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Lei Xue

# Face Numbers of Polytopes, Posets, and Complexes

Lei Xue

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Reading Committee:

Isabella Novik, Chair

Steven Klee

Farbod Shokrieh

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**Abstract**

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Lei Xue

Chair of the Supervisory Committee:

Isabella Novik

Department of Mathematics

A key tool that combinatorialists use to study simplicial complexes and polytopes is the *f*-**vector** (or face vector), which records the number of faces of each dimension. In order to better understand the face numbers, relations involving both equalities and inequalities on *f*-vectors have been extensively studied. In this dissertation we discuss the author's contributions to these topics.

The classical Dehn–Sommerville relations assert that the *h*-vector of an Eulerian simplicial complex is symmetric. In Chapter 2, we establish three generalizations of the Dehn–Sommerville relations: one for the *h*-vectors of pure simplicial complexes, another one for the flag *h*-vectors of balanced simplicial complexes and graded posets, and yet another one for the toric *h*-vectors of graded posets with restricted singularities. In all of these cases, we express any failure of symmetry in terms of “errors coming from the links.” For simplicial complexes, this further extends Klee's semi-Eulerian relations.

In Chapters 3 and 4, we change our focus from equalities to inequalities on *f*-vectors. In 1967, Grünbaum conjectured that any *d*-dimensional polytope with  $d + s \leq 2d$  vertices has at least

$$\phi_k(d + s, d) = \binom{d + 1}{k + 1} + \binom{d}{k + 1} - \binom{d + 1 - s}{k + 1}$$

$k$ -faces. In Chapter 3, we prove this conjecture and also characterize the cases in which equality holds.

In Chapter 4, several extensions of Grünbaum's conjecture are established. Specifically, it is proved that every lattice with diamond property and  $d + s \leq 2d$  atoms has at least  $\phi_k(s)$  elements of rank  $k + 1$ . Furthermore, in the case of lattices that are face lattices of strongly regular CW complexes representing normal pseudomanifolds with up to  $2d$  vertices, a characterization of equality cases is given. Finally, sharp lower bounds on the number of  $k$ -faces of strongly regular CW complexes representing normal pseudomanifolds with  $2d + 1$  vertices are obtained. These bounds are given by the face numbers of certain polytopes with  $2d + 1$  vertices.

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## DEDICATION

to my parents, 秋桔 and 南冬.

## Chapter 1

### INTRODUCTION

A **polytope** is the convex hull of a finite set of points in a Euclidean space. The foundation of the study of polytopes was laid by the Greeks over 2000 years ago. Yet many interesting properties of polytopes were only discovered in recent centuries or still remain unknown. While polytopes have many applications to areas such as statistics, optimization, and even mineralogy, it also occupies a central place in combinatorics and discrete geometry. At the same time, from the earliest days of topology, simplicial and CW complexes have been very useful for computing various invariants of spaces, such as homology. Simplicial complexes have been studied by combinatorialists, topologists, and algebraists (through Stanley–Reisner rings). This resulted in many beautiful theorems, among them the complete characterization of the face numbers of simplicial polytopes, known as the  $g$ -theorem.

A key tool that combinatorialists use to study simplicial complexes and polytopes is the  **$f$ -vector** (or face vector), which records the number of faces of each dimension. The central problem in this area can be stated as follows: *find necessary and sufficient conditions for a given integer vector to be the  $f$ -vector of some polytope.* In order to better understand the face numbers, relations involving both equalities and inequalities on  $f$ -vectors have been extensively studied. In this dissertation we will describe our contributions to these topics.

#### ***Generalized Dehn–Sommerville relations***

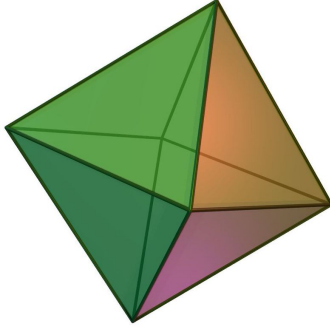
Among the most fundamental results about linear relations on  $f$ -vectors of simplicial complexes are the Dehn–Sommerville relations. These relations take on the most elegant form when described in terms of the  **$h$ -vectors**.

Throughout this section and Chapter 2, we will use the following notation. Let  $\Delta$  be a

(non-empty)  $(d - 1)$ -dimensional simplicial complex. The  $f$ -vector of  $\Delta$  is  $f(\Delta) = (f_{-1} = 1, f_0, f_1, \dots, f_{d-1})$ , where  $f_i$  denotes the number of  $i$ -dimensional faces of  $\Delta$ . The  $h$ -vector  $h(\Delta) = (h_0, h_1, \dots, h_d)$  is obtained from the  $f$ -vector by the following invertible linear transformation.

$$\sum_{i=0}^d h_i(\Delta)x^{d-i} = \sum_{i=0}^d f_{i-1}(\Delta)(x-1)^{d-i}.$$

The **reduced Euler characteristic** of  $\Delta$  is  $\tilde{\chi}(\Delta) = \sum_{i=-1}^{d-1} (-1)^i f_i$ . In particular,  $h_d(\Delta) = (-1)^{d-1} \tilde{\chi}(\Delta)$ . For example, if  $\Delta$  is the boundary complex of an octahedron, then  $f(\Delta) = (1, 6, 12, 8)$ ,  $h(\Delta) = (1, 3, 3, 1)$ .



The **link** of a face  $F$  is a certain subcomplex of  $\Delta$  that encodes the information about the local structure “near”  $F$ . When  $\Delta$  is (combinatorially equivalent to) the boundary of a polytope, then so are all of its links. We say that  $\Delta$  is a **simplicial sphere** if the geometric realization of  $\Delta$  is homeomorphic to a sphere. If  $\Delta$  is a triangulation of a  $(d - 1)$ -dimensional sphere, then  $\tilde{\chi}(\Delta) = (-1)^{d-1}$ .

In 1964, Klee defined Eulerian and semi-Eulerian simplicial complexes as combinatorial analogs of spheres and manifolds. He then proved that their  $h$ -vectors are (almost) symmetric, see [17]. More precisely, the  $h$ -vector of a  $(d - 1)$ -dimensional Eulerian simplicial complex  $\Delta$  (for example, a simplicial sphere) satisfies  $h_i(\Delta) = h_{d-i}(\Delta)$  for all  $i$ , while the  $h$ -vector of a  $(d - 1)$ -dimensional semi-Eulerian complex  $\Gamma$  (such as the boundary of a simplicial manifold) satisfies  $h_{d-i}(\Gamma) = h_i(\Gamma) + (-1)^i \binom{d}{i} [\tilde{\chi}(\Gamma) - (-1)^{d-1}]$ .

Since then these relations have played a very important role in the  $f$ -vector theory, e.g., in the proof of the Upper Bound Theorem, see [18], [26], and [20]. Stanley [29] (see also [31]) extended Klee's definition of Eulerian and semi-Eulerian complexes to (finite) graded posets. The **flag** and **toric  $h$ -vectors** are two generalizations of the usual  $h$ -vectors to arbitrary graded posets. The flag  $h$ -vector (up to an invertible linear transformation) counts the number of chains of a graded poset according to the ranks of elements in the chain. The toric  $h$ -vector has a recursive definition introduced by Stanley [29]. If a poset  $P$  is the poset of faces of a rational polytope  $Q$ , then the toric  $h$ -vector of  $P$  is related to the intersection cohomology of the toric variety associated to  $Q$ . In 1985, Bayer and Billera proved a version of the Dehn–Sommerville relations for flag vectors of **Eulerian posets**, see [4]. The Bayer–Billera relations played an instrumental role in Fine's definition of the **cd-index** (see [6], [31]). In 1987, Stanley [29] proved that the toric  $h$ -vector of any Eulerian poset is symmetric. This result was extended in 2009 by Swartz [35] to **semi-Eulerian posets**.

In Chapter 2 we extend all the above results from Eulerian/semi-Eulerian case to the case of arbitrary posets/complexes. This is a joint work with Connor Sawaske (see [25]). We establish three generalizations of the Dehn–Sommerville relations in [25]. Specifically, we extend:

- *Klee's relations on the  $h$ -vectors of Eulerian/semi-Eulerian simplicial complexes to relations on the  $h$ -vectors of **all** pure simplicial complexes, see Theorem 2.3.1.*
- *Bayer–Billera's relations on the flag  $h$ -vectors of Eulerian graded posets, and similar relations on balanced simplicial complexes, to relations on the flag  $h$ -vectors of **all** graded posets and **all** balanced simplicial complexes, see Theorem 2.4.1.*
- *Stanley's and Swartz's relations on the toric  $h$ -vectors of Eulerian and semi-Eulerian posets to relations on the toric  $h$ -vectors of graded posets with restricted singularities, see Theorems 2.5.6 and 2.6.14.*

In all these cases, we express any failure of symmetry in terms of “errors coming from the links.”

### **Grünbaum’s lower bound conjecture on general polytopes**

In the last fifty years a lot of effort has gone into trying to answer questions about **minimal face numbers** of polytopes. For instance, Barnette (see [3], [1], and [2]) proved the Lower Bound Theorem for *simplicial* polytopes; his result provides tight lower bounds on the number of  $k$ -faces a  $d$ -dimensional simplicial polytope with  $n$  vertices can have. Furthermore, in 1980, Billera and Lee [7] and Stanley [28] completely characterized the face numbers of all simplicial (and by duality also simple) polytopes. Their result is known as the  $g$ -theorem.

Despite these spectacular advances in the study of *simplicial* polytopes, to date no Lower Bound Theorem is known for *general*  $d$ -dimensional polytopes with an arbitrary number of vertices; in fact, there is not even a plausible conjecture. However for general polytopes with up to  $2d$  vertices, Grünbaum made the following conjecture in [13, p. 184].

**Conjecture 1.0.1 (Grünbaum, 1967).** Let  $P$  be a  $d$ -dimensional polytopes with  $d+s \leq 2d$  vertices. The number of  $k$ -dimensional faces of  $P$  is at least

$$\phi_k(d+s, d) = \binom{d+1}{k+1} + \binom{d}{k+1} - \binom{d+1-s}{k+1}.$$

He proved this conjecture for the cases of  $s = 2, 3, 4$ . The conjecture remained completely open for  $s \geq 5$  until very recently Pineda-Villavicencio, Ugon and Yost [22] proved that this conjecture holds for the number of edges, i.e., they verified the  $k = 1$  case.

In Chapter 3, we prove this long-standing conjecture in full generality. We also classify all the cases of equality. Our main results can be summarized as follows.

**Theorem 3.3.2. (Former Grünbaum’s Conjecture)** *Let  $P$  be a  $d$ -polytope with  $d+s$  vertices where  $s \geq 2$  and  $d \geq s$ . Then  $f_k(P) \geq \phi_k(d+s, d)$  for every  $k$ .*

**Theorem 3.4.3.** *Let  $P$  be a  $d$ -polytope with  $d+s$  vertices where  $s \geq 2$  and  $d \geq s$ . If  $f_k(P) = \phi_k(d+s, d)$  for some  $1 \leq k \leq d-2$ , then  $P$  is  $\text{Py}^{d-s}(\Delta^1 \times \Delta^{s-1})$  — the polytope that is a  $(d-s)$ -fold pyramid over a prism over an  $(s-1)$ -simplex.*

The main novelty of the approach in Chapter 3 is that instead of focusing on contributions coming from facets, we look at sets of potentially unrelated vertices and bound the number of  $k$ -faces containing one or more of them.

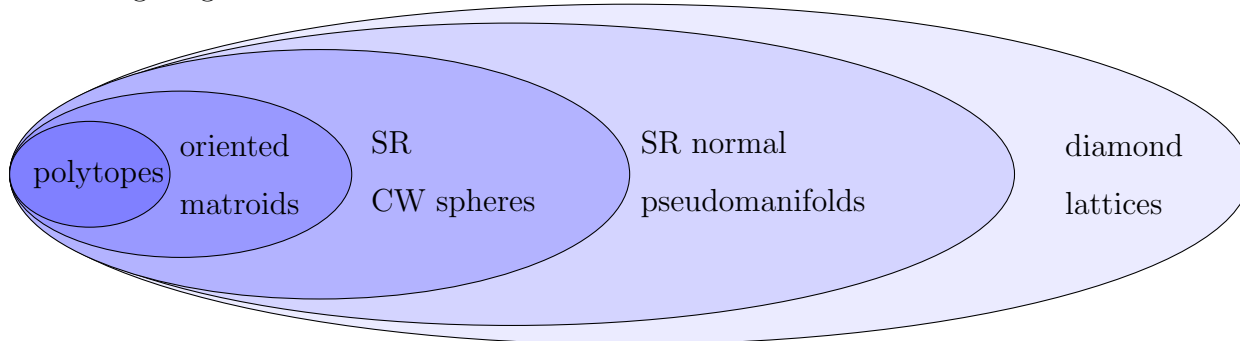
***A Lower Bound Theorem for strongly regular CW spheres with up to  $2d + 1$  vertices***

Now that Grünbaum's conjecture is proved, two of the most natural new questions are: 1). Can these results be extended from polytopes to larger classes of objects? 2). What can we say about  $d$ -dimensional polytopes with more than  $2d$  vertices? Chapter 4 is devoted to answering these questions. Specifically, we extend results from Chapter 3 to the following much more general classes of posets and complexes.

We begin with some necessary definitions. A **regular CW sphere** is called **strongly regular (SR for short)** if the intersection of any two faces is a face (possibly empty). For example, the boundary complex of a  $d$ -polytope is a strongly regular CW  $(d-1)$ -sphere. Also, any **oriented matroid sphere** (as defined in [9]) is a strongly regular CW sphere. Finally, a graded lattice is called a **diamond lattice** if every interval of rank-2 is diamond-shaped, i.e., every interval of rank 2 has exactly four elements. We have the following hierarchy.

$$\left\{ \text{polytopes} \right\} \subset \left\{ \begin{array}{l} \text{oriented} \\ \text{matroids} \end{array} \right\} \subset \left\{ \begin{array}{l} \text{SR} \\ \text{CW spheres} \end{array} \right\} \subset \left\{ \begin{array}{l} \text{SR normal} \\ \text{pseudomanifolds} \end{array} \right\} \subset \left\{ \begin{array}{l} \text{diamond} \\ \text{lattices} \end{array} \right\}.$$

The following diagram allows us to visualize these inclusions.



Our main results are the following.

**Theorem 4.3.6.** *Theorem 3.3.2 (Former Grünbaum's Conjecture) holds in the generality*

of all **diamond lattices**. In other words, for any diamond lattice  $L$  of rank  $d + 1$  with  $d + s \leq 2d$  atoms, the number of elements of  $L$  of rank  $k + 1$  is at least  $\phi_k(d + s, d)$ .

**Theorem 4.3.8.** *Theorem 3.4.3 (The equality cases) holds for all face lattices of **strongly regular (SR) CW complexes representing normal  $(d - 1)$ -pseudomanifolds with up to  $2d$  vertices.***

In particular, we show that any strongly regular normal  $(d - 1)$ -pseudomanifold with  $d + 2$  facets and any number of vertices is combinatorially equivalent to the boundary complex of a  $d$ -polytope.

We also generalize the above two results to objects with  $2d + 1$  vertices:

*In the class of **SR normal  $(d - 1)$ -pseudomanifolds with  $2d + 1$  vertices**, we determined the minimum value of the number of  $k$ -dimensional faces for all  $1 \leq k \leq d - 1$ . These minimum values are achieved by certain polytopes with  $d + 2$  and  $d + 3$  facets (the lower bound has two parts).*

Some of the results just mentioned, but only for the case of *polytopes* with  $2d + 1$  vertices, were also obtained by Pineda-Villavicencio and Yost in [24].

All chapters of this dissertation are based on research papers: each chapter is self-contained and can be read independently of the others.

## Chapter 2

### NON-EULERIAN DEHN–SOMMERVILLE RELATIONS

#### 2.1 Introduction

In this paper we generalize Dehn–Sommerville relations in three ways: the first one relates to the  $h$ -vectors of all pure simplicial complexes, the second one deals with the flag  $h$ -vectors of balanced simplicial complexes and graded posets, and the third one concerns the toric  $h$ -vectors.

In 1964, Klee defined Eulerian and Semi-Eulerian simplicial complexes and proved that their  $h$ -vectors are almost symmetric, see [17]. More precisely, the  $h$ -vector of a  $(d - 1)$ -dimensional Eulerian simplicial complex  $\Delta$  (for example, a simplicial sphere) satisfies  $h_i(\Delta) = h_{d-i}(\Delta)$  for all  $i$ , while the  $h$ -vector of a  $(d - 1)$ -dimensional semi-Eulerian complex  $\Gamma$  (such as the boundary of a simplicial manifold) satisfies  $h_{d-i}(\Gamma) = h_i(\Gamma) + (-1)^i \binom{d}{i} [\tilde{\chi}(\Gamma) - (-1)^{d-1}]$ , where  $\tilde{\chi}$  is the reduced Euler characteristic of  $\Gamma$ . Since then these relations have played a very important role in the  $f$ -vector theory, e.g., in the proof of the Upper Bound Theorem, see [18], [26], and [20]. In 2012, Novik and Swartz derived similar results for *psuedo-manifolds with isolated singularities* as defined in [21].

A  $(d - 1)$ -dimensional simplicial complex  $\Delta$  is balanced if it has a vertex coloring in  $d$  colors such that no two vertices in the same face are colored with the same color. (It is standard to label the colors by elements of  $[d]$ .) A refinement of the usual  $f$ - and  $h$ -vectors for balanced complexes are called flag  $f$ - and flag  $h$ -vectors. The flag  $f$ -vector of a  $(d - 1)$ -dimensional balanced simplicial complex  $\Delta$ , denoted  $\{f_S(\Delta)\}_{S \subseteq [d]}$ , counts the number of faces of  $\Delta$  according to the color sets of their vertices. The flag  $h$ -vector of  $\Delta$ , denoted  $\{h_S(\Delta)\}_{S \subseteq [d]}$ , is the image of the flag  $f$ -vector under a certain invertible linear transformation. Bayer and Billera proved the Dehn–Sommerville relations on flag  $f$ -vectors

of Eulerian balanced simplicial complexes, see [5] (also see [33, Thm. 3.16.6] for the proof of the flag  $h$ -vector version). The Bayer–Billera relations played an instrumental role in Fine’s definition of the cd-index (see [6], [31]).

Stanley [29] (see also [31]) extended Klee’s definition of Eulerian and semi-Eulerian complexes to (finite) graded partially ordered sets (posets for short). He also introduced a notion of toric  $h$ - and  $g$ -vectors of posets and proved that the toric  $h$ -vector of any Eulerian poset is symmetric. This result was extended by Swartz [35] to semi-Eulerian posets.

Here we provide generalizations of these three types of Dehn–Sommerville relations. Our results can be summarized as follows; for all undefined terminology and notations, see Section 3.2.

- For an arbitrary  $(d - 1)$ -dimensional pure simplicial complex  $\Delta$ , we express  $h_{d-i}(\Delta) - h_i(\Delta)$  in terms of the Euler characteristics of links of faces, see Theorem 2.3.1. The result is also generalized to simplicial posets, see Corollary 2.3.6.
- For an arbitrary  $(d - 1)$ -dimensional balanced simplicial complex  $\Delta$  and  $S \subseteq [d]$ , we express  $h_S(\Delta) - h_{[d]-S}(\Delta)$  in terms of the Euler characteristics of links of faces whose color sets are contained in  $S$ , see Theorem 2.4.1.
- For finite posets, we define the notion of  $j$ -**Singular** posets (see Section 2.6) such that
  - $j = -1$  recovers Eulerian posets;
  - $j = 0$  recovers semi-Eulerian posets;
  - $j = 1$  is analogous to complexes with isolated singularities (see the definition in Section 2.5).
- Extending the results of Stanley and Swartz, for a 1-Singular poset  $P$  of rank  $d + 1$ , we express  $\hat{h}(P, x) - x^d \hat{h}(P, \frac{1}{x})$  in terms of the Möbius functions of intervals  $[s, t]$  in  $P$  of length greater than or equal to  $d - 1$ , see Theorem 2.5.6. Here  $\hat{h}$  denotes the toric  $h$ -polynomial.

- We extend this result further and obtain a similar formula for a  $j$ -Singular poset  $P$  with  $j < \lfloor \frac{d}{2} \rfloor$  (see Theorems 2.6.9 and 2.6.14).

The structure of this paper is as follows: Section 3.2 introduces some basic results and definitions pertaining to simplicial complexes and posets. Section 2.3 is devoted to establishing the generalization of Dehn–Sommerville relations for pure simplicial complexes and simplicial posets. Section 2.4 proves the flag Dehn–Sommerville relations for balanced simplicial complexes and graded posets. Sections 2.5 and 2.6 establish the toric generalizations of Dehn–Sommerville formulas. Our proofs build on methods used by Klee, Stanley, and Swartz.

## 2.2 Preliminaries

### 2.2.1 Simplicial complexes

In this section we review some definitions pertaining to simplicial complexes. Let  $V$  be a finite set. A **simplicial complex**  $\Delta$  with vertex set  $V$  is a collection of subsets of  $V$  that is closed under inclusion. We call each element of  $\Delta$  a **face** of  $\Delta$ , and each face  $F \in \Delta$  has a **dimension** defined by  $\dim(F) = |F| - 1$ . Similarly, the dimension of  $\Delta$  is defined by  $\dim(\Delta) = \max\{\dim F : F \in \Delta\}$ . If all maximal faces of  $\Delta$  (with respect to inclusion) have the same dimension, then  $\Delta$  is called **pure**. We denote the collection of faces of  $\Delta$  of a specific dimension  $i$  by

$$\Delta_i := \{F \in \Delta : \dim(F) = i\}.$$

Lastly, the **link** of a face  $F$  of  $\Delta$ , denoted  $\text{lk}_\Delta F$ , is defined by

$$\text{lk}_\Delta F := \{G \in \Delta : F \cup G \in \Delta \text{ and } F \cap G = \emptyset\}.$$

Let  $\Delta$  be a simplicial complex of dimension  $d - 1$ . The  **$f$ -vector** of  $\Delta$  is defined by  $f(\Delta) := (f_{-1}(\Delta), f_0(\Delta), f_1(\Delta), \dots, f_{d-1}(\Delta))$ , where  $f_i(\Delta) = |\Delta_i|$ . We further define the  **$h$ -**

**vector** of  $\Delta$  by  $h(\Delta) := (h_0(\Delta), h_1(\Delta), \dots, h_d(\Delta))$ , with entries determined by the equation

$$\sum_{i=0}^d h_i(\Delta) x^{d-i} = \sum_{i=0}^d f_{i-1}(\Delta) (x-1)^{d-i}.$$

For the remainder of this section, we will assume that  $\Delta$  is a pure simplicial complex of dimension  $d-1$ .

Each simplicial complex  $\Delta$  admits a geometric realization  $\|\Delta\|$  that contains a geometric  $i$ -simplex for each  $i$ -face of  $\Delta$ . We say that  $\Delta$  is a simplicial sphere (manifold, respectively) if  $\|\Delta\|$  is homeomorphic to a sphere (manifold, respectively).

The (reduced) **Euler characteristic** of  $\Delta$  is

$$\tilde{\chi}(\Delta) := \sum_{i=-1}^{d-1} (-1)^i f_i(\Delta),$$

and by the Euler-Poincaré formula,  $\tilde{\chi}(\Delta)$  is a topological invariant of  $\Delta$ , or more precisely, of its geometric realization  $\|\Delta\|$ . For instance, if  $\Gamma$  is an  $(i-1)$ -dimensional simplicial sphere, then  $\tilde{\chi}(\Gamma) = (-1)^{i-1}$ .

Given two simplicial complexes  $\Delta_1$  and  $\Delta_2$  on disjoint vertex sets, their **simplicial join**,  $\Delta_1 * \Delta_2$ , is defined as

$$\Delta_1 * \Delta_2 := \{F \cup G : F \in \Delta_1, G \in \Delta_2\}.$$

In particular,  $\Delta_1 * \Delta_2$  is a simplicial complex of dimension  $\dim \Delta_1 + \dim \Delta_2 + 1$ .

Central to many classifications of simplicial complexes is the notion of the link of a face having the same Euler characteristic as that of a sphere of the appropriate dimension. To that end, we measure potential failures of this condition by defining an error function  $\varepsilon_\Delta(F)$  on faces  $F$  of a pure  $(d-1)$ -dimensional simplicial complex  $\Delta$  as

$$\varepsilon_\Delta(F) := \tilde{\chi}(\text{lk}_\Delta F) - (-1)^{d-1-|F|}$$

(note that  $\dim(\text{lk}_\Delta F) = d-1-|F|$ , so  $(-1)^{d-1-|F|}$  is the same as the reduced Euler characteristic of a sphere of dimension  $\dim(\text{lk}_\Delta F)$ ). In addition, we form the set of faces with non-trivial error as

$$\mathfrak{E}(\Delta) := \{F \in \Delta : \varepsilon(F) \neq 0\}.$$

We say that  $\Delta$  is **Eulerian** if  $\mathfrak{E}(\Delta) = \emptyset$ , and that it is **semi-Eulerian** if  $\mathfrak{E}(\Delta) = \{\emptyset\}$ . In line with these definitions, we refer to  $\mathfrak{E}(\Delta)$  as the **non-Eulerian** part of  $\Delta$ .

**Example 2.2.1.** If  $\Delta$  is a simplicial sphere, then  $\mathfrak{E}(\Delta) = \emptyset$ . More generally, if  $\Delta$  is a simplicial manifold, then  $\mathfrak{E}(\Delta) \subseteq \{\emptyset\}$ .

### 2.2.2 Graded posets

The notions of being Eulerian and semi-Eulerian can also be defined in the graded poset settings. Throughout this paper, we let every finite graded poset  $P$  have unique bottom and top elements  $\hat{0}$  and  $\hat{1}$ . Let  $\rho : P \rightarrow \mathbb{N}$  be the rank function. The rank of  $P$ ,  $\rho(P)$ , is defined as  $\rho(\hat{1})$ .

Let  $\mu_P$  denote the Möbius function of poset  $P$ . If for all proper intervals  $[s, t] \subsetneq P$ ,  $\mu_P(s, t) = (-1)^{\rho(t) - \rho(s)}$ , then  $P$  is called **semi-Eulerian**. If in addition,  $\mu_P(\hat{0}, \hat{1}) = (-1)^{\rho(P)}$ , then  $P$  is **Eulerian**.

For each poset  $P$ , there is a simplicial complex associated with  $P$ ; it is called the (reduced) **order complex** of  $P$  and denoted by  $O(P)$ , see for instance [8]. The complex  $O(P)$  has the set  $P \setminus \{\hat{0}, \hat{1}\}$  as its vertex set, and the (finite) chains in the open interval  $(\hat{0}, \hat{1})$  as its faces. A chain  $C = \{t_1 < t_2 < \dots < t_k\}$  in  $P \setminus \{\hat{0}, \hat{1}\}$  of size  $k$  corresponds to a face in  $O(P)$  of dimension  $k - 1$ ; this face is sometimes denoted by  $F_C$ . In particular, if  $\rho(P) = d + 1$ , then  $\dim O(P) = d - 1$ .

**Remark 2.2.2** (Relations between  $\mu$  and  $\tilde{\chi}$ ). For a poset  $P$  and a chain  $C$  in it, the following formulas hold:

$$\tilde{\chi}(O(P)) = \mu_P(\hat{0}, \hat{1}); \quad (2.2.1)$$

$$\tilde{\chi}(\text{lk}_{O(P)} F_C) = (-1)^{|F|} \mu_P(\hat{0}, t_1) \cdot \mu_P(t_1, t_2) \cdot \dots \cdot \mu_P(t_{k-1}, t_k) \mu_P(t_k, \hat{1}). \quad (2.2.2)$$

The first formula is well known, see, for instance, [33]. To prove the second equality, note that

$$\text{lk}_{O(P)} F = O(\hat{0}, t_1) * O(t_1, t_2) \cdot \dots \cdot O(t_{k-1}, t_k) * O(t_k, \hat{1}),$$

where  $O(s, t)$  is the order complex of the open interval  $(s, t) = \{x : s < x < t\}$ . Using this and the fact that  $\tilde{\chi}(\Delta_1 * \Delta_2) = (-1)\tilde{\chi}(\Delta_1)\tilde{\chi}(\Delta_2)$ , we obtain

$$\tilde{\chi}(\text{lk}_{O(P)} F) = (-1)^{|F|} \tilde{\chi}(O(\hat{0}, t_1)) \cdot \tilde{\chi}(O(t_1, t_2)) \cdots \tilde{\chi}(O(t_{k-1}, t_k)) \tilde{\chi}(O(t_k, \hat{1})).$$

This together with (2.2.1) implies (2.2.2).

Since each face  $F_C \in O(P)$  corresponds to a chain  $C \in P \setminus \{\hat{0}, \hat{1}\}$ , we can define a **chain error** in  $P$  that corresponds to the face error in  $O(P)$ : Let  $C = \{t_1 < \cdots < t_i\}$  be a chain in  $P \setminus \{\hat{0}, \hat{1}\}$ . Define

$$\mu_P(C) := \mu_P(\hat{0}, t_1) \mu_P(t_1, t_2) \cdots \mu_P(t_i, \hat{1}),$$

and

$$\varepsilon_P(C) := (-1)^{|C|} [\mu_P(C) - (-1)^{d+1}].$$

We call  $C \mapsto \varepsilon_P(C)$  the **error function for chains** in a poset  $P$ . However this is not a “new” error function: comparing it with  $\varepsilon_{O(P)}(-)$  and using (2.2.2), we obtain:

**Remark 2.2.3.**  $\varepsilon_P(C) = \varepsilon_{O(P)}(F_C)$ .

### 2.2.3 Balanced simplicial complexes and flag vectors

For a specific type of simplicial complexes, called the balanced simplicial complexes, there exists a certain refinement of  $f$ - and  $h$ -vectors. A  $(d-1)$ -dimensional pure simplicial complex  $\Delta$  is **balanced** if it is equipped with a vertex coloring  $\kappa : V \rightarrow [d]$  such that no two vertices in the same face have the same color. These complexes were introduced by Stanley in [27].

For any subset  $S \subseteq [d]$ , the  **$S$ -rank selected subcomplex of  $\Delta$**  is

$$\Delta_S = \{F \in \Delta : \kappa(F) \subseteq S\}.$$

Define  $f_S(\Delta)$  as the number of faces in  $\Delta$  with  $\kappa(F) = S$ . The numbers  $f_S(\Delta)$  are called the flag  $f$ -numbers of  $\Delta$  and the collection  $(f_S(\Delta))_{S \subseteq [d]}$  is called the **flag  $f$ -vector of  $\Delta$** . Similarly, the flag  $h$ -numbers of  $\Delta$  are defined as

$$h_T(\Delta) = \sum_{S \subseteq T} (-1)^{|T|-|S|} f_S(\Delta) \quad \text{for } T \subseteq [d].$$

Note that the flag  $f$ - and  $h$ -numbers refine the ordinary  $f$ - and  $h$ -numbers:

$$f_{i-1}(\Delta) = \sum_{S \subseteq [d], |S|=i} f_S(\Delta) \quad \text{and} \quad h_j(\Delta) = \sum_{T \subseteq [d], |T|=j} h_T(\Delta).$$

One common example of a balanced complex is the order complex of a graded poset: let  $P$  be a graded poset of rank  $d + 1$ , then  $O(P)$  is balanced w.r.t. the coloring given by the rank function of  $P$ . Moreover, the flag vectors can also be defined in the setting of graded posets. For  $S \subseteq [d]$ , we define

$$P_S = \{x \in P : \rho(x) \in S \cup \{0, d + 1\}\}$$

considered as a subposet of  $P$ . The poset  $P_S$  is called the **S-rank selected subposet** of  $P$  and if  $\Delta = O(P)$ , then  $\Delta_S = O(P_S)$ . We let  $\alpha_P(S)$  be the number of maximal (w.r.t. inclusion) chains in  $P_S$ . The function  $S \mapsto \alpha_P(S)$  is the **flag  $f$ -vector of  $P$** . We also consider the function

$$S \mapsto \beta_P(S), \quad \beta_P(S) = \sum_{T \subseteq S} (-1)^{|S|-|T|} \alpha_P(T)$$

and call it the **flag  $h$ -vector of  $P$** . It is easy to see that

$$\alpha_P(S) = f_S(O(P)) \quad \text{and} \quad \beta_P(S) = h_S(O(P)). \quad (2.2.3)$$

#### 2.2.4 The Stanley-Reisner ring

An equivalent way to define the  $h$ - and flag  $h$ -vectors is through the Stanley-Reisner ring (for more details see [32, Ch. II.1]). Let  $\Delta$  be a  $(d - 1)$ -dimensional simplicial complex with vertex set  $V = [n]$ . Let  $\mathbb{k}$  be a field and let  $R = \mathbb{k}[x_1, \dots, x_n]$ . The **Stanley-Reisner ring of  $\Delta$**  is  $\mathbb{k}[\Delta] = R/I_\Delta$ , where

$$I_\Delta = \langle x_{i_1} x_{i_2} \cdots x_{i_k} : \{i_1, \dots, i_k\} \notin \Delta \rangle.$$

Let  $\mathbb{k}[\Delta]_i$  be the  $i$ -th homogeneous component of  $\mathbb{k}[\Delta]$ . **The Hilbert series of  $\mathbb{k}[\Delta]$**  is

$$F(\Delta, \lambda) = \sum_{i=0}^{\infty} \dim_{\mathbb{k}} \mathbb{k}[\Delta]_i \lambda^i.$$

The  $h$ -vector of  $\Delta$  can be easily obtained from the Hilbert series of  $\mathbb{k}[\Delta]$  using the following relation (see, for example, [32, Ch. II.2]):

$$F(\Delta, \lambda) = \frac{\sum_{i=0}^d h_i(\Delta) \lambda^i}{(1 - \lambda)^d}. \quad (2.2.4)$$

If  $\Delta$  is balanced with vertex coloring  $\kappa : [n] \rightarrow [d]$ , then  $\mathbb{k}[\Delta]$  has a natural  $\mathbb{Z}^d$ -grading which is induced by this coloring. For  $i = 1, 2, \dots, d$ , let  $e_i \in \{0, 1\}^d$  be the  $i$ -th coordinate unit vector, and for  $j \in [n]$ , define  $\deg(x_j) = e_{\kappa(j)}$ . This gives a  $\mathbb{Z}^d$  grading of  $\mathbb{k}[x_1, \dots, x_n]$ . Let  $\lambda = (\lambda_1, \dots, \lambda_d)$  be a  $d$ -tuple of variables. For  $a \in \mathbb{Z}_{\geq 0}^d$ , we let  $\lambda^a = \lambda_1^{a_1} \lambda_2^{a_2} \dots \lambda_d^{a_d}$ . The **fine Hilbert series** of  $\mathbb{k}[\Delta]$  with respect to this  $\mathbb{Z}^d$  grading is

$$F(\Delta, \lambda) = \sum_{a \in \mathbb{Z}_{\geq 0}^d} (\dim_{\mathbb{k}} \mathbb{k}[\Delta]_a) \cdot \lambda^a.$$

The flag  $h$ -vectors can be obtained from this fine Hilbert series (see [27]):

$$F(\Delta, \lambda) = \frac{1}{\prod_{i=1}^d (1 - \lambda_i)} \sum_{S \subseteq [d]} h_S \lambda^S \quad (2.2.5)$$

where  $\lambda^S = \prod_{j \in S} \lambda_j$ .

### 2.2.5 Toric vectors of graded posets

We will encounter toric vectors only in Sections 2.5 and 2.6, so the reader may skip this subsection for now and come to it later.

As in Subsection 2.2.2, we let  $P$  be a finite graded poset with unique bottom and top elements  $\hat{0}$  and  $\hat{1}$ . If  $P$  has only one element, i.e., when  $\hat{0} = \hat{1}$ , then we call  $P$  the trivial poset and denote it as  $P = \mathbb{1}$ . Let  $\rho : P \rightarrow \mathbb{N}$  be the rank function. Let  $\tilde{P} = \{[\hat{0}, t] : t \in P\}$  be the poset of lower intervals of  $P$  ordered by inclusion. Define two polynomials  $\hat{h}(P, x)$  and  $\hat{g}(P, x)$  recursively as follows.

- $\hat{h}(\mathbb{1}, x) = \hat{g}(\mathbb{1}, x) = 1$ .

- If  $P$  has rank  $d + 1$ , then  $\deg \hat{h}(P, x) = d$ . We first write

$$\hat{h}(P, x) = \hat{h}_d + \hat{h}_{d-1}x + \hat{h}_{d-2}x^2 + \cdots + \hat{h}_0x^d.$$

We then define  $\hat{g}(P, x)$  as

$$\hat{g}(P, x) := \hat{h}_d + (\hat{h}_{d-1} - \hat{h}_d)x + (\hat{h}_{d-2} - \hat{h}_{d-1})x^2 + \cdots + (\hat{h}_{d-m} - \hat{h}_{d-m+1})x^m,$$

where  $m = \deg \hat{g}(P, x) = \lfloor \frac{d}{2} \rfloor$ .

- Finally, for a poset  $P$  of rank  $d + 1$ , define

$$\hat{h}(P, x) := \sum_{Q \in \tilde{P}, Q \neq P} \hat{g}(Q, x)(x - 1)^{d - \rho(Q)}.$$

The coefficients of these polynomials, arranged as vectors, are called the **toric  $h$ -vector** and the **toric  $g$ -vector**, respectively.

**Remark 2.2.4.** We follow the convention of Swartz [35], and so our  $\hat{h}_i$  is  $\hat{h}_{d-i}$  in Stanley's paper [29].

## 2.3 Dehn–Sommerville relations

### 2.3.1 Pure Simplicial Complexes

The main result of this section is the following generalization of Dehn–Sommerville relations (see [18]) to all pure simplicial complexes. We will discuss two proofs of this result; the third one is sketched in Section 2.4 (see Remark 2.4.4).

**Theorem 2.3.1.** *Let  $\Delta$  be a pure  $(d - 1)$ -dimensional simplicial complex. Then*

$$h_{d-j}(\Delta) - h_j(\Delta) = (-1)^j \sum_{F \in \Delta} \binom{d - |F|}{j} \varepsilon_\Delta(F) \quad \text{for } j = 0, \dots, d. \quad (2.3.1)$$

*Proof.* Note that  $(-1)^{d-1-i} f_{i-1}(\Delta) - \sum_{F \in \Delta_{i-1}} (-1)^{d-1-|F|} = 0$ , and hence

$$\sum_{F \in \Delta_{i-1}} \tilde{\chi}(\text{lk}_\Delta F) = (-1)^{d-1-i} f_{i-1}(\Delta) + \sum_{F \in \Delta_{i-1}} [\tilde{\chi}(\text{lk}_\Delta F) - (-1)^{d-1-|F|}]. \quad (2.3.2)$$

On the other hand, since each  $(j - 1)$ -dimensional face of  $\Delta$  contains exactly  $\binom{j}{i}$  faces of dimension  $i - 1$ ,

$$\sum_{F \in \Delta_{i-1}} \tilde{\chi}(\text{lk}_\Delta F) = \sum_{j=i}^d (-1)^{j-i-1} \binom{j}{i} f_{j-1}(\Delta). \quad (2.3.3)$$

Setting the right-hand sides of (2.3.2) and (2.3.3) equal to each other and multiplying throughout by  $(-1)^{d-1-i}$  yields

$$\sum_{j=i}^d (-1)^{d-j} \binom{j}{i} f_{j-1}(\Delta) = f_{i-1}(\Delta) + (-1)^{d-1-i} \sum_{F \in \Delta_{i-1}} \varepsilon_\Delta(F). \quad (2.3.4)$$

Now we multiply both sides of (2.3.4) by  $(\lambda - 1)^{d-i}$  and sum the result over  $i$ :

$$\sum_{i=0}^d \left[ \sum_{j=i}^d (-1)^{d-j} \binom{j}{i} f_{j-1}(\Delta) \right] (\lambda - 1)^{d-i} = \sum_{i=0}^d \left[ f_{i-1}(\Delta) + (-1)^{d-1-i} \sum_{F \in \Delta_{i-1}} \varepsilon_\Delta(F) \right] (\lambda - 1)^{d-i}. \quad (2.3.5)$$

The left hand-side of equation (2.3.5) may be rewritten as

$$\begin{aligned} \sum_{j=0}^d (-1)^{d-j} (\lambda - 1)^{d-j} f_{j-1}(\Delta) \left( \sum_{i=0}^d \binom{j}{i} (\lambda - 1)^{j-i} \right) &= \sum_{j=0}^d f_{j-1}(\Delta) (1 - \lambda)^{d-j} \lambda^j \\ &= \sum_{j=0}^d h_j(\Delta) \lambda^j. \end{aligned}$$

The right hand-side of equation (2.3.5) can be broken up as

$$\begin{aligned} \sum_{i=0}^d \left[ f_{i-1}(\Delta) + (-1)^{d-1-i} \sum_{F \in \Delta_{i-1}} \varepsilon_\Delta(F) \right] (\lambda - 1)^{d-i} &= \sum_{i=0}^d f_{i-1}(\Delta) (\lambda - 1)^{d-i} \\ &\quad + \sum_{i=0}^d (-1)^{d-1-i} \left( \sum_{F \in \Delta_{i-1}} \varepsilon_\Delta(F) \right) (\lambda - 1)^{d-i}. \end{aligned}$$

We will analyze each of these terms on the right independently. Firstly,

$$\sum_{i=0}^d f_{i-1}(\Delta) (\lambda - 1)^{d-i} = \sum_{i=0}^d h_i(\Delta) \lambda^{d-i}.$$

For the second term,

$$\begin{aligned}
& \sum_{i=0}^d (-1)^{d-1-i} \left( \sum_{F \in \Delta_{i-1}} \varepsilon_{\Delta}(F) \right) (\lambda - 1)^{d-i} \\
&= \sum_{i=0}^d (-1)^{d-1-i} \left( \sum_{F \in \Delta_{i-1}} \varepsilon_{\Delta}(F) \right) \left( \sum_{j=0}^{d-i} (-1)^{d-i-j} \binom{d-i}{j} \lambda^j \right) \\
&= \sum_{i=0}^d \left( \sum_{F \in \Delta_{i-1}} \varepsilon_{\Delta}(F) \right) \left( \sum_{j=0}^{d-i} (-1)^{j-1} \binom{d-i}{j} \lambda^j \right) \\
&= \sum_{j=0}^d (-1)^{j-1} \left( \sum_{i=0}^d \binom{d-i}{j} \left( \sum_{F \in \Delta_{i-1}} \varepsilon_{\Delta}(F) \right) \right) \lambda^j.
\end{aligned}$$

By equating coefficients in equation (2.3.5) we obtain

$$h_{d-j}(\Delta) - h_j(\Delta) = (-1)^j \left( \sum_{i=0}^d \binom{d-i}{j} \sum_{F \in \Delta_{i-1}} \varepsilon_{\Delta}(F) \right),$$

and the summations on the right may be re-written as in the statement of the theorem.  $\square$

Since  $\varepsilon_{\Delta}(F) = 0$  unless  $F \in \mathfrak{E}(\Delta)$ , we have the following corollary that phrases the relationship between  $h_j(\Delta)$  and  $h_{d-j}(\Delta)$  in terms of the non-Eulerian part of  $\Delta$ .

**Corollary 2.3.2.** *Let  $\Delta$  be a pure  $(d-1)$ -dimensional simplicial complex. Then*

$$h_{d-j}(\Delta) - h_j(\Delta) = (-1)^j \sum_{F \in \mathfrak{E}(\Delta)} \binom{d-|F|}{j} \varepsilon_{\Delta}(F).$$

**Example 2.3.3.** When  $\Delta$  is semi-Eulerian (so that  $\mathfrak{E}(\Delta) = \{\emptyset\}$ ),

$$\sum_{F \in \mathfrak{E}(\Delta)} \binom{d-|F|}{j} \varepsilon_{\Delta}(F) = \binom{d}{j} [\tilde{\chi}(\Delta) - (-1)^{d-1}]$$

and

$$h_{d-j}(\Delta) - h_j(\Delta) = (-1)^j \binom{d}{j} [\tilde{\chi}(\Delta) - (-1)^{d-1}].$$

Thus, in this case Theorem 2.3.1 reduces to Klee's Dehn–Sommerville equations in [18].

**Example 2.3.4.** As for  $h$ -vectors of complexes with  $\mathfrak{E}(\Delta)$  containing faces of dimension larger than  $-1$ , consider the case in which  $\|\Delta\| = \mathbb{S}^1 * \|M\|$ , where  $\mathbb{S}^1$  denotes the 1-dimensional sphere and  $M$  is some  $(d-3)$ -dimensional simplicial manifold with  $\varepsilon_\Delta(\emptyset) = \tilde{\chi}(\Delta) - (-1)^{d-1} \neq 0$ . Then  $\mathfrak{E}(\Delta)$  forms a cycle (in the graph theory sense), say of length  $n$ , and

$$\begin{aligned} & \sum_{i=0}^d \binom{d-i}{j} \left( \sum_{F \in \mathfrak{E}(\Delta)_{i-1}} \varepsilon_\Delta(F) \right) \\ &= \binom{d}{j} [\tilde{\chi}(M) - (-1)^{d-1}] + n \binom{d-1}{j} [-\tilde{\chi}(M) - (-1)^{d-2}] + n \binom{d-2}{j} [\tilde{\chi}(M) - (-1)^{d-3}] \\ &= (\tilde{\chi}(M) + (-1)^d) \left[ \binom{d}{j} - n \binom{d-2}{j-1} \right], \end{aligned}$$

and so

$$h_{d-j}(\Delta) - h_j(\Delta) = (-1)^j (\tilde{\chi}(M) + (-1)^d) \left[ \binom{d}{j} - n \binom{d-2}{j-1} \right] \quad (2.3.6)$$

for  $j = 0, \dots, d$ . In particular, if  $M$  is a triangulation of the torus, then  $d = 5$  and  $\tilde{\chi}(M) = -1$ , and so

$$h_{5-j}(\Delta) - h_j(\Delta) = (-1)^j (-2) \left[ \binom{5}{j} - n \binom{3}{j-1} \right].$$

**Remark 2.3.5.** One consequence of (2.3.6) is that, with  $\tilde{\chi}(M)$  known, the exact number of non-Eulerian edges in any triangulation  $\Delta$  of  $\mathbb{S}^1 * M$  is determined by just the face numbers  $f_i(\Delta)$  up to a dimension about  $\frac{d}{2}$ .

The following is an alternative proof of Theorem 2.3.1. This proof uses *short  $h$ -numbers*, and we define them as follows.

For  $0 \leq i \leq d-1$ , the  *$i$ -th short  $h$ -number* (defined by Hersh and Novik) is

$$h_i^*(\Delta) = \sum_{v \in V(\Delta)} h_i(\text{lk}_\Delta v).$$

These numbers go back to McMullen's proof of the Upper Bound Theorem (see [19]), but were formalized by Hersh and Novik in [15]. The following formula that connects the short  $h$ -numbers to the usual ones was verified by McMullen for simplicial polytopes (see [19, pg. 183]) and by Swartz for pure simplicial complexes (see [34, Prop. 2.3]):

$$h_{i-1}^* = ih_i + (d-i+1)h_{i-1} \quad \text{for all } 1 \leq i \leq d. \quad (2.3.7)$$

*Another Proof of Theorem 2.3.1.* We will prove (2.3.1) by double induction: first we induct on the dimension of  $\Delta$ , and for each  $(d-1)$ -dimensional simplicial complex we induct on  $i \geq 0$  (using the validity of the statement for  $h_{d-i+1} - h_{i-1}$  to derive its validity for  $h_{d-i} - h_i$ ).

If  $\dim(\Delta) = 0$ , it is easy to check that (2.3.1) holds.

If  $\dim(\Delta) = d-1$  for  $d > 1$  and  $\Delta$  is pure, then  $\text{lk}_\Delta v$  is a pure  $(d-2)$ -dimensional simplicial complex (for any vertex  $v$ ). By the inductive hypothesis,

$$h_{d-i-1}(\text{lk}_\Delta v) - h_i(\text{lk}_\Delta v) = (-1)^i \sum_{F \in \text{lk}(v)} \binom{d-1-|F|}{i} \varepsilon_{\text{lk}(v)}(F), \quad (2.3.8)$$

and by summing over  $v \in V(\Delta)$  on both sides, we obtain that

$$h_{d-i-1}^*(\Delta) - h_i^*(\Delta) = (-1)^i \sum_{v \in V(\Delta)} \sum_{F \in \text{lk}(v)} \binom{d-1-|F|}{i} \varepsilon_{\text{lk}(v)}(F). \quad (2.3.9)$$

Assume that (2.3.1) of Theorem 2.3.1 holds for all  $d' < d$ . The base case of the induction on  $i$  is when  $i = 0$ . By the definition of  $h$ -vectors and Euler's formula,

$$h_d(\Delta) - h_0(\Delta) = (-1)^{d-1} \tilde{\chi}(\Delta) - 1. \quad (2.3.10)$$

On the other hand, the expression on the right-hand side of (2.3.1) can be rewritten as

$$\begin{aligned} & (-1)^0 \sum_{F \in \Delta} \binom{d-1-|F|}{0} [\tilde{\chi}(\text{lk}_\Delta F) - (-1)^{d-1-|F|}] \\ &= \sum_{i=0}^d \sum_{F \in \Delta_{i-1}} \tilde{\chi}(\text{lk}_\Delta F) - \sum_{i=0}^d \sum_{F \in \Delta_{i-1}} (-1)^{d-1-i} \\ &= \sum_{i=0}^d \sum_{j=i}^d (-1)^{j-i-1} \binom{j}{i} f_{j-1}(\Delta) - \sum_{i=0}^d (-1)^{d-1-i} f_{i-1}(\Delta) \\ &= - \sum_{j=0}^d f_{j-1}(\Delta) \left[ \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} \right] - \sum_{i=0}^d (-1)^{d-1-i} f_{i-1}(\Delta) \\ &\stackrel{(\boxtimes)}{=} -1 + (-1)^{d-1} \sum_{i=0}^d (-1)^{i-1} f_{i-1}(\Delta) \\ &= -1 + (-1)^{d-1} \tilde{\chi}(\Delta), \\ &\stackrel{(2.3.10)}{=} h_d(\Delta) - h_0(\Delta) \end{aligned}$$

where the equality ( $\bowtie$ ) follows from the following (well-known) binomial identity.

$$\sum_{i=0}^j (-1)^{j-i} \binom{j}{i} = \begin{cases} 1, & \text{if } j = 0, \\ 0, & \text{if } j > 0. \end{cases}$$

This completes the proof of the  $i = 0$  case.

Let  $i > 0$  and assume  $h_{d-i+1}(\Delta) - h_{i-1}(\Delta)$  satisfies the formula in the statement of the theorem. Then by (2.3.9) and (2.3.7), we have

$$\begin{aligned} i(h_{d-i} - h_i) &\stackrel{(2.3.7)}{=} (-1)(d-i+1)[h_{d-i+1} - h_{i-1}] + (h_{d-i}^* - h_{i-1}^*) \\ &\stackrel{\text{ind. hyp.}}{=} (-1)(d-i+1) \left[ (-1)^{i-1} \sum_{F \in \Delta} \binom{d-|F|}{i-1} \varepsilon_{\Delta}(F) \right] + (h_{d-i}^* - h_{i-1}^*) \\ &\stackrel{(2.3.9)}{=} (-1)(d-i+1) \left[ (-1)^{i-1} \sum_{F \in \Delta} \binom{d-|F|}{i-1} \varepsilon_{\Delta}(F) \right] \\ &\quad + (-1)^{i-1} \left( \sum_{v \in V(\Delta)} \sum_{F \in \text{lk}_{\Delta} v} \binom{d-1-|F|}{i-1} \varepsilon_{\text{lk}(v)}(F) \right). \end{aligned}$$

Therefore

$$\begin{aligned} h_{d-i}(\Delta) - h_i(\Delta) &= (-1)^i \frac{d-i+1}{i} \sum_{F \in \Delta} \binom{d-|F|}{i-1} \varepsilon_{\Delta}(F) \\ &\quad + \frac{(-1)^{i-1}}{i} \sum_{v \in V(\Delta)} \sum_{F \in \text{lk}_{\Delta} v} \binom{d-1-|F|}{i-1} \varepsilon_{\text{lk}(v)}(F). \end{aligned} \quad (2.3.11)$$

To show (2.3.11) equals the right-hand side of (2.3.1), we need to show:

$$\frac{d-i+1}{i} \sum_{F \in \Delta} \binom{d-|F|}{i-1} \varepsilon_{\Delta}(F) - \sum_{v \in V(\Delta)} \sum_{F \in \text{lk}_{\Delta} v} \frac{1}{i} \binom{d-1-|F|}{i-1} \varepsilon_{\text{lk}(v)}(F) = \sum_{F \in \Delta} \binom{d-|F|}{i} \varepsilon_{\Delta}(F). \quad (2.3.12)$$

Notice that for each  $v \in V(\Delta)$  and  $F \in \text{lk}_{\Delta}(v)$ ,  $\text{lk}_{\text{lk}(v)}(F) = \text{lk}_{\Delta}(F \cup v)$ , therefore  $\varepsilon_{\text{lk}(v)} F = \varepsilon_{\Delta}(F \cup v)$ , and so

$$\sum_{v \in V(\Delta)} \sum_{F \in \text{lk}_{\Delta} v} \frac{1}{i} \binom{d-1-|F|}{i-1} \varepsilon_{\text{lk}(v)}(F) = \sum_{G \in \Delta} |G| \cdot \frac{1}{i} \binom{d-|G|}{i-1} \varepsilon_{\Delta}(G).$$

Plugging this expression into the left-hand side of (2.3.12), we obtain that the left-hand side

of (2.3.12) can be rewritten as

$$\begin{aligned}
& \frac{d-i+1}{i} \sum_{F \in \Delta} \binom{d-|F|}{i-1} \varepsilon_{\Delta}(F) - \sum_{F \in \Delta} |F| \cdot \frac{1}{i} \binom{d-|F|}{i-1} \varepsilon_{\Delta}(F) \\
&= \sum_{F \in \Delta} \left[ \frac{d-i+1}{i} - \frac{|F|}{i} \right] \binom{d-|F|}{i-1} \varepsilon_{\Delta}(F) \\
&= \sum_{F \in \Delta} \binom{d-|F|}{i} \varepsilon_{\Delta}(F),
\end{aligned}$$

and so (2.3.12) does hold. This completes the proof of the theorem.  $\square$

### 2.3.2 Simplicial posets

In this subsection we will show that Theorem 2.3.1 can be generalized to **simplicial posets** (for more details see [30]). A graded poset  $P$  (with the unique bottom and top elements  $\hat{0}$  and  $\hat{1}$ ) is **simplicial** if every proper lower interval  $[\hat{0}, t]$  is a Boolean lattice. Given a simplicial complex  $\Delta$ , the poset of faces of  $\Delta$  ordered by inclusion is a simplicial poset. (It is called the **face lattice of  $\Delta$** .) Therefore simplicial posets are generalizations of simplicial complexes. Many notions and structures on simplicial complexes can be generalized to simplicial posets.

Given a simplicial poset  $P$  of rank  $d+1$ , for  $-1 \leq i \leq d-1$ , define  $f_i = f_i(P)$  as the number of elements in  $P$  with rank  $i+1$ . The vector  $f(P) = (f_{-1}, f_0, f_1, \dots, f_{d-1})$  is called the  **$f$ -vector of  $P$** . Similar to the definition of  $h$ -vectors in Subsection 2.2.1, we can define  $h_0, h_1, \dots, h_d$  by

$$\sum_{i=0}^d h_i x^{d-i} = \sum_{i=0}^d f_{i-1} (x-1)^{d-i}.$$

The vector  $h(P) = (h_0, h_1, \dots, h_d)$  is the  **$h$ -vector of  $P$** . When  $P$  is the face lattice of a simplicial complex  $\Delta$ , then  $f(P) = f(\Delta)$  and  $h(P) = h(\Delta)$ .

The notion of links can also be generalized from simplicial complexes to simplicial posets. Let  $P$  be a simplicial poset and  $t \in P$ , the **link of  $t$  in  $P$**  is simply the upper interval  $[t, \hat{1}]$ . It is easy to see that  $[t, \hat{1}]$  is also a simplicial poset.

With these notions in hand, both proofs of Theorem 2.3.1 can be easily adapted to the more general setting of simplicial posets and result in the following corollary:

**Corollary 2.3.6.** *Let  $P$  be a graded simplicial poset of rank  $d + 1$ . Then*

$$h_{d-j}(P) - h_j(P) = (-1)^j \sum_{t \in P} \binom{d - \rho(t)}{j} \left[ \mu_P(t, \hat{1}) - (-1)^{d-1-\rho(t)} \right] \quad \text{for } j = 0, \dots, d.$$

## 2.4 Flag Dehn–Sommerville relations

The goal of this section is to generalize the Bayer–Billera theorem [5] on flag  $h$ -vectors of Eulerian balanced simplicial complexes (see also [33, Cor. 3.16.6] for the poset version). This result states that if  $\Delta$  is an Eulerian balanced simplicial complex of dimension  $d - 1$ , then for all  $S \subseteq [d]$ ,  $h_S(\Delta) = h_{[d]-S}(\Delta)$ . Recall that the error at face  $F \in \Delta$  is defined as

$$\varepsilon_\Delta(F) = \tilde{\chi}(\text{lk}_\Delta F) - (-1)^{d-1-|F|}.$$

The main result of this section is the following. We will provide two proofs.

**Theorem 2.4.1.** *Let  $\Delta$  be a  $(d-1)$ -dimensional balanced simplicial complex with the coloring map  $\kappa : V(\Delta) \rightarrow [d]$ . Let  $S \subseteq [d]$ . Then*

$$h_S(\Delta) - h_{S^c}(\Delta) = (-1)^{d-|S|} \sum_{F \in \Delta_S} \varepsilon_\Delta(F). \quad (2.4.1)$$

Our first proof will rely on the following proposition.

**Proposition 2.4.2.** *Let  $\Delta$  be a  $(d-1)$ -dimensional balanced simplicial complex with the coloring map  $\kappa : V(\Delta) \rightarrow [d]$ . Let  $S$  be a subset of  $[d]$  and  $i \notin S$ , then*

$$\sum_{v: \kappa(v)=i} h_S(\text{lk}_\Delta(v)) = h_{S \cup \{i\}}(\Delta) + h_S(\Delta). \quad (2.4.2)$$

We delay the proof of Proposition 2.4.2 until after the proof of Theorem 2.4.1.

*Proof of Theorem 2.4.1.* Similar to the second proof of Theorem 2.3.1, we will verify (2.4.1) by double induction: first on the dimension of  $\Delta$ , and then on the size of  $S$ . If  $\dim \Delta = 0$ , it is easy to check that (2.4.1) holds.

Assume  $\dim \Delta = d - 1 > 0$ . The base case when  $S = \emptyset$  follows from Theorem 2.3.1:

$$h_{\emptyset}(\Delta) - h_{[d]}(\Delta) = h_0(\Delta) - h_d(\Delta) \stackrel{\text{Thm. 3.1}}{=} (-1)^d \varepsilon_{\Delta}(\emptyset).$$

For the inductive step, fix  $j \in [d - 1]$  and assume that for all  $k > j$  and  $|S| = d - k$ , the equality (2.4.1) holds. Up to reordering of the colors, it suffices to show that

$$h_{[j+1,d]}(\Delta) - h_{[j]}(\Delta) = (-1)^j \sum_{F \in \Delta: k(F) \subseteq [j+1,d]} \varepsilon_{\Delta}(F). \quad (2.4.3)$$

By Proposition 2.4.2,

$$\begin{aligned} & \sum_{\kappa(v)=j+1} \left[ h_{[j+1,d]-\{j+1\}}(\text{lk}_{\Delta}(v)) - h_{[j]}(\text{lk}_{\Delta}(v)) \right] \\ &= \left[ h_{[j+1,d]}(\Delta) + h_{[j+1,d]-\{j+1\}}(\Delta) \right] - \left[ h_{[j+1]}(\Delta) + h_{[j]}(\Delta) \right] \\ &= \left[ h_{[j+1,d]}(\Delta) - h_{[j]}(\Delta) \right] + \left[ h_{[j+1,d]-\{j+1\}}(\Delta) - h_{[j+1]}(\Delta) \right] \\ &\stackrel{\text{Ind. Hyp.}}{=} \left[ h_{[j+1,d]}(\Delta) - h_{[j]}(\Delta) \right] + \left[ (-1)^{j+1} \sum_{\substack{F \in \Delta, \\ \kappa(F) \subseteq [j+1,d]-\{j+1\}}} \varepsilon_{\Delta}(F) \right]. \end{aligned} \quad (2.4.4)$$

On the other hand, for each vertex  $v$  such that  $\kappa(v) = j + 1$ ,  $\text{lk}_{\Delta}(v)$  is a  $(d - 2)$ -dimensional balanced simplicial complex with  $\kappa : V(\text{lk}_{\Delta}(v)) \rightarrow [d] - \{j + 1\}$ . By the inductive hypothesis on  $\text{lk}_{\Delta}(v)$ , we have

$$h_{[j+1,d]-\{j+1\}}(\text{lk}_{\Delta}(v)) - h_{[j]}(\text{lk}_{\Delta}(v)) = (-1)^j \sum_{\substack{F \in \text{lk}_{\Delta} v, \\ k(F) \subseteq [j+1,d]-\{j+1\}}} \varepsilon_{\text{lk}_{\Delta}(v)}(F). \quad (2.4.5)$$

Therefore,

$$\begin{aligned} \sum_{\kappa(v)=j+1} \left[ h_{[j+1,d]-\{j+1\}}(\text{lk}_{\Delta}(v)) - h_{[j]}(\text{lk}_{\Delta}(v)) \right] &\stackrel{(2.4.5)}{=} \sum_{\kappa(v)=j+1} (-1)^j \sum_{\substack{F \in \text{lk}_{\Delta} v, \\ \kappa(F) \subseteq [j+1,d]-\{j+1\}}} \varepsilon_{\text{lk}_{\Delta}(v)}(F) \\ &\stackrel{(*)}{=} (-1)^j \sum_{\substack{\kappa(v)=j+1; F \in \text{lk}_{\Delta} v; \\ k(F \cup v) \subseteq [j+1,d]}} \varepsilon_{\Delta}(F \cup v). \end{aligned} \quad (2.4.6)$$

where  $(\star)$  holds since for any  $v \in V(\Delta)$  and  $F \in \text{lk}_\Delta(v)$ ,  $\text{lk}_{\text{lk}_\Delta(v)}(F) = \text{lk}_\Delta(F \cup v)$ .

Comparing (2.4.4) and (2.4.6), we obtain

$$\begin{aligned} h_{[j+1,d]}(\Delta) - h_{[j]}(\Delta) &= (-1)^j \left[ \sum_{\substack{F \in \Delta, \\ \kappa(F) \subseteq [j+1,d] - \{j+1\}}} \varepsilon_\Delta(F) + \sum_{\substack{\kappa(v)=j+1; F \in \text{lk}_\Delta v; \\ k(F \cup v) \subseteq [j+1,d]}} \varepsilon_\Delta(F \cup v) \right] \\ &= (-1)^j \sum_{\substack{F \in \Delta, \\ \kappa(F) \subseteq [j+1,d]}} \varepsilon_\Delta(F), \end{aligned}$$

Therefore (2.4.3) holds.  $\square$

*Proof of Proposition 2.4.2.* Recall that  $i$  and  $S$  are fixed and that  $i \notin S$ . The proof is a routine computation that relies on the definition of flag  $h$ -numbers in terms of flag  $f$ -numbers:

$$\begin{aligned} \sum_{v:\kappa(v)=i} h_S(\text{lk}_\Delta v) &= \sum_{v:\kappa(v)=i} \sum_{R \subseteq S} (-1)^{|S-R|} f_R(\text{lk}_\Delta v) \\ &= \sum_{R \subseteq S} (-1)^{|S-R|} \sum_{v:\kappa(v)=i} f_R(\text{lk}_\Delta v) \\ &= \sum_{R \subseteq S} (-1)^{|S \cup \{i\}| - |R \cup \{i\}|} f_{R \cup \{i\}}(\Delta) \\ &= h_{S \cup \{i\}}(\Delta) - \sum_{T \subseteq S} (-1)^{|S \cup \{i\}| - |T|} f_T(\Delta) \\ &= h_{S \cup \{i\}}(\Delta) + \sum_{T \subseteq S} (-1)^{|S| - |T|} f_T(\Delta) \\ &= h_{S \cup \{i\}}(\Delta) + h_S(\Delta). \end{aligned} \tag{2.4.7}$$

$\square$

Our second proof of Theorem 2.4.1 uses the Hilbert series. The idea of this proof is similar to that of [35, Theorem 3.8], which uses the following theorem by Stanley. Recall that  $F(\Delta, \lambda)$  is the Hilbert series of  $\mathbb{k}[\Delta]$  (w.r.t. the  $\mathbb{Z}^d$ -grading), that for  $F \in \Delta$ ,  $\kappa(F)$  is the set of colors of vertices of  $F$ , and that for any subset  $S \subseteq [d]$ ,  $\lambda^S$  denotes  $\prod_{i \in S} \lambda_i$ . The following theorem is a corollary of [32, II, Thm. 7.1].

**Theorem 2.4.3.** *Let  $\Delta$  be a  $(d-1)$ -dimensional balanced simplicial complex and let  $F(k[\Delta], 1/\lambda)$  be the Hilbert series. Then*

$$(-1)^d F(k[\Delta], 1/\lambda) = (-1)^{d-1} \tilde{\chi}(\Delta) + \sum_{F \in \Delta, F \neq \emptyset} (-1)^{d-|F|-1} \tilde{\chi}(\text{lk } F) \cdot \lambda^{\kappa(F)} \cdot \prod_{v \in F} \frac{1}{1 - \lambda_{\kappa(v)}}.$$

We will also use the following relation that follows easily from the definition of the flag  $h$ -numbers, see, the proof of [35, Theorem 3.8]:

$$\sum_{F \in \Delta, F \neq \emptyset} \lambda^{\kappa(F)} \prod_{i \in [d] - \kappa(F)} (1 - \lambda_i) = \sum_{S \subseteq [d]} \left[ h_S(\Delta) - (-1)^{|S|} \right] \cdot \lambda^S. \quad (2.4.8)$$

*Second proof of Theorem 2.4.1 .*

$$\begin{aligned} & (-1)^d F(k[\Delta], 1/\lambda) \\ \stackrel{\text{Thm. 2.4.3}}{=} & (-1)^{d-1} \tilde{\chi}(\Delta) + \sum_{F \in \Delta, F \neq \emptyset} (-1)^{d-|F|-1} \tilde{\chi}(\text{lk } F) \cdot \lambda^{\kappa(F)} \cdot \prod_{v \in F} \frac{1}{1 - \lambda_{\kappa(v)}} \\ = & (-1)^{d-1} \tilde{\chi}(\Delta) + \left( \prod_{j=1}^d \frac{1}{1 - \lambda_j} \right) \cdot \sum_{F \in \Delta, F \neq \emptyset} (-1)^{d-|F|-1} \underbrace{\tilde{\chi}(\text{lk } F)}_{=(-1)^{d-1-|F|+\varepsilon_\Delta(F)}} \lambda^{\kappa(F)} \prod_{i \in [d] - \kappa(F)} (1 - \lambda_i) \\ = & (-1)^{d-1} \tilde{\chi}(\Delta) + \left( \prod_{j=1}^d \frac{1}{1 - \lambda_j} \right) \cdot \sum_{F \in \Delta, F \neq \emptyset} \lambda^{\kappa(F)} \prod_{i \in [d] - \kappa(F)} (1 - \lambda_i) \\ & + \left( \prod_{j=1}^d \frac{1}{1 - \lambda_j} \right) \cdot \sum_{F \in \Delta, F \neq \emptyset} (-1)^{d-|F|-1} \cdot \varepsilon_\Delta(F) \lambda^{\kappa(F)} \prod_{i \in [d] - \kappa(F)} (1 - \lambda_i) \\ \stackrel{(2.4.8)}{=} & (-1)^{d-1} \tilde{\chi}(\Delta) + \left( \prod_{j=1}^d \frac{1}{1 - \lambda_j} \right) \cdot \sum_{S \subseteq [d]} \left[ h_S(\Delta) - (-1)^{|S|} \right] \cdot \lambda^S \quad (2.4.9) \\ & + \left( \prod_{j=1}^d \frac{1}{1 - \lambda_j} \right) \cdot \sum_{F \in \Delta, F \neq \emptyset} (-1)^{d-|F|-1} \cdot \varepsilon_\Delta(F) \lambda^{\kappa(F)} \prod_{i \in [d] - \kappa(F)} (1 - \lambda_i). \end{aligned}$$

On the other hand,

$$(-1)^d F(k[\Delta], 1/\lambda) = \left( \prod_{j=1}^d \frac{1}{1 - \lambda_j} \right) \cdot \sum_{S \subseteq [d]} h_S(\Delta) \lambda^{[d]-S}. \quad (2.4.10)$$

Comparing (2.4.9) with (2.4.10) and multiplying both sides by  $\prod_{j=1}^d (1 - \lambda_j)$ , we obtain

$$\begin{aligned} (-1)^{d-1} \tilde{\chi}(\Delta) \cdot \prod_{j=1}^d (1 - \lambda_j) + \sum_{S \subseteq [d]} \left[ h_S(\Delta) - (-1)^{|S|} \right] \cdot \lambda^S \\ + \sum_{F \in \Delta, F \neq \emptyset} (-1)^{d-|F|-1} \cdot \varepsilon_\Delta(F) \lambda^{\kappa(F)} \cdot \prod_{i \in [d] - \kappa(F)} (1 - \lambda_i) = \sum_{S \subseteq [d]} h_S(\Delta) \lambda^{[d]-S}. \end{aligned}$$

Therefore

$$\begin{aligned} (-1)^{d-1} \tilde{\chi}(\Delta) \cdot \prod_{j=1}^d (1 - \lambda_j) + \sum_{S \subseteq [d]} (-1)^{|S|-1} \cdot \lambda^S + \sum_{F \in \Delta, F \neq \emptyset} (-1)^{d-|F|-1} \cdot \varepsilon_\Delta(F) \lambda^{\kappa(F)} \cdot \prod_{i \in [d] - \kappa(F)} (1 - \lambda_i) \\ = \sum_{S \subseteq [d]} \left[ h_S(\Delta) \lambda^{[d]-S} - h_S(\Delta) \lambda^S \right]. \end{aligned}$$

The coefficient of  $\lambda^S$  on the RHS is  $h_{[d]-S} - h_S$ . The coefficient of  $\lambda^S$  on the LHS is:

$$\begin{aligned} (-1)^{d-|S|-1} \tilde{\chi}(\Delta) + (-1)^{|S|-1} + \sum_{F \in \Delta_S, F \neq \emptyset} (-1)^{d-|F|-1} \cdot \varepsilon_\Delta(F) \cdot (-1)^{|S|-|F|} \\ = (-1)^{d-|S|-1} \sum_{F \in \Delta_S} \varepsilon_\Delta(F). \end{aligned}$$

Together we obtain

$$h_{[d]-S} - h_S = (-1)^{d-|S|-1} \sum_{F \in \Delta_S} \varepsilon_\Delta(F).$$

□

**Remark 2.4.4.** A similar argument, but using the coarse Hilbert series, provides yet another proof of Theorem 2.3.1.

Observe that Theorem 2.4.1 refines Theorem 2.3.1: summing eq. (2.4.1) over all subsets  $S \subseteq [d]$  of size  $i$ , we obtain

$$\sum_{|S|=i} h_S(\Delta) - \sum_{|S|=i} h_{S^c}(\Delta) = (-1)^{i-1} \sum_{F \in \Delta} \binom{d-|F|}{i} \varepsilon_\Delta(F),$$

which is equivalent to eq. (2.3.1).

Recall from Section 3.2 that for a graded poset  $P$ , the complex  $O(P)$  is always balanced w.r.t. the coloring given by the rank function, and that for  $S \subseteq [d]$ ,  $\beta_P(S) = h_S(O(P))$ .

Moreover, by Remark 2.2.3, the chain error of a poset ( $\varepsilon_P(-)$ ) is the same as the link error of its order complex ( $\varepsilon_{O(P)}(-)$ ). The following corollary now follows directly from Theorem 2.4.1:

**Corollary 2.4.5.** *Let  $P$  be a graded poset with rank  $d + 1$  and let  $S \subseteq [d]$ . Then*

$$\beta_P(S) - \beta_P(S^c) = (-1)^{d-|S|} \sum_{C \in \mathcal{C}(P_S)} \varepsilon_P(C).$$

where  $P_S$  is the  $S$ -selected subposet of  $P$ , and  $\mathcal{C}(P_S)$  denotes the set of all chains in  $P_S \setminus \{\hat{0}, \hat{1}\}$ .

## 2.5 Posets with isolated singularities

Stanley extended the Dehn–Sommerville relations for Eulerian simplicial complexes to the generality of toric  $h$ -vectors of Eulerian posets. The goal of this and the following sections is to further generalize these relations to more general posets.

We start by defining the **error function for intervals in posets**. Let  $P$  be a graded poset of rank  $(d + 1)$  and let  $[s, t]$  be an interval in  $P$ . The error of  $[s, t]$  is defined as

$$e_P([s, t]) := \mu_P(s, t) - (-1)^{\rho(t) - \rho(s)}.$$

From now on we will use  $e_P(s, t)$  as the abbreviation for  $e_P([s, t])$ .<sup>1</sup>

**Definition 2.5.1.** A graded poset  $P$  with  $\rho(P) = d + 1$  has **singularities of rank 1** or is **1-Sing** if all intervals  $[s, t]$  in  $P$  of length  $\rho(t) - \rho(s) \leq d - 1$  are Eulerian.

**Proposition 2.5.2.** *A poset  $P$  of rank  $\rho(P) = d + 1$  is 1-Sing if and only if its reduced order complex  $O(P)$  satisfies the following condition: for all faces  $F \in O(P)$  with  $\dim(F) \geq 1$ ,  $\tilde{\chi}(\text{lk}_{O(P)} F) = (-1)^{d-1-|F|}$ .*

We omit the proof as we will prove a generalization of this result in Proposition 2.6.5. Our work in the rest of this section is motivated by the following theorem of Stanley [29] and its generalization due to Swartz (see [35, Theorem 3.15]).

---

<sup>1</sup>We have already defined the error function for chains  $\varepsilon_P(C)$ , but to study toric  $h$ -vectors it is easier to use interval errors rather than link errors. The connection between the two will be discussed later in the proof of Corollary 2.6.12.

**Theorem 2.5.3** (Stanley). *Let  $P$  be an Eulerian poset of rank  $d + 1$ . Then  $\hat{h}_i(P) = \hat{h}_{d-i}(P)$  for all  $0 \leq i \leq d$ .*

**Theorem 2.5.4** (Swartz). *Let  $P$  be a semi-Eulerian poset  $P$  of rank  $d + 1$  and let  $O(P)$  be its reduced order complex. Then for all  $0 \leq i \leq d$ ,*

$$\hat{h}_{d-i}(P) - \hat{h}_i(P) = (-1)^{d-i+1} \binom{d}{i} [\tilde{\chi}(O(P)) - (-1)^{d-1}] = (-1)^{d-i+1} \binom{d}{i} \cdot e_P(\hat{0}, \hat{1}).$$

**Remark 2.5.5.** The formula given by Swartz in [35, Theorem 3.15] is equivalent to the statement above. Indeed, when  $d$  is even,  $P$  is Eulerian, and so the right hand-side is zero. If  $d$  is odd, then  $d - i + 1$  and  $i$  have the same parity and the formula above agrees with the one in Swartz's Theorem 3.15.

Using ideas from Swartz's and Stanley's proofs, we establish the following generalization of Theorems 2.5.3 and 2.5.4 for 1-Sing posets. For the rest of this section, we let  $y = x - 1$ .

**Theorem 2.5.6.** *Let  $P$  be a graded 1-Sing poset, and let  $\rho(P) = d + 1$ . Then for  $i > \lfloor \frac{d}{2} \rfloor$ ,*

$$\hat{h}_{d-i}(P) - \hat{h}_i(P) = (-1)^{d-i+1} \left[ \binom{d}{i} e_P(\hat{0}, \hat{1}) + \binom{d}{i} \sum_{\rho(t)=d} e_P(\hat{0}, t) + \binom{d-1}{i-1} \sum_{\rho(s)=1} e_P(s, \hat{1}) \right].$$

*Proof.* The left hand-side is the coefficient of  $x^i$  in the polynomial  $\hat{h}(P) - x^d \cdot \hat{h}(P, 1/x)$ . We first prove the following lemma related to this polynomial. From now on we use  $\hat{h}(P)$  to abbreviate  $\hat{h}(P, x)$  and  $\hat{g}(P)$  to abbreviate  $\hat{g}(P, x)$ .

**Lemma 2.5.7.** *Let  $P$  be a graded poset with  $\rho(P) = d + 1$  and let  $y = x - 1$ . Then*

$$\begin{aligned} \hat{h}(P) - x^d \cdot \hat{h}(P, 1/x) &= -[\mu_P(\hat{0}, \hat{1}) - (-1)^{d+1}] y^d \\ &+ \sum_{\substack{Q=[0, q] \in \tilde{P} \\ 1 \leq \rho(Q) \leq d}} \left( \left[ -y^{d-\rho(Q)} (\hat{g}(Q) + y \hat{h}(Q)) \cdot \mu_P(q, \hat{1}) \right] - \left[ (-y)^{d-\rho(Q)} \hat{g}(Q, 1/x) \cdot x^{\rho(Q)} \right] \right). \end{aligned} \tag{2.5.1}$$

*Proof.* By definitions, for  $P \neq \mathbb{1}$ ,

$$x^d \cdot \hat{h}(P, 1/x) = \sum_{Q \in \tilde{P}, Q \neq P} (-y)^{d-\rho(Q)} \hat{g}(Q, 1/x) \cdot x^{\rho(Q)}, \quad (2.5.2)$$

and equivalently,

$$\hat{h}(P) = \sum_{Q \in \tilde{P}, Q \neq P} \hat{g}(Q) y^{d-\rho(Q)}. \quad (2.5.3)$$

Multiplying equation (2.5.3) by  $y$  and adding  $\hat{g}(P)$  to both sides, we obtain that for  $P \neq \mathbb{1}$ ,

$$\hat{g}(P) + y\hat{h}(P) = \sum_{Q \in \tilde{P}} \hat{g}(Q) y^{\rho(P)-\rho(Q)}.$$

Therefore for  $P \neq \mathbb{1}$ ,

$$y^{-\rho(P)} \cdot (\hat{g}(P) + y\hat{h}(P)) = \sum_{Q \in \tilde{P}} \hat{g}(Q) y^{-\rho(Q)}.$$

By Möbius inversion,

$$\hat{g}(P) y^{-\rho(P)} = \mu_P(\hat{0}, \hat{1}) + \sum_{\substack{Q=[0,q] \in \tilde{P} \\ 1 \leq \rho(q) \leq d+1}} y^{-\rho(q)} \cdot (\hat{g}(Q) + y\hat{h}(Q)) \cdot \mu_P(q, \hat{1}). \quad (2.5.4)$$

Multiplying (2.5.4) by  $y^{\rho(P)}$  and then subtracting  $\hat{g}(P) + y\hat{h}(P)$  yields

$$-y\hat{h}(P) = \mu_P(\hat{0}, \hat{1}) y^{\rho(P)} + \sum_{\substack{Q=[0,q] \in \tilde{P} \\ 1 \leq \rho(q) \leq d}} y^{\rho(P)-\rho(q)} (\hat{g}(Q) + y\hat{h}(Q)) \cdot \mu_P(q, \hat{1}),$$

and so

$$\hat{h}(P) = -\mu_P(\hat{0}, \hat{1}) y^d - \sum_{\substack{Q=[0,q] \in \tilde{P} \\ 1 \leq \rho(q) \leq d}} y^{d-\rho(q)} (\hat{g}(Q) + y\hat{h}(Q)) \cdot \mu_P(q, \hat{1}).$$

This, together with equation (2.5.2), proves the lemma.  $\square$

Next we prove the following lemma, which helps us further simplify equation (2.5.1) for semi-Eulerian posets.

**Lemma 2.5.8.** *Let  $Q$  be a semi-Eulerian poset with  $\rho(Q) = r + 1$ , let  $s = \lfloor \frac{r}{2} \rfloor$ , and let  $y = x - 1$ . Then*

$$\hat{g}(Q) + y\hat{h}(Q) = x^{\rho(Q)}\hat{g}(Q, 1/x) + \underbrace{\sum_{k=r-s+1}^{r+1} (-1)^{r-k} \binom{r+1}{k} e_Q(Q)}_{\gamma_k} \cdot x^k,$$

where  $\sum^*$  means that, if  $r$  is odd, then there is an extra summand,  $\frac{1}{2}\gamma_k x^k$ , for  $k = r - s$ .

*Proof.* Recall that  $\hat{h}(Q) = \hat{h}_r + \hat{h}_{r-1}x + \cdots + \hat{h}_0x^r$ . This together with the definition of  $\hat{g}(Q)$  implies

$$\hat{g}(Q) + y\hat{h}(Q) = (\hat{h}_{r-s} - \hat{h}_{r-s-1})x^{s+1} + (\hat{h}_{r-s-1} - \hat{h}_{r-s-2})x^{s+2} + \cdots + (\hat{h}_1 - \hat{h}_0)x^r + \hat{h}_0x^{r+1},$$

while

$$x^{\rho(Q)}\hat{g}(Q, 1/x) = (\hat{h}_{r-s} - \hat{h}_{r-s+1})x^{r-s+1} + (\hat{h}_{r-s+1} - \hat{h}_{r-s+2})x^{r-s+2} + \cdots + (\hat{h}_{r-1} - \hat{h}_r)x^r + \hat{h}_r x^{r+1}.$$

By Theorem 2.5.4, if  $Q$  is semi-Eulerian, then  $\hat{h}_{r-k} = \hat{h}_k + (-1)^{r-k+1} \binom{r}{k} e_Q(\hat{0}, \hat{1})$ . Hence for  $k < r$ ,

$$\hat{h}_{r-k} - \hat{h}_{r-k-1} = (\hat{h}_k - \hat{h}_{k+1}) + (-1)^{r-k+1} \cdot \left[ \binom{r}{k} + \binom{r}{k+1} \right] e_Q(\hat{0}, \hat{1}),$$

and since  $\binom{r}{k} + \binom{r}{k+1} = \binom{r+1}{k+1}$ , we infer that

$$\hat{h}_{r-k} - \hat{h}_{r-k-1} = (\hat{h}_k - \hat{h}_{k+1}) + \underbrace{(-1)^{r-k+1} \binom{r+1}{k+1}}_{\gamma_{k+1}(Q)} \cdot e_Q(\hat{0}, \hat{1}).$$

Comparing the coefficients of  $x^{k+1}$  in  $\hat{g}(Q) + y\hat{h}(Q)$  and  $x^{\rho(Q)}\hat{g}(Q, 1/x)$ , yields the lemma.  $\square$

Now we resume the proof of Theorem 2.5.6. For any lower interval  $Q = [\hat{0}, q]$  in  $P$ , if  $1 < \rho(q) < d$ , then  $Q$  is Eulerian. If  $\rho(q) = d$ , then  $Q$  is semi-Eulerian. By Swartz's result and the lemma above,

$$\hat{g}(Q) + y\hat{h}(Q) = \begin{cases} x^{\rho(q)}\hat{g}(Q, 1/x) & \text{if } 1 \leq \rho(q) < d \\ x^{\rho(q)}\hat{g}(Q, 1/x) + \sum_{i=\lfloor \frac{d}{2} \rfloor + 1}^d \gamma_i(Q) \cdot x^i & \text{if } \rho(q) = d. \end{cases}$$

We can now simplify equation (2.5.1):

$$\begin{aligned} \hat{h}(P) - x^d \cdot \hat{h}(P, 1/x) &= -e_P(\hat{0}, \hat{1})y^d \\ &\quad - \sum_{\rho(Q)=1} y^{d-1}x[\mu_P(q, \hat{1}) - (-1)^d] + \sum_{\rho(Q)=d} \sum_{i=\lfloor \frac{d}{2} \rfloor + 1}^d \gamma_i(Q) \cdot x^i, \end{aligned} \quad (2.5.5)$$

and when  $d$  is even, the last summation  $\sum^*$  on the right hand-side has an extra summand  $\frac{1}{2}\gamma_i x^i$  for  $i = \lfloor \frac{d}{2} \rfloor$ . Comparing like-terms from both sides: for  $i > \lfloor \frac{d}{2} \rfloor$ ,

$$\hat{h}_{d-i} - \hat{h}_i = (-1)^{d-i+1} \left[ \binom{d}{i} e_P(\hat{0}, \hat{1}) + \binom{d-1}{i-1} \sum_{\rho(q)=1} e_P(q, \hat{1}) + \binom{d}{i} \sum_{\rho(q)=d} e_P(\hat{0}, q) \right] \quad (2.5.6)$$

as desired.  $\square$

The following special case is worth mentioning: if  $d$  is even and  $i = \lfloor \frac{d}{2} \rfloor = \frac{d}{2} = d - i$ , the left-hand side of (2.5.6) is simply zero, and hence so is the right-hand side. This observation leads to the following Corollary. We will generalize it later in Corollaries 2.6.11 and 2.6.12.

**Corollary 2.5.9.** *Let  $P$  be a 1-Sing poset with odd rank  $\rho(P) = d + 1$ , and let  $O(P)$  be the reduced order complex of  $P$ . Then*

$$2(\tilde{\chi}(O(P)) + 1) = \sum_{\rho(q)=1 \text{ or } d} 1 - \sum_{\rho(q)=1 \text{ or } d} \tilde{\chi}(\text{lk}_{O(P)} v_q)$$

where  $v_q$  is the vertex in  $O(P)$  that corresponds to the element  $q \in P$ .

*Proof.* If  $d$  is even, and  $i = \lfloor \frac{d}{2} \rfloor$ , then  $d - i = i$ . Equation (2.5.6) gives:

$$\begin{aligned} 0 = \hat{h}_{\frac{d}{2}} - \hat{h}_{\frac{d}{2}} &= (-1)^{\frac{d}{2}+1} \binom{d}{\frac{d}{2}} [\mu_P(\hat{0}, \hat{1}) - (-1)] \\ &\quad + \sum_{\rho(q)=1} (-1)^{\frac{d}{2}+1} \binom{d-1}{\frac{d}{2}-1} [\mu_P(q, \hat{1}) - 1] + \frac{1}{2} \sum_{\rho(q)=d} (-1)^{\frac{d}{2}-1} \binom{d}{\frac{d}{2}} [\mu_P(\hat{0}, q) - 1]. \end{aligned} \quad (2.5.7)$$

Since  $\binom{d-1}{\frac{d}{2}-1} = \frac{1}{2} \binom{d}{\frac{d}{2}}$ , this can be simplified to

$$0 = [\mu_P(\hat{0}, \hat{1}) + 1] + \frac{1}{2} \sum_{\rho(q)=1} [\mu_P(q, \hat{1}) - 1] + \frac{1}{2} \sum_{\rho(q)=d} [\mu_P(\hat{0}, q) - 1].$$

Hence

$$\begin{aligned}
2(\mu_P(\hat{0}, \hat{1}) + 1) &= \sum_{\rho(q)=1} [1 - \mu_P(q, \hat{1})] + \sum_{\rho(q)=d} [1 - \mu_P(\hat{0}, q)] \\
&= \sum_{\rho(q)=1, d} 1 + \sum_{\rho(q)=1, d} \mu_P(\hat{0}, q) \cdot \mu_P(q, \hat{1}) = \sum_{\rho(q)=1, d} 1 - \sum_{\rho(q)=1, d} \tilde{\chi}(\text{lk}_{O(P)} F_{\{q\}}).
\end{aligned}$$

□

**Remark 2.5.10.** The above corollary is a generalization of a result from [21] asserting that for a  $(d - 1)$ -dimensional simplicial pseudomanifold  $\Delta$  with isolated singularities,

$$2(\tilde{\chi}(\Delta) + 1) = |V| - \sum_{v \in V} \tilde{\chi}(\text{lk}_{\Delta} v).$$

## 2.6 Posets with singularities of higher degrees

This section generalizes several results from Section 2.5, most notably Theorem 2.5.6.

### 2.6.1 $j$ -Sing Posets

We start with a definition of posets with singularities of degree at most  $j$ . A similar notion was introduced in [10].

**Definition 2.6.1.** For a finite graded poset  $P$  of rank  $\rho(P) = d + 1$ , we recursively define the notion of  $j$ -**Sing**:

$P$  is  $(-1)$ -**Sing** if  $P$  is Eulerian.

$P$  is  $0$ -**Sing** if  $P$  is semi-Eulerian.

$P$  is  $j$ -**Sing** if every interval of length  $\leq d$  in  $P$  is  $(j - 1)$ -**Sing**.

A few remarks are in order.

**Remark 2.6.2.** A poset  $P$  is  $j$ -Sing if and only if for all  $s \leq j + 1$ , every interval of length  $\leq d + 1 - s$  in  $P$  is  $(j - s)$ -Sing. Also,  $P$  is  $j$ -Sing if and only if every interval of length  $\leq d - j$  in  $P$  is Eulerian, i.e., every such interval  $[s, t]$  has  $e_P(s, t) = 0$ .

**Remark 2.6.3.** When  $j$  is odd (even, resp.), all  $j$ -Sing posets  $P$  with even (odd, resp.) rank are in fact  $(j - 1)$ -Sing. This follows from Definition 2.6.1 and the fact that every semi-Eulerian poset of odd rank is actually Eulerian.

**Definition 2.6.4.** A  $(d - 1)$ -dimensional pure simplicial complex  $\Delta$  is called a  $j$ -**singular complex** if  $\varepsilon_\Delta(F) = 0$  for every face  $F \in \Delta$  of  $\dim(F) \geq j$ , i.e.,  $\tilde{\chi}(\text{lk}_\Delta F) = (-1)^{d-1-|F|}$ .

**Proposition 2.6.5.** *The following are equivalent:*

- (1). *A poset  $P$  is  $j$ -Sing.*
- (2). *The order complex of  $P$ ,  $O(P)$ , is a  $j$ -singular complex.*
- (3). *For every chain  $C = \{\hat{0} = t_0 < t_1 < \dots < t_{i-1} < t_i = \hat{1}\}$  in  $P$  such that  $i > j + 1$ , the (chain) error  $\varepsilon_P(C) = 0$ .*

*Proof.* Assume  $P$  has rank  $d + 1$ , and so  $\dim O(P) = d - 1$ .

(2) $\iff$ (3) This is clear since by definitions,  $\varepsilon_P(C) = \varepsilon_{O(P)}(F_C)$ , and every chain  $C$  in (3) corresponds to a face  $F_C \in O(P)$  of dimension  $\geq j$ .

(1) $\implies$ (3)  $P$  is  $j$ -Sing if and only if every interval of length  $\leq d - j$  in  $P$  is Eulerian. Therefore, for any interval  $[s, t]$  in  $P$  with  $\rho(t) - \rho(s) \leq d - j$ ,  $\mu_P(s, t) = (-1)^{\rho(t) - \rho(s)}$ . For any chain  $C$  as in (3),

$$\rho(t_k) - \rho(t_{k-1}) \leq d - j \quad \text{for all } 1 \leq k \leq i,$$

therefore every interval  $[t_{k-1}, t_k]$  is Eulerian. This implies  $\mu_P(C) = (-1)^{d+1}$  and so  $\varepsilon_P(C) = 0$ .

(3) $\implies$ (1) Pick any interval  $[s, t]$  in  $P$  with  $\rho(t) - \rho(s) \leq d - j$ . Consider a maximal chain  $(\hat{0} = t_0 < t_1 < \dots < t_{\rho(s)} = s)$  in  $[\hat{0}, s]$  and a maximal chain  $(t = t_{\rho(s)+1} < t_{\rho(s)+2} < \dots < t_k = \hat{1})$  in  $[t, \hat{1}]$ , and let  $C$  be the union of these two chains:

$$C = (\hat{0} = t_0 < t_1 < \dots < s < t < \dots < t_{k-1} < t_k = \hat{1}).$$

In particular,  $\rho(t_i) - \rho(t_{i-1}) = 1$  unless  $i = \rho(s) + 1$ . Then  $C$  has length  $k \geq j$ , and so by (3),

$$\mu_P(\hat{0}, t_1)\mu_P(t_1, t_2) \dots \mu_P(s, t) \dots \mu_P(t_{i-1}, \hat{1}) = (-1)^{d+1}.$$

Since intervals of length 1 always have the Möbius value  $-1$ , this forces  $\mu_P(s, t) = (-1)^{\rho(t) - \rho(s)}$ . All intervals of length  $\leq d - j$  are Eulerian, therefore  $P$  is  $j$ -Sing. □

To make the exposition cleaner, we introduce the following notation:

**Definition 2.6.6.** For any  $j$ -Sing poset  $P$  of rank  $\rho(P) = d + 1$ , define

$$A_k^{(j)}(P) := \hat{h}_{d-k}(P) - \hat{h}_k(P).$$

Note that any graded poset of rank  $d + 1$  is automatically  $(d - 1)$ -Sing. The following claim on  $A_0^{(j)}(P)$  can be shown by an easy induction on  $j$ .

**Proposition 2.6.7.** For any graded poset  $P$  of rank  $d + 1$ ,

$$A_0^{(j)}(P) = (-1)^d \cdot e_P(\hat{0}, \hat{1}).$$

### 2.6.2 Dehn–Sommerville relations

We are now in a position to generalize Theorem 2.5.6. First, by comparing the polynomials  $\hat{g}(Q) + (x - 1)\hat{h}(Q)$  and  $x^{\rho(Q)}\hat{g}(Q, \frac{1}{x})$  (as in the proof of Lemma 2.5.8), we obtain the following extension of Lemma 2.5.8. We omit the proof.

**Lemma 2.6.8.** Let  $Q$  be a  $j$ -Sing poset with rank  $r + 1$ , and let  $y = x - 1$ . Then

$$\hat{g}(Q) + y\hat{h}(Q) = x^{\rho(Q)}\hat{g}(Q, \frac{1}{x}) + \sum_{k=\lfloor \frac{r+1}{2} \rfloor + 1}^{r+1} \left[ A_{k-1}^{(j)}(Q) - A_k^{(j)}(Q) \right] x^k,$$

where  $\sum^*$  means that, if  $r$  is odd, then there is an extra summand,  $\frac{1}{2}[A_{k-1}^{(j)}(Q) - A_k^{(j)}(Q)]x^k$  for  $k = \lfloor \frac{r+1}{2} \rfloor$ , which equals  $-A_{\lfloor \frac{r+1}{2} \rfloor}^{(j)}(Q) \cdot x^{\lfloor \frac{r+1}{2} \rfloor}$ .

The first main result of this section is the following generalization of Theorem 2.5.6:

**Theorem 2.6.9.** *Let  $P$  be a  $j$ -Sing poset with rank  $d + 1$ , where  $-1 \leq j \leq d$ . For  $q \in P$ , denote by  $Q$  the interval  $[\hat{0}, q]$ . Then*

$$\begin{aligned} \hat{h}(P) - x^d \hat{h}(P, \frac{1}{x}) &= - \sum_{\rho(q) \leq j} \hat{g}(Q, \frac{1}{x}) \cdot e_P(q, \hat{1}) \cdot y^{d-\rho(q)} \cdot x^{\rho(q)} \\ &- \sum_{d-j < \rho(q) \leq d} \sum_{k=\lfloor \frac{\rho(q)}{2} \rfloor + 1}^{\rho(q)} \left( A_{k-1}^{(j+\rho(q)-d-1)}(Q) - A_k^{(j+\rho(q)-d-1)}(Q) \right) \cdot \mu(q, \hat{1}) \cdot y^{d-\rho(q)} x^k. \end{aligned} \quad (2.6.1)$$

Before proving this formula, we first show that for small  $j$ 's, equation (2.6.1) reduces to previous results.

**Example 2.6.10.**

1. If  $P$  is  $(-1)$ -Sing, then  $P$  is Eulerian. Hence by [29],  $A_k^{(-1)}(P) = 0$  for every  $k$ . This matches equation (2.6.1) as both sums on the right hand-side of (2.6.1) are empty.
2. If  $P$  is 0-Sing, then  $P$  is semi-Eulerian, so that all proper intervals  $Q \in \tilde{P}$  are  $(-1)$ -Sing (i.e., Eulerian). In this case, by [35],

$$A_k^{(0)}(P) = (-1)^{d-k+1} \binom{d}{k} e_P(\hat{0}, \hat{1}).$$

This coincides with equation (2.6.1) as the summand corresponding to  $Q$  of rank 0 is the only term showing up on the right hand-side of (2.6.1) for  $j = 0$ .

3. If  $P$  is 1-Sing, then equation (2.6.1) implies that

$$\begin{aligned} A_k^{(1)}(P) &= (-1)^{d-k+1} \binom{d}{k} e_P(\hat{0}, \hat{1}) + \sum_{\rho(Q)=1} (-1)^{d-k+1} \binom{d-1}{k-1} e_P(q, \hat{1}) + \sum_{\rho(Q)=d} \left( A_{k-1}^{(0)}(Q) - A_k^{(0)}(Q) \right) \\ &= (-1)^{d-k+1} \binom{d}{k} e_P(\hat{0}, \hat{1}) + \sum_{\rho(q)=1} (-1)^{d-k+1} \binom{d-1}{k-1} e_P(q, \hat{1}) + \sum_{\rho(q)=d} (-1)^{d-k+1} \binom{d}{k} e_P(\hat{0}, q), \end{aligned} \quad (2.6.2)$$

where for the last step we used that, if  $\rho(q) = d$ , then the interval  $[\hat{0}, q]$  is semi-Eulerian.

This agrees with our formula in Theorem 2.5.6.

*Proof of Theorem 2.6.9.* By Lemma 2.5.7, the following equation holds for an arbitrary graded poset  $P$  of rank  $d + 1$ .

$$\begin{aligned} \hat{h}(P, x) - x^d \hat{h}(P, \frac{1}{x}) &= -y^d e_P(\hat{0}, \hat{1}) \\ &+ \sum_{\substack{Q=[\hat{0}, q] \in \tilde{P} \\ 1 \leq \rho(Q) \leq d}} \underbrace{\left( \left[ -y^{d-\rho(Q)}(\hat{g}(Q) + y\hat{h}(Q))\mu_P(q, \hat{1}) \right] - \left[ (-y)^{d-\rho(Q)}\hat{g}(Q, \frac{1}{x})x^{\rho(Q)} \right] \right)}_{C(Q)}. \end{aligned}$$

From now on we assume  $P$  to be  $j$ -Sing of rank  $d + 1$ . We are interested in  $A_k^{(j)}(P)$ , which is the coefficient of  $x^k$  in  $\hat{h}(P, x) - x^d \hat{h}(P, \frac{1}{x})$ . Since  $P$  is  $j$ -Sing, Remark 2.6.2 implies that each interval  $Q = [\hat{0}, q] \subseteq P$  of rank  $r + 1$  is  $(j + r - d)$ -Sing. In particular,

$$\mu(q, \hat{1}) = (-1)^{d-\rho(q)+1} \quad \text{for } \rho(q) > j.$$

Using this observation together with Lemma 2.6.8, we conclude that the following holds for any interval  $Q$  of rank  $r + 1$ :

Case 1:  $j < \lfloor \frac{d}{2} \rfloor \leq d - j$ .

$$C(Q) = \begin{cases} -\hat{g}(Q, \frac{1}{x}) \cdot e_P(q, \hat{1}) \cdot y^{d-\rho(Q)} x^{\rho(Q)} & \text{for } \rho(Q) \leq j \\ 0 & \text{for } \rho(Q) \in (j, d - j] \\ \sum_{k=\lfloor \frac{r+1}{2} \rfloor + 1}^{r+1} \left( A_{k-1}^{(j+r-d)}(Q) - A_k^{(j+r-d)}(Q) \right) x^k \cdot (-y)^{d-\rho(Q)} & \text{for } \rho(Q) > d - j. \end{cases}$$

Case 2:  $j \geq \lfloor \frac{d}{2} \rfloor$ .

$$C(Q) = \begin{cases} -\hat{g}(Q, \frac{1}{x}) \cdot e_P(q, \hat{1}) \cdot y^{d-\rho(Q)} x^{\rho(Q)}, & \text{for } \rho(Q) \leq d-j \\ -\hat{g}(Q, \frac{1}{x}) \cdot e_P(q, \hat{1}) \cdot y^{d-\rho(Q)} x^{\rho(Q)} \\ \quad - \sum_{k=\lfloor \frac{r+1}{2} \rfloor + 1}^{r+1} \left( A_{k-1}^{(j+r-d)}(Q) - A_k^{(j+r-d)}(Q) \right) \cdot \mu(q, \hat{1}) y^{d-\rho(Q)} x^k, & \text{for } \rho(Q) \in (d-j, j] \\ - \sum_{k=\lfloor \frac{r+1}{2} \rfloor + 1}^{r+1} \left( A_{k-1}^{(j+r-d)}(Q) - A_k^{(j+r-d)}(Q) \right) \cdot \underbrace{\mu(q, \hat{1})}_{=(-1)^{d-r}} \cdot y^{d-\rho(Q)} x^k, & \text{for } \rho(Q) \geq j+1. \end{cases}$$

In both cases, comparing the coefficients on both sides, yields the statement.  $\square$

This shows that the difference between  $\hat{h}_{d-k}(P)$  and  $\hat{h}_k(P)$  is a “weighted” sum of the error functions of the intervals in  $P$ . Unfortunately, as  $j$  gets larger, the length of our formula expands very quickly. In the rest of this section, we will simplify this formula for  $j > \frac{d}{2}$  and  $k > \frac{d+j}{2}$ , in Theorem 2.6.14. Our main tool is the following result, that might be of interest on its own.

**Corollary 2.6.11.** *Let  $P$  be a  $j$ -Sing poset of rank  $d+1$ .*

- *If  $d$  is even, then*

$$2e_P(\hat{0}, \hat{1}) = - \sum_{1 \leq \rho(t) \leq j} e_P(t, \hat{1}) - \sum_{d-j+1 \leq \rho(t) \leq d} e_P(\hat{0}, t).$$

- *If  $d$  is odd, then*

$$\sum_{1 \leq \rho(t) \leq j} e_P(t, \hat{1}) = \sum_{d-j+1 \leq \rho(t) \leq d} e_P(\hat{0}, t).$$

*Proof.* We will only treat the case of  $j \geq \lfloor \frac{d}{2} \rfloor$  since the case of  $j < \lfloor \frac{d}{2} \rfloor$  is very similar.

By Theorem 2.6.9,

$$\begin{aligned}
& \hat{h}_0(P) - \hat{h}_d(P) \\
&= \sum_{0 \leq \rho(t) \leq j} (-1)^{\rho(t)+1} \cdot \mu(\hat{0}, t) \cdot e_P(t, \hat{1}) + \sum_{d-j+1 \leq \rho(t) \leq d} (-1)^{d-\rho(t)} \cdot [\hat{h}_0([\hat{0}, t]) - \hat{h}_{\rho(t)-1}([\hat{0}, t])] \\
&\stackrel{*}{=} \sum_{0 \leq \rho(t) \leq j} (-1)^{\rho(t)+1} \cdot \mu(\hat{0}, t) \cdot e_P(t, \hat{1}) + \sum_{d-j+1 \leq \rho(t) \leq d} (-1)^{d-\rho(t)} \cdot (-1)^{\rho(t)+1} e_P(\hat{0}, t) \\
&= - \sum_{0 \leq \rho(t) \leq d-j} e_P(t, \hat{1}) + \sum_{d-j+1 \leq \rho(t) \leq j} [(-1)^{d+1} \mu(\hat{0}, t) - \mu(t, \hat{1})] + \sum_{j+1 \leq \rho(t) \leq d} (-1)^{d+1} e_P(\hat{0}, t) \\
&= - \sum_{0 \leq \rho(t) \leq j} e_P(t, \hat{1}) + \sum_{d-j+1 \leq \rho(t) \leq d} (-1)^{d+1} \cdot e_P(\hat{0}, t)
\end{aligned} \tag{2.6.3}$$

where the equality “ $\stackrel{*}{=}$ ” follows from Proposition 2.6.7. However by Proposition 2.6.7, the left hand-side should also equal  $(-1)^d e_P(\hat{0}, \hat{1})$ . Comparing it with the last line of (2.6.3) yields the result.  $\square$

If  $j < \lfloor d/2 \rfloor$ , Corollary 2.6.11 is equivalent to the following geometric interpretation that generalizes Corollary 2.5.9.

**Corollary 2.6.12.** *Let  $P$  be a  $j$ -Sing poset of rank  $d + 1$  with  $j < \lfloor d/2 \rfloor$ . Let  $\varepsilon_{O(P)}$  be the error function associated to  $O(P)$ .*

- If  $d$  is even, then

$$2\varepsilon_{O(P)}(\emptyset) = - \sum_{F \in O(P), |F| \leq j} \varepsilon_{O(P)}(F).$$

- If  $d$  is odd, and for every face  $F \in O(P)$  we let  $F^{top}$  and  $F_{bot}$  denote the top and the bottom element of  $F$  (viewing  $F$  as a chain in  $P \setminus \{\hat{0}, \hat{1}\}$ ), then

$$\sum_{\substack{F \in O(P) \\ \rho(F^{top}) \leq j}} \varepsilon_{O(P)}(F) = \sum_{\substack{F \in O(P) \\ \rho(F_{bot}) \geq d-j+1}} \varepsilon_{O(P)}(F).$$

*Proof.* First notice that when  $j < \lfloor d/2 \rfloor \leq d - j$ , any face  $F$  with  $\rho(F^{top}) > j$  or  $\rho(F_{bot}) < d - j + 1$  will have  $\varepsilon_{O(P)}(F) = 0$  since all the intervals defined by this chain are Eulerian.

This means

$$\sum_{F \in O(P), |F| \leq j} \varepsilon_{O(P)}(F) = \sum_{\substack{F \in O(P) \\ \rho(F^{top}) \leq j}} \varepsilon_{O(P)}(F) + \sum_{\substack{F \in O(P) \\ \rho(F_{bot}) \geq d-j+1}} \varepsilon_{O(P)}(F).$$

Now it suffices to show the following claim: for  $q, t \in P$ ,  $\rho(q) \leq j$  and  $\rho(t) \geq d - j + 1$ ,

$$\sum_{\substack{F \in O(P) \\ F^{\text{top}}=q}} \varepsilon_{O(P)}(F) = e_P(q, \hat{1}) \quad \text{and} \quad \sum_{\substack{F \in O(P) \\ F^{\text{bot}}=t}} \varepsilon_{O(P)}(F) = e_P(\hat{0}, t).$$

Recall that every face  $F$  in  $O(P)$  corresponds to a chain in  $P \setminus \{\hat{0}, \hat{1}\}$ , therefore by abusing notation, we can write  $F = \{t_1 < t_2 < \dots < t_k = q\}$ . If  $\rho(q) \leq j$ , then all intervals  $[t_i, t_{i+1}]$  are Eulerian, and so

$$\tilde{\chi}(\text{lk}_{O(P)} F) = (-1)^{|F|} \cdot \mu(\hat{0}, t_1) \mu(t_1, t_2) \dots \mu(t_k, \hat{1}) = (-1)^{|F|+\rho(q)} \cdot \mu(q, \hat{1}).$$

Hence  $\varepsilon_{O(P)}(F) = (-1)^{|F|+\rho(q)} e_P(q, \hat{1})$ . In particular, if  $F$  is a facet in  $O([\hat{0}, q])$ , i.e., a saturated chain in  $[\hat{0}, q]$ , then  $\rho(q) = |F|$  and  $\varepsilon_{O(P)}(F) = e_P(q, \hat{1})$ . If  $F$  a ridge in  $O([\hat{0}, q])$ , then  $\varepsilon_{O(P)}(F) = -e_P(q, \hat{1})$ , etc. This, together with the fact that  $O([\hat{0}, q])$  is Eulerian, implies

$$\sum_{\substack{F \in O(P) \\ F^{\text{top}}=q}} \varepsilon_{O(P)}(F) = (-1)^{\rho(q)} \cdot \tilde{\chi}(O([\hat{0}, q])) \cdot e_P(q, \hat{1}) = e_P(q, \hat{1}).$$

A symmetric argument takes care of the other half of the claim. Our statement now follows from Corollary 2.6.11 □

**Remark 2.6.13.** The polynomial in (2.6.1) is a symmetric polynomial with half of its coefficients negated. In particular, it is the following (here  $A_k$  stands for  $A_k(P)$ ):

$$A_d x^d + A_{d-1} x^{d-1} + \dots + A_k x^k + \dots - A_k x^{d-k} - \dots - A_{d-1} x - A_d.$$

Therefore the coefficients of  $x^k$  and  $x^{d-k}$  add up to zero for every  $k \neq \frac{d}{2}$ . The case of  $k = d$  was used to obtain Corollaries 2.6.11 and 2.6.12. A natural open problem is the following: can we make use of other equalities arising from comparing the coefficients of  $x^k$  and  $x^{d-k}$  for  $k \neq d$  of this polynomial?

The next main theorem in this section will give an explicit formula for  $A_k^{(j)}(P)$ . Before stating the theorem, we first introduce some notation. Given a poset  $T = [\hat{0}, t]$  and integers  $u$  and  $v$ , we define the following function.

$$C(T, u, v) := \binom{u - \rho(t)}{v} - \hat{g}_{\rho(t)-2}(T) \cdot \binom{u - \rho(t)}{v-1} + \dots + (-1)^m \hat{g}_{\rho(t)-1-m}(T) \cdot \binom{u - \rho(t)}{v-m}$$

where  $m = \lfloor (\rho(t) - 1)/2 \rfloor$ .

Note that by Pascal's rule on binomial coefficients, for each  $u, v$ , we have

$$C(T, u, v) + C(T, u, v + 1) = C(T, u + 1, v + 1). \quad (2.6.4)$$

**Theorem 2.6.14.** *Let  $P$  be a  $j$ -Sing poset of rank  $d + 1$  (with  $d > 2j$ ), and let  $k > (d + j)/2$ . For each element  $t \in P$ , let  $T := [\hat{0}, t]$ , then*

$$A_k^{(j)}(P) = (-1)^k \sum_{\rho(t) \leq j} e_P(t, \hat{1}) \cdot C(T, d, k) \quad \text{for } k > (d + j)/2. \quad (2.6.5)$$

*Proof.* The proof is by induction on  $j$ . The base cases of  $j = -1$  and  $0$  are immediate from Stanley's and Swartz's results, see Theorems 2.5.3 and 2.5.4. The  $j = 1$  case can be easily checked using (2.6.2) and Corollary 2.6.11. The inductive hypothesis is that for all  $j' < j$ , all  $j'$ -Sing posets  $Q$  of rank  $d' + 1$ , and all  $k > \frac{d'+j'}{2}$ ,

$$A_k^{(j')}(Q) = (-1)^k \sum_{\rho(t) \leq j'} e_P(t, \hat{1}) \cdot C(T, d', k). \quad (2.6.6)$$

If  $P$  is a  $j$ -Sing poset and  $k > \frac{d+j}{2}$ , then since  $A_k^{(j)}(P)$  is the coefficient of  $x^k$  in Equation (2.6.1),

$$\begin{aligned} A_k^{(j)}(P) &= (-1)^{d-k+1} \sum_{\rho(t) \leq j} e(t, \hat{1}) \cdot C(T, d, d - k) \\ &\quad - \sum_{\substack{\rho(q) = d - j + b \\ 0 < b \leq j \\ k - (d - \rho(q)) - 1 \leq a \leq k - 1}} (-1)^{k-a} \cdot \binom{d - \rho(q)}{k - (a + 1)} \cdot \left( A_a^{(b-1)}(Q) - A_{a+1}^{(b-1)}(Q) \right). \end{aligned} \quad (2.6.7)$$

We now check that the inductive hypothesis applies to all of the summands in the second summation in (2.6.7). For each  $q$  with  $\rho(q) = d - j + b$ ,  $Q = [\hat{0}, q]$  is a  $(b - 1)$ -Sing poset. By (2.6.6), for all  $a > \frac{(d-j+b-1)+(b-1)}{2} = \frac{d-j}{2} + b - 1$ ,

$$A_a^{(b-1)}(Q) = (-1)^a \sum_{\rho(t) \leq b-1} e_Q(t, \hat{1}) \cdot C(T, \rho(Q) - 1, a). \quad (2.6.8)$$

In (2.6.7), the second summation is a sum over  $q \in P$  such that  $\rho(q) = d - j + b$  and

$$a \geq k - d + \rho(q) - 1 = k - j + b - 1 > \frac{d+j}{2} - j + b - 1 = \frac{d-j}{2} + b - 1.$$

Therefore (2.6.8) holds for all  $A_a^{(b-1)}(Q)$ 's in (2.6.7), and so

$$\begin{aligned} A_k^{(j)}(P) &= (-1)^{d-k+1} \sum_{\substack{\rho(t)=r+1 \leq j \\ m=\lfloor r/2 \rfloor}} e(t, \hat{1}) \cdot C(T, d, d-k) \\ &\quad - (-1)^k \sum_{\substack{\rho(q)=d-j+b \\ 0 < b \leq j \\ a \in [k-(d-\rho(q))-1, k-1]}} \binom{d-\rho(Q)}{k-(a+1)} \sum_{\rho(t) \leq b-1} e_Q(t, \hat{1}) \left[ C(T, \rho(Q)-1, a) + C(T, \rho(Q)-1, a+1) \right]. \end{aligned}$$

Note that  $e_Q(t, \hat{1}) = e_P(t, q)$ , and by recurrence relation (2.6.4), we obtain:

$$\begin{aligned} A_k^{(j)}(P) &= (-1)^{d-k+1} \sum_{\substack{\rho(t)=r+1 \leq j \\ m=\lfloor r/2 \rfloor}} e_P(t, \hat{1}) \cdot C(T, d, d-k) \\ &\quad - (-1)^k \sum_{\substack{\rho(q)=d-j+b \\ 0 < b \leq j \\ k-(d-\rho(q))-1 \leq a \leq k-1}} \sum_{\rho(t) \leq b-1} e_P(t, q) \binom{d-\rho(Q)}{k-(a+1)} \cdot C(T, \rho(Q), a+1). \end{aligned}$$

Since each  $C(T, \rho(Q), a+1)$  is an alternating sum of multiples of the binomial coefficients  $\binom{\rho(Q)-\rho(T)}{(a+1)-c}$  (for some  $c$ 's), we can use the Chu-Vandermonde identity to conclude that:

$$\sum_{a=k-(d-\rho(q))-1}^{k-1} \binom{d-\rho(Q)}{k-(a+1)} \cdot \binom{\rho(Q)-\rho(T)}{(a+1)-c} = \binom{d-\rho(T)}{k-c}.$$

This shows that (for  $k > \frac{d+j}{2}$ ),

$$\begin{aligned} A_k^{(j)}(P) &= (-1)^{d-k+1} \sum_{\rho(t) \leq j} e_P(t, \hat{1}) C(T, d, d-k) - (-1)^k \sum_{\substack{\rho(q)=d-j+b \\ 0 < b \leq j}} \sum_{\rho(t) \leq b-1} e_P(t, q) \cdot C(T, d, k). \\ &= (-1)^{d-k+1} \sum_{\rho(t) \leq j} e_P(t, \hat{1}) \cdot C(T, d, d-k) + (-1)^{k+1} \sum_{\substack{\rho(t) \in [0, \rho(q)-(d-j+1)] \\ \rho(q) \in [d-j+1, d]}} e_P(t, q) \cdot C(T, d, k). \end{aligned} \tag{2.6.9}$$

The second equality holds because in the second summation,  $e_P(t, q) = 0$  for all intervals  $[t, q]$  of length  $\leq d-j$ .

The next step is to apply Corollary 2.6.11 to all intervals  $[t, \hat{1}]$  with  $0 \leq \rho(t) \leq j-1$  to replace the summands in (2.6.9) that involve  $e_P(t, q)$  with sums of multiples of  $e_P(t', \hat{1})$  where  $0 \leq \rho(t') \leq j$ .

The cases of even and odd  $j$ 's are slightly different because of the two cases in Corollary 2.6.11. Here we assume  $j$  is odd (and hence  $d$  is assumed to be even, see Remark 2.6.3). The proof for the case of even  $j$  is similar; we omit it.

As  $j$  is odd and  $d$  is even, Corollary 2.6.11 implies that

$$\begin{aligned} & (-1)^{k+1} \sum_{\substack{\rho(u)=i \\ d-j+1 \leq \rho(q) \leq d}} e_P(u, q) C(U, d, k) \\ &= \begin{cases} (-1)^k \sum_{\rho(t)=i} 2e_P(t, \hat{1}) \cdot C(T, d, k) + (-1)^k \sum_{\substack{\rho(u)=i \\ i+1 \leq \rho(t) \leq j \\ u < t}} e_P(t, \hat{1}) \cdot C(U, d, k) & \text{if } i \text{ is even} \\ (-1)^{k+1} \sum_{\substack{\rho(u)=i \\ i+1 \leq \rho(t) \leq j \\ u < t}} e_P(t, \hat{1}) \cdot C(U, d, k) & \text{if } i \text{ is odd.} \end{cases} \end{aligned}$$

After these substitutions, all of the summands in equation (2.6.9) that involve  $e_P(u, q)$  with  $q \neq \hat{1}$  are replaced with sums of multiples of  $e_P(t, \hat{1})$  for some  $t \geq u$ . Now we have:

$$\begin{aligned} & (-1)^{k+1} \sum_{\substack{0 \leq \rho(t) \leq j-1 \\ d-j+1 \leq \rho(q) \leq d}} e(t, q) C(T, d, k) \\ &= (-1)^k \sum_{i \text{ even}} \left[ \sum_{\rho(t)=i} 2e_P(t, \hat{1}) C(T, d, k) + \sum_{\substack{\rho(u)=i \\ i+1 \leq \rho(t) \leq j \\ u < t}} e_P(t, \hat{1}) C(U, d, k) \right] + (-1)^{k+1} \sum_{i \text{ odd}} \sum_{\substack{\rho(u)=i \\ i+1 \leq \rho(t) \leq j \\ u < t}} e_P(t, \hat{1}) C(U, d, k) \\ &= (-1)^k \sum_{\rho(t) \leq j} e_P(t, \hat{1}) \left[ \sum_{u < t} C(U, d, k) \cdot (-1)^{\rho(u)} \right] + (-1)^k \sum_{\rho(t) \text{ is even}} e_P(t, \hat{1}) 2C(T, d, k). \end{aligned}$$

Together with (2.6.9) and the assumption that  $d$  is even, this shows

$$A_k^{(j)}(P) = \sum_{\rho(t) \leq j} (-1)^k e_P(t, \hat{1}) \cdot \left[ -C(T, d, d-k) + \sum_{u < t} (-1)^{\rho(u)} C(U, d, k) \right] + (-1)^k \sum_{\substack{\rho(t) \leq j \\ \rho(t) \text{ even}}} e_P(t, \hat{1}) 2C(T, d, k).$$

Comparing this with (2.6.5), it suffices to show that for each  $t \in P$  with  $\rho(t) \leq j$  and  $T = [\hat{0}, t]$ ,

$$(-1)^{k+1} C(T, d, d-k) + (-1)^k \sum_{u < t} (-1)^{\rho(u)} C(U, d, k) = \begin{cases} (-1)^k C(T, d, k) & \text{if } \rho(t) \text{ is odd,} \\ (-1)^{k+1} C(T, d, k) & \text{if } \rho(t) \text{ is even.} \end{cases} \quad (2.6.10)$$

Observe that since  $d$  is even,

- $(-1)^{k+1}C(T, d, d-k) =$  the coefficient of  $x^k$  in  $-(x-1)^{d-\rho(t)} \cdot x^{\rho(t)} \cdot \hat{g}(T, \frac{1}{x})$ ,
- $(-1)^k \sum_{u < t} (-1)^{\rho(u)} C(U, d, k) =$  the coefficient of  $x^k$  in  $\sum_{u < t} (x-1)^{d-\rho(u)} \cdot \hat{g}(U, x)$ ,
- $(-1)^k C(T, d, k) =$  the coefficient of  $x^k$  in  $(-1)^{\rho(t)} (x-1)^{d-\rho(t)} \cdot \hat{g}(T, x)$ .

Therefore, (2.6.10) is equivalent to:

$$-(x-1)^{d-\rho(t)} \cdot x^{\rho(t)} \cdot \hat{g}(T, \frac{1}{x}) + \sum_{u < t} (x-1)^{d-\rho(u)} \cdot \hat{g}(U, x) = -(x-1)^{d-\rho(t)} \cdot \hat{g}(T, x). \quad (2.6.11)$$

This equality holds since by the definition of  $\hat{h}(T, x)$ , the left hand-side is

$$\begin{aligned} & -(x-1)^{d-\rho(t)} \cdot x^{\rho(t)} \cdot \hat{g}(T, \frac{1}{x}) + (x-1)^{d-\rho(t)+1} \cdot \hat{h}(T, x) \\ &= (x-1)^{d-\rho(t)} \left[ -x^{\rho(t)} \cdot \hat{g}(T, \frac{1}{x}) + (x-1) \cdot \hat{h}(T, x) \right] \\ &= -(x-1)^{d-\rho(t)} \cdot \hat{g}(T, x). \end{aligned}$$

where the last equality holds since  $T$  is Eulerian.

This completes the proof that  $A_k^j = (-1)^k \sum_{0 \leq \rho(t) \leq j} e(t, \hat{1}) C(t)$  when  $j$  is odd. When  $j$  is even, the proof is very similar; we omit it.  $\square$

### 2.6.3 The lower Eulerian case

In this subsection we assume that  $P$  is **lower Eulerian**, i.e., all intervals  $[0, t]$  are Eulerian for  $t \neq \hat{1}$ . This is an important subclass of graded posets. For instance, the face posets of all regular CW complexes are lower Eulerian. If  $P$  is  $j$ -Sing and lower Eulerian, the formula of Theorem 2.6.9 takes on the following simpler form:

$$A_k^{(j)}(P) = \sum_{0 \leq \rho(q) = r+1 \leq j} \sum_{l=0}^{\lfloor r/2 \rfloor} (-1)^{d-k-l+1} \binom{d-\rho(q)}{k-\rho(q)-l} \cdot e_P(q, \hat{1}) \cdot \hat{g}_{r-l}([\hat{0}, q]).$$

**Example 2.6.15.** Let  $P$  be a  $j$ -Sing poset of rank  $d + 1$ , where  $j \leq 2$ . If  $P$  is also lower Eulerian, then

$$\hat{h}_{d-k}(P) - \hat{h}_k(P) = (-1)^{d-k+1} \sum_{\rho(q) \leq 2} \binom{d - \rho(q)}{k - \rho(q)} e_P(q, \hat{1}).$$

**Remark 2.6.16.** Unfortunately for larger  $j$  the situation becomes more complicated. For instance if  $j = 3$ , then using the fact (easy to check) that

$$\hat{g}(Q) = -\mu(Q) + [f_1(Q) + \mu(Q) - 2]x \quad \text{for any poset } Q \text{ with } \rho(Q) = 3$$

and the assumption that the poset  $P$  is lower Eulerian, one can show that

$$\begin{aligned} A_k^{(3)}(P) &= (-1)^{d-k+1} \sum_{\rho(q) \leq 2} \binom{d - \rho(q)}{k - \rho(q)} e(q, \hat{1}) \\ &\quad + (-1)^{d-k+1} \sum_{\rho(q)=3} \underbrace{\left[ \binom{d-3}{k-3} - \binom{d-3}{k-4} \cdot (f_1([0, q]) - 3) \right]}_{\heartsuit} \cdot e_P(q, \hat{1}). \end{aligned}$$

Observe that “ $\heartsuit$ ” equals  $\binom{d-3}{k-3}$  if and only if  $f_1(Q) = 3$ , which is the case when  $Q$  is the face poset of a simplex.

**Definition 2.6.17.** A pure graded poset  $P$  is  $k$ -lower simplicial if for all  $t \in P$  with  $\rho(t) \leq k$ , the interval  $[0, t]$  is a Boolean lattice.

**Corollary 2.6.18.** Let  $P$  be a  $j$ -Sing lower Eulerian poset  $P$  of rank  $d + 1$ . If  $P$  is also  $j$ -lower simplicial, then for  $k > \lfloor \frac{d}{2} \rfloor$ ,

$$\hat{h}_{d-k}(P) - \hat{h}_k(P) = (-1)^{d-k+1} \sum_{\rho(q) \leq j} \binom{d - \rho(q)}{k - \rho(q)} \cdot e_P(q, \hat{1}).$$

**Remark 2.6.19.** In the case that  $P$  is the face poset of a simplicial complex  $\Delta$  and  $P$  is  $j$ -Sing, Corollary 2.6.18 agrees with the formula in Theorem 2.3.1.

#### 2.6.4 Open problems

The most natural open problem is to find a “nice” formula for  $A_k^{(j)}(P)$  when  $j \geq \frac{d}{2}$ . Here  $\rho(P) = d + 1$  and  $\frac{d}{2} \leq k \leq \frac{d+j}{2}$ .

To state the next problem, let  $P$  be a  $j$ -Sing poset, and let  $P^*$  be the dual poset of  $P$ . By definition,  $P^*$  is also a  $j$ -Sing poset. Recall that

- When  $j = -1$ ,  $A_k^{(-1)}(P) = A_k^{(-1)}(P^*) = 0$ .

- When  $j = 0$ ,

$$A_k^{(0)}(P) = (-1)^{d-k+1} \binom{d}{k} e_P(\hat{0}, \hat{1}) = A_k^{(0)}(P^*).$$

When  $d$  is odd and  $k = \lfloor \frac{d}{2} \rfloor$ , this means

$$\hat{g}_{\lfloor \frac{d}{2} \rfloor}(P) = \hat{g}_{\lfloor \frac{d}{2} \rfloor}(P^*).$$

- When  $j = 1$  (and assuming  $d$  is even), after cancellations, we obtain

$$A_k^{(1)}(P) - A_k^{(1)}(P^*) = (-1)^{d-k} \binom{d-1}{k} \left[ \sum_{\rho(q)=1} e_P(q, \hat{1}) - \sum_{\rho(Q)=d} e_P(\hat{0}, q) \right].$$

This leads to the following question:

**Question 2.6.20.** *How do the numbers  $A_k^{(j)}(P)$  compare with  $A_k^{(j)}(P^*)$ ?*

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## Chapter 3

# A PROOF OF GRÜNBAUM'S LOWER BOUND CONJECTURE FOR GENERAL POLYTOPES

## 3.1 Introduction

The paper is devoted to the proof of Grünbaum's general lower bound conjecture for polytopes with few vertices.

In the last fifty years a lot of effort has gone into trying to understand face numbers of polytopes. For instance, McMullen [19] established the Upper Bound Theorem in 1970, which provides tight upper bounds on the number of  $k$ -faces a  $d$ -dimensional polytope with  $n$  vertices can have. A couple of years later, Barnette (see [3], [1], and [2]) proved the Lower Bound Theorem for *simplicial* polytopes; his result provides tight lower bounds on the number of  $k$ -faces a  $d$ -dimensional simplicial polytope with  $n$  vertices can have. Furthermore, in 1980, Billera and Lee [7] and Stanley [28] completely characterized the face numbers of all simplicial (and by duality also simple) polytopes. Their result is known as the  $g$ -theorem. Billera and Lee [7] established sufficiency of the conditions while Stanley [28] proved their necessity.

Despite these spectacular advances, to date no Lower Bound Theorem is known for general  $d$ -dimensional polytopes with an arbitrary number of vertices; in fact, there is not even a plausible conjecture. However for general  $d$ -dimensional polytopes with  $d + s \leq 2d$  vertices, Grünbaum conjectured in [13, p. 184] (see also [14, p. 265]) that the number of  $k$ -faces is at least

$$\phi_k(d + s, d) = \binom{d + 1}{k + 1} + \binom{d}{k + 1} - \binom{d + 1 - s}{k + 1}.$$

He proved this conjecture for the cases of  $s = 2, 3$ , and  $4$ . The conjecture remained completely open for  $s \geq 5$  until very recently Pineda-Villavicencio, Ugon and Yost [22] proved

this conjecture for the number of edges, i.e., they verified the  $k = 1$  case.

In this paper we prove the conjecture in full generality. Our results can be summarized as follows;

**Theorem 3.3.2** *Let  $P$  be a  $d$ -polytope with  $d + s$  vertices where  $s \geq 2$  and  $d \geq s$ . Then  $f_k(P) \geq \phi_k(d + s, d)$  for every  $k$ .*

**Theorem 3.4.3** *Let  $P$  be a  $d$ -polytope with  $d + s$  vertices where  $s \geq 2$  and  $d \geq s$ . If  $f_k(P) = \phi_k(d + s, d)$  for some  $1 \leq k \leq d - 2$ , then  $P$  is  $\text{Pyr}^{d-s}(\Delta^1 \times \Delta^{s-1})$  — the polytope that is a  $(d - s)$ -fold pyramid over a prism over an  $(s - 1)$ -simplex.*

The main novelty of our approach is that instead of focusing on contributions coming from facets, we look at sets of potentially unrelated vertices and bound the number of  $k$ -faces containing one or more of them.

### 3.2 Background and Preliminaries

Before starting the proof we recall some definitions and introduce some notation. We refer the reader to books by Grünbaum [13] and Ziegler [39] for all undefined notions. By a polytope we mean the convex hull of finitely many points in  $\mathbb{R}^d$ . A  **$d$ -simplex**, denoted as  $\Delta^d$ , is the convex hull of  $d + 1$  affinely independent points. A **face** of a polytope  $P$  is the intersection of  $P$  with a supporting hyperplane. It is known that a face of a polytope is a polytope. The dimension of a polytope is the dimension of its affine span. For brevity, we refer to a  $k$ -dimensional face as a  $k$ -face and to a  $d$ -dimensional polytope as a  $d$ -polytope. The 0-faces are called vertices. The  $(d - 1)$ -faces of a  $d$ -polytope are called facets. We denote by  $f_k(P)$  the number of  $k$ -faces of a polytope  $P$ .

Let  $P \subset \mathbb{R}^d$  be a  $d$ -polytope and  $v$  a vertex of  $P$ . The **vertex figure of  $P$  at  $v$** ,  $P/v$ , is obtained by intersecting  $P$  with a hyperplane  $H$  that separates  $v$  from the rest of the vertices of  $P$ . One property of vertex figures that will be very useful for us is that  $(k - 1)$ -faces of

$P/v$  are in bijection with  $k$ -faces of  $P$  that contain  $v$ .

### 3.3 The proof of the inequality

We start with the following formulas, most of which are straightforward consequences of Pascal's relation:  $\binom{n}{m} = \binom{n-1}{m-1} + \binom{n-1}{m}$ . For all integers  $k$ ,  $d$ , and  $a > b$ ,

$$\phi_k(d+a, d) - \phi_k(d+b, d) = \binom{d+1-b}{k+1} - \binom{d+1-a}{k+1} = \sum_{i=1}^{a-b} \binom{d+1-b-i}{k}; \quad (3.3.1)$$

$$\phi_k(d+1, d) = \binom{d+1}{k+1}; \quad (3.3.2)$$

$$\begin{aligned} \phi_k(d+s, d-1) + \phi_{k-1}(d-1, d-2) + \phi_{k-1}(d, d-1) \\ = \phi_k(d+s, d-1) + \binom{d-1}{k} + \binom{d}{k} = \phi_k(d+s+2, d). \end{aligned} \quad (3.3.3)$$

Let  $\mathcal{P}(d+s, d)$  be the set of all  $d$ -polytopes with  $d+s$  vertices. The main ingredient of the proof is the following.

**Proposition 3.3.1.** *Let  $P$  be a  $d$ -polytope and let  $\{v_1, v_2, \dots, v_m\}$  be a subset of vertices of  $P$ , where  $m \leq d$ . Then the number of  $k$ -faces of  $P$  that contain at least one of the  $v_i$ 's is bounded from below by  $\sum_{i=1}^m \binom{d-i+1}{k}$ .*

*Proof.* We induct on  $m$  to show that there exists a sequence of faces,  $\{F_1, \dots, F_m\}$ , such that

- (1). each  $F_i$  has dimension  $d-i+1$ ,
- (2).  $F_i$  contains  $v_i$  but does not contain any  $v_j$  with  $j < i$ .

The base case is  $m = 1$ , and we simply pick  $F_1 = P$ . Inductively we assume that for every  $p \leq m-1$  and any  $p$ -set of vertices  $\{v_1, \dots, v_p\}$ , there exists a sequence  $\{F_1, \dots, F_p\}$  such that for  $1 \leq i \leq p$ , conditions (1) and (2) are satisfied.

Let  $m > 1$  and let  $v_1, \dots, v_m$  be  $m$  given vertices of  $P$ . By the inductive hypothesis, for  $\{v_1, \dots, v_{m-1}\}$  there exist faces  $F_1, \dots, F_{m-1}$  satisfying conditions (1) and (2). Similarly, by considering  $\{v_1, \dots, v_{m-2}, v_m\}$ , there also exists a  $(d - m + 2)$ -face  $F$  that contains  $v_m$  but not  $v_1, \dots, v_{m-2}$ . Regardless of whether  $v_{m-1}$  is in  $F$  or not, there must exist a facet of  $F$ , call it  $F_m$ , that contains  $v_m$  but not  $v_{m-1}$ . Then  $v_i \in F_m$  if and only if  $i = m$ , and  $F_1, \dots, F_{m-1}, F_m$  is a desired sequence.

For each  $i$ , the  $k$ -faces of  $F_i$  that contain  $v_i$  correspond to the  $(k - 1)$ -faces of the vertex figure  $F_i/v_i$ . Since  $\dim(F_i/v_i) = \dim(F_i) - 1 = d - i$ , we obtain that

$$\begin{aligned} \# \text{ } k\text{-faces of } P \text{ that contain some } v_i \text{ (} 1 \leq i \leq m \text{)} &\geq \# \bigcup_{i=1}^m \{k\text{-faces of } F_i \text{ containing } v_i\} \\ &\geq \# \bigcup_{i=1}^m \{(k-1)\text{-faces of } F_i/v_i\} \\ &\geq \sum_{i=1}^m \binom{d-i+1}{k}. \end{aligned}$$

The result follows. □

We are now ready to prove our first main result.

**Theorem 3.3.2.** *Let  $s \geq 3$  and  $d \geq s$ . Then for all  $d$ -polytopes  $P$  with  $d + s$  vertices and for all  $1 \leq k \leq d - 1$ ,  $f_k(P) \geq \phi_k(d + s, d)$ .*

The statement clearly holds for  $s = 1$ , and the cases of  $s = 2, 3, 4$  were proved by Grünbaum (see [13, 10.2.2]),

*Proof.* The proof is by induction on  $s$ . We fix  $s \geq 2$ . The following argument will show that if the statement holds for all pairs  $(s', d')$  such that  $s' < s$  and  $d' \geq s'$ , then for all  $d \geq s$ , it also holds for the pair  $(s, d)$ . Thus, consider  $d \geq s$ , and let  $P \in \mathcal{P}(d + s, d)$ .

If there exists a facet  $Q$  of  $P$  with  $d \leq f_0(Q) \leq d + s - 2$ , or equivalently if  $Q \in \mathcal{P}(d + s - m, d - 1)$  where  $2 \leq m \leq s$ , then there are  $m$  vertices of  $P$  outside of  $Q$ . We denote them by  $\{v_1, v_2, \dots, v_m\}$ . The  $k$ -faces of  $P$  fall into two disjoint categories:

the  $k$ -faces of  $Q$  and the  $k$ -faces of  $P$  that contain some  $v_i$ . By the inductive hypothesis,  $f_k(Q) \geq \phi_k(d + s - m, d - 1)$ . Therefore by Proposition 3.3.1,

$$\begin{aligned}
f_k(P) &\stackrel{(\diamond)}{\geq} \phi_k(d + s - m, d - 1) + \sum_{i=1}^m \binom{d - i + 1}{k} \\
&\stackrel{(\text{by (3.3.3)})}{=} \phi_k(d + s - m + 2, d) + \sum_{i=3}^m \binom{d - i + 1}{k} \\
&\stackrel{(\text{by (3.3.1)})}{=} \left[ \phi_k(d + s, d) - \sum_{j=1}^{m-2} \binom{d - s + m - 1 - j}{k} \right] + \sum_{i=3}^m \binom{d - i + 1}{k} \quad (3.3.4) \\
&= \phi_k(d + s, d) + \underbrace{\sum_{j=1}^{m-2} \left[ -\binom{d - j - 1 - (s - m)}{k} + \binom{d - j - 1}{k} \right]}_{\geq 0 \text{ (since } s \geq m)} \\
&\stackrel{(\infty)}{\geq} \phi_k(d + s, d).
\end{aligned}$$

This completes the proof of this case. The inequalities  $(\diamond)$  and  $(\infty)$  will be discussed later in the proof of Theorem 3.4.3.

Otherwise, all facets in  $P$  have  $d + s - 1$  vertices. But this implies that each vertex of  $P$  is not in the affine span of the rest, hence the vertex set of  $P$  is affinely independent, and so  $P$  can only be a simplex, contradicting our assumption that  $P$  has  $d + s$  vertices and  $s \geq 2$ .  $\square$

### 3.4 Treatment of equality

In this section we discuss the cases of equality in the Lower Bound Theorem. We first review some definitions relevant to the proof below. For more details, see for example [39, Chapter 1]. Let  $P \subset \mathbb{R}^{d+1}$  be a  $d$ -polytope, and let  $x \in \mathbb{R}^{d+1}$  be a point that does not lie in the affine hull of  $P$ . The **pyramid over  $P$  with apex  $x$**  is the convex hull of  $P \cup \{x\}$ . A pyramid over a  $d$ -polytope  $P$  is a  $(d + 1)$ -polytope, denoted as  $\text{Pyr}(P)$ . An  **$s$ -fold pyramid over  $P$**  is a pyramid over an  $(s - 1)$ -fold pyramid over  $P$ , denoted as  $\text{Pyr}^s(P)$ .

For every  $d$ -polytope  $P$ , there exists a polytope of the same dimension, denoted by  $P^*$ , whose face lattice is the opposite of the face lattice of  $P$ . In particular, vertices of  $P^*$  correspond to facets of  $P$ . The polytope  $P^*$  (or more precisely, its combinatorial type) is

called the **dual polytope of  $P$** .

Let  $P \subseteq \mathbb{R}^d$  be a  $d$ -polytope with  $n$  vertices and  $P' \subseteq \mathbb{R}^{d'}$  be a  $d'$ -polytope with  $n'$  vertices. Then the **product**  $P \times P'$  is a  $(d + d')$ -polytope with  $n \cdot n'$  vertices, defined as  $P \times P' = \{(v, u) \in \mathbb{R}^{d+d'} \mid v \in P, u \in P'\}$ . Assume that both  $P$  and  $P'$  have the origin in their interiors, then the **direct sum**  $P \oplus P'$  is the following  $(d + d')$ -polytope with  $n + n'$  vertices.

$$P \oplus P' = \text{conv}\left(\left\{\{(v, 0) \in \mathbb{R}^{d+d'} \mid v \in P\} \cup \{(0, u) \in \mathbb{R}^{d+d'} \mid u \in P'\}\right\}\right).$$

When both  $P$  and  $P'$  have the origin in their interiors, the product and the direct sum are “dual constructions”. In particular,  $(P \oplus P')^* = P^* \times P'^*$ .

A vertex of a  $d$ -polytope is **simple** if it is contained in exactly  $d$  facets (equivalently, if it is adjacent to exactly  $d$  vertices). A polytope  $P$  is simple if all vertices of  $P$  are simple. The dual polytope of a simple polytope is a simplicial polytope, and vice versa.

Grünbaum [13, Section 6.1] proved the following results, which will be used in the proof of the main result of this section. Recall that  $\Delta^d$  is a  $d$ -simplex.

**Lemma 3.4.1.** *If  $P$  is a simplicial  $d$ -polytope with  $d + 2$  vertices, then  $P \simeq \Delta^m \oplus \Delta^{d-m}$  for some  $1 \leq m \leq d - 1$ .<sup>1</sup>*

**Lemma 3.4.2.** *For all  $0 \leq k \leq d - 1$ ,*

$$f_k(\text{Pyr}^{d-a}(\Delta^m \oplus \Delta^{d-m})) = \binom{d+2}{d-k+1} - \binom{d-a+m+1}{d-k+1} - \binom{d-m+1}{d-k+1} + \binom{d-a+1}{d-k+1}.<sup>2</sup>$$

*In particular,  $f_{d-1}(\Delta^m \oplus \Delta^{d-m}) = d + 1 + m(d - m)$ .*

Now we are ready to state the main result of this section.

**Theorem 3.4.3.** *Let  $P \in \mathcal{P}(d + s, d)$  where  $s \geq 2$  and  $d \geq s$ . If  $f_k(P) = \phi_k(d + s, d)$  for some  $k$  with  $1 \leq k \leq d - 2$ , then  $P \simeq \text{Pyr}^{d-s}(\Delta^1 \times \Delta^{s-1})$ .*

<sup>1</sup>The polytope  $\Delta^m \oplus \Delta^{d-m}$  is denoted by  $T_m^d$  in [13, Section 6.1].

<sup>2</sup>The polytope  $\text{Pyr}^{d-a}(\Delta^m \oplus \Delta^{d-m})$  is denoted by  $T_m^{d,d-a}$  in [13, Section 6.1].

First it is easy to verify that

$$f_k(\text{Pyr}^{d-s}(\Delta^1 \times \Delta^{s-1})) = \phi_k(d+s, d) \quad \text{for all } d \geq s, 1 \leq k \leq d-1.^3$$

Assuming  $f_k(P) = \phi_k(d+s, d)$  for some  $k$  with  $1 \leq k \leq d-2$ , we will prove this theorem using the following corollary of Theorem 3.3.2.

**Corollary 3.4.4** (Corollary of Theorem 3.3.2). *If  $f_k(P) = \phi_k(d+s, d)$  for some  $k$  with  $1 \leq k \leq d-2$ , then each facet of  $P$  has  $d$ ,  $d+s-2$ , or  $d+s-1$  vertices, and  $P$  has  $d+2$  facets.*

*Proof.* Notice that (3.3.4) holds independently of the choice of a facet (with at most  $d+s-2$  vertices) in  $P$  or the ordering of the vertices that lie outside of this facet. Thus for  $f_k(P) = \phi_k(d+s, d)$  to hold, both inequalities in (3.3.4) must be satisfied as equalities for any chosen facet with at most  $d+s-2$  vertices. The inequality ( $\diamond\diamond$ ) of (3.3.4) holds as equality for some  $k < d-1$  if and only if  $m = 2$  or  $s$ . This implies that for the equality to hold, each facet of  $P$  can only have  $d$ ,  $d+s-2$ , or  $d+s-1$  vertices. The first inequality ( $\diamond$ ) in (3.3.4) holds as equality only if, for every facet in  $P$  that has at most  $d+s-2$  vertices and for each of the remaining vertices  $v_1, v_2, \dots$ ,

$$\#\{k\text{-faces containing } v_i \text{ but not any } v_j \text{ for } j < i \text{ in } P\} = \binom{d-i+1}{k}. \quad (3.4.1)$$

Particularly, the number of  $k$ -faces containing  $v_1$  is  $\binom{d}{k}$ , hence the number of edges containing  $v_1$  is  $\binom{d}{1} = d$ , so  $v_1$  is simple. Since  $v_1$  is arbitrary, this means that **all** of the vertices that are not in the chosen facet are simple. For each vertex  $v$  of  $P$  that is not an apex, there is a facet (of size  $\leq d+s-2$ ) that does not contain  $v$ , so we conclude that every non-apex vertex of  $P$  is simple.

We saw in the proof of Theorem 3.3.2 that it is impossible for all facets of  $P$  to contain  $d+s-1$  vertices. This means that there must exist some facet with  $d+s-p$  vertices where  $2 \leq p \leq s$ . Pick such a facet  $F$ , and label the vertices outside of  $F$  as  $v_1, v_2, \dots, v_p$ . We will show that  $f_{d-1}(P) = d+2$ . The facets of  $P$  fall into the following disjoint categories:

---

<sup>3</sup>Notice that  $\text{Pyr}^{d-a}(\Delta^m \oplus \Delta^{a-m}) \simeq (\text{Pyr}^{d-a}(\Delta^m \times \Delta^{a-m}))^*$ .

- (0).  $F$ ;
- (1). facets containing  $v_1$ ;
- (2). facets containing  $v_2$ , but not  $v_1$ ;
- (3). facets containing  $v_3$ , but not  $v_1, v_2$ ;
- ...
- (p). facets containing  $v_p$ , but not  $v_1, \dots, v_{p-1}$ .

Since  $v_1$  is simple, it is contained in  $d$  facets. These facets together with  $F$  account for  $d + 1$  facets of  $P$ . Next we show that there is a unique facet in category (2), i.e., a unique facet that contains  $v_2$ , but not  $v_1$ . Suppose not, and let  $F_2$  and  $F'_2$  be two distinct facets that contain  $v_2$  but not  $v_1$ . Then there must be a  $k$ -face of  $F'_2$  that contains  $v_2$  and is not a face of  $F_2$ . Therefore

$$\begin{aligned} \#\{k\text{-faces containing } v_2 \text{ but not } v_1 \text{ in } P\} &> \#\{k\text{-faces containing } v_2 \text{ but not } v_1 \text{ in } F_2\} \\ &\geq \binom{d-1}{k}, \end{aligned}$$

which contradicts our assumption in (3.4.1).

We have shown that the number of facets of  $P$  in categories (0), (1), and (2) is  $1 + d + 1 = d + 2$ . If  $F$  has  $d + s - 2$  vertices (and so  $p = 2$ ), we are done. In the case that  $p > 2$ , it suffices to show that for all  $v_i$  with  $i \geq 3$ , there exist no facets that contain  $v_i$  but not  $v_1$  and  $v_2$ . Moreover, by reordering the vertices, it suffices to prove this statement for  $v_3$ .

Suppose there exists a facet  $G$  that contains  $v_3$ , but not  $v_1, v_2$ . Then

$$\begin{aligned}
\#\{k\text{-faces containing } v_3 \text{ but not } v_1, v_2 \text{ in } P\} &\geq \#\{k\text{-faces containing } v_3 \text{ in } G\} \\
&= \#\{(k-1)\text{-faces of } G/v_3\} \\
&\geq \binom{d-1}{k}. \\
&> \binom{d-2}{k} \quad (\text{since } 1 \leq k \leq d-2).
\end{aligned}$$

This again contradicts our assumption in (3.4.1).  $\square$

Now we are ready to prove Theorem 3.4.3.

*Proof of Theorem 3.4.3.* Suppose that  $P$  has  $d-a$  facets that have  $d+s-1$  vertices. By Corollary 3.4.4,  $P$  has  $d+2$  facets. Then  $P$  is a  $(d-a)$ -fold pyramid over an  $a$ -polytope  $Q$  with  $a+s$  vertices and  $a+2$  facets. Since the vertex set of  $Q$  consists of non-apex vertices of  $P$ , by the argument above,  $Q$  is a simple polytope that is not a simplex. According to Lemma 3.4.1,  $Q \simeq \Delta^m \times \Delta^{a-m}$  for some  $1 \leq m \leq a-1$ . We will show that  $Q \simeq \Delta^1 \times \Delta^{s-1}$ .

Since each facet of  $P$  has  $d+s-1$ ,  $d+s-2$ , or  $d$  vertices, and since  $Q$  is not a pyramid, each facet of  $Q$  has either  $a+s-2$  or  $a$  vertices. Since  $Q \simeq \Delta^m \times \Delta^{a-m}$ , there are only two possible types of facets of  $Q$ :  $\Delta^{m-1} \times \Delta^{a-m}$  and  $\Delta^m \times \Delta^{a-m-1}$ . Let  $F$  be a facet of  $Q$  with  $a+s-2$  vertices (such a facet must exist as  $Q$  is not simplicial). As for any two polytopes  $P_1, P_2$ ,  $f_0(P_1 \times P_2) = f_0(P_1) \cdot f_0(P_2)$ , it follows that  $f_0(Q) = (m+1)(a-m+1)$  and  $f_0(F) = (m+1)(a-m)$  or  $m(a-m+1)$ . But we also know that  $f_0(Q) = a+s$  and  $f_0(F) = a+s-2$ . Comparing the equalities, we conclude that  $m=1$  or  $a-1$  and  $a=s$ , so  $Q \simeq \Delta^1 \times \Delta^{s-1}$ . Therefore  $P \simeq \text{Pyr}^{d-s}(\Delta^1 \times \Delta^{s-1})$  as desired.  $\square$

**Remark 3.4.5.** Our proof shows that for  $d+2 \leq s \leq d$  and  $1 \leq k \leq d-2$ ,  $\text{Pyr}^{d-s}(\Delta^1 \times \Delta^{s-1})$  is the unique polytope in  $\mathcal{P}(d+s, d)$  that has  $\phi_k(d+s, d)$  many  $k$ -faces. This is in general not true in the case of  $k=d-1$  (where  $\phi_{d-1}(d+s, d) = d+2$ ), i.e., there might exist more than one polytope  $\text{Pyr}^{d-a}(\Delta^m \times \Delta^{a-m}) \in \mathcal{P}(d+s, d)$  with  $d+2$  facets. Among those polytopes,

by the theorem above,  $\text{Pyr}^{d-s}(\Delta^1 \times \Delta^{s-1})$  has the componentwise minimal  $f$ -vector. This result was also proved in [22, Theorem 24].

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## Chapter 4

## A LOWER BOUND THEOREM FOR STRONGLY REGULAR CW SPHERES WITH UP TO $2D + 1$ VERTICES

### 4.1 Introduction

This paper is devoted to the study of the minimal face numbers of strongly regular CW complexes representing normal pseudomanifolds. A great deal of research has been done on the face numbers of polytopes in the past fifty years. In 1970, McMullen [19] established the Upper Bound Theorem, providing tight upper bounds on the number of  $k$ -faces a  $d$ -dimensional polytope with  $n$  vertices can have. Later, Barnette (see [3], [1], and [2]) proved the Lower Bound Theorem for *simplicial* polytopes; his result provides tight lower bounds on the number of  $k$ -faces a  $d$ -dimensional simplicial polytope with  $n$  vertices can have. Moreover, the lower bound theorem was proved to even hold in the generality of simplicial complexes that are normal pseudomanifolds (see [16], [36], and [12]). In 1980, Billera and Lee [7] and Stanley [28] completely characterized the face numbers of all simplicial (and by duality also simple) polytopes. Their result is known as the  $g$ -theorem. Billera and Lee [7] established sufficiency of the conditions while Stanley [28] proved their necessity.

Despite many advances on face numbers of simplicial polytopes, much less is known about the face numbers of general polytopes. In fact, the only plausible conjecture providing the lower bounds on face numbers was made by Grünbaum in [13, p. 184] (see also [14, p. 265]). He conjectured that for general  $d$ -dimensional polytopes with  $d + s \leq 2d$  vertices, the number of  $k$ -faces is at least

$$\phi_k(d + s, d) = \binom{d + 1}{k + 1} + \binom{d}{k + 1} - \binom{d + 1 - s}{k + 1}.$$

He proved this conjecture for the cases of  $s = 2, 3$ , and  $4$ . The conjecture remained completely open for  $s \geq 5$  until Pineda-Villavicencio, Ugon and Yost [22] proved this conjecture

for the number of edges, i.e., they verified the  $k = 1$  case. In 2020, we proved (see [37]) the conjecture in full generality and characterized equality cases.

In this paper we extend our results to the following much more general classes of posets and complexes. For all undefined terminology and notation, see Section 2.

- We prove that Grünbaum’s conjecture holds in the generality of lattices with the diamond property. In other words, for any diamond lattice  $L$  of rank  $d + 1$  with  $d + s \leq 2d$  atoms, the number of elements of  $L$  of rank  $k + 1$  is at least  $\phi_k(d + s, d)$ , see Theorem 4.3.6.
- For the case of face lattices of strongly regular CW complexes representing normal  $(d - 1)$ -pseudomanifolds with up to  $2d$  vertices, we characterize the equality cases, see Theorem 4.3.8. Along the way, we show that any strongly regular normal  $(d - 1)$ -pseudomanifold with  $d + 2$  facets is combinatorially equivalent to the boundary complex of a  $d$ -polytope, see Theorem 4.3.10.
- Finally, in the class of strongly regular normal  $(d - 1)$ -pseudomanifolds with  $2d + 1$  vertices, we determine the minimum value of the number of  $k$ -dimensional faces for all  $1 \leq k \leq d - 1$ . These minimum values are achieved by certain polytopes with  $d + 2$  and  $d + 3$  facets. See Section 4.5. The lower bound for these pseudomanifolds has two parts. Specifically, let  $P$  be such a pseudomanifold. Then the following holds.

If  $P$  has  $d + 2$  facets, then  $P$  is combinatorially equivalent to the boundary complex of a  $d$ -polytope (see Theorem 4.3.10), and for  $1 \leq k \leq d - 2$ ,

$$f_k(P) \geq \binom{d+1}{k+1} + \binom{d}{k+1} + \binom{d-1}{k+1} - \binom{\lceil \frac{d}{2} \rceil}{k+1} - \binom{\lceil \frac{d}{2} \rceil - 1}{k+1}, \text{ see Lemma 4.4.3.} \quad (4.1.1)$$

If  $P$  has at least  $d + 3$  facets, then for  $1 \leq k \leq d - 2$ ,

$$f_k(P) \geq \binom{d+1}{k+1} + \binom{d}{k+1} + \binom{d-1}{k}, \text{ see Theorem 4.5.1.} \quad (4.1.2)$$

Some of these results but only for the case of *polytopes* with  $2d+1$  vertices were obtained by Pineda-Villavicencio and Yost in [24]. At the end of [24] they conjectured (based on computational data up to  $d = 100$ ) that for any  $d$ -polytope with  $2d + 1$  vertices, the lower bound for the numbers of  $m$ -dimensional faces has two parts. The bound is (4.1.2) when  $1 \leq m \leq \lceil d/2 \rceil - 1$ . And when  $m$  is larger, the lower bound comes from another polytope with  $d + 2$  facets. The first part of their conjecture could be easily confirmed by comparing the two formulas (4.1.1) and (4.1.2) (see Appendix).

## 4.2 Basics on polytopes and strongly regular CW complexes

We will first recall some definitions and introduce some notation. We refer the reader to Günbaum's and Ziegler's books (see [13] and [39]) for all undefined notions related to polytopes, and to Björner's survey paper [8] for notions related to CW complexes. By a **polytope** we mean the convex hull of finitely many points in  $\mathbb{R}^d$ . A  **$d$ -simplex**, denoted as  $\Delta^d$ , is the convex hull of  $d + 1$  affinely independent points. A **face** of a polytope  $P$  is the intersection of  $P$  with a supporting hyperplane. It is known that a face of a polytope is a polytope. The dimension of a polytope is the dimension of its affine span. For brevity, we refer to a  $k$ -dimensional face as a  $k$ -face and to a  $d$ -dimensional polytope as a  $d$ -polytope. The 0-faces are called vertices. The 1-faces are called edges. The  $(d - 1)$ -faces of a  $d$ -polytope are called **facets**. We denote by  $f_k(P)$  the number of  $k$ -faces of a polytope  $P$ .

Let  $P \subset \mathbb{R}^d$  be a  $d$ -polytope and  $v$  a vertex of  $P$ . The **vertex figure of  $P$  at  $v$** ,  $P/v$ , is obtained by intersecting  $P$  with a hyperplane  $H$  that separates  $v$  from the rest of the vertices of  $P$ . One property of vertex figures that will be very useful for us is that  $(k - 1)$ -faces of  $P/v$  are in bijection with  $k$ -faces of  $P$  that contain  $v$ . If a vertex  $v \in P$  is contained in exactly  $d$  edges, then  $v$  is a **simple** vertex, otherwise  $v$  is in more than  $d$  edges and it is called **nonsimple**.

A **CW complex** is any space  $X$  which can be built in the following "hierarchical" way: start with a (finite) collection of isolated points called  $X^0$ , then attach one-dimensional disks  $D^1$  to  $X^0$  along the boundaries  $S^0$ , denote this object by  $X^1$ , then attach two-dimensional

disks  $D^2$  to  $X^1$  along their boundaries  $S^1$ , writing  $X^2$  for the new space, and so on, giving spaces  $X^k$  for every  $k$ . A CW complex is any space that has this sort of decomposition into subspaces  $X^k$  (the  $X^k$ s must exhaust all of  $X$ ). The (closed)  $k$ -disks used to build  $X$  are called the  $k$ -dimensional **faces**<sup>1</sup> or ( $k$ -faces). A CW complex  $X$  is **pure** if all of its facets have the same dimension.

A CW complex  $X$  is a **CW sphere** if it is homeomorphic to a sphere. And it is **regular** if every attaching map is a homeomorphism. Moreover, we call a CW sphere **strongly regular** if, in addition, the intersection of any two faces is a face (possibly empty). For example, the boundary complex of a  $d$ -polytope is a strongly regular CW  $(d - 1)$ -sphere. Also, any oriented matroid sphere (as defined in [9]) is a strongly regular CW sphere.

Let  $L = (L, \leq)$  be a poset (partially ordered set). Any (totally) ordered subset  $t_0 < t_1 < \dots < t_i$  is a **chain** in  $L$  of length  $i$ . A finite poset is **graded** if all the maximal chains have the same length. Throughout this paper, every poset we discuss will be finite and graded. Every poset will also be **bounded**, i.e., have unique bottom and top elements  $\hat{0}$  and  $\hat{1}$ . Let  $\rho : L \rightarrow \mathbb{N}$  be the rank function. For the bottom element, the rank is  $\rho(\hat{0}) = 0$ . The rank of  $L$ ,  $\rho(L)$ , is defined as  $\rho(\hat{1})$ . Let  $x, y$  be two elements of  $L$  where  $x \leq y$ , the **interval**  $[x, y]$  is the sub-poset of  $L$  that contains all elements  $z \in L$  such that  $x \leq z \leq y$  (in particular, we can write  $L$  itself as the interval  $[\hat{0}, \hat{1}]$ ). The pair  $\{x, y\}$  is in above/under relation if  $x < y$ . If in addition there exists no  $z \in L$  such that  $x < z < y$ , then it is also a **covering** relation ( $y$  **covers**  $x$ ). A poset is a **lattice** if for every pair of elements, both the least upper bound and the greatest lower bound exist.

If a graded lattice  $L = [\hat{0}, \hat{1}]$  satisfies the condition that all rank-2 intervals are of the diamond shape (i.e., have four elements), then we call  $L$  a **diamond lattice** or  $\diamond$ -**lattice**. In a lattice of rank  $d$ , the rank-1 elements are called the **atoms**, while the rank- $(d - 1)$  elements are called the **coatoms**. A lattice  $L$  is called **coatom-distinguishable** if given any two elements  $t, s \in L \setminus \{\hat{0}, \hat{1}\}$  of the same rank, there exists a coatom  $F \in L$  such that

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<sup>1</sup>The disks here are usually called (closed) **cells** in the literature. We call them faces in this paper to be consistent with the notion of faces of polytopes.

$t \leq F$  while  $s \not\leq F$ . We will show that  $\diamond$ -lattices are coatom-distinguishable (see Proposition 4.3.1).

The **open face poset**  $L^\circ(X)$  of a CW complex  $X$  is the collection of its proper faces ordered by inclusion, together with the unique bottom element  $\hat{0}$ . The **face poset**  $L(X)$  is  $L^\circ(X)$  to which the unique top element  $\hat{1}$  was added. If  $X$  is a regular CW complex, then (1) the face poset  $L(X)$  is graded, and (2) for every  $x \in X$ , the interval  $[\hat{0}, x]$  is the face poset of a CW sphere. If, in addition,  $X$  is strongly regular (in other words,  $X$  satisfies the intersection property), then  $L(X)$  is a lattice. The face lattice of a  $d$ -simplex is called the **Boolean lattice** of rank  $d$ , denoted by  $B^d$ . Every interval of a Boolean lattice is a Boolean lattice.

We have the following hierarchy relating face lattices of polytopes, strongly regular (SR) CW spheres/pseudomanifolds, and diamond lattices.

$$\{\text{polytopes}\} \subset \{\text{SR CW spheres}\} \subset \{\text{SR normal pseudomanifolds}\} \subset \{\text{diamond lattices}\}.$$

Let  $X$  be a strongly regular CW complex and let  $L = L(X)$  be its face lattice. For any  $k$ -face  $F \in X$ , the rank of  $F$  in  $L$  is  $\rho(F) = k + 1$ . We denote by  $f_k(X)$  or  $f_k(L)$  the number of  $k$ -faces of  $X$ . In this paper, we will use the  $f$ -number notation for all graded lattices, not just those that are face lattices. (Note that  $f_k(L) = W_{k+1}(L)$  where  $W_k$  is the  $k$ -th Whitney number of the second type.) The **pyramid** operation on a finite bounded poset  $L$  (as defined in [11]) is the Cartesian product with the Boolean lattice  $B^1 = \{\hat{0}, \hat{1}\}$ . That is, for a poset  $L = [\hat{0}, \hat{1}]$  we let  $\text{Pyr}(L) := L \times B^1$  (with the componentwise partial order).

Let  $X$  be a regular CW complex and  $L = L(X)$  be its face poset. The **order complex**  $O(X)$  (or  $O(L)$ ) of  $X$  is the following simplicial complex. The vertices of  $O(X)$  are the nonempty proper faces of  $X$ , and the faces of  $O(X)$  are the chains of nonempty proper faces of  $X$ . One advantage of working with a regular CW complex  $X$  is that (the geometric realization of) the order complex  $O(X)$  is homeomorphic to  $X$  (see [8]).

A **pseudomanifold** is a pure regular CW complex of dimension  $d - 1$  such that every  $(d - 2)$ -face is contained in exactly two  $(d - 1)$ -faces. It is easy to check that this property is

preserved under taking barycentric subdivisions. A  $(d - 1)$ -dimensional pseudomanifold  $X$  is called **normal** if  $X$  is connected and  $H_{d-1}(X)$  and  $H_{d-1}(X, X - x)$  are 1-dimensional for all  $x \in X$ . Here  $H_i(X)$  and  $H_i(X, X - x)$  are singular and relative homologies, respectively, computed with coefficients in  $\mathbb{Z}/2\mathbb{Z}$ . For any simplicial complex  $\Delta$  and a face  $\tau \in \Delta$ , the **link of  $\tau$  in  $\Delta$** , denoted by  $\text{lk}_\Delta \tau$  is a subcomplex of  $\Delta$  defined as follow.

$$\text{lk}_\Delta \tau = \{\sigma \mid \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Delta\}.$$

If  $X$  is a normal pseudomanifold and  $T$  is a triangulation of  $X$ , then  $\text{lk}_T(\tau)$  is a normal pseudomanifold for each simplex  $\tau \in T$ .

### 4.3 From polytopes to lattices

Theorem 3.2 of [37] established Grünbaum's conjecture for all polytopes. The goal of this section is to extend this result to a much larger class of objects, namely, to all diamond lattices.

First we point that the key ingredient of the proofs in [37] relied on following property of polytopes: *if  $P$  is a polytope and  $u, v$  are two arbitrary distinct vertices of  $P$ , then there always exists a facet  $F$  such that  $v \in F$ , but  $u \notin F$* . The notion of coatom-distinguishability introduced in Section 2 was motivated by this property of polytopes.

The following few propositions hold for all  $\diamond$ -lattices. All of them can be easily proved by induction on rank. We include the proofs here for the sake of completeness but with fewer and fewer details as these proofs are very similar to each other.

**Proposition 4.3.1.** *Every  $\diamond$ -lattice is coatom-distinguishable.*

*Proof.* Let  $L = [\hat{0}, \hat{1}]$  be a  $\diamond$ -lattice of rank  $d + 1$ . We will prove the statement by induction on  $d$ . When  $d = 1$ ,  $L$  is a diamond, and diamonds are coatom-distinguishable. Assume  $d \geq 2$ , and let  $t, s \in L$  be elements of the same rank. If  $s$  and  $t$  are coatoms or if  $s \vee t = \hat{1}$ , there is nothing to prove. Otherwise, let  $F$  be a coatom that is above  $s \vee t$ . Since the interval  $[\hat{0}, F]$  is a  $\diamond$ -lattice of rank  $d$ , there must be a rank- $(d - 1)$  element  $G < F$  such that  $t < G$

while  $s \not\prec G$ . Since the interval  $[G, \hat{1}]$  has rank 2, by the diamond property there exists another coatom  $F'$  of  $L$  such that  $G = F \wedge F'$ . This implies that  $t < F'$  and  $s \not\prec F'$ .  $\square$

**Proposition 4.3.2.** *Let  $L = [\hat{0}, \hat{1}]$  be a  $\diamond$ -lattice of rank  $d + 1$ . For every element  $t \in L$ , the interval  $[t, \hat{1}]$  has at least the same number of elements in each rank as the Boolean lattice  $B^{d+1-\rho(t)}$  (where  $\rho(t)$  is the rank of  $t$ ).*

*Proof.* The proof is again by induction on  $d$  and  $\rho(t)$ . When  $d = 1, 2$ , the statement is clear. Let  $d \geq 3$ . If  $\rho(t) = d + 1$  then  $t = \hat{1}$  and so  $[t, \hat{1}] = B^0$ . If  $\rho(t) = k < d$ , there must exist (at least) two elements  $s_1, s_2$  that cover  $t$  in  $[t, \hat{1}]$ . Since  $P$  is coatom-distinguishable by the previous proposition, there is a coatom  $F$  such that  $s_1 < F$  and  $s_2 \not\prec F$ . By the inductive hypothesis, the two intervals  $[t, F]$  and  $[s_2, \hat{1}]$  both have at least the same number of elements in each rank as the Boolean lattice  $B^{d-\rho(t)}$  and they are disjoint from each other. Since  $f_k([t, \hat{1}]) \geq f_k([t, F]) + f_{k-1}([s_2, \hat{1}])$  for  $k \geq 1$ , the result follows by the Pascal triangle recursion.  $\square$

**Proposition 4.3.3.** *Let  $L = [\hat{0}, \hat{1}]$  be a  $\diamond$ -lattice of rank  $d + 1$ . Consider any lower or upper rank- $k$  interval in  $L$ . If this interval has exactly  $k$  atoms or exactly  $k$  coatoms, then it is the Boolean lattice  $B^k$ .*

*Proof.* We will, once again, induct on the rank of  $L$ . The base case is clear. Now let  $\rho(L) = d + 1 > 2$ . Notice that if any proper lower or upper interval of  $L$  satisfies the condition in the statement, then this interval is a Boolean lattice by the inductive hypothesis. Now suppose  $L$  has  $d + 1$  atoms. Then each atom is below at most  $d$  rank-2 elements, and hence, by Proposition 4.3.2, it is below exactly  $d$  such elements. By the inductive hypothesis, each proper upper interval of  $L$  is a Boolean lattice. Now since  $L$  has  $d + 1$  atoms, it is the Boolean lattice  $B^{d+1}$ .

The argument for the case when  $L$  has  $d + 1$  coatoms is very similar, hence we omit it.  $\square$

**Proposition 4.3.4.** *Let  $L = [\hat{0}, \hat{1}]$  be a  $\diamond$ -lattice of rank  $d + 1$ . Consider any lower or upper interval  $[x, y]$  in  $L$ . If every coatom of this interval contains all the atoms of  $L$  but one, then  $[x, y]$  is a Boolean lattice.*

*Proof.* The proof is, again, by induction on the rank of the interval. The base case is trivial. Suppose the statement is true for all intervals of rank at most  $k - 1$  and let  $[x, y]$  be a rank- $k$  interval satisfying the conditions in the statement above. Then for any coatom  $z$  of  $[x, y]$ , the interval  $[x, z]$  also satisfies conditions of the statement, and hence by the inductive hypothesis  $[x, z] = B^{k-1}$ . Since there is only one atom of  $[x, y]$  that is not below  $z$ , this forces  $[x, y]$  to have exactly  $k$  atoms. By the previous proposition,  $[x, y] = B^k$ .  $\square$

Now we are ready to obtain the key ingredient of the proof of Grünbaum's Conjecture, this time, for  $\diamond$ -lattices. The following proofs are adaptations of the original proofs from [37] to the generality of diamond lattices. We will only be using part (i) of Proposition 4.3.5 in this section. Parts (ii) and (iii) will become useful later in Section 4.5.

**Proposition 4.3.5.** *Let  $L = [\hat{0}, \hat{1}]$  be a  $\diamond$ -lattice of rank  $d + 1$ , and let  $\mathcal{S} = \{v_1, v_2, \dots, v_m\}$  be a subset of atoms of  $L$ . Let  $|L_{k+1}^{\mathcal{S}}|$  denote the number of rank- $(k + 1)$  elements of  $L$  that are above at least one of the  $v_i$ 's in  $\mathcal{S}$ . Then,*

$$(i). \quad |L_{k+1}^{\mathcal{S}}| \geq \sum_{i=1}^m \binom{d-i+1}{k}.$$

$$(ii). \quad \text{If one of the atoms in } \mathcal{S} \text{ is covered by more than } d \text{ elements of rank 2, then } |L_{k+1}^{\mathcal{S}}| \geq \left( \sum_{i=1}^m \binom{d+1-i}{k} \right) + \binom{d-2}{k-1}.$$

$$(iii). \quad \text{If there exists a coatom } F \in L \text{ such that } F \text{ is above at least one and not more than } |S| - 2 \text{ elements of } S, \text{ then } |L_{k+1}^{\mathcal{S}}| \geq \binom{d}{k} + \binom{d-1}{k} + \sum_{i=1}^{m-2} \binom{d-i}{k}.$$

*Proof.* We will only prove part (i) since the proofs of (ii) and (iii) are essentially the same.

We induct on  $m$  to show that there exists a sequence of elements,  $\{F_1, \dots, F_m\}$ , such that

$$(1). \quad \text{each } F_i \text{ has rank } d - i + 2,$$

$$(2). \quad v_i \leq F_i, \text{ but } v_j \not\leq F_i \text{ for any } j < i.$$

The base case is  $m = 1$ , and we simply pick  $F_1 = \hat{1}$ . Inductively we assume that for every  $p \leq m - 1$  and any  $p$ -element set of vertices  $\{v_1, \dots, v_p\}$ , there exists a sequence  $\{F_1, \dots, F_p\}$  such that for  $1 \leq i \leq p$ , conditions (1) and (2) are satisfied.

Let  $m > 1$  and let  $v_1, \dots, v_m$  be  $m$  given atoms of  $L$ . By the inductive hypothesis, for  $\{v_1, \dots, v_{m-1}\}$  there exist elements  $F_1, \dots, F_{m-1}$  satisfying conditions (1) and (2). Similarly, by considering  $\{v_1, \dots, v_{m-2}, v_m\}$ , there also exists a rank- $(d - m + 3)$  element  $F$  such that  $v_m \leq F$  but  $v_1, \dots, v_{m-2} \not\leq F$ . Regardless of whether  $v_{m-1}$  is under  $F$  or not, by the coatom-distinguishability of  $L$ , there must exist a coatom of  $[\hat{0}, F]$ , call it  $F_m$ , that satisfies  $v_m \leq F_m$  but  $v_{m-1} \not\leq F_m$ . Then  $v_i \in F_m$  if and only if  $i = m$ , and  $F_1, \dots, F_{m-1}, F_m$  is a desired sequence.

For each  $i$ , the rank- $(k+1)$  elements in  $F_i$  that are above  $v_i$  correspond to rank- $k$  elements in the interval  $[v_i, F_i]$ . From this we obtain that

$$\begin{aligned} & \# \text{rank-}(k+1) \text{ elements of } L \text{ that contain some } v_i \text{ (} 1 \leq i \leq m \text{)} \\ & \geq \# \bigcup_{i=1}^m \{\text{rank-}(k+1) \text{ elements of } F_i \text{ containing } v_i\} \\ & \geq \# \bigcup_{i=1}^m \{\text{rank-}k \text{ elements in the lattice } [v_i, F_i]\} \\ & \geq \sum_{i=1}^m \binom{d-i+1}{k}, \end{aligned}$$

where the last inequality follows from Proposition 4.3.2. The result follows.  $\square$

With Proposition 4.3.5 at our disposal, we are now ready to prove the first main theorem of this section. Recall that

$$\phi_k(d+s, d) = \binom{d+1}{k+1} + \binom{d}{k+1} - \binom{d+1-s}{k+1}.$$

**Theorem 4.3.6.** *Let  $L$  be a rank- $(d+1)$   $\diamond$ -lattice that has  $d+s$  atoms where  $d \geq s \geq 1$ . Then  $f_k(L) \geq \phi_k(d+s, d)$  for every  $k$ .*

*Proof.* The statement holds for  $s = 1$  by Proposition 4.3.2. The proof is by induction on  $s$ . We fix  $s \geq 2$ . The following argument will show that if the statement holds for all pairs

$(s', d')$  such that  $s' < s$  and  $d' \geq s'$ , then it also holds for the pair  $(s, d)$  for all  $d \geq s$ . Now consider  $d \geq s$ , and let  $L = [\hat{0}, \hat{1}]$  be a rank- $(d+1)$   $\diamond$ -lattice with  $d+s$  atoms. If there exists a coatom  $z$  of  $L$  such that  $d \leq f_0([\hat{0}, z]) \leq d+s-2$ , or equivalently if  $[\hat{0}, z]$  is a  $\diamond$ -lattice with  $(d+s-m)$  atoms such that  $2 \leq m \leq s$ , then there are  $m$  atoms of  $L$  that are not under the element  $z$ . We denote  $m$  of them by  $\{v_1, v_2, \dots, v_m\}$ . The rank- $k$  elements of  $L$  fall into two disjoint categories: those that are under  $z$  and those that are above some  $v_i$ . By the inductive hypothesis on  $[\hat{0}, z]$ ,  $f_k([\hat{0}, z]) \geq \phi_k(d+s-m, d-1)$ . Therefore by Proposition 4.3.5,

$$\begin{aligned}
f_k(P) &\stackrel{(\star)}{\geq} \phi_k(d+s-m, d-1) + \sum_{i=1}^m \binom{d-i+1}{k} \\
&= \phi_k(d+s-m+2, d) + \sum_{i=3}^m \binom{d-i+1}{k} \\
&= \left[ \phi_k(d+s, d) - \sum_{j=1}^{m-2} \binom{d-s+m-1-j}{k} \right] + \sum_{i=3}^m \binom{d-i+1}{k} \quad (4.3.1) \\
&= \phi_k(d+s, d) + \underbrace{\sum_{j=1}^{m-2} \left[ -\binom{d-j-1-(s-m)}{k} + \binom{d-j-1}{k} \right]}_{\geq 0 \text{ (since } s \geq m)} \\
&\stackrel{(\star\star)}{\geq} \phi_k(d+s, d).
\end{aligned}$$

This completes the proof of this case.

Otherwise, all coatoms in  $P$  contain all but one atom. By Proposition 4.3.4 the interval  $[x, y]$  is the Boolean lattice.  $\square$

Using the inequalities  $(\star)$  and  $(\star\star)$  in the proof above, we can obtain the following corollary. The proof is the same as in [37] hence we omit it.

**Corollary 4.3.7.** *If  $f_k(L) = \phi_k(d+s, d)$  for some  $k$  with  $1 \leq k \leq d-2$ , then each coatom of  $L$  lies above  $d, d+s-2$ , or  $d+s-1$  atoms, and  $L$  has  $d+2$  coatoms.*

Our next goal is to extend the treatment of equality cases from the class of polytopes to the class of strongly regular normal pseudomanifolds. The next main theorem in this section

is the following. We denote by  $\text{Pyr}^{d-s}(\Delta^1 \times \Delta^{s-1})$  the polytope that is a  $(d-s)$ -fold pyramid over a prism over an  $(s-1)$ -simplex.

**Theorem 4.3.8.** *Let  $L$  be the face lattice of a strongly regular normal  $(d-1)$ -pseudomanifold  $S$  with  $d+s$  vertices where  $s \geq 2$  and  $d \geq s$ . If  $f_k(L) = \phi_k(d+s, d)$  for some  $1 \leq k \leq d-2$ , then  $S$  is the boundary of  $\text{Pyr}^{d-s}(\Delta^1 \times \Delta^{s-1})$ .*

By [37, Thm. 4.3], the statement holds when  $S$  is (the boundary of) a polytope. By Corollary 4.3.7,  $S$  has  $d+2$  facets. Therefore it suffices to show that all strongly regular normal  $(d-1)$ -pseudomanifolds with  $d+2$  facets are boundary complexes of polytopes. Before proving this, we will prove the following useful proposition.

**Proposition 4.3.9.** *Let  $L = L(P)$  be the face lattice of a regular CW complex  $P$ . If all proper upper intervals of  $L$  are Boolean lattices, then its dual lattice  $L^*$  is the face lattice of a simplicial complex that is homeomorphic to  $P$ .*

*Proof.* Since all proper upper intervals of  $L$  are Boolean lattices, so are all proper lower intervals of  $L^*$ . Therefore  $L^*$  is the face lattice of a pure, connected simplicial complex  $\Delta$ . Now the two order complexes  $O(L)$  and  $O(L^*)$  are homeomorphic to  $P$  and  $\Delta$  respectively. Since  $O(L) = O(L^*)$ , it follows that  $P$  and  $\Delta$  are homeomorphic.  $\square$

We can now prove the following theorem, which is the last piece in the proof of Theorem 4.3.8.

**Theorem 4.3.10.** *Every strongly regular  $(d-1)$ -dimensional normal pseudomanifold that has  $d+2$  facets is combinatorially equivalent to the boundary complex of a  $d$ -polytope.*

*Proof.* Let  $P$  be a strongly regular  $(d-1)$ -dimensional normal pseudomanifold that has  $d+2$  facets. This implies that every  $k$ -face of  $P$  has at most  $k+2$  facets. We will, one more time, induct on  $d$ . The base cases ( $d \leq 2$ ) are trivial. Now for  $d > 2$ , we first consider the case when  $L(P)$  is a pyramid over  $L(F)$  for some facet  $F$  of  $P$ . By the inductive hypothesis  $F$  is combinatorially equivalent to a polytope, hence so is  $P$ .

Next we assume that  $P$  is not a pyramid, which implies that, for any vertex  $v$  there exist at least two facets that do not contain  $v$ . Let  $L$  be the face lattice of  $P$ . We now look at its dual lattice,  $L^*$ , which has  $d + 2$  atoms. Since  $P$  is not a pyramid, for every coatom  $v \in L^*$ , the lower interval  $[\hat{0}^*, v]$  has rank  $d$  and at most  $d$  atoms. This forces  $[\hat{0}^*, v]$  to be the Boolean lattice, and by Proposition 4.3.9,  $L^*$  is the face lattice of a simplicial normal  $(d-1)$ -pseudomanifold with  $d+2$  vertices, which must be the join of the boundary complexes of two simplices whose dimensions add up to  $d$  (see, for example, [38, Lemma 4.1]). Thus  $L^*$  is the face lattice of a polytope, and hence so is  $L$ .

□

#### 4.4 Polytopes with $d + 2$ facets

In the previous section, we extended all the results from [37] about polytopes with up to  $2d$  vertices to the generality of diamond lattices and strongly regular normal pseudomanifolds. Our goal in the rest of the paper is to provide another generalization: this time to strongly regular normal pseudomanifolds with  $2d + 1$  vertices. To do so, in this section we will temporarily go back to the world of polytopes. We start by using the face numbers to define a certain partial order on the  $d$ -polytopes.

Given two  $d$ -polytopes  $P$  and  $Q$ , we write  $P \leq Q$  if  $f_i(P) \leq f_i(Q)$  for all  $0 \leq i \leq d - 1$ ; if, in addition,  $f(P) \neq f(Q)$ , then we write  $P < Q$ . If  $\mathcal{S}$  is a set of  $d$ -polytopes, and there exists a polytope  $P \in \mathcal{S}$  such that for all other polytopes  $Q \in \mathcal{S}$ ,  $P < Q$ , then  $P$  is called **componentwise minimal** in  $\mathcal{S}$  and  $f(P)$  is called the **componentwise minimal  $f$ -vector in  $\mathcal{S}$** . If moreover we are able to order all polytopes in  $\mathcal{S}$  so that  $\mathcal{S} = \{P_1 < P_2 < P_3 < \dots\}$ , then  $\mathcal{S}$  is **completely ordered by  $f$ -vectors**. Notice that this is a very strong condition, and it implies that any nonempty subset of  $\mathcal{S}$  contains a polytope that is componentwise minimal.

Let  $\mathcal{P}(d + s, d, d + 2)$  be the set of all  $d$ -polytopes that have  $d + s$  vertices and  $d + 2$  facets. The main result in this section is the following.

**Theorem 4.4.1.** *Polytopes in  $\mathcal{P}(d + s, d, d + 2)$  are completely ordered by  $f$ -vectors.*

The set  $\mathcal{P}(d + s, d, d + 2)$  consists of polytopes dual to those with  $d + 2$  vertices and  $d + s$  facets. The set of all  $d$ -polytopes with  $d + 2$  vertices, denoted  $\mathcal{P}(d + 2, d)$ , was fully characterized in [13, 6.1]. In particular, every simplicial polytope in this class is obtained from some  $(d - m)$ -simplex by adding one additional vertex and placing it beyond exactly  $m$  of its facets for some  $1 \leq m \leq \lfloor \frac{d}{2} \rfloor$ . Grünbaum denoted such a simplicial polytope as  $T_m^d$ . Moreover, every  $P \in \mathcal{P}(d + 2, d)$  is a  $(d - i)$ -fold pyramid over  $T_m^i$  for some  $2 \leq i \leq d$  and  $1 \leq m \leq \lfloor \frac{i}{2} \rfloor$ . Such a polytope is denoted as  $T_m^{d, d-i}$ . Grünbaum showed in [13, 6.1.4] that,

$$f_k(T_m^{d, d-i}) = \binom{d+2}{d-k+1} - \binom{d-i+m+1}{d-k+1} - \binom{d-m+1}{d-k+1} + \binom{d-i+1}{d-k+1} \quad (4.4.1)$$

for all  $0 \leq k \leq d - 1$ . In particular,

$$f_{d-1}(T_m^{d, d-i}) = d + 1 + m(i - m). \quad (4.4.2)$$

Equation (4.4.1) implies the following inequalities on the face numbers of polytopes in  $\mathcal{P}(d + 2, d)$  (also see [13, 6.1]):

(i) For  $i$  and  $m$  as above,

$$f_k(T_m^{d, d-i}) \leq f_k(T_{m+1}^{d, d-i}), \quad (4.4.3)$$

with strict inequality if and only if  $m \leq k + 1$ .

(ii) For  $i$  and  $m$  as above,

$$f_k(T_m^{d, d-i}) \leq f_k(T_m^{d, d-i-1}), \quad (4.4.4)$$

with strict inequality if and only if  $d \leq d - i + k + m$ .

We are now ready to prove Theorem 4.4.1.

*Proof of Theorem 4.4.1.* Let  $\leq$  be the partial order on polytopes defined at the start of this section. Note that if two polytopes  $P$  and  $Q$  satisfy  $P \leq Q$ , then so do their duals, that is,  $P^* \leq Q^*$ . Thus to prove the statement, it suffices to show that  $\mathcal{P}(d+2, d, d+s)$  is completely ordered by  $f$ -vectors. By (4.4.2), for any  $i \neq i'$ ,  $T_m^{d, d-i}$  and  $T_{m'}^{d, d-i'}$  have different numbers of facets. Therefore two (distinct) polytopes  $P = T_m^{d, d-i}$  and  $P' = T_{m'}^{d, d-i'}$  are both in  $\mathcal{P}(d+2, d, d+s)$  only if  $m \neq m'$  (similarly, if  $m = m'$ , then  $i = i'$  for polytopes in this set). This implies that each polytope  $P$  in  $\mathcal{P}(d+2, d, d+s)$  is of the form  $T_m^{d, d-i}$  where all  $m$  (and all  $i$ ) are distinct (that is, the functions  $P \rightarrow m$  and  $P \rightarrow i$  are one-to-one). Hence, in order to prove the statement, it suffices to show that if  $\mathcal{P}(d+2, d, d+s)$  contains both  $T_{m'}^{d, d-i'}$  and  $T_m^{d, d-i}$  with  $m < m'$ , then

$$f_k(T_m^{d, d-i}) \leq f_k(T_{m'}^{d, d-i'}) \quad \text{for any } 1 \leq k \leq d-2.$$

Since  $T_m^{d, d-i}$  and  $T_{m'}^{d, d-i'}$  have the same number of facets, and the inequalities in (4.4.3) and (4.4.4) are both strict when  $k = d-1$ , the fact that  $m < m'$  implies  $i > i'$ .

For any  $1 \leq k \leq d-2$ , by (4.4.1),

$$\begin{aligned} & f_k(T_{m'}^{d, d-i'}) - f_k(T_m^{d, d-i}) \\ &= \underbrace{\left[ -\binom{d-i'+m'+1}{d-k+1} + \binom{d-i+m+1}{d-k+1} \right]}_{(=A)} + \underbrace{\left[ -\binom{d-m'+1}{d-k+1} + \binom{d-m+1}{d-k+1} \right]}_{(=B)} \\ & \quad + \underbrace{\left[ \binom{d-i'+1}{d-k+1} - \binom{d-i+1}{d-k+1} \right]}_{(=C)}. \end{aligned}$$

By repeatedly applying the Pascal identity, we obtain the following (known) formula.

$$\binom{n}{k+1} - \binom{n-a}{k+1} = \sum_{p=1}^a \binom{n-p}{k}.$$

Therefore

$$A = - \sum_{p=1}^{(i-i')+(m'-m)} \binom{d-i'+m'+1-p}{d-k},$$

$$B = \sum_{p=1}^{m'-m} \binom{d-m+1-p}{d-k},$$

$$C = \sum_{p=1}^{i-i'} \binom{d-i'+1-p}{d-k}.$$

By shifting the summation index of  $A$  and then cancelling with the overlapping terms in  $C$ , we obtain

$$\begin{aligned} A + C &= - \sum_{p=1-m'}^0 \binom{d-i'+1-p}{d-k} + \sum_{p=(i-i')-m+1}^{i-i'} \binom{d-i'+1-p}{d-k} \\ &= - \sum_{p=1}^{m'} \binom{d-i'+p}{d-k} + \sum_{p=1}^m \binom{d-i+p}{d-k} \\ &= - \sum_{p=1}^{m'-m} \binom{d-i'+p+m}{d-k} - \sum_{p=1}^m \left[ \binom{d-i'+p}{d-k} - \binom{d-i+p}{d-k} \right], \text{ while} \end{aligned}$$

$$\begin{aligned} B + A + C &= \sum_{p=1}^{m'-m} \left[ \binom{d-m+1-p}{d-k} - \binom{d-i'+p+m}{d-k} \right] - \sum_{p=1}^m \left[ \binom{d-i'+p}{d-k} - \binom{d-i+p}{d-k} \right] \\ &= \sum_{p=1}^{m'-m} \left[ \sum_{q=1}^{i'+1-2p-2m} \binom{d-m+1-p-q}{d-k-1} \right] - \sum_{p=1}^m \left[ \sum_{q=1}^{i-i'} \binom{d-i'+p-q}{d-k-1} \right] \end{aligned} \tag{4.4.5}$$

$$\begin{aligned} &\stackrel{(\text{X})}{\geq} \left[ \sum_{p=1}^{m'-m} (i'+1-2p-2m) \right] \binom{d-m+1-(1)-(i'-1-2m)}{d-k-1} \\ &\quad - (m)(i-i') \binom{d-i'+m-1}{d-k-1} \\ &= (m'-m)(i'-m'-m) \binom{d-i'+m+1}{d-k-1} - (m)(i-i') \binom{d-i'+m-1}{d-k-1}. \end{aligned}$$

By (4.4.2),

$$d + s = d + 1 + m(i - m) = d + 1 + m'(i' - m'),$$

which implies that  $m(i - m) = m'(i' - m')$ , and so

$$(m' - m)(i' - m' - m) = (m)(i - i'). \quad (4.4.6)$$

Therefore

$$f_k(T_{m'}^{d,d-i'}) - f_k(T_m^{d,d-i}) = B + A + C \geq \underbrace{(m)(i - i')}_{>0} \underbrace{\left[ \binom{d-i'+m+1}{d-k-1} - \binom{d-i'+m-1}{d-k-1} \right]}_{\geq 0} \quad (4.4.7)$$

$$\geq 0.$$

The result follows.  $\square$

Recall that the simplicial polytope  $T_1^s$  is a bipyramid over an  $(s-1)$ -simplex, and  $T_1^{d,d-s}$  is a  $(d-s)$ -fold pyramid over this bipyramid. When  $s \leq d$ , the  $d$ -polytope  $T_1^{d,d-s}$  exists and is contained in the set  $\mathcal{P}(d+2, d, d+s)$ . By Theorem 4.4.1, it has the componentwise minimal  $f$ -vector in the completely ordered set  $\mathcal{P}(d+2, d, d+s)$ . Moreover, the inequality  $(\boxtimes)$  in (4.4.5) is equality only if  $k < m - 2$ . Therefore when  $m = 1$ , the inequality is strict. This means that any other polytope  $P \in \mathcal{P}(d+2, d, d+s)$  has strictly more  $k$ -faces than  $T_1^{d,d-s}$  for any  $1 \leq k \leq d-2$ . The dual statement, together with the fact that  $f_k\left((T_1^{d,d-s})^*\right) = \phi_k(d+s, d)$ , implies the following corollary.

**Corollary 4.4.2.** *For any  $P \in \mathcal{P}(d+s, d, d+2)$ ,  $2 \leq s \leq d$ , and  $1 \leq k \leq d-2$ ,*

$$f_k(P) \geq \phi_k(d+s, d).$$

*Moreover, equality holds for some  $k$  if and only if  $P = (T_1^{d,d-s})^*$ .*

When  $s > d$ , the polytope  $T_1^{d,d-s}$  does not exist. By Theorem 4.4.1 the “next-best” polytopes would be  $T_2^{d,d-i}$  (for some  $i$ ) when they exist. If  $d$  is even and  $s = d+1$ , then the polytope  $T_2^{d, \frac{d}{2}-2}$  has the componentwise minimal  $f$ -vector in  $\mathcal{P}(d+2, d, 2d+1)$ . However, when  $d$  is odd,  $\frac{d}{2}$  is not an integer. Yet in the proof above, the inequality (4.4.7) continues

to hold even though the equality in (4.4.6) becomes “ $\geq$ ”. Hence we only need to slightly adjust the  $f$ -vector formula of  $T_2^{d, \frac{d}{2}-2}$  by using the floor or ceiling function as needed to keep the inequalities in the proof valid, so the proof above still works, and the face numbers of hypothetical  $T_2^{d, \frac{d}{2}-2}$  (obtained from (4.4.1) and shown below) form the lower bounds.

**Lemma 4.4.3.** *Let  $P$  be a  $d$ -polytope with  $f_0(P) \geq 2d + 1$  and  $f_{d-1}(P) = d + 2$ . Then for  $m \geq 1$ ,*

$$f_m(P) \geq \binom{d+1}{m+1} + \binom{d}{m+1} + \binom{d-1}{m+1} - \binom{\lceil \frac{d}{2} \rceil}{m+1} - \binom{\lceil \frac{d}{2} \rceil - 1}{m+1}. \quad (4.4.8)$$

When  $f_0(P) = 2d + 1$  and  $d$  is even, this lower bound is sharp, and it is attained for every  $m \geq 1$  when  $P = (T_2^{d, \frac{d}{2}-2})^*$ .

#### 4.5 Pseudomanifolds with at least $d + 3$ facets

In this section, we will find the lower bounds on the  $f$ -numbers of polytopes and strongly regular normal pseudomanifolds with at least  $2d + 1$  vertices and  $d + 3$  facets. In particular, we will show that in this case, there is a polytope  $\nabla$  (constructed below) that has the componentwise minimal  $f$ -vector.

Let  $P$  be a polytope that has a facet  $F$  that is a simplex. The operation that builds a shallow pyramid over  $F$  while maintaining the convexity is called **stacking** on  $P$  over  $F$ . Let  $T_1^{d, d-2}$  be the polytope with  $d + 2$  vertices as defined in Section 4.4. Stack over one arbitrary facet of  $T_1^{d, d-2}$  that is a simplex (note that  $T_1^{d, d-2}$  is a  $(d - 2)$ -fold pyramid over a square), then take the dual of this polytope. We will obtain the polytope  $\nabla := (\text{Stack}(T_1^{d, d-2}))^*$ , which has  $d + 3$  facets. It is also easy to check that  $f_0(\nabla) = 2d + 1$  and

$$f_m(\nabla) = \binom{d+1}{m+1} + \binom{d}{m+1} + \binom{d-1}{m} \quad \text{for } 0 < m \leq d-1. \quad (4.5.1)$$

The main theorem of this section is the following.

**Theorem 4.5.1.** *Let  $P$  be a  $(d - 1)$ -dimensional strongly regular normal pseudomanifold with  $f_0(P) \geq 2d + 1$  and  $f_{d-1}(P) \geq d + 3$ . Then  $P \geq \nabla$ .*

The proof of this theorem consists of several cases. We first use the Lower Bound Theorem to take care of the case when  $P$  is simple, i.e., every vertex is in  $d$  edges.

**Theorem 4.5.2.** *Let  $P$  be a pseudomanifold as above. If, in addition,  $P$  is simple, then  $P \geq \nabla$ .*

*Proof.* Let  $L = L(P)$  be the face lattice of  $P$ . Since every vertex of  $P$  is in exactly  $d$  edges, the dual lattice  $L^*$  is the face lattice of a pure simplicial complex. Since  $L$  is the face lattice of a normal pseudomanifold, by Proposition 4.3.9,  $L^*$  is the face lattice of a simplicial normal pseudomanifold, call it  $P^*$ . Since  $f_0(P^*) \geq d + 3$ , by the Lower Bound Theorem (see [12], [16], and [36]), the  $f$ -vector of  $P^*$  is componentwise bounded by the  $f$ -vector of the stacked polytope with  $d + 3$  vertices, denoted by  $\text{Stack}(d + 3, d)$ . For each  $m \geq 1$ ,

$$\begin{aligned}
f_m(P) &\geq f_{d-1-m}(\text{Stack}(d + 3, d)) \\
&= \binom{d}{m+1} \cdot (d + 3) - \binom{d+1}{m+1} \cdot (d - 1 - m) \\
&= \binom{d+1}{m+1} + \binom{d}{m+1} + \left[ \binom{d}{m+1} \cdot (d + 2) - \binom{d+1}{m+1} \cdot (d - m) \right] \\
&= \binom{d+1}{m+1} + \binom{d}{m+1} + \binom{d}{m+1} \\
&= f_m(\nabla) + \binom{d-1}{m+1} \\
&\geq f_m(\nabla).
\end{aligned} \tag{4.5.2}$$

The result follows. □

*Proof of Theorem 4.5.1.* Let  $P$  be a  $(d-1)$ -dimensional strongly regular normal pseudomanifold with  $f_0(P) = 2d + k$  ( $k \geq 1$ ) and  $f_{d-1}(P) \geq d + 3$ . By Theorem 4.5.2, we now only need to consider the case where there is a vertex  $v \in P$  that is **nonsimple**, i.e.,  $v$  is in at least  $d + 1$  edges. In the rest of the proof  $v$  always denotes a non-simple vertex of  $P$ . The proof that Theorem 4.5.1 holds in this case is by induction on  $d$ .

The statement clearly holds for  $d = 2$ . For the inductive step, we treat the following four cases.

Case 1:  $P$  is the pyramid with apex  $v$ , that is, there exists a facet  $F_1$  of  $P$  such that  $P = F_1 * v$ .

In this case, the face lattice of  $F_1$  is the face lattice of a strongly regular CW  $(d-2)$ -sphere with at least  $d+2$  facets and  $2d+k-1 (> 2(d-1)+1)$  vertices. By the inductive hypothesis, for  $m \geq 2$ , we have

$$\begin{aligned} f_m(P) &= f_m(F_1) + f_{m-1}(F_1) \\ &\geq \left[ \binom{d}{m+1} + \binom{d-1}{m+1} + \binom{d-2}{m} \right] + \left[ \binom{d}{m} + \binom{d-1}{m} + \binom{d-2}{m-1} \right] \\ &= \binom{d+1}{m+1} + \binom{d}{m+1} + \binom{d-1}{m} = f_m(\nabla). \end{aligned}$$

And for  $m = 1$ ,

$$\begin{aligned} f_1(P) &= f_1(F_1) + f_0(F_1) \\ &\geq \left[ \binom{d}{2} + \binom{d-1}{2} + \binom{d-2}{1} \right] + 2d + k - 1 \\ &= \binom{d+1}{2} + \binom{d}{2} + \binom{d-1}{1} + \underbrace{k-1}_{\geq 0} \\ &\geq f_1(\nabla). \end{aligned}$$

Case 2: There exists a facet  $F_1$  of  $P$  that does not contain  $v$  and such that  $2d-1 \leq f_0(F_1) \leq f_0(P) - 2$  and  $f_{d-2}(F_1) \geq (d-1) + 3$ . Then  $F_1$  satisfies the inductive hypothesis. Furthermore, there are at least two vertices of  $P$  outside of  $F_1$  including a non-simple vertex  $v$ . By Proposition 4.3.5(ii) we have that for all  $m \geq 1$ ,

$$\begin{aligned} f_m(P) &\geq f_m(F_1) + \binom{d}{m} + \binom{d-1}{m} + \binom{d-2}{m-1} \\ &\geq \underbrace{\left[ \binom{d}{m+1} + \binom{d-1}{m+1} + \binom{d-2}{m} \right]}_{(*)} + \binom{d}{m} + \binom{d-1}{m} + \binom{d-2}{m-1} \quad (4.5.3) \\ &= f_m(\nabla). \end{aligned}$$

Case 3: There exists a facet  $F_1$  of  $P$  that does not contain  $v$  and such that  $2d-1 \leq f_0(F_1) \leq$

$f_0(P) - 2$  and  $f_{d-2}(F_1) = (d-1) + 2$ . In this case, by Theorem 4.3.10,  $F_1$  is combinatorially equivalent to (the boundary of) a polytope. By Lemma 4.4.3, for  $m \geq 1$ ,

$$f_m(F_1) \geq \binom{d}{m+1} + \binom{d-1}{m+1} + \binom{d-2}{m+1} - \binom{\lceil \frac{d-1}{2} \rceil}{m+1} - \binom{\lceil \frac{d-1}{2} \rceil - 1}{m+1}. \quad (4.5.4)$$

Recall that as in the previous case, there are two vertices outside of  $F_1$  and one of them is nonsimple. Thus, using (4.5.3), when  $1 \leq m < \lceil \frac{d}{2} \rceil - 1$  it suffices to show that  $f_m(F_1) \geq \binom{d}{m+1} + \binom{d-1}{m+1} + \binom{d-2}{m}$ . Recall that any polytope with  $n$  vertices and  $d+2$  facets is of the form  $(T_m^{d,d-i})^*$  for some  $2 \leq i \leq d$ ,  $m < \lfloor \frac{i}{2} \rfloor$ , and  $n = d+1 + m(i-m)$ . It is easy to see that there does not exist a 5-dimensional polytope with 11 vertices and 7 facets. Therefore when  $d = 6$ , the facet  $F_1$  has at least 12 vertices. According to the ordering of all 5-polytopes with 7 facets as in Theorem 4.4.1, the minimizer should be  $(T_2^{5,0})^*$ , hence  $f_m(F_1) \geq f_m((T_2^{5,0})^*)$ . In this case if  $m < \lceil \frac{5}{2} \rceil - 1 = 2$ , then  $m = 1$ . The result then follows since

$$\binom{d}{m+1} + \binom{d-1}{m+1} + \binom{d-2}{m} = 29 < 30 = f_1((T_2^{5,0})^*) \leq f_1(F_1).$$

When  $d \geq 7$  and  $m < \lceil \frac{d-1}{2} \rceil - 1$ , the lower bound in (4.5.4) is greater than  $\binom{d}{m+1} + \binom{d-1}{m+1} + \binom{d-2}{m}$  in (4.5.3). The computations to verify this are purely combinatorial, hence included in Appendix.

Now it suffices to focus on the cases when  $m \geq \lceil \frac{d-1}{2} \rceil - 1$ . If  $f_0(F_1) \leq f_0(P) - 3$ , then there exist at least three vertices outside  $F$ . By Proposition 4.3.5,

$$\begin{aligned} f_m(P) &\geq f_m(F_1) + \binom{d}{m} + \binom{d-1}{m} + \binom{d-2}{m} + \binom{d-2}{m-1} \\ &\geq \left[ \binom{d}{m+1} + \binom{d-1}{m+1} + \binom{d-2}{m+1} - \binom{\lceil \frac{d-1}{2} \rceil - 1}{m} \right] + \binom{d}{m} + 2 \binom{d-1}{m} \\ &= \binom{d+1}{m+1} + \binom{d}{m+1} + \binom{d-1}{m} + \underbrace{\left[ \binom{d-2}{m+1} - \binom{\lceil \frac{d-1}{2} \rceil - 1}{m} \right]}_{\geq 0 \text{ since } m \geq \lceil \frac{d-1}{2} \rceil - 1} \\ &\geq f_m(\nabla). \end{aligned}$$

Assume now that  $f_0(F_1) = f_0(P) - 2$ , and let  $v$  and  $w$  be the only two vertices of  $P$  outside of  $F_1$ . If  $v$  has at least  $d + 2$  neighbors, then the vertex figure  $P/v$  has at least  $d + 2$  vertices. Then we use similar arguments and computations in Proposition 4.3.5 to obtain:

$$\#\{m\text{-faces of } P \text{ containing } v\} \geq \binom{d}{m} + \binom{d-1}{m} - \binom{d-3}{m}.$$

Therefore

$$\begin{aligned} f_m(P) &\geq f_m(F_1) + \binom{d}{m} + 2\binom{d-1}{m} - \binom{d-3}{m} \\ &\geq \left[ \binom{d}{m+1} + \binom{d-1}{m+1} + \binom{d-2}{m+1} - \binom{\lceil \frac{d-1}{2} \rceil - 1}{m} \right] + \binom{d}{m} + 2\binom{d-1}{m} - \binom{d-3}{m} \\ &= \binom{d+1}{m+1} + \binom{d}{m+1} + \binom{d-1}{m} + \left[ \binom{d-2}{m+1} - \binom{d-3}{m} - \binom{\lceil \frac{d-1}{2} \rceil - 1}{m} \right] \\ &= \binom{d+1}{m+1} + \binom{d}{m+1} + \binom{d-1}{m} + \underbrace{\left[ \binom{d-3}{m+1} - \binom{\lceil \frac{d-1}{2} \rceil - 1}{m} \right]}_{\geq 0 \text{ since } m \geq \lceil \frac{d-1}{2} \rceil - 1} \\ &\geq f_m(\nabla). \end{aligned}$$

Otherwise,  $v$  and  $w$  each have at most  $d + 1$  neighbors. We will now show that this is impossible. First notice that  $\{v, w\}$  must be an edge. Since  $f_0(F_1) = f_0(P) - 2 \geq 2d - 1$ , there exist at least  $2d - 1$  edges connecting vertices of  $F_1$  to  $v$  and  $w$ . If at least  $d + 1$  of these edges go to either  $v$  or  $w$ , then since  $\{v, w\}$  is also an edge, one of  $v, w$  has at least  $d + 2$  neighbors. This takes us back to the previous case. Therefore there has to be at most  $2d$  edges going out of  $F_1$  (hence  $f_0(F_1) = 2d - 1$  or  $2d$ ). Since  $v$  has degree  $d + 1$ ,  $d$  of the vertices of  $F_1$  will have edges connecting to  $v$ , and the other vertices of  $F$  must be connected to  $w$ . Therefore there exists at most one vertex of  $F_1$  that is connected to both  $v$  and  $w$ . On the other hand, since every other facet of  $P$  must intersect with  $F_1$ , but  $f_{d-2}(F_1) = d + 1$  while  $f_{d-1}(P) \geq d + 3$ , there has to be at least one facet  $F_2$  that does not intersect with  $F_1$  at a ridge. The only possibility is that  $F_2 \cap F_1 = G$  is a  $(d - 3)$ -dimensional, and  $F_2 = G * v * w$ . This means that every vertex

of  $G$  has an edge to both  $v$  and  $w$ . By our assumptions it must be  $f_0(G) = 1$ , and so  $d = 3$ ,  $f_0(P) = 7$  and  $F_1$  is a pentagon. Recall that  $f(\nabla) = (7, 11, 6)$  when  $d = 3$ . Any 3-polytope with 7 vertices that has a pentagon ( $F_1$ ) and a triangle ( $F_2$ ) that intersect at a vertex will need at least 12 edges and seven 2-faces. Therefore  $P > \nabla$ .

We are now done with Case 3. From now on we can assume that **for each nonsimple vertex  $v \in P$ , each facet that does not contain  $v$  has size at most  $2d - 2$ .**

Case 4: There is a facet  $F_1$  such that  $v \notin F_1$  and  $d + 1 \leq f_0(F_1) = (d - 1) + s \leq 2d - 2$ , i.e.,  $2 \leq s \leq d - 1$ . First label the vertices of  $P$  that are not in  $F_1$  as  $v = x_1, x_2, \dots, x_{d-s+2}$ . By Proposition 4.3.5,

$$\begin{aligned}
f_m(P) &= f_m(F) + \#\{m\text{-faces of } P \text{ that contain some } x_i\} \\
&= \left[ \binom{d}{m+1} + \binom{d-1}{m+1} - \binom{d-s}{m+1} \right] \\
&\quad + \left[ \sum_{i=1}^{d-s+2} \binom{d+1-i}{m} + \binom{d-2}{m-1} \right] \quad (\text{by [37, Thm. 3.2].}) \\
&= \binom{d+1}{m+1} + \binom{d}{m+1} + \binom{d-1}{m} + \sum_{i=4}^{d-s+2} \binom{d+1-i}{m} - \binom{d-s}{m+1} \\
&= f_m(\nabla) + \sum_{i=4}^{d-s+2} \binom{d+1-i}{m} - \sum_{i=m}^{d-s-1} \binom{i}{m} \\
&= f_m(\nabla) + \sum_{i=s-1}^{d-3} \binom{i}{m} - \sum_{i=m}^{d-s-1} \binom{i}{m} \tag{4.5.5} \\
&\geq f_m(\nabla) + \sum_{i=(s-1)-(s-2)}^{(d-3)-(s-2)} \binom{i}{m} - \sum_{i=m}^{d-s-1} \binom{i}{m} \\
&= f_m(\nabla) + \underbrace{\sum_{i=1}^{d-s-1} \binom{i}{m} - \sum_{i=m}^{d-s-1} \binom{i}{m}}_{=0 \text{ (since } m \geq 1)} \\
&= f_m(\nabla).
\end{aligned}$$

Case 5: Every facet that does not contain a nonsimple vertex has  $d$  vertices.

Let  $F_1$  be a facet that does not contain a nonsimple vertex  $v$ , and let  $\mathcal{S}_1 = \{x_0 = v, x_1, x_2, \dots\}$  be the set of vertices outside of  $F_1$ . By our assumption,  $|\mathcal{S}_1| \geq d + 1$ . If there exists some other facet  $F_2$  such that  $0 < |F_2 \cap \mathcal{S}_1| \leq |\mathcal{S}_1| - 2$  (in other words,  $F_2$  contains some  $x_i$ 's but also excludes at least two vertices from  $\mathcal{S}_1$ .) Then once again by Proposition 4.3.5(iii) we obtain  $f_m(P) \geq f(\nabla)$ .

Assuming such a facet  $F_2$  does not exist, the following must be true:

- (1).  $|\mathcal{S}_1| = d + 1$ ;
- (2).  $f_0(P) = 2d + 1$ ;
- (3). there is a unique facet  $F_2$  that contains some  $x_i$  but not  $v$  and  $F_2 = \{x_1, \dots, x_d\}$ .

Let  $\mathcal{S}_2$  be the set of vertices not in  $F_2$  (notice that  $\mathcal{S}_2 = V(F_1) \cup v$  and so  $|\mathcal{S}_2| = d + 1$ ). Besides  $F_1$  and  $F_2$ , every other facet of  $P$  must contain the vertex  $v$ , and so every other facet contains some vertices from  $\mathcal{S}_1$  and  $\mathcal{S}_2$ . Let  $F_3$  be one of them. Then  $F_3$  needs to contain at least  $d$  vertices (hence exactly  $d$  vertices) from each of  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , so  $|F_3| = 2d - 1$  and there are exactly two vertices of  $P$  outside of  $F_3$ . This subcase is checked previously in Case 3.

This completes the proof. □

In summary, we now have the full picture of the lower bounds on face numbers of normal CW  $(d - 1)$ -pseudomanifolds with at least  $2d + 1$  vertices. The picture is split into two parts by restrictions on the number of facets:

- (i). If the number of facets is  $d + 2$ , by Lemma 4.4.3, the face vector of the polytope,  $(T_2^{d, \frac{d}{2}-2})^*$ , is componentwise minimal (when it exists). We know this by completely understanding all possible polytopes (hence all pseudomanifolds by Theorem 4.3.10) with  $d + 2$  facets and the fact that all polytopes in this set are completely ordered by  $f$ -numbers (by Theorem 4.4.1).

- (ii). If the number of facets is greater than  $d + 2$ , then by Theorem 4.5.1, the face vector of a polytope with  $d + 3$  facets,  $\nabla = (\text{Stack}(T_1^{d,d-2}))^*$ , is componentwise minimal.

While the results (i) and (ii) are stated and proved for normal pseudomanifolds with **at least**  $2d + 1$  vertices, they are sharp only for normal pseudomanifolds with **exactly**  $2d + 1$  vertices. The reason we need the “at least”  $2d + 1$  part in the statements is to enable us to apply induction. The problem of obtaining sharp lower bounds on the face vectors of  $d$ -polytopes with  $2d + 2$  or more vertices is wide open at present: there is not even a plausible conjecture. For polytopes with exactly  $2d + 2$  vertices, the only result known is the minimal number of edges proved by Pineda-Villavicencio, Ugon, and Yost in 2020 (see [23]).

#### 4.6 Concluding remarks

The most natural open problem is to find the lower bounds for the face numbers of polytopes, strongly regular CW spheres, or even strongly regular normal pseudomanifolds with  $2d + 2$  and more vertices. In view of our results for  $2d + 1$  vertices, one can conjecture that the bounds in the case of  $2d + 2$  vertices are also given by the face numbers of certain polytopes. An approach that is similar to the proof of Theorem 4.5.1 might be feasible but would probably be lengthy and contain quite a few cases.

Another question is the following. By proving Grünbaum’s conjecture in [37] we showed the existence of the componentwise minimal  $f$ -factors of  $d$ -polytopes with at most  $2d$  vertices. We now know that such  $f$ -vector does not exist (in general) when there are  $2d + 1$  vertices since the lower bound has two parts. Notice that if in addition we also restrict the number of facets, then the componentwise minimal  $f$ -factors still exist. Whether restrictions on the number of facets helps reducing the “complexity” of the lower bounds for  $f$ -vectors would be an interesting problem. This might be more plausible when the number of vertices is relatively small (up to  $3d - 2$ ).

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## Appendix A

In this section we will show that the following purely combinatorial inequality (see Proposition A.0.1) holds. Let  $d \geq 3$  and  $1 \leq m \leq \lceil \frac{d}{2} \rceil - 2$ , define the following two functions  $A(m, d)$  and  $B(m, d)$ .

$$A(m, d) = \binom{d+1}{m+1} + \binom{d}{m+1} + \binom{d-1}{m+1} - \binom{\lceil \frac{d}{2} \rceil}{m+1} - \binom{\lceil \frac{d}{2} \rceil - 1}{m+1}, \text{ as in Lemma 4.4.3.} \quad (\text{A.0.1})$$

$$B(m, d) = \binom{d+1}{m+1} + \binom{d}{m+1} + \binom{d-1}{m}, \text{ as in Theorem 4.5.1.} \quad (\text{A.0.2})$$

We will show that  $B(m, d)$  is smaller than  $A(m, d)$  roughly for the first half of values of  $m$ .

**Proposition A.0.1.** *When  $1 \leq m < \lceil \frac{d}{2} \rceil - 1$  and  $d \geq 6$ ,  $B(m, d) \leq A(m, d)$ .*

*Proof.* We will prove the statement by induction on  $d$ . First let

$$\delta(m, d) = A(m, d) - B(m, d) = \binom{d-1}{m+1} - \binom{d-1}{m} - \binom{\lceil \frac{d}{2} \rceil}{m+1} - \binom{\lceil \frac{d}{2} \rceil - 1}{m+1}. \quad (\text{A.0.3})$$

When  $d = 6$  and  $m = 1$ ,  $\delta(m, d) = 1 \geq 0$ .

Now let  $d \geq 7$ , and assume that  $\delta(m, d-1) \geq 0$  for all  $m < \lceil \frac{d-1}{2} \rceil - 1$ . Consider  $m < \lceil \frac{d}{2} \rceil - 1$ . By Pascal's identity,  $\delta(m, d) = \delta(m, d-1) + \delta(m-1, d-1)$ . If both  $m-1, m < \lceil \frac{d-1}{2} \rceil - 1$ , then by the inductive hypothesis  $\delta(m, d) \geq 0$ . The only case that is not covered in the inductive hypothesis is the following.

$$\lceil \frac{d-1}{2} \rceil - 1 = m < \lceil \frac{d}{2} \rceil - 1. \quad (\text{A.0.4})$$

It occurs only when  $d$  is odd (write it as  $d = 2a + 1$ ) and  $m = \lceil \frac{2a}{2} \rceil - 1 = a - 1$ . Note that  $a \geq 4$  since  $d \geq 7$ . In this case,

$$\begin{aligned} \delta(a-1, 2a+1) &= \binom{2a}{a} - \binom{2a}{a-1} - \binom{a+1}{a} - \binom{a}{a} \\ &\stackrel{(*)}{=} \frac{1}{2a+1} \binom{2a+1}{a} - (a+2) \\ &= \left[ \frac{(2a)(2a-1)\dots(a+3)}{a!} - 1 \right] \cdot (a+2). \end{aligned} \quad (\text{A.0.5})$$

where  $(*)$  is by the binomial identity  $\binom{n}{k} - \binom{n}{k-1} = \frac{n+1-2k}{n+1} \binom{n+1}{k}$ .

Since  $a \geq 4$ ,

$$\frac{(2a)(2a-1)\dots(a+3)}{a!} = \frac{2a}{a} \cdot \frac{2a-1}{a-1} \cdot \dots \cdot \frac{a+3}{3} \cdot \frac{1}{2} \quad (\text{A.0.6})$$

$$= \frac{2a-1}{a-1} \cdot \dots \cdot \frac{a+3}{3} \quad (\text{A.0.7})$$

$$\geq 1 \quad \text{since every term is at least 1.} \quad (\text{A.0.8})$$

Plugging this into (A.0.5), we obtain that  $\delta(a-1, 2a+1) \geq 0$ , which completes the proof.

□

## VITA

**Ph.D. student, Mathematics**, University of Washington, *2016 - Summer 2022*

**B.S., Pure Mathematics**: University of California, Los Angeles *2016*

**Undergraduate student, Mathematics**: University of California, Davis *2011 - 2013*