An Extremal Property of the Square Lattice

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Abstract

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ABSTRACT. Motivated by a 2019 result of Faulhuber-Steinerberger [7] on the hexagonal lattice Λ, we demonstrate that the square lattice $\mathbb{Z}^2$ exhibits the same local extremal property as Λ, where distances of lattice points from the barycenters of natural fundamental domains increase under perturbation. These two lattices are very special lattices in $\mathbb{R}^2$, as they have nontrivial symmetries. Precisely, we show the following: let $p = (1/2, 1/2)$ denote the center of the standard square fundamental domain $[0,1]^2$ for $\mathbb{Z}^2$ acting on $\mathbb{R}^2$, and let A_r denote the set of lattice points that are at distance exactly r from p. If Δ is a small perturbation of $\mathbb{Z}^2$ in the space of unimodular lattices, consider C_r, the set of points in A_r shifted to Δ. Then,

$$\sum_{\delta \in C_r} \|p - \delta\| - \sum_{\lambda \in A_r} \|p - z\| \geq r\,|A_r|\,d(\Delta, \mathbb{Z}^2)^2,$$

where $d(\Delta, \mathbb{Z}^2)$ denotes the distance between the lattices, measured by, for example, the distances between basis vectors of Δ and those of $\mathbb{Z}^2$.

As mentioned above, this says that the distances of lattice points from the barycenter of the fundamental domain *strictly* increase under perturbation, and we give an explicit bound for the minimum increase. Further, we conjecture that many higher-dimensional symmetric lattices will exhibit similar extremal properties.

1. Introduction

In this section we introduce the space of lattices and state our problem. Lattices in Euclidean spaces are ubiquitous in many fields of math, for example, group theory, cryptography [6], and the study of Lie groups and Lie algebras [1]. A *lattice* Γ in Euclidean space $\mathbb{R}^n$ is a discrete, additive subgroup of finite covolume. Every lattice Γ can be expressed as a set of *integer* linear combinations of a basis $\{v_1, v_2, \dots, v_n\}$ of $\mathbb{R}^n$. Symbolically we can express Γ as

$$\Gamma = \left\{ \sum_{i=1}^{n} m_i v_i : m_i \in \mathbb{Z} \right\}.$$

The convex hull of the v_i is a *fundamental domain* for Γ acting on $\mathbb{R}^n$.

Using the column vectors v_i we can form a matrix g; we can use this to express Γ as $\Gamma = g\mathbb{Z}^n$. Restricting to the case where $g \in SL(n,\mathbb{R})$ is the same as considering lattices of unit covolume. The *space of unimodular lattices*, denoted $L(\mathbb{R}^n)$, is then given by the quotient $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$, where the coset $gSL(n,\mathbb{Z})$ is identified with the lattice $g\mathbb{Z}^n$. Note that this assignment is well-defined, since for any $h \in SL(n,\mathbb{Z})$, $h\mathbb{Z}^n = \mathbb{Z}^n$. In this paper, we are principally concerned with lattices in $n = 2$ dimensions.

Notation. Here is a brief summary of the notation used throughout this work. First, we remark that though we are always working with column vectors, for ease of writing, we will write them as row vectors.

- $SL(n,\mathbb{R})$ is the group of $n \times n$ real matrices with determinant 1; $SL(n,\mathbb{Z})$ has integer entries.
- $L(\mathbb{R}^n) = SL(n,\mathbb{R})/SL(n,\mathbb{Z})$ is the space of unimodular lattices in $\mathbb{R}^n$.
- Γ will refer to an arbitrary point $\Gamma = \mathbb{Z}a + \mathbb{Z}b$ in $L(\mathbb{R}^2)$ where a, b is a basis for $\mathbb{R}^2$ with $\det(a,b) = 1$.
- If we take the standard basis vectors $v = (1,0)$ and $w = (0,1)$, then we write $\mathbb{Z}^2 = \mathbb{Z}v + \mathbb{Z}w$.
- Δ is a unimodular lattice given by a small perturbation of the square lattice $\mathbb{Z}^2$, that is,
$$\Delta = \mathbb{Z}v' + \mathbb{Z}w'$$
where $|w - w'|$ and $|v - v'|$ are small.
- Λ is the unit covolume *hexagonal* lattice,
$$\Lambda = \mathbb{Z}\frac{\sqrt{2}}{3^{1/4}}(1,0) + \mathbb{Z}\frac{1}{3^{1/4}\sqrt{2}}(1,\sqrt{3}).$$
- Given a lattice $\Gamma = (\mathbb{Z}a + \mathbb{Z}b) \in L(\mathbb{R}^2)$ and a point $q \in \mathbb{R}^2$, we define
$$A_r(\Gamma, q) = \{ma + nb : m, n \in \mathbb{Z}, |ma + nb - q| = r\}$$
to be the set of lattice points exactly distance r from q. We will denote it just as A_r when the lattice is understood.
- For $\Gamma' = (\mathbb{Z}a' + \mathbb{Z}b')$ a fixed small perturbation of Γ, we define $C_r(\Gamma, p)$ (with shorthand C_r) to be the set of perturbations of lattice points in Γ which are distance r from p; that is
$$C_r = \{ma' + nb' : m, n \in \mathbb{Z}, |ma + nb - p| = r\}.$$

1.1. **Distances from deep holes.**

1.1.1. *Deep holes.* We now fix our problem: Let $p = (1/2, 1/2)$ denote the center of the standard square fundamental domain $[0,1]^2$ of $\mathbb{Z}^2$; in the terminology of [7], this center is called a *deep hole* in the lattice. Let $A_r(\mathbb{Z}, p) = \{mv + nw \in \mathbb{Z}^2 : |mv + nw - p| = r\}$ be the set of integer lattice points of distance r from p. Let $\Delta = (v'w')Z^2 = \mathbb{Z}v' + \mathbb{Z}w'$ represent a small perturbation of $\mathbb{Z}^2$ in the space of unimodular lattices. Then, $\det(v'w') = 1$, and $|v - v'|$ and $|w - w'|$ are small. Next, we define $C_r(\mathbb{Z}^2, p) = \{mv' + nw' : |mv + nw - p| = r\}$ to be the set of perturbations of lattice points which were originally at distance r from p in $\mathbb{Z}^2$. Note that it is equivalent to express C_r in terms of A_r: $C_r = \{mv' + nw' : mv + nw \in A_r\}$. We want to compare the distances of the lattice points C_r in the perturbed lattice Δ from the deep hole p to the distances of the points A_r in the original lattice. Symbolically, we want to compare the quantities

$$\sum_{\delta \in C_r} |p - \delta| \text{ and } \sum_{z \in A_r} |p - z|.$$

1.1.2. *The main result.*

Theorem 1.
If Δ is sufficiently close to $\mathbb{Z}^2$ with respect to the Euclidean metric, then for a fixed deep hole p

$$\sum_{\delta\in C_r} |p-\delta| - \sum_{z\in A_r} |p-z| \gtrsim r\,|A_r|\, d(\mathbb{Z}^2,\Delta)^2.$$

If $\phi : \mathbb{R}_+ \to \mathbb{R}$ is any monotonically increasing, convex function, then

$$\sum_{\mu\in C_r} \phi(\|p-\mu\|) - \sum_{\lambda\in A_r} \phi(\|p-\lambda\|) \gtrsim r\,\phi'(r)\,|A_r|\,d(\Lambda,\Gamma)^2.$$

The distance function $d(\mathbb{Z}^2,\Delta)$ is given by $\sqrt{|v-v'|^2+|w-w'|^2}$. Taking $d(\mathbb{Z}^2,\Delta) = \|(v\ w)-(v'\ w')\|$, where $\|\cdot\|$ is any other norm on $\mathbb{R}^4$, would yield an equivalent result, up to constants. Our proof of this result relies on explicit computations of derivatives.

2. Symmetries

Following [7], we first show that points in lattice $\mathbb{Z}^2$ at a fixed distance r from the deep hole $p = (1/2, 1/2)$ are invariant under rotation by R and occur naturally in quadruples. In the following lemma, let $\mathbb{Z}^2 \in \mathbb{R}^2$ have basis

$$v = (1,0) \text{ and } w = (0,1)\,.$$

By $p = \frac{v+w}{2} = (1/2, 1/2)$ we denote the center point of the unit square, and by $R = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ the rotation matrix by $\frac{\pi}{2}$.

Lemma 1. *For any $q \in \mathbb{R}^2$ such that $p+q \in \mathbb{Z}^2$, the images $R^i q$ are in the set $\{p+q, p+Rq,\ p+R^2q\ p+R^3q\} \subset \mathbb{Z}^2$.*

Proof. We explicitly calculate the quantities $p+Rq$, $p+R^2q$, $p+R^3q$, and argue that they are contained in $\mathbb{Z}^2$. If $p+q \in \mathbb{Z}^2$, then

$$p+q = kv + lw \text{ for some } k, l \in \mathbb{Z}$$

and therefore we can write

$$q = \left(k-\frac{1}{2}\right)v + \left(l-\frac{1}{2}\right)w.$$

Using that $Rv = w$ and $Rw = -v$, we have that

$$\begin{aligned}
p+Rq &= p + R\left(\left(k-\frac{1}{2}\right)v + \left(l-\frac{1}{2}\right)w\right)\\
&= p + \left(k-\frac{1}{2}\right)Rv + \left(l-\frac{1}{2}\right)Rw\\
&= \frac{v+w}{2} + \left(k-\frac{1}{2}\right)w + \left(l-\frac{1}{2}\right)(-v)\\
&= \frac{1}{2}v + \frac{1}{2}w + kw\\
&= \frac{1}{2}w - lw + \frac{1}{2}v\\
&= (1-l)\,v + kw \in \mathbb{Z}^2
\end{aligned}$$

Similarly,

$$p + R^2q = (1-k)\,v + (1-l)\,w \in \mathbb{Z}^2$$

and

$$p + R^3q = lv + (1-k)\,w \in \mathbb{Z}^2.$$

□

3. Derivatives and Hessians

We are now ready to prove our main theorem, following the strategy of [7], using explicit computation of derivatives. To prove Theorem 1, we need to show that for any lattice Δ sufficiently close $\mathbb{Z}^2$,

$$\sum_{\delta\in C_r} \|p-\delta\| - \sum_{z\in A_r} \|p-z\| \gtrsim r\,|A_r|\,d(\mathbb{Z}^2,\Delta)^2;$$

we note that "sufficiently close" is with respect to the Euclidean metric on the space of lattices. We first prove the case of squared distances, and proceed to the linear case. Considered together, these two cases yield our result.

3.1. **Squared Distances.** We now prove Theorem 1. First, note that since the problem we are considering is rotationally invariant, any covolume one unimodular lattice in $\mathbb{R}^2$ can be rotated to have a horizontal basis; we fix this as a convention. Since Γ has covolume 1, we can then write its basis in the form

$$v_1(x,y) = y^{\frac{-1}{2}}(1,0) \qquad \text{and} \qquad w_1(x,y) = y^{\frac{-1}{2}}(x,y).$$

Note that $\mathbb{Z}^2$ corresponds to a parameter choice of $x=0, y=1$, so $\mathbb{Z}^2$ is generated by $v=(1,0)$, $w=(0,1)$. We take $p=\left(\frac{1}{2},\frac{1}{2}\right)$ to be the center of the standard fundamental domain $[0,1]^2$. An arbitrary lattice point $z\in\mathbb{Z}^2$ is given by the expression $z=kv+lw$ for $k,l\in\mathbb{Z}$. It naturally has three distinct associated points by a rotation of $\frac{\pi}{2}$ around p. These associated points have the following expression:

$$z' = p + Rq = p + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}(z-p),$$

$$z'' = p + \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}(z-p) \text{ and}$$

$$z''' = p + \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}(z-p).$$

Now, take $z\in\mathbb{Z}^2$ and it's associated quadruple $\{z,z',z'',z'''\}$. We will investigate this quadruple under perturbation. Let Δ be a perturbation of $\mathbb{Z}^2$ and $(\delta,\delta',\delta'',\delta''')$ be the perturbation in Δ of our quadruple (z,z',z'',z''') in $\mathbb{Z}^2$. The previous lemma implies that each of these perturbed lattice points has the following form:

$$\delta = kv_1 + lw_1,$$

$$\delta' = (1-l)\,v_1 + kw_1,$$

$$\delta'' = (1-k)\,v_1 + (1-l)\,w_1 \text{and}$$

$$\delta''' = lv_1 + (1-k)\,w_1.$$

3.1.1. *Defining f.* We will show that, in total, the squared distance of a perturbed quadruple to our fixed p strictly increases. That is, if Δ has parameters x and y as discussed above, we want to understand the behavior of the function $f(x,y)$ given by

$$f(x,y) = \|\delta-p\|^2 + \|\delta'-p\|^2 + \|\delta''-p\|^2 + \|\delta'''-p\|^2.$$

There are many ways to express and simplify f. We use the form

$$f(x,y) = \left(\left(\frac{-1}{2}+\frac{k}{\sqrt{y}}+\frac{lx}{\sqrt{y}}\right)^2 + \left(\frac{1}{2}+(-1+k)\sqrt{y}\right)^2 + \left(\frac{-1}{2}+k\sqrt{y}\right)^2\right)4y + \left(-2+2k+2(-1+l)\,x+\sqrt{y}\right)^2$$
$$+\left(\frac{1}{2}+(-1+l)\sqrt{y}\right)^2 + \left(\frac{-1}{2}+l\sqrt{y}\right)^2 + \left(-2l+2(-1+k)\,x+\sqrt{y}\right)^2 + \left(-2+2l-2kx+\sqrt{y}\right)^2.$$

3.1.2. *Partial derivatives of f.* We compute partial derivatives of the squared distance function f to show that $\mathbb{Z}^2$ is a critical point in $L(R^2)$. We have

$$\partial_x f =$$

$$\frac{(k-1)\left(-2l+2(k-1)x+\sqrt{y}\right)}{y}+\frac{(l-1)\left(-2+2k+2(l-1)x+\sqrt{y}\right)}{y}$$
$$-\frac{k\left(-2+2l-2kx+\sqrt{y}\right)}{y}+\frac{2l\left(-\frac{1}{2}+\frac{k}{\sqrt{y}}+\frac{lx}{\sqrt{y}}\right)}{\sqrt{y}}$$

and

$$\partial_y f =$$

$$2\left(-\frac{k}{2y^{3/2}}-\frac{lx}{2y^{3/2}}\right)\left(-\frac{1}{2}+\frac{k}{\sqrt{y}}+\frac{lx}{\sqrt{y}}\right)-\frac{\left(-2l+2(-1+k)x+\sqrt{y}\right)^2}{4y^2}-\frac{\left(-2+2l-2kx+\sqrt{y}\right)^2}{4y^2}$$
$$-\frac{\left(-2+2k+2(-1+l)x+\sqrt{y}\right)^2}{4y^2}+\frac{-2l+2(-1+k)x+\sqrt{y}}{4y^{3/2}}+\frac{-2+2l-2kx+\sqrt{y}}{4y^{3/2}}+\frac{-2+2k+2(-1+l)x+\sqrt{y}}{4y^{3/2}}$$
$$+\frac{(-1+k)\left(\frac{1}{2}+(-1+k)\sqrt{y}\right)}{\sqrt{y}}+\frac{k\left(-\frac{1}{2}+k\sqrt{y}\right)}{\sqrt{y}}+\frac{(-1+l)\left(\frac{1}{2}+(-1+l)\sqrt{y}\right)}{\sqrt{y}}+\frac{l\left(-\frac{1}{2}+l\sqrt{y}\right)}{\sqrt{y}}.$$

To show that $\mathbb{Z}^2$ is a critical point of $L(\mathbb{R}^2)$, we evaluate $\partial_x f, \partial_y f$ at $x=0, y=1$ and show that $\partial_x f|_{x=0,y=1} = \partial_y f|_{x=0,y=1} = 0$, independent of the values of k and l. We evaluate $\partial_x f$ at p, so $x=0, y=1$; then

$$\begin{aligned}
&\partial_x(0,1) = \\
&\quad (-1+k)(1-2l)+(-1+2k)(-1+l)+2\left(-\left(\frac{1}{2}\right)+k\right)l-k(-1+2l)\\
&\quad = (-1+k+2l-2kl)+(1-2k-l+2kl)+(k-l)\\
&\quad = 0
\end{aligned}$$

We proceed in the same way to examine $\partial_y f(p)$.

$$\begin{aligned}
&\partial_y(0,1) = \\
&\quad ((-1+k)\left(-\frac{1}{2}+k\right)+\frac{1}{4}(-1+2k)-\frac{1}{4}(-1+2k)^2+\frac{1}{4}(1-2l)-\frac{1}{4}(1-2l)^2\\
&\quad +(-1+l)\left(-\frac{1}{2}+l\right)+-\frac{l}{2}-l^2+\frac{1}{4}(-1+2l)-\frac{1}{4}(-1+2l)^2)\\
&\quad =\left(\frac{1}{2}-\frac{3k}{2}+k^2\right)+\left(-\frac{1}{2}+\frac{3k}{2}-k^2\right)+\left(\frac{l}{2}-l^2\right)+\left(\frac{1}{2}-\frac{3l}{2}+l^2\right)+\left(-\frac{1}{2}+l\right)\\
&\quad = 0
\end{aligned}$$

We have now shown $\mathbb{Z}^2$ is a critical point with respect to the squared distance metric f on the space of lattices.

3.1.3. *The Hessian of f.* To understand the nature of this critical point, we compute the Hessian for f at $x=0, y=1$. The Hessian has the generic form

$$H(k,l) = \begin{bmatrix} \partial_{xx} & \partial_{xy} \\ \partial_{yx} & \partial_{yy} \end{bmatrix}.$$

In our case, we have

$$H(k,l) = D^2 f|_{x=0,y=1} = \begin{bmatrix} h_1(k,l) & -1 \\ -1 & h_3(k,l) \end{bmatrix}$$

where $h_1(k,l) = 4\left(1-k+k^2-l+l^2\right)$ and $h_3(k,l) = 3-4k+4k^2-4l+4l^2$. It is important to note here that this is a key difference between $\mathbb{Z}^2$ and Δ, the triangular lattice in [7]. There the Hessian was a symmetric matrix for Δ. Another difference is that for Δ, $H_1 = h_3$; here, $h_3 = h_1 - 1$. The characteristic polynomial of this matrix is

$$P(\gamma) = -1 + (3-4k+4k^2-4l+4l^2-\gamma)(4(1-k+k^2-l+l^2)-\gamma)$$

with roots

$$\begin{aligned} \gamma &= \frac{1}{2}\left(7 \pm \sqrt{5} - 8k + 8k^2 - 8l + 8l^2\right) \\ &= \frac{1}{2}(7 \pm \sqrt{5} + 2h_1(k,l) - 8) \\ &= \frac{1}{2}(-1 \pm \sqrt{5} + 2h_1(k,l)). \end{aligned}$$

Let $\lambda_{min}(k,l)$ denote the behavior of smaller of the two eigenvaules, here $h_1 + \frac{-1-\sqrt{5}}{2}$; we want to minimize this with respect to k and l. For all values of $(k,l) \in \mathbb{Z}^2$, $h_1 \geq 4$ with equality achieved at $(k,l) \in \{(0,0),(0,1),(1,0),(1,1)\}$; so, our minimum is $\lambda_{min} = 4 - \frac{-1-\sqrt{5}}{2}$. We note that the same conditions for k,l hold for the larger root, so λ_{max} is $h_1(k,l) + \frac{\sqrt{5}-1}{2}$. Then, our growth is bounded away from 0. Note that for nonzero radii contained in the closure of the fundamental domain, $r = \frac{1}{\sqrt{2}}$. Since $\lambda_{min} = 4 - \frac{-1-\sqrt{5}}{2}$, we have that $\lambda_{min} > \frac{1}{2} = r^2 = k^2 + l^2$.

Lastly, we consider the asymptotic behavior of λ_{min}. We note that both h_1 and h_3 are positive definite quadratic forms, and that the off-diagonal terms are fixed at -1. Thus, $\lambda_{min} \geq \frac{1}{2}(9 - \sqrt{2} + \sqrt{5})$ We now have an explicit lower bound for growth under perturbation: $(4 - \frac{-1-\sqrt{5}}{2}) - \sqrt{\frac{1}{2}} =\simeq 4.9109\ldots$.
This implies the result in the case of squared distances:

$$\sum_{\delta \in C_r} |p-\delta|^2 - \sum_{z \in A_r} |p-z|^2 \gtrsim r^2 |A_r| d(\mathbb{Z}^2, \Delta)^2.$$

□

3.2. **Proof of Theorem 1.** We are now finally ready to prove the main theorem. This proof of the linear case follows the one preceding for squared distances, but in this case things look a little more complicated with the square root being taken over each summed term of $f(x,y)$. We begin with

$$g(x,y) = \|\gamma - p\| + \|\gamma' - p\| + \|\gamma'' - p\| + \|\gamma''' - p\|.$$

By substituting in λ, λ', λ'', λ''' in $\mathbb{Z}^2$ for γ, γ', γ'', γ''', and the value of p, we re-express g as the following:

$$\begin{aligned} g(x,y) &= \left|\left(\frac{2k+2lx-\sqrt{y}}{2\sqrt{y}}, \frac{2l\sqrt{y}-1}{2}\right)\right| \\ &+ \left|\left(\frac{2(1-l)+2kx-\sqrt{y}}{2\sqrt{y}}, \frac{2k\sqrt{y}-1}{2}\right)\right| \\ &+ \left|\left(\frac{2(1-k)+2x(1-l)-\sqrt{y}}{2\sqrt{y}}, \frac{2(1-l)\sqrt{y}}{2}\right)\right| \\ &+ \left|\left(\frac{2l+2(1-k)x-\sqrt{y}}{2\sqrt{y}}, \frac{2(1-k)\sqrt{y}-1}{2}\right)\right|. \end{aligned}$$

The next lines are the result of expanding the norm on each term.

$$
\begin{aligned}
g(x,y) = &\left(\sqrt{\left(-\frac{1}{2}+\frac{k}{\sqrt{y}}+\frac{lx}{\sqrt{y}}\right)^2+\left(-\frac{1}{2}+l\sqrt{y}\right)^2}\right)\\
&+\left(\sqrt{\left(-\frac{1}{2}+k\sqrt{y}\right)^2+\frac{\left(-2+2l-2kx+\sqrt{y}\right)^2}{4y}}\right)\\
&+\left(\sqrt{\left(\frac{1}{2}+(-1+l)\sqrt{y}\right)^2+\frac{\left(-2+2k+2(-1+l)x+\sqrt{y}\right)^2}{4y}}\right)\\
&+\left(\sqrt{\left(\frac{1}{2}+(-1+k)\sqrt{y}\right)^2+\frac{\left(-2l+2(-1+k)x+\sqrt{y}\right)^2}{4y}}\right).
\end{aligned}
$$

3.2.1. *Partial Derivatives.* The partial $\partial_x g$ comes out to be

$$
\begin{aligned}
&\partial_x g = \\
&\frac{(-1+k)\left(-2l+2(-1+k)x+\sqrt{y}\right)}{2\sqrt{\left(\frac{1}{2}+(-1+k)\sqrt{y}\right)^2+\frac{\left(-2l+2(-1+k)x+\sqrt{y}\right)^2}{4y}}\,y}-\frac{k\left(-2+2l-2kx+\sqrt{y}\right)}{2\sqrt{\left(-\frac{1}{2}+k\sqrt{y}\right)^2+\frac{\left(-2+2l-2kx+\sqrt{y}\right)^2}{4y}}\,y}\\
&+\frac{(-1+l)\left(-2+2k+2(-1+l)x+\sqrt{y}\right)}{2\sqrt{\left(\frac{1}{2}+(-1+l)\sqrt{y}\right)^2+\frac{\left(-2+2k+2(-1+l)x+\sqrt{y}\right)^2}{4y}}\,y}+\frac{l\left(-\frac{1}{2}+\frac{k}{\sqrt{y}}+\frac{lx}{\sqrt{y}}\right)}{\sqrt{\left(-\frac{1}{2}+\frac{k}{\sqrt{y}}+\frac{lx}{\sqrt{y}}\right)^2+\left(-\frac{1}{2}+l\sqrt{y}\right)^2}\sqrt{y}}.
\end{aligned}
$$

Evaluated at $(0,1)$, we see that

$$
\begin{aligned}
&\partial_x g(0,1) = \\
&\frac{(-1+k)(-2l+1)}{2\sqrt{\left(\frac{1}{2}+(-1+k)\right)^2+\frac{(-2l+2(-1+k)0+1)^2}{4}}}-\frac{k(-2+2l+)}{2\sqrt{\left(-\frac{1}{2}+k\right)^2+\frac{(-2+2l+1)^2}{4}}}\\
&+\frac{(-1+l)(-2+2k+1)}{2\sqrt{\left(\frac{1}{2}+(-1+l)\right)^2+\frac{(-2+2k+1)^2}{4}}}+\frac{l\left(-\frac{1}{2}+k\right)}{\sqrt{\left(-\frac{1}{2}+k\right)^2+\left(-\frac{1}{2}+l\right)^2}}\\
&= 0.
\end{aligned}
$$

Next, we compute the partial of g with respect to y, which gives us

$$
\begin{aligned}
\partial_y g = {} & \\
& \frac{-\frac{\left(-2l+2(-1+k)x+\sqrt{y}\right)^2}{4y^2}+\frac{-2l+2(-1+k)x+\sqrt{y}}{4y^{3/2}}+\frac{(-1+k)\left(\frac{1}{2}+(-1+k)\sqrt{y}\right)}{\sqrt{y}}}{2\sqrt{\left(\frac{1}{2}+(-1+k)\sqrt{y}\right)^2+\frac{\left(-2l+2(-1+k)x+\sqrt{y}\right)^2}{4y}}} \\
& + \frac{-\frac{\left(-2+2l-2kx+\sqrt{y}\right)^2}{4y^2}+\frac{-2+2l-2kx+\sqrt{y}}{4y^{3/2}}+\frac{k\left(-\frac{1}{2}+k\sqrt{y}\right)}{\sqrt{y}}}{2\sqrt{\left(-\frac{1}{2}+k\sqrt{y}\right)^2+\frac{\left(-2+2l-2kx+\sqrt{y}\right)^2}{4y}}} + \frac{2\left(-\frac{k}{2y^{3/2}}-\frac{lx}{2y^{3/2}}\right)\left(-\frac{1}{2}+\frac{k}{\sqrt{y}}+\frac{lx}{\sqrt{y}}\right)+\frac{l\left(-\frac{1}{2}+l\sqrt{y}\right)}{\sqrt{y}}}{2\sqrt{\left(-\frac{1}{2}+\frac{k}{\sqrt{y}}+\frac{lx}{\sqrt{y}}\right)^2+\left(-\frac{1}{2}+l\sqrt{y}\right)^2}} \\
& + \frac{-\frac{\left(-2+2k+2(-1+l)x+\sqrt{y}\right)^2}{4y^2}+\frac{-2+2k+2(-1+l)x+\sqrt{y}}{4y^{3/2}}+\frac{(-1+l)\left(\frac{1}{2}+(-1+l)\sqrt{y}\right)}{\sqrt{y}}}{2\sqrt{\left(\frac{1}{2}+(-1+l)\sqrt{y}\right)^2+\frac{\left(-2+2k+2(-1+l)x+\sqrt{y}\right)^2}{4y}}}.
\end{aligned}
$$

Evaluating at $(0,1)$ we again get 0:

$$
\begin{aligned}
\partial_y g(0,1) = {} & \\
& \frac{-\frac{\left(-2l+\sqrt{y}\right)^2}{4}+\frac{-2l+\sqrt{y}}{4}+\frac{(-1+k)\left(\frac{1}{2}+(-1+k)\right)}{1}}{2\sqrt{\left(\frac{1}{2}+(-1+k)\right)^2+\frac{(-2l+1)^2}{4}}} + \frac{-\frac{(-2+2l+1)^2}{4}+\frac{-2+2l+1}{4}+\frac{k\left(-\frac{1}{2}+k\right)}{1}}{2\sqrt{\left(-\frac{1}{2}+k\right)^2+\frac{(-2+2l+1)^2}{4}}} \\
& + \frac{-\frac{(-2+2k+1)^2}{4}+\frac{-2+2k+1}{4}+\frac{(-1+l)\left(\frac{1}{2}+(-1+l)\right)}{\sqrt{y}}}{2\sqrt{\left(\frac{1}{2}+(-1+l)\right)^2+\frac{(-2+2k+1)^2}{4}}} + \frac{2\left(-\frac{k}{2}\right)\left(-\frac{1}{2}+k\right)+\frac{l\left(-\frac{1}{2}+l\right)}{1}}{2\sqrt{\left(-\frac{1}{2}+\frac{k}{1}\right)^2+\left(-\frac{1}{2}+l\right)^2}} \\
& = 0.
\end{aligned}
$$

Therefore $\mathbb{Z}^2$ is a critical point for the function g. We lastly compute the mixed partial $\partial g_{xy}|_{x=0,y=1}$, which evaluates to:

$$
\frac{-1+k^2(10-24l)+6l-14l^2+8l^3+8k^3(-1+2l)-2k\left(1-12l^2+8l^3\right)}{2\sqrt{2}\left(1-2k+2k^2-2l+2l^2\right)^{3/2}}.
$$

3.2.2. *The Hessian.* Our next step will be to form the hessian to establish $\mathbb{Z}^2$ as a local minima or maxima. This matrix A has the form

$$
H(k,l) = D^2 g|_{x=0,y=1} = \begin{bmatrix} h_1(k,l) & h_2(k,l) \\ h_2(k,l) & h_3((k,l)) \end{bmatrix},
$$

where

$$
\begin{aligned}
h_1(k,l) &= \frac{\sqrt{2}(1-3k+7k^2-8k^3+4k^4-3l+7l^2-8l^3+4l^4)}{(1-2k+2k^2-2l+2l^2)^{\frac{3}{2}}} \\
h_2(k,l) &= \frac{-1+k^2(10-24l)+6l-14l^2+8l^3+8k^3(-1+2l)-2k(1-12l^2+8l^3)}{2\sqrt{2}(1-2k+2k^2-2l+2l^2)^{\frac{3}{2}}} \\
\text{and} \quad h_3(k,l) &= \frac{5-16k^3+8k^4-18l+26l^2-16l^3+8l^4-6k(3-8l+8l^2)+k^2(26-48l+48l^2)}{2\sqrt{2}(1-2k+2k^2-2l+2l^2)^{\frac{3}{2}}}.
\end{aligned}
$$

It is again important to note here that this is a key difference between $\mathbb{Z}^2$ and Λ, the triangular lattice in [7]. There the Hessian was a symmetric matrix for Λ. The determinant of this Hessian is

$$\frac{19-192k^5+64k^6-82l+194l^2-304l^3+320l^4-192l^5+64l^6+64k^4(5-3l+3l^2)}{8(1-2k+2k^2-2l+2l^2)^2}$$
$$-\frac{16k^3(21-26l+24l^2)+2k^2(121-264l+336l^2-192l^3+96l^4)-2k(49-144l+216l^2-176l^3+96l^4))}{8(1-2k+2k^2-2l+2l^2)^2}.$$

This expression is always greater than 0 for any values of k and l. The minimum value of this function over the reals is 1.5526 corresponds to $(k,l)=(0.0955547, 0.614894)$. As $k,l\in\mathbb{Z}$, the true minimum is 2.375 or $\frac{19}{8}$, achieved by the three triples $\{(0,0),(0,1),(1,0)\}$. We note that $h_1>0$ for all values of k,l; in fact, the minimum is achieved at $k=1, l=1$. From this we know that $\mathbb{Z}^2$ is a local minimum! With respect to the bound on the growth of distances under perturbation, we give the following computation.

The characteristic polynomial for $H(k,l)$ is $(h_1-z)*(h_3-z)-h_2^2=0$, which expanded has a frankly hilarious form taking 10 printed lines, and so we omit it. The roots of this polynomial are equally bulky when expanded, so we leave them in short form:

$$z=\frac{1}{2}(h_1+h_3\pm\sqrt{h_1^2+4h_2^2-2h_1h_3+h_3^2}).$$

With respect to k and l, these roots are always strictly positive. Their values over the reals are given below.

Root ± Max/Min	Decimal Approx.
Max. (+)	$z=5.46182*10^25$ at $k=1.3024*10^24, l=2.7278*10^25$
Min (+)	$z=1.6231$ at $k=1.1530, l=0.8641$
Max (−)	$z=7.3155*10^33$ at $k=6.5972*10^33, l=3.1615*10^33$
Min (−)	$z=0.618034$ at $k=0.584444, l=0.797255$

The smaller of the two eigenvalues is $z=\frac{1}{2}\left(h_1+h_3-\sqrt{h_1^2+4h_2^2-2h_1h_3+h_3^2}\right)$; it is minimized at $(0,0)=(1,1)$ where $z=(\frac{9-\sqrt{5}}{(2\sqrt{2})}$, or roughly $0.6180\ldots$. Thus, we have a positive bound for the smallest growth in total distance of lattice points from p under perturbation. We note that this eigenvalue is undefined at $(0.5,0.5)$, but this value is inadmissible and so does not affect our computation.

□

Proof of (1). It remains to study the case

$$\sum_{\mu\in C_r}\phi(\|p-\mu\|)-\sum_{\lambda\in A_r}\phi(\|p-\lambda\|).$$

where ϕ is a convex function. For $\lambda\in A_r$ and its corresponding point under perturbation, $\mu\in C_r$, consider the quantity

$$\|p-\mu\|=\|p-\lambda\|+\varepsilon_\mu,\qquad \varepsilon_\mu\in\mathbb{R}.$$

By substitution, we have

$$\|p-\lambda\|+\varepsilon_\mu=r+\varepsilon_\mu.$$

Summing, we see

$$\sum_{\mu\in C_r}(\|p-\mu\|)-\sum_{\lambda\in A_r}(\|p-\lambda\|)=\sum_{\mu\in C_r}(\varepsilon_\mu).$$

Then

$$\sum_{\mu\in C_r}(\|p-\mu\|)=\sum_{\lambda\in A_r}(\|p-\lambda\|)+\sum\varepsilon_\mu=\sum_{\mu\in C_r}(r+\varepsilon_\mu)$$

A Taylor expansion of $\sum_{\mu \in C_r} \phi(r+\varepsilon_\mu)$ around $\varepsilon_\mu = 0$ shows that

$$\sum_{\mu \in C_r} \phi(\|p-\mu\|) - \sum_{\lambda \in A_r} \phi(\|p-\lambda\|) = \sum_{\mu \in C_r} (r+\varepsilon_\mu)$$

$$= \phi(r+\epsilon_\mu) + \phi'(r) \sum_{\mu \in C_r} \varepsilon_\mu + \frac{\phi''(r)}{2} \sum_{\mu \in C_r} \varepsilon_\mu^2 + o(d(\Lambda,\Gamma)^2),$$

where the error term is allowed to depend on r and A_r. Our first result gives us a bound for $r|A_r|d(\Lambda,\Gamma)^2$, so we have

$$\sum_{\mu \in C_r} \varepsilon_\mu \gtrsim r|A_r|d(\Lambda,\Gamma)^2.$$

Thus, we have shown that for a convex function ϕ,

$$\sum_{\mu \in C_r} \phi(\|p-\mu\|) - \sum_{\lambda \in A_r} \phi(\|p-\lambda\|) \gtrsim r\phi'(r)|A_r|d(\Lambda,\Gamma)^2. \tag{1}$$

□

4. Further research

4.1. Aysmptotics. From [7], a function $h : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be admissible if:

(i) h is compactly supported
(ii) h is monotonically decreasing on $(0.5, \infty)$, and
(iii) h satisfies

$$r f''(r) \leq -c f'(r),$$

for some universal $c > 0$ in a neighborhood of $\{\|\lambda - p\| : \lambda \in \Lambda\} \subset \mathbb{R}_+$.

In Theorem 2, Faulhuber and Steinerberger showed that if h is admissible and we suppose that

$$\min_{z \in \mathbb{R}^2} \sum_{\lambda \in \Lambda} h(\|z-\lambda\|) = \sum_{\lambda \in \Lambda} h(\|p-\lambda\|),$$

then the hexagonal lattice Λ is a strict, local maximizer of

$$\max_{\Gamma} \min_{z \in \mathbb{R}^2} \sum_{\gamma \in \Gamma} h(\|z-\gamma\|).$$

Here, we are still requiring that Γ has unit area. Because the hexagonal lattice Λ is a global minimum in $L(\mathbb{R}^2)$, Faulhuber and Steinerberger conjecture that this result is not optimal. In the case of $\mathbb{Z}^2$, a local minimum for $L(\mathbb{R}^2)$ but not a global minimum, we can reasonably expect the result of Theorem 2 to hold; these calculations are underway. The main obstruction here is that admissible functions are a specific class of functions for which the hexagonal lattice Δ locally has the largest minimum and that $\mathbb{Z}^2$ is only a local minimum of $L(\mathbb{R}^2)$.

4.2. Higher dimensions. This result gives promising indications of generalization. In particular, we conjecture that the 3-dimensional analog of the result for some lattices which are optimal for sphere packing will hold. Our main obstacle in this endeavor is the growth of the dimension of the space of lattices. We construct $L(\mathbb{R}^n)$ as $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$, a quotient space with dimension n^2-1. The space $L(\mathbb{R}^2)$ considered above has dimension 3, but if we consider lattices only up to rotation, then the quotient space is 2 dimensional. The dimension of $L(\mathbb{R}^3)$ is 8; even if we quotient by rotations again, the resulting space is 5-dimensional.

4.3. **Connections to Sphere Packings.** Given a lattice Γ, we can associate a *sphere packing* $\mathcal{B}$ by putting spheres of the same radius around each lattice point so that the resulting spheres are mutually tangent. Informally, optimal sphere packings in Euclidean spaces are arrangements of (disjoint) spheres of the same size which cover as much of the space as possible. More precisely, let $B_r(x)$ denote a Euclidean ball of radius r around the point x. We define $B_r(x, \Lambda) := B_r(x) \cap \mathcal{B}$. Then the ratio of the volumes

$$\frac{B_r(x, \Lambda)}{B_r(x)}$$

is called the *density* of the packing. An *optimal* packing maximizes

$$r_{\mathcal{B}} = \lim_{r \to \infty} \frac{B_r(x, \Lambda)}{B_r(x)}.$$

The sphere packings associated to the square and hexagonal lattices in $\mathbb{R}^2$ are critical points for this notion of density, and as we showed above, the lattices themselves are critical points in the space of lattices. It is then natural to ask whether those lattices in $L(\mathbb{R}^n)$ which are associated to optimal sphere packings are also extremal in our sense. Generally, we conjecture that any optimal sphere packing unimodular lattice in $\mathbb{R}^n$ will exhibit our extremal property.

4.4. **Other point sets.** There are other naturally occurring families of point sets in Euclidean spaces, arising from various geometric and dynamical constructions. Examples include sets of *holonomy vectors of saddle connections on translation surfaces* [3], and *cut-and-project quasicrystals* [4]. In both examples there are versions of the question we have considered above about *deep holes* in these point sets; however, as we saw with the growth on dimension of $L(\mathbb{R}^n)$, understanding optimal configurations is a challenging question due to the higher-dimensional nature of the associated spaces of configurations. An additional consideration would be the lack of an obvious additive structure. Intuition may be gained from first examining examples like the sets of saddle connections associated to *Veech surfaces* and well-known tilings [2], like the Penrose tiling[5].

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