NON-INTERSECTION EXPONENTS FOR BROWNIAN PATHS PART I. EXISTENCE AND AN INVARIANCE PRINCIPLE

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Abstract. Let X and Y be independent 3-dimensional Brownian motions, $X(0) = (0,0,0)$, $Y(0) = (1,0,0)$ and let $p_r = P(X[0,r] \cap Y[0,r] = \emptyset)$. Then the "non-intersection exponent" $\lim_{r\to\infty} -\log p_r / \log r$ exists and is equal to a similar "non-intersection exponent" for random walks. Analogous results hold in $\mathbb{R}^2$ and for more than 2 paths.

1. Introduction and main results. Let $n = 2$ or 3 and let $X^1, X^2, \dots, X^k$, $Y^1, Y^2, \dots, Y^m$, be independent n-dimensional Brownian motions,

$$X^1(0) = X^2(0) = \dots = X^k(0) = 0,$$

$$Y^1(0) = Y^2(0) = \dots = Y^m(0) = (1,0) \text{ or } (1,0,0).$$

Let $S^1, S^2, \dots, S^k, U^1, U^2, \dots, U^m$ be independent n-dimensional simple random walks on $\mathbb{Z}^n$,

$$S^1(0) = S^2(0) = \dots = S^k(0) = 0,$$

$$U^1(0) = U^2(0) = \dots = U^m(0) = (1,0) \text{ or } (1,0,0).$$

Denote

$$T_Z(r) = \inf\{t \ge 0 : |Z(t)| \ge r\},$$

where $r > 0$ and Z may be any of the processes X^j, Y^j, S^j or U^j. Let

$$p_r = P(\bigcup_{j=1}^k X^j[0, T_{X^j}(r)] \cap \bigcup_{j=1}^m Y^j[0, T_{Y^j}(r)] = \emptyset),$$

$$q_r = P(\bigcup_{j=1}^k S^j[0, T_{S^j}(r)] \cap \bigcup_{j=1}^m U^j[0, T_{U^j}(r)] = \emptyset),$$

$$a_r = P(\bigcup_{j=1}^k X^j[0, r] \cap \bigcup_{j=1}^m Y^j[0, r] = \emptyset),$$

$$b_r = P(\bigcup_{j=1}^k S^j[0, r] \cap \bigcup_{j=1}^m U^j[0, r] = \emptyset).$$

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Brownian paths intersect with positive probability in 2 and 3 dimensions (Dvoretzky, Erdös and Kakutani (1950)) and, therefore, $0 < p_r < 1$ and $0 < a_r < 1$ for $r > 1$. Obviously, $0 < q_r < 1$ and $0 < b_r < 1$ for $r > 1$.

Theorem 1.1. *For every $n = 2$ or $3, k \geq 1$, $m \geq 1$, the limits*

$$\lim_{r\to\infty} -\log p_r / \log r \stackrel{\text{df}}{=} \xi = \xi(n,k,m) \tag{1.1}$$

and

$$\lim_{r\to\infty} -\log q_r / \log r \tag{1.2}$$

exist and are equal.

Theorem 1.2. *For every $n = 2$ or 3, $k \geq 1$, $m \geq 1$, we have*

$$\lim_{r\to\infty} -\log a_r / \log r = \lim_{r\to\infty} -\log b_r / \log r = \xi/2.$$

The constant $\xi/2$ was denoted ζ in other articles on the subject. The proof of Theorem 1.2 will be omitted since it follows easily from Theorem 1.1 and the following standard estimate

$$P(r^{1-\xi} \leq T_Z(\sqrt{r}) \leq r^{1+\xi}) = 1 - \mathcal{O}(\exp(-cr^a)),$$

where $\xi > 0$, $c = c(\xi) > 0$, $a = a(\xi) > 0$ and Z is any of the processes X^j, Y^j, S^j or U^j.

Until now, the limits in (1.1)-(1.2) were known to exist only in the case $n = 2$ or 3, $k = 2$, $m = 1$ (Lawler (1989)).

It has been conjectured that

$$q_r \sim cr^{-\xi} \tag{1.3}$$

or

$$q_r \sim f(r) r^{-\xi}$$

where f is a slowly varying function. The constants ξ may be called "non-intersection exponents" in view of Theorems 1.1-1.2 and (1.3). The asymptotic form (1.3) of q_r is usually assumed when numerical estimates of the exponent ξ are sought (Duplantier and Kwon (1988), Burdzy, Lawler and Polaski (1989)). Lawler (1989) proved a result stronger than (1.2) but weaker than (1.3) for $n = 2$ or 3, $k = 2$, $m = 1$.

Jaime San Martin has pointed out to us that the existence of the limit in (1.1) can be proved in an alternative way, using properties of subadditive functions. However, we know of no easy proof that the limit in (1.2) exists or that the two limits are the same. Our proof makes use of the Harnack principle for random walks (Lawler (1988)).

Our results will enable us in part II of this paper (Burdzy and Lawler (1988)) to apply to Brownian motion estimates obtained earlier for random walks. We will derive some new estimates of non-intersection exponents and will apply them to estimates of Hausdorff dimension of a fractal and to some local geometric properties of Brownian paths.

Some non-intersection exponents are known, for example $\xi(2,2,1) = 2$ and $\xi(3,2,1) = 1$ (Lawler (1989)). This implies that

$$\frac{4-n}{2} \le \xi(n,1,1) \le 4-n, \qquad n = 2,3.$$

This is the best rigorous bound known for $n = 3$, but for $n = 2$ it has been proved that $1 < \xi(2,1,1) \le 3/2$ (Burdzy, Lawler and Polaski (1989)). Duplantier and Kwon (1988) have given a number of nonrigorous exact values including the conjecture $\xi(2,1,1) = 5/4$. Monte Carlo simulations suggest that $\xi(3,1,1) \approx .57$. The interest in non-intersection problems stems in part from the relationship between such problems and the quantum field theory (Aizenman (1982)). For further information about random walk intersection problems, the reader is referred to Lawler (1989).

The next section contains notation (somewhat different from the above one) and a review of some useful results. Theorem 1.1 will be rephrased and proved in Section 3. We will treat only the case $n = 3$, $k = 1$, $m = 1$. The case $k > 1$ or $m > 1$ poses no new problems but requires more complicated notation and, therefore, is omitted. The case $n = 2$ is not different from $n = 3$; our proofs remain valid in the 2-dimensional space.

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2. Preliminaries. In order to save space, we will limit ourselves to introducing notation without any discussion of the concepts. The reader is advised to consult Doob (1984) and Williams (1979) concerning Brownian motion, h-processes, etc.

The symbols $\mathbb{R}(\mathbb{R}_+)$ and $\mathbb{Z}(\mathbb{Z}_+)$ will stand for the sets of all real (non-negative) numbers and (non-negative) integers.

$0 = (0,0,0), 1 = (1,0,0)$,

$B(x,r) = \{y \in \mathbb{R}^3 : |x-y| < r\}$,

$B(r) = B(0,r)$,

$K(x,r) = \{y \in \mathbb{R}^3 : |x_j - y_j| \le r, j = 1,2,3\}$,

$K_{\mathbb{Z}}(x,r) = K(x,r) \cap \mathbb{Z}^3$,

$K(r) = K(0,r)$,

$K_{\mathbb{Z}}(r) = K_{\mathbb{Z}}(0,r)$,

$\partial K_{\mathbb{Z}}(x,r) = \{y \in K_{\mathbb{Z}}(x,r) : \max_{j=1,2,3} |x_j - y_j| = r\}$.

We will always tacitly assume that $x \in \mathbb{Z}^3$ and $r \in \mathbb{Z}$ when using the notation $K_{\mathbb{Z}}(x,r)$, $K_{\mathbb{Z}}(r)$, etc.

For a subset A of $\mathbb{R}^3$, its closure, boundary and complement will be denoted $\overline{A}, \partial A$ and A^c. The usual distance between sets (or a set and a point) in $\mathbb{R}^3$ will be denoted $\mathrm{dist}(\,\cdot\,,\,\cdot\,)$.

Let Ω be the space of all paths $\omega : [0,\infty) \to \mathbb{R}^3 \cup \{\delta\}$ such that ω is continuous on $[0,R)$ and equal to δ otherwise. The "lifetime" $R = R(\omega)$ may be infinite. The state δ is an isolated trap in $\mathbb{R}^3 \cup \{\delta\}$. We will often define processes on a finite interval only; it is understood that they are equal to δ for the remaining values of the time parameter.

The canonical process will be defined by $X^1(t) = X^1_\omega(t) = \omega(t)$ for all ω and t. We will use an identical copy of X^1, denoted X^2, so that (X^1, X^2) will be defined on the product space.

We will need several measures on Ω and Ω^2. Under P^0, X^1 will be a Brownian motion with variance parameter 1/3 (i.e. the limiting process for $n^{-1/2}S[nt]$ where S is a three dimensional simple random walk). The P^0-distribution of X^1+x will be denoted P^x. We note that the non-intersection exponents ξ under $P^{\cdot}$ are the same as for a standard Brownian motion. All h-processes (i.e. conditioned Brownian motions) and killed Brownian motions will be defined relative to $P^{\cdot}$ rather than standard Brownian distributions. The symbol P^x_h will stand for the distribution of an h-process starting from x, for a positive superharmonic function h defined in some Greenian domain $D \subset \mathbb{R}^3$. The measure $P^{x,y}_{h_1,h_2}$ on Ω^2 will make X^1 and X^2 independent, with distributions $P^x_{h_1}$ and $P^y_{h_2}$. The symbols $P^\mu_h, P^{\mu_1,\mu_2}_{h_1,h_2}$, where μ, μ_1 and μ_2 are measures on $\mathbb{R}^3$, will be interpreted in the usual way.

Let Ω_1 be the space of paths $\omega : \mathbb{Z}_+ \to \mathbb{Z}^3 \cup \{\delta\}$, such that $\omega(t) \in \mathbb{Z}^3$ for $t < R$ and $\omega(t) = \delta$ otherwise. The lifetime $R = R(\omega)$ may be infinite. Let $S^1(t) = S^1_\omega(t) = \omega(t)$. The process S^2 will be an identical copy of S^1.

Under Q^x, S^1 will be a simple random walk on $\mathbb{Z}^3$ starting from x.

A function h defined on $D \subset \mathbb{Z}^3$ will be called harmonic if for each $x \in D$, $h(x)$ is the average of its values at the 6 adjacent points y (i.e. $|x-y| = 1$).

The symbol Q^x_h will denote the distribution of the random walk conditioned by h; such a process jumps from x to one of its nearest neighbors y ($|x-y|=1$) with probability $\frac{1}{6}h(y)/h(x)$. The symbols $Q^{x,y}_{h_1,h_2}$, $Q^{\mu_1,\mu_2}_{h_1,h_2}$, etc, will have the meaning analogous to $P^{x,y}_{h_1,h_2}$, $P^{\mu_1,\mu_2}_{h_1,h_2}$ etc.

For a continuous time process Z we will write

$$T_Z(A) = \inf\{t > 0 : \lim_{s \uparrow\uparrow t} Z(s) \in A\},$$

and for a discrete time process Z,

$$T_Z(A) = \min\{t \geq 0 : Z(t) \in A\}.$$

The uniform probability measure on $\partial B(r)$ will be denoted σ_r.

$$\nu_r(dx) = P^0(X^1(T_{X^1}(\partial K(r))) \in dx),$$

$$\eta_r(A) = Q^0(S^1(T_{S^1}(\partial K_{\mathbb{Z}}(r))) \in A),$$

$$\sigma = \sigma_1, \nu = \nu_1, \eta = \eta_1.$$

For a (Greenian) domain $D \subset \mathbb{R}^3$, its Green function will be denoted $G_D(\,\cdot\,,\,\cdot\,)$.

The trace of a process will be denoted as

$$X^1(s,t] = \{x \in \mathbb{R}^3 : \exists\, u \in (s,t] \text{ such that } X(u) = x\},$$

and $X^2[s,t], S^1[s,t)$, etc. will have a similar interpretation.

Lemma 2.1. *(Brownian scaling) Suppose that h is a superharmonic and positive function in a Greenian domain $D \subset \mathbb{R}^3$, $x \in \overline{D}$ and $A \subset \Omega$. Denote*

$D_c = \{y \in \mathbb{R}^3 : \exists\, z \in D \text{ such that } cz = y\}$,

$h_c(z) = h(z/c)$,

$A_c = \{\omega \in \Omega : \exists\, \omega_1 \in A \;\forall\, t \quad c\omega_1(t) = \omega(c^2 t)\}$.

Then $P^{cx}_{h_c}(A_c) = P^x_h(A)$.

Proof. The lemma follows immediately from the scaling properties of Brownian motion and the definition of an h-process (Doob (1984)). □

Lemma 2.2. *(Discrete Harnack principle; Lawler (1988)) There exist $c > 0$ such that for all positive harmonic functions h in $K_{\mathbb{Z}}(2n)$ and all $x, y \in K_{\mathbb{Z}}(n)$ we have*

$$h(x)/h(y) > c. \quad \square$$

Standard methods allow to generalize the discrete Harnack principle to domains different from $K_{\mathbb{Z}}(n)$ and $K_{\mathbb{Z}}(2n)$.

We will use repeatedly h-processes and, therefore, we present here a short review of some of their properties. The proofs may be found in Doob (1984) and Meyer, Smyth and Walsh (1972).

Let $D \subset \mathbb{R}^3$ be a Greenian domain and h be a positive superharmonic function in D. Let $p^D_t(x, y)$ be the transition density for Brownian motion killed at $T_{X^1}(D^c)$ and

$$p^h_t(x, y) = p^D_t(x, y)h(y)/h(x).$$

Any process with the p^h_t-transition densities will be called an h-process (conditioned Brownian motion).

Suppose that M is a closed subset of D and let

$$L = \sup\{t < R : X^1(t) \in M\}$$

be the last exist time from M. Denote

$Y^1(t) = X^1(t), \quad t \in [0, T_{X^1}(M))$,

$Y^2(t) = X^1(T_{X^1}(M) + t), \quad t \in [0, R - T_{X^1}(M))$,

$Y^3(t) = X^1(t), \quad t \in [0, L)$,

$Y^4(t) = X^1(L + t), \quad t \in [0, R - L)$,

$Y^5(t) = X^1(R - t), \quad t \in (0, R)$.

Under P^x_h, each process Y^k is an h_k-process in a domain D_k.

$D_1 = D_4 = D \backslash M$,

$D_2 = D_3 = D_5 = D$.

$h_1 = h_2 = h$.

h_3 is a potential supported by ∂M.

h_4 has the boundary values 0 on ∂M and the same boundary values as h on $\partial D \backslash \partial M$.

h_5 is the Green function $G_D(x,\,\cdot\,)$ if $x \in D$ or a harmonic function with a pole at x if $x \in \partial D$.

The initial distributions of Y^1 and Y^3 are concentrated on $\{x\}$. For the remaining initial distributions see Doob (1984).

If $\mu(dy)$ is the P^x-distribution of

$$X^1(\inf\{t < T_{X^1}(D^c) : X^1(t) \in M\})$$

then the P^x_h-distribution of this random variable is $\mu(dy)h(y)/h(x)$. We will use the last fact not only in the context of Brownian motion but random walk as well.

3. Proofs. First we will rephrase Theorem 1.1. Recall that we plan to discuss only the case of 2 paths in 3 dimensions. Denote

$$p_r = P^{0,1}(X^1[0, T_{X^1}(B^c(r))] \cap X^2[0, T_{X^2}(B^c(r))] = \emptyset),$$
$$q_r = Q^{0,1}(S^1[0, T_{S^1}(B^c(r))] \cap S^2[0, T_{S^2}(B^c(r))] = \emptyset),$$
$$\xi_0 = \liminf_{r\to\infty} -\log p_r / \log r,$$
$$\xi_1 = \limsup_{r\to\infty} -\log p_r / \log r.$$

Theorem 3.1.

$$\limsup_{r\to\infty} -\log q_r / \log r \le \xi_0, \tag{3.1}$$

$$\liminf_{r\to\infty} -\log q_r / \log r \ge \xi_1. \tag{3.2}$$

It is obvious that (3.1) and (3.2) imply

$$\lim_{r\to\infty} -\log p_r / \log r = \lim_{r\to\infty} -\log q_r / \log r;$$

in particular, both limits exist. The proof of Theorem 3.1 will consist of several lemmas.

Lemma 3.1. *Let h be the harmonic function in $B(1)\backslash\overline{B}(r)$ which has the boundary values 1 on $\partial B(1)$ and 0 on $\partial B(r)$. Denote*

$$a_1 = \operatorname{dist}(X^1[0, T_{X^1}(\partial B(1))), X^2(T_{X^2}(\partial B(1)))),$$
$$a_2 = \operatorname{dist}(X^2[0, T_{X^2}(\partial B(1))), X^1(T_{X^1}(\partial B(1)))),$$
$$a = \min(a_1, a_2).$$

For every $p > 0$ there exists $\varepsilon > 0$ such that for all $r \le 1/4$ and all $x, y \in \partial B(1/2)$ we have

$$P^{x,y}_{h,h}(a < \varepsilon) < p.$$

Proof. The $P^{0,0}$-distributions of $X^1(T_{X^1}(\partial B(1)))$ and $X^2(T_{X^2}(\partial B(1)))$ (i.e. σ and σ) are non-atomic so, by independence,

$$|X^1(T_{X^1}(\partial B(1))) - X^2(T_{X^2}(\partial B(1)))| > 0 \quad P^{0,0}\text{-a.s.}$$

This and the continuity of paths imply that for every $p_1 > 0$ there is an $\varepsilon > 0$ such that

$$P^{0,0}(a < \varepsilon) < p_1.$$

Denote

$$L_j = \sup\{t < T_{X^j}(\partial B(1)) : X^j(t) \in B(r)\},$$

$$T_j = \inf\{t > L_j : X^j(t) \in \partial B(3/4)\},$$

$$Y^j(t) = X^j(T_j + t), \quad t \in [0, T_{X^j}(\partial B(1)) - T_j),$$

$$\lambda = \sigma_{3/4}.$$

The process (Y^1, Y^2) has the distribution $P^{\lambda,\lambda}_{h,h}$. Since the paths of Y's are subsets of the paths of X's, we have

$$P^{\lambda,\lambda}_{h,h}(a < \varepsilon) < p_1. \tag{3.3}$$

Let $\mu(x)$ denote the P^x_h-distribution of $X^1(T_{X^1}(\partial B(3/4)))$. Obviously, we may limit ourselves to $\varepsilon < 1/4$. Then the event $\{a < \varepsilon\}$ is determined by the parts of the paths in $B(1)\backslash\overline{B}(3/4)$ and, therefore,

$$P^{x,y}_{h,h}(a < \varepsilon) = P^{\mu(x),\mu(y)}_{h,h}(a < \varepsilon)$$

for $x, y \in \partial B(1/2)$.

An easy application of the Harnack principle in $B(5/8)\backslash\overline{B}(3/8)$ shows that the density of $\mu(x)$ with respect to λ is in an interval (c^{-1}, c), $c < \infty$, for all $r \le 1/4$ and $x \in \partial B(1/2)$. Thus, the Radon-Nikodym derivative $dP^{\mu(x),\mu(y)}_{h,h}/dP^{\lambda,\lambda}_{h,h}$ is less or equal to c^2. This combined with (3.3) yields

$$P^{\mu(x),\mu(y)}_{h,h}(a < \varepsilon) = P^{x,y}_{h,h}(a < \varepsilon) < p_1 c^2.$$

Now take $p_1 = p/c^2$ and the proof is complete. □

Lemma 3.2. *Suppose that h is a harmonic function in $B(r)\backslash\overline{B}(1)$ with the boundary values 1 on $\partial B(1)$ and 0 on $\partial B(r)$. Let a be defined as in Lemma 3.1. Then for every $p > 0$ there exists $\varepsilon > 0$ such that for all $r \ge 4$ and all $x, y \in \partial B(2)$ we have*

$$P^{x,y}_{h,h}(a < \varepsilon) < p. \quad \square$$

We omit the proof of the above lemma as it is completely analogous to that of Lemma 3.1. We will apply the last two lemmas in the spirit of "Brownian scaling" (see Section 2).

Lemma 3.3. *Let h be harmonic in $B(r)\backslash\overline{B}(1)$ with the boundary values 1 on $\partial B(r)$ and 0 on $\partial B(1)$. Denote*

$$A(r) = \{X^1[0, T_{X^1}(\partial B(r))] \cap X^2[0, T_{X^2}(\partial B(r))] = \emptyset\}.$$

Then if $v_r = P^{\sigma,\sigma}_{h,h}(A(r))$,

$$\liminf_{r\to\infty} \frac{-\log v_r}{\log r} \le \xi_0.$$

Proof. Let $x = (1/2, 0, 0)$. Then, by Brownian scaling,

$$P^{0,x}(A(r)) = p_{2r}. \tag{3.4}$$

Let

$$L_j = \sup\{t < T_{X^j}(\partial B(r)) : X^j(t) \in B(1)\},\ j = 1, 2,$$

and let $\mu_1 \times \mu_2$ be the distribution of $(X^1(L_1), X^2(L_2))$ under $P^{0,x}$. Note that $\mu_1 = \sigma$. Under $P^{0,x}$, the process

$$\{X^1(L_1 + t), \quad t \in [0, T_{X^1}(\partial B(r)) - L_1)\}$$

has the distribution $P_h^{\mu_1}$ and a similar statement holds for X^2. In view of (3.4) we have

$$P_{h,h}^{\sigma,\mu_2}(A(r)) = P_{h,h}^{\mu_1,\mu_2}(A(r)) \geq p_{2r}.$$

It is easy to see that the density of μ_2 with respect to σ is less than some $c_1 < \infty$ and, therefore,

$$P_{h,h}^{\sigma,\sigma}(A(r)) \geq p_{2r}/c_1,$$

from which the lemma follows. □

Lemma 3.4. *Let h and $A(r)$ be as in Lemma 3.3. Denote*

$$\begin{aligned} A_1^\varepsilon(r) = \{&\mathrm{dist}(X^1(0), X^2[0, T_{X^2}(\partial B(r)))) > \varepsilon, \\ &\mathrm{dist}(X^2(0), X^1[0, T_{X^1}(\partial B(r)))) > \varepsilon, \\ &\mathrm{dist}(X^1(T_{X^1}(\partial B(r))), X^2[0, T_{X^2}(\partial B(r)))) > \varepsilon r, \\ &\mathrm{dist}(X^2(T_{X^2}(\partial B(r))), X^1[0, T_{X^1}(\partial B(r)))) > \varepsilon r\}. \end{aligned}$$

There exists $\varepsilon > 0$ such that if $u_r^\varepsilon = P_{h,h}^{\sigma,\sigma}(A(r) \cap A_1^\varepsilon(r))$,

$$\liminf_{r\to\infty} \frac{-\log u_r^\varepsilon}{\log r} \leq \xi_0. \tag{3.5}$$

Proof. Take any $\xi > \xi_0$. By the previous lemma, for each $r_0 < \infty$ we can find $r > r_0$ with

$$v_r > r^{-\xi}. \tag{3.6}$$

First we will show that for each $r_0 < \infty$ there is $r > r_0$ such that

$$v_r < 2 \cdot 4^\xi v_{4r} \tag{3.7}$$

and

$$v_{4r} > (4r)^{-\xi}. \tag{3.8}$$

Note that $r \to v_r$ is a non-increasing function. If

$$v_{r_0 4^k} > (r_0 4^k)^{-\xi} \tag{3.9}$$

is satisfied for only a finite number of $k \in \mathbb{Z}_+$ then Lemma 3.3 is contradicted. Thus, (3.9) holds for infinitely many k.

Now suppose that

$$v_{r_0 4^{k-1}} < 2 \cdot 4^{\xi}\, v_{r_0 4^k} \tag{3.10}$$

holds for no $k \geq 1$. Then, again, the function $r \to v_r$ would decay faster than $r^{-\xi}$ (use its monotonicity) which contradicts (3.6). It follows that (3.10) holds for some $k \geq 1$.

If (3.9) holds for all $k \geq 1$ then (3.7) and (3.8) are satisfied by $r = r_0 4^{k-1}$, where k satisfies (3.10). If (3.9) does not hold for all $k \geq 1$, then choose a k so that (3.9) is satisfied by k but not by $k-1$. Then (3.7) and (3.8) hold for $r = r_0 4^{k-1}$. This completes the proof of (3.7)–(3.8).

Choose $r > r_0$, $r > 4$, such that it satisfies (3.7) and (3.8). Let $D_1 = B(4r)\backslash \overline{B}(1)$, $D_2 = B(2r)\backslash \overline{B}(2)$, $D_3 = B(4r)\backslash \overline{B}(2)$. Denote by h_k the harmonic function in D_k with the boundary values 1 on

$$\begin{aligned} \partial B(1) &\text{ for } k = 1, \\ \partial B(2r) &\text{ for } k = 2, \\ \partial B(4r) &\text{ for } k = 3, \end{aligned}$$

and 0 elsewhere. Let h_4 be harmonic in D_1 with the boundary values 1 on $\partial B(4r)$ and 0 elsewhere. Denote for $j = 1, 2$,

$$\begin{aligned} &L_j = \sup\{t < T_{X^j}(\partial B(4r)) : X^j(t) \in \partial B(2)\}, \\ &T_j = \inf\{t > L_j : X^j(t) \in \partial B(2r)\}, \\ &Y_1^j(t) = X^j(L_j - t), \quad t \in (0, L_j), \\ &Y_2^j(t) = X^j(L_j + t), \quad t \in (0, T_j - L_j), \\ &Y_3^j(t) = X^j(T_j + t), \quad t \in (0, T_{X^j}(\partial B(4r)) - T_j). \end{aligned}$$

By the strong Markov property, last exit decomposition and time reversal, the processes (Y_1^1, Y_1^2), (Y_2^1, Y_2^2) and (Y_3^1, Y_3^2) are independent under $P^{\sigma,\sigma}_{h_4,h_4}$ given $X^1(L_1), X^2(L_2)$, $X^1(T_1)$ and $X^2(T_2)$ and their distributions are $P^{\sigma_2,\sigma_2}_{h_1,h_1}$, $P^{\sigma_2,\sigma_2}_{h_2,h_2}$ and $P^{\sigma_{2r},\sigma_{2r}}_{h_3,h_3}$. It follows that given an event F in the σ-field $\sigma(Y_2^1, Y_2^2)$, the processes (Y_1^1, Y_1^2) and (Y_3^1, Y_3^2) have distributions $P^{\mu_1,\mu_2}_{h_1,h_2}$ and $P^{\mu_3,\mu_4}_{h_3,h_3}$ for some measures μ_1 and μ_2 on $\partial B(2)$ and μ_3 and μ_4 on $\partial B(2r)$. Using Lemmas 3.1 and 3.2 (and Brownian scaling), we see that, given any $F \in \sigma(Y_2^1, Y_2^2)$, the $P^{\sigma,\sigma}_{h_4,h_4}$-probability of $(A_1^{\varepsilon}(4r))^c$ is less than $(4^{-\xi/2}/2)^2$, for a suitable $\varepsilon > 0$. Now let

$$F = \{Y_2^1(0, T_{Y_2^1}(\partial B(2r))) \cap Y_2^2(0, T_{Y_2^2}(\partial B(2r))) = \emptyset\}.$$

Then, by (3.7) and Brownian scaling,

$$\begin{aligned} P^{\sigma,\sigma}_{h_4,h_4}(A(4r) \cap (A_1^{\varepsilon}(4r))^c) &\leq P^{\sigma,\sigma}_{h_4,h_4}(F \cap (A_1^{\varepsilon}(4r))^c) \\ &\leq P^{\sigma_2,\sigma_2}_{h_2,h_2}(A(2r))4^{-\xi}/4 \\ &= v_r \cdot 4^{-\xi}/4 \\ &\leq 2 \cdot 4^{\xi} v_{4r} 4^{-\xi}/4 \\ &\leq v_{4r}/2. \end{aligned}$$

Since $P^{\sigma,\sigma}_{h_4,h_4}(A(4r)) = v_{4r}$, we have by (3.8),

$$\begin{aligned} P^{\sigma,\sigma}_{h_4,h_4}(A(4r)\cap A_1^\varepsilon(4r)) &\geq v_{4r} - v_{4r}/2 \\ &= v_{4r}/2 \\ &\geq (4r)^{-\xi}/2, \end{aligned}$$

and the lemma follows. □

Lemma 3.5. *Let*

$$\begin{aligned} A_2(r) = \{&X^1[0,T_{X^1}(\partial B(r))]\cap B(1) \subset B(X^1(0),\varepsilon/8), \\ &X^2[0,T_{X^2}(\partial B(r))]\cap B(1) \subset B(X^2(0),\varepsilon/8), \\ &\mathrm{dist}(X^1(T_{X^1}(\partial B(r))), X^2[0,T_{X^2}(\partial B(r))]) > \varepsilon r, \\ &\mathrm{dist}(X^2(T_{X^2}(\partial B(r))), X^1[0,T_{X^1}(\partial B(r))]) > \varepsilon r\}. \end{aligned}$$

There exists $\varepsilon > 0$ such that for every $\xi > \xi_0$ and $r_0 < \infty$ there exists $r > r_0$ such that for all $x, y \in \partial B(1)$ with $|x-y| > \varepsilon/4$ we have

$$P^{x,y}(A(r)\cap A_2(r)) > r^{-\xi}. \tag{3.11}$$

Proof. Let h be harmonic in $B(r)\backslash\overline{B}(2)$ with boundary values 1 on $\partial B(r)$ and 0 otherwise. Denote for $j = 1, 2$,

$$\begin{aligned} &L_j = \sup\{t < T_{X^j}(\partial B(r)) : X^j(t) \in B(2)\}, \\ &Y_1^j(t) = X^j(t), \quad t \in [0, L_j), \\ &Y_2^j(t) = X^j(L_j + t), \quad t \in (0, T_{X^j}(\partial B(r)) - L_j). \end{aligned}$$

Let $\mu(x)\times\mu(y)$ be the distribution of $(X^1(L_1), X^2(L_2))$ under $P^{x,y}$. Then the distribution of (Y_2^1, Y_2^2) is $P^{\mu(x),\mu(y)}_{h,h}$. It is easy to see that the Radon-Nikodym derivative of $\mu(x)\times\mu(y)$ with respect to $\sigma_2\times\sigma_2$ is bounded below by $c > 0$, uniformly for all $x, y \in \partial B(1)$.

Let $A_3(r)$ be defined as $A(r)\cap A_1(r)$ except that X^1 and X^2 should be replaced by Y_2^1 and Y_2^2. Then Lemma 3.4 and Brownian scaling imply that for every $\xi_2 \in (\xi_0, \xi)$, $r_0 < \infty$ and some $r > r_0$ we have for all $x, y \in \partial B(1)$

$$\begin{aligned} P^{x,y}(A_3(r)) &= P^{\mu(x),\mu(y)}_{h,h}(A(r)\cap A_1(r)) \\ &\geq cP^{\sigma,\sigma}_{h,h}(A(r)\cap A_1(r)) \\ &\geq c(r/2)^{-\xi_2}. \end{aligned} \tag{3.12}$$

Suppose that $x, y \in \partial B(1), |x-y| > \varepsilon/4$, and assume that (X^1, X^2) has the distribution $P^{x,y}$ conditioned by $A_3(r)$ and $\{X^1(L_1) = u^1, X^2(L_2) = u^2\}$; in particular, $|u^1 - u^2| > \varepsilon$. Then the process (Y_1^1, Y_1^2) has the distribution $P^{x,y}_{g_1,g_2}$ where $g_j(\,\cdot\,) = G_{B(r)}(u^j, \,\cdot\,), j = 1, 2$. It is easy to see that there is an integer $N = N(\varepsilon)$ such that for all $x, y \in \partial B(1)$, $u^1, u^2 \in \partial B(2)$, $|x-y| > \varepsilon/4, |u^1-u^2| > \varepsilon$, there exist sequences of balls $\{B_j^1\}_{j=1}^{k_1}, \{B_j^2\}_{j=1}^{k_2}$ with the following properties.

$B_1^1 = B(x, \varepsilon/8)$,

$$
\begin{aligned}
&B^1_k = B(z^k, \varepsilon/8), \quad z^k \in \partial B^1_{k-1}, \quad k \geq 2,\\
&B^1_{k_1} = B(u^1, \varepsilon/8), \quad k_1 \leq N,\\
&B^2_1 = B(y, \varepsilon/8),\\
&B^2_k = B(v^k, \varepsilon/8), \quad v^k \in \partial B^2_{k-1}, \quad k \geq 2,\\
&B^2_{k_2} = B(u^2, \varepsilon/8), \quad k_2 \leq N,\\
&C^1 = \bigcup_{k=1}^{k_1} B^1_k, \;\; C^2 = \bigcup_{k=1}^{k_2} B^2_k,\\
&\overline{C^1} \cap \overline{C^2} = \emptyset.
\end{aligned}
$$

A standard argument shows that

$$
P^{x,y}_{g_1,g_2}(X^1[0, R_1) \subset C^1, X^2[0, R_2) \subset C^2) > p = p(N(\varepsilon)) > 0 \tag{3.13}
$$

where R_1 and R_2 are lifetimes of X^1 and X^2 under $P^{x,y}_{g_1,g_2}$. Note that if $Y^1_1[0, L_1) \subset C^1, Y^2_1[0, L_2) \subset C^2$ and $A_3(r)$ holds then $A(r) \cap A_2(r)$ holds. Thus, by (3.12)–(3.13),

$$
P^{x,y}(A(r) \cap A_2(r)) > c(r/2)^{-\xi_2} p.
$$

This implies (3.11) for sufficiently large r, since $\xi_2 < \xi$. □

Lemma 3.6. *Denote*

$$
\begin{aligned}
A_4(r) = \{&S^1[0, T_{S^1}(B^c(mr))] \cap S^2[0, T_{S^2}(B^c(mr))] = \emptyset,\\
&S^1[0, T_{S^1}(B^c(mr))] \cap \overline{B}(m+1) \subset B(S^1(0), m\varepsilon/4),\\
&S^2[0, T_{S^2}(B^c(mr))] \cap \overline{B}(m+1) \subset B(S^2(0), m\varepsilon/4),\\
&\mathrm{dist}(S^1(T_{S^1}(B^c(mr))), S^2[0, T_{S^2}(B^c(mr))]) > mr\varepsilon/2,\\
&\mathrm{dist}(S^2(T_{S^2}(B^c(mr))), S^1[0, T_{S^1}(B^c(mr))]) > mr\varepsilon/2\}.
\end{aligned}
$$

There exists $\varepsilon > 0$ such that for each $\xi > \xi_0$ and $r_0 < \infty$ there exist $m_0 > 0$ and $r > r_0$ such that for all $m > m_0$ and all $x, y \in \mathbb{Z}^3$ with $\mathrm{dist}(x, \partial B(m)) \leq 1$, $\mathrm{dist}(y, \partial B(m)) \leq 1, |x - y| > m\varepsilon/2$, *we have*

$$
Q^{x,y}(A_4(r)) > r^{-\xi}.
$$

Proof. Let $\xi_2 \in (\xi_0, \xi)$. It follows immediately from the invariance principle (see e.g. its strong version in Csörgö and Révész (1981) Theorem 2.2.1) and Lemma 3.5 that

$$
Q^{x,y}(A_4(r)) > r^{-\xi_2}/2
$$

for large m and x, y, ε, etc. as specified in the lemma. For large r, the right hand side is greater than $r^{-\xi}$. □

Proof of (3.1). For $k \in \mathbb{Z}_+$, let F_k be the event

$$
\begin{aligned}
&\{S^1[T_{S^1}(B^c(r^k)), T_{S^1}(B^c(r^{k+1}))] \cap S^2[T_{S^2}(B^c(r^k)), T_{S^2}(B^c(r^{k+1}))] = \emptyset,\\
&\; S^j[T_{S^j}(B^c(r^k)), T_{S^j}(B^c(r^{k+1}))] \cap B(r^k + 1) \subset B(S^j(T_{S^j}(B^c(r^k))), r^k\varepsilon/4), \;\; j = 1, 2,\\
&\; \mathrm{dist}(S^j(T_{S^j}(B^c(r^{k+1}))), S^{3-j}[T_{S^{3-j}}(B^c(r^k)), T_{S^{3-j}}(B^c(r^{k+1}))] > r^{k+1}\varepsilon/2, \;\; j = 1, 2\},
\end{aligned}
$$

with the special convention when $k = 0$: $T_{S^j}(B^c(r^0)) \overset{\mathrm{df}}{=} 0$ and the middle pair of conditions is (by definition) always satisfied when $k = 0$.

Choose any $\xi > \xi_0$ and $r > 0$ so that the last lemma is satisfied for large m. Then, for large k, the $Q^{0,1}$-conditional probability of F_k given $\bigcap_{n=0}^{k-1} F_n$ is at least $r^{-\xi}$. It follows that

$$\liminf_{k\to\infty} Q^{0,1}\Big(\bigcap_{n=0}^{k} F_n\Big) r^{k\xi} > 0.$$

Note that we have $Q^{0,1}$-a.s.

$$\bigcap_{n=0}^{k} F_n \subset \{S^1[0, T_{S^1}(B^c(r^{k+1}))] \cap S^2[0, T_{S^2}(B^c(r^{k+1}))] = \emptyset\}.$$

This, and the monotonicity of q_r, prove (3.1). $\square$

Now we will present several lemmas in preparation for the proof of (3.2). It will be more convenient to work with cubes K rather than balls B. Recall the relevant definitions from Section 2.

Lemma 3.7. *Denote*

$$A(r) = \{X^1[0, T_{X^1}(\partial K(r))] \cap X^2[0, T_{X^2}(\partial K(r))] = \emptyset\}.$$

Then

$$\liminf_{r\to\infty} \frac{-\log P^{\nu,\nu}(A(r))}{\log r} \geq \xi_1.$$

Proof. Let

$$A_1(r) = \{X^1[0, T_{X^1}(\partial B(x, r/2))] \cap X^2[0, T_{X^2}(\partial B(y, r/2))] = \emptyset\}.$$

For large r and all $x \in \partial K(1)$ we have $B(x, r/2) \subset K(r)$, so

$$P^{x,y}(A(r)) < P^{x,y}(A_1(r)) \tag{3.14}$$

for $x, y \in \partial K(1)$ and large r. By scaling,

$$P^{x,y}(A_1(r)) = p_{r/(2|x-y|)}$$

and, by the monotonicity of $r \to p_r$,

$$P^{x,y}(A_1(r)) \leq p_{r/8} \tag{3.15}$$

for $x, y \in \partial K(1)$, which completes the proof. $\square$

Lemma 3.8. *Let*

$$A_2 = \{S^1[0, T_{S^1}(\partial K_{\mathbb{Z}}(m))] \cap S^2[0, T_{S^2}(\partial K_{\mathbb{Z}}(nm))] \neq \emptyset\}.$$

For each n and k, there exists a $p > 0$ such that if $x \in \partial K_{\mathbb{Z}}(km)$,

$$Q^{0,x}(A_2) > p.$$

Proof. The lemma is a version of a result of Erdös and Taylor (1960). See also Lawler (1989). $\square$

Lemma 3.9. *For each $r > 0$, the $P^{\nu,\nu}$-probability of*

$$\{\exists\, x \in \mathbb{R}^3 \ \exists\, \varepsilon > 0 : B(x,\varepsilon) \cap X^1[0, T_{X^1}(\partial K(r))] \cap X^2[0, T_{X^2}(\partial K(r))] = \{x\}\}$$

is zero.

Proof. Suppose that $M \subset \mathbb{R}^3$. A point $x \in M$ is called irregular if $P^x(T_{X^1}(M) > 0) > 0$. A set $K \subset \mathbb{R}^3$ is called polar if $P^x(T_{X^1}(K) < \infty) = 0$ for all $x \in \mathbb{R}^3$. The set of all irregular points of M is polar, by Theorem 2.6.3 of Port and Stone (1978). The strong Markov property applied at $T_{X^1}(M)$ shows that

$$P^x(\inf\{t > T_{X^1}(M) : X^1(t) \in M\} = T_{X^1}(M)) = 1.$$

In other words, the first hitting time of M is not an isolated point of $\{t \geq 0 : X^1(t) \in M\}$. A similar argument applies to every process $\{X^1(t),\ t \in [a,\infty)\}$ under P^x. It follows that, with P^x-probability 1, the first hitting time of M by $\{X^1(t),\ t \in [a,\infty)\}$ is not an isolated point of $\{t \in [a,\infty) : X^1(t) \in M\}$, for all rational a simultaneously. This implies that the set $\{t \geq 0 : X^1(t) \in M\}$ does not have isolated points P^x-a.s.

The above argument may be applied to almost every set $X^2[0, T_{X^2}(\partial K(r))]$ in place of M. From this, the result follows easily. □

Lemma 3.10. *Suppose that $y \in \partial K_{\mathbb{Z}}(8m)$ and $v \in \partial K_{\mathbb{Z}}(32m)$. Let h_y be the harmonic function in $K_{\mathbb{Z}}(8m)$ defined by*

$$h_y(x) = Q^x(S^1(T_{S^1}(\partial K_{\mathbb{Z}}(8m))) = y)$$

and let h_v be defined in $K_{\mathbb{Z}}(32m)$ by

$$h_v(x) = Q^x(S^2(T_{S^2}(\partial K_{\mathbb{Z}}(32m))) = v).$$

Denote

$$A_3 = \{S^1[0, T_{S^1}(\partial K_{\mathbb{Z}}(8m))] \cap S^2[0, T_{S^2}(\partial K_{\mathbb{Z}}(32m))] \neq \emptyset\}.$$

For some $p > 0$ and all $y \in \partial K_{\mathbb{Z}}(8m)$, $v \in \partial K_{\mathbb{Z}}(32m)$, $z \in \partial K_{\mathbb{Z}}(16m)$, we have

$$Q^{0,z}_{h_y,h_v}(A_3) > p.$$

Proof. Let

$$A_4 = \{S^1[0, T_{S^1}(\partial K_{\mathbb{Z}}(m))] \cap S^2[0, T_{S^2}(\partial K_{\mathbb{Z}}(20m))] \neq \emptyset\},$$

$$H_y = \{S^1(T_{S^1}(\partial K_{\mathbb{Z}}(8m))) = y\},$$

$$H_v = \{S^2(T_{S^2}(\partial K_{\mathbb{Z}}(32m))) = v\}.$$

By Lemma 3.8, there exists a $p_1 > 0$ such that for $z \in \partial K_{\mathbb{Z}}(16m)$

$$Q^{0,z}(A_4) > p_1.$$

The Harnack principle (Lemma 2.2) implies for $x \in K_{\mathbb{Z}}(m)$ and $u \in K_{\mathbb{Z}}(20m)$,

$$\begin{aligned}
&Q^{x,u}(H_y \cap H_v)[Q^{0,z}(H_y \cap H_v)]^{-1} \\
&= Q^x(H_y)Q^u(H_v)[Q^0(H_y)Q^z(H_v)]^{-1} \\
&\geq c.
\end{aligned}$$

By the Markov property applied at $T_1 \stackrel{\mathrm{df}}{=} T_{S^1}(\partial K_{\mathbb{Z}}(m))$ and $T_2 \stackrel{\mathrm{df}}{=} T_{S^2}(\partial K_{\mathbb{Z}}(20m))$,

$$\begin{aligned}
Q^{0,z}_{h_y,h_v}(A_3) &= Q^{0,z}(A_3 \cap H_y \cap H_v)[Q^{0,z}(H_y \cap H_v)]^{-1} \\
&\geq Q^{0,z}(A_4 \cap H_y \cap H_v)[Q^{0,z}(H_y \cap H_v)]^{-1} \\
&= Q^{0,z}(A_4)Q^{0,z}(H_y \cap H_v \mid A_4)[Q^{0,z}(H_y \cap H_v)]^{-1} \\
&= Q^{0,z}(A_4)E_{Q^{0,z}}(Q^{S^1(T_1),S^2(T_2)}(H_y \cap H_v) \mid A_4)[Q^{0,z}(H_y \cap H_v)]^{-1} \\
&\geq p_1 c,
\end{aligned}$$

and the lemma follows. □

Lemma 3.11. *Denote*

$$\begin{aligned}
A_X &= \{X^1[0, T_{X^1}(\partial K(r))] \cap X^2[0, T_{X^2}(\partial K(r))] \neq \emptyset\}, \\
A_S &= \{S^1[0, T_{S^1}(\partial K_{\mathbb{Z}}(mr))] \cap S^2[0, T_{S^2}(\partial K_{\mathbb{Z}}(mr))] \neq \emptyset\}.
\end{aligned}$$

For every $r > 1$, we have

$$\lim_{m\to\infty} Q^{\eta_m,\eta_m}(A_S) = P^{\nu,\nu}(A_X).$$

Proof. Fix some $r > 1$ and let $\beta = P^{\nu,\nu}(A_X)$.

The event A_X^c is an increasing limit of the events

$$\{\mathrm{dist}(X^1[0, T_{X^1}(\partial K(r))], X^2[0, T_{X^2}(\partial K(r))]) > \varepsilon\}$$

and the invariance principle may be applied separately to each of these events, thus showing that

$$\limsup_{m\to\infty} Q^{\eta_m,\eta_m}(A_S) \leq \beta.$$

Take any $\beta_1 < \beta$. By Lemma 3.9, there are no isolated intersection points of X^1 and X^2. Therefore, for each $k > 0$ it is possible to find $\varepsilon > 0$ such that with $P^{\nu,\nu}$-probability at least β_1 there exist points $x^1, x^2, \dots, x^k$ such that

$$x^j \in X^1[0, T_{X^1}(\partial K(r))] \cap X^2[0, T_{X^2}(\partial K(r))],$$

$$|x^j - x^n| > \varepsilon \quad \text{for} \quad j \neq n, \ 1 \leq j, n \leq k.$$

By the invariance principle, for every $\varepsilon_1 > \varepsilon$ and $\beta_2 < \beta_1$, for all large m, it is possible, with Q^{η_m,η_m}-probability at least β_2, to find some points $x^1, x^2, \dots, x^k$ such that

$$|x^j - x^n| > m\varepsilon_1/2 \qquad \text{for } j \neq n, 1 \leq j, n \leq k, \tag{3.16}$$

$$\mathrm{dist}(x^j, S^n[0, T_{S^n}(\partial K_{\mathbb{Z}}(mr))]) < m\varepsilon_1/32, \quad j = 1, 2, \dots, k, \quad n = 1, 2. \tag{3.17}$$

Denote $U_0 = 0$,

$$U_j = \inf\{t > U_{j-1} : S^1(t) \in \partial K_{\mathbb{Z}}(S^1(U_{j-1}), m\varepsilon_1/32)\}, \quad j \geq 1,$$

$$M = \bigcup_{j=1}^{\infty} \partial K_{\mathbb{Z}}(S^1(U_j), m\varepsilon_1/16),$$

$$V_1 = \inf\{t > 0 : S^2(t) \in M\},$$

$$V_j = \inf\{t > 0 : S^2(t) \in M \setminus \bigcup_{n<j} K_{\mathbb{Z}}(S^2(V_n), m\varepsilon_1/8)\}, \quad j \geq 2.$$

By (3.16)–(3.17), for large m,

$$Q^{\eta_m,\eta_m}(V_j < \infty \quad \text{for all } j \leq k) > \beta_2. \tag{3.18}$$

Let j_n be the smallest j such that

$$S^2(V_n) \in \partial K_{\mathbb{Z}}(S^1(U_j), m\varepsilon_1/16)$$

and let

$$\tilde{V}_n = T_{S^2}(\partial K_{\mathbb{Z}}(S^1(U_{j_n}), m\varepsilon_1/8)).$$

Given $\{S^1(U_j), j \geq 1\}$ and $\{S^2(V_j), S^2(\tilde{V}_j), j \geq 1\}$, the processes

$$\{S^1(U_{j-1} + t), \quad t \in [0, U_j - U_{j-1}]\}, \quad j \geq 1,$$

$$\{S^1(V_j + t), \quad t \in [0, \tilde{V}_j - V_j]\}, \quad j \geq 1,$$

are independent under Q^{η_m,η_m} and each one is an h-process. Then, conditioning on $\{S^1(U_j), j \geq 1\}$, $\{S^2(V_j), S^2(\tilde{V}_j), j \geq 1\}$, $\{V_k < \infty\}$, and using Lemma 3.10, we see that each event

$$A_n \stackrel{\mathrm{df}}{=} \{S^1[U_{j_n}, U_{j_{n+1}}] \cap S^2[V_n, \tilde{V}_n] \neq \emptyset\}$$

has the Q^{η_m,η_m}-probability at least p, for large m. By the previous remark about conditional independence, in view of (3.18),

$$Q^{\eta_m,\eta_m}(\bigcup_{n=1}^{k-1} A_n) \geq [1 - (1-p)^{k-1}]\beta_2,$$

for large m. By a suitable choice of k the last quantity may be made arbitrarily close to β_2. Since β_2 and β_1 are arbitrary numbers smaller than β, the lemma follows. □

Proof of (3.2). Fix an arbitrary $\xi_2 < \xi_1$. Denote

$$U_j^k = T_{S^k}(\partial K_{\mathbb{Z}}(r^j 2^{j-1})), \quad j \geq 1, \quad k = 1, 2,$$

$$V_j^k = T_{S^k}(\partial K_{\mathbb{Z}}(r^j 2^j)), \quad j \geq 1, \quad k = 1, 2.$$

Choose r according to Lemma 3.7, so that

$$P^{\nu,\nu}(A_X^c) = P^{\nu,\nu}(A(r)) < r^{-\xi_2}.$$

Note that A_X, defined in Lemma 3.11, is the complement of $A(r)$, defined in Lemma 3.7.

By the Harnack principle, for each $x \in \partial K_{\mathbb{Z}}(r^j 2^{j-1})$, the Radon-Nikodym derivative of the Q^x-distribution of $S^1(T_{S^1}(\partial K_{\mathbb{Z}}(r^j 2^j)))$ with respect to $\eta_{r^j 2^j}$ is bounded from above by $c_1 < \infty$, and a similar statement holds for S^2. By the strong Markov property applied at U_j^k and V_j^k, and the last lemma, the conditional $Q^{0,1}$-probability, given $S^1[0, U_j^1]$ and $S^2[0, U_j^2]$, that the paths of S^1 and S^2 do not intersect between V_j^k and U_{j+1}^k, is at most $2c_1^2 r^{-\xi_2}$, for large j. It follows that

$$Q^{0,1}(S^1[0, U_j^1] \cap S^2[0, U_j^2] = \emptyset) < c_2 (2c_1^2)^j r^{-\xi_2 j}$$

for some c_2 and all large j. This implies that

$$q_{r^j 2^j} < c_2 (2c_1^2)^j r^{-\xi_2 j}$$

for large j, or, equivalently,

$$\frac{\log q_{r^j 2^j}}{\log(r^j 2^j)} < \frac{\log c_2 + j \log(2c_1^2) - \xi_2 j \log r}{j \log r + j \log 2}.$$

It follows that

$$\liminf_{j \to \infty} - \frac{\log q_{r^j 2^j}}{\log(r^j 2^j)} \geq \frac{-\log(2c_1^2) + \xi_2 \log r}{\log r + \log 2}.$$

By choosing large r, we may make the last expression arbitrarily close to ξ_2. This and monotonicity of $r \to q_r$ yields

$$\liminf_{r \to \infty} - \log q_r / \log r \geq \xi_2.$$

Since ξ_2 is an arbitrary number less than ξ_1, (3.2) follows. $\square$

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