Homological tools, intersection theory, and tropical abelian varieties

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Abstract

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We develop tropical homological tools and apply them to the study of certain tautological cycles in the Jacobian of a metric graph.

First, in joint work with Junaid Hasan and Farbod Shokrieh, we construct an explicit tropical de Rham map from superforms to tropical cochains and prove that it induces an isomorphism of cohomology rings that is compatible with Poincaré duality on both sides. Applying this to a principally polarized tropical abelian variety (X, ϑ), we produce a canonical $(1,1)$-superform ν_ϑ whose cohomology class is Poincaré-dual to the theta divisor. Specializing to the Jacobian of a metric graph Γ, we show that integrating powers of ν_ϑ over the effective loci $\widetilde{W}_d$ recovers Foster's theorem and Kirchhoff's matrix-tree theorem as special cases. We also give an alternative proof of the tropical Poincaré formula.

Second, we study the tropical Ceresa cycle $\widetilde{W}_1 - \widetilde{W}_1^-$ in $\mathrm{Jac}(\Gamma)$. We define a graph invariant $\alpha(G)$ called the Ceresa period which acts as an obstruction to $\widetilde{W}_1$ and $\widetilde{W}_1^-$ being algebraically equivalent. We prove that $\alpha(G)$ vanishes if and only if G is of hyperelliptic type, which has a forbidden minor characterization in terms of the graphs K_4 and L_3. As a consequence, a very general metric graph overlying a non-hyperelliptic-type graph has algebraically nontrivial tropical Ceresa cycle.

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DEDICATION

for my father, who taught me about x,

and my mother, who made me do math problems to keep quiet

Chapter 1

INTRODUCTION

In algebraic geometry, one studies varieties, which are sets of points cut out by polynomial equations. For example, in the Euclidean plane $\mathbb{R}^2$, the equation $y = 3x + 1$ describes a line, while $y^2 = x^2(x + 1)$ describes a nodal cubic plane curve. The problem of describing explicit solutions to a set of equations very quickly becomes intractable as one increases the number of variables, the number of equations, or the degrees of those equations; instead, one studies geometric properties of these varieties, with a basic goal of being able to tell when two varieties, given by different sets of equations, are in some sense the same variety.

Tropical geometry is the study of geometric spaces that are locally modeled by rational polyhedral sets in $\mathbb{R}^n$. This subject has been studied from many different perspectives, but the one of most interest to an algebraic geometer is the following. Given certain nice families of algebraic varieties $X \to T$, one can gain information about a general fiber X_t by studying a special fiber X_0 which has simpler geometry. One associates to X a tropical variety $\mathrm{Trop}(X)$ which captures the combinatorial structure of X_0 and encodes how the family degenerates to X_0. In this way, one can use tools from combinatorics and polyhedral geometry to study $\mathrm{Trop}(X)$ and learn something about X_t, e.g., its cohomology or intersection theory.

An oft-repeated slogan is that "tropical geometry is algebraic geometry over the $(\min, +)$-semiring." This is not a baseless comparison; indeed, one finds that tropical geometry, despite involving a completely different set of spaces and tools, shares many parallels with algebraic geometry. In a striking example, graphs take the place of curves and rational functions are supplanted by piecewise-linear ones—and yet, miraculously, the Riemann–Roch theorem continues to hold in this context [BN07]. In this dissertation, we continue in the tradition of strengthening the parallels between these two fields by introducing several tools from complex algebraic geometry into the tropical setting and examining their combinatorial implications.

1.1 Preliminaries

The basic object of study in tropical geometry is the rational polyhedral space. Like a manifold, it has a local description in terms of charts and some global restrictions on the topology, but rather than having charts that map into open subsets of $\mathbb{R}^n$, a rational polyhedral space X looks locally like the support of a polyhedral complex, all of whose faces admit an integral tangent basis. The "tropical" structure arises from the requirement that transition maps between charts of X should take the form $x \mapsto Ax + b$ for some $A\colon \mathbb{Z}^m \to \mathbb{Z}^n$ and $b \in \mathbb{R}^n$, thereby preserving rational directions. As a consequence, a tropical space X comes equipped with a natural sheaf of abelian groups Ω^p_X analogous to the sheaf of Kähler differentials of degree p on a variety (see §§2.2.6 and 2.2.8). The dual space $T^p_{X,x}$ is the p-th integral tangent space at $x \in X$. Following the lead of [GS23a], we shall emphasize the use of these intrinsic objects, rather than chart-dependent tools, as we build up our definitions in Chapter 2.

It should be noted that, in the most general setting, one should work with polyhedral complexes in $\mathbb{T}^n$, where $\mathbb{T} := \mathbb{R} \cup \{\infty\}$ is regarded as a semiring with ∞ as its "zero" element (i.e., $\min(\infty, a) = a$ and $\infty + a = \infty$ for all $a \in \mathbb{T}$). For ease of exposition, and because it suffices for our intended applications, we shall focus our attention on rational polyhedral spaces "without boundary", meaning that their charts map to polyhedral complexes in $\mathbb{R}^n$.

The rational polyhedral space X is a good starting point for the foundations of tropical geometry, but for many applications to algebraic geometry, we need a more restrictive notion of "tropical space". This is achieved by placing a local condition on the points of X called the balancing (or "zero tension") condition; see §2.4.1. Balanced polyhedral spaces, possibly with weights on their facets, are the output of the tropicalization functor Trop; accordingly, they occupy a special place in the theory. Some authors (e.g., [MR18]) require also that X be of finite type, regular at infinity, or smooth; we make no mention of these properties except the last, which we briefly discuss in §2.4.5.

As in the classical setting, curves and abelian varieties have been a focus for tropical geometers, both because they are tractable and because they serve as a model for what can happen in more general situations. Graphs serve as models for tropical curves. In detail, given a graph G and edge lengths $\ell\colon E \to \mathbb{R}_{>0}$, we obtain a metric space Γ by identifying each edge e with the closed interval $[0, \ell(e)] \subset \mathbb{R}$, gluing these intervals together at their endpoints according to the adjacency relations in G, and endowing the resulting topological graph with the shortest-path metric. The spaces so obtained are called metric graphs, and they are the tropical analogue of smooth algebraic curves. Meanwhile, polarized tropical abelian varieties are determined by two lattices N and Λ of equal rank along with an embedding $\Lambda \subset \mathbb{R} \otimes N$ and a polarization ϑ—a symmetric, positive-definite bilinear form on $\mathbb{R} \otimes N$ with respect to which N and Λ are dual lattices. These data determine a real torus $(\mathbb{R} \otimes N)/\Lambda$ with integral structure N. A principal polarization induces a tropical theta divisor Θ.

To a metric graph Γ we may associate a principally polarized tropical abelian variety $\mathrm{Jac}(\Gamma)$ called its tropical Jacobian. By identifying tropical 1-forms on Γ with integral flows on the underlying graph G, we obtain

$$\mathrm{Jac}(\Gamma) := \mathrm{Hom}(H^0(\Gamma, \Omega^1_\Gamma), \mathbb{R})/H_1(\Gamma; \mathbb{Z})$$

for some natural embedding of cycles as linear functionals on the space of flows. For each $d \geq 1$, there exists an Abel–Jacobi map $\Phi^{(d)}\colon \Gamma^d \to \mathrm{Jac}(\Gamma)$, whose image we denote by $\widetilde{W}_d$. The principal polarization ϑ on $\mathrm{Jac}(\Gamma)$ arises from the natural identification of cycles and flows on G.

1.2 Tropical de Rham isomorphism

Fix a rational polyhedral space X. In §2.3, we describe the spaces $\mathcal{A}^{p,q}(X)$ of (p,q)-superforms introduced by [Lag12]. Taken together, $\mathcal{A}^{\bullet,\bullet}(X)$ mimics the Dolbeault complex for a complex manifold, with natural differential operators d' and d'' analogous to ∂ and $\bar{\partial}$. In §2.3.2, we define the groups $C^{\mathrm{sing}}_{p,q}(X;\mathbb{Z})$ and $C^{p,q}_{\mathrm{sing}}(X;\mathbb{Z})$ of singular tropical (p,q)-chains and cochains, respectively, which first appeared in [IKMZ19]. These behave like the usual singular q-chain and cochain groups, with coefficients of degree p coming from the tropical structure on X.

The cohomology groups of $\mathcal{A}^{p,\bullet}(X)$ and $C^{p,\bullet}_{\text{sing}}(X;\mathbb{Z})$ are written $H^{p,q}_{d''}(X)$ and $H^{p,q}_{\text{sing}}(X;\mathbb{Z})$, respectively, while the homology of $C^{\text{sing}}_{p,\bullet}(X;\mathbb{Z})$ is given by $H^{\text{sing}}_{p,q}(X;\mathbb{Z})$. We likewise have Dolbeault cohomology with compact support $H^{p,q}_{d'',c}(X)$, tropical Borel–Moore homology $H^{\text{sing,lf}}_{p,q}(X;\mathbb{Z})$ (see §2.3.4), and versions of tropical chain homology $H^{\text{sm}}_{p,q}(X;\mathbb{Z})$ and $H^{\text{sm,lf}}_{p,q}(X;\mathbb{Z})$ using smooth simplices in place of singular ones (see §2.3.3).

In [JSS19], the authors show that $H^{p,q}_{d''}(X)$ and $H^{p,q}_{\text{sing}}(X;\mathbb{R})$ are abstractly isomorphic as real vector spaces. Separately, they describe a Poincaré duality isomorphism

$$\text{PD}\colon H^{p,q}_{d''}(X) \to H^{n-p,n-q}_{d'',c}(X)^\vee$$

on tropical Dolbeault cohomology. An analogous isomorphism

$$\text{PD}'\colon H^{p,q}_{\text{sing}}(X;\mathbb{Z}) \to H^{\text{sing,lf}}_{n-p,n-q}(X;\mathbb{Z})$$

is obtained on the level of tropical singular cohomology in [JRS18]. Chapter 3 presents joint work with Junaid Hasan and Farbod Shokrieh that complements these results as follows.

First, in the spirit of the classical de Rham theorem, we describe a way of integrating a superform along a smooth tropical chain, thereby inducing a map $\text{dR}\colon \mathcal{A}^{p,q}(X) \mapsto C^{p,q}_{\text{sm}}(X;\mathbb{R})$. This descends to a chain map by a tropical version of Stokes formula:

Lemma 1.1. *For all $c \in C^{\text{sm}}_{p,q}(X;\mathbb{R})$ and $\omega \in \mathcal{A}^{p,q-1}(X)$, we have that*

$$\int_c d''\omega = \int_{\partial c} \omega.$$

We also define cup and cap products on tropical (co)homology, making $H^{\bullet,\bullet}_{\text{sm}}(X;\mathbb{Z})$ into a ring and $H^{\text{sm}}_{\bullet,\bullet}(X;\mathbb{Z})$ into a right module over it. Finally, we prove that

Theorem 1.2. *The induced map on cohomology $\text{dR}\colon H^{\bullet,\bullet}_{d''}(X) \to H^{\bullet,\bullet}_{\text{sm}}(X;\mathbb{R})$ is an isomorphism of rings. Moreover, this map is compatible with Poincaré duality on either side, in the sense that the following diagram commutes:*

$$\begin{tikzcd}
H^{p,q}_{d''}(X) \arrow[r, "\text{dR}"] \arrow[d, "\text{PD}"] & H^{p,q}_{\text{sm}}(X;\mathbb{R}) \arrow[d, "\text{PD}'"] \\
H^{n-p,n-q}_{d'',c}(X)^\vee & H^{\text{sm,lf}}_{n-p,n-q}(X;\mathbb{R}) \arrow[l, "\text{dR}^\vee"']
\end{tikzcd}.$$

1.3 Canonical superform on tropical Jacobians

Chapter 4 describes joint work with Junaid Hasan and Farbod Shokrieh that applies the tropical de Rham theorem to the study of tropical abelian varieties, and especially to tropical Jacobians. Fix a polarized tropical abelian variety (X, ϑ). We use Theorem 1.2 to find a closed $(1,1)$-form $\nu_\vartheta \in \mathcal{A}^{1,1}(X)$ for which $\mathrm{dR}(\nu_\vartheta)$ coincides with a natural $(1,1)$-cocycle Poincaré-dual to the theta divisor Θ, as described by [GS23b].

Specializing to the case where $X = \mathrm{Jac}(\Gamma)$ for some metric graph Γ, we find that

Theorem 1.3. *For all $d \in \{0, \ldots, g\}$,*

$$\int_{\widetilde{W}_d} \frac{1}{d!} \nu_\vartheta^{\wedge d} = \binom{g}{d}.$$

This recovers the classical result with ν_ϑ replaced by the $(1,1)$-form associated to the Kähler metric on the Jacobian of a Riemann surface [Wen90]. Pulling back ν_ϑ to Γ via the Abel–Jacobi map yields a $(1,1)$-form closely related to the Zhang measure defined in [Zha93]; in particular, the coefficient on the edge e is $\ell(e)^{-1}F(e)$, where $F(e)$ is the Foster coefficient that features prominently in the study of random spanning trees and electrical networks [BF11, dJS22b]. Consequently, Theorem 1.3 has interesting combinatorial implications: the special case $d = 1$ is Foster's theorem, and $d = g$ is Kirchhoff's matrix-tree theorem.

Let $\|\Theta\|$ denote the normalized tropical Riemann theta function introduced by [dJS22a], and write $\delta_{X \cdot \|\Theta\|}$ for the supercurrent of integration associated to the tropical Weil divisor $X \cdot \Theta$; see §4.2 for details. In a further demonstration that ν_ϑ is the "correct" superform to consider, we prove that it satisfies a tropical version of the Poincaré–Lelong formula with respect to $\|\Theta\|$:

Proposition 1.4. *As supercurrents on X, we have that*

$$d'd''\|\Theta\| = \nu_\vartheta - \delta_{\|\Theta\| \cdot X}.$$

We describe one final application of ν_ϑ. The tropical Poincaré formula proved by [GS23b] says that

$$[\widetilde{W}_d] = \frac{[\Theta]^{g-d}}{(g-d)!} \tag{1.1}$$

in tropical homology. Using the description in [GST22] of $\widetilde{W}_d$ as a polyhedral complex, we obtain an alternative proof of this result by demonstrating the corresponding statement in superform cohomology:

Theorem 1.5. *Given an algebraic k-cycle Z in $\mathrm{Jac}(\Gamma)$, let ν_Z denote a $(g-k, g-k)$-superform whose Poincaré dual is the linear functional $\int_Z \cdot$. Then $\nu_\vartheta = \nu_\Theta$, and*

$$\nu_{\widetilde{W}_d} = \frac{\nu_\Theta^{\wedge(g-d)}}{(g-d)!}.$$

By Theorem 1.2, this identity is equivalent to (1.1).

1.4 Tropical Ceresa period

Let C be a smooth complex projective curve. There are two natural ways to embed C into its Jacobian: as the image $\widetilde{W}_1$ of the Abel–Jacobi map and as its inversion $[-1]_*\widetilde{W}_1$. The Ceresa cycle is the algebraic 1-cycle defined by their difference,

$$\widetilde{W}_1 - [-1]_*\widetilde{W}_1. \tag{1.2}$$

While it is homologically trivial, [Cer83] shows that it is algebraically nontrivial for very general curves of genus at least 3. It is straightforward to show that the Ceresa cycle is algebraically trivial if C is hyperelliptic, but understanding whether the converse holds is an active area of study, with recent results demonstrating that there are families of non-hyperelliptic curves with algebraically *torsion* Ceresa cycle [BS21, BLLS23, LS23, LS24, LS25].

The Ceresa cycle was first studied tropically by [Zha15]. As in the classical setting, we define the tropical Ceresa cycle of a metric graph Γ as the tropical 1-cycle in $\mathrm{Jac}(\Gamma)$ given by (1.2). We find that the tropical Ceresa cycle is trivial in $H_{1,1}^{\mathrm{sing}}(\mathrm{Jac}(\Gamma); \mathbb{Z})$. Moreover, there is a notion of algebraic equivalence in the tropical setting due to [Zha15] and investigated further by [GS23b]. As a partial result in the vein of [Cer83], [Zha15] showed that a very general metric graph Γ has algebraically nontrivial Ceresa cycle provided that Γ overlies the complete graph K_4. Chapter 5, following [Rit25], generalizes this result to obtain a complete characterization of algebraic nontriviality in the very general case.

As one might guess from the classical picture, the characterization of nontriviality is strongly tied to hyperellipticity; however, the story is complicated by the fact that there are two notions of hyperellipticity for metric graphs. Strictly speaking, a metric graph Γ of genus at least 1 is hyperelliptic if it has a divisor of degree 2 and rank 1, or equivalently, if it has an involution ι for which the quotient Γ/ι is a tree [Cha13]. However, because the tropical Torelli map is not injective, it is useful to consider a coarser notion: we say that Γ is of hyperelliptic type if there exists a hyperelliptic metric graph Γ' for which $\mathrm{Jac}(\Gamma) \cong \mathrm{Jac}(\Gamma')$. By [Cor21], being of hyperelliptic type is an invariant of the underlying graph and is minor-closed with forbidden minors K_4 and L_3 (see Fig. 5.4). Accordingly, we define a graph-theoretic version of the tropical Ceresa cycle and a corresponding homological invariant $\alpha(G)$, which we call the Ceresa period. Let R denote the integral polynomial ring generated by edge length variables x_e, N the cohomology $H^1(G;R)$, and $n_1, \ldots, n_g$ the basis for N that is dual to a basis of cycles $\gamma_1, \ldots, \gamma_g$ of G. Then we describe a particularly nice representative of $\alpha(G)$, which lives in a quotient of $R \otimes \bigwedge^3 N$, as follows:

Proposition 1.6. *Fix a graph G. Then $\alpha(G)$ is represented by*

$$\sum_{\substack{(e,e') \\ (i,j,k)}} a_{i,j,k}(e,e') x_e x_{e'}\, n_i \wedge n_j \wedge n_k,$$

where $a_{i,j,k}(e,e')$ takes values in $\{0, \pm 2\}$ and is nonzero precisely when, up to relabeling (i,j,k), γ_i contains both e and e', γ_j contains e but not e', and γ_k contains e' but not e.

Proposition 1.6 is the main computational tool from which all our results follow. With it, we compute $\alpha(K_4)$ and $\alpha(L_3)$ directly and show that they are nontrivial; likewise, we are able to show that the property of having trivial Ceresa period is closed under certain edge deletions and contractions. With these results in hand, we prove that

Theorem 1.7. *$\alpha(G) = 0$ if and only if G is of hyperelliptic type.*

Using observations of [Zha15], one can show that a metric graph Γ has algebraically trivial Ceresa cycle if the underlying graph G is of hyperelliptic type. With Theorem 1.7, we obtain the converse in the generic case:

Corollary 1.8. *If G is not of hyperelliptic type, then a very general metric graph Γ overlying G has algebraically nontrivial Ceresa cycle.*

As for the classical Ceresa cycle, it remains open whether the "very general" hypothesis can be removed, i.e., must a metric graph with algebraically trivial Ceresa cycle be of hyperelliptic type?

The Ceresa period is related to two other recently defined invariants. In [CL24], Corey and Li define a Ceresa–Zharkov class $\mathbf{w}_\tau(G)$ using mapping class groups and the Johnson homomorphism. Just as for $\alpha(G)$, triviality of $\mathbf{w}_\tau(G)$ shares the same forbidden minor characterization in terms of K_4 and L_3. Moreover, when G is equal to either graph, there is a choice of hyperelliptic involution τ of the surface associated to G for which $\alpha(G) = \mathbf{w}_\tau(G)$. Accordingly, we believe this equality holds for all graphs:

Conjecture 1.9. *Let G be a graph of genus g. Then there exists a hyperelliptic involution τ of the surface of genus g for which $\alpha(G) = \mathbf{w}_\tau(G)$.*

Separately, [ACM25] introduces an invariant called the tropical Ceresa class, defined as the Abel–Jacobi image of the tropical Ceresa cycle in a particular intermediate Jacobian of $\mathrm{Jac}(\Gamma)$. The authors expect that our $\alpha(\Gamma)$, obtained by specializing $\alpha(G)$ for particular edge lengths $x_e \mapsto \ell(e)$, coincides with the image of the tropical Ceresa class under the tropical monodromy map.

Chapter 2

FOUNDATIONS

2.1 *Basic notation and conventions*

Given $m \le n \in \mathbb{Z}$, let $[m,n] := \{m, m+1, \ldots, n\}$, and abbreviate $[n] := [1,n]$. For any totally ordered set A, we write $\binom{A}{k} \subset A^k$ for the collection of k-tuples whose elements are in strictly ascending order. Let $\mathfrak{S}_k$ denote the permutation group of $[k]$. Then for $I = (i_1, \ldots, i_k) \in \binom{A}{k}$ and $\mathfrak{s} \in \mathfrak{S}_k$, we let $I^{\mathfrak{s}} := (i_{\mathfrak{s}(1)}, \ldots, i_{\mathfrak{s}(k)}) \in A^k$.

If $I \in \binom{[m]}{k}$ and $J \in \binom{[n]}{l}$ and $\mathbf{M}$ is an $m \times n$ matrix, we write $\mathbf{M}_{I,J}$ for the $k \times l$ submatrix of $\mathbf{M}$ whose rows and columns are indexed in order by the elements of I and J, respectively. Likewise, we abbreviate $dx_I := dx_{i_1} \wedge \ldots \wedge dx_{i_k}$. We extend this notation to permutations of the tuples I and J in the natural way.

Given $v \in \mathbb{R}^n$, we write $[0,v]$ for the closed line segment $\{tv \mid 0 \le t \le 1\}$ and $(0,v)$ for the open line segment $\{tv \mid 0 < t < 1\}$ (and likewise for the half-open intervals $[0,v)$ and $(0,v]$). Given subsets $X, Y \subseteq \mathbb{R}^n$, $X + Y$ shall denote their Minkowski sum $\{x + y \mid x \in X, y \in Y\}$. Likewise, if $S \subseteq \mathbb{R}$, we define $SX := \{sx \mid s \in S, x \in X\}$.

Suppose that R is a commutative unital ring. Let $R^\times$ denote the group of units of R. We let $\langle \cdot, \cdot \rangle \colon M \otimes M^\vee \to R$ denote the evaluation pairing of an R-module M with its dual $M^\vee := \operatorname{Hom}_R(M,R)$. Given a ring homomorphism $R \to S$, we write $M_S := S \otimes_R M$ for the extension of scalars and often abbreviate its primitive elements $s \otimes m$ by sm.

On a topological space X, we let $\underline{R}$ denote the constant sheaf with values in a ring R. Given a $\underline{R}$-module $\mathcal{F}$ on X, we denote its sections over an open set U variously by $\mathcal{F}(U)$, $\Gamma(\mathcal{F}, U)$, or $H^0(U, \mathcal{F})$. We also define its *dual sheaf* $\mathcal{F}^\vee$ by using "sheaf" (or "internal") Hom:

$$\mathcal{F}^\vee := \mathscr{H}\!om_{\underline{R}}(\mathcal{F}, \underline{R}).$$

When there is no chance of confusion, we shall write $e_1, \ldots, e_n$ for the standard basis of $\mathbb{R}^n$. We let $\delta_{i,j}$ denote the Kronecker delta function, and write $\mathbf{I}$ for the identity matrix.

2.2 Tropical spaces

2.2.1 Rational polyhedra

We assume basic familiarity with polyhedral geometry and emphasize only a few points here. For a comprehensive overview, see [Ful93, §§1.2 and 1.4].

An integral vector $x \in \mathbb{Z}^n \subset \mathbb{R}^n$ is *primitive* if $tx \notin \mathbb{Z}^n$ for all $t \in (0,1)$. We call $x \in \mathbb{R}^n$ *rational* if it is a scalar multiple of an integral vector. A polyhedron in $\mathbb{R}^n$ is *rational* if it may be expressed as a finite intersection of halfspaces with rational normal vector. In what follows, we shall assume that all polyhedra are rational unless explicitly stated otherwise. It is straightforward to check that the familiar constructions for polyhedra carry over without issue for rational polyhedra; in particular, faces of rational polyhedra are again rational.

The *affine span* of a polyhedron σ is the smallest affine subspace $V \subseteq \mathbb{R}^n$ that contains it. Then the *dimension* of σ is $\dim(\sigma) := \dim(V)$, while its *relative interior* $\mathrm{ri}(\sigma)$ is its interior as a subset of V. If τ is a face or proper face of σ, we write $\tau \preceq \sigma$ or $\tau \prec \sigma$, respectively. A face of codimension 1 is called a *facet*. We exclude the empty set as a face.

A *cone* is a nonempty subset of $\mathbb{R}^n$ that is closed under multiplication by nonnegative scalars, while a *polyhedral cone* is a polyhedron that is also a cone. Such a polyhedral cone σ necessarily has a unique minimal face τ, which one may check is equal to the lineality space of σ. If $\tau = \{0\}$, we say that σ is *strongly convex*.

2.2.2 Polyhedral sets

A *polyhedral complex* is a locally finite collection $\mathcal{C}$ of polyhedra in $\mathbb{R}^n$ satisfying

- if $\sigma \in \mathcal{C}$ and $\tau \preceq \sigma$, then $\tau \in \mathcal{C}$, and
- if $\sigma, \tau \in \mathcal{C}$ and $\sigma \cap \tau \neq \emptyset$, then $\sigma \cap \tau \preceq \sigma$ and $\sigma \cap \tau \preceq \tau$.

An element of $\mathcal{C}$ is called a *face*; faces that are maximal with respect to inclusion are *facets*. If every face of $\mathcal{C}$ is a cone, then we call $\mathcal{C}$ a *fan*. As a consequence of local finiteness at the origin, a fan has only finitely many faces.

The *support* of a polyhedral complex $\mathcal{C}$, denoted by $|\mathcal{C}|$, is the union of its faces. The *(closed) star* of a face σ of $\mathcal{C}$ is the polyhedral subcomplex defined by

$$\mathrm{star}_{\mathcal{C}}(\sigma) := \{\tau \in \mathcal{C} \mid \exists \upsilon \in \mathcal{C},\, \tau \preceq \upsilon \succeq \sigma\}.$$

Likewise, the *star* $\mathrm{star}_{\mathcal{C}}(x)$ of a point $x \in |\mathcal{C}|$ is equal to the star of the unique face σ for which $x \in \mathrm{ri}(\sigma)$.

A *(closed) polyhedral set* is a subset $X \subseteq \mathbb{R}^n$ that is the support of some polyhedral complex; if $\mathcal{C}$ is a polyhedral complex (resp., fan) such that $|\mathcal{C}| = X$, then we call it a *face structure* (resp., *fan structure*) on X. An *open polyhedral set* is an open subset of a polyhedral set.

The set of face structures on a polyhedral set is partially ordered by refinement as follows. We say that $\mathcal{C}_1$ is a *refinement* of $\mathcal{C}_2$, denoted by $\mathcal{C}_1 \leq \mathcal{C}_2$, if, for every $\sigma_1 \in \mathcal{C}_1$, there exists a $\sigma_2 \in \mathcal{C}_2$ such that $\sigma_1 \subseteq \sigma_2$. Any two face structures $\mathcal{C}_1$ and $\mathcal{C}_2$ on X have a common refinement $\mathcal{C}_1 \wedge \mathcal{C}_2$ whose faces are intersections of the form $\sigma_1 \cap \sigma_2$ for $\sigma_i \in \mathcal{C}_i$. A maximal element with respect to this partial order is precisely a face structure that is a refinement only of itself. We caution that a polyhedral set need not have a unique maximal face structure (see Example 2.1).

2.2.3 Local cones

Given a subset $X \subseteq \mathbb{R}^n$ and a point $x \in X$, the *local cone* of X at x is the cone in $\mathbb{R}^n$ defined by

$$\mathrm{lc}_X(x) := \{tv \mid t \in \mathbb{R}_{\geq 0},\, v \in \mathbb{R}^n,\, [x, x+v] \subseteq X\}.$$

If σ is a polyhedron containing x, then $\mathrm{lc}_\sigma(x)$ is a polyhedral cone, with defining halfspaces $H - x$ as H ranges over the defining halfspaces of σ that contain x on their boundary. More generally, if X is a polyhedral set, then so is $\mathrm{lc}_X(x)$: each face structure $\mathcal{C}$ on X determines a fan structure $\mathrm{lc}_{\mathcal{C}}(x)$ on $\mathrm{lc}_X(x)$ whose faces are of the form $\mathrm{lc}_\sigma(x)$ for each face σ of $\mathcal{C}$ that contains x. We illustrate this construction in Fig. 2.1.

We call an open neighborhood U of x in X a *star neighborhood* if U is *star-convex* with respect to x, i.e., $[x, x'] \subseteq U$ whenever $x' \in U$. This implies that $U - x$ is an open neighborhood of 0 in $\mathrm{lc}_X(x)$. Using local finiteness of any face structure on X and the fact that polyhedra are convex, one can show that any point $x \in X$ has a neighborhood basis of star neighborhoods.

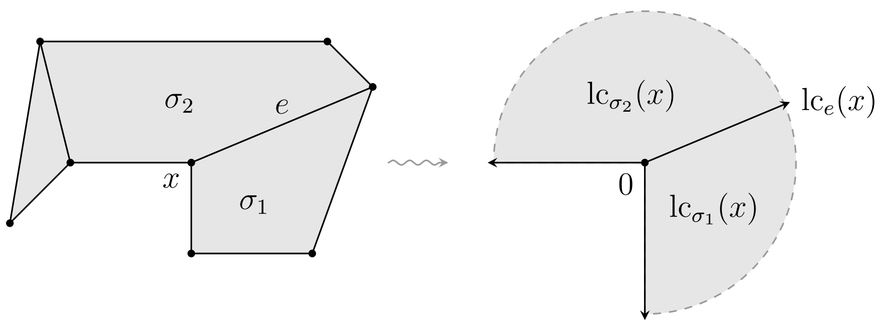


Figure 2.1: A face structure $\mathcal{C}$ on a polyhedral set X and the corresponding fan structure $\mathrm{lc}_{\mathcal{C}}(x)$ on the local cone $\mathrm{lc}_X(x)$

2.2.4 Local cone stratification

Fix a polyhedral set X. We define a relation on points of X by declaring $x \sim x'$ whenever X has a face structure for which x and x' are in the relative interior of the same face; one checks from the definition of the local cone that $\mathrm{lc}_X(x) = \mathrm{lc}_X(x')$. Evidently, this relation is reflexive and symmetric. As Example 2.1 shows, however, it is not necessarily transitive. Therefore, we close it under transitivity to define an equivalence relation on X, which in turn defines the *local cone stratification* on X by locally closed subsets that are connected components of points at which X has the same local cone. We write $\mathrm{lc}_X(\sigma) := \mathrm{lc}_X(x)$ for any $x \in \sigma$.

Example 2.1. Let X be the polyhedral set depicted in Fig. 2.1. Then the unique 2-dimensional stratum of X is its interior X°, each of whose points has local cone equal to $\mathbb{R}^2$.

However, because X° is non-convex, we can find a pair of points x_1 and x_2 in X° such that no face structure of X has x_1 and x_2 in the relative interior of the same face. As another consequence of non-convexity, we note that X does not have a unique maximal face structure, nor $\mathrm{lc}_X(x)$ a unique maximal fan structure.

Let $\mathcal{X}$ be the local cone stratification of X. Given strata $\sigma, \tau \in \mathcal{X}$, we write $\tau \preceq \sigma$ if $\tau \cap \overline{\sigma} \neq \emptyset$ and $\tau \prec \sigma$ if, moreover, $\tau \neq \sigma$. If $\tau \preceq \sigma$, then $\mathrm{lc}_X(\sigma)$ can be determined from $\mathrm{lc}_X(\tau)$. Indeed, choosing any $x \in \tau \cap \overline{\sigma}$ and a star neighborhood U of x, there exists an $x' \in \sigma \cap U$. Then we compute

$$\mathrm{lc}_X(\sigma) = \mathrm{lc}_X(x') = \mathrm{lc}_U(x') = \mathrm{lc}_{U-x}(x'-x) = \mathrm{lc}_{\mathrm{lc}_X(x)}(x'-x) = \mathrm{lc}_{\mathrm{lc}_X(\tau)}(x'-x).$$

Since all points of τ have the same local cone, they are all limits of points that have local cone $\mathrm{lc}_X(\sigma)$ and hence are in σ. As a result, $\tau \subseteq \overline{\sigma}$, implying that $\preceq$ is transitive and that $\overline{\sigma}$ is a union of strata.

Let X^{gen} denote the subset of points of X that are locally Euclidean. One checks from the definitions above that X^{gen} is precisely the (disjoint) union of the maximal strata of $\mathcal{X}$.

2.2.5 Integral affine functions

Given polyhedral sets $X \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^n$, a continuous map $\phi\colon X \to Y$ is *integral affine* if it is the restriction to X of $x \mapsto Ax + b$ for some integral linear transformation $A\colon \mathbb{Z}^m \to \mathbb{Z}^n$ and $b \in \mathbb{R}^n$. Integral affine functions $\phi\colon X \to \mathbb{R}$ define a presheaf of abelian groups on X whose sheafification, denoted Aff_X, consists of continuous functions that are *locally* integral affine (cf. [GS23a], where "integral affine linear" is by definition a local condition). Identifying $c \in \mathbb{R}$ with the constant map $x \mapsto c$, we obtain the constant sheaf $\underline{\mathbb{R}}$ as a subsheaf of Aff_X.

It is straightforward to show that any locally integral affine function on a star neighborhood is, in fact, integral affine. It follows that $\mathrm{Aff}_{X,x} \cong \mathrm{Aff}_{\mathrm{lc}_X(x)}(\mathrm{lc}_X(x))$.

2.2.6 Integral tangent space

Let X be a rational polyhedral set. Following [MZ08, GS23a], we define the sheaf of *tropical* 1*-forms* Ω_X^1 by the following exact sequence, called the *tropical exponential sequence*:

$$0 \to \underline{\mathbb{R}} \to \mathrm{Aff}_X \xrightarrow{d} \Omega_X^1 \to 0. \tag{2.1}$$

By construction, the quotient map d satisfies the familiar property $dc = 0$ whenever c is locally constant.

We now describe the stalk $\Omega_{X,x}$ at a point x explicitly as follows. Let U be a star neighborhood of x, and fix the standard coordinates $x_1, \ldots, x_n$ on $\mathbb{R}^n$. Then $\mathrm{Aff}_{X,x}$ may be identified with the group of integral affine functions on $\mathrm{lc}_X(x)$ via translation by $-x$. Such functions are written in the form $\sum_{i=1}^n a_i x_i + c$ for $a_i \in \mathbb{Z}$ and $c \in \mathbb{R}$ and vanish on $\mathrm{lc}_X(x)$ precisely when they vanish on the (linear) span V of $\mathrm{lc}_X(x)$. Consequently, writing $k := \dim(V)$, then by rationality of V, we may choose integral linear functionals $f_1, \ldots, f_{n-k}$ that generate the saturated subgroup $\mathrm{lc}_X(x)^\perp = V^\perp \subseteq (\mathbb{Z}^n)^\vee$. Then

$$\Omega_{X,x}^1 = \Omega_{\mathrm{lc}_X(x),0}^1 = \mathbb{Z}\langle dx_1, \ldots, dx_n \rangle / \langle df_1, \ldots, df_{n-k} \rangle \cong (\mathbb{Z}^k)^\vee, \tag{2.2}$$

where the first equality holds because translation by $-x$ modifies affine functions by a constant. We define $\Omega_{X,\mathbb{R}}^1 := \underline{\mathbb{R}} \otimes_{\underline{\mathbb{Z}}} \Omega_X^1$ the sheaf of *real-valued tropical* 1*-forms*, whose stalk at x is isomorphic to $(\mathbb{R}^k)^\vee$.

The *integral tangent space* of X at x is defined by

$$T_{X,x}^1 := \mathrm{Hom}_{\mathbb{Z}}(\Omega_{X,x}^1, \mathbb{Z}),$$

Likewise, the *tangent space* is given by $T_{X,x,\mathbb{R}}^1 := \mathrm{Hom}_{\mathbb{Z}}(\Omega_{X,x}^1, \mathbb{R})$. For x as above, we see immediately that $T_{X,x,\mathbb{R}}^1 \cong \mathbb{R}^k$ and contains $T_{X,x}^1 \cong \mathbb{Z}^k$ as a lattice. Then we may identify $\mathrm{lc}_X(x)$ with a cone inside of $T_{X,x,\mathbb{R}}^1$ via $v \mapsto \langle v, \cdot \rangle$. (Here, $\langle v, df \rangle$ is well-defined for any integral affine function f, because $f(v) - f(0)$ does not depend on the constant term of f.) We obtain, therefore, a more intrinsic definition of $\mathrm{lc}_X(x)$ that does not depend on the way

X sits inside of $\mathbb{R}^n$, and that agrees with the local cone as defined in [GS23a, §2.2]. By [GS23a, Corollary 2.6], $\mathrm{lc}_X(x)$ spans $T^1_{X,x,\mathbb{R}}$.

Let $\mathcal{X}$ be the local cone stratification on X. Because the (embedded) local cone at a point is constant on each stratum $\sigma \in \mathcal{X}$, it follows from the description of $\Omega^1_{X,x}$ above that Ω^1_X is in turn locally constant on σ; in other words, Ω^1_X is a constructible sheaf. Accordingly, we write $\Omega^1_X(\sigma)$ for its stalk at any point $x \in \sigma$, and likewise, $T^1_X(\sigma)$ for $T^1_{X,x}$. If $\tau \in \mathcal{X}$ with $\tau \preceq \sigma$, then there is a canonical surjection $\Omega^1_X(\tau) \to \Omega^1_X(\sigma)$ given by restricting sections over a star neighborhood of a point in τ to a star neighborhood of a point in σ. By dualizing, we obtain an injection $T^1_X(\sigma) \to T^1_X(\tau)$. We refer to both of these maps only implicitly in what follows.

2.2.7 Dimension

Fix a polyhedral set X with its local cone stratification $\mathcal{X}$. We define the *local dimension* of X at a point x by $\dim_x(X) := \dim(T^1_{X,x})$. This is constant over strata $\sigma \in \mathcal{X}$, so we also write $\dim_\sigma(X)$ for the local dimension at any $x \in \sigma$. The *(global) dimension* of X is given by

$$\dim(X) := \sup_{\substack{\sigma \in \mathcal{X} \\ \text{maximal}}} \dim_\sigma(X),$$

and we say that X is *pure* if $\dim(X) = \dim_\sigma(X)$ for all maximal strata σ. We regard the empty set as a polyhedral space of every dimension.

Any stratum σ of X is locally Euclidean, so we define $\dim(\sigma)$ to be its dimension as a manifold; if X is pure, then we also define $\operatorname{codim}(\sigma) := \dim(X) - \dim(\sigma)$. Recall that X is locally Euclidean at any point in a maximal stratum σ, so that $\dim(\sigma) = \dim_\sigma(X)$. By contrast, at any non-maximal stratum τ, we necessarily have that $\dim(\tau) < \dim_\tau(X)$, and moreover, $\dim_\tau(X) > \dim_\sigma(X)$ for each $\sigma \succ \tau$. In particular, it is possible for the local dimension of X at a point to exceed $\dim(X)$.

2.2.8 Tropical p-forms

Let X be a polyhedral set with its local cone stratification $\mathcal{X}$. Then we define the p-th exterior power $\bigwedge^p \Omega^1_X$ as the sheafification of the presheaf defined on star neighborhoods by $U \mapsto \bigwedge^p \Omega^1_X(U)$. As [GS23a, Example 2.9] shows, this sheaf can be nonzero for $p > \dim X$, so it is not a good candidate for Ω^p_X. One remedies this as follows. Write $\iota\colon X^{\mathrm{gen}} \to X$ for the inclusion map. Then we define the sheaf of *tropical p-forms* Ω^p_X as the image sheaf of the natural morphism

$$\bigwedge^p \Omega^1_X \to \iota_* \left(\bigwedge^p \iota^* \Omega^1_X \right), \tag{2.3}$$

with $\Omega^0_X := \underline{\mathbb{Z}}$ by convention. (This definition is equivalent to [GS23a, Definition 2.7].) One sees that Ω^p_X is also constructible.

Write K for the kernel of the homomorphism of $\underline{\mathbb{Z}}$-modules in (2.3). Then we have an exact sequence

$$0 \to K \to \bigwedge^p \Omega^1_X \to \Omega^p_X \to 0.$$

Fix a stratum $\tau \in \mathcal{X}$. Taking stalks at some $x \in \tau$ and dualizing, we get another exact sequence

$$0 \to T^p_X(\tau) \to \bigwedge^p T^1_X(\tau) \to K(\tau)^\vee \to 0,$$

where $T^p_{X,x} := {\Omega^p_{X,x}}^\vee$. By definition of Ω^p_X, a section $\omega \in \bigwedge^p \Omega^1_X(\tau)$ lies in $K(\tau)$ precisely when its restriction to X^{gen}, or equivalently, its restriction to each maximal stratum $\sigma \succeq \tau$, vanishes. Meanwhile, $v \in \bigwedge^p T^1_X(\tau)$ lies in $T^p_X(\tau)$ if $\langle v, \omega \rangle = 0$ for all $\omega \in K$. This certainly holds if $v \in \bigwedge^p T^1_X(\sigma)$, and one may check that such vectors generate $T^p_X(\tau)$, up to taking the saturation:

$$T^p_X(\tau) = \left(\bigoplus_{\substack{\sigma \succeq \tau \\ \text{maximal}}} \bigwedge^p T^1_X(\sigma) \right)^{\mathrm{sat}}.$$

We note that this agrees with the multi-tangent space $\mathbf{F}^{\mathbb{Z}}_p(\tau)$ of [JRS18, §2.B].

Let $v \in T^{p+p'}_X(\sigma)$ and $\omega \in \Omega^p_X(\sigma)$. Then we define the contraction $v \smile \omega \in T^{p'}_X(\sigma)$ as being adjoint to the wedge product on superforms with respect to evaluation:

$$\langle v \smile \omega, \omega' \rangle = \langle v, \omega \wedge \omega' \rangle \tag{2.4}$$

for all $\omega' \in \Omega_X^{p'}(\sigma)$. This is well-defined for any choice of representative for ω in $\bigwedge^p \Omega_X^1(\sigma)$. Indeed, if $\omega \in K$, then one checks locally on each maximal stratum that $\omega \wedge \omega' \in K$, so that the right-hand side of (2.4) vanishes. The computation of $v \smile \omega$ with respect to a basis reduces to the classical computation for vectors and covectors in $\mathbb{R}^n$.

Given $\tau \preceq \sigma$, the natural map $\Omega_X^1(\tau) \to \Omega_X^1(\sigma)$ determines $\Omega_X^p(\tau) \twoheadrightarrow \Omega_X^p(\sigma)$ and its dual $T_X^p(\sigma) \hookrightarrow T_X^p(\tau)$ for each p. We also define spaces of real-valued tropical p-forms and p-vectors by $\Omega^p_{X,\mathbb{R}} := \Omega^p_X \otimes_{\underline{\mathbb{Z}}} \underline{\mathbb{R}}$ and $T^p_{X,x,\mathbb{R}} := {\Omega^p_{X,x,\mathbb{R}}}^\vee$.

2.2.9 Polyhedral spaces

We closely follow [GS23a, JRS18, MZ14, MR18, AR10] for the definition of a polyhedral space.

A *polyhedral space* is a paracompact, second-countable, Hausdorff topological space X endowed with a sheaf Aff_X of continuous functions that is locally modeled by open polyhedral sets. More precisely, we require that, for each $x \in X$, there exists $n \in \mathbb{Z}_{\geq 0}$, an open polyhedral set $V \subseteq \mathbb{R}^n$, an open neighborhood U of x, and a homeomorphism $\psi \colon U \to V$ that induces via pullback an isomorphism $\mathrm{Aff}_V \to \psi_* \mathrm{Aff}_X|_U$ fixing $\underline{\mathbb{R}}$. These data together constitute a *chart* for x. As a consequence of the definition, given two charts $\psi \colon U \to V$ and $\psi' \colon U \to V'$, the transition map $\psi' \circ \psi^{-1} \colon V \to V'$ is locally integral affine.

Let X be a polyhedral space. A *polyhedron* σ in X is a closed subset such that there exists a chart $\psi \colon U \to V \subseteq \mathbb{R}^n$ containing σ for which $\psi(\sigma)$ is a polyhedron in $\mathbb{R}^n$. We define a *polyhedral complex* in X as in $\mathbb{R}^n$ (see §2.2.2). A *local face structure* on X at x is a polyhedral complex $\mathcal{C}$ in X such that $x \in |\mathcal{C}|^\circ$, $\mathrm{star}_{\mathcal{C}}(x) = \mathcal{C}$, and there exists a chart containing $|\mathcal{C}|$. By contrast, a *(global) face structure* on X is a polyhedral complex whose support is all of X and for which $\mathrm{star}_{\mathcal{C}}(x)$ is a local face structure at x for all $x \in X$. Because X locally looks like a polyhedral set and face structures on polyhedral sets can be refined to have faces of arbitrarily small diameter, it follows that every point of X has a local face structure. However, while all local face structures on a polyhedral set may be extended globally, the same is not true a priori for a general polyhedral space.

Local notions on a polyhedral set (e.g., tropical forms, tangent spaces, and local cones) are imparted to polyhedral spaces through charts. We may likewise define the local cone stratification on a polyhedral space X, except we must allow ourselves the flexibility that comes from using local face structures rather than global ones. In detail, we declare $x \sim x'$ whenever there is a local face structure at x for which x and x' are in the relative interior of the same face. The equivalence relation generated by $\sim$ determines the *local cone stratification* on X. It is straightforward to show that any chart $\psi\colon U \to V$ as above maps the stratification on U to the stratification on V.

2.2.10 Morphisms

A *morphism* of polyhedral spaces is a continuous map $f\colon X \to Y$ that induces a homomorphism $f^*\colon \mathrm{Aff}_Y \to f_*\mathrm{Aff}_X$, or equivalently, that is locally integral affine in all charts for X and Y. Pullback preserves locally constant functions and so determines maps on tropical forms $\Omega^1_Y \to f_*\Omega^1_X$ and $\Omega^p_Y \to f_*\Omega^p_X$, both of which are also represented by f^*. Taking stalks and dualizing, we obtain a pushforward on tangent spaces $T^1_{X,x} \to T^1_{Y,f(x)}$ and $T^1_{X,x,\mathbb{R}} \to T^1_{Y,f(x),\mathbb{R}}$, which we denote by the familiar notation d_xf.

A morphism $f\colon X \to Y$ of polyhedral spaces is *proper* if it is proper as a map of topological spaces, i.e., if $f^{-1}(K)$ is compact for all compact $K \subseteq Y$.

A morphism $\iota\colon Z \to X$ is an *embedding* if it is an embedding of topological spaces and the induced map $d_z\iota\colon T^1_{Z,z} \to T^1_{X,\iota(z)}$ is injective for every $z \in Z$. A subset Z of X is a *polyhedral subspace* if the inclusion map $Z \hookrightarrow X$ is an embedding of polyhedral spaces.

2.3 Tropical cohomology

Fix a polyhedral space X with its local cone stratification. A notion of tropical homology on X was first introduced by [IKMZ19], and has been further studied in, e.g., [JRS18, JSS19, GS23a, CLD25]. In this section, we elaborate on the theories of tropical Dolbeault cohomology and tropical singular cohomology on X. In Chapter 3, we shall prove a tropical analogue of the de Rham theorem that links these two together.

2.3.1 Tropical Dolbeault cohomology

We say that a continuous function $f\colon X \to \mathbb{R}$ is *smooth* if every point $x \in X$ has a star neighborhood U so that $f|_U$ extends to a smooth function on $T^1_{X,x,\mathbb{R}} \cong \mathbb{R}^n$. Denote by C^∞ the sheaf of smooth functions. Then we define the sheaf of (p,q)-*superforms* by the quotient

$$\mathcal{A}^{p,q} := (C^\infty \otimes \Omega^p_X \otimes \Omega^q_X)/K, \tag{2.5}$$

where the $\underline{\mathbb{Z}}$-submodule K is defined as follows. Given $x \in X$ with star neighborhood U, any other point $x' \in U$, a simple tensor $\omega := f \otimes \alpha \otimes \beta \in C^\infty(U) \otimes \Omega^p_X(U) \otimes \Omega^q_X(U)$, and tangent vectors $v \in T^p_{X,x'}$ and $w \in T^q_{X,x'}$, we let

$$\langle v, w; \omega(x') \rangle := f(x') \langle v, \alpha \rangle \langle w, \beta \rangle \in \mathbb{R} \tag{2.6}$$

and extend $\mathbb{Z}$-linearly in ω and $v \otimes w$. We declare $K(U)$ to consist of sections ω for which $\langle v, w; \omega(x') \rangle = 0$ for all $x' \in U$, $v \in T^p_{X,x'}$, and $w \in T^q_{X,x'}$. Example 2.2 shows that K can be nonzero.

Example 2.2. Let X be the polyhedral set in $\mathbb{R}^2$ given as the union of rays

$$X := \mathbb{R}_{\geq 0} e_1 \cup \mathbb{R}_{\geq 0} e_2.$$

Then although $x_1 + x_2$ is nonzero in $C^\infty(X)$ and dx_1 and dx_2 are each nonzero in $\Omega^1_X(X)$, we claim that $(x_1 + x_2) \otimes dx_1 \otimes dx_2 \in K$. Indeed, at $x = 0$, $x_1 + x_2 = 0$; at any point $x \in X$ with $x_1 > 0$, $x_2 = 0$ and so $dx_2 = 0$; and whenever $x_2 > 0$, we likewise find that $dx_1 = 0$. Consequently, $\langle v, w; (x_1 + x_2) \otimes dx_1 \otimes dx_2 \rangle = 0$ for all $x \in X$ and $v, w \in T^1_{X,x}$.

One may check that our definition of $\mathcal{A}^{p,q}$ agrees with that of [JSS19]. By convention, a simple (p,q)-superform whose expression in local coordinates $x_1, \ldots, x_n$ is $f \otimes dx_I \otimes dx_J$ is notated $f d'x_I \wedge d''x_J$.

We define exterior derivatives $d'\colon \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q}$ and $d''\colon \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}$ on simple (p,q)-forms on a star neighborhood according to

$$\begin{aligned} d'\left(f d'x_I \wedge d''x_J\right) &:= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} d'x_i \wedge d'x_I \wedge d''x_J. \\ d''\left(f d'x_I \wedge d''x_J\right) &:= (-1)^p \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} d'x_I \wedge d''x_j \wedge d''x_J. \end{aligned} \tag{2.7}$$

Then $d'^2 = d''^2 = 0$, thereby making $(\mathcal{A}^{\bullet,q}, d')$ and $(\mathcal{A}^{p,\bullet}, d'')$ into cochain complexes of $\underline{\mathbb{Z}}$-modules on X. By taking global sections, we obtain cochain complexes of $\mathbb{Z}$-modules, whose k-th cohomology groups we denote by $H^{k,q}_{d'}(X)$ and $H^{p,k}_{d''}(X)$, respectively. In what follows, we focus our attention on cohomology with respect to d'', as this agrees with the singular cohomology discussed in §2.3.2. A straightforward computation shows that $d'd'' + d''d' = 0$.

There is also a wedge product $\wedge\colon \mathcal{A}^{p_1,q_1} \otimes_{C^\infty} \mathcal{A}^{p_2,q_2} \to \mathcal{A}^{p_1+p_2,q_1+q_2}$ given locally by

$$(f_1 \otimes \alpha_1 \otimes \beta_1) \wedge (f_2 \otimes \alpha_2 \otimes \beta_2) := (-1)^{q_1 p_2} f_1 f_2 \otimes \alpha_1 \wedge \alpha_2 \otimes \beta_1 \wedge \beta_2. \tag{2.8}$$

A basic consequence is that simple superforms $\omega_i \in \mathcal{A}^{p_i,q_i}(X)$ satisfy a Leibniz formula with respect to both d' and d'', i.e.,

$$d''(\omega_1 \wedge \omega_2) = d''\omega_1 \wedge \omega_2 + (-1)^{p_1+q_1} \omega_1 \wedge d''\omega_2. \tag{2.9}$$

The pullback of a simple superform by a morphism of polyhedral spaces $\psi\colon Y \to X$ is defined by

$$\psi^*(f \otimes \alpha \otimes \beta) := (f \circ \psi) \otimes \psi^*\alpha \wedge \psi^*\beta, \tag{2.10}$$

where we use the pullback of tropical forms described in §2.2.10. One finds that pullback is compatible with both the wedge product and the exterior derivatives.

Let $\mathcal{A}^{p,\bullet}_c$ be the subcomplex of $\mathcal{A}^{p,\bullet}$ whose sections are compactly supported superforms; then we obtain cohomology groups $H^{p,q}_{d'',c}(X)$ analogously to $H^{p,q}_{d''}(X)$. Suppose that X is pure of dimension n. Given $\omega \in \mathcal{A}^{n,n}_c(X)$ with compact support contained inside a star neighborhood U of some maximal stratum σ, we may write $\omega|_U = f d'x_{[n]} \wedge d''x_{[n]}$ in local coordinates $x_1, \dots, x_n$ for $T^1_{X,\mathbb{R}}(\sigma) \cong \mathbb{R}^n$. Following [Lag12], we define the *integral* of ω over X by

$$\int_X \omega := (-1)^{\binom{n}{2}} \int_X f dx_{[n]}, \tag{2.11}$$

where the right-hand side is the usual smooth integral in $\mathbb{R}^n$ with the standard orientation. If, more generally, ω is not supported inside a single star neighborhood, then we may use partitions of unity to define the integral; see [JSS19, §4.1] for details.

2.3.2 Singular tropical cohomology

The *standard q-simplex* Δ^q is the polyhedron in $\mathbb{R}^{q+1}$ defined by

$$\Delta^q := \left\{ (t_0, \dots, t_q) \in \mathbb{R}^{q+1} \;\middle|\; \sum_{i=0}^q t_i = 1 \right\}.$$

A *(singular) q-simplex* is a continuous map $f\colon \Delta^q \to X$. We say that f is *admissible* if it maps the relative interior of each face of Δ^q into a single stratum of X; we denote by σ_f the stratum into which f maps $\mathrm{ri}(\Delta^q)$. Then the group of *(singular tropical)* (p,q)*-chains* on X (with coefficients in a free abelian group R) is given by

$$C^{\mathrm{sing}}_{p,q}(X;R) := \bigoplus_{\substack{f\colon \Delta^q \to X \\ \text{admissible}}} T^p_X(\sigma_f)_R. \tag{2.12}$$

We write $v \otimes f$ for the element of $C^{\mathrm{sing}}_{p,q}(X;R)$ associated to the q-simplex f and we call v its *framing vector*.

For each $i \in [0,q]$, there is a *face map* $\delta^q_i\colon \Delta^{q-1} \to \Delta^q$ sending

$$(t_0, \dots, t_{q-1}) \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{q-1}).$$

We abbreviate $f_i := f \circ \delta^q_i$ for any q-simplex f. By admissibility and continuity, we have $\sigma_{f_i} \preceq \sigma_f$, so that $T^p_X(\sigma_f) \hookrightarrow T^p_X(\sigma_{f_i})$. This allows us to define a boundary map $\partial\colon C^{\mathrm{sing}}_{p,q}(X;R) \to C^{\mathrm{sing}}_{p,q-1}(X;R)$ via

$$\partial(v \otimes f) := (-1)^p \sum_{i=0}^q (-1)^i v \otimes f_i. \tag{2.13}$$

It is straightforward to check that $\partial^2 = 0$, making $(C^{\mathrm{sing}}_{p,\bullet}(X;R), \partial)$ into a chain complex with q-th homology group denoted by $H^{\mathrm{sing}}_{p,q}(X;R)$.

Remark 2.3. The sign of $(-1)^p$ in (2.13) is analogous to that of (2.7). While this sign does not appear in the literature, it is necessary to ensure compatibility with $\mathcal{A}^{p,\bullet}(X)$ under the de Rham map defined in §3.1.2: without it, Lemma 3.2 would fail. Intuitively, it results from moving the boundary operator ∂ past the framing vector v, which has degree p.

We further define a complex $(C^{p,\bullet}_{\mathrm{sing}}(X;R), \partial^*)$ of *(singular tropical)* (p,q)*-cochains* by

$$C^{p,\bullet}_{\mathrm{sing}}(X;R) := \mathrm{Hom}(C^{\mathrm{sing}}_{p,\bullet}(X;R), R),$$

with coboundary map ∂^* given by precomposition with ∂. Given $\alpha \in C^{p,q}_{\mathrm{sing}}(X;R)$ and a q-simplex f, we write $\alpha_f \in \Omega^p_X(\sigma_f)_R$ for the tropical p-form satisfying, for all $v \in T^p_X(\sigma_f)_R$,

$$\langle v, \alpha_f \rangle = \langle v \otimes f, \alpha \rangle. \tag{2.14}$$

We obtain a presheaf $C^{p,\bullet}_{\mathrm{sing},R}$ with sections $C^{p,\bullet}_{\mathrm{sing},R}(U) := C^{p,\bullet}_{\mathrm{sing}}(U;R)$ and the natural restriction maps. Let $\mathrm{sh}\colon C^{p,q}_{\mathrm{sing},R} \to \underline{C}^{p,q}_{\mathrm{sing},R}$ denote the sheafification map. By analogy with [JSS19, Lemma 3.14] (replacing the face structure in their definition by the local cone stratification), one may show that sh is a quasi-isomorphism in the category of presheaves. The quotient presheaf is simply the presheaf of quotients, so the induced map on global sections $C^{p,\bullet}_{\mathrm{sing}}(X;R) \to \underline{C}^{p,\bullet}_{\mathrm{sing}}(X;R)$ descends to an isomorphism $H^{p,q}_{\mathrm{sing}}(X;R) \cong \underline{H}^{p,q}_{\mathrm{sing}}(X;R)$ of their q-th cohomologies.

2.3.3 Smooth tropical cohomology

In order to make sense of integration on tropical chains, we must restrict to smooth simplices. First, endow Δ^q with the structure of an oriented smooth manifold with piecewise-smooth boundary as follows. Fix

$$\widetilde{\Delta}^q := \left\{ (t_1, \dots, t_q) \,\middle|\, t_i \geq 0, \sum_{i=1}^{q} t_i \leq 1 \right\} \subset \mathbb{R}^q,$$

with the standard orientation $\widetilde{\nu} := \frac{\partial}{\partial t_1} \wedge \ldots \wedge \frac{\partial}{\partial t_q}$. We parameterize Δ^q by $\widetilde{\Delta}^q$ using the smooth map $\phi\colon \mathbb{R}^q \to \mathbb{R}^{q+1}$ defined by

$$\phi(t_1, \ldots, t_q) := \left(1 - \sum_{i=1}^{q} t_i, t_1, \ldots, t_q\right). \tag{2.15}$$

A straightforward integral computation shows that $\mathrm{Vol}(\widetilde{\Delta}^q) = \frac{1}{q!}$. For later reference, we define

$$\nu := \left(\frac{\partial}{\partial t_q} - \frac{\partial}{\partial t_0}\right) \wedge \ldots \wedge \left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_0}\right). \tag{2.16}$$

We declare that ϕ is orientation-preserving, so that the orientation on Δ^q is given by $d\phi(\widetilde{\nu}) = (-1)^{\binom{q}{2}}\nu$.

We say that a (p,q)-simplex $v \otimes f$ is *smooth* (resp., *affine*) if $f \circ \phi$ is smooth (resp., affine) as a map $\widetilde{\Delta}^q \to X$, where the closed stratum $\overline{\sigma_f}$ into which f maps is endowed with its usual smooth (resp., affine) structure by charts on X. We define $C^{\mathrm{sm}}_{p,\bullet}(X;R)$ as the subcomplex of $C^{\mathrm{sing}}_{p,\bullet}(X;R)$ generated by smooth tropical simplices, with q-th homology $H^{\mathrm{sm}}_{p,q}(X;R)$. Just as for singular tropical homology, this determines a presheaf of cochain complexes $C^{p,\bullet}_{\mathrm{sm},R}$ and its sheafification $\underline{C}^{p,\bullet}_{\mathrm{sm},R}$; both have the same q-th cohomology on global sections, denoted $H^{p,q}_{\mathrm{sm}}(X;R)$. Analogous constructions apply to the group $C^{\mathrm{aff}}_{p,\bullet}(X;R)$ of affine tropical simplices; we then have a chain of inclusions

$$C^{\mathrm{aff}}_{p,\bullet}(X;R) \subseteq C^{\mathrm{sm}}_{p,\bullet}(X;R) \subseteq C^{\mathrm{sing}}_{p,\bullet}(X;R).$$

One can show that $H^{\mathrm{sing}}_{p,q}(X;R) \cong H^{\mathrm{sm}}_{p,q}(X;R)$ by imitating the classical proof, wherein one constructs a chain homotopy inverse for the inclusion map on chains by Whitney approximation. However, a simpler method that works in this context is to show instead that $H^{\mathrm{sing}}_{p,q}(X;R) \cong H^{\mathrm{aff}}_{p,q}(X;R)$ using the affinization map. In detail, an affine q-simplex is determined by where it sends the vertices of Δ^q. Therefore, we let $\mathrm{aff}\colon C^{\mathrm{sing}}_{p,\bullet}(X;R) \to C^{\mathrm{sm}}_{p,\bullet}(X;R)$ denote the *affinization* map sending $v \otimes f$ to the unique affine (p,q)-simplex $v \otimes f_{\mathrm{aff}}$ satisfying $f_{\mathrm{aff}}(e_i) = f(e_i)$ for all $i \in [0,q]$. Necessarily, $\sigma_{f_{\mathrm{aff}}} \preceq \sigma_f$. It is straightforward to check that aff is a chain map (i.e., $\mathrm{aff} \circ \partial = \partial \circ \mathrm{aff}$) and that the dual map on cochains induces a cochain map of presheaf complexes $\mathrm{aff}^*\colon C^{p,\bullet}_{\mathrm{sm},R} \to C^{p,\bullet}_{\mathrm{sing},R}$.

2.3.4 Tropical Borel–Moore homology

If we modify the definitions of $C^{\mathrm{sing}}_{p,q}(X;R)$ and $C^{\mathrm{sm}}_{p,q}(X;R)$ to allow formal sums of q-simplices that are locally finite, we obtain chain groups $C^{\mathrm{sing,lf}}_{p,q}(X;R)$ and $C^{\mathrm{sm,lf}}_{p,q}(X;R)$, respectively. The q-th homology groups of the corresponding chain complexes are denoted $H^{\mathrm{sing,lf}}_{p,q}(X;R)$ and $H^{\mathrm{sm,lf}}_{p,q}(X;R)$, respectively. As in §2.3.3, one finds that the natural inclusion $C^{\mathrm{sm,lf}}_{p,\bullet}(X;R) \to C^{\mathrm{sing,lf}}_{p,\bullet}(X;R)$ induces an isomorphism

$$H^{\mathrm{sm,lf}}_{p,q}(X;R) \cong H^{\mathrm{sing,lf}}_{p,q}(X;R).$$

Our $H^{\mathrm{sing,lf}}_{p,q}(X;R)$ agrees with the Borel–Moore homology described in [JRS18, Definition 2.7] and with the corresponding sheaf-theoretic notion described by [GS23a, Theorem A].

2.4 Tropical intersection theory

Fix a polyhedral space X with its local cone stratification. In this section, we define tropical cycles on X and highlight some of their basic properties and different equivalence relations that one can consider on the group of k-cycles $\mathscr{Z}_k(X)$.

2.4.1 Balanced polyhedral spaces

A *weight function* on X is a locally constant map $w_X \colon X^{\mathrm{gen}} \to \mathbb{Z}^\times$. We call the pair (X, w_X) a *weighted polyhedral space*, although we will omit w_X from the notation when it is clear from context. Recall that X^{gen} is the disjoint union of the maximal strata of X, each of which is connected. Accordingly, the *weight* $w_X(\sigma)$ of a maximal stratum σ is the value $w_X(x)$ at any point $x \in \sigma$.

Given a stratum σ of X, let $L(\sigma)$ denote the intersection of the lineality space $L(\sigma)_{\mathbb{R}}$ of $\mathrm{lc}_X(\sigma)$ with the integral tangent space $T^1_X(\sigma)$. By rationality, $\operatorname{rank} L(\sigma) = \dim(\sigma)$. Moreover, given $\tau \prec \sigma$ where $\dim(\tau) = \dim(\sigma) - 1$, one finds that $L(\tau)$ is a saturated subgroup of $L(\sigma)$, so that we have a split short exact sequence

$$0 \to L(\tau) \to L(\sigma) \to \mathbb{Z} \to 0. \tag{2.17}$$

Choose an integral vector $n_{\sigma/\tau} \in L(\sigma)$ whose image generates $\mathbb{Z}$ and that points into σ from τ, i.e., for all $x \in \sigma$ and $\varepsilon > 0$ sufficiently small, $x + \varepsilon n_{\sigma/\tau}$ lies inside of σ under the identification of a neighborhood of 0 in $\mathrm{lc}_X(\tau)$ with a star neighborhood of x in X. Necessarily, $n_{\sigma/\tau}$ is a primitive vector.

A weighted, pure polyhedral space Z with a stratum τ of codimension 1 is *balanced at* τ if

$$\sum_{\sigma \succ \tau} w_Z(\sigma) n_{\sigma/\tau} \in L(\tau), \tag{2.18}$$

where the sum is taken inside of $T_Z^1(\tau) \supset L(\sigma)$. Likewise, Z is *balanced* if it is balanced at every such τ.

Fix a balanced polyhedral space Z of dimension k. Then the *integral* of a compactly supported superform $\omega \in \mathcal{A}_c^{k,k}(Z)$ is defined by

$$\int_Z \omega := \sum_{\sigma} w_Z(\sigma) \int_\sigma \omega, \tag{2.19}$$

where the sum ranges over the maximal strata σ of Z and the integral on the right is given by (2.11).

2.4.2 Tropical cycles

A *(tropical algebraic) k-cycle* on X is a balanced, closed polyhedral subspace of X of dimension k. A k-cycle is *effective* if all of its weights are positive. We denote the set of k-cycles on X by $\mathscr{Z}_k(X)$.

We now outline how to make $\mathscr{Z}_k(X)$ into an abelian group; see [AR10, Construction 2.13] for details. Given polyhedral subspaces Z_1 and Z_2 of X, their union $Z_1 \cup Z_2$ again has the structure of a polyhedral subspace, whose local cone stratification is a refinement of the stratifications on Z_1 and Z_2. Suppose now that $Z_1, Z_2 \in \mathscr{Z}_k(X)$. Then $Z_1 \cup Z_2$ is pure of dimension k. For each maximal stratum σ of the union, we let σ_i denote the unique stratum of Z_i containing σ, if it exists. Define $w(\sigma) := w_{Z_1}(\sigma_1) + w_{Z_2}(\sigma_2)$, where $w(\sigma_i) = 0$ if σ_i does not exist. Then $Z_1 + Z_2$ is the weighted polyhedral subspace of X given by the closure of $\bigcup_{w(\sigma)\neq 0} \sigma$ with weight function w. One can show that $Z_1 + Z_2$ is in fact balanced.

Following [AR10, §4], a proper morphism $f\colon X \to Y$ of polyhedral spaces induces a *pushforward* homomorphism $f_*\colon \mathscr{Z}_k(X) \to \mathscr{Z}_k(Y)$ as follows. Fix $Z \in \mathscr{Z}_k(X)$. As a polyhedral subspace of Y, f_*Z is the closure of the set of points $f(z)$ for which $z \in Z$ and $d_z f|_{T^1_{Z,z}}$ is injective. In particular, for any maximal stratum σ of Z, if $d_z f|_{T^1_{Z,z}}$ is injective at some $z \in \sigma$, then it holds for all $z \in \sigma$. Then $f|_\sigma$ is a local embedding, so that f_*Z is indeed pure of dimension k. The weight function on $f_*Z\colon (f_*Z)^{\mathrm{gen}} \to \mathbb{Z}^\times$ is defined by

$$w(y) := \sum_{z \in Z \cap f^{-1}(y)} [T^1_{f_*Z,y} : d_z f(T^1_{Z,z})] w_Z(z),$$

where the fact that f is proper ensures that the sum is finite. We defer to [GKM09, Proposition 2.25] for the proof that f_*Z is balanced.

If X is compact, then the constant map $c\colon X \to \mathbb{R}^0$ is proper. Identifying $\mathscr{Z}_0(\mathbb{R}^0) \cong \mathbb{Z}$, we define the *degree map* $\deg\colon \mathscr{Z}_0(X) \to \mathbb{Z}$ by sending a 0-cycle Z to c_*Z.

2.4.3 Tropical Cartier divisors

A continuous function $\phi\colon X \to \mathbb{R}$ is *rational* if every point in X has a local face structure $\mathcal{C}$ such that $\phi|_\sigma$ is integral affine for every $\sigma \in \mathcal{C}$. Such functions define a $\underline{\mathbb{Z}}$-module $\mathcal{K}_X$ that naturally contains Aff_X as a $\underline{\mathbb{Z}}$-submodule. We define

$$\mathrm{Div}(X) := \Gamma(\mathcal{K}_X/\mathrm{Aff}_X, X)$$

and call its elements *(tropical) Cartier divisors*. The quotient map $\mathcal{K}_X \to \mathcal{K}_X/\mathrm{Aff}_X$ induces a map $\mathrm{div}\colon \Gamma(\mathcal{K}_X, X) \to \mathrm{Div}(X)$, whose image $\mathrm{Prin}(X)$ consists of *principal divisors*.

Suppose that Z is a balanced polyhedral space of dimension k. Then there is a map $\mathrm{Div}(Z) \to \mathscr{Z}_{k-1}(Z)$ that sends D to its *associated Weil divisor* $Z \cdot D$, which we describe as follows. First, it is straightforward to show that, at any point $z \in Z$, the restriction of D to a star neighborhood U of z is represented modulo $\mathrm{Aff}_Z(U)$ by a single rational function $\phi \in \mathcal{K}_Z(U)$. Fixing an isomorphism $T^1_{Z,z,\mathbb{R}} \cong \mathbb{R}^n$ and choosing a chart U to an open subset of $\mathrm{lc}_Z(z) \subseteq T^1_{Z,z,\mathbb{R}}$, we may choose a fan structure $\mathcal{C}$ on $\mathrm{lc}_Z(z)$ so that the restriction of ϕ to each face $\tau \in \mathcal{C}$ agrees with some integral affine function ϕ_τ. If τ has codimension 1, we define its weight by

$$w_D(\tau) := \sum_{\sigma \succ \tau} w_Z(\sigma) \langle n_{\sigma/\tau}, d\phi_\tau - d\phi_\sigma \rangle. \tag{2.20}$$

Notice that if ϕ itself is integral affine, then we can choose $\phi_\tau = \phi_\sigma = \phi$, and subsequently, $w_D(\tau) = 0$. In particular, the weights are independent of the representative of $D|_U \in \mathcal{K}_Z(U)/\mathrm{Aff}_Z(U)$. Then $(Z \cdot D)|_U$ is the weighted, pure polyhedral subset of U of dimension $k-1$ with weight function w_D and support equal to the closure of $\bigcup_{w_D(\tau)\neq 0} \tau$. One can show that it is in fact balanced. Moreover, this definition is compatible with transition maps on Z, since those are given by integral affine maps; consequently, the local $(k-1)$-cycles $(Z \cdot D)|_U$ glue to a $(k-1)$-cycle $Z \cdot D$ on Z.

Remark 2.4. Let Γ_ϕ denote the graph of ϕ. As described in [AR10, Construction 3.3], the motivation behind (2.20) is that $\Gamma_\phi \subset \mathbb{R}^{n+1}$ becomes balanced at its face $\Gamma_{\phi|_\tau}$ when we add the facet $\Gamma_{\phi|_\tau} + \mathbb{R}_{\geq 0} e_{n+1}$ with weight $w_D(\tau)$. Our choice to extend these facets in the e_{n+1}-direction (rather than towards $-e_{n+1}$ as in [AR10]) is compatible with the "min"-convention, whereby, e.g., $\min\{0, x\}$ should have a zero rather than a pole at $0 \in \mathbb{R}$.

A morphism $f\colon X \to Y$ of polyhedral spaces determines a natural pullback homomorphism $\mathcal{K}_Y \to f_*\mathcal{K}_X$, which restricts to the map $\mathrm{Aff}_Y \to f_*\mathrm{Aff}_X$ described in §2.2.10. This descends to the quotient, thereby inducing a pullback on Cartier divisors $f^*\colon \mathrm{Div}(Y) \to \mathrm{Div}(X)$. This allows us to define a homomorphism $\mathscr{Z}_k(X) \otimes \mathrm{Div}(X) \to \mathscr{Z}_{k-1}(X)$ that sends $Z \otimes D$ to the *intersection* $Z \cdot D := \iota_*(Z \cdot \iota^* D)$, where $\iota\colon Z \to X$ is the inclusion morphism. We define the following notation inductively:

$$Z \cdot D^k := (Z \cdot D^{k-1}) \cdot D. \tag{2.21}$$

2.4.4 Stable intersection

Suppose that Z is a balanced polyhedral space of dimension n. Given two cycles $Z_i \in \mathscr{Z}_{n-k_i}(Z)$, their *(stable) intersection* $Z_1 \cdot Z_2$ is an $(n - k_1 - k_2)$-cycle; in other words, their codimensions add. For details on the construction of $Z_1 \cdot Z_2$, we refer to [AR10]. For our purposes, it suffices to note that, for any $D \in \mathrm{Div}(Z)$ and any cycle Z' in Z, the stable intersection of cycles $Z' \cdot (Z \cdot D)$ agrees with the intersection with a Cartier divisor $Z' \cdot D$ defined in §2.4.3. Accordingly, from (2.21), we see that $(Z \cdot D)^{\cdot k} = Z \cdot D^k$.

2.4.5 Smoothness

Let $h\colon \mathbb{R}^n \to \mathbb{R}$ be the rational function $h(x) := \min\{0, x_1, \ldots, x_n\}$, whose associated Weil divisor $H := \mathbb{R}^n \cdot \mathrm{div}(h)$ is called the *standard tropical hyperplane*. Any k-cycle Z in $\mathbb{R}^n$ has a well-defined *degree* given by

$$\deg(Z) := \deg(H^{\cdot k}),$$

where the right-hand side computes the degree of the 0-cycle $H^{\cdot k}$ as in §2.4.2. We caution that $\deg(Z)$ need not be invariant under integral affine isomorphism.

Following [MR18, §7.4], we say that a balanced polyhedral space Z is *smooth* at a point z if the local cone $\mathrm{lc}_Z(z)$, regarded as a tropical cycle in $T^1_{Z,z,\mathbb{R}}$, has degree 1 with respect to some identification $T^1_{Z,z} \cong \mathbb{Z}^{\dim_z(Z)}$. Likewise, Z is *smooth* if it is smooth at all $z \in Z$. In particular, if Z is smooth, then it must have constant weight 1 on every maximal stratum.

2.4.6 Rational equivalence

Fix algebraic k-cycles A, B, and C on X. Following [AHR16, Definition 3.1], we say that A and B are *rationally equivalent*, denoted $A \sim_{\mathrm{rat}} B$, if there exists a $(k+1)$-cycle $\iota\colon Z \to X$ and a rational function ϕ on Z such that

$$A - B = \iota_*(Z \cdot \mathrm{div}(\phi)).$$

It follows immediately from the definition that $A \sim_{\mathrm{rat}} B$ implies that $A + C \sim_{\mathrm{rat}} B + C$; it is also straightforward to show that $\sim_{\mathrm{rat}}$ is an equivalence relation. We say that A is *rationally trivial* if $A \sim_{\mathrm{rat}} 0$, and we denote the subgroup of such cycles by $\mathscr{Z}_k^{\mathrm{rat}}(X)$.

2.4.7 Algebraic equivalence

Algebraic equivalence of tropical cycles was first defined in [Zha15] and explored further in [GS23b, Section 5.1]. Given k-cycles A and B on X, we write $A \sim B$, if there exists a tropical curve Γ (see §2.5), points $v_0, v_1 \in \Gamma$, and a $(k+1)$-cycle Z on $X \times \Gamma$ such that

$$A - B = \pi_{X*}(Z \cdot \pi_\Gamma^*(v_1 - v_0)),$$

where $\pi_X \colon X \times \Gamma \to X$ and $\pi_\Gamma \colon X \times \Gamma \to \Gamma$ are the standard projections and $v_1 - v_0 \in \mathrm{Div}(\Gamma) \cong \mathscr{Z}_0(\Gamma)$. Let $\sim_{\mathrm{alg}}$ be the equivalence relation generated by $\sim$ by closing under transitivity. We say that A and B are *algebraically equivalent* if $A \sim_{\mathrm{alg}} B$, and that A is *algebraically trivial* if $A \sim_{\mathrm{alg}} 0$. Let $\mathscr{Z}_k^{\mathrm{alg}}(X)$ denote the subgroup of algebraically trivial k-cycles.

Remark 2.5. In the definition of $\sim_{\mathrm{alg}}$, we allow the base curve Γ to have boundary points, i.e., charts to $\mathbb{T}$ or, in the graph-theoretic language of §2.5, leaves of infinite length; we refer to [MR18, §7.1] for the relevant definitions. This realizes rational equivalence as a special case of algebraic equivalence, namely, where Γ must be the tropical projective line $\mathbb{TP}^1$. Consequently, $\mathscr{Z}_k^{\mathrm{rat}}(X) \subseteq \mathscr{Z}_k^{\mathrm{alg}}(X)$.

2.4.8 Cycle class map

We follow [JRS18, §4.C] and [GS23a, Section 5]. Given a balanced polyhedral space Z of dimension k, there exists a (not necessarily rational) triangulation $\mathcal{T}$ of Z that refines the local cone stratification and that comes equipped with a total order $<$ on its vertices. For each facet $\tau \in \mathcal{T}$, define $w_Z(\tau) := w_Z(\sigma_\tau)$ where σ_τ is the maximal stratum of Z containing $\mathrm{ri}(\tau)$. Let f_τ denote the unique affine k-simplex parameterizing τ for which $f_\tau(e_0) < \ldots < f_\tau(e_k)$. Define ν_τ to be the generator of $T_Z^k(\sigma_\tau) \cong \mathbb{Z}$ that is a positive multiple $df_\tau(\nu) \in T_{Z,\mathbb{R}}^k(\sigma_\tau) \cong \mathbb{R}$, where ν is given by (2.16). Then the *fundamental chain* of Z with respect to $\mathcal{T}$ is the locally finite affine (k, k)-chain defined by

$$\mathrm{ch}_{\mathcal{T}}(Z) := \sum_{\tau \in \mathcal{T}} w_Z(\tau)\nu_\tau \otimes f_\tau. \tag{2.22}$$

In contexts where $\mathcal{T}$ is implicit, we will omit it from the notation and simply write $\mathrm{ch}(Z)$.

Lemma 2.6. *For any triangulation $\mathcal{T}$, we have that $\partial\, \mathrm{ch}_{\mathcal{T}}(Z) = 0$.*

Proof. The boundary of each f_τ is supported on the codimension-1 skeleton of $\mathcal{T}$, so it suffices to show vanishing at each face υ of $\mathcal{T}$ of codimension 1. Let $x_0 < \ldots < x_{k-1}$ be the vertices of υ in order. Each facet $\tau \succ \upsilon$ has vertices $x_0, \ldots, x_{k-1}, y_\tau$; let $o(\tau) \in [0, k]$ be the index such that $x_{o(\tau)-1} < y_\tau < x_{o(\tau)}$, where we formally define $x_{-1} := -\infty$ and $x_q := \infty$. Let f_υ be the unique affine $(k-1)$-simplex satisfying $f_\upsilon(e_i) = x_i$. Then from the definition of ∂ in (2.13), we compute

$$\partial\, \mathrm{ch}_{\mathcal{T}}(Z)|_\upsilon = (-1)^k \sum_{\tau \succ \upsilon} (-1)^{o(\tau)} w_Z(\tau) \nu_\tau \otimes f_\upsilon. \tag{2.23}$$

Suppose first that σ_υ is a stratum of codimension 1, so that the strata $(\sigma_\tau)_{\tau \succ \upsilon}$ are all pairwise distinct and precisely the maximal strata σ of Z for which $\sigma \succ \sigma_\upsilon$. Let ν_υ be the generator of $\bigwedge^{k-1} L(\sigma_\upsilon) \cong \mathbb{Z}$ that is a positive multiple $df_\upsilon(\nu)$. Identifying the orientation ν on Δ^q with $(e_q - e_0) \wedge \ldots \wedge (e_1 - e_0)$, we find that, for some $c_\upsilon > 0$ and $c_\tau > 0$ for each $\tau \succ \upsilon$,

$$\nu_\upsilon = c_\upsilon (x_{k-1} - x_0) \wedge \ldots \wedge (x_1 - x_0)$$

and

$$\nu_\tau = \begin{cases} c_\tau (x_{k-1} - w_\tau) \wedge \ldots \wedge (x_0 - w_\tau) & \text{if } o(\tau) = 0 \\ (-1)^{o(\tau)-1} c_\tau (x_{k-1} - x_0) \wedge \ldots \wedge (x_1 - x_0) \wedge (w_\tau - x_0) & \text{otherwise} \end{cases},$$

where we regard $x_1 - x_0$ as a vector inside of $T^1_{Z,\mathbb{R}}(\sigma_\upsilon)$, etc. In the case where $o(\tau) = 0$, we rewrite $x_i - w_\tau = (x_i - x_0) - (w_\tau - x_0)$ in each factor except the last so that, in any case, we have that

$$\nu_\tau = (-1)^{o(\tau)-1} \frac{c_\tau}{c_\upsilon} \nu_\upsilon \wedge (w_\tau - x_0).$$

Because $w_\tau - x_0$ and $n_{\sigma_\tau/\sigma_\upsilon}$ (defined in §2.4.2) both point into σ_τ from σ_υ, it follows that

$$\nu_\tau = (-1)^{o(\tau)-1} c'_\tau \nu_\upsilon \wedge n_{\sigma_\tau/\sigma_\upsilon} \tag{2.24}$$

for some $c'_\tau > 0$. However, ν_τ is a generator of $T^k_Z(\sigma_\tau)$, and from splitting of (2.17), it follows that the same holds for $\nu_\upsilon \wedge n_{\sigma_\tau/\sigma_\upsilon}$. This forces $c'_\tau = 1$. Substituting (2.24) into (2.23) and applying the balancing condition (2.18) at σ_υ, we obtain the desired vanishing result:

$$\begin{aligned}\partial\,\mathrm{ch}_{\mathcal{T}}(Z)|_\upsilon &= (-1)^k \sum_{\tau \succ \upsilon} (-1)^{o(\tau)} w_Z(\tau) \left((-1)^{o(\tau)-1}\nu_\upsilon \wedge n_{\sigma_\tau/\sigma_\upsilon}\right) \otimes f_\upsilon \\ &= (-1)^{k-1}\nu_\upsilon \wedge \sum_{\sigma_\tau \succ \sigma_\upsilon} w_Z(\sigma_\tau) n_{\sigma_\tau/\sigma_\upsilon} \otimes f_\upsilon \\ &= 0.\end{aligned}$$

If, instead, σ_υ is maximal, then we are in an easier situation: there are exactly two facets τ_1 and τ_2 of $\mathcal{T}$ for which $\tau_i \succ \upsilon$, and they satisfy $\sigma_{\tau_1} = \sigma_{\tau_2} = \sigma_\upsilon$. One finds in a manner analogous to the previous case that $(-1)^{o(\tau_1)}\nu_{\tau_1} = -(-1)^{o(\tau_2)}\nu_{\tau_2}$, with the sign coming from the fact that $w_{\tau_1} - x_0$ and $w_{\tau_2} - x_0$ point in opposite directions relative to υ. Then (2.23) again vanishes, as desired. □

The *cycle class map* is the group homomorphism $\mathrm{cyc}\colon \mathscr{Z}_k(X) \to H^{\mathrm{sing,lf}}_{k,k}(X;\mathbb{Z})$ defined by sending Z to the homology class with representative $\mathrm{ch}_{\mathcal{T}}(Z)$ for some $\mathcal{T}$. In principle, one could show that this is well-defined by proving that any two triangulations of Z have a common refinement, and that if $\mathcal{T}'$ refines $\mathcal{T}$, then $\mathrm{ch}_{\mathcal{T}'}(Z)$ and $\mathrm{ch}_{\mathcal{T}}(Z)$ differ by a boundary. For our purposes, it suffices to have well-definedness in the torsion-free case; see Corollary 3.11.

Remark 2.7. As explained in [GS23a, §5.1], the definition of the cycle class map depends on how one chooses to identify the two notions of orientation given by $\mathbf{F}_k(\mathbb{R}^k)$ and $H_k(\mathbb{R}^k, \mathbb{R}^k \backslash \{0\})$. This choice is captured by our requirement that f_σ map ν to v_σ (up to positive scaling); ν itself was chosen so as to ensure that no sign appears in Lemma 3.10, and differs from the choice made in [JRS18, §4.B].

Fix cycles $A, B \in \mathscr{Z}_k(X)$. Then A and B are *homologically equivalent*, denoted $A \sim_{\mathrm{hom}} B$, if $A - B \in \ker(\mathrm{cyc})$, and A is *homologically trivial* if $A \sim_{\mathrm{hom}} 0$. Identifying cyc with the sheaf-theoretic cycle class map defined in [GS23a], it follows from [GS23b, Proposition 5.8] that $\mathscr{Z}^{\mathrm{alg}}_k(X) \subseteq \mathscr{Z}^{\mathrm{hom}}_k(X)$.

2.5 Tropical curves

A *tropical curve* is a compact, connected, smooth polyhedral space of dimension 1. We study tropical curves combinatorially through the lens of metric graphs.

2.5.1 Graphs

Let G be a graph with vertex and edge sets V and E, respectively. The *valence* of a vertex v, denoted $\mathrm{val}(v)$, is the number of half-edges incident to v. Unless explicitly stated otherwise, we assume going forward that all graphs are finite and connected, allow parallel edges and loops, and have no vertices of valence 1. In particular, we may define the *genus* of G to be $g(G) := \#E - \#V + 1$.

An *orientation* $\mathcal{O}$ on G is the data of *head* and *tail* functions $E \to V$, denoted by $e \mapsto e^+$ and $e \mapsto e^-$, respectively, so that $e = \{e^+, e^-\}$ as a multiset. As sets, we let $\mathcal{O} := E$, although we often simply write E when the orientation is clear from context. Elements $e \in \mathcal{O}$ are called *oriented edges*. Every orientation $\mathcal{O}$ has a *reverse* orientation $\bar{\mathcal{O}}$ obtained by swapping its head and tail functions; if e is an oriented edge in $\mathcal{O}$ (resp., in $\bar{\mathcal{O}}$), then $\bar{e}$ denotes the corresponding oriented edge in $\bar{\mathcal{O}}$ (resp., in $\mathcal{O}$), so that $\bar{e}^+ = e^-$ and $\bar{e}^- = e^+$.

Given $u, v \in V$, a *path from u to v* is a list $e_1, \ldots, e_k$ of oriented edges in $\mathcal{O} \sqcup \bar{\mathcal{O}}$, not necessarily distinct, such that $e_1^- = u$, $e_k^+ = v$, and $e_i^+ = e_{i+1}^-$ for all $i \in [k-1]$; we call it a *cycle* if, moreover, $u = v$. Fix a basepoint $q \in V$ and a spanning tree $T \subset E$. Given $v \in V$, we let ζ_v denote the unique minimal path in T from q to v. Likewise, given an oriented edge $e \notin T$, we let γ_e denote the unique minimal cycle using only e and edges from T. There is a natural identification

$$C_1(G, \mathbb{Z}) \cong \mathbb{Z}\langle \mathcal{O} \sqcup \bar{\mathcal{O}} \rangle / \langle \bar{e} + e \mid e \in \mathcal{O} \rangle \cong \mathbb{Z}^{\mathcal{O}},$$

so that $\bar{e} \mapsto -e$. By abuse of notation, we conflate paths and cycles with their images in $C_1(G, \mathbb{Z})$. Under this identification, $\{\gamma_e \mid e \in E \setminus T\}$ forms a basis of $H_1(G, \mathbb{Z})$.

2.5.2 Metric graphs

Suppose that G is equipped with a *length function* $\ell\colon E \to \mathbb{R}_{>0}$. This determines a metric space Γ as follows. First, fix an arbitrary orientation $\mathcal{O}$ on G. Then as a topological space,

$$\Gamma := \bigsqcup_{e\in E}[0,\ell(e)]\Big/\sim, \tag{2.25}$$

where $[0,\ell(e)] \subset \mathbb{R}$ is a closed interval of length $\ell(e)$ and the gluing relations on endpoints are given by the incidence relations in G, i.e., for all $e, f \in E(G)$ not necessarily distinct,

$$\begin{cases} (0,e) \sim (0,f) & \text{if } e^- = f^- \\ (0,e) \sim (\ell(f),f) & \text{if } e^- = f^+ \\ (\ell(e),e) \sim (\ell(f),f) & \text{if } e^+ = f^+, \end{cases}$$

where (t,e) denotes the point t in the closed interval $[0,\ell(e)]$ associated to e in the disjoint union. We make Γ into a metric space by defining the distance between two points as the infimum of the lengths of all paths between them. A metric space Γ so obtained is called a *metric graph*. We further say that the edge-weighted graph (G,ℓ) is a *model* for Γ and that Γ *overlies* G. Observe that Γ is independent of the choice of orientation $\mathcal{O}$. The valence function on V extends to a function $\mathrm{val}\colon \Gamma \to \mathbb{Z}_{>0}$ by defining the valence of any interior point of an edge to be 2.

A single metric graph overlies many different graphs. More precisely, consider the relation on pairs (G,ℓ) generated by a single edge refinement: pick an edge e in G and subdivide it into two edges, e_1 and e_2, to obtain a new graph G'; fix the length function ℓ' on G' that equals ℓ away from e and satisfies $\ell'(e_1) + \ell'(e_2) = \ell(e)$. The inverse operation is performed at vertices of valence two. If (G,ℓ) is a model for Γ, then every other model is obtained from (G,ℓ) by a sequence of refinements and inverse refinements. The *genus* of Γ is its first Betti number, or equivalently, the genus of any model. A metric graph of genus at least two has a unique minimal model with no vertices of valence two.

A continuous function $\phi\colon \Gamma \to \mathbb{R}$ is *harmonic* if there exists a model G of Γ so that the restriction of f to each edge is linear, with the sum of incoming slopes at every vertex equal to zero. A metric graph, equipped with its sheaf of harmonic functions, may be identified with a tropical curve as follows. Every $x \in \Gamma$ has local charts to open subsets of the smooth 1-dimensional fan in $\mathbb{R}^{\mathrm{val}(x)-1}$ defined by the union of rays

$$\bigcup_{i=0}^{\mathrm{val}(x)-1} \mathbb{R}_{\geq 0}\langle e_i \rangle, \tag{2.26}$$

where $e_0 := -\sum_{i=1}^{\mathrm{val}(x)-1} e_i$, subject to the condition that harmonic functions should pull back to integral affine functions on the fan. The natural stratification of Γ is given by the (open) edges and vertices of its minimal model, and the local fan at a vertex v is precisely given by the star-shaped set in (2.26).

2.5.3 *Integral flows*

We now determine Ω^1_Γ and $T^1_{\Gamma,x}$ for a metric graph Γ. Fixing a model (G,ℓ) with orientation $\mathcal{O}$, the interior of each edge e has an orientation-preserving chart to $(0,\ell(e)) \subset \mathbb{R}$. Let t denote the standard coordinate on $\mathbb{R}$; then we abuse notation by writing de for the tropical 1-form on e which coincides with dt in this chart and $\frac{\partial}{\partial e}$ for the dual vector $\frac{d}{dt}$. Specializing to the local fan at a vertex v, we see that

$$\sum_{e^+=v} \frac{\partial}{\partial e} = \sum_{e^-=v} \frac{\partial}{\partial e}. \tag{2.27}$$

By (2.2), any $\omega \in \Omega^1_\Gamma(\Gamma)$ is uniquely determined by its stalks at the minimal strata of Γ, i.e., the vertices. However, these data determine the stalks ω_e at the edges; using (2.27) and the identification $\omega_e = \frac{\partial}{\partial e}\omega$, we find that

$$\sum_{e^+=v} \omega_e = \sum_{e^-=v} \omega_e, \tag{2.28}$$

and it is straightforward to show that any collection $(\omega_e)_{e\in E} \in \mathbb{Z}^E$ satisfying (2.28) determines a unique tropical 1-form. In other words, we may identify $H^0(\Gamma,\Omega^1_\Gamma)$ with the space of integral flows on G. Likewise, there is a canonical identification with $H_1(G,\mathbb{Z})$, implying that $\operatorname{rank} H^0(\Gamma,\Omega^1_\Gamma) = g(\Gamma)$.

2.6 Tropical abelian varieties

In complex algebraic geometry, abelian varieties correspond to complex tori that admit a polarization (or equivalently, a Riemann form). In the tropical setting, we work instead with real tori that have a fixed integral structure, thereby making them into polyhedral spaces. We describe the tropical analogue of a polarization, and we discuss how we may associate to each metric graph a tropical abelian variety called its Jacobian that comes equipped with a canonical principal polarization.

2.6.1 Real tori with integral structures

Fix two finitely-generated free abelian groups Λ and N of rank g and an embedding $\Lambda \subset N_{\mathbb{R}}$. Notice that N and Λ are both lattices inside of $N_{\mathbb{R}} \cong \Lambda_{\mathbb{R}}$. We obtain a real torus X of dimension g as the quotient $N_{\mathbb{R}}/\Lambda$. The quotient map $\pi\colon N_{\mathbb{R}} \to X$ is a local homeomorphism, allowing us to define charts $\pi_U^{-1}\colon \pi(U) \to U$ for any sufficiently small open subset $U \subset N_{\mathbb{R}}$. Any identification $N \cong \mathbb{Z}^g$ makes $N_{\mathbb{R}}$ into a polyhedral space with integral tangent space

$$N \cong T^1_{N_{\mathbb{R}}}(N_{\mathbb{R}}) \subset T^1_{N_{\mathbb{R}},\mathbb{R}}(N_{\mathbb{R}}) \cong N_{\mathbb{R}}.$$

The transition maps $\pi_U^{-1} \circ \pi_{U'}$ act by translation, so they preserve $T^1_{N_{\mathbb{R}}}(N_{\mathbb{R}})$, thereby endowing X with the structure of a polyhedral space. We call X a *real torus with integral structure* N. Naturally, at any point $x \in X$ we may identify $T_{X,x} \cong N$ and likewise $\Omega_{X,x} \cong N^\vee$.

2.6.2 Tropical homology of a real torus

Fix a real torus $X = N_{\mathbb{R}}/\Lambda$ with integral structure N. Because the local cone of X at any point is isomorphic to $N_{\mathbb{R}}$, the singular tropical chain complex defined by (2.12) reduces to the simpler form

$$C^{\mathrm{sing}}_{p,\bullet}(X;R) \cong \bigwedge^p N_R \otimes C^{\mathrm{sing}}_{\bullet}(X;\mathbb{Z}),$$

where $C^{\mathrm{sing}}_{\bullet}(X;\mathbb{Z})$ is the usual singular chain complex except with a signed boundary map compatible with (2.13). Because the singular homology groups of X are free, the universal coefficient theorem and Künneth theorem together yield

$$H_{p,q}^{\mathrm{sing}}(X;R) \cong \bigwedge^{p} N_R \otimes \bigwedge^{q} \Lambda. \tag{2.29}$$

and subsequently,

$$H^{p,q}_{\mathrm{sing}}(X;R) \cong \bigwedge^{p} N_R^{\vee} \otimes \bigwedge^{q} \Lambda^{\vee}.$$

Analogous equalities hold for smooth and affine simplices, and the classical identification of the corresponding homology groups applies here:

$$H_{p,q}^{\mathrm{aff}}(X;R) \cong H_{p,q}^{\mathrm{sm}}(X;R) \cong H_{p,q}^{\mathrm{sing}}(X;R). \tag{2.30}$$

Let $g := \dim(X)$. Given points $u_i \in N_{\mathbb{R}}$, we write $(u_0, \ldots, u_q)$ for the unique affine q-simplex $\Delta^q \to X$ sending $e_i \mapsto u_i$. Observe that, for each $\lambda \in \Lambda$, $(u_0 + \lambda, \ldots, u_q + \lambda)$ and $(u_0, \ldots, u_q)$ represent the same simplex. Fixing bases $n_1, \ldots, n_g$ for N and $\lambda_1, \ldots, \lambda_g$ for Λ, the Eilenberg–Zilber map for affine homology determines explicit generators

$$\begin{cases} H_{1,1}^{\mathrm{aff}}(X;\mathbb{Z}) = \mathbb{Z}\langle n_i \otimes (0, \lambda_j) \mid i,j \in [g]\rangle \\ H_{1,2}^{\mathrm{aff}}(X;\mathbb{Z}) = \mathbb{Z}\big\langle n_i \otimes \big((0,\lambda_j,\lambda_j+\lambda_k) - (0,\lambda_k,\lambda_j+\lambda_k)\big) \;\big|\; i,j,k \in [g],\, j<k\big\rangle. \end{cases} \tag{2.31}$$

2.6.3 Polarizations

Fix a real torus $X = N_{\mathbb{R}}/\Lambda$ with integral structure N. A *polarization* on X is a homomorphism $\vartheta\colon N \to \Lambda^{\vee}$ for which the induced bilinear form $[\cdot,\cdot]_{\vartheta}\colon N \otimes \Lambda \to \mathbb{Z}$ given by

$$[n,\lambda]_{\vartheta} := \langle \lambda, \vartheta(n)\rangle$$

is symmetric and positive-definite on $N_{\mathbb{R}} \otimes N_{\mathbb{R}}$ (via the canonical identification $\Lambda_{\mathbb{R}} \cong N_{\mathbb{R}}$). In situations where ϑ is clear from context, we shall simply write $[\cdot,\cdot]$ for this bilinear form. If X admits a polarization, we call it a *tropical abelian variety*. If we fix a particular ϑ, then we call the pair (X,ϑ) a *polarized tropical abelian variety*. Any polarization ϑ is necessarily injective; it is *principal* if it is an isomorphism, in which case (X,ϑ) is said to be *principally polarized*.

Using the identification in (2.29) and applying the universal coefficient theorem for cohomology, we find that

$$\mathrm{Hom}(N,\Lambda^\vee) \xrightarrow{\cong} \mathrm{Hom}(N\otimes\Lambda,\mathbb{Z}) \xleftarrow{\cong} H^{1,1}_{\mathrm{aff}}(X;\mathbb{Z}) \tag{2.32}$$

where the left isomorphism sends a map ϑ to $[\cdot,\cdot]_\vartheta$ and the right isomorphism sends a cohomology $(1,1)$-class represented by α to the pairing $n\otimes\lambda \mapsto \langle n\otimes[0,\lambda],\alpha\rangle$. Given a polarization ϑ, we denote its corresponding class in $H^{1,1}_{\mathrm{aff}}(X;\mathbb{Z})$ by h_ϑ.

2.6.4 Tropical Jacobians

Fix a metric graph Γ, a model (G,ℓ) of Γ, and an orientation $\mathcal{O}$ on G. Then the *length pairing* on G is the homomorphism $[\cdot,\cdot]_G\colon C_1(G;\mathbb{Z})\otimes C_1(G;\mathbb{Z})\to\mathbb{R}$ defined on edges by

$$[e,e']_G := \begin{cases} \ell(e) & \text{if } e=e' \\ 0 & \text{otherwise} \end{cases}. \tag{2.33}$$

This restricts in either entry to $H_1(G;\mathbb{Z})\subseteq C_1(G;\mathbb{Z})$. Let

$$N := H^1(G;\mathbb{Z}) \cong H_1(G;\mathbb{Z})^\vee.$$

Define $\pi\colon C_1(G;\mathbb{Z})\to N_\mathbb{R}$ by $\pi(c) := [c,\cdot]_G$. One finds that π is injective on $H_1(G;\mathbb{Z})$ and so defines a lattice Λ in $N_\mathbb{R}$ by

$$\Lambda := \pi(H_1(G;\mathbb{Z})).$$

As in §2.6.1, these data determine a real torus $\mathrm{Jac}(\Gamma) := N_\mathbb{R}/\Lambda$ with integral structure N. We equip $\mathrm{Jac}(\Gamma)$ with the principal polarization $\vartheta\colon N\to\Lambda^\vee$ given by the dual of $\pi|^{-1}_{H_1(G;\mathbb{Z})}\colon\Lambda\to H_1(G;\mathbb{Z})$. Then $\mathrm{Jac}(\Gamma)$ is the *tropical Jacobian* of Γ. Recall that we may identify $H_1(\Gamma;\mathbb{Z})\cong H_1(G;\mathbb{Z})$. One checks that $\mathrm{Jac}(\Gamma)$ is invariant under the edge subdivision operation described in §2.5.2, and therefore does not depend on the choice of model (G,ℓ).

We now relate the pairings $[\cdot,\cdot]_G$ and $[\cdot,\cdot]$ explicitly. In particular, for $\gamma\in H_1(G;\mathbb{Z})$,

$$[n,\pi(\gamma)] = \langle\pi(\gamma),\vartheta(n)\rangle = \langle\pi(\gamma), n\circ\pi|^{-1}_{H_1(G;\mathbb{Z})}\rangle = \langle\gamma,n\rangle. \tag{2.34}$$

If $c \in C_1(G;\mathbb{R})$ satisfies $\pi(c) = n$, then

$$[\pi(c), \pi(\gamma)] = \langle \gamma, \pi(c) \rangle = [c, \gamma]_G. \tag{2.35}$$

2.6.5 Polyhedral structure on $\widetilde{W}_d$

Given a tropical curve Γ with model (G, ℓ) and a basepoint $q \in \Gamma$, the d-th *Abel–Jacobi map* $\Phi_q^{(d)} \colon \Gamma^d \to \mathrm{Jac}(\Gamma)$ is the morphism of polyhedral spaces defined by

$$\Phi_q^{(d)}(p_1, \ldots, p_d) := \sum_{i=1}^{d} \pi(\zeta_{p_i}), \tag{2.36}$$

where ζ_{p_i} is any path from q to p_i in some refinement (G_i, ℓ_i) of (G, ℓ) containing both points as vertices. We claim that this is well-defined. Indeed, the definition does not depend on the chosen refinements (G_i, ℓ_i) because the length pairing is preserved under edge subdivision. Moreover, it is independent of the choice of path ζ_{p_i}: given any other path ζ'_{p_i} with the same endpoints, we have that $\zeta'_{p_i} - \zeta_{p_i} \in H_1(\Gamma, \mathbb{Z})$, hence $\pi(\zeta'_{p_i}) \equiv \pi(\delta) \pmod{\Lambda}$. When $d = 1$ we abbreviate $\Phi_q := \Phi_q^{(1)}$. Notice that $\Phi_q^{(d)}$ factors through $\Gamma^d/\mathfrak{S}_d$, the d-th symmetric power of Γ; by abuse of notation, we denote the factorization also by $\Phi_q^{(d)}$.

We call $\widetilde{W}_d := \Phi_q^{(d)}(\Gamma^d)$ the d-th *effective locus* of Γ. By [GST22, Theorem D], $\widetilde{W}_d$ defines a tropical d-cycle in $\mathrm{Jac}(\Gamma)$ and comes equipped with a canonical face structure. In particular, according to the discussion in [GST22, §8.2], the facets of $\widetilde{W}_d$ are parameterized by subsets $\prod \mathscr{E} \subset \Gamma^d/\mathfrak{S}_d$, where $\mathscr{E} \in \binom{E}{d}$ is a set of d distinct edges of G for which the finite graph $G \setminus \mathscr{E}$ is connected. The images of these parameterizations intersect in sets of measure zero. Moreover, if $G \setminus \mathscr{E}$ is not connected, then the image $\Phi_q^{(d)}(\prod \mathscr{E})$ has measure zero. Consequently, the integral of a superform $\omega \in \mathcal{A}^{d,d}(\mathrm{Jac}(\Gamma))$ over $\widetilde{W}_d$ may be computed as follows:

$$\int_{\widetilde{W}_d} \omega = \int_{\Gamma^d/\mathfrak{S}_d} (\Phi_q^{(d)})^* \omega = \sum_{\mathscr{E} \in \binom{E}{d}} \int_{\prod \mathscr{E}} (\Phi_q^{(d)})^* \omega. \tag{2.37}$$

Chapter 3

TROPICAL DE RHAM THEOREM

This chapter is based on joint work with Junaid Hasan and Farbod Shokrieh. For smooth manifolds, there is an integration pairing of differential forms with smooth chains. By the Stokes formula, this pairing descends to a pairing of de Rham cohomology and singular homology with real coefficients. The content of de Rham's theorem is that this is a perfect pairing, and so induces an isomorphism of de Rham cohomology with real singular cohomology via the universal coefficient theorem. By [JSS19, Theorem 3.22], for a polyhedral space X equipped with a face structure, there is an analogous isomorphism

$$H^{p,q}_{d''}(X) \cong H^{p,q}_{\mathrm{sing}}(X;\mathbb{R}).$$

This follows from the fact that $\mathcal{A}^{p,\bullet}$ and $\underline{C}^{p,\bullet}_{\mathrm{sing},\mathbb{R}}$ are both $\Gamma(\cdot, X)$-acyclic resolutions of $\Omega^p_{X,\mathbb{R}}$, so both complexes abstractly compute the sheaf cohomology $H^q(X, \Omega^p_{X,\mathbb{R}})$. We generalize this result to a polyhedral space without reference to a face structure and make the isomorphism more explicit by defining an integration pairing in the spirit of the classical de Rham pairing, as suggested by [JSS19, Remark 3.25].

3.1 Isomorphism of cohomology groups

We introduce the tropical de Rham map dR and prove that it descends to an isomorphism of cohomology groups.

3.1.1 Integration over smooth chains

We define the integral of a (p, q)-form ω over a smooth (p, q)-simplex $v \otimes f$. Assume first that f maps $\mathrm{ri}(\Delta^q)$ into a single chart U. By admissibility, we may assume that U is in fact a star neighborhood for σ_f. Identify U with an open polyhedral set in $\mathbb{R}^n$ and fix coordinates $x_1, \dots, x_n$. Then locally, we may write $\omega|_U = \sum_{I,J} h_{I,J} d'x_I \wedge d''x_J$. Because $U \cap \sigma_f$ is identified with an open set inside the linear subspace $L(\sigma_f)_{\mathbb{R}}$ of $\mathbb{R}^n$, we may regard $h_{I,J} dx_J$ as a differential q-form. Then we define

$$\int_{v\otimes f} \omega := \sum_{I,J} \langle v, dx_I \rangle \int_f h_{I,J} dx_J, \tag{3.1}$$

where the integral over a smooth simplex f is defined by pullback:

$$\int_f h_{I,J} dx_J := \int_{\Delta^q} f^*(h_{I,J} dx_J). \tag{3.2}$$

In the general case, by repeated barycentric subdivision, we may replace $v \otimes f$ by a (p,q)-chain $\sum_k v \otimes f_k$ such that every f_k maps $\mathrm{ri}(\Delta^q)$ into a single chart. Then we define $\int_{v\otimes f} \omega := \sum_k \int_{v\otimes f_k} \omega$. Integration extends linearly over any smooth (p,q)-chain. If ω is compactly supported, then its support intersects only finitely many simplices in a locally finite chain. Accordingly, we define integration in this setting as well.

Well-definedness of $\int_{v\otimes f} \omega$ follows from several observations. First, the integral does not depend on the number of times we take the barycentric subdivision because the usual integral of a smooth form is invariant under reparameterizations of the domain. Second, if the local expression of $\omega|_U$ is replaced by another, then their difference lies in $K(U)$, where K is the $\underline{\mathbb{Z}}$-submodule that appears in (2.5); then it follows from Lemma 3.1 that their integrals over $v \otimes f$ agree. Third, if, within the expression of $\omega|_U$ as a linear combination $h \otimes \alpha \otimes \beta$ inside of $C^\infty(U) \otimes \Omega^p_X(U) \otimes \Omega^q_X(U)$, we replace β with an equivalent $\beta' \in \bigwedge^p \Omega^1_X(U)$, then, by the discussion in §2.2.8, their difference satisfies $\langle v, \beta' - \beta \rangle = 0$ for all $v \in T^p_{X,x}$ at all points $x \in U$. Then $h \otimes \alpha \otimes (\beta' - \beta)$ lies in $K(U)$ by definition, reducing well-definedness to the previous case.

Lemma 3.1. *Fix a star neighborhood U of some stratum σ and a superform $\omega \in \mathcal{A}^{p,q}(U)$. Suppose that $\langle v, w; \omega(x) \rangle = 0$ for all $x \in U$, $v \in T^p_{X,x}$, and $w \in T^q_{X,x}$. Then $\int_{v\otimes f} \omega = 0$ for all $v \otimes f \in C^{\mathrm{sing}}_{p,q}(X)$ such that $\sigma_f = \sigma$ and $f(\mathrm{ri}(\Delta^q)) \subseteq U$.*

Proof. We identify U with an open polyhedral set in $\mathbb{R}^n$ with coordinates $x_1, \ldots, x_n$ and write $\omega = \sum_{I,J} h_{I,J} d'x_I \wedge d''x_J$. Making use of the parameterization $\phi\colon \widetilde{\Delta}^q \to \Delta^q$ defined by (2.15) and writing $(f \circ \phi)^*(h_{I,J} dx_J) = g_{I,J} dt_{[q]}$ for some smooth function $g_{I,J}$, it suffices to show that $\sum_{I,J} \langle v, dx_I \rangle \int_{\widetilde{\Delta}^q} g_{I,J} dt_{[q]} = 0$.

Fix $\varepsilon > 0$. By iterated barycentric subdivision, we may choose a triangulation $\mathcal{D}$ of $\widetilde{\Delta}^q$ such that, fixing a vertex p_τ of each facet $\tau \in \mathcal{D}$, we have that

$$|g_{I,J}(t) - g_{I,J}(p_\tau)| < \varepsilon q!$$

for all $I \in \binom{[n]}{p}$, $J \in \binom{[n]}{q}$, $\tau \in \mathcal{D}$, and $t \in \tau$. Then

$$\begin{aligned}
\left| \int_{\widetilde{\Delta}^q} g_{I,J} dt_{[q]} - \sum_\tau \int_\tau g_{I,J}(p_\tau) dt_{[q]} \right| &= \left| \sum_\tau \int_\tau (g_{I,J} - g_{I,J}(p_\tau)) dt_{[q]} \right| \\
&\leq \sum_\tau \int_\tau |g_{I,J} - g_{I,J}(p_\tau)| dt_{[q]} \\
&< \sum_\tau \int_\tau \varepsilon q! dt_{[q]} \\
&= \varepsilon q! \sum_\tau \operatorname{Vol}(\tau) \\
&= \varepsilon q! \operatorname{Vol}(\widetilde{\Delta}^q) \\
&= \varepsilon. \qquad (3.3)
\end{aligned}$$

Let $w_\tau \in \bigwedge^q \mathbb{R}^n$ be the positively-oriented q-vector whose component vectors span the edges of τ starting from p_τ. Another copy of $\widetilde{\Delta}^q$ may be used to give an affine parameterization of τ, sending the standard basis to w_τ. Then

$$\begin{aligned}
\int_\tau g_{I,J}(p_\tau) dt_{[q]} &= g_{I,J}(p_\tau) \operatorname{Vol}(\tau) \\
&= g_{I,J}(p_\tau) \operatorname{Vol}(\widetilde{\Delta}^q) \langle w_\tau, dt_{[q]} \rangle \\
&= \frac{1}{q!} \langle w_\tau, g_{I,J}(p_\tau) dt_{[q]} \rangle \\
&= \frac{1}{q!} \langle w_\tau, (f \circ \phi)^*(h_{I,J} dx_J)(p_\tau) \rangle
\end{aligned}$$

$$= \frac{1}{q!} h_{I,J}(f \circ \phi(p_\tau)) \langle (f \circ \phi)_* w_\tau, dx_J \rangle. \tag{3.4}$$

By the triangle inequality, (3.3), and (3.4), we find that

$$\begin{aligned}
&\left| \sum_{I,J} \langle v, dx_I \rangle \int_{\widetilde{\Delta}^q} g_{I,J} dt_{[q]} \right| \\
&\leq \sum_{I,J} |\langle v, dx_I \rangle| \left| \int_{\widetilde{\Delta}^q} g_{I,J} dt_{[q]} - \sum_\tau \int_\tau g_{I,J}(p_\tau) dt_{[q]} \right| + \left| \sum_{I,J} \langle v, dx_I \rangle \sum_\tau \int_\tau g_{I,J}(p_\tau) dt_{[q]} \right| \\
&< \sum_{I,J} |\langle v, dx_I \rangle| \varepsilon + \left| \sum_\tau \sum_{I,J} \langle v, dx_I \rangle \frac{1}{q!} h_{I,J}(f \circ \phi(p_\tau)) \langle (f \circ \phi)_* w_\tau, dx_J \rangle \right| \\
&= \sum_{I,J} |\langle v, dx_I \rangle| \varepsilon + \left| \frac{1}{q!} \sum_\tau \langle v, (f \circ \phi)_* w_\tau ; \omega(f \circ \phi(p_\tau)) \rangle \right|.
\end{aligned}$$

Since $f \circ \phi(p_\tau) \in U \cap \sigma$, $v \in T_X^p(\sigma)$, and $(f \circ \phi)_* w_\tau \in T_{X,\mathbb{R}}^q(\sigma)$, by our assumption on ω, this simplifies to

$$= \sum_{I,J} |\langle v, dx_I \rangle| \varepsilon.$$

We may choose ε arbitrarily small, proving the desired equality. □

3.1.2 Tropical de Rham maps

Integration over smooth tropical chains and locally finite chains respectively yields pairings

$$C_{p,q}^{\mathrm{sm}}(X;\mathbb{R}) \otimes_{\mathbb{R}} \mathcal{A}^{p,q}(X) \to \mathbb{R} \quad \text{and} \quad C_{p,q}^{\mathrm{sm,lf}}(X;\mathbb{R}) \otimes_{\mathbb{R}} \mathcal{A}_c^{p,q}(X) \to \mathbb{R}$$

via the assignment $(v \otimes f) \otimes \omega \mapsto \int_{v \otimes f} \omega$. Then we obtain a natural map of presheaves, called the *(tropical) de Rham map*, via

$$\mathrm{dR} \colon \mathcal{A}^{p,\bullet} \to C_{\mathrm{sm},\mathbb{R}}^{p,\bullet},$$

as well as dual homomorphisms

$$\mathrm{dR}^\vee \colon C_{p,\bullet}^{\mathrm{sm}}(X,\mathbb{R}) \to \mathcal{A}^{p,\bullet}(X)^\vee \quad \text{and} \quad \mathrm{dR}^\vee \colon C_{p,\bullet}^{\mathrm{sm,lf}}(X;\mathbb{R}) \to \mathcal{A}_c^{p,\bullet}(X)^\vee.$$

It follows from Lemma 3.2 that these are chain maps and so descend to (co)homology.

Lemma 3.2 (Stokes formula)**.** *For all $c \in C^{\mathrm{sm}}_{p,q}(X;\mathbb{R})$ and $\omega \in \mathcal{A}^{p,q-1}(X)$, we have that*

$$\int_c d''\omega = \int_{\partial c} \omega.$$

The same holds if c is locally finite, provided that ω has compact support.

Proof. By bilinearity, it suffices to prove the statement (in both the finite and locally finite cases) for $c = v \otimes f$ and $\omega = h d'x_I \wedge d''x_J$ in some chart with coordinates $x_1, \ldots, x_n$. We compute

$$\begin{aligned}
\int_c d''\omega &= \int_{v\otimes f} d''(h d'x_I \wedge d''x_J) \\
&= (-1)^p \sum_{j=1}^n \int_{v\otimes f} \frac{\partial h}{\partial x_j} d'x_I \wedge d''x_j \wedge d''x_J \\
&= (-1)^p \sum_{j=1}^n \langle v, dx_I\rangle \int_f \frac{\partial h}{\partial x_j} dx_j \wedge dx_J \\
&= (-1)^p \langle v, dx_I\rangle \int_f d(h dx_J),
\end{aligned} \tag{3.5}$$

where $d(hdx_J)$ indicates the usual exterior derivative of a differential form hdx_J. Meanwhile, computing the boundary of $v \otimes f$ using (2.13), we have that

$$\begin{aligned}
\int_{\partial c} \omega &= \int_{\partial(v\otimes f)} h d'x_I \wedge d''x_J \\
&= (-1)^p \sum_{i=0}^q (-1)^i \int_{v\otimes f_i} h d'x_I \wedge d''x_J \\
&= (-1)^p \sum_{i=0}^q (-1)^i \langle v, dx_I\rangle \int_{f_i} h dx_J \\
&= (-1)^p \langle v, dx_I\rangle \int_{\partial f} h dx_J,
\end{aligned}$$

which coincides with (3.5) by the classical Stokes formula applied to Δ^q. $\square$

3.1.3 Resolutions of $\Omega^p_{X,\mathbb{R}}$

By [JSS19], $\mathcal{A}^{p,\bullet}$ and $\underline{C}^{p,\bullet}_{\mathrm{sing},\mathbb{R}}$ are $\Gamma(\cdot, X)$-acyclic resolutions of the same sheaf, which, as noted in [GS23a, Remark 2.8], is isomorphic to $\Omega^p_{X,\mathbb{R}}$. Recall that $\Omega^0_X = \underline{\mathbb{Z}}$. Then explicitly, the embedding $\Omega^p_{X,\mathbb{R}} \hookrightarrow \mathcal{A}^{p,0}$ is given by composing the canonical map $\underline{\mathbb{R}} \otimes \Omega^p_X \to C^\infty \otimes \Omega^p_X \otimes \underline{\mathbb{Z}}$ that sends $r \otimes \omega \mapsto r \otimes \omega \otimes 1$ (on star neighborhoods) with the quotient map whose kernel K is defined in terms of the contraction in (2.6). One sees that this composition is in fact injective by noting that, for $r \in \mathbb{R}^\times$, $r \otimes \omega \otimes 1$ lands in K precisely when $\langle v, \omega \rangle = 0$ for all $v \in T^p_{X,x}$ and $x \in X$; this forces $\omega = 0$ in Ω^p_X.

Meanwhile, identifying $C^{p,0}_{\text{sing}}$, which is already a sheaf, with its sheafification $\underline{C}^{p,0}_{\text{sing}}$, we define $\Omega^p_X \hookrightarrow \underline{C}^{p,0}_{\text{sing}}$ by sending ω to the linear functional $v \otimes x \mapsto \langle v, \omega \rangle$. This map is injective for the same reason as before. Tensoring with $\underline{\mathbb{R}}$, we obtain the desired embedding $\Omega^p_{X,\mathbb{R}} \hookrightarrow \underline{C}^{p,0}_{\text{sing},\mathbb{R}}$. Identifying ω with the corresponding $(p,0)$-cochain $\langle \cdot, \omega \rangle$ and using the notation in (2.14), we have that

$$\omega_x = \omega \tag{3.6}$$

for all $x \in X$.

3.1.4 Tropical de Rham isomorphism

In §§2.3.2 and 2.3.3, we defined the sheafification $\text{sh}\colon C^{p,\bullet}_{\text{sing},\mathbb{R}} \to \underline{C}^{p,\bullet}_{\text{sing},\mathbb{R}}$ of the singular cochain complex and an affinization map $\text{aff}^*\colon C^{p,\bullet}_{\text{sm},\mathbb{R}} \to C^{p,\bullet}_{\text{sing},\mathbb{R}}$. We abbreviate $\widetilde{\text{dR}} := \text{sh} \circ \text{aff}^* \circ \text{dR}$.

Lemma 3.3. *The composition* $\widetilde{\text{dR}}\colon \mathcal{A}^{p,\bullet} \to \underline{C}^{p,\bullet}_{\text{sing},\mathbb{R}}$ *is a quasi-isomorphism.*

Proof. We have seen that each of sh, aff^*, and dR are chain maps, so the same is true of $\widetilde{\text{dR}}$. Both $\mathcal{A}^{p,\bullet}$ and $C^{p,\bullet}_{\text{sing},\mathbb{R}}$ are exact in degrees $q > 0$; accordingly, both have trivial q-th cohomology, and there is nothing to prove. At $q = 0$, the cohomology of each complex is equal to the kernel of the coboundary map, which we identify with $\Omega^p_{X,\mathbb{R}}$ as in §3.1.3. It is left to us to show that $\widetilde{\text{dR}}$ in fact identifies $\Omega^p_{X,\mathbb{R}} \subseteq \mathcal{A}^{p,0}$ with $\Omega^p_{X,\mathbb{R}} \subseteq \underline{C}^{p,0}_{\text{sing},\mathbb{R}} \cong C^{p,0}_{\text{sing},\mathbb{R}}$, i.e., that the following diagram commutes:

$$\begin{tikzcd}
\Omega^p_{X,\mathbb{R}} \arrow[r, hook] \arrow[dr, hook] & \mathcal{A}^{p,0} \arrow[d, "\text{aff}^* \circ \text{dR}"] \\
 & C^{p,0}_{\text{sing},\mathbb{R}}
\end{tikzcd} \tag{3.7}$$

Indeed, on any chart U, a real-valued tropical p-form $r\omega \in \Omega_{X,\mathbb{R}}(U)$ corresponds to the $(p,0)$-superform with representative $r \otimes \omega \otimes 1$. Then because $\mathrm{aff}(v \otimes x) = v \otimes x$ whenever x is a 0-simplex, we find that

$$\begin{aligned}\langle v \otimes x, \mathrm{aff}^* \circ \mathrm{dR}(r \otimes \omega \otimes 1)\rangle &= \int_{v\otimes x} r \otimes \omega \otimes 1 \\ &= \langle v, \omega\rangle \int_x r \\ &= \langle v \otimes x, r\omega\rangle.\end{aligned}$$

Because $v \otimes x$ was arbitrary, $\mathrm{aff}^* \circ \mathrm{dR}(r \otimes \omega \otimes 1)$ and $r\omega$ agree in $C^{p,0}_{\mathrm{sing},\mathbb{R}}$, as desired. □

Theorem 3.4. *The induced maps on (co)homology given by* $\mathrm{aff}^* \circ \mathrm{dR}\colon H^{p,q}_{d''}(X) \to H^{p,q}_{\mathrm{sing}}(X;\mathbb{R})$ *and* $\mathrm{dR}^\vee \circ \mathrm{aff}\colon H^{\mathrm{sing,lf}}_{p,q}(X;\mathbb{R}) \to H^{p,q}_{d'',c}(X)^\vee$ *are isomorphisms.*

Proof. The global sections functor $\Gamma(\cdot, X)$ is left-exact; by [JSS19, Lemmas 2.15 and 3.14], $\mathcal{A}^{p,q}$ is fine and $\underline{C}^{p,q}_{\mathrm{sing},\mathbb{R}}$ is flasque, hence both are $\Gamma(\cdot,X)$-acyclic. By a well-known result of homological algebra, it follows that $\Gamma(\cdot, X)$ sends $\widetilde{\mathrm{dR}}\colon \mathcal{A}^{p,\bullet} \to \underline{C}^{p,\bullet}_{\mathrm{sing},\mathbb{R}}$, which is a quasi-isomorphism by Lemma 3.3, to a quasi-isomorphism $\mathcal{A}^{p,\bullet}(X) \to \underline{C}^{p,\bullet}_{\mathrm{sing}}(X;\mathbb{R})$. Consequently, $\widetilde{\mathrm{dR}}$ induces an isomorphism $H^{p,q}_{d''}(X) \to \underline{H}^{p,q}_{\mathrm{sing}}(X;\mathbb{R})$. From §2.3.2, we recall that sh induces an isomorphism $H^{p,q}_{\mathrm{sing}}(X;\mathbb{R}) \to \underline{H}^{p,q}_{\mathrm{sing}}(X;\mathbb{R})$. Since $\widetilde{\mathrm{dR}} = \mathrm{sh} \circ (\mathrm{aff}^* \circ \mathrm{dR})$, it follow that $\mathrm{aff}^* \circ \mathrm{dR}\colon H^{p,q}_{d''}(X) \to H^{p,q}_{\mathrm{sing}}(X;\mathbb{R})$ must itself be an isomorphism, as desired.

Following the same arguments for $\mathcal{A}^{p,\bullet}_c$ and $\underline{C}^{p,\bullet}_{\mathrm{sing},c,\mathbb{R}}$ (defined in [JSS19, Definition 3.4]) and dualizing, we obtain the second isomorphism. □

3.2 Isomorphism of cohomology rings

We now upgrade the tropical de Rham isomorphism $\mathrm{aff}^* \circ \mathrm{dR}$ on cohomology to an isomorphism of graded rings. While $H^{\bullet,\bullet}_{d''}(X)$ has a graded ring structure inherited from the wedge product of superforms (2.8), we must first define a graded ring structure on $H^{\bullet,\bullet}_{\mathrm{sing}}(X;\mathbb{R})$ that mimics the classical cup product.

3.2.1 *Cup and cap products*

Let $\mathrm{fr}_q\colon \Delta^q \to \Delta^{q+q'}$ and $\mathrm{ba}_{q'}\colon \Delta^{q'} \to \Delta^{q+q'}$ be the inclusion maps of the front and back faces of $\Delta^{q+q'}$, respectively. Fix $\alpha \in C^{p,q}_{\mathrm{sing}}(X;R)$ and $\beta \in C^{p',q'}_{\mathrm{sing}}(X;R)$. Using the notation of (2.14), we let

$$(\alpha \smile \beta)_f := (-1)^{qp'} \alpha_{f\circ \mathrm{fr}_q} \wedge \beta_{f\circ \mathrm{ba}_{q'}}. \tag{3.8}$$

This defines a $(p+p', q+q')$-cochain $\alpha \smile \beta$. Extending bilinearly, we obtain the *(tropical) cup product* $C^{p,q}_{\mathrm{sing}}(X;R) \otimes C^{p',q'}_{\mathrm{sing}}(X;R) \to C^{p+p',q+q'}_{\mathrm{sing}}(X;R)$ on singular tropical cochains. Following the standard argument for the cup product of singular chains, one may verify that

$$\partial^*(\alpha \smile \beta) = \partial^*\alpha \smile \beta + (-1)^{p+q} \alpha \smile \partial^*\beta. \tag{3.9}$$

Consequently, the cup product descends to cohomology.

Likewise, we define the *(tropical) cap product* $C^{\mathrm{sing}}_{p+p',q+q'}(X;R) \otimes C^{p,q}_{\mathrm{sing}}(X;R) \to C^{\mathrm{sing}}_{p',q'}(X;R)$ via

$$(v \otimes f) \frown \alpha := (-1)^{qp'} v \frown \alpha_{f\circ \mathrm{fr}_q} \otimes f \circ \mathrm{ba}_{q'}, \tag{3.10}$$

where the $\frown$ inside the definition is the contraction defined as an adjoint to $\wedge$ by (2.4). Using this property and (3.8) together, one finds that the cap product itself is adjoint to the cup product, in the sense that

$$\langle (v \otimes f) \frown \alpha, \beta \rangle = \langle v \otimes f, \alpha \smile \beta \rangle \tag{3.11}$$

for all $\beta \in C^{p',q'}_{\mathrm{sing}}(X;R)$. Then it follows immediately that

$$(v \otimes f) \frown \alpha \frown \beta = (v \otimes f) \frown (\alpha \smile \beta) \tag{3.12}$$

wherever this makes sense, i.e., $\beta \in C^{p'',q''}_{\mathrm{sing}}(X;R)$ for $p'' \leq p'$ and $q'' \leq q'$. Likewise, it follows formally from (3.9) and (3.11) that

$$\partial((v \otimes f) \frown \alpha) = (-1)^{p+q}\big(\partial(v \otimes f) \frown \alpha - (v \otimes f) \frown \partial^*\alpha\big),$$

hence the cap product descends to (co)homology.

If $c = \sum_i v_i \otimes f_i$ is locally finite, then we define $c \frown \alpha$ as the locally finite chain defined by $\sum_i (v_i \otimes f_i) \frown \alpha$ using (3.10). This defines a map $C^{\mathrm{sing,lf}}_{p+p',q+q'}(X;R) \otimes C^{p,q}_{\mathrm{sing}}(X;R) \to C^{\mathrm{sing,lf}}_{p',q'}(X;R)$ that satisfies all the same properties as the finite cap product.

3.2.2 Tropical de Rham isomorphism preserves the graded ring structure

We introduce the derived category of X following [GS23a, §4], although note that we work over $\underline{\mathbb{R}}$ rather than $\underline{\mathbb{Z}}$. Let $\mathcal{K}(\underline{\mathbb{R}})$ and $\mathcal{D}(\underline{\mathbb{R}})$ denote respectively the homotopy and derived categories of chain complexes of $\underline{\mathbb{R}}$-modules on X. Given a complex $(A^\bullet, \delta_A)$, we let $A^\bullet[q]$ denote the shifted complex with entries $(A^\bullet[q])^i = A^{q+i}$ and coboundary map $(-1)^q\delta_A$. In what follows, we conflate a sheaf with the corresponding complex concentrated in degree 0.

We may identify the sheaf cohomology $H^q(X, \Omega^p_{X,\mathbb{R}})$ with $\mathrm{Hom}_{\mathcal{D}(\underline{\mathbb{R}})}(\underline{\mathbb{R}}, \Omega^p_{X,\mathbb{R}}[q])$. Since $\mathcal{A}^{p,\bullet}$ is a $\Gamma(\cdot, X)$-acyclic resolution of $\Omega^p_{X,\mathbb{R}}$, a cohomology class $\alpha \in H^{p,q}_{d''}(X)$ determines a roof diagram in $\mathcal{K}(\underline{\mathbb{R}})$, and hence a morphism in $\mathcal{D}(\underline{\mathbb{R}})$, via

$$\underline{\mathbb{R}} \xrightarrow{\alpha} \mathcal{A}^{p,\bullet}[q] \xleftarrow{\simeq} \Omega^p_{X,\mathbb{R}}[q]. \tag{3.13}$$

The first map is determined by the map of global sections $\mathbb{R} \to \mathcal{A}^{p,q}(X)$ sending $1 \mapsto \alpha$, and is independent of the choice of representative for α up to chain homotopy; the second map, a quasi-isomorphism, is the resolution described in §3.1.3. A singular tropical cohomology class $\alpha \in H^{p,q}_{\mathrm{sing}}(X;\mathbb{R})$ likewise determines a roof diagram.

The *derived cup product* $H^q(X, \Omega^p_{X,\mathbb{R}}) \otimes H^{q'}(X, \Omega^{p'}_{X,\mathbb{R}}) \to H^{q+q'}(X, \Omega^{p+p'}_{X,\mathbb{R}})$ sending $\alpha \otimes \beta \mapsto \alpha \smile \beta$ is defined by the following composition in $\mathcal{D}(\underline{\mathbb{R}})$:

$$\underline{\mathbb{R}} \xrightarrow{\cong} \underline{\mathbb{R}} \otimes \underline{\mathbb{R}} \xrightarrow{\alpha\otimes\beta} \Omega^p_{X,\mathbb{R}} \otimes \Omega^{p'}_{X,\mathbb{R}}[q+q'] \xrightarrow{\wedge} \Omega^{p+p'}_{X,\mathbb{R}}[q+q']. \tag{3.14}$$

Lemma 3.5. *The wedge product* $\wedge\colon H^{p,q}_{d''}(X) \otimes H^{p',q'}_{d''}(X) \to H^{p+p',q+q'}_{d''}(X)$ *of cohomology classes of superforms coincides with the derived cup product of the corresponding morphisms in* $\mathcal{D}(\underline{\mathbb{R}})$.

Proof. Fix $\alpha \in H^{p,q}_{d''}(X)$ and $\beta \in H^{p',q'}_{d''}(X)$. Each class α, β, and $\alpha \wedge \beta$ naturally corresponds to a roof diagram in the manner of (3.13). Meanwhile, we may construct the derived cup product $\alpha \smile \beta$ according to (3.14). Comparing $\alpha \wedge \beta$ with this composition leads to the following diagram in the derived category:

$$\begin{tikzcd} \underline{\mathbb{R}} \otimes \underline{\mathbb{R}} \arrow[r, "\alpha\otimes\beta"] & \mathcal{A}^{p,\bullet}(X) \otimes \mathcal{A}^{p',\bullet}(X)[q+q'] \arrow[d, "\wedge"] & \Omega^{p}_{X,\mathbb{R}} \otimes \Omega^{p'}_{X,\mathbb{R}}[q+q'] \arrow[l, "\simeq"'] \arrow[d, "\wedge"] \\ \underline{\mathbb{R}} \arrow[u, "\cong"] \arrow[r, "\alpha\wedge\beta"] & \mathcal{A}^{p+p',\bullet}(X)[q+q'] & \Omega^{p+p'}_{X,\mathbb{R}}[q+q'] \arrow[l, "\simeq"'] \end{tikzcd} . \tag{3.15}$$

The desired result follows if we can show that this diagram commutes in $\mathcal{K}(\underline{\mathbb{R}})$, since this would imply that the morphisms $\underline{\mathbb{R}} \to \Omega^{p+p'}_{X,\mathbb{R}}[q+q']$ in $\mathcal{D}(\underline{\mathbb{R}})$ given by following either the top row or the bottom are the same.

Commutativity of the left square in (3.15) is trivial. For the right, one must check that the wedge product of tropical forms is compatible with the wedge product of the corresponding superforms. Indeed, fix a chart U and tropical forms $r\omega \in \Omega^{p}_{X,\mathbb{R}}(U)$ and $r'\omega' \in \Omega^{p'}_{X,\mathbb{R}}(U)$. Then by §3.1.3, their corresponding superforms are $r \otimes \omega \otimes 1$ and $r' \otimes \omega' \otimes 1$. From (2.8), we see that the wedge product of these superforms is $rr' \otimes \omega \wedge \omega' \otimes 1$, which is precisely the superform corresponding to $r\omega \wedge r'\omega' \in \Omega^{p+p'}_{X,\mathbb{R}}(U)$. □

Lemma 3.6. *The cup product* $\smile\colon H^{p,q}_{\mathrm{sing}}(X;\mathbb{R}) \otimes H^{p',q'}_{\mathrm{sing}}(X;\mathbb{R}) \to H^{p+p',q+q'}_{\mathrm{sing}}(X;\mathbb{R})$ *of singular tropical cohomology classes coincides with the derived cup product of the corresponding morphisms in* $\mathcal{D}(\underline{\mathbb{R}})$.

Proof. By analogy with Lemma 3.5, it suffices to prove commutativity of the diagram in (3.15) with $\underline{C}^{p,\bullet}_{\mathrm{sing}}(X;\mathbb{R})$ taking the place of $\mathcal{A}^{p,\bullet}(X)$, etc., and $\smile$ replacing $\wedge$. Once again, commutativity of the left square is straightforward. For the right, we need to check that the wedge product of tropical forms $\omega \in \Omega^{p}_{X}(U)$ and $\omega' \in \Omega^{p'}_{X}(U)$ is compatible with the cup product of the corresponding singular tropical cohomology classes, which we also denote by ω and ω', respectively. Indeed, for all $x \in X$ and $v \in T^{p}_{X}(\sigma_x)$, we have by (3.6) and (3.8) that

$$\langle v, (\omega \smile \omega')_x \rangle = \langle v, \omega_x \wedge \omega'_x \rangle = \langle v, \omega \wedge \omega' \rangle = \langle v, (\omega \wedge \omega')_x \rangle.$$

This implies that $\omega \smile \omega'$ and $\omega \wedge \omega'$ agree inside of $\underline{C}^{p+p',0}_{\mathrm{sing}}(U)$, as desired. $\square$

Theorem 3.7. *The isomorphism on cohomology $H^{p,q}_{d''}(X;\mathbb{R}) \to H^{p,q}_{\mathrm{sing}}(X;\mathbb{R})$ induced by $\mathrm{aff}^* \circ \mathrm{dR}$ takes the wedge product to the cup product.*

Proof. By Lemmas 3.5 and 3.6, the wedge product and cup product each agree with the derived cup product. Therefore, it suffices to show that the map that $\widetilde{\mathrm{dR}}$ induces on the derived category is the identity. In other words, we need to check that the following diagram commutes in $\mathcal{K}(\underline{\mathbb{R}})$:

$$\begin{tikzcd} \underline{\mathbb{R}} \arrow[r] \arrow[dr] & \mathcal{A}^{p,\bullet}[q] \arrow[d, "\widetilde{\mathrm{dR}}"] & \Omega^p_{X,\mathbb{R}}[q] \arrow[l, "\simeq"'] \arrow[dl, "\simeq"'] \\ & \underline{C}^{p,\bullet}_{\mathrm{sing},\mathbb{R}}[q] & \end{tikzcd} .$$

The maps from $\underline{\mathbb{R}}$ define the cohomology classes, and so commute by construction. Commutativity on the right follows precisely from commutativity of the diagram in (3.7). $\square$

Corollary 3.8. *Let $\alpha_i \in H^{p_i,q_i}_{\mathrm{sing}}(X;\mathbb{R})$. Then*

$$\alpha_1 \smile \alpha_2 = (-1)^{(p_1+q_1)(p_2+q_2)} \alpha_2 \smile \alpha_1.$$

Moreover, for any chain $c \in H^{\mathrm{sing}}_{p,q}(X;\mathbb{R})$ with $p \geq p_1 + p_2$ and $q \geq q_1 + q_2$,

$$c \frown \alpha_1 \frown \alpha_2 = (-1)^{(p_1+q_1)(p_2+q_2)} c \frown \alpha_2 \frown \alpha_1.$$

Proof. The first statement follows immediately from Theorem 3.7 and the fact that wedge product itself is graded-commutative. The second statement follows from the first by (3.12). $\square$

3.3 Poincaré duality

Lemma 3.9. *Given $c \in H^{\mathrm{sing}}_{p+p',q+q'}(X;\mathbb{R})$ and $\alpha \in H^{p,q}_{d''}(X)$, we have that*

$$\mathrm{dR}^\vee \circ \mathrm{aff}(c \frown \mathrm{aff}^* \circ \mathrm{dR}(\alpha)) = \int_{\mathrm{aff}(c)} \alpha \wedge \cdot \tag{3.16}$$

in $H^{p',q'}_{d''}(X)^\vee$. If c is locally finite, then (3.16) holds in $H^{p',q'}_{d'',c}(X)^\vee$.

Proof. Fix $\beta \in H^{p',q'}_{d''}(X)$ (with compact support if c is locally finite). Using (3.11) and Theorem 3.7, we compute

$$\begin{aligned}\langle \mathrm{dR}^\vee \circ \mathrm{aff}(c \frown \mathrm{aff}^* \circ \mathrm{dR}(\alpha)), \beta\rangle &= \langle c \frown \mathrm{aff}^* \circ \mathrm{dR}(\alpha), \mathrm{aff}^* \circ \mathrm{dR}(\beta)\rangle \\ &= \langle c, (\mathrm{aff}^* \circ \mathrm{dR}(\alpha)) \smile (\mathrm{aff}^* \circ \mathrm{dR}(\beta))\rangle \\ &= \langle c, \mathrm{aff}^* \circ \mathrm{dR}(\alpha \wedge \beta)\rangle \\ &= \int_{\mathrm{aff}(c)} \alpha \wedge \beta,\end{aligned}$$

as desired. $\square$

Now we fix a balanced polyhedral space Z of dimension k. Let $\mathcal{T}$ be a triangulation of Z as in §2.4.8.

Lemma 3.10. *For any $\omega \in \mathcal{A}^{k,k}_c(Z)$, we have that*

$$\int_Z \omega = \int_{\mathrm{ch}_{\mathcal{T}}(Z)} \omega.$$

Proof. The integral defined by (2.19) may be computed using parameterizations and remains unchanged if we remove sets of measure zero. In particular, it remains unchanged if, instead of using partitions of unity, we sum over the integrals over each facet $\tau \in \mathcal{T}$. So

$$\int_Z \omega = \sum_{\tau \in \mathcal{T}} w_Z(\sigma_\tau) \int_\tau \omega, \tag{3.17}$$

where σ_τ is the maximal stratum of Z that contains $\mathrm{ri}(\tau)$. Likewise, by (2.22), we may write

$$\int_{\mathrm{ch}_{\mathcal{T}}(Z)} \omega = \sum_{\tau \in \mathcal{T}} w_Z(\tau) \int_{\nu_\tau \otimes f_\tau} \omega, \tag{3.18}$$

where we use the notation introduced in §2.4.8. The integrals in both (3.17) and (3.18) are invariant under subdivision of the domain, so we may assume without loss of generality that $\mathcal{T}$ is fine enough that every facet τ is contained in a single star neighborhood. After fixing an isomorphism $T^1_Z(\sigma_\tau) \cong \mathbb{Z}^k$, we obtain coordinates $x_1, \ldots, x_k$ on τ. Permuting the coordinates if necessary, we may assume that $f_\tau\colon \Delta^k \to \tau$ is positively-oriented with respect to the smooth orientation form $dx_{[k]}$ on $T^1_{Z,\mathbb{R}}(\sigma_\tau)$. It then suffices to show that

$$\int_\tau h d'x_{[k]} \wedge d''x_{[k]} = \int_{\nu_\tau \otimes f_\tau} h d'x_{[k]} \wedge d''x_{[k]}.$$

We may compute the integral over τ in terms of a smooth integral using the definition in (2.11). We then pull back to $\widetilde{\Delta}^k$ along the orientation-preserving parameterization $f_\tau \circ \phi\colon \widetilde{\Delta}^k \to \tau$, where ϕ is given by (2.15). This yields

$$\begin{aligned} \int_\tau h d'x_{[k]} \wedge d''x_{[k]} &= (-1)^{\binom{k}{2}} \int_\tau h dx_{[k]} \\ &= (-1)^{\binom{k}{2}} \int_{\widetilde{\Delta}^k} (f_\tau \circ \phi)^*(h dx_{[k]}). \end{aligned} \tag{3.19}$$

Meanwhile, expanding the integral over $\nu_\tau \otimes f_\tau$ using (3.1) and (3.2), we find that

$$\begin{aligned} \int_{\nu_\tau \otimes f_\tau} h d'x_{[k]} \wedge d''x_{[k]} &= \langle \nu_\tau, dx_{[k]} \rangle \int_{f_\tau} h dx_{[k]} \\ &= \langle \nu_\tau, dx_{[k]} \rangle \int_{\Delta^k} f_\tau^*(h dx_{[k]}) \\ &= \langle \nu_\tau, dx_{[k]} \rangle \int_{\widetilde{\Delta}^k} (f_\tau \circ \phi)^*(h dx_{[k]}). \end{aligned} \tag{3.20}$$

By definition, ν_τ is the generator of $T^k_Z(\sigma_\tau) \cong \mathbb{Z}$ equal to $c df_\tau(\nu)$ for some $c > 0$. But the orientation ν on Δ^k satisfies $\nu = (-1)^{\binom{k}{2}} d\phi(\widetilde{\nu})$, where $\widetilde{\nu}$ defines the orientation on $\widetilde{\Delta}^k$. It follows that

$$\begin{aligned} \langle \nu_\tau, dx_{[k]} \rangle &= \langle \nu_\tau, dx_{[k]} \rangle \\ &= (-1)^{\binom{k}{2}} c \langle d(f_\tau \circ \phi)(\widetilde{\nu}), dx_{[k]} \rangle \\ &= (-1)^{\binom{k}{2}} c \langle \widetilde{\nu}, (f_\tau \circ \phi)^*(dx_{[k]}) \rangle \\ &= (-1)^{\binom{k}{2}} cc', \end{aligned}$$

where $c' > 0$ because $f_\tau \circ \phi$ is orientation-preserving. Then $\langle \nu_\tau, dx_{[k]} \rangle \in \{\pm 1\}$ because both entries are generators, forcing $\langle \nu_\tau, dx_{[k]} \rangle = (-1)^{\binom{k}{2}}$. Consequently, (3.19) and (3.20) agree, as desired. $\square$

Corollary 3.11. *The real cycle class* $\operatorname{cyc}_{\mathbb{R}}(Z) \in H^{\mathrm{sing,lf}}_{k,k}(Z;\mathbb{R})$ *does not depend on the triangulation* $\mathcal{T}$*. If* $H^{\mathrm{sing,lf}}_{k,k}(Z;\mathbb{Z})$ *is torsion-free, then the same holds for* $\operatorname{cyc}(Z)$*.*

Proof. By Theorem 3.4, $\mathrm{dR}^\vee \circ \mathrm{aff}\colon C^{\mathrm{sing,lf}}_{k,k}(Z;\mathbb{R}) \to \mathcal{A}^{k,k}_c(Z)^\vee$ induces an isomorphism on homology. Because $\mathrm{ch}_{\mathcal{T}}(Z)$ is already affine by definition, its image under $\mathrm{dR}^\vee \circ \mathrm{aff}$ is the linear functional $\int_{\mathrm{ch}_{\mathcal{T}}(Z)} \cdot$. Given another triangulation $\mathcal{T}'$, we have by Lemma 3.10 that

$$\int_{\mathrm{ch}_{\mathcal{T}}(Z)} \cdot = \int_Z \cdot = \int_{\mathrm{ch}_{\mathcal{T}'}(Z)} \cdot.$$

In particular, the integrals determine the same homology class in $H^{k,k}_{d'',c}(Z)^\vee$. By injectivity, this forces $\mathrm{ch}_{\mathcal{T}}(Z)$ and $\mathrm{ch}_{\mathcal{T}'}(Z)$ themselves to agree in real homology, proving the first claim.

If $H^{\mathrm{sing,lf}}_{k,k}(Z;\mathbb{Z})$ is torsion free, then the natural map $H^{\mathrm{sing,lf}}_{k,k}(Z;\mathbb{Z}) \to H^{\mathrm{sing,lf}}_{k,k}(Z;\mathbb{R})$ is injective. Therefore, $\mathrm{ch}_{\mathcal{T}}(Z)$ and $\mathrm{ch}_{\mathcal{T}'}(Z)$ must also agree in integral homology, proving the second claim. $\square$

Following [JSS19, Definition 4.11], there is a Poincaré duality map $\mathrm{PD}\colon \mathcal{A}^{p,q}(Z) \to \mathcal{A}^{k-p,k-q}(Z)^\vee$ on superforms defined by

$$\mathrm{PD}(\alpha) := \epsilon_{p,q} \int_Z \alpha \wedge \cdot,$$

where $\epsilon_{p,q}$ is a sign chosen so that $\epsilon_{p,q} = \epsilon_{p,q+1}(-1)^{p+q+1}$. The sign is necessary in order to ensure that PD descends to a chain map, and results directly from the Stokes formula in Lemma 3.2 and the Leibniz formula in (2.9). By [JSS19, Theorem 2], PD descends to an isomorphism on (co)homology.

We define a corresponding map $\mathrm{PD}'_{\mathcal{T}}\colon C^{p,q}_{\mathrm{sing}}(Z;R) \to C^{\mathrm{sing,lf}}_{k-p,k-q}(Z;R)$ on R-valued singular tropical cochains by

$$\mathrm{PD}'_{\mathcal{T}}(\alpha) := \epsilon_{p,q}\, \mathrm{ch}_{\mathcal{T}}(Z) \frown \alpha.$$

That we can take the same sign $\epsilon_{p,q}$ follows from the fact that we chose our cup product on singular tropical cochains so that the corresponding Leibniz formula given by (3.9) has the same sign as (2.9). In (co)homology, $\mathrm{PD}'_{\mathcal{T}}(\alpha) = \epsilon_{p,q}\operatorname{cyc}(Z) \frown \alpha$, where we have used Corollary 3.11. We therefore drop the $\mathcal{T}$ subscript and just write PD'. Although we will not need it, we note that this map agrees with the Poincaré duality map on the derived category described in [GS23a, Corollary 6.9].

Theorem 3.12. *The following diagram commutes:*

$$\begin{tikzcd} H^{p,q}_{d''}(Z) \arrow[r, "\mathrm{aff}^* \circ \mathrm{dR}"] \arrow[d, "\mathrm{PD}"] & H^{p,q}_{\mathrm{sing}}(Z;\mathbb{R}) \arrow[d, "\mathrm{PD}'"] \\ H^{k-p,k-q}_{d'',c}(Z)^\vee & H^{\mathrm{sing,lf}}_{k-p,k-q}(Z;\mathbb{R}) \arrow[l, "\mathrm{dR}^\vee \circ \mathrm{aff}"'] \end{tikzcd} .$$

Proof. This follows immediately from Lemmas 3.9 and 3.10. □

Chapter 4

INTEGRATION OF CYCLES ON TROPICAL JACOBIANS

This chapter is based on joint work with Junaid Hasan and Farbod Shokrieh. We use the tropical de Rham formalism from Chapter 3 to define a canonical $(1,1)$-form ν_ϑ on a polarized tropical abelian variety (X,ϑ). In the case that X is the tropical Jacobian of a metric graph Γ, we connect ν_ϑ to existing constructions on Γ and show that integrating ν_ϑ and its powers over tautological cycles in $\mathrm{Jac}(\Gamma)$ leads to interesting connections between geometry and graph theory. For our purposes, it suffices to work with affine tropical chains, but all statements involving $H^{\mathrm{aff}}_{\bullet,\bullet}(X;R)$ hold equally true for $H^{\mathrm{sing}}_{\bullet,\bullet}(X;R)$.

4.1 Canonical superform

Fix a polarized tropical abelian variety (X,ϑ), where X is the real torus $N_\mathbb{R}/\Lambda$ with integral structure N and dimension g.

4.1.1 Distinguished bases for polarized tropical abelian varieties

We recall from §2.6.3 the basic notation associated with polarizations. Because ϑ is injective and N and Λ have equal rank, the quotient group $\Lambda^\vee/\vartheta(N)$ is torsion. We write $e_1,\ldots,e_g$ for the elementary divisors of ϑ and $\mathbf{E}$ for the corresponding diagonal matrix. The product $\deg\vartheta := \prod_{i=1}^g e_i$ is called the *degree* of ϑ; one sees that ϑ is principal just if $\deg\vartheta = 1$. By writing ϑ in Smith normal form, we obtain bases $n_1,\ldots,n_g$ of N and $\lambda_1,\ldots,\lambda_g$ of Λ satisfying $\vartheta(n_i) = e_i\lambda_i^\vee$ for all i, hence

$$[n_i,\lambda_j] = e_i\delta_{i,j} = \mathbf{E}_{i,j}. \tag{4.1}$$

We choose coordinates $x_1,\ldots,x_g$ on the universal cover $N_\mathbb{R}$ associated to the basis $n_1,\ldots,n_g$. The corresponding tropical 1-forms $dx_1,\ldots,dx_g$, which we identify with smooth forms in the obvious way, descend to X.

Let $\mathbf{G}$ denote the Gram matrix of $[\cdot,\cdot]$ with respect to the basis for Λ. Then $\mathbf{G}$ is symmetric and positive-definite with entries

$$\mathbf{G}_{i,j} = [\lambda_i, \lambda_j] \tag{4.2}$$

Since $\Lambda \subset N_{\mathbb{R}}$, we may write each λ_j as a linear combination of the n_i; one may apply (4.1) and (4.2) to check that

$$\lambda_j = \sum_{i=1}^{g} (\mathbf{E}^{-1}\mathbf{G})_{i,j} n_i. \tag{4.3}$$

In the case that $X = \mathrm{Jac}(\Gamma)$ for some metric graph Γ according to §2.6.4, we choose our bases as follows. Fix a basis $\gamma_1, \ldots, \gamma_g$ for $H_1(G;\mathbb{Z})$ and let $n_1, \ldots, n_g$ denote the corresponding dual basis for $N = H^1(G;\mathbb{Z})$. Define also a basis for Λ via $\lambda_i := \pi(\gamma_i)$. Then by (2.34),

$$[n_i, \lambda_j] = \langle \lambda_j, n_i \rangle = \delta_{i,j};$$

in particular, (4.1) holds for our choice of bases with $\mathbf{E} = \mathbf{I}$. In this setting, we may regard $\mathbf{G}$ as the matrix representing the linear map $\pi|_{H_1(G;\mathbb{Z})} \colon H_1(G;\mathbb{Z}) \to N_{\mathbb{R}}$ with respect to the bases $\gamma_1, \ldots, \gamma_g$ and $n_1, \ldots, n_g$, respectively. By (2.35) and (4.2), it has entries $\mathbf{G}_{i,j} = [\gamma_i, \gamma_j]_G$.

4.1.2 Defining the canonical superform

Recall that we may canonically identify N with the integral tangent space $T^1_{X,x}$ at any point $x \in X$. Likewise, we have that $N^\vee \cong \Omega^1_{X,x} \cong \Omega^1_X(X)$.

We extend the diagram in (2.32) as follows:

$$\begin{tikzcd}
N^\vee \otimes \Lambda^\vee \arrow[r, "\cong"] \arrow[d, hook] & \mathrm{Hom}(N, \Lambda^\vee) \arrow[r, "\cong"] & \mathrm{Hom}(N \otimes \Lambda, \mathbb{Z}) \arrow[d, hook] & H^{1,1}_{\mathrm{aff}}(X;\mathbb{Z}) \arrow[l, "\cong"'] \arrow[d, hook] \\
\mathbb{R} \otimes N^\vee \otimes \Lambda^\vee \arrow[rr, "\cong"] \arrow[d, "\cong"] & & \mathrm{Hom}(N \otimes \Lambda, \mathbb{R}) & H^{1,1}_{\mathrm{aff}}(X;\mathbb{R}). \arrow[l, "\cong"'] \\
\mathbb{R} \otimes N^\vee \otimes N^\vee \arrow[r, "\cong"] & H^{1,1}_{d''}(X) \arrow[urr, bend right, "\cong", "\mathrm{dR}"'] & &
\end{tikzcd} \tag{4.4}$$

The second row is obtained from the first by tensoring by $\mathbb{R}$, so we see that the upper half of the diagram is commutative. Meanwhile, the isomorphism $\mathbb{R} \otimes N^\vee \otimes \Lambda^\vee \to \mathbb{R} \otimes N^\vee \otimes N^\vee$ is given by identifying $\Lambda_{\mathbb{R}}$ with $N_{\mathbb{R}}$; using (4.3), one finds that, with respect to the distinguished bases in §4.1.1, the composition of this map with the following one to $H^{1,1}_{d''}(X)$ is given by

$$1 \otimes n_i^\vee \otimes \lambda_j^\vee \mapsto \sum_k (\mathbf{G}^{-1}\mathbf{E})_{j,k} \otimes n_i^\vee \otimes n_k^\vee \mapsto \sum_k (\mathbf{G}^{-1}\mathbf{E})_{j,k} d'x_i \wedge d''x_k. \tag{4.5}$$

We recall the definition of the integration pairing in (3.1). One checks that $\mathrm{dR}(d'x_i \wedge d''x_k)$ is represented by the affine $(1,1)$-cochain sending

$$n \otimes [v_0, v_1] \mapsto \langle n, dx_i \rangle \int_{[v_0,v_1]} dx_k = \langle n, dx_i \rangle \langle v_1 - v_0, dx_k \rangle.$$

The tropical de Rham map dR is $\mathbb{R}$-linear, so this computation may be applied to (4.5). Then it is straightforward to show that following the diagram in (4.4) from $\mathbb{R} \otimes N^\vee \otimes \Lambda^\vee$ to $\mathrm{Hom}(N \otimes \Lambda, \mathbb{R})$ along the bottom row sends $1 \otimes n_i^\vee \otimes \lambda_j^\vee$ to the linear functional

$$n \otimes \lambda \mapsto \sum_k (\mathbf{G}^{-1}\mathbf{E})_{j,k} \langle n, dx_i \rangle \langle \lambda, dx_k \rangle.$$

One shows using basis elements along with (4.3) that this agrees with the linear functional $n \otimes \lambda \mapsto \langle n, n_i^\vee \rangle \langle \lambda, \lambda_j^\vee \rangle$, which is the image of $1 \otimes n_i^\vee \otimes \lambda_j^\vee$ along the other map. This proves commutativity of the lower part of the diagram. The fact that any superform cohomology $(1,1)$-class on X is uniquely identified by a representative $\mathbb{R} \otimes N^\vee \otimes N^\vee$ follows immediately (although this can also be shown directly).

In (4.4), the polarization $\vartheta \in \mathrm{Hom}(N, \Lambda^\vee)$ corresponds to the bilinear form $[\cdot,\cdot] \in \mathrm{Hom}(N \otimes \Lambda, \mathbb{Z})$, the affine cohomology class $h_\vartheta \in H^{1,1}_{\mathrm{aff}}(X;\mathbb{Z})$, and a superform cohomology class $\nu_\vartheta \in H^{1,1}_{d''}(X)$. By abuse of notation, the symbol ν_ϑ will also denote the particular representative in $\mathbb{R} \otimes N^\vee \otimes N^\vee \subset \mathcal{A}^{1,1}(X)$, as follows. In $N^\vee \otimes \Lambda^\vee$, ϑ takes the form

$$\sum_{i=1}^{g} e_i n_i^\vee \otimes \lambda_i^\vee.$$

Applying the computation in (4.5), we find that

$$\nu_\vartheta = \sum_{i,j} (\mathbf{E}\mathbf{G}^{-1}\mathbf{E})_{i,j} d'x_i \wedge d''x_j. \tag{4.6}$$

We remark that ν_ϑ parallels a similar construction for complex abelian varieties; see, e.g., [Wil17, §2.1].

4.1.3 Powers of the canonical superform

Before we specialize to the tropical Jacobian case, we compute the higher powers of the canonical superform and show that the integral of $\nu_\vartheta^{\wedge g}$ over X recovers $\deg \vartheta$.

Lemma 4.1. *We have that*

$$\nu_\vartheta^{\wedge d} = (-1)^{\binom{d}{2}} d! \sum_{I,J \in \binom{[g]}{d}} \det((\mathbf{E}\mathbf{G}^{-1}\mathbf{E})_{I,J}) d'x_I \wedge d''x_J.$$

Proof. To simplify notation, let $\mathbf{A} := \mathbf{E}\mathbf{G}^{-1}\mathbf{E}$. Then

$$\begin{aligned}
\nu_\vartheta^{\wedge d} &= \bigwedge_{\alpha=1}^{d} \sum_{i_\alpha, j_\alpha} \mathbf{A}_{i_\alpha, j_\alpha} d'x_{i_\alpha} \wedge d''x_{j_\alpha} \\
&= \left(\sum_{i_1, j_1} \mathbf{A}_{i_1, j_1} d'x_{i_1} \wedge d''x_{j_1} \right) \wedge \ldots \wedge \left(\sum_{i_d, j_d} \mathbf{A}_{i_d, j_d} d'x_{i_d} \wedge d''x_{j_d} \right) \\
&= \sum_{i_1, j_1, \ldots, i_d, j_d} \left(\prod_{\alpha=1}^{d} \mathbf{A}_{i_\alpha, j_\alpha} \right) \bigwedge_{\alpha=1}^{d} d'x_{i_\alpha} \wedge d''x_{j_\alpha} \\
&= (-1)^{\binom{d}{2}} \sum_{i_1, j_1, \ldots, i_d, j_d} \left(\prod_{\alpha=1}^{d} \mathbf{A}_{i_\alpha, j_\alpha} \right) \left(\bigwedge_{\alpha=1}^{d} d'x_{i_\alpha} \right) \wedge \left(\bigwedge_{\alpha=1}^{d} d''x_{j_\alpha} \right).
\end{aligned}$$

Each (d, d)-superform is nonzero precisely when all of the i_α are distinct and all of the j_α are distinct. Therefore, we may reindex $i_\alpha \mapsto i_{\mathfrak{s}(\alpha)}$ and $j_\alpha \mapsto j_{\mathfrak{t}(\alpha)}$ over all *ascending* tuples $I = (i_1, \ldots, i_d)$ and $J = (j_1, \ldots, j_d)$ in $\binom{[g]}{d}$ and all permutations $\mathfrak{s}, \mathfrak{t} \in \mathfrak{S}_d$ and then reorder to obtain

$$\begin{aligned}
&= (-1)^{\binom{d}{2}} \sum_{I,J,\mathfrak{s},\mathfrak{t}} \left(\prod_{\alpha=1}^{d} \mathbf{A}_{i_{\mathfrak{s}(\alpha)}, j_{\mathfrak{t}(\alpha)}} \right) d'x_{I^{\mathfrak{s}}} \wedge d''x_{J^{\mathfrak{t}}} \\
&= (-1)^{\binom{d}{2}} \sum_{I,J,\mathfrak{s}} \operatorname{sgn}(\mathfrak{s}) \left(\sum_{\mathfrak{t}} \operatorname{sgn}(\mathfrak{t}) \prod_{\alpha=1}^{d} \mathbf{A}_{i_{\mathfrak{s}(\alpha)}, j_{\mathfrak{t}(\alpha)}} \right) d'x_I \wedge d''x_J.
\end{aligned}$$

Applying properties of the determinant and using the fact that $\#\mathfrak{S}_d = d!$, we find that

$$= (-1)^{\binom{d}{2}} \sum_{I,J,\mathfrak{s}} \operatorname{sgn}(\mathfrak{s}) \det(\mathbf{A}_{I^{\mathfrak{s}},J}) d'x_I \wedge d''x_J$$

$$
\begin{aligned}
&= (-1)^{\binom{d}{2}} \sum_{I,J,\mathfrak{s}} \operatorname{sgn}(\mathfrak{s})^2 \det(\mathbf{A}_{I,J}) d'x_I \wedge d''x_J \\
&= (-1)^{\binom{d}{2}} \sum_{I,J,\mathfrak{s}} \det(\mathbf{A}_{I,J}) d'x_I \wedge d''x_J \\
&= (-1)^{\binom{d}{2}} d! \sum_{I,J} \det(\mathbf{A}_{I,J}) d'x_I \wedge d''x_J. \qquad \square
\end{aligned}
$$

The following computation mimics the corresponding classical results of [CL06, §2.3] and [Zha95, Theorem 1.4 and Theorem 1.7].

Proposition 4.2. *We have that*

$$\int_X \frac{1}{g!} \nu_\vartheta^{\wedge g} = \deg \vartheta.$$

Proof. Recall that the coordinates $x_1, \dots, x_g$ on $N_\mathbb{R}$ are defined in terms of the integral basis $n_1, \dots, n_g$. We fix half-open parallelepipeds $P_N := \sum_j [0, n_j)$ and $P_\Lambda := \sum_j [0, \lambda_j)$ in $N_\mathbb{R}$. Notice that P_N is the unit cube with respect to the smooth volume form $dx_{[g]}$ while P_Λ is a fundamental domain for X. The linear automorphism $N_\mathbb{R} \to N_\mathbb{R}$ sending $n_j \mapsto \lambda_j$ takes P_N to P_Λ; by the coordinate transformation in (4.3), it is represented in the $n_1, \dots, n_g$ basis by the matrix $\mathbf{E}^{-1}\mathbf{G}$. Using Lemma 4.1 to rewrite $\nu_\vartheta^{\wedge g}$ and (2.11) to reduce the superform integral to a smooth integral, we compute

$$
\begin{aligned}
\int_X \frac{1}{g!} \nu_\vartheta^{\wedge g} &= \det(\mathbf{E}\mathbf{G}^{-1}\mathbf{E})(-1)^{\binom{g}{2}} \int_X d'x_{[g]} \wedge d''x_{[g]} \\
&= \det(\mathbf{E}\mathbf{G}^{-1}\mathbf{E})(-1)^{\binom{g}{2}} \int_{P_\Lambda} d'x_{[g]} \wedge d''x_{[g]} \\
&= \det(\mathbf{E}\mathbf{G}^{-1}\mathbf{E}) \int_{P_\Lambda} dx_{[g]} \\
&= \det(\mathbf{E}\mathbf{G}^{-1}\mathbf{E}) \int_{P_N} \det(\mathbf{E}^{-1}\mathbf{G}) dx_{[g]} \\
&= \det(\mathbf{E}) \operatorname{Vol}(P_N) \\
&= \deg \vartheta. \qquad \square
\end{aligned}
$$

4.2 Connection to the Riemann theta function

4.2.1 Tropical cycles with smooth weights

Let X be a weighted, pure polyhedral space. As we alluded to in §2.4.1, we may identify w_X with the data of the weights $w_X(\sigma) \in \mathbb{Z}^\times$ on each maximal stratum σ of X. In [GK17, §1], the authors generalize this idea to allow for smooth weights. We give only a brief exposition here.

For each σ, fix a continuous function $w_\sigma \colon \overline{\sigma} \to \mathbb{R}$ that is smooth in every chart. Then the balancing condition defined at a stratum τ of codimension 1 for constant weights in (2.18) is replaced by the condition that

$$\sum_{\sigma \succ \tau} w_\sigma(x) n_{\sigma/\tau} \in L_{\mathbb{R}}(\tau)$$

for all $x \in \overline{\tau}$. We call X a *tropical cycle with smooth weights* if it satisfies this condition at every such τ.

Now suppose that $\phi \colon X \to \mathbb{R}$ is *piecewise-smooth*, in the sense that X has at every point a local face structure $\mathcal{C}$ for which $\phi|_\sigma$ is smooth for all every face $\sigma \in \mathcal{C}$. Then the same formalism for rational functions developed in §2.4.3 works for piecewise-smooth functions. In particular, choosing a smooth extension ϕ_σ of $\phi|_\sigma$ to $T^1_{X,\mathbb{R}}(\sigma)$ for each face $\sigma \in \mathcal{C}$, we can define a Weil divisor $X \cdot \operatorname{div}(\phi)$ on the codimension-1 skeleton of $\mathcal{C}$ with smooth weights

$$w_\tau(x) := \sum_{\sigma \succ \tau} w_\sigma(x) \langle n_{\sigma/\tau}, d_x\phi_\tau - d_x\phi_\sigma \rangle. \tag{4.7}$$

Here, w_σ is an abuse of notation that refers to the smooth weight function on the unique maximal stratum of X that contains $\operatorname{ri}(\sigma)$. Specializing to the case where X is a real torus with integral structure and constant weight $w_X \equiv 1$, each face $\tau \in \mathcal{C}$ of codimension 1 is contained in exactly two facets σ and σ'. Then $n_{\sigma'/\tau} = -n_{\sigma/\tau}$, and we can always choose $f_\tau = f_\sigma$. Then (4.7) reduces to

$$w_\tau(x) = \langle n_{\sigma/\tau}, d_x\phi_{\sigma'} - d_x\phi_\sigma \rangle. \tag{4.8}$$

Remark 4.3. Our weight function given by (4.7) agrees with the formula in [GK17, Definition 1.10] up to a sign. The reason for this discrepancy is due to our adoption of the "min"-convention, as explained in Remark 2.4.

4.2.2 Tropical line bundles

Taking sheaf cohomology of the tropical exponential sequence in (2.1), we obtain a long exact sequence

$$\ldots \to H^1(X,\underline{\mathbb{R}}) \to H^1(X,\mathrm{Aff}_X) \to H^1(X,\Omega^1_X) \to H^2(X,\underline{\mathbb{R}}) \to \ldots. \tag{4.9}$$

We define the group of *(tropical) line bundles* by $H^1(X,\mathrm{Aff}_X)$. For a geometric construction, we refer to [GS23b, §3B]. Suffice it to say that, at least on a smooth polyhedral space X, a Cartier divisor $D \in \mathrm{Div}(X)$ uniquely determines a line bundle $\mathcal{L}(D)$ along with a rational section on that line bundle, and vice versa.

By following the arguments in [JSS19, Proposition 3.11], one can show, analogous to the observations in §3.1.3, that $\underline{C}^{p,\bullet}_{\mathrm{aff}}$ is a $\Gamma(\cdot,X)$-acyclic resolution of Ω^p_X. In particular, $H^{p,q}_{\mathrm{aff}}(X;R)$ computes the sheaf cohomology $H^q(X,\Omega^p_{X,R})$, so that (4.9) becomes

$$\ldots \to H^{0,1}_{\mathrm{aff}}(X;\mathbb{R}) \to H^1(X,\mathrm{Aff}_X) \xrightarrow{c_1} H^{1,1}_{\mathrm{aff}}(X;\mathbb{Z}) \to H^{0,2}_{\mathrm{aff}}(X;\mathbb{R}) \to \ldots.$$

The map c_1 sends a line bundle $\mathcal{L}$ to its *first Chern class* $c_1(\mathcal{L})$.

On a real torus $X = N_{\mathbb{R}}/\Lambda$ with integral structure N, under the identifications in (2.29) and by the universal coefficient in cohomology, we may identify the image of c_1 with the kernel of the homomorphism

$$\mathrm{Hom}(N\otimes\Lambda,\mathbb{Z}) \to \mathrm{Hom}\left(\bigwedge^2 \Lambda,\mathbb{R}\right).$$

By [JRS18, Proposition 3.5] and [ACM25, Proposition 4.1], this map sends a bilinear pairing $[\cdot,\cdot]$ to the linear functional $\lambda\wedge\lambda' \mapsto [\lambda',\lambda]-[\lambda,\lambda']$, where we use the fact that $N_{\mathbb{R}} \cong \Lambda_{\mathbb{R}}$. In other words, the image of c_1 consists precisely in the symmetric bilinear pairings in $\mathrm{Hom}(N\otimes\Lambda,\mathbb{Z})$. Recall that such pairings correspond to polarizations on X precisely when they are positive-definite on $N_{\mathbb{R}}\otimes N_{\mathbb{R}}$.

4.2.3 Normalized tropical Riemann theta function

Fix a principally polarized tropical abelian variety (X, ϑ). Following [dJS22a, §8.1], the cohomology class h_ϑ associated to ϑ is the first Chern class c_1 of a tropical line bundle on X whose space of global sections has dimension 1 and is generated by the *normalized tropical Riemann theta function* $\|\Theta\|$ defined on the universal cover $N_\mathbb{R}$ by

$$\|\Theta\|(x) := \frac{1}{2} \min_{\lambda \in \Lambda} [x - \lambda, x - \lambda]. \tag{4.10}$$

Observe that $\|\Theta\|$ is invariant under translation by Λ and therefore descends to X. We note also that (4.10) differs from the corresponding definition in [MZ08, §5.2] by a sign.

Let Θ denote the Weil divisor with smooth weights defined by $X \cdot \operatorname{div}\|\Theta\|$. By Lemma 4.4, Θ in fact has constant integral weights and so fits into the intersection theory that we established in §2.4. One can also show that $c_1(\mathcal{L}(\operatorname{div}\|\Theta\|)) = h_\vartheta$.

Lemma 4.4. *$X \cdot \operatorname{div}\|\Theta\|$ has constant weight 1 on each facet.*

Proof. Since X and $N_\mathbb{R}$ are locally isomorphic, we may compute the weights in $N_\mathbb{R}$. Interpreting $[x - \lambda, x - \lambda]$ as measuring the distance squared from x to λ, we see that the singular locus of $\|\Theta\|$ induces a Voronoi decomposition $\mathcal{C}$ of $N_\mathbb{R}$; we write σ_λ for the facet where the minimum in (4.10) is achieved at λ. Evidently, $\sigma_\lambda = \sigma_0 + \lambda$, so σ_0 is a fundamental domain for X (up to removing half of its proper faces). The boundary between σ_0 and σ_λ is a subset of the hyperplane

$$H_\lambda := \left\{ x \in N_\mathbb{R} \;\middle|\; [x, \lambda] = \frac{1}{2}[\lambda, \lambda] \right\};$$

if nonempty, this boundary determines a face τ_λ of $\mathcal{C}$ of codimension 1, and all such faces are of this form. In fact, it suffices to consider primitive vectors $\lambda \in \Lambda$: if $\lambda = c\lambda'$ for some $c \geq 2$, then $[x - \lambda', x - \lambda'] < [x, x]$ for all $x \in H_\lambda$, so that $\sigma_0 \cap H_\lambda$ is empty.

Fix a primitive vector $\lambda \in \Lambda$. Since ϑ is principal, we may associate to λ an integral vector $n_\lambda \in N$ so that $[n_\lambda, -\lambda] = 1$. The pairing $[\cdot, \cdot]$ takes values in $\mathbb{Z}$, which implies that n_λ descends to a generator of $N/L(\tau_\lambda)$. Moreover, $-\lambda$ points into σ_0 from τ_λ, so n_λ, which satisfies $[n_\lambda, -\lambda] > 0$, must as well. Taken together, these facts imply that we may choose n_λ as the primitive vector $n_{\sigma_0/\tau_\lambda}$ that appears in the balancing condition.

On σ_0, $\|\Theta\|$ agrees with the smooth function defined by $\|\Theta\|_{\sigma_0}(x) = \frac{1}{2}[x, x]$. Choosing coordinates as in §4.1.1 and writing $x = \sum_i x_i n_i$, we find that

$$\begin{aligned}
\langle n_k, d_x(\|\Theta\|_{\sigma_0})\rangle &= \frac{\partial}{\partial x_k}\|\Theta\|_{\sigma_0}(x) \\
&= \frac{\partial}{\partial x_k}\left(\frac{1}{2}\sum_{i,j} x_i x_j [n_i, n_j]\right) \\
&= \frac{1}{2}\sum_{i,j}(\delta_{i,k}x_j + x_i\delta_{j,k})[n_i, n_j] \\
&= \frac{1}{2}\left(\sum_j x_j[n_k, n_j] + \sum_i x_i[n_i, n_k]\right) \\
&= \sum_j x_j[n_k, n_j] \\
&= [n_k, x].
\end{aligned}$$

This holds for all $k \in [g]$; by $\mathbb{R}$-linearity, it follows that

$$\langle \cdot, d_x(\|\Theta\|_{\sigma_0})\rangle = [\cdot, x]$$

as linear functionals on $N_{\mathbb{R}}$. Then by Λ-periodicity of $\|\Theta\|$, we compute the smooth weight at $x \in \tau_\lambda$ defined in (4.8) as follows:

$$\begin{aligned}
w_{\tau_\lambda}(x) &= \langle n_{\sigma_0/\tau_\lambda}, d_x\|\Theta\|_{\sigma_\lambda} - d_x\|\Theta\|_{\sigma_0}\rangle \\
&= \langle n_\lambda, d_{x-\lambda}\|\Theta\|_{\sigma_0} - d_x\|\Theta\|_{\sigma_0}\rangle \\
&= [n_\lambda, (x-\lambda) - x] \\
&= 1.
\end{aligned}$$

□

4.2.4 Supercurrents

Following [Lag12], the space of (p, q)-*supercurrents* $\mathcal{D}_{p,q}(X)$ on a polyhedral space X is defined to be the topological dual of $\mathcal{A}_c^{p,q}(X)$ with respect to a certain topology; we refer to

[Dem12, §I.2]. Boundary operators d' and d'' are induced on $\mathcal{D}_{\bullet,\bullet}(X)$ by adjointness with the corresponding exterior derivatives on $\mathcal{A}_c^{\bullet,\bullet}(X)$. A wedge product is defined similarly. Consequently, we have a *Laplacian*

$$\mathrm{dd^c} := d'd'' \colon \mathcal{D}_{p,q}(X) \to \mathcal{D}_{p-1,q-1}(X).$$

A (p,q)-superform α may be regarded as a $(g-p,g-q)$-supercurrent by the rule $\beta \mapsto \int_X \alpha \wedge \beta$. Likewise, a tropical k-cycle Z in X determines a (k,k)-supercurrent δ_Z via $\beta \mapsto \int_Z \beta$ according to (2.19). Any polyhedron σ of dimension k similarly determines a (k,k)-supercurrent δ_σ by integration.

Proposition 4.5. *Let (X,ϑ) be a principally polarized tropical abelian variety. Then we have the following equality in $\mathcal{D}_{g-1,g-1}(X)$:*

$$\mathrm{dd^c}\|\Theta\| = \nu_\vartheta - \delta_\Theta.$$

Proof. Recall from Remark 4.3 that the corner locus as we have defined it differs from that of [GK17] by a sign. Taking this into account, it follows from [GK17, Corollary 3.19] that, at least in $\mathcal{D}_{g-1,g-1}(N_\mathbb{R})$, we have that

$$\mathrm{dd^c}\|\Theta\| = d'_\mathrm{P} d''_\mathrm{P}\|\Theta\| - \delta_\Theta.$$

Here, d'_P and d''_P are the "polyhedral derivatives" defined in [GK17, Definition 2.3]. We use [GK17, Remarks 1.7 and 2.8] to compute

$$d'_\mathrm{P} d''_\mathrm{P}\|\Theta\| = \sum_{\lambda\in\Lambda} d'd''(\|\Theta\|_{\sigma_\lambda}) \wedge \delta_{\sigma_\lambda}.$$

In other words, we compute the derivatives only at the smooth points of $\|\Theta\|$. Since $\|\Theta\|$ is periodic, we may restrict our attention to σ_0 and, using (4.2) and (4.6), find that

$$d'd''(\|\Theta\|_{\sigma_0}) = \sum_{i,j} (\mathbf{G}^{-1})_{i,j} d'x_i \wedge d''x_j = \nu_\vartheta.$$

This implies that the desired statement holds on the universal cover $N_\mathbb{R}$.

It remains to show this equality of currents descends to the real torus X. Choose a finite atlas of charts $(U_i)_i$ for X, identifying each U_i with an open subset of $N_{\mathbb{R}}$ via a section of the quotient map $N_{\mathbb{R}} \to X$. Choose also a smooth partition of unity $(\phi_i)_i$ subordinate to $(U_i)_i$. Then we can write any $\omega \in \mathcal{A}^{g-1,g-1}(X)$ as $\omega = \sum_i \phi_i \omega$, where $\phi_i \omega$ is a superform with compact support in U_i. Supercurrents are linear functionals, so it suffices to show that the desired equality of supercurrents holds for $\phi_i \omega$. This follows by the corresponding statement in $N_{\mathbb{R}}$. $\square$

4.3 Integration of cycles

In what follows, we fix a metric graph Γ with an oriented model (G, ℓ) and consider its tropical Jacobian $\mathrm{Jac}(\Gamma)$ with the canonical principal polarization ϑ described in §2.6.4 and bases chosen as in §4.1.1. We fix also a basepoint $q \in V$, with respect to which we define the Abel–Jacobi maps $\Phi_q^{(d)}$ as in (2.36).

4.3.1 Pullback formula

Recalling from (2.25) the definition of Γ in terms of the graph G and some fixed orientation, we define for each edge $e \in E$ the inclusion

$$\iota_e \colon [0, \ell(e)] \to \Gamma.$$

One checks on the charts of Γ to the star-shaped sets of (2.26) that ι_e is a morphism of polyhedral spaces. Let t_e denote the coordinate function on $[0, \ell(e)] \subset \mathbb{R}$. Fix an ordering on E so that $\binom{E}{d}$ is well-defined. Then for any $\mathscr{E} \in \binom{E}{d}$, the product map

$$\iota_{\mathscr{E}} := \prod_{e \in \mathscr{E}} \iota_e \colon \prod_{e \in \mathscr{E}} [0, \ell(e)] \to \Gamma^d / \mathfrak{S}_d \tag{4.11}$$

parameterizes $\prod \mathscr{E} \subseteq \Gamma^d / \mathfrak{S}_d$. Let $\mathbf{M}$ be the integral $g \times \#E$ matrix whose transpose $\mathbf{M}^\intercal$ represents the inclusion $H_1(G; \mathbb{Z}) \hookrightarrow C_1(G; \mathbb{Z})$ with respect to the bases $\gamma_1, \ldots, \gamma_g$ and E, respectively. Explicitly, we have that

$$\gamma_i = \sum_e \mathbf{M}_{i,e} e. \tag{4.12}$$

Lemma 4.6. *We have that*

$$(\Phi_q^{(d)} \circ \iota_{\mathscr{E}})^* \left(\frac{1}{d!}\nu_\vartheta^{\wedge d}\right) = (-1)^{\binom{d}{2}} \det((\mathbf{M}^\intercal \mathbf{G}^{-1}\mathbf{M})_{\mathscr{E},\mathscr{E}}) d't_{\mathscr{E}} \wedge d''t_{\mathscr{E}}.$$

Proof. Let δ_e be a path in $C_1(G,\mathbb{Z})$ from v to e^-. Then $\delta_e + e$ is a path to e^+. In particular, $\Phi_q(e^-) = \pi(\delta_e)$ and $\Phi_q(e^+) = \pi(\delta_e + e)$. One sees that $\Phi_q \circ \iota_e$ is linear, with $\Phi_q \circ \iota_e(t_e) = \pi(\delta_e) + \ell(e)^{-1}\pi(e)t_e$. Consequently,

$$\Phi_q^{(d)} \circ \iota_{\mathscr{E}}((t_e)_{e\in\mathscr{E}}) = \sum_{e\in\mathscr{E}} \pi(\delta_e) + \sum_{e\in\mathscr{E}} \ell(e)^{-1}\pi(e)t_e. \tag{4.13}$$

Using (4.12), it is straightforward to check that $\mathbf{M}_{i,e} = \ell(e)^{-1}[e,\gamma_i]_G$. It follows that $\pi(e) = \ell(e)\sum_i \mathbf{M}_{i,e}n_i$, so (4.13) may be rewritten as

$$\Phi_q^{(d)} \circ \iota_{\mathscr{E}}((t_e)_{e\in\mathscr{E}}) = \sum_{e\in\mathscr{E}} \pi(\delta_e) + \sum_i \sum_{e\in\mathscr{E}} \mathbf{M}_{i,e}t_e n_i.$$

Using this, we compute the pullback of dx_i via

$$(\Phi_q^{(d)} \circ \iota_{\mathscr{E}})^* dx_i = \sum_{e\in\mathscr{E}} \mathbf{M}_{i,e} dt_e.$$

Given $I \in \binom{[g]}{d}$, one may use the same techniques as in the proof of Lemma 4.1 to show that

$$(\Phi_q^{(d)} \circ \iota_{\mathscr{E}})^* dx_I = \det(\mathbf{M}_{I,\mathscr{E}}) dt_{\mathscr{E}}$$

Then by the definition of the pullback of superforms in (2.10),

$$(\Phi_q^{(d)} \circ \iota_{\mathscr{E}})^*(d'x_I \wedge d''x_J) = \det(\mathbf{M}_{I,\mathscr{E}})\det(\mathbf{M}_{J,\mathscr{E}}) d't_{\mathscr{E}} \wedge d''t_{\mathscr{E}}. \tag{4.14}$$

By Lemma 4.1, (4.14), and the Cauchy–Binet formula, we have that

$$\begin{aligned}(\Phi_q^{(d)} \circ \iota_{\mathscr{E}})^* \left(\frac{1}{d!}\nu_\vartheta^{\wedge d}\right) &= (-1)^{\binom{d}{2}} \sum_{I,J\in\binom{[g]}{d}} \det((\mathbf{G}^{-1})_{I,J})(\Phi_q^{(d)} \circ \iota_{\mathscr{E}})^*(d'x_I \wedge d''x_J) \\ &= (-1)^{\binom{d}{2}} \sum_{I,J\in\binom{[g]}{d}} \det((\mathbf{G}^{-1})_{I,J})\det(\mathbf{M}_{I,\mathscr{E}})\det(\mathbf{M}_{J,\mathscr{E}}) d't_{\mathscr{E}} \wedge d''t_{\mathscr{E}} \\ &= (-1)^{\binom{d}{2}} \det((\mathbf{M}^\intercal\mathbf{G}^{-1}\mathbf{M})_{\mathscr{E},\mathscr{E}}) d't_{\mathscr{E}} \wedge d''t_{\mathscr{E}}.\end{aligned}$$

∎

Let $\mathbf{D}$ be the $\#E \times \#E$ diagonal matrix with $\mathbf{D}_{e,e} := \ell(e)$, regarded as representing the linear map $C_1(E;\mathbb{Z}) \to C^1(E;\mathbb{R})$ sending $e \mapsto \ell(e)e^\vee$. From (4.2) and (4.12), one can show that

$$\mathbf{G} = \mathbf{M}\mathbf{D}\mathbf{M}^\intercal. \tag{4.15}$$

Define $\mathbf{P} = \mathbf{M}^\intercal\mathbf{G}^{-1}\mathbf{M}\mathbf{D}$. Then $\mathbf{P}$ represents a linear map $C_1(G;\mathbb{R}) \to H_1(G;\mathbb{R}) \subseteq C_1(G;\mathbb{R})$ with respect to the E basis on both sides.

Lemma 4.7. *The matrix $\mathbf{P}$ represents the orthogonal projection $C_1(G;\mathbb{R}) \to C_1(G;\mathbb{R})$ onto $H_1(G;\mathbb{R})$ with respect to the inner product $[\cdot,\cdot]_G$.*

Proof. Using (4.15), we see that

$$\mathbf{P}^2 = \mathbf{M}^\intercal\mathbf{G}^{-1}\mathbf{M}\mathbf{D}\mathbf{M}^\intercal\mathbf{G}^{-1}\mathbf{M}\mathbf{D} = \mathbf{M}^\intercal\mathbf{G}^{-1}\mathbf{G}\mathbf{G}^{-1}\mathbf{M}\mathbf{D} = \mathbf{M}^\intercal\mathbf{G}^{-1}\mathbf{M}\mathbf{D} = \mathbf{P},$$

so $\mathbf{P}$ is a projection. To prove orthogonality, we show that $\mathbf{P} = \mathbf{P}^*$, where $\mathbf{P}^*$ is the adjoint matrix satisfying $[e, \mathbf{P}^*(f)]_G = [\mathbf{P}(e), f]_G$ for all $e, f \in E$. We compute

$$(\mathbf{P}^*)_{e,f} = \ell(e)^{-1}[e, \mathbf{P}^*(f)]_G = \ell(e)^{-1}[\mathbf{P}(e), f]_G = \ell(e)^{-1}\mathbf{P}_{f,e}\ell(f) = (\mathbf{D}^{-1}\mathbf{P}^\intercal\mathbf{D})_{e,f},$$

where we use the fact that $e^\vee = \ell(e)^{-1}[e,\cdot]_G$ in $C^1(G;\mathbb{R})$ and likewise for $f^\vee$. Then $\mathbf{P}^* = \mathbf{D}^{-1}\mathbf{P}^\intercal\mathbf{D}$. Since $\mathbf{G}$ and $\mathbf{D}$ are both symmetric, we indeed find that

$$\mathbf{P}^* = \mathbf{D}^{-1}(\mathbf{M}^\intercal\mathbf{G}^{-1}\mathbf{M}\mathbf{D})^\intercal\mathbf{D} = \mathbf{D}^{-1}\mathbf{D}\mathbf{M}^\intercal\mathbf{G}^{-1}\mathbf{M}\mathbf{D} = \mathbf{M}^\intercal\mathbf{G}^{-1}\mathbf{M}\mathbf{D} = \mathbf{P}.$$

It remains to show that $\mathbf{P}$ acts as the identity on $H_1(G;\mathbb{R})$. By (4.12), the columns of $\mathbf{M}^\intercal$ form a basis for $H_1(G;\mathbb{R})$ (in E coordinates). Then we use (4.15) again to check

$$\mathbf{P}\mathbf{M}^\intercal = \mathbf{M}^\intercal\mathbf{G}^{-1}\mathbf{M}\mathbf{D}\mathbf{M}^\intercal = \mathbf{M}^\intercal\mathbf{G}^{-1}\mathbf{G} = \mathbf{M}^\intercal,$$

as desired. $\square$

4.3.2 Foster coefficients

It follows immediately from the definition of $\mathbf{P}$ and Lemma 4.6 that

$$(\Phi_q \circ \iota_e)^* \nu_\vartheta = \ell(e)^{-1} \mathbf{P}_{e,e} d' t_e \wedge d'' t_e. \tag{4.16}$$

By Lemma 4.7 and [BF11, §6.1], $\mathbf{P}_{e,e}$ equals the Foster coefficient $F(e)$, which is intimately tied to the theory of random spanning trees, random walks, and electrical circuits on G. We only briefly describe these connections here and refer to [LP17] for a more comprehensive examination of this rich subject.

We regard the metric graph Γ as an electrical network in which each edge e is a uniform wire of total resistance $\ell(e)$. For points $v, w \in \Gamma$, the *effective resistance* $r(v, w)$ between v and w is defined as the voltage drop between v and w when a unit current is injected at v and extracted at w. The *Foster coefficient* $F(e)$ is then defined by

$$r(e^-, e^+) = (1 - F(e))\ell(e).$$

In what turns out to be an equivalent formulation, if we take a random walk on G where, at e^-, we move to e^+ with probability proportional to $\ell(e)^{-1}$ (the conductance of e in the electrical network interpretation), then the probability that a random walk starting at e^- reaches e^+ without using the edge e is equal to $F(e)$.

Given a spanning tree T in G, we define its *weight* to be $w(T) := \prod_{e \notin T} \ell(e)$. Then for the random spanning tree model $\mathscr{T}$ that chooses a spanning tree T with probability proportional to its weight, the probability that a given edge e does not appear in $\mathscr{T}$ is equal to $F(e)$. More generally, a dual, weighted version of the transfer-current theorem of [BP93], which can be attributed to [Lyo03], states that $\mathscr{T}$ is a determinantal point process with matrix $\mathbf{P}$, i.e., for a subset $E' \subset E$, the probability that E' is disjoint from $\mathscr{T}$ is precisely $\det(\mathbf{P}_{E',E'})$.

The following theorem was first proved in [Fos49]; meanwhile, [BF11, Corollary 6.5] provides a proof that highlights the connection to weighted spanning trees. In §4.3.3, we give a geometric proof.

Theorem 4.8 (Foster's theorem)**.** *Fix an electrical network G of genus g where each edge e represents a resistor with resistance $\ell(e)$. Then*

$$\sum_{e \in E} F(e) = g.$$

4.3.3 Volumes of the effective loci

Lemma 4.9. *We have*

$$\int_{\widetilde{W}_d} d' x_I \wedge d'' x_J = (-1)^{\binom{d}{2}} \det(\mathbf{G}_{I,J}).$$

Proof. By (2.37), the integral over $\widetilde{W}_d$ may be pulled back to $\Gamma^d / \mathfrak{S}_d$ and computed as a sum of integrals over subsets of the form $\prod \mathscr{E}$, which in turn are parameterized by d-cubes $\prod_{e \in \mathscr{E}} [0, \ell(e)] \subset \mathbb{R}^d$ according to (4.11). Using (4.14) to compute the pullback, we find that

$$\begin{aligned}
\int_{\widetilde{W}_d} d' x_I \wedge d'' x_J &= \sum_{\mathscr{E} \in \binom{E}{d}} \int_{\prod \mathscr{E}} (\Phi_q^{(d)})^* (d' x_I \wedge d'' x_J) \\
&= \sum_{\mathscr{E} \in \binom{E}{d}} \int_{\prod_{e \in \mathscr{E}} [0, \ell(e)]} (\Phi_q^{(d)} \circ \iota_{\mathscr{E}})^* (d' x_I \wedge d'' x_J) \\
&= \sum_{\mathscr{E} \in \binom{E}{d}} \int_{\prod_{e \in \mathscr{E}} [0, \ell(e)]} \det(\mathbf{M}_{I,\mathscr{E}}) \det(\mathbf{M}_{J,\mathscr{E}}) d' t_{\mathscr{E}} \wedge d'' t_{\mathscr{E}} \\
&= (-1)^{\binom{d}{2}} \sum_{\mathscr{E} \in \binom{E}{d}} \det(\mathbf{M}_{I,\mathscr{E}}) \det(\mathbf{M}_{J,\mathscr{E}}) \int_{\prod_{e \in \mathscr{E}} [0, \ell(e)]} dt_{\mathscr{E}} \\
&= (-1)^{\binom{d}{2}} \sum_{\mathscr{E} \in \binom{E}{d}} \det(\mathbf{M}_{I,\mathscr{E}}) \det(\mathbf{M}_{J,\mathscr{E}}) \prod_{e \in \mathscr{E}} \ell(e) \\
&= (-1)^{\binom{d}{2}} \sum_{\mathscr{E} \in \binom{E}{d}} \det(\mathbf{M}_{I,\mathscr{E}}) \det(\mathbf{M}_{J,\mathscr{E}}) \det(\mathbf{D}_{\mathscr{E},\mathscr{E}}).
\end{aligned}$$

Since $\det(\mathbf{D}_{\mathscr{E},\mathscr{F}}) = 0$ for $\mathscr{E} \neq \mathscr{F}$, we expand the summation and then apply the Cauchy–Binet formula to simplify:

$$= (-1)^{\binom{d}{2}} \sum_{\mathscr{E}, \mathscr{F} \in \binom{E}{d}} \det(\mathbf{M}_{I,\mathscr{E}}) \det(\mathbf{M}_{J,\mathscr{F}}) \det(\mathbf{D}_{\mathscr{E},\mathscr{F}})$$

$$= (-1)^{\binom{d}{2}} \sum_{\mathscr{E},\mathscr{F} \in \binom{E}{d}} \det(\mathbf{M}_{I,\mathscr{E}}) \det(\mathbf{D}_{\mathscr{E},\mathscr{F}}) \det((\mathbf{M}^\intercal)_{\mathscr{F},J})$$
$$= (-1)^{\binom{d}{2}} \det((\mathbf{M}\mathbf{D}\mathbf{M}^\intercal)_{I,J})$$
$$= (-1)^{\binom{d}{2}} \det(\mathbf{G}_{I,J}). \qquad \square$$

Theorem 4.10. *We have that*

$$\int_{\widetilde{W}_d} \frac{1}{d!} \nu_\vartheta^{\wedge d} = \binom{g}{d}.$$

Proof. Using Lemmas 4.1 and 4.9, we compute

$$\begin{aligned}
\int_{\widetilde{W}_d} \frac{1}{d!} \nu_\vartheta^{\wedge d} &= (-1)^{\binom{d}{2}} \sum_{I,J \in \binom{[g]}{d}} \det((\mathbf{G}^{-1})_{I,J}) \int_{\widetilde{W}_d} d' x_I \wedge d'' x_J \\
&= \sum_{I,J \in \binom{[g]}{d}} \det((\mathbf{G}^{-1})_{I,J}) \det(\mathbf{G}_{I,J}) \\
&= \sum_{I,J \in \binom{[g]}{d}} \det((\mathbf{G}^{-1})_{I,J}) \det(\mathbf{G}_{J,I}) \\
&= \sum_{I \in \binom{[g]}{d}} \det((\mathbf{G}^{-1}\mathbf{G})_{I,I}) \\
&= \sum_{I \in \binom{[g]}{d}} \det(\mathbf{I}_{I,I}) \\
&= \sum_{I \in \binom{[g]}{d}} 1 \\
&= \binom{g}{d}. \qquad \square
\end{aligned}$$

We now show that Theorem 4.10 interpolates between classical graph-theoretic results, with $d = 1$ corresponding to Foster's theorem and $d = g$ corresponding to a dual, weighted version of Kirchhoff's matrix-tree theorem.

Proof of Theorem 4.8. Let Γ be the metric graph with model (G, ℓ). In the case $d = 1$, Theorem 4.10 says that

$$\int_{\widetilde{W}_1} \nu_\vartheta = g.$$

On the other hand, by (4.16), we can pull the integral back to Γ and split it over each edge, much as in the proof of Lemma 4.9:

$$\int_{\widetilde{W}_1} \nu_\vartheta = \sum_{e\in E} \int_{[0,\ell(e)]} \ell(e)^{-1}\mathbf{P}_{e,e} d't_e \wedge d''t_e = \sum_{e\in E} \ell(e)^{-1}\mathbf{P}_{e,e} \int_0^{\ell(e)} dt_e = \sum_{e\in E} F(e). \qquad \square$$

We define the weight of a spanning tree as in §4.3.2. One sees by specializing to the case where G has all unit edge lengths that the following theorem indeed gives a version of the matrix-tree theorem (with the reduced graph Laplacian replaced by the Gram matrix of the length pairing). A similarly geometric proof of this theorem was given in [ABKS14].

Theorem 4.11 (Kirchhoff's matrix-tree theorem)**.** *Let G be a graph with edge weight function ℓ. Then*

$$\sum_T w(T) = \det(\mathbf{G}),$$

where the summation is taken over all spanning trees of G.

Proof. Let Γ be the metric graph with model (G,ℓ). Recall from §2.6.5 that $\widetilde{W}_g = \mathrm{Jac}(\Gamma)$ has a polyhedral decomposition in terms of tuples $\mathscr{E} \in \binom{E}{g}$ for which $E \setminus \mathscr{E}$ is connected. Because $\mathscr{E}$ consists of g distinct edges, this forces $E \setminus \mathscr{E}$ to be a spanning tree. In particular, the facets of the polyhedral decomposition of $\mathrm{Jac}(\Gamma)$ are indexed by the spanning trees of G, a fact which was first shown in [ABKS14, §3.2].

By Theorem 4.10 with $d = g$, we have that

$$\int_{\widetilde{W}_g} \frac{1}{g!}\nu_\vartheta^{\wedge g} = 1.$$

On the other hand, pulling the integral back to $\Gamma^g/\mathfrak{S}_g$ with Lemma 4.6 and using Cauchy–Binet and the correspondence between $\binom{E}{g}$ and spanning trees of G, we compute

$$\begin{aligned}\int_{\widetilde{W}_g} \frac{1}{g!}\nu_\vartheta^{\wedge g} &= \sum_{\mathscr{E}\in\binom{E}{g}} \int_{\prod_{e\in\mathscr{E}}[0,\ell(e)]} (-1)^{\binom{g}{2}} \det((\mathbf{M}^\intercal\mathbf{G}^{-1}\mathbf{M})_{\mathscr{E},\mathscr{E}}) d't_{\mathscr{E}} \wedge d''t_{\mathscr{E}} \\ &= \sum_{\mathscr{E}\in\binom{E}{g}} \det((\mathbf{M}^\intercal\mathbf{G}^{-1}\mathbf{M})_{\mathscr{E},\mathscr{E}}) \prod_{e\in\mathscr{E}} \ell(e)\end{aligned}$$

$$= \sum_{\mathscr{E} \in \binom{E}{g}} \det(\mathbf{M}^{\intercal}_{\mathscr{E},[g]}) \det(\mathbf{G}^{-1}) \det(\mathbf{M}_{[g],\mathscr{E}}) \prod_{e \in \mathscr{E}} \ell(e)$$
$$= \det(\mathbf{G})^{-1} \sum_{T} \det(\mathbf{M}_{[g],E \setminus T})^2 w(T).$$

It suffices to show that $\det(\mathbf{M}_{[g],E\setminus T}) \in \{\pm 1\}$. Indeed, notice that changing our basis of $H_1(G;\mathbb{Z})$ (via row operations) changes $\det(\mathbf{M}_{[g],E\setminus T})$ at most by a sign. Therefore, for a particular spanning tree T, enumerate $E \setminus T = (e_1, \ldots, e_g)$ (in order) and suppose that we have picked the basis of cycles $\gamma_1, \ldots, \gamma_g$ so that $\gamma_i = \gamma_{e_i}$ is the fundamental cycle of e_i described in §2.5.1. Then it is immediate that $\mathbf{M}_{[g],E\setminus T} = \mathbf{I}$. □

4.3.4 Tropical Poincaré formula

Given a tropical k-cycle Z in X, its cycle class $\mathrm{cyc}(Z) \in H^{\mathrm{aff}}_{k,k}(X;\mathbb{Z})$ determines the linear functional $\int_Z \cdot$ in $H^{k,k}_{d''}(X)^\vee$ by the dual of the tropical de Rham map. By Poincaré duality, we may choose a cohomology class $\nu_Z \in H^{g-k,g-k}_{d''}(X)$ such that

$$\int_X \nu_Z \wedge \alpha = \int_Z \alpha$$

for all d''-closed $\alpha \in \mathcal{A}^{k,k}(X)$. Then it follows from commutativity of dR and PD in Theorem 3.12 that $\mathrm{cyc}(X) \frown \mathrm{dR}(\nu_Z) = \mathrm{cyc}(Z)$.

Lemma 4.12. *For all $k \in \mathbb{Z}_{\geq 0}$, we have that*

$$\mathrm{cyc}(\Theta^{\cdot k}) = \mathrm{cyc}(X) \frown \mathrm{dR}(\nu_\Theta^{\wedge k}). \tag{4.17}$$

Proof. We prove this by induction. The base case of $k = 0$ is trivial, since both sides equal $\mathrm{cyc}(X)$. Suppose now that (4.17) holds for a particular k. By [GS23a, Proposition 5.12],

$$\mathrm{cyc}(Z \cdot D) = \mathrm{cyc}(Z) \frown c_1(\mathcal{L}(D)) \tag{4.18}$$

for any tropical cycle Z and Cartier divisor D. Taking $Z = \Theta^{\cdot k}$ and $D = \mathrm{div}\|\Theta\|$ and applying the induction hypothesis, we find that

$$\mathrm{cyc}(\Theta^{\cdot(k+1)}) = \mathrm{cyc}(\Theta^{\cdot k} \cdot \mathrm{div}\|\Theta\|)$$

$$= \mathrm{cyc}(\Theta^{\cdot k}) \frown c_1(\mathcal{L}(\mathrm{div}\|\Theta\|))$$
$$= \mathrm{cyc}(X) \frown \mathrm{dR}(\nu_\Theta^{\wedge k}) \frown c_1(\mathcal{L}(\mathrm{div}\|\Theta\|)).$$

By graded-commutativity of the cup product from Corollary 3.8 and another application of (4.18), we obtain

$$= \mathrm{cyc}(X) \frown c_1(\mathcal{L}(\mathrm{div}\|\Theta\|)) \frown \mathrm{dR}(\nu_\Theta^{\wedge k})$$
$$= \mathrm{cyc}(\Theta) \frown \mathrm{dR}(\nu_\Theta^{\wedge k})$$
$$= \mathrm{cyc}(X) \frown \mathrm{dR}(\nu_\Theta) \frown \mathrm{dR}(\nu_\Theta^{\wedge k}).$$

Finally, because dR sends the wedge product to the cup product by Theorem 3.7, the desired result follows:

$$= \mathrm{cyc}(X) \frown \mathrm{dR}(\nu_\Theta^{\wedge (k+1)}). \qquad \square$$

One also finds that ν_Θ agrees with the canonical $(1,1)$-form ν_ϑ. To see why, recall from §§4.1.2 and 4.2.3 that $\mathrm{dR}(\nu_\vartheta) = h_\vartheta = c_1(\mathcal{L}(\mathrm{div}\|\Theta\|))$. Then following the same argument as in the proof of Lemma 4.12, we compute

$$\mathrm{cyc}(X) \frown \mathrm{dR}(\nu_\vartheta) = \mathrm{cyc}(X) \frown c_1(\mathcal{L}(\mathrm{div}\|\Theta\|)) = \mathrm{cyc}(\Theta) = \mathrm{cyc}(X) \frown \mathrm{dR}(\nu_\Theta).$$

Because $\mathrm{PD}' \circ \mathrm{dR}$ is injective, we get that $\nu_\vartheta = \nu_\Theta$. With these tools in hand, we can now prove the following theorem, which was first demonstrated by [GS23b, Theorem A].

Theorem 4.13 (Poincaré formula)**.** *In $H^{\mathrm{aff}}_{d,d}(X)$, we have that*

$$\mathrm{cyc}(\widetilde{W}_d) = \frac{\mathrm{cyc}(\Theta^{\cdot (g-d)})}{(g-d)!}.$$

Proof. Writing $\mathrm{cyc}(\widetilde{W}_d) = \mathrm{cyc}(X) \frown \mathrm{dR}(\nu_{\widetilde{W}_d})$ and $\mathrm{cyc}(\Theta^{\cdot(g-d)}) = \mathrm{cyc}(X) \frown \mathrm{dR}(\nu_\Theta^{\wedge(g-d)})$ by Lemma 4.12, we see that it suffices to show that

$$\nu_{\widetilde{W}_d} = \frac{\nu_\Theta^{\wedge(g-d)}}{(g-d)!}.$$

Taking the Poincaré dual of both sides and identifying $\nu_\Theta = \nu_\vartheta$, we will instead show that

$$\int_{\widetilde{W}_d} \cdot = \int_X \frac{\nu_\vartheta^{\wedge(g-d)}}{(g-d)!} \wedge \cdot$$

as linear functionals on $H^{d,d}_{d''}(X)$. We check this on basis elements $d'x_I \wedge d''x_J$ for $I, J \in \binom{[g]}{d}$.

Applying Lemma 4.1 yields

$$\begin{aligned}
&\int_X \frac{\nu_\Theta^{\wedge(g-d)}}{(g-d)!} \wedge d'x_I \wedge d''x_J \\
&= (-1)^{\binom{g-d}{2}} \sum_{I',J' \in \binom{[g]}{g-d}} \det((\mathbf{G}^{-1})_{I',J'}) \int_X d'x_{I'} \wedge d''x_{J'} \wedge d'x_I \wedge d''x_J \\
&= (-1)^{\binom{g-d}{2}+(g-d)d} \det((\mathbf{G}^{-1})_{[g]\setminus I,[g]\setminus J}) \int_X d'x_{[g]\setminus I} \wedge d'x_I \wedge d''x_{[g]\setminus J} \wedge d''x_J \\
&= (-1)^{\binom{g}{2}+\binom{d}{2}} \operatorname{sgn}(\mathfrak{s}_I) \operatorname{sgn}(\mathfrak{s}_J) \det((\mathbf{G}^{-1})_{[g]\setminus I,[g]\setminus J}) \int_X d'x_{[g]} \wedge d''x_{[g]},
\end{aligned}$$

where $\mathfrak{s}_I$ is the permutation $([g] \setminus I, I)$ (and likewise for J). Computing the integral as in the proof of Proposition 4.2, we obtain

$$= (-1)^{\binom{d}{2}} \operatorname{sgn}(\mathfrak{s}_I) \operatorname{sgn}(\mathfrak{s}_J) \det((\mathbf{G}^{-1})_{[g]\setminus I,[g]\setminus J}) \det(\mathbf{G}).$$

By a well-known formula for complementary matrix minors, this equals

$$= (-1)^{\binom{d}{2}+\sum I+\sum J} \operatorname{sgn}(\mathfrak{s}_I) \operatorname{sgn}(\mathfrak{s}_J) \det(\mathbf{G}_{J,I}).$$

It suffices to show that $(-1)^{\sum I+\sum J} \operatorname{sgn}(\mathfrak{s}_I) \operatorname{sgn}(\mathfrak{s}_J) = 1$, since then symmetry of $\mathbf{G}$ and Lemma 4.9 together yield the desired equality.

We justify the sign as follows. Fix a $(g-d,d)$-shuffle $\mathfrak{s}$, i.e., a permutation of $[g]$ of the form $\mathfrak{s} = (a_1, \ldots, a_{g-d}, b_1, \ldots, b_d)$ with $a_1 < \ldots < a_{g-d}$ and $b_1 < \ldots < b_d$, and define a function ϕ on such shuffles by $\phi\colon \mathfrak{s} \mapsto \sum_{i=1}^d b_i$. Suppose that $a_i + 1 = b_j$. Let $\mathfrak{t}_{a_i}$ denote the transposition that flips a_i and $a_i + 1$. By the assumption on $\mathfrak{s}$, we see that $\mathfrak{t}_{a_i} \circ \mathfrak{s}$ is again a $(g-d,d)$-shuffle. Moreover, $\phi(\mathfrak{t}_{a_i} \circ \mathfrak{s}) = \phi(\mathfrak{s}) - 1$, while $\operatorname{sgn}(\mathfrak{t}_{a_i} \circ \mathfrak{s}) = -\operatorname{sgn}(\mathfrak{s})$; it follows that the quantity $(-1)^{\phi(\mathfrak{s})} \operatorname{sgn}(\mathfrak{s})$ remains constant when we apply $\mathfrak{t}_{a_i}$. It is not hard to see that one can reach any shuffle $\mathfrak{s}$ from the identity permutation id by a series of transpositions as above (only switching entries from both "sides" of the shuffle). Therefore,

$$\begin{aligned}(-1)^{\sum I+\sum J}\operatorname{sgn}(\mathfrak{s}_I)\operatorname{sgn}(\mathfrak{s}_J) &= (-1)^{\phi(\mathfrak{s}_I)}\operatorname{sgn}(\mathfrak{s}_I)(-1)^{\phi(\mathfrak{s}_J)}\operatorname{sgn}(\mathfrak{s}_J)\\ &= (-1)^{\phi(\mathrm{id})}\operatorname{sgn}(\mathrm{id})(-1)^{\phi(\mathrm{id})}\operatorname{sgn}(\mathrm{id})\\ &= 1. \qquad \square\end{aligned}$$

Chapter 5
CERESA PERIOD

This chapter is based on [Rit25]. For the sake of consistency, we use the more traditional version of the boundary map on singular (p,q)-chains that does not introduce a sign of $(-1)^p$; see Remark 2.3. This choice has no bearing on the final results.

5.1 Ceresa period of a tropical curve

Fix a tropical curve Γ, a model (G,ℓ) for Γ, and an orientation $\mathcal{O}$ on G. Following [Zha15], we define the tropical Ceresa cycle $\mathbf{Cer}_v(\Gamma)$ and construct a tropical homological invariant $\alpha(\Gamma)$, called the Ceresa period, which obstructs algebraic triviality of $\mathbf{Cer}_v(\Gamma)$.

5.1.1 Tropical Ceresa cycle

Let $[-1]\colon \operatorname{Jac}(\Gamma) \to \operatorname{Jac}(\Gamma)$ denote the inversion map. Then the *(tropical) Ceresa cycle* based at $v \in \Gamma$ is the 1-cycle $\mathbf{Cer}_v(\Gamma) \in \mathscr{Z}_1(\Gamma)$ defined by

$$\mathbf{Cer}_v(\Gamma) := \widetilde{W}_1 - [-1]_*\widetilde{W}_1,$$

where $\widetilde{W}_1 = \Phi_{v,*}\Gamma$ (see §2.6.5). Given another point $v' \in \Gamma$, $\Phi_{v,*}\Gamma$ and $\Phi_{v',*}\Gamma$ differ by a translation. Then it follows from [GS23b, Proposition 5.5] that $\mathbf{Cer}_v(\Gamma)$ does not depend on the choice of basepoint v modulo algebraic equivalence.

The model (G,ℓ) determines a face structure (or equivalently, a triangulation) on Γ. Recalling the definition of the fundamental chain in (2.22), we define

$$\operatorname{Cer}_{v,G}(\Gamma) := \Phi_{v,*}\operatorname{ch}_G(\Gamma) - [-1]_*\Phi_{v,*}\operatorname{ch}_G(\Gamma) \in C_{1,1}^{\mathrm{aff}}(\operatorname{Jac}(\Gamma);\mathbb{Z}).$$

Because the pushforward on cycles is compatible with the pushforward on tropical homology under cyc (see [GS23a, §5.4]), we find that $\operatorname{Cer}_{v,G}(\Gamma)$ is a representative for the homology class $\mathrm{cyc}(\mathbf{Cer}_v(\Gamma)) \in H_{1,1}^{\mathrm{aff}}(\mathrm{Jac}(\Gamma);\mathbb{Z})$. Using the generators given by (2.31), it is clear that $[-1]_*$ acts as the identity on $H_{1,1}^{\mathrm{aff}}(\mathrm{Jac}(\Gamma);\mathbb{Z})$. It follows that the Ceresa cycle is homologically trivial:

$$\begin{aligned}\mathrm{cyc}\left(\mathbf{Cer}_v(\Gamma)\right) &= \mathrm{cyc}(\widetilde{W}_1) - [-1]_*\,\mathrm{cyc}(\widetilde{W}_1)\\ &= \mathrm{cyc}(\widetilde{W}_1) - \mathrm{cyc}(\widetilde{W}_1)\\ &= 0.\end{aligned}$$

Using the notation for affine simplices in $\mathrm{Jac}(\Gamma)$ and the length pairing introduced in §2.6.2 and §2.6.4, respectively, a straightforward computation shows that

$$\mathrm{Cer}_{v,G}(\Gamma) = \sum_{e\in E} \ell(e)^{-1}\pi(e) \otimes \big((\pi(\delta_e), \pi(\delta_e + e)) + (-\pi(\delta_e), -\pi(\delta_e + e))\big), \tag{5.1}$$

where δ_e is a path in G from v to e^-. This formula will be the key to defining a universal version of the Ceresa cycle in §5.2.3.

5.1.2 Integration map

Let $n_1, \dots, n_g$ denote a basis for N and $x_1, \dots, x_g$ the associated coordinate functions on $N_{\mathbb{R}}$; then the tropical 1-forms dx_i on $N_{\mathbb{R}}$ descend to $\mathrm{Jac}(\Gamma)$. It follows from §2.6.2 that $C_{1,2}^{\mathrm{sm}}(\mathrm{Jac}(\Gamma);\mathbb{Z})$ is generated by elements of the form $n \otimes f$, where $n \in N$ and $f\colon \Delta^2 \to \mathrm{Jac}(\Gamma)$ is a smooth 2-simplex.

Mimicking the "determinantal 2-form" for K_4 introduced by [Zha15], we define the *integration map* $\Psi\colon C_{1,2}^{\mathrm{sm}}(\mathrm{Jac}(\Gamma);\mathbb{Z}) \to \bigwedge^3 N_{\mathbb{R}}$ sending $n \otimes f$ to

$$2\sum_{I\in\binom{[g]}{3}} \int_{n\otimes f} \left(d'x_{i_1}\wedge d''x_{i_2}\wedge d''x_{i_3} - d'x_{i_2}\wedge d''x_{i_1}\wedge d''x_{i_3} + d'x_{i_3}\wedge d''x_{i_1}\wedge d''x_{i_2}\right) n_I. \tag{5.2}$$

Because the $(1,2)$-superform appearing in the integrand has constant coefficients, the Stokes formula in Lemma 3.2 implies that Ψ is trivial on boundaries. Consequently, Ψ descends to a map on homology $H_{1,2}^{\mathrm{sm}}(\mathrm{Jac}(\Gamma);\mathbb{Z}) \to \bigwedge^3 N_{\mathbb{R}}$. Let

$$\mathscr{P} := \Psi(H_{1,2}^{\mathrm{sm}}(\mathrm{Jac}(\Gamma);\mathbb{Z})) \subset \bigwedge^3 N_{\mathbb{R}}.$$

In the case that $f = (u_0, u_1, u_2)$ is affine, (5.2) simplifies to

$$\Psi(n \otimes (u_0, u_1, u_2)) = \sum_{I \in \binom{[g]}{3}} \det(\mathbf{M}_{I,[3]}) n_I,$$

where $\mathbf{M}$ is the $g \times 3$ matrix whose columns are the entries of n, $u_1 - u_0$, and $u_2 - u_0$ with respect to the basis $n_1, \ldots, n_g$. We see that Ψ is coordinate-independent at least on affine $(1, 2)$-chains by rewriting this expression in terms of exterior products:

$$\Psi(n \otimes (u_0, u_1, u_2)) = n \wedge (u_1 - u_0) \wedge (u_2 - u_0). \tag{5.3}$$

Recall that, by (2.30), we may identify affine and smooth tropical homology. Then we apply (5.3) to the generating set of $H_{1,2}^{\mathrm{aff}}(\mathrm{Jac}(\Gamma); \mathbb{Z})$ established in (2.31) to find that

$$\mathscr{P} = \mathbb{Z}\langle 2n_i \wedge \lambda_j \wedge \lambda_k \mid i, j, k \in [g],\ j < k\rangle,$$

for some fixed basis $\lambda_1, \ldots, \lambda_g$ for Λ.

5.1.3 Ceresa period

We saw in §5.1.1 that $\mathrm{Cer}_{v,G}(\Gamma)$ is trivial in $H_{1,1}^{\mathrm{aff}}(\mathrm{Jac}(\Gamma); \mathbb{Z})$, so we may choose some $\Sigma \in C_{1,2}^{\mathrm{aff}}(\mathrm{Jac}(\Gamma); \mathbb{Z})$ for which $\partial\Sigma = \mathrm{Cer}_{v,G}(\Gamma)$. Define the *Ceresa period* $\alpha(\Gamma) \in \bigwedge^3 N_{\mathbb{R}}/\mathscr{P}$ via

$$\alpha(\Gamma) := \Psi(\Sigma) \pmod{\mathscr{P}}.$$

As the notation suggests, $\alpha(\Gamma)$ is independent of the choice of v, Σ, and G. We prove independence from the basepoint v in Corollary 5.13. Given $\Sigma' \in C_{1,2}^{\mathrm{aff}}(\mathrm{Jac}(\Gamma); \mathbb{Z})$ with the same boundary as Σ, we have that $\Sigma' - \Sigma \in H_{1,2}^{\mathrm{aff}}(\mathrm{Jac}(\Gamma); \mathbb{Z})$, hence $\Psi(\Sigma') \equiv \Psi(\Sigma) \pmod{\mathscr{P}}$. Finally, any other model (G', ℓ') of Γ is related to (G, ℓ) by a series of edge refinements, so it suffices to prove independence in the case that G' is obtained from G by subdividing a single edge e into edges e_1 and e_2 as described in §2.5.2. Orient e_1 and e_2 in the same direction as e, so that $e^- = e_1^-$, $e_1^+ = e_2^-$, and $e_2^+ = e^+$. Then we have both that $\ell'(e_1)^{-1}\pi(e_1) = \ell'(e_2)^{-1}\pi(e_2) = \ell(e)^{-1}\pi(e)$, and that $\delta_{e_1} = \delta_e$ and $\delta_{e_2} = \delta_e + e_1$. Given Σ as above, let

$$\Sigma' := \Sigma + \ell(e)^{-1}\pi(e) \otimes \big((\pi(\delta_e), \pi(\delta_e + e_1), \pi(\delta_e + e)) + (-\pi(\delta_e), -\pi(\delta_e + e_1), -\pi(\delta_e + e))\big).$$

Comparing the e term in $\mathrm{Cer}_{v,G}(\Gamma)$ to the e_1 and e_2 terms in $\mathrm{Cer}_{v,G'}(\Gamma)$, one sees that $\partial\Sigma' = \mathrm{Cer}_{v,G'}(\Gamma)$. Moreover, since $\pi(\delta_e)$, $\pi(\delta_e + e_1)$, and $\pi(\delta_e + e)$ are collinear in $N_{\mathbb{R}}$, both of the $(1,2)$-simplices in $\Sigma' - \Sigma$ vanish under the integration map Ψ. This forces $\Psi(\Sigma) = \Psi(\Sigma')$, as desired.

Given the data of an algebraic equivalence between Z_1 and Z_2 in $\mathscr{Z}_1(\mathrm{Jac}(\Gamma))$, Zharkov constructs in [Zha15, §3.1] an affine $(1,2)$-chain $\Sigma := \sum_i n_i \otimes f_i$ that satisfies $\partial\Sigma = \mathrm{ch}(Z_1) - \mathrm{ch}(Z_2)$ [Zha15, Lemma 4]. As noted in [Zha15, Lemma 5], by construction, each n_i lies in the affine hull of the corresponding simplex $f_i(\Delta^2)$. The following result is a straightforward generalization of [Zha15, Lemma 5].

Proposition 5.1. *If the tropical Ceresa cycle of Γ is algebraically trivial, then $\alpha(\Gamma) = 0$.*

Proof. Suppose that $\widetilde{W}_1$ and $[-1]_*\widetilde{W}_1$ are algebraically equivalent. Then we construct Σ as above so that

$$\partial\Sigma = \mathrm{ch}\left(\widetilde{W}_1\right) - \mathrm{ch}\left([-1]_*\widetilde{W}_1\right).$$

Up to adding degenerate simplices, on which Ψ vanishes, this expression agrees with $\mathrm{Cer}_{v,G}(\Gamma)$. The fact that n_i lies in the affine hull of $f_i(\Delta^2)$ forces $\Psi(n_i \otimes f_i) = 0$ by (5.3). This implies that $\Psi(\Sigma) = 0$, as desired. □

5.2 Ceresa period of a graph

We would like to work with graphs rather than particular tropical curves. To that end, we define "universal" versions of the Jacobian, homology, the Ceresa cycle, and the Ceresa period using edge weights in a polynomial ring. We conclude this section by showing that these universal objects specialize to their tropical counterparts when we fix specific edge lengths.

5.2.1 Universal Jacobian

We associate to an oriented graph G of genus g a polynomial ring $R := \mathbb{Z}[x_e \mid e \in E]$. The universal Jacobian of G is constructed by analogy with the tropical Jacobian of §2.6.4, with $\mathbb{R}$ replaced by R. In particular, we take $N := H^1(G;\mathbb{Z})$ and consider the free R-module

$$N_R \cong H^1(G;R) \cong \operatorname{Hom}(H_1(G;\mathbb{Z}), R).$$

We define a pairing $[\cdot,\cdot]\colon C_1(G;\mathbb{Z}) \times C_1(G;\mathbb{Z}) \to R$ via

$$[e,e'] = \begin{cases} x_e & \text{if } e = e' \\ 0 & \text{otherwise} \end{cases},$$

which induces a homomorphism $\pi\colon C_1(G;\mathbb{Z}) \to N_R$ by $e \mapsto [e,\cdot]$. Then $\Lambda := \pi(H_1(G;\mathbb{Z}))$ is a free $\mathbb{Z}$-submodule of N_R of rank g. Formally, we define the *universal Jacobian* $\operatorname{Jac}(G)$ of G to be the triple $(N_R/\Lambda, N_R, N)$, although we shall conflate $\operatorname{Jac}(G)$ with the quotient group N_R/Λ whenever the other data are clear from context.

It should be noted that, although we use the symbol $\operatorname{Jac}(G)$, our universal Jacobian is distinct from the Jacobian of a finite graph, also known as the abelian sandpile group or critical group, which has been studied in various contexts in physics, arithmetic geometry, and graph theory. For more on this subject, see, for instance, [BTW88, Lor89, Dha90, Gab93, BLHN97, Nag97, BN07].

5.2.2 Universal homology

We define homology theories on $\operatorname{Jac}(G)$ as follows. Let $C_q(\operatorname{Jac}(G))$ denote the free abelian group generated by $N_R^{q+1}/\sim$, equivalence classes of ordered q-simplices, where we identify

$$(u_0 + \lambda, \dots, u_q + \lambda) \sim (u_0, \dots, u_q) \tag{5.4}$$

for all $\lambda \in \Lambda$. This becomes a chain complex $C_\bullet(\operatorname{Jac}(G))$ via the usual boundary maps

$$\partial(u_0, \dots, u_q) = \sum_{i=0}^{q} (-1)^i (u_0, \dots, \hat{u}_i, \dots, u_q),$$

where $\hat{\cdot}$ means that the corresponding entry is omitted. Likewise, let $H_q(\mathrm{Jac}(G))$ denote the q-th homology of $C_\bullet(\mathrm{Jac}(G))$. We further define $C_{p,\bullet}(\mathrm{Jac}(G)) := \bigwedge^p N \otimes C_\bullet(\mathrm{Jac}(G))$, with corresponding q-th homology $H_{p,q}(\mathrm{Jac}(G))$. We call a (p,q)-chain *degenerate* if every q-simplex that it contains has repeated entries.

The universal coefficient theorem yields

$$H_{p,q}(\mathrm{Jac}(G)) \cong \bigwedge^p N \otimes_{\mathbb{Z}} H_q(\mathrm{Jac}(G)). \tag{5.5}$$

Comparing the following technical result to (2.31), this says that $\mathrm{Jac}(G)$ has the homology we would expect from a g-dimensional real torus.

Lemma 5.2. *Fix bases $n_1, \ldots, n_g$ of N and $\lambda_1, \ldots, \lambda_g$ of Λ. Then*

(a) $H_{1,1}(\mathrm{Jac}(G)) = \mathbb{Z}\langle n_i \otimes (0, \lambda_j) \mid i, j \in [g]\rangle$ *and*

(b) $H_{1,2}(\mathrm{Jac}(G)) = \mathbb{Z}\langle n_i \otimes ((0, \lambda_j, \lambda_j + \lambda_k) - (0, \lambda_k, \lambda_j + \lambda_k)) \mid i, j, k \in [g],\ j < k\rangle$.

Proof. By restriction of scalars, the R-module N_R inherits the structure of a free $\mathbb{Z}$-module of infinite rank, which we denote by M. Then $M_\mathbb{R}$ is an infinite-dimensional $\mathbb{R}$-vector space that naturally contains M as a $\mathbb{Z}$-submodule. We endow the g-dimensional subspace $\Lambda_\mathbb{R}$ of $M_\mathbb{R}$ with the Euclidean topology. Choose a subspace W complementary to $\Lambda_\mathbb{R}$ and give it the Euclidean norm with respect to some basis $w_1, w_2, \ldots$. Give $M_\mathbb{R} = \Lambda_\mathbb{R} \times W$ the product topology. The action of Λ on $\Lambda_\mathbb{R}$ by translation extends to an action on $M_\mathbb{R}$; define $J := M_\mathbb{R}/\Lambda$ with the quotient topology. Equivalently, we may write $J = \Lambda_\mathbb{R}/\Lambda \times W$. It is a straightforward exercise to show that $W \cong \varinjlim_n \mathbb{R}^n$ is contractible via a straight-line homotopy to 0; therefore, the singular homology $H_q(J)$ of J is isomorphic to that of the g-dimensional real torus $\Lambda_\mathbb{R}/\Lambda$. In particular, just as in (2.31), one can show using the Eilenberg–Zilber map that $H_1(J)$ is freely generated by the affine simplices $(0, \lambda_i)$ in $M_\mathbb{R}$ for each i, while $H_2(J)$ is freely generated by $(0, \lambda_i, \lambda_i + \lambda_j) - (0, \lambda_j, \lambda_i + \lambda_j)$ for $i < j$.

Let $S_\bullet$ denote the singular chain complex functor. There is a natural surjection $S_\bullet(M_\mathbb{R}) \to S_\bullet(J)$ that identifies simplices in $M_\mathbb{R}$ that are Λ-translates of each other, so we may refer to simplices of J by their representatives in $M_\mathbb{R}$. There is an injective chain map

$\Phi\colon C_\bullet(\mathrm{Jac}(G)) \to S_\bullet(J)$ sending the ordered q-simplex $(u_0, \ldots, u_q)$ to the affine singular simplex $(u_0, \ldots, u_q)$; we naturally identify $C_\bullet(\mathrm{Jac}(G)) \cong \operatorname{im}\Phi$. It is clear that the given generators for $H_q(J)$ for $q \in \{1, 2\}$ are in $\operatorname{im}\Phi$, so the induced map $H_q(\mathrm{Jac}(G)) \to H_q(J)$ is surjective. We claim that it is in fact an isomorphism.

It suffices to show that the injection Φ splits, since then it must induce an injection on homology. Indeed, let $A_\bullet(J) \subseteq S_\bullet(J)$ denote the subcomplex generated by affine simplices; by construction, $\operatorname{im}\Phi \subseteq A_\bullet(J)$. We define chain maps $F\colon S_\bullet(J) \to A_\bullet(J)$ sending a singular simplex to the affine simplex with the same vertices and $G\colon A_\bullet(J) \to \operatorname{im}\Phi$ sending $(u_0, \ldots, u_q) \mapsto (\lfloor u_0 \rfloor, \ldots, \lfloor u_q \rfloor)$, where $\lfloor \cdot \rfloor$ applies the usual floor function to each coordinate of the argument with respect to the basis $\lambda_1, \ldots, \lambda_g, w_1, w_2, \ldots$ of $M_{\mathbb{R}}$.

$$
\begin{tikzcd}
C_\bullet(\mathrm{Jac}(G)) \arrow[d, "\cong"] \arrow[drr, "\Phi"] & & \\
\operatorname{im}\Phi \arrow[r, hook] & A_\bullet(J) \arrow[r, hook] \arrow[l, bend left, "G"] & S_\bullet(J) \arrow[l, bend left, "F"]
\end{tikzcd}
$$

The restriction of GF to $\operatorname{im}\Phi$ is the identity, as desired.

Since $H_q(\mathrm{Jac}(G)) \to H_q(J)$ is an isomorphism for $q \in \{1, 2\}$, the chosen generators of $H_q(J)$ pull back to corresponding generators for $H_q(\mathrm{Jac}(G))$. The final result then follows from (5.5). $\square$

Suppose that G and G' are graphs with universal Jacobians $\mathrm{Jac}(G) = N_R/\Lambda$ and $\mathrm{Jac}(G') = N'_{R'}/\Lambda'$, respectively. Then any $\mathbb{Z}$-linear map $f\colon N_R \to N'_{R'}$ for which $f(\Lambda) \subseteq \Lambda'$ and $f(N) \subseteq N'$ induces a chain map $f_*\colon C_{p,\bullet}(\mathrm{Jac}(G)) \to C_{p,\bullet}(\mathrm{Jac}(G'))$ via

$$f_*(n_1 \wedge \ldots \wedge n_p \otimes (u_0, \ldots, u_q)) := f(n_1) \wedge \ldots \wedge f(n_p) \otimes (f(u_0), \ldots, f(u_q)). \tag{5.6}$$

5.2.3 Universal Ceresa cycle

Fix a basepoint $v \in V$ and define the *Abel–Jacobi map* $\Phi_v\colon V \to \mathrm{Jac}(G)$ by sending $w \mapsto \pi(\delta)$, where δ is a path from v to w. Just as for the Abel–Jacobi map associated to a tropical curve, Φ_v is well-defined modulo $\Lambda = \pi(H_1(G;\mathbb{Z}))$. By an abuse of notation, we also define $\Phi_v \colon E \to C_{1,1}(\mathrm{Jac}(G))$ by

$$\Phi_v(e) := x_e^{-1}\pi(e) \otimes (\pi(\delta_e), \pi(\delta_e + e)),$$

where δ_e is a path from v to e^-. Notice that $x_e^{-1}\pi(e) \in N$ and that $\pi(\delta_e)$ and $\pi(\delta_e + e)$ are representatives in N_R of $\Phi_v(e^-)$ and $\Phi_v(e^+)$ respectively.

Taking (5.1) as inspiration, we define the *(universal) Ceresa cycle* of G based at v by

$$\mathrm{Cer}_v(G) := \sum_{e\in E} \Phi_v(e) - [-1]_* \left(\sum_{e\in E} \Phi_v(e) \right), \tag{5.7}$$

where the inversion map $[-1] \colon N_R \to N_R$ induces a pushforward as in (5.6).

Lemma 5.3. $\mathrm{Cer}_v(G)$ *is independent of the orientation on the edges up to adding the boundary of a degenerate* $(1,2)$*-chain.*

Proof. Let $\bar{e}$ be the edge e with the opposite orientation as in §2.5.1. In particular, $x_{\bar{e}} = x_e$, and we may take $\delta_{\bar{e}} = \delta_e + e$. Consequently,

$$\begin{aligned}
\Phi_v(\bar{e}) &= x_{\bar{e}}^{-1}\pi(\bar{e}) \otimes (\pi(\delta_{\bar{e}}), \pi(\delta_{\bar{e}} + \bar{e})) \\
&= -x_e^{-1}\pi(e) \otimes (\pi(\delta_e + e), \pi(\delta_e + e + \bar{e})) \\
&= -x_e^{-1}\pi(e) \otimes (\pi(\delta_e + e), \pi(\delta_e)) \\
&= \Phi_v(e) - \partial\left(x_e^{-1}\pi(e) \otimes \left((\pi(\delta_e), \pi(\delta_e + e), \pi(\delta_e)) + (\pi(\delta_e), \pi(\delta_e), \pi(\delta_e)) \right) \right). \qquad \square
\end{aligned}$$

Lemma 5.4. *The* $(1,1)$*-chain* $\sum_{e\in E} \Phi_v(e)$ *has trivial boundary.*

Proof. We compute

$$\partial\Phi_v(e) = x_e^{-1}\pi(e) \otimes ((\pi(\delta_e + e)) - (\pi(\delta_e))).$$

Then $\partial\left(\sum_{e\in E} \Phi_v(e)\right)$ is supported on the subset $\Phi_v(V) \subseteq \mathrm{Jac}(G)$, so it suffices to fix $w \in V$ and show that the part of the boundary supported on $\Phi_v(w)$ vanishes. The only edges that have a boundary component at $\Phi_v(w)$ are those that are incident to w. We further restrict our attention to the non-loop edges, since if e is a loop edge, $\pi(e) \in \Lambda$, hence $\partial\Phi_v(e) = 0$.

Let δ be a path from v to w, and label the non-loop edges adjacent to w by $e_1, \ldots, e_n$. By Lemma 5.3, up to adding a boundary, we may assume that each e_i is oriented away from w. We have that $\Phi_v(e_i) = x_{e_i}^{-1}\pi(e_i) \otimes (\pi(\delta), \pi(\delta + e_i))$. This contributes a boundary component of $-x_{e_i}^{-1}\pi(e_i) \otimes (\pi(\delta))$ at $\Phi_v(w)$, so we need only show that $\sum_{i=1}^n x_{e_i}^{-1}\pi(e_i) = 0$, or equivalently, that $\sum_{i=1}^n x_{e_i}^{-1}[e_i, \gamma] = 0$ for all $\gamma \in H_1(G;\mathbb{Z})$. Indeed, we may write $\gamma = \sum_{i=1}^n c_i e_i + \gamma'$, where γ' is supported away from the edges $e_1, \ldots, e_n$. The fact that $\partial\gamma = 0$ forces $\sum_{i=1}^n c_i = 0$, so

$$\sum_{i=1}^n x_{e_i}^{-1}[e_i, \gamma] = \sum_{i=1}^n c_i = 0. \qquad \square$$

Immediately, we get that $\mathrm{Cer}_v(G)$ defines a $(1,1)$-cycle, and so descends to a homology class in $H_{1,1}(\mathrm{Jac}(G))$. In fact, more is true:

Lemma 5.5. $\mathrm{Cer}_v(G)$ *is trivial in* $H_{1,1}(\mathrm{Jac}(G))$.

Proof. By (5.7) and Lemma 5.4, it suffices to show that $[-1]_*$ acts as the identity on $H_{1,1}(\mathrm{Jac}(G))$. We need only check this on the generators described in Lemma 5.2.

$$\begin{aligned}
[-1]_*(n_i \otimes (0, \lambda_j)) &= -n_i \otimes (0, -\lambda_j) \\
&= -n_i \otimes (\lambda_j, 0) \\
&= n_i \otimes (0, \lambda_j) - \partial\left(n_i \otimes (0, \lambda_j, 0) + n_i \otimes (0, 0, 0)\right),
\end{aligned}$$

The first equality follows by definition of the pushforward from (5.6), while the second follows from the equivalence of simplices under Λ-translation defined in (5.4). $\square$

5.2.4 Universal Ceresa period

We take exterior powers of $N_R \cong R^g$ with respect to its R-module structure. We no longer have a topology, let alone a smooth structure for integration, but we can still mimic the application of Ψ on affine simplices given by (5.3). Accordingly, we define $\Psi\colon C_{1,2}(\mathrm{Jac}(G)) \to \bigwedge^3 N_R$ by

$$\Psi(n \otimes (u_0, u_1, u_2)) := n \wedge (u_1 - u_0) \wedge (u_2 - u_0).$$

This is well-defined because it remains unchanged if we replace the simplex (u_0, u_1, u_2) with any of its Λ-translates $(u_0 + \lambda, u_1 + \lambda, u_2 + \lambda)$.

A direct computation shows that $\Psi \circ \partial = 0$, so $\mathscr{P} := \Psi(H_{1,2}(\mathrm{Jac}(G)))$ is well-defined. Let $\Sigma \in C_{1,2}(\mathrm{Jac}(G))$ be such that $\partial\Sigma = \mathrm{Cer}_v(G)$. Then the *(universal) Ceresa period* of G is the image of $\Psi(\Sigma)$ in $\bigwedge^3 N_R/\mathscr{P}$. Just as for $\alpha(\Gamma)$ in §5.1.3, $\alpha(G)$ depends neither on the choice of Σ nor on the basepoint $v \in V$, although the latter fact will not be proved until Corollary 5.12. Although we will need the tools of §5.3 to compute $\alpha(G)$ efficiently, see §5.4.1 for the minimal examples of nontrivial Ceresa periods.

5.2.5 Evaluation

Fix a tropical curve Γ and a model (G, ℓ). We let $[\cdot,\cdot]$ continue to denote the length pairing with values in R. To avoid confusion, we adopt the notation $[\cdot,\cdot]_\Gamma$ for the pairing defined by (2.33). Likewise, we adopt the notations π_Γ, Λ_Γ, Ψ_Γ, and $\mathscr{P}_\Gamma$ for the "real edge length" analogues of the "universal" π, Λ, Ψ, and $\mathscr{P}$, respectively. The length function ℓ determines an *evaluation* map $\mathrm{ev}\colon R \to \mathbb{R}$ via $x_e \mapsto \ell(e)$. This in turn induces maps $N_R \to N_{\mathbb{R}}$ and $\bigwedge^3 N_R \to \bigwedge^3 N_{\mathbb{R}}$, which we also denote by ev.

Proposition 5.6. *If Γ is a tropical curve with underlying graph G, then $\alpha(\Gamma) = \mathrm{ev}(\alpha(G))$.*

Proof. By definition, $\mathrm{ev}(N) = N$. Moreover, it follows from the definitions given in §§2.6.4 and 5.2.1 that $[\cdot,\cdot]_\Gamma = \mathrm{ev} \circ [\cdot,\cdot]$, hence $\pi_\Gamma = \mathrm{ev} \circ \pi$ and $\Lambda_\Gamma = \mathrm{ev}(\Lambda)$. As a result, we obtain a chain map $\mathrm{ev}_*\colon C_\bullet(\mathrm{Jac}(G)) \to C_\bullet^{\mathrm{aff}}(\mathrm{Jac}(\Gamma); \mathbb{Z})$ sending $(u_0, \ldots, u_q)$ to the affine q-chain $(\mathrm{ev}(u_0), \ldots, \mathrm{ev}(u_q))$, which extends to (p, q)-chains by preserving the $\bigwedge^p N$-component. Comparing (5.1) and (5.7), one sees that $\mathrm{Cer}_{v,G}(\Gamma) = \mathrm{ev}_* \mathrm{Cer}_v(G)$.

Now fix $\Sigma \in C_{1,2}(\mathrm{Jac}(G))$ for which $\partial\Sigma = \mathrm{Cer}_v(G)$. Then the affine $(1, 2)$-chain $\Sigma_\Gamma := \mathrm{ev}_* \Sigma$ satisfies $\partial\Sigma_\Gamma = \mathrm{Cer}_{v,G}(\Gamma)$. One sees also that ev_* sends the generators of $H_{1,2}(\mathrm{Jac}(G))$ given by Lemma 5.2 to the generators of $H_{1,2}(\mathrm{Jac}(\Gamma))$ in (2.31). Moreover, (5.3) ensures that $\Psi_\Gamma \circ \mathrm{ev}_* = \mathrm{ev} \circ \Psi$; in particular, $\mathscr{P}_\Gamma = \mathrm{ev}(\mathscr{P})$ and $\Psi_\Gamma(\Sigma_\Gamma) = \mathrm{ev}(\Psi(\Sigma))$. Then $\alpha(\Gamma) = \mathrm{ev}(\alpha(G))$, as desired. $\square$

Corollary 5.7. *If $\alpha(G) = 0$, then $\alpha(\Gamma) = 0$.*

Proof. This follows immediately from Proposition 5.6. □

We naturally associate to a graph G a moduli space of isomorphism classes of tropical curves overlying G as follows (but for more general constructions, see, e.g., [Cha13, Section 4.1]). The group $\operatorname{Aut}(G)$ acts on E; this induces an action on $\mathbb{R}^E_{>0}$ by permuting the coordinates. Then let

$$\mathscr{M}_g^{\mathrm{tr}} := \mathbb{R}^E_{>0} / \operatorname{Aut}(G).$$

Each point of $\mathscr{M}_g^{\mathrm{tr}}$ determines a length function ℓ on G and hence the tropical curve with (G, ℓ) as a model. In analogy to the classical setting, we say that a property is true of a *very general* tropical curve overlying G if it is true of every tropical curve corresponding to a point on $\mathscr{M}_g^{\mathrm{tr}} \setminus H$, where H is some countable union of hypersurfaces. Then Corollary 5.7 has the following partial converse.

Proposition 5.8. *If $\alpha(G) \neq 0$, then a very general tropical curve overlying G has nontrivial Ceresa period.*

Proof. Let Γ be a tropical curve with (G, ℓ) as a model, and adopt the notation used in the proof of Proposition 5.6. If we choose generators $p_1, \ldots, p_r$ of $\mathscr{P}$, then $\operatorname{ev}(p_1), \ldots, \operatorname{ev}(p_r)$ generate $\mathscr{P}_\Gamma$. Suppose that $\alpha(\Gamma) = 0$. Then by definition, $\Psi_\Gamma(\Sigma_\Gamma) \in \mathscr{P}_\Gamma$, so we may write

$$\operatorname{ev}(\Psi(\Sigma)) = \Psi_\Gamma(\Sigma_\Gamma) = \sum_{i=1}^{r} a_i \operatorname{ev}(p_i) = \operatorname{ev}\left(\sum_{i=1}^{r} a_i p_i\right)$$

for some $a_i \in \mathbb{Z}$, hence

$$S := \Psi(\Sigma) - \sum_{i=1}^{r} a_i p_i \in \ker \operatorname{ev}.$$

Observe that $S \in \bigwedge^3 N_R$. Since $\bigwedge^3 N_R \cong R \otimes_{\mathbb{Z}} \bigwedge^3 N$ is a free R-module of finite rank, we may fix a basis and write S as an R-linear combination of the basis elements. Then $S \in \ker \operatorname{ev}$ precisely when all of its finitely many R-coefficients vanish under ev. The fact that $\alpha(G) \neq 0$ implies that $S \neq 0$, so these coefficients cut out a subvariety of $\mathbb{R}^E_{>0}$ of positive codimension.

The desired statement follows by considering all possible choices of the a_i, of which there are countably many. $\square$

Corollary 5.9. *If $\alpha(G) \neq 0$, then a very general tropical curve overlying G has algebraically nontrivial Ceresa cycle.*

Proof. The result follows immediately from Propositions 5.1 and 5.8. $\square$

5.3 Explicit computation of the Ceresa period

5.3.1 Explicit $(1,2)$-chain Υ whose boundary is $\mathrm{Cer}_v(G)$

Given a graph G with basepoint $v \in V$, we shall define a $(1,2)$-chain Υ for which $\partial\Upsilon = \mathrm{Cer}_v(G)$. To that end, we fix an orientation $\mathcal{O}$ on G and a spanning tree T and label the edges of $G \setminus T$ by $e_1, \dots, e_n$. Each e_i determines a simple cycle $\gamma_i := \gamma_{e_i}$ as in §2.5.1. Order the edges of γ_i cyclically. Then $H_1(G;\mathbb{Z})$ has $\gamma_1, \dots, \gamma_g$ as a basis; let $n_1, \dots, n_g$ be the dual basis for N, i.e.,

$$n_i(\gamma_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

For $e \in E$, the integer $f_i(e) := x_e^{-1}[e, \gamma_i]$ counts the multiplicity of e in γ_i; by our choice of homology basis, $f_i(e) \in \{0, \pm 1\}$. Observe also that

$$\pi(e) = \sum_{i=1}^{g} [e, \gamma_i] n_i = x_e \sum_{i=1}^{g} f_i(e) n_i.$$

From (5.7), we write

$$\begin{aligned}
\mathrm{Cer}_v(G) &= \sum_{e \in E} (\Phi_v(e) - [-1]_* \Phi_v(e)) \\
&= \sum_{e \in E} \big(x_e^{-1}\pi(e) \otimes (\pi(\delta_e), \pi(\delta_e + e)) + x_e^{-1}\pi(e) \otimes (-\pi(\delta_e), -\pi(\delta_e + e)) \big) \\
&= \sum_{e \in E} \sum_{i=1}^{g} f_i(e) n_i \otimes \big((\pi(\delta_e), \pi(\delta_e + e)) + (-\pi(\delta_e), -\pi(\delta_e + e)) \big) \\
&= \sum_{i=1}^{g} n_i \otimes \sum_{e \in E} f_i(e) \big((\pi(\delta_e), \pi(\delta_e + e)) + (-\pi(\delta_e), -\pi(\delta_e + e)) \big).
\end{aligned} \tag{5.8}$$

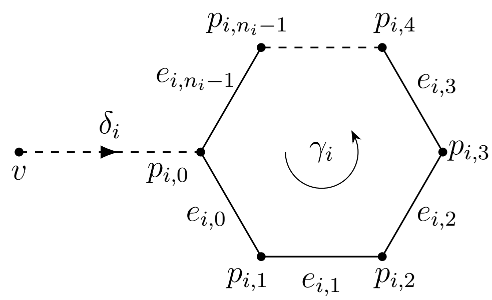


Figure 5.1: Notation for the cycle γ_i

Fix the unique path δ_i from v to a vertex $p_{i,0}$ in γ_i so that δ_i is contained in T and does not share any edges with γ_i. Following the cyclic orientation of γ_i, label the subsequent vertices by $p_{i,1}, p_{i,2}, \ldots$ with $p_{i,n_i} = p_{i,0}$. Write $e_{i,j}$ for the edge in γ_i with vertices $p_{i,j}$ and $p_{i,j+1}$ (see Fig. 5.1). For $0 \leq j \leq n_i$, we define

$$u_{i,j} := \pi\left(\delta_i + \sum_{k=0}^{j-1} f_i(e_{i,k})e_{i,k}\right).$$

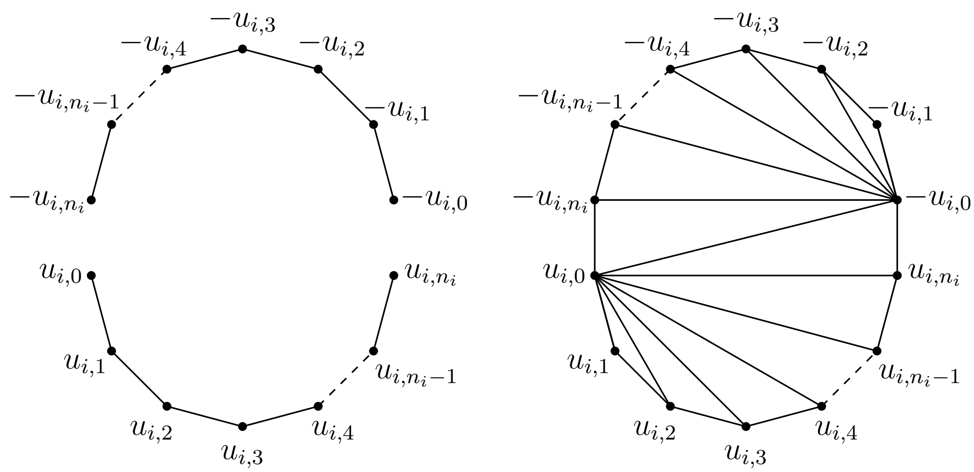
Notice that $\delta_i + \sum_{k=0}^{j-1} f_i(e_{i,k})e_{i,k}$ is a path from v to $p_{i,j}$ that passes through $p_{i,0}, p_{i,1}, \ldots, p_{i,j-1}$; in particular, $u_{i,j}$ is a lift of $\Phi_v(p_{i,j})$ to N_R, and (5.8) reduces to

$$\mathrm{Cer}_v(G) = \sum_{i=1}^{g} n_i \otimes \sum_{j=0}^{n_i-1} \begin{cases} (u_{i,j}, u_{i,j+1}) + (-u_{i,j}, -u_{i,j+1}) & \text{if } f_i(e_{i,j}) = 1 \\ -(u_{i,j+1}, u_{i,j}) - (-u_{i,j+1}, -u_{i,j}) & \text{if } f_i(e_{i,j}) = -1 \end{cases}, \tag{5.9}$$

the n_i-component of which is depicted in Fig. 5.2(a). This cumbersome notation is necessary because, unlike for simplicial homology, we are not able to permute the vertices of a simplex at the cost of introducing a sign. The alternative that works in the current context is the following:

$$\begin{aligned} -(u_{i,j+1}, u_{i,j}) - (-u_{i,j+1}, -u_{i,j}) = {} & (u_{i,j}, u_{i,j+1}) + (-u_{i,j}, -u_{i,j+1}) \\ & - \partial(u_{i,j}, u_{i,j+1}, u_{i,j}) - \partial(u_{i,j}, u_{i,j}, u_{i,j}) \\ & - \partial(-u_{i,j}, -u_{i,j+1}, -u_{i,j}) - \partial(-u_{i,j}, -u_{i,j}, -u_{i,j}). \end{aligned} \tag{5.10}$$

In other words, we must add the boundary of a degenerate $(1,2)$-chain in order to permute the vertices of each $(1,1)$-simplex.



(a) Support of the n_i-component of $\mathrm{Cer}_v(G)$ in N_R

(b) Support of Υ_i

Figure 5.2

Now that we have expressed $\mathrm{Cer}_v(G)$ more explicitly, we define $\Upsilon := \sum_{i=1}^{g} n_i \otimes \Upsilon_i$, where

$$\begin{aligned}\Upsilon_i := \sum_{j=1}^{n_i-1} &((u_{i,0}, u_{i,j}, u_{i,j+1}) + (-u_{i,0}, -u_{i,j}, -u_{i,j+1})) + (u_{i,0}, u_{i,n_i}, -u_{i,0}) + (u_{i,0}, -u_{i,0}, -u_{i,n_i}) \\ &- \sum_{j|f_i(e_{i,j})=-1} ((u_{i,j}, u_{i,j+1}, u_{i,j}) + (u_{i,j}, u_{i,j}, u_{i,j}) + (-u_{i,j}, -u_{i,j+1}, -u_{i,j}) + (-u_{i,j}, -u_{i,j}, -u_{i,j}))\,.\end{aligned} \tag{5.11}$$

Using the fact that $u_{i,n_i} = u_{i,0} + \lambda_i$, a straightforward computation yields

$$\begin{aligned}\partial\Upsilon_i = &\sum_{j=0}^{n_i-1} ((u_{i,j}, u_{i,j+1}) + (-u_{i,j}, -u_{i,j+1})) \\ &- \sum_{j|f_i(e_{i,j})=-1} \partial\left((u_{i,j}, u_{i,j+1}, u_{i,j}) + (u_{i,j}, u_{i,j}, u_{i,j}) + (-u_{i,j}, -u_{i,j+1}, -u_{i,j}) + (-u_{i,j}, -u_{i,j}, -u_{i,j})\right),\end{aligned}$$

which by (5.10) equals the n_i-component of (5.9). This is depicted without orientations or degenerate simplices in Fig. 5.2. We conclude that $\partial\Upsilon = \mathrm{Cer}_v(G)$.

Remark 5.10. Notice that each $u_{i,j}$ is independent of the orientations on the edges of G. Consequently, Υ is also independent of the orientations.

5.3.2 Explicit representative $\Psi(\Upsilon)$ for $\alpha(G)$

Since Ψ kills degenerate $(1,2)$-simplices, we may ignore the second summation in (5.11) in the following computation:

$$\begin{aligned}
\Psi(\Upsilon) &= \sum_{i=1}^{g} \Psi(n_i \otimes \Upsilon_i) \\
&= \sum_{i=1}^{g} n_i \wedge \left(\sum_{j=1}^{n_i-1} ((u_{i,j} - u_{i,0}) \wedge (u_{i,j+1} - u_{i,0}) + (-u_{i,j} + u_{i,0}) \wedge (-u_{i,j+1} + u_{i,0})) \right. \\
&\qquad\qquad \left. + (u_{i,n_i} - u_{i,0}) \wedge (-2u_{i,0}) + (-2u_{i,0}) \wedge (-u_{i,n_i} - u_{i,0}) \right) \\
&= \sum_{i=1}^{g} n_i \wedge \left(2 \sum_{j=1}^{n_i-1} (u_{i,j} - u_{i,0}) \wedge (u_{i,j+1} - u_{i,j}) - 4(u_{i,n_i} - u_{i,0}) \wedge u_{i,0} \right) \\
&= \sum_{i=1}^{g} n_i \wedge \left(2 \sum_{j=1}^{n_i-1} \left(\sum_{k=0}^{j-1} \pi(f_i(e_{i,k}) e_{i,k}) \right) \wedge \pi(f_i(e_{i,j}) e_{i,j}) - 4\pi(\gamma_i) \wedge \pi(\delta_i) \right) \\
&= \sum_{i=1}^{g} n_i \wedge \left(2 \sum_{0 \le k < j \le n_i - 1} \pi(f_i(e_{i,k}) e_{i,k}) \wedge \pi(f_i(e_{i,j}) e_{i,j}) - 4\pi(\gamma_i) \wedge \pi(\delta_i) \right). \qquad (5.12)
\end{aligned}$$

Since we have bases $n_1, \ldots, n_g$ for N and $\pi(\gamma_1), \ldots, \pi(\gamma_g)$ for Λ, Lemma 5.2 tells us that

$$H_{1,2}(\mathrm{Jac}(G)) = \mathbb{Z}\langle n_i \otimes ((0, \pi(\gamma_j), \pi(\gamma_j + \gamma_k)) - (0, \pi(\gamma_k), \pi(\gamma_j + \gamma_k))) \mid i, j, k \in [g],\ j < k \rangle.$$

Applying Ψ, we find that

$$\mathscr{P} = \mathbb{Z}\langle 2n_i \wedge \pi(\gamma_j) \wedge \pi(\gamma_k) \mid i, j, k \in [g],\ j < k \rangle. \qquad (5.13)$$

Since $\pi(\gamma_j) = \sum_{p=1}^{g} [\gamma_j, \gamma_p] n_p$, we may rewrite

$$2n_i \wedge \pi(\gamma_j) \wedge \pi(\gamma_k) = 2 \sum_{p=1}^{g} \sum_{q=1}^{g} [\gamma_j, \gamma_p][\gamma_k, \gamma_q] n_i \wedge n_p \wedge n_q$$

$$= 2\sum_{p<q} \left([\gamma_j, \gamma_p][\gamma_k, \gamma_q] - [\gamma_j, \gamma_q][\gamma_k, \gamma_p]\right) n_i \wedge n_p \wedge n_q. \tag{5.14}$$

In other words, the $n_i \wedge n_p \wedge n_q$-coefficient of $2n_i \wedge \pi(\gamma_j) \wedge \pi(\gamma_k)$ is twice the $(j,k) \times (p,q)$-minor of the Gram matrix of the pairing $[\cdot, \cdot]$.

5.3.3 Basepoint independence of the Ceresa period

Proposition 5.11. *Fix a graph G and let Υ be the $(1,2)$-chain defined by (5.11). Then $\Psi(\Upsilon)$ does not depend on the choice of basepoint.*

Proof. Since G is connected, it suffices to show that $\Psi(\Upsilon)$ remains unchanged whether computed using v as the basepoint or some vertex v' separated from v by an edge a. Without loss of generality, assume that a is oriented from v' to v.

We observe that, in the computations in §5.3.1, as well as in deriving (5.12), we do not use any properties of δ_i other than that it is a path from v to *some* vertex $p_{i,0}$ in the cycle γ_i. In particular, if we construct Υ' as in §5.3.1 using the basepoint v' and the paths $\delta_i' := \delta_i + a$, then the same argument shows that $\partial\Upsilon' = \mathrm{Cer}_{v'}(G)$. Moreover, the points $p_{i,0}$ have not changed, nor have the edge labels $e_{i,j}$, so it follows from (5.12) that

$$\begin{aligned}
\Psi(\Upsilon') &= \sum_{i=1}^{g} n_i \wedge \left(2 \sum_{0 \le k < j \le n_i - 1} \pi(f_i(e_{i,k})e_{i,k}) \wedge \pi(f_i(e_{i,j})e_{i,j}) - 4\pi(\gamma_i) \wedge \pi(\delta_i + a) \right) \\
&= \Psi(\Upsilon) - 4\sum_{i=1}^{g} n_i \wedge \pi(\gamma_i) \wedge \pi(a) \\
&= \Psi(\Upsilon) - 4\sum_{i=1}^{g} n_i \wedge \sum_{j=1}^{g} [\gamma_i, \gamma_j] n_j \wedge \pi(a) \\
&= \Psi(\Upsilon) - 4\sum_{i,j} [\gamma_i, \gamma_j] n_i \wedge n_j \wedge \pi(a).
\end{aligned}$$

Ordering so that $i < j$ and using the fact that $n_j \wedge n_i = -n_i \wedge n_j$, we obtain

$$= \Psi(\Upsilon) - 4\sum_{i<j} \left([\gamma_i, \gamma_j] - [\gamma_j, \gamma_i]\right) n_i \wedge n_j \wedge \pi(a).$$

Finally, $[\cdot, \cdot]$ is symmetric, so the summation vanishes and we are left with $\Psi(\Upsilon') = \Psi(\Upsilon)$, as desired. $\square$

Corollary 5.12. *The Ceresa period $\alpha(G)$ does not depend on the choice of basepoint.*

Proof. This follows immediately from Proposition 5.11 and the fact that $\Psi(\Upsilon)$ is a representative of $\alpha(G)$. $\square$

Corollary 5.13. *Let Γ be a tropical curve overlying G. Then $\alpha(\Gamma)$ does not depend on the choice of basepoint.*

Proof. Fix $v, v' \in \Gamma$. After refining G if necessary, we may assume that $v, v' \in V$. We recall from Proposition 5.6 that $\alpha(\Gamma) = \mathrm{ev}(\alpha(G))$. But Corollary 5.12 implies that $\alpha(G)$ is the same whether computed using v or v', so the same must be true for $\alpha(\Gamma)$. $\square$

5.3.4 *Rewriting $\Psi(\Upsilon)$ in terms of edge pairs*

Lemma 5.14. *The variable x_{e_i} does not appear in any coefficient of $\Psi(\Upsilon)$ for any $i \in [g]$.*

Proof. For each edge e, $\pi(e) = x_e \sum_{i=1}^{g} f_i(e) n_i \in x_e N$. In particular, the variable x_{e_i} appears in $\pi(e)$ only for $e = e_i$. Since no δ_j contains e_i, and γ_j contains e_i only for $j = i$, the only computation $\Psi(n_j \otimes \Upsilon_j)$ in which $\pi(e_i)$ appears is for $j = i$ (see (5.12)). But every term of $\Psi(n_i \otimes \Upsilon_i)$ with $\pi(e_i) = x_{e_i} n_i$ as a factor also has n_i as a factor, so the alternating property implies that these terms all vanish. $\square$

It is not hard to see from (5.12) that $\Psi(\Upsilon)$ is homogeneous of degree 2 in the variables x_e. We now develop a characterization for when a pair of edges e and e' contributes a monomial $x_e x_{e'}\, n_i \wedge n_j \wedge n_k$ to $\Psi(\Upsilon)$. Recall from §5.3.1 that

$$f_i(e) := x_e^{-1}[e, \gamma_i] = \begin{cases} 0 & \text{if } e \notin |\gamma|_i \\ 1 & \text{if } e \text{ and } \gamma_i \text{ have the same orientation} \\ -1 & \text{if } e \text{ and } \gamma_i \text{ have opposite orientations.} \end{cases}$$

Likewise, we define

$$g_i(e) := x_e^{-1}[e, \delta_i] = \begin{cases} 0 & \text{if } e \notin |\delta|_i \\ 1 & \text{if } e \text{ and } \delta_i \text{ have the same orientation} \\ -1 & \text{if } e \text{ and } \delta_i \text{ have opposite orientations.} \end{cases}$$

Given distinct edges e and e' in γ_i, say that $e <_i e'$ if e occurs before e' in the ordering determined by the endpoint of δ_i and the cyclic orientation of γ_i. Equivalently, in the notation of Fig. 5.1, we declare $e_{i,j} <_i e_{i,j+1}$ for $0 \leq j \leq n_i - 2$. Then let

$$h_i(e, e') := \begin{cases} 0 & \text{if } e \notin |\gamma|_i,\ e' \notin |\gamma|_i, \text{ or } e = e' \\ 1 & \text{if } e <_i e' \\ -1 & \text{if } e' <_i e. \end{cases}$$

Fix a subset $S \subseteq E \times E$ that contains exactly one element from $\{(e, e'), (e', e)\}$ for each pair of distinct edges e and e'. Likewise, in lieu of requiring that $i < j < k$, we instead fix a subset $S' \subseteq [g]^3$ that contains exactly one permutation of each tuple $(i, j, k) \in [g]^3$ of all distinct indices.

Proposition 5.15. *Fix a graph G and let Υ be the $(1,2)$-chain defined by (5.11). Then we may write*

$$\Psi(\Upsilon) = \sum_{\substack{(e,e') \in S \\ (i,j,k) \in S'}} a_{i,j,k}(e, e') x_e x_{e'}\, n_i \wedge n_j \wedge n_k,$$

where $a_{i,j,k}(e, e')$ takes values in $\{0, \pm 2\}$ and is nonzero precisely when, up to relabeling (i, j, k), γ_i contains both e and e', γ_j contains e but not e', and γ_k contains e' but not e. In particular,

$$a_{i,j,k}(e, e') = 2 f_j(e) f_k(e') \left(f_i(e) f_i(e') h_i(e, e') + 2 f_i(e) g_j(e') - 2 f_i(e') g_k(e) \right), \tag{5.15}$$

and reversing the cyclic orientation of any one of the cycles γ_i, γ_j, or γ_k flips the sign.

Proof. From (5.12), we may write

$$\Psi(\Upsilon) = \sum_{i=1}^{g} n_i \wedge \left(2 \sum_{0 \leq k < j \leq n_i - 1} \pi(f_i(e_{i,k}) e_{i,k}) \wedge \pi(f_i(e_{i,j}) e_{i,j}) - 4\pi(\gamma_i) \wedge \pi(\delta_i) \right)$$

$$= 2\sum_{i=1}^{g} n_i \wedge \left(\sum_{\substack{e,e' \in |\gamma|_i \\ e <_i e'}} f_i(e)\pi(e) \wedge f_i(e')\pi(e') - 2\pi(\gamma_i) \wedge \pi(\delta_i) \right)$$

$$= 2\sum_{i=1}^{g} n_i \wedge \left(\sum_{(e,e') \in S} f_i(e) f_i(e') h_i(e,e') \pi(e) \wedge \pi(e') - 2\pi(\gamma_i) \wedge \pi(\delta_i) \right)$$

We may write $\gamma_i = \sum_{e \in E} f_i(e) e$ and $\delta_i = \sum_{e \in E} g_i(e) e$, so that

$$= 2\sum_{i=1}^{g} n_i \wedge \left(\sum_{(e,e') \in S} f_i(e) f_i(e') h_i(e,e') \pi(e) \wedge \pi(e') - 2 \sum_{e,e' \in E} f_i(e) g_i(e') \pi(e) \wedge \pi(e') \right)$$

$$= \sum_{(e,e') \in S} \sum_{i=1}^{g} 2\left(f_i(e) f_i(e') h_i(e,e') - 2 f_i(e) g_i(e') + 2 f_i(e') g_i(e) \right) n_i \wedge \pi(e) \wedge \pi(e').$$

Expanding $\pi(e) = x_e \sum_{j=1}^{g} f_j(e) n_j$ and $\pi(e') = x_{e'} \sum_{k=1}^{g} f_k(e') n_k$, we obtain

$$= \sum_{(e,e') \in S} \sum_{i,j,k} 2\left(f_i(e) f_i(e') h_i(e,e') - 2 f_i(e) g_i(e') + 2 f_i(e') g_i(e) \right) f_j(e) f_k(e') x_e x_{e'}\, n_i \wedge n_j \wedge n_k.$$

We reindex by S' to get

$$= \sum_{(e,e') \in S} \sum_{(i,j,k) \in S'} a_{i,j,k}(e,e') x_e x_{e'}\, n_i \wedge n_j \wedge n_k,$$

where

$$a_{i,j,k}(e,e') := 2(A_i B_{j,k} - A_j B_{i,k} + A_k B_{i,j})$$

$$A_t := f_t(e) f_t(e') h_t(e,e') - 2 f_t(e) g_t(e') + 2 f_t(e') g_t(e)$$

$$B_{m,n} := f_m(e) f_n(e') - f_m(e') f_n(e).$$

As one might expect, reordering the tuple (e,e') does not affect $a_{i,j,k}(e,e')$, while permuting (i,j,k) changes it by the sign of the permutation. We record here the possible combinations of values of $|f_{\cdot}(\cdot)|$ up to such reordering:

$$\begin{array}{c|ccc} \text{(a)} & i & j & k \\ \hline e & 1 & 1 & * \\ e' & 1 & 1 & * \end{array} \qquad \begin{array}{c|ccc} \text{(b)} & i & j & k \\ \hline e & 1 & 1 & 0 \\ e' & 1 & 0 & 1 \end{array} \qquad \begin{array}{c|ccc} \text{(c)} & i & j & k \\ \hline e & * & 1 & 1 \\ e' & * & 0 & 0 \end{array} \qquad \begin{array}{c|ccc} \text{(d)} & i & j & k \\ \hline e & * & * & 0 \\ e' & * & * & 0 \end{array}.$$

A $*$ indicates that the corresponding entry can have value either 0 or 1. While these cases are not all pairwise disjoint, they do cover all the possibilities. Indeed, up to permuting the rows and columns, if zero or one of the six values is 0, then we are in case (a). If exactly two values are 0, then we fall into one of (b), (c), or (d). If three are 0, then we are in cases (c) or (d). Finally, if at least four values are 0, then case (d) applies.

We claim that the $a_{i,j,k}(e,e') \neq 0$ only in case (b). As a shortcut, we may consider only edges e and e' in T; by Lemma 5.14, the remaining edges do not contribute terms. Then $T \setminus \{e, e'\}$ has three connected components; let G' be the graph obtained from G by contracting each of these components to a vertex and removing the additional edges e_t for $t \notin \{i,j,k\}$. We depict in Fig. 5.3 the resulting graph for each of the cases (a), (b), and (c) with the edges e_i, e_j, e_k drawn as necessary. The basepoint v descends to one of the three vertices of G', but by Proposition 5.11, we may fix it arbitrarily. We remark that the values of f_t, g_t, and h_t on e and e' remain unchanged for $t \in \{i,j,k\}$ when passing from G to G'.

The following observations will be helpful in further narrowing down the cases. By Remark 5.10, $\Psi(\Upsilon)$ is independent of the orientations on the edges of G. Then without loss of generality, we may orient the edges of G' as shown in Fig. 5.3. Meanwhile, replacing γ_i with the cycle $\bar{\gamma}_i := -\gamma_i$ having the opposite cyclic orientation of edges (i.e., the reverse of the partial order $<_i$) changes the sign of f_i and h_i. Hence, only A_i, $B_{i,k}$, and $B_{i,j}$ change sign, thereby causing $a_{i,j,k}(e,e')$ also to change sign. The same applies to the indices j and k. Notably, this means that the choice for each t of γ_t versus $\bar{\gamma}_t$ does not affect whether or not $a_{i,j,k}(e,e') = 0$, so in Fig. 5.3, we orient γ_t (not depicted) in the same direction as e_t. Finally, recall from §5.3.1 that the path δ_t is defined uniquely by the fact that it lies in T and has disjoint support from the cycle γ_t.

To reiterate, all of the choices that went into drawing Fig. 5.3 were made without loss of generality up to a sign. Therefore, one may read off the values of f_t, g_t, and h_t on e and e' for each of the relevant indices of t from $\{i,j,k\}$; one finds that $a_{i,j,k}(e,e')$ vanishes in cases (a) and (c) and equals -2 in case (b). That $a_{i,j,k}(e,e')$ vanishes in case (d) is not hard to see, since $f_k(e) = f_k(e') = 0$ forces $B_{j,k} = B_{i,k} = A_k = 0$. The particular expression for the

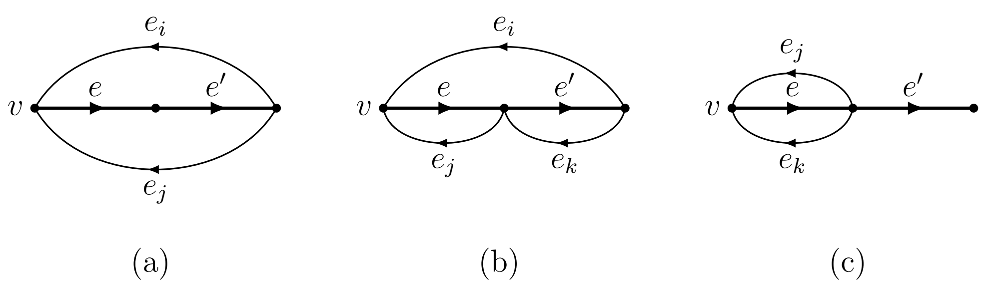


Figure 5.3

coefficient given by (5.15) in the statement of the result follows from case (b) by plugging into $a_{i,j,k}(e,e')$ only the values of the indicator functions that vanish. $\square$

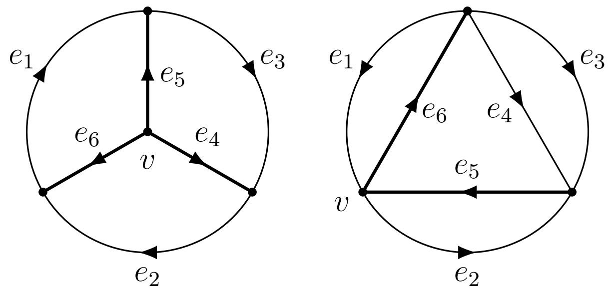
5.4 Forbidden minor characterization of nontriviality of the Ceresa period

With the powerful tools of §5.3 in hand, we now endeavor to prove Theorem 1.7 by following the approach of [CL24]. In §5.4.1, we show that K_4 and L_3, the graphs depicted in Fig. 5.4, have nontrivial Ceresa period. In §§5.4.2 and 5.4.3, we show that having nontrivial Ceresa period is preserved under certain contraction and deletion operations, respectively. Finally, in §5.4.4, we show that graphs of hyperelliptic type have trivial Ceresa period and that there are no other cases left to consider.

Throughout this section, whenever we are working with two graphs G and G', for each object O associated to G that was defined in §§5.2 and 5.3, we let O' denote the corresponding object for G'.

5.4.1 Base cases

Example 5.16. Let $G := K_4$ be the graph with basepoint v and oriented edges e_i as depicted in Fig. 5.4(a), with $T = \{e_4, e_5, e_6\}$, paths $\delta_i = 0$ for all i, and the cyclic orientation of γ_i



(a) K_4 (b) L_3

Figure 5.4: Minimal graphs with nontrivial Ceresa period

chosen to match the orientation of e_i. Let $x_i := x_{e_i}$. Then $[\cdot,\cdot]$ has Gram matrix

$$\begin{pmatrix} x_1+x_5+x_6 & -x_6 & -x_5 \\ -x_6 & x_2+x_4+x_6 & -x_4 \\ -x_5 & -x_4 & x_3+x_4+x_5. \end{pmatrix}.$$

One can show using either (5.12) or Proposition 5.15 that

$$\Psi(\Upsilon) = 2(x_4x_5 + x_4x_6 + x_5x_6)\, n_1 \wedge n_2 \wedge n_3.$$

Meanwhile, (5.13) and (5.14) imply that $\mathscr{P}$ is generated by the elements

$$2(x_2x_5 + x_4x_5 + x_4x_6 + x_5x_6)\, n_1 \wedge n_2 \wedge n_3,$$
$$2(-x_4x_5 - x_3x_6 - x_4x_6 - x_5x_6)\, n_1 \wedge n_2 \wedge n_3,$$
$$2(x_2x_3 + x_2x_4 + x_3x_4 + x_2x_5 + x_4x_5 + x_3x_6 + x_4x_6 + x_5x_6)\, n_1 \wedge n_2 \wedge n_3,$$
$$2(x_1x_4 + x_4x_5 + x_4x_6 + x_5x_6)\, n_1 \wedge n_2 \wedge n_3,$$
$$2(-x_1x_3 - x_1x_4 - x_1x_5 - x_3x_5 - x_4x_5 - x_3x_6 - x_4x_6 - x_5x_6)\, n_1 \wedge n_2 \wedge n_3,$$
$$2(x_4x_5 + x_3x_6 + x_4x_6 + x_5x_6)\, n_1 \wedge n_2 \wedge n_3,$$
$$2(x_1x_2 + x_1x_4 + x_2x_5 + x_4x_5 + x_1x_6 + x_2x_6 + x_4x_6 + x_5x_6)\, n_1 \wedge n_2 \wedge n_3,$$
$$2(-x_1x_4 - x_4x_5 - x_4x_6 - x_5x_6)\, n_1 \wedge n_2 \wedge n_3,$$

$$2(x_2x_5 + x_4x_5 + x_4x_6 + x_5x_6)\, n_1 \wedge n_2 \wedge n_3.$$

Simplifying, we are left with six generators:

$$\begin{aligned}
2(x_1x_4 - x_3x_6)\, n_1 \wedge n_2 \wedge n_3,\\
2(x_2x_5 - x_3x_6)\, n_1 \wedge n_2 \wedge n_3,\\
2(x_1x_2 + x_2x_5 + x_1x_6 + x_2x_6)\, n_1 \wedge n_2 \wedge n_3,\\
2(x_1x_3 + x_1x_5 + x_3x_5 + x_3x_6)\, n_1 \wedge n_2 \wedge n_3,\\
2(x_2x_3 + x_2x_4 + x_3x_4 + x_3x_6)\, n_1 \wedge n_2 \wedge n_3,\\
2(x_4x_5 + x_3x_6 + x_4x_6 + x_5x_6)\, n_1 \wedge n_2 \wedge n_3.
\end{aligned}$$

Suppose that we could obtain $\Psi(\Upsilon)$ as some $\mathbb{Z}$-linear combination of these generators. Since x_1x_4, x_1x_2, x_1x_3, and x_2x_3 each appear in only one generator and not in $\Psi(\Upsilon)$, those generators cannot contribute. This leaves the second and the sixth. Continuing along the same lines, the second generator uniquely contains the term x_2x_5, so it must be trivial. Now the sixth generator uniquely contains x_3x_6, so it also vanishes, a contradiction. In other words, $\alpha(K_4) \neq 0$, with

$$\alpha(K_4) = 2(x_4x_5 + x_4x_6 + x_5x_6)\, n_1 \wedge n_2 \wedge n_3 \pmod{\mathscr{P}}.$$

Example 5.17. Consider the graph $G := L_3$ with basepoint v and oriented edges e_i as shown in Fig. 5.4(b), with $T = \{e_5, e_6\}$, paths $\delta_i = 0$ for all i, and the cyclic orientation of γ_i matching e_i. We again abbreviate $x_i := x_{e_i}$. Then $[\cdot,\cdot]$ has Gram matrix

$$\begin{pmatrix}
x_1 + x_6 & 0 & x_6 & x_6\\
0 & x_2 + x_5 & x_5 & x_5\\
x_6 & x_5 & x_3 + x_5 + x_6 & x_5 + x_6\\
x_6 & x_5 & x_5 + x_6 & x_4 + x_5 + x_6
\end{pmatrix}.$$

One can show that

$$\Psi(\Upsilon) = 2x_5x_6(n_1 \wedge n_2 \wedge n_4 + n_1 \wedge n_2 \wedge n_3).$$

By Corollary 5.7, to prove that $\alpha(L_3) \neq 0$, it suffices to show that some tropical curve overlying L_3 has nontrivial Ceresa period. Indeed, let Γ be the tropical curve obtained by evaluating every edge length x_i to 1, with the corresponding map $\mathrm{ev}\colon N_R \to N_{\mathbb{R}}$. Then $\mathscr{P}_\Gamma$ is generated by the four elements

$$\begin{aligned} 2n_2 \wedge n_3 \wedge n_4 \qquad\qquad\qquad &+ 2n_1 \wedge n_2 \wedge n_4 + 2n_1 \wedge n_2 \wedge n_3, \\ 2n_1 \wedge n_3 \wedge n_4 &+ 2n_1 \wedge n_2 \wedge n_4 + 2n_1 \wedge n_2 \wedge n_3, \\ &\ 4n_1 \wedge n_2 \wedge n_4 \qquad\qquad\qquad, \\ &\qquad\qquad\qquad\quad 4n_1 \wedge n_2 \wedge n_3. \end{aligned}$$

Let $\Upsilon_\Gamma := \mathrm{ev}_* \Upsilon$. Then $\Psi(\Upsilon_\Gamma) = 2n_1 \wedge n_2 \wedge n_4 + 2n_1 \wedge n_2 \wedge n_3$, which is clearly not in $\mathscr{P}_\Gamma$, as desired. Therefore, $\alpha(L_3) \neq 0$, with

$$\alpha(L_3) = 2x_5x_6(n_1 \wedge n_2 \wedge n_4 + n_1 \wedge n_2 \wedge n_3) \pmod{\mathscr{P}}.$$

5.4.2 Contraction

Given a graph G with a non-loop edge a, we let G/a denote the graph obtained from G by contracting a. We identify $E(G) = E(G/a) \sqcup \{a\}$ in the usual way.

Proposition 5.18. *Let G be a graph with a non-loop edge a. If $\alpha(G) = 0$, then $\alpha(G/a) = 0$. Moreover, if a is a separating edge, then the converse also holds.*

Although the explicit $(1,2)$-chain Υ is not strictly necessary for proving the first part of this result, it is nonetheless helpful in simplifying the argument.

Proof. Let $G' := G/a$. We shall declare $v := a^-$ to be the basepoint of both G and G'. Fix a spanning tree T of G containing a and let $T' := T/a$ be the corresponding spanning tree in G'. Let $\gamma_1, \ldots, \gamma_g$ be the basis of $H_1(G;\mathbb{Z}) \cong H_1(G';\mathbb{Z})$ determined by T in G and T' in G'. Finally, choose orientations of the edges and cycles in G; these descend to G', allowing us to define both Υ and Υ' as in §5.3.1.

Let $\iota\colon R' \to R$ be the natural inclusion, which misses x_a, and define a projection map $\rho\colon R \to R'$ via $x_a \mapsto 0$ and $x_e \mapsto x_e$ for $e \neq a$. Notice that $\rho \circ \iota = \mathrm{id}$. Then the induced maps $\iota\colon N'_{R'} \to N_R$ and $\rho\colon N_R \to N'_{R'}$ making the natural identification $N \cong N'$ also satisfy $\rho \circ \iota = \mathrm{id}$. We write $c\colon C_1(G;\mathbb{Z}) \to C_1(G';\mathbb{Z})$ for the homomorphism sending $a \mapsto 0$ and $e \mapsto e$ for $e \neq a$.

It is straightforward to check that $\rho([e,e']) = [c(e), c(e')]'$ for all $e, e' \in E$, hence $\rho \circ \pi = \pi' \circ c$. In particular,

$$\rho(n_i \wedge \pi(\gamma_j) \wedge \pi(\gamma_k)) = n_i \wedge \pi'(\gamma_j) \wedge \pi'(\gamma_k);$$

by (5.13), ρ induces a group isomorphism $\mathscr{P} \cong \mathscr{P}'$. Moreover, it is clear from the edge pair characterization in Proposition 5.15 that, for edges $e, e' \in E'$, the indicator functions f_t, g_t, and h_t are the same in G' as they are in G. In other words,

$$\Psi(\Upsilon) = \iota(\Psi'(\Upsilon')) + \sum_{\substack{e \in E \\ (i,j,k) \in S'}} a_{i,j,k}(e,a) x_e x_a n_i \wedge n_j \wedge n_k.$$

Then $\rho(\Psi(\Upsilon)) = \Psi'(\Upsilon')$, proving the first part of the statement. If a is a separating edge, then a is not part of any cycle. This implies that x_a does not appear in $\pi(\gamma_t)$ for any t, so ι induces $\mathscr{P}' \cong \mathscr{P}$. Likewise, $a_{i,j,k}(e,a) = 0$ for all $e \in E$, so $\iota(\Psi'(\Upsilon')) = \Psi(\Upsilon)$. This proves the remaining part of the statement. ∎

5.4.3 Deletion

Fix a graph G and let Υ be the explicit $(1,2)$-chain defined in §5.3.1. If $\alpha(G) = 0$, then by definition, we can write $\Psi(\Upsilon)$ as some $\mathbb{Z}$-linear combination of the generators in (5.13). In fact, not all of the generators are necessary.

Lemma 5.19. *If $\alpha(G) = 0$, then $\Psi(\Upsilon)$ is a $\mathbb{Z}$-linear combination of the generators*

$$\{2n_i \wedge \pi(\gamma_i) \wedge \pi(\gamma_j) \mid i, j \in [g],\ i \neq j\}.$$

Proof. Fix indices i, j, and k all distinct with $j < k$. By (5.14), the generator $2n_i \wedge \pi(\gamma_j) \wedge \pi(\gamma_k)$ contributes a term

$$2\left([\gamma_j, \gamma_j][\gamma_k, \gamma_k] - [\gamma_j, \gamma_k]^2\right) n_i \wedge n_j \wedge n_k,$$

which contains $2x_{e_j}x_{e_k}\,n_i \wedge n_j \wedge n_k$. Since e_t appears only in γ_t, it is straightforward to show that this is in fact the only generator that contains a nonzero multiple of $2x_{e_j}x_{e_k}n_i \wedge n_j \wedge n_k$. Moreover, by Lemma 5.14, such a term also does not appear in $\Psi(\Upsilon)$; as there is no way to cancel this term out with a different generator, we conclude that $n_i \wedge \pi(\gamma_j) \wedge \pi(\gamma_k)$ cannot contribute to a combination equaling $\Psi(\Upsilon)$. In other words, we must have either $i = j$ or $i = k$. ☐

Proposition 5.20. *Let G be a graph with an edge a. If either*

(a) a is a loop edge or

(b) a is parallel to an edge a',

then $\alpha(G) = 0$ implies that $\alpha(G \setminus a) = 0$.

Proof. Let $G' := G \setminus a$. In either case a or b, a is part of some cycle. Without loss of generality, we may choose the spanning tree T and the labeling on the edges of $G \setminus T$ so that $e_g = a$. We also fix a basepoint v and orientations on the edges and cycles. These choices descend to G'. Identifying edges of G' with the corresponding edges of G, there is a natural inclusion $\phi\colon R' \to R$ that misses x_{e_g}. Likewise, identifying the cycles of G' with the corresponding cycles in G, we let $\iota\colon H_1(G';\mathbb{Z}) \to H_1(G;\mathbb{Z})$ denote the induced inclusion on homology and $\rho\colon H_1(G;\mathbb{Z}) \to H_1(G';\mathbb{Z})$ the projection that kills γ_g. Define

$$\begin{array}{ccc} N'_{R'} & \xrightarrow{\nu} & N_R \\ u & \longmapsto & \phi \circ u \circ \rho \end{array} \qquad \begin{array}{ccc} N_R & \xrightarrow{\xi} & N_R \\ u & \longmapsto & u \circ \iota \circ \rho \end{array}.$$

In coordinates, we may identify $N'_{R'} \cong R'\langle n_1, \ldots, n_{g-1}\rangle$ and $N_R \cong R\langle n_1, \ldots, n_g\rangle$; then ν is the inclusion of R'-modules sending $n_i \mapsto n_i$, while ξ is the projection of R-modules sending $n_i \mapsto n_i$ for $i < g$ and $n_g \mapsto 0$.

Because x_{e_g} appears in $[e, \gamma_i]$ only for $e = e_g$ and $i = g$, it is straightforward to check that

$$\nu(\pi'(e)) = \xi(\pi(e))$$

for all edges $e \in E' \subseteq E$. We claim that $\nu_* \Upsilon' = \xi_* \Upsilon$. Indeed, observe first that Υ' does not have an n_g-component, and that the n_g-component of Υ is killed by ξ_*. Fixing $i \neq g$, we recall that the paths δ_i defined in §5.3.1 remain in $T = T'$, so $\delta_i' = \delta_i$. The definition of Υ' given by (5.11) depends only on the points $u_{i,j}' = \pi'\left(\delta_i + \sum_{k=0}^{j-1} f_i(e_{i,k})e_{i,k}\right) \in N_{R'}'$. Then $\nu(u_{i,j}') = \xi(u_{i,j})$ for all j, so the definition of the pushforward in (5.6) implies that $\nu_*(n_i \otimes \Upsilon_i') = \xi_*(n_i \otimes \Upsilon_i)$. The claim follows. Consequently,

$$\nu(\Psi'(\Upsilon')) = \Psi(\nu_* \Upsilon') = \Psi(\xi_* \Upsilon) = \xi(\Psi(\Upsilon)).$$

A similar computation on generators of $H_{1,2}(\mathrm{Jac}(G))$ shows that $\nu(\mathscr{P}') \subseteq \xi(\mathscr{P})$, but we may not have equality in general. Explicitly,

$$\nu(\mathscr{P}') = \mathbb{Z}\langle 2n_i \wedge \xi(\pi(\gamma_j)) \wedge \xi(\pi(\gamma_k)) \mid i, j, k \in [g-1],\, j < k\rangle,$$

leaving open the possibility that ξ fails to map generators of the form $2n_i \wedge \pi(\gamma_j) \wedge \pi(\gamma_g)$ into $\nu(\mathscr{P}')$. Here we have restricted our attention to the case where $i < g$, since $\xi(n_g) = 0$. Our assumption that $\alpha(G) = 0$ means that we may write $\Psi(\Upsilon)$ explicitly as a $\mathbb{Z}$-linear combination of the generators in Lemma 5.19. Therefore, if we can show that $2n_i \wedge \xi(\pi(\gamma_i)) \wedge \xi(\pi(\gamma_g))$ lies in $\nu(\mathscr{P}')$ for any such generator appearing in the linear combination, then we will have shown that $\nu(\Psi'(\Upsilon')) \in \nu(\mathscr{P}')$; by injectivity of ν, this would imply in turn that $\Psi'(\Upsilon') \in \mathscr{P}'$, as desired.

For case a, this is straightforward because $\pi(\gamma_g) = \pi(e_g) = x_{e_g} n_g$, so $\xi(\pi(\gamma_g)) = 0$. In case b, recall that e_g is parallel to an edge a'. Without loss of generality, assume that e_g and a' have the same orientation. If a' is a separating edge in G', then $\gamma_g = -a' + e_g$ and a' is not part of any other cycle γ_i. In particular, $\pi(\gamma_g) = (x_{a'} + x_{e_g})n_g$, so we again have that $\xi(\pi(\gamma_g)) = 0$. Otherwise, a' is not separating in G', so we may choose T so that $e_{g-1} = a'$. We write explicitly

$$\Psi(\Upsilon) \equiv 2\sum_{i=1}^{g-1} a_i n_i \wedge \pi(\gamma_i) \wedge \pi(\gamma_g) \pmod{\xi^{-1}(\nu(\mathscr{P}'))}$$

for $a_i \in \mathbb{Z}$, where we have omitted from the linear combination the generators of $H_{1,2}(\mathrm{Jac}(G))$ that we already know map to $\nu(\mathscr{P}')$ under ξ, i.e., those that do not contain $\pi(\gamma_g)$ as a factor or that have n_g as the first factor.

The fact that $\gamma_g = \gamma_{g-1} - e_{g-1} + e_g$ allows us to rewrite

$$\Psi(\Upsilon) \equiv -2x_{e_{g-1}} \sum_{i=1}^{g-1} a_i n_i \wedge \pi(\gamma_i) \wedge n_{g-1} + 2x_{e_g} \sum_{i=1}^{g-1} a_i n_i \wedge \pi(\gamma_i) \wedge n_g \pmod{\xi^{-1}(\nu(\mathscr{P}'))}. \tag{5.16}$$

Since x_{e_g} appears as a coefficient in $\pi(\gamma_j)$ only for $j = g$, the only place where it appears in (5.16) is where we have explicitly written it before the second summation. In particular, x_{e_g} does not appear in any of the omitted generators of the form $2n_g \wedge \pi(\gamma_g) \wedge \pi(\gamma_j)$ because the leading factor of n_g kills the n_g-term of $\pi(\gamma_g)$, nor does it appear in $\Psi(\Upsilon)$ by Lemma 5.14. We conclude that

$$\begin{aligned} 0 &= \sum_{i=1}^{g-1} a_i n_i \wedge \pi(\gamma_i) \wedge n_g \\ &= \sum_{i=1}^{g-1} a_i n_i \wedge \left(\sum_{j=1}^{g} [\gamma_i, \gamma_j] n_j \right) \wedge n_g \\ &= \sum_{i=1}^{g-1} \sum_{j=1}^{g-1} [\gamma_i, \gamma_j] a_i n_i \wedge n_j \wedge n_g, \\ &= \sum_{1 \le i < j \le g-1} [\gamma_i, \gamma_j](a_i - a_j) n_i \wedge n_j \wedge n_g, \end{aligned}$$

hence $[\gamma_i, \gamma_j](a_i - a_j) = 0$ for all $i, j \in [g-1]$. Applying ξ to (5.16) kills terms that contain n_g as a factor, so we obtain

$$\begin{aligned} \xi(\Psi(\Upsilon)) &\equiv \xi\left(-2x_{e_{g-1}} \sum_{i=1}^{g-1} a_i n_i \wedge \pi(\gamma_i) \wedge n_{g-1} \right) \pmod{\nu(\mathscr{P}')} \\ &= \xi\left(-2x_{e_{g-1}} \sum_{i=1}^{g-1} \sum_{j=1}^{g} [\gamma_i, \gamma_j] a_i n_i \wedge n_j \wedge n_{g-1} \right) \\ &= \xi\left(-2x_{e_{g-1}} \sum_{i=1}^{g-1} \sum_{j=i+1}^{g} [\gamma_i, \gamma_j](a_i - a_j) n_i \wedge n_j \wedge n_{g-1} \right) \\ &= \xi\left(-2x_{e_{g-1}} \sum_{i=1}^{g-1} [\gamma_i, \gamma_g](a_i - a_g) n_i \wedge n_g \wedge n_{g-1} \right) \end{aligned}$$

$$= 0,$$

completing the proof. □

5.4.4 *Graphs of hyperelliptic type*

Given a graph G, we write G^2 to mean the *2-edge-connectivization* of G, obtained by contracting each of the separating edges of G.

Corollary 5.21. $\alpha(G) = 0$ *if and only if* $\alpha(G^2) = 0$.

Proof. This follows immediately from repeated application of Proposition 5.18. □

Lemma 5.22. *Let G_1 and G_2 be graphs, with $G := G_1 \vee G_2$ the wedge sum. If G_1 and G_2 are both period-trivial, then so is G. In particular, a graph has trivial Ceresa period if each of its maximal 2-connected components is.*

Proof. The second statement follows trivially from the first. To prove the first statement, let $v_1 \in V(G_1)$ and $v_2 \in V(G_2)$ be the two vertices identified in G. For $t \in \{1, 2\}$, fix v_t as the basepoint of G_t with spanning tree T_t. Notice that $T := T_1 \vee T_2$ is a spanning tree of G; let $v_1 = v_2$ be the basepoint for G. Let $\gamma_{t,1}, \ldots, \gamma_{t,g_t}$ be the simple cycles determined by T_t with arbitrary orientation; then $\gamma_{1,1}, \ldots, \gamma_{1,g_1}, \gamma_{2,1}, \ldots, \gamma_{2,g_2}$ are the simple cycles of G. Consequently, we identify $H_1(G; \mathbb{Z}) \cong H_1(G_1; \mathbb{Z}) \oplus H_1(G_2; \mathbb{Z})$. The paths $\delta_{t,i}$ in T_t descend to the corresponding paths in T, allowing us to define $(1, 2)$-chains Υ_1, Υ_2, and Υ using the explicit construction given in §5.3.1.

We may regard R as the coproduct of $\mathbb{Z}$-algebras $R \cong R_1 \oplus_{\mathbb{Z}} R_2$. Then $N_R \cong N_{1,R} \oplus N_{2,R}$ as R-modules. It is straightforward to see using Proposition 5.15 that $\Psi(\Upsilon)$ may be obtained as $\Psi(\Upsilon_1) + \Psi(\Upsilon_2)$. Indeed, no pair of edges (e_1, e_2) with $e_t \in E(G_t)$ share a common cycle, so they do not contribute to $\Psi(\Upsilon)$. Meanwhile, any pairs (e, e') with both edges coming from the same subgraph G_1 have the same $f_{1,i}$, $g_{1,i}$, and $h_{1,i}$ values in G as they do in G_1, with $f_{2,i}$, $g_{2,i}$, and $h_{2,i}$ values all zero (and vice versa for edge pairs coming from G_2). Likewise, one can see from the generators given by (5.13) that $\mathscr{P}_1 \oplus \mathscr{P}_2 \subseteq \mathscr{P}$; it follows that if $\Psi(\Upsilon_t) \in \mathscr{P}_t$ for both t, then $\Psi(\Upsilon) \in \mathscr{P}$, as desired. $\square$

Let G and G' be graphs. We say that G' is a *permissible minor* of G if we may obtain G' from G by deleting only loops or parallel edges and contracting only non-loop edges. Then Propositions 5.18 and 5.20 immediately imply that $\alpha(G) = 0$ only if $\alpha(G') = 0$.

A tropical curve Γ is *hyperelliptic* if it admits an involution ι for which the quotient Γ/ι is a tree. More generally, it is of *hyperelliptic type* if its Jacobian is isomorphic to that of a hyperelliptic tropical curve. We say that a graph G is of *hyperelliptic type* if some choice of edge lengths makes it into a hyperelliptic-type tropical curve. For more details on these notions, we refer to [BN09, ABBR15a, ABBR15b, Cha13]. By [Cor21, Proposition 3.3], the property of being of hyperelliptic type does not in fact depend on the choice of edge lengths. Following [Cor21], we say that G is *strongly of hyperelliptic type* if some choice of edge lengths yields a hyperelliptic tropical curve.

Let T be a tree of maximal valence 3 and fix a disjoint copy T' of T. Let $\iota\colon T \to T'$ be the involution that identifies each vertex of T with the corresponding vertex of T'. Following [Cha13, Definition 4.7], we define the *ladder* over T to be the graph $L(T)$ obtained by adding $3 - \mathrm{val}(v)$ parallel edges between v and $\iota(v)$ for each $v \in V(T)$. We call the edges added in this way *vertical edges*.

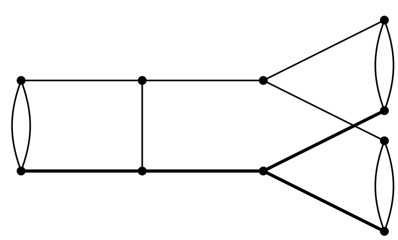


Figure 5.5: The graph $L(T)$ for the tree T marked with bold edges

Lemma 5.23. *Let G be a 2-edge-connected graph that is strongly of hyperelliptic type. Then $\alpha(G) = 0$.*

Proof. We claim that G is a permissible minor of $L(T)$ for some tree T of maximal valence 3. Indeed, this follows almost immediately from [Cha13, Theorem 4.9], which states that ladders are precisely the maximal cells of the moduli space of 2-edge-connected hyperelliptic tropical curves. However, since we do not allow weighted vertices, we must delete loops rather than contracting them. Then without loss of generality, we may assume that $G = L(T)$.

We show that $\Psi(\Upsilon)$ is identically zero for $L(T)$. Fix a 1-valent vertex v_0 of T and an edge e_0 of $L(T)$ from v_0 to $\iota(v_0)$. Let S be the spanning tree of $L(T)$ with edges $E(T) \cup E(\iota(T)) \cup \{e_0\}$. Each remaining vertical edge determines a unique cycle in $L(T)$; label these edges arbitrarily and orient them from T to $\iota(T)$. Orient each $e \in E(T)$ away from v_0; there is a partial order on $E(T)$ given by declaring that $e < e'$ whenever the unique path in T from v to the tail of e' contains e. Rephrasing Proposition 5.15, nonzero terms of $\Psi(\Upsilon)$ correspond to pairs of distinct edges e and e' satisfying:

($*$) e and e' share a cycle and each is part of another cycle that the other is not in.

We claim that ($*$) is not satisfied for any pair of edges in $L(T)$. Indeed, neither edge can be any of the vertical edges: e_0 is part of every cycle, while each other vertical edge is part of only one cycle. If both e and e' lie in T, then either $e < e'$ or the two edges are incomparable. In the first case, every cycle containing e' also contains e. In the second case, there are no common cycles between e and e'. By symmetry, ($*$) also fails if both e and e' lie in T'. If $e' = \iota(e)$, then e' and e form a separating pair; in particular, they are contained in precisely the same cycles. If $e \in T$ and $e' \in T'$ with $e' \neq \iota(e)$, then the fact that ($*$) fails for e and $\iota(e')$ implies that it also fails for e and e'. This proves the claim, so $\Psi(\Upsilon) = 0$, as desired. □

We now prove Theorem 1.7.

Theorem 5.24. *G has trivial Ceresa period if and only if G is of hyperelliptic type.*

Proof. Suppose first that G is not of hyperelliptic type. By [Cor21, Theorem 1.1], G contains $G' \in \{K_4, L_3\}$ as a minor. In fact, [CL24, Lemma 5.10] implies that, in this special case,

G' must be a permissible minor of G. That $\alpha(G) \neq 0$ follows from our computations in Examples 5.16 and 5.17 showing that $\alpha(G') \neq 0$ in either case.

Conversely, suppose that G is of hyperelliptic type. By [Cor21, Theorem 1.1], G does not contain K_4 or L_3 as a minor. Then neither do the maximal 2-connected components of the 2-edge-connectivization G^2, so any such component is still of hyperelliptic type. In particular, Corollary 5.21 and Lemma 5.22 imply that we may assume that G itself is 2-connected. By definition, there exists a 2-connected tropical curve Γ of hyperelliptic type with underlying graph G. Then by [Cor21, Theorem 4.5], there exists a hyperelliptic tropical curve Γ' of which Γ is a permissible minor. Let G' denote the underlying graph of Γ'. Contracting any separating edges, the resulting tropical curve Γ'^2 is hyperelliptic by [Cha13, Corollary 3.11]. Then G'^2 is 2-edge-connected and strongly of hyperelliptic type, so Lemma 5.23 implies that $\alpha(G'^2) = 0$. By Corollary 5.21, $\alpha(G') = 0$; since G is a permissible minor of G', we have that $\alpha(G) = 0$. $\square$

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