, should it exist, to $Tu = 0$ with the boundary condition $u\big|_{\Gamma_-} = u_-$. The albedo operator is defined to be $\mathcal{A} : u_- \mapsto u\big|_{\Gamma_+}$. We will assume knowledge of an *average* over outgoing directions of u at $x \in \partial X$. To be more precise, for $x \in \partial X$, define

$$\Omega_x^\pm X = \{v \in \Omega_x X \,:\, \pm\langle v, \nu_x\rangle > 0\}$$

(and so $\Gamma_\pm$ are the disjoint unions of the $\Omega_x^\pm X$ over $x \in \partial X$). We will also use the somewhat unconventional notation

$$\Omega^2 X := \{(x, v, w) \, : \, x \in X, \; v, w \in \Omega_x X\}.$$

We shall denote by T^{-1} the solution operator to the boundary value problem $Tu = 0, \; u|_{\Gamma_-} = u_-$, that is, $u = T^{-1}u_-$.

Definition 3.1.1. The data of which we will be assuming knowledge consists of *angularly averaged* measurements on ∂X, weighted with respect to a prescribed function $m(x, v)$. Specifically, given u_- on Γ_- and $u = T^{-1}u_-$ we define $\mathcal{M} : L^1(\Gamma_-, d\mu) \to L^1(\partial X)$ by (the mapping properties will be verified later)

$$\mathcal{M}u_-(x, v) := \int_{\Omega_x^+ X} u(x, v) m(x, v) \, dv.$$

We require that both $m \in L^1(\Gamma_+)$ and that m^{-1} is bounded on Γ_+.

The function m corresponds to the limitations of the measurement apparatus. It may represent a limited aperture or, for example, when $m(x, v) = \langle v, \nu_x \rangle$, the measurement is power flux exiting the boundary. As X is strictly convex, this choice of m is indeed bounded below.

When the ambient metric is Euclidean, this problem was introduced and studied in [24] where an analysis of a singular decomposition of the Schwarz kernel of $\mathcal{M}$ (into an infinite series) is performed. From the most singular term, corresponding to the ballistic particles which do not scatter, the extinction coefficient $\sigma(x)$ is recovered. To recover k, additional assumptions (on σ, m and k) are made that allow one to view the term linear in k as a weighted X-ray transform and apply results from [23] to show injectivity of this map. Under a smallness assumption on k, injectivity of the full nonlinear map (as well as stability) is then proved. The precise formulation is very similar to what will be presented in the present article where we obtain analogous results when the background metric is Riemannian.

Here we state our basic assumptions on the coefficients in (3.1). First, notice that we assume σ depends only on position. This slightly restrictive (but physically realistic) assumption on σ is necessary: If $k \equiv 0$, and g is Euclidean, then this problem reduces to the usual X-ray transform. One can then show that if σ depends on the direction v (as opposed to $|v|$), no boundary measurement can uniquely determine σ (see the introduction of [20]). The "monochromatic" assumption $|v|_g = 1$ greatly simplifies the problem, and is commonly used. We will also assume that $0 \leq \sigma \in L^\infty(X)$, $k \in L^\infty(\Omega^2 X)$ are bounded functions. Solution to the forward problem also requires some sort of bound on energy production (and therefore k), which is sometimes given in relation to σ, [?]. Here we assume $\|k\|_{L^\infty} < (|\mathbb{S}^{n-1}|\text{diam}(X))^{-1}$. Unique recovery of the scattering kernel requires the assumption that it takes the form $k(x)\Theta(x, v', v)$ for some known function Θ (this is not assumed for recovery of σ). This assumption is reasonable (and quite common) and Θ can model the known underlying scattering process, whereas k models the density of scattering objects. Both of our main results require additional smallness assumptions on k, Our uniqueness theorem for k requires that m, σ, and g are all close to real analytic. These last two assumptions will be made precise in the statement of theorems 3.2.1, and 3.2.2. As in the Euclidean case, chapter 2, smoothness assumptions about σ imply smoothness assumptions about k and vice-versa.

We must also place restrictions on the geometry of (X, g). Recovery of both σ and k relies on injectivity of the (weighted) geodesic x-ray transform. It is for this reason that we assume the metric is "simple" (assumption X3 below).

X1. For every $x \in X$ and for every 2-plane Π in $T_x X$, the sectional curvature κ of (X, g) satisfies $\kappa_m \leq \kappa(\Pi) \leq \kappa_M$.

X2. If $\kappa_M > 0$ then the diameter of (X, g) satisfies $\text{diam}X < \pi/\sqrt{\kappa_M}$.

X3. (X, g) is simple: for any $x \in \bar{X}$ the exponential map $\exp_x : \exp_x^{-1}(\bar{X}) \to \bar{X}$ is a diffeomorphism (and consequently X is diffeomorphic to a ball).

To state the precise results it is necessary to introduce the volume forms on $\Omega_x X$ and $\Gamma_\pm$. On X we have the naturally defined volume form of the metric. At any $x \in X$, the volume form dv on $\Omega_x X$ is the form induced from the Euclidean volume on $T_x X$ defined by the metric g at x. The resulting form on ΩX is the Liouville form and is preserved under the geodesic flow of g. We denote by $d\mu$ the induced volume form on $\Gamma_\pm$ which has the property that $dt\, d\mu(x', v')$ is the pull-back of the Liouville form by the geodesic flow. Equivalently, we have the induced volume form of ∂X included in $\bar{X}$; if x are local coordinates for ∂X and dx is this volume form on ∂X, then it holds that

$$d\mu(x,v) = |\langle v, \nu_x \rangle|\, dv\, dx.$$

When not explicitly stated otherwise, measures on (for example) ∂X and X will be the volume forms defined by the metric g. Thus we shall write, for example, $L^1(\partial X)$ and $L^1(X)$ without further reference to the measures on these spaces. Similarly, for example, $L^1(\Omega X)$ is with respect to the Liouville form, and $L^1(\Gamma_\pm)$ is with respect to $d\mu$.

3.2 Statement of the main results

If $\mathcal{H} \subset \Gamma_-$, then by $\Gamma(\mathcal{H})$ we mean the set of geodesics with initial data in $\mathcal{H}$:

$$\Gamma(\mathcal{H}) = \{\gamma_{(x',v')}(t) \,:\, (x',v') \in \mathcal{H},\ 0 \le t \le \tau_+(x',v')\}.$$

Our curvature bounds (X1) result in the appearance of two constants C_{κ_m} and C_{κ_M} which reflect bounds on the possible size of Jacobi fields. See Lemma 3.4.4.

Theorem 3.2.1 (Recovery of σ). *Suppose that*

$$\|k\|_{L^\infty} < \min\left\{[(C_{\kappa_m} C_{\kappa_M})^{n-1} diam X |\mathbb{S}^{n-1}|]^{-1}, [diam X |\mathbb{S}^{n-1}|]^{-1}\right\},$$

and that $\mathcal{H}_\sigma \subset \Gamma_-$ is such that the geodesic x-ray transform restricted to $\Gamma(\mathcal{H}_\sigma)$ is injective. Then σ is uniquely determined by $\{(u_-, \mathcal{M}u_-) : u_- \in L^1(\mathcal{H}_\sigma)\}$.

Fixing $D > 0$ and making the definition

$$\mathcal{K}_\varepsilon^D := \{k \in L^\infty(X) : \text{dist}(\text{supp}(k), \partial X) > D, \|k\|_{L^\infty} \le \varepsilon\}$$

we also have the following uniqueness result for k.

Theorem 3.2.2 (Recovery of k). *Suppose the scattering kernel has the form $k(x)\Theta(x, v', v)$, with m and Θ both non-vanishing and with (g, m, σ, Θ) known and real analytic. Let $\mathcal{H}_k \subset$*

$\partial_- SX$ be open, and assume that for every $(x,v) \in TX \setminus \{0\}$, there exists $\gamma \in \Gamma(\mathcal{H}_k)$ through x and normal to v at x. Then there exits ε sufficiently small that for a.e. *$x \in \partial X$, knowledge of $\{(u_-, \mathcal{M}u_-(x)) : u_- \in L^1(\mathcal{H}_k)\}$ uniquely determines k within the class $\mathcal{K}_\varepsilon^D$.*

Furthermore, ε may be chosen such that this result holds in some C^2 neighborhood of (m, σ) and a C^∞ neighborhood of (Θ, g).

Remark. For Theorem 3.2.2 to hold, we at least require $\varepsilon < \left[\|\Theta\|_{L^\infty} \mathrm{diam} X |\mathbb{S}^{n-1}|\right]^{-1}$. As mentioned earlier, the Schwarz kernel term linear in k is a weighted X-ray transform. To "invert" the transform, one uses the open mapping theorem. This results in a constant (and therefore a smallness requirement on ε) that cannot be calculated explicitly. We take ε small enough to meet both of these requirements. The smoothness requirements on Θ and g could be relaxed to C^N for some $N < \infty$. Since real analytic functions are dense, we have a uniqueness result for an open and dense set of (m, g, σ, Θ).

3.3 The forward problem

Given $(x, v) \in \Omega X$ we denote by $\gamma_{(x,v)}(\cdot)$ the geodesic uniquely determined by $\gamma_{(x,v)}(0) = x,\ \dot\gamma_{(x,v)}(0) = v$. We will use the shorthand notation

$$\vec\gamma_{(x,v)}(t) := (\gamma_{(x,v)}(t), \dot\gamma_{(x,v)}(t)).$$

Define the "distance to boundary" functions

$$\tau_\pm(x, v) = \min\{t \geq 0 \ : \ \gamma_{(x,v)}(\pm t) \in \partial X\}.$$

Since (X, g) is simple, these functions are well-defined and finite. The operator $\mathcal{D}$ in (3.1) is the derivative along the geodesic flow and is defined by

$$\mathcal{D}u(x, v) = \frac{\partial}{\partial t}\Big|_{t=0} u(\gamma_{(x,v)}(t), \dot\gamma_{(x,v)}(t)). \tag{3.2}$$

Given two points $x,\ y \in X$ there is a unique geodesic from x to y; let $d(x, y) = d_g(x, y)$ be the geodesic distance between x and y and let $v(x, y)$ be the tangent vector to this geodesic at x. Define

$$E(x, y) := \exp\Big\{ - \int_0^{d(x,y)} \sigma\big(\gamma_{(x,v(x,y))}(t)\big)\, dt \Big\}.$$

Note, using the fact $\gamma_{(y,v(y,x))}(d(y,x)-s) = \gamma_{(x,v(x,y))}(s)$ we have

$$E(x,y) = E(y,x).$$

We define the following operators which arise naturally from (3.1). When $k \equiv 0$, the solution operator to $Tu = 0$, $u\big|_{\Gamma_-} = u_-$ is

$$Ju_-(x,v) := E\big(x, \gamma_{(x,v)}(-\tau_-(x,v))\big)u_-\big(\vec{\gamma}_{(x,v)}(-\tau_-(x,v))\big).$$

Define

$$T_1 f(x,v) := \int_{\Omega_x X} k(x,v',v) f(x,v')\, dv'$$

and

$$\begin{aligned} Kf(x,v) &:= \int_0^{\tau_-(x,v)} E\big(x, \gamma_{(x,v)}(t-\tau_-(x,v))\big) T_1 f\big(\vec{\gamma}_{(x,v)}(t-\tau_-(x,v))\big)\, dt \\ &= \int_0^{\tau_-(x,v)} E\big(x, \gamma_{(x,v)}(t-\tau_-(x,v))\big) \\ &\quad \times \int_{\Omega_{\gamma_{(x,v)}(t-\tau_-(x,v))}X} k\big(\gamma_{(x,v)}(t-\tau_-(x,v)), v', \dot{\gamma}_{(x,v)}(t-\tau_-(x,v))\big) \\ &\qquad \times f(\gamma_{(x,v)}(t-\tau_-(x,v)), v')\, dv'\, dt. \end{aligned}$$

The solution to the boundary value problem $Tu = 0$, $u\big|_{\Gamma_-} = u_-$ then satisfies

$$(I-K)u = Ju_-.$$

If $u_- \in L^\infty(\Gamma_-)$ then clearly $\|Ju_-\|_{L^\infty(\Omega X)} \le \|u_-\|_{L^\infty(\Gamma_-)}$. Further, if $f \in L^\infty(\Omega X)$ then

$$\begin{aligned} |Kf(x,v)| &\le \|k\|_{L^\infty(\Omega^2 X)} \|f\|_{L^\infty(\Omega X)} \Big| \int_0^{\tau_-(x,v)} \int_{\Omega_x X} dv'\, dt \Big| \\ &\le (\mathrm{diam} X) |\mathbb{S}^{n-1}| \|k\|_{L^\infty(\Omega^2 X)} \|f\|_{L^\infty(\Omega X)}. \end{aligned}$$

Thus, our assumptions on $\|k\|_{L^\infty(\Omega^2 X)}$ guarantee that $K : L^\infty(\Omega X) \to L^\infty(\Omega X)$ with operator norm $\|K\| < 1$ and so

$$u = T^{-1}u_- = \sum_{i=0}^{\infty} K^i J u_- \in L^\infty(\Omega X).$$

3.4 The averaged albedo operator

The goal of this section is to establish Theorem 3.4.1 where we show that the averaged albedo operator $\mathcal{M}$ can be expressed as the infinite sum of the operators $\mathcal{M}_i$ (see Definition 3.4.1), and where we calculate the Schwartz kernels for each $\mathcal{M}_i$. We first obtain a series expansion for the albedo operator $\mathcal{A}$ (see Lemma 3.4.3) for boundary fluxes $u_- \in L^\infty(\Gamma_-, d\mu)$. The definition of the measure on Γ_- makes immediate the following lemma which facilitates a useful and repeatedly used change of variables in integration.

Lemma 3.4.1 (Change of Variables). *If* $u \in L^1(\Omega X)$ *then*

$$\int_X \int_{\Omega_x X} u(x,v)\, dv_x\, dv = \int_{\Gamma_\pm} \int_0^{\tau_{mp}(x',v')} u(\gamma_{(x',v')}(t), \dot{\gamma}_{(x',v')}(t))\, dt\, d\mu(x',v').$$

Lemma 3.4.2. *Let* $u_- \in L^\infty(\Gamma_-, d\mu)$ *and* $u = T^{-1}u_-$*; for every* $(x,v) \in \Omega X$*, the map* $t \mapsto u(\vec{\gamma}_{(x,v)}(t))$ *is Lipschitz continuous with constant uniform in* (x,v).

Proof. Since $(I-K)u = Ju_-$, for any $0 \le l \le \tau_+(x,v)$ we may write

$$\begin{aligned} u(\vec{\gamma}_{(x,v)}(l)) = \int_0^{\tau_-(x,v)+l} & E\big(\gamma_{(x,v)}(l), \gamma_{(x,v)}(t-\tau_-(x,v))\big) \\ & \times \int_{\Omega_{\gamma_{(x,v)}(t-\tau_-(x,v))}X} k\big(\gamma_{(x,v)}(t-\tau_-(x,v)), v', \dot{\gamma}_{(x,v)}(t-\tau_-(x,v))\big) \\ & \quad \times u(\gamma_{(x,v)}(t-\tau_-(x,v)), v')\, dv'\, dt \\ & + E\big(\gamma_{(x,v)}(l), \gamma_{(x,v)}(-\tau_-(x,v))\big) u_-\big(\vec{\gamma}_{(x,v)}(-\tau_-(x,v))\big). \end{aligned}$$

Thus

$$\begin{aligned} & u(\vec{\gamma}_{(x,v)}(l)) - u(\vec{\gamma}_{(x,v)}(0)) \\ & \quad = \int_0^{\tau_-(x,v)} \big\{ E\big(\gamma_{(x,v)}(l), \gamma_{(x,v)}(t-\tau_-(x,v))\big) - E\big(x, \gamma_{(x,v)}(t-\tau_-(x,v))\big) \big\} \\ & \qquad\qquad \times \int_{\Omega_{\gamma_{(x,v)}(t-\tau_-(x,v))}X} k(\cdot)u(\cdot)\, dv'\, dt \\ & \qquad + \int_0^l E\big(\gamma_{(x,v)}(l), \gamma_{(x,v)}(t-\tau_-(x,v))\big) \int_{\Omega_{\gamma_{(x,v)}(t)}X} k(\cdot)u(\cdot)\, dv'\, dt \\ & \qquad + \big\{ E\big(\gamma_{(x,v)}(l), \gamma_{(x,v)}(-\tau_-(x,v))\big) - E\big(x, \gamma_{(x,v)}(-\tau_-(x,v))\big) \big\} u_-\big(\gamma_{(x,v)}(-\tau_-(x,v))\big) \end{aligned}$$

and so

$$\left|u(\vec{\gamma}_{(x,v)}(l)) - u(\vec{\gamma}_{(x,v)}(0))\right| \leq l\big\{|\mathbb{S}^{n-1}|(\mathrm{diam}X\|\sigma\|_{L^\infty(X)} + 1)\|k\|_{L^\infty(\Omega^2X)}\|u\|_{L^\infty(\Omega X)} + \|\sigma\|_{L^\infty(X)}\|u_-\|_{L^\infty(\Gamma_-)}\big\}.$$

□

Lemma 3.4.3. *Let k satisfy $\|k\|_{L^\infty(\Omega^2X)} < \left(|\mathbb{S}^{n-1}|diamX\right)^{-1}$. If $u_- \in L^\infty(\Gamma_-, d\mu)$ then the albedo operator $\mathcal{A}$ has the expression*

$$\mathcal{A}u_-(x,v) := (T^{-1}u_-)\big|_{\Gamma_+} = \sum_{i=0}^{\infty} K^i J u_-(x,v)\Big|_{\Gamma_+} \in L^1(\Gamma_+, d\mu)$$

and the series for $\mathcal{A}$ extends uniquely to a continuous operator $\mathcal{A} : L^1(\Gamma_-, d\mu) \to L^1(\Gamma_+, d\mu)$.

Proof. We extend the metric g to $\tilde{g}$, defined on a domain $\tilde{X}$ slightly larger than X, and so that $(\tilde{X}, \tilde{g})$ is again strictly convex and simple. Extend σ and k to be zero outside X. For every $(x,v) \in \Gamma_-$ define

$$\tilde{u}_-\left(\vec{\tilde{\gamma}}_{(x,v)}(-\tilde{\tau}_-(x,v))\right) = u_-(x,v)$$

and $\tilde{u}(\tilde{x},\tilde{v}) = 0$ if $(\tilde{x},\tilde{v}) \neq \tilde{\gamma}_{(x,v)}(-\tilde{\tau}_-(x,v))$ for any $(x,v) \in \Gamma_-$. This defines $\tilde{u}_-$ on $\tilde{\Gamma}_-$ by "back-propagating" u_- from Γ_-. One has, for all $(x,v) \in \Omega X$,

$$\tilde{K}f(x,v) = Kf(x,v), \qquad \text{and} \qquad \tilde{J}\tilde{u}_-(x,v) = Ju_-(x,v).$$

Since $\tilde{u}_- \in L^\infty(\tilde{\Gamma}_-)$, $\tilde{u} = \sum_{i=1}^{\infty} \tilde{K}^i \tilde{J}\tilde{u}_- \in L^\infty(\Omega\tilde{X})$ and by Lemma 3.4.2 the map $t \mapsto \tilde{u}(\tilde{\gamma}_{(x,v)}(t))$ is Lipschitz continuous on $\Omega\tilde{X}$ and hence on $\Omega\bar{X}$. Thus we may define the restriction of $T^{-1}u_-$ to Γ_+ by, for $(x,v) \in \Gamma_+$,

$$u_-(x,v) = \sum_{i=0}^{\infty} \tilde{K}^i \tilde{J}\tilde{u}_-(x,v) = \sum_{i=0}^{\infty} K^i J u_-(x,v).$$

It remains to show that this restriction is a continuous map. We use the ideas of [CS2] (see also [M1] and [L]). Define the function space $\mathcal{W}$ via the norm

$$\|f\|_{\mathcal{W}} := \|\mathcal{D}f\|_{L^1(\Omega X)} + \|(\tau_+ + \tau_-)^{-1} f\|_{L^1(\Omega X)}.$$

Since $u_- \in L^\infty(\Gamma_-) \subset L^1(\Gamma_-)$, Proposition 2.6 of [M1] (together with the proof given there) yields $u = T^{-1}u_- \in \mathcal{W}$, with $\|u\|_{\mathcal{W}} \leq C_{X,\sigma,k}\|u_-\|_{L^1(\Gamma_-)} \leq C'_{X,\sigma,k}\|u_-\|_{L^\infty(\Gamma_-)}$. For any $x \in \partial X$, $v \in \Omega_x^+ X$, since $h(t) = u(\gamma_{(x,v)}(-t))$ is absolutely continuous on $[0, \tau_-(x,v)]$, $h' \in L^1([0,\tau_-(x,v)])$ and so $h(0) = -\int_0^t h'(s)\,ds + h(t)$. It follows that

$$|u(x,v)| \leq \int_0^{\tau_-(x,v)} \left|\mathcal{D}_s u(\vec{\gamma}_{(x,v)}(-s))\right| ds + \frac{1}{\tau_-(x,v)} \int_0^{\tau_-(x,v)} \left|u(\vec{\gamma}_{(x,v)}(-s))\right| ds.$$

Thus, upon utilizing Lemma 3.4.1 and integrating over Γ_+,

$$\left\|u|_{\Gamma_+}\right\|_{L^1(\Gamma_+)} \leq \|u\|_{\mathcal{W}} \leq C\|u_-\|_{L^\infty(\Gamma_-)}.$$

Finally, since $L^\infty(\Gamma_-)$ is dense in $L^1(\Gamma_-)$ we have the claim of the lemma. □

In order to simplify the presentation of what follows, we introduce some notation. If $x_1, \ldots, x_j$ are points in X, then

$$E(x_1, x_2, \ldots, x_j) := E(x_1,x_2)E(x_2,x_3)\cdots E(x_{j-1},x_j) = \prod_{i=1}^{j-1} E(x_i, x_{i+1}). \tag{3.3}$$

If $y \in X$, consider $z = z_y(t,v) = \gamma_{(y,v)}(t)$ defined from the "polar" coordinates $(t,v) \in \mathbb{R} \times \Omega_y X$. Let $J_y(z)$ denote the Jacobian determinant $|\det \partial z/\partial(t,v)|^{-1}$ of this change of variables. For a given z, let (t,v) be its polar coordinate expression, and let $\{v^1, \ldots, v^n\}$ be an orthonormal basis for $T_x X$ with $v^1 = v$. Let $Y_{y,z,i}$ be the Jacobi field along $\gamma_{(y,v)}(\cdot)$ with initial data $Y_{y,z,i}(0) = 0$, $\dot{Y}_{y,z,i} = v^i$, $2 \leq i \leq n$. Then $J_y(z)$ is given by the expression

$$J_y(z) = \prod_{i=2}^{n} |Y_{y,z,i}(d(y,z))|^{-1}. \tag{3.4}$$

The notation is chosen to remind the reader that J_y comes from "geodesic" polar coordinates based at y. We will frequently need products of such change of volume elements and so introduce the notation

$$\mathcal{J}(y_1, \ldots, y_j) := \prod_{i=2}^{j} J_{y_i}(y_{i-1}). \tag{3.5}$$

Lemma 3.4.4. *Let the sectional curvature κ of (X,g) satisfy $\kappa_m \leq \kappa(\Pi) \leq \kappa_M$ for all 2-planes Π in $T_y X$, for all $y \in X$. Then*

$$J_y(z) \leq \frac{(C_{\kappa_M})^{n-1}}{d(y,z)^{n-1}} \tag{3.6}$$

where

$$C_{\kappa_M} = \begin{cases} \frac{\sqrt{\kappa_M}\, diamX}{\sin(\sqrt{\kappa_M}\, diamX)}, & \kappa_M > 0, \\ 1, & \kappa_M \le 0. \end{cases}$$

Further, if Y is any Jacobi field along a geodesic $\gamma(t)$ with $Y(0) = 0$ and $\dot{Y}(0) \in \dot{\gamma}^{\perp}(0) \subset \Omega_{\gamma(0)}X$, then

$$\frac{|Y(t)|}{t} \le C_{\kappa_m} = \begin{cases} \frac{\sinh(\sqrt{-\kappa_m}\, diamX)}{\sqrt{-\kappa_m}\, diamX}, & \kappa_m < 0, \\ 1, & \kappa_m \ge 0 \end{cases} \tag{3.7}$$

for all $0 \le t \le \tau_+(\gamma(0), \dot{\gamma}(0))$.

Proof. Let Y_i be a Jacobi field of the form described in the second part of the lemma. From the upper bound for κ, Rauch comparison gives

$$|Y_i(d(y,z))| \ge \begin{cases} \frac{1}{\sqrt{\kappa_M}} \sin(\sqrt{\kappa_M}\, d(y,z)), & \kappa_M > 0, \\ d(y,z), & \kappa_M = 0, \\ \frac{1}{\sqrt{-\kappa_M}} \sinh(\sqrt{-\kappa_M}\, d(y,z)), & \kappa_M < 0. \end{cases}$$

If $\kappa_M > 0$, since $\mathrm{diam}X < \pi/\sqrt{\kappa_M}$,

$$\frac{\sin(\sqrt{\kappa_M}\, d)}{\sqrt{\kappa_M}\, d} \ge \frac{\sin(\sqrt{\kappa_M}\, \mathrm{diam}X)}{\sqrt{\kappa_M}\, \mathrm{diam}X}$$

for $0 \le d \le \mathrm{diam}X$. If $\kappa_M < 0$ then $\sinh(\sqrt{-\kappa_M}\, d(y,z)) > \sqrt{-\kappa_M}\, d(y,z)$. Given the expression (3.4) for $J_y(z)$, these estimates yield (3.6).

For (3.7), Theorem 4.5.2 of [?] gives

$$|Y(t)| \le \begin{cases} \frac{1}{\sqrt{\kappa_m}} \sin(\sqrt{\kappa_m}\, t), & \kappa_m > 0, \\ t, & \kappa_m = 0, \\ \frac{1}{\sqrt{-\kappa_m}} \sinh(\sqrt{-\kappa_m}\, t), & \kappa_m < 0 \end{cases}$$

which, as above, in turn implies (3.7). □

Definition 3.4.1. For each $i \ge 1$ we define the operator $\mathcal{M}_i : L^1(\Gamma_-) \to L^1(\partial X)$ by

$$\mathcal{M}_i u_-(x) := \int_{\Omega_x^+ X} K^i J u_-(x,v) m(x,v)\, dv.$$

Theorem 3.4.1. *Let* $\|k\|_{L^\infty(\Omega^2 X)} < \left(|\mathbb{S}^{n-1}| diam X\right)^{-1}$. *Then* $\mathcal{M} : L^1(\Gamma_-, d\mu) \to L^1(\partial X)$ *continuously, and for almost every* $x \in \partial X$ *has the expansion*

$$\begin{aligned}\mathcal{M}u_-(x) &= \sum_{j=0}^{\infty} \int_{\Omega_x^+ X} K^j J u_-(x,v) m(x,v)\, dv \\ &= \sum_{j=0}^{\infty} \int_{\Gamma_-} \alpha_j(x,x',v') u_-(x',v')\, d\mu(x',v')\end{aligned}$$

where $\alpha_0(x,\cdot,\cdot)$ *is a distribution supported on a manifold of dimension* $n-1$, *and for* $j \geq 1$, $\alpha_j(x,x',v')$ *are the following Schwartz kernels: when* $j = 1$,

$$\alpha_1(x,x',v') = \int_0^{\tau_+(x',v')} k(\vec{\gamma}_{(x',v')}(t), \bar{w}_1) E(x', \gamma_{(x',v')}(t), x) m(x, \hat{w}_1) J_x(\gamma_{(x',v')}(t))\, dt \quad \textbf{(3.8)}$$

where $\bar{w}_1, \hat{w}_1$ *are the initial and final tangent vectors of the geodesic from* $\gamma_{(x',v')}(t)$ *to* x, *and* E *and* J_x *are defined in (3.3), and (3.4); for* $j \geq 2$,

$$\begin{aligned}\alpha_j(x,x',v') = \int_0^{\tau_+(x',v')} \int_X \cdots \int_X k(\vec{\gamma}_{(x',v')}(t), \bar{w}_1) \prod_{i=2}^{j} k(y_i, \hat{w}_{i-1}, \bar{w}_i) \\ E(x', \gamma_{(x',v')}(t), y_2, \ldots, y_j, x) \mathcal{J}\, m(x, \hat{w}_j)\, dy_j \cdots dy_2\, dt \quad \textbf{(3.9)}\end{aligned}$$

where:

- $\bar{w}_1$, $\hat{w}_1$ *are the initial and final tangent vectors of the geodesic joining* $\gamma_{(x',v')}(t)$ *to* y_2,
- $\bar{w}_i$, $\hat{w}_i$ *are the initial and final tangent vectors of the geodesic joining* y_i *to* y_{i+1} *for* $i = 2, \ldots, j-1$,
- $\bar{w}_j$, $\hat{w}_j$ *are the initial and final tangent vectors of the geodesic joining* y_j *to* x, *and*
- $\mathcal{J} = \mathcal{J}(\gamma_{(x',v')}(t), y_2, \ldots, y_j, x)$.

Proof. First, if $u_- \in L^1(\Gamma_-, d\mu)$ then since $m(x,v)|\langle v, \nu_x\rangle|^{-1} \in L^\infty(\Gamma_+)$,

$$\begin{aligned}\|\mathcal{M}u_-\|_{L^1(\partial X)} &\leq \int_{\partial X} \int_{\Omega_x^+ X} |\mathcal{A}u_-(x,v)| \frac{|m(x,v)|}{|\langle v, \nu_x\rangle|} |\langle v, \nu_x\rangle|\, dv\, dx \\ &\leq \left\| \frac{m(x,v)}{|\langle v, \nu_x\rangle|} \right\|_{L^\infty(\Gamma_+)} \|\mathcal{A}u_-\|_{L^1(\Gamma_+, d\mu)}\end{aligned}$$

and so by Lemma 3.4.3, $\mathcal{M} : L^1(\Gamma_-, d\mu) \to L^1(\partial X)$ continuously.

Next by Lemma 3.4.3,

$$\mathcal{M}u_-(x) := \int_{\Omega_x^+ X} \mathcal{A}u_-(x,v) m(x,v)\, dv = \int_{\Omega_x^+ X} \sum_{j=0}^{\infty} K^j J u_-(x,v) m(x,v)\, dv.$$

Now

$$\begin{aligned}\Big\| \sum_{j=0}^{\infty} K^j J u_- m \Big\|_{L^1(\Gamma_+, dv_x\, dx)} &= \Big\| \sum_{j=0}^{\infty} K^j J u_- \frac{m}{|\langle v, \nu_x \rangle|} \Big\|_{L^1(\Gamma_+, d\mu)} \\ &\leq \Big\| \frac{m(x,v)}{|\langle v, \nu_x\rangle|} \Big\|_{L^\infty(\Gamma_+)} \|\mathcal{A}u_-\|_{L^1(\Gamma_+, d\mu)}\end{aligned}$$

and so for almost every $x \in \partial X$, $\sum_{j=0}^{\infty} K^j J u_-(x,\cdot) m(x,\cdot) \in L^1(\Omega_x^+ X)$. Thus, since K and J preserve non-negativity, we may use monotone convergence to interchange the integration with the summation in the above to obtain

$$\mathcal{M}u_-(x) = \sum_{j=0}^{\infty} \int_{\Omega_x^+ X} K^j J u_-(x,v) m(x,v)\, dv = \sum_{j=0}^{\infty} \mathcal{M}_j u_-(x).$$

We let this last inequality define the operator $\mathcal{M}_j$.

Next, for each j, we claim that $\mathcal{M}_j$ is a bounded map $C_c^\infty(\Gamma_-) \to \mathcal{D}'(\partial X)$ and so has a Schwarz kernel $\alpha_j(x, x', v') \in \mathcal{D}'(\partial X \times \Gamma_-)$, that is

$$\mathcal{M}_j u_-(x) = \int_{\Gamma_-} \alpha_j(x, x', v') u_-(x', v')\, d\mu.$$

To see this for $j \geq 1$, let $u_-^n \to 0$ in $C_c^\infty(\Gamma_-)$ (as $n \to \infty$); for any $\varphi \in C_c^\infty(\partial X)$

$$\begin{aligned}\langle \mathcal{M}_j u_-^n, \varphi \rangle &= \int_{\partial X} \int_{\Omega_x^+ X} K^j J u_-^n(x,v) m(x,v)\, dv\, \varphi(x)\, dx \\ &= \int_{\Gamma_+} K^j J u_-^n(x,v) \frac{m(x,v)}{|\langle \nu_x, v\rangle|} \varphi(x)\, d\mu(x,v) \\ &\leq \Big\| \frac{m(x,v)}{|\langle \nu_x, v\rangle|} \varphi(x) \Big\|_{L^\infty(\Gamma_+)} \|K^j J u_-^n\|_{L^1(\Gamma_+, d\mu)} \to 0\end{aligned}$$

since $u_-^n \to 0$ in $C_c^\infty(\Gamma_-)$ implies that $u_-^n \to 0$ in $L^1(\Gamma_-)$, which in turn implies that $K^i J u_-^n \to 0$ in $L^1(\Gamma_+)$ by Lemma 3.4.3.

When $j = 0$, for fixed $x \in \partial X$, $\alpha_0(x, x', v')$ is a singular distribution. To see this, let $F = \{(x', v') \in \Gamma_- \,:\, \gamma_{(x',v')}(\tau_+(x', v')) = x\}$, let $U \subset F$ be open in F, and let u_n be a sequence of smooth functions on Γ_- such that $u_n\big|_U = 1$ and $u_n \to \chi_U$ as $n \to \infty$ (in C^0

norm), where χ_U is the characteristic function of U. Note that F is an $n-1$ dimensional sub-manifold lying in the $2n-2$ dimensional Γ_-. Then

$$\begin{aligned}\int_{\Gamma_-} \alpha_0(x,x',v')u_n(x',v')\,d\mu &= \int_{\Omega_x^+X} Ju_n(x,v)m(x,v)\,dv \\ &= \int_{\Omega_x^+X} E\big(x,\gamma_{(x,v)}(-\tau_-(x,v))\big)u_n\big(\vec{\gamma}_{(x,v)}(-\tau_-(x,v))\big)m(x,v)\,dv \\ &\geq \int_{\{v\in\Omega_x^+X\,:\,\vec{\gamma}_{(x,v)}(-\tau_-(x,v))\in U\}} E\big(x,\gamma_{(x,v)}(-\tau_-(x,v))\big)m(x,v)\,dv \\ &\neq 0\end{aligned}$$

for every n. Now $u_n \to 0$ in every $L^p(\Gamma_-)$, $1 \leq p \leq \infty$ and so $\alpha_0 \notin L^p(\Gamma_-)$ for every such p. The above also shows that the support of $\alpha_0(x,\cdot,\cdot)$ is at least F; further, if $\varphi \in C_c^\infty(\Gamma_-)$ is such that $\operatorname{supp}\varphi \cap F = \emptyset$ then, as above,

$$\int_{\Gamma_-} \alpha_0(x,x',v')\varphi(x',v')\,d\mu = 0$$

and so $\operatorname{supp}\alpha_0(x,\cdot,\cdot) = F$.

We now derive the expressions (3.8) and (3.9). When $j=1$, let ϕ_- be a function on Γ_-. We have

$$\begin{aligned}\mathcal{M}_1\phi_-(x) &= \int_{\Omega_x^+X} KJ\phi_-(x,v)\,dv \\ &= \int_{\Omega_x^+X}\int_0^{\tau_-(x,v)} E\big(x,\gamma_{(x,v)}(t-\tau_-(x,v))\big)T_1J\phi_-(\vec{\gamma}_{(x,v)}(t-\tau_-(x,v))\,dt\,m(x,v)\,dv \\ &= \int_{\Omega_x^+X}\int_0^{\tau_-(x,v)} E\big(x,\gamma_{(x,v)}(t-\tau_-(x,v))\big) \\ &\qquad\times\int_{\Omega_yX} k(y,\hat{w},\bar{w}_1)E\big(\gamma_{(x,v)}(t-\tau_-(x,v)),\gamma_{(y,\hat{w})}(-\tau_-(y,\hat{w}))\big) \\ &\qquad\quad\times \phi_-(\vec{\gamma}_{(y,\hat{w})}(-\tau_-(y,\hat{w}))\,d\hat{w}\,dt\,m(x,v)\,dv\end{aligned}$$

where $(y,\bar{w}_1) = \vec{\gamma}_{(x,v)}(t-\tau_-(x,v))$

$$\begin{aligned}&= \int_X\int_{\Omega_yX} E\big(x,y,\gamma_{(y,\hat{w})}(-\tau_-(y,\hat{w}))\big)k(y,\hat{w},\bar{w}_1)\phi_-\big(\vec{\gamma}_{(y,\hat{w})}(-\tau_-(y,\hat{w}))\big)\,d\hat{w} \\ &\qquad\times\; m(x,\hat{w}_1)J_x(y)\,dy\end{aligned}$$

where $\bar{w}_1$, $\hat{w}_1$ are the initial and final tangent vectors (respectively) of the geodesic from y to x,

$$= \int_{\partial_- SX} \int_0^{\tau_+(x',v')} k(\vec{\gamma}_{(x',v')}(t), \bar{w}_1) E(x', \gamma_{(x',v')}(t), x) m(x, \hat{w}_1) J_x(\gamma_{(x',v')}(t))\, dt$$
$$\times\ \phi_-(x', v')\, d\mu(x', v').$$

This proves (3.8).

For brevity of exposition, we shall cease to write out the arguments of the functions E writing simply $E(\cdot)$ (recall also the convention (3.3)). Toward deriving the result for $j = 2$, we first compute

$$\begin{aligned}
&T_1 K J \phi_-(y_2, \bar{w}_2)\\
&= \int_{\Omega_{y_2} X} k(y_2, \hat{w}_1, \bar{w}_2) \int_0^{\tau_-(y_2, \hat{w}_1)} E(\cdot) T_1 J \phi_-\big(\vec{\gamma}_{(y_2,\hat{w}_1)}(t_1 - \tau_-(y_2, \hat{w}_1))\big)\, dt_1\, d\hat{w}_1\\
&= \int_{\Omega_{y_2} X} k(y_2, \hat{w}_1, \bar{w}_2) \int_0^{\tau_-(y_2, \hat{w}_1)} \int_{\Omega_{y_1} X} k(y_1, \hat{w}, \bar{w}_1) E(\cdot) \phi_-(\vec{\gamma}_{(y_1,\hat{w})}(-\tau_-(y_1, \hat{w}))\, d\hat{w}\, dt_1\, d\hat{w}_1
\end{aligned}$$

where $(y_1, \bar{w}_1) = \vec{\gamma}_{(y_2,\hat{w}_1)}(t_1 - \tau_-(y_2, \hat{w}_1))$

$$= \int_X \int_{\Omega_{y_1} X} E(\cdot) k(y_2, \hat{w}_1, \bar{w}_2) k(y_1, \hat{w}, \bar{w}_1) \phi_-\big(\vec{\gamma}_{(y_1,\hat{w})}(-\tau_-(y_1, \hat{w}))\big)\, d\hat{w}\, J_{y_2}(y_1)\, dy_1.$$

Now,

$$\mathcal{M}_2 \phi_-(x) = \int_{\Omega_x^+ X} \int_0^{\tau_-(x,v)} E(\cdot) T_1 K J \phi_-(y_2, \bar{w}_2)\, dt_2\, m(x, v)\, dv$$

where $(y_2, \bar{w}_2) = \vec{\gamma}_{(x,v)}(t_2 - \tau_-(x, v))$,

$$\begin{aligned}
&= \int_X E(\cdot) T_1 K J \phi_-(y_2, \bar{w}_2) J_x(y_2) m(x, \hat{w}_2)\, dy_2\\
&= \int_X \int_X \int_{\Omega_{y_1} X} E(\cdot) k(y_2, \hat{w}_1, \bar{w}_2) k(y_1, \hat{w}, \bar{w}_1) \phi_-\big(\vec{\gamma}_{(y_1,\hat{w})}(-\tau_-(y_1, \hat{w}))\big)\, d\hat{w}\\
&\qquad \times\ J_{y_2}\, dy_1 J_x(y_2) m(x, \hat{w}_2)\, dy_2\\
&= \int_{\partial_- SX} \int_0^{\tau_+(x',v')} \int_X E(\cdot) k(\vec{\gamma}_{(x',v')}(t), \bar{w}_1) k(y_2, \hat{w}_1, \bar{w}_2) \mathcal{J}(\gamma_{(x',v')}(t), y_2, x)\\
&\qquad \times m(x, \hat{w}_2)\, dy_2\, dt \phi_-(x', v')\, d\mu(x', v').
\end{aligned}$$

This proves (3.9) for $j = 2$.

An inductive argument on j first shows that

$$T_1 K^{j-1} J\phi_-(y_j, \bar{w}_j) = \int_X \cdots \int_X \int_{\Omega_{y_1} X} E(\cdot) k(y_1, \hat{w}, \bar{w}_1) \prod_{i=2}^{j} k(y_i, \hat{w}_{i-1}, \bar{w}_i)$$
$$\times \phi_-\big(\vec{\gamma}_{(y_1,\hat{w})}(-\tau_-(y_1,\hat{w}))\big) \mathcal{J}(y_1, \ldots, y_j)\, d\hat{w}\, dy_1 \cdots dy_{j-1}$$

where $\bar{w}_i$, $\hat{w}_i$ are the initial and final tangent vectors of the geodesic joining y_i to y_{i+1} for $i = 1, \ldots, j-1$, and where $\mathcal{J}$ is as in (3.5). Combining this with an inductive argument based on the above proof of (3.9) when $j = 2$ yields (3.9) for arbitrary $j \geq 2$. □

3.5 Recovering the extinction coefficient

We recall some definitions relevant to this section. The open set $\mathcal{H}_\sigma \subset \Gamma_-$ is such that the geodesic x-ray transform restricted to the set of geodesics with initial data in $\mathcal{H}_\sigma$ is injective. The notation indicates that this is the subset of geodesics which is used in the recovery of the extinction coefficient σ. The weight function m is assumed to be bounded away from zero on Γ_+, and $m \in L^1(\mathcal{H}_\sigma)$. The main result of this section is Theorem 3.5.1 which shows that, due to the singular nature of α_0, we can use an approximate identity as our prescribed flux to obtain the extinction coefficient in a limiting process. The key estimate used in the proof of Theorem 3.5.1 is that of Proposition 3.5.1 which enables us to show that the contribution due to α_j for $j \geq 1$ tends to zero.

Let $0 < \mu < 1$ and set $p = (n-\mu)/(n-1)$, $q = (1 - 1/p)^{-1}$.

Lemma 3.5.1. *It holds that $d(x, \cdot)^{n-1} \in L^p(X)$ with*

$$\left\| \frac{1}{d(x,\cdot)^{n-1}} \right\|_{L^p(X)} \leq C_d^{1/p}. \tag{3.10}$$

Proof. We compute

$$\int_X \frac{1}{d(x,y)^{(n-1)p}}\, dy = \int_X \frac{1}{d(x,y)^{n-\mu}}\, dy = \int_{\Omega_x X} \int_0^{\tau_+(x,v')} \frac{1}{t^{n-\mu}} J_x(\gamma_{(x,v')}(t))^{-1}\, dt$$
$$\leq (C_{\kappa_m})^{n-1} \int_{\Omega_x X} \int_0^{\tau_+(x,v')} t^{\mu-1}\, dt \leq C_d, \text{ say,}$$

by (3.7). □

Definition 3.5.1. Let $\mathcal{T}$ be the operator with kernel $d(x,\cdot)^{n-1}$, that is

$$\mathcal{T}f(x) := \int_X \frac{f(y)}{d(x,y)^{n-1}}\,dy.$$

Lemma 3.5.1 (with $p=1$) shows that $\mathcal{T}: L^p(X) \to L^p(X)$ continuously, $1 \le p \le \infty$ with $\|\mathcal{T}\| \le (C_{\kappa_m})^{n-1}|\mathbb{S}^{n-1}|\text{diam}X$ (see [T], Prop. 5.1, Appendix A).

We also define the analogous operator $\tilde{\mathcal{T}}$,

$$\tilde{\mathcal{T}}f(x) := \int_X f(y)J_y(x)\,dy. \tag{3.11}$$

Since

$$\int_X J_y(x)\,dy = \int_{\Omega_x X}\int_0^{\tau_+(x,v)} dt\,dv \le |\mathbb{S}^{n-1}|\text{diam}X, \tag{3.12}$$

$\tilde{\mathcal{T}}: L^p(X) \to L^p(X)$, $1 \le p \le \infty$, with $\|\tilde{\mathcal{T}}\| \le |\mathbb{S}^{n-1}|\text{diam}X$.

Proposition 3.5.1. *Let $p = (n-\mu)/(n-1)$, $0<\mu<1$, $q = (1-1/p)^{-1}$, and $\|k\|_{L^\infty(\Omega^2 X)} < \left[(C_{\kappa_m}C_{\kappa_M})^{n-1}\text{diam}X|\mathbb{S}^{n-1}|\right]^{-1}$, where $C_{\kappa_m}, C_{\kappa_M}$ are as defined in Lemma 3.4.4. Then for almost every $x \in \partial X$,*

$$\Big|\sum_{j=1}^{\infty}\int_{\Gamma_-}\alpha_j(x,x',v')f(x',v')\,d\mu(x',v')\Big| \le C_0\|f\|_{L^q(\Gamma_-,d\mu)} \tag{3.13}$$

where $C_0>0$ depends on κ_m, κ_M, $\|k\|_{L^\infty(\Omega^2 X)}$, $\|m\|_{L^\infty(\Omega X)}$ and diam$X$.

Proof. When $j=1$,

$$\begin{aligned}
&\Big|\int_{\Gamma_-}\alpha_1(x,x',v')f(x',v')\,d\mu(x',v')\Big| \\
&\quad= \Big|\int_{\Gamma_-}\int_0^{\tau_+(x',v')} k(\vec{\gamma}_{(x',v')}(t),\bar{w})E(\cdot)m(x,\hat{w}_1)J_x(\gamma_{(x',v')}(t))\,dt\,f(x',v')\,d\mu(x',v')\Big| \\
&\quad\le \|k\|_\infty\|m\|_\infty(C_{\kappa_M})^{n-1}\int_X\int_{\Omega_y X}\frac{|f(\vec{\gamma}_{(y,v)}(-\tau_-(y,v)))|}{d(x,y)^{n-1}}\,dv\,dy \qquad \text{by (3.6)} \\
&\quad\le (C_{\kappa_M})^{n-1}\|k\|_\infty\|m\|_\infty\Big(\int_{\Omega X}\frac{1}{d(x,y)^{(n-1)p}}\,dv\,dy\Big)^{\frac{1}{p}} \\
&\qquad\times\Big(\int_{\Omega X}|f(\vec{\gamma}_{(y,v)}(-\tau_-(y,v)))|^q\,dv\,dy\Big)^{\frac{1}{q}} \qquad \text{by Hölder's inequality} \\
&\quad\le (C_{\kappa_M})^{n-1}\|k\|_\infty\|m\|_\infty|\mathbb{S}^{n-1}|^{1/p}C_d^{1/p}(\text{diam}X)^{1/q}\|f\|_{L^q(\Gamma_-,d\mu)} \qquad \text{by (3.10)} \\
&\quad= C\|k\|_\infty\|m\|_\infty(\text{diam}X)^{1/q}\|f\|_{L^q(\Gamma_-,d\mu)}, \text{ say.}
\end{aligned}$$

The terms for $j \geq 2$ are slightly different from the above; first when $j = 2$,

$$\begin{aligned}
&\Big|\int_{\Gamma_-} \alpha_2(x,x',v') f(x',v')\, d\mu(x',v')\Big| \\
&\quad = \Big|\int_{\Gamma_-} \int_0^{\tau_+(x',v')} \int_X E(\cdot)^2 k(\cdot)^2 m(\cdot) J_{y_2}(\gamma_{(x',v')}(t)) J_x(y_2)\, dt\, f(x',v')\, d\mu(x',v')\Big| \\
&\quad \leq (C_{\kappa_M})^{2n-2} \|m\|_\infty \|k\|_\infty^2 \int_X \int_{\Omega_{y_1}X} \int_X \frac{\big|f\big(\vec{\gamma}_{(y_1,v_1)}(-\tau_-(y_1,v_1))\big)\big|}{d(y_1,y_2)^{n-1} d(y_2,x)^{n-1}}\, dy_2\, dv_1\, dy_1 \\
&\quad = (C_{\kappa_M})^{2n-2} \|m\|_\infty \|k\|_\infty^2 \int_X \int_{\Omega_{y_1}X} \big|f\big(\vec{\gamma}_{(y_1,v_1)}(-\tau_-(y_1,v_1))\big)\big| (\mathcal{T}h)(y_1)\, dv_1\, dy_1
\end{aligned}$$

where $h(y_2) = d(y_2,x)^{1-n}$,

$$\begin{aligned}
&\leq (C_{\kappa_M})^{2n-2} \|m\|_\infty \|k\|_\infty^2 \Big(\int_{\Omega X} |(\mathcal{T}h)(y_1)|^p\, dv_1\, dy_1\Big)^{\frac{1}{p}} \\
&\qquad \times \Big(\int_{\Omega X} \big|f\big(\vec{\gamma}_{(y_1,v_1)}(-\tau_-(y_1,v_1))\big)\big|^q\, dv_1\, dy_1\Big)^{\frac{1}{q}} \qquad \text{by Hölder's inequality,} \\
&\leq (C_{\kappa_M})^{2n-2} \|m\|_\infty \|k\|_\infty^2 |\mathbb{S}^{n-1}|^{1/p} \|\mathcal{T}h\|_{L^p(X)} (\text{diam}X)^{1/q} \|f\|_{L^q(\Gamma_-,d\mu)} \\
&\leq (C_{\kappa_M})^{2n-2} \|m\|_\infty \|k\|_\infty^2 |\mathbb{S}^{n-1}|^{1/p} \|(C_{\kappa_m})^{n-1} |\mathbb{S}^{n-1}| (\text{diam}X) C_d^{1/p} \\
&\qquad \times (\text{diam}X)^{1/q} \|f\|_{L^q(\Gamma_-,d\mu)}
\end{aligned}$$

by (3.10). Finally, for any $j \geq 2$,

$$\begin{aligned}
&\Big|\int_{\Gamma_-} \alpha_j(x,x',v') f(x',v')\, d\mu(x',v')\Big| \\
&\quad = \Big|\int_{\Gamma_-} \int_0^{\tau_+(x',v')} \int_X \cdots \int_X E(\cdot)^{j+1} k(\cdot)^j \mathcal{J}(\gamma_{(x',v')}(t), y_2, \ldots, y_j, x) \\
&\qquad \times m(\cdot)\, dy_j \cdots dy_2\, dt\, f(x',v')\, d\mu(x',v')\Big| \\
&\quad \leq (C_{\kappa_M})^{j(n-1)} \|m\|_\infty \|k\|_\infty^j \\
&\qquad \times \int_X \int_{\Omega_{y_1}} \int_X \cdots \int_X \frac{\big|f\big(\vec{\gamma}_{(y_1,v_1)}(-\tau_-(y_1,v_1))\big)\big|}{d(y_j,x)^{n-1} \prod_{l=1}^{j-1} d(y_l,y_{l+1})^{n-1}}\, dy_j \cdots dy_2\, dv_1\, dy_1 \\
&\quad = (C_{\kappa_M})^{j(n-1)} \|m\|_\infty \|k\|_\infty^j \int_X \int_{\Omega_{y_1}} \big|f\big(\vec{\gamma}_{(y_1,v_1)}(-\tau_-(y_1,v_1))\big)\big| (T^{j-1}h)(y_1)\, dv_1\, dy_1
\end{aligned}$$

where $h(y) = d(y,x)^{1-n}$,

$$\begin{aligned}
&\le (C_{\kappa_M})^{j(n-1)}\|m\|_\infty\|k\|_\infty^j|\mathbb{S}^{n-1}|^{1/p}\|T^{j-1}h\|_{L^p(X)} \\
&\quad\times\Big(\int_{\Omega X}\big|f\big(\vec{\gamma}_{(y_1,v_1)}(-\tau_-(y_1,v_1))\big)\big|^q\,dv_1\,dy_1\Big)^{\frac{1}{q}} \\
&\le (C_{\kappa_M})^{j(n-1)}\|m\|_\infty\|k\|_\infty^j\big[(C_{\kappa_m})^{n-1}(\mathrm{diam}X)|\mathbb{S}^{n-1}|\big]^{j-1}C_d^{1/p} \\
&\quad\times(\mathrm{diam}X)^{1/q}\|f\|_{L^q(\Gamma_-,d\mu)} \\
&= C\big[\|k\|_\infty(C_{\kappa_M}C_{\kappa_m})^{n-1}(\mathrm{diam}X)|\mathbb{S}^{n-1}|\big]^{j-1}\|f\|_{L^q(\Gamma_-,d\mu)}
\end{aligned}$$

where $C = \big[C_d^{1/p}(\mathrm{diam}X)^{1/q}(C_{\kappa_M})^{n-1}\|m\|_\infty\|k\|_\infty\big]$. Our smallness assumption on k ensures that the sum converges, and we have the claim of the proposition. □

Before proceeding to Theorem 3.5.1 we must construct an appropriate approximate identity to use as our in-going boundary flux. Let $x \in X$ be fixed; $\Omega_x X$ is the unit sphere in $T_x X$ with respect to the metric $g(x)$ on $T_x X$. Endow $\Omega_x X$ with the metric induced from the inner product on $T_x X$. Fix $v \in \Omega_x X$ and let $\exp_v : T_v\Omega_x X \to \Omega_x X$ be the exponential map for $\Omega_x X$ based at v. If $\hat{w} = \hat{w}(w) = \exp_v^{-1}(w)$ let $\mathcal{J}_{(x,v)}(\hat{w})$ be the determinant of the Jacobian of this change of variables.

Let $\varphi \in C_c^\infty(T_v\Omega_x X)$ be such that $0 \le \varphi \le 1$, $\varphi(0) = 1$, $\varphi(\hat{w}) = 0$ for $|\hat{w}| \ge \varepsilon_0$ (ε_0 fixed and sufficiently small), and $\int_{\mathbb{R}^{n-1}}\varphi(\hat{w})\,d\hat{w} = 1$. Define $\psi_\eta : \Omega_x X \to \mathbb{R}$ by

$$\psi_\eta(w) = \frac{1}{\eta^{n-1}}\varphi\Big(\frac{\exp_v^{-1}(w)}{\eta}\Big).$$

Note that if $f : \Omega_x X \to \mathbb{R}$ is continuous at v then

$$\begin{aligned}
\int_{\Omega_x X} f(w)\psi_\eta(w)\,dw &= \int_{\mathbb{R}^{n-1}} f(\exp_v(\hat{w}))\frac{1}{\eta^{n-1}}\varphi\Big(\frac{\hat{w}}{\eta}\Big)\mathcal{J}_{(x,v)}(\hat{w})\,d\hat{w} \\
&\to f(\exp_v(0))\mathcal{J}_{(x,v)}(0) = f(v)
\end{aligned}$$

as $\eta \to 0$. Now we will use such functions to construct $f_\eta \in L^1(\Gamma_-)$. Fix a point $(x^*,v^*) \in \Gamma_+$. Define $\psi_{x^*,\eta} : \Omega_{x^*}^+ X \to \mathbb{R}$ by

$$\psi_{x^*,\eta}(v) = \frac{1}{\eta^{n-1}}\varphi\Big(\frac{\exp_{v^*}^{-1}(v)}{\eta}\Big).$$

Let $\rho \in C_c^\infty(\mathbb{R})$ be such that $0 \le \rho \le 1$, $\rho(0) = 1$, and $\rho(t) = 0$ for $|t| \ge \varepsilon_0$; for $\lambda > q - 1$ define $\chi_\eta : \partial X \to \mathbb{R}$ by

$$\chi_\eta(x) = \rho\Big(\frac{d_{\partial X}(x, x^*)}{\eta^\lambda}\Big)$$

where $d_{\partial X}(x, y)$ is distance in the boundary of X. Now if we denote by $\mathcal{P}(v; x, x^*)$ the parallel translation of v along the geodesic (in X) joining x to x^*, then we can define $\psi_{x,\eta} : \Omega_x X \to \mathbb{R}$ by

$$\psi_{x,\eta}(v) = \psi_{x^*,\eta}(\mathcal{P}(v; x, x^*)).$$

This is simply a smooth way of extending the definition of $\psi_{x^*,\eta}$ from $\Omega^+_{x^*} X$ to $\Omega_x X$ for x near x^*. Finally, define $f_\eta \in L^1(\Gamma_-)$ by

$$f_\eta(x', v') = \frac{1}{m\big(\vec{\gamma}_{(x',v')}(\tau_+(x', v'))\big)} \chi_\eta\big(\gamma_{(x',v')}(\tau_+(x', v'))\big) \psi_{\gamma_{(x',v')}(\tau_+(x',v')),\eta}\big(\dot{\gamma}_{(x',v')}(\tau_+(x', v'))\big). \tag{3.14}$$

In the following theorem we show that $\mathcal{M}$ determines the integral of σ along any given geodesic of (X, g) with data in $\mathcal{H}_\sigma$, and hence the geodesic x-ray transform of σ, restricted to these geodesics. By definition of $\mathcal{H}_\sigma$, this uniquely determines the function σ, and proves Theorem 3.2.1.

Theorem 3.5.1. *Let (σ, k, m) satisfy the hypothesis of Theorem 3.2.1. Let x^* be such that Theorem 3.4.1 holds at x^* (this happens* a.e.*), let $v^* \in \Omega^+_{x^*} X$ be fixed, and assume that $(x^*, v^*) \in \mathcal{H}_\sigma$; let $f_\eta \in L^1(\Gamma_-)$ be as defined in* (3.14) *above. Then*

$$\lim_{\eta \to 0} \mathcal{M} f_\eta(x^*) = E\big(\gamma_{(x^*,v^*)}(-\tau_-(x^*, v^*)), x^*\big).$$

Proof. From Theorem 3.4.1 we have $\mathcal{M} f_\eta = \sum_{j=0}^\infty \int_{\Gamma_-} \alpha_j f_\eta \, d\mu$. When $j = 0$,

$$\begin{aligned}
&\int_{\Gamma_-} \alpha_0(x^*, x', v') f_\eta(x', v') \, d\mu \\
&\qquad = \int_{\Omega^+_x X} E\big(x^*, \gamma_{(x^*,v)}(-\tau_-(x^*, v))\big) f_\eta\big(\vec{\gamma}_{(x^*,v)}(-\tau_-(x^*, v))\big) \, dv \\
&\qquad = \int_{\Omega^+_{x^*} X} E\big(x^*, \gamma_{(x^*,v)}(-\tau_-(x^*, v))\big) \chi_\eta(x^*) \psi_{x^*,\eta}(v) \, dv \\
&\qquad \to E\big(\gamma_{(x^*,v^*)}(-\tau_-(x^*, v^*)), x^*\big)
\end{aligned}$$

as $\eta \to 0$. It remains to show that the rest of the series tends to zero. First,

$$\|f_\eta\|^q_{L^q(\Gamma_-,d\mu)} = \int_{\Gamma_-} |f_\eta(x',v')|^q \, d\mu = \int_{\Gamma_+} |f_\eta(\vec{\gamma}_{(x,v)}(-\tau_-(x,v))|^q \, d\mu$$

where we have changed variables to $(x,v) = \vec{\gamma}_{(x',v')}(\tau_+(x',v'))$; it is easy to check that the volume elements are the same,

$$\begin{aligned}
&= \int_{\partial X}\int_{\Omega_x^+ X} \chi_\eta(x)^q \psi_{x,\eta}(v)^q \frac{|\langle v,\nu_x\rangle|}{|m(x,v)|^q} \, dv \, dx \\
&\le C \int_{\partial X}\int_{\Omega_x^+ X} \chi_\eta(x)^q \psi_{x,\eta}(v)^q \, dv \, dx \\
&= C \int_{\partial X} \chi_\eta(x)^q \int_{\Omega_x^+ X} \frac{1}{\eta^{(n-1)q}} \varphi\Big(\frac{\exp_{v^*}^{-1}(\mathcal{P}(v;x,x^*))}{\eta}\Big)^q \, dv \, dx
\end{aligned}$$

where we use the fact that $1/m$ is bounded. In a sufficiently small neighborhood $N \subset \partial X$ of x^*, we shall make the change of variables $(x,v) \mapsto (x,\hat{w}_x)$ where $(x,v) \in \cup_{x\in N}\Omega_x X$, defined by

$$(x,\hat{w}_x) = \big(x, \exp_{v^*}^{-1}(\mathcal{P}(v;x,x^*))\big) \in \partial X \times \mathbb{R}^{n-1}$$

where $\exp_{v^*} : T_{v^*}\Omega_{x^*}X \to \Omega_{x^*}X$. This is a smooth map and so the Jacobian determinant is bounded on N by M, say. We thus have

$$\begin{aligned}
\|f_\eta\|^q_{L^q(\Gamma_-,d\mu)} &\le CM \int_{\partial X} \chi_\eta(x)^q \int_{\mathbb{R}^{n-1}} \frac{1}{\eta^{(n-1)q}} \varphi\Big(\frac{\hat{w}_x}{\eta}\Big)^q \, d\hat{w}_x \, dx \\
&= CM \int_{\partial X} \chi_\eta(x)^q \int_{\mathbb{R}^{n-1}} \frac{1}{\eta^{(n-1)(q-1)}} \varphi(\hat{w})^q \, d\hat{w} \, dx \\
&\le M' \frac{1}{\eta^{(n-1)(q-1)}} \int_{\partial X} \rho\Big(\frac{d_{\partial X}(x,x*)}{\eta^\lambda}\Big)^q \, dx \\
&\le M' \frac{1}{\eta^{(n-1)(q-1)}} \operatorname{Vol}_{\partial X}(\{x \in \partial X : d_{\partial X}(x,x*) < \varepsilon_0 \eta^\lambda\}) \\
&\le M' \frac{1}{\eta^{(n-1)(q-1)}} \operatorname{Vol}_{\mathbb{R}^{n-1}}(\{|x| < \varepsilon_0\eta^\lambda\}) \\
&= \frac{M'|\mathbb{S}^{n-2}|\varepsilon_0^{n-1}}{n-1} \eta^{\lambda(n-1)-(n-1)(q-1)} \to 0
\end{aligned}$$

as $\eta \to 0$ for any $\lambda > q-1$. The last inequality follows from the fact that the curvature of the boundary is positive and so the volume of a ball of radius r is bounded above by the volume of a ball of radius r in the constant zero curvature Euclidean space. This is a corollary to Bishop's volume comparison theorem. (See [39], Corollary 3.2, Section 4.)

To complete the proof of the theorem, by Proposition 3.5.1, we have

$$\Big|\sum_{j=1}^{\infty}\int_{\Gamma_-}\alpha_j(x,x',v')f_\eta(x',v')\,d\mu(x',v')\Big| \le C_0\|f_\eta\|_{L^q(\Gamma_-,d\mu)} \to 0$$

as $\eta \to 0$ from the above computation. □

3.6 Recovering the scattering kernel

Throughout this section, the measurement point x will be fixed. We assume here that the scattering kernel is less general than in the previous section. Precisely, we assume that it is of the form $k(x)\Theta(x,v',v)$, where $\Theta(x,v',v)$ is *a-priori* known. We prove in this setting that the spatial distribution $k(x)$ is uniquely determined by the averaged albedo operator $\mathcal{M}$, and in fact prove that this is so from knowledge of measurements at the single fixed measurement point x.

Definition 3.6.1. Given a complete Riemannian manifold (X,g) with geodesics $\gamma_{(x,v)}(t)$, and functions $\eta : \Omega X \to \mathbb{R}$, and $\beta \in C^\infty(\Gamma_-)$ we may define the *weighted geodesic transform* by, for $f : X \to \mathbb{R}$,

$$I_{\eta,\beta}f(x',v') := \beta(x',v')\int_0^{\tau_+(x',v')} f(\gamma_{(x',v')}(t))\eta(\vec{\gamma}_{(x',v')}(t))\,dt. \tag{3.15}$$

We also have the L^2 adjoint, for $f : \Gamma_- \to \mathbb{R}$,

$$I^*_{\eta,\beta}f(x) = \int_{\Omega_x X} f\big(\gamma_{(x,v)}(-\tau_-(x,v))\big)\beta\big(\gamma_{(x,v)}(-\tau_-(x,v))\big)\eta(x,v)\,dv. \tag{3.16}$$

Under appropriate assumptions to be stated shortly, the kernel α_1 which represents single scattering is a weighted x-ray transform of the unknown function $k(x)$. We make use of the injectivity results of [23] for such an x-ray transform to prove unique identifiability of k. The results of [23] require a sufficiently rich set of curves (geodesics in our case) along which the integral transform is known. The inclusion of the factor β in Definition 3.15 essentially serves the purpose of restricting the transform to a (possibly proper) subset of geodesics. This is made clearer in the following definition:

Definition 3.6.2. We say that Γ is a *regular* family of curves (for the metric g) if for any $(x,v) \in T^*X\backslash\{0\}$ there exists $\gamma \in \Gamma$ through x, normal to v, and such that γ has no conjugate points.

We say that $\beta \in C^\infty(\Gamma_-)$ is *regular* if there exists a set $\mathcal{H} \subset \{(x',v') \,:\, \beta(x',v') \neq 0\}$ such that $\Gamma(\mathcal{H})$ is a regular family.

Note that in our setting of simplicity of (X,g), all geodesics are without conjugate points. In dimension $n=2$, one must take *all* geodesics of X in order to satisfy regularity as defined in Definition 3.6.2; when $n \geq 3$, one may take proper subsets of the set of all geodesics and still satisfy regularity.

The following two theorems are proven (in greater generality) in [23].

Theorem 3.6.1. *([23]) Let $\beta \in C^\infty(\Gamma_-)$ be a regular function for the real-analytic metric g, as in Definition 3.6.2. Suppose that $\eta : \Omega X \to \mathbb{R}$ is real-analytic and non-vanishing on U where $U \subset \bar{U} \subset X$. Then $I_{\eta,\beta}$ is injective on $\mathcal{D}'(U)$.*

Theorem 3.6.2. *([23]) Let β and U be as in Theorem 3.6.1, and let $\eta \in C^\infty(\Omega X)$. Then we may find a constant C such that:*

(a) If $I_{\eta,\beta}$ is injective on $L^2(U)$, then

$$\frac{1}{C}\|f\|_{L^2(U)} \leq \|I^*_{\eta,\beta} I_{\eta,\beta} f\|_{H^1(X)} \leq C\|f\|_{L^2(U)}.$$

(b) There exists a C^2 neighborhood of (η,β), and a C^∞ neighborhood of g on which the above estimate remains true, with a uniform constant C.

To apply these results to our problem, we first recall that that $k(x)\Theta(x,v,v')$ and $\tilde{k}(x)\Theta(x,v,v')$ are two scattering kernels with $k,\tilde{k} \in \mathcal{K}^D_\varepsilon$; i.e. $\text{dist}(\text{supp}(k),\partial X) > D$ and $\|k\|_{L^\infty(X)} \leq \varepsilon$, and similarly for $\tilde{k}$. Let $\mathcal{M}$, $\widetilde{\mathcal{M}}$, and α, $\tilde{\alpha}$ be the averaged albedo operators and Schwarz kernels associated to k and $\tilde{k}$ respectively. We set $\Delta k = k - \tilde{k}$ and $\Delta\alpha_j = \alpha_j - \tilde{\alpha}_j$, $j = 1,2,\ldots$.

For an appropriately defined weight w, we have $\alpha_1(x,x',v') = I_{w,1}k(x',v')$. To this end, let $\chi \in C^\infty_c(X)$ with $\chi \equiv 1$ on $\{y \in X \,:\, \text{dist}(y,\partial X) > D\}$. If $x_1 \in X$, let $\bar{v} = \bar{v}(x_1) \in \Omega_{x_1}X$ and $v = v(x_1) \in \Omega^+_x X$ be the initial and final tangent vectors, respectively, of the geodesic joining x_1 to x. Then for $v_1 \in \Omega_{x_1}X$ define the weight function

$$w(x_1,v_1) := \Theta(x_1,v_1,\bar{v})E\big(\gamma_{(x_1,v_1)}(-\tau_-(x_1,v_1)),x_1,x\big)m(x,v)J_x(x_1)\chi(x_1). \tag{3.17}$$

With this definition, we see from (3.8) and (3.15) that indeed $\alpha_1(x, x', v') = I_{w,1}k(x', v')$. Assuming that the metric g is real-analytic, J_x is then real-analytic; if further Θ, σ and m are real-analytic, then w is a real-analytic, non-vanishing weight function in $\{y \in X : \mathrm{dist}(y, \partial X) > D\}$. If $\beta \in C^\infty(\Gamma_-)$ is a regular function for g then Theorems 3.6.1 and 3.6.2 give us:

Lemma 3.6.1. *Let (m, Θ, σ, g) be fixed and real analytic, and suppose $\beta \in C^\infty(\Gamma_-)$ is a regular function for g. Then there exits $C > 0$, independent of $k, \tilde{k} \in \mathcal{K}_\varepsilon^D$ such that*

$$\|\Delta k\|_{L^2(X)} \leq C\|I^*_{w,\beta} I_{w,\beta} \Delta k\|_{H^1(X)}, \tag{3.18}$$

with the above estimate holding in a C^2 neighborhood of (m, Θ, σ), and a C^∞ neighborhood of g.

Proposition 3.6.1. *Let (m, Θ, σ, g) be fixed and real analytic, and suppose $\beta \in C^\infty(\Gamma_-)$ is a regular function for g. Furthermore, suppose $\mathcal{H}_k \subset \{(x', v') : \beta(x', v') \neq 0\}$ be such that $\Gamma(\mathcal{H}_k)$ is a regular set of geodesics. Suppose that $\mathcal{M} = \widetilde{\mathcal{M}}$ on $L^1(\mathcal{H}_k, d\mu)$. Then there exists $C > 0$ such that for all $k, \tilde{k} \in \mathcal{K}_\varepsilon^D$*

$$\|\Delta k\|_{L^2(X)} \leq C\|I^*_{w,\beta} I_{w,\beta} \Delta k\|_{H^1(X)} = C\|I^*_{w,\beta} \beta \Delta\alpha_1\|_{H^1(X)} = C\left\|I^*_{w,\beta} \beta \sum_{j=2}^{\infty} \Delta\alpha_j\right\|_{H^1(X)},$$

with the above estimate holding in a C^2 neighborhood of (m, σ, Θ) and a C^∞ neighborhood of g.

Proof. The first inequality is (3.18); next, $\beta I_{w,1} = I_{w,\beta}$ and w has been defined so that $I_{w,1}\Delta k = \Delta\alpha_1$; finally, since $\sigma = \tilde{\sigma}$, $\alpha_0 = \tilde{\alpha}_0$ and so $\mathcal{M} = \widetilde{\mathcal{M}}$ implies that $\sum_{j=1}^{\infty} \Delta\alpha_j = 0$ which proves the final equality. $\square$

In order to prove unique identifiability of k we proceed to prove that $\|\Delta k\|_{L^2(X)} \leq \varepsilon C\|\Delta k\|_{L^2(X)}$, and so for sufficiently small $\varepsilon > 0$, we will have $\Delta k = 0$. We begin with Proposition 3.6.2 which is very similar to Proposition 4 in [23]. However, to use that result here we would have to assume that $(\eta_1, \beta_1, \eta_2, \beta_2)$ were all small.

Proposition 3.6.2. *There is $C > 0$ such that for all $f \in L^2(X)$ with $\operatorname{supp} f \subset \{y \in X : d(x, \partial X) > D\}$,*

$$\|I^*_{\eta_1,\beta_1} I_{\eta_2,\beta_2} f\|_{H^1(X)} \leq C\|\eta_1\eta_2\|_{C^2(\Omega X)}\|\beta_1\beta_2\|_{C^2(\Gamma_-)}\|f\|_{L^2(X)}, \tag{3.19}$$

with the above estimate holding in a C^∞ neighborhood of g.

Proof. We will compute in a fixed global coordinate system $\hat{x} = \{\hat{x}^1, \ldots, \hat{x}^n\}$ for X. Let $\hat{v} = \{\hat{v}_1, \ldots, \hat{v}_n\} = \{\partial_{\hat{x}^1}, \ldots, \partial_{\hat{x}^n}\}$ be the naturally induced coordinates on $T_x X$. We define C^s and H^s norms with respect to this fixed coordinate system.

We shall also need to change to coordinates in which for a fixed $x \in X$, $\Omega_x X$ is $\mathbb{S}^{n-1}$. To this end, let $x \in X$ and given $v \in \Omega_x X$ and $0 \leq t \leq \tau_+(x,v)$ let $\hat{y} = \hat{y}(t, v; x) \in \mathbb{R}^n$ be the coordinate expression for $y = \exp_x(tv)$. Define $\omega \in \mathbb{S}^{n-1}$ by $\omega = \omega(t, v; x) = (\hat{y} - \hat{x})/|\hat{y} - \hat{x}|$ and $r = r(t, v; x) \geq 0$ by $r = |\hat{y} - \hat{x}|$. If $\hat{v}(v)$ is the coordinate expression of v, we define $\omega(0, v) = \hat{v}/|\hat{v}|$. We claim that the map $(t, v) \mapsto (r, \omega)$ is smooth, including at $t = 0$, and that for sufficiently small $t > 0$ it is a diffeomorphism onto its range. We demonstrate this latter property by showing that at $t = 0$ the differential is of full rank.

To see that the map is smooth, it is convenient (as in [23]) to define $m(t, v; x) := (\hat{y}(t, v) - \hat{x})/t$. Then it holds that

$$m(0, v; x) := \lim_{t \to 0} m(t, v; x) = \hat{v} \neq 0$$

on $\Omega_x X$, and further, expanding $\hat{y}$ in a Taylor series about $t = 0$,

$$m(t, v; x) = \frac{1}{t}(\hat{y}(t, v) - \hat{x}) = \frac{1}{t}(t\hat{v} + t^2 Y(t, v; x)) = \hat{v} + tY(t, v; x)$$

with $Y(t, v; x) \in C^\infty$, and so $m(t, v; x)$ is smooth and never vanishes (at least in a neighborhood of $t = 0$). It follows that

$$\omega(t, v; x) = \frac{\hat{y} - \hat{x}}{|\hat{y} - \hat{x}|} = \frac{m(t, v; x)}{|m(t, v; x)|}$$

is smooth, including at $t = 0$. Again, since m never vanishes, $r = t|m(t, v; x)|$ is smooth, including at $t = 0$.

We now demonstrate that at $t = 0$ (and hence in a neighborhood) the map is of full rank. In coordinates, the map $\{\sum_{ij} \hat{v}_i g_{ij}(x) \hat{v}_j = 1\} \ni \hat{v} \mapsto \omega \in \mathbb{S}^{n-1}$ is given by $\omega := g^{1/2}\hat{v}$ where for every x, $g^{1/2}(x)$ is a square root of the matrix $g(x)$. We denote by $\frac{\partial \omega}{\partial v}$ the differential of this map, and use the same notation for the $(n-1) \times (n-1)$ matrix representation in a choice of coordinates. Next,

$$\frac{\partial r}{\partial t}\Big|_{t=0} = \frac{\partial}{\partial t}\Big|_{t=0} t|m(t,v;x)| = |\hat{v}| + \frac{\partial}{\partial t}|m(t,v;x)|\Big|_{t=0} = |\hat{v}|.$$

And finally

$$\frac{\partial r}{\partial v}\Big|_{t=0} = \frac{\partial}{\partial v}\Big|_{t=0} t|m(t,v;x)| = 0.$$

Combining these, in in the $(\hat{x}, \hat{v})$ coordinate system,

$$\frac{\partial(\omega, r)}{\partial(v,t)}\Big|_{t=0} = \begin{pmatrix} \frac{\partial \omega}{\partial v} & \frac{\partial \omega}{\partial t} \\ 0 & |\hat{v}| \end{pmatrix}$$

is full-rank. Thus, $|\det \frac{\partial(\omega,r)}{\partial(v,t)}| \neq 0$ at $t = 0$ and there exists $\varepsilon_1(x)$ such that for $0 \leq t < \varepsilon_1(x)$ the change of variables $\psi : (t,v) \mapsto (r,\omega)$ is a diffeomorphism onto its range. However, the domain in $\mathbb{R}^n$ described by the polar coordinates $(r,w) \in \psi([0,\varepsilon_1(x)) \times \mathbb{S}^{n-1})$ need not be star-shaped with respect to the origin; or put another way, if $(r_0,\omega_0) \in \psi([0,\varepsilon_1(x)) \times \mathbb{S}^{n-1})$, it is not necessarily true that the same holds for all $0 \leq r \leq r_0$. But there does exist $r_m(x) > 0$ such that for all $\omega \in \mathbb{S}^{n-1}$ and $0 \leq r \leq r_m(x)$ we have $(r,\omega) \in \psi([0,\varepsilon_1(x)) \times \mathbb{S}^{n-1})$. Let $r_m(x)$ be the largest such radius for which this holds. Then it is clear that there exists $0 < R_m \leq r_m(x)$ for all x; indeed, let $\{B_x\}$ be an open cover of the compact set $\{x \in X : \operatorname{dist}(x, \partial X) \geq D/2\}$ by balls of radius $r_m(x)$, and let R_m be the Lebesgue number associated with that cover. We then have that any ball of radius $\leq R_m$ is contained in one member of this cover, and therefore in a convex neighborhood which is the image of the diffeomorphism ψ.

With these preparations complete, let f be supported as in the statement of the proposition. Then, re-parameterizing in the t variable,

$$\begin{aligned} I^*_{\beta_1,\eta_1} I_{\beta_2,\eta_2} f(x) &= \int_{\Omega_x X} \beta_1\big(\vec{\gamma}_{(x,v)}(-\tau_-(x,v))\big)\eta_1(x,v)\beta_2\big(\vec{\gamma}_{(x,v)}(-\tau_-(x,v))\big) \\ &\qquad \times \int_{-\tau_-(x,v)}^{\tau_+(x,v)} \eta_2(\vec{\gamma}_{(x,v)}(t)) f(\gamma_{(x,v)}(t))\, dt\, dv \\ &= \int_{\Omega_x X} \int_{-\tau_-(x,v)}^{\tau_+(x,v)} A(x,t,v) f(\gamma_{(x,v)}(t))\, dt\, dv = (I_1), \text{ say,} \end{aligned}$$

with

$$A(x,t,v) := \eta_1(x,v)\eta_2(\vec{\gamma}_{(x,v)}(t))\beta_1\big(\vec{\gamma}_{(x,v)}(-\tau_-(x,v))\big)\beta_2\big(\vec{\gamma}_{(x,v)}(-\tau_-(x,v))\big).$$

If $A(x,t,\cdot)$ is odd, then it is easily seen that the integral vanishes and so if $A_e(x,t,v) := A(x,t,v) + A(x,t,-v)$,

$$\begin{aligned}
(I_1) &= \int_{\Omega_x X}\int_{-\tau_-(x,v)}^{\tau_+(x,v)} \frac{1}{2}A_e(x,t,v)f(\gamma_{(x,v)}(t))\,dt\,dv \\
&= \int_{\Omega_x X}\int_{0}^{\tau_+(x,v)} A_e(x,t,v)f(\gamma_{(x,v)}(t))\,dt\,dv \\
&= \int_{\Omega_x X}\int_{0}^{\tau_+(x,v)} \chi(t)A_e(x,t,v)f(\gamma_{(x,v)}(t))\,dt\,dv \\
&\quad + \int_{\Omega_x X}\int_{0}^{\tau_+(x,v)} (1-\chi(t))A_e(x,t,v)f(\gamma_{(x,v)}(t))\,dt\,dv = (I_2) + (I_3), \quad \text{say.}
\end{aligned}$$

Here, $\chi(t) \in C^\infty(\mathbb{R})$, $\chi(0) = 1$, and is supported in $[0,\delta)$ with $\delta < \min\{R_m, D\}$. Rewriting (I_3) in terms of spatial variables,

$$(I_3) = \int_X \big(1 - \chi(d(x,y))\big)A_e(x,d(x,y),v(x,y))f(y)J_x(y)\,dy$$

where $v(x,y)$ is the initial tangent vector of the geodesic joining x to y, and $\mathcal{J}_x(y)$ is the Jacobian determinant of the change of variables. We trivially have

$$\|(I_3)\|_{H^1(X)} \le \sup_{x,y\in X,\ d(x,y)\ge\delta} \sum_{|\alpha|\le 1} \big|\partial_{\hat{x}}^\alpha A_e(\hat{x}, d(\hat{x},\hat{y}), \hat{v}(\hat{x},\hat{y}))J_{\hat{x}}(\hat{y})\big|\|f\|_{L^2(X)}. \tag{3.20}$$

Note that the Jacobian $J_{\hat{x}}$ depends continuously on the magnitude of Jacobi fields, which in turn depend continuously on one derivative of the Christoffel symbols, which in turn depend on one derivative of the metric. To treat (I_2), we express the integral in terms of our chosen coordinates. Let $\hat{y}$ denote the coordinate form of $\gamma_{(x,v)}(t)$, $d\hat{v}$ be the volume form dv expressed in these coordinates, and $\hat{\mathbb{S}} = \{\hat{v} \,:\, \hat{v}^i g_{ij}(x)\hat{v}^j = 1\} \subset \mathbb{R}^n$. Then

$$(I_2) = \int_{\hat{\mathbb{S}}}\int_0^{\hat{\tau}_+(x,\hat{v})} \chi(t)A_e(\hat{x},t,\hat{v})f(\hat{y})\,dt\,d\hat{v}.$$

Changing coordinates to transform $\hat{\mathbb{S}}$ to $\mathbb{S}^{n-1}$ and letting $d\omega$ be the standard volume form on $\mathbb{S}^{n-1}$, we have

$$\begin{aligned}
(I_2) &= \int_{\mathbb{S}^{n-1}}\int_0^{R_m} \chi(t)A_e(\hat{x},t,\hat{v})f(\hat{x}+r\omega)J_2(\hat{x},t,\hat{v})\big|_{t=t(\hat{x},r,\omega),\ \hat{v}=\hat{v}(\hat{x},r,\omega)}\,dr\,d\omega \\
&= \int_{\mathbb{S}^{n-1}}\int_0^{\infty} G_e(\hat{x},r,\omega)f(\hat{x}+r\omega)\,dr\,d\omega
\end{aligned}$$

where

$$G_e = \frac{1}{2}\big(G(\hat{x}, r, \omega) + G(\hat{x}, r, -\omega)\big),$$

$$G(\hat{x}, r, \omega) = \chi(t) A_e(\hat{x}, t, \hat{v}) J_2(\hat{x}, t, \hat{v})\big|_{t=t(\hat{x},r,\omega),\ \hat{v}=\hat{v}(\hat{x},r,\omega)}.$$

Expanding in a Taylor series, $G_e(\hat{x}, r, \omega) = G_0(\hat{x}, \omega) + r\, G_1(\hat{x}, r, \omega)$. We are left needing to estimate $(I_2) = (I_4) + (I_5)$ where

$$(I_4) = \int_{\mathbb{S}^{n-1}} \int_0^\infty G_0(x, \omega) f(\hat{x} + r\omega)\, dr\, d\omega = \int_{\mathbb{R}^n} G_0\Big(\hat{x}, \frac{\hat{y} - \hat{x}}{|\hat{y} - \hat{x}|}\Big) \frac{f(\hat{y})}{|\hat{y} - \hat{x}|^{n-1}}\, d\hat{y},$$

and

$$(I_5) = \int_{\mathbb{S}^{n-1}} \int_0^\infty G_1(x, r, \omega) f(\hat{x} + r\omega)\, dr\, d\omega = \int_{\mathbb{R}^n} G_1\Big(\hat{x}, |\hat{y} - \hat{x}|, \frac{\hat{y} - \hat{x}}{|\hat{y} - \hat{x}|}\Big) \frac{f(\hat{y})}{|\hat{y} - \hat{x}|^{n-2}}\, d\hat{y}.$$

Derivatives of the kernel of (I_4) yield a non-integrable kernel; however the resulting integral makes sense as a principal value integral, and the mapping properties of such a singular integral operator are understood and studied in [31] to which we will refer for all relevant results. We take care of (I_4) in detail after first noticing that (I_5) causes no problems. We may differentiate inside the integral in (I_5) obtaining

$$\frac{\partial}{\partial \hat{x}^i}(I_5) = \int_{\mathbb{R}^n} \Big(\big(\frac{\partial}{\partial \hat{x}^i} G_1(\cdot)\big) \frac{f(\hat{y})}{|\hat{y} - \hat{x}|^{n-2}} + G_1(\cdot) \frac{f(\hat{y})(\hat{y}^i - \hat{x}^i)}{|\hat{y} - \hat{x}|^n} \Big)\, d\hat{y}$$

from which we have (in a manner similar to our bounds on $\mathcal{T}$, see definition 3.5.1)

$$\|(I_5)\|_{H^1(X)} \le C \|G_1\|_{C^1(\Omega X)} \|f\|_{L^2(X)}. \tag{3.21}$$

Now G_1 consists of derivatives of order one of G, and hence of A_e. From this we obtain the estimate (3.19) for (I_5). Notice that the constant C will exhibit continuous dependence on g with respect to the C^∞ topology.

Toward (I_4) we use the notation of [31],

$$(I_4) = \int_{\mathbb{R}^n} \frac{G_0(\hat{x}, \omega)}{r^{n-1}} f(\hat{y})\, d\hat{y}$$

where $r = |\hat{y} - \hat{x}|$ and $\omega = (\hat{y} - \hat{x})/|\hat{y} - \hat{x}|$. Let D_i' denote differentiation with respect to $\hat{x}^i$, assuming that ω and r are independent of $\hat{x}^i$, and D_i'' denote differentiation assuming that *only* ω and r depend on $\hat{x}^i$. Then

$$D_i' \int_{\mathbb{R}^n} \frac{G_0(\hat{x}, \omega)}{r^{n-1}} f(\hat{y})\, d\hat{y} = \int_{\mathbb{R}^n} \big(D_i' G_0(\hat{x}, \omega)\big) \frac{f(\hat{y})}{r^{n-1}}\, d\hat{y}$$

and

$$\left\| D_i' \int_{\mathbb{R}^n} \frac{G_0(\hat{x},\omega)}{r^{n-1}} f(\hat{y})\, d\hat{y} \right\|_{L^2(X)} \le C\|G_0\|_{C^1(\Omega X)}\|f\|_{L^2(X)}. \tag{3.22}$$

Next, by [31] (Theorem 11.1, Section 11, Chapter XI),

$$\begin{aligned} &\left\| \frac{\partial}{\partial \hat{x}^i} \int_{\mathbb{R}^n} \frac{G_0(\hat{x},\omega)}{r^{n-1}} f(\hat{y})\, d\hat{y} \right\|_{L^2(X)} \\ &\qquad \le \left\| \int_{\mathbb{R}^n} \frac{D_i' G_0(\hat{x},\omega)}{r^{n-1}} f(\hat{y})\, d\hat{y} \right\|_{L^2(X)} + \left\| \int_{\mathbb{R}^n} D_i''\Big(\frac{G_0(\hat{x},\omega)}{r^{n-1}}\Big) f(\hat{y})\, d\hat{y} \right\|_{L^2(X)} \\ &\qquad\quad + C\|G_0\|_{C^1(\Omega X)}\|f\|_{L^2(X)}. \end{aligned}$$

By (3.22) the first of these two integrals is bounded by $C\|G_0\|_{C^1(\Omega X)}\|f\|_{L^2(X)}$. For the second, we observe that

$$D_i''\Big(\frac{G_0(\hat{x},\omega)}{r^{n-1}}\Big) = \frac{\tilde{G}_0(\hat{x},\omega)}{r^n}$$

with $\|\tilde{G}_0\|_{C^0(\Omega X)} \le C\|G_0\|_{C^1(\Omega X)}$ and $\int_{\mathbb{S}^{n-1}} \tilde{G}_0(\hat{x},\omega)\, d\omega = 0$ (see [31] Section 7, Chapter IX). Now

$$\int_{\mathbb{S}^{n-1}} |\tilde{G}_0(\hat{x},\omega)|^2\, d\omega \le C\|G_0\|_{C^1(\Omega X)}$$

so we may apply the Calderon-Zygmund theorem ([31], Theorem 3.1, Chapter XI) to conclude that

$$\left\| \int_{\mathbb{R}^n} D_i''\Big(\frac{G_0(\hat{x},\omega)}{r^{n-1}}\Big) f(\hat{y})\, d\hat{y} \right\|_{L^2(X)} \le C\|G_0\|_{C^1(\Omega X)}\|f\|_{L^2(X)}. \tag{3.23}$$

Combining the estimates (3.20), (3.21), (3.22) and (3.23) above, $(I_1) = (I_4) + (I_5) + (I_3)$ satisfies

$$\|(I_1)\|_{H^1(X)} \le C(\|G_0\|_{C^1} + \|G_1\|_{C^1})\|f\|_{L^2(X)} \le C\|\eta_1\beta_1\eta_2\beta_2\|_{C^2(\Omega X)}\|f\|_{L^2(X)}.$$

□

For each $i > 1$ we will write $\Delta\alpha_i$ as an infinite sum of weighted x-ray transforms; in order to have properly defined weights mapping $\Omega X \to \mathbb{R}$ we expand one instance of the kernel Θ occurring in $\Delta\alpha_i$ in a manner based on spherical harmonic expansions of functions on $\mathbb{S}^{n-1}$. We present the statement of the lemma demonstrating this expansion and postpone its proof until after we apply it to prove Proposition 3.6.3.

Lemma 3.6.2. *Let $\Theta(x,v',v) \in C^\infty(\Omega^2 X)$. There exist $\Theta_j(x,v'),\ \varphi_j(x,v) \in C^\infty(\Omega X)$ such that*

$$\Theta(x,v',v) = \sum_{j=1}^{\infty} \Theta_j(x,v')\varphi_j(x,v)$$

with

$$\|\Theta_j\|_{C^2(\Omega X)} \le \frac{C}{1+j^2}, \qquad \text{and} \qquad \|\varphi_j(x,v)\|_{L^\infty(\Omega X)} \le 1,$$

with above estimate holding in a C^∞ neighborhood of (Θ, g).

Proposition 3.6.3. *Fix (m,g,Θ,σ) with $m,\sigma \in C^2$, and $g,\Theta \in C^\infty$. Suppose that $\|k\|_\infty < [\|\Theta\|_\infty \mathrm{diam} X|\mathbb{S}^{n-1}|]^{-1}$, $k,\ \tilde{k} \in \mathcal{K}_\varepsilon^D$, and $\beta \in C^\infty(\Gamma_-)$. Then there is $C > 0$ such that*

$$\Big\| I^*_{w,\beta}\beta \sum_{i=2}^{\infty} \Delta\alpha_i \Big\|_{H^1(X)} \le C\varepsilon \|\Delta k\|_{L^2(X)},$$

with the above estimate holding in a C^2 neighborhood of (m,σ), and a C^∞ neighborhood of (Θ, g).

Proof. For $(x,v) \in \Omega X$ we define $\mathcal{E}(x,v) = E(\gamma_{(x,v)}(-\tau_-(x,v)), x)$. From Theorem 3.4.1,

$$\begin{aligned}
&\Delta\alpha_2(x,x',v') \\
&= \int_0^{\tau_+(x',v')} E(x', \gamma_{(x',v')}(t)) \int_X \big[\Delta k(\gamma_{(x',v')}(t))k(y_2) - \tilde{k}(\gamma_{(x',v')}(t))\Delta k(y_2)\big] \\
&\qquad \times \Theta(\vec{\gamma}_{(x',v')}(t), \bar{w}_1)\Theta(y_2, \hat{w}_1, \bar{w}_2) E(\gamma_{(x',v')}(t), y_2) E(y_2, x) \mathcal{J}(\gamma_{(x',v')}(t), y_2, x)\, dy_2\, dt
\end{aligned}$$

and from Lemma 3.6.2 we have (formally at least),

$$\begin{aligned}
&\Delta\alpha_2(x,x',v') \\
&= \sum_{l=1}^{\infty} \int_0^{\tau_+(x',v')} \mathcal{E}(\vec{\gamma}_{(x',v')}(t))\Theta_l(\vec{\gamma}_{(x',v')}(t)) \\
&\qquad \times \int_X \big[\Delta k(\gamma_{(x',v')}(t))k(y_2) + \tilde{k}(\gamma_{(x',v')}(t))\Delta k(y_2)\big] \phi_l(\gamma_{(x',v')}(t), \bar{w}_1)\Theta(y_2, \hat{w}_1, \bar{w}_2) \\
&\qquad \times E(\gamma_{(x',v')}(t), y_2) E(y_2, x) \mathcal{J}(\gamma_{(x',v')}(t), y_2, x) m(x, \hat{w}_2)\, dy_2\, dt \\
&= \sum_{l=1}^{\infty} I_{\Theta_l \mathcal{E},1}\big[\Psi_{2,1,l} + \Psi_{2,2,l}\big](x',v')
\end{aligned}$$

where

$$\Psi_{2,1,l}(z) = \int_X \Delta k(z) k(y_2) \varphi_l(z, \bar{w}_1(z,y_2)) \Theta(y_2, \hat{w}_1(z,y_2), \bar{w}_2(y_2,x)) E(z,y_2) E(y_2,x) \times \mathcal{J}(z,y_2,x) m(x, \hat{w}_2(y_2,x))\, dy_2,$$

$$\Psi_{2,2,l}(z) = \int_X \tilde{k}(z) \Delta k(y_2) \varphi_l(z, \bar{w}_1(z,y_2)) \Theta(y_2, \hat{w}_1(z,y_2), \bar{w}_2(y_2,x)) E(z,y_2) E(y_2,x) \times \mathcal{J}(z,y_2,x) m(x, \hat{w}_2(y_2,x))\, dy_2.$$

In a similar manner, with $y_1 = \gamma_{(x',v')}(t)$,

$$\begin{aligned}
&\Delta\alpha_j(x,x',v') \\
&= \sum_{l=1}^{\infty} \int_0^{\tau_+(x',v')} \mathcal{E}(\vec{\gamma}_{(x',v')}(t)\Theta_l(\vec{\gamma}_{(x',v')}(t)) \\
&\qquad \times \int_X \cdots \int_X \Big(\sum_{i=1}^{j} \tilde{k}(y_1) \cdots \tilde{k}(y_{i-1}) \Delta k(y_i) k(y_{i+1}) \cdots k(y_j) \Big) \varphi_l(\gamma_{(x',v')}(t), \bar{w}_1) \\
&\qquad \times \Big(\prod_{i=2}^{j} \Theta(y_i, \hat{w}_{i-1}, \bar{w}_i) E(y_{i-1}, y_i) \Big) E(y_j, x) \mathcal{J}(\gamma_{(x',v')}(t), y_2, \ldots, y_j, x) \\
&\qquad \times m(x, \hat{w}_j)\, dy_j \cdots dy_2\, dt \\
&= \sum_{l=1}^{\infty} \sum_{i=1}^{j} I_{\Theta_l \mathcal{E},1} \Psi_{j,i,l}(x',v').
\end{aligned}$$

Explicitly,

$$\Psi_{j,i,l}(y_1) = \int_X \cdots \int_X \tilde{k}(y_1) \cdots \tilde{k}(y_{i-1}) \Delta k(y_i) k(y_{i+1}) \cdots k(y_j) \varphi_l(y_1, \bar{w}_1) \times \Big(\prod_{i=2}^{j} \Theta(y_i, \hat{w}_{i-1}, \bar{w}_i) E(y_{i-1}, y_i) \Big) E(y_j, x) \mathcal{J}(\cdot)\, m(x, \hat{w}_j)\, dy_j \cdots dy_2.$$

It is now necessary to estimate $\|\Psi_{j,i,l}\|_{L^2(X)}$. In what follows, $\|\cdot\|_\infty$ is shorthand for $\|\cdot\|_{L^\infty(Z)}$ where Z is simply the necessary domain of the object for which we are taking the L^∞ norm. Let $X_k = \operatorname{supp} k \cup \operatorname{supp} \tilde{k}$, and $D = \operatorname{dist}(X_k, x) > 0$, and recall that $\max\{\|k\|_\infty, \|\tilde{k}\|_\infty\} < \varepsilon$. First, since $E \le 1$ and $\|\varphi_l\|_{\infty(\Omega X)} \le 1$,

$$\|\Psi_{2,1,l}\|_{L^2(X)} \le \varepsilon \|\Theta\|_\infty \|m\|_\infty \Big\| \int_{X_k} \Delta k(z) \mathcal{J}(z, y_2, x)\, dy_2 \Big\|_{L^2(X)}.$$

Since on X_k $d(x,y_2)\geq D$, by Lemma 3.4.4,

$$\begin{aligned}\Big|\int_{X_k}\Delta k(z)\mathcal{J}(z,y_2,x)\,dy_1\Big| &= |\Delta k(z)|\int_{X_k}J_x(y_2)J_{y_2}(z)\,dy_2\\ &\leq |\Delta k(z)|\Big(\frac{C_{\kappa_M}}{D}\Big)^{n-1}\int_{X_k}J_{y_2}(z)\,dy_2\\ &= |\Delta k(z)|\Big(\frac{C_{\kappa_M}}{D}\Big)^{n-1}(\tilde{\mathcal{T}}1)(z)\end{aligned}$$

(see (3.11)). Thus

$$\|\Psi_{2,1,l}\|_{L^2(X)}\leq\varepsilon\|\Theta\|_\infty\|m\|_\infty\Big(\frac{C_{\kappa_M}}{D}\Big)^{n-1}(\text{diam}X)|\mathbb{S}^{n-1}|\|\Delta k\|_{L^2(X)}.$$

The estimate for $\Psi_{2,2,l}$ is the same although it is derived in a slightly different manner:

$$\begin{aligned}\|\Psi_{2,2,l}\|_{L^2(X)} &\leq \varepsilon\|\Theta\|_\infty\|m\|_\infty\Big\|\int_{X_k}\Delta k(y_2)J_x(y_2)J_{y_2}(z)\,dy_2\Big\|_{L^2(X)}\\ &\leq \varepsilon\|\Theta\|_\infty\|m\|_\infty\Big(\frac{C_{\kappa_M}}{D}\Big)^{n-1}\|\tilde{\mathcal{T}}\Delta k\|_{L^2(X)}\\ &\leq \varepsilon\|\Theta\|_\infty\|m\|_\infty\Big(\frac{C_{\kappa_M}}{D}\Big)^{n-1}(\text{diam}X)|\mathbb{S}^{n-1}|\|\Delta k\|_{L^2(X)}.\end{aligned}$$

For general j when $i=1$,

$$\begin{aligned}\|\Psi_{j,1,l}\|_{L^2(X)} &\leq \varepsilon^{j-1}\|\Theta\|_\infty^{j-1}\|m\|_\infty\Big\|\Delta k(z)\int_{X_k}\cdots\int_{X_k}\mathcal{J}(z,y_2,\ldots,y_j,x)\,dy_j\ldots dy_2\Big\|_{L^2(X)}\\ &\leq \varepsilon^{j-1}\|\Theta\|_\infty^{j-1}\|m\|_\infty\Big(\frac{C_{\kappa_M}}{D}\Big)^{n-1}\|\tilde{\mathcal{T}}^{j-1}1\|_\infty\|\Delta k\|_{L^2(X)}\\ &\leq \varepsilon C(\varepsilon\|\Theta\|_\infty(\text{diam}X)|\mathbb{S}^{n-1}|)^{j-2}\|m\|_\infty\Big(\frac{C_{\kappa_M}}{D}\Big)^{n-1}\|\Delta k\|_{L^2(X)}\end{aligned}$$

where $C=\|\Theta\|_\infty(\text{diam}X)|\mathbb{S}^{n-1}|$. For general j and $i>1$,

$$\begin{aligned}\|\Psi_{j,i,l}\|_{L^2(X)} &\leq \varepsilon^{j-1}\|\Theta\|_\infty^{j-1}\|m\|_\infty\Big\|\int_{X_k}\cdots\int_{X_k}\Delta k(y_i)\mathcal{J}(z,y_1,\ldots,y_j,x)\,dy_j\ldots dy_2\Big\|_{L^2(X)}\\ &\leq \varepsilon^{j-1}\|\Theta\|_\infty^{j-1}\|m\|_\infty\Big(\frac{C_{\kappa_M}}{D}\Big)^{n-1}\|\tilde{\mathcal{T}}^{j-i+1}(\Delta k\tilde{\mathcal{T}}^{i-2}1)\|_{L^2(X)}\\ &\leq \varepsilon^{j-1}\|\Theta\|_\infty^{j-1}\|m\|_\infty\Big(\frac{C_{\kappa_M}}{D}\Big)^{n-1}\|\tilde{\mathcal{T}}\|^{j-i+1}\|\Delta k\|_{L^2(X)}\|\tilde{\mathcal{T}}^{i-2}1\|_\infty\\ &\leq \varepsilon C(\varepsilon\|\Theta\|_\infty(\text{diam}X)|\mathbb{S}^{n-1}|)^{j-2}\|m\|_\infty\Big(\frac{C_{\kappa_M}}{D}\Big)^{n-1}\|\Delta k\|_{L^2(X)}.\end{aligned}$$

Now, using the relation $\beta I_{\Theta_j\mathcal{E},1} = I_{\Theta_j\mathcal{E},\beta}$, we have

$$\begin{aligned}\left\| I^*_{w,\beta}\beta\sum_{j=2}^{\infty}\Delta\alpha_j\right\|_{H^1(X)} &\leq \sum_{j=2}^{\infty}\sum_{i=1}^{j}\sum_{l=1}^{\infty}\|I^*_{w,\beta}I_{\Theta_j\mathcal{E},\beta}\Psi_{j,i,l}\|_{H^1(X)} \\ &\leq \sum_{j=2}^{\infty}\sum_{i=1}^{j}\sum_{l=1}^{\infty}\frac{C}{1+l^2}\|\Psi_{j,i,l}\|_{L^2(X)}\end{aligned}$$

from Proposition 3.6.2 and Lemma 3.6.2,

$$\begin{aligned}&\leq \sum_{j=2}^{\infty}\sum_{i=1}^{j}\sum_{l=1}^{\infty}\frac{C'}{1+l^2}\varepsilon(\varepsilon\|\Theta\|_\infty(\mathrm{diam}X)|\mathbb{S}^{n-1}|)^{j-2}\|\Delta k\|_{L^2(X)} \\ &\leq \varepsilon C''\|\Delta k\|_{L^2(X)}\sum_{j=2}^{\infty}j(\varepsilon\|\Theta\|_\infty(\mathrm{diam}X)|\mathbb{S}^{n-1}|)^{j-2}.\end{aligned}$$

Note that C depends continuously on the C^2 norm of $(w, \mathcal{E}, \beta, \Theta_j)$, and thus the C^2 norm of $(\sigma, m, \beta, \Theta_j)$, as well as the C^4 norm of the metric g. The metric enters because the weight w contains the Jacobian, which is written in terms of Jacobi fields. In coordinates, these fields are solutions to a differential equation with coefficients involving one derivative of the Christoffel symbols. The C^2 norm of Θ is bounded in lemma 3.6.2. This bound is uniform in a C^∞ neighborhood of g. We may therefore choose C (and thus C'') to remain fixed in a C^2 neighborhood of (σ, m) and a C^∞ neighborhood of g.

We see that for sufficiently small ε this series converges (thus justifying the formal computations performed above). □

Proof of theorem 3.2.2. First, fix real analytic (m, g, σ, Θ). Given the hypothesis of theorem 3.2.2 we are ensured that both Proposition 3.6.1 and Proposition 3.6.3 hold. Combining them, we have the existence of a constant C such that

$$\|\Delta k\|_{L^2(X)} \leq C\varepsilon\|\Delta k\|_{L^2(X)}$$

and so for $0 \leq \varepsilon < C^{-1}$, we must have $k = \tilde{k}$. □

Proof of Lemma 3.6.2. Given a fixed coordinate system for X (and hence for ΩX), there is a well defined smooth bijection $\theta = \theta_x : \Omega_x X \to \mathbb{S}^{n-1}$, which is smooth in the x variable and depends continuously on g in the C^∞ topology. We define $\tilde{\Theta} \in C^\infty(\Omega X \times \mathbb{S}^{n-1})$

$$\tilde{\Theta}(x, v', \theta) = \Theta(x, v', v(\theta)).$$

We shall make use of expansions in terms of spherical harmonics. If $f \in H_k$, the space of spherical harmonics of order k, then it can be shown that $\Delta_S f = -k(k+n-2)f$, where Δ_S is the Laplacian on $\mathbb{S}^{n-1}$. Denote by $Z_k^x(\theta)$ the so-called zonal harmonics for which $f(x) = \langle f, Z_k^x(\theta)\rangle_{L^2(\mathbb{S}^{n-1})}$ for all $f \in H_k$. Then one has (see, for example, [?]),

$$\dim(H_k) = d_k \leq c_n(k^{n-2} + 1)$$
$$\|Z_k^x\|_{L^2(\mathbb{S}^{n-1})} = c_n'\sqrt{d_k} \quad \text{for all } x \in \mathbb{S}^{n-1}.$$

Let $\{\tilde{\psi}_{kl}\}_{l=1}^{d_k}$ be an $L^2(\mathbb{S}^{n-1})$ orthonormal basis for H_k and define $\psi_{kl} = \|\tilde{\psi}_{kl}\|_{L^\infty(\mathbb{S}^{n-1})}^{-1}\tilde{\psi}_{kl}$. Then for each $(x, v') \in \Omega X$,

$$\tilde{\Theta}(x, v', \theta) = \sum_{k=0}^{\infty}\sum_{l}^{d_k} \Theta_{kl}(x, v')\psi_{kl}(\theta)$$

with

$$\Theta_{kl}(x, v') = \|\tilde{\psi}_{kl}\|_{L^\infty(\mathbb{S}^{n-1})} \int_{\mathbb{S}^{n-1}} \tilde{\Theta}(x, v', \theta)\tilde{\psi}_{kl}(\theta)\, d\theta.$$

For each θ,

$$|\tilde{\psi}_{kl}(\theta)| = |\langle\tilde{\psi}_{kl}, Z_k^\theta\rangle| \leq \|\tilde{\psi}_{kl}\|_{L^2(\mathbb{S}^{n-1})}\|Z_k^\theta\|_{L^2(\mathbb{S}^{n-1})} = c_n'\sqrt{d_k}.$$

Next, since $\tilde{\psi}_{kl} \in H_k$, for any $N \in \mathbb{N}$ and $k \geq 1$,

$$\begin{aligned}
|\Theta_{kl}(x, v')| &= \|\tilde{\psi}_{kl}\|_{L^\infty(\mathbb{S}^{n-1})}\Big|\int_{\mathbb{S}^{n-1}} \tilde{\Theta}(x, v', \theta)\tilde{\psi}_{kl}(\theta)\, d\theta\Big| \\
&= \|\tilde{\psi}_{kl}\|_{L^\infty(\mathbb{S}^{n-1})}\Big|\int_{\mathbb{S}^{n-1}} \tilde{\Theta}(x, v', \theta)\frac{(\Delta_S)^N\tilde{\psi}_{kl}(\theta)}{(-k(k+n-2))^N}\, d\theta\Big| \\
&\leq \frac{c_n'\sqrt{d_k}}{(-k(k+n-2))^N}\Big|\int_{\mathbb{S}^{n-1}} \tilde{\psi}_{kl}(\Delta_S)^N\tilde{\Theta}(x, v', \theta)\, d\theta\Big| \\
&\leq \frac{c_n'' k^{(n-2)/2}}{(-k(k+n-2))^N}\|(\Delta_S)^N\tilde{\Theta}(x, v', \cdot)\|_{L^2(\mathbb{S}^{n-1})},
\end{aligned}$$

where c_n'' depends only on the dimension n. Thus for sufficiently large N (in fact $N \geq 2n$), there is $c_{n,N}$ such that

$$|\Theta_{kl}(x, v')| \leq \frac{c_{n,N}}{1 + k^{2n}}\|(\Delta_S)^N\tilde{\Theta}(x, v', \cdot)\|_{L^2(\mathbb{S}^{n-1})}.$$

Now renumber the collection of coefficient functions and define new basis functions as follows: with $j = d_0 + d_1 + \cdots + d_{k-1} + l$, set

$$\Theta_j(x, v') = \Theta_{kl}(x, v'), \text{ and } \varphi_j(x, v) := \psi_{kl}(\theta_x(v)).$$

Then

$$\Theta(x, v', v) = \sum_{j=1}^{\infty} \Theta_j(x, v')\varphi_j(x, v).$$

Now

$$j \leq \sum_{m=0}^{k} d_m \leq c_n(k+1)(k^{n-2}+1) \leq \hat{c}_n k^n$$

so

$$\begin{aligned} |\Theta_j(x, v')| = |\Theta_{kl}(x, v')| &\leq \frac{c_{n,N}}{1+k^{2n}} \|(\Delta_S)^N \tilde{\Theta}(x, v', \cdot)\|_{L^2(\mathbb{S}^{n-1})} \\ &\leq \frac{\tilde{c}_{n,N}}{1+j^2} \|(\Delta_S)^N \tilde{\Theta}(x, v', \cdot)\|_{L^2(\mathbb{S}^{n-1})}. \end{aligned}$$

If one applies the same decomposition to $\partial_x^\alpha \Theta(x, v', v)$, $\alpha \in \mathbb{N}^m$, one finds that the coefficients are nothing more than $\partial_x^\alpha \Theta_j(x, v')$ and these satisfy

$$|\partial_x^\alpha \Theta_j(x, v')| \leq \frac{\tilde{c}_{n,N}}{1+j^2} \|(\Delta_S)^N \partial_x^\alpha \tilde{\Theta}(x, v', \cdot)\|_{L^2(\mathbb{S}^{n-1})}.$$

□

Chapter 4

DIFFUSION TYPE MEASUREMENTS

4.1 Introduction

The purpose of this chapter is to consider the reconstruction of optical parameters from isotropic sources and angularly averaged measurements, i.e., from $2(n-1)$ dimensional data. In dimension $n = 2$, this corresponds to 2-dimensional data sets. The optical parameters therefore realistically need be 2-dimensional as well, i.e., may only depend on the spatial variable x.

The problem thus resembles that of electrical impedance tomography and of the reconstruction of diffusion coefficients from Cauchy data [16, 19, 45]. Diffusion equations may also be used to model solutions of transport equations in highly scattering media [21]. This explains why diffusion models are very popular in optical tomography [5]. In this chapter, we are interested in cases where the diffusion approximation does not hold, and yet only diffusion-type measurements are available.

Our results require somewhat non-practical simplifications. We know in the diffusive regime that only one of the optical parameters k and σ may be reconstructed from Cauchy data; see e.g. [6]. In this chapter, we are interested in reconstructing the scattering coefficient. The first simplification is thus to assume that $\sigma(x)$ has already been obtained, for instance by angularly dependent measurements of Radon transform type; see [40]. We also assume that the scattering coefficient $k = k(x)$ is independent of the direction of scattering. Our techniques do not allow for the reconstruction of scattering coefficients of the form $f(v, v')k(x)$ (see [40]), even when the phase function $f(v, v')$ is known. The second simplification is that both the known $\sigma(x)$ and the unknown $k(x)$ are sufficiently small. How small they have to be will be made more explicit in the next section. Smallness is required because our inversion formula is based on a linearization of the inverse transport problem, which we can invert explicitly only in the limit $\sigma = 0$. The nonlinear reconstruction is then based on applying standard fixed point arguments.

Surprisingly enough, the solution of the linearized inverse transport problem is performed by using the same complex geometrical optics (CGO) solutions as in the inverse diffusion problem treated in its linearized form in [16] and in its full non-linear form in [45]; see also [49]. As in [16], the main tool used in this inversion is the construction of harmonic CGOs, which allow us to have access to the Fourier transform of $k(x)$ thanks to the density of products of harmonic functions. As a result, as in [16], the linearized inverse generates a severely ill-posed problem with exponential-type stability. These very negative results should be contrasted with the situation where phase space measurements are available, which allow us in many settings to obtain much better behaved Hölder-type stability estimates [?, 40, 51].

The rest of the chapter is structured as follows. We state our main hypotheses and main results in section 4.2. In section 4.3 we describe the forward problem, in particular the measurements (4.5) and isotropic sources in section 4.3.1. In section 4.3.2, the so-called half-adjoint operator is introduced. This emerges as a result of our measurements, and is the reason we are able to use harmonic solutions. In section 4.4, we use the half-adjoint operator to solve the linearized inverse problem. In section 4.5 we define a regularized inverse and use it to obtain our main results. Some conclusions are offered in section 4.6.

4.2 Statement of the main results

Let $X \subset \mathbb{R}^n, n \geq 2$ be an open bounded strictly convex set with C^2 boundary ∂X. Denote $\partial_\pm SX = \{(x,v) \in \partial X \times \mathbb{S}^{n-1} : \pm\nu_x \cdot v > 0\}$, where ν_x is the outer normal to ∂X at $x \in \partial X$. The stationary linear transport equation for the density $u(x,v)$ we consider in this chapter is defined as:

$$\begin{aligned} v \cdot \nabla_x u(x,v) + \sigma(x)u(x,v) - k(x)\int_{\mathbb{S}^{n-1}} u(x,v')dS(v') = 0, \\ u|_{\partial_- SX} = u_-. \end{aligned} \tag{4.1}$$

Following our discussion in the introduction, we consider the simplified setting of isotropic scattering $k = k(x)$. An existence theory for (4.1) is recalled in the next section.

We assume that $k, \sigma \in L^\infty(X)$ are bounded functions, extended by 0 outside of X.

Smallness assumptions on $k(x)$ are necessary for (4.1) to admit a solution. Indeed creation of particles need be compensated by absorption and leakage of particles at the domain's boundary in order for (4.1) to admit a physical solution [21, 32]. Our inversion algorithm requires that we make additional smallness assumptions on $k(x)$ and $\sigma(x)$.

Let us define the operator $L_{\sigma,k}$ as:

$$L_{\sigma,k}h(x) := \int_X \frac{e^{-\int_0^{|x-y|} \sigma(x+s\frac{y-x}{|y-x|})ds}}{|x-y|^{n-1}} k(y)h(y)\,dy. \tag{4.2}$$

Existence of a solution to (4.1) is guaranteed provided that the spectral radius $\rho(L_{\sigma,k}) < 1$ [32], which is always satisfied provided that k is sufficiently small. We also define $L_\sigma = L_{\sigma,1}$, where $k \equiv 1$. We then verify that

$$\|L_\sigma\|_p \leq \sup_{x\in X} \int_X \frac{e^{-\int_0^{|x-y|} \sigma(x+sv)ds}}{|x-y|^{n-1}}\,dy, \quad p \in [1,\infty], \tag{4.3}$$

where $\|\cdot\|_p$ denotes the norm in $\mathcal{L}(L^p, L^p)$. The above bound is straightforward to verify for $p = 1$ and $p = \infty$ and comes for other values of p by the Riesz-Thorin interpolation theorem [13] (see also lemma 4.3.2 below for a simple proof).

Our first smallness assumption on k is that

$$\|k\|_{L^\infty}\|L_\sigma\| < 1, \tag{4.4}$$

where we denote by $\|L_\sigma\| = \|L_\sigma\|_2$. This assumption, which is less optimal than the condition $\rho(L_{\sigma,k}) < 1$, will allow us to write the transport solution as an infinite series corresponding to increasing orders of scattering and to conveniently estimate the influence of high orders of scattering. This is the only necessary assumption in our first result, theorem 4.2.1. Additional assumptions will be made explicit in order to prove our non-linear inversion result in theorem 4.2.2.

Our measurements are constructed as follows. For each isotropic source $u_- = f(x)$ at the domain's boundary, we measure the current at point $x \in \partial X$ given by:

$$\int_{\nu_x\cdot v>0} u(x,v)\nu_x \cdot v\,dS(v), \tag{4.5}$$

where ν is outward the unit normal, and u solves (4.1). After integrating the current over ∂X with weight function $g(x)$, we obtain the following type of measurements:

$$\mathcal{M}(f,g) := \int_{\partial X} g(x) \int_{\nu_x \cdot v > 0} u(x,v)(\nu_x \cdot v)\, dS(v) d\mu(x). \tag{4.6}$$

The crux of the inversion is to reconstruct $k(x)$ from the contribution in (4.6) that is linear in k, which corresponds to the single scattering term. When $\sigma = 0$, we show that this contribution to our measurement is equivalent to the integral of k against the product of two harmonic functions. Under suitable regularity conditions, these harmonic functions are arbitrary. Borrowing then techniques from the problem of electrical impedance tomography, we are able to solve this problem with the classical harmonic solutions of Calderón [16]. As in the Calderón problem, however, this inversion is severely ill-posed.

Specifically, the recovery of higher frequencies comes with an error growing exponentially with frequency. We therefore propose a regularized inverse, which attempts to recover $P_\chi k$ with:

$$P_\chi k := \int_{\mathbb{R}^n} \hat{k}(\xi)\chi(\xi)e^{i\xi\cdot x}\, d\xi.$$

Here χ must decay sufficiently fast for large ξ. In particular, we require

$$\chi(\cdot)e^{|\cdot|\operatorname{diam}(X)} \in L^1(\mathbb{R}^n). \tag{4.7}$$

Our error will depend on the bound $\|T^\chi\| := \|T^\chi\|_{L^2((\partial X)^2)\to L^\infty(X)}$, for an operator T^χ that will be introduced in (4.41) below. If χ is the characteristic function of the ball of radius M, the result (4.42) gives us

$$\|T^\chi\| \le C_X \int_{|\xi|<M} e^{|\xi|\operatorname{diam}(X)}\, d\xi,$$

with C_X depending only on X. In this case, P_χ is the orthogonal projection of an L^2 function onto its low frequency content ($|\xi| < M$). When computing the Fourier transform of k, we extend it to be zero outside of X. Since then the support of k is compact, k always has some high frequency content.

Our main results are the following:

Theorem 4.2.1 (Recovery of $P_\chi k$). *Suppose that* (4.4) *holds. Then there exists a constant* $C_X > 0$, *depending only on* X *and* $\|L_\sigma\|$ *such that for all* χ *satisfying* (4.7), *the measurements* $\{\mathcal{M}(f,g),(f,g) : f \in L^1(\partial X), g \in L^1(\partial X)\}$ *determine* $P_\chi k$ *up to an error, bounded in* L^∞ *by*

$$C_X \|T^\chi\| \|k\|_{L^\infty} \left(\|\sigma\|_{L^\infty} + \|k\|_{L^\infty}\right).$$

Using an iterative scheme, and a smallness assumption on k and σ, we are able to improve this result as follows.

Theorem 4.2.2 (Iterative improvement). *Suppose that* (4.4) *holds. Given* χ *satisfying* (4.7), $c_1 \in (0,1)$, *and* σ *such that* $\|\sigma\|_{L^\infty} < \frac{c_1}{C_X\|T^\chi\|}$, *there exists* $\varepsilon > 0$ *such that for* $\|k\|_{L^\infty(X)} < \varepsilon$, *the measurements* $\{\mathcal{M}(f,g),(f,g) : f \in L^1(\partial X), g \in L^1(\partial X)\}$ *determine* $P_\chi k$ *up to an error bounded in* L^∞ *by*

$$\frac{c_1}{1-c_1}\|(I-P_\chi)k\|_{L^\infty(X)}.$$

Our iterative method requires a certain constant of contraction in order to converge. Hence the c_1 constant. The constant ε depends on $c_1, \sigma, C_X, \|T^\chi\|$, though when $\|\sigma\|_{L^\infty} \ll \frac{c_1}{C_X\|T^\chi\|}$, we can approximate this constant with the following expression independent of $\|\sigma\|_{L^\infty}$:

$$\varepsilon \approx \frac{c_1(1-c_1)}{2\|L_\sigma\|(C_X\|T^\chi\|)^2 + c_1(1-c_1)}.$$

The general expression of this constant can be established with equations (4.48) and (4.50). These constants are not necessarily optimal since, while proving the theorem, we look for sufficient conditions.

4.3 The forward problem

We now return to the forward model (4.1) and present well-known properties that will be useful in subsequent sections. These properties follow exactly as in chapter 2. We repeat them here in brief.

We begin with some notation. For $(x, v) \in X \times \mathbb{S}^{n-1}$, let $\tau_\pm(x, v)$ be the distance from x to ∂X traveling in the direction of $\pm v$, and $x_\pm(x, v) = x \pm \tau_\pm(x, v)v$ be the boundary point encountered when we travel from x in the direction of $\pm v$. We also define $\tau = \tau_+ + \tau_-$. We give $\partial_\pm SX$ the measure $d\xi(x, v) = |\nu_x \cdot v| d\mu(x) dS(v)$, where $d\mu, dS$ are the volume forms on $\partial X, \mathbb{S}^{n-1}$ respectively.

We first define

$$
\begin{aligned}
Ju_-(y, v) &:= E(x_-(y, v), y)u_-(x_-(y, v), v), && (4.8)\\
Kf(x, v) &:= \int_0^{\tau_-(x,v)} E(x, x - tv) \int_{\mathbb{S}^{n-1}} k(x - tv) f(x - tv, v')\, dS(v') dt, && (4.9)\\
E(x, y) &:= \exp\left(- \int_0^{|y-x|} \sigma\Big(x + \frac{y - x}{|y - x|} s\Big) ds \right). && (4.10)
\end{aligned}
$$

For future reference, we also define iteratively:

$$E(a_1, \dots, a_{i+1}) := E(a_1, \dots, a_i) E(a_i, a_{i+1}). \tag{4.11}$$

We solve the forward problem with a Neumann series,

$$u = (I - K)^{-1} Ju_- = \sum_{m=0}^{\infty} K^m Ju_-. \tag{4.12}$$

Note that the restriction of u on Γ_+ is well defined in $L^1(\Gamma_+, \tau d\xi)$.

4.3.1 The surface distribution model

Suppose our incoming flux $u_-(x, v) = f(x)$ is independent of the angular variable v. Then, the contribution due to flux at point $y \in \bar{X}$ due to incoming flux coming directly from ∂X is given by

$$Jf(x, v) := E(x_-(x, v), x) f(x_-(x, v)).$$

We will often need to integrate this flux over all directions $v \in \mathbb{S}^{n-1}$ and change variables from $v \in S^{n-1}$ to the boundary point $x_0 \equiv x_-(x, v) \in \partial X$ (at x fixed). The change of variables from the sphere to the convex boundary ∂X is given formally by

$$dS(v) = \frac{|\nu_{x_0} \cdot v|}{|x - x_0|^{n-1}} d\mu(x_0). \tag{4.13}$$

The above change of variables is justified in the following result, whose proof is postponed to the appendix:

Proposition 4.3.1 (Change of variables from $\mathbb{S}^{n-1}$ to ∂X). *Let* $\mathbf{S}$ *be the* C^2 *boundary of a convex domain in* $\mathbb{R}^n$.

1. *Pick any* y *enclosed by* $\mathbf{S}$. *Then for* $f \in L^1(\mathbf{S})$,

$$\int_{\mathbf{S}} f(x) \frac{\nu_x \cdot (x-y)}{|x-y|^n} \, d\mathbf{S}(x) = \int_{\mathbb{S}^{n-1}} f(x_-(y,v)) \, dS(v). \tag{4.14}$$

Here ν_x *is the outward unit normal to* $\mathbf{S}$ *at* x, *and* $d\mathbf{S}$, dS *are the volume forms on* $\mathbf{S}$, $\mathbb{S}^{n-1}$ *respectively.*

2. *Moreover, if* $\mathbf{S}$ *is the boundary of a strictly convex domain, we have, for any* $y \in \mathbf{S}$ *and* $f \in L^\infty(\mathbf{S})$,

$$\int_{\mathbf{S}} f(x) \frac{\nu_x \cdot (x-y)}{|x-y|^n} \, d\mathbf{S}(x) = \int_{\substack{v \cdot \nu_y > 0 \\ v \in \mathbb{S}^{n-1}}} f(x_-(y,v)) \, dS(v), \tag{4.15}$$

with the same notation as in (4.14).

Recall that our averaged measurements at point $x \in \partial X$ are given by

$$\int_{\nu_x \cdot v > 0} u(x,v) \nu_x \cdot v \, dS(v),$$

where ν is outward the unit normal, and u solves (4.1). We take measurements all along ∂X, and compute the weighted average of them with weighting function g. This gives:

$$\int_{\partial X} g(x) \int_{\nu_x \cdot v > 0} u(x,v)(\nu_x \cdot v) \, dS(v) d\mu(x). \tag{4.16}$$

Suppose now that $u_-(x_0, v_0) = f(x_0)$, i.e., our incoming flux is the same in every direction. We then obtain the following type of measurement:

$$\mathcal{M}(f,g) := \int_{\partial X} g(x) \int_{\nu_x \cdot v > 0} u(x,v)(\nu_x \cdot v) \, dS(v) d\mu(x),$$

where u solves (4.1) and $u(x,v) = f(x)$ on $\partial_- SX$.

The contribution to the outgoing flux density at position x, in direction v, that has not scattered (a.k.a the ballistic contribution) will be

$$Ju_-(x,v) = Jf(x,v) = E(x_-(x,v), x) f(x_-(x,v)).$$

The ballistic contribution to $\mathcal{M}(f,g)$ is therefore,

$$\begin{aligned}
&\int_{\partial X} g(x) \int_{\nu_x\cdot v>0} Jf(x,v)|\nu_x\cdot v|\, dS(v) d\mu(x) \\
&= \int_{\partial X} g(x) \int_{\nu_x\cdot v>0} E(x_-(x,v),x) f(x_-(x,v))|\nu_x\cdot v|\, dS(v) d\mu(x) \\
&= \int_{\partial X} g(x) \int_{\partial X} f(x_0) \left(\frac{E(x_0,x)|\nu_x\cdot v||\nu_{x_0}\cdot v|}{|x_0-x|^{n-1}}\right) d\mu(x_0) d\mu(x)\Big|_{v=\frac{x-x_0}{|x-x_0|}} \\
&= \langle T_0, f\otimes g\rangle_{L^2((\partial X)^2)}, \qquad T_0(x_0,x) := \frac{E(x_0,x)|\nu_x\cdot v||\nu_{x_0}\cdot v|}{|x_0-x|^{n-1}},
\end{aligned} \tag{4.17}$$

where T_0 is known since σ is. T_0 is integrable due to the fact that for $x, y \in \partial X$, [22, Lemma 3.15]

$$|(x-y)\cdot\nu_y| \le c|x-y|^2, \tag{4.18}$$

so that the ballistic contribution to $\mathcal{M}(f,g)$ is defined. The second equality is due to the change of variables (4.13) or equation (4.15) in proposition 4.3.1.

The contribution to the outgoing flux density at position x, in direction v, that has scattered once will be $KJf(x,v)$. The single scattering contribution to (4.5) at point x is then:

$$\begin{aligned}
&\int_{\nu_x\cdot v_1>0} KJf(x,v_1)(\nu_x\cdot v_1)\, dS(v_1) \\
&= \int_{\nu_x\cdot v_1>0}\int_0^{\tau_-(x,v_1)} E(x, x-t_1v_1) \int_{\mathbb{S}^{n-1}} k(x-t_1v_1) Jf(x-t_1v_1, v_0)\, dS(v_0) dt (\nu_x\cdot v_1)\, dS(v_1) \\
&= \int_{\nu_x\cdot v_1>0}\int_0^{\tau_-(x,v_1)} \int_{\mathbb{S}^{n-1}} k(x-t_1v_1) E(x, x-t_1v_1, x-t_1v_1-t_0v_0) \\
&\qquad\qquad \times f(x-t_1v_1-t_0v_0) dS(v_0) dt (\nu_x\cdot v)\, dS(v_1) \\
&= \int_X \int_{\mathbb{S}^{n-1}} k(x_1) f(x_-(x_1,v_0)) E(x_-(x_1,v_0),x_1,x) \frac{|\nu_x\cdot v_1|}{|x_1-x|^{n-1}}\, dS(v_0) dx_1 \\
&= \int_X \int_{\partial X} f(x_0) E(x_0,x_1,x) k(x_1) \frac{|\nu_{x_0}\cdot v_0||\nu_x\cdot v_1|}{|x_0-x_1|^{n-1}|x_1-x|^{n-1}}\, dx_1 d\mu(x_0).
\end{aligned} \tag{4.19}$$

The third equality comes from the change of variables $x_1 = x - t_1v_1,\ dx_1 = t_1^{n-1} dt_1 dS(v_1)$. The last equality is due to equation (4.14) of proposition 4.3.1.

The total single scattering contribution to $\mathcal{M}(f,g)$ is therefore

$$\begin{aligned}
&\int_{\partial X} g(x) \int_{\partial X} f(x_0) \int_X E(x_0,x_1,x) k(x_1) (t_0t_1)^{1-n} (\nu_{x_0}\cdot v_0)(\nu_x\cdot v_1)\, dx_1 d\mu(x_0) d\mu(x) \\
&= \int_X k(x_1) \int_{\partial X} f(x_0) \frac{E(x_0,x_1)|\nu_{x_0}\cdot v_0|}{|x_0-x_1|^{n-1}}\, d\mu(x_0) \int_{\partial X} g(x) \frac{E(x_1,x)|\nu_x\cdot v_1|}{|x_1-x|^{n-1}}\, d\mu(x)\ dx_1,
\end{aligned} \tag{4.20}$$

where $v_0 = (x_1 - x_0)|x_1 - x_0|^{-1}$, and $v_1 = (x - x_1)|x - x_1|^{-1}$.

An inductive argument shows that for $m \geq 2$,

$$K^m h(x,v) = \int_0^{\tau_-(x,v)} \int_{X^{m-1}} \int_{\mathbb{S}^{n-1}} E(x, x - t_m v, x_{m-1}, x_{m-2}, \dots, x_1) k(x - t_m v) \\ \times k(x_{m-1}) \dots k(x_1)(t_1, \dots, t_{m-1})^{1-n} h(x_1, v_0)\, dS(v_0) dx_1 \dots dx_{m-1} dt_m.$$

Following a procedure similar to (4.19), we are able to represent the contribution to our measurements due to m scattering events in a compact form. To this end, denote by $\langle \cdot, \cdot \rangle$ an L^2 inner product, conjugate linear in the first variable, and linear in the second. We have the contribution to $\mathcal{M}(f,g)$ due to m scattering events equal to

$$\langle T_m(k), f \otimes g \rangle_{L^2((\partial X)^2)},$$

where we define T_m as follows.

Definition 4.3.1 (m^{th} scattering kernel).

$$T_m(k)(x_0, x) := \int_{X^m} k(x_1) \cdots k(x_m) \frac{E(x_0, \dots, x_m, x)|\nu_{x_0} \cdot v_0||\nu_x \cdot v_m|}{|x_0 - x_1|^{n-1} \cdots |x_m - x|^{n-1}} dx_1 \dots dx_m.$$

Note that $T_m(k)$ is real valued. This leads to

$$\mathcal{M}(f,g) = \langle T_0, f \otimes g \rangle_{L^2((\partial X)^2)} + \sum_{m=1}^{\infty} \langle T_m(k), f \otimes g \rangle_{L^2((\partial X)^2)}. \qquad \textbf{(4.21)}$$

Note that T_0 and $T_m(k)$, taken at points x and x_0, are the measurements given source $f = \delta_{\{x_0\}}$, and weight $g = \delta_{\{x\}}$. Here, $\delta_{\{x\}}$ is the distribution on ∂X defined by: $\int_{\partial X} \delta_{\{x\}}(y) f(y) d\mu(y) = f(x)$.

The measurement viewpoint (4.21) will play the dominant role from now on. We will attempt to find suitable boundary functions f and g to extract the necessary information on the unknown parameter $k(x)$.

4.3.2 The half-adjoint operator

In this section we introduce the "half-adjoint" operator along with some of its basic properties. We first recall that the Newton potential, the fundamental solution of $\Delta_y N(x,y) =$

$\delta_x(y)$ is given by

$$N(x,y) := \frac{1}{c_n|x-y|^{n-2}} \ (n>2); \quad \frac{1}{c_2}\log|x-y| \ (n=2), \tag{4.22}$$

where

$$c_n := (2-n)\omega_n \ (n \geq 3); \quad c_2 := 2\pi \ (n=2), \quad \omega_n = \mathrm{Vol}(\mathbb{S}^{n-1}).$$

Given $x \in \partial X$, $y \in \mathbb{R}^n$, one can check that

$$\partial_{\nu_x} N(x,y) = \frac{\nu_x \cdot (x-y)}{\omega_n |x-y|^n}.$$

Since the above kernel is central to the following calculations, we recall that:

$$\int_{\partial X} \partial_\nu N(x,y)\,dx = \begin{cases} 0, & y \in \mathbb{R}^n - \bar{X} \\ 1, & y \in X \\ \frac{1}{2}, & y \in \partial X \end{cases}, \tag{4.23}$$

so that care must be taken with boundary values; see the sections on "double layer potentials" in e.g. [2, 22].

We are now ready to define the following.

Definition 4.3.2 (Half-Adjoint Operator). For $y \in X$,

$$Af(y) := \omega_n \int_{\partial X} f(x)E(x,y)\partial_\nu N(x,y)\,d\mu(x). \tag{4.24}$$

When $E = 1$, $\omega_n^{-1}A$ is a harmonic function called the "double layer potential." $\omega_n^{-1}Af$ is a "moment" of the "double layer potential." For $y \notin \partial X$, $\omega_n^{-1}Af(y)$ is equal to the potential at y due to a distribution f of dipoles on ∂X.

We now note that:

$$T_1(k)(x_0,x) := \int_X k(x_1)\frac{E(x_0,x_1,x)|\nu_{x_0}\cdot v_0||\nu_x \cdot v_1|}{|x_0-x_1|^{n-1}|x_1-x|^{n-1}}\,dx_1. \tag{4.25}$$

With our definitions of A, T_1 we may re-write our single-scattering measurement (4.20) as

$$\langle T_1(k), f\otimes g\rangle_{L^2((\partial X)^2)} = \langle k, Af\,Ag\rangle_{L^2(X)}. \tag{4.26}$$

We also have that the resultant contribution to $\mathcal{M}(f,g)$ due to two scattering events is:

$$\begin{aligned}
&\langle T_2(k), f\otimes g\rangle_{L^2((\partial X)^2)}\\
&=\int_X\int_X k(x_1)k(x_2)\frac{E(x_1,x_2)}{|x_1-x_2|^{n-1}}Af(x_1)Ag(x_2)\,dx_1dx_2,\\
&=\int_X k(x)Af(x)L_{\sigma,k}Ag(x)dx.
\end{aligned}\tag{4.27}$$

and from m scattering events,

$$\begin{aligned}
&\langle T_m(k), f\otimes g\rangle_{L^2((\partial X)^2)}\\
&=\int_X\cdots\int_X \frac{k(x_1)\cdots k(x_m)E(x_1,\ldots,x_m)}{|x_1-x_2|^{n-1}\cdots|x_{m-1}-x_m|^{n-1}}Af(x_1)Ag(x_m)\,dx_1\cdots dx_m\\
&=\int_X k(x)Af(x)L_{\sigma,k}^{m-1}Ag(x)dx.
\end{aligned}\tag{4.28}$$

The relation (4.26) suggests an inversion based on finding boundary values f and g such that their products $AfAg$ are dense. As it turns out, this is possible when $\sigma=0$ because the product of harmonic functions is dense. Proving this requires some facts about double-layer potentials.

Let A_0 be the half-adjoint operator A defined in (4.24) when $E\equiv 1$. Now, when $f\in L^1(\partial X)$, the defining property of the Newton potential shows that $A_0f\in C(X)$ is harmonic. The main question now is whether any harmonic function in X may be prescribed as A_0f for a suitable boundary term f. That this is possible is based on the following classical result on the jump conditions of the double layer potential:

$$\lim_{X\ni x'\to y\in\partial X}\omega_n^{-1}A_0f(x')=\frac{1}{2}f(y)+\omega_n^{-1}\mathscr{A}_0f(y),\tag{4.29}$$

where

$$\mathscr{A}_0f(y)=\omega_n\int_{\partial X}f(x)\partial_{\nu_x}N(x,y)\,d\mu(x),\tag{4.30}$$

where the integral is defined in the usual sense; see [22], (4.18), and the proof of Lemma 4.3.3 below. When X has boundary ∂X of class C^1, the operator $\mathscr{A}_0$ is compact on $L^2(\partial X)$ and does not admit $-\frac{1}{2}$ as an eigenvalue [2, §2.2]. This shows that

$$f\mapsto\frac{1}{2}f+\omega_n^{-1}\mathscr{A}_0f,\qquad\text{is an isomorphism}\quad L^2(\partial X)\to L^2(\partial X).\tag{4.31}$$

We have thus obtained that the $L^2(\partial X)$-valued trace of any harmonic function in, say $H^{\frac{1}{2}}(X)$, may be written as $\lim_{X \ni x' \to y \in \partial X} A_0 f(x')$ for some $f \in L^2(\partial\Omega)$. More formally, we have:

Lemma 4.3.1 (Pseudo-inverse for the A_0 operator). *The operator defined by*

$$A_0^\dagger : \begin{cases} H^{\frac{1}{2}}(X) & \to \quad L^2(\partial X) \\ \quad u & \mapsto \quad A_0^\dagger u := (\frac{1}{2}I + \omega_n^{-1}\mathscr{A}_0)^{-1}(\omega_n^{-1}u|_{\partial X}) \end{cases},$$

is continuous and such that $A_0 A_0^\dagger u = u|_X$ for all harmonic function $u \in H^{\frac{1}{2}}(X)$.

Proof. The above isomorphism (4.31) shows that $A_0^\dagger$ is a well-posed operator. Now for u harmonic, for $f = A_0^\dagger u$, we have $A_0 f$ harmonic and $A_0 f$ and u having the same trace on ∂X. Thus $A_0 f = u$. □

When $\partial X = \mathbb{S}^1$, we can find an explicit expression for $A_0^\dagger$. First, the Poisson kernel on $\{x \in \mathbb{R}^n : |x| \leq 1\}$ is given by

$$P(x,y) = \frac{1-|y|^2}{\omega_n |x-y|^n}, \quad x \in \mathbb{S}^{n-1},\ y \in \text{unit ball}. \tag{4.32}$$

If $n = 2$, we can show (using the fact $|x| = 1$) that

$$P(x,y) + \frac{1}{2\pi} = 2\partial_{\nu_x} N(x,y),$$

which leads to

$$\begin{aligned} A_0 f(y) &= \frac{2\pi}{2} \int_{\mathbb{S}^1} P(x,y) f(x)\, dx + \frac{1}{2} \int_{\mathbb{S}^1} f(x)\, dx \\ &= \pi(\tilde{f}(y) + \tilde{f}(0)). \end{aligned} \tag{4.33}$$

Where $\tilde{f}$ satisfies $\Delta \tilde{f} = 0$ in X, and $\tilde{f}|_{\partial X} = f$. Here we have used the defining property of the Poisson kernel, and the mean value theorem for harmonic functions. This analysis tells us that in this special geometry,

$$A_0^\dagger u(x) = \frac{2u(x) - u(0)}{2\pi}, \quad x \in \mathbb{S}^1.$$

We close this section with some properties on the operators A and T_i that will be useful in later sections. First we state and prove the following lemma.

Lemma 4.3.2. *Suppose that for every $y \in Y$,*

$$\int_X |k(x,y)|\,dx < C_1, \quad \text{and for every } x \in X, \ \int_Y |k(x,y)|\,dy < C_2.$$

Then for $1/p + 1/q = 1$, $p \in [1,\infty]$,

$$\left\| \int_X k(x,\cdot) f(x)\,dx \right\|_{L^p(Y)} \le C_1^{1/p} C_2^{1/q} \|f\|_{L^p(X)}.$$

Proof. The proof is classical and may be found e.g. in [47] when $X = Y$. For completeness, we recall it here. First consider the case $p \in (1,\infty)$. The identity $ab \le a^p/p + b^q/q$ applied to $f(x)g(y)$ shows that

$$\Big|\int_Y \int_X k(x,y) f(x) g(y)\,dxdy\Big| \le C_1 \frac{\|f\|_{L^p}^p}{p} + C_2 \frac{\|g\|_{L^q}^q}{q}.$$

Now the same computation with f replaced by tf, and g replaced by $t^{-1}g$, $t > 0$, and simple calculus minimization show that

$$\Big|\int_Y \int_X k(x,y) f(x) g(y)\,dxdy\Big| \le C_1^{1/p} C_2^{1/q} \|f\|_{L^p(X)} \|g\|_{L^q(Y)}.$$

This proves the lemma for the case $p \in (1,\infty)$. The special cases $p = 1$ and $p = \infty$ are easily checked. □

This allows us to prove the following

Lemma 4.3.3 (L^p mapping property of half-adjoint operator). *There exists C_X depending on X such that for $p \in [1,\infty]$,*

$$\|Af\|_{L^p(X)} \le C_X \|f\|_{L^p(\partial X)}.$$

Proof. Recall that the kernel of the integral operator A is (up to some constant) $\partial_{\nu_y} N(x,y)$, with $y \in \partial X$, $x \in X$. Since ∂X is of class C^2, lemma 3.20 from [22] and (4.18) show that there exists C such that

$$\int_{\partial X} |\partial_{\nu_y} N(x,y)|\,dy < C, \ \forall x \in X, \ \text{and} \ \int_X |\partial_{\nu_y} N(x,y)|\,dx < C, \ y \in \partial X.$$

The result therefore is a direct application of lemma 4.3.2. □

Lemma 4.3.4 (Mapping property of T_m). *There exists C_X, depending only on X such that for $m \geq 1$, we have*

$$\|T_m(k)\|_{L^2((\partial X)^2)} \leq C_X \|k\|_{L^\infty} \|L_{\sigma,k}\|_2^{m-1} \leq C_X \|k\|_{L^\infty}^m \|L_\sigma\|_2^{m-1}. \tag{4.34}$$

Proof. When $m = 1$, the proof follows from (4.26) and lemma 4.3.3. For $m \geq 2$, we find that

$$\begin{aligned}
&|\langle T_m(k), f \otimes g\rangle_{L^2((\partial X)^2)}| \\
&\leq \|k\|_{L^\infty} \int_X \cdots \int_X k(x_1) \cdots k(x_{m-1}) \frac{Af(x_1)E(x_1, \ldots, x_m)Ag(x_i)}{|x_1 - x_2|^{n-1} \cdots |x_{m-1} - x_m|^{n-1}}\, dx_i \cdots dx_1 \\
&= \|k\|_{L^\infty} \int_X Af(x_1) L_{\sigma,k}^{m-1} Ag(x_1)\, dx_1 \\
&\leq \|k\|_{L^\infty} \|L_{\sigma,k}\|_2^{m-1} \|Af\|_{L^2} \|Ag\|_{L^2}.
\end{aligned}$$

The proof then follows from lemma 4.3.3 and the obvious bound $\|L_{\sigma,k}\|_2 \leq \|k\|_{L^\infty}\|L_\sigma\|_2$. □

4.4 The linearized inverse problem

Lemma 4.3.1 and (4.26) motivate an attempt to invert the operator T_1 by finding dense products of harmonic functions. For the disk, one such choice would be (in polar coordinates) $r^k e^{ik\theta}$. This choice, along with its stability has been explored in [1]. Here we opt for the more general, and familiar complex geometrical optics (CGO) solutions of Calderón [16]. So let $\mathbb{C}^n \ni \rho = -\frac{1}{2}(\xi + i\eta)$, where $\xi, \eta \in \mathbb{R}^n$, $\xi \cdot \eta = \sum_{i=1}^n \xi_i \eta_i = 0$, and $|\xi| = |\eta|$. Then the functions $e^{i\rho\cdot x}$, and $e^{i\bar\rho\cdot x}$ are harmonic, and $e^{i\rho\cdot x}e^{i\bar\rho\cdot x} = e^{-i\xi\cdot x}$.

Definition 4.4.1 (Oscillatory boundary values). With ρ as above, we define

$$f_\xi(x) := A_0^\dagger e^{i\rho\cdot x}, \text{ and } g_\xi(x) := A_0^\dagger e^{i\bar\rho\cdot x}, \quad x \in \partial X.$$

Both functions are in $L^2(\partial X)$. Assuming that a coordinate system is chosen such that $|e^{\eta\cdot x}| \leq e^{|\eta|\text{diam}(X)} = e^{|\xi|\text{diam}(X)}$, from the construction of $A_0^\dagger$, these functions satisfy the following estimate:

$$\|f_\xi\|_{L^2(\partial X)}, \|g_\xi\|_{L^2(\partial X)} \leq \frac{\alpha_0}{\omega_n} |\partial X|^{\frac{1}{2}} e^{|\xi|\frac{\text{diam}(X)}{2}}, \tag{4.35}$$

where $\alpha_0 = \|(\frac{1}{2}I + \omega_n^{-1}\mathscr{A}_0)^{-1}\|$ and $|\partial X|$ is the (Lebesgue) measure of ∂X.

We have a pseudo-inverse for A_0. However for $\sigma \neq 0$, $\langle T_1(k), f \otimes g\rangle \neq \langle k, A_0 f A_0 g\rangle$. We therefore introduce the following notation to deal with non vanishing absorption, which we treat as a perturbation.

Let us define:

$$[T_1^0(k)](x_0, x) = \int_X k(x_1) \frac{|\nu_{x_0} \cdot v_0||\nu_x \cdot v_1|}{|x_0 - x_1|^{n-1}|x_1 - x|^{n-1}}\, dx_1,$$
$$[T_1^\sigma(k)](x_0, x) = \int_X k(x_1)\,[E(x_0, x_1, x) - 1]\, \frac{|\nu_{x_0} \cdot v_0||\nu_x \cdot v_1|}{|x_0 - x_1|^{n-1}|x_1 - x|^{n-1}}\, dx_1,$$

so that $T_1(k) = T_1^0(k) + T_1^\sigma(k)$. We can prove a mapping property of T_1^σ, similar to lemma 4.3.4.

Lemma 4.4.1. *There exists C_X, depending only on X such that*

$$\|T_1^\sigma(k)\|_{L^2((\partial X)^2)} \leq C_X \|\sigma\|_{L^\infty} \|k\|_{L^\infty}. \tag{4.36}$$

Proof. The proof is identical to that of lemma 4.3.4, except that we use the relation $|e^{-a} - 1| \leq a$ (valid for $a \geq 0$) to show that $|E(x_0, x_1, x) - 1| \leq 2\mathrm{diam}(X)\|\sigma\|_{L^\infty}$. □

This yields a refined version of (4.26):

$$\begin{aligned}\langle T_1(k), f \otimes g\rangle_{L^2((\partial X)^2)} &= \langle T_1^0(k), f \otimes g\rangle_{L^2((\partial X)^2)} + \langle T_1^\sigma(k), f \otimes g\rangle_{L^2((\partial X)^2)} \\ &= \langle k, A_0 f A_0 g\rangle_{L^2(X)} + \langle T_1^\sigma(k), f \otimes g\rangle_{L^2(X)}.\end{aligned} \tag{4.37}$$

Now setting boundary values equal to f_ξ and g_ξ, (4.26) gives

$$\langle T_1^0(k), f_\xi \otimes g_\xi\rangle_{L^2((\partial X)^2)} = \langle k, A_0 f_\xi\, A_0 g_\xi\rangle_{L^2(X)} = \langle k, e^{-i(\xi, \cdot)}\rangle_{L^2(X)} = \hat{k}(\xi).$$

Here we are using (ξ, y) to denote the dot product. This leads to:

$$k(x) = \int_{\mathbb{R}^n} \hat{k}(\xi) e^{i\xi\cdot x}\, \frac{d\xi}{(2\pi)^n} = \int_{\mathbb{R}^n} \langle T_1^0(k), f_\xi \otimes g_\xi\rangle e^{i\xi\cdot x}\, \frac{d\xi}{(2\pi)^n}.$$

We may also define the following operator, whose domain contains the range of T_1^0:

$$\begin{aligned}(T_1^0)^{-1} h(x) &:= \int_{\mathbb{R}^n} \widehat{(T_1^0)^{-1} h}(\xi) e^{i\xi\cdot x}\, \frac{d\xi}{(2\pi)^n} \\ &= \int_{\mathbb{R}^n} \langle h, f_\xi \otimes g_\xi\rangle_{L^2((\partial X)^2)} e^{i\xi\cdot x}\, \frac{d\xi}{(2\pi)^n}.\end{aligned} \tag{4.38}$$

This formal inverse operator to T_1^0 will be useful in the analysis of the nonlinear inversion problem.

4.5 The nonlinear problem

4.5.1 The regularized inverse

Inverting T_1^0 is a severely ill-posed problem. To show this, put

$$L^2_D(X) := \{h \in L^2(X) : \mathrm{dist}\,(\mathrm{supp}(h), \partial X) \geq D > 0\}.$$

It is easy enough to see that $T_1^0 : L^2_D(X) \to H^s((\partial X)^2)$ is bounded for every s. Therefore $T_1^0 : L^2_D \to H^s$ is compact for every s. We may then construct a sequence of unit vectors $h_n \in L^2(X)$ such that $T_1^0 h_n \to 0$ in H^s.

We therefore look for some regularized version of $(T_1^0)^{-1}$ that is a bounded operator from $L^2((\partial X)^2) \to L^\infty(X)$. To this end, using the bound (4.35), we notice that

$$\begin{aligned} |\langle h, f_\xi \otimes g_\xi\rangle| &\leq \|h\|_{L^2((\partial X)^2)} \|f_\xi\|_{L^2(\partial X)} \|g_\xi\|_{L^2(\partial X)} \\ &\leq \|h\|_{L^2((\partial X)^2)} \frac{\alpha_0^2}{\omega_n^2} |\partial X| e^{|\xi|\mathrm{diam}(X)}. \end{aligned}$$

Our measurements and knowledge of σ give us access to:

$$\begin{aligned} \mathcal{M}(f_\xi, g_\xi) - \langle T_0, f_\xi \otimes g_\xi\rangle &= \sum_{m=1}^{\infty} \langle T_m(k), f_\xi \otimes g_\xi\rangle \\ &= \langle T_1^0(k), f_\xi \otimes g_\xi\rangle + \langle T_1^\sigma k, f_\xi \otimes g_\xi\rangle + \sum_{m=2}^{\infty} \langle T_m(k), f_\xi \otimes g_\xi\rangle \\ &= \hat{k}(\xi) + R(\xi), \end{aligned} \tag{4.39}$$

where

$$|R(\xi)| \leq C_X \alpha_0^2 |\partial X| \left(\|\sigma\|_{L^\infty} \|k\|_{L^\infty} + \|k\|_{L^\infty}^2 \right) e^{|\xi|\mathrm{diam}(X)}. \tag{4.40}$$

This bound was obtained by using lemmas 4.3.4 and 4.4.1. To deal with error that will grow exponentially with frequency, we introduce a cutoff χ and define the following regularized version of T_0^{-1} in (4.38):

Definition 4.5.1 (Regularized Inverse). We define:

$$T^\chi h(x) := \int_{\mathbb{R}^n} \langle h, f_\xi \otimes g_\xi\rangle_{L^2((\partial X)^2)} \chi(\xi) e^{i\xi\cdot x} \frac{d\xi}{(2\pi)^n}, \tag{4.41}$$

where we require that:

$$\chi(\xi) e^{|\xi|\mathrm{diam}(X)} \in L^1(\mathbb{R}^n).$$

The cut-off $\chi(\xi)$ should be seen as a function equal to 1 for small values of ξ corresponding to frequencies that we wish to invert accurately, and equal to 0 for large values of ξ corresponding to frequencies that cannot be reconstructed because of noise in the data. The simplest example of such a function is the indicatrix function χ_M equal to 1 for $|\xi| < M$ and equal to 0 elsewhere.

The following results are immediate:

$$\begin{gathered} T^\chi T_1^0(k) = P_\chi k, \quad \text{where} \quad P_\chi h(x) := \int_{\mathbb{R}^n} \hat{h}(\xi)\chi(\xi)e^{ix\cdot\xi}\,d\xi, \\ \|T^\chi h\|_{L^\infty(X)} \le \alpha_0^2 |\partial X| \|\chi(\cdot)e^{|\cdot|\mathrm{diam}(X)}\|_{L^1(\mathbb{R}^n)} \|h\|_{L^2((\partial X)^2)}. \end{gathered} \tag{4.42}$$

We now illustrate a step-by-step procedure for implementing the regularized inverse. First, set the boundary values to $f = f_\xi$ and $g = g_\xi$, with $\xi \in \mathbb{R}^n$. Our measurement is then

$$\mathcal{M}(f_\xi, g_\xi) = \left\langle T_0 + \sum_{m=1}^{\infty} T_m(k), f_\xi \otimes g_\xi \right\rangle. \tag{4.43}$$

Subtracting the known contribution $\langle T_0, f_\xi \otimes g_\xi\rangle$, and multiplying by $\chi(\xi)$ we have thereby computed

$$\left(T^\chi \sum_{m=1}^{\infty} T_m(k)\right)^\wedge (\xi).$$

We do the same thing for all $\xi \in \mathrm{supp}(\chi)$. Then after multiplying by $e^{ix\cdot\xi}$, and integrating over $\mathrm{supp}(\chi)$ we have computed

$$\begin{gathered} T^\chi \left(\sum_{m=1}^{\infty} T_m(k)\right) = k_\chi + T^\chi \left(T_1^\sigma(k) + \sum_{m=2}^{\infty} T_m(k)\right), \\ \text{where} \\ k_\chi := P_\chi k. \end{gathered} \tag{4.44}$$

We thus obtain from our measurements a reconstruction of $P_\chi k$, the low-frequency part of k, up to an error that is formally quadratic in (k, σ). In the next section, we make this statement more precise and prove our main theorems 4.2.1 and 4.2.2.

4.5.2 Proofs of main results

Using this framework, we may prove theorem 4.2.1.

Proof of theorem 4.2.1. (4.44) shows that our measurements determine k_χ up to the error term $T^\chi(T_1^\sigma k + \sum_{m=2}^\infty T_m(k))$. Using lemmas 4.3.4, 4.4.1, and (4.42) we have

$$\left\| T^\chi T_1^\sigma(k) + T^\chi \sum_{m=2}^{\infty} T_m(k) \right\|_{L^\infty(X)} \leq C_X \|T^\chi\| \left(\|\sigma\|_{L^\infty} \|k\|_{L^\infty} + \sum_{m=2}^{\infty} \|L_\sigma\|^{m-1} \|k\|_{L^\infty}^m \right).$$

Our smallness assumption on k that $\|k\|_\infty \|L_\sigma\| < 1$ ensures that the series converges, and the theorem is proved. □

Our iterative scheme is motivated by (4.44), and allows us to improve our estimate of k_χ.

Proof of theorem 4.2.2. In this proof, C_X denotes the maximum of the constants from lemma 4.3.4, and lemma 4.4.1. Each depends only on X. Defining $D := T^\chi(\sum_{m=1}^\infty T_m(k))$, we arrive at our iterative scheme.

$$\begin{aligned} k_\chi^0 &= D, \\ k_\chi^{\nu+1} &= T^\chi \left(\sum_{m=1}^{\infty} T_m(k) \right) - T^\chi \left(T_1^\sigma(k_\chi^\nu) + \sum_{m=2}^{\infty} T_m(k_\chi^\nu) \right) \\ &= D - F(k_\chi^\nu) \quad \nu = 0, 1, 2, \ldots \end{aligned} \tag{4.45}$$

The idea is that $\sum_{m=1}^\infty T_m(k)$ is our measured data, and F is a mapping we are able to compute since σ is known. To show convergence of the scheme, we will use the contraction mapping principle. To this end, we define the operator

$$G(k_\chi) := D - F(k_\chi),$$

and we will show that, under certain conditions on $\|k\|_{L^\infty}$ and $\|\sigma\|_{L^\infty}$, we can find a closed set $B \subset L^\infty(X)$, such that G is a contraction mapping on B and $G(B) \subset B$. Moreover, if $\|k\|_{L^\infty}$ is sufficiently small so that $D \in B$, then the iterated scheme (4.45) will converge.

Step 1: Condition for G to be a contraction. We first prove the following estimate:

Lemma 4.5.1. *For all $k_\chi, \tilde{k}_\chi, \sigma \in L^\infty(X)$, the following inequality holds:*

$$\|F(k_\chi) - F(\tilde{k}_\chi)\|_{L^\infty} \leq \|T^\chi\| C_X \|k_\chi - \tilde{k}_\chi\|_{L^\infty} \left(\|\sigma\|_{L^\infty} + \frac{M\|L_\sigma\|(2 - M\|L_\sigma\|)}{(1 - M\|L_\sigma\|)^2} \right), \tag{4.46}$$

where $M = \max(\|k_\chi\|_{L^\infty}, \|\tilde{k}_\chi\|_{L^\infty})$.

Proof. We start with the inequality:

$$\|F(k_\chi) - F(\tilde{k}_\chi)\|_{L^\infty} \leq \|T^\chi\| \left(\|T_1^\sigma(k_\chi - \tilde{k}_\chi)\|_{L^2} + \left\| \sum_{m=2}^{\infty} T_m(k_\chi) - T_m(\tilde{k}_\chi) \right\|_{L^2} \right). \quad (4.47)$$

Repeated use of the relation $ab - \tilde{a}\tilde{b} = (a-\tilde{a})b + \tilde{a}(b-\tilde{b})$ shows that the term $T_m(k)_\chi - T_m(\tilde{k}_\chi)$ looks like $T_m(k)$, except that instead of $k(x_1)\cdots k(x_m)$ it has the term

$$\begin{aligned} &k_\chi(x_1)\cdots k_\chi(x_m) - \tilde{k}_\chi(x_1)\cdots\tilde{k}_\chi(x_m) \\ &\qquad = \sum_{j=1}^{m} \tilde{k}_\chi(x_1)\cdots\tilde{k}_\chi(x_{j-1})[k_\chi(x_j) - \tilde{k}_\chi(x_j)]k_\chi(x_{j+1})\cdots k_\chi(x_m), \end{aligned}$$

so we may use lemma 4.3.4 to see that

$$\|T_m(k_\chi) - T_m(\tilde{k}_\chi)\|_{L^2} \leq mC_X\|L_\sigma\|^{m-1}M^{m-1}\|k_\chi - \tilde{k}_\chi\|_{L^\infty}.$$

Now since

$$\sum_{m=2}^{\infty} mx^m = \frac{x(2-x)}{(1-x)^2}, \quad |x| < 1,$$

we can sum the last equation for $m = 2\ldots\infty$ provided that $M\|L_\sigma\| < 1$. The result is the following inequality:

$$\sum_{m=2}^{\infty} \|T_m(k_\chi) - T_m(\tilde{k}_\chi)\|_{L^2} \leq C_X\|k_\chi - \tilde{k}_\chi\|_{L^\infty}\frac{M\|L_\sigma\|(2 - M\|L_\sigma\|)}{(1 - M\|L_\sigma\|)^2}.$$

We then use lemma 4.4.1 and sum the terms in (4.47) to obtain the desired result. □

Since D is constant, we will have exactly the same estimate if we replace F by G. Let us fix now $c_1 \in (0,1)$. G will be a c_1-contraction as soon as

$$\|\sigma\|_{L^\infty} + \frac{M\|L_\sigma\|(2 - M\|L_\sigma\|)}{(1 - M\|L_\sigma\|)^2} \leq \frac{c_1}{T^\chi C_X}.$$

Now we fix $\|\sigma\|_{L^\infty} < c_1(\|T^\chi\|C_X)^{-1}$, and the previous condition becomes

$$M \leq \rho_{c_1}, \quad \text{where } \rho_{c_1} := \frac{1}{\|L_\sigma\|}\left(1 - \left(1 + \frac{c_1}{\|T^\chi\|C_X} - \|\sigma\|_{L^\infty}\right)^{-\frac{1}{2}}\right). \quad (4.48)$$

In other words, if we call B the $\|\cdot\|_{L^\infty(X)}$-ball of radius ρ_{c_1} and center 0, then lemma 4.5.1 under the condition (4.48) ensures that G is a c_1-contraction on B. Let us find now the condition under which $G(B) \subset B$.

Step 2: condition for B to be G-stable. We look for a bound on $\|D\|_{L^\infty}$ such that B is G-stable. For any $k_\chi \in B$, provided that F and G are c_1-contractions on B and $F(0) = 0$, we have

$$\|G(k_\chi)\|_{L^\infty} \leq \|D\|_{L^\infty} + \|F(k_\chi)\|_{L^\infty} \leq \|D\|_{L^\infty} + c_1\rho_{c_1}.$$

Thus in order to get $\|G(k_\chi)\|_{L^\infty} \leq \rho_{c_1}$, we need that

$$\|D\|_{L^\infty} \leq (1 - c_1)\rho_{c_1}. \tag{4.49}$$

Using lemma 4.3.4 and the fact that $\|k\|_{L^\infty}\|L_\sigma\| < 1$, we have the following estimate on D

$$\|D\|_{L^\infty} \leq C_X\|T^\chi\|\frac{\|k\|_{L^\infty}}{1 - \|L_\sigma\|\|k\|_{L^\infty}}.$$

Thus (4.49) will hold if k satisfies

$$\|k\|_{L^\infty} \leq \frac{(1 - c_1)\rho_{c_1}}{C_X\|T^\chi\| + (1 - c_1)\rho_{c_1}\|L_\sigma\|}. \tag{4.50}$$

As a result of the first two steps, lemma 4.5.1, and equation (4.48), we see that the hypothesis (4.50) ensures G is a contraction mapping on B, and that $D \in B$. Thus in virtue of the contraction mapping principle, the iterated scheme (4.45) will converge to an element $k_\chi^* \in B$ such that

$$k_\chi^* = D - F(k_\chi^*). \tag{4.51}$$

Step 3: Error estimates We now show that the difference between k_χ^* and k_χ is small if $k_\chi^\perp := k - k_\chi$ is small. If we define $D_0 = T^\chi \sum_{i=1}^\infty T_i(k_\chi)$, some straightforward calculations show that

$$k_\chi = D_0 - F(k_\chi), \tag{4.52}$$

which relies on the fact that

$$T^\chi T_1^0 k_\chi = P_\chi k_\chi = k_\chi.$$

Subtracting (4.51) from (4.52) we have

$$\|k_\chi - k_\chi^*\|_{L^\infty} \leq \|D_0 - D\|_{L^\infty} + c_1\|k_\chi - k_\chi^*\|_{L^\infty}.$$

Since $c_1 < 1$, we may absorb the second term on the right hand side into the left hand side, yielding

$$\begin{aligned}
\|k_\chi - k_\chi^*\|_{L^\infty} &\leq \frac{1}{1-c_1}\|D_0 - D\|_{L^\infty} = \frac{1}{1-c_1}\left\|T^\chi \sum_{m=1}^{\infty}(T_m(k_\chi) - T_m(k_\chi + k_\chi^\perp))\right\|_{L^\infty} \\
&= \frac{1}{1-c_1}\left\|T^\chi\left(T_1(k_\chi) - T_1(k_\chi + k_\chi^\perp) + \sum_{m=2}^{\infty}(T_m(k_\chi) - T_m(k_\chi + k_\chi^\perp))\right)\right\|_{L^\infty} \\
&= \frac{1}{1-c_1}\left\|T^\chi\left(T_1^\sigma(k_\chi) - T_1^\sigma(k_\chi + k_\chi^\perp) + \sum_{m=2}^{\infty}(T_m(k_\chi) - T_m(k_\chi + k_\chi^\perp))\right)\right\|_{L^\infty} \\
&= \frac{1}{1-c_1}\|F(k_\chi) - F(k_\chi + k_\chi^\perp)\|_{L^\infty} \leq \frac{c_1}{1-c_1}\,\|k_\chi^\perp\|_{L^\infty}.
\end{aligned}$$

Here, the third equality comes from the decomposition $T_1 = T_1^0 + T_1^\sigma$ and the property $T^\chi T_1^0 k = P_\chi k$ (see 4.42). □

4.6 Conclusions

We have shown that the reconstruction of the smooth part of the scattering coefficient in a transport equation could be obtained when arbitrary isotropic sources are used and the corresponding angularly averaged outgoing currents are measured, i.e., in the setting of "diffusion-type" measurements. This corresponds to the practical setting in many applications of inverse transport [5, 34, 53]. The accuracy of the reconstruction is proportional to the size of the non-smooth part of the scattering coefficient. However, we have assumed that the total absorption coefficient σ could be reconstructed by other means and was sufficiently small. These two hypotheses are very constraining from a practical viewpoint. Nonetheless, the results we have presented give a realistic theoretical backbone to practical reconstructions of optical parameters from diffusion-type measurements.

The measurements are similar to what is available in the reconstruction of diffusion coefficients from boundary measurements, as in the application to electrical impedance tomography. [16, 49]. They are thus of the same type as the measurements available in the diffusion approximation to the transport equation, which arises in the limit of vanishing mean free path and is overwhelmingly used in optical tomography [5, 21]. Our reconstructions, however, work for small values of k and σ, i.e., in a transport regime where the diffusion approximation is not valid. It is therefore somewhat surprising that the same complex geometric optics solutions may be used in both our context and that of the reconstruction of diffusion coefficients.

We would like to stress that the reconstruction of optical parameters from diffusion-type measurements is a severely ill-posed problem. More precisely, whenever σ is smooth, the forward map $k \mapsto \sum_{i=0}^{\infty} T_i k$ takes k supported inside X to a C^∞ function on $(\partial X)^2$. This explains why the stability estimates we have obtained are of exponential type, as in Calderón's problem [16]. These results are in sharp contrast to the results obtained when either the source or the measurements (or both) are allowed to depend on the angular variable. In such instances, better stability estimates of Hölder type, which render the reconstruction a mildly ill-posed problem, are available in many settings [?, 41, 51].

Appendix

Proof of Proposition 4.3.1. Proof of equation (4.14) :
First, assume $f \in C^1(\mathbf{S})$, and extend it to $\tilde{f} \in C^1(\mathbb{R}^2 \setminus \{y\})$, where $\tilde{f}$ is constant along rays originating at y. We can think of $\mathbb{S}^{n-1}$ as a spherical surface ${\mathbb{S}^{n-1}}_y \subset \mathbb{R}^n$, centered at y. We then show

$$\int_{\mathbf{S}} \tilde{f}(x) \frac{\nu_x \cdot (x-y)}{|x-y|^n} \, d\mathbf{S}(x) = \int_{{\mathbb{S}^{n-1}}_y} \tilde{f}(x) \, dS(x).$$

The proposition then follows for C^1 f since $\tilde{f}|_{\mathbf{S}} = f$, and $\tilde{f}(x) = f(x_-(y,v))$ when $v = (x-y)|x-y|^{-1}$. We can extend the result to $f \in L^1$ by density since for fixed y, $x \mapsto \nu_x \cdot (x-y)|x-y|^{-1}$ is bounded.

Proceeding, we note that

$$\nabla \tilde{f}(x) \cdot (x-y) = 0. \tag{4.53}$$

We first prove that for any C^1 hypersurfaces S_1 and S_2 such that S_1 encloses y and S_2 encloses S_1, the following equality holds:

$$\int_{S_1} \tilde{f}(x) \frac{\nu_x \cdot (x-y)}{|x-y|^n} d\boldsymbol{S}(x) = \int_{S_2} \tilde{f}(x) \frac{\nu_x \cdot (x-y)}{|x-y|^n} d\boldsymbol{S}(x). \tag{4.54}$$

Indeed, if V denotes the volume between S_1 and S_2, the divergence theorem applied to the function $\tilde{f}(x)\nabla N(x,y)$ in the volume V yields

$$\begin{aligned}\int_V \nabla\cdot(\tilde{f}(x)\nabla N(x,y))\,dx &= \int_{\partial V}(\tilde{f}(x)\nabla N(x,y))\cdot\nu_x\,d\boldsymbol{S}(x)\\ &= \int_{S_2}\tilde{f}(x)\partial_{\nu_x}N(x,y)\,d\boldsymbol{S}(x) - \int_{S_1}\tilde{f}(x)\partial_{\nu_x}N(x,y)\,d\boldsymbol{S}(x).\end{aligned}$$

We now show that the left-hand side of the previous equation is zero: After writing

$$\nabla\cdot(\tilde{f}(x)\nabla N(x,y)) = \nabla\tilde{f}(x)\cdot\nabla N(x,y) + \tilde{f}(x)\Delta N(x,y),$$

(all the operators apply on the x variable), we first notice that $\Delta N(x,y)=0$ for all $x\neq y$, in particular on V. Second, as $N(x,y)$ is a radial function of x with respect to y then its gradient is collinear to the vector $x-y$. Using (4.53), we see that the scalar product $\nabla\tilde{f}(x)\cdot\nabla N(x,y)$ is zero on V. Finally, since $\omega_n\partial_{\nu_x}N(x,y)=\frac{\nu_x\cdot(x-y)}{|x-y|^n}$, the equality (4.54) holds.

If S is either enclosing or enclosed in $\mathbb{S}_y^{n-1}$, then the proof is done. Otherwise, pick any hypersurface S' which encloses both S and $\mathbb{S}_y^{n-1}$ then applying the first part of this proof to S and S', then to S' and $\mathbb{S}_y^{n-1}$, yields by transitivity

$$\int_S \tilde{f}(x)\frac{\nu_x\cdot(x-y)}{|x-y|^n}dS(x) = \int_{\mathbb{S}_y^{n-1}}\tilde{f}(x)\frac{\nu_x\cdot(x-y)}{|x-y|^n}dS(x) = \int_{\mathbb{S}_y^{n-1}}\tilde{f}(x)d\boldsymbol{S}(x),$$

hence the result.

Proof of equation (4.15) : As in the proof of equation (4.14), we extend $f\in C^1$ to $\tilde{f}$, which is constant along rays originating at y, and zero on all rays $\vec{r}$ such that $\vec{r}\cdot\nu_y>0$. The strict convexity of $\mathbf{S}$ allows us to do that. We then show that

$$\int_{\mathbf{S}}\tilde{f}(x)\frac{\nu_x\cdot(x-y)}{|x-y|^n}\,d\mathbf{S}(x) = \int_{\mathbb{S}^{n-1}{}_y}\tilde{f}(x)\,dS(x).$$

The corollary then follows for $f\in C^1(\mathbf{S})$ by the same reasoning as in the proof of equation (4.14). We may then extend the result for L^∞ f since for $y\in\partial X$, $x\mapsto\nu_x\cdot(x-y)|x-y|^{-1}\in L^1(\partial X)$ as can be seen using (4.18).

Let B_ε be the ball of radius ε centered at y, and $\mathbf{S}_\varepsilon=\mathbf{S}-B_\varepsilon$. $\mathbf{S}_\varepsilon$ is not a closed surface, so we cannot directly apply proposition 4.3.1 to it. Form a new closed C^2 surface $\mathbf{S}'_\varepsilon$ closing

$\mathbf{S}_\varepsilon$ in such a way that all but a small part (whose volume is $O(\varepsilon^n)$) of $\mathbf{S} \setminus \mathbf{S}_\varepsilon$ lies on the side of the tangent plane to $\mathbf{S}$ at y on which $\tilde{f}$ is identically zero. This is possible since the surface is C^2. Call this small part P_ε. Using the proof of equation (4.14)

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}{}_y} \tilde{f}(x)\, dS(x) &= \int_{\mathbf{S}'_\varepsilon} \tilde{f}(x) \frac{\nu_x \cdot (x-y)}{|x-y|^n}\, d\mathbf{S}(x) \\
&= \int_{\mathbf{S}_\varepsilon} \tilde{f}(x) \frac{\nu_x \cdot (x-y)}{|x-y|^n}\, d\mathbf{S}(x) + \int_{P_\varepsilon} \tilde{f}(x) \frac{\nu_x \cdot (x-y)}{|x-y|^n}\, d\mathbf{S}(x) \\
&= \int_{\mathbf{S}_\varepsilon} \tilde{f}(x) \frac{\nu_x \cdot (x-y)}{|x-y|^n}\, d\mathbf{S}(x) + O(\varepsilon).
\end{aligned}$$

Using (4.18), we see that the integral over $\mathbf{S}_\varepsilon$ becomes the integral over the entire boundary $\mathbf{S}$ as $\varepsilon \to 0$. □

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4.6.1 Vita

VITA

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