Inverse Problems for Linear and Non-linear Elliptic Equations

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Abstract

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An inverse problem is a mathematical framework that is used to obtain information about a physical object or system from observed measurements. A typical inverse problem is to recover the coefficients of a partial differential equation from measurements on the boundary of the domain. The study of conditions under which such a recovery is possible is of considerable interest and has seen a lot of work in the past few years.

This thesis research makes two primary contributions to uniqueness aspects of elliptic inverse problems. First, we prove that the knowledge of Dirichlet-to-Neumann map for a rough first order perturbation of the poly-harmonic operator in a bounded domain uniquely determines the perturbation. This is relevant as the result generalizes previous work on unique recovery of perturbations of the poly-harmonic operator.

Second, we show that for a quasi-linear elliptic equation, a perfect cloak can be obtained via a singular change of variables scheme and an approximate cloak can be achieved via a regular change of variables scheme. These approximate cloaks turn out to be anisotropic. We also show that it is possible to get isotropic regular approximate cloaks using a homogenization framework. This work generalizes to quasi-linear settings previous work on cloaking in the context of Electrical Impedance Tomography for the conductivity equation.

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GLOSSARY

Ω: Open bounded subset in $\mathbb{R}^N$ where $N \geq 2$;

$\partial\Omega$: Boundary of Ω;

$H^s(\Omega)$, $s \geq 0$: L^2 based Sobolev space of order s on Ω;

$H_0^s(\Omega)$, $s \geq 0$: Completion of $C_0^\infty(\Omega)$ under H^s norm;

$H^{-s}(\Omega)$, $s \geq 0$: Dual of $H_0^s(\Omega)$;

$H^{\frac{1}{2}}(\partial\Omega) := \{u|_{\partial\Omega}$ where $u \in H^1(\Omega)\}$;

$H^{-\frac{1}{2}}(\partial\Omega)$: Dual of $H^{\frac{1}{2}}(\partial\Omega)$;

$\mathcal{E}'(\bar{\Omega})$: Space of compactly supported distributions, supported in $\bar{\Omega}$;

$N_{A,q}$: Dirichlet-to-Neumann map associated with the operator $(-\Delta)^m + A.D + q$;

$W^{s,p}(\mathbb{R}^N)$, $s \geq 0$, $p > 1$: L^p based Sobolev space of order s on $\mathbb{R}^N$;

$W^{-s,p'}(\mathbb{R}^N)$, $s \geq 0$, $p' > 1$: Dual of $W^{s,p}(\mathbb{R}^N)$, where $\frac{1}{p} + \frac{1}{p'} = 1$;

$H^s_{scl}(\mathbb{R}^N)$, $s \in \mathbb{R}$; L^2 based semi-classical Sobolev space of order s on $\mathbb{R}^N$;

Λ_A: Dirichlet-to-Neumann map associated with $-div(A(x,u)\nabla u)$;

Y: Unit cube $[0,1]^N$ in $\mathbb{R}^N$;

$H^1_{loc}(\mathbb{R}^N)$: Space of functions u for which there exists $\phi \in C_0^\infty(\mathbb{R}^N)$ such that $\phi u \in H^1(\mathbb{R}^N)$;

$H^1_\#(Y) := \{f \in H^1_{loc}(\mathbb{R}^N) :\ y \mapsto f(y)$ is Y periodic$\}$;

$H^1_{\#,0}(Y) := \{f \in H^1_\#(Y) : \int_Y f = 0\}$.

Chapter 1

INTRODUCTION

Inverse problems is a field of study dealing with inversion of models or data. The fundamental problem in an *inverse* problem is to recover the unknown parameters of a hidden black-box system from external observations. Applications of such problems include a number of medical imaging techniques (used for instance in early detection of cancer and pulmonary edema [83]), geophysical imaging techniques (used for instance, in finding location of oil and mineral deposits in the Earth's interior [72]), detection of leaks in pipes [47], creation of astrophysical images from telescope data [10], shape optimization [20], modeling in life sciences among others.

The class of inverse problems we are interested in arise from a physical situation modeled by a partial differential equation. The inverse problem is to determine coefficients of the equation, given some information about the solutions. A prototypical example of an inverse boundary problem is the classical Calderón problem [16], forming the basis of Electrical Impedance Tomography (EIT). Alberto Calderón, motivated by a problem on geophysical prospection, proposed this problem in the mathematical literature in 1980. In EIT one attempts to determine the electrical conductivity of a medium by making voltage and current measurements at the boundary of the medium. This information is encoded in the so-called Dirichlet-to-Neumann (DN) map associated to the conductivity equation. Different concepts and questions associated with an inverse boundary problem can be best understood by looking at the formulation of the Calderón problem.

Calderón problem has been extensively studied and various developments related to this problem have spawned off investigation in inverse problems modeled by other partial differential equations, and it is worthwhile to look at the Calderón problem in more detail to understand the developments in this field.

The concept of a Dirichlet-to-Neumann map (DN map) is central to this thesis. Let us first take a look at the mathematical formulation behind the Calderón problem, to understand the notion of DN map.

1.1 Calderón problem

Consider a smooth open bounded region $\Omega \subset \mathbb{R}^N$, $N \geq 2$, and suppose that there is a conductor filling Ω with conductivity $\gamma(x)$. Let us apply a voltage f on the boundary $\partial\Omega$ and measure the current that flows out of the region. By measuring the rate at which the charge is leaving various parts of $\partial\Omega$, we are measuring the current flux through $\partial\Omega$, and that determines the quantity $\gamma(x)\nabla u(x) \cdot \hat{n(x)} = \gamma(x)\partial_\nu u(x)$ on $\partial\Omega$ where ∂_ν is the normal derivative of u and $u(x)$ is the voltage at a point $x \in \partial\Omega$.

For a given (reasonably smooth) γ and f, it is well known that the boundary value problem

$$\begin{aligned} \nabla \cdot (\gamma(x)\nabla u(x)) = 0 \text{ in } \Omega, \\ u = f \text{ on } \partial\Omega. \end{aligned} \tag{1.1}$$

has a unique solution u in Ω.

For a given γ and f, let $\Lambda_\gamma(f)$ be the function $\gamma\partial_{\nu u}|_{\partial\Omega}$. Note that Λ_γ is a 'boundary-to-boundary' operator and maps the Dirichlet boundary value to Neumann boundary value. The map $f \to \Lambda_\gamma(f)$ is called the *Dirichlet-to-Neumann* map. This map encodes the electrical boundary measurements for all possible functions f on the boundary and forms the 'external observations' part of the inverse problem. For reasonably smooth γ, the operator Λ_γ is a bounded linear map between two Banach spaces on $\partial\Omega$, *The inverse problem of Calderón, also called the inverse conductivity problem, is to determine the conductivity function γ from the knowledge of Λ_γ.*

In connection with any inverse boundary problem, there are a number of different questions that are of interest. The mathematical formulation for these questions can be best understood by looking at different aspects of the Calderón problem.

1.1.1 Different aspects of the Calderón problem

Boundary uniqueness: Does the knowledge of the DN map determine γ uniquely on the boundary? That is, does $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ imply $\gamma_1|_{\partial\Omega} = \gamma_2|_{\partial\Omega}$? [51]

Interior uniqueness: Does the knowledge of the DN map determine γ uniquely in the interior? That is, does $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ imply $\gamma_1 = \gamma_2$ in Ω? [77]

Stability: One would ideally like to have an efficient numerical implementation of the reconstruction algorithm in practice. But the problem is ill-posed in the sense that small errors in the measurements may lead to a large error in the reconstructed function. The *stability* aspect of the inverse conductivity problem quantifies the degree of ill-posedness. [2]

Reconstruction : Is there a convergent algorithm for determining γ from Λ_γ? [65]

Partial Data: The previous results considered the case of full data, where one can do measurements on the whole boundary $\partial\Omega$. In practice this is often not possible. It is therefore of interest to consider partial data problems. For the Calderón problem, this means that one has knowledge of $\Lambda_\gamma f$ on some subset of the boundary, for functions f supported in some subset of the boundary. We can ask questions about uniqueness, stability and reconstruction for this partial data problem. [49, 62, 17]

Other aspects:

We can also look at the practical side of things, where only finitely many measurements at finitely many points are available (as opposed to the mathematical idealizations we considered above, where infinitely many measurements were made for all possible boundary voltage f), and hence discrete versions of the problem could be studied. [26]

One also needs a careful modeling of how the measurements are implemented. There will always be some noise in the measured DN map, and the stability results need to take the noisy version of the DN map in to account. Therefore, an improved stability analysis possibly including a regularization strategy for the numerical algorithm would be of interest. [18]

All the above aspects have similar counterparts for inverse problems modeled on other partial differential equations. The focus of this thesis will be on the *interior uniqueness* aspect of the inverse boundary problem with full boundary data, for two different families of elliptic partial differential equations.

1.2 Dirichlet-to-Neumann map

Let us now define the DN map for a general second order elliptic equation in a rigorous manner. The first concept we need is that of Sobolev spaces on the boundary $\partial\Omega$ for a bounded open set $\Omega \subset \mathbb{R}^N$.

Consider the Hilbert space $H^1(\Omega)$ (as defined in [1]) and define $H_0^1(\Omega)$ as the closure of $C_0^\infty(\Omega)$ under the $H^1(\Omega)$ norm. We can think of $H_0^1(\Omega)$ as the set of functions in Ω whose boundary value, called *trace*, vanishes on $\partial\Omega$. This motivates the following definition of an abstract trace space of $H^1(\Omega)$.

Definition 1.2.1. *Define $H^{\frac{1}{2}}(\partial\Omega)$ as the quotient space*

$$H^{\frac{1}{2}}(\partial\Omega) = H^1(\Omega)/H_0^1(\Omega).$$

The elements of $H^{\frac{1}{2}}(\Omega)$ are the equivalence classes $[u] = \{u + \phi : \phi \in H_0^1(\Omega)\}$, where u runs through all the elements of $H^1(\Omega)$. Also, define the trace operator

$$R : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega), R(u) = [u].$$

We also write $R(u) = u|_{\partial\Omega}$.

We now define the negative order Sobolev space $H^{-\frac{1}{2}}(\partial\Omega)$ as the dual to $H^{\frac{1}{2}}(\partial\Omega)$.

Definition 1.2.2. *Define $H^{\frac{1}{2}}(\partial\Omega)$ as the quotient space*

$$H^{-\frac{1}{2}}(\partial\Omega) = \{L : H^{\frac{1}{2}}(\partial\Omega) \to \mathbb{C} \text{ bounded linear functional } \}$$

1.2.1 Weak solutions

Consider Ω a bounded open set in $\mathbb{R}^N$, $N \geq 2$. To motivate the idea of weak solution to an second order elliptic partial differential equation (PDE), consider a more restrictive second order differential operator L, acting on functions u on Ω, of the form

$$Lu = -\sum_{j,k=1}^{N} \frac{\partial}{\partial x_j}\left(a^{jk}\frac{\partial u}{\partial x_k}\right) + qu, \tag{1.2}$$

where the coefficients are assumed to satisfy the following conditions:

1. $a^{jk}, q \in L^\infty(\Omega)$ are real valued;
2. $a^{jk} = a^{kj}$ for all $j, k = 1, 2, .., n$;
3. $\sum_{j,k=1}^{N} a^{jk}\xi_j xi_k \geq c|\xi|^2$, for a.e $x \in \Omega$ and all $\xi \in \mathbb{R}^N$, where $c > 0$.

The last condition is an ellipticity condition for the operator L, and will ensure that the operator L will have similar properties to that of the Laplace operator $-\Delta$. Note that the operator L is in *divergence form*, and such operators are better suited to functional analytic methods related to weak solutions.

Let us now define the concept of a weak solution associated with the operator L.

Definition 1.2.3. *Let L be the differential operator as defined in (1.2). The sesqui-linear form associated with L is given by*

$$B(u, v) := \int_\Omega \left(\sum_{j,k=1}^{N} a^{jk}\partial_j u \bar{\partial_k} v\right) dx + qu\bar{v}\, dx, \quad u, v \in H^1(\Omega). \tag{1.3}$$

If $F \in H^{-1}(\Omega)$ and $f \in H^{\frac{1}{2}}(\partial\Omega)$, we say that $u \in H^1(\Omega)$ is a weak solution of the boundary value problem

$$\begin{aligned} Lu &= F \text{ in } \Omega, \\ u &= f \text{ on } \partial\Omega, \end{aligned} \tag{1.4}$$

if one has

$$B(u, v) = F(\bar{v}) \text{ for all } v \in H^1_0(\Omega)$$

and if $Ru = f$, where R is trace operator as defined in (1.2.1).

It can be shown, using usual functional analytic arguments that the boundary value problem (1.4) has a weak solution $u \in H^1(\Omega)$ with $u|_{\partial\Omega} = f$. (If $q \geq 0$ a.e, then a unique weak solution is guaranteed. Otherwise, we need to assume that 0 is not an eigenvalue for the operator L to guarantee uniqueness.) A detailed treatment on different assumptions to guarantee unique weak solutions to (1.4) can be found in [34]. (For a more general second order divergence form elliptic operator, the proof is analogous. We might need more assumptions on lower order terms to guarantee existence and uniqueness.)

1.2.2 DN map and Calderón problem reformulated

With the notion of a weak solution to L, we can now formally define the DN operator. The DN map for the operator L is formally given by

$$\Lambda_L : f \to \sum_{j,k=1}^{N} a^{jk}(\partial_j u)\nu_k|_{\partial\Omega}. \tag{1.5}$$

Motivation for (1.5)

Let us now see how definition (1.5) arises naturally.

Let us assume that Ω has a smooth boundary and $a^{jk} \in C^\infty(\bar{\Omega})$, and let u_f solve the Dirichlet problem $Lu = 0$ in Ω with $u_f = f$ on $\partial\Omega$ for some $f \in C^\infty(\partial\Omega)$. By elliptic regularity, $u_f \in C^\infty(\bar{\Omega})$ and we can define $\Lambda_L f$ as in the right hand side of (1.5), as a function in $C^\infty(\partial\Omega)$. Let $g \in C^\infty(\partial\Omega)$ and let $e_g \in C^\infty(\bar{\Omega})$ be any function such that $e_g|_{\partial\Omega} = g$. Integration by parts combined with the fact that all quantities are smooth, shows that

$$\begin{aligned}
\int_{\partial\Omega} (\Lambda_L f) g \, dS &= \sum_{j,k=1}^{N} \int_{\partial\Omega}^{n} a^{jk} (\partial_j u_f) e_g \nu_k \, dS \\
&= \sum_{j,k=1}^{N} \int_{\Omega} \partial_k \left(a^{jk} (\partial_j u_f) e_g \right) \, dx \\
&= \int_{\Omega} \left[\sum_{j,k=1}^{N} a^{jk} \partial_j u_f \partial_k e_g + q u_f e_g \right] dx. \qquad (1.6)
\end{aligned}$$

Note that in the last step we used that $Lu = 0$. The quantity in the last line of (1.6) is just $B(u_f, \bar{e}_g)$, where B is the sesqui-linear form associated with L. This expression is well defined when a^{jk}, $q \in L^\infty(\Omega)$ and $u_f, e_g \in H^1(\Omega)$. We use this observation to define the DN map in a weak sense even when the quantity $a_{jk}(\partial_j u)\nu_k$ may not be defined point-wise.

Let us use this observation and define the DN map in a weak sense when the quantity $a^{jk}(\partial_j u)\nu_k$ cannot be defined point-wise. Since $H^{-\frac{1}{2}}(\partial\Omega)$ is the dual space of $H^{\frac{1}{2}}(\Omega)$, we can express the duality by

$$\langle f, g \rangle = f(g), \quad g \in H^{\frac{1}{2}}(\Omega).$$

The following theorem defines the DN map for the operator L in a weak sense. The proof of the theorem is fairly straightforward and thus we skip it, stating only the result.

Theorem 1.2.1. *Let Ω be a bounded open set in $\mathbb{R}^N$ and let L satisfies the conditions specified right after (1.2). There is a unique bounded linear map*

$$\Lambda_L : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega),$$

which satisfies

$$\begin{aligned}\langle \Lambda_L f, g\rangle &= B(u_f, \bar{e_g}) \\ &= \int_\Omega \sum_{j,k=1}^{N} \partial_j u_f \partial_k e_g \, dx, \end{aligned} \tag{1.7}$$

where $u_f \in H^1(\Omega)$ *is the unique solution of* $Lu = 0$ *in* Ω *with* $u|_{\partial\Omega} = f$*, and* e_g *is any function in* $H^1(\Omega)$ *with* $e_g|_{\partial\Omega} = g$*.*

If Ω *has* C^∞ *boundary and* $\gamma \in C^\infty(\bar{\Omega})$*, and if* $f, g \in C^\infty(\partial\Omega)$*, then by elliptic regularity* $u_f \in C^\infty(\partial\Omega)$ *and* $\Lambda_L f$ *can be identified with the* C^∞ *function* $\sum_{j,k=1}^{N} a^{jk}(\partial_j u_f)\nu_k|_{\partial\Omega}$*.*

Such a weak definition of a DN map can in fact be extended to any second or higher order elliptic operator. Once we can guarantee the existence and uniqueness for the Dirichlet boundary value problem associated with a more general second or higher order operator, we can provide a weak definition of a DN map. Thus, in absence of high regularity of co-efficients of an elliptic partial differential equation, it is possible to extend the definition of a DN map as an operator acting between Sobolev spaces on the boundary.

Now that we know how to define the DN operator, let us reformulate the Calderón problem. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with smooth boundary. We think of Ω as an electrical conductor, and assume that the conductivity at each point of Ω is given by a function $\gamma \in L^\infty(\Omega)$ satisfying $\gamma(x) \geq c > 0$ a.e. in Ω. It is easy to show that there is a bounded linear map $\Lambda_\gamma : H^{\frac{1}{2}}(\partial\Omega) \to H^{-\frac{1}{2}}(\partial\Omega)$, which formally associates to a function $f \in H^{\frac{1}{2}}(\partial\Omega)$ an element $\Lambda_\gamma f \in H^{\frac{1}{2}}(\partial\Omega)$, that may be thought of as the electrical current $\gamma\partial_\nu u_f|_{\partial\Omega}$ corresponding to boundary voltage f. (If Ω has smooth boundary and $\gamma \in C^\infty(\bar{\Omega})$, we saw before that $\Lambda_\gamma f = \gamma\partial_\nu u_f|_{\partial\Omega}$ for any $f \in C^\infty(\partial\Omega)$ in the classical sense.)

We will rigorously define the DN map for each operator considered in this thesis.

1.3 Outline of Thesis

This thesis concerns inverse boundary problem for two different families of elliptic partial differential equations. We focus on the uniqueness aspect for these inverse problems. For a given elliptic operator $\mathcal{L}$, we consider the associated Dirichlet-to-Neumann operator $\mathcal{L}$ and study conditions under which unique recovery for co-efficients of $\mathcal{L}$ is possible.

In *Chapter 2*, we consider the following perturbed poly-harmonic operator

$$\mathcal{L}_{A,q}u = (-\Delta)^m u(x) + A(x) \cdot Du(x) + q(x)u(x).$$

We prove that the knowledge of Dirichlet-to-Neumann map for rough A and q in $\mathcal{L}_{A,q}$ for $m \geq 2$ for a bounded domain in $\mathbb{R}^N$ for $N \geq 3$ determines A and q uniquely. This unique identifiability is proved via construction of complex geometrical optics solutions with sufficient decay of remainder terms, by using property of products of functions in Sobolev spaces.

Chapter 2 is based on a joint work with Yernat Assylbekov [6], currently under review.

In *Chapter 3*, we consider the following quasi-linear operator

$$\mathcal{L}_A u = \nabla \cdot (A(x, u(x))\nabla u(x)),$$

where $A(x,t)$ is a matrix valued conductivity. We consider cloaking for this quasi-linear elliptic partial differential equation of divergence type defined on a bounded domain in $\mathbb{R}^N$ for $N = 2, 3$. We prove that a perfect cloak can be obtained via a singular change of variables scheme and an approximate cloak can be achieved via a regular change of variables scheme. These approximate cloaks, though non-degenerate, are anisotropic. This work thus shows *non-uniqueness* in the context of inverse problems.

We also show, within the framework of homogenization, that it is possible to get isotropic regular approximate cloaks. This work generalizes to quasi-linear settings previous work on cloaking in the context of Electrical Impedance Tomography for the conductivity equation.

The results in Chapter 3 are based on a joint work with Tuhin Ghosh which is published in [32]. Copyright 2018 of American Institute of Mathematical Sciences.

Chapter 4 is a brief foray in to the uniqueness aspect of the inverse boundary problem considered in Chapter 3, but for a scalar valued conductivity. We provide conditions under which the operator $A \to \mathcal{L}_A$ is injective. This particular result was cited in [80] without proof. In this chapter, we provide a short proof of this uniqueness result and generalize the class of coefficients for which the uniqueness holds.

Chapters 3 and 4 are related and independent from Chapter 2. We provide the necessary background details with regards to the Dirichlet-to-Neumann map in the relevant sections in Chapters 2 and 3. The material in each of the chapters is self-contained and the interested reader may read the chapters independently of each other.

1.4 Notes

1. I thank Prof. Uhlmann for generously sharing his unpublished draft of notes from which sections of the first chapter were inspired.
2. We only briefly touched on the history and development in Calderón problem. A more mathematically detailed treatment can be found in an excellent survey article, [80].

Chapter 2

UNIQUENESS FOR ROUGH PERTURBATION OF POLY-HARMONIC OPERATOR

2.1 Introduction and Statement of Results

2.1.1 Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ be a bounded open set with C^∞ boundary. Consider the poly-harmonic operator $(-\Delta)^m$ where $m \geq 1$ is an integer. The operator $(-\Delta)^m$ is positive and self-adjoint on $L^2(\Omega)$ with domain $H^{2m}(\Omega) \cap H_0^m(\Omega)$, where $H_0^m(\Omega) = \{u \in H^m(\Omega) : \gamma u = 0\}$.

This operator can be obtained as the Friedrichs extension starting from the space of test functions; see, for example, [45]. Here and in what follows γ is the Dirichlet trace operator

$$\gamma : H^m(\Omega) \to \prod_{j=0}^{m-1} H^{m-j-1/2}(\partial\Omega), \quad \gamma u = (u\big|_{\partial\Omega}, \partial_\nu u\big|_{\partial\Omega}, ..., \partial_\nu^{m-1} u\big|_{\partial\Omega}),$$

where ν is the unit outer normal to the boundary $\partial\Omega$, and $H^s(\Omega)$ and $H^s(\partial\Omega)$ are the standard L^2 based Sobolev spaces on Ω and $\partial\Omega$ respectively for $s \in \mathbb{R}$.

Let us first consider the perturbed polyharmonic operator $\mathcal{L}_{A,q} = (-\Delta)^m + A \cdot D + q$ where A and q are sufficiently smooth and $D = -i\nabla$.
For $f = (f_0, f_1, ..., f_{m-1}) \in \prod_{j=0}^{m-1} H^{2m-j-1/2}(\partial\Omega)$, consider the Dirichlet problem

$$\mathcal{L}_{A,q} u = 0 \text{ in } \Omega \quad \text{and} \quad \gamma u = f \text{ on } \partial\Omega. \tag{2.1}$$

If 0 is not in the spectrum of $\mathcal{L}_{A,q}$, it can be shown that the Dirichlet problem (2.1) has a unique solution $u \in H^{2m}(\Omega)$. We can then define the Dirichlet-to-Neumann $\mathcal{N}_{A,q}$ map as

$$\mathcal{N}_{A,q}f = (\partial_\nu^m u\big|_{\partial\Omega}, ..., \partial_\nu^{2m-1} u\big|_{\partial\Omega}) = \tilde{\gamma} u \in \prod_{j=m}^{2m-1} H^{2m-j-1/2}(\partial\Omega).$$

The inverse boundary value problem for the perturbed poly-harmonic operator $\mathcal{L}_{A,q}$ is to determine A and q in Ω from the knowledge of the Dirichlet to Neumann map $\mathcal{N}_{A,q}$.

Before we proceed, we need to fix some notations. Here and in what follows, $\mathcal{E}'(\bar{\Omega}) = \{u \in \mathcal{D}'(\mathbb{R}^N) : supp(u) \subseteq \bar{\Omega}\}$ and $W^{s,p}(\mathbb{R}^N)$ is the standard L^p based Sobolev space on $\mathbb{R}^N$, $s \in \mathbb{R}$ and $1 < p < \infty$, which is defined via the Bessel potential operator. We can also define the analogous spaces $W^{s,p}(\Omega)$ for Ω a bounded open set with smooth boundary. We refer the reader to [1] for properties of these spaces.

The study of inverse problems for such first order perturbations of the poly-harmonic operator was initiated in [52]. The authors tackled the question of unique recovery of A and q from the knowledge of the Dirichlet-to-Neumann map. More precisely, they show that for $m \geq 2$, the set of Cauchy data $C_{A,q} = \{(\gamma u, \tilde{\gamma} u) : u \in H^{2m}(\Omega) \text{ with } \mathcal{L}_{A,q} u = 0\}$ determines A and q uniquely provided $A \in W^{1,\infty}(\Omega, \mathbb{C}^N) \cap \mathcal{E}'(\bar{\Omega}, \mathbb{C}^N)$ and $q \in L^\infty(\Omega)$. Regularities of A and q were relaxed in [5] to $A \in W^{-\frac{m-2}{2}, \frac{2N}{m}}(\mathbb{R}^N) \cap \mathcal{E}'(\bar{\Omega})$ and $q \in W^{-\frac{m}{2}+\delta, \frac{2N}{m}}(\mathbb{R}^N) \cap \mathcal{E}'(\bar{\Omega})$, $0 < \delta < 1/2$, for the case $m < N$. A natural question that remained open was the problem of uniqueness in this inverse problem when the regularity of the coefficients is significantly lowered.

The goal of this chapter is to tackle the above question and improve the results of [52] and [5] in several directions. We show that the restriction $m < N$ in [5] is not necessary and that the uniqueness can in fact be proved for any $N \geq 3$ and any $m \geq 2$. Second, we substantially relax the regularity and integrability conditions for A and q, and we prove the uniqueness result for the inverse problem for a much broader class of coefficients. Third, we show how careful book-keeping in fact improves the result in [5] for $m < N$. Along the way, we also reason how the class of coefficients for which uniqueness in this inverse problem can be answered using this technique is as broad as possible and cannot be further improved.

Let us remark that the problem considered in this chapter can be considered as a generalization of Calderón's inverse conductivity problem [16], also known as electrical impedance tomography, for which the question of reducing regularity has been studied extensively. In the fundamental paper by Sylvester and Uhlmann [77] it was shown that C^2 conductivities can be uniquely determined from boundary measurements. Successive papers have focused on weakening the regularity for the conductivity; see [15, 66, 43, 46, 19] for more details.

As was observed in [73], for the case $m = 1$ in (2.1), there is a gauge invariance that prohibits uniqueness and therefore we can hope to recover A and q only modulo such a gauge transformation. It was shown in [73] that such uniqueness modulo a gauge invariance is possible provided that $A \in W^{2,\infty}$, $q \in L^\infty$ and dA satisfy a smallness condition. There have been many successive papers which have weakened the regularity assumptions on A and q for the case $m = 1$. The reader is referred to [70, 53, 64, 79] for details.

Inverse problems for higher order operators have been considered in [52, 54, 31, 33, 5, 7] where unique recovery actually becomes possible. Higher order poly-harmonic operators occur in the areas of physics and geometry such as the study of the Kirchoff plate equation in the theory of elasticity, and the study of the Paneitz-Branson operator in conformal geometry; for more details see monograph [30].

2.1.2 Statement of Result

Throughout this chapter we assume $m \geq 2$ and $N \geq 3$.

Suppose that the first order perturbation A be in $W^{-\frac{m}{2}+1,p'}(\mathbb{R}^N) \cap \mathcal{E}'(\bar{\Omega})$, where p' satisfies

$$\begin{cases} p' \in [2N/m, \infty) & \text{if} \quad m < N, \\ p' \in (2, \infty) & \text{if} \quad m = N \quad \text{or} \quad m = N+2, \\ p' \in [2, \infty) & \text{otherwise.} \end{cases} \tag{2.2}$$

For a fixed δ with $0 < \delta < \frac{1}{2}$, suppose that the zeroth order perturbation q be in

$W^{-\frac{m}{2}+\delta, r'}(\mathbb{R}^N) \cap \mathcal{E}'(\bar{\Omega})$, where r' satisfies

$$\begin{cases} r' \in [2N/(m-2\delta), \infty), & \text{if} \quad m < N, \\ r' \in [2N/(m-2\delta), \infty), & \text{if} \quad m = N, \\ r' \in [2, \infty), & \text{if} \quad m \geq N+1. \end{cases} \tag{2.3}$$

Before stating the main result, we consider the bi-linear forms B_A and m_q on $H^m(\Omega)$ which are defined by

$$B_A(u,v) := B_A^{\mathbb{R}^N}(\tilde{u}, \tilde{v}) := \langle A, \tilde{v} D\tilde{u}\rangle, \quad b_q(u,v) := b_q^{\mathbb{R}^N}(\tilde{u}, \tilde{v}) := \langle q, \tilde{u}\tilde{v}\rangle \tag{2.4}$$

for all $u, v \in H^m(\Omega)$, where $\langle \cdot, \cdot \rangle$ denotes the distributional duality on $\mathbb{R}^N$ such that $\langle \cdot, \bar{\cdot} \rangle$ naturally extends $L^2(\mathbb{R}^N)$-inner product, and $\tilde{u}, \tilde{v} \in H^m(\mathbb{R}^N)$ are any extensions of u and v, respectively. In Section 2.4.1, we show that these definitions are well defined i.e. independent of the choice of extensions $\tilde{u}, \tilde{v}$. Using a property of multiplication of functions in Sobolev spaces, we show that the forms B_A and b_q are bounded on $H^m(\Omega)$. We also adopt the convention that for any $z > 1$, the number z' is defined by $z' = z/(z-1)$.

Consider the operator D_A, which is formally $A \cdot D$ where $D_j = -i\partial_j$, and the operator m_q of multiplication by q. To be precise, for $u \in H^m(\Omega)$, $D_A(u)$ and $m_q(u)$ are defined as

$$\langle D_A(u), \psi \rangle_\Omega = B_A(u, \psi) \quad \textit{and} \quad \langle m_q(u), \psi\rangle_\Omega = b_q(u, \psi), \quad \psi \in C_0^\infty(\Omega),$$

where $\langle \cdot, \cdot \rangle_\Omega$ is the distribution duality on Ω such that $\langle \cdot, \bar{\cdot} \rangle_\Omega$ naturally extends $L^2(\Omega)$-inner product. The operators D_A and m_q are shown in Section 2.4.1 to be bounded from $H^m(\Omega) \to H^{-m}(\Omega)$ and hence, standard arguments show that the operator
$\mathcal{L}_{A,q} = (-\Delta)^m + D_A + m_q : H^m(\Omega) \to H^{-m}(\Omega) = (H_0^m(\Omega))'$ is a Fredholm operator with index 0.

For $f = (f_0, f_1, ..., f_{m-1}) \in \prod_{j=0}^{m-1} H^{m-j-1/2}(\partial\Omega)$, consider the Dirichlet problem (2.1). If 0 is not in the spectrum of $\mathcal{L}_{A,q}$, it is shown in Section 2.4.2 that the Dirichlet problem (2.1) has a unique solution $u \in H^m(\Omega)$. We define the Dirichlet-to-Neumann map $\mathcal{N}_{A,q}$ weakly as follows

$$\langle N_{A,q}f, \bar{h}\rangle_{\partial\Omega} = \sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha u, D^\alpha v_h)_{L^2(\Omega)} + B_A(u, \bar{v}_h) + b_q(u, \bar{v}_h), \tag{2.5}$$

where $h = (h_0, h_1, ..., h_{m-1}) \in \prod_{j=0}^{m-1} H^{m-j-1/2}(\partial\Omega)$, $v_h \in H^m(\Omega)$ is any extension of h so that $\gamma v_h = h$, and where $\langle\cdot,\cdot\rangle_{\partial\Omega}$ is the distribution duality on $\partial\Omega$ such that $\langle\cdot,\bar{\cdot}\rangle_{\partial\Omega}$ naturally extends $L^2(\partial\Omega)$-inner product. It is shown in Section 2.4.2 that $\mathcal{N}_{A,q}$ is a well-defined bounded operator

$$\prod_{j=0}^{m-1} H^{m-j-1/2}(\partial\Omega) \to \left(\prod_{j=0}^{m-1} H^{m-j-1/2}(\partial\Omega)\right)' = \prod_{j=0}^{m-1} H^{-m+j+1/2}(\partial\Omega).$$

Our main result is as follows.

Theorem 2.1.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ be a bounded open set with C^∞ boundary, and let $m \geq 2$ be an integer. Let $0 < \delta < 1/2$. Suppose that A_1, A_2 satisfy (2.2) and q_1, q_2 satisfy (2.3) and 0 is not in the spectra of $\mathcal{L}_{A_1,q_1}$ and $\mathcal{L}_{A_2,q_2}$. If $\mathcal{N}_{A_1,q_1} = \mathcal{N}_{A_2,q_2}$, then $A_1 = A_2$ and $q_1 = q_2$.*

Detailed explanation for the assumption $\delta > 0$ is given in Remark 4.

The key ingredient in the proof of Theorem 2.1.1 is the construction of complex geometric optics solutions for the operator $\mathcal{L}_{A,q}$ with correct decay for the remainder term. We use the method of Carleman estimates which is based on the corresponding Carelman estimates for the Laplacian with a gain of two derivatives, due to Salo and Tzou [71], and chain it with Proposition 2.2.2, which gives property of products of functions in various Sobolev spaces, to obtain the desired decay.

The idea of constructing such complex geometric optics solutions to elliptic operators goes back to the fundamental paper by Sylvester and Uhlmann [77] and has been extensively used to show unique recovery of coefficients in many inverse problems

The rest of the chapter is organized as follows. We construct complex geometrical optics solutions for the perturbed poly-harmonic operator $\mathcal{L}_{A,q}$ with A and q as defined in (2.2) and (2.3) respectively. The proof of Theorem 2.1.1 is given in section 2.3. In Section 2.4.1, we study mapping properties of D_A and m_q. Section 2.4.2 is devoted to the well-posedness of the Dirichlet problem $\mathcal{L}_{A,q}$ with A satisfying (2.2) and q satisfying (2.3). In Section 2.4.3, we specify why we use Bessel potential to define fractional Sobolev spaces.

2.2 Carleman estimate and CGO solutions

Let us first derive Carleman estimates for the operator $\mathcal{L}_{A,q}$. We first recall the Carleman estimates for the semi-classical Laplace operator $-h^2\Delta$ with a gain of two derivatives, established in [71]. Let $\tilde{\Omega}$ be an open set in $\mathbb{R}^N$ such that $\bar{\Omega} \subset\subset \tilde{\Omega}$ and let $\phi \in C^\infty(\tilde{\Omega}, \mathbb{R})$. Consider the conjugated operator $P_\phi = e^{\phi/h}(-h^2\Delta)e^{-\phi/h}$, and it's semi classical principal symbol $p_\phi(x,\xi) = \xi^2 + 2i\nabla\phi\cdot\xi - |\nabla\phi|^2$, $x \in \tilde{\Omega}$, $\xi \in \mathbb{R}^N$. Following [49], we say that ϕ is a limiting Carleman weight for $-h^2\Delta$ in $\tilde{\Omega}$, if $\nabla\phi \neq 0$ in $\tilde{\Omega}$ and the Poisson bracket of $Re\, p_\phi$ and $Im\, p_\phi$ satisfies $\{Re\, p_\phi, Im\, p_\phi\}(x,\xi) = 0$ when $p_\phi(x,\xi) = 0$, $(x,\xi) \in \tilde{\Omega} \times \mathbb{R}^N$.

Before we state the Carleman estimates in [71], we define the semi-classical Sobolev norms on $\mathbb{R}^N$,

$$||u||_{H^s_{scl}(\mathbb{R}^N)} := ||\langle hD\rangle^s u||_{L^2(\mathbb{R}^N)},$$

where $\langle\xi\rangle = (1+|\xi|^2)^{1/2}$ and $s \in \mathbb{R}$.

The following Carleman estimate with a gain of two derivatives for the semi-classical Laplacian was established in [71].

Proposition 2.2.1. *Let ϕ be a limiting Carleman weight for $-h^2\Delta$ in $\tilde{\Omega}$ and let $\phi_\epsilon = \phi + \frac{h}{2\epsilon}\phi^2$. Then for $0 < h \ll \epsilon \ll 1$ and $s \in \mathbb{R}$, we have*

$$\frac{h}{\sqrt{\epsilon}}||u||_{H^{s+2}_{scl}(\mathbb{R}^N)} \leq C||e^{\phi_\epsilon/h}(-h^2\Delta)e^{-\phi_\epsilon/h}u||_{H^s_{scl}(\mathbb{R}^N)}, \quad C > 0$$

for all $u \in C_0^\infty(\Omega)$.

We now state a theorem on products of functions in Sobolev spaces; see Theorem 1 and Theorem 2 in [69, Section 4.4.4].

Proposition 2.2.2. *Let $0 < s_1 \leq s_2$. Suppose* *(a)* $p^{-1} \leq p_1^{-1} + p_2^{-1} \leq 1$;

(b) either

$$\frac{N}{p} - s_1 > \begin{cases} (\frac{N}{p_1} - s_1)^+ + (\frac{N}{p_1} - s_2)^+ & \text{if } max_i(\frac{N}{p_i} - s_i) > 0, \\ max_i(\frac{N}{p_i} - s_i) & \text{otherwise} \end{cases}$$

or

$$\frac{N}{p} - s_1 = \begin{cases} (\frac{N}{p_1} - s_1)^+ + (\frac{N}{p_1} - s_2)^+ & \text{if } max_i(\frac{N}{p_i} - s_1) > 0, \\ max_i(\frac{N}{p_i} - s_1) & \text{otherwise} \end{cases}$$

$$\{i \in \{1, 2\} : s_i = N/p_i \text{ and } p_i > 1\} = \varnothing.$$

If $u \in W^{s_1,p_1}(\mathbb{R}^N)$ and $v \in W^{s_2,p_2}(\mathbb{R}^N)$, then $uv \in W^{s_1,p}(\mathbb{R}^N)$. Moreover, the point-wise multiplication of functions is a continuous bi-linear map $W^{s_1,p_1}(\mathbb{R}^N) \cdot W^{s_2,p_2}(\mathbb{R}^N) \hookrightarrow W^{s_1,p}(\mathbb{R}^N)$ and

$$||uv||_{W^{s_1,p}(\mathbb{R}^N)} \leq C||u||_{W^{s_1,p_1}(\mathbb{R}^N)}||v||_{W^{s_2,p_2}(\mathbb{R}^N)} \tag{2.6}$$

where the constant C depends only on the various indices.

We now have the tools to derive Carleman estimates for the perturbed operator $\mathcal{L}_{A,q}$ when A and q are as in (2.2) and (2.3) respectively. We have the following result.

Proposition 2.2.3. *Let ϕ be a limiting Carleman weight for $-h^2\Delta$ in $\tilde{\Omega}$ and suppose A and q satisfy (2.2) and (2.3), respectively. Then for $0 < h \ll 1$,we have*

$$||u||_{H^{m/2}_{scl}(\mathbb{R}^N)} \lesssim \frac{1}{h^m}||e^{\phi/h}(h^{2m}\mathcal{L}_{A,q})e^{-\phi/h}u||_{H^{-3m/2}_{scl}(\mathbb{R}^N)}, \tag{2.7}$$

for all $u \in C_0^\infty(\Omega)$.

Proof. Iterate the Carleman estimate in Proposition 2.2.1 m times with $s = -3m/2$ and a fixed $\epsilon > 0$ sufficiently small and independent of h to get the estimate

$$\frac{h^m}{\epsilon^{m/2}}||u||_{H^{m/2}_{scl}(\mathbb{R}^N)} \leq C||e^{\phi_\epsilon/h}(-h^2\Delta)^m e^{-\phi_\epsilon/h}u||_{H^{-3m/2}_{scl}(\mathbb{R}^N)}, \tag{2.8}$$

for all $u \in C_0^\infty(\Omega)$ and $0 < h \ll \epsilon \ll 1$.

Let $\psi \in C_0^\infty(\mathbb{R}^N)$. By duality and Proposition 2.2.2, we have for any $m \geq 2$,

$$\begin{aligned} |\langle e^{\phi_\epsilon/h} h^{2m} m_q(e^{-\phi_\epsilon/h}u), \psi\rangle| &\leq h^{2m}||q||_{W^{-\frac{m}{2}+\delta,r'}(\mathbb{R}^N)}||u\psi||_{W^{\frac{m}{2}-\delta,r}(\mathbb{R}^N)} \\ &\leq Ch^{2m}||q||_{W^{-\frac{m}{2}+\delta,r'}(\mathbb{R}^N)}||u||_{H^{\frac{m}{2}-\delta}(\mathbb{R}^N)}||\psi||_{H^{\frac{m}{2}-\delta}(\mathbb{R}^N)} \\ &\leq Ch^m||u||_{H^{m/2}_{scl}(\mathbb{R}^N)}||\psi||_{H^{m/2}_{scl}(\mathbb{R}^N)} \\ &\leq Ch^m||u||_{H^{m/2}_{scl}(\mathbb{R}^N)}||\psi||_{H^{3m/2}_{scl}(\mathbb{R}^N)}. \end{aligned} \tag{2.9}$$

Remark 1. *Estimate (2.9) actually goes through even for $\delta = 0$. For $m < N$, in Proposition 2.2.2, we choose $p_1 = p_2 = 2$, $r = p \in (1, \frac{2N}{2N-m}]$, $s_1 = s_2 = \frac{m}{2}$. For $m = N$, we choose $p_1 = p_2 = 2$, $r = p \in (1, 2)$, $s_1 = \frac{m}{2}, s_2 = \frac{m}{2}$. Finally, for $m > N$, we choose $p_1 = p_2 = 2$, $r = p \in (1, 2]$, $s_1 = s_2 = \frac{m}{2}$. It is also easy to see that we, in fact, have a stronger decay of $\mathcal{O}(h^{m+2\delta})$ in (2.9).*

By definition of dual norm,

$$||e^{\phi_\epsilon/h} h^{2m} m_q(e^{-\phi_\epsilon/h}u)||_{H^{-3m/2}_{scl}(\mathbb{R}^N)} \leq Ch^m||q||_{W^{-\frac{m}{2}+\delta,r'}(\mathbb{R}^N)}||u||_{H^{m/2}_{scl}(\mathbb{R}^N)}. \tag{2.10}$$

For $m > 2$, by duality, we have

$$\begin{aligned} &|\langle e^{\phi_\epsilon/h} h^{2m} D_A(e^{-\phi_\epsilon/h}u), \psi\rangle| \\ &= |\langle h^{2m}A, e^{\phi_\epsilon/h}\psi D(e^{-\phi_\epsilon/h}u)\rangle| \\ &\leq |\langle h^{2m-1}A, \psi[-u(1+h\phi/\epsilon)D\phi + hDu]\rangle \\ &\leq Ch^{2m-1}||A||_{W^{-\frac{m}{2}+1,p'}(\mathbb{R}^N)}||-u(1+h\phi/\epsilon)D\phi\psi + hDu\psi||_{W^{\frac{m}{2}-1,p}(\mathbb{R}^N)}. \end{aligned}$$

Using Proposition 2.2.2, we have

$$\begin{aligned} &||-u(1+h\phi/\epsilon)D\phi\psi + hDu\psi||_{W^{\frac{m}{2}-1,p}(\mathbb{R}^N)} \\ &\leq C||-u(1+h\phi/\epsilon)D\phi + hDu||_{H^{\frac{m-2}{2}}(\mathbb{R}^N)}||\psi||_{H^{\frac{m}{2}}(\mathbb{R}^N)} \\ &\leq C||-u(1+h\phi/\epsilon)D\phi + hDu||_{H^{\frac{m-2}{2}}(\mathbb{R}^N)}||\psi||_{H^{m/2}(\mathbb{R}^N)} \\ &\leq Ch^{-m+1}||u||_{H^{\frac{m}{2}}_{scl}(\mathbb{R}^N)}||\psi||_{H^{\frac{m}{2}}_{scl}(\mathbb{R}^N)} \\ &\leq Ch^{-m+1}||u||_{H^{\frac{m}{2}}_{scl}(\mathbb{R}^N)}||\psi||_{H^{\frac{3m}{2}}_{scl}(\mathbb{R}^N)}. \end{aligned}$$

For $m = 2$, we get

$$
\begin{aligned}
|\langle e^{\phi_\epsilon/h} h^{2m} D_A(e^{-\phi_\epsilon/h}u), \psi\rangle| &= |\langle h^{2m} A, e^{\phi_\epsilon/h}\psi D(e^{-\phi_\epsilon/h}u)\rangle| \\
&\leq |\langle h^{2m-1} A, \psi[-u(1 + h\phi/\epsilon)D\phi + hDu]\rangle| \\
&\leq Ch^{2m-1}||A||_{L^N(\mathbb{R}^N)}|| - u(1 + h\phi/\epsilon)D\phi\psi + hDu\psi||_{L^{N'}(\Omega)}.
\end{aligned}
$$

We now use Hölder's inequality and Sobolev Embedding Theorem to get

$$
\begin{aligned}
&|\langle e^{\phi_\epsilon/h} h^{2m} D_A(e^{-\phi_\epsilon/h}u), \psi\rangle| \\
&\leq Ch^{2m-1}||A||_{L^N(\mathbb{R}^N)}\ || - u(1 + h\phi/\epsilon)D\phi + hDu||_{L^2(\mathbb{R}^N)}||\psi||_{L^{\frac{2N}{N-2}}(\mathbb{R}^N)} \\
&\leq Ch^{2m-1}||A||_{L^N(\mathbb{R}^N)}\ || - u(1 + h\phi/\epsilon)D\phi + hDu||_{H^{\frac{m-2}{2}}(\mathbb{R}^N)}||\psi||_{H^1(\mathbb{R}^N)} \\
&\leq Ch^m||u||_{H^{m/2}_{scl}(\mathbb{R}^N)}||\psi||_{H^{m/2}_{scl}(\mathbb{R}^N)} \leq Ch^m||u||_{H^{m/2}_{scl}(\mathbb{R}^N)}||\psi||_{H^{3m/2}_{scl}(\mathbb{R}^N)}.
\end{aligned}
$$

Thus, for any $m \geq 2$, by definition of dual norm we have

$$||e^{\phi_\epsilon/h} h^{2m} D_A(e^{-\phi_\epsilon/h}u)||_{H^{-3m/2}_{scl}(\mathbb{R}^N)} \leq Ch^m||A||_{W^{-\frac{m}{2}+1,p'}(\mathbb{R}^N)}||u||_{H^{m/2}_{scl}(\mathbb{R}^N)}.$$

Combining this together with (2.8) and (2.10), for small enough $h > 0$ and $m \geq 2$, we get

$$||u||_{H^{m/2}_{scl}(\mathbb{R}^N)} \lesssim \frac{1}{h^m}||e^{\phi_\epsilon/h}(h^{2m}\mathcal{L}_{A,q})e^{-\phi_\epsilon/h}u||_{H^{-3m/2}_{scl}(\mathbb{R}^N)}. \tag{2.11}$$

Since $e^{-\phi_\epsilon/h}u = e^{-\phi/h}e^{-\phi^2/2\epsilon}u$ and ϕ is smooth, we obtain (2.7). ■

Remark 2. *Note that the Carleman estimate in Proposition 2.2.1 is valid for any $\tilde{t} \in \mathbb{R}$. We have in particular chosen $s = -3m/2$ so that $s + 2m = m/2$. The main motivation for choosing this particular value of s is to get bounds on $H^{m/2}_{scl}(\mathbb{R}^N)$ norm of u. Though the direct problem has a solution in $H^m(\Omega)$, we only need Carleman estimates in the $H^{m/2}_{scl}$ norm.*

A natural question would be, why in particular has s been chosen so that $s + 2m = m/2$. If we choose $s + 2m < m/2$ or $s + 2m > m/2$, then in the former case we will have to take more regular A and q to ensure that we have the correct decay essentially as dictated by the *hypotheses in Proposition 2.2.2, or in the latter case* ($s+2m > m/2$) *we can no longer ensure a decay of at least* $\mathcal{O}(h^m)$ *for* $||e^{\phi/h}(h^{2m}\mathcal{L}_{A,q})e^{-\phi/h}u||_{H^s_{scl}(\mathbb{R}^N)}$*, and* $\mathcal{O}(h^m)$ *decay is crucially used in the construction of complex geometric optics solutions.*

We now use the above proved Carleman estimate to first establish an existence and uniqueness result for the inhomogeneous partial differential equation. Let $\phi \in C^\infty(\tilde{\Omega}, \mathbb{R})$ be a limiting Carleman weight for $-h^2\Delta$. Set

$$\mathcal{L}_\phi := e^{\phi/h}(h^{2m}\mathcal{L}_{A,q})e^{-\phi/h}.$$

By Proposition (2.4.3), we have

$$\langle \mathcal{L}_\phi u, \bar{v}\rangle_\Omega = \langle u, \overline{\mathcal{L}^*_\phi v}\rangle_\Omega, \quad u, v \in C^\infty_0(\Omega),$$

where $\mathcal{L}^*_\phi = e^{-\phi/h}(h^{2m}\mathcal{L}_{\bar{A},\, \bar{q}+D\cdot\bar{A}})e^{\phi/h}$ is the formal adjoint of $\mathcal{L}_\phi$. Note that the zeroth order coefficient of the adjoint operator $\mathcal{L}^*_{A,q}$ comprises of two terms $\bar{q}$ and $D\cdot\bar{A}$. The Carleman estimate for $\bar{q}$ is the same as (2.9) as $\bar{q}$ lies in the same class as q. However $D\cdot\bar{A} \in W^{-\frac{m}{2},p'}(\mathbb{R}^N) \cap \mathcal{E}'(\bar{\Omega})$ where $p' \geq \frac{2N}{m}$ if $m < N$; $p' > 2$ if $m = N$ or $m = N+2$; and $p' \geq 2$ otherwise. As mentioned in Remark 1, estimate (2.9) goes through for zeroth order perturbation belonging to this class too. Hence, estimate (2.7) is valid for $\mathcal{L}^*_\phi$, since $-\phi$ is a limiting Carleman weight as well.

We now convert the Carleman estimate (2.7) for $\mathcal{L}^*_\phi$ in to a solvability result for $\mathcal{L}_\phi$.

For $s \geq 0$, we define semi-classical Sobolev norms on a smooth bounded domain Ω as

$$||u||_{H^s_{scl}(\Omega)} := \inf_{v\in H^s_{scl}(\mathbb{R}^N),\, v|_\Omega = u} ||v||_{H^s_{scl}(\mathbb{R}^N)},$$
$$||u||_{H^{-s}_{scl}(\Omega)} := \sup_{0\neq\phi\in C^\infty_0(\Omega)} \frac{|\langle u, \bar{\phi}\rangle|}{||\phi||_{H^s_{scl}(\Omega)}}.$$

Proposition 2.2.4. *Let* A *and* q *be as defined in (2.2) and (2.3) respectively and let* ϕ *be a limiting Carleman weight for* $-h^2\Delta$ *on* $\tilde{\Omega}$*. If* $h > 0$ *is small enough, then for any* $v \in H^{-\frac{m}{2}}(\Omega)$*, there is a solution* $u \in H^{\frac{m}{2}}(\Omega)$ *of the equation*

$$e^{\phi/h}(h^{2m}\mathcal{L}_{A,q})e^{-\phi/h}u = v \quad in \quad \Omega,$$

which satisfies

$$||u||_{H^{\frac{m}{2}}_{scl}(\Omega)} \lesssim \frac{1}{h^m}||v||_{H^{-\frac{m}{2}}_{scl}(\Omega)}.$$

Proof. Let $D = \mathcal{L}_\phi^*(C_0^\infty(\Omega))$ and consider the linear functional $L : D \to \mathbb{C}$, $L(\mathcal{L}_\phi^* w) = \langle w, v\rangle_\Omega$ for $w \in C_0^\infty(\Omega)$. By Carleman estimate (2.7),

$$|L(\mathcal{L}_\phi^* w)| \le ||w||_{H^{m/2}_{scl}(\mathbb{R}^N)}||v||_{H^{-m/2}_{scl}(\Omega)} \le \frac{C}{h^m}||\mathcal{L}_\phi^* w||_{H^{-m/2}_{scl}(\mathbb{R}^N)}||v||_{H^{-m/2}_{scl}(\Omega)}.$$

Hahn-Banach theorem ensures that there is a bounded linear functional $\tilde{L} : H^{-m/2}(\mathbb{R}^N) \to \mathbb{C}$ satisfying $\tilde{L} = L$ on D and $||\tilde{L}|| \le Ch^{-m}||v||_{H^{-m/2}_{scl}(\Omega)}$. By the Riesz Representation theorem, there is $u \in H^{m/2}(\mathbb{R}^N)$ such that for all $\psi \in H^{-m/2}(\mathbb{R}^N)$, $\tilde{L}(\psi) = \langle \psi, \bar{u}\rangle_{\mathbb{R}^N}$ and

$$||u||_{H^{m/2}_{scl}(\mathbb{R}^N)} \le \frac{C}{h^m}||v||_{H^{-m/2}_{scl}(\Omega)}.$$

Let us now show $\mathcal{L}_\phi u = v$ in Ω.

For arbitrary $w \in C_0^\infty(\Omega)$,

$$\langle \mathcal{L}_\phi u, \bar{w}\rangle_\Omega = \langle u, \overline{\mathcal{L}_\phi^* w}\rangle_{\mathbb{R}^N} = \overline{\tilde{L}(\mathcal{L}_\phi^* w)} = \overline{L(\mathcal{L}_\phi^* w)} = \overline{\langle w, v\rangle}_\Omega = \langle v, \bar{w}\rangle_\Omega.$$

This finishes the proof. ∎

We now wish to construct complex geometric optics solutions for the equation $\mathcal{L}_{A,q}u = 0$ in Ω with A and q as defined in (2.2) and (2.3) respectively using the solvability result Proposition 2.2.4. These are solutions of the form

$$u(x,\zeta;h) = e^{\frac{ix\cdot\zeta}{h}}(a(x,\zeta;h) + h^{m/2}r(x,\zeta;h)), \tag{2.12}$$

where $\zeta \in \mathbb{C}^N$ is such that $\zeta\cdot\zeta = 0$, $|\zeta| \sim 1$, $a \in C^\infty(\bar{\Omega})$ is an amplitude, r is a correction term, and $h > 0$ is a small parameter.

Conjugating $h^{2m}\mathcal{L}_{A,q}$ by $e^{\frac{ix\cdot\zeta}{h}}$, we get

$$e^{-\frac{ix\cdot\zeta}{h}}h^{2m}\mathcal{L}_{A,q}e^{\frac{ix\cdot\zeta}{h}} = (-h^2\Delta - 2i\zeta\cdot h\nabla)^m + h^{2m}D_A + h^{2m-1}m_{A\cdot\zeta} + h^{2m}m_q. \tag{2.13}$$

Following [53], we shall consider ζ depending slightly on h, i.e $\zeta = \zeta_0 + \zeta_1$ with ζ_0 independent of h and $\zeta_1 = \mathcal{O}(h)$ as $h \to 0$. We also assume that $|Re\,\zeta_0| = |Im\zeta_0| = 1$. Then we can write (2.13) as

$$e^{-\frac{ix\cdot\zeta}{h}} h^{2m} \mathcal{L}_{A,q} e^{\frac{ix\cdot\zeta}{h}} = (-h^2\Delta - 2i\zeta_0 \cdot h\nabla - 2i\zeta_1 \cdot h\nabla)^m + h^{2m} D_A + h^{2m-1} m_{A\cdot(\zeta_1+\zeta_0)} + h^{2m} m_q.$$

Observe that (2.12) is a solution to $\mathcal{L}_{A,q} = 0$ if and only if

$$e^{-\frac{ix\cdot\zeta}{h}} h^{2m} \mathcal{L}_{A,q}(e^{\frac{ix\cdot\zeta}{h}} h^{m/2} r) = -e^{-\frac{ix\cdot\zeta}{h}} h^{2m} \mathcal{L}_{A,q}(e^{\frac{ix\cdot\zeta}{h}} a),$$

and hence if and only if

$$\begin{aligned} & e^{-\frac{ix\cdot\zeta}{h}} h^{2m} \mathcal{L}_{A,q}(e^{\frac{ix\cdot\zeta}{h}} h^{m/2} r) \\ & \qquad = -\sum_{k=0}^{m} \frac{m!}{k!(m-k)!} (-h^2\Delta - 2i\zeta_1 \cdot h\nabla)^{m-k} (-2i\zeta_0 \cdot h\nabla)^k a \\ & \qquad\quad - h^{2m} D_A a - h^{2m-1} m_{A\cdot(\zeta_0+\zeta_1)} a - h^{2m} m_q a. \end{aligned} \tag{2.14}$$

Our goal is get a decay of at least $\mathcal{O}(h^{m+m/2})$ in $H^{-m/2}_{scl}(\Omega)$ norm on the right-hand side of (2.14). The terms $h^{2m} D_A a$, $h^{2m-1} m_{A\cdot(\zeta_0+\zeta_1)} a$ and $h^{2m} m_q a$ will eventually give us a decay of $\mathcal{O}(h^{m+m/2})$ provided $m \geq 2$.

(For a smooth enough first order perturbation of the polyharmonic operator we only need an $\mathcal{O}(h^{m+1})$ decay, but here we need a stronger decay of $\mathcal{O}(h^{m+m/2})$ essentially because our coefficients are less regular. See Remark 3 for more details.)

If $a \in C^\infty(\bar{\Omega})$ satisfies

$$(\zeta_0 \cdot \nabla)^j a = 0 \quad \text{in} \quad \Omega,$$

for some $j \geq 1$, then since $\zeta_1 = \mathcal{O}(h)$, the lowest order of h on the right-hand side of (2.14) is $j - 1 + 2(m - j + 1) = 2m - j + 1$ provided $j \geq 2$. We will hence obtain an overall decay of $\mathcal{O}(h^{m+m/2})$ on the right-hand side of (2.14) provided $j \leq 1 + m/2$. Since $m \geq 2$, we choose $j = 2$ to get the following transport equation,

$$(\zeta_0 \cdot \nabla)^2 a = 0 \quad \text{in} \quad \Omega. \tag{2.15}$$

Such choice of a is clearly possible. We thus obtain the following equation for r:

$$\begin{aligned}
&e^{-\frac{ix\cdot\zeta}{h}} h^{2m} \mathcal{L}_{A,q}(e^{\frac{ix\cdot\zeta}{h}} h^{m/2} r) \\
&\qquad = -(-h^2\Delta - 2i\zeta_1 \cdot h\nabla)^m a - m(-h^2\Delta - 2i\zeta_1 \cdot h\nabla)^{m-1}(-2i\zeta_0 \cdot h\nabla)a \\
&\qquad\quad - h^{2m} D_A(a) - h^{2m-1} m_{A\cdot(\zeta_0+\zeta_1)}(a) - h^{2m} m_q(a) := g.
\end{aligned}$$

We complete the proof by showing $||g||_{H^{-m/2}_{scl}(\Omega)} = \mathcal{O}(h^{m+m/2})$. We will estimate each term separately.

Suppose that $\psi \in C_0^\infty(\Omega)$ and $\psi \neq 0$. By Cauchy-Schwarz inequality and the fact that $\zeta_1 = \mathcal{O}(h)$ and $\zeta_0 = \mathcal{O}(1)$ we get

$$\begin{aligned}
&\left|(-(-h^2\Delta - 2i\zeta_1 \cdot h\nabla)^m a - m(-h^2\Delta - 2i\zeta_1 \cdot h\nabla)^{m-1}(-2i\zeta_0 \cdot h\nabla)a, \psi)_{L^2(\Omega)}\right| \\
&\qquad = \mathcal{O}(h^{m+m/2})||\psi||_{L^2(\Omega)} = \mathcal{O}(h^{m+m/2})||\psi||_{H^{m/2}_{scl}(\Omega)}.
\end{aligned} \tag{2.16}$$

For $m > 2$, we have

$$\begin{aligned}
|\langle h^{2m-1} m_{A\cdot(\zeta_0+\zeta_1)}(a), \psi\rangle_\Omega| &\leq Ch^{2m-1}||A||_{W^{-\frac{m}{2}+1,p'}(\mathbb{R}^N)}||a\psi||_{W^{\frac{m}{2}-1,p}(\mathbb{R}^N)} \\
&\leq Ch^{2m-1}||a\psi||_{W^{\frac{m}{2}-1,p}(\mathbb{R}^N)} \text{ (as } A \in W^{-\frac{m}{2}+1,p'}(\mathbb{R}^N)) \\
&\leq Ch^{m+m/2+m/2-1}||\psi||_{H^{\frac{m}{2}-1}(\mathbb{R}^N)} \text{ (by Proposition 2.2.2)} \\
&= \mathcal{O}(h^{m+m/2})||\psi||_{H^{m/2}_{scl}(\mathbb{R}^N)} = \mathcal{O}(h^{m+m/2})||\psi||_{H^{m/2}_{scl}(\Omega)}.
\end{aligned} \tag{2.17}$$

(Here, we have used Proposition 2.2.2 for $m < N$ with $p_1 = p_2 = 2$, $p \in (1, \frac{2N}{2N-m})$, $s_1 = \frac{m}{2} - 1, s_2 = \frac{m}{2}$. For $m = N$, we choose $p_1 = p_2 = 2$, $p \in (1, 2)$, $s_1 = \frac{m}{2} - 1, s_2 = \frac{m}{2}$. And for $m > N$, we choose $p_1 = p_2 = 2$, $p \in (1, 2]$, $s_1 = \frac{m}{2} - 1, s_2 = \frac{m}{2}$.)

Thus (2.17) is justified for all $m > 2$.

Similarly, for $m > 2$, we also have

$$\begin{aligned}
|\langle h^{2m} D_A(a), \psi\rangle_\Omega| &\leq Ch^{2m}||A||_{W^{-\frac{m}{2}+1,p'}(\mathbb{R}^N)}||\psi Da||_{W^{\frac{m}{2}-1,p}(\mathbb{R}^N)} \\
&\leq Ch^{m+m/2+m/2}||A||_{W^{-\frac{m}{2}+1,p'}(\mathbb{R}^N)}||\psi||_{H^{m/2}(\mathbb{R}^N)} \text{ (by Proposition 2.2.2)} \\
&= \mathcal{O}(h^{m+m/2})||\psi||_{H^{m/2}_{scl}(\Omega)}.
\end{aligned} \tag{2.18}$$

For $m = 2$, we have

$$\begin{aligned}
|\langle h^{2m-1} m_{A\cdot(\zeta_0+\zeta_1)}(a), \psi\rangle_\Omega| &\le Ch^{2m-1}||A||_{L^N(\mathbb{R}^N)}||a\psi||_{L^{N'}(\Omega)} \\
&\le Ch^{2m-1}||a\psi||_{L^{N'}(\Omega)} \\
&\le Ch^{2m-1}||\psi||_{L^2(\Omega)} \text{ (by Hölder's inequality)} \\
&= \mathcal{O}(h^{m+m/2})||\psi||_{H^{m/2}_{scl}(\Omega)},
\end{aligned}$$

and

$$\begin{aligned}
|\langle h^{2m} D_A(a), \psi\rangle_\Omega| &\le Ch^{2m}||A||_{L^N(\mathbb{R}^N)}||\psi D a||_{L^{N'}(\Omega)} \\
&\le Ch^{m+m/2+m/2}||A||_{L^N(\mathbb{R}^N)}||\psi||_{L^2(\Omega)} \text{ (by Hölder's inequality)} \\
&= \mathcal{O}(h^{m+m/2})||\psi||_{H^{m/2}_{scl}(\Omega)}.
\end{aligned} \tag{2.19}$$

We also have, for any $m \ge 2$,

$$\begin{aligned}
|\langle h^{2m} m_q a, \psi\rangle_\Omega| &\le Ch^{2m}||q||_{W^{-\frac{m}{2}+\delta,r'}(\mathbb{R}^N)}||a\psi||_{W^{\frac{m}{2}-\delta,r}(\mathbb{R}^N)} \text{ (as } q \in W^{-\frac{m}{2}+\delta,r'}(\mathbb{R}^N)) \\
&\le Ch^{m+m/2+m/2}||q||_{W^{-\frac{m}{2}+\delta,r'}(\mathbb{R}^N)}||\psi||_{H^{\frac{m}{2}}(\mathbb{R}^N)} \text{ (by Proposition 2.2.2)} \\
&= \mathcal{O}(h^{m+m/2})||\psi||_{H^{m/2}_{scl}(\mathbb{R}^N)} = \mathcal{O}(h^{m+m/2})||\psi||_{H^{m/2}_{scl}(\Omega)}.
\end{aligned} \tag{2.20}$$

Combining the estimates (2.16 - 2.20) we conclude that for any $m \ge 2$,

$$||g||_{H^{-m/2}_{scl}(\Omega)} = \mathcal{O}(h^{m+m/2}).$$

Using this and Propostion 2.2.4, for $h > 0$ small enough, there exists $r \in H^{m/2}(\Omega)$ solving

$$e^{-\frac{ix\cdot\zeta}{h}} h^{2m}\mathcal{L}_{A,q}(e^{\frac{ix\cdot\zeta}{h}} h^{m/2} r) = -e^{-\frac{ix\cdot\zeta}{h}} h^{2m}\mathcal{L}_{A,q}(e^{\frac{ix\cdot\zeta}{h}} a),$$

such that

$$||h^{m/2} r||_{H^{m/2}_{scl}(\Omega)} \lesssim \frac{1}{h^m}||e^{\frac{-ix\cdot\zeta}{h}} h^{2m}\mathcal{L}_{A,q}(e^{\frac{ix\cdot\zeta}{h}} a)||_{H^{-m/2}_{scl}(\Omega)} \lesssim \frac{1}{h^m}||g||_{H^{-m/2}_{scl}(\Omega)} = \mathcal{O}(h^{m/2}).$$

Therefore, $||r||_{H^{m/2}_{scl}(\Omega)} = \mathcal{O}(1)$. We have thus proved the following result.

Proposition 2.2.5. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ be a bounded open set with smooth boundary and let m be an integer so that $m \geq 2$. Suppose A and q satisfy (2.2) and (2.3), respectively, and let $\zeta \in \mathbb{C}^N$ be such that $\zeta \cdot \zeta = 0$, $\zeta = \zeta_0 + \zeta_1$ with ζ_0 independent of h and $\zeta_1 = \mathcal{O}(h)$ as $h \to 0$. Then for all $h > 0$ small enough, there exists a solution $u(x, \zeta; h) \in H^{m/2}(\Omega)$ to the equation $\mathcal{L}_{A,q} u = 0$ of the form*

$$u(x, \zeta; h) = e^{\frac{ix\cdot\zeta}{h}}(a(x, \zeta_0) + h^{m/2} r(x, \zeta; h)),$$

where $a(\cdot, \zeta_0) \in C^\infty(\bar{\Omega})$ satisfies (2.15) and the correction term r is such that $||r||_{H^{m/2}_{scl}(\Omega)} = \mathcal{O}(1)$ as $h \to 0$.

2.3 Integral Identity

We first perform a standard reduction to a larger domain. For the proof, we follow [53, Proposition 3.2].

Proposition 2.3.1. *Let Ω, $\Omega' \subset \mathbb{R}^N$ be two bounded open sets such that $\Omega \subset\subset \Omega'$ and $\partial\Omega$ and $\partial\Omega'$ are smooth. Let A_1, A_2 and q_1, q_2 satisfy (2.2) and (2.3), respectively. If $\mathcal{N}_{A_1,q_1} = \mathcal{N}_{A_2,q_2}$, then $\mathcal{N}'_{A_1,q_1} = \mathcal{N}'_{A_2,q_2}$ where $\mathcal{N}'_{A_j,q_j}$ denotes the set of the Dirichlet-to-Neumann map for $\mathcal{L}_{A_j,q_j}$ in Ω', $j = 1, 2$.*

Proof. Let $f' \in \prod_{j=0}^{m-1} H^{m-j-1/2}(\partial\Omega)$ and let $v_1' \in H^m(\Omega')$ be the unique solution (which exists by the results in section 2.4.2) to $\mathcal{L}_{A_1,q_1} v_1' = 0$ in Ω' with $\gamma' v_1' = f'$ on $\partial\Omega'$ where γ' denotes the Dirichlet trace on $\partial\Omega'$. Let $v_1 = v_1'\big|_\Omega \in H^m(\Omega)$ and let $f = \gamma v_1$. By the well-posedness result in section 2.4.2, we can guarantee the existence of a unique $v_2 \in H^m(\Omega)$ so that $\mathcal{L}_{A_2,q_2} v_2 = 0$ and $\gamma v_2 = \gamma v_1 = f$. Thus $\phi = v_2 - v_1 \in H^m_0(\Omega) \subset H^m_0(\Omega')$. Define

$$v_2' = v_1' + \phi \in H^m(\Omega').$$

Note that $v_2' = v_2$ in Ω and $\gamma' v_2' = \gamma' v_1' = f'$ on $\partial\Omega'$.

We now show that $\mathcal{L}_{A_2,q_2}v_2' = 0$ in Ω'. Let $\psi \in C_0^\infty(\Omega')$. We then have

$$\langle \mathcal{L}_{A_2,q_2}v_2', \bar{\psi}\rangle_{\Omega'} = \sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha v_2', D^\alpha \psi)_{L^2(\Omega')} + \langle D_{A_2}(v_2'), \bar{\psi}\rangle_{\Omega'} + \langle m_{q_2}(v_2'), \bar{\psi}\rangle_{\Omega'}.$$

Since A_2 and q_2 are compactly supported in $\bar{\Omega}$ and $\phi \in H_0^m(\Omega)$, we can rewrite the above equality as

$$\begin{aligned}\langle \mathcal{L}_{A_2,q_2}v_2', \bar{\psi}\rangle_{\Omega'} &= \sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha v_1', D^\alpha \psi)_{L^2(\Omega')} + \sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha \phi, D^\alpha (\psi\big|_\Omega)_{L^2(\Omega)} \\ &\quad + B_{A_2}(v_2', \bar{\psi}\big|_\Omega) + b_{q_2}(v_2', \bar{\psi}\big|_\Omega) \\ &= \sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha v_1', D^\alpha \psi)_{L^2(\Omega')} - \sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha v_1, D^\alpha (\psi\big|_\Omega))_{L^2(\Omega)} \\ &\quad + \sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha v_2, D^\alpha (\psi\big|_\Omega))_{L^2(\Omega)} + B_{A_2}(v_2', \bar{\psi}\big|_\Omega) + b_{q_2}(v_2', \bar{\psi}\big|_\Omega).\end{aligned}$$

Note that

$$\langle \mathcal{N}_{A_2,q_2}f, \overline{\gamma(\psi\big|_\Omega)}\rangle_{\partial\Omega} = \sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha v_2, D^\alpha (\psi\big|_\Omega)_{L^2(\Omega)} + B_{A_2}(v_2', \bar{\psi}\big|_\Omega) + b_{q_2}(v_2', \bar{\psi}\big|_\Omega).$$

Hence, we have

$$\begin{aligned}\langle \mathcal{L}_{A_2,q_2}v_2', \bar{\psi}\rangle_{\Omega'} = \sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha v_1', D^\alpha \psi)_{L^2(\Omega')} - \sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha v_1, D^\alpha (\psi\big|_\Omega))_{L^2(\Omega)} \\ + \langle \mathcal{N}_{A_2,q_2}f, \gamma(\psi\big|_\Omega)\rangle_{\partial\Omega}.\end{aligned}$$

Since

$$\langle \mathcal{N}_{A_2,q_2}f, \overline{\gamma(\psi\big|_\Omega)}\rangle_{\partial\Omega} = \langle \mathcal{N}_{A_1,q_1}f, \gamma(\psi\big|_\Omega)\rangle_{\partial\Omega}$$

and

$$\langle \mathcal{N}_{A_1,q_1}f, \overline{\gamma(\psi\big|_\Omega)}\rangle_{\partial\Omega} = \sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha v_1, D^\alpha (\psi\big|_\Omega))_{L^2(\Omega)} + B_{A_1}(v_1, \bar{\psi}\big|_{\partial\Omega}) + b_{q_1}(v_1, \bar{\psi}\big|_\Omega).$$

We get

$$\langle \mathcal{L}_{A_2,q_2}v_2', \bar{\psi}\rangle_{\Omega'} = \sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha v_1', D^\alpha \psi)_{L^2(\Omega')} + B_{A_1}(v_1, \bar{\psi}\big|_{\partial\Omega}) + b_{q_1}(v_1, \bar{\psi}\big|_\Omega).$$

Using the fact A_1 and q_1 are compactly supported in $\bar{\Omega}$, we obtain

$$\begin{aligned}\langle \mathcal{L}_{A_2,q_2} v_2', \bar{\psi}\rangle_{\Omega'} &= \sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha v_1', D^\alpha \psi)_{L^2(\Omega')} + \langle D_{A_1}(v_1'), \bar{\psi}\rangle_{\Omega'} + \langle m_{q_1}(v_1'), \bar{\psi}\rangle_{\Omega'} \\ &= \langle \mathcal{L}_{A_1,q_1} v_1', \bar{\psi}\rangle_{\Omega'} = 0.\end{aligned}$$

Using exact same arguments, one can show that $\mathcal{N}'_{A_2,q_2} f' = \mathcal{N}'_{A_1,q_1} f'$ on $\partial\Omega'$, which finishes the proof. ■

We now derive the following integral identity based on the assumption that $\mathcal{N}_{A_1,q_1} = \mathcal{N}_{A_2,q_2}$.

Proposition 2.3.2. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ be a bounded open set with smooth boundary. Assume that A_1, A_2 and q_1, q_2 satisfy (2.2) and (2.3), respectively. If $\mathcal{N}_{A_1,q_1} = \mathcal{N}_{A_2,q_2}$, then the following integral identity holds*

$$\langle D_{A_2-A_1}(u_2), \bar{v}\rangle_\Omega + \langle m_{q_2-q_1}(u_2), \bar{v}\rangle_\Omega = 0,$$

for any $u_2, v \in H^m(\Omega)$ satisfying $\mathcal{L}_{A_2,q_2} u_2 = 0$ in Ω and $\mathcal{L}^*_{A_1,q_1} v = 0$ in Ω, respectively. Recall that $\mathcal{L}^*_{A,q} = \mathcal{L}_{\bar{A},\,\bar{q}+D\cdot\bar{A}}$ is the formal adjoint of $\mathcal{L}_{A,q}$.

Proof. Let v satisfy $\mathcal{L}^*_{A_1,q_1} v = 0$ in Ω. Let u_1 solve $\mathcal{L}_{A_1,q_1} u_1 = 0$ in Ω. Since $\mathcal{N}_{A_1,q_1} = \mathcal{N}_{A_2,q_2}$, we choose $u_2 \in H^m(\Omega)$ solving $\mathcal{L}_{A_2,q_2} u_2$ so that $\gamma u_1 = \gamma u_2$ and $\mathcal{N}_{A_1,q_1}\gamma u_1 = \mathcal{N}_{A_2,q_2}\gamma u_2$. It then follows that

$$\mathcal{L}_{A_1,q_1}(u_1 - u_2) = D_{A_2-A_1}(u_2) + m_{q_2-q_1}(u_2).$$

Since

$$\langle \mathcal{L}_{A,q} u, \bar{v}\rangle_\Omega = \langle u, \overline{\mathcal{L}^*_{A,q} v}\rangle_\Omega,$$

we get the desired identity. ■

To show $A_1 = A_2$, we will need to use Poincaré lemma for currents [23] which requires the domain to be simply connected. Therefore, we reduce the problem to larger simply connected domain, in particular to a ball.

Let B be an open ball in $\mathbb{R}^N$ such that $\Omega \subset\subset B$. According to Proposition (2.3.1), we know that $\mathcal{N}'^{B}_{A_1,q_1} = \mathcal{N}'^{B}_{A_2,q_2}$, where $\mathcal{N}'^{B}_{A_j,q_j}$ denotes the Dirichlet-to-Neumann map for $\mathcal{L}_{A_j,q_j}$ in B, $j = 1, 2$. By Proposition (2.3.2), the following integral identity holds.

$$B^B_{A_2-A_1}(u_2, \bar{v}) + b^B_{q_2-q_1}(u_2, \bar{v}) = 0, \tag{2.21}$$

for any $u_2, v \in H^m(B)$ satisfying $\mathcal{L}_{A_2,q_2} u_2 = 0$ in B and $\mathcal{L}^*_{A_1,q_1} v = 0$ in B, respectively. Here and in what follows, by $B^B_{A_2-A_1}$ and $b^B_{q_2-q_1}$ we denote the bi-linear forms corresponding to $A_2 - A_1$ and $q_2 - q_1$ (as defined in Section 2.4.1) in the ball B.

The key idea now is to use complex geometric optics solutions u_2 to $\mathcal{L}_{A_2,q_2} u_2 = 0$ in B and v to $\mathcal{L}^*_{A_1,q_1} v = 0$ in B, and plug them in the integral identity (2.21). In order to construct these solutions, consider $\xi, \mu_1, \mu_2 \in \mathbb{R}^N$ such that $|\mu_1| = |\mu_2| = 1$ and $\mu_1 \cdot \mu_2 = \xi \cdot \mu_1 = \xi \cdot \mu_2 = 0$. (We use $N \geq 3$ at this step.) For $h > 0$, set

$$\zeta_2 = \frac{h\xi}{2} + \sqrt{1 - h^2 \frac{|\xi|^2}{4}}\mu_1 + i\mu_2, \quad \zeta_1 = -\frac{h\xi}{2} + \sqrt{1 - h^2 \frac{|\xi|^2}{4}}\mu_1 - i\mu_2.$$

Note that we have $\zeta_2 = \mu_1 + i\mu_2 + \mathcal{O}(h)$, $\zeta_1 = \mu_1 - i\mu_2 + \mathcal{O}(h)$, $\zeta_j \cdot \zeta_j = 0$, $j = 1, 2$ and $\zeta_2 - \bar{\zeta}_1 = h\xi$.

By Proposition 2.2.5, for all $h > 0$ small enough, there are solutions $u_2(\cdot, \zeta_2; h)$ and $v(\cdot, \zeta_1; h)$ in $H^m(B)$ to the equations $\mathcal{L}_{A_2,q_2} u_2 = 0$ and $\mathcal{L}^*_{A_1,q_1} v = 0$ in B, respectively, and of the form

$$v(x, \zeta_1; h) = e^{\frac{ix\cdot\zeta_1}{h}}(a_1(x, \mu_1 + i\mu_2) + h^{m/2} r_1(x, \zeta_1; h)),$$
$$u_2(x, \zeta_2; h) = e^{\frac{ix\cdot\zeta_2}{h}}(a_2(x, \mu_1 - i\mu_2) + h^{m/2} r_2(x, \zeta_1; h)),$$

where the amplitudes $a_1(x, \mu_1 + i\mu_2)$, $a_2(x, \mu_1 - i\mu_2) \in C^\infty(\bar{B})$ satisfy the transport equations

$$\begin{aligned} ((\mu_1 + i\mu_2) \cdot \nabla)^2 a_2(x, \mu_1 + i\mu_2) = 0 \quad \text{in } B, \\ ((\mu_1 - i\mu_2) \cdot \nabla)^2 a_1(x, \mu_1 - i\mu_2) = 0 \quad \text{in } B, \end{aligned} \tag{2.22}$$

and the remainder terms $r_1(., \zeta_1; h)$ and $r_2(., \zeta_2; h)$ satisfy

$$||r_j||_{H^{m/2}_{scl}(B)} = \mathcal{O}(1), \quad j = 1, 2.$$

We substitute u_2 and v in to the (2.21) and get

$$\begin{aligned} 0 = &\frac{1}{h} b^B_{\zeta_2\cdot(A_2-A_1)}(a_2 + h^{m/2} r_2, e^{ix\cdot\xi}(\bar{a}_1 + h^{m/2}\bar{r}_1)) \\ &+ B^B_{A_2-A_1}(a_2 + h^{m/2} r_2, e^{ix\cdot\xi}(\bar{a}_1 + h^{m/2}\bar{r}_1)) \\ &+ b^B_{q_2-q_1}(a_2 + h^{m/2} r_2, e^{ix\cdot\xi}(\bar{a}_1 + h^{m/2}\bar{r}_1)). \end{aligned} \tag{2.23}$$

Multiply by h throughout and let $h \to 0$ to get

$$b^B_{(\mu_1+i\mu_2)\cdot(A_2-A_1)}(a_2, e^{ix\cdot\xi}\bar{a}_1) = 0. \tag{2.24}$$

Let us justify how we get (2.24). We use Proposition 2.4.2 to show

$$\begin{aligned} &|B^B_{A_2-A_1}(a_2 + h^{m/2} r_2, e^{ix\cdot\xi}(\bar{a}_1 + h^{m/2} h\bar{r}_1))| \\ &\quad \le C||A_1 - A_2||_{W^{-\frac{m}{2}+1,p'}(\mathbb{R}^N)} ||a_2 + h^{m/2} r_2||_{H^{m/2}(B)}\, ||\bar{a}_1 + h^{m/2}\bar{r}_1||_{H^{m/2}(B)} \\ &\quad \le C(||a_1||_{H^{m/2}(B)} + ||h^{m/2} r_1||_{H^{m/2}(B)})(||\bar{a}_2||_{H^{m/2}(B)} + ||h^{m/2}\bar{r}_2||_{H^{m/2}(B)}) \\ &\quad \le C(||a_1||_{H^{m/2}(B)} + ||r_1||_{H^{m/2}_{scl}(B)})\,(||\bar{a}_2||_{H^{m/2}(B)} + ||\bar{r}_2||_{H^{m/2}_{scl}(B)}) = \mathcal{O}(1). \end{aligned}$$

Hence

$$h|B^B_{A_2-A_1}(a_2 + h^{m/2} r_2, e^{ix\cdot\xi}(\bar{a}_1 + h^{m/2}\bar{r}_1))| = \mathcal{O}(h). \tag{2.25}$$

We also have for any $m \ge 2$, using Proposition 2.4.2,

$$\begin{aligned} &|b^B_{q_1-q_2}(a_1 + h^{m/2} r_1, e^{ix\cdot\xi}(\bar{a}_2 + h^{m/2}\bar{r}_2))| \\ &\quad \le C||q_1 - q_2||_{W^{-\frac{m}{2}+\delta,r'}(\mathbb{R}^N)} ||a_1 + h^{m/2} r_1||_{H^{\frac{m}{2}-\delta}(B)}\, ||\bar{a}_2 + h^{m/2}\bar{r}_2||_{H^{\frac{m}{2}-\delta}(B)} \\ &\quad \le C(||a_1||_{H^{m/2}(B)} + ||r_1||_{H^{m/2}_{scl}(B)})\,(||\bar{a}_2||_{H^{m/2}(B)} + ||\bar{r}_2||_{H^{m/2}_{scl}(B)}) = \mathcal{O}(1). \end{aligned}$$

Hence,

$$h|b^B_{q_1-q_2}(a_1 + h^{m/2} r_1, e^{ix\cdot\xi}(\bar{a}_2 + h^{m/2}\bar{r}_2))| = \mathcal{O}(h). \tag{2.26}$$

Thus, we see that after multiplying (2.23) by h, the latter 2 terms in (2.23) go to zero as $h \to 0$.

We also need to justify that

$$\begin{aligned} |b^B_{\zeta_2\cdot(A_2-A_1)}(a_2, e^{ix\cdot\xi}h^{m/2}\bar{r}_1)| = \mathcal{O}(h), \quad |b^B_{\zeta_2\cdot(A_2-A_1)}(h^{m/2}r_2, e^{ix\cdot\xi}h^{m/2}\bar{a}_1)| = \mathcal{O}(h), \\ |b^B_{\zeta_2\cdot(A_2-A_1)}(h^{m/2}r_2, e^{ix\cdot\xi}h^{m/2}\bar{r}_1)| = \mathcal{O}(h). \end{aligned} \tag{2.27}$$

We only show why $b^B_{\zeta_2\cdot(A_2-A_1)}(a_2, e^{ix\cdot\xi}h^{m/2}\bar{r}_1) = \mathcal{O}(h)$. The proof for other two terms follows similarly. By Proposition 2.2.2, we have for any $m \geq 2$,

$$\begin{aligned} |b^B_{\zeta_2\cdot(A_2-A_1)}(a_2, e^{ix\cdot\xi}h^{m/2}\bar{r}_1)| &\leq ||A_2 - A_1||_{W^{-\frac{m}{2}+1,p'}(\mathbb{R}^N)}||h^{m/2}\bar{r}_1||_{H^{\frac{m}{2}-1}(B)}||a_2||_{H^{\frac{m}{2}}(B_2)} \\ &= \mathcal{O}(h)||r_1||_{H^{\frac{m}{2}-1}_{scl}(B)} = \mathcal{O}(h). \end{aligned}$$

From (2.25), (2.26) and (2.27), we see that (2.24) is indeed justified.

Remark 3. *Observe that because our A and q are rough, by duality and Sobolev multiplication, we get estimates in $H^{m/2}(\Omega)$ norm and hence we need a decay of $h^{m/2}$ so that the $H^{m/2}_{scl}$ norm of the correction term is $\mathcal{O}(1)$. If we had just used an $\mathcal{O}(h)$ decay then we would eventually have to use Sobolev estimates in $H^1(\Omega)$, which would require A and q to have higher regularity.*

Now plug in $a_1 = a_2 = 1$ in (2.24) to obtain

$$\langle(\mu_1 + i\mu_2)\cdot(A_1 - A_2), e^{ix\cdot\xi}\rangle = 0.$$

We can run the whole argument starting from the construction of ζ_1 and ζ_2, this time with the triple $(\mu_1, -\mu_2, \xi)$ to obtain

$$\langle(\mu_1 - i\mu_2)\cdot(A_1 - A_2), e^{ix\cdot\xi}\rangle = 0.$$

The last two equalities then imply

$$\mu\cdot(\hat{A}_2(\xi) - \hat{A}_1(\xi)) = 0 \quad \text{for all} \quad \mu, \xi \in \mathbb{R}^N \quad \text{with} \quad \mu\cdot\xi = 0. \tag{2.28}$$

For each $\xi = (\xi_1, \xi_2, ..., \xi_N)$ and for $j \neq k$, $1 \leq j, k \leq N$, consider the vector $\mu = \mu(\xi, j, k)$ such that $\mu_j = -\xi_k$, $\mu_k = \xi_j$ and all other components equal to zero. Therefore, μ satisfies $\mu\cdot\xi = 0$. Hence, from (2.28), we obtain

$$\xi_j\cdot(\hat{A}_{1,k}(\xi) - \hat{A}_{2,k}(\xi)) - \xi_k\cdot(\hat{A}_{1,j}(\xi) - \hat{A}_{2,j}(\xi)) = 0,$$

which proves

$$\partial_j(A_{1,k} - A_{2,k}) - \partial_k(A_{1,j} - A_{2,j}) = 0 \quad \text{in} \quad \Omega, \quad 1 \le j, k \le N,$$

in the sense of distributions.

To prove $A_1 = A_2$, we consider $A_1 - A_2$ as a 1- current and using the Poincaré lemma for currents, we conclude that there is a $g \in \mathcal{D}'(\mathbb{R}^N)$ such that $\nabla g = A_1 - A_2$; see [23]. Note that g is a constant outside $\bar{B}$ since $A_1 - A_2 = 0$ in $\mathbb{R}^N \setminus \bar{B}$ (also near ∂B). Considering $g - c$ instead of g, we may instead assume $g \in \mathcal{E}'(\bar{B})$.

To show $A_1 = A_2$, consider (2.24) with $a_1(\cdot, \mu_1 - i\mu_2) = 1$ and $a_2(\cdot, \mu_1 + i\mu_2)$ satisfying

$$(\mu_1 + i\mu_2) \cdot \nabla)a_2(x, \mu_1 + i\mu_2) = 1 \quad \text{in} \quad B.$$

Such a choice of $a_2(\cdot, \mu_1 + i\mu_2)$ is possible because of (2.22). The previous equation is an inhomogeneous $\bar{\partial}$-equation and we can solve it by setting

$$a_2(x, \mu_1 + i\mu_2) = \frac{1}{2\pi} \int\limits_{\mathbb{R}^2} \frac{\chi(x - y_1\mu_1 - y_2\mu_2)}{y_1 + iy_2}\, dy_1\, dy_2,$$

where $\chi \in C_0^\infty(\mathbb{R}^N)$ is such that $\chi = 1$ near $\bar{B}$; see [70, Lemma 4.6].

From (2.24), we have

$$b^B_{(\mu_1+i\mu_2)\cdot\nabla g}(a_2, e^{ix\cdot\xi}) = 0.$$

Now, use the fact that $\mu_1 \cdot \xi = \mu_2 \cdot \xi = 0$ to get

$$\begin{aligned} 0 = -b^B_{(\mu_1+i\mu_2)\cdot\nabla g}(a_2, e^{ix\cdot\xi}) &= -\langle(\mu_1 + i\mu_2) \cdot \nabla g, e^{ix\cdot\xi} a_2\rangle_B \\ &= \langle g, e^{ix\cdot\xi}(\mu_1 + i\mu_2) \cdot \nabla a_2\rangle_B = \langle g, e^{ix\cdot\xi}\rangle_B. \end{aligned}$$

Since g is compactly supported, this gives $g = 0$ in $\mathbb{R}^N$, and in B in particular, implying $A_1 = A_2$.

To show $q_1 = q_2$, substitute $A_1 = A_2$ and $a_1 = a_2 = 1$ in to the identity (2.21) to obtain

$$b^B_{q_2-q_1}(1 + h^{m/2} r_2, (1 + h^{m/2}\bar{r}_1)e^{ix\cdot\xi}) = 0. \tag{2.29}$$

Let $h \to 0$ to get $\hat{q}_1(\xi) - \hat{q}_2(\xi) = 0$ for all $\xi \in \mathbb{R}^N$. To justify this we need to show that

$$b^B_{q_2-q_1}(h^{m/2}r_1, e^{ix\cdot\xi}) \to 0,\ b^B_{q_2-q_1}(h^{m/2}r_2, e^{ix\cdot\xi}) \to 0,\ b^B_{q_2-q_1}(h^{m/2}r_2, h^{m/2}\bar{r}_1 e^{ix\cdot\xi}) \to 0$$

as $h \to 0$. We will only consider the term $b^B_{q_2-q_1}(h^{m/2}r_1, e^{ix\cdot\xi})$. The justification for the other two terms follows similarly. We have for any $m \geq 2$

$$\begin{aligned}|b^B_{q_2-q_1}(h^{m/2}r_1, e^{ix\cdot\xi})| &\leq C||q_2 - q_1||_{W^{-\frac{m}{2}+\delta,r'}(\mathbb{R}^N)}||e^{ix\cdot\xi}||_{H^{\frac{m}{2}-\delta}(B)}||h^{\frac{m}{2}}r_1||_{H^{\frac{m}{2}-\delta}(B)}\\ &= \mathcal{O}(h^\delta)||r_1||_{H^{\frac{m}{2}-\delta}_{scl}(B)} = \mathcal{O}(h^\delta)||r_1||_{H^{\frac{m}{2}}_{scl}(B)} = \mathcal{O}(h^\delta).\end{aligned}$$

Since $\hat{q}_1(\xi) - \hat{q}_2(\xi) = 0$ for all $\xi \in \mathbb{R}^N$, we get $q_1 = q_2$ in B.

Remark 4. *If we take $\delta = 0$, then we see that all we can say using Propostion 2.2.2 is that $|b^B_{q_2-q_1}(h^{m/2}r_1, e^{ix\cdot\xi})| = \mathcal{O}(1)$. This is why we impose slightly higher regularity for q.*

2.4 Appendix: Forward Problem

2.4.1 Properties of D_A and m_q

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded open set with smooth boundary, and $m \geq 2$ be an integer. Let A and q satisfy (2.2) and (2.3), respectively. As before, in what follows, $W^{s,p}$ is the standard L^p based Sobolev space on $\mathbb{R}^N$, $s \in \mathbb{R}$ and $1 < p < \infty$ defined using Bessel potential.

We start by considering the bi-linear forms

$$B^{\mathbb{R}^N}_A(\tilde{u}, \tilde{v}) = \langle A, \tilde{v}D\tilde{u}\rangle, \quad b^{\mathbb{R}^N}_q(\tilde{u}, \tilde{v}) = \langle q, \tilde{u}\tilde{v}\rangle, \qquad \tilde{u}, \tilde{v} \in H^m(\mathbb{R}^N).$$

The following result shows that the forms $B^{\mathbb{R}^N}_A$ and $b^{\mathbb{R}^N}_q$ are bounded on $H^m(\mathbb{R}^N)$. The proof is based on a property of multiplication of functions in Sobolev spaces.

Proposition 2.4.1. *The bi-linear forms $B^{\mathbb{R}^N}_A$ and $b^{\mathbb{R}^N}_q$ on $H^m(\mathbb{R}^N)$ are bounded and satisfy for any $m \geq 2$,*

$$\begin{aligned}|b^{\mathbb{R}^N}_q(\tilde{u}, \tilde{v})| &\leq C||q||_{W^{-\frac{m}{2}+\delta,r'}(\mathbb{R}^N)}||\tilde{u}||_{H^{\frac{m}{2}}(\mathbb{R}^N)}||\tilde{v}||_{H^{\frac{m}{2}}(\mathbb{R}^N)},\\ |B^{\mathbb{R}^N}_A(\tilde{u}, \tilde{v})| &\leq C||A||_{W^{-\frac{m}{2}+1+,p'}(\mathbb{R}^N)}||\tilde{u}||_{H^{\frac{m}{2}}(\mathbb{R}^N)}||\tilde{v}||_{H^{\frac{m}{2}}(\mathbb{R}^N)}.\end{aligned}$$

Proof. Using the duality between $W^{-\frac{m}{2}+\delta,r'}(\mathbb{R}^N)$ and $W^{\frac{m}{2}-\delta,r}(\mathbb{R}^N)$, we conclude from Proposition 2.2.2 that for all $\tilde{u},\tilde{v}\in H^m(\mathbb{R}^N)$ with $m\geq 2$,

$$\begin{aligned}|b_q^{\mathbb{R}^N}(\tilde{u},\tilde{v})| &\leq C||q||_{W^{-\frac{m}{2}+\delta,r'}(\mathbb{R}^N)}||\tilde{u}\tilde{v}||_{W^{\frac{m}{2}-\delta,r}(\mathbb{R}^N)}\\ &\leq C||q||_{W^{-\frac{m}{2}+\delta,r'}(\mathbb{R}^N)}||\tilde{u}||_{H^{\frac{m}{2}-\delta}(\mathbb{R}^N)}||\tilde{v}||_{H^{\frac{m}{2}-\delta}(\mathbb{R}^N)}\\ &\leq C||q||_{W^{-\frac{m}{2}+\delta,r'}(\mathbb{R}^N)}||\tilde{u}||_{H^{\frac{m}{2}}(\mathbb{R}^N)}||\tilde{v}||_{H^{\frac{m}{2}}(\mathbb{R}^N)}.\end{aligned}$$

(The hypotheses for Proposition 2.2.2 are satisfied for $m\leq N$ with $p_1=p_2=2$, $r=p\in(1,\frac{2N}{2N-m+2\delta}]$, $s_1=\frac{m}{2}-\delta$, $s_2=\frac{m}{2}-\delta$. For $m>N$, we choose $p_1=p_2=2$, $r=p\in(1,2]$, $s_1=\frac{m}{2}-\delta$, $s_2=\frac{m}{2}-\delta$.)

We now give the proof for the bi-linear form $B_A^{\mathbb{R}^N}$. Using the duality between $W^{-\frac{m}{2}+1,p'}(\mathbb{R}^N)$ and $W^{\frac{m}{2}-1,p}(\mathbb{R}^N)$, we conclude from Proposition 2.2.2 that for all $\tilde{u},\tilde{v}\in H^m(\mathbb{R}^N)$, and for $m>2$ we have

$$\begin{aligned}|B_A^{\mathbb{R}^N}(\tilde{u},\tilde{v})| &\leq C||A||_{W^{-\frac{m}{2}+1,p'}(\mathbb{R}^N)}\,||D\tilde{u}\tilde{v}|_{W^{\frac{m}{2}-1,p}(\mathbb{R}^N)}\\ &\leq C||A||_{W^{-\frac{m}{2}+1,p'}(\mathbb{R}^N)}\,||D\tilde{u}||_{H^{\frac{m}{2}-1}(\mathbb{R}^N)}||\tilde{v}||_{H^{\frac{m}{2}}(\mathbb{R}^N)}\\ &\leq C||A||_{W^{-\frac{m}{2}+1,p'}(\mathbb{R}^N)}\,||\tilde{u}||_{H^{\frac{m}{2}}(\mathbb{R}^N)}||\tilde{v}||_{H^{\frac{m}{2}}(\mathbb{R}^N)}.\end{aligned}$$

(The hypotheses for Proposition 2.2.2 are satisfied for $m<N$ with $p_1=p_2=2$, $p\in(1,\frac{2N}{2N-m}]$, $s_1=\frac{m}{2}-1$, $s_2=\frac{m}{2}$. For $m=N$ and $m=N+2$, we choose $p_1=p_2=2$, $p\in(1,2)$, $s_1=\frac{m}{2}-1$, $s_2=\frac{m}{2}$. Finally, for other m, we choose $p_1=p_2=2$, $p\in(1,2]$, $s_1=\frac{m}{2}-1$, $s_2=\frac{m}{2}$.)

For the case $m=2$, using Hölder's inequality and Sobolev embedding we get

$$\begin{aligned}|B_A^{\mathbb{R}^N}(\tilde{u},\tilde{v})| &\leq C||A||_{L^N(\mathbb{R}^N)}\,||D\tilde{u}\tilde{v}|_{L^{N'}(\Omega)}\\ &\leq C||A||_{L^N(\mathbb{R}^N)}\,||D\tilde{u}||_{H^{\frac{m}{2}-1}(\Omega)}||\tilde{v}||_{H^{\frac{m}{2}}(\Omega)}\\ &\leq C||A||_{L^N(\mathbb{R}^N)}\,||\tilde{u}||_{H^{\frac{m}{2}}(\mathbb{R}^N)}||\tilde{v}||_{H^{\frac{m}{2}}(\mathbb{R}^N)}.\end{aligned}$$

The proof is thus complete. ∎

Now, we show that the operators B_A and b_q defined in (2.4) are indeed well defined. Recall that

$$B_A(u,v) := B_A^{\mathbb{R}^N}(\tilde{u},\tilde{v}), \quad b_q(u,v) := b_q^{\mathbb{R}^N}(\tilde{u},\tilde{v}), \qquad u,v \in H^m(\Omega),$$

where $\tilde{u},\tilde{v} \in H^m(\mathbb{R}^N)$ are any extensions of u and v, respectively. We want to show that this definition is independent of the choice of extensions $\tilde{u},\tilde{v}$. Indeed, let $u_1, u_2 \in H^m(\mathbb{R}^N)$ be such that $u_1 = u_2 = u$ in Ω, and let $v_1, v_2 \in H^m(\mathbb{R}^N)$ be such that $v_1 = v_2 = v$ in Ω. It is enough to show that for all $w \in H^m(\mathbb{R}^N)$,

$$B_A^{\mathbb{R}^N}(u_1,w) = B_A^{\mathbb{R}^N}(u_2,w), \quad B_A^{\mathbb{R}^N}(w,v_1) = B_A^{\mathbb{R}^N}(w,v_2)$$

and

$$b_q^{\mathbb{R}^N}(u_1,w) = b_q^{\mathbb{R}^N}(u_2,w), \quad b_q^{\mathbb{R}^N}(w,v_1) = b_q^{\mathbb{R}^N}(w,v_2).$$

Since A and q are supported in $\bar{\Omega}$ and since $u_1 = u_2$ and $v_1 = v_2$ in Ω, we have

$$B_A^{\mathbb{R}^N}(u_1-u_2,w) = \langle A, D(u_1-u_2)w\rangle = 0, \quad b_q^{\mathbb{R}^N}(w,v_1-v_2) = \langle q, w(v_1-v_2)\rangle = 0$$

and

$$B_A^{\mathbb{R}^N}(w,v_1-v_2) = \langle A, (v_1-v_2)Dw\rangle = 0, \quad b_q^{\mathbb{R}^N}(u_1-u_2,w) = \langle q, (u_1-u_2)w\rangle = 0.$$

The next result shows that the bi-linear forms B_A and b_q are bounded on $H^m(\Omega)$.

Proposition 2.4.2. *The bi-linear forms B_A and b_q are bounded on $H^m(\Omega)$ are bounded and satisfy for any $m \geq 2$*

$$|b_q(u,v)| \leq C||q||_{W^{-\frac{m}{2}+\delta,r'}(\mathbb{R}^N)}||u||_{H^{\frac{m}{2}}(\Omega)}||v||_{H^{\frac{m}{2}}(\Omega)}$$
$$|B_A(u,v)| \leq C||A||_{W^{-\frac{m}{2}+1,p'}(\mathbb{R}^N)}||u||_{H^{\frac{m}{2}}(\Omega)}||v||_{H^{\frac{m}{2}}(\Omega)}$$

for all $u,v \in H^m(\Omega)$.

Proof. Let $u,v \in H^m(\Omega)$ and let $\tilde{\Omega}$ be any bounded Lipschitz domain so that $\bar{\Omega} \subset\subset \tilde{\Omega}$. Then there is a bounded linear map $E : H^m(\Omega) \to H_0^m(\tilde{\Omega})$ such that $E = Id$ on Ω, (Theorem 6.44

in [28]). According to estimates proven in Proposition 2.4.1, we obtain for any $m \geq 2$,

$$
\begin{aligned}
|b_q(u,v)| &= |b_q^{\mathbb{R}^N} E(u), E(v)| \\
&\leq C||q||_{W^{-\frac{m}{2}+\delta,r'}(\mathbb{R}^N)} ||E(u)||_{H^{\frac{m}{2}}(\tilde{\Omega})} ||E(v)||_{H^{\frac{m}{2}}(\tilde{\Omega})} \\
&\leq C||q||_{W^{-\frac{m}{2}+\delta,r'}(\mathbb{R}^N)} ||u||_{H^{\frac{m}{2}}(\Omega)} ||v||_{H^{\frac{m}{2}}(\Omega)} \qquad (2.30)
\end{aligned}
$$

and

$$
\begin{aligned}
|B_A(u,v)| &= |B_A^{\mathbb{R}^N}(E(u), E(v)| \\
&\leq C||A||_{W^{-\frac{m}{2}+1,p'}(\mathbb{R}^N)} ||E(u)||_{H^{\frac{m}{2}}(\tilde{\Omega})} ||E(v)||_{H^{\frac{m}{2}}(\tilde{\Omega})} \\
&\leq C||A||_{W^{-\frac{m}{2}+1,p'}(\mathbb{R}^N)} ||u||_{H^{\frac{m}{2}}(\Omega)} ||v||_{H^{\frac{m}{2}}(\Omega)} \qquad (2.31)
\end{aligned}
$$

■

Now, for $u \in H^m(\Omega)$, we define $D_A(u)$ and $m_q(u)$ for any $v \in H_0^m(\Omega)$ by

$$
\langle D_A(u), v\rangle_\Omega := B_A(u,v), \quad \langle m_q(u), v\rangle_\Omega := b_q(u,v).
$$

The following result, which is an immediate corollary of Proposition 2.4.2, implies that D_A and m_q are bounded operators from $H^m(\Omega) \to H^{-m}(\Omega)$. The norm on $H^{-m}(\Omega)$ is the usual dual norm given by

$$
||v||_{H^{-m}(\Omega)} = \underset{0\neq\phi\in H_0^m(\Omega)}{sup} \frac{|\langle v, \bar{\phi}\rangle_\Omega|}{||\phi||_{H^m(\Omega)}}.
$$

Corollary 2.4.2.1. *The operators D_A and m_q are bounded from $H^m(\Omega) \to H^{-m}(\Omega)$ and satisfy*

$$
\begin{aligned}
&||m_q(u)||_{H^{-m}(\Omega)} \leq C||q||_{W^{-\frac{m}{2}+\delta,r'}(\mathbb{R}^N)} ||u||_{H^m(\Omega)} \text{ and} \\
&||D_A(u)||_{H^{-m}(\Omega)} \leq C||A||_{W^{-\frac{m}{2}+1,p'}(\mathbb{R}^N)} ||u||_{H^m(\Omega)} \qquad (2.32)
\end{aligned}
$$

for all $u \in H^m(\Omega)$.

Finally, we state the following identities which are useful for defining the adjoint of $\mathcal{L}_{A,q}$.

Proposition 2.4.3. *For any $u, v \in H^m(\Omega)$, the forms B_A and b_q satisfy the following identities*

$$B_A(u,v) = -B_A(v,u) - b_{D \cdot A}(u,v) \quad \textit{and} \quad b_q(u,v) = b_q(v,u).$$

Proof. Choose extensions $\tilde{u}, \tilde{v} \in H_0^m(\tilde{\Omega})$ of u, v.

$$b_q(u,v) = b_q^{\mathbb{R}^N}(\tilde{u},\tilde{v}) = \langle q, \tilde{u}\tilde{v}\rangle = \langle q, \tilde{v}\tilde{u}\rangle = b_q^{\mathbb{R}^N}(\tilde{u},\tilde{v}) = b_q(v,u) \tag{2.33}$$

Using product rule for $\tilde{u}, \tilde{v} \in C_0^\infty(\tilde{\Omega})$ and then applying continuity for B_A and density arguments gives us

$$\begin{aligned} B_A(u,v) &= B_A^{\mathbb{R}^N}(\tilde{u},\tilde{v}) = \langle A, \tilde{v} D\tilde{u}\rangle = -\langle A, \tilde{u} D\tilde{v}\rangle + \langle A, D(\tilde{u}\tilde{v})\rangle \\ &= -\langle A, \tilde{u} D\tilde{v}\rangle - \langle D \cdot A, \tilde{u}\tilde{v}\rangle = -B_A(v,u) - b_{D.A}(u,v) \end{aligned} \tag{2.34}$$

■

2.4.2 Well-posedness and Dirichlet-to-Neumann map

Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ be a bounded open set with smooth boundary, and let A and q be as in (2.2) and (2.3) respectively with $m \geq 2$. For $f = (f_0, f_1, ..., f_m - 1) \in \prod_{j=0}^{m-1} H^{m-j-1/2}(\partial\Omega)$, consider the Dirichlet problem

$$\mathcal{L}_{A,q} u = 0 \text{ in } \Omega \quad \text{and} \quad \gamma u = f \text{ on } \partial\Omega, \tag{2.35}$$

where γ is the Dirichlet trace operator $\gamma : H^m(\Omega) \to \prod_{j=0}^{m-1} H^{m-j-1/2}(\partial\Omega)$ which is bounded and surjective; see [45, Theorem 9.5].

First aim of this section is to use standard variational arguments to show well-posedness of problem (2.35). We start with the following inhomogeneous problem.

$$\mathcal{L}_{A,q} u = F \text{ in } \Omega \quad \text{and} \quad \gamma u = 0 \text{ on } \partial\Omega, \quad F \in H^{-m}(\Omega). \tag{2.36}$$

To define a sesquilinear form a associated to the problem (2.36), for $u, v \in C_0^\infty(\Omega)$, we can integrate by parts and get

$$\langle \mathcal{L}_{A,q} u, \bar{v} \rangle_\Omega = \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int\limits_\Omega D^\alpha u \, \overline{D^\alpha v} \, dx + \langle D_A(u), \bar{v} \rangle_\Omega + \langle m_q(u), \bar{v} \rangle_\Omega := a(u,v).$$

Hence, we define a on $H_0^m(\Omega)$ by

$$a(u,v) := \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int\limits_\Omega D^\alpha u \, \overline{D^\alpha v} \, dx + \langle D_A(u), \bar{v} \rangle_\Omega + \langle m_q(u), \bar{v} \rangle_\Omega, \qquad u, v \in H_0^m(\Omega).$$

We now show that this sesqui-linear form a is bounded on $H_0^m(\Omega)$. Using duality and Proposition 2.4.2, for $u, v \in H_0^m(\Omega)$, we obtain

$$|a(u,v)| \leq C ||u||_{H^m(\Omega)} ||v||_{H^m(\Omega)}.$$

thereby showing boundedness of a. Also, Poincare's inequality for $u \in H_0^m(\Omega)$ gives

$$||u||^2_{H^m(\Omega)} \leq C \sum_{|\alpha|=m} ||D^\alpha u||^2_{L^2(\Omega)}.$$

Split $q = q^\sharp + (q - q^\sharp)$ with $q^\sharp \in L^\infty(\Omega, \mathbb{C})$ and $||q - q^\sharp||_{W^{-\frac{m}{2}+\delta, r'}(\mathbb{R}^N)}$ small enough, and split $A = A^\sharp + (A - A^\sharp)$ with $A^\sharp \in L^\infty(\Omega, \mathbb{C}^N)$ and $||A - A^\sharp||_{W^{-\frac{m}{2}+1, p'}(\mathbb{R}^N)}$ small enough. Using Poincaré inequality and Proposition 2.4.2, we obtain,

$$\begin{aligned}
Re\, a(u,u) &\geq \sum_{|\alpha|=m} \frac{m!}{\alpha!} ||D^\alpha u||^2_{L^2(\Omega)} - |B_A(u, \bar{u})| - |b_q(u, \bar{u})| \\
&\geq C \sum_{|\alpha|=m} ||D^\alpha u||^2_{L^2(\Omega)} - |B_{A^\sharp}(u, \bar{u})| - |b_{q^\sharp}(u, \bar{u})| - |B_{A-A^\sharp}(u, \bar{u})| - |b_{q-q^\sharp}(u, \bar{u})| \\
&\geq C ||u||^2_{H^m(\Omega)} - ||A^\sharp||_{L^\infty(\Omega)} ||Du||_{L^2(\Omega)} ||u||_{L^2(\Omega)} - ||q^\sharp||_{L^\infty(\Omega)} ||u||^2_{L^2(\Omega)} \\
&\quad - C' ||A - A^\sharp||_{W^{-\frac{m}{2}+1, p'}(\mathbb{R}^N)} ||u||^2_{H^{\frac{m}{2}}(\Omega)} - C' ||q - q^\sharp||_{W^{-\frac{m}{2}+\delta, r'}(\mathbb{R}^N)} ||u||^2_{H^{\frac{m}{2}}(\Omega)} \\
&\geq C ||u||^2_{H^m(\Omega)} - ||A^\sharp||_{L^\infty(\Omega)} \frac{\epsilon}{2} ||Du||^2_{L^2(\Omega)} - ||A^\sharp||_{L^\infty(\Omega)} \frac{1}{2\epsilon} ||u||^2_{L^2(\Omega)} \\
&\quad - ||q^\sharp||_{L^\infty(\Omega)} ||u||^2_{L^2(\Omega)} - C' ||A - A^\sharp||_{W^{-\frac{m}{2}+1, p'}(\mathbb{R}^N)} ||u||^2_{H^{\frac{m}{2}}(\Omega)} \\
&\quad - C' ||q - q^\sharp||_{W^{-\frac{m}{2}+\delta, r'}(\mathbb{R}^N)} ||u||^2_{H^{\frac{m}{2}}(\Omega)}.
\end{aligned}$$

Choose $\epsilon > 0$ to be sufficiently small to get

$$Re\, a(u,u) \geq C||u||^2_{H^m(\Omega)} - C_0||u||^2_{L^2(\Omega)} \quad C, C_0 > 0, \quad u \in H^m_0(\Omega).$$

Therefore, the seqsui-linear form a is coercive on $H^m_0(\Omega)$. Compactness of the embedding $H^m_0(\Omega) \hookrightarrow H^{-m}(\Omega)$ together with positivity of bounded operator $\mathcal{L}_{A,q} + C_0 I : H^m_0(\Omega) \to H^{-m}(\Omega)$ implies that $\mathcal{L}_{A,q} : H^m_0(\Omega) \to H^{-m}(\Omega)$ is Fredholm with zero index and hence Fredholm alternative holds for $\mathcal{L}_{A,q}$; see [61, Theorem 2.33]. (2.36) thus has a unique solution $u \in H^m_0(\Omega)$ if 0 is outside the spectrum of $\mathcal{L}_{A,q}$.

Now, consider the Dirichlet problem (2.35) and assume 0 is not in the spectrum of $\mathcal{L}_{A,q}$. We know that there is a $w \in H^m(\Omega)$ such that $\gamma w = f$. According to the Corollary (2.4.2.1), we have $\mathcal{L}_{A,q} w \in H^{-m}(\Omega)$. Therefore $u = v + w$ with $v \in H^m_0(\Omega)$ being the unique solution of the equation $\mathcal{L}_{A,q} v = -\mathcal{L}_{A,q} w \in H^{-m}(\Omega)$ is the unique solution of the Dirichlet problem (2.35).

Under the assumption that 0 is not in the spectrum of $\mathcal{L}_{A,q}$, the Dirichlet-to-Neumann map is defined as follows: Let $f, h \in \prod_{j=0}^{m-1} H^{m-j-1/2}(\partial\Omega)$. Set

$$\langle \mathcal{N}_{A,q} f, \bar{h} \rangle_{\partial\Omega} := \sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha u, D^\alpha v_h)_{L^2(\Omega)} + B_A(u, \bar{v}_h) + b_q(u, \bar{v}_h), \tag{2.37}$$

where u is the unique solution of the Dirichlet problem (2.35) and $v_h \in H^m(\Omega)$ is an extension of h, that is $\gamma v_h = h$. To see that this definition is independent of v_h, let $v_{h,1}, v_{h,2} \in H^m(\Omega)$ be such that $\gamma v_{h,1} = v_{h,2} = h$. Since $w = v_{h,1} - v_{h,2} \in H^m_0(\Omega)$ and u solves the Dirichlet problem (2.35), we have,

$$0 = \langle \mathcal{L}_{A,q} u, \bar{w} \rangle_\Omega = \sum_{|\alpha|=m} \frac{m!}{\alpha!}(D^\alpha u, D^\alpha w)_{L^2(\Omega)} + B_A(u, \bar{w}) + b_q(u, \bar{w}).$$

This shows that the definition (2.37) is independent of the extension v_h. Next we show that $\mathcal{N}_{A,q}$ is a bounded operator from $\prod_{j=0}^{m-1} H^{m-j-1/2}(\partial\Omega)$ into $\prod_{j=0}^{m-1} H^{-m+j+1/2}(\partial\Omega)$.

From the boundedness of the sesquilinear form a it follows that

$$|\langle N_{A,q} f, \bar{h} \rangle_{\partial\Omega}| \leq C||u||_{H^m(\Omega)}||v_h||_{H^m(\Omega)} \leq C||f||_{\prod_{j=0}^{m-1} H^{m-j-1/2}(\partial\Omega)}||h||_{\prod_{j=0}^{m-1} H^{m-j-1/2}(\partial\Omega)},$$

where

$$||g||_{\prod_{j=0}^{m-1} H^{-m+j+1/2}(\partial\Omega)} = (||g_0||^2_{H^{m-1/2}(\partial\Omega)} + ... + ||g_{m-1}||^2_{H^{1/2}(\partial\Omega)})^{1/2}.$$

is the product norm on the space $\prod_{j=0}^{m-1} H^{m-j-1/2}(\partial\Omega)$. Here we have used the fact that the extension operator $\prod_{j=0}^{m-1} H^{m-j-1/2}(\partial\Omega) \ni h \mapsto v_h \in H^m(\Omega)$ is bounded; see [45, Theorem 9.5]. Hence, $\mathcal{N}_{A,q}$ maps $\prod_{j=0}^{m-1} H^{m-j-1/2}(\partial\Omega)$ continuously into $\big(\prod_{j=0}^{m-1} H^{m-j-1/2}(\partial\Omega)\big)' = \prod_{j=0}^{m-1} H^{-m+j+1/2}(\partial\Omega)$.

2.4.3 Bessel potential spaces versus Slobodeckij spaces

In this section, we show why it is important to consider the Sobolev spaces as defined by the Bessel potential.

There is an alternative, non-equivalent way to generalize the definition of an integer valued Sobolev space to allow fractional exponents. We can define Sobolev spaces with non-integer exponents as Slobodeckij spaces, i.e. if $s = k + \theta$ with $k \in \mathbb{N}_0$ and $\theta \in (0, 1)$, then for $p \in [1, \infty)$,

$$H^{s,p}(\mathbb{R}^N) = \{u \in W^{k,p}(\mathbb{R}^N) : ||u||_{H^{s,p}(\mathbb{R}^N)} < \infty\},$$

where

$$||u||_{H^{s,p}(\mathbb{R}^N)} := ||u||_{W^{k,p}(\mathbb{R}^N)} + \left(\sum_{|\alpha|=k} \int\int_{\mathbb{R}^N\times\mathbb{R}^N} |\frac{\partial^\alpha u(x) - \partial^\alpha u(y)|}{|x-y|^{N+\theta p}}\, dx\, dy\right)^{1/p}$$

If $s < 0$ and $p \in (1, \infty)$, we define $H^{s,p}(\mathbb{R}^N) = (H^{-s,p/(p-1)}(\mathbb{R}^N)^*$.

We use the Bessel potential definition in this chapter as that definition gives more flexibility with regards to multiplication as the following result shows.

Proposition 2.4.4. *Suppose $s, s_1 \geq 0$, $s \notin \mathbb{Z}$ and $p_1, p_2, p > 1$. If the point-wise multiplication of functions is a continuous bi-linear map $H^{s,p_1}(\mathbb{R}^N) \times H^{s_1,p_1}(\mathbb{R}^N) \hookrightarrow H^{s,p}(\mathbb{R}^N)$, then $p_1 \leq p$.*

Proof. Proof for this result can be found in [12, Proposition 4.3]. ■

Chapter 3

CLOAKING FOR A QUASI-LINEAR ELLIPTIC PDE

3.1 Introduction and Definitions

The topic of cloaking has long been fascinating and has recently attracted a lot of attention within the mathematical and broader scientific community. A region of space is said to be cloaked if its contents along with the existence of the cloak are invisible to wave detection. One particular route to cloaking that has received considerable interest is that of transformation optics, the use of changes of variables to produce novel optical effects on waves or to facilitate computations.

A transformational optics approach to cloaking using the invariance properties of the conductivity equation was first discovered by Greenleaf, Lassas and Uhlmann [42, 44] in 2003. In 2006, Pendry, Schurig and Smith in [67] used a transformation optics approach, using invariance properties of the governing Maxwell's equations to design invisibility cloaks at microwave frequencies while Leonhardt in [57] used an optical conformal mapping based cloaking scheme. The ideal/perfect invisibility cloaking considered in [67, 57] is a singular 'blow-up-a-point' transformation. The cloaking media achieved in this way inevitably have singular materials parameters and require design of *metamaterials*. The singularity poses a formidable challenge to both theoretical analysis and practical construction. While several proof-of-concept prototypes have been proposed as cloaks, many challenges still remain in developing fully functional devices capable of fully cloaking objects. A lot of current academic and industrial research in material science is focused on development of such metamaterials from proof-of-concept prototypes to practical devices. See [27] for more details on this topic.

In order to avoid the singular structure, it is natural to introduce regularizations into the construction, and instead of a perfect cloak, one considers an approximate cloak or near-cloak. In order to handle the singular structure from the perfect cloaking constructions, the papers [37, 36, 68] used a truncation of singularities method to approach the nearly cloaking theory, whereas other papers regularize the 'blow-up-a-point' transformation to 'blow-up-a-small-region' transformation. The small-inclusion-blowup method was studied in [4, 50] for the conductivity model. Other approaches like blowing up a *crack* (namely, a curve in $\mathbb{R}^3$) or a *screen* (namely, a surface in $\mathbb{R}^3$) were also considered, respectively in [38] and [58] and have applications in the so-called wormholes and carpet-cloak respectively.

The papers [42, 44] considered the case of electrostatics, which is optics at frequency zero. These papers provide counter examples to uniqueness in Calderón Problem, which is the inverse problem for electrostatics and lies at the heart of Electrical Impedance Tomography [EIT]. EIT consists of determining the electrical conductivity of a medium filling a region Ω by making voltage and current measurements at the boundary $\partial\Omega$ and was first proposed in [16]. The fundamental mathematical idea behind cloaking is using the invariance of a coordinate transformation for specific systems, such as conductivity, acoustic, electromagnetic, and elasticity systems. We refer readers to the article [81] for a nice overview of development in EIT and cloaking for electrostatics. We also refer the readers to [37, 40, 41, 59, 24, 25] for the theory behind cloaking in various systems and related developments.

In this chapter we will focus our attention on cloaking in electrostatics and consider the following divergence type quasi-linear elliptic boundary value problem.

$$\begin{aligned} -div(A(x,u)\nabla u) &= 0 \text{ in } \Omega, \\ u &= f \text{ on } \partial\Omega. \end{aligned} \tag{3.1}$$

Here $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded open set with smooth enough boundary and $A(x,t)$ is a non-negative symmetric matrix valued function in $\Omega \times \mathbb{R}$ which satisfies certain structure conditions. Equation (3.1) is a generalization of the conductivity equation considered in Electrical Impedance Tomography.

Let us now introduce the basic mathematical set up. Consider $\Omega \subset \mathbb{R}^N$, $N \geq 2$, a bounded open set with Lipschitz boundary. Let $\mathcal{M}(\alpha, \beta, L; \Omega \times \mathbb{R})$ with $0 < \alpha < \beta < \infty$ and $L > 0$ denote the set of all real $N \times N$ symmetric matrices $A(x,t)$ of functions defined almost everywhere on $\Omega \times \mathbb{R}$ such that if $A(x,t) = [a_{kl}(x,t)]_{1 \leq k,l \leq N}$ then

1. $a_{kl}(x,t) = a_{lk}(x,t)\ \forall l,k = 1,..,N$,
2. $(A(x,t)\xi, \xi) \geq \alpha|\xi|^2,\ |A(x,t)\xi| \leq \beta|\xi|,\quad \forall \xi \in \mathbb{R}^N,\quad$ a.e. $x \in \Omega$ and,
3. $|a_{kl}(x,t) - a_{kl}(x,s)| \leq L|t-s|$ for a.e $x \in \Omega$, any t, s in $\mathbb{R}$ and $\forall l, k = 1, ..., N$.

Under the above conditions, we show in Theorem 3.5.1 that the following boundary value problem has a unique solution $u \in H^1(\Omega)$,

$$\begin{aligned} -div(A(x,u)\nabla u) = 0 \text{ in } \Omega, \\ u = f \in H^{1/2}(\partial\Omega). \end{aligned} \tag{3.2}$$

The Dirichlet-to-Neumann map for the boundary value problem (3.2) is defined formally as the map $f :\rightarrow \Lambda_A f$

$$\Lambda_A f = \nu \cdot A(x,f)\nabla u|_{\partial\Omega}$$

where ν is the outer unit normal to $\partial\Omega$. It is shown in section 3.5 that one can define the Dirichlet-to-Neumann (DN) map for the equation (3.2) in a weak sense as a mapping

$$\Lambda_A : H^{\frac{1}{2}}(\partial\Omega) \rightarrow H^{-\frac{1}{2}}(\partial\Omega).$$

The inverse problem is to recover the quasi-linear coefficient matrix $A(x,t)$, also called the conductivity, from the knowledge of Λ_A. Before we provide the definition of cloaking, since the problem of cloaking is essentially that of non-uniqueness, we digress a bit and mention some previous work regarding uniqueness in the inverse problem for the equation considered in (3.2).

In the isotropic case, that is, $A(x,t) = a(x,t)I_{N\times N}$ where $I_{N\times N}$ denotes the $N \times N$ identity matrix and a is a positive $C^{2,\gamma}(\overline{\Omega} \times \mathbb{R})$ function having a uniform positive lower bound on

$\overline{\Omega} \times [-s, s]$ for each $s > 0$, the Dirichlet to Neumann map Λ_a determines uniquely the scalar coefficient $a(x, t)$ on $\overline{\Omega} \times \mathbb{R}$. This uniqueness result was first proved for the linear case (i.e when a is a function of x alone) in the fundamental paper [77] for $N \geq 3$ for, in [63] for $N = 2$; and in [74] for the quasi-linear case.

For the anisotropic/ matrix valued case, it is well known that one cannot recover the coefficient $A(x, t)$ itself because of the following invariance property for the DN map.

Choose a smooth diffeomorphism $\Phi : \Omega \to \Omega$ such that $\Phi(x) = x$ on $\partial\Omega$ and define

$$\Phi_* A(x, t) = \frac{D\Phi(x)^T A(x, t) D\Phi(x)}{|D\Phi|} \circ \Phi^{-1}(x). \tag{3.3}$$

We make the change of variables $y = \Phi(x)$ in (3.2) to get

$$\int_\Omega \sum_{i,j=1}^N a_{ij}(x, u) \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j}\, dx = \int_\Omega \sum_{i,j=1}^N a_{ij}(x, u) \frac{\partial u}{\partial y^k} \frac{\partial y^k}{\partial x^i} \frac{\partial u}{\partial y^l} \frac{\partial y^l}{\partial x^j} det \left(\frac{\partial x}{\partial y} \right)\, dy.$$

We can write this more compactly as

$$\int_\Omega \langle A(x, u) \nabla_x u, \nabla_x u \rangle \; dx = \int_\Omega \langle \Phi_* A(y, u) \nabla_y u, \nabla_y u \rangle \; dy,$$

where $\Phi_* A$ is as defined in (3.3).

Since Φ is identity on $\partial\Omega$, the change of variables does not affect the Dirichlet data and we obtain

$$\Lambda_A = \Lambda_{\Phi_* A}. \tag{3.4}$$

Thus for matrix valued coefficients $A(x, t)$, one can expect uniqueness only modulo such a diffeomorphism. For dimension 2, in the linear case, such uniqueness up to difeomorphsim has been proved in [76, 9] and for dimension 3 and higher in [56]. For the quasi-linear case, Sun and Uhlmann in [75] showed uniqueness up to diffeomorphism in dimension 2 assuming $C^{2,\gamma}$, $0 < \gamma < 1$ smoothness of the coefficients A. They also proved uniqueness up to diffeomorphism for $N \geq 3$ for real analytic coefficients $A(x, t)$. Whether uniqueness up to diffeomorphism can be shown for less regular anisotropic quasi-linear coefficient $A(x,t)$ is an interesting question and remains open.

Equations of the form (3.1) are important and arise in many applications (eg, the stationary form of Richards equation [11], the modeling of thermal conductivity of the Earth's crust [82] or heat conduction in composite materials [48]). More specifically, steady state heat conduction in an inhomogeneous anisotropic nonlinear medium with Dirichlet boundary conditions is governed by equations of the form (3.1). One of the goals of our work is to extend the result obtained in [42, 44, 50] to the quasi-linear elliptic equation (3.2). We propose a change of variable scheme, similar to the one in [42, 44] and show how one can, in principle, obtain perfect cloaking using singular change of variables in the context of the equation considered in (3.2) and approximate cloaking using a regular change of variables. For approximate cloaking we use the small inclusion blow up method as in [50, 4]. The singularity and extreme anisotropy resulting from a singular change of variables pose a great challenge in manufacturing invisibility devices. The construction of approximate cloaks using regular change of variables is more tractable. However, these approximate cloaks, though non-singular are still anisotropic.

The other major goal of this chapter is to construct approximate isotropic cloaks. In a further regularization of the structures as a next step, it is natural to introduce isotropic and nonsingular cloaking material parameters to approximate the anisotropic material parameters. In a realistic sense, this provides a very flexible tool in order to design and implement numerical assimilation to achieve the desired cloaking. This will be accomplished using techniques of homogenization [3, 78].

We first construct approximate anisotropic cloaks using regular change of variables. Next, within the framework of homogenization, we approximate each approximate regular anisotropic cloak by a sequence of regular isotropic cloaks. Homogenization process for constructing isotropic regular approximate cloaks was first considered for the linear conductivity equation in [39]. We wish to extend the construction in [39] to the quasi-linear equation considered in (3.1).

3.1.1 Definition of Cloaking

We now provide a mathematical definition of cloaking for the quasi-linear elliptic partial differential equation considered in (3.2).

Definition 3.1.1. *Let $E \subset \Omega$ be fixed and let $\sigma_c : \Omega \setminus E \times \mathbb{R}$ be a non negative matrix valued function defined on $\Omega \setminus E \times \mathbb{R}$. We say σ_c cloaks E if its any extension across E of of following form*

$$\sigma_A(x,t) = \begin{cases} A(x,t) & (x,t) \in E \times \mathbb{R}, \\ \sigma_c(x,t) & (x,t) \in \Omega \setminus E \times \mathbb{R} \end{cases}$$

produces the same Dirichlet-to-Neumann map as a uniform isotropic region irrespective of the choice of $A(x,t) \in \mathcal{M}(\alpha, \beta, L; \Omega \times \mathbb{R})$.

That is, σ_c cloaks E in the sense of Definition 3.1.1 if $\Lambda_{\sigma_A} = \Lambda_1$ regardless of the choice of the extension $A(x,t)$.

Suppose $\sigma_c(x,t)$ cloaks E in the sense of Definition 3.1.1 and let Ω' be any domain containing Ω. Then the Dirichlet-to-Neumann map for

$$\sigma_{\tilde{A}(x,t)} = \begin{cases} A(x,t) & (x,t) \in E \times \mathbb{R}, \\ \sigma_c(x,t) & (x,t) \in \Omega \setminus E \times \mathbb{R}, \\ I_{N\times N} & (x,t) \in \Omega' \setminus \Omega \times \mathbb{R}, \end{cases}$$

is independent of A and is equal to Λ_1 (or equivalently $\Lambda_{I_{N\times N}}$). This extension argument produces many other examples. Indeed, if σ_c cloaks E in the sense of Definition 3.1.1, then the extension of σ_c by $I_{N\times N}$ outside Ω cloaks E in any larger domain Ω' containing Ω. If cloaking is possible, measurements made on the boundary i.e the knowledge of Dirichlet-to-Neumann map is not enough to detect the presence of an arbitrary inclusion inside the cloaked region. Since $A(x,t)$ can be arbitrary, an equality of the form (3.4) cannot possibly hold and thus cloaking is essentially a non-uniqueness result.

This chapter is organized as follows. In section 3.2, we introduce a regular change of variables scheme which will give us the desired approximate cloaking. In section 3.3, we introduce a singular change of variables and show how perfect cloaking can be achieved. The analysis in this section is essentially a simple extension of the arguments in [42, 44, 50]. Section 3.4 is devoted to using homogenization techniques for constructing regular isotropic cloaks. We begin this section by recalling the basic notions of H-convergence in the linear case. Following that, we perform periodic homogenization in quasi-linear settings. This enables us to construct regular isotropic cloak in the sense made precise in section 3.4. In section 3.5, we prove existence and uniqueness for the boundary value problem (3.2) and show that is possible to define, in a weak sense, the DN map associated with (3.2). (The well-posedness of the boundary value problem (3.2) is standard and follows 'freezing a variable' technique. Since we will use the well-posedness result quite a few times, we include a proof for sake of completeness.) Moreover, we also state a result on higher regularity for the solutions to (3.1) which will be used in section 3.4. Henceforth, we consider the physical dimensions $N = 2, 3$.

3.2 Regular Change of Variables

In this section, we apply a regular change of variables and nearly cloak E in the sense made precise below. For simplicity, we let $\Omega = B_2$ and restrict our attention the case when $B_1 = E$ needs to be nearly cloaked. We extend the result to non-radial domains later.

The basic premise is as follows. Consider a small ball of radius r, B_r centered at 0 where $r < 1$. Construct a map $F^r(x) : B_2 \to B_2$ with the following properties:

1) F^r is continuous and piecewise smooth;

2) F^r expands B_r to B_1 and maps B_2 to itself;

3) $F^r(x) = x$ on ∂B_2.

It is easy to see that the following candidate for F^r satisfies the above properties.

$$F^r(x) = \begin{cases} \frac{x}{r} & |x| \le r, \\ \left(\frac{2-2r}{2-r} + \frac{1}{2-r}|x|\right)\frac{x}{|x|} & r \le |x| \le 2. \end{cases}$$

Note that F^r is continuous, piecewise smooth and non-singular and $(F^r)^{-1}$ is also continuous and piecewise smooth. Consider

$$\sigma_A^r(x,t) = \begin{cases} A(x,t) & (x,t) \in B_1 \times \mathbb{R}, \\ F_*^r 1 & (x,t) \in B_2 \setminus B_1 \times \mathbb{R}, \end{cases} \tag{3.5}$$

where $A(x,t) \in \mathcal{M}(\alpha,\beta,L;B_2 \times \mathbb{R})$. By nearly cloaking, we mean that the following must hold

$$|\langle \Lambda_{\sigma_A^r} f, g\rangle - \langle \Lambda_1 f, g\rangle| = \mathrm{o}(1)||f||_{H^{\frac{1}{2}}(\partial B_2)}||g||_{H^{\frac{1}{2}}(\partial B_2)} \text{ for any } f,g \in H^{\frac{1}{2}}(\partial B_2), \tag{3.6}$$

where the o(1) term is independent of f and g. (3.6) is equivalent to

$$\begin{aligned} ||\Lambda_{\sigma_A^r} - \Lambda_1||_{H^{\frac{1}{2}}(\partial B_2)\times H^{-\frac{1}{2}}(\partial B_2)} &= \sup_{f,\,g \in H^{\frac{1}{2}}(\partial B_2)} |\langle \Lambda_{\sigma_A^r} f, g\rangle - \langle \Lambda_1 f, g\rangle| \\ &= \mathrm{o}(1)||f||_{H^{\frac{1}{2}}(\partial B_2)}||g||_{H^{\frac{1}{2}}(\partial B_2)}. \end{aligned} \tag{3.7}$$

By (3.4), the DN map for σ_A^r is identical to that of $(F^r)_*^{-1}\sigma_A^r$. We will show

$$|\langle \Lambda_{(F^r)_*^{-1}\sigma_A^r} f, g\rangle - \langle \Lambda_1 f, g\rangle| = \mathrm{o}(1)||f||_{H^{\frac{1}{2}}(\partial B_2)}||g||_{H^{\frac{1}{2}}(\partial B_2)} \text{ for any } f,g \in H^{\frac{1}{2}}(\partial B_2),$$

where the o(1) term is independent of f and g and where

$$(F^r)_*^{-1}\sigma_A^r = \begin{cases} (F^r)_*^{-1}A = \widetilde{A}^r(x,t) & (x,t) \in B_r \times \mathbb{R}, \\ 1 & (x,t) \in (B_2 \setminus B_r) \times \mathbb{R}. \end{cases}$$

Let us first explicitly calculate $\widetilde{A}^r(x,t)$. Note that for $\Phi(x) = (F^r)^{-1}(x)$, $D\Phi(x) = rI$ for $x \in B_1$. This implies that $|D\Phi| = r^N$. From (3.3), it follows that

$$\widetilde{A}^r(x,t) = (F^r)_*^{-1}A = \frac{1}{r^{N-2}}A\left(\frac{x}{r}, t\right) \text{ for } x \in B_r.$$

Hence, if $A \in \mathcal{M}(\alpha, \beta, L; B_1 \times \mathbb{R})$ then $\widetilde{A}^r(x,t) \in \mathcal{M}(\frac{\alpha}{r^{N-2}}, \frac{\beta}{r^{N-2}}, L; B_r \times \mathbb{R})$. This ultimately implies that for $r \ll 1$, $F_*^{r-1}\sigma_A^r \in \mathcal{M}(1, \frac{\beta}{r^{N-2}}, L; B_r \times \mathbb{R})$.

Let us now fix $f \in H^{\frac{1}{2}}(\partial B_2)$. Let $u^{r,f} \in H^1(B_2)$ uniquely solve

$$\begin{aligned} -div((F^r)_*^{-1}\sigma_A^r(x, u^{r,f})\nabla u^{r,f}) &= 0 \text{ in } B_2, \\ u^{r,f} &= f \text{ on } \partial B_2. \end{aligned} \tag{3.8}$$

(Unique solution to (3.8) is indeed guaranteed by the existence and uniqueness result proved in Theorem 3.5.1.)

Let $v^f \in H^1(B_2)$ solve

$$\begin{aligned} -\Delta v^f &= 0 \text{ in } B_2, \\ v^f &= f \text{ on } \partial B_2. \end{aligned} \tag{3.9}$$

Note that $u^{r,f} - v^f \in H_0^1(B_2)$. Using coercivity for $(F^r)_*^{-1}\sigma_A$ gives us

$$||\nabla(u^{r,f} - v^f)||_{L^2(B_2)}^2 \leq |\int_{B_2} (F^r)_*^{-1}\sigma_A^r(x, u^{r,f})\nabla(u^{r,f} - v^f) \cdot \nabla(u^{r,f} - v^f)\, dx|. \tag{3.10}$$

Now, r.h.s of (3.10)

$$\begin{aligned} &= |\int_{B_2} (F^r)_*^{-1}\sigma_A^r(x, u^{r,f})\nabla(u^{r,f}) \cdot \nabla(u^{r,f} - v^f) dx \\ &\quad - \int_{B_2} (F^r)_*^{-1}\sigma_A^r(x, u^{r,f})\nabla(v^f) \cdot \nabla(u^{r,f} - v^f)\, dx| \\ &= | - \int_{B_2} (F^r)_*^{-1}\sigma_A^r(x, u^{r,f})\nabla v^f \cdot \nabla(u^{r,f} - v^f)\, dx| \text{ (as } u^{r,f} \text{ solves (3.8))} \\ &= | - \int_{B_2} ((F^r)_*^{-1}\sigma_A^r(x, u^{r,f}) - I)\nabla v^f \cdot \nabla(u^{r,f} - v^f)\, dx| \text{ (as } v^f \text{ solves (3.9))} \\ &= |\int_{B_r} ((F^r)_*^{-1}\sigma_A^r(x, u^{r,f}) - I)\nabla v^f \cdot \nabla(u^{r,f} - v^f)\, dx|) \\ &= |\int_{B_r} (\widetilde{A}^r(x, u^{r,f}) - I)\nabla v^f \cdot \nabla(u^{r,f} - v^f)\, dx|. \end{aligned} \tag{3.11}$$

We note that $||\widetilde{A}^r(x,t)||_{L^\infty(B_r)} \leq \frac{C}{r^{N-2}}$ where the constant C is independent of r. We apply Hölder's inequality to the last line in (3.11) to obtain

$$||\nabla(u^{r,f} - v^f)||_{L^2(B_2)}^2 \leq Cr^{\frac{N}{p_1} - N + 2}||\nabla v^f||_{L^{p_2}(B_r)}||\nabla(u^{r,f} - v^f)||_{L^2(B_r)},$$

where C is independent of r and $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{2}$. We thus have

$$\begin{aligned}||\nabla(u^{r,f} - v^f)||^2_{L^2(B_2)} &\leq Cr^{\frac{N}{p_1}-N+2}||\nabla v^f||_{L^{p_2}(B_r)}||\nabla(u^{r,f} - v^f)||_{L^2(B_r)} \\ &\leq Cr^{\frac{N}{p_1}-N+2}||\nabla v^f||_{L^{p_2}(B_r)}||\nabla(u^{r,f} - v^f)||_{L^2(B_2)}.\end{aligned}$$

By Poincaré inequality, we can say that

$$||u^{r,f} - v^f||_{H^1(B_2)} \leq Cr^{\frac{N}{p_1}-N+2}||\nabla v^f||_{L^{p_2}(B_r)}. \tag{3.12}$$

By Corollary 6.3 in [35], we can say that

$$||\nabla v^f||_{L^\infty(B_r)} \leq C||v^f||_{L^\infty(B_1)}, \tag{3.13}$$

where C is independent of r and f.

We now use [35, Theorem 8.24] to conclude that

$$||v^f||_{L^\infty(B_1)} \leq C||v^f||_{L^2(B_2)}, \tag{3.14}$$

where C is independent of r and f. (3.13) and (3.14) together imply

$$||\nabla v^f||_{L^{p_2}(B_r)} \leq C\, r^{\frac{N}{p_2}}||v^f||_{L^2(B_2)} \leq C\, r^{\frac{N}{p_2}}||f||_{H^{\frac{1}{2}}(\partial B_2)}. \tag{3.15}$$

(3.12) and (3.15) hence imply

$$\begin{aligned}||u^{r,f} - v^f||_{H^1(B_2)} &\leq Cr^{\frac{N}{p_1}+\frac{N}{p_2}-N+2}||f||_{H^{\frac{1}{2}}(\partial B_2)} \\ &= Cr^{-\frac{N}{2}+2}||f||_{H^{\frac{1}{2}}(\partial B_2)},\end{aligned} \tag{3.16}$$

where C is independent of r and also f.

Let $g \in H^{\frac{1}{2}}(\partial B_2)$ be arbitrary. We know that there exists a unique $v^g \in H^1(B_2 \setminus B_1)$ which solves the following boundary value problem

$$\begin{aligned}-\Delta v^g &= 0 \text{ in } B_2 \setminus B_1, \\ v^g &= 0 \text{ on } \partial B_1, \\ v^g &= g \text{ on } \partial B_2\end{aligned} \tag{3.17}$$

such that

$$||v^g||_{H^1(B_2\setminus B_1)} \leq C||g||_{H^{\frac{1}{2}}(\partial B_2)}, \tag{3.18}$$

where C is independent of g.

From (3.8), (3.9), (3.17), (3.16), (3.18) and the definition of $(F^r)_*^{-1}\sigma_A^r$, we see that

$$\begin{aligned}
|\langle \Lambda_{(F^r)_*^{-1}\sigma_A^r} f, g\rangle - \langle \Lambda_1 f, g\rangle| &= |\int_{\partial B_2} \frac{\partial u^{r,f}}{\partial \nu} v^g \, dS - \int_{\partial B_2} \frac{\partial v^f}{\partial \nu} v^g \, dS| \\
&= |\int_{B_2\setminus B_1} (\nabla u^{r,f} \cdot \nabla v^g - \nabla v^f \cdot \nabla v^g) \, dx| \\
&\leq \int_{B_2\setminus B_1} ||\nabla u^{r,f} - \nabla v^f||_{L^2(B_2\setminus B_1)} ||v^g||_{H^1(B_2\setminus B_1)} \\
&\leq Cr^{-\frac{N}{2}+2} ||f||_{H^{\frac{1}{2}}(\partial B_2)} ||g||_{H^{\frac{1}{2}}(\partial B_2)},
\end{aligned} \tag{3.19}$$

where C is independent of r, f and g.

(3.19) implies that for $N = 2, 3$, we obtain (3.7).

3.2.1 Faster decay

In this subsection we derive an improved rate of convergence in (3.19) at the cost of choosing smoother boundary data. Fix $f \in H^{\frac{3}{2}}(\partial B_2)$.

Since $u^{r,f}$ solves (3.8) and v^f solves (3.9), we have for any $w \in H_0^1(B_2)$

$$\int_{B_2} \nabla(u^{r,f} - v^f) \cdot \nabla w \, dx = \int_{B_r} (I_{N\times N} - \widetilde{A}^r(x, u^{r,f}))\nabla u^{r,f} \cdot \nabla w \, dx. \tag{3.20}$$

Choose w which uniquely solves

$$\begin{aligned}
-\Delta w &= u^{r,f} - v^f \text{ in } B_2, \\
w &= 0 \text{ on } \partial B_2.
\end{aligned} \tag{3.21}$$

Since $u^{r,f} - v^f \in L^2(B_2)$, $w \in H^2(B_2)$ with

$$||w||_{H^2(B_2)} \leq C||u^{r,f} - v^f||_{L^2(B_2)} \tag{3.22}$$

where C is independent of r.

By Sobolev embedding, this implies $\nabla w \in L^q(B_2)$ for any $1 \leq q < \infty$ for $N = 2$ and $\nabla w \in L^q(B_2)$ for any $1 \leq q \leq \frac{2N}{N-2}$ for $N = 3$. We have

$$\begin{aligned}\int_{B_2} (u^{r,f} - v^f)^2\, dx &= \int_{B_r} (I_{N\times N} - \widetilde{A}^r(x, u^{r,f}))\nabla u^{r,f} \cdot \nabla w\, dx \\ &\leq \frac{C}{r^{N-2}} ||\nabla u^{r,f}||_{L^{p'}(B_r)} ||\nabla w||_{L^p(B_r)} \quad \text{where } \tfrac{1}{p} + \tfrac{1}{p'} = 1,\ p > 2 \\ &\leq C r^{\frac{N}{q'} - N + 2} ||\nabla u^{r,f}||_{L^{p'}(B_r)} ||\nabla w||_{L^q(B_2)} \quad \text{where } \tfrac{1}{q} + \tfrac{1}{q'} = \tfrac{1}{p}.\end{aligned}$$

Note that we have the following conditions

$$\begin{aligned} &2 < p < q < \infty \quad \text{ for } N = 2, \\ &2 < p < q \leq \frac{2N}{N-2} \text{ for } N = 3. \end{aligned} \tag{3.23}$$

Then by using (3.22) we get

$$\begin{aligned} ||u^{r,f} - v^f||_{L^2(B_2)} &\leq C r^{\frac{N}{q'} - N + 2} ||\nabla u^{r,f}||_{L^{p'}(B_r)} \\ &\leq C r^{\frac{N}{q'} - N + 2} \left[||\nabla (u^{r,f} - v^f)||_{L^{p'}(B_r)} + ||\nabla v^f||_{L^{p'}(B_r)} \right] \\ &\leq C r^{\frac{N}{q'} - N + 2} ||\nabla (u^{r,f} - v^f)||_{L^2(B_r)}\, ||\mathbf{1}||_{L^s(B_r)} \;+ \\ &\quad\; C r^{\frac{N}{q'} - N + 2} ||\nabla v^f||_{L^\infty(B_r)} ||\,\mathbf{1}||_{L^{p'}(B_r)} \end{aligned}$$

where $\frac{1}{p'} = \frac{1}{2} + \frac{1}{s}$. By using (3.13), (3.14) and (3.16) together, we have

$$||u^{r,f} - v^f||_{L^2(B_2)} \leq C r^{\frac{N}{q'} - N + 2} \left(r^{\frac{-N}{2} + 2 + \frac{N}{s}} ||f||_{H^{\frac{1}{2}}(\partial B_2)} + r^{\frac{N}{p'}} ||f||_{H^{\frac{1}{2}}(\partial B_2)} \right)$$

or,

$$\begin{aligned} ||u^{r,f} - v^f||_{L^2(B_2)} &\leq C r^{N(1 - \frac{1}{p})} ||f||_{H^{\frac{1}{2}}(\partial B_2)} \\ &= \mathcal{O}(r^{-N - \frac{N}{q} + 4}). \end{aligned} \tag{3.24}$$

(3.24) implies that for $N = 3$, we have a decay rate of $\mathcal{O}(r^{\frac{q-3}{q}})$. From (3.23), we can optimize this decay rate to $\mathcal{O}(r^{\frac{6-3}{6}}) = \mathcal{O}(r^{\frac{1}{2}})$. This is of the same order as in (3.16) for $N = 3$.

However, from (3.24) and (3.23), we can conclude that For $N = 2$, the best possible decay is $\mathcal{O}(r^{2-\frac{2}{q}})$ for any $q > 2$. This decay rate is faster than the one we obtained in (3.16).

Now, by [35, Theorem 9.13], since $u^{r,f} - v^f$ is harmonic in $B_2 \setminus \overline{B_r}$ and $u^{r,f} - v^f = 0$ on ∂B_2, we have

$$||u^{r,f} - v^f||_{H^2(\Omega')} \leq C||u^{r,f} - v^f||_{L^2(B_2 \setminus \overline{B_r})}, \tag{3.25}$$

where $\Omega' = B_2 \setminus \overline{B_{r+\delta}}$ and $\delta > 0$ is small enough. (3.24) and (3.25) along with the trace theorem for Sobolev spaces imply

$$\Big\|\frac{\partial u^{r,f}}{\partial \nu} - \frac{\partial v^f}{\partial \nu}\Big\|_{H^{1/2}(\partial B_2)} = \mathrm{o}(1)||f||_{H^{3/2}(\partial B_2)}. \tag{3.26}$$

The decay estimates for nearly cloaking here are weaker than the one in [50] where a decay of $\mathcal{O}(r^N)$ is obtained. In our case, though we have a slower decay rate, it is sufficient to show approximate cloaking.

So far, we focused on the radial setting because of its explicit character. Our arguments in the previous proof did not use symmetry in any essential manner. A similar argument as to the one provided in this section in fact proves

Corollary 3.2.0.1. *Let $H : B_2 \to \Omega$ be a Lipschitz continuous map with Lipschitz continuous inverse and let $E = H(B_1)$. Then $G^r = H \circ F^r \circ H^{-1} : \Omega \to \Omega$ is piecewise smooth with the following properties.*

1. *G^r expands B_r to E.*

2. *$G^r(x) = x$ on $\partial\Omega$.*

Let

$$\sigma_A^r(x,t) = \begin{cases} A(x,t) & (x,t) \in E \times \mathbb{R}, \\ G_*^r 1 & (x,t) \in \Omega \setminus E \times \mathbb{R}, \end{cases} \tag{3.27}$$

where $A(x,t) \in \mathcal{M}(\alpha, \beta, L;\ \Omega \times \mathbb{R})$. Then the following holds

$$|\langle \Lambda_{\sigma_A^r} f, g\rangle - \langle \Lambda_1 f, g\rangle| = \mathrm{o}(1)||f||_{H^{\frac{1}{2}}(\partial\Omega)}||g||_{H^{\frac{1}{2}}(\Omega)} \text{ for any } f, g \in H^{\frac{1}{2}}(\partial\Omega). \tag{3.28}$$

In other words, G^r_*1 approximately/nearly cloaks E.

3.3 Perfect cloaking

We now show how $E \subset \Omega$ can be perfectly cloaked using a singular change of variables. For simplicity we first take $E = B_1$ and $\Omega = B_2$. The analysis in this section mirrors that in section 4 of [50].

Let us define

$$F(x) = \left(1 + \frac{|x|}{2}\right)\frac{x}{|x|}, \quad x \in B_2. \tag{3.29}$$

F has the following properties:

1) F is smooth except at $x = 0$;

2) F expands 0 to B_1 and maps B_2 to itself;

3) $F(x) = x$ on ∂B_2.

Our candidate for perfect cloaking will be F_*1. Let us calculate F_*1 explicitly. Note that

$$DF = \left(\frac{1}{2} + \frac{1}{|x|}\right) I_{N\times N} - \frac{1}{|x|}\hat{x}\hat{x}^T \tag{3.30}$$

for $|x| > 0$, where $I_{N\times N}$ is the identity matrix and $\hat{x} = \frac{x}{|x|}$. DF is a symmetric matrix such that

a) $\hat{x}$ is an eigen-vector with eigen-value $1/2$;

b) $\hat{x}^\perp$ is a (N-1) dimensional eigen-space with eigen-value $\frac{1}{2} + \frac{1}{|x|}$.

The determinant is thus

$$det(DF) = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{|x|}\right)^{N-1}. \tag{3.31}$$

Hence, whenever $1 < |y| < 2$, we get,

$$F_*1(y) = \frac{2^N}{(2+|x|)^{N-1}} \left[\left(\frac{|x|^{N-1}}{4} + |x|^{N-2} + |x|^{N-3} \right) (I_{N\times N} - \hat{x}\hat{x}^T) \right] + \frac{2^N}{(2+|x|)^{N-1}} \left[\frac{1}{4}|x|^{N-1}\hat{x}\hat{x}^T \right], \tag{3.32}$$

where the right hand side is evaluated at $x = F^{-1}(y) = 2(|y|-1)\frac{y}{|y|}$.

As $|y| \to 1$, F_*1 becomes singular. In fact, following the same arguments as in [50], we get

* When $N = 2$, one eigen-value of $F_*1 \to 0$ and the other tends to ∞ as $|y| \to 1$.
* For $N = 3$, one eigen-value goes to 0 while others remain finite as $|y| \to 1$.

Consider σ_A defined as

$$\sigma_A(y,t) = \begin{cases} A(y,t) & (y,t) \in B_1 \times \mathbb{R}, \\ F_*1 & (y,t) \in B_2 \setminus B_1 \times \mathbb{R}, \end{cases} \tag{3.33}$$

where $A(y,t) \in \mathcal{M}(\alpha, \beta, L; B_1 \times \mathbb{R})$.

Define v by

$$v(y) = \begin{cases} u(0) & y \in B_1, \\ u(x) & y \in B_2 \setminus B_1, \end{cases} \tag{3.34}$$

where $u \in H^1(B_2)$ is the harmonic function in B_2 with trace $f \in H^{1/2}(\partial B_2)$, and $x = F^{-1}(y)$.

We will show that v solves

$$\begin{aligned} -div_y(\sigma_A(y, v(y))\nabla_y v(y)) = 0 \text{ in } B_2, \\ v = f \text{ on } \partial B_2. \end{aligned} \tag{3.35}$$

Since F_*1 is degenerate near $|y| = 1$, it is not immediately clear if (3.35) has a weak solution.

We need to put some constraints to get a unique weak solution for (3.35) since $\sigma_A(y,t)$ is not uniformly elliptic. As F_*1 is smooth for $|y| > 1$, by elliptic regularity, v will be uniformly bounded in any compact subset of $B_2 \setminus \overline{B_1}$. Since F_*1 becomes degenerate near $|y| = 1$, we ask that any solution $v(y)$ not diverge as $|y| \to 1$. That is, we ask that

$$|v(y)| \leq C \text{ for } |y| \leq \rho \tag{3.36}$$

for some finite C and $1 < \rho < 2$.

Let us first prove a lemma which identifies the value of any solution v to (3.35) on $B_2 \setminus \overline{B_1}$.

Lemma 3.3.1. *Any solution v to (3.35) satisfying (3.36) is such that*

$$v(y) = u(x) \textit{ for } 1 < |y| < 2$$

where $x = F^{-1}(y)$ and u is the harmonic function on B_2 with the same Dirichlet data as v.

Proof. For any compactly supported test function ϕ in $B_2 \setminus \overline{B_1}$, by change of variables, we have

$$0 = \int_{B_2 \setminus \overline{B_1}} \sigma_A(y, v) \nabla_y v \cdot \nabla_y \phi \, dy = \int_{B_2 \setminus \{0\}} \nabla_x v(F(x)) \cdot \nabla_x \phi(F(x)) \, dx.$$

We thus see that $v(F(x))$ is weakly harmonic in the punctured ball $B_2 \setminus \{0\}$. Elliptic regularity implies $v(F(x))$ is strongly harmonic in the punctured ball.

Since we demand that v should satisfy (3.36), $u(x) = v(F(x))$ has a removable singularity at 0. Thus $u(0)$ is determined by continuity and the extended u is harmonic in the entire ball B_2.

Clearly, since $F(x) = x$ on ∂B_2, u has the same Dirichlet data as v on ∂B_2. The conclusion of the lemma thus holds. ∎

We now show that v as defined in (3.34) solves (3.35).

Lemma 3.3.2. *v as defined in (3.34) solves*

$$-div_y(\sigma_A(y, v(y)) \nabla_y v) = 0 \textit{ in } B_2,$$
$$v = f \textit{ on } \partial B_2$$

where u is the harmonic function on B_2 with Dirichlet data f.

Proof.

We first show that $|\nabla v|$ is uniformly bounded in B_r for every $r < 2$. To see this, note that by chain rule and symmetry of DF, we have

$$\nabla_y v = (DF^{-1})^T \nabla_x u = DF^{-1} \nabla_x u,$$

for $1 < |y| < 2$. By (3.30), the matrix DF^{-1} is uniformly bounded and so is $\nabla_x u$, except perhaps near ∂B_2, as u is harmonic in whole B_2. Hence $|\nabla_y v|$ is bounded on $B_r \setminus B_1$ for any $1 \leq r < 2$. Moreover v is constant on B_1, and continuous across ∂B_1. Thus $|\nabla v|$ is uniformly bounded on B_r for $r < 2$.

Next we prove that $|\sigma_A(y, v(y))\nabla_y v|$ is also uniformly bounded on B_r for every $r < 2$. For $1 < |y| < 2$, using definition of σ_A and chain rule and symmetry of DF, we have

$$\sigma_A(y, v)\nabla_y v = F_*1\,(DF^{-1})\nabla_x u. \tag{3.37}$$

The symmetric matrices F_*1 and $(DF)^{-1}$ have the same eigenvectors, namely $\hat{x}$ and $\hat{x}^\perp$.

Taking $N = 2$, we see that the eigen-value of F_*1 in direction $\hat{x}^\perp$ behaves like $|x|^{-1}$, while that of $(DF)^{-1}$ behaves like $|x|$. The eigen-values of both matrices in direction $\hat{x}$ are bounded. Thus the product $F_*1(DF)^{-1}$ is bounded. This proves $|\sigma_A(y, v(y))\nabla_y v|$ is also uniformly bounded on B_r for every $r < 2$, since $\nabla_x u$ is bounded away from ∂B_2 and $\sigma_A(y, v)\nabla v = 0$ for $y \in B_1$ as v is defined to be a constant in B_1.

For $N = 3$, this follows directly from the above proved fact that $|\nabla v|$ is uniformly bounded on B_r for every $r < 2$. Since F_*1 is uniformly bounded and by (3.30), DF^{-1} is uniformly bounded we get that $\sigma_A(y, v(y))\nabla_y v$ is uniformly bounded in B_r for every $r < 2$.

Next we show $\sigma_A(y, v)\nabla_y v \cdot \nu \to 0$ uniformly as $|y| \to 1$ where ν is the unit outer normal to ∂B_1. We have, $\frac{y}{|y|} = \frac{x}{|x|} = \hat{x}$ and $|y| \to 1 \equiv x \to 0$. We thus need to show that $\hat{x}$ component of (3.37) goes to 0 as $|x| \to 0$. Since $F_*1(DF^{-1})$ is symmetric and $\hat{x}$ is an eigen-vector, it is enough to show that the corresponding eigen-value tends to 0.

We have, from (3.31) and (3.32), that the eigen-value corresponding to $\hat{x}$ is

$$\frac{2^{N-1}}{(2+|x|)^{N-1}}|x|^{N-1} \leq |x|^{N-1},$$

which tends to zero for any $N \geq 2$ as $x \to 0$.

We will use the fact that a bounded vector field X is weakly divergence free on B_2 iff it is weakly divergence free on $B_2 \setminus \overline{B_1}$ and B_1 and the normal flux $\nu \cdot X$ is continuous across ∂B_1 to show that $\sigma_A \nabla v$ is divergence-free. We have shown above that the vector field $\sigma(y,v)\nabla_y v$ is uniformly bounded away from ∂B_2, and next we showed that its normal flux $\sigma_A(y,v)\nabla v \cdot \nu$ is continuous across ∂B_1. Moreover, it is obvious that $\sigma(y,v)\nabla_y v$ is divergence free in B_1 as v is constant there. By Lemma 3.3.1, $\sigma(y,v)\nabla_y v$ is weakly divergence free on $B_2 \setminus \overline{B_1}$ and thus we can conclude that $-div_y(\sigma_A(y,v)\nabla_y v) = 0$ weakly in B_2. ■

We have shown in Lemma 3.3.2 that v as defined in (3.34) solves (3.35). We have also identified in Lemma 3.3.1 the values of v in $B_2 \setminus B_1$. Since σ_A is degenerate, uniqueness can fail. For instance, if σ_A is identically 0 in B_1 then v can be arbitrary in B_1. However, such a possibility does not arise here as the degeneracy for σ_A is only near ∂B_1.

To show that $v = u(0)$ in B_1, we need to restrict further the class in which v belongs. We assumed earlier that v is uniformly bounded near ∂B_1. We need a condition to make v continuous across ∂B_1 and a hypothesis on $\sigma_A(y, v(y))\nabla v(y)$ for the PDE (3.35) to make sense. We thus further assume

$$\nabla v \in L^2(B_2) \text{ and } \sigma_A(y,v)\nabla v \in L^2(B_2). \tag{3.38}$$

Lemma 3.3.3. *If v is a weak solution of (3.35) which satisfies (3.36) and (3.38), then v must be given by (3.34).*

Proof. By Lemma 3.3.1, $v(y) = u(x)$ for $1 < |y| < 2$. By assumption (3.36), since $v(y)$ is uniformly bounded away from $|y| = 1$, $u(x)$ has a removable singularity at 0. In particular, it is continuous at 0. As F^{-1} maps ∂B_1 to 0, $v(y) \to u(0)$ as y approaches ∂B_1 from outside.

Since $\nabla v \in L^2(B_2)$ by assumption, the trace of v on ∂B_1 is well defined. Since $v(y) \to u(0)$ as y approaches ∂B_1 from outside, the restriction of v on ∂B_1 must be equal to $u(0)$. We can thus conclude that $v = u(0)$ in B_1 by the uniqueness result in Theorem 3.5.1 for the boundary value problem

$$\begin{aligned} -div_y(A(y, v(y))\nabla_y v) = 0 \text{ in } B_1, \\ v = u(0) \text{ on } \partial B_1. \end{aligned}$$

■

3.3.1 Equality of DN maps

We now show that the singular cloak F_1^* cloaks B_1 in the sense of Definition 3.1.1.

Theorem 3.3.4. *Assume $\sigma_A(x,t)$ is as defined in (3.33), where F is given by (3.29) and $A \in \mathcal{M}(\alpha, \beta, L; \Omega \times \mathbb{R})$. Then the associated Dirichlet-to-Neumann map Λ_{σ_A} is equal to Λ_1.*

Proof. By Lemma 3.3.1, Lemma 3.3.2 and Lemma 3.3.3, the Dirichlet-to-Neumann map Λ_{σ_A} is well defined. Let $f, g \in H^{\frac{1}{2}}(\partial B_2)$. Let $\phi_g \in H^1(B_2)$ be such that $\phi_g|_{\partial B_2} = g$. We have

$$\begin{aligned} \langle \Lambda_{\sigma_A} f, g\rangle &= \int_{B_2 \setminus B_1} \sigma_A(y, v)\nabla_y v \cdot \nabla_y \phi_g \, dy \\ &= \int_{B_2 \setminus \{0\}} \nabla_x u \cdot \nabla_x \phi_g(F(x)) \, dx \\ &= \langle \Lambda_1 f, g\rangle. \end{aligned} \tag{3.39}$$

This completes the proof for perfect cloaking. ■

We have focused on the radial setting because of its explicit character. The analysis extends similarly to non-radial domains.

Corollary 3.3.4.1. *Let $H : B_2 \to \Omega$ be a Lipschitz continuous map with Lipschitz continuous inverse, and suppose $E = H(B_1)$. Then $G = H \circ F \circ H^{-1} : \Omega \to \Omega$ is identity on Ω and H*

"expands" the point $H(0)$ to E. Let

$$\sigma_A(x,t) = \begin{cases} A(x,t) & (x,t) \in E \times \mathbb{R}, \\ G_* 1 & (x,t) \in \Omega \setminus E \times \mathbb{R} \end{cases} \tag{3.40}$$

where $A(x,t) \in \mathcal{M}(\alpha, \beta, L;\ \Omega \times \mathbb{R})$. Then we have

$$\Lambda_{\sigma_A} = \Lambda_1. \tag{3.41}$$

In other words, $G_* 1$ perfectly cloaks E.

Proof. The proof in Theorem 4 in [50] goes through, almost word by word, for the quasi-linear equation (3.2). For brevity, we omit the details and refer the reader to Theorem 4 in [50]. ■

Let us remark here that (3.41) does not contradict the uniqueness up to diffeomorphism result of Sun and Uhlmann in [75]. The result in [75] crucially depends on the ellipticity of the matrix $A(x,t)$ in (3.2) which we violate by considering a singular change of variables that makes $A(x,t)$ degenerate. In other words, 'non-uniqueness' for the Calderón problem for the quasi-linear elliptic equation (3.2) is possible if we allow $A(x,t)$ to be degenerate.

3.4 Homogenization Scheme

In Section 3.2, we showed how it is possible to nearly cloak $E \subset \Omega$. The approximate cloak, though non-degenerate, is anisotropic. What we would like to do in this section is to construct near cloaks which are isotropic. This will be done within the framework of homogenization.

This section is organized as follows. In section 3.4.1, we develop the tools needed to prove homogenization for the quasi-linear PDE (3.2) with inhomogeneous boundary conditions for locally periodic micro-structures. In section 3.4.2, we use the results of section 3.4.1 to construct explicit isotropic regular approximate cloaks for radial domains.

The main idea of homogenization process [3, 78] is to provide a (macro scale) approximation to a problem with heterogeneities/micro-structures (at micro scale) by suitably averaging out small scales and by incorporating their effects on large scales. These effects are quantified by the so-called homogenized coefficients.

Here we are concerned with the notion of H-convergence for quasi-linear PDEs of the form

$$\begin{aligned} -div\left(A^{\epsilon}(x, u^{\epsilon}(x))\nabla u^{\epsilon}(x)\right) = 0 \text{ in } \Omega, \\ u^{\epsilon} = f \text{ on } \partial\Omega. \end{aligned} \tag{3.42}$$

We begin by recalling the notion of H-convergence [3, 78] in the linear case.

Let $\mathcal{M}(\alpha, \beta; \Omega)$ with $0 < \alpha < \beta$ denote the set of all real $N \times N$ symmetric matrices $A(x)$ of functions defined almost everywhere on a bounded open subset Ω of $\mathbb{R}^N$ such that if $A(x) = [a_{kl}(x)]_{1 \leq k,l \leq N}$, then

$$\begin{aligned} &a_{kl}(x) = a_{lk}(x) \; \forall l, k = 1, .., N \text{ and } \; (A(x)\xi, \xi) \geq \alpha|\xi|^2, \\ &|A(x)\xi| \leq \beta|\xi|, \; \forall \xi \in \mathbb{R}^N, \quad \text{a.e. } x \in \Omega. \end{aligned}$$

Let A^{ϵ} and A^* belong to $\mathcal{M}(\alpha, \beta; \Omega)$. We say $A^{\epsilon} \xrightarrow{H} A^*$ or H-converges to a homogenized matrix A^* if $A^{\epsilon}\nabla u^{\epsilon} \rightharpoonup A^*\nabla u$ in $L^2(\Omega)^N$ weak, for all test sequences u^{ϵ} satisfying

$$\begin{aligned} u^{\epsilon} \rightharpoonup u \quad \text{weakly in } H^1(\Omega) \text{ and} \\ -div(A^{\epsilon}\nabla u^{\epsilon}) \text{ is strongly convergent in } H^{-1}(\Omega). \end{aligned}$$

In particular, we consider the homogenization of linear PDEs of the form

$$\begin{aligned} -div\left(A^{\epsilon}(x)\nabla u^{\epsilon}(x)\right) = 0 \text{ in } \Omega, \\ u^{\epsilon} = f \text{ on } \partial\Omega \text{ with } f \in H^{\frac{1}{2}}(\partial\Omega). \end{aligned}$$

Then as $\epsilon \to 0$ we say $A^{\epsilon} \xrightarrow{H} A^*$, whenever

$$\begin{aligned} u^{\epsilon} \rightharpoonup u \quad \text{weakly in } H^1(\Omega) \text{ and} \\ A^{\epsilon}\nabla u^{\epsilon} \rightharpoonup A^*\nabla u \quad \text{weakly in } L^2(\Omega)^N, \end{aligned}$$

where $u \in H^1(\Omega)$ solves

$$-div\,(A^*(x)\nabla u(x)) = 0 \text{ in } \Omega,$$
$$u = f \text{ on } \partial\Omega.$$

Homogenization with periodic micro-structures (linear case): Let us give an example in the class of periodic micro-structures and its homogenization. Let Y denote the unit cube $[0,1]^N$ in $\mathbb{R}^N$.

Define $A : Y \mapsto \mathbb{R}^{N\times N}$ as $A(y) = [a_{kl}(y)]_{1\le k,l\le N} \in \mathcal{M}(\alpha,\beta;Y)$ such that $a_{kl}(y)$ are Y-periodic functions $\forall k,l = 1,2..,N$; implying $a_{kl}(y+z) = a_{kl}(y)$ whenever $z \in \mathbb{Z}^N$ and $y \in Y$. We now set

$$A^\epsilon(x) = [a^\epsilon_{kl}(x)] = [a_{kl}(\frac{x}{\epsilon})]$$

and extend it to the whole of $\mathbb{R}^N$ by ϵ-periodicity with a small period of scale ϵ via scaling the coordinate $y = \frac{x}{\epsilon}$. The restriction of A^ϵ on Ω is known as periodic micro-structures.

In this classical case, the homogenized conductivity $A^* = [a^*_{kl}]$ is a constant matrix and can be defined by its entries (see [3, 13, 22]) as

$$a^*_{kl} = \int_Y a_{ij}(y)\frac{\partial}{\partial y_i}(\chi_k(y) + y_k)\frac{\partial}{\partial y_j}(\chi_l(y) + y_l)dy,$$

where we define the χ_k through the so-called cell-problems. For each canonical basis vector e_k, consider the following conductivity problem in the periodic unit cell :

$$-div_y\; A(y)(\nabla_y\chi_k(y) + e_k) = 0 \quad \text{in } \mathbb{R}^N, \quad y \to \chi_k(y) \quad \text{is } Y\text{-periodic}.$$

Let us now generalize the above case and consider a *locally periodic* function $A : \Omega \times Y \mapsto \mathbb{R}^{N\times N}$ defined as $A(x,y) = [a_{kl}(x,y)]_{1\le k,l\le N} \in \mathcal{M}(\alpha,\beta;\Omega\times Y)$ such that $a_{kl}(\cdot,y)$ are Y-periodic functions with respect to the second variable $\forall k,l = 1,2..,N$ and for almost every x in Ω. We now set

$$A^\epsilon(x) = [a^\epsilon_{kl}(x)] = [a_{kl}(x,\frac{x}{\epsilon})].$$

The homogenized conductivity $A^*(x) = [a^*_{kl}(x)]$ is then defined by its entries as (see [13])

$$a^*_{kl}(x) = \int_Y a_{ij}(x,y)\frac{\partial}{\partial y_i}(\chi_k(x,y) + y_k)\frac{\partial}{\partial y_j}(\chi_l(x,y) + y_l)dy, \tag{3.43}$$

where $\chi_k(\cdot, y) \in H^1(Y)$ solves the following cell problem for almost every x in Ω;

$$-div_y\ A(x,y)(\nabla_y \chi_k(x,y) + e_k) = 0 \quad \text{in } \mathbb{R}^N, \quad y \to \chi_k(x,y) \quad \text{is } Y\text{-periodic.}$$

We end the discussion on homogenization for the linear case by mentioning the following localization result [3].

Proposition 3.4.1. *Let $A^\epsilon(x)$ H-converge to $A^*(x)$ in Ω. Let ω be an open subset of Ω. Then $A^\epsilon|_\omega$ (restrictions of A^ϵ to ω) H-converge to $A^*|_\Omega$.*

Based on the above localization result, let us present the following example.

Example: Let Ω be a domain which is subdivided into domains Ω^z, $z = 1, 2..., m$ with Lipschitz boundaries. Let $A^z(x,y)$ be periodic functions in the y variable with periods Y^z for $z = 1, 2, ..., m$. Let us define for any $\epsilon > 0$,

$$\widetilde{A}^\epsilon(x) = A^z\left(x, \frac{x}{\epsilon}\right) \text{ if } x \in \Omega^z.$$

Then

$$\widetilde{A}^\epsilon(x) \text{ } H\text{-converges to } \widetilde{A}^*(x) \text{ in } \Omega \tag{3.44}$$

where

$$\widetilde{A}^*(x) = A^{*,z}(x) \text{ if } x \in \Omega^z$$

and $A^{*,z}(x) = [a^{*,z}_{kl}(x)]$ is defined as in (3.43) over the periodic cell Y^z as

$$a^{*,z}_{kl}(x) = \frac{1}{|Y^z|}\int_{Y^z} a^z_{ij}(x,y)\frac{\partial}{\partial y_i}(\chi^z_k(x,y) + y_k)\frac{\partial}{\partial y_j}(\chi^z_l(x,y) + y_l)dy, \ \ z = 1, .., m.$$

Here $\chi^z_k(\cdot, y) \in H^1(Y^z)$ solves the following cell problem for a.e x in Ω^z;

$$-div_y\ A^z(x,y)(\nabla_y \chi^z_k(x,y) + e_k) = 0 \quad \text{in } \mathbb{R}^N, \quad y \to \chi^z_k(x,y) \quad \text{is } Y^z\text{-periodic.}$$

This finishes the example.

We will now turn our attention to the quasi-linear equation (3.42). For each fixed $\epsilon > 0$, we consider $A^\epsilon(x,t) \in \mathcal{M}(\alpha, \beta, L; \Omega \times \mathbb{R})$ where α, β, L are positive, finite and independent of ϵ.

It is shown in Theorem 3.5.1, that, for all fixed $\epsilon > 0$, the weak form of (3.42) with $f \in H^{\frac{1}{2}}(\partial\Omega)$ has a unique solution $u^\epsilon \in H^1(\Omega)$ satisfying the estimate

$$||u^\epsilon||_{H^1(\Omega)} \leq C||f||_{H^{\frac{1}{2}}(\partial\Omega)}, \tag{3.45}$$

where $C = C(N, \Omega, \alpha, \beta, L)$ is independent of ϵ. Thus, standard compactness arguments imply that up to a subsequence (still denoted by ϵ)

$$u^\epsilon \rightharpoonup u \text{ weakly in } H^1(\Omega).$$

Our goal is to get the limiting equation for $u \in H^1(\Omega)$. We remain in the class of periodic micro-structures and derive the homogenization result in quasi-linear settings.

3.4.1 Periodic homogenization for quasi-linear PDEs

Let Y denote the unit cube $[0,1]^N$ in $\mathbb{R}^N$. Let $A : \Omega \times Y \times \mathbb{R} \mapsto \mathbb{R}^{N\times N}$ and $A(x, y, t) = [a_{ij}(x, y, t)] \in \mathcal{M}(\alpha, \beta, L; \Omega \times Y \times \mathbb{R})$ be such that

$$y \mapsto a_{ij}(x, y, t), \text{ are } Y\text{-periodic functions for a.e } (x, t) \in \Omega \times \mathbb{R} \text{ and } i, j = 1, 2.., N.$$

We now set

$$A^\epsilon(x, t) = [a_{ij}(x, \frac{x}{\epsilon}, t)], \quad (x, t) \in \Omega \times \mathbb{R}.$$

This is known as periodic micro-structures in quasi-linear settings.

We will show that in this case the homogenized conductivity
$A^*(x, t) = [a^*_{ij}(x, t)] \in \mathcal{M}(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{L}; \Omega \times \mathbb{R})$ can be defined by its entries (see [60, 29, 14])

$$a^*_{kl}(x, t) = \int_Y a_{ij}(x, y, t)\frac{\partial}{\partial y_i}(\chi_k(x, y, t) + y_k)\frac{\partial}{\partial y_j}(\chi_l(x, y, t) + y_l)dy, \tag{3.46}$$

where $\chi_k(x, y, t) \in H^1_\#(Y)$ for almost every $(x, t) \in \Omega \times \mathbb{R}$ are the solutions of the so-called cell-problems:

For each canonical basis vector $e_k \in \mathbb{R}^N$, $\chi_k(x,y,t)$ satisfy the following problem for $y \in Y$, where Y is the periodic unit cell and for almost every $(x,t) \in \Omega \times \mathbb{R}$:

$$\begin{aligned} -\, div_y\ A(x,y,t)(\nabla_y \chi_k(x,y,t) + e_k) = 0 \quad \text{in } \mathbb{R}^N, \\ y \to \chi_k(x,y,t) \quad \text{is } Y\text{-periodic for all } (x,t) \in \Omega \times \mathbb{R}. \end{aligned} \tag{3.47}$$

The above problem (3.47) has a unique solution in $H^1_{\#}(Y)/\mathbb{R}$ where

$$H^1_{\#}(Y) = \{f \in H^1_{loc}(\mathbb{R}^N) :\ y \mapsto f(y) \text{ is } Y \text{ periodic.}\}$$

We further assume that,

$$\int_Y \chi_k(x,y,t)dy = 0 \quad \forall (x,t) \in \Omega \times \mathbb{R}$$

in order to get a unique solution $\chi(x,y,t) \in H^1_{\#,0}(Y)$ for almost every $(x,t) \in \Omega \times \mathbb{R}$, where $H^1_{\#,0}(Y) = \{f \in H^1_{\#}(Y) : \int_Y f = 0\}$.

Note that, from (3.46) it follows that $a^*_{kl} = a^*_{lk}$ for $k,l = 1,..,N$ and there exist $0 < \widetilde{\alpha} < \widetilde{\beta} < \infty$ such that, $A^*(x,t) \in \mathcal{M}(\widetilde{\alpha}, \widetilde{\beta}, \widetilde{L}; \Omega \times \mathbb{R})$. We will in fact show that $t \mapsto A^*(x,t)$ is uniformly Lipschitz in Ω.

Before proving the homogenization result, we first discuss few properties of the expected homogenized matrix $A^*(x,t)$ defined in (3.46).

Lemma 3.4.2. *Let $A(x,y,t) \in \mathcal{M}(\alpha,\beta,L;\Omega \times Y \times \mathbb{R})$ be such that*

$$y \mapsto A(x,y,t) = [a_{ij}(x,y,t)] \quad \textit{is } Y \textit{ periodic for almost every } (x,t) \in \Omega \times \mathbb{R}$$

and $t \mapsto A(x,y,t)$ is uniformly Lipschitz for almost every $(x,y) \in \Omega \times Y$, i.e.

$$|a_{ij}(x,y,t_1) - a_{ij}(x,y,t_2)| \leq L|t_1 - t_2|, \quad i,j = 1,..,N. \tag{3.48}$$

Then the unique solution $\chi_k(x,\cdot,t) \in H^1_{\#,0}(Y)$, $k = 1,...,N$ to (3.47) is such that $t \mapsto \chi_k(x,y,t)$ is uniformly Lipschitz for almost every $(x,y) \in \Omega \times Y$ i.e.

$$||\chi_k(x,\cdot,t_1) - \chi_k(x,\cdot,t_2)||_{H^1_{\#,0}(Y)} \leq C_L|t_1 - t_2|, \quad k = 1,...,N \tag{3.49}$$

and this holds for any $t_1, t_2 \in \mathbb{R}$, where C_L is a constant independent of x, y, t_1, t_2.

Proof. By using the fact $A(x,y,t) \in \mathcal{M}(\alpha,\beta,L;\Omega\times Y\times\mathbb{R})$ it can be easily seen that,

$$||\chi_k(x,y,t)||_{H^1_{\#,0}} \le C \quad \text{almost every } (x,t)\in\Omega\times\mathbb{R}, \quad k=1,..,N \tag{3.50}$$

where C is independent of x,y,t,k. Let $\chi_k(x,y,t_1)\in H^1_{\#,0}(Y)$ and $\chi_k(x,y,t_2)\in H^1_{\#,0}(Y)$ be solutions to (3.47) for two pairs of points (x,t_1) and (x,t_2) in $\Omega\times\mathbb{R}$, it is easy to see that we have the relation

$$\begin{aligned}&\int_Y A(x,y,t_1)\nabla\left(\chi_k(x,y,t_1)-\chi_k(x,y,t_2)\right)\cdot\nabla\left(\chi_k(x,y,t_1)-\chi_k(x,y,t_2)\right)dy\\ &=\int_Y \left(A(x,y,t_2)-A(x,y,t_1)\right)\left(\nabla\chi_k(x,y,t_2)+e_k\right)\cdot\nabla\left(\chi_k(x,y,t_1)-\chi_k(x,y,t_2)\right)dy.\end{aligned}$$

By using the coercivity of A along with (3.50) and the Lipschitz criteria (3.48) we have for $k=1,2,...,N$

$$\begin{aligned}&\alpha||\chi_k(x,\cdot,t_1)-\chi_k(x,\cdot,t_2)||^2_{H^1_{\#,0}(Y)}\\ &\le\int_Y A(x,y,t_1)\nabla\left(\chi_k(x,y,t_1)-\chi_k(x,y,t_2)\right)\cdot\nabla\left(\chi_k(x,y,t_1)-\chi_k(x,y,t_2)\right)dy\\ &=\int_Y \left(A(x,y,t_2)-A(x,y,t_1)\right)\left(\nabla\chi_k(x,y,t_2)+e_k\right)\cdot\nabla\left(\chi_k(x,y,t_1)-\chi_k(x,y,t_2)\right)dy\\ &\le\left(\sum_{i,j=1}^N ||a_{ij}(x,y,t_1)-a_{ij}(x,y,t_2)||_{L^\infty(Y)}\right)\left(||\chi_k(x,\cdot,t_2)||_{H^1_{\#,0}(Y)}+\sqrt{|Y|}\right)\\ &\qquad\qquad\times\left(||\chi_k(x,\cdot,t_1)-\chi_k(x,\cdot,t_2)||_{H^1_{0,\#}(Y)}\right).\end{aligned} \tag{3.51}$$

(3.49) thus follows from (3.51). ∎

Recalling the definition of $A^*=[a^*_{ij}(x,t)]$, (see (3.46)), then following Lemma 3.4.2 we have the following result.

Lemma 3.4.3. *Let the assumptions of Lemma 3.4.2 be satisfied. Then $t\mapsto A^*(x,t)$ is uniformly Lipschitz, i.e.*

$$|a^*_{ij}(x,t_1)-a^*_{ij}(x,t_2)|\le\widetilde{L}|t_1-t_2|, \quad i,j=1,2,...,N \tag{3.52}$$

holds for almost every $x\in\Omega$ and $t_1,t_2\in\mathbb{R}$ where $\widetilde{L}$ is independent of x,t_1,t_2.

Proof. Let us write (3.46) as

$$a_{kl}^*(x,t) = \langle a_{il}(x,y,t) \frac{\partial}{\partial y_i} \chi_k(x,y,t) \rangle + \langle a_{kl}(x,y,t) \rangle,$$

where $\langle\rangle$ denotes the average over the periodic cell Y.

Let us prove the Lipschitz property for the first term in the above formula. The proof for the second term is straightforward.

We write

$$\begin{aligned} &\left| \left\langle a_{il}(x,\cdot,t_1) \frac{\partial \chi_k}{\partial y_i}(x,\cdot,t_1) - a_{il}(x,\cdot,t_2) \frac{\partial \chi_k}{\partial y_i}(x,\cdot,t_2) \right\rangle \right| \\ &\leq \left| \left\langle a_{il}(x,\cdot,t_2) \left(\frac{\partial \chi_k(x,\cdot,t_1)}{\partial y_i} - \frac{\partial \chi_k(x,\cdot,t_2)}{\partial y_i} \right) \right\rangle \right| + \\ &\quad \left| \left\langle \left(a_{il}(x,\cdot,t_1) - a_{il}(x,\cdot,t_2) \right) \frac{\partial \chi_k(x,\cdot,t_1)}{\partial y_i} \right\rangle \right|. \end{aligned}$$

Then by using that $t \mapsto a_{il}(x,y,t)$ and $t \mapsto \chi_k(x,y,t)$ are uniformly Lipschitz functions, it follows that $t \mapsto a_{il}^*(x,y,t)$ is uniformly Lipschitz for almost every $(x,y) \in \Omega \times Y$. We thus obtain (3.52). ∎

Next we present the local characterization in quasi-linear settings analogous to the local case in the linear setting mentioned in Proposition 3.4.1.

Lemma 3.4.4. *Let $A^\epsilon(x,t)$ governed by the periodic micro-structures H-converge to $A^*(x,t)$ in $\Omega \times \mathbb{R}$. Let ω be an open subset of Ω and assume that $A^\epsilon|_{\omega\times\mathbb{R}}$ (restrictions of A^ϵ to $\omega \times \mathbb{R}$) are independent of t i.e*

$$A^\epsilon(x,t) = A^\epsilon(x) \quad \textit{whenever } (x,t) \in \omega \times \mathbb{R}.$$

Then the homogenized limit $A^*(x,t)|_{\omega\times\mathbb{R}}$ is also independent of t, i.e.

$$A^*(x,t) = A^*(x) \quad \textit{whenever, } (x,t) \in \omega \times \mathbb{R}.$$

Proof. The proof is straightforward from the locality of the second order PDE (3.47) satisfied by $\chi_k(x,y,t)$; as under the assumption $\chi_k(x,y,t) = \chi_k(x,y)$ whenever $(x,y,t) \in \omega \times Y \times \mathbb{R}$ and by using (3.46) we can conclude that $A^*(x,t) = A^*(x)$ whenever $(x,t) \in \omega \times \mathbb{R}$. ∎

We now prove a homogenization result for a quasi-linear PDE with inhomogeneous boundary conditions for locally periodic micro-structures. Similar result for example, with homogeneous boundary condition and globally periodic micro-structures can be found in [60, 29]. We first choose Ω to be a smooth enough domain and later relax the assumptions on Ω.

Theorem 3.4.5. *Let Ω be a bounded domain in $\mathbb{R}^N$ with $C^{2,\gamma}$ boundary where $0 < \gamma < 1$. Let the matrix $A(x, y, t)$ satisfy:*

1. *$A(x, y, t) \in \mathcal{M}(\alpha, \beta, L; \Omega \times Y \times \mathbb{R})$, where α, β, L are independent of x, y, t.*
2. *$(x, y, t) \mapsto A(x, y, t) = [a_{ij}(x, y, t)]$ is $C^{1,\gamma}(\overline{\Omega} \times Y \times \mathbb{R})$.*

Let us consider the sequence of matrices $\{A^\epsilon(x, t)\}_{\epsilon>0}$ for $(x, t) \in \Omega \times \mathbb{R}$ given by

$$A^\epsilon(x, t) := A\left(x, \frac{x}{\epsilon}, t\right)$$

and consider the following inhomogeneous quasi-linear PDEs with $f \in C^{2,\gamma}(\overline{\Omega})$:

$$\begin{aligned} -div\left(A^\epsilon(x, u^\epsilon)\nabla u^\epsilon(x)\right) &= 0 \text{ in } \Omega, \\ u^\epsilon &= f \text{ on } \partial\Omega. \end{aligned} \tag{3.53}$$

Then up to a subsequence the corresponding solutions $\{u^\epsilon\}_\epsilon \in H^1(\Omega)$ of (3.53) satisfy

$$u^\epsilon \rightharpoonup u \quad \text{weakly in } H^1(\Omega) \text{ and}$$
$$A^\epsilon(x, u^\epsilon)\nabla u^\epsilon \rightharpoonup A^*(x, u)\nabla u \quad \text{weakly in } L^2(\Omega)^N;$$

where $u \in H^1(\Omega)$ is the unique solution of the so-called homogenized problem

$$\begin{aligned} -div\left(A^*(x, u(x))\nabla u(x)\right) &= 0 \text{ in } \Omega, \\ u &= f \text{ on } \partial\Omega. \end{aligned}$$

We say $A^\epsilon(x, t) \xrightarrow{H} A^*(x, t)$ in $\Omega \times \mathbb{R}$, where the homogenized matrix $A^*(x, t)$ is defined as in (3.46).

Proof. For $\epsilon > 0$ fixed, let us consider $f \in C^{2,\gamma}(\overline{\Omega})$, then the problem (3.53) has a unique solution $u^\epsilon \in C^{2,\gamma}(\overline{\Omega})$ satisfying (see Lemma 3.5.2), for $x, y \in \overline{\Omega}$,

$$|u^\epsilon(x) - u^\epsilon(y)| \leq C|x - y|^\lambda$$

where $C = C(N, \Omega, \alpha, \beta, f)$ and $\lambda = \lambda(N, \Omega, \alpha, \beta, \gamma)$ are independent of ϵ. Thus it follows that the sequence $\{u^\epsilon\}_{\epsilon>0}$ satisfies the assumption of the Arzelá-Ascoli theorem. Hence there exists a subsequence still denoted by $\{u^\epsilon\}_{\epsilon>0}$ that converges strongly to u in $C(\overline{\Omega})$. We also know that subsequence $\{u^\epsilon\}_{\epsilon>0}$ satisfies (see (3.45))

$$||u^\epsilon||_{H^1(\Omega)} \leq C||f|_{\partial\Omega}||_{H^{\frac{1}{2}}(\partial\Omega)},$$

where C is independent of ϵ.

Thus it has a weakly convergent subsequence still denoted as $\{u^\epsilon\}_{\epsilon>0}$ in $H^1(\Omega)$ and the subsequential limit is same as $u \in H^1(\Omega) \cap C(\overline{\Omega})$. Let $\delta > 0$ be arbitrary, then the domain Ω can be divided in to sub-domains Ω^z, $z = 1, 2..., m$ with Lipschitz boundaries and there exists a function u^δ constant on every sub-domain Ω^z such that for sufficiently small ϵ, we have

$$|u^\epsilon(x) - u^\delta(x)| \leq \delta \quad \text{ for } x \in \Omega. \tag{3.54}$$

This immediately implies

$$|u(x) - u^\delta(x)| \leq \delta \quad \text{ for } x \in \Omega. \tag{3.55}$$

Let $u^{\delta,\epsilon} \in H^1(\Omega)$ be a sequence of solutions to the linear PDEs

$$\begin{aligned} -div\left(A^\epsilon(x, u^\delta(x))\nabla u^{\delta,\epsilon}(x)\right) &= 0 \text{ in } \Omega, \\ u^{\delta,\epsilon} &= f \text{ on } \partial\Omega. \end{aligned} \tag{3.56}$$

Then we have the following identity which follows from (3.53) and (3.56);

$$\begin{aligned} &\int_\Omega A^\epsilon(x, u^\epsilon)\nabla\left(u^\epsilon - u^{\delta,\epsilon}\right) \cdot \nabla\left(u^\epsilon - u^{\delta,\epsilon}\right) dx \\ &= \int_\Omega \left(A^\epsilon(x, u^\delta) - A^\epsilon(x, u^\epsilon)\right) \nabla u^{\delta,\epsilon} \cdot \nabla\left(u^\epsilon - u^{\delta,\epsilon}\right) dx. \end{aligned} \tag{3.57}$$

Using ellipticity for A^ϵ gives us

$$
\begin{aligned}
\alpha||u^\epsilon - u^{\delta,\epsilon}||^2_{H^1_0(\Omega)} &\leq \int_\Omega A^\epsilon(x,u^\epsilon)\nabla\left(u^\epsilon - u^{\delta,\epsilon}\right)\cdot\nabla\left(u^\epsilon - u^{\delta,\epsilon}\right)dx \\
&= \int_\Omega \left(A^\epsilon(x,u^\delta) - A^\epsilon(x,u^\epsilon)\right)\cdot\nabla u^{\delta,\epsilon}\cdot\nabla\left(u^\epsilon - u^{\delta,\epsilon}\right)dx \\
&\leq \left(\sum_{i,j=1}^N ||a^\epsilon_{ij}(x,u^\delta) - a^\epsilon_{ij}(x,u^\epsilon)||_{L^\infty(\Omega)}\right)\times \\
&\qquad ||u^{\delta,\epsilon}||_{H^1(\Omega)}||u^\epsilon - u^{\delta,\epsilon}||_{H^1_0(\Omega)}. \qquad (3.58)
\end{aligned}
$$

We use fact that $u^{\delta,\epsilon}\in H^1(\Omega)$ solves (3.56) with $||u^{\delta,\epsilon}||_{H^1(\Omega)}\leq C||f||_{H^{\frac{1}{2}}(\partial\Omega)}$ where C is independent of δ and ϵ, and(3.58) to obtain

$$
||u^\epsilon - u^{\delta,\epsilon}||_{H^1_0(\Omega)} \leq C\left(\sum_{i,j=1}^N ||a^\epsilon_{ij}(x,u^\delta) - a^\epsilon_{ij}(x,u^\epsilon)||_{L^\infty(\Omega)}\right). \qquad (3.59)
$$

As $A^\epsilon(\cdot,t)$ is uniformly Lipschitz in x and ϵ, therefore using (3.54) gives us

$$
\left(\sum_{i,j=1}^N ||a^\epsilon_{ij}(x,u^\delta) - a^\epsilon_{ij}(x,u^\epsilon)||_{L^\infty(\Omega)}\right) \leq L\delta.
$$

Consequently, from (3.59) it follows that,

$$
||u^\epsilon - u^{\delta,\epsilon}||_{H^1_0(\Omega)} \leq C\delta \qquad (3.60)
$$

where C is independent of ϵ and δ and this inequality holds for sufficiently small ϵ.

Since $u^\delta(x)$ are constant in each Ω^z for $z = 1, 2..., m$, then applying the H-convergence result of (3.44) where we take $\widetilde{A^\epsilon}(x) = A^\epsilon(x,u^\delta(x)) = A\left(x,\frac{x}{\epsilon},u^\delta(x)\right) = \widetilde{A}_\delta(x,\frac{x}{\epsilon})$ in Ω^z, we have

$$
\widetilde{A}_\delta\left(x,\frac{x}{\epsilon}\right) \ H\text{-converges to } A^*_\delta(x) \quad \text{in } \Omega^z, \quad z = 1,2,...,m
$$

where $A^*_\delta(x) = A^*(x,u^\delta(x))$ is defined as in (3.46) with constant $t = u^\delta(x)$ for $x\in\Omega^z$, $z = 1,2,...,m$. Hence in the whole domain Ω we have

$$
A^\epsilon(x,u^\delta(x)) \text{ H-converges to } A^*(x,u^\delta(x)) \text{ in } \Omega.
$$

This means that we have

$$u^{\delta,\epsilon} \rightharpoonup u^{\delta,*} \text{ in } H^1(\Omega) \text{ and}$$
$$A^{\epsilon}(x,u^{\delta})\nabla u^{\delta,\epsilon} \rightharpoonup A^{*}(x,u^{\delta})\nabla u^{\delta,*} \text{ in } L^2(\Omega)^N \tag{3.61}$$

where $u^{\delta,*} \in H^1(\Omega)$ is the solution to the linear PDE

$$\begin{aligned} -div\left(A^{*}(x,u^{\delta}(x))\nabla u^{\delta,*}(x)\right) &= 0 \text{ in } \Omega, \\ u^{\delta,*} &= f \text{ on } \partial\Omega. \end{aligned} \tag{3.62}$$

Let $w \in H^1(\Omega)$ be the solution to linear PDE

$$\begin{aligned} -div\left(A^{*}(x,u(x))\nabla w(x)\right) &= 0 \text{ in } \Omega, \\ w &= f \text{ on } \partial\Omega. \end{aligned} \tag{3.63}$$

Then similar to (3.57) we have the following identity which follows from the equations (3.62) and (3.63).

$$\begin{aligned} &\int_{\Omega} A^{*}(x,u^{\delta})\nabla\left(w-u^{\delta,*}\right)\cdot\nabla\left(w-u^{\delta,*}\right)dx \\ &= \int_{\Omega}\left(A^{*}(x,u)-A^{*}(x,u^{\delta})\right)\nabla u^{\delta,*}\cdot\nabla\left(w-u^{\delta,*}\right)dx. \end{aligned}$$

Then together with (3.55) and the fact that $A^{*}(\cdot,t)$ is uniformly Lipschitz in x we get the following estimate

$$||u^{\delta,*}-w||_{H_0^1(\Omega)} \leq C\delta \tag{3.64}$$

where C is independent of δ.

Combining (3.60), (3.61) and (3.64) and by writing

$$\langle u^{\epsilon}-w,v\rangle = \langle (u^{\epsilon}-u^{\delta,\epsilon}),v\rangle + \langle (u^{\delta,\epsilon}-u^{\delta,*}),v\rangle + \langle (u^{\delta,*}-w^{*}),v\rangle,$$

where $\langle,\rangle$, denotes the usual scalar product on $H^1(\Omega)$ and v in $H^1(\Omega)$ is arbitrary, we obtain

$$|\langle u^{\epsilon}-w,v\rangle| \leq C\delta||v||_{H^1(\Omega)} \tag{3.65}$$

where C is independent of ϵ, δ, v and the estimate holds for sufficiently small ϵ. From (3.65), since $\delta > 0$ is arbitrary we can conclude that as $\epsilon \to 0$

$$u^\epsilon \rightharpoonup w \text{ in } H^1(\Omega)$$

and by uniqueness of the weak limit we have $w = u$. Hence we obtain the homogenized equation as

$$\begin{aligned} -div\left(A^*(x, u(x))\nabla u(x)\right) &= 0 \text{ in } \Omega, \\ u &= f \text{ on } \partial\Omega. \end{aligned} \tag{3.66}$$

We note that as $A^*(\cdot, t)$ is uniformly Lipschitz in x (By Lemma 3.4.3), so from Theorem 3.5.1, it follows that the above problem (3.66) has the unique solution $u \in H^1(\Omega)$.

Next we prove that the L^2 weak limit of the flux $A^\epsilon(x, u^\epsilon)\nabla u^\epsilon$ is $A^*(x, u)\nabla u$. Consider

$$\begin{aligned} ||A^\epsilon(x, u^\epsilon)\nabla u^\epsilon - A^\epsilon(x, u^\delta)\nabla u^{\delta,\epsilon}||_{L^2(\Omega)^N} \leq{}& ||A^\epsilon(x, u^\epsilon)\nabla\left(u^\epsilon - u^{\delta,\epsilon}\right)||_{L^2(\Omega)^N} + \\ & ||\left(A^\epsilon(x, u^\epsilon) - A^\epsilon(x, u^\delta)\right)\nabla u^{\delta,\epsilon}||_{L^2(\Omega)^N}. \end{aligned} \tag{3.67}$$

Then from the right hand side of (3.67) we get

$$\begin{aligned} ||A^\epsilon(x, u^\epsilon)\nabla\left(u^\epsilon - u^{\delta,\epsilon}\right)||_{L^2(\Omega)^N} &\leq \left(\sum_{j=1}^N ||a_{ij}(x, u^\epsilon)||_{L^\infty(\Omega)}\right) ||u^\epsilon - u^{\delta,\epsilon}||_{H^1(\Omega)} \\ &\leq C\delta. \end{aligned} \tag{3.68}$$

Using the Lipschitz continuity of $A^\epsilon(\cdot, t)$ we get

$$\begin{aligned} ||\left(A^\epsilon(x, u^\epsilon) - A^\epsilon(x, u^\delta)\right)\nabla u^{\delta,\epsilon}||_{L^2(\Omega)^N} &\leq \left(\sum_{j=1}^N ||a^\epsilon_{ij}(x, u^\epsilon) - a^\epsilon_{ij}(x, u^\delta)||_{L^\infty(\Omega)}\right) \times \\ &\qquad ||u^{\delta,\epsilon}||_{H^1(\Omega)} \\ &\leq C\delta. \end{aligned} \tag{3.69}$$

Thus from (3.67), (3.68) and (3.69) we have

$$||A^\epsilon(x, u^\epsilon)\nabla u^\epsilon - A^\epsilon(x, u^\delta)\nabla u^{\delta,\epsilon}||_{L^2(\Omega)^N} \leq C\delta \tag{3.70}$$

where C is independent of ϵ, δ and the estimates hold for sufficiently small ϵ.
In a similar manner, we can prove the estimates

$$||A^*(x,u^\delta)\nabla u^{\delta,*} - A^*(x,u)\nabla u||_{L^2(\Omega)^N} \leq C\delta \tag{3.71}$$

where C is independent of δ.

Finally we consider

$$\begin{aligned}\langle (A^\epsilon(x,u^\epsilon)\nabla u^\epsilon - A^*(x,u)\nabla u)\,, v\rangle = {}& \langle \big(A^\epsilon(x,u^\epsilon)\nabla u^\epsilon - A^\epsilon(x,u^\delta)\nabla u^{\delta,\epsilon}\big)\,, v\rangle + \\ & \langle \big(A^\epsilon(x,u^\delta)\nabla u^{\delta,\epsilon} - A^*(x,u^\delta)\nabla u^{\delta,*}\big)\,, v\rangle + \\ & \langle \big(A^*(x,u^\delta)\nabla u^{\delta,*} - A^*(x,u)\nabla u\big)\,, v\rangle. \end{aligned} \tag{3.72}$$

Here $\langle , \rangle$ denotes the usual inner product on $L^2(\Omega)^N$ and $v \in L^2(\Omega)^N$ is arbitrary. Now by using (3.61), (3.70), and (3.71) we conclude from (3.72) that

$$|\langle (A^\epsilon(x,u^\epsilon)\nabla u^\epsilon - A^*(x,u)\nabla u)\,, v\rangle| \leq C\delta \tag{3.73}$$

where C is independent of ϵ, δ and $v \in L^2(\Omega)^N$. This inequality holds for sufficiently small ϵ. If we consider $\delta > 0$ to be arbitrary, then (3.73) yields

$$A^\epsilon(x,u^\epsilon)\nabla u^\epsilon \rightharpoonup A^*(x,u)\nabla u \quad \text{weakly in } L^2(\Omega)^N.$$

This completes the discussion of our proof. ∎

Remark 5. *Let us now consider a Lipschitz domain Ω' such that $\Omega \subset\subset \Omega'$. We extend the homogenization coefficients $A^\epsilon(x,t)$ defined in Theorem 3.4.5 by identity in $\Omega' \setminus \Omega \times \mathbb{R}$ and still denote the extended coefficients A^ϵ. Fix $f \in H^{\frac{1}{2}}(\partial\Omega')$. We consider the following equation*

$$\begin{aligned} -div(A^\epsilon(x,u^\epsilon)\nabla u^\epsilon) = 0 \;\; &in \;\Omega', \\ u^\epsilon = f \;\; &on \;\partial\Omega'. \end{aligned}$$

Note that by interior regularity $u^\epsilon \in C^{2,\gamma}(\overline{\Omega}) \cap H^1(\Omega')$.

Since the homogenization coefficients $A^\epsilon(x,t) = I_{N\times N}$ in the ring $\Omega' \setminus \Omega \times \mathbb{R}$ for any $\epsilon > 0$, the conclusion of Theorem 3.4.5 still holds and passing through the limit in ϵ we obtain

$$A^\epsilon(x,t) \text{ } H\text{-converges to } A^*(x,t) \text{ in } \Omega' \times \mathbb{R}$$

where, by localization principle, $A^*(x,t) = I_{N\times N}$ in $\Omega' \setminus \Omega \times \mathbb{R}$ and $A^*(x,t)$ in $\Omega \times \mathbb{R}$ is in (3.46).

Moreover, we also have

$$u^\epsilon \rightharpoonup u \quad \text{weakly in } H^1(\Omega') \text{ and}$$
$$A^\epsilon(x,u^\epsilon)\nabla u^\epsilon \rightharpoonup A^*(x,u)\nabla u \quad \text{weakly in } L^2(\Omega')^N$$

where $u \in H^1(\Omega')$ is the unique solution to

$$-div\,(A^*(x,u(x))\nabla u(x)) = 0 \text{ in } \Omega',$$
$$u = f \text{ on } \partial\Omega'.$$

3.4.2 Regular isotropic approximate cloak in $\mathbb{R}^N$

In this section we approximate the anisotropic approximate (or near) cloaks $\sigma_A^r(x,t)$ as defined in (3.5) by isotropic conductivities, which then will themselves be approximate cloaks. We restrict our attention to the case when $\Omega = B_2$ and $E = B_1$ needs to be cloaked.

We will be considering isotropic conductivities of the form

$$A^\epsilon(x,t) = \sigma\left(x, \frac{|x|}{\epsilon}, t\right) I_{N\times N}, \quad (x,t) \in \Omega \times \mathbb{R}, \tag{3.74}$$

where $\sigma(x,r',t)$ is a smooth, scalar valued function such that

A1) $0 < \alpha \le \sigma(x,r',t) \le \beta < \infty$ for all $(x,r',t) \in \Omega \times [0,1] \times \mathbb{R}$;

A2) $t \mapsto \sigma(x,r',t)$ is uniformly Lipschitz for $(x,r') \in \Omega \times [0,1]$;

A3) σ is periodic in r' with period 1 i.e $\sigma(x,r'+1,t) = \sigma(x,r',t)$, where $(x,r',t) \in \Omega \times [0,1] \times \mathbb{R}$.

We would like to find the homogenized coefficient $A^*(x,t)$ given in (3.46) for this choice of $A^\epsilon(x,t)$ in (3.74). In order to do that, we introduce polar coordinates. Let $s_1 = (r, \theta_1, .., \theta_N)$ and $s_2 = (r', \theta'_1, .., \theta'_N)$ be spherical coordinates corresponding to two different scales. Next we homogenize the conductivity in the $(r', \theta'_1, .., \theta'_N)$-coordinates. Let us consider the canonical basis vectors $e_1, .., e_N$ of $\mathbb{R}^N$ in r', $\theta'_1, ..., \theta'_N$ directions, respectively. Then for almost every $(s_1, t) \in \Omega \times \mathbb{R}$, let $\chi_k(s_1, s_2, t)$, $k = 1, .., N$ be the solutions to

$$\begin{aligned} &- div_{s_2}(\sigma(s_1, r', t)(\nabla_{s_2}\chi_k(s_1, s_2, t) + e_k) = 0 \text{ in } \mathbb{R}^N, \\ &s_2 = (r', \theta'_1, .., \theta'_N) \to \chi_k(s_1, s_2, t) \quad \text{is 1-periodic in each of coordinate } r', \theta'_1, .., \theta'_N. \end{aligned} \tag{3.75}$$

Further, it is assumed that

$$\int_Y \chi_k(s_1, s'_2, t) ds'_2 = 0, \tag{3.76}$$

where, $s'_2 = (r', \theta'_1, .., \theta'_N)$ and $ds'_2 = dr' d\theta'_1 ... d\theta'_N$.

Since $\sigma(s_1, r', t)$ is independent of $\theta'_1, .., \theta'_N$, therefore from (3.75) together with (3.76), it implies that $\chi_k = 0$ for $k = 2, .., N$.

Let us consider the equation (3.75) for χ_1 which satisfies

$$\frac{\partial}{\partial r'}\left(\sigma(s_1, r', t)\frac{\partial \chi_1(s_1, s_2, t)}{\partial r'}\right) = -\frac{\partial \sigma(s_1, r', t)}{\partial r'}. \tag{3.77}$$

From the fact that χ_1 is 1-periodic with respect to the variables $\theta'_1, ..., \theta'_N$ it implies that χ_1 is independent of $\theta'_1, ..., \theta'_N$. Moreover, from (3.77) we get

$$\frac{\partial \chi_1}{\partial r'} = -1 + \frac{C}{\sigma(s_1, r', t)} \tag{3.78}$$

where the constant C can be found by using the periodicity of χ_1 with respect to r' as

$$C = \frac{1}{\int_0^1 \sigma^{-1}(s_1, r', t) dr'} := \underline{\sigma}(s_1, t)$$

where $\underline{\sigma}(s_1, t)$ denotes the harmonic mean of $\sigma(s_1, \cdot, t)$ in the second variable.

Let $\overline{\sigma}(s_1, t)$ denote the arithmetic mean of $\sigma(s_1, \cdot, t)$ in the second variable defined as

$$\overline{\sigma}(s_1, t) = \int_0^1 \sigma(s_1, r', t) dr'.$$

Then from (3.46) the homogenized conductivity, say $\widehat{\sigma}_{kl}(s_1, t)$, turns out to be

$$\widehat{\sigma}_{kl}(s_1, t) = \int_Y \sigma(s_1, r', t) \left(\frac{\partial \chi_k(s_1, s_2', t)}{\partial (s_2')_l} + \delta_{lk} \right) ds_2',$$

and can be written as

$$\widehat{\sigma}(x, t) = \underline{\sigma}(x, t)\Pi(x) + \overline{\sigma}(x, t)(I - \Pi(x)),$$

where $\Pi(x) : \mathbb{R}^N \to \mathbb{R}^N$ is the projection on to the radial direction, defined by

$$\Pi(x)\, v = \left(v \cdot \frac{x}{|x|} \right) \frac{x}{|x|},$$

i.e., $\Pi(x)$ is represented by the matrix $|x|^{-2} x x^t$, cf. [39].

Next, we give more explicit construction of the regular isotropic cloak. This construction is a generalization of the linear case presented in [39]. Let us consider functions $\phi : \mathbb{R} \to \mathbb{R}$ and $\phi_M : \mathbb{R} \to \mathbb{R}$, for some positive integer M, given by

$$\phi(t) = \begin{cases} 0 & t < 0, \\ \frac{1}{2} t^2 & 0 \le t < 1, \\ 1 - \frac{1}{2}(2 - t)^2 & 1 \le t < 2, \\ 1 & t \ge 2 \end{cases} \tag{3.79}$$

and

$$\phi_M(t) = \begin{cases} 0 & t < 0, \\ \phi(t) & 0 \le t < 2, \\ 1 & 2 \le t < M - 2, \\ \phi(M - t) & t \ge M - 2. \end{cases} \tag{3.80}$$

Let us define

$$\sigma(x, r', t) = \left[1 + a^1(x, t)\zeta_1\left(\frac{r}{\epsilon}\right) - a^2(x, t)\zeta_2\left(\frac{r}{\epsilon}\right) \right]^2,$$

where $a^k(x, t)$, $k = 1, 2$ are chosen positive smooth functions such that $\sigma(x, r', t)$ satisfies the conditions immediately following (3.74) (namely A1), A2) and A3)). In particular, we choose a_1 and a_2 to satisfy conditions A1) and A2), for some possibly different choice of constants α, β, L. And for some positive integer M, we define $\zeta_j : \mathbb{R} \to \mathbb{R}$ to be $1-$periodic functions,

$$\begin{aligned}
\zeta_1(t) &= \phi_M\Big(2Mt\Big) \quad 0 \le t < 1, \\
\zeta_2(t) &= \phi_M\Big(2M(t-0.5)\Big) \quad 0 \le t < 1,
\end{aligned}$$

where ϕ_M is as defined in (3.80).

Let us temporarily fix $R > 1$. In order to guarantee that the conductivity $\hat{\sigma}$ is smooth enough, we piece together the cloaking conductivity in the exterior domain $|x| > R$ and the homogeneous conductivity in the cloaked domain in a smooth manner. To this end, we introduce a new parameter $\eta > 0$ and solve for each (x,t) the parameters $a^1(x,t), a^2(x,t)$ from the following expressions of the equations for the harmonic and arithmetic averages for $1 < R < 2$ respectively.

$$\begin{aligned}
&\int_0^1 [1 + a^1(x,t)\zeta_1(r') - a^2(x,t)\zeta_2(r')]^{-2} dr' \\
&= \begin{cases} 2R^{-2}(R-1)^2(1-\phi(\frac{R-r}{\eta})) + \psi(x,t)\phi\left(\frac{R-r}{\eta}\right) & \text{if } |x| < R, \\ 2r^{-2}(r-1)^2 & \text{if } R < |x| < 2, \\ 1 & \text{if } |x| > 2. \end{cases} \\
&\int_0^1 [1 + a^1(x,t)\zeta_1(r') - a^2(x,t)\zeta_2(r')]^{2} dr' \\
&= \begin{cases} 2(1-\phi(\frac{R-r}{\eta})) + \psi(x,t)\phi(\frac{R-r}{\eta}) & \text{if } |x| < R, \\ 2 & \text{if } R < |x| < 2, \\ 1 & \text{if } |x| > 2 \end{cases}
\end{aligned}$$

where, $\psi(x,t) \in C^\infty(B(0,2)\times\mathbb{R}) \ge C > 0$.

Thus by obtaining $a^1(x,t) = a^1_{R,\eta}(x,t)$ and $a^2(x,t) = a^2_{R,\eta}(x,t)$, the homogenized conductivity can be written as

$$\widehat{\sigma}(x,t) = \sigma_{R,\eta}(x,t) = \begin{cases} \pi_R(1-\phi(\frac{R-r}{\eta})) + \psi(x,t)\phi(\frac{R-r}{\eta}) & \text{if } |x| < R, \\ \pi_R & \text{if } R < |x| < 2, \\ 1 & \text{if } |x| > 2 \end{cases}$$

where

$$\pi_R = 2R^{-2}(R-1)^2\Pi(x) + 2(1-\Pi(x)).$$

Note that the term $\psi(x,t)\phi(\frac{R-r}{\eta})$ connects the exterior conductivity smoothly to the interior conductivity $\psi(x,t)$.

We first let $\epsilon \to 0$, then $\eta \to 0$ and finally $R \searrow 1$. The obtained homogenized conductivities approximate better and better the cloaking conductivity σ_A (cf (3.33)). We hence choose appropriate sequences $R_n \searrow 1$, $\eta_n \to 0$ and $\varepsilon_n \to 0$ and denote

$$\sigma_n(x,t) := \left[1 + a^1_{R_n,\eta_n}(x,t)\zeta_1\left(\frac{r}{\varepsilon_n}\right) - a^2_{R_n,\eta_n}(x,t)\zeta_2\left(\frac{r}{\varepsilon_n}\right)\right]^2, \quad r = |x|.$$

Let $\Omega' = B_3$. The above sequence $\sigma_n(x,t)$ is the desired regular isotropic sequence which approximates the cloaking for quasi-linear problem in the following sense:

Let $u_n \in H^1(\Omega')$ solve

$$-div(\sigma_n(x,u_n)\nabla u_n(x)) = 0 \quad \text{in } \Omega',$$
$$u_n = f \quad \text{on } \partial\Omega' \text{ with } f \in H^{\frac{1}{2}}(\partial\Omega').$$

Then as $n \to \infty$,

$$u_n \text{ weakly converges to } u \text{ in } H^1(\Omega') \tag{3.81}$$

where $u \in H^1(\Omega')$ solves

$$-div(\sigma_A(x,u)\nabla u(x)) = 0 \quad \text{in } \Omega',$$
$$u = f \quad \text{on } \partial\Omega',$$

and $\sigma_A(x,t)$ is as defined in (3.33) in Ω and we assume $\sigma_A(x,t) = I_{N\times N}$ on $\Omega' \setminus \Omega \times \mathbb{R}$.

3.4.3 Convergence of DN map

Let $\Omega' = B(0,3)$. Recall that we extend our isotropic regular cloaks $\sigma_n(x,t)$ and the perfect cloak $\sigma_A(x,t)$ by 1 outside $\Omega = B_2$. In particular, for some $0 < \kappa < 1$

$$\sigma_n(x,t) = \sigma_A(x,t) = 1 \quad \forall (x,t) \in B(0, 2+\kappa, 3) \times \mathbb{R}.$$

The Dirichlet-to-Neumann map is now given by

$$\Lambda_n, \Lambda : H^{\frac{3}{2}}(\partial\Omega') \to H^{\frac{1}{2}}(\partial\Omega') \quad \text{defined as}$$
$$\Lambda_n(f) := \frac{\partial u_n}{\partial \nu} \text{ and } \Lambda(f) := \frac{\partial u}{\partial \nu} \text{ respectively.}$$

We then consider $w_n = (u_n - u)$, which satisfy

$$-\Delta w_n = 0 \quad \text{ in } B(0, 2+\kappa, 3),$$
$$w_n = 0 \quad \text{on } \partial B(0,3).$$

From the elliptic regularity theory, we conclude $w_n \in H^2(\widetilde{\Omega})$ where
$\widetilde{\Omega} \subset B(0, 2+\kappa, 3)) \cup \partial B(0,3)$. Moreover, from [35, Theorem 9.13], we have

$$||w_n||_{H^2(\widetilde{\Omega})} \le C||w_n||_{L^2(B(0,2+\kappa,3))}.$$

Now if $f \in H^{\frac{3}{2}}(\partial\Omega')$ then

$$\left\|\frac{\partial u_n}{\partial \nu} - \frac{\partial u}{\partial \nu}\right\|_{H^{\frac{1}{2}}(\partial\Omega')} \le ||u_n - u||_{H^2(\widetilde{\Omega})} \le C||u_n - u||_{L^2(B(0,2+\kappa,3))}. \tag{3.82}$$

As we see from (3.81) by using the Rellich compactness theorem we have

$$u_n \text{ strongly converges to } u \text{ in } L^2(\Omega'). \tag{3.83}$$

Thus from (3.82), for $f \in H^{\frac{3}{2}}(\partial\Omega')$ we obtain

$$||(\Lambda_n - \Lambda)(f)||_{H^{\frac{1}{2}}(\partial\Omega')} = \left\|\frac{\partial u_n}{\partial \nu} - \frac{\partial u}{\partial \nu}\right\|_{H^{\frac{1}{2}}(\partial\Omega')} \to 0 \quad \text{ as } n \to \infty,$$

which gives the required strong convergence of the DN map.

3.5 Appendix: Forward Problem

In this section, we show the well-posedness of the boundary value problem (3.2) and define the Dirichlet-to-Neumann map associated to (3.2).

Theorem 3.5.1 (Existence and Uniqueness)**.** *Consider $\Omega \subset \mathbb{R}^N$, $N \geq 2$, a bounded open set with Lipschitz boundary. Let $\mathcal{M}(\alpha, \beta, L; \Omega \times \mathbb{R})$ with $0 < \alpha < \beta < \infty$ and $L > 0$ denote the set of all real $N \times N$ symmetric matrices $A(x,t)$ of functions defined almost everywhere on $\Omega \times \mathbb{R}$ such that if $A(x,t) = [a_{kl}(x,t)]_{1 \leq k,l \leq N}$ then*

1. *$a_{kl}(x,t) = a_{lk}(x,t)\ \forall l, k = 1, .., N,$*
2. *$(A(x,t)\xi, \xi) \geq \alpha|\xi|^2,\ |A(x,t)\xi| \leq \beta|\xi|, \quad \forall \xi \in \mathbb{R}^N, \quad$ a.e. $x \in \Omega$ and,*
3. *$|a_{kl}(x,t) - a_{kl}(x,s)| \leq L|t-s|$ for a.e $x \in \Omega$ and any t, s in $\mathbb{R}$.*

Then the boundary value problem (3.2) has a unique solution $u_f \in H^1(\Omega)$ for any $f \in H^{\frac{1}{2}}(\partial\Omega)$ with $||u_f||_{H^1(\Omega)} \leq C||f||_{H^{1/2}(\partial\Omega)}$ where $C = C(\Omega, N, L, \alpha, \beta)$.

Proof. Let us fix $f \in H^{1/2}(\partial\Omega)$ and $\hat{v} \in L^2(\Omega)$. We consider the linear boundary value problem

$$\begin{aligned} -div(A(x,\hat{v})\nabla u) &= 0 \text{ in } \Omega, \\ u &= f \text{ on } \partial\Omega. \end{aligned} \tag{3.84}$$

Since $A(x,\hat{v}) \in L^\infty(\Omega)$ we use Lax Milgram theorem and conclude that there exists a unique u that solves the linearized boundary value problem (3.84) with $||u||_{H^1(\Omega)} \leq C||f||_{H^{1/2}(\partial\Omega)}$ where C depends only on the parameters mentioned in the statement of the theorem.

We consider an operator $T : L^2(\Omega) \to L^2(\Omega)$ defined by $T(\hat{v}) = u$ where u is a solution to (3.84). By existence and uniqueness theory for linear elliptic equations, T is well defined. Note that T maps $S = \left(u \in H^1(\Omega) : ||u||_{H^1(\Omega)} \leq C||f||_{H^{1/2}(\partial\Omega)}\right)$ to itself. C may change line by line but henceforth will depend only on $(\Omega, \alpha, \beta, L)$.

The existence of a solution to (3.2) will be proved if we show T has a fixed point in S. We will use Schauder fixed point theorem to to prove this. For that, we need to show T is a continuous operator and that S is closed and convex subset of $L^2(\Omega)$.

Let $\hat{v_n} \to \hat{v}$ in $L^2(\Omega)$. Let $T(\hat{v_n}) = u_n$ and $T(\hat{v}) = u$. Since $(u_n - u) \in H_0^1(\Omega)$ we get

$$\int_\Omega (A(x, \hat{v_n})\nabla(u_n - u)) \cdot \nabla(u_n - u)\, dx$$
$$= \int_\Omega ((A(x, \hat{v}) - \sigma(x, \hat{v_n}))\nabla u) \cdot \nabla(u_n - u)\, dx.$$

From Poincaré inequality, Hölder's inequality and the uniform ellipticity assumption on $A(x, t)$ we get,

$$\alpha||u_n - u||_{L^2(\Omega)} \leq C||(A(x, \hat{v_n}) - A(x, \hat{v}))\nabla u||_{L^2(\Omega)}. \tag{3.85}$$

Let ϵ be a positive real number. Let $E_n = (x \in \Omega : |\hat{v_n(x)} - \hat{v(x)}| \geq \sqrt{\epsilon})$. Since $\hat{v_n} \to \hat{v}$ in $L^2(\Omega)$, $meas(E_n) \to 0$ as $n \to \infty$. (Convergence in $L^2(\Omega)$ implies a.e convergence along a subsequence. We rename this subsequence $\hat{v}_n$ and proceed.) Moreover, there exists a $\delta > 0$ such that for any $E \subset \Omega$ with $meas(E) < \delta$, we have $\int_E |\nabla u|^2\, dx < \epsilon$. Choose a n_0 so that for $n \geq n_0$, $meas(E_n) < \delta$.

For $n > n_0$, we then have

$$\int_{\Omega\setminus E_n} |(A(x, \hat{v}) - A(x, \hat{v_n}))\nabla u|^2\, dx \leq \epsilon L||f||^2_{H^{\frac{1}{2}}(\partial\Omega)} \text{ and}$$
$$\int_{E_n} |(A(x, \hat{v}) - A(x, \hat{v_n}))\nabla u|^2\, dx \leq C\epsilon.$$

Since f is fixed and ϵ is arbitrary, $||(A(x, \hat{v_n}) - A(x, \hat{v}))\nabla u||_{L^2(\Omega)} \to 0$ as $n \to \infty$. By (3.85), $u_n \to u$ in $L^2(\Omega)$ proving the continuity of T. (Actually this only shows that there is a subsequence of u_n that converges in $L^2(\Omega)$ to u. We run the entire argument with any arbitrary subsequence of u_n and thereby get by the above argument that every subsequence of u_n has a subsequence that converges in $L^2(\Omega)$ to u. Hence we can say $u_n \to u$ in $L^2(\Omega)$.) Note that T is a compact operator in L^2 topology by Rellich compactness theorem.

S is clearly a convex subset of $L^2(\Omega)$. To show S is closed, we let $u_n \to u$ in $L^2(\Omega)$ and $u_n \in S$. The sequence u_n is bounded in $H^1(\Omega)$. By Banach Alouglu theorem, there exists a subsequence u_{nk} so that $u_{nk} \to w$ weakly in $H^1(\Omega)$. We also have

$$||w||_{H^1(\Omega)} \leq \liminf_{k\to\infty} \ ||u_{nk}||_{H^1(\Omega)} \leq C||f||_{H^{1/2}(\partial\Omega)}.$$

Weak convergence in $H^1(\Omega)$ implies weak convergence in $L^2(\Omega)$. Uniqueness of limits implies $u = w$, thereby proving S is closed.

Let us now prove uniqueness. Let there exist 2 solutions u_1 and u_2 to (3.2). For $\epsilon > 0$ define F_ϵ as

$$F_\epsilon(x) = \begin{cases} \int_0^\epsilon \frac{dt}{L^2t^2} & x \geq \epsilon, \\ 0 & x \leq \epsilon. \end{cases}$$

F_ϵ is a piecewise continuous C^1 function such that

$$|F'_\epsilon(x)| \leq \frac{1}{L^2\epsilon^2} \quad \forall x \in \mathbb{R}.$$

By Corollary 2.14 in [21],

$$F_\epsilon(u_1 - u_2) \in H^1_0(\Omega).$$

We thus have, for any $v \in H^1_0(\Omega)$,

$$\int_\Omega A(x, u_1)\nabla(u_1 - u_2) \cdot \nabla v \, dx = \int_\Omega (A(x, u_2) - \sigma(x, u_1))\nabla u_2 \cdot \nabla v \, dx.$$

Choose $v = F_\epsilon(u_1 - u_2)$ to get

$$\begin{aligned} &\int_{\{u_1 - u_2 > \epsilon\}} A(x, u_1) \frac{|\nabla(u_1 - u_2)|^2}{L^2(u_1 - u_2)^2} \, dx \\ &= \int_{\{u_1 - u_2 > \epsilon\}} \frac{(A(x, u_2) - A(x, u_1))\nabla u_2 \cdot \nabla(u_1 - u_2)}{L^2(u_1 - u_2)^2} \, dx. \end{aligned} \tag{3.86}$$

From (3.86), the fact that $t \mapsto A(., t)$ is uniformly Lipschitz and Cauchy-Schwartz inequality, we obtain

$$
\begin{aligned}
&\alpha \int_{\{u_1-u_2>\epsilon\}} \frac{|\nabla(u_1-u_2)|^2}{L^2(u_1-u_2)^2\,dx} \\
&\leq \int_{\{u_1-u_2>\epsilon\}} \frac{|(A(x,u_2)-A(x,u_1)|)|\nabla u_2||\nabla(u_1-u_2)|}{L^2(u_1-u_2)^2}\,dx \\
&\leq \int_{\{u_1-u_2>\epsilon\}} \frac{|\nabla u_2||\nabla(u_1-u_2)|}{L|(u_1-u_2)|}\,dx \\
&\leq \left[\int_{\{u_1-u_2>\epsilon\}} \frac{|\nabla(u_1-u_2)|^2}{L^2(u_1-u_2)^2}\,dx\right]^{1/2}\left[\int_{\{u_1-u_2>\epsilon\}} |\nabla u_2|^2\,dx\right]^{1/2}.
\end{aligned}
$$

From this it follows that

$$
\begin{aligned}
&\left[\int_{\{u_1-u_2>\epsilon\}} \frac{|\nabla(u_1-u_2)|^2}{L^2(u_1-u_2)^2}\,dx\right] \\
&\leq \frac{1}{\alpha^2}\int_{\{u_1-u_2>\epsilon\}} |\nabla u_2|^2\,dx \\
&\leq \frac{1}{\alpha^2}\int_{\Omega} |\nabla u_2|^2\,dx.
\end{aligned} \tag{3.87}
$$

Set

$$
G_\epsilon(x) = \begin{cases} \int_0^\epsilon \frac{dt}{Lt} & x \geq \epsilon, \\ 0 & x \leq \epsilon. \end{cases}
$$

With this definition, (3.87) becomes

$$
\int_\Omega |\nabla G_\epsilon(u_1-u_2)|^2\,dx \leq \frac{1}{\alpha^2}\int_\Omega |\nabla u_2|^2\,dx. \tag{3.88}
$$

Note that G_ϵ is a continuous piecewise C^1 function satisfying the assumptions of Corollary 2.15 in [21]. Thus by Corollary 2.15 in [21] we obtain

$$
G_\epsilon(u_1-u_2) \in H_0^1(\Omega). \tag{3.89}
$$

Use Poincaré inequality in (3.88) to get

$$
\int_\Omega |G_\epsilon(u_1-u_2)^2\,dx \leq C\int_\Omega |\nabla u_2|^2\,dx.
$$

We now pass to the limit in ϵ and use Fatou's lemma to get

$$\int_\Omega \liminf_{\epsilon\to 0} |G_\epsilon(u_1-u_2)^2\, dx \le C\int_\Omega |\nabla u_2|^2\, dx.$$

Hence

$$\liminf_{\epsilon\to 0} |G_\epsilon(u_1-u_2)^2 < +\infty \text{ a.e } x\in\Omega.$$

The definition of G_ϵ then implies that

$$u_1 - u_2 \le 0 \text{ a.e } x\in\Omega.$$

Switch the roles of u_1 and u_2 to conclude that

$$u_1 = u_2 \text{ a.e } x\in\Omega.$$

■

Theorem 3.5.1 allows us to define the Dirichlet-to-Neumann map weakly. The DN map

$$\Lambda_A : H^{1/2}(\partial\Omega) \to H^{-1/2}(\partial\Omega)$$

is defined weakly as

$$\langle \Lambda_A(f), g\rangle = \int_\Omega A(x,u)\nabla u\cdot\nabla v\, dx$$

where u is the unique solution to (3.2) and v is any $H^1(\Omega)$ function with trace g. This is the natural generalization of the Dirichlet-to-Neumann map to a quasi-linear equation of divergence type.

We end this section by stating a result on higher global regularity of solutions to the quasi-linear elliptic equation (3.2).

Lemma 3.5.2 (Regularity)**.** *Let $\Omega\subset\mathbb{R}^N$, $N\ge 2$ be a bounded domain with $C^{2,\gamma}$ boundary, $0<\gamma<1$. Let $A(x,t) = [a_{ij}(x,t)]_{N\times N}\in\mathcal{M}(\alpha,\beta,L;\overline{\Omega}\times\mathbb{R})$ and we further assume that $A\in C^{1,\gamma}(\overline{\Omega}\times\mathbb{R})$, then the quasi-linear boundary value problem*

$$\nabla\cdot A(x,u)\nabla u = 0 \text{ in } \Omega,$$

$$u|_{\partial\Omega} = f. \tag{3.90}$$

has a unique solution $u \in C^{2,\gamma}(\overline{\Omega})$ *for any* $f \in C^{2,\gamma}(\overline{\Omega})$ *satisfying for all* $x, y \in \overline{\Omega}$

$$|u(x) - u(y)| \leq C_f |x - y|^{\lambda}$$

where $C_f = C(N, \Omega, \alpha, \beta, f)$ *and* $\lambda = \lambda(N, \Omega, \alpha, \beta, \gamma)$.

Proof. The existence and uniqueness result for the quasi-linear boundary value problem considered in (3.90) can be found in [35] or [55].

For proving the Hölder estimates on u, we let $v = u - f$ and consider the linear boundary value problem

$$\nabla \cdot A(x, u)\nabla v = -\nabla \cdot A(x, u)\nabla f,$$
$$v|_{\partial\Omega} = 0.$$

Let $\varphi = -\nabla \cdot A(x, u)\nabla f$ and note that $\varphi \in C^{\gamma}(\overline{\Omega})$. By considering $\varphi \in L^p(\Omega)$ for some $p > \frac{N}{2}$ we obtain, for all $x, y \in \overline{\Omega}$

$$|v(x) - v(y)| \leq C||\varphi||_{L^p(\Omega)}|x - y|^{\lambda'},$$

where $C = C(N, \Omega, \alpha, \beta)$ and $\lambda' = \lambda'(N, \Omega, \alpha, \beta)$. By using triangle inequality on $u = v + f$ we conclude that u is also Hölder continuous with the exponent $\lambda = min\{\lambda', \gamma\}$. ∎

Chapter 4

UNIQUENESS FOR A QUASI-LINEAR ELLIPTIC PDE

4.1 Introduction

In the last chapter, we showed how in the absence of ellipticity, we can construct approximate and perfect cloaks for the quasi-linear equation 3.2. The underlying coefficient $A(x,t)$ was anisotropic and we first constructed anisotropic approximate cloaks, and then used homogenization techniques to construct isotropic approximate cloaks.

The focus of this chapter will be slightly different. We will state and prove a simple uniqueness result. More specifically, for the equation 3.2, under the condition $A(x,t) = a(x,t)I_{N\times N}$, we prove injectivity of the DN map. The statements of this chapter hold for dimensions $N \geq 2$.

4.2 Main result

We now state and prove the main result of this chapter.

Theorem 4.2.1. *Consider $\Omega \subset \mathbb{R}^N$, $N \geq 2$, a bounded open set with Lipschitz boundary. Let $\mathcal{R}(\alpha, \beta, L; \Omega \times \mathbb{R})$ with $0 < \alpha < \beta < \infty$ and $L > 0$ denote the set of all realvalued functions $a(x,t)$ defined everywhere on $\Omega \times \mathbb{R}$ such that*

1. $a(x,t) \geq \alpha,\ |a(x,t)| \leq \beta, \quad$ *a.e.* $x \in \Omega$ *and,*
2. $|a(x,t) - a(x,s)| \leq L|t-s|$ *for a.e* $x \in \Omega$ *and any* t, s *in* $\mathbb{R}$.
3. *For* $N \geq 3$*, assume that for each* t*,* $x \to a(x,t) \in C^{0,1}(\bar{\Omega})$.

If $a_1(x,t)$, $a_2(x,t)$ satisfy (4.2.1) and $\Lambda_{a_1} = \Lambda_{a_2}$, then $a_1(x,t) = a_2(x,t)$ a.e in $\Omega \times \mathbb{R}$.

(Note that for $N \geq 3$, assume that $a(x,t)$ is defined on $\bar{\Omega} \times \mathbb{R}$ and Properties 1 and 2 above hold for all $x \in \bar{\Omega}$.)

Proof. To start with, let us first linearize the Dirichlet to Neumann map around constant functions. Fix $t \in \mathbb{R}$ and $f \in H^{\frac{1}{2}}(\partial\Omega)$. Let $a(x,t) \in \mathcal{R}(\alpha, \beta, L; \Omega \times \mathbb{R})$.

Let us denote the Dirichlet-to-Neumann map (which was shown to well-defined in Section 3.5) corresponding to $a(x,t)$, where we freeze t, by Λ_{a^t}.

Consider

$$\nabla \cdot (a(x,u)\nabla u) = 0 \quad \text{in} \quad \Omega \quad \textit{with} \quad u\big|_{\partial\Omega} = f, \tag{4.1}$$

where $f \in H^{1/2}(\partial\Omega)$.

Let $u^s(x)$ be the unique solution of (4.1) with boundary data $t + sf$ in place of f. We can then write $u^s(x) = t + sv^s(x)$ where $v^s(x)$ is the unique solution to

$$\nabla \cdot (a(x, t + sv^s(x))\nabla v^s(x)) = 0, \quad v^s\big|_{\partial\Omega} = f. \tag{4.2}$$

1. We will first show $v_s \to v_0$ in $H^1(\Omega)$ as $s \to 0$, where v^0 is the unique solution to (4.2) for $s = 0$.

   To that end, let $w^s = v^s - v^0$. Note that, w^s solves

   $$\nabla \cdot (a(x,u^s)\nabla w^s) = \nabla \cdot \left((a(x,t) - a(x,u^s))\nabla v^0\right), \quad w^s\big|_{\partial\Omega} = 0. \tag{4.3}$$

   Using the standard estimates for linear elliptic equations, we obtain

   $$||w^s||_{H^1(\Omega)} \leq C \left(|| \left(a(x,t) - a(x,u^s)\right) \nabla v^0\right) ||_{L^2(\Omega)}, \tag{4.4}$$

   where C depends only on Ω, L, α, β. The proof of the existence and uniqueness result for the direct problem, as in Section 3.5, allows us to conclude that $||u^s - t||_{H^1(\Omega)} \to 0$ as $s \to 0$. Thus there exists a subsequence, still denoted by u^s that goes to t point wise a.e in Ω. Using Lebesgue dominated convergence theorem and continuity of a in the second parameter, we get $||(a(x,t) - a(x,u^s))\nabla v^0\big)\,||_{L^2(\Omega)} \to 0$ as $s \to 0$. Hence

$$||v^s - v^0||_{H^1(\Omega)} \to 0 \quad \text{as} \quad s \to 0. \tag{4.5}$$

2. Fix a g in $H^{1/2}(\partial\Omega)$. We will now show

$$||a(x,t+sv^s)\nabla(v^s)\cdot\nabla v^g - a(x,t)\nabla(v^0)\cdot\nabla v^g||_{L^1(\Omega)} \to 0 \text{ as } s \to 0, \tag{4.6}$$

where $v^g \in H^1(\Omega)$ has g.

The choice of v_g is in our hands. In particular, let us choose $v^g \in H^1(\Omega)$ such that v^g uniquely solves

$$\nabla\cdot(\sigma(x,t)\nabla v^g) = 0 \quad v^g\big|_{\partial\Omega} = g.$$

Thus $||v^g||_{H^1(\Omega)} \leq C||g||_{H^{1/2}(\partial\Omega)}$.

Now,

$$\begin{aligned}
&||a(x,t+sv^s)\nabla(v^s)\cdot\nabla v^g - a(x,t)\nabla(v^0)\cdot\nabla v^g\;||_{L^1(\Omega)} \\
&\leq ||a(x,t+sv^s)\nabla(v^s - v^0)\nabla v^g|||_{L^1(\Omega)} + \\
&||(a(x,t+sv^s) - a(x,t))\nabla v^0\cdot\nabla v^g||_{L^1(\Omega)}.
\end{aligned} \tag{4.7}$$

By Holder's inequality, the assumptions on a, (4.5), and the bound on v^g, we get (4.6).

3. Using the weak definition of the DN map Λ_a, (4.6) is enough to imply that for each t,

$$\left|\frac{1}{s}\Lambda_a(t+sf)(g) - \Lambda_{a^t}(f)(g)\right| \;\to\; 0 \quad \text{as } s \to 0. \tag{4.8}$$

4. Equation (4.8) implies that if $\Lambda_{a_1} = \Lambda_{a_2}$, then for each t, $\Lambda_{a_1^t} = \Lambda_{a_2^t}$.

5. From the main result in [19] (for $N \geq 3$) and [8] (for $N = 2$), we can conclude that for each t, $a_1^t(x) = a_2^t(x)$ a.e in Ω.

6. This proves the desired result

$\blacksquare$

Remark 6. *The proof of the above theorem is a simple linearization argument, and was proved under more restricted assumptions in Theorem 1.7 in [81] for* $N \geq 3$*. Moreover, for* $N = 2$*, a similar result to the one proved above was also alluded to. In this chapter, we gave a short self-contained proof for uniqueness in the Calderón problem for scalar valued quasi-linear conductivities.*

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