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Boundary Rigidity and the Geodesic X-Ray Transform in Low Regularity

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Abstract

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We prove that the geodesic X-ray transform is injective on L^2 when the Riemannian metric is simple but only finitely differentiable. This result was a collaborative effort with Joonas Ilmavirta and Antti Kykkänen [12]. The number of derivatives needed depends explicitly on the dimension of the manifold, and in dimension two we assume $g \in C^{10}$. Our proof is based on microlocal analysis of the normal operator; we establish ellipticity and a smoothing property in a suitable sense. We then use a recent injectivity result on Lipschitz functions. When the metric tensor is C^k, the Schwartz kernel is not smooth but C^{k-2} off the diagonal, which makes standard smooth microlocal analysis inapplicable.

With the result above, we also prove that on a simple surface where the metric is C^{17}, the scattering relation determines the Dirichlet to Neumann map (DN map) [15] - a result proved in [31] for the case when the metric is smooth. For metrics with finite differentiability we had to modified each technical result used in the original proof; such as properties of the exit time function and the characterization of C_α space (Theorem 5.1.1 [30]). Moreover, surjectivity of I^* in the original proof required the use of microlocal analysis of the normal operator I^*I; which is not a standard pseudodifferential operator when the metric only has finite regularity- this was addressed in [12]. Finally, using the injectivity of I on Lipschitz one forms for simple $C^{1,1}$ manifolds by [11] we prove an equivalent characterization of harmonic conjugacy using operators determined by the scattering relation (Theorem 1.6 [31]) to prove the titular result. We also prove that the boundary distance function determines the metric at the boundary (which in turns determines the scattering relation) for a closed disk even when the metric is only $C^{1,1}$ and the exponential map is only Lipschitz and does not preserve tangent vectors or differentials pointwise. We also provide a report on the partial progress for the Calderón problem for $C^{1,1}$ metrics.

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DEDICATION

to my grandmother, in loving memory

Chapter 1

INTRODUCTION

Given two simple metrics g on a compact surface with boundary M, it was proved in [31] that the boundary distance function determines the metric up to a boundary fixing diffeomorphism; in other word simple surfaces are boundary rigid. The key step in the proof of boundary rigidity involves showing that the boundary distance function determines the Dirichlet-to-Neumann (DN map) for simple metrics. In this paper we prove that the same is true when the metric has sufficiently high but finite regularity.

We prove several intermediate results with various regularity requirements for the metric g, from which it follows that for simple C^{17} metrics the DN map is determined by the boundary distance function.

The three main results in this paper are the following: 1. For $C^{1,1}$ metrics on the closed disk $\mathbb{D}$, the boundary distance function determines the metric on the boundary up to a boundary fixing gauge (which fixes the boundary distance function). 2. The adjoint of the geodesic x ray transform I^* is surjective for simple metrics g that are C^{17}. 3. Given a simple metric g of class C^{17} on a compact surface with boundary, the boundary distance function determines the DN map. These results are summarized in the section below.

Let M be a manifold with boundary, and g a Riemannian metric on M, that is, g is a symmetric covariant 2-tensor field that is positive definite; in other word, g is a smooth choice of inner product for the tangent space T_pM at each $p \in M$. In local coordinates $g|_p$ is given by g_{ij} that is a symmetric, positive definite matrix. A manifold M equipped with a Riemannian metric g is called a Riemannian manifold. A Riemannian manifold (M, g) equipped with a metric gives rise to the notion of length of vectors, which then defines the notion of length of a curve on M. We can then define the notion of distance between any two points x, y on M by the infimum of the length of all admissible curves connecting x, y, we call this function $d_g : M \times M \to \mathbb{R}$ the Riemannian distance function on M. The natural question to ask is then the following: Suppose we have two Riemannian metrics g_1, g_2 on a manifold with boundary M such that the induced Riemannian distance function are equal on ∂M, then does this follow that the metrics $g_1 = g_2$? This is known as the boundary rigidity problem.

The boundary rigidity problem has a natural obstruction, since any boundary fixing diffeomorphism acts invariantly on the classes of curves connecting any pair of points on ∂M, the boundary distance function is the same for a metric and it's pullback under any such diffeomorphism. Then the next question to ask is whether this is the only source of obstructions for boundary rigidity. The answer remains negative, consider yet another example, let (M, g) be the upper half of S^2 with the induced metric from $\mathbb{R}^3$, then all geodesics realizing the minimum distance between any two points in ∂M (that is, the equator) will remain in ∂M. So one can increase the weight of g in the interior of the manifold arbitrarily without affecting this property (such as multiplying g by a function that is 1 on the boundary and ≥ 1 otherwise. This shows that further restrictions must be imposed on our manifold to expect boundary rigidity.

This leads us to the definition of a simple manifold. A Riemannian manifold with boundary is called simple if all geodesics starting in the interior leaves the manifold in finite amount of time, geodesics starting at the boundary in a direction tangent to the boundary leaves the manifold for small time, and there is no non trivial variations through geodesics. A typical example of a simple manifold is a closed ball $\mathbb{B}^n$ with the induced $\mathbb{R}^n$ metric. Michel conjectured in [21] that simple manifolds are boundary rigid in the sense that boundary fixing diffeomorphisms are the only gauge invariant, more precisely: Suppose g_1, g_2 are two simple metrics on a manifold with boundary M such that $d_{g_1}|_{\partial M \times \partial M} = d_{g_2}|_{\partial M \times \partial M}$, then there exists a boundary fixing diffeomorphism $\phi : M \to M$ with $\phi|_{\partial M} = id$ such that $g_1 = \phi^* g_2$. In 1977 Mukhometov [27] proved boundary rigidty for simple surfaces within the same conformal class, a result that was later proved for higher dimensions in 1978 by Bernstein and

Gerver [3] , and by Mukhometov in 1981 [28]. A shorter proof following [4] for the higher dimensional case was given by integrating the conformal factor using Santalós formula, this requires the need to define the Sasaki metric - which is the natural metric on TM (or SM) for a Riemannian manifold (M, g), for a precise definition, see [30].

Theorem 1 (Santalós formula)**.** *Let G be the Sasaki metric on SM, $d\Sigma^{2n-1}$ be the induced volume form on SM, $d\mu$ be the volume form on $\partial_+ SM$ of $G|_{\partial_+ SM}$, then for any $f \in L^2(SM)$, we have:*

$$\int_{SM} f d\Sigma^{2n-1} = \int_{\partial_+ SM} d\mu(x,v) \int_0^{\tau(x,v)} f(\phi_t(x,v))dt$$

Where τ is the exit-time function and ϕ_t is the geodesic flow (to be defined)

The Santalós formula is extremely useful, for example, it is used to prove that the geodesic X-ray transform is an bounded operator on L^2 functions.

A major breakthrough occurred in 2005, when Leonid Pestov and Gunther Uhlmann proved that two-dimensional simple surfaces are boundary rigid [31]. The general case for higher dimensions is largely unsolved. In this paper we will generalize the result in [31] to the case where the metric is C^{17}. As we will soon see, one advantage with simple manifolds is the fact that they are diffeomorphic to closed balls via the exponential map [[30]] which makes computations simpler in global coordinates and analogue of polar coordinates.

The first step to proving simple surfaces are boundary rigid is to show that two metrics having the same boundary distance function agrees at the boundary. Assuming $d_{g_1}|_{\partial M \times \partial M} = d_{g_2}|_{\partial M \times \partial M}$. in the smooth case, one can prove that the metrics agree in the direction tangent to ∂M by considering the following: Let $(x, v) \in T\partial M$, consider the quotient

$$\lim_{s \to 0} \frac{d(p, \tau(s))}{s}$$

where $p \in \partial M$ and $\tau(s) : (-\epsilon, \epsilon) \to \partial M$ is a smooth curve on the boundary with $\tau(0) = p$ and $\tau'(0) = v$. Taking the limit of the quotient result in $g_i|_p(v, v)$ which is the square of the length of v with respect to the g_i metric. Since the quotient is determined entirely by the boundary distance function it also determines the length of any such vector $v \in T\partial M$. This proof however depends on sufficient differentiability of the exponential map since the computation is performed in normal coordinates, and C^2 differentiability is required to ensure the differential of the normal coordinate is continuous so that tangent vectors are preserved. We establish another proof in the case when the metric is only $C^{1,1}$ in this paper: by utilizing some recent results on the existence of normal coordinates, convex balls and regularity of squared distance function for $C^{1,1}$ metrics in [22]. It was established in [22] that the following results are true for $C^{1,1}$ metrics:.

Theorem 2 (Minguzzi, 2014, theorem 4)**.** *Let M be a $C^{2,1}$ manifold endowed with a $C^{1,1}$ metric, let O be an open neighborhood of $p \in M$. Then there is a reversible striclty convex normal neighborhood C of p contained in O, such that exp establishes a Bi-Lipschitz homeomorphism between an open star-shaped subset of TC and $C \times C$.*

Theorem 3 (Minguzzi, 2014, theorem 6)**.** *Let (M, g) be a $C^{2,1}$ Riemannian manifold for which the metric g is $C^{1,1}$. Let N be a normal neighborhood of $p \in M$. Then:*
Let $\sigma : [0, 1] \to N$, $s \to \sigma(s)$, be any admissible curve starting from p, then its length is larger than that of the (unique) geodesic connecting its endpoints, unless its image coincides with that of that geodesic.

Note that the original results in [22] were stated for much more general settings, for Finsler manifolds with Finsler spray, as well as Lorentzian-Finsler manifolds, for more details, see [22].

Using these results by [22], we can show that the boundary distance function determines the tangent direction of the metric g at the boundary ∂M without computations in normal coordinates (we do however need the existence of normal neighborhood). We can also prove the existence of a local diffeomorphism $\phi : U \to U$ for an open set $U \subset M$ with $\partial M \subset U$ such that $\phi^* \nu_{g_1} = \nu_{g_2}$, where ν_{g_i} is the outward pointing normal vector with respect to the metric g_i using the result by [1]. Then, following [30](Chapter 11.2), one can extend the diffeomorphism ϕ to all of M using a smooth Riemannian metric on M and its induced exponential map. This gives the following result:

Theorem 4. *Let $M \subset \mathbb{R}^2$ be the closed unit disk, g_1, g_2 be $C^{1,1}$ metrics on M such that $d_{g_1}|_{\partial M\times\partial M} = d_{g_2}|_{\partial M\times\partial M}$, then there exists a $C^{2,1}$ diffeomorphism $\psi : M \to M$ with $\psi|_{\partial M} = id$ such that $\psi^* g_1|_{\partial M} = g_2|_{\partial M}$*

The proof of the determination of the DN map from scattering relation goes as follow:

$$d_{g_1}|_{\partial M\times\partial M} = d_{g_2}|_{\partial M\times\partial M}$$

$$=> g_1|_{\partial M} = \phi^* g_2|_{\partial M}, \qquad \alpha_{g_1} = \alpha_{g_2}$$

$$=> \Lambda_{g_1} = \Lambda_{g_2}$$

where $\alpha_{g_i}, \Lambda_{g_i}$ are the scattering relation and Dirichlet to Neumann map with respect to the metric g_i. Which we define in Chapter 2 below.

We have just established that

$$d_{g_1}|_{\partial M\times\partial M} = d_{g_2}|_{\partial M\times\partial M} => g_1|_{\partial M} = \phi^* g_2|_{\partial M}$$

even when the metric is only $C^{1,1}$. Next we will prove that the scattering relation is also determined by the boundary distance function when g is at least C^{17}. We follow the proof in [30]. Assuming we have a C^2 metrics, the same argument in [30] (Chapter 11.3) works to show that the scattering relation is also determined by the metric at the boundary and the boundary distance function. The argument goes as follow; first, using the fact that the exponential map is a diffeomorphism for a simple manifold, then together with Gauss lemma we can prove the following lemma:

Lemma 5. *Let (M, g) be a simple manifold. Given $x \in M$, let $f : M \to \mathbb{R}$ be the function $f(y) = d_g(x, y)$. For any $y \in M$ with $y \neq x$, let $\eta_{x,y}$ be the unique unit speed geodesic* *connecting x to y and let $l_{x,y} > 0$ be such that $\eta_{x,y}(l_{x,y}) = y$. Then*

$$\nabla f(y) = \dot{\eta_{x,y}}(l_{x,y})$$

Now consider two points $x, y \in \partial M$, since M is simple, there exists a unique geodesic connecting the two points with length $l_{x,y} = d_{g_i}(x, y)$ (which are equal for $i = 1, 2$ by assumption). Since we have

$$\alpha_{g_i}(x, \dot{\gamma^i}_{x,y}(0)) = (y, \dot{\gamma^i}_{x,y}(l))$$

it suffices to prove that

$$\dot{\gamma^1}_{x,y}(0) = \dot{\gamma^2}_{x,y}(0)$$

and

$$\dot{\gamma^1}_{x,y}(l) = \dot{\gamma^2}_{x,y}(l)$$

Consider the function $h := f|_{\partial M}$, then it is easy to show that $\nabla h(y)$ is the orthogonal projection of ∇f onto $T_y \partial M$, since ∇f is unit length and outward pointing it is completely determined by ∇h (in 2 dimensions). By the lemma above we then have

$$\dot{\gamma^1}_{x,y}(l) = \nabla f_1(y) = \nabla f_2(y) = \dot{\gamma^2}_{x,y}(l)$$

which is what we wanted.

Notice that this proof works in the smooth case as well as for C^2 metrics, where the exponential map (and hence the distance function) is continuously differentiable. This reduces boundary rigidity to the question of determination of the DN map by the scattering relation. In [31] (Theorem 1.6) it was shown using an application of Hilbert transform that the question can be reduced to the surjectivity of the adjoint operator of the geodesic X-ray transform with respect to the L^2 norm. The proof of the reduction consists of two parts: proving that the X-ray transform is injective on $L^2(M)$, and showing that the normal operator $I^*I : L^2(M) \to L^2(M)$ is an elliptic pseudodifferential operator of order -1. In the case when the metric g is only C^8, we prove injectivity of the X-ray transform and show that the normal operator is an elliptic pseudodifferential operator in the Marschall sense ([20]). Since the metric is only finitely differentiable, it turns out that the normal operator is an elliptic pseduodifferential operator where the symbol only has finite differentiability in x and ξ (To be precise, it is smooth in ξ but it does not satisfies the estimates in ξ for symbols for standard pseudodifferential operators up to a certain derivative). In 1996, Jurgen Marschall [20] studied these types of operators and their Sobolev boundedness properties, as well as commutative formulas that laid foundation for the symbol calculus for these operators which we use in this paper for construction of parametrix for the normal operator.

In [10] the notion of $C^{1,1}$ simplicity is defined and it was proved that when the metric is at least C^2, this definition of simplicity agrees with the usual definition of simplicity for C^∞. Furthermore, they proved that for any $C^{1,1}$ simple manifolds, the geodesic X-ray transform is injectivity on Lipschitz functions and Lipschitz 1-forms. provided that the functions have 0 boundary values - a condition that was later removed in a strengthened result in [11]. The idea for proving injectivity of the geodesic X-ray transform for metrics at finite regularity is then to prove that the normal operator is an elliptic pseudodifferntial operator in the Marshcall sense. We then construct a 1 degree smoothing parametrix explicitly to establish elliptic regularity for the normal operator, with which we can apply a bootstrapping argument to establish elliptic regularity - allowing us to conclude that a function $f \in L^2(M)$ such that $If = 0$ (and hence $I^*If = 0$) is in fact Lipschitz by Sobolev embedding theorem for a metric with sufficiently high regularity, we can then apply the injectivity of I on Lipschitz functions to conclude that $f = 0$. This was the essence of the work in [12] and Chapter 2.

Together with injectivity of the geodesic X-ray transform and the existence of a parametrix for the normal operator for C^{17} metrics, we can prove the adjoint operator is surjective using various modification of the original argument laid out in [31] and [30]. The regularity requirement of C^{17} arises from various regularity requirement for the exit time function near the boundary which will be discussed in details in chapter 2. From the surjectivity of the adjoint we then show that the scattering relation determines the DN map using theorem 1.6 in [31]. The idea is to apply the surjectivity of the adjoint operator to show that all harmonic conjugate of the harmonic continuation of an arbitrary function on the boundary ∂M have the same boundary values if their scattering relations are equal.

The main results are summarized as follow:

1.0.1 Main results

Theorem 6. *Let (M, g) be a simple manifold, $n := \dim M \geq 2$ and $g \in C^k(M)$ for some $k \geq 7 + \frac{n}{2}$. Then if $f \in H^s_c(M)$ for some $s > -k + 6 + \frac{n}{2}$ and $Nf = 0$, we have $f \in H^r_c(M)$ for all $s < r < k - 6 - \frac{n}{2}$.*

Theorem 15 can be applied to geodesic X-ray tomography in low metric regularity assuming that the X-ray transform I acts on $L^2(M)$, since then $N = I^*I$ is in fact the normal operator for the X-ray transform I.

Proposition 7. *Let (M, g) be a simple manifold with $g \in C^k(M)$ for some $k \geq 2$. Then $I^*I = N$ on $L^2(M)$.*

Theorem 8. *Let (M, g) be a simple manifold, $n := \dim M \geq 2$ and $g \in C^k(M)$ for some $k \geq 8 + n$. Then the X-ray transform I is injective on $L^2(M)$.*

Theorem 9. *Let $M \subset \mathbb{R}^2$ be the closed unit disk, g_1, g_2 be $C^{1,1}$ metrics on M such that $d_{g_1}|_{\partial M\times\partial M} = d_{g_2}|_{\partial M\times\partial M}$, then there exists a $C^{2,1}$ diffeomorphism $\psi : M \to M$ with $\psi|_{\partial M} = id$ such that $\psi^* g_1|_{\partial M} = g_2|_{\partial M}$*

Theorem 10. *Let (M, g) be a simple surface with $g \in C^k$ with $k \geq 10$, let $f \in C^l(M)$ with $1 < m + 1 < l - 1 < k - 7$, $m, l, k \in \mathbb{N}$, then there exists $w \in C^{\min(k-4,m)}(\partial_+ SM)$ with $w^\# \in C^{\min(k-4,m)}(SM)$, such that $I^*w = f$.*

Theorem 11. *Let (M, g_1) and (M, g_2) be a simple surfaces with $g_1, g_2 \in C^{17}$, with $d_{g_1} = d_{g_2}$, then the DN maps $\Lambda : C^{2,\alpha}(\partial M) \to C^{1,\alpha}(\partial M)$ determined by g_1, g_2 are equal for all $0 < \alpha < 1$.*

The proof of theorem 1 relies on some recent results by [22] and [1], theorem 2 relies on some microlocal analysis at low regularity studied in [20] and [12], all of which were derived from the proofs in [30], and finally we use theorem 1, 2 and a modification of the proof in [31] to prove the titular result.

1.1 Preliminaries

We introduce the geometric preliminaries and the operators used throughout this paper in this section.

1.1.1 Simple manifolds

Definition 12. Let $k \in \mathbb{Z}$ and assume that $k \geq 2$. Let M be a compact smooth manifold with a smooth boundary and equip M with a C^k smooth Riemannian metric g. We say that (M, g) is simple if M is C^k-diffeomorphic to the closed Euclidean unit ball in $\mathbb{R}^n$ and the following hold:

1. The boundary ∂M is strictly convex in the sense of the second fundamental form.
2. The manifold is non-trapping i.e. all geodesics hit the boundary in a finite time.
3. There are no conjugate points in M.

When the Riemannian metric g is C^∞-smooth definition 12 is equivalent to any standard definition of a simple manifold.

1.1.2 Function spaces

Let (M, g) be a simple manifold where $g \in C^k(M)$ for some $k \geq 2$. Since M is C^{k-1}-diffeomorphic to the closed Euclidean unit ball $\overline{B} \subseteq \mathbb{R}^n$ we take $M = \overline{B}$ from now on and all computations are to be interpreted via a C^{k-1}-diffeomorphism as explained in (Theorem 3.8.5, [30]) We use smooth global coordinates $(x^1, \dots, x^n)$ in the definitions of our functions spaces. We use the Riemannian volume for dVol_g to define $L^2(M)$ in the standard way i.e. $L^2(M) = L^2(M, \mathrm{dVol}_g)$.

For $s > 0$ we denote by $H^s_c(M)$ the space of compactly supported functions in $H^s(M)$ (Here by compactly supported we mean $f = \phi f$ for some $\phi \in C^\infty_c(M)$). For $s > 0$ we let $H^{-s}(M)$ be the continuous dual of $H^s(M)$ and $H^{-s}_c(M)$ be the subspace of compactly supported distributions.

1.1.3 Non smooth operators and symbol

We recall the basics of a non-smooth pseudodifferential calculus introduced in [20]. We rerecord the results that are relevant to the current work for the convenience of the reader.

Let $m \in \mathbb{R}$ and $r, L \in \mathbb{N}$ be given. Multi-indices in $\mathbb{N}^n$ are denoted by α and β. For all $\rho, \delta \in [0,1]$ the symbol class $S^m_{\rho\delta}(r, L)$ consists of continuous functions $p \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying the estimates

$$\left|\partial^\alpha_\xi p(x,\xi)\right| \le C_\alpha (1 + |\xi|)^{m-\rho|\alpha|} \tag{1.1}$$

and

$$\left\|\partial^\alpha_\xi p(\,\cdot\,,\xi)\right\|_{C^r_*} \le C_{\alpha r} (1 + |\xi|)^{m+r\delta-\rho|\alpha|} \tag{1.2}$$

for all $|\alpha| \le L$.

Given a symbol $p \in S^m_{\rho\delta}(r, L)$ the corresponding operator $\mathrm{Op}(p)$ is defined by its action

$$\mathrm{Op}(p) f(x) = \int_{\mathbb{R}^n} e^{ix\cdot\xi} p(x,\xi) \hat{f}(\xi) \,\mathrm{d}\xi \tag{1.3}$$

on functions f in $L^2(\mathbb{R}^n)$. The identity operator Id is the operator corresponding to the constant symbol 1.

We also include two useful results on the operators of class $\Psi^m(r, L)$. For the proofs of the lemmas we refer the reader to [20].

Lemma 13 ([20] Theorem 2.1.). *Let $p \in S^m_{\rho\delta}(r, L)$ and consider the operator $P \coloneqq \mathrm{Op}(p)$. Suppose that $\rho, \delta \in [0,1]$ and $r, L > 0$ satisfy*

$$\delta \leq \rho, \quad L > \frac{n}{2}, \quad r > \frac{1-\rho}{1-\delta}\frac{n}{2}. \tag{1.4}$$

Then the operator $P\colon H^{s+m}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ is bounded when

$$(1-\rho)\frac{n}{2} - (1-\delta)r < s < r. \tag{1.5}$$

Lemma 14 ([20] Theorem 3.5.). *Let $p \in S^{m_1}_{\rho_1\delta_1}(r, L)$ and $q \in S^{m_2}_{\delta_2\rho_2}(r, L + \frac{n}{2} + 1)$ and suppose that $\delta_1 < \rho_2$ and $L > \frac{n}{2}$. Denote the corresponding operators by $P \coloneqq \mathrm{Op}(p)$ and $Q \coloneqq \mathrm{Op}(q)$. Let $\tau \in (0,1]$ be such that $0 < \tau < r$. Define*

$$\delta \coloneqq \max\{\delta_1 + (\rho_1 - \delta_2)\tau, \delta_2\} \quad \textit{and} \quad \rho \coloneqq \min\{\rho_1, \rho_2\}. \tag{1.6}$$

Assume that $\delta \leq \rho$ and in the case $\rho < 1$ suppose in addition that $r > \frac{1-\rho}{1-\delta}\frac{n}{2} + \tau$. Then the commutator

$$QP - \mathrm{Op}(qp)\colon H^{s+m_1+m_2-(\rho_1-\delta_2)\tau}(\mathbb{R}^n) \to H^s(\mathbb{R}^n) \tag{1.7}$$

is bounded when

$$\max\{-m_2, 0\} + (1-\rho)\frac{n}{2} - (1-\delta)(r-\tau) < s < r - \max\{m_2, 0\}. \tag{1.8}$$

1.1.4 Geodesic X-ray transforms

Let (M, g) be a simple manifold where $g \in C^k(M)$ for some $k \geq 2$. For a given unit vector $v \in T_xM$ there is a unique geodesic $\gamma_{x,v}$ corresponding to the initial conditions $\gamma_{x,v}(0) = x$ and $\dot\gamma_{x,v}(0) = v$. Since the manifold is non-trapping, the geodesic $\gamma_{x,v}$ is defined on a maximal interval of existence $[-\tau_-(x,v), \tau_+(x,v)]$ where $\tau_\pm(x,v) \geq 0$ and we abbreviate $\tau \coloneqq \tau_+$.

The X-ray transform If of a function $f \in L^2(M)$ is defined for all inwards pointing unit vectors $(x,v) \in \partial_+SM$ by the formula

$$If(x,v) \coloneqq \int_0^{\tau(x,v)} f(\gamma_{x,v}(t))\,\mathrm{d}t. \tag{1.9}$$

For $g \in C^5$, the same proof of Prop 4.1.2 in [30] works to show that $I : L^2(M) \to L^2(\partial_+ SM)$ is bounded (The proof relies on the continuity of the term defined in lemma 3.2.8, which requires the odd extension of τ being C^1 in lemma 3.2.6, which requires $g \in C^5$). The backprojection I^*h of a function h on $L^2(\partial_+(SM)))$ is defined for all $x \in M$ by the formula

$$I^*h(x) := \int_{S_x M} h(\phi_{-\tau(x,-v)})\, \mathrm{d}S_x(v). \tag{1.10}$$

Finally, we define the operator N which we will call the normal operator. The normal operator is defined on $L^2(M)$ by the formula

$$Nf(x) = 2\int_{S_x M} \int_0^{\tau(x,v)} f(\gamma_{x,v}(t)))\, \mathrm{d}t\, \mathrm{d}S_x(v). \tag{1.11}$$

It is proved in [Prop 8.1.5 [30]] that N agrees with the composition I^*I on $L^2(M)$ and hence the name normal operator.

In the case that M is diffeomorphic a closed ball (which is the case if the metric g is C^k simple for $k \geq 2$). Then we can also consider the operator $\phi I^* I \phi$ with $\phi \in C_c^\infty(M)$, it is shown in lemma 11 of [12] (See 39 below) that $\phi I^* I \phi$ is actually a pseudodifferential operator with non smooth symbol.

Chapter 2 contains the proof of the injectivity of I for C^k metrics in details. Chapter 3 contains the proof that the scattering relation determines the DN map of C^{17} simple manifolds. We also discuss the partial progress of a possible approach to proving the Calderón problem in dimension two for $C^{1,1}$ metrics in chapter 4.

Chapter 2

INJECTIVITY OF THE X-RAY TRANSFORM FOR C^K SIMPLE METRICS

2.1 Introduction

In the collaborative work with Joonas Ilmavirta and Antti Kykkänen [12], we show that on a simple Riemannian manifold (M, g) where $g \in C^k$ for a finite and explicit k the geodesic X-ray transform is injective on L^2 (Theorem 17). This chapter contains the proof of this injectivity result in details from our joint work. We do this using a typical two-step approach, first showing that a function in the kernel of the transform is smoother than assumed a priori and then showing that injectivity holds for smooth functions. Both of the two steps of the proof have to be adapted to low regularity. The "smooth" injectivity (on Lipschitz functions) was established in [10], so it remains to prove that a function in the kernel of the X-ray transform has to be Lipschitz.

This regularity result (Theorem 15) is based on microlocal analysis of the normal operator. This normal operator is not a pseudodifferential operator in the usual sense because the "smooth" off-diagonal part of the Schwartz kernel is only C^{k-2}. Also, when the metric tensor is not infinitely differentiable, the Sobolev scale of H^s spaces only makes sense for a bounded range of indices s in both the positive and the negative direction. These two issues mean that the concepts of ellipticity, smoothing, and a parametrix need careful treatment.

2.1.1 Main results

We consider two operators: The X-ray transform I and its normal operator N. These are defined separately, and we only prove that $N = I^*I$ when acting on L^2 functions. Precise definitions of the operators and spaces we employ are given in section 1.1 below.

We prove two main theorems. Theorem 15 concerns functions in the kernel of the operator N and proves that they have, a priori, improved regularity. Theorem 17 can be compared to a recent result in [10]. We prove that the X-ray transform is injective on $L^2(M)$ while requiring more metric regularity whereas [10, Theorem 1] proves that the X-ray transform is injective only on Lipschitz functions.

Theorem 15. *Let (M, g) be a simple manifold, $n := \dim M \geq 2$ and $g \in C^k(M)$ for some $k \geq 7 + \frac{n}{2}$. Then if $f \in H^s_c(M)$ for some $s > -k + 6 + \frac{n}{2}$ and $Nf = 0$, we have $f \in H^r_c(M)$ for all $s < r < k - 6 - \frac{n}{2}$.*

Theorem 15 can be applied to geodesic X-ray tomography in low metric regularity assuming that the X-ray transform I acts on $L^2(M)$, since then $N = I^*I$ is in fact the normal operator for the X-ray transform I.

Proposition 16. *Let (M, g) be a simple manifold with $g \in C^k(M)$ for some $k \geq 2$. Then $I^*I = N$ on $L^2(M)$.*

Theorem 17. *Let (M, g) be a simple manifold, $n := \dim M \geq 2$ and $g \in C^k(M)$ for some $k \geq 8 + n$. Then the X-ray transform I is injective on $L^2(M)$.*

The proofs of the theorems rely on microlocal tools. We study the so-called normal operator $N = I^*I$ related to the X-ray transform I. We prove that N is a non-smooth elliptic operator and construct a principal parametrix with an error term smoothing of order $\tau \in (0, 1)$. The construction and its implications use a non-smooth microlocal calculus developed in [20] and, in particular, we use the non-smooth symbol and operator classes, continuous Sobolev mapping properties and a commutator theorem there introduced. The details are recalled in section 1.1.

2.1.2 Related results

The geodesic X-ray transform on a Riemannian manifold has been studied in a variety of contexts and with a variety of tools [30, 32, 13, 29]. The current article focus on the aspect of not studying the X-ray transform directly but via the related normal operator. This approach has seen plenty of applications in C^∞-smooth metric regularity.

In [31] it was proved that the normal operator on a simple Riemannian manifold is an elliptic pseudodifferential operator in the interior of the manifold — a result that is essential in their proof that all two dimensional simple Riemannian manifolds are boundary rigid. The normal operator has also played a role in later developments in boundary rigidity [34, 35]. Microlocal methods in relation to the normal operator are useful in geometries permitting conjugate points [36, 37, 9]. More recently, there has been interest in isomorphic mapping properties of the normal operator and its variants between suitably weighted function spaces [23, 24].

Microlocal analysis of the normal operator in the X-ray tomography is in non-smooth geometries virtually unexplored. However, injectivity for the X-ray transform of Lipschitz scalar and $C^{1,1}$ tensor fields on simple $C^{1,1}$ manifolds was proved in two recent articles [10, 11], and injectivity is known for the scalar transform on spherically symmetric $C^{1,1}$ manifolds satisfying the Herglotz condition [5].

The current article uses non-smooth microlocal methods. As references on pseudodifferential operators with symbols non-smooth in both variables we mention [14, 20] and as references to paradifferential methods we mention [39].

2.2 Parametrix construction for the normal operator

This section provides a detailed analysis of the operator N culminating in a leading order parametrix construction in the non-smooth symbol calculus setting. The parametrix construction is the main tool used in the proofs of our main theorems.

2.2.1 The Schwartz kernel and the symbol

The objective of this section is to study the operator N as a non-smooth elliptic pseudodifferential operator. We begin from the Schwartz kernel of the operator and analyse its symbol by dissecting it into manageable parts. The end result containing the principal part of the symbol is presented in corollary 26.

Lemma 18. *Let (M,g) be a simple manifold with $g \in C^k(M)$ for some $k \geq 2$. Let $a(x,y) = \det(\mathrm{d}\exp_x|_{\exp_x^{-1}(y)})^{-1}$. Then for all $f \in L^2(M)$ we have*

$$Nf(x) = 2\int_M a(x,y) d_g(x,y)^{1-n} f(y)\,\mathrm{dVol}_g(y). \tag{2.1}$$

Proof. The same formula is derived in [30, Lemma 8.1.10] when $g \in C^\infty(M)$. The computation works when $g \in C^k(M)$ with $k \geq 2$. □

The Schwartz kernel of the operator N is

$$K(x,y) = 2a(x,y)d_g(x,y)^{1-n} \tag{2.2}$$

on $M \times M$. We will construct leading order parametrices for operators on $\mathbb{R}^n$ related to the Schwartz kernels of the form

$$\tilde{K}(x,y) := \psi(x)2a(x,y)d_g(x,y)^{1-n}\det(g(y))^{\frac{1}{2}}\phi(y) \tag{2.3}$$

where ψ and ϕ are suitable cut-off functions in $\mathbb{R}^n$.

Consider $\Omega \subseteq M$ and consider $f \in H^s_c(M)$ so that $\operatorname{supp} f \subseteq \Omega$. We can choose a cut-off function $\phi \in C_c^\infty(M)$ so that $\phi f = f$ on M. Then if $\psi \in C_c^\infty(M)$ is to that $\psi = 1$ on Ω we have for all $x \in \Omega$ that

$$\begin{aligned} Nf(x) &= \int_{\mathbb{R}^n} \psi(x)K(x,y)\det(g(y))^{\frac{1}{2}}\phi(y)f(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n} \tilde{K}(x,y)f(y)\,\mathrm{d}y. \end{aligned} \tag{2.4}$$

We let $\tilde{N}$ be the operator corresponding to the kernel $\tilde{K}$. Then $Nf(x) = \tilde{N}f(x)$ on Ω which shows that it is enough to only consider operators with kernel of the form (2.3). For the details see the proof of Theorem 15 in section 2.3. From now on we let $N = \tilde{N}$ to avoid cluttered notation and we keep the cut-off functions ψ and ϕ fixed for the remainder of this section.

We will prove that $N \in \Psi^{-1}(k-s, s-4)$ for all $s \in \mathbb{N}$ with $4 \le s \le k$. This is accomplished by studying the operator in the global coordinates of the Euclidean unit ball and by computing the symbol of the operator. By [30, Lemma 8.1.12] we can write in the coordinates that

$$\tilde{K}(x,y) = \psi(x) \frac{2a(x,y) \det(g(y))^{1/2}}{[G_{jk}(x,y)(x-y)^j(x-y)^k]^{\frac{n-1}{2}}} \phi(y) \tag{2.5}$$

for some functions G_{jk} with $G_{jk}(x,x) = g_{jk}(x)$.

Lemma 19. *Let (M,g) be a simple manifold with $g \in C^k(M)$ for some $k \ge 3$. Then $\tilde{K} \in C^{k-2}(\mathbb{R}^n \times \mathbb{R}^n \setminus \Delta)$ where $\Delta := \{(x,x) : x \in \mathbb{R}^n\}$ is the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$.*

Proof. The kernel $\tilde{K}$ can be expressed in the form

$$\tilde{K}(x,y) = \psi(x) 2a(x,y) d_g(x,y)^{1-n} \det(g(y))^{\frac{1}{2}} \phi(y). \tag{2.6}$$

By standard ODE theory the geodesic flow has C^{k-1} smooth initial value dependence when $g \in C^k(M)$, and thus the exponential function is also C^k. It follows that $a \in C^{k-2}(M \times M)$. In addition, since $d_g(x, \exp_x(v)) = |v|_g$ for $(x,v) \in TM$ it follows that $d_g(x,y) \in C^{k-1}(M \times M \setminus \Delta)$. Finally, since the determinant term in (2.6) is C^k we see that $\tilde{K}$ is C^{k-2} off diagonal as claimed. □

By denoting

$$k(x,z) := \tilde{K}(x, x-z) \tag{2.7}$$

and letting

$$a(x,\xi) := \int_{\mathbb{R}^n} e^{-iz\cdot\xi} k(x,z)\,\mathrm{d}z \tag{2.8}$$

the normal operator on $L^2(M)$ can be brought to the form

$$Nf(x) = \int_{R^n} e^{ix\cdot\xi} a(x,\xi) \hat{f}(\xi)\,\mathrm{d}\xi. \tag{2.9}$$

The following lemma is a finite regularity adaptation of the classical result [38, Chapter VI.7.4].

Lemma 20. *Let $m < 0$ and suppose that $\kappa \in C_c^l(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ where $l \in \mathbb{N}$ satisfies estimates*

$$\left|\partial_x^\alpha \partial_z^\beta \kappa(x,z)\right| \le C_{\alpha\beta} \left|z\right|^{-m-n-|\beta|}, \quad z \neq 0, \tag{2.10}$$

for $|\alpha| + |\beta| \le l$. Then the function on $\mathbb{R}^n \times \mathbb{R}^n$ defined by

$$b(x,\xi) \coloneqq \int_{\mathbb{R}^n} e^{-iz\cdot\xi} \kappa(x,z)\,\mathrm{d}z \tag{2.11}$$

is a symbol in the class $S^m(l-s, s-2)$ for all $s \in \mathbb{N}$ with $2 \le s \le l$.

Proof. Since by assumption

$$\left|\partial_z^\beta \kappa(x,z)\right| \le C_\beta \left|z\right|^{-m-n-|\beta|}, \quad z \neq 0, \tag{2.12}$$

holds for all $|\beta| \le l$ and since κ is compactly supported, it can be shown by using [38, VI 4.5.] as in [38, VI 7.4.] that b is a continuous function on $\mathbb{R}^n \times \mathbb{R}^n$ and

$$\left|\partial_\xi^\beta b(x,\xi)\right| \le C_\beta (1 + |\xi|)^{m-|\beta|} \tag{2.13}$$

for all $|\beta| \le l-2$, which is the first estimate we set out to prove.

Then let $s \in [2,l]$ be an integer. Since κ is compactly supported we have

$$\partial_x^\alpha b(x,\xi) = \int_{\mathbb{R}^n} e^{-iz\cdot\xi} \partial_x^\alpha \kappa(x,z)\,\mathrm{d}z. \tag{2.14}$$

Let us denote $\kappa_\alpha(x,z) = \partial_x^\alpha \kappa(x,z)$. Then it holds that

$$\left|\partial_z^\beta \kappa_\alpha(x,z)\right| \le C_{\alpha\beta} \left|z\right|^{-m-n-|\beta|}, \quad z \neq 0, \tag{2.15}$$

for all $|\beta| \le l - |\alpha|$. Therefore by a similar application of [38, VI 4.5] we have

$$\left|\partial_\xi^\beta \partial_x^\alpha b(x,\xi)\right| \le C_{\alpha\beta} (1 + |\xi|)^{m-|\beta|} \tag{2.16}$$

for all $|\alpha| + |\beta| \le l-2$. Then it follows that

$$\left\|\partial_\xi^\beta b(\,\cdot\,,\xi)\right\|_{C_*^{l-s}} \le \left\|\partial_\xi^\beta b(\,\cdot\,,\xi)\right\|_{C^{l-s}} \le C_{\alpha\beta} (1 + |\xi|)^{m-|\beta|}, \tag{2.17}$$

which uses compactness of the support of κ again. By estimates (2.13) and (2.17) we have shown $b \in S^m(l-s, s-2)$ for all integers $s \in [2,l]$ as claimed. ∎

Lemma 21. *Let* (M, g) *be a simple manifold with* $g \in C^k(M)$ *for some* $k \geq 5$*. Then the function* a *defined by* (2.8) *belongs to* $S^{-1}(k-s, s-4)$ *for all* $s \in [4, k]$ *with* $4 \leq s \leq k$*.*

Proof. We write the kernel in (2.7) in the form

$$k(x, z) = |z|^{1-n}\, \psi(x) \frac{2a(x, x-z) \det(g(x-z))^{\frac{1}{2}}}{[G_{jk}(x, x-z)\frac{z^j}{|z|}\frac{z^k}{|z|}]^{\frac{n-1}{2}}} \phi(x-z) \tag{2.18}$$

and denote

$$k_0(x, z) = \psi(x) \frac{2a(x, x-z) \det(g(x-z))^{\frac{1}{2}}}{[G_{jk}(x, x-z)\frac{z^j}{|z|}\frac{z^k}{|z|}]^{\frac{n-1}{2}}} \phi(x-z). \tag{2.19}$$

Then $k_0(x, z)$ is C^{k-2} for $z \neq 0$ and its derivatives $\partial_x^\alpha \partial_z^\beta k_0(x, z)$, $z \neq 0$, are bounded for all $|\alpha| + |\beta| \leq k-2$ since $k_0(x, z)$ is compactly supported. Thus k satisfies estimates

$$\left|\partial_x^\alpha \partial_z^\beta k(x, z)\right| \leq C_{\alpha\beta}\, |z|^{1-n-|\beta|}\,, \quad z \neq 0, \tag{2.20}$$

for all $|\alpha| + |\beta| \leq k-4$ and the claim follows from lemma 20. □

Remark 22. The symbol $a(x, \xi)$ is smooth in ξ but our argument does not prove that $a(x, \xi)$ satisfies the estimates of the class $S^m(k, L)$ for all orders L of ξ-derivatives. Thus we cannot use paradifferential calculus to study N.

Lemma 21 shows that $N \in \Psi^{-1}(k-s, s-4)$ for all $s \in \mathbb{N}$ with $4 \leq s \leq k$ when the Riemannian metric is in $C^k(M)$ when $k \geq 5$. The rest of this section is devoted to computing the principal symbol of the normal operator. We start by writing the kernel k as

$$k(x, z) = |z|^{1-n}\, h\left(x, z, \frac{z}{|z|}\right) \tag{2.21}$$

where h is a function on $\mathbb{R}^n \times [0, \infty) \times S^{n-1}$ defined by

$$h(x, r, \omega) = \psi(x) \frac{2a(x, x-r\omega) \det(g(x-r\omega))^{\frac{1}{2}}}{[G_{jk}(x, x-r\omega)\omega^j\omega^k]^{\frac{n-1}{2}}} \phi(x-r\omega). \tag{2.22}$$

Since $G_{jk}(x, x-r\omega)\omega^j\omega^k$ is non-vanishing we see that $h \in C^{k-2}(\mathbb{R}^n \times [0, \infty) \times S^{n-1})$. By the Fundamental theorem of calculus

$$h(x, r, \omega) = h(x, 0, \omega) + r\int_0^1 \partial_r h(x, rt, \omega)\, \mathrm{d}t \tag{2.23}$$

and we can decompose $k(x,z) = k_{-1}(x,z) + r(x,z)$ where

$$k_{-1}(x,z) := |z|^{1-n}\, h\left(x, 0, \frac{z}{|z|}\right) \tag{2.24}$$

and

$$r(x,z) := |z|^{2-n} \int_0^1 \partial_r h\left(x, |z|\, t, \frac{z}{|z|}\right) \,\mathrm{d}t. \tag{2.25}$$

Since $k(x,z)$ is compactly supported in z we can choose a cut-off function $\chi(z)$ so that $0 \leq \chi \leq 1$ and $\chi = 1$ near the origin so that

$$k(x,z) = \chi(z)k(x,z) = \chi(z)k_{-1}(x,z) + \chi(z)r(x,z). \tag{2.26}$$

Now the full symbol of N is decomposed as

$$\begin{aligned} a(x,\xi) &= \mathcal{F}(\chi(\,\cdot\,)k_{-1}(x,\,\cdot\,))(\xi) + \mathcal{F}(\chi(\,\cdot\,)r(x,\,\cdot\,))(\xi) \\ &=: a_{-1}(x,\xi) + c(x,\xi). \end{aligned} \tag{2.27}$$

Lemma 23. *Let (M,g) be a simple manifold with $g \in C^k(M)$ for some $k \geq 5$. Then $a_{-1} \in S^{-1}(k-s, s-4)$ for all $s \in \mathbb{N}$ with $4 \leq s \leq k$ and $c \in S^{-2}(k-s, s-5)$ for all $s \in \mathbb{N}$ with $5 \leq s \leq k$.*

Proof. Since $h \in C^{k-2}(\mathbb{R}^n \times [0,\infty) \times S^{n-1})$ is compactly supported in x and r and S^{n-1} is compact, we can extend h to a compactly supported function on $\mathbb{R}^n \times [0,\infty) \times S^{n-1}$. Thus $\partial_x^\alpha \partial_r^l \partial_\omega^\beta h(x,r,\omega)$ is continuous and compactly supported for all $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^{n-1}$ and $l \in \mathbb{N}$ for which we have $|\alpha| + l + |\beta| \leq k-2$.

First, we prove the claim about the Fourier transform c of the remainder. Since derivatives of h are continuous and compactly supported, a simple computation using the chain rule shows that

$$\left| \partial_x^\alpha \partial_{z^j} \partial_r h\left(x, |z|\, t, \frac{z}{|z|}\right) \right| \leq C\, |z|^{-1} \tag{2.28}$$

near $z = 0$ and for all $t \in [0,1]$ when $|\alpha| + 2 \leq k-2$. Therefore by iteration

$$\left| \partial_x^\alpha \partial_z^\beta \partial_r h\left(x, |z|\, t, \frac{z}{|z|}\right) \right| \leq C_{\alpha\beta}\, |z|^{-|\beta|} \tag{2.29}$$

near $z = 0$ when $|\alpha| + |\beta| + 1 \leq k - 2$. The above estimate applied to the remainder term $r(x, z)$ yields

$$\left|\partial_x^\alpha \partial_z^\beta \int_0^1 \partial_r h\left(x, |z|\, t, \frac{z}{|z|}\right) \mathrm{d}t\right| \leq C_{\alpha\beta}\, |z|^{-|\beta|} \tag{2.30}$$

near $z = 0$ when $|\alpha| + |\beta| + 1 \leq k - 2$, which implies that

$$\left|\partial_x^\alpha \partial_z^\beta (\chi(z) r(x, z))\right| \leq C_{\alpha\beta}\, |z|^{2-n-|\beta|} \tag{2.31}$$

for all z and $|\alpha| + |\beta| \leq k - 3$ since the cut-off $\chi(z)$ implies that only have to derive the estimate near $z = 0$. It follows from lemma 20 that $c \in S^{-2}(k - s, s - 5)$ for all $s \in \mathbb{N}$ with $5 \leq s \leq k$.

By a similar computation we see that k_{-1} satisfies estimates

$$\left|\partial_x^\alpha \partial_z^\beta (\chi(z) k_{-1}(x, z))\right| \leq C_{\alpha\beta}\, |z|^{1-n-|\beta|} \tag{2.32}$$

for all z and $|\alpha| + |\beta| \leq k - 2$. Thus, again, by lemma 20 we have $a_{-1} \in S^{-1}(k - s, s - 4)$ for all $s \in \mathbb{N}$ with $4 \leq s \leq k$, which finishes the proof. □

In the next section we construct a leading order parametrix for N. To this end we need to find a more explicit representation for a_{-1}. We write $\chi(z)k_{-1}(x, z) = k_{-1}(x, z) - (1 - \chi(z))k_{-1}(x, z)$ and analyze the Fourier transforms of the parts separately.

Lemma 24. *For a dimensional constant C it holds that*

$$\int_{\mathbb{R}^n} e^{-iz\cdot\xi} k_{-1}(x, z)\, \mathrm{d}z = C\psi(x)\, |\xi|_{g(x)}^{-1}\, \phi(x). \tag{2.33}$$

Proof. The Fourier transform of

$$k_{-1}(x, z) = \psi(x) \frac{2 \det(g(x))^{1/2}}{(g_{jk}(x) z^j z^k)^{\frac{n-1}{2}}} \phi(x) \tag{2.34}$$

in z is computed in [30, Chapter 8.1]. The only difference is regularity in x, which does not affect the computation. □

Lemma 25. *Let (M, g) be a simple manifold with $g \in C^k(M)$ for some $k \geq 3$. Let $b(x, \xi) \coloneqq \mathcal{F}((1-\chi(\,\cdot\,))k_{-1}(x,\,\cdot\,))(\xi)$. Then $b \in C_x^{k-2} C_\xi^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$. Moreover, b has a singularity of type $|\xi|_g^{-1}$ at the origin, and satisfies $|b(x, \xi)| \leq C\, |\xi|^{2-k}$ when $|\xi|$ is large enough.*

Proof. The fact that $b(x,\xi)$ has a singularity of type $|\xi|_{g(x)}^{-1}$ at the origin follows from the fact that $b(x,\xi) = a(x,\xi) - C\psi(x)\,|\xi|_{g(x)}^{-1}\,\phi(x)$ near $\xi = 0$ and $a \in C_x^{k-2}C_\xi^\infty(\mathbb{R}^n \times \mathbb{R}^n)$.

Next, we prove the claim about the decay of b away from $\xi = 0$. Since $(1-\chi(z))k_{-1}(x,z) = 0$ for z near the origin and since

$$(1-\chi(z))k_{-1}(x,z) = (1-\chi(z))\psi(x)\frac{2\det(g(x))^{1/2}}{(g_{jk}(x)z^j z^k)^{\frac{n-1}{2}}}\phi(x) \tag{2.35}$$

for $z \neq 0$, we know that $(1-\chi)k_{-1}$ is in $C^{k-2}(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ and compactly supported in x. As in the proof of lemma 23 we can use boundedness of the derivatives of $h(x,0,z\,|z|^{-1})$ to prove that

$$|\partial_z^\alpha k_{-1}(x,z)| = \left|\partial_z^\alpha\left((1-\chi(z))\,|z|^{1-n}\,h\left(x,0,\frac{z}{|z|}\right)\right)\right| \leq C_\alpha\,|z|^{-n-1} \tag{2.36}$$

for $2 \leq |\alpha| \leq k-2$ which proves that for a fixed x we have $\partial_z^\alpha k_{-1}(x,z) \in L^1(\mathbb{R}^n)$. Therefore by the Riemann–Lebesgue lemma we conclude that

$$\begin{aligned} &|\xi^\alpha \mathcal{F}((1-\chi(\,\cdot\,))k_{-1}(x,\,\cdot\,))(\xi)| \\ &\quad= |\mathcal{F}(\partial_z^\alpha((1-\chi(\,\cdot\,))k_{-1}(x,\,\cdot\,)))(\xi)| \\ &\quad\to 0 \end{aligned} \tag{2.37}$$

for all $2 \leq |\alpha| \leq k-2$ as $|\xi| \to \infty$. Thus since 2.37 holds for all $2 \leq |\alpha| \leq k-2$ we have $|b(x,\xi)| \leq C\,|\xi|^{2-k}$ for $|\xi|$ large enough. □

Lemmas 23, 24 and 25 together prove the following corollary.

Corollary 26. *Let (M,g) be a simple manifold with $g \in C^k(M)$ for some $k \geq 5$. Then $N \in \Psi^{-1}(k-s,s-4)$ for all $s \in \mathbb{N}$ with $4 \leq s \leq k$. The principal symbol of N is*

$$a_{-1}(x,\xi) = C\psi(x)\,|\xi|_g^{-1}\,\phi(x) - b(x,\xi) \in S^{-1}(k-s,s-4) \tag{2.38}$$

where b is as in lemma 25 and $s \in \mathbb{N}$ with $4 \leq s \leq k$, in particular this shows that N is elliptic of order -1 in the sense of principal symbol.

The function a_{-1} is a function on the whole cotangent bundle and thus b has to have a singularity of type $|\xi|_{g(x)}^{-1}$ at $\xi = 0$ to cancel out the singularity in $C\psi(x)\,|\xi|_g^{-1}\,\phi(x)$.

2.2.2 Parametrix construction

In this section we construct a leading order parametrix for the normal operator. The construction is based on a commutator result in [20]. We define $p(x,\xi) := C^{-1}\zeta(\xi)\,|\xi|_{g(x)}$ for some $\zeta \in C^\infty(\mathbb{R}^n)$ so that $0 \le \zeta \le 1$, $\zeta = 0$ near $\xi = 0$ and $\zeta = 1$ for large ξ, and where C is the same dimensional constant as in lemma 24. We will prove that the operator corresponding to the symbol p which is in $S^1(k-s, N)$ for all $s \in \mathbb{N}$ with $4 \le s \le k$ and $N \in \mathbb{N}$ provides the parametrix to the leading order.

Lemma 27. *Let (M,g) be a simple manifold with $g \in C^k(M)$ for some $k \ge 7 + \frac{n}{2}$. Let $P = \mathrm{Op}(p)$. If $\tau \in (0,1]$ is fixed then the operator*

$$PN - \mathrm{Op}(pa)\colon H^{t-\tau}(\mathbb{R}^n) \to H^t(\mathbb{R}^n) \tag{2.39}$$

is continuous when $-(1-\tau)(k-5-\frac{n}{2}-\tau) < t < k-6-\frac{n}{2}$.

Proof. Choose $s \in \mathbb{N}$ so that $s \in (4+\frac{n}{2}, k-1)$ which is possible since $k \ge 7+\frac{n}{2}$. Let $L := s-4$ and let $r := k-s$. Then $L > \frac{n}{2}$ and $r > 1 \ge \tau$. By lemma 21 we have $N \in \Psi^{-1}(r, L)$ and also it holds that $P \in \Psi^1(r, L+1+\frac{n}{2})$, which means that we are in the setting of lemma 14. For δ and ρ as the lemma it holds that

$$\delta = \tau, \quad \rho = 1 \quad \text{and} \quad \delta < \rho. \tag{2.40}$$

Thus since $m_1 = -1$ and $m_2 = 1$ in the lemma the commutator

$$PN - \mathrm{Op}(pa)\colon H^{t-\tau}(\mathbb{R}^n) \to H^t(\mathbb{R}^n) \tag{2.41}$$

is continuous for

$$\max\{-1,0\} - (1-\tau)(r-\tau) < t < r - \max\{1,0\} \tag{2.42}$$

which simplifies to

$$-(1-\tau)(k-s-\tau) < t < k-s-1. \tag{2.43}$$

To have a non-empty range of indices t we must have

$$k - s > \frac{1+(1-\tau)\tau}{2-\tau} \tag{2.44}$$

which is satisfied since $s < k-1$ and by an elementary computation it holds that $\frac{1+(1-\tau)\tau}{2-\tau} \leq 1$ for all $\tau \in (0,1]$.

Finally to conclude the proof we note that if

$$-(1-\tau)(k-5-\frac{n}{2}-\tau) < t < k-6-\frac{n}{2} \tag{2.45}$$

there is $s_t \in \mathbb{N}$ so that $s_t \in [4+\frac{n}{2}, k-1)$ and

$$-(1-\tau)(k-s_t-\tau) < t < k-s_t-1 \tag{2.46}$$

since $k \geq 7+\frac{n}{2}$ and thus the operator $PN - \mathrm{Op}(pa) \colon H^{t-\tau}(\mathbb{R}^n) \to H^t(\mathbb{R}^n)$ is continuous as claimed. ☐

Lemma 28. *Let (M, g) be a simple manifold with $g \in C^k(M)$ for some $k \geq 7+\frac{n}{2}$. Then* $\mathrm{Op}(pa_{-1}) = \mathrm{Id} + R_1$ *where* Id *is an operator acting as the identity on elements in $H^{t+2-k}(\mathbb{R}^n)$ which are supported in the set where $\psi = 1 = \phi$ and the remainder*

$$R_1 \colon H^{t+2-k}(\mathbb{R}^n) \to H^t(\mathbb{R}^n) \tag{2.47}$$

is continuous when $-k+2 < t < k-2$.

Proof. By corollary 26 the principal symbol a_{-1} of N can be decomposed as

$$a_{-1}(x,\xi) = C\psi(x) \left|\xi\right|_{g(x)}^{-1} \phi(x) - b(x,\xi) \tag{2.48}$$

where $b(x,\xi)$ is in $C_x^{k-2} C_\xi^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ which is compactly supported in x and decays faster than $|\xi|^{2-k}$ in ξ. Therefore

$$\begin{aligned} p(x,\xi)a_{-1}(x,\xi) &= \zeta(\xi)\psi(x)\phi(x) - C^{-1}\zeta(\xi)b(x,\xi) \\ &= \psi(x)\phi(x) - (1-\zeta(\xi))\psi(x)\phi(x) - C^{-1}\zeta(\xi)b(x,\xi). \end{aligned} \tag{2.49}$$

Since $(1-\zeta(\xi))\psi(x)\phi(x)$ is smooth and compactly supported, it decays faster than $|\xi|^{-l}$ for any $l \in \mathbb{N}$. Since $\psi(x)\phi(x)$ equals to 1 on in the set where $\psi = 1 = \phi$ the corresponding operator acts as the identity on functions in $H^{t+2-k}(\mathbb{R}^n)$ which are supported in this set.

Also, by lemma 25 the function $\zeta(\xi)b(x,\xi)$ decays faster than $|\xi|^{2-k}$. Therefore, since the support in x is compact and $b \in C_x^{k-2}C_\xi^\infty(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ and $\zeta(\xi) = 0$ near $\xi = 0$ it follows from the definitions that

$$\tilde{b}(x,\xi) := -(1-\zeta(\xi))\psi(x)\phi(x) - C^{-1}\zeta(\xi)b(x,\xi) \tag{2.50}$$

is a symbol in the class $S^{2-k}(k-2, 1+\lfloor \frac{n}{2} \rfloor)$. Therefore by lemma 13 that $\mathrm{Op}(\tilde{b})\colon H^{t+2-k}(\mathbb{R}^n) \to H^t(\mathbb{R}^n)$ for all $-k+2 < t < k-2$ since $1 + \lfloor \frac{n}{2} \rfloor > \frac{n}{2}$, which proves the claim. □

Lemma 29. *Let (M,g) be a simple manifold with $g \in C^k(M)$ for some $k \geq 7 + \frac{n}{2}$. Then the operator*

$$\mathrm{Op}(pc)\colon H^{t-1}(\mathbb{R}^n) \to H^t(\mathbb{R}^n) \tag{2.51}$$

is continuous when $-k+6+\frac{n}{2} < t < k-6-\frac{n}{2}$.

Proof. Let $s \in \mathbb{N}$ be so that $s \in (5+\frac{n}{2}, k)$. Since p is in $S^1(k-s, s-5)$ and c is in $S^{-2}(k-s, s-5)$ by lemma 23 the product pc is in $S^{-1}(k-s, s-5)$. Furthermore, since $s-5 > \frac{n}{2}$ it follows from lemma 13 that $\mathrm{Op}(pc)$ continuously maps from $H^{t-1}(\mathbb{R}^n)$ to $H^t(\mathbb{R}^n)$ for all $-k+s < t < k-s$. To see that the continuous mapping property holds for all $-k+6+\frac{n}{2} < t < k-6-\frac{n}{2}$, we note that given any such t we can choose any $s_t \in \mathbb{N}$ so that $s_t \in (5+\frac{n}{2}, k-t)$ when $t \geq 0$ or $s_t \in (5+\frac{n}{2}, k+t)$ when $t < 0$ and it holds that $s_t \in \mathbb{N}$ with $s_t \in (5+\frac{n}{2}, k)$ and $-k+s_t < t < k-s_t$. This finishes the proof. □

Lemma 30. *Let (M,g) be a simple manifold with $g \in C^k(M)$ for some $k \geq 7 + \frac{n}{2}$. Let $P = \mathrm{Op}(p)$. Then there is $\varepsilon > 0$ so that $PN = \mathrm{Id} + R$ where Id is an operator acting as the identity on elements in $H^{t-\tau}(\mathbb{R}^n)$ which are supported in the set where $\psi = 1 = \phi$ and the remainder*

$$R\colon H^{t-\tau}(\mathbb{R}^n) \to H^t(\mathbb{R}^n) \tag{2.52}$$

is continuous whenever $0 < \tau \leq \varepsilon$ and

$$-k+6+\frac{n}{2} < t < k-6-\frac{n}{2}. \tag{2.53}$$

Proof. By lemma 23 we may write

$$\begin{aligned} PN &= \mathrm{Op}(pa) + (PN - \mathrm{Op}(pa)) \\ &= \mathrm{Op}(pa_{-1}) + (PN - \mathrm{Op}(pa)) + \mathrm{Op}(pc). \end{aligned} \tag{2.54}$$

Let $\tau \in (0,1]$. Then by lemma 27 we have that

$$PN - \mathrm{Op}(pa)\colon H^{t-\tau}(\mathbb{R}^n) \to H^t(\mathbb{R}^n) \tag{2.55}$$

is continuous for $-(1-\tau)(k-5-\frac{n}{2}-\tau) < t < k-6-\frac{n}{2}$. By lemmas 28 and 29 the operator $\mathrm{Op}(pa_{-1})$ is the identity up to an operator R_1 that is smoothing by 2 degrees and $\mathrm{Op}(pc)$ is smoothing by 1 degree, and therefore R_1 and $\mathrm{Op}(pc)$ are also smoothing by τ degrees. More precisely, $\mathrm{Op}(pa_{-1}) = \mathrm{Id} + R_1$, and we have that

$$R_1\colon H^{t-\tau}(\mathbb{R}^n) \to H^t(\mathbb{R}^n) \tag{2.56}$$

is continuous for $-k+2 < t < k-2$ and

$$\mathrm{Op}(pc)\colon H^{t-\tau}(\mathbb{R}^n) \to H^t(\mathbb{R}^n) \tag{2.57}$$

is continuous for $-k+6+\frac{n}{2} < t < k-6-\frac{n}{2}$. Letting R be the sum of the operators in (2.55), (2.56) and (2.57) we find that $PN = \mathrm{Id} + R$. Now suppose that τ is close enough to zero. Then the remainder continuously maps $H^{t-\tau}(\mathbb{R}^n)$ to $H^t(\mathbb{R}^n)$ for

$$-k+6+\frac{n}{2} < t < k-6-\frac{n}{2} \tag{2.58}$$

since $k-6-\frac{n}{2}$ is the smallest among the upper bound requirements and when τ is close to zero $-k+6+\frac{n}{2}$ is the largest of the lower bound. This proves the claimed identity and the mapping properties. □

2.3 Proofs of main theorems

In the last section we show that the parametrix construction in lemma 30 in combination with the recent result [10, Theorem 1] can be used to prove our main results.

Proof of theorem 15. Let $f \in H^s_c(M)$ for some $s > -k+6+\frac{n}{2}$ and assume that $Nf = 0$. Let $\operatorname{supp} f \subseteq \Omega$. There is a cut-off function $\phi \in C^\infty_c(M)$ so that $\phi f = f$ and moreover there is a cut-off $\psi \in C^\infty_c(M)$ with $\psi = 1$ on Ω so that $Nf(x) = (\psi N \phi)f(x) = 0$ for all $x \in M$. The operator $\psi N \phi$ has Schwartz kernel of the form (2.3) so by lemma 30 there is an operator P and $\varepsilon > 0$ so that $P(\psi N \phi) = \mathrm{Id} + R$ where Id acts as the identity on elements in $H^t(\mathbb{R}^n)$ with support in Ω and $R\colon H^t_c(M) \to H^{t+\tau}(\mathbb{R}^n)$ is continuous for $\tau \in (0, \varepsilon]$ and

$$-k + 6 + \frac{n}{2} - \tau < t < k - 6 - \frac{n}{2} - \tau. \tag{2.59}$$

We may choose τ so small that $s > -k + 6 + \frac{n}{2} - \tau$. Then $f \in H^s_c(M)$ and

$$\phi f = P(\psi N \phi) f - Rf = -Rf. \tag{2.60}$$

Thus $\phi f \in H^{t+\tau}(\mathbb{R}^n)$ and therefore $f \in H^{t+\tau}_c(M)$.

Then let $s < r < k - 6 - \frac{n}{2}$. By possibly choosing τ to be even smaller we may assume that there is $m \in \mathbb{N}$ so that $r < s + m\tau < k - 6 - \frac{n}{2} - \tau$. Then by iterating m times the argument in the previous paragraph we see that $f \in H^{s+m\tau}_c(M) \subseteq H^r_c(M)$ as claimed in the theorem. □

Proof of proposition 16. The composition of I and I^* was computed in [30, Lemma 8.1.5] for $g \in C^\infty(M)$. The same computation works for $g \in C^k(M)$ when $k \geq 2$. □

Proof of theorem 17. Let $(\tilde{M}, \tilde{g})$ be a simple extension of (M, g) and let $\tilde{I}$ be the X-ray transform of $(\tilde{M}, \tilde{g})$. Suppose that $f \in L^2(M)$ and $If = 0$. Then zero extension of f to $\tilde{M}$ still denoted by f satisfies $\tilde{I}f = 0$. Therefore $\tilde{N}f = \tilde{I}^*\tilde{I}f = 0$ by proposition 16 where $\tilde{N}$ and $\tilde{I}^*$ are the operators on $\tilde{M}$ defined by (1.11) and (1.10) with all objects replaced by corresponding objects of $(\tilde{M}, \tilde{g})$. Therefore by theorem 15 applied to the simple extension $(\tilde{M}, \tilde{g})$ implies that $f \in H^r_c(\tilde{M})$ for all $s < r < k - 6 + \frac{n}{2}$. Since $k \geq n + 8$ there is some $r \in \mathbb{R}$ so that $\lceil 1 + \frac{n}{2} \rceil < r < k - 6 + \frac{n}{2}$ and $f \in H^r_c(\tilde{M})$. Sobolev embedding yields

$$H^r_c(\tilde{M}) \subseteq W^{1,\infty}(\tilde{M}) = \mathrm{Lip}(\tilde{M}). \tag{2.61}$$

Thus $f \in \mathrm{Lip}(\tilde{M})$ and since f vanishes in $\tilde{M} \setminus M$ we have $f \in \mathrm{Lip}_0(M)$. We see that $f = 0$ since I is injective on $\mathrm{Lip}(M)$ by [10, Theorem 1.] which finishes the proof. $\square$

Chapter 3

DETERMINATION OF THE DN MAP BY SCATTERING RELATION

3.1 Introduction

Given two simple metrics g on a compact surface with boundary M, it was proved in [31] that the boundary distance function determines the metric up to a boundary fixing diffeomorphism; in other word simple surfaces are boundary rigid. The key step in the proof of boundary rigidity involves showing that the boundary distance function determines the Dirichlet-to-Neumann (DN map) for simple metrics. In this paper we prove that the same is true when the metric has sufficiently high but finite regularity.

We prove several intermediate results with various regularity requirements for the metric g, from which it follows that for simple C^{17} metrics DN map is determined by the boundary distance function.

The three main results in this paper are the following: 1. For $C^{1,1}$ metrics on the closed disk $\mathbb{D}$, the boundary distance function determines the metric on the boundary up to a boundary fixing gauge (which fixes the boundary distance function). 2. The adjoint operator I^* of the geodesic x ray transform is surjective for simple metrics g that are C^{17}. 3. Given a simple metric g of class C^{17} on a compact surface with boundary, the boundary distance function determines the DN map. These results are summarized in the section below.

3.1.1 Main results

Theorem 31. *Let $M \subset \mathbb{R}^2$ be the closed unit disk, g_1, g_2 be $C^{1,1}$ metrics on M such that $d_{g_1}|_{\partial M\times\partial M} = d_{g_2}|_{\partial M\times\partial M}$, then there exists a $C^{2,1}$ diffeomorphism $\psi : M \to M$ with $\psi|_{\partial M} = id$*

such that $\psi^* g_1|_{\partial M} = g_2|_{\partial M}$

Theorem 32. *Let* (M, g) *be a simple surface with* $g \in C^k$ *with* $k \geq 10$*, let* $f \in C^l(M)$ *with* $1 < m+1 < l-1 < k-7$*,* $m, l, k \in \mathbb{N}$*, then there exists* $w \in C^{\min(k-4,m)}(\partial_+ SM)$ *with* $w^\# \in C^{\min(k-4,m)}(SM)$*, such that* $I^* w = f$*.*

Theorem 33. *Let* (M, g_1) *and* (M, g_2) *be a simple surfaces with* $g_1, g_2 \in C^{17}$*, with* $d_{g_1} = d_{g_2}$*, then the DN maps* $\Lambda : C^{2,\alpha}(\partial M) \to C^{1,\alpha}(\partial M)$ *determined by* g_1, g_2 *are equal for all* $0 < \alpha < 1$*.*

The proof of theorem 1 relies on some recent results by [22] and [1], theorem 2 relies on some microlocal analysis at low regularity studied in [20] and [12], all of which derived from the proofs in [30], and finally we use theorem 1, 2 and a modification of the proof in [31] to prove the titular result.

3.2 Boundary distance function determines the scattering relation

In this section we prove theorem 1 (31) for $C^{1,1}$, then we show that for C^2 metrics the boundary distance function also determines the scattering relation.

We first prove two technical lemmas before proving theorem 1, for the rest of the proof of theorem 1 we assume $M = \mathbb{D} \subset \mathbb{R}^2$.

Lemma 34. *Let* g *be a* $C^{1,1}$ *metric on* M*, there exists a* $C^{2,1}$ *diffeomorphism* $\phi : U \to U'$ *with* U, U' *neighborhoods of* ∂M *such that* $\phi|_{\partial M} = id$ *and* $\phi^* \nu = \bar{\nu}$ *where* ν *is the unit normal vector with respect to* g *and* $\bar{\nu}$ *is the Euclidean normal on* ∂M*.*

Proof. Consider $U \subset \mathbb{R}^2 = (a, b) \times (1-\epsilon, 1+\epsilon)$ a neighborhood of a segment of ∂M in polar coordinates $\psi(\theta, r)$ for small $\epsilon < 1$ and $b - a < 2\pi$. Suppose such a $C^{2,1}$ diffeomorphism exists for U, then

$$\nu = d\phi(\bar{\nu}) = d\phi(dr) = \frac{1}{|dr - \frac{g_{12}}{g_{11}} d\theta|_g} dr - \frac{\frac{g_{12}}{g_{11}}}{|dr - \frac{g_{12}}{g_{11}} d\theta|_g} d\theta \tag{3.1}$$

where g_{ij} is the metric component of g in the coordinates of ψ. In this case the differential $d\phi$ at the boundary (at $r = 1$) must be of the form

$$d\phi|_{\partial M} = \begin{pmatrix} \frac{d\phi_1}{d\theta} & \frac{d\phi_1}{dr} \\ \frac{d\phi_2}{d\theta} & \frac{d\phi_2}{dr} \end{pmatrix} |_{\partial M} = \begin{pmatrix} 1 & \frac{-g_{12}}{|dr - \frac{g_{12}}{g_{11}} d\theta|_g} \\ 0 & \frac{1}{|dr - \frac{g_{12}}{g_{11}} d\theta|_g} \end{pmatrix} |_{\partial M} \tag{3.2}$$

We will construct a diffeomorphism on a collar neighborhood of ∂M in $\mathbb{R}^2$ with differential 3.2. Observe that the differential above has coefficients that are $C^{1,1}$, by [lemma 3.3.1, [1]] we may choose

$\tilde{\phi_1}, \tilde{\phi_2} \in C^{2,1}(M)$ such that $\tilde{\phi_1}|_{\partial M} = \tilde{\phi_2}|_{\partial M} = 0$ and

$$\frac{\partial \tilde{\phi_1}}{\partial r} = \frac{-g_{12}}{|dr - \frac{g_{12}}{g_{11}} d\theta|_g}, \frac{\partial \tilde{\phi_2}}{\partial r} = \frac{1}{|dr - \frac{g_{12}}{g_{11}} d\theta|_g} \tag{3.3}$$

Now define a $C^{2,1}$ map ϕ on $U := (0, 2\pi] \times (1 - \epsilon, 1 + \epsilon)$ in polar coordinates for some small ϵ by $\phi(\theta, r) = (\phi_1, \phi_2) =: (\theta + \tilde{\phi_1}, 1 + \tilde{\phi_2})$, then ϕ fixes ∂M $(r = 1)$ and maps U into a neighborhood of ∂M in $\mathbb{R}^2$ with the differential (number) at the boundary. Since $\tilde{\phi_2}(\theta, 1) = 0$ and $\frac{\partial \tilde{\phi_2}}{\partial r}(\theta, 1) > 0$ for any fixed θ, $\tilde{\phi_2}(\theta, r) < 0$ for sufficiently small $1 - r > 0$, by compactness of ∂M we may choose sufficiently small ϵ so that ϕ maps $[0, 2\pi) \times (1 - \epsilon, 1]$ into M. Furthermore, since the differential at the boundary is clearly invertible, ϕ is a local diffeomorphism near $r = 1$. Hence for a sufficiently small ϵ, ϕ is a local diffeomorphism that maps U into a neighborhood of ∂M in M which fixes ∂M and $\phi^*(\nu) = \bar{\nu}$.

An argument similar to that of the proof of (Theorem 5.25 [18]) shows that ϕ is injective on an possibly even smaller neighborhood of ∂M, so it restricts to a $C^{2,1}$ diffeomorphism from some neighborhood U of ∂M to another such neighborhood U'.

□

Lemma 35. *Suppose the $C^{1,1}$ metrics g_1 and g_2 on M induces the same boundary distance functions, then the metrics agree in the tangential direction at the boundary.*

Proof. It suffices to show that the boundary distance function determines the metric in the tangential direction. In other word, we prove that given a $C^{1,1}$ metric g on M, $d_g|_{\partial M\times\partial M}$ determines $g|_{\partial M}$ in the tangential direction. Let $p\in\partial M$ and $v\in T\partial M$, and a smooth curve $\tau:(-\epsilon,\epsilon)\to\partial M$ with $\tau(0)=p$ and $\tau'(0)=v$.

Consider a local coordinate $((x,y),U)$ centered at p with $\partial M\subset\{y=0\}$ near $p=0$. By applying the appropriate linear transformations, we may assume g is euclidean at 0. Denote $\bar{g}$ the euclidean metric in the local coordinates $((x,y),U)$, then we know $\bar{g}|_0=g|_0$.

Consider $\lim_{s\to 0}\frac{d(p,\tau(s))}{s}$, then since

$$\frac{d(p,\tau(s))}{s}\leq\frac{\int_0^s|\tau'(t)|_g dt}{s}$$

and

$$|v|_g=\lim_{s\to 0}\frac{\int_0^s|\tau'(t)|_g dt}{s}$$

, we have

$$\lim_{s\to 0}\frac{d(p,\tau(s))}{s}\leq|v|_g \tag{3.4}$$

We now prove the equality. For any $(x,y)\in U$, consider the change of basis matrix from the $\frac{\partial}{\partial x},\frac{\partial}{\partial y}$ basis to the orthonormal basis with respect to g, denoted as $T(x)$, then we have for any $(x,v)\in TM$,

$$|v|_{\bar{g}}\leq||T^{-1}(x)|||v|_g \tag{3.5}$$

where $||T^{-1}(x)||$ is the operator norm of T^{-1}. Since $T(x)\to I$ for $x\to 0$, $||T^{-1}(x)||\to 1$. So for a fixed ϵ, there is a sufficiently small neighborhood U' near 0 so that

$$|v|_{\bar{g(x)}}\leq|v|_{g(x)}(1+\epsilon) \tag{3.6}$$

for all $(x,v)\in TU'$.

By (Theorem 6 in [22]), for $C^{1,1}$ metrics, for every normal neighborhood N of a point p, every absolutely continuous curve starting from p connecting to another point $q\in N$ must have length larger than the geodesic connecting them. So choose a geodesic ball V of small radius center at p (Theorem 4 in [22]), as a consequence of Theorem 6 in [22] all length minimizing geodesics connecting p with points in V lie in V. Since we are taking limits of s with $\tau(s)$ converging to p we may assume U lie in such a V. So we have

$$(1+\epsilon)\frac{d(0,\tau(s))}{s} = \frac{\int_0^\alpha (1+\epsilon)|\gamma_s'(t)|_g dt}{s} \geq \frac{\int_0^\alpha |\gamma_s'(t)|_{\bar{g}}}{s} \tag{3.7}$$

Where $\gamma_s : [0,\alpha] \to V$ is the g-geodesic connecting p with $\tau(s)$. But in Euclidean metrics the shortest curve between two points must be a straight line, since τ is a straight line lying on $y = 0$, we must also have

$$\frac{\int_0^\alpha |\gamma_s'(t)|_{\bar{g}}}{s} \geq \frac{\int_0^s |\tau'(s)|_{\bar{g}} dt}{s}$$

So we have the following chains of inequalities

$$\limsup_{s\to 0}(1+\epsilon)\frac{d(0,\tau(s))}{s} \geq \limsup_{s\to 0}\frac{\int_0^s |\tau'(s)|_{\bar{g}} dt}{s} = \lim_{s\to 0}\frac{\int_0^s |\tau'(s)|_{\bar{g}} dt}{s} = |v|_g$$

But since ϵ was arbitrary, we have $\limsup_{s\to 0}\frac{d(0,\tau(s))}{s} \geq |v|_g$.

So we have

$$|v|_g \geq \limsup_{s\to 0}\frac{d(0,\tau(s))}{s} \geq |v|_g$$

. Since we also know $\liminf_{s\to 0}\frac{\int_0^s |\tau'(s)|_{\bar{g}} dt}{s} = \lim_{s\to 0}\frac{\int_0^s |\tau'(s)|_{\bar{g}} dt}{s}$ (since $\lim_{s\to 0}\frac{\int_0^s |\tau'(s)|_{\bar{g}} dt}{s}$ converges), the same inequalities as above holds true if we replace lim sup with lim inf, so we have

$$\limsup_{s\to 0}\frac{d(0,\tau(s))}{s} = \liminf_{s\to 0}\frac{d(0,\tau(s))}{s} = |v|_g$$

which shows

$$\lim_{s\to 0}\frac{d(0,\tau(s))}{s} = |v|_g$$

This shows that $|v|_g$ is completely determined by the distance function for a $C^{1,1}$ metric.

□

We are now ready to prove theorem 1:

Proof of theorem 1. By 34 there exists $\phi : U \to U'$ that is a boundary fixing diffeomorphism between neighborhoods U, U' of ∂M such that $\phi(\nu) = \bar{\nu}$. Using the the exponential map with resepct to the Euclidean metric on $M = \mathbb{D}$, the proof of (Prop 11.2.5 [30]) remains valid for C^2 diffeomorphism near the boundary, from which we can conclude there exists a $C^{2,1}$ diffeomorphism $\Phi : M \to M$ that restricts to ϕ near the boundary (The regularity of this diffeomorphism will be one order higher than that of the metrics). Furthermore, Φ is a diffeomorphism such that $\Phi * (\nu_1) = \bar{\nu}$. We can find another such Φ_2 so that $\Phi_2^*(\nu_2) = \bar{\nu}$, then $\Phi =: \Phi_2^{-1}\Phi_1$ is also boundary fixing and $\Phi^*(\nu_1) = \nu_2$. Since boundary distance functions are invariant under boundary fixing diffeomorphism, $\Phi^* g_1$ and g_2 has the same unit normal vector field at ∂M and boundary distance function , and by 35 they agree in the tangential direction at the boundary. □

Corollary 36. *Suppose g_1 and g_2 are two simple C^3 metrics with the same boundary distance function on a simple manifold, then they have the same scatter relation (defined above).*

Proof. It is proved in [30] that C^k simple manifolds are C^{k-1} diffeomorphic to closed ball, so we may without loss of generality assume g_1 and g_2 are two C^2 metrics on a closed ball. the abscence of conjugate points , non trapping and strict convexisty are C^2 diffeomorphism invariant conditions, so g_1 and g_2 are C^2 simple metrics on the euclidean disk. So we can apply (Lemma 11.3.2 [30]) to conclude that the scattering relations and exit time functions are equal and (Lemma 11.2.6 [30]) to conclude that the volume form are equal. □

Remark 37. For $C^{1,1}$ metrics on a closed disk, if we assume the non trapping condition and strict convexity (which are defined for $C^{1,1}$ metrics, and furthermore if we assume there exists x in the interior of M with so that $exp_x : D_x \to M$ is a lipscthiz homeomorphism, then (Lemma 11.3.2, [30]) applies almost everywhere to conclude that the scattering relations are equal almost everywhere. For Simple $C^{1,1}$ simple manifold (Defined in [10]), the Santaló formula holds by (lemma 24, [11]), so $g_1|_{\partial M} = g_2|_{\partial M}$ and $\tau_{g_1} = \tau_{g_2}$ together implies that the Vol(M, g_1) = Vol(M,g_2).

3.3 Surjectivity of the backprojection operator

We now prove main theorem 2 following a modification of the argument in (Theorem 8.2.5 [30]). Throughout the rest of the paper we will assume (M, g) is a simple surface.

Similar to (Lemma 3.1.8 in [30]), we embed M into a closed manifold isometrically of the same dimension with metric also in C^k for $k \geq 2$. Cover N with finitely many simple open sets M_j with $M \subset U_1$ and $M \bigcap \bar{U}_j$ for $j \geq 2$, and consider a smooth partition of unity ϕ_j subordinate to this cover. We now consider the operator $A := L^2(N) \to L^2(N)$ defined by $Af = \sum_j^n \phi_j I_j^* I_j \phi_j f$, where I_j is the geodesic X-ray transform for the simple manifold $\bar{M}_j$ for each j. We first we state several technical lemmas from ([12]):

Lemma 38 ([20] Theorem 2.1.)**.** *Let $p \in S^m_{\rho\delta}(r, L)$ and consider the operator $P := \mathrm{Op}(p)$. Suppose that $\rho, \delta \in [0, 1]$ and $r, L > 0$ satisfy*

$$\delta \leq \rho, \quad L > \frac{n}{2}, \quad r > \frac{1-\rho}{1-\delta}\frac{n}{2}. \tag{3.8}$$

Then the operator $P\colon H^{s+m}(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$ is bounded when

$$(1-\rho)\frac{n}{2} - (1-\delta)r < s < r. \tag{3.9}$$

Lemma 39 ([12], Lemma 11)**.** *Let (M, g) be a simple manifold with $g \in C^k(M)$ for some $k \geq 5$. Then for each j the operator $\phi_j I_j^* I_j \phi_j$ belongs to $S^{-1}(k-s, s-4)$ for all $s \in [4, k]$ with $4 \leq s \leq k$.*

Lemma 40 ([12] Lemma 20)**.** *Let (M, g) be a simple manifold with $g \in C^k(M)$ for some $k \geq 7 + \frac{n}{2}$. Consider the operator $B := \phi I^* I \phi$ where $\phi \in C_c^\infty(M)$. Then there is an operator P (That is, a left parametrix for B) and $\varepsilon > 0$ so that $PB = \mathrm{Id} + R$ where Id is an operator acting as the identity on elements in $H^{t-\tau}(\mathbb{R}^n)$ which are supported in the set where $\psi = 1 = \phi$ and the remainder*

$$R\colon H^{t-\tau}(\mathbb{R}^n) \to H^t(\mathbb{R}^n) \tag{3.10}$$

is continuous whenever $0 < \tau \leq \varepsilon$ *and*

$$-k + 6 + \frac{n}{2} < t < k - 6 - \frac{n}{2}. \tag{3.11}$$

We need to first prove some properties of A.

Theorem 41. *(a) Let* $k > 10$ *and* $7 - k < 0 < t < l - 1 < k - 7$*, then if* $Au = f$ *with* $f \in H^l(N)$*, and* $u \in L^2(N)$*, then* $u \in H^{l-1}(N)$*.*
(b) $Au = f$ *for* $f \in L^2(N)$ *has a solution* $u \in H^{-1}(N)$ *iff* $\langle f, w\rangle_{L^2} = 0$ *for all* $w \in Ker(A^*)$

Proof. To prove (a), apply 40 to the operator $\phi_j I_j^* I_j \phi_j$ and obtain the parametrix $L_j : H^1 \to L^2$ 38 such that then $L_j\phi_j I_j^* I_j \phi_j u = \phi_j u + R_j \phi_j u = L_j \phi_j f \in H^{l-1}(\bar{U}_j)$ where $R_j : H^{t-\tau}(\bar{U}_j) : H^t(\bar{U}_j)$ for all $-k + 7 < t < k - 7$ and a small $\tau < 1 \in \mathbb{Q}$, so that we can conclude that $\phi_j u \in H^{0+\tau}(\bar{U}_j)$, by a bootstrapping-like argument we can then conclude that $\phi_j u \in H^{l-1}(\bar{U}_j)$, which implies $u \in H^{l-1}(N)$.

To prove (b), we consider the space

$$Y := \{f \in L^2(N) | \langle f, w\rangle_{L^2} = 0 \ \forall \ w \in ker(A^*)\}$$

(where A^* is the L^2 adjoint).

We will show that the range of A is surjective onto Y. Note that $A : H^{-1} \to L^2$ is bounded By 39 and 38 (Also see remark preceding remark 12 in [12]). Given a fixed $w \in ker(A^*)$, for any test functions $u \in L^2$ (So $u \in H^{-1}$) we have:

$$\langle Au, w\rangle_{L^2} = \langle u, A^* w\rangle_{L^2} = 0$$

which means $Au \in Y$.

Equip Y with the L^2 inner product. Suppose the range of $A : H^{-1} \to L^2$ is not dense in Y, then by orthogonal projection there is an element $f \in Y$ such that $\langle f, Au\rangle_{L^2} = 0$ for all $u \in H^{-1}$, in particular any $u \in L^2$. But this means for all $u \in L^2$ we have $\langle A^* f, u\rangle_{L^2} = 0$, so $f \in ker(A^*)$, by definition of Y we then have $\langle f, f\rangle_{L^2} = 0$ so $f = 0$.

Now we show A has closed range in Y; we will show that there exists some $C > 0$ such that for all $u \in H^{-1}(N)$ with $u \perp Ker(A)$, we have $|u|_{H^{-1}} \leq C|Au|_{L^2}$. Suppose not, then by increasing choices of C and normalizing $|u|_{H^{-1}}$ we obtain a sequence of u_i such that $|u_i|_{H^{-1}} = 1$ and $|Au_i|_{L^2} \to 0$. Apply the operator $L := \sum_j^n \phi_j L_j \phi_j : L^2(N) \to H^{-1}(N)$(remark preceding lemma 17 and lemma 6 in [12] and 38) to $Au_i \in L^2(N)$ and obtain $u_i + \sum_j^n \phi_j R_j \phi_j u_i \to 0 \in H^{-1}$, since $|u_i|_{H^{-1}}$ are bounded, and each R_j are compact operators, we get from Rellich theorem [7] that for each j there exists a sub-sequence u_{i_k} so that each $\phi_j R_j \phi_j u_{i_k}$ converges in $H^{-1+\tau}$ for some small positive τ, since there's only finitely many j this gives a sub-sequence u_{i_k} such that $\sum_j^n \phi_j R_j \phi_j u_{i_k}$ converges in $H^{-1+\tau}$ and hence also in H^{-1}. Then we have u_{i_k} also converges to some $u \in H^{-1}$. For any test functions ψ (suffice to take $\psi \in L^2$), then consider $\langle u, A^*\psi\rangle$ (as distributional pairing). This makes sense since A^* is the L^2 adjoint of A which is just equal to A since $A : L^2 \to L^2$ is self adjoint, which means A^* is also one Sobolev degree smoothing(A is by 39)) so $A^*\psi \in H^1$, and we have:

$$\langle u, A^*\psi\rangle = \lim\langle u_{i_k}, A^*\psi\rangle = \lim\langle Au, \psi\rangle = 0$$

This shows that $u \in Ker(A)$, but since each $u_{i_k} \perp Ker(A)$ we have $u \perp Ker(A)$ by continuity of inner product, so $u \in Ker(A)$ and $u \perp Ker(A)$ so $u = 0$, but this contradicts with $|u|_{H^{-1}} = 1$, so we are done.

This shows that there exists some $C > 0$ such that for all $u \in H^{-1}(N)$ with $u \perp Ker(A)$, we have $|u|_{H^{-1}} \leq C|Au|_{L^2}$, let u_i be any sequence such that Au_i converges in Y, then consider the $\tilde{u}_i := u_i - proj_{ker(A)}u_i$, then $A(\tilde{u}_i) = A(u_i)$, so Au_i being Cauchy implies $\tilde{u}_i$ is Cauchy, let u be the limit of u_i, then $Au = \lim Au_i$. so indeed A has closed range in Y.

So A has closed range that's dense in Y, so (b) follows.

□

Proof of surjectivity of A. We are now in a position to prove that $A : L^2(N) \to H^1(N)$ is in fact surjective. We do so by first proving that $A : L^2 \to L^2$ is injective; suppose $Af = 0$, then $\langle Af, f\rangle_{L^2} = 0$, then by definition of A we have

$$Af = \sum_j^n \phi_j I_j^* I_j \phi_j f = 0$$

so

$$0 = \langle \sum_j^n \phi_j I_j^* I_j \phi_j f, f\rangle_{L^2(N)} = \sum_j^n \langle I_j^* I_j \phi_j f, \phi_j f\rangle_{L^2(\bar{U}_j)}$$
$$= \sum_j^n \langle I_j \phi_j f, I_j \phi_j f\rangle_{L^2(\bar{U}_j)} = \sum_j^n |I_j \phi_j f|^2_{L^2(\bar{U}_j)}$$

so each $I_j \phi_j f = 0$, by the injectivity of I_j on L^2 [12], we have that $\phi_j f$ is 0 for all j so $f = 0$.

This shows that A is injective on L^2, since $A : L^2(N) \to L^2(N)$ is self adjoint, A^* is also injective, by 41 (b) we have $A : L^2(N) \to H^1(N)$ is surjective. □

Proof of main theorem. :

Let $f \in C^l(M)$ with $1 < m + 1 < l - 1 < k - 7$, $m, l, k \in \mathbb{N}$. Extend f to $C^l(N)$ and still denote it f, so that it is in $H^l(N)$, so that in particular $f \in H^1(N)$. By the preceding result there exists $h \in L^2(N)$ such that $Ah = f$, by 41(a) since $f \in H^l(N)$, $h \in H^{l-1}(N)$, by Sobolev embedding $h \in C^m(N)$.

Define

$$w_1 := I_1 \phi_1 h = \int_0^{\tau_1(x,v)} \phi_1(h(\varphi_{1,(x,v)}(t))dt \tag{3.12}$$

where τ_1 and $\varphi_{(1,\cdot)}$ are the exit time function and geodesic flow with respect to $\bar{M}_1$. The geodesic flow of a C^k metric has $k - 1$ regularity so $\varphi_1 \in C^{k-1}(\partial_+ SM_1)$. By an argument identical to that in (Theorem 3.2.6 [30]) for finite regularity k, the odd extension of τ_1, $\bar{\tau_1} \in C^{k-4}(\partial SM_1)$, $\bar{\tau_1}|_{\partial_+ SM_1} = \tau_1|_{\partial_+ SM_1} \in C^{k-4}(\partial_+ SM_1)$. This shows that $w_1 \in C^{\min(m,k-4)}(\partial_+ SM_1)$.

Since SM is away from $\partial_0 SM_1$, we have that $\tau_1|_{SM} \in C^{k-1}(SM)$. Consider $w_1^\# = w_1(\varphi_{(1,\tau_1(x,v))})|_{SM} \in C^{\min(m,k-4)}(SM)$. Define $w := w_1^\#|_{\partial_+ SM}$, clearly $w^\# = w_1^\#|_{SM}$ since they both agree on $\partial_+ SM$ and are constant along geodesics, this shows that $w^\# \in C^{\min m, k-4}(SM)$, and so $w \in C^m_\alpha(\partial_+ SM)$.

Now it remains to prove that $I^* w = f$, we have that for all $x \in M$

$$I^* w(x) = \int_{S_x M} w^\#(x,v) dS_x(v) = \int_{S_x M} w_1^\#(x,v) dS_x(v)$$
$$= (I_1^* w_1)(x) = I_1^* I_1 \phi_1 h(x) = Ah(x) = f(x)$$

.

□

3.4 Boundary determination from scattering relation

We are now ready to prove theorem 3. We first prove the finite regularity version of (Theorem 5.1.1 of [30]] and (Theorem 1.6 in [31]).

3.4.1 Geometric Preliminaries (cont.)

Definition 42. Let ν be the inward pointing normal vector. Define $\partial_\pm SM := \{(x,v) \in \partial SM | \pm \langle v, \nu \rangle \geq 0\}$, also define the *glancing region* $\partial_0 SM = \partial_+ SM \bigcap \partial_- SM$.

Remark 43. Note that if we have two metrics g_1 and g_2 with the same boundary distance function, then by virtue of 31 the sets above are all the same.

Definition 44 (Exit time function)**.** Let (M, g) be a simple surface. Define $\tau(x,v) : SM \to \mathbb{R}$ the *exit time function*, defined by the length of the (unique) geodesic starting at x in the direction of $v \in S_x M$ and ends at the boundary.

The non-trapping condition of a simple manifold says precisely that the exit time function is bounded. And the strict convexity condition implies $\tau(x,v) = 0$ for $(x,v) \in \partial_0 SM$.

An argument similar to (lemma 3.2.3 in [30]) shows that τ is C^{k-1} away from $\partial_0 SM$.

Definition 45 (Scattering relation)**.** Define the *odd extension of the exit time function* $\tilde{\tau}(x,v) = \tau(x,v) - \tau(x,-v)$

Define the *scattering relation* $\alpha(x,v) : \partial SM \to \partial M$ to be

$$\alpha(x,v) := (\varphi_{\tilde{\tau}(x,v)}(x,v), \varphi'_{\tilde{\tau}(x,v)}(x,v))$$

where (φ, φ') is the geodesic flow on SM. Clearly $\alpha : \partial_+ SM \to \partial_- SM$ and vice versa, and $\alpha^2 = id$.

Definition 46. Let $w \in C(\partial_+ SM)$, define $w^\# \in C(SM)$ by $w(\varphi_{\tau(x,v)}(x,v))$.

Also define the odd and even continuation of w:

$$A_\pm w(x,v) = \begin{cases} w(x,v) & (x,v) \in \partial_+ SM \\ \pm w \circ \alpha(x,v) & (x,v) \in \partial_- SM \end{cases}$$

Equip $\partial_+ SM$ with the L^2 inner product $\int_{\partial_+ SM} uv\mu d\Sigma$, with $\mu = \langle \xi, \nu \rangle$ and $d\Sigma = d(\partial M) \wedge d(S_x M)$ ([31])

Also equip ∂SM with a similar L^2 structure with $\int_{\partial_+ SM} uv|\mu| d\Sigma$. Then it can be shown (Lemma 9.4.5 [30]) that $A_\pm : L^2_\mu(\partial_+ SM) \to L^2_\mu(\partial SM)$ is a bounded operator, and the adjoint A^* is given by $A^*_\pm u = (u \pm u \circ \alpha)|_{\partial_+ SM}$.

Definition 47. Define the spaces

$$C^j_\beta(\partial_+ SM) := \{w \in C^j(\partial_+ SM) : A_+ w \in C^j(\partial SM)\}$$

$$C^j_\alpha(\partial_+ SM) := \{w \in C^j(\partial_+ SM) : w^\# \in C^j(SM)\}$$

Definition 48. We define the *Hilbert transform*

$$Hu(x,\xi) = \frac{1}{2\pi} \int_{S_x M} \frac{1 + (\xi, \eta)}{(\xi_\perp, \eta)}, \qquad \xi \in S_x M$$

Also denote the odd and even part of the Hilbert transform H_+ and H_- respectively, note that $H_+ u = H u_+$ and $H_- u = H u_-$.

Definition 49. For a C^k metric g We also define the Geodesic vector field $X : C^m(SM) \to C^{\min m,k-1}(SM)$ given by

$$Xu(x,\xi) = \frac{d}{dt}(u(\varphi_t(x,v)))|_{t=0}$$

where φ is the geodesic flow.

Also define $X_\perp : C^m(SM) \to C^{\min m,k-1}(SM)$ given by

$$X_\perp u(x,\xi) = \frac{d}{dt}(u(\psi_t(x,v)))|_{t=0}$$

where $\psi_t(x,v) = (\gamma_{x,v_\perp}(t), W(t))$, where $v_\perp$ is the 90° clockwise rotation (This is well defined since our manifold is orientable and 2D), and $W(t)$ is the parallel transport of v along the geodesic $\gamma_{x,v_\perp}$

Finally we define the DN map:

Definition 50. Since we are working with simple surfaces which are diffeomorphic to closed disk $\mathbb{D} \subset \mathbb{R}^2$, we may assume a global coordinate on M. Let $0 < \lambda < 1$ and $f \in C^{2,\lambda}(\partial M)$ and assume the metric g is at least C^3, then by theorem 6.14 [8] there is a unique harmonic $u \in C^{2,\alpha}(M)$ with

$$\Delta u = 0, \qquad u|_{\partial M} = f$$

Define the Dirichlet to Neumann (DN) map $\Lambda : C^{2,\lambda}(\partial M) \to C^{1,\lambda}(\partial M)$ by $\Lambda f = \partial_\nu u$

Remark 51. Note that by virtue of 31, if two metrics have the same boundary distance function it also implies (after possibly applying a boundary fixing diffeomorphism with one regularity higher than that of the metric) they have the same inward pointing normal vector.

3.4.2 From surjectivity of I^* to scattering relation

Since $A_+ w = w^{\#}|_{\partial_S M}$, it is clear that $C^j_\alpha(\partial_+ SM) \subset C^j_\beta(\partial_+ SM)$, the theorem below shows a partial converse:

Theorem 52. *Let $g \in C^k$, $k > 5$, then $C^{2j}_\beta(\partial_+ SM) \subset C^{\lfloor \frac{\min j, k-5}{2} \rfloor}_\alpha(\partial_+ SM)$*

We need a couple of technical lemmas for the proof of theorem 6.

Lemma 53 (lemma 3.2.9 in [30])**.** *Let g be a C^k metric on a compact smooth manifold with boundary M, let $(x_0, v_0) \in \partial_0 SM$, and let ∂M be strictly convex near x_0. Assume that M is embedded in a compact manifold N without boundary. Then, near (x_0, v_0) in SM, one has*

$$\tau(x,v) = Q(\sqrt{a(x,v)}, x, v), -\tau(x,-v) = Q(-\sqrt{a(x,v)}, x, v) \tag{3.13}$$

Where Q is C^{k-5} near $(0, x_0, v_0) \in \mathbb{R} \times SN$ and a is C^{k-2} near $(x_0, v_0) \in SN$.

Proof. The lemma follows from a simple regularity counting argument in the proof of the smooth metric setting. □

Lemma 54 (Whitney([40])**.** *Suppose $f \in C^{2k}(\mathbb{R})$ and $f(t) = f(-t)$ for all $t \in \mathbb{R}$, then there exists $h \in C^k$ with $f(t) = h(t^2)$ for all $t \in \mathbb{R}$.*

Proof. This follows from Whitney's proof in [40] that if $f \in C^{2k}$ and even then $f(\sqrt{x}) \in C^k$, and the fact that for every sequence r_i, there exists a smooth function whose *ith* derivative at 0 is r_i (Exercise 8C.2 [7].

□

Proof of theorem 52. This will be a modified version of the proof in the smooth setting for Theorem 5.1.1 in [30]. We embed (M, g) isometrically into a closed manifold (N, g) with the same dimension with metric of the same regularity. Let $A_+ w \in C^j(\partial_+ SM)$, extend $A_+ w$ to some $W \in C^j$. Consider $F(t, x, v) = \frac{1}{2} W(\varphi_t(x, v))$, then

$$\begin{aligned} w^\#(x,v) &= \frac{1}{2}[W(\varphi_{\tau(x,v)}(x,v) + W(\varphi_{-\tau(x,-v)}(x,v) \\ &= F(\tau(x,v), x, v) + F(-\tau(x,v), x, v) \end{aligned}$$

. A similar proof for that of (Lemma 3.2.3 [30]) show that τ is C^{k-2} (for $k > 2$) away from the glancing region $\partial_0 SM$, so the regularity of $w^\#$ is determined by that near the glancing region.

Fix some $(x_0, v_0) \in \partial_0 SM$, by 53 above, for (x, v) near (x_0, v_0) we can write $w^{\#}(x, v) = F(Q(\sqrt{a(x, v)}, x, v)) + F(Q(-\sqrt{a(x, v)}, x, v)$ with Q being C^{k-5} near $(0, x_0, v_0) \in \mathbb{R} \times SN$ and a is C^{k-2} near $(x_0, v_0) \in SN$.

Set $G := F(Q(r, x, v), x, v)$ so we have that near (x_0, v_0) we have $w^{\#}(x, v) = G(\sqrt{a(x, v)}, x, v) + G(-\sqrt{a(x, v)}, x, v)$. Clearly $G(r, x, v) + G(-r, x, v)$ is $C^{\min k-5, m}(\mathbb{R})(\mathbb{R} \times SN)$ near $(0, x_0, v_0)$ and even in r, so we may apply 54 above to obtain $H \in C^{\lfloor \frac{\min j, k-5}{2} \rfloor}(\mathbb{R} \times SN)$ near $(0, x, v,)$ such that $G(r, x, v) + G(-r, x, v) = H(r^2, x, v)$, which implies near (x_0, v_0) we have

$$w^{\#}(x, v) = G(\sqrt{a(x, v)}, x, v) + G(-\sqrt{a(x, v)}, x, v) = H(a(x, v), x, v)$$

which shows that $w^{\#}$ is $C^{\lfloor \frac{\min j, k-5}{2} \rfloor}(SM)$ near (x_0, v_0) in SM. The regularity away from the glancing region is $k - 1$, so $w^{\#} \in C^{\lfloor \frac{\min j, k-5}{2} \rfloor}(SM)$.

$\square$

We now state one final technical theorem we need to prove theorem 3. Following the set up for (Theorem 1.6 [31]). Let $w \in C_{\alpha}^{(2,\lambda)}(\partial_+ SM)$, if we assume g is a simple C^3 metric the argument in Pestov Uhlmann remains valid to show that for $f \in C^{(2,\lambda)}(M)$

$$IXf = -A_-^* f^0 \tag{3.14}$$

where $f^0 = f|_{\partial M}$, from an application of the Hilbert transform (Theorem 1.5 in [31]) we also have:

$$2\pi A_-^* H_+ A_+ w = IX_\perp I^* w \tag{3.15}$$

If $h = I^* w \in C^{(2,\lambda)}(M)$, and h_* its harmonic conjugate, then $IX_\perp h = IXh_*$, so 3.14 and 3.15 together gives

$$2\pi A_-^* H_+ A_+ w = -A_-^* h_*^0 \tag{3.16}$$

We now prove a converse of this result, following Theorem 1.6 in [31].

Theorem 55. *Suppose* $w \in C_\alpha^{(2,\lambda)}(\partial_+ SM)$. $h_* \in C^{(2,\lambda)}(M)$ *the harmonic continuation of* $h_*^0 \in C^{(2,\lambda)}(\partial M)$. *Then* $h := I^* w$ *and* h_* *are harmonic conjugates if and only if 3.16 holds.*

Proof. By 3.14 and 3.15, 3.16 above is equivalent to $IX_\perp h = IXq$ where q is any $C^{(2,\lambda)}$ continuation of h_*^0. So $I(\nabla q + \nabla_\perp h) = 0$, since g is C^3 simple, it is in particular $C^{1,1}$ simple ([11]), so by the injectivity of Lipschitz 1 form for $C^{1,1}$ simple manifolds, the vector field $\nabla q + \nabla_\perp h = \nabla p$ for $p \in C^{1,1}(M)$ and $p|_{\partial M} = 0$. (Note : The injectivity of Lipschitz 1-form with arbitrary boundary conditions on $C^{1,1}$ simple manifolds follows from theorem 1 (b) in [10] and lemma 2 in [11], see proof of theorem 1 in [11]).

Since q and h are $C^{2,\lambda)}$, their gradients are $C^{1,\lambda}$, which implies $p \in C^{(2,\lambda)}(M)$ as well. Now consider the function $h_* := q - p$, then h_* is in $C^{(2,\lambda)}$ and is the harmonic continuation of h_*^0 since $h_*^0|_{\partial M} = q|_{\partial M} = h_*^0$, and h and h_* are harmonic conjugates by constructions. □

We are now ready to prove theorem 3.

Proof of theorem 3.

Let g_1 and g_2 be two C^{17} simple metrics on a compact two dimensional manifold with boundary M, so that $d_{g_1} = d_{g_2}$. By theorem 1 there exists a boundary fixing gauge $\Phi : M \to M$ such that $g_1|_{\partial M} = \Phi^* g_2|_{\partial M}$. Since DN map is invariant in two-dimensional under such a gauge (See the beginning of 11.6 in [30]), we will denote $\Phi^* g_2$ simply by g_2 from here on (Since proving DN maps for g_1 and $\Phi^* g_2$ are equal implies equality for DN map of g_2). From here on out we will use subscript $1, 2$ to denote all geometric objects and operators that depend on the metrics.

Suppose $l = 9$, $m = 6$ so that $k - 7 = 9 > l - 1 = 8 > +1 = 7$. Given $h_*^0 \in C^{10}(\partial M)$, let $h_{*,1} \in C^{10}(M)$ its harmonic continuation with respect to g_1 and $h_1 \in C^{10}(M)$ its harmonic conjugate. By theorem 2 32 we can find $w \in C_\alpha^6(\partial_+ SM))$ such that $I_1^* w = h_1$. By the analysis above we have that 3.16 holds for g_1.

Note that A, A_-^*, A_+ are all determined by the scattering relation, so by assumption they are the same for both metrics. H_+ applied to the function $(A_+ w) \in C(\partial SM)$ is an integral over S_xM which is the same for both metrics since $g_1|_{\partial M} = g_2|_{\partial M}$ by 31.

So 3.16 holds for g_2 as well.

Clearly $w \in C^6_\alpha(\partial_+ SM) \subset C^6_\beta(\partial_+ SM)$ for g_1, but $C^6_\beta(\partial_+ SM)$ is the same for both metrics since it is determined by the regularity of A_+w on ∂SM, so we have that $w \in C^6_\beta(\partial_+ SM)$ for g_2 as well, now apply 52 and conclude that for g_2 we have:

$$w \in C^6_\beta(\partial_+ SM) \subset C^3_\alpha(\partial_+ SM)$$

Since 3.16 holds, we can apply theorem 55 to conclude that the function $I_2^* w \in C^3(M)$, and any C^3 harmonic continuation $h_{*,2}$ with respect to g_2 of h^0_* are $C^{(2,\lambda)}$ harmonic conjugates.

We know that $h^0_i := h_i|_{\partial M}$. Now the g_1 DN map applied to h^0_* is

$$\Lambda_1 h^0_* = \langle \nu, \nabla h_{*,1}|_{\partial M} \rangle = \langle \nu_\perp, \nabla h^0_1|_{\partial M} \rangle = \partial_{\nu_\perp} h^0_1$$

.

But note that $h^0_1 = h^0_2$ since $I_i^* w|_{\partial M} = \int_{S_xM} A_+ w(x,\xi) dS_xM$ which is the same for both metrics, this shows that as $C^{1,\lambda}(\partial M)$ functions, $\partial_\nu h_{1,*} = \partial_\nu h_{2,*}$ are equal, hence equal as functions of their maximum regularity. This concludes theorem 3.

□

Remark 56. We conclude this chapter by noting that the only step left for proving boundary rigidity is the Calderón problem for C^{17} metrics in two dimension. The Calderón problem in 2D was resolved by Lassas and Uhlmann [17] in 2D and for real analytic metrics in higher dimension, this result was later generalized to complete manifolds in [16]. One promising approach for proving boundary rigidity in the 2D case, is to note that in [17] they make use of a result in [19] to show that the DN map determines the metric at the boundary in the tangential direction- which is not needed if we are only interested in boundary rigidity by 31. Another possible approach can be used to prove the full Calderón problem in 2D for metrics with regularity as low as $C^{1,\alpha}$, is to modify the later proof by Belishev [2], which characterized the complex structure on a manifold by the algebra of holomorphic functions, which in turns determines the conformal classes of metric on M. Both proofs can be adapted to the $C^{1,\alpha}$ case since they both rely on the existence of isothermal coordinates [33] and the analysis of the induced complex structures. It is currently in work at the time of writing of this paper. We also outline another possibility of the Calderón problem in the next chapter.

Chapter 4

CALDERÓN PROBLEM

4.1 Introduction

In this chapter we analyze the proof of the Calderón problem for real analytic complete manifolds with $n > 2$ in (*The Dirichlet-to-Neumann map for complete Riemannian manifolds with boundary* (Lassas, Taylor, Uhlmann, 2003)), to investigate the possibility of a different proof for the Calderón problem for compact Riemannian surfaces, formulated by theorem 1 below:

Theorem 1: Let (M, g_1) and (M, g_2) be compact, connected Riemannian surfaces with boundary, and $\Lambda_{g_1} = \Lambda_{g_2}$, then $g_1 = c\psi^* g_2$ where $c \in C^\infty(M), c > 0$ and $\psi : M \to M$ is a diffeomorphism such that $\psi|_{\partial M} = id$ (That is, (M, g_1) and (M, g_2) are conformal).

We then analyze this proof to address the challenges and applicability for the same result for $C^{(1,1)}$ metrics.

This result was already proved by Lassas and Uhlmann (2001)[17] using sheaves theory. Here we merely mimic the proof in Lassas, Taylor, Uhlmann (2003)[16] instead (which was originally only provided for complete manifolds with $n > 2$ and real analytic metrics), in order to reformulate the proof in [17] without using the language of sheaves.

The outline of the proof goes as the follow: we first extend (M, g_1) and (M, g_2) to larger manifolds with Lipschitz metrics. For each manifold we then use the DN map to obtain the Greens function for the Dirichlet problem for the Laplace-Beltrami operator. We then show that these Green functions induces embeddings from the extended manifolds into a certain Sobolev space. We would like to show that the composition of one Green function and the inverse of the other is a conformal diffeomorphism.

After we have proved the case for Riemannian surfaces, we will then analyze the feasibility of extending our result to compact connected surfaces with $C^{(1,1)}$ metrics.

4.2 Manifold Extension

We outline the procedure of extending the manifolds in (Lassas, Uhlmann 2001). We use boundary normal coordinates near the boundary, so that for x near ∂M, $x = (s, h)$ where s is the nearest point in ∂M to x and $h = dist(x, s)$, in local coordinates $x = (\xi, h) \in \mathbf{R} \times \mathbf{R}^+$. so that $g = g_{ij}(\xi, h)$ with $i, j = 1, 2$ in local coordinates. We now extend the manifold (M, g_1) and (M, g_2) by attaching a collar to M, for sufficiently small ϵ, we have that $\phi : \partial M \times [0, \epsilon] \to M$ is an embedding, so we can glue $\partial M \times [-\epsilon, 0]$ to M by identifying the points in $\phi(s, 0) \in \partial M \times [0, \epsilon]$ with $(s, 0) \in \partial M \times [-\epsilon, 0]$ for all s in ∂M (See Theorem 9.29, Introduction to Smooth Manifold, John M Lee). We denote this extended manifold $\tilde{M}$.

We can also extend the metric g to $\tilde{M}$ by using the product metric on $\partial M \times [-\epsilon, 0]$; It is clear that by gluing $\partial M \times [-\epsilon, 0]$ to M along ∂M we have a smooth manifold $\tilde{M}$, furthermore, we also know that the product metric on $\partial M \times [-\epsilon, 0]$ agree with the metric g (g_1 or g_2) on ∂M by the choice of collar and the construction of $\tilde{M}$, if we use (s, h) as coordinatse for $\partial M \times [-\epsilon, 0]$, then $\frac{\partial}{\partial s}|_{(x,0)}$ for any $(x, 0) \in \partial M \times [-\epsilon, 0]$ corresponds to the unit tangent vector $\frac{\partial}{\partial s}|_x \in \partial M$, and $\frac{\partial}{\partial h}|_{(x,0)}$ corresponds to the outward pointing unit normal vector $\frac{\partial}{\partial h}|_x$ (See Theorem 9.25, Introduction to Smooth Manifold, John M Lee, this is because Φ in the proof of the said theorem is the identity in boundary normal coordinates for the collar in M).

And because we are using the product metric for $\partial M \times [-\epsilon, 0]$, the metric on $\tilde{M}$ restricted to the attached collar is uniquely determined by the tangential component of the metric g in ∂M. By (Lee and Uhlmann) [19], Λ_g determines the tangential component of the metric on the boundary, that is , $g_{11}(\xi, h)$ is determined by the DN map, hence (M, g_1) and (M, g_2) having the same D-N map implies the extended Manifold $\tilde{M}_1$ and $\tilde{M}_2$ are isometric on the attached collar $\partial M \times [-\epsilon, 0]$.

Finally we note that this metric is Lipschitz; that is, there exists coordinate charts covering $\tilde{M}$ such that in each coordinate $g_{ij} \in C^{(0,1)}(\tilde{M})$. (One such choice of covering would be charts that cover the manifold M together with the boundary normal coordinates covering the collar of M and the attached collar), we note that this metric is only known to be Lipschitz as the normal derivatives of g_{11} will not necessarily be continuous at ∂M.

We conclude this section with the first challenge we face in the case of $C^{1,1}$ metric; namely, the determination of the metric in the tangential direction at the boundary in [19] is only known for smooth metrics, it is currently unknown if the same techniques can be adapted to $C^{1,1}$ metrics in dimension two.

4.3 Properties of the Green functions

We denote the extended metric also by g, and let

$$U \subset \tilde{M} \backslash M$$

be the closure of a an open neighborhood of $\partial \tilde{M}$ and

$$U_r = \{x \in \tilde{M} : \quad d(x, U) < r\}, \quad 0 < r < r_0$$

be a family of open neighborhoods $U_r \subset \tilde{M} \backslash M$ for sufficiently small r_0.

Consider the Green's functions of the Laplace-Beltrami operator which are solutions of the equations

$$\Delta_g G(\cdot, y) = \delta_y \quad \in \quad \tilde{M}, \qquad G(\cdot, y)|_{\partial \tilde{M}} = 0$$

where $y \in \tilde{M}$. We now consider y as a parameter and use the notation

$$h_y(x) = G(x, y)$$

Lemma 57. *Every point $x \in \tilde{M} \backslash U$ has a coordinate neighborhood (W, ϕ), $\quad \phi : W \to \mathbf{R}^n$ where the Green's functions $h_y \circ \phi^{-1}, y \in U$, are real analytic. This is true for any U that is* *the closure of any open neighborhood of $\partial\tilde{M}$ In fact, every point $x \in \tilde{M}^o$ has such coordinate neighrborhood so that the Green's function is real analytic in $x \in \tilde{M}^o \backslash y$*

Proof: (See Lemma 2.1 in Lassas Uhlmann 2001)

Lemma 58. *The D-N map on ∂M and the metric g on $\tilde{M}\backslash M$ determine the values of Green's functions $h_y(x)$ for $x, y \in \tilde{M}\backslash M$.*

Proof: (See Lemma 2.2 in Lassas Uhlmann 2001).

Because the extended metric is only Lipschitz(on the boundary of the , the Laplacian is only defined weakly

At last, we mention a standard result that shows the asymptotic behavior of the Green's function with respect to $d(x, y)$ which we will use in the proof below, that is:

$$|G(x, y)| = c_n \log(d(x, y)) + O(1) \qquad (3.1)$$

.

By construction, the attached collar $\tilde{M}_2\backslash M_2$ and $\tilde{M}_1\backslash M_1$ are isometric (in fact the same set), from now on we identify the collar as O for both $\tilde{M}_1$ and $\tilde{M}_2$ We now define the maps

$$G_j : \tilde{M} \to H^s(O)$$

by:

$$G_j(x)(y) = G(x, y), \quad x \in \tilde{M}_j, \quad y \in O$$

for any $s < 2 - \frac{n}{2} = 1$ (because $\delta_y \in H^s(O)$ for $s < -\frac{n}{2} = -1$).

Because δ_x depends on x continuously, $G_j(x) \in H^s(O)$ also depends on x continuously and for $s < 1 - \frac{n}{2} = 0$ we have that $G_j(x, y) \in C^1$, so from now we restrict ourselves to $s < 0$, now we can consider the derivative $DG_j(x)$:

$$DG_j : T_x\tilde{M}_j \to H^s(O)$$

which is given by

$$DG_j(x)v = vG_j(x,\cdot) = v^k \frac{\partial}{\partial x^k} G_j(x,\cdot)|_x$$

in local coordinates (one can see this by taking derivative with respect to t of $G_j(x(t), y)$), where $v = v^k(\frac{\partial}{\partial x^k}) \in T_x\tilde{M}$ in local coordinates. Note that we regard $H^s(O)$ a Banach manifold when defining the differential DG_j.

From now on we fix an $r < r_0$

Lemma 59. *$DG_j(x)$ is injective for each $x \in \tilde{M}^o$*

Proof. We can just use Lemma 3.2 in Lassas Uhlmann directly, which uses lemma 2.1 in the same paper (3.1 above). Lemma 3.2 in [Lau] shows that there exists c_0 so that for every $(x,\xi) \in S(\tilde{M}\backslash U_r)$, there exists $y \in \tilde{M}\backslash M$ such that

$$|DG_j(x,y)| > c_0$$

, this shows directly that DG_j is injective for $x \in \tilde{M}\backslash U_r$, since r is arbitrary, the statement is true for all $x \in \tilde{M}^o$ OR We could use the fact used in proof of lemma 2.1 and show that for every point in the interior of $\tilde{M}$ has a isothermal coordinate neighborhood such that G_j is real analytic (See 2nd paragraph of proof of lemma 2.1 on [Lau] paper);

Now suppose $DG_j(x)v = 0$ for some $x \in \tilde{M}^o$ and non 0 $v \in T_x\tilde{M}^o$, then we have that in local coordinates

$$v^k \frac{\partial}{\partial x^k} G_j(x,\cdot)|_x = 0$$

for all $y \in \tilde{M}\backslash M$, but by real analyticity of G_j (in the isothermal coordinates) and hence the analyiticity of $v^k \frac{d}{dx^k} G_j$, $v^k \frac{d}{dx^k} G_j = 0$ for all $y \in \tilde{M}\backslash\{x\}$, by considering a smooth curve $x(t)$ with $x(0) = y$ we obtain a contradiction with the asymptotic result in (3.1).

□

Lemma 60. *The map $G_j : \tilde{M}^o \to H^s(O)$ is an embedding.*

Proof. By preceding lemma (injectivity of DG_j on $\tilde{M}^o$) and lemma 3.1 in Lassas Uhlmann (Injectivity of G_j on $\tilde{M}$ so in particular on $\tilde{M}^o$)

□

4.4 Feasibility of proof of conformal diffeomorphism

We now analyze the proof in [16] for feasibility of modification to show that the metrics $C^{1,1}$ metrics g_1 and g_2 are conformally related instead of isometric as in the original proof, we would like to prove the following result:

Theorem 61. *Assume G_1 and G_2 coincide in $\tilde{M \setminus M}$. Then the sets $G_1(\tilde{M_1}^o)$ and $G_2(\tilde{M_2}^o)$ are identical subsets of $H^s(O)$. Moreover, the map $G_2 \circ G_1^{-1} : \tilde{M_1}^o \to \tilde{M_2}^o$ is a conformal diffeomorphism.*

However, it is not clear at this time if this is true, not even for smooth metrics. The original result (Theorem 3.3 in [16]) for the case of a real analytic complete manifold for $n > 2$ has the same statement with isometry instead of conformal diffeomorphism, we outline the proof of the original result (Theorem 3.3 in [16], as well as a few other technical lemmas in [16] that are crucial steps to the proof of isometry in the original paper (that are currently in work for generalization to $n = 2$ for $C^{1,1}$ metrics). for the rest of the section we assume our metrics are real analytic and $n > 2$.

Proof. (Proof of theorem 3.3 [16])

We first show that $G_1(\tilde{M_1}^o) \subset G_1(\tilde{M_2}^o)$; fix an $\epsilon_0 > 0$ and define:

$$N(\epsilon_0) = \{x \in \tilde{M_1} \ : \ d_{\tilde{M_1}}(x, \tilde{M}^0)) \leq \epsilon_0\}, \quad C(\epsilon_0) = \{x \in \tilde{M_1} \ : \ d_{\tilde{M_1}}(x, (\partial(\tilde{M}^0)) > \epsilon_0\}$$

For ϵ_0 sufficiently small so that $C(\epsilon_0)$ is connected.

Let $x_0 \in O \bigcap C(\epsilon_0)$ and $B_1 \subset C(\epsilon_0)$ be the largest open subset containing x_0 such that $G_1(x) = G_2(x)$ for all $x \in B_1$, and we know that B_1 is non empty by Lemma 3.2.

Now define

$$J := G_2^{-1} \circ G_1 : B_1 \to \tilde{M}_2$$

and let $D_1 \subset B_1$ be the largest open set so that J is an isometry (It is in fact an isometry in O by lemma 3.2 in [16] so D_1 is non-empty and open). Let x_1 be the closet point in $(\tilde{M}^0)\backslash(N(\epsilon_0) \bigcup D_1))^o$ to x_1, note that $x_1 \in \partial D_1$.

□

We now prove another technical lemma:

Lemma 62. *(lemma 3.4 [16]) There exists $x_2 \in \tilde{M}_2{}^o$ such that $G_1(x_1) = G_2(x_2)$,. Moreover, there is a sequence $p_k \in D_1$ such that*

$$\lim_{k\to\infty} p_k = x_1, \qquad \lim_{k\to\infty} J(p_k) = x_2$$

Proof. Let $p_k \, inD_1$ be a sequence converging to x_1, since $D_1 \subset B_1$ so by definition $G_1(p_k) = G_2(q_k)$ for some $q_k \in \tilde{M}_2$, if some some such $\{q_k\}$ we have q_k converges to $x_2 \in \tilde{M}_2{}^o$ then we are done by continuity of G_1 and G_2. If no such q_k exists, then for every sequence $\{q_k\}$ either we have

$$d_{\tilde{M}_2}(q_k, x_0) \to \infty \qquad (4.1)$$

or

$$q_k \to q_0 \in \partial\tilde{M}_2 \qquad (4.2)$$

If the first case is true, then consider the length minimizing curve $\gamma : [0,1] \to \tilde{M}_1$ such that $\gamma(0) = x_1$, $\gamma(l) = x_0$, with $((0,l]) \subset D_1$. Let $p_k = \gamma(l - \frac{1}{k})$, and $q = J(p_k)$, then $d_{\tilde{M}_2}(q_k, x_0) < J(\gamma([0,1]))$, contradicting (4.1). Hence there exists at least one such sequence $\{q_k\}$ where (4.1) is not the case. Suppose for such sequence (4.2) is true, then $q_k \to q_o \in \partial\tilde{M}_2$, hence $G_2(q_k, y) \to G_2(q_o, y) = 0 \in H^s(O)$, which yields that $G_2(q_k) \to 0$, and hence $G_1(p_k) \to 0$, so $G_1(x_1) = 0 \in H^s(O)$, which implies

$$G_1(x_1, y) = 0$$

for all $y \in O$. By unique continuation (by the real analyticity of $G_1(x_1, y)$ in $x \in \tilde{M}_1 \backslash \{y\}$ see, Lemma 3.1), we conclude that $G_1(x, y) = 0$ for all $y \in \tilde{M}_1 \backslash \{x_1\}$. However, by letting y approach x_1, we obtain a contradiction with the asymptotic behavior in (3.1), hence the limit point $x_2 \in \tilde{M}_2{}^o$ exists and the limits are valid.

□

We now state another technical lemma. By the unique continuation:

Lemma 63. *(lemma 3.5 [16]) Let $\Omega \in O$ be an non empty open set, if we compose the map*

$$G_j(x) \to H^s(O)$$

with the restriction operation

$$R : H^s(O) \to H^s(\Omega)$$

then

$$G_j^\Omega =: R \circ G_j : \tilde{M}_j \to H^s(\Omega)$$

is an embedding, and these maps are real-analytic on $\tilde{M}_j \backslash \bar{\Omega}$, and for $x_j \in \tilde{M}_j$ we have $G_1(x_1) = G_2(M_2)$ if and only if $G_1^\Omega(x_1) = G_2^\Omega(x_2)$.

Proof. From now on x_j (for $j = 1, 2$) will denote the corresponding x_j in Lemma 4.2. By lemma 4.3, we have $G_1^\Omega(x_1) = G_2^\Omega(x_2) =: u \in H^s(\Omega)$. Pick $\Omega \subset O$ that is disjoint from $x_1 \in M_1$ and $x_2 \in M_2$. Then G_j^Ω is an analytic embedding in a neighborhood of x_j. Denote $R(G_j)$ the image of the map G_j^Ω in $H^s(\Omega)$.

We show next that the tangent spaces also coincide at u, that is

$$T_u R(G_1^\Omega) = T_u R(G_2^\Omega) \subset H^s(\Omega)$$

For any point q in the interior of D_1, let $v = G_1(q)$ and $p = J(q) \in \tilde{M}_2$ be the point for which $G_2(p) = v$. By assumption we know G_1^Ω and $G_2^\Omega \circ J$ coincide in D_1, hence their differential coincide. Let p_k, q_k be the sequence in Lemma 4.2, denote $v_k =: G_1^\Omega(p_k)$, then we have

$$T_{v_k}R(G_1^\Omega) = DG_1^\Omega(T_{p_k}\tilde{M}_1) = DG_2^\Omega(T_{q_k}\tilde{M}_2) = T_{v_k}R(G_2^\Omega)$$

By continuity of $DG_j^\Omega : \tilde{M}_j \to H^s(\Omega)$, we have the desire result.

$$T_uR(G_1^\Omega) = DG_1^\Omega(T_{x_1}\tilde{M}_1) = DG_2^\Omega(T_{x_2}\tilde{M}_2) = T_uR(G_2^\Omega)$$

.

Denote $V =: T_uR(G_1^\Omega) = T_uR(G_2^\Omega)$, let L denote the linear subspace orthorgonal to V in $H^s(\Omega)$. and

$$P : H^s(\Omega) \to V$$

be the orthogonal projection to the space $V \subset H^s(\Omega)$.

Now consider the map

$$PG_j^\Omega : \tilde{M}_j \to V, \qquad x \to P(G_j^\Omega(x, \cdot))$$

.

The differential of PG_j^Ω is $P \circ DG_j^\Omega$ which is surjective, and for dimensional reason is invertible, hence by inverse function theorem there exists an open neighborhood $U \subset V$ of Pu and a real analytic map $H_j : U \to \tilde{M}_j$ such that

$$P(G_j^\Omega(H_j(v), \cdot)) = v$$

Thus we can represent the graph of the function $G_j^\Omega(M_j)$ locally near u as graphs of the real analytic functions

$$\Phi_j : U \to L, \qquad \Phi_j(v) = G_j^\Omega(H_j(v), \cdot)$$

The real analytic maps Φ_j coincide in an open subset $PG_1(D_1) \subset U$ and thus in the whole set U, This shows that $x_1 = H_1(Pu)$ is an interior point of B_1, hence in the domain of J. Moreover, the maps G_1^Ω and $G_2^\Omega \circ J$ coincide near x_1, since $J = H_2 \circ H_1^{-1}$ and it is real analytic. We have shown that $G_1(x, y)$ and $G_2(J(x), J(y)$ coincide when x is near x_1 and $y \in \Omega$. Since J is real analytic in D_1 and near x_1 and the Green functions real analytic, it follows that $G_1(x, y)$ and $G_2(J(x), J(y))$ coincide when x and y are near x_1.

□

4.5 Determining conformal class of metric with Greens functions

Assuming the theorems in the preceding section hold for smooth metrics (with isometry being replaced with conformal diffeomorphism) in dimension two. We outline a possible strategy to prove g_1 and g_2 are conformally related. We may mimic the technique in [16] to show that near x_1, J is a conformal diffeomorphism. Since $x_1 \in B_1$ we know x_1 is in the domain of J, and by the above H_j are diffeomorphisms onto open subsets of $\tilde{M}_j$ containing x_j, so we can conclude that $J = H_2 \circ H^{-1}$ is a diffeomorphism from an open subset in $\tilde{M}_1$ containing x_1 onto its image in $\tilde{M}_2$.

We now mimic the approach in [17] to describe the process of how we may recover the conformal class of metric from the Green's functions in $n = 2$, let $h = (h_1, h_2)$ be a coordinate near x_1, further restrict to a neighborhood U of x_1 so that 1. h is a coordinate on $\bar{U}$, 2. J is an orientation preserving (or reversing) diffeomorphism U onto its image in $\tilde{M}_2$, 3. $G_1(x, y) = G_2(J(x), J(y))$, 4.$\bar{U}$ is a 2-manifold with boundary such that J is a diffeomorphism from $\bar{U} \to J(\bar{U})$.

Now choose a smooth positive measure μ_1 on M_1, so that $\mu_1 = \sigma_1 dV_{g_1}$ where dV_{g_1} is the Riemannian volume measure on $\tilde{M}_1$ and $\sigma_1 > 0$ is a smooth positive function on $\tilde{M}_1$. Now we have

$$u(x) = \int_{\tilde{M}_1} G_1(x, z) f(z) d\mu_1(z)$$

satisfying $\Delta_{\sigma_1 g_1} u = f$.

By choosing f with $supp(u) \subset U$ and $v \in C_0^\infty(U)$, we find that

$$\int_U g_1^{\frac{1}{2}} g_1^{ij} \frac{\partial}{\partial h^i} u \frac{\partial}{\partial h^j} v = (\Delta_{\sigma_1 g_1}(u), v)_{L^2(\tilde{M}_1, \sigma_1 g_1)} = \int_U f(z) v(z) d\mu_1(z))$$

.

By choosing f and v such that the support of u and v shrink to one point, we can find the function $g_1^{\frac{1}{2}} g_1^{ij}$, so we see that knowing the Greens function $G_1(x, y)$ allows us to recover the conformal class of the metric g_1.

Now consider the metric g_1 and the measure μ_1 restricted to U, note that

$$u(x) = \int_U G_1(x, z) f(z) d\mu_1(z)$$

is still a solution to $\Delta_{\sigma_1 g_1} u(x) = f(x)$ on U for f with $supp(f) \subset U$. Because we only consider f with $supp(u) \subset U$, we may also assume $supp(f) \subset U$ as well.

Since $G_1(x, y) = G_2(J(x), J(y))$ on U, we also have

$$u(x) = \int_U G_2(J(x), J(z)) f(z) d\mu_1(z)$$

Now consider the pullback metric $J^*(g_2)$ on U, we have

$$\Delta_{J^*(g_2)} u = \Delta_{J^*(g_2)}(J^*(J^{-1})^* u) = J^*(\Delta_{g_2}((J^{-1})^* u))$$

where

$$(J^{-1})^* u(x) = u(J^{-1}(x)) = \int_U G_2(x, J(z) f(z) d\mu_1(z) = \int_{J(U)} G_2(x, w)((J^{-1})^* f) d\mu_1((J^{-1})^*(w))$$

where the last equality is by diffeomorphism invariance of integral.

Because μ_1 is a smooth positive measure, so is $(J^{-1})^*(\mu_1)$. Furthermore, we know $(J^{-1})^* f$ has support contained in $J(U)$, so by the same reasoning as above we can conclude that $\Delta_{\sigma_2 g_2}(J^{-1})^* u = ((J^{-1})^* f)$ for some smooth positive function $\sigma_2 > 0$ on $J(U)$. Thus we can conclude that

$$\Delta_{g_2}(J^{-1})^* u = \sigma_2((J^{-1})^* f)$$

for some smooth positive σ_2 on $J(U)$.

then

$$\Delta_{J^*(g_2)} u = \Delta_{J^*(g_2)}(J^*(J^{-1})^* u) = J^*(\Delta_{g_2}((J^{-1})^* u)) = J^*(\sigma_2((J^{-1})^* f) = \lambda f$$

for some smooth positive function λ on U.

which yield

$$\Delta_{\lambda J^*(g_2)} u = f$$

for all f with $supp(f) \subset U$

Now apply the conformal metric recovery procedure as above, we see that

$$\int_U (\lambda J^*(g_2))^{\frac{1}{2}} (\lambda J^*(g_2))^{ij} \frac{\partial}{\partial h^i} u \frac{\partial}{\partial h^j} v = (\Delta_{(\lambda J^*(g_2))}(u), v)_{L^2(\tilde{M}_1, (\lambda J^*(g_2))} = \int_U f(z) v(z) d\mu_1(z))$$

which is equal to

$$\int_U g_1^{\frac{1}{2}} g_1^{ij} \frac{\partial}{\partial h^i} u \frac{\partial}{\partial h^j} v$$

for f and v with support sufficiently shrunk to a point. which implies on U we have $g_1^{\frac{1}{2}} g_1^{ij} = \lambda(J^*(g_2)^{\frac{1}{2}})(\lambda J^*(g_2)^{ij})$, that is, the two metrics are in the same conformal class.

Hence J is a conformal diffeomorphism near x_1, contradicting the assumption that $x_1 \in \partial D_1$. Since ϵ_0 (following Theorem 4.1) can be chosen arbitrarily small, this proves that J is in fact a conformal diffeomorphism, which proves theorem 4.1.

Since the manifolds M_1 and M_2 are isometrically embedded into $\tilde{M}_1$ and $\tilde{M}_2$ respectively, this proves M_1 and M_2 are conformally related.

Now all that remains is to verify that the technical lemmas in the previous section hold true even for $C^{1,1}$ metrics on a compact surface with boundary. This is a promising approach as the smoothness requirement of the metric is weak for the arguments in this section, so it may work as well in $C^{1,1}$ as in the case for smooth metrics, provided that the result in [19] can be adapted to show that the DN map also determines $C^{1,1}$ metrics in the tangential direction at the boundary.

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Appendix A
APPENDIX

A.1 Stein's lemma

This section contains the details of proposition 3 in chapter 6, section 4.5 in Stein's "Harmonic analysis":

A.1.1 Fourier transform of Schwartz kernels of pseudodifferential operators

Define a *normalized bump function* to be a smooth function ϕ supported in $|x| < 1$ and for some large N satisfies the inequalities

$$|\frac{\partial^\alpha \phi}{\partial x^\alpha}| \leq 1$$

for $0 \leq |\alpha| \leq N$.

Lemma 64 (Stein)**.** *Let K be a distribution which equals a function away from the origin. Suppose that $|\partial_z^\alpha K(z)| \leq C\,|z|^{-n-\alpha}$ for all $|\alpha| \leq 1$. Then $\hat{K}$ is bounded iff there is A such that $\left|K(\phi^R)\right| \leq A$ for all normalized test functions and for all radii R.*

An application: Suppose that $\left|\partial_z^\beta k(z)\right| \leq C\,|z|^{1-n-|\beta|}$ for all β. Let $a(\xi) = \hat{k}(\xi)$. We will show that $\left|\xi^\gamma \partial_\xi^\alpha a(\xi)\right| \leq C$ for all γ with $\|\gamma| \leq |\alpha| + 1$ and where the constant $C > 0$ only depends on α and γ. Let K be such that $\hat{K}(\xi) = \xi^\gamma \partial_\xi^\alpha a(\xi)$. Then $|K(z)| = |\partial_z^\gamma(z^\alpha k(z))|$. Now we show the assumption required to use Stein's lemma holds, that is,

$$|\partial^\theta K(z)| \leq C|z|^{-n-|\theta|}$$

for $|\theta| \leq 1$.

Indeed, by the estimates $\partial_z^\beta k(z)^{-n+1-|\beta|}$ and the Leibniz rule, we have

$$|K(\xi)| = |\partial_z^\theta \partial_z^\gamma(z^\alpha k(x,z))| \leq C|z|^{-n-|\theta|}$$

for z near the origin, and the inequality holds true for all z for some choice of C if k has compact support.

Now we can apply Stein's lemma and conclude that $\hat{K}$ is bounded if there is $C > 0$ such that $\left|K(\phi^R)\right| \leq C$ is bounded for all normalized test functions ϕ and for all R. Indeed, we have

$$\begin{aligned}\left|\left\langle K, \phi^R\right\rangle\right| &= \left|\left\langle z^\alpha k, \partial_z^\beta \phi^R\right\rangle\right| \\ &\leq \int_{\{|z|<R\}} |z^\alpha|\, |k(z)| \left|\partial_z^\beta \phi^R(z)\right| \mathrm{d}z \\ &\leq C \int_{\{|z|<R\}} |z|^{|\alpha|} |z|^{1-n} R^{-|\beta|} \mathrm{d}z \\ &= CR^{-|\beta|} \int_0^R r^{|\alpha|} r^{1-n} r^{n-1} \mathrm{d}r \\ &= CR^{-|\beta|} R^{|\alpha|+1} = C.\end{aligned} \tag{A.1}$$

which is what we wanted.

Now we have shown that $\left|\xi^\gamma \partial_\xi^\alpha a(\xi)\right| \leq C$ for all γ with $|\gamma| \leq |\alpha| + 1$ and where the constant $C > 0$ only depends on α and γ, since $|\xi|^{|\gamma|} \leq C \sum_{u \leq \gamma} |\xi^u|$ for some C, we can conclude that

$$\left|(1 + |\xi|^{|\gamma|}) \partial_\xi^\alpha a(\xi)\right| = \left|(1 + |\xi|^{|\alpha|+1}) \partial_\xi^\alpha a(\xi)\right| \leq C$$

Observe that this application is valid for all α and γ so that k is differentiable and the estimates $\partial_z^\beta k(z)^{-n+1-|\beta|}$ is valid, so Stein's lemma can be applied finitely many times to conclude our symbol $a(x, \xi)$ is indeed in $S_{10}^{-1}(r, N)$

Now we can simply apply a straight-forward induction to include derivative of x, and the maximum differentiation of derivative of x and ξ in terms of the regularity of the metric to conclude that a is in $S^m(r, N)$.