A Mathematical Theory for Optimal Marital Interactions

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Abstract

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The study of marriage dynamics and of strategies to reduce the likelihood of divorce has been an important research area for decades. Gottman's [15] research on successful marriages revealed three interaction styles: conflict-avoiding, validating, and volatile. There has not been progress in explaining how couples evolve into these styles of interactions and why failure to do so leads to failed marriages. The first chapter shows that the ubiquitous conflict-avoider style naturally arises through a couple maximizing a goal in their marriage when they do not consider an emotional cost. This leads to a mathematical optimal-control problem. In the second chapter, we present a differential-game-theory model where we explore what happens when spouses have different marriage goals. We also show that validating interaction-styles arise from the psychological cost of interacting with and/or ignoring one's spouse. In the final chapter, we present strategies for marriage repair.

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Finally, I thank my Lord and Savior, Jesus Christ.

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DEDICATION

to my parents, Toni and Antonio Henson, my sister and best friend Kobi Henson, and my village

Chapter 1

WHY SOME COUPLES ARE CONFLICT AVOIDING

1.1 Introduction

Marriage and love are integral parts of the human experience. From a young age, we see movies and other media that romanticize finding "the one." In reality, the real work begins after the "happily ever after." Because of the importance of marriage in my society, researchers across fields have studied the institution of marriage and relationships to explain the complex social structures of love.

In particular, many mathematical models have been developed to study marriage. Researchers have used stochastic models [3, 5], economic models [10, 23, 26, 19], and dynamical systems [31, 2, 39, 41, 29, 35] to describe some aspect of the intricacies of love.

One of the first mathematical models of love is the Romeo-and-Juliet model of Strogatz [43]. Strogatz intended this model to be an educational tool for students learning differential equations; however, it has been seminal for work in this area. Rinaldi [36] expanded the Strogatz model to build a more realistic model. Rinaldi's linear model is based on the initial appeal in a relationship, reciprocation of love, and forgetting. Rinaldi et al. [38, 37] wrote a series of papers and a book detailing the dynamics of love and appeal. Sprott [42] expanded on the Romeo-and-Juliet model by defining romantic styles of "eager beaver," "narcissistic nerd," "cautious love," and "hermit" to introduce a model based on these personality traits.

Becker [6] introduced concepts from economics to define a "marriage market" that defines success in terms of the utility of each couple. Utility increases with the consumption of household goods, as well as with income, education, and physical beauty. Becker [7] introduced the component of marital "caring" and quantified its effect in the "marriage market." This work is highly influential in papers in economics studying love.

My methodology is based on research done by Gottman [17, 14, 15] who spent decades interviewing couples. In this research, he characterizes 15-minute interviews between spouses with the Rapid Couple Scoring System (RCISS). This system measures the number of positive comments minus negative comments during each turn at speech. Gottman identifies three types of stable interaction-styles that predict successful marriages. These styles are conflict-avoiding, validating, and volatile. A validating couple interacts in both the positive and negative range of emotion. Conflict-avoiding couples have close to zero interaction in their negative range of emotions and only interact when they feel positive. Finally, the volatile couple has passionate fights in their negative range of emotions but does not interact when they are positive.

Cook et al. [13] conducted significant research that bridged the gap between psychology and mathematics. They wrote a set of difference equations for spousal behavior in an attempt to develop a mathematical theory behind the research of Gottman[15]. In this work, the authors convert RCISS code into theoretical broken-stick functions representing each type of interaction style.

Tung [44] rewrote the Cook et al. model as a coupled, continuous-time differential-equation model representing the "positivity" of each spouse over time. In this model, a positive or negative solution of the differential equation predicts whether or not a marriage is successful.

In this paper, I develop Tung's model into an optimal-control problem in which a couple maximizes a shared goal to find the best strategy for their marriage. The work of Gottman [15] influenced me heavily in that I seek to explain how couples evolve into the interaction styles found in his work.

My work is novel in that very few researchers [28, 29] have taken optimal-control approaches, and those that have focused on an approach rooted in economic concepts within the framework of a "marriage stock." Instead, I base my optimal-control model on a psychological framework. Furthermore, this paper is different from previous approaches because it maximizes the happiness of a spouse based on social interactions within the marriage as opposed to optimizing marital capital based on utility [28, 29]. Preliminary work applying optimal control to Tung's model was completed by Hemmi [21] in an undergraduate thesis.

In section 1.2, I introduce my model. In section 1.3, I present the essential definitions and theorems used in the mathematical theory of optimal control that I use. I also introduce my numerical methods. In section 1.4, I apply the theory from section 1.3 to the model and present my results. I also provide examples that help to interpret the results. Finally, in section 1.7, I summarize my main results, outline my model's limitations, and describe some directions for future work.

1.2 Model

My main goal is to determine how couples evolve into their interaction styles. I propose a model where $x_1(t)$ and $x_2(t)$ represent the positivity of each spouse over time. These are the state variables.

The state equations,

$$\frac{dx_1}{dt} = r_1(\bar{x}_1 - x_1) + u_1 x_2, \tag{1.1a}$$
$$\frac{dx_2}{dt} = r_2(\bar{x}_2 - x_2) + u_2 x_1, \tag{1.1b}$$

account for the internal and external influences on each spouse. The state equations are the same for each spouse, and so without loss of generality, I describe the equation for spouse 1. The internal-influence portion of the equation for spouse 1 consists of a natural disposition, $\bar{x}_1$, and a specific return rate to their natural disposition, r_1. The external portion,

$$I_1(x_2) = u_1 x_2, \tag{1.2}$$

is called the influence function and represents the amount of influence spouse 2 has on spouse 1's positivity. The variable u_1 is called the control. Spouse 1 decides the value of u_1 by maximizing the objective functional, J,

$$\max_{0 \le u_1 \le u_{1,max}} [J = \int_0^T F(x_1, x_2, t)dt]. \tag{1.3}$$

I assume $u_1 \ge 0$ because a negative reaction to a negative spouse could unnaturally increase dx_1/dt. I impose $u_{1,max}$ because infinitely high interaction with one's spouse is unrealistic. Each spouse can only devote so much energy to responding to their spouse.

Maximizing the objective functional can be considered the common marriage goal of the relationship. The goal could be to prioritize both spouses' happiness or, if one spouse needs more support, the couple could focus on supporting that spouse. The function F accounts for the relative importance of the two spouses through the parameter α,

$$F = \alpha x_1 + (1 - \alpha) x_2. \tag{1.4}$$

If the couple wants to prioritize spouse 1's happiness,

$$\alpha = 1. \tag{1.5}$$

If the couple wants to prioritize average happiness,

$$\alpha = \frac{1}{2}. \tag{1.6}$$

If the couple wants to prioritize spouse 2's happiness,

$$\alpha = 0. \tag{1.7}$$

The parameter α can have any value in the closed interval $[0, 1]$.

Other, more flexible payoffs do exist. One could, for example, proceed in analogy with Keyfitz [25] and consider a harmonic mean that automatically weighs the least happy spouse more heavily. For now, however, I work with linear equation (3) for simplicity.

1.3 Method: Optimal Control

I have formulated an optimal-control problem. The study of optimal-control problems originated with Pontryagin's group [34]. I borrow from a recent treatment by Lenhart and Workman [27].

Definition 1 (Standard two-state-variable, two-control-variable optimal-control problem)**.** *We wish to determine*

$$\max_{u_1, u_2} \; [J = \int_0^T F(x_1, x_2, u_1, u_2, t)dt] \tag{1.8a}$$

subject to

$$\tag{1.8b}$$

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t), u_1(t), u_2(t), t), \tag{1.8c}$$

$$\dot{x}_2(t) = f_2(x_1(t), x_2(t), u_1(t), u_2(t), t), \tag{1.8d}$$

$$u_1(t) \in U_1 = [0, u_{1,max}], \tag{1.8e}$$

$$u_2(t) \in U_2 = [0, u_{2,max}], \tag{1.8f}$$

and the initial conditions $x_1(0) = x_1^0, x_2(0) = x_2^0$.

Definition 2. *The Hamiltonian is*

$$H(t, x_1, x_2, u_1, u_2, \lambda_1, \lambda_2) = F(t, x_1, x_2, u_1, u_2) + \sum_{k=1}^{2} \lambda_k f_k(t, x_1, x_2, u_1, u_2). \tag{1.9}$$

Definition 3. *The maximized Hamiltonian, M, is*

$$M(x_1, x_2, \lambda_1, \lambda_2, t) = \max_{u_1 \in U_1, u_2 \in U_2} H(x_1, x_2, \lambda_1, \lambda_2, t). \tag{1.10}$$

Theorem 1 (Sufficiency Conditions, Mangasarian [30])**.** *If M is concave and continuously differentiable with respect to x for all $t \in [0, T]$ and if there exists a continuously differentiable λ_i such that*

$$\lambda_i'(t) = -\frac{\partial H}{\partial x_i} \tag{1.11}$$

i = 1,2, (adjoint equations), and

$$\lambda_i(T) = 0 \tag{1.12}$$

(terminal condition) are satisfied, then u_i is an optimal path.

The most important part of this theorem is the existence of the adjoint equations. I can find the optimal path for each player through these adjoint equations. If the objective functional represents profit or cost, the adjoint equation represents the shadow or highest price a player could reasonably be willing to pay for a marginal increase in the state variable (Sethi [40]). Since, in my context, there is no monetary value, the adjoint equation represents more of an emotional cost. The Hamiltonian also has a traditional economic interpretation. It is the sum of the rate of change of assets and dividends accruing from the state variable (Dorfman [12]). In my case, the assets are the happiness of a spouse or partner accrued from their marriage.

Model 1.15 has a linear dependence on the control variables.

Definition 4 (Switching Function, Lenhart and Workman [27])**.** *For a Hamiltonian that is linear in both control variables, the switching function is*

$$\Psi_i(t) = \frac{\partial H}{\partial u_i}, \tag{1.13}$$

$i = 1, 2.$

To differentiate between the general control variables and the optimal controls, I use the notation ϕ_1, ϕ_2 when I mean the optimal control path.

Definition 5. *If the Hamiltonian has linear dependence on the control variables, the optimal control path is*

$$\phi_i = \begin{cases} 0, \Psi_i < 0, \\ u_{i,s}, \Psi_i = 0, \\ u_{i,max}, \Psi_i = 0, \end{cases} \tag{1.14}$$

for $i = 1, 2$.

The values of $u_{i,s}$ exist if the switching function is equal to 0 over an interval in time. The exact value $u_{i,s}$ is revealed in the problem analysis.

1.4 Analysis

I now apply the sufficiency condition to the problem of maximizing the objective functional J,

$$\max_{0 \le u_1 \le u_{1,max}, 0, \le u_2 \le u_{2,max}} [J = \int_0^T \alpha x_i + (1-\alpha) x_j dt] \tag{1.15a}$$

subject to

$$\dot{x_1} = r_1(\bar{x}_1 - x_1) + u_1 x_2, \tag{1.15b}$$

$$\dot{x_2} = r_2(\bar{x}_2 - x_2) + u_2 x_1, \tag{1.15c}$$

$$x_1(0) = x_1^0, x_2(0) = x_2^0. \tag{1.15d}$$

The Hamiltonian is

$$H = \alpha x_1 + (1-\alpha) x_2 + \lambda_1 [r_1(\bar{x}_1 - x_1) + u_1 x_2] + \lambda_2 [r_2(\bar{x}_2 - x_2) + u_2 x_1], \tag{1.16}$$

and the switching function is given by

$$\frac{\partial H}{\partial u_i} = \Psi_i = \lambda_i x_j. \tag{1.17}$$

In addition, the optimal control for spouse i is given by

$$\phi_i = \begin{cases} 0, \ \ \Psi_i < 0, \\ u_{i,s}, \ \ \Psi_i = 0, \\ u_{i,max}, \ \ \Psi_i > 0, \end{cases} \tag{1.18}$$

and the adjoint equations are given by

$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = -\alpha + \lambda_1 r_1 - \lambda_2 \phi_2, \tag{1.19}$$

$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -(1-\alpha) - \lambda_1 \phi_1 + \lambda_2 r_2. \tag{1.20}$$

The scope of the parameters for which the controls are singular is very narrow. Since controls are generally not singular, I will proceed as if they are not. The cases for which they are singular are included in the appendix. From now on, assume

$$\phi_i = \begin{cases} 0, \ \ \Psi_i < 0, \\ u_{i,max}, \ \ \Psi_i > 0, \end{cases} \tag{1.21}$$

for $i = 1, 2$.

Solving the maximized system of equations

$$\dot{x}_1^* = r_1(\bar{x}_1 - x_1^*) + \phi_1 x_2^* \tag{1.22a}$$

$$\dot{x}_2^* = r_2(\bar{x}_2 - x_2^*) + \phi_2 x_1^* \tag{1.22b}$$

$$\dot{\lambda}_1 = -\alpha + \lambda_1 r_1 - \lambda_2 \phi_2 \tag{1.22c}$$

$$\dot{\lambda}_2 = -(1-\alpha) - \lambda_1 \phi_1 + \lambda_2 r_2, \tag{1.22d}$$

with conditions $x_1(0) = x_1^0, x_2(0) = x_2^0$ and $\lambda_i(T) = 0$, gives us the solution.

1.5 Numerical Method

I use numerical methods to verify my analytical results. However, finding numerical solutions comes with challenges because the state equations have initial conditions, while the adjoint equations have terminal conditions. This means I have a two-point, boundary-value problem. I have built my solvers using the Forward-Backward Sweep (FBS) algorithm.

FBS specifically solves problems in optimal control when the Maximum Principle formulates solutions. The version presented here is based on Lenhart and Workman[27], which is based on Wolfgang [20]. McAsey et al. [32] proved the convergence of this method for optimal control problems. The steps of the algorithm are:

1. Discretize time and initialize step size. Guess an initial control.
2. Use the initial condition to solve the state system forward in time.
3. Use the terminal condition to solve the adjoint system backward in time.
4. Update the control using the state and adjoint solutions previously calculated.
5. Repeat this process until the differences of each variable between iterations are within a specified tolerance.

Any appropriate numerical method for solving ordinary differential equations that return values at the specified grid points can be used in steps 2 and 3. I chose to use Runge-Kutta 4. More information on this method can be found in Butcher [9].

1.6 Results

I examine the implications of this model through a few numerical examples to verify my analytical result that model 1.15 gives rise to conflict-avoiding marriages. The first example below examines a naturally positive couple where one spouse is twice as positive as the other spouse. In example 2, I observe the state trajectory of a couple with opposite dispositions. In the final example, I observe the results when one spouse is twice as positive as their partner is negative. The numerical results show that all of these couples are conflict-avoiding.

Example 1.6.1

I verify my analytical results with numerical results obtained by applying the method outlined in section 1.5. Let us try an example with specific parameters. Let $u_{i,max} = 0.5$, $r_1 = 0.82 = r_2, \bar{x}_1 = 0.2, \bar{x}_2 = 0.4, x_i^0 = -\bar{x}_i, \alpha = 1$, and $T = 10$.

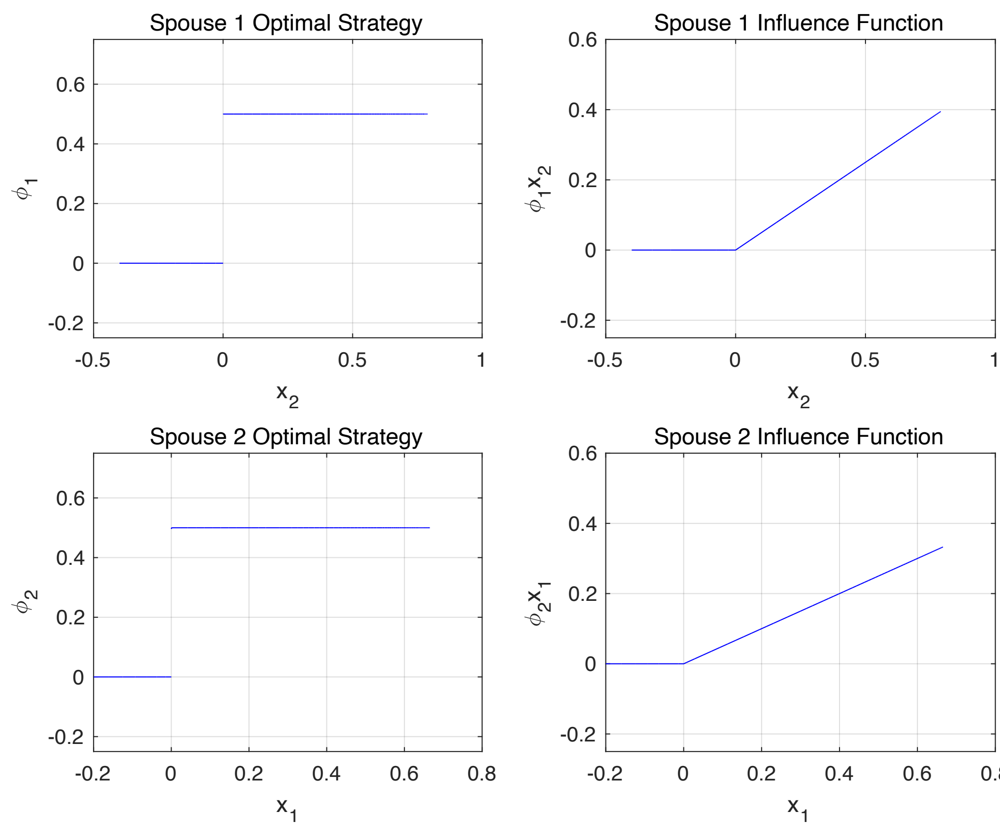


Figure 1.1: Sample optimal controls

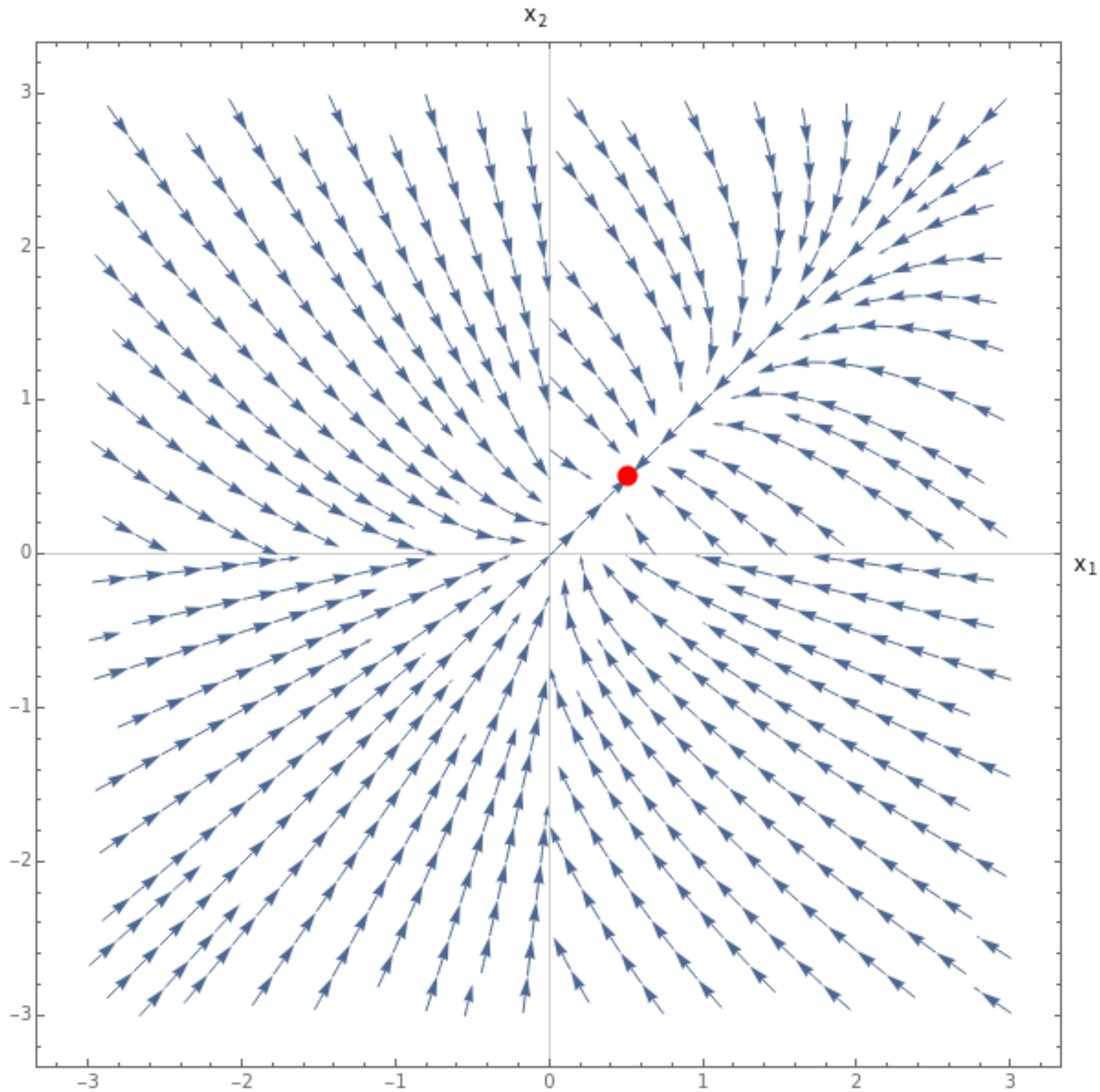


Figure 1.2: Example state-space with equilibrium point for a couple with positive natural disposition

In figure 1.1, each spouse's control variable ϕ_i switches from 0 to 1 as their partner's positivity switches from negative to positive. These are conflict-avoiding interaction-styles. In figure 1.2, I see the couple heading to an equilibrium point in the first quadrant.

Example 1.6.2

Let us try another example with specific parameters. Let $u_{i,max} = 0.5$, $r_1 = 0.82 = r_2, \bar{x}_1 = 0.26, \bar{x}_2 = -0.26, x_i^0 = \bar{x}_i$, and $T = 10$. This is a couple with opposite natural dispositions of equal magnitude.

In figure 1.3, I see that spouse 1 stays at their natural disposition, but spouse 2 is above their natural disposition, but still negative. Figure 1.4 is the phase portrait for this system. The equilibrium point is in the fourth quadrant.

Example 1.6.3

This example has the same parameters as example 2, except that spouse one is now twice as positive as spouse two is negative. That is, $\bar{x}_1 = 0.4$ and $\bar{x}_2 = -0.26$. Figure 1.6 shows that both spouses end up at points above their natural dispositions. Figure 1.7 shows that they have an equilibrium point in the first quadrant.

The main result of this model is that I have recovered the conflict-avoiding interaction-style as defined by Gottman [15]. This style is characterized by avoiding your spouse when negative and only interacting with them in their positive range of emotions.

Figure 1.5 is a sample phase portrait of the adjoint space for λ_2 versus λ_1 for the parameters in example 1.6.3. It is clear that to satisfy the terminal condition $\vec{\lambda}(T) = 0$ within the time period, the trajectory must begin in the first quadrant above the stable manifold. This means that $\lambda_1, \lambda_2 \geq 0$. Therefore, from equation 1.21, the optimal control is 0 when $x_2 < 0$ and u_{max} when $x_2 > 0$. This is conflict avoiding. I prove that this is generally the case in the appendix. Still, the gist of the proof consists of exhausting every possible combination of ϕ_1 and ϕ_2 and showing that $\lambda_1, \lambda_2 \geq 0$ is the only possible outcome. When ϕ_1 and ϕ_2 are non-zero, I prove that $\lambda_1, \lambda_2 > 0$ through stability analysis of the adjoint space. The quadrant location of the equilibrium and stability require λ_1 and λ_2 to begin and remain in the first quadrant. When $\phi_1 = u_{1,max}$ and $\phi_2 = 0$, I show that $\lambda_1 > 0$ always. λ_2 does not matter because spouse 2 will head to their natural disposition without interaction with their spouse. Finally, when $\phi_1 = 0$, both spouses head to their natural disposition, and there is no interaction. The same can be said for spouse 2 when

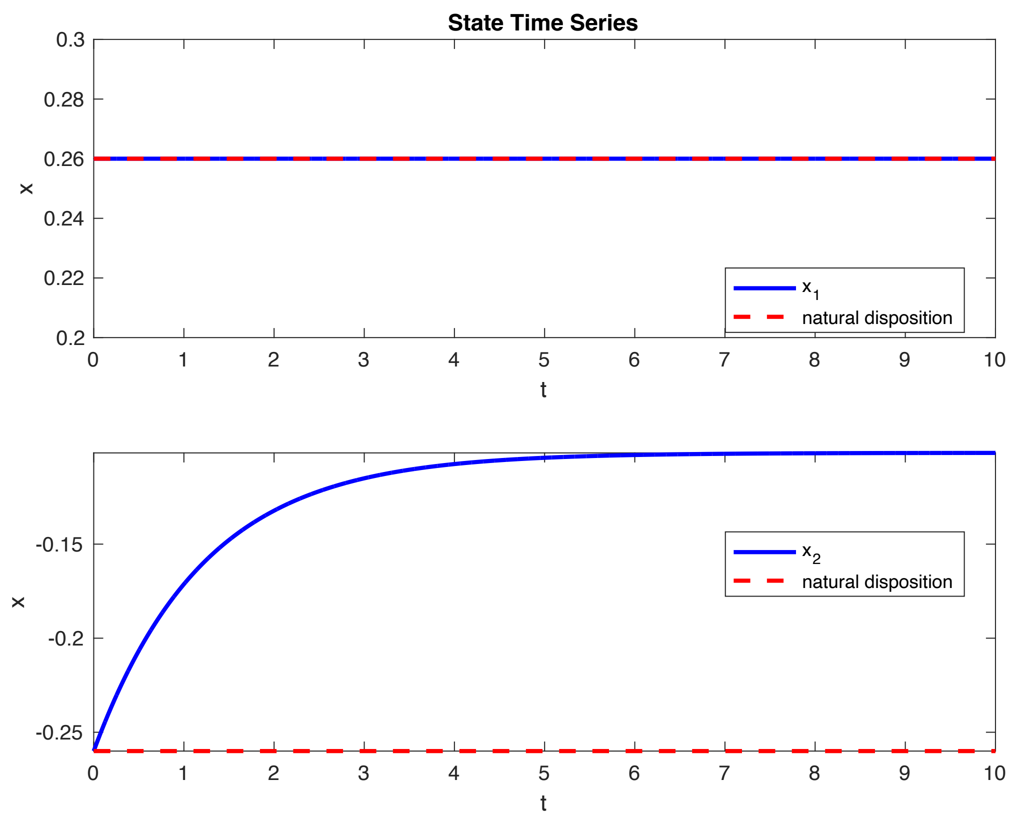


Figure 1.3: State time-series for a couple with equal and opposite dispositions

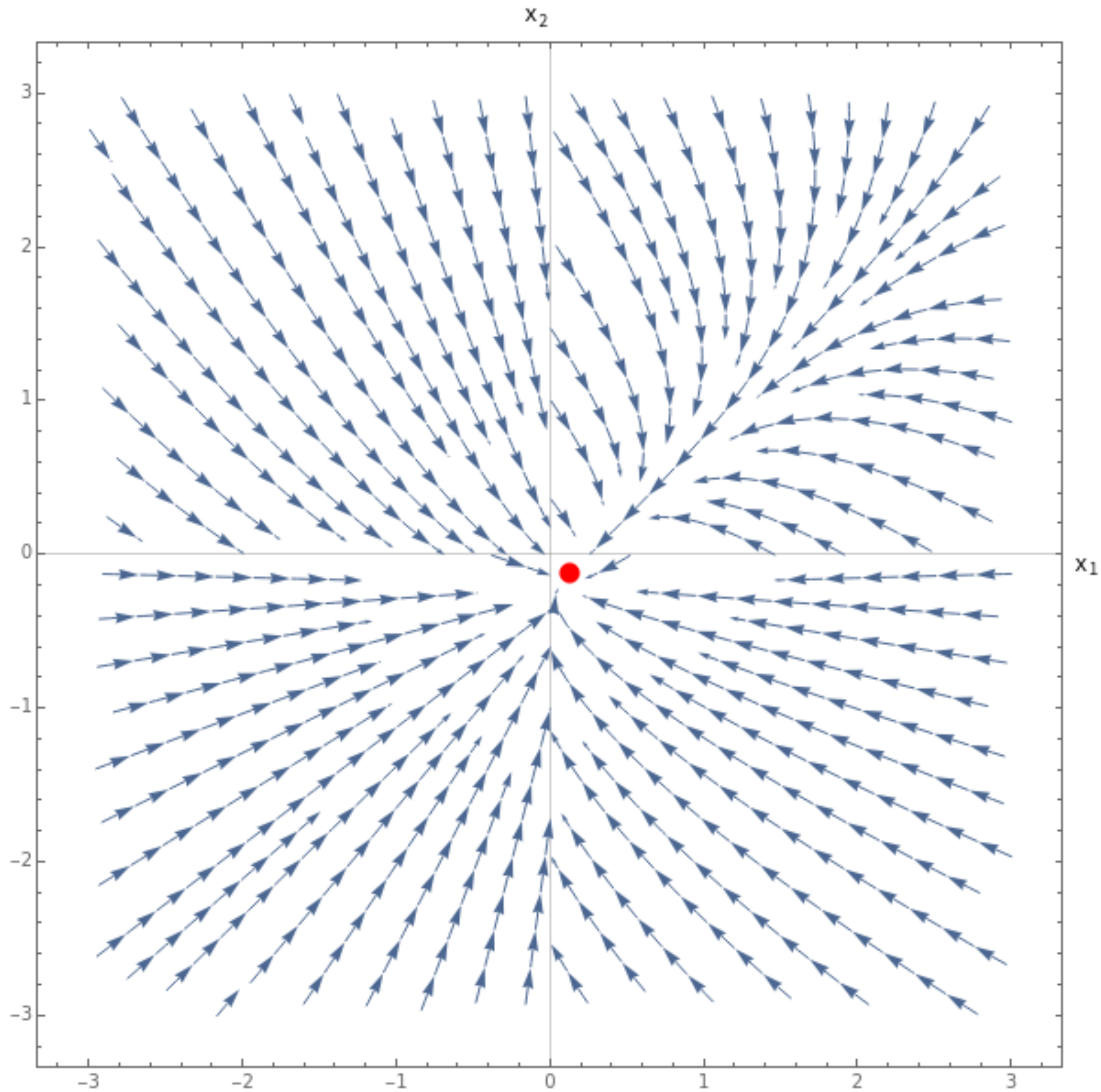


Figure 1.4: Phase portrait for a couple with equal and opposite dispositions

$\phi_2 = u_{2,max}$ and $\phi_1 = 0$.

I define three terms to assess the state of a marriage. First, I call a marriage successful if, in the marriage, the couple is more positive than their natural dispositions at the end of the problem. That is, $x_i(T) > \bar{x}_i$. I call a marriage semi-successful if $x_i(T) \geq \bar{x}_i$. Finally, I call a marriage stable if both spouses are positive at the end of the problem. That is,

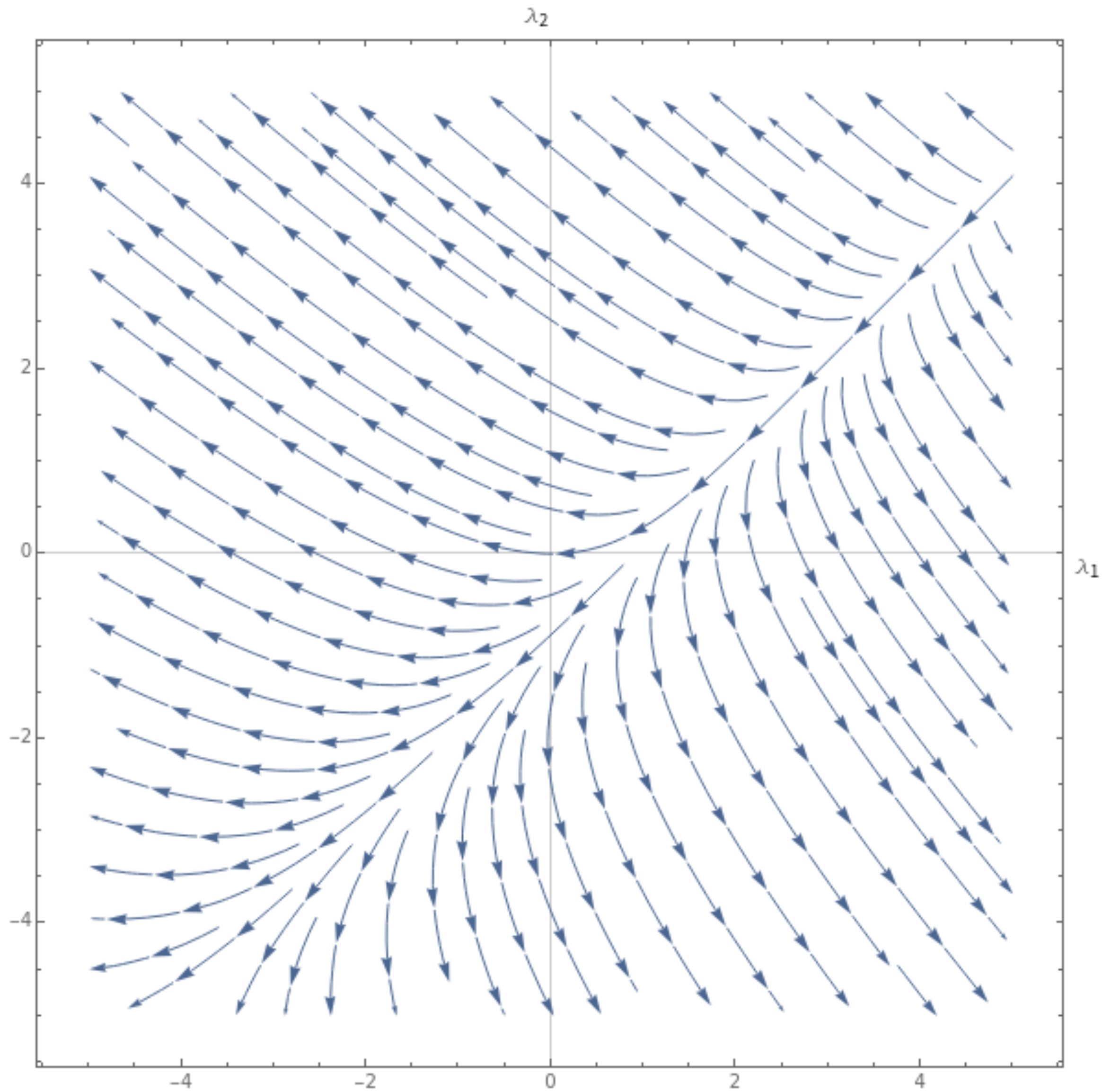


Figure 1.5: Sample phase-portrait for the adjoint space.

$x_i(T) > 0$.

My results show that conflict avoidance is the optimal strategy in a marriage in this model. This is true regardless of the marriage goal. Tung [44] shows that conflict-avoider marriages are successful and preferable for only slightly positive couples. It is also successful for couples with opposite dispositions when the cheerful spouse is more positive than the negative spouse is negative. The couple in example 1 is both successful and stable. They are both naturally positive and benefit from being married to each other. The couple in example 2, who have equal magnitude but opposite dispositions, is semi-successful because spouse 1 is no better in marriage in marriage than alone, but spouse 2 two benefits from the marriage. The couple in example 3, where the positive spouse is twice as positive as the negative spouse is negative, is both stable and successful.

1.7 Discussion

My main result in this chapter is that regardless of the marriage goal, conflict-avoidance may be an effective strategy if the natural disposition of each spouse is positive enough.

There are a few limitations model 1.15. First, I assume that $u_i \geq 0$ because allowing a spouse to gain positivity through interacting negatively with an opposing spouse would be a toxic relationship. There, however, may be situations where allowing u_i to be negative may be realistic. This brings us to the first limitation of this model. Spouses can only respond to their spouses with their control variable. If a spouse could directly influence their spouse's positivity, a negative control variable could be interpreted as cajoling their spouse when unhappy. This would mean changing the state equations to

$$\dot{x_1} = r_1(\bar{x}_1 - x_1) + u_2 x_2, \tag{1.23a}$$

$$\dot{x_2} = r_2(\bar{x}_2 - x_2) + u_1 x_1, \tag{1.23b}$$

$$x_1(0) = x_1^0, x_2(0) = x_2^0. \tag{1.23c}$$

Notice the location of the control variables has switched. I have begun to look into what occurs when a spouse can influence their spouse, and I have some preliminary results.

Another direction for future work is to examine how changing the mean in the objective function affects the prognosis for couples. I already know that the optimal strategy of the conflict would remain bang-bang, but that does not necessarily mean that the outcome would remain the same or that the strategy would remain exactly conflict-avoiding because there would be a different, more complicated adjoint equation. A potential mean to consider would be the harmonic mean. Kefitz [25] examined this mean in a paper on how population distributions relate to age, marriage, and sex. This means more weight would be given to the spouse who needs more support. This might be useful in studying marriage repair.

A limitation of model 1.15 is that bang-bang is the only strategy achievable in this model. There is also no penalty for interacting with or ignoring one's spouse. I would expect some energy expenditure for ignoring a negative spouse or interacting with a positive spouse. This would change how the couple interacts with one another. This scenario is addressed in the next chapter. The chapter introduces a penalty for interaction and explains how a validating interaction-style develops. Using a differential-game-theory approach, I examine what happens when couples have different marital goals.

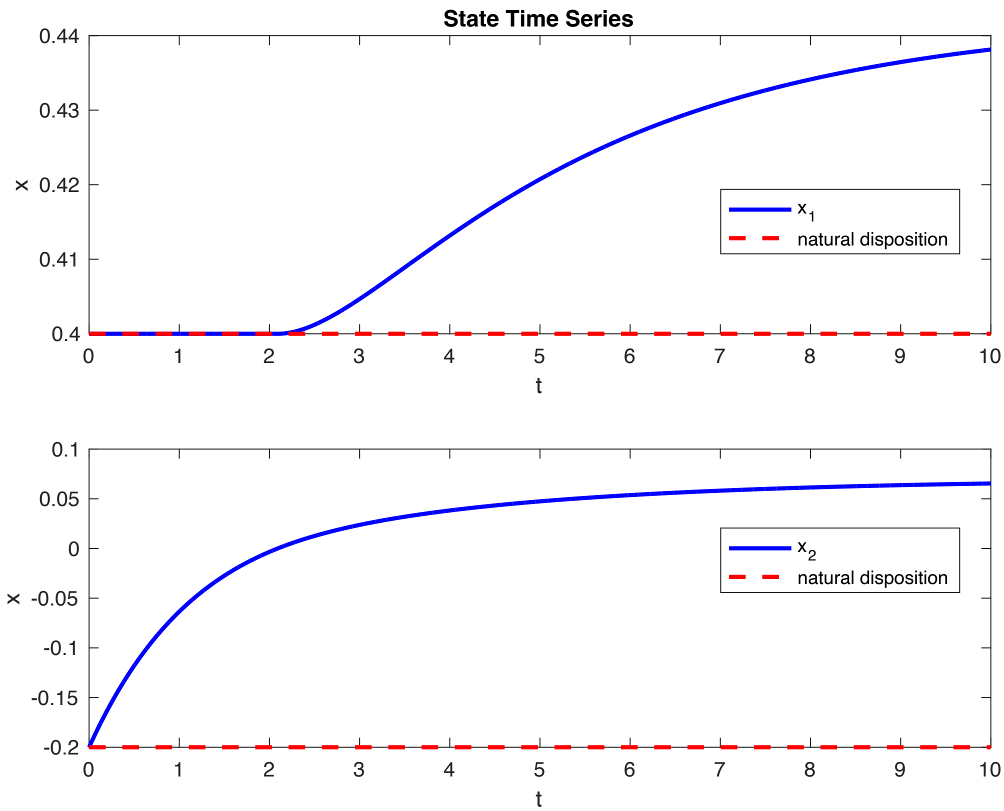


Figure 1.6: State time-series for a couple with opposite dispositions, positive spouse more positive

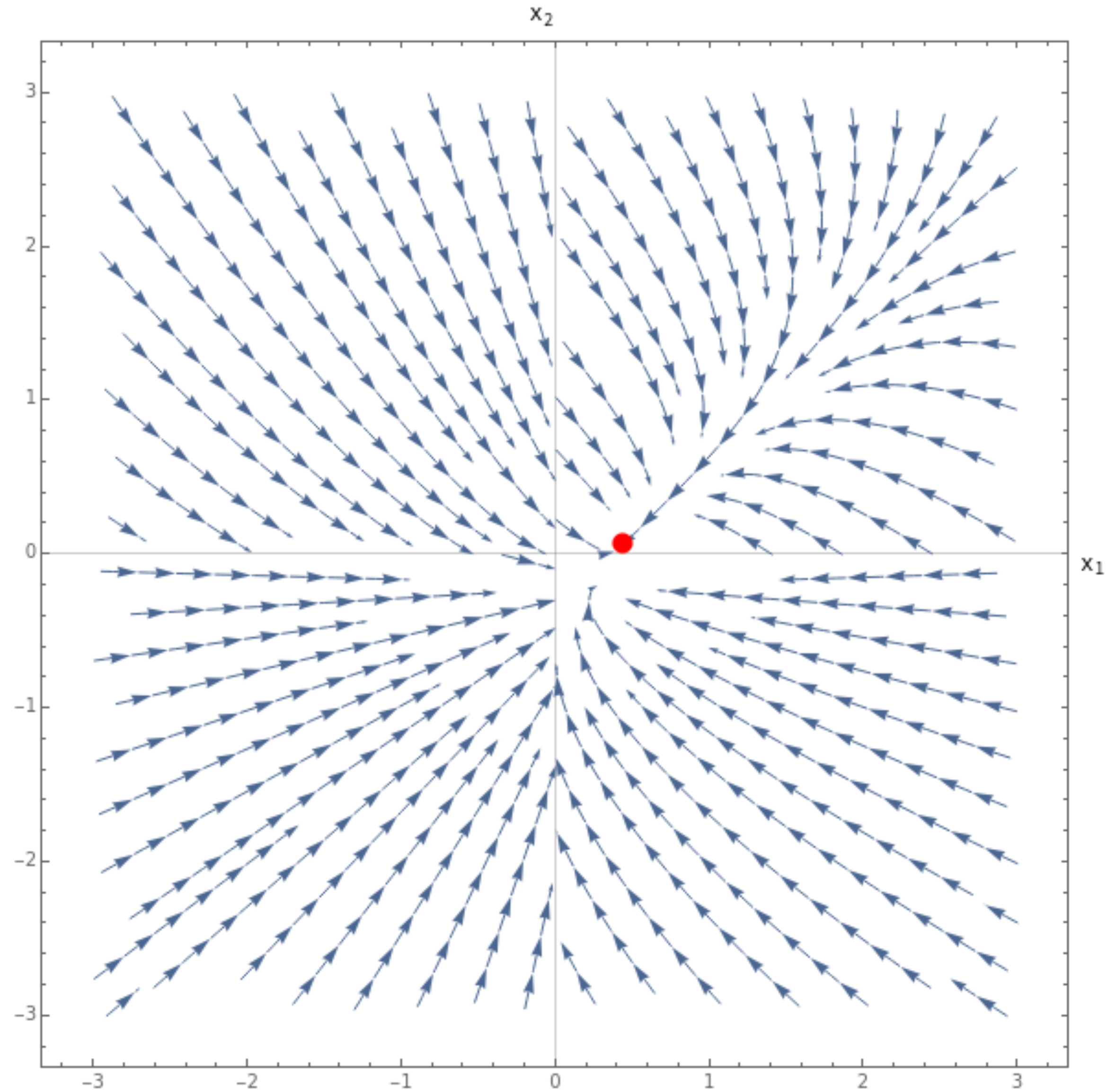


Figure 1.7: Phase portrait for a couple with opposite dispositions, positive spouse more positive

Chapter 2

WHY SOME COUPLES ARE VALIDATING TYPE

2.1 Introduction

Marriage is a foundational pillar of culture and society with political and personal implications [24]. For this reason, researchers across many fields have chosen to study questions surrounding marriage and how it evolves, how it is sustained, and why it dissolves. Social scientists have undoubtedly led the charge in studying marriage. Still, increasingly, they have turned to mathematics and mathematicians to fill in the gaps where experiments and data do not exist and are too costly to obtain.

A seminal work in the study of marriage through mathematics came from Cook et al.[13] and was expanded on in their book *The Mathematics of Marriage: Dynamic Nonlinear Models* [18]. They built mathematical theory behind the phenomena discovered in the research of Gottman [15, 16], Gottman and Levenson [14], and Gottman and Krokoff [17].

My research builds on the work of Gottman [15], most particularly on his designation of interaction styles in marriages. These interaction styles consist of the stable types of validating, avoiding, or volatile, and the unstable types of hostile and hostile-detached. Cook et al. [13] conjecture that marriages are successful when they have matched interaction-styles and unsuccessful when they have mismatched interaction-styles.

My paper is novel because I use differential game theory to present a possible explanation for why couples evolve into a validating interaction-style. In contrast, Gottman found these styles through observation. My contribution is essential because psychologists may use it to offer strategies for marriage repair. This paper is the second part of this series. In the first paper, I presented a possible explanation for why some couples evolve into a conflict-avoiding interaction-style.

Few researchers have taken an optimal-control or differential-game-theory approach to studying marriage. The work in [28, 29] focuses on "marriage stock" and does not consider any psychological framework. Rey [35] considered a psychological framework for an optimal-control model. Herrara de la Cruz and Rey [22] expanded on that model. They simplified marriage by assuming that a marriage's happiness and success or failure exist as a unit and not as a relationship between two people interacting, as I do in my research.

In section 2.2, I introduce my model and explain its parameters. In section 2.3, I summarize the mathematical theory I use to approach my problem by stating relevant theorems and definitions from differential game theory. In section 2.4, I apply the theory to my model. In section 2.6, I show several examples of the model and offer a possible interpretation of the results. In the final section, I discuss the model's implications and limitations and present potential directions for future work.

2.2 Model

My main goals are to determine how couples evolve into their interaction styles and how best to repair marriages if repair is possible.

I propose a model where $x_1(t)$ and $x_2(t)$ represent the positivity of each spouse over time. The dynamics of the state variables are governed by the differential equations,

$$\frac{dx_1}{dt} = r_1(\bar{x}_1 - x_1) + u_1 x_2, \tag{2.1a}$$

$$\frac{dx_2}{dt} = r_2(\bar{x}_2 - x_2) + u_2 x_1, \tag{2.1b}$$

consisting of internal influence plus external influence for each spouse.

The internal-influence portion consists of the natural disposition $\bar{x}_i$ and the rate constant, r_1, which affects the return rate to that natural disposition. The external portion comes from interaction between the spouses. I call the external-influence portion the influence function for each spouse . I define these functions

$$I_1(x_2) = u_1 x_2 \tag{2.2a}$$

$$I_2(x_1) = u_2 x_1. \tag{2.2b}$$

Equation 2.2a is the influence function for spouse 1 while equation 2.2b is the influence function for spouse 2.

The values of u_1 and u_2 are nonnegative and are determined by maximizing the objective functional J_i, for each spouse $i = 1, 2$,

$$\max_{0 \leq u_i \leq u_{i,max}} [J_i = \int_0^T F_i(x_1, x_2, u_1, u_2, t) dt], \tag{2.3}$$

with

$$F_i = \alpha_i [x_i - \frac{1}{2\epsilon}(u_i - u_0)^2] + (1 - \alpha_i)[x_j - \frac{1}{2\epsilon}(u_j - u_0)^2], \tag{2.4}$$

where F_i is chosen based on the goal of spouse i in the marriage and spouse j is the other spouse.

If the spouse is selfish and wants to prioritize their own happiness,

$$\alpha_i = 1. \tag{2.5}$$

If the spouse is balanced and wants to prioritize joint happiness,

$$\alpha_i = \frac{1}{2} \tag{2.6}$$

If the spouse is altruistic and wants to prioritize their spouse's happiness,

$$\alpha_i = 0. \tag{2.7}$$

The parameter α_i is a relative weight to determine the priorities of spouse i for their happiness versus their spouses and can be any value in the interval $[0, 1]$.

The second half of equation 2.4, the portion that begins with $1/(2\epsilon)$, is the emotional cost spouse i incurs from interaction with their spouse. The parameter $u_0 \geq 0$ is the ideal, effortless, natural level of interaction between the spouses. The parameter $\epsilon > 0$ measures the emotional effort of interacting with (or ignoring) one's spouse. It is unknown by the couples. As $\epsilon \to 0$, the problem becomes a nonlinear problem with a high emotional cost between spouses. This is less deliberate effort for the spouse. As $\epsilon \to \infty$, I get a linear problem. In this linear problem, the emotional cost goes to zero, but this represents more effort for the spouse. This problem is explored in part I. Furthermore, the terms are squared so that larger differences between u_i and u_0 are penalized more than smaller differences.

2.3 Methods

2.3.1 Differential Game Theory

The problem that I have defined is a differential-game-theory problem. Game theory is a highly active field in many fields, especially economics. Many Nobel prizes in economics have been awarded for results in the field [1]. However, differential game theory has its roots in mathematics as it extends linear and nonlinear programming and optimal control theory to multiple players [4]. In this section, I present essential definitions and theorems from differential game theory that are relevant to my model.

Definition 6 (Standard Differential Game for 2 Players and 2 State Variables)**.** *The following conditions define a standard differential game.*

1. *There is a state system x from X, a subset of $\mathbb{R}^2$. There is a set of players $M = \{1, 2\}$. Each player i from the set M chooses a control u_i from U_i. U_i is called a feasible set and is a subset of $\mathbb{R}$.*

2. *The state evolves via*

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t), u_1(t), u_2(t), t), \;\; x_1(0) = x_1^0, \tag{2.8a}$$

$$\dot{x}_2(t) = f_2(x_1(t), x_2(t), u_1(t), u_2(t), t), \;\; x_2(0) = x_2^0. \tag{2.8b}$$

3. *Each player seeks to maximize (or minimize) their own objective functional*

$$\max_{u_1 \geq 0} \;[J_1 = \int_0^T F_1(x_1(t), x_2(t), u_1(t), u_2(t), t)dt], \tag{2.9a}$$

$$\max_{u_2 \geq 0} \;[J_2 = \int_0^T F_2(x_1(t), x_2(t), u_1(t), u_2(t), t)dt]. \tag{2.9b}$$

   While each objective functional depends on the control paths of the other players, J_1 is only maximized with respect to u_1, and J_2 is only maximized with respect to u_2.

(Modified from Basar et al. [4])

Definition 7 (Open-Loop Strategy [11])**.** *A player chooses an open-loop strategy if their decision depends on the current time t and initial state conditions.*

Definition 8 (Open-Loop Nash Equilibrium [11])**.** *The vector (ϕ_1, ϕ_2) of functions is called an open-loop Nash equilibrium if, for each player i, an optimal control path $u_i(t)$ of the standard differential game exists and is given by the open-loop strategy $u_i(t) = \phi_i(t)$.*

There are two main types of strategies. The open-loop strategy defined above and the closed-loop strategy. In a closed-loop strategy, the player makes their decision based on the current time and current state conditions, not the initial conditions, as in the open-loop strategy. Closed-loop differential games are difficult problems that usually cannot be solved. I choose open-loop strategies for ease but also because there is precedent for using this strategy. The research I build my model on by Gottman [17, 15, 18] are open-loop problems. Using information from initial couple interviews, Gottman is able to make predictions about the marriage years down the road. Thus, I are justified in choosing open-loop strategies here.

The Nash equilibrium is a fundamental concept in economics. It represents the solution(s) of the mutual best response for each player. In other words, player i cannot do any better if they chose a different path, and the same is said of player j. If all players $j \neq i$ use open strategies $u_j(t) = \phi_j(t)$, then player i's problem is an optimal control problem [11].

Definition 9 (Player i's Optimal Control Problem [11])**.** *Player i maximizes their objective functional*

$$\max_{u_i(t)\in U_i} [J_i = \int_0^T F_i(x_1, x_2, u_i, \phi_j, t)dt] \tag{2.10a}$$

subject to

$$\dot{x}_1(t) = f_1(x_1(t), x_2(t), u_i(t), \phi_j(t), t), \tag{2.10b}$$

$$\dot{x}_2(t) = f_2(x_1(t), x_2(t), u_i(t), \phi_j(t), t), \tag{2.10c}$$

where J_i is only maximized with respect to u_i and ϕ_j is an open loop strategy with initial conditions $x_1(0) = x_1^0$ and $x_2(0) = x_2^0$.

Definition 10. *The Hamiltonian, H is*

$$H_i(t, x_1, x_2, u_i, \phi_j, \lambda_{i1}, \lambda_{i2}) = F_i(t, x_1, x_2, u_i, \phi_j) + \sum_{k=1}^{2} \lambda_{ik} f_k(t, x_1, x_2, u_i, \phi_j). \tag{2.11}$$

Definition 11. *The maximized Hamiltonian is*

$$M_i(x_1, x_2, \lambda_{i1}, \lambda_{i2}, t) = \max_{u_i\in U_i} H_i(x_1, x_2, u_i, \lambda_{i1}, \lambda_{i2}, t). \tag{2.12}$$

Theorem 2 (Sufficiency Condition, Magnasarian [30])**.** *If M_i is concave and continuously differentiable with respect to x for all $t \in [0, T]$ and if there exists a continuously differentiable $\lambda_{i,k}$ such that*

$$\lambda'_{i,k}(t) = -\frac{\partial M_i}{\partial x_k}, \tag{2.13}$$

k = 1,2, (adjoint equations), and

$$\lambda_{i,k}(T) = 0 \tag{2.14}$$

(terminal condition) are satisfied, then, u_i is an optimal path (Modified from Dockner et al. [11].).

The most important part of this theorem is the existence of the adjoint equations. I can find the optimal path for each player through these adjoint equations. Traditionally, the adjoint equation represents the shadow price or the highest price a player could reasonably be willing to pay for a marginal increase in the state variable. In my case, the adjoint equation represents the highest emotional price a spouse would be willing to pay for a marginal increase in their happiness.

2.3.2 Perturbation Theory

I use perturbation theory to find analytical solutions for my differential-game-theory problem. Perturbation theory includes asymptotic expansion of differential equations. It is a method for finding approximate analytical solutions to differential equations. The method has a long and storied history in applied mathematics, dating back to Henry Poincare in 1886 [33]. The procedure presented here is adapted from [8].

Consider the differential equation

$$\frac{dx}{dt} + p(t)x(t) = g(t) \tag{2.15}$$

with initial condition $x(0) = A$.

The variable $x(t)$ can be expanded via a Taylor series in the parameter $\epsilon \to 0$,

$$x(t;\epsilon) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t) + ... = \sum_{n=0}^{\infty} \epsilon^n x_n(t). \tag{2.16}$$

If I plug this Taylor expansion into equation 2.15 and compare coefficients of ϵ_n, I end up with an infinite system of differential equations that may (or may not) be easier to solve:

$$x_0'(t) + p(t)x_0(t) = g(t),\ x_0(t) = A, \tag{2.17}$$

$$x_1'(t) + p(t)x_1(t) = 0,\ x_1(t) = 0 \tag{2.18}$$

$$\vdots \tag{2.19}$$

Truncating the Taylor expansion of x at some finite n gives an approximate analytical solution to differential equation 2.15.

2.4 Analysis

Next, I apply the differential-game-theory method to the problem. The result does not change for the altruistic case because each spouse only maximizes their control, and the linear dependence on their control remains. Observe that when $\alpha_i = 0$,

$$H_i = x_j - \frac{1}{2\epsilon}(u_j - u_0)^2 + \lambda_{i,i}[r_i(\bar{x}_i - x_i) + u_i x_j] + \lambda_{i,j}[r_j(\bar{x}_j - x_j) + u_j x_i], \tag{2.20}$$

and I see that there is only linear dependence on u_i. Thus,

$$\phi_i = \begin{cases} 0,\ \lambda_{i,i}x_j < 0, \\ u_{i,max},\ \lambda_{i,i}x_j > 0. \end{cases} \tag{2.21}$$

The implications of this problem were explored in part I of the series.

I obtain completely new controls for the non-zero α-values and apply Pontryagin's maximum principle to my model. For spouse i's problem, I have

$$\max_{u_i \geq 0} \ \{J_i = \int_0^T \alpha_i[x_i - \frac{1}{2\epsilon}(u_i - u_0)^2] + (1-\alpha_i)[x_j - \frac{1}{2\epsilon}(u_j - u_0)^2]dt\} \tag{2.22}$$

subject to

$$\dot{x_1} = r_1(\bar{x}_1 - x_1) + u_1 x_2 \tag{2.23a}$$

$$\dot{x_2} = r_2(\bar{x}_2 - x_2) + \phi_2 x_1 \tag{2.23b}$$

$$x_1(0) = x_1^0, x_2(0) = x_2^0 \tag{2.23c}$$

We form the Hamiltonian,

$$H_i = \alpha_i[x_i - \frac{1}{2\epsilon}(u_i - u_0)^2] + (1-\alpha_i)[x_j - \frac{1}{2\epsilon}(u_j - u_0)^2] + \lambda_{i,i}[r_i(\bar{x}_i - x_i) + u_i x_j]$$
$$+ \lambda_{i,j}[r_j(\bar{x}_j - x_j) + u_j x_i]. \tag{2.24}$$

I find the optimal path for u_1 by taking the derivative of the Hamiltonian concerning u_1, setting it equal to zero, and solving for u_1,

$$\frac{\partial H_i}{\partial u_i} = -\frac{\alpha_i}{\epsilon}(u_i - u_0) + \lambda_{i,i} x_j = 0. \tag{2.25}$$

Thus,

$$\phi_i = \frac{\epsilon}{\alpha_i}\lambda_{i,i} x_j + u_0 \tag{2.26}$$

I call the optimal path ϕ_1 because it is a Markovian Nash equilibrium. The adjoint equations are the same as in the previous section.

Finally, the optimal system is given by

$$\dot{x_1} = r_1(\bar{x}_1 - x_1) + (\frac{\epsilon}{\alpha_1}\lambda_{1,1} x_2 + u_0)x_2, \tag{2.27a}$$

$$\dot{x_2} = r_2(\bar{x}_2 - x_2) + (\frac{\epsilon}{\alpha_2}\lambda_{2,2} x_1 + u_0)x_1, \tag{2.27b}$$

$$\dot{\lambda}_{1,1} = -\alpha_1 + \lambda_{1,1} r_1 - \lambda_{1,2}(\frac{\epsilon}{\alpha_2}\lambda_{2,2} x_1 + u_0) \tag{2.27c}$$

$$\dot{\lambda}_{1,2} = -(1-\alpha_1) - \lambda_{1,1}(\frac{\epsilon}{\alpha_1}\lambda_{1,1} x_2 + u_0) + \lambda_{1,2} r_2 \tag{2.27d}$$

$$\dot{\lambda}_{2,1} = -(1-\alpha_2) + \lambda_{2,1} r_1 - \lambda_{2,2}(\frac{\epsilon}{\alpha_2}\lambda_{2,2} x_1 + u_0) \tag{2.27e}$$

$$\dot{\lambda}_{2,2} = -\alpha_2 - \lambda_{2,1}(\frac{\epsilon}{\alpha_1}\lambda_{1,1} x_2 + u_0) + \lambda_{2,2} r_2, \tag{2.27f}$$

with conditions $x_1(0) = x_1^0, x_2(0) = x_2^0, \lambda_{i,k}(T) = 0$.

2.4.1 $\epsilon \to 0$

Now, I can implement perturbation theory to obtain a solution. Let $\lambda_{1,k} = \lambda_k$ and $\lambda_{2,k} = \mu_k$. I begin by assuming solutions as perturbation series

$$x_1(t) = \sum_{n=0}^{\infty} \epsilon^n x_{1,n}, \;\; x_2(t) = \sum_{n=0}^{\infty} \epsilon^n x_{2,n}, \tag{2.28a}$$

$$\lambda_1(t) = \sum_{n=0}^{\infty} \epsilon^n \lambda_{1,n}, \;\; \lambda_2(t) = \sum_{n=0}^{\infty} \epsilon^n \lambda_{2,n}, \tag{2.28b}$$

$$\mu_1(t) = \sum_{n=0}^{\infty} \epsilon^n \mu_{1,n}, \;\; \mu_2(t) = \sum_{n=0}^{\infty} \epsilon^n \mu_{2,n}. \tag{2.28c}$$

Zeroth order

I look at terms with no ϵ parameter for this order. This gives us a system,

$$\dot{x}_{1,0} = r_1(\bar{x}_1 - x_{1,0}) + u_0 x_{2,0}, \tag{2.29a}$$

$$\dot{x}_{2,0} = r_2(\bar{x}_2 - x_{2,0}) + u_0 x_{1,0}, \tag{2.29b}$$

$$\dot{\lambda}_{1,0} = -\alpha_1 + \lambda_{1,0} r_1 - \lambda_{2,0} u_0, \tag{2.29c}$$

$$\dot{\lambda}_{2,0} = -(1-\alpha_1) - \lambda_{1,0} u_0 + \lambda_{2,0} r_2 \tag{2.29d}$$

$$\dot{\mu}_{1,0} = -(1-\alpha_2) + \mu_{1,0} r_1 - \mu_{2,0} u_0, \tag{2.29e}$$

$$\dot{\mu}_{2,0} = -\alpha_2 - \mu_{1,0} u_0 + +\mu_{2,0} r_2, \tag{2.29f}$$

for obtaining a solvable, analytical solution.

The zeroth-order system of equations is now a system of three, uncoupled pairs of equations that can each be solved. For the zeroth-order state equations, we get

$$\begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix} = a_1 \begin{bmatrix} u_0 \\ r_1 + \eta_1 \end{bmatrix} e^{\eta_1 t} + a_2 \begin{bmatrix} u_0 \\ r_1 + \eta_2 \end{bmatrix} e^{\eta_2 t} + \begin{bmatrix} \frac{\bar{x}_1 r_1 r_2 + \bar{x}_2 r_2 u_0}{r_1 r_2 - u_0^2} \\ \frac{r_1 r_2 \bar{x}_2 + r_1 \bar{x}_1 u_0}{r_1 r_2 - u_0^2} \end{bmatrix} \tag{2.30}$$

where

$$\eta_1, \eta_2 = \frac{1}{2}[-(r_1 + r_2) \pm \sqrt{(r_1 + r_2)^2 - 4(r_1 r_2 - u_0^2)}], \tag{2.31}$$

and initial conditions

$$\begin{bmatrix} x_{1,0}(0) \\ x_{2,0}(0) \end{bmatrix} = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}. \tag{2.32}$$

For the two sets of adjoint equations, we have

$$\begin{bmatrix}\lambda_{1,0}\\ \lambda_{2,0}\end{bmatrix} = b_1\begin{bmatrix}u_0\\ r_1-\rho_1\end{bmatrix}e^{\rho_1 t} + b_2\begin{bmatrix}u_0\\ r_1-\rho_2\end{bmatrix}e^{\rho_2 t} + \begin{bmatrix}\frac{u_0-u_0\alpha_1+\alpha_1 r_2}{r_1r_2-u_0^2}\\ \frac{u_0\alpha_1+(1-\alpha_1)r_1}{r_1r_2-u_0^2}\end{bmatrix}, \tag{2.33}$$

and

$$\begin{bmatrix}\mu_{1,0}\\ \mu_{2,0}\end{bmatrix} = c_1\begin{bmatrix}u_0\\ r_1-\rho_1\end{bmatrix}e^{\rho_1 t} + c_2\begin{bmatrix}u_0\\ r_1-\rho_2\end{bmatrix}e^{\rho_2 t} + \begin{bmatrix}\frac{u_0\alpha_2+(1-\alpha_2)r_2}{r_1r_2-u_0^2}\\ \frac{u_0\alpha_2-u_0-\alpha_2 r_1}{r_1r_2-u_0^2}\end{bmatrix}, \tag{2.34}$$

where

$$\rho_{1,2} = \frac{1}{2}[r_1+r_2 \pm \sqrt{(r_1+r_2)^2-4(r_1r_2-u_0^2)}], \tag{2.35}$$

with terminal conditions

$$\begin{bmatrix}\lambda_{1,0}(T)\\ \lambda_{2,0}(T)\\ \mu_{1,0}(T)\\ \mu_{2,0}(T)\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}. \tag{2.36}$$

As you can see, the first-order solution has many terms and parameters. While it can be exactly solved, it becomes messy immediately. Higher-order terms depend on the solutions from the previous order, as you will see in the next section.

First Order

This system of equations comes from the coefficients of terms with ϵ raised to one,

$$\dot{x}_{1,1} = -r_1x_{1,1} + \frac{1}{\alpha_1}\lambda_{1,0}x_{2,0}^2 + u_0x_{2,1} \tag{2.37a}$$

$$\dot{x}_{2,1} = -r_2x_{2,1} + \frac{1}{\alpha_2}\mu_{2,0}x_{1,0}^2 + u_0x_{1,1}, \tag{2.37b}$$

$$\dot{\lambda}_{1,1} = \lambda_{1,1}r_1 - \frac{1}{\alpha_2}\lambda_{2,0}\mu_{2,0}x_{1,0} - u_0\lambda_{2,1}, \tag{2.37c}$$

$$\dot{\lambda}_{2,1} = -\frac{1}{\alpha_1}\lambda_{1,0}^2x_{2,0} - u_0\lambda_{1,1} + \lambda_{2,1}r_2 \tag{2.37d}$$

$$\dot{\mu}_{1,1} = -\frac{1}{\alpha_2}\mu_{2,0}^2x_{1,0} + \mu_{1,1}r_1 - u_0\mu_{2,1}, \tag{2.37e}$$

$$\dot{\mu}_{2,1} = -\frac{1}{\alpha_1}\mu_{1,0}\lambda_{1,0}x_{2,0} - u_0\mu_{1,1} + \mu_{2,1}r_2. \tag{2.37f}$$

It only depends on zeroth order terms. All of these equations can be written

$$\dot{\vec{x}} + A\vec{x} = \vec{f}(t) \tag{2.38a}$$

$$\dot{\vec{\lambda}} - A\vec{\lambda} = \vec{g}(t) \tag{2.38b}$$

$$\dot{\vec{\mu}} - A\vec{\mu} = \vec{h}(t) \tag{2.38c}$$

where

$$A = \begin{bmatrix} r_1 & -u_0 \\ -u_0 & r_2, \end{bmatrix} \tag{2.39}$$

with conditions $\dot{\vec{x}}(0) = 0$, $\dot{\vec{\lambda}}(T) = 0$, and $\dot{\vec{\mu}}(T) = 0$. Each equation can be solved with any two-dimensional, linear, differential-equation method. The exact solution of the first-order scheme contains many terms and parameters, so I omit it. Higher-order perturbation systems can systematically be determined and solved.

Overall, using the perturbation scheme, I can recover a validating interaction-style by considering small values of ϵ. This penalty for ignoring one's spouse differentiates between avoiding and validating marriages. As $\epsilon \to 0$, there is a massive cost of interaction, and I see in equation 2.26 that the first term of these equations also goes to zero, and the only interaction between the spouses comes from u_0. The zeroth-order perturbation scheme verifies this. In equations 2.29a and 2.29b, I see a fixed, positive response of u_0 between spouses. This is equivalent to a validating interaction-style.

The higher orders of the perturbation scheme show how the different marriage goals affect the relationship because of the presence of the α_i values. They are deviations from the validating marriage. The goal of the marriage determines precisely how significant that deviation is. In equation 2.37, a spouse with a balanced marriage goal gets twice the increase from their spouse's positivity compared to the selfish spouse.

I return to equation 2.25 for the altruistic spouse. If $\alpha_i = 0$, I get the same switching function as model I. The altruistic spouse never becomes validating. The optimal strategy remains conflict-avoiding regardless of the values of ϵ. The altruist decides to accept the higher energetic cost that comes with ignoring their spouse. I explore how marriage goals affect the marriage through specific numerical examples.

2.5 Numerical Method

The perturbation scheme is sufficient in understanding the model for $\epsilon \to 0$ and $\epsilon \to \infty$ To understand $0 < \epsilon < \infty$, I will need to solve this system of equations, which is no easy feat. It is six-dimensional and nonlinear, which makes it impossible to achieve an exact analytical solution; as such, I solve each system numerically. However, finding the numerical solutions comes with challenges because the state equations have initial conditions, and the adjoint equations have terminal conditions. This means I have a two-point, boundary-value problem. I cannot use any traditional solvers, and I have thus built my solvers using an algorithm called Forward-Backward Sweep (FBS).

FBS was developed to tackle optimal-control problems that arise from the Maximum Principle. The version used here is adapted from Lenhart and Workman [27], which is based on Wolfgang [20]. Convergence of FBS for the specific type of problem used in my model was proved in McAsey et al. [32]. The steps of the algorithm are:

1. Discretize time and initialize step size. Guess an initial control.
2. Use the initial condition to solve the state system forward in time.
3. Use the terminal condition to solve the adjoint system backward in time.
4. Update the control using the state and adjoint solutions previously calculated.
5. Repeat this process until the differences of each variable between iterations are within a specified tolerance.

2.6 Numerical Results

The choice of terminal time and the time scale affects the solution. There are technical limitations in the numerical solver that I used. A T that is too large will break the code. However, Gottman and Levenson [14] could make long-term predictions based on 15-minute conversations between couples, so by selecting a time scale of $T = 1$, would make $T = 3$ sufficient to capture this research. The program can process terminal times of this length.

This section defines a few terms to interpret the numerical results. First, a marriage is called successful if each spouse in the couple is happier in marriage than alone at the terminal time T. That is, $x(T) > \bar{x}$. A marriage is semi-successful if $x_i(T) = \bar{x}_i$ and $x_j(T) > \bar{x}_j$. A marriage is stable if both spouses are positive at the terminal time, $x(T) > 0$.

2.6.1 Example: Verification of Numerical Results

To verify the integrity of my numerical codes, I present a specific example and use the perturbation scheme to compare. Let $r_1 = r_2 = 0.82$, $\bar{x}_1 = 0.26$, $\bar{x}_2 = -0.26$, $u_0 = 1$, $T = 1$, $x_i^0 = \bar{x}_i$, and $\epsilon = 0.1$. Using selfish goals, the zeroth-order system is a set of three, two-dimensional, linear equations that can be solved exactly. These equations are given by

$$\dot{x}_{1,0} = 0.8(0.25 - x_{1,0}) + x_{2,0} \tag{2.40a}$$

$$\dot{x}_{2,0} = 0.8(-0.25 - x_{2,0}) + x_{1,0} \tag{2.40b}$$

$$\dot{\lambda}_{1,0} = -1 + 0.8\lambda_{1,0} - \lambda_{2,0} \tag{2.40c}$$

$$\dot{\lambda}_{2,0} = \lambda_{1,0} - 0.8\lambda_{2,0} \tag{2.40d}$$

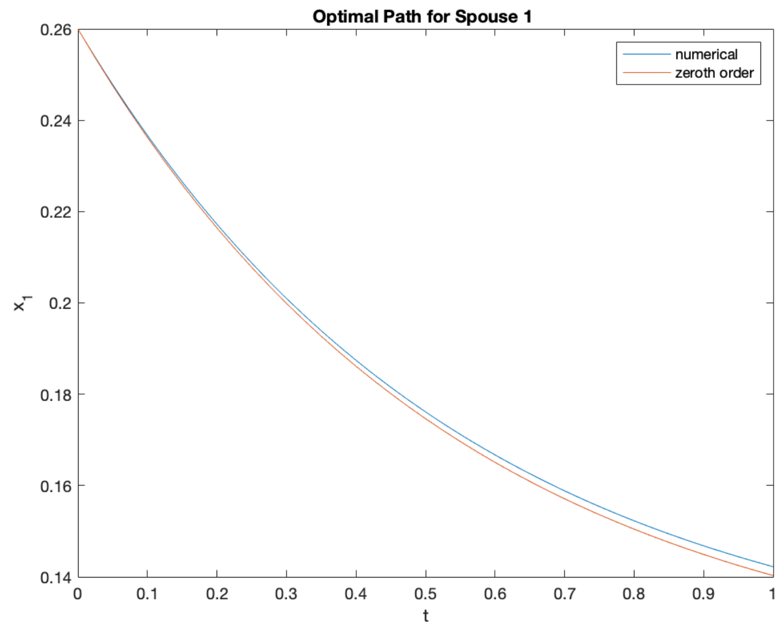
$$\dot{\mu}_{1,0} = 0.8\mu_{1,0} - \mu_{2,0} \tag{2.40e}$$

$$\dot{\mu}_{2,0} = -1 + \mu_{1,0} - 0.8\mu_{2,0} \tag{2.40f}$$

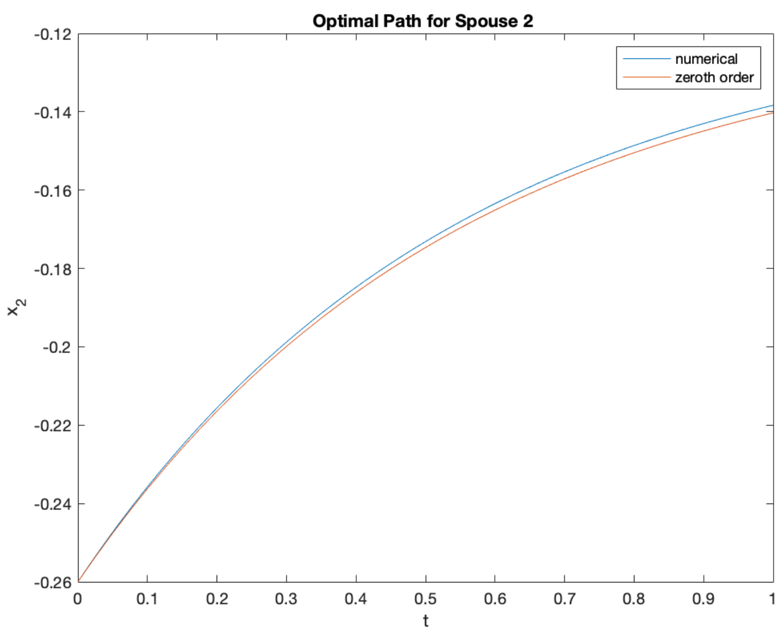
with appropriate terminal conditions. In figures 2.1 and 2.2, I have plotted the solutions for each spouse, λ_1, and μ_2. The numerical solution and zeroth-order perturbation scheme give close solutions. For the adjoint variables, they are virtually indistinguishable. It is important to remember that the perturbation scheme is only accurate for small ϵ. As ϵ increases, the accuracy of the zeroth order estimate decreases.

2.6.2 Example: Effect of Changing ϵ

The numerical results verify what I saw when studying the asymptotics. The interaction styles are validating for small ϵ, and as ϵ increases, the interaction style becomes more avoiding. Using the parameters from section 2.6.1, figures 2.3 and 2.4 demonstrate this result.

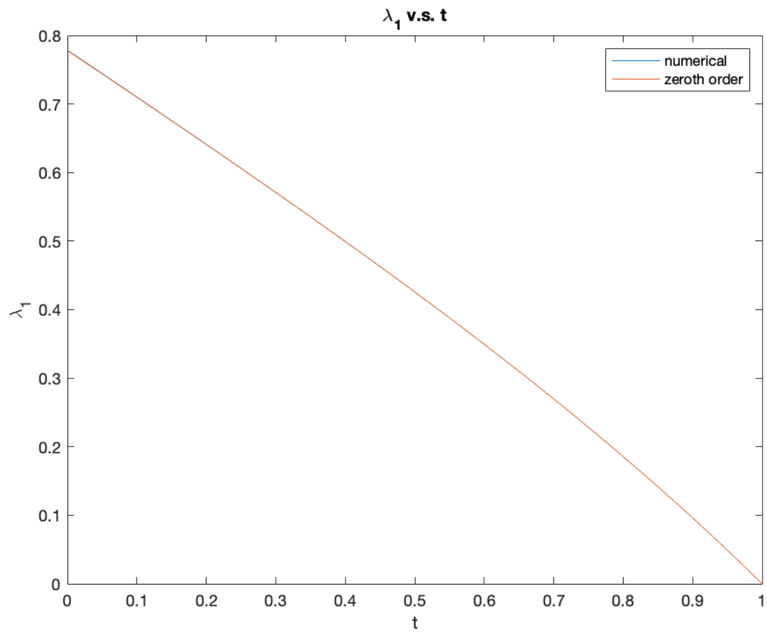


(a) Spouse one state-space for $\bar{x}_1 = 0.25$.

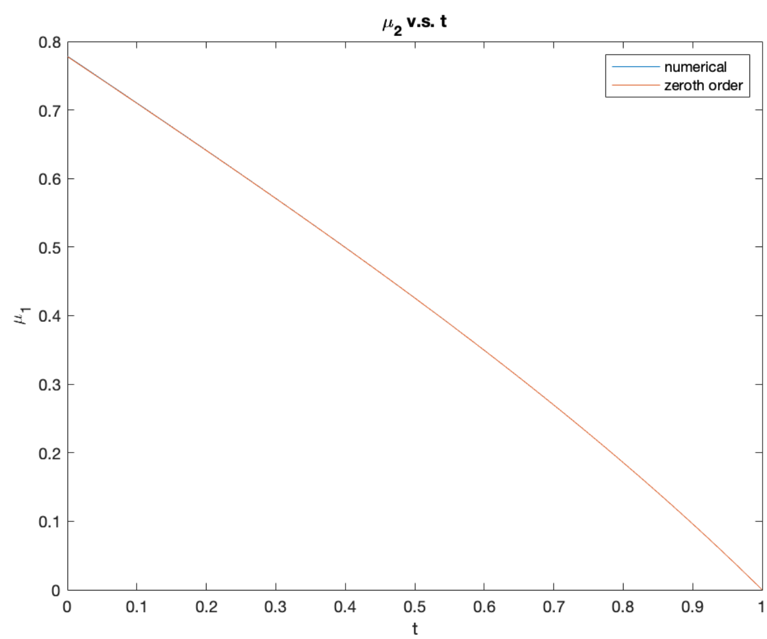


(b) Spouse two state-space for $\bar{x}_2 = -0.25$.

Figure 2.1: Comparison of numerical results to perturbation scheme for the state space.



(a) λ_1 space



(b) μ_2 space

Figure 2.2: Comparison of numerical results to perturbation scheme for two adjoint variables.

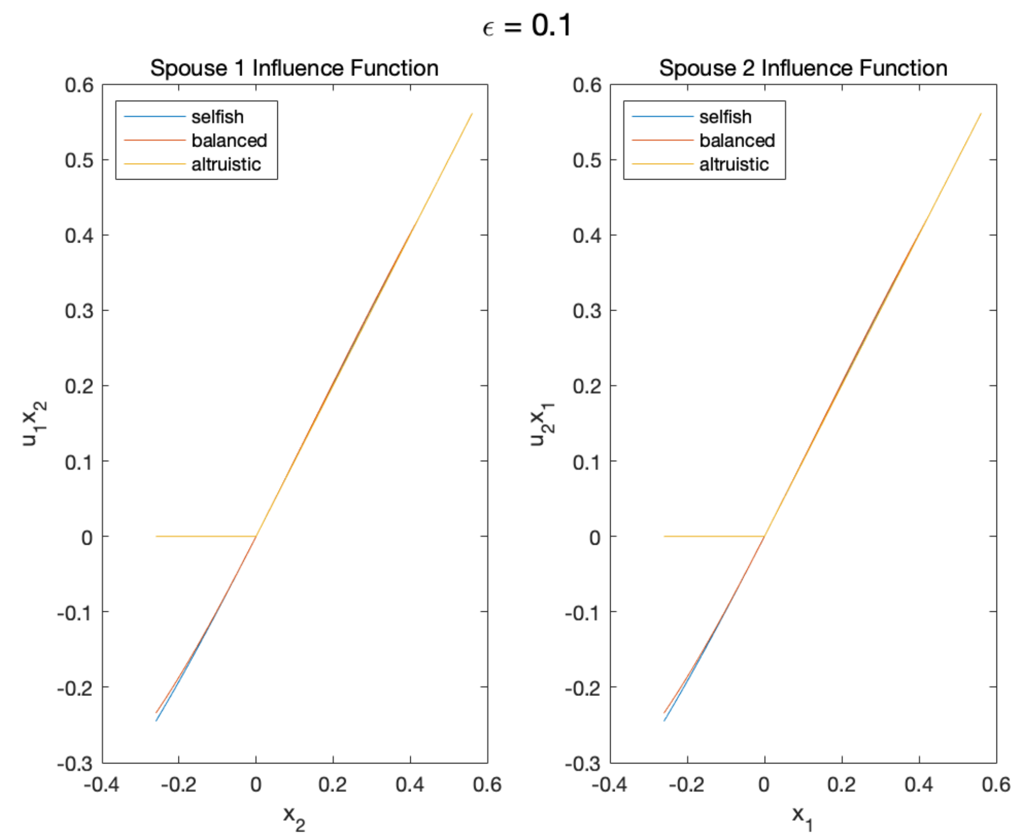


Figure 2.3: Influence functions when $\epsilon = 0.1$

For relationships when both couples have positive natural dispositions, the marriage is successful regardless of the marriage goal. Increasing ϵ only increases how happy the selfish and balanced couples become. This can be seen in figure 2.5.

More interesting results occur when one of the spouses has a negative disposition. The success of the marriage depends on the goal and the effort the spouse is willing to devote to marital interactions. Consider an example with parameters $r_1 = r_2 = 0.82$, $\bar{x}_1 = 0.26$, $\bar{x}_2 = -0.26$, $u_0 = 1$, $T = 3$, $x_i^0 = \bar{x}_i$. In figure 2.6a, with $\epsilon = 0.1$, only the altruistic marriage, where ϵ has no effect, is successful. The altruistic couple absorbs the emotional cost that comes from ignoring their spouse. In the long run, they can benefit.

Balanced and selfish marriages cannot overcome the emotional cost of ignoring their

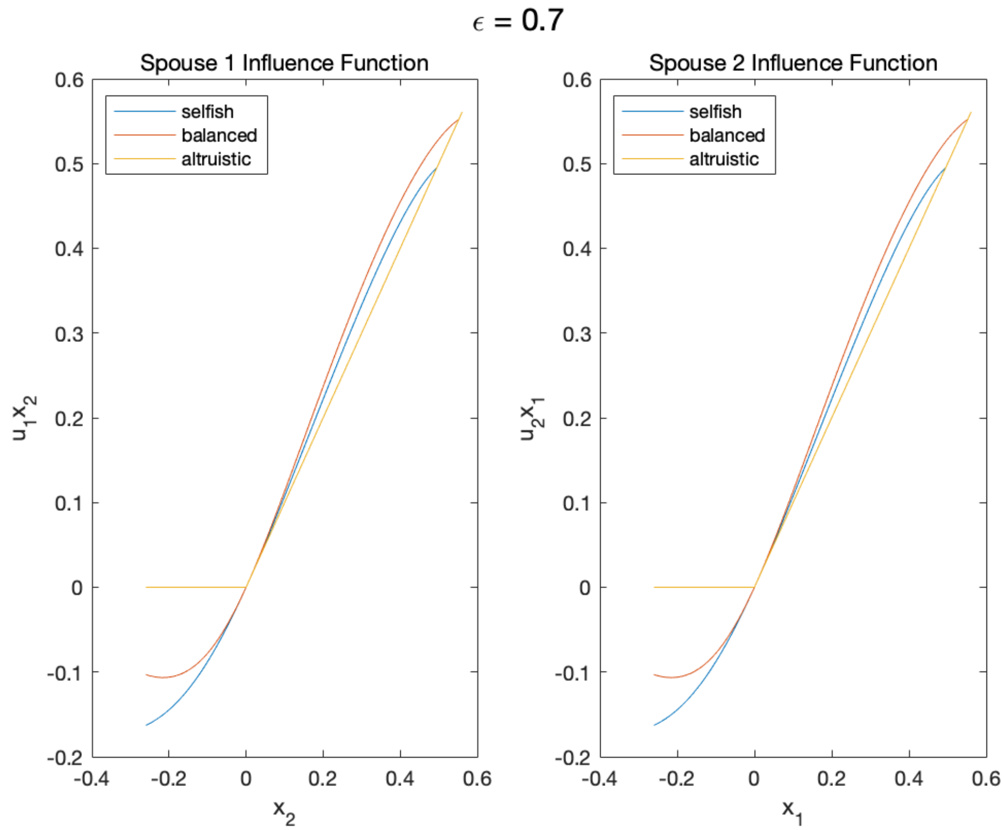


Figure 2.4: Influence functions when $\epsilon = 0.7$

spouse and are unsuccessful. However, they are still stable marriages because both couples remain positive; even though the naturally cheerful spouse is below their natural disposition, they sacrifice some of their happiness so the naturally negative spouse can be happy. This couple may not divorce because both spouses can eventually become positive.

Once I increase ϵ to 0.7, I see in figure 2.6b that the balanced marriage also becomes successful. The couple benefits more by also interacting with their negative range of emotions. Figure 2.7 shows how ϵ affects each marriage goal in a state space plot. For the higher value of ϵ, there is a more significant difference between the selfish and balanced relationship-goals.

2.7 Discussion

2.7.1 How Robust is the Model?

In Figure 2.8, I plot the average slope of the influence function for different initial conditions for intervals of time $t = 1$ for terminal times of $T = 5$ (in blue) and $T = 10$ (in red). Recall the optimal strategy for spouse each spouse in equation 2.26. The terminal condition $\lambda(T) = 0$ means that $\phi_i(T) = u_0$, regardless of the parameters. I did not want my results to be sensitive to this. For the work in this paper, I ran simulations with a terminal time of $T = 5$, but I only used values up to $t = 3$ to obtain the strategy for each spouse. The simulation shows that the slope is fairly consistent up to $t = 3$ for a terminal time of $T = 5$ and $t = 6$ for a terminal time of $T = 10$. This means that regardless of whether the conversation is 15 minutes or 30 minutes, the couples develop similar interaction styles and have the same outcome for the marriage. This is further evidence that Gottman's original research [17] using short conversations to predict the success of a marriage is a valid assumption.

Furthermore, Figure 2.8 shows a lot of variability in the slopes for times near the terminal time. Considering the optimal control always collapses to u_0, this may seem a bit surprising. The terminal condition allows for this variability. Figure 2.9 shows the influence function of spouse one on spouse two for two different sets of initial conditions. It confirms the results from figure 2.8 that early in the time interval, the slopes are similar, but at the terminal time, the slopes are quite disparate. One slope is -13 while the other is 5 while still exhibiting the drop due to the terminal condition. Figures 2.8 and 2.9 show that I are justified in calculating my slope values on sub-interval of $[0, 3]$ for $T = 5$.

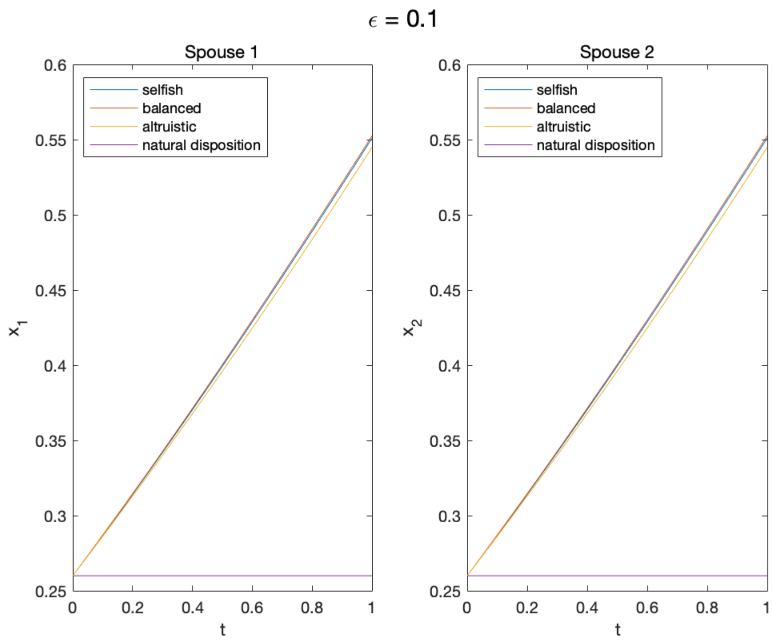
2.7.2 Concluding Remarks

My research aims to recover the interaction styles identified by [15] so that I are able to understand how couples evolve into their styles. These interaction styles were developed from research [17] observing how couples interact during a conflict. Three styles are considered stable, and two are considered unstable, meaning these couples are more likely to get divorced. The three stable relationships are volatile, avoiding, and validating. Volatile couples are characterized by interactions in their negative range of emotions and few during their positive range of emotions. Validating couples interact in both their positive and negative emotions, avoiders interact during their positive range of emotions, and few interact in their negative range. The two unstable relationship types are hostile and hostile detached. Hostile couples have one spouse who is a validator and the other who is an avoider. Hostile-detached couples have one spouse who is a validator and one who is volatile.

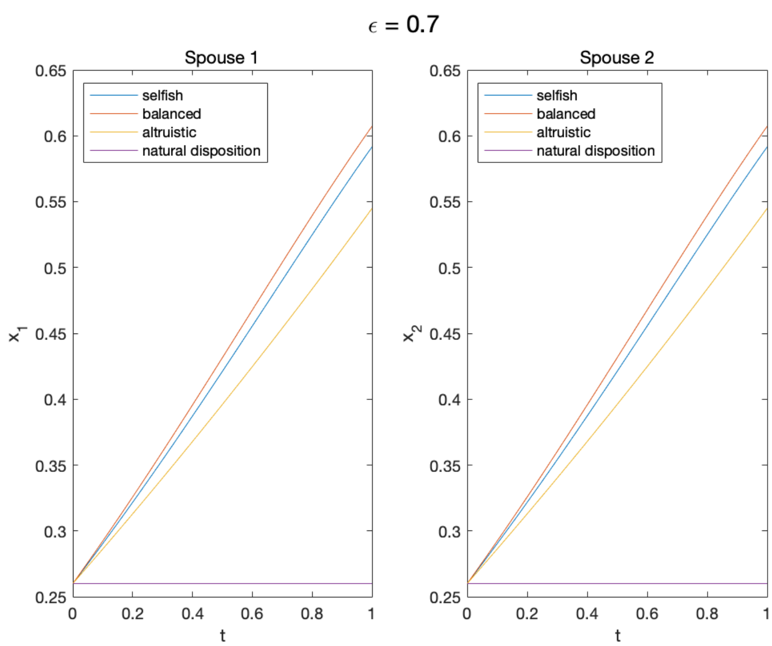
In the first paper of this series, I recovered the conflict-avoiding interaction-style. In this paper, I recover the validating interaction-style. By plotting the external portion of the optimal path of one spouse versus the positivity of the other spouse, I obtain the interaction style of a spouse. I use the qualitative descriptions of each type of couple and compare that to my graphs to classify the type of relationship my models have. I have shown that the penalty term ϵ is the key to differentiating between avoiding and validating marriages. While the natural disposition of the spouses is undoubtedly crucial in determining how happy a couple can be, the balance between the cost of interacting and the cost of ignoring one's spouse determines what interaction style a couple will evolve into.

My model does not include Gottman's volatile interaction-style, which has been observed to be present in couples with the highest natural disposition. Although this style is not an optimal solution, it is still capable of maintaining a stable marriage due to the high natural disposition of the couples. However, adopting one of my optimal strategies would have led to a happier marriage for these couples.

One limitation is that spouses can only respond to their spouses with their control variable. If a spouse can also influence their spouse's positivity directly, a negative u_i could be interpreted as cajoling their spouse when they are unhappy. I have begun to look into what occurs when a spouse can influence their spouse, and I have some preliminary results. In the next part of the paper, I use the model from this section to find strategies for marriage repair.

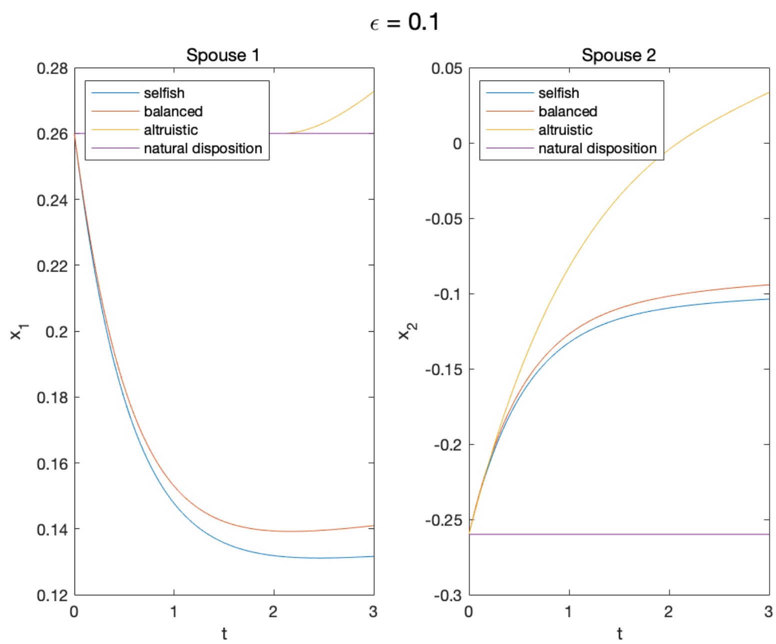


(a) $\epsilon = 0.1$

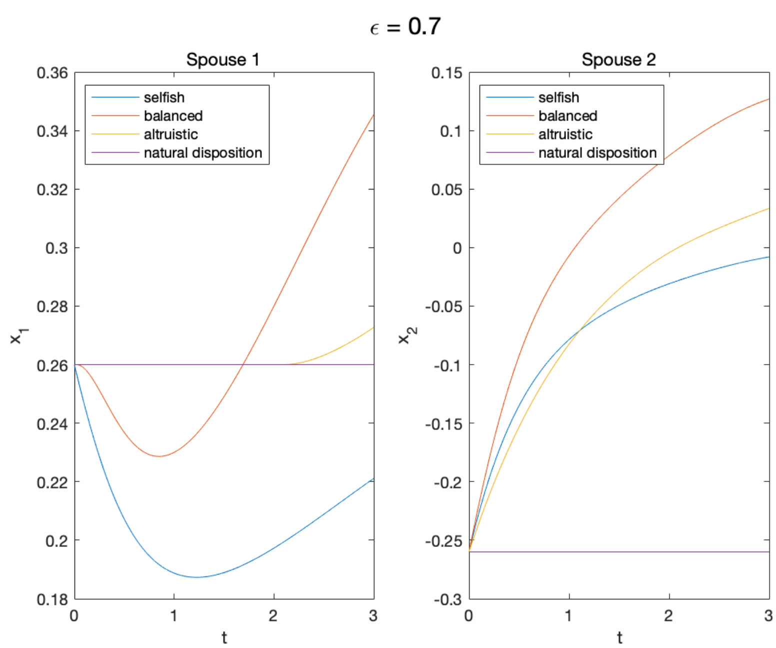


(b) $\epsilon = 0.7$

Figure 2.5: Comparison of state space for different values of ϵ for two naturally positive spouses.



(a) $\epsilon = 0.1$



(b) $\epsilon = 0.7$

Figure 2.6: Time series for state space of couple with opposite dispositions for different values of ϵ

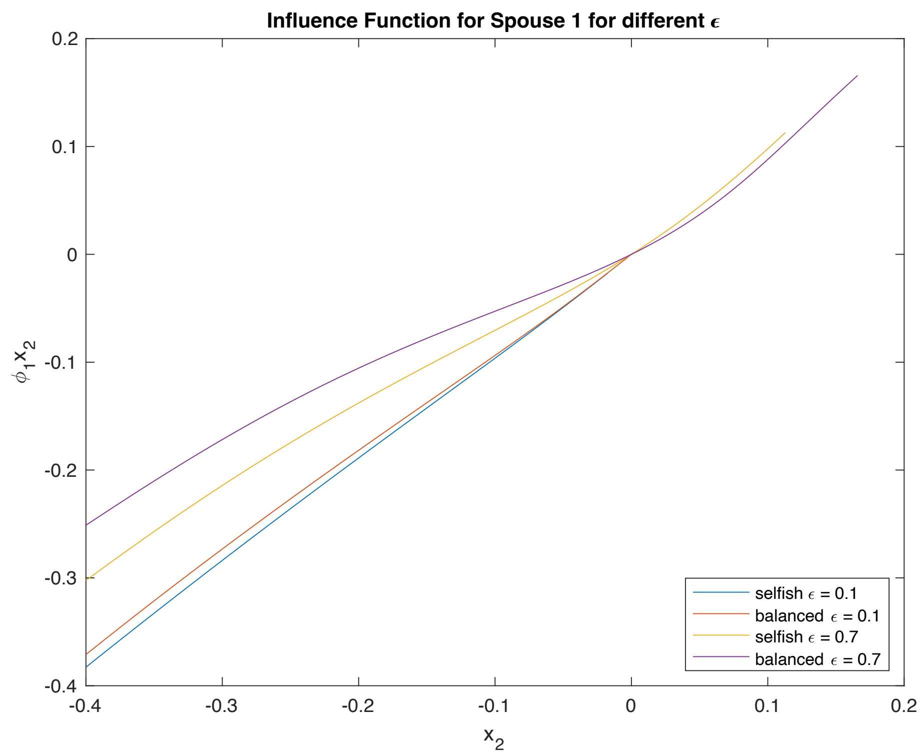


Figure 2.7: Influence functions for different ϵ for spouse 1.

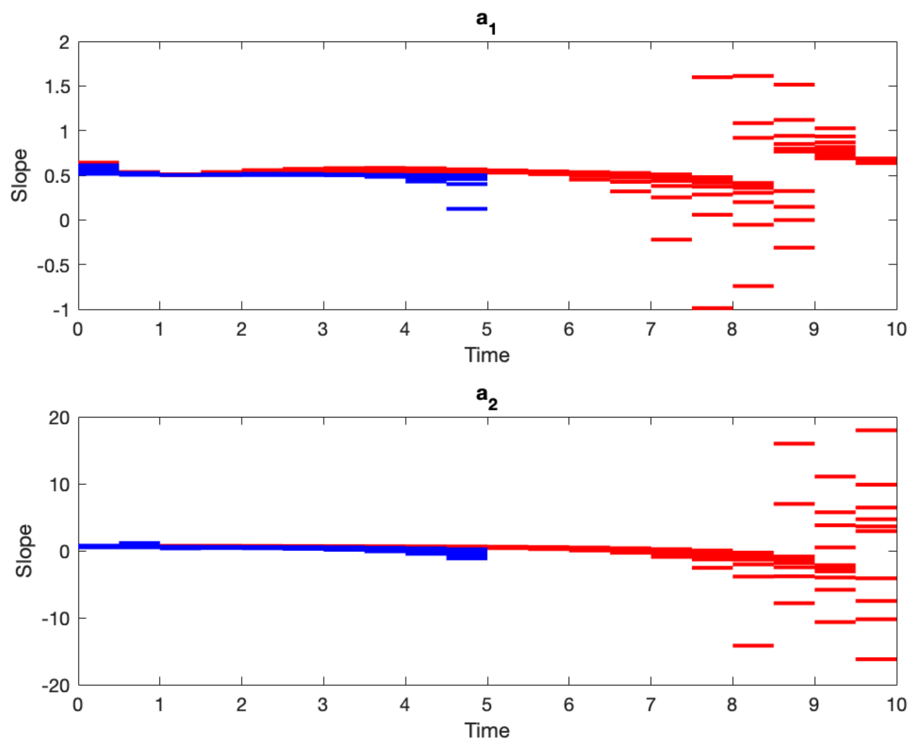


Figure 2.8: The slope calculated for time intervals of size 1 for terminal times of T = 5 and T = 10

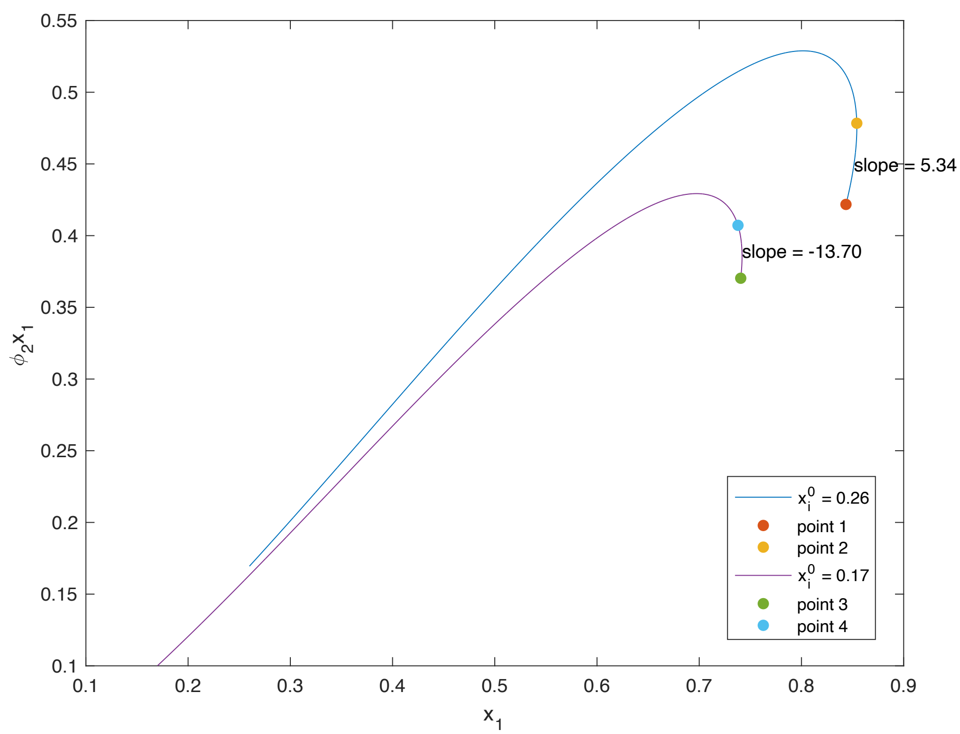


Figure 2.9: Influence function on spouse 2 for two sets of initial conditions.

Chapter 3

STRATEGIES FOR MARRIAGE REPAIR

3.1 Introduction

Now that I have established a model for determining how couples evolve into their interaction styles, I can use it to identify marriage-repair strategies. The methodology in this part captures decades of marital research. I can simulate couple interviews, determine interaction styles, and identify strategies for marriage repair, all with one model.

Cook et al. [13] constructed a mathematical theory to account for the research of Gottman and Levenson [14] that showed that initial observation of couples in interviews is sufficient to predict marital dissolution. Cook et al. hypothesized that marriages fail because they have differing interaction-styles. Their model is also highly sensitive to initial conditions, meaning that the positivity (or negativity) of the couple's interactions determines the conversation's outcome, as specified by the steady states. I improve on this method because I determine an equilibrium point that is consistent across initial conditions. This is an improvement because the interaction styles fall naturally out of the model presented in chapter 2 instead of being predetermined as they are in the Cook et al. model.

Recall that I developed a model for couples with validating interaction-styles in part II. I determined that including a term for emotional cost in the objective functional explains how couples may develop a validating style. In this part, I use the influence functions that arise for a spectrum of initial conditions to determine, on average, how a validating spouse with predetermined natural disposition and rate of return to natural disposition parameters responds to their spouse. Using these average values, I can find stable equilibrium-points that predict where the marriage is headed. An equilibrium in the first quadrant represents a stable marriage. An equilibrium point in the second or fourth quadrants may or may not be successful, depending on the natural dispositions of the spouses. An equilibrium in the third quadrant represents a failed marriage.

It is important to make the distinction in the terminology here. A stable equilibrium-point is different than a stable marriage. All of the equilibrium points I calculate here are asyptotically stable, even the equilibrium points of the failed marriages. I will be sure to make it clear when talking about the equilibrium point versus the stability of the marriage.

In this chapter, I find that if a couple has an interaction style that leads to an equilibrium point in the second or fourth quadrants, there may be an opportunity for the marriage to be repaired by changing their marriage goal and developing a different type of validating interaction-style. In section 3.2, I recap the model from part II and discuss how I will adapt it for marriage repair. In section 3.3, I outline the steps I use to get average interaction-style values and present previous research I use to determine the equilibrium points. In section 3.4, I show how couples who evolve into a conflict-avoiding interaction-style cannot repair their marriage without changing their interaction style to allow for an emotional cost. I also present an example of how a counselor could use my model to augment client sessions. Finally, in section 3.5, I summarize my results and present directions for future work.

3.2 Model

The model for this section is identical to the model in part II. The model is summarized here for convenience, but please refer to part II for a more detailed explanation.

The model is designed to be a differential-game-theory problem that consists of two objective functionals, one for each spouse, that are governed by state variables about the positivity of each spouse.

The goal of maximizing the objective functional for spouse i, $i = 1, 2$, is given by

$$\max_{u_i \geq 0} \; [J_i = \int_0^T F_i(x_1, x_2, u_1, u_2, t) dt], \tag{3.1}$$

where F_i is defined based

$$F_i = \alpha_i x_i + (1 - \alpha_i) x_j - \frac{1}{2\epsilon} [\alpha_i (u_i - u_0)^2 + (1 - \alpha_i)(u_j - u_0)^2], \tag{3.2}$$

subject to state variables,

$$\frac{dx_1}{dt} = r_1(\bar{x}_1 - x_1) + u_1 x_2, \tag{3.3a}$$

$$\frac{dx_2}{dt} = r_2(\bar{x}_2 - x_2) + u_2 x_1, \tag{3.3b}$$

with initial conditions. I know this model results in the optimal control given by

$$\phi_i = \frac{\epsilon}{\alpha_i} \lambda_{i,i} x_j + u_0 \tag{3.4}$$

and is a validating interaction-style.

The optimal control changes based on the initial conditions provided to the model. So, I want to find the average optimal control across a range of initial conditions to determine the underlying influence function. More precisely, I define

$$I_i(x_j) = \begin{cases} a_i x_j, x_j > 0 \\ \\ b_i x_j, x_j < 0, \end{cases} \tag{3.5}$$

and my optimal state variables are given by

$$\frac{dx_1}{dt} = r_1(\bar{x}_1 - x_1) + I_1(x_2), \tag{3.6a}$$

$$\frac{dx_2}{dt} = r_2(\bar{x}_2 - x_2) + I_2(x_1). \tag{3.6b}$$

I must use my simulation to find the a_i and b_i parameters. They are found using the average response across initial conditions. The stable equilibrium point of this system will predict the future of the marriage.

3.3 Method

The goal is to model Gottman's couple interviews to obtain values for interaction style. Then, use the interaction style to predict the long-term success of the marriage. This is a multi-step process.

1. Using the numerical method outlined in part II, I run the simulation for 100 initial condition pairs where x_i^0 is in the interval $[-2\bar{x}_i, 2\bar{x}_i]$. This is so that I can determine how each spouse reacts to their spouse in both the positive and negative ranges of emotion.

2. I plot the resulting influence functions for each of these initial condition pairs. The influence functions are given by $\phi_j x_i$ versus x_i. This function shows how spouse j reacts to spouse i throughout spouse i's range of emotions.

3. Next, I find the linear best-fit lines for $x_i < 0$ and $x_i > 0$ separately from the influence function graphs. Let a_i be the slope of the best-fit line for the influence function of spouse i when $x_j < 0$. Let b_i be the slope of the best-fit line for the influence function of spouse i when $x_j > 0$. Therefore, define the optimal strategy for spouse i be

$$\phi_i = \begin{cases} a_i, & x_j < 0 \\ b_i, & x_j > 0. \end{cases} \tag{3.7}$$

4. Finally, I calculate the asymptotically stable equilibrium-point for each couple to find the predicted state of the marriage. From Tung [44], I know a set of mutually exclusive conditions determines the stable equilibrium point:

   (a) If $\bar{x}_1 + \frac{a_1}{r_1}\bar{x}_2 > 0$ and $\bar{x}_2 + \frac{a_2}{r_2}\bar{x}_1 > 0$, then

   $$\left(\frac{\bar{x}_1 + \frac{a_1}{r_1}\bar{x}_2}{1 - \frac{a_1 a_2}{r_1 r_2}}, \frac{\bar{x}_2 + \frac{a_2}{r_2}\bar{x}_1}{1 - \frac{a_1 a_2}{r_1 r_2}} \right) \tag{3.8}$$

   is the equilibrium point.

   (b) If $\bar{x}_1 + \frac{a_1}{r_1}\bar{x}_2 < 0$ and $\bar{x}_2 + \frac{b_2}{r_2}\bar{x}_1 > 0$, then

   $$\left(\frac{\bar{x}_1 + \frac{a_1}{r_1}\bar{x}_2}{1 - \frac{a_1 b_2}{r_1 r_2}}, \frac{\bar{x}_2 + \frac{b_2}{r_2}\bar{x}_1}{1 - \frac{a_1 b_2}{r_1 r_2}} \right) \tag{3.9}$$

   is the equilibrium point.

   (c) If $\bar{x}_1 + \frac{b_1}{r_1}\bar{x}_2 < 0$ and $\bar{x}_2 + \frac{b_2}{r_2}\bar{x}_1 > 0$, then

   $$\left(\frac{\bar{x}_1 + \frac{b_1}{r_1}\bar{x}_2}{1 - \frac{b_1 b_2}{r_1 r_2}}, \frac{\bar{x}_2 + \frac{b_2}{r_2}\bar{x}_1}{1 - \frac{b_1 b_2}{r_1 r_2}} \right) \tag{3.10}$$

   is the equilibrium point.

   (d) If $\bar{x}_1 + \frac{b_1}{r_1}\bar{x}_2 > 0$ and $\bar{x}_2 + \frac{a_2}{r_2}\bar{x}_1 > 0$, then

   $$\left(\frac{\bar{x}_1 + \frac{b_1}{r_1}\bar{x}_2}{1 - \frac{b_1 a_2}{r_1 r_2}}, \frac{\bar{x}_2 + \frac{a_2}{r_2}\bar{x}_1}{1 - \frac{b_1 a_2}{r_1 r_2}} \right) \tag{3.11}$$

is the equilibrium point.

3.4 Results

To analyze the effectiveness of this model, I define some terms. I use precise mathematical definitions for stability and success for each spouse. Each spouse in the relationship can have a different interpretation of the marriage. A spouse will consider the marriage successful if they are happier in marriage than alone. That is, their stable equilibrium point x_i^* is greater than their natural disposition $\bar{x}_i$. A spouse will consider their marriage stable if their equilibrium point is positive.

As explained in part 2, this model always gives rise to a validating interaction-style. The validating style is always successful for couples where both spouses are naturally positive. This is clear from the equilibrium points determined in Tung [44] that are defined in the method section. However, when couples have opposite interaction-styles, changing the marriage goal can determine whether or not this validating couple is successful.

Ideally, both spouses consider the marriage to be successful. That will not always be possible when couples have opposite dispositions. It is unlikely that the positive spouse will have an equilibrium point greater than their natural disposition if they marry someone with a negative natural disposition. This does not automatically mean the couple is destined for divorce. If the positive spouse considers the marriage stable while the negative spouse considers the marriage successful, I predict that the couple will not divorce. I examine a specific example of this type later on in the section. I also explore the robustness of the model, but first, I examine why conflict-avoiding couples cannot repair their marriage while maintaining their interaction style.

3.4.1 Case 1: No cost, no repair

Recall from part 1 that when I do not consider the cost of spousal interaction in the objective functional, I get an optimal strategy of

$$\phi_i = \begin{cases} 0, \ \ x_j < 0, \\ u_{i,max}, \ \ x_j > 0. \end{cases} \tag{3.12}$$

This corresponds to a marriage goal of

$$F = \alpha x_1 + (1-\alpha)x_2 - \frac{1}{2\epsilon}[\alpha(u_1 - u_0)^2 + (1-\alpha)(u_2 - u_0)^2] \tag{3.13}$$

where $\epsilon \to \infty$, thus eliminating the terms containing u_1 and u_2 which represent the emotional cost. This type of marriage is more work for the spouse because it requires the most amount of effort. It is not that they do not have an emotional cost, it is that they do not care what the cost is. In this case with $\epsilon \to \infty$, the optimal strategy is the same regardless of the marriage goal. It is a conflict avoiding marriage. Therefore, there are no opportunities for marriage repair via changing the marriage goal.

3.4.2 Case 2: Can opposites attract?

A couple with opposite dispositions ($\bar{x}_1 = -0.26, \bar{x}_2 = 0.12$) comes in to see a counselor. Spouse 2 initiated the decision, but neither spouse is delighted with the marriage. After speaking with the couple, the counselor predicts that the marriage is destined for failure and wants to make suggestions for repair.

The counselor can turn to my simulation to augment their recommendation. From the session, the counselor gleaned that the couple has a marriage goal

$$F_i = \frac{9}{10}x_1 + \frac{1}{10}x_2 - \frac{1}{2\epsilon}[\frac{9}{10}(u_1 - u_0)^2 + \frac{1}{10}(u_2 - u_0)^2] \tag{3.14}$$

with $\epsilon = 0.3, T = 5, r_1 = r_2 = 0.8$, and $u_0 = 0.5$. Spouse two is overcompensating for spouse one's negativity. After the simulation, the counselor verifies their prediction and determines that the couple is probably headed for an equilibrium point in the third quadrant at $(-0.27, -0.01)$. Figure 3.1 shows the couple's strategy to reach this point in their marriage, as determined by the simulation.

The simulation reveals that the best marriage this couple could achieve would be for the marriage to be successful for spouse one and stable for spouse two. More specifically, spouse one is happier in marriage than alone, and spouse two maintains a natural disposition that is already positive. This result occurs whenever $\alpha_2 = 0$ and α_1 is any feasible value, and the marriage goal is

$$F_2 = x_1 - \frac{1}{2\epsilon}(u_1 - u_0)^2, \tag{3.15}$$

for spouse two, which I know from part II, has an optimal control that is conflict-avoiding. Figure 3.2 shows the strategy that evolves from this marriage goal. This is an intuitive result, and I expect the positive spouse to support their negative spouse to result in the best possible marriage. However, I know from part I that this strategy is difficult and takes the most effort, especially for a couple who naturally adopted a validating interaction-style.

The subsequent best outcome for spouse two occurs for $\alpha_1 = 1$ and $\alpha_2 = 1$, where spouses prioritize their happiness. On the surface, this is surprising. Both spouses being selfish does not sound like a healthy way to repair a marriage. However, in this example, the positive spouse is less positive than the negative spouse is negative. It makes sense that if the positive spouse can prioritize their own happiness and let their natural positivity do its job, they will have better interactions with their spouse and make for a happier marriage. This is evidenced in the equilibrium point of $(-0.24, 0.03)$. Like the best outcome, this relationship is successful for spouse one and stable for spouse two. Spouse one can adopt any style while spouse two maintains $\alpha_2 \geq 0.4$ to achieve this same result.

Now, armed with this extra information from the simulation, the counselor is able to offer a number of solutions to the couple. From continued sessions, they can determine what is the best suggestion to help repair this marriage.

3.5 Discussion

I sought to use the models I developed in parts I and II to identify strategies for marriage repair for couples with opposite natural dispositions. I explained that couples who do not care about emotional costs do not have a way to repair their marriage. I provided an example of how marriage counselors could use my model to better support their clients by instructing them on how changing their priorities can improve their marriage.

Updating the strategy by observing the spouse's mood would provide more robust marriage repair strategies. However, closed-loop problems are much more challenging if they can be solved. As a next step, I would look more into a dynamic-programming approach to this problem. That still may not be solved, so I must reformulate the model into a solvable closed-loop problem. Furthermore, the Gottman [17] experiments are open-loop since they make predictions from initial conditions observed in the couples' interviews; thus, I can directly compare my results to this study.

There is much opportunity for future work from what I have presented her. Firstly, there are more asymptotic solutions to explore. The problem where ϵ is not exactly infinity but near infinity is a singular perturbation problem. Many approaches could be applied to solving these types of problems. Next, I only studied inter-spousal communication for external influences. Many other factors in a marriage may affect each spouse's positivity, such as other family members, finances, and careers. I could study how these factors influence how spouses interact with one another.

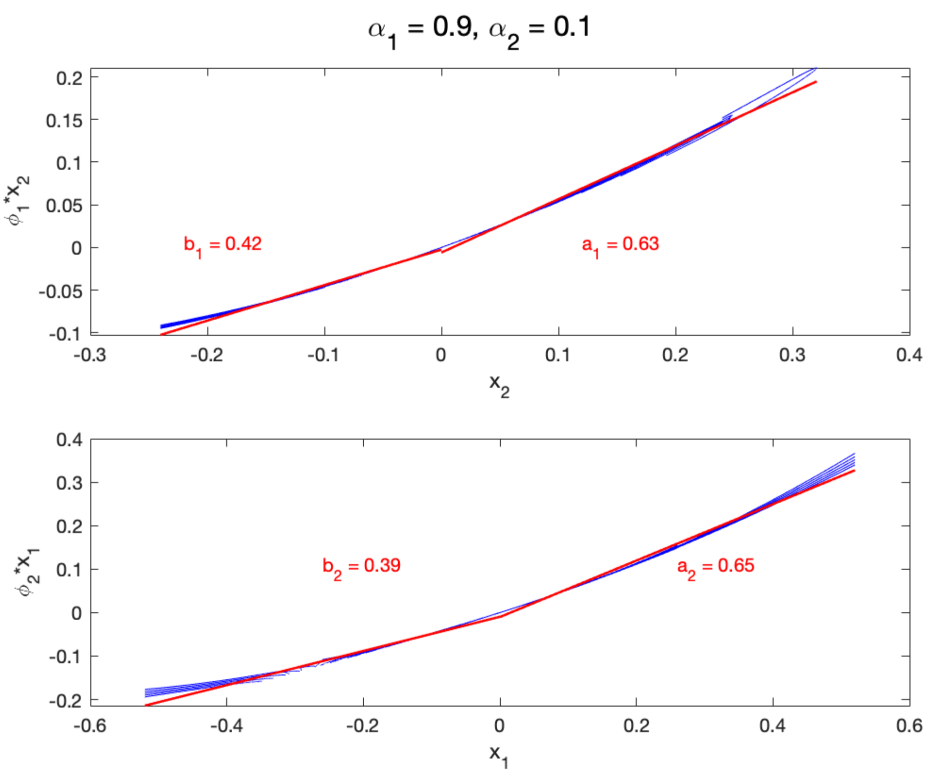


Figure 3.1: This is the couple's strategy when they come to therapy.

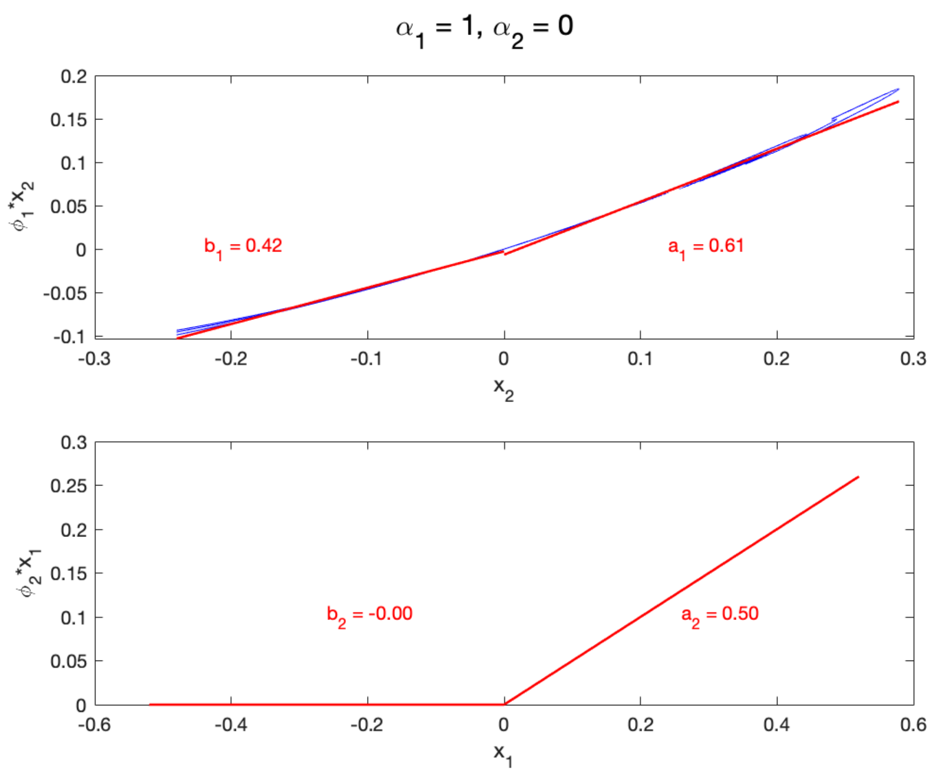


Figure 3.2: This strategy results when the more positive spouse adopts an altruistic goal. It is the best possible outcome but the hardest to achieve.

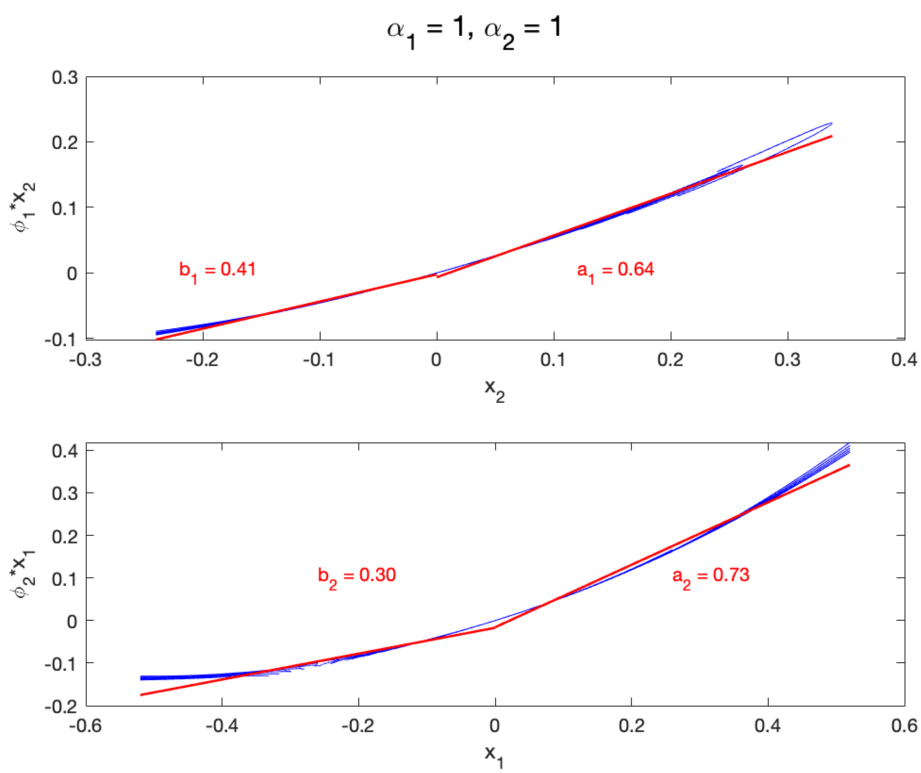


Figure 3.3: This is the easiest stable marriage the couple can achieve.

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Appendix A
APPENDIX

A.1 *Proof that* $\lambda_i > 0$

We prove this fact by exhausting all possible solutions of this problem. This can be broken into three cases:

1. $\phi_i = u_{i,max}$
2. $\phi_i = u_{i,max}, \phi_j = 0$
3. $\phi_i = 0,$

with $i = 1, 2$ and $j \neq i$.

A.1.1 Case 1: $\phi_i = u_{i,max}$

To prove this case, we perform stability analysis on the adjoint system in equations 1.22c and 1.22d. This system has an equilibrium point at

$$\left(\frac{(1-\alpha)\phi_2 + \alpha r_2}{r_1 r_2 - \phi_1\phi_2}, \frac{(1-\alpha)r_1 + \alpha\phi_1}{r_1 r_2 - \phi_1\phi_2}\right). \tag{A.1}$$

The adjoint system has characteristic polynomial $\mu^2 - (r_1 + r_2)\mu + r_1 r_2 - ua = 0$. The roots of the characteristic polynomial are

$$\mu_{\pm} = \frac{r_1 + r_2 \pm \sqrt{(r_1 + r_2)^2 - 4(r_1 r_2 - \phi_1\phi_2)}}{2}. \tag{A.2}$$

The corresponding eigenvectors s are given by

$$v = \begin{bmatrix} \frac{\phi_2}{r_1 - \mu_+} \\ 1 \end{bmatrix}, \tag{A.3}$$

$$w = \begin{bmatrix} \frac{\phi_2}{r_1 - \mu_-} \\ 1 \end{bmatrix}. \tag{A.4}$$

There are three possible classifications of the equilibrium point. They are defined by what we call classification regions:

1. Where $\phi_1\phi_2 > r_1r_2$, the equilibrium point is a saddle point since this is where $\mu_+ > 0$ and $\mu_- < 0$.

2. Where $\frac{(r_1-r_2)^2}{-4} < \phi_1\phi_2 < r_1r_2$, the equilibrium point is an unstable node because this is the region for which $\mu_\pm$ are positive.

3. Where $\phi_1\phi_2 < \frac{(r_1-r_2)^2}{-4}$, the equilibrium point is an unstable spiral. This is where $\mu_\pm$ are complex.

Because $u_{i,max} > 0$, we are only concerned with regions 1 and 2.

$u_{1,max}u_{2,max} > r_1r_2$

We have a saddle point. The numerator of the equilibrium point is always positive, but the sign of the denominator depends on the classification of the point. Here, the denominator is negative, so we can conclude that the point must be in the third quadrant. The eigenvalues given in equation A.2 have $\mu_+ > 0$ and $\mu_- < 0$. This tells us that eigenvector v gives us the equation of the unstable manifold, and w gives is the equation of the stable manifold. Therefore, v has a negative slope, and w has a positive slope.

The slopes of the stable and unstable manifolds and the location of the equilibrium point in the third quadrant make it clear that any trajectory that will satisfy that transversality condition must originate in the first quadrant.

$0 < u_{1,max}u_{2,max} < r_1r_2$

These conditions tell us that the equilibrium point A.1 is an unstable node in the first quadrant. In order to satisfy the terminal conditions, λ_1 and λ_2 must begin in the first quadrant because beginning in any other quadrant would send the trajectories away from the first quadrant, and by extension also the origin.

A.1.2 Case 2: $\phi_i = u_{i,max}, \phi_j = 0$

Without loss of generality, assume $i = 1$ and $j = 2$. The sign of λ_2 does not matter because spouse 2 is just heading to their natural disposition. Thus, we just need to show that $\lambda_1 > 0$. We can rewrite the adjoint equation to reflect this conditions. We get

$$\dot{\lambda}_1 = -\alpha + \lambda_1 r_1 \tag{A.5}$$

Solving equation A.5 with the terminal condition,

$$\lambda_1(t) = \alpha \frac{1 - e^{r_1(t-T)}}{r_1}. \tag{A.6}$$

If $\lambda_1 < 0$, then

$$\begin{aligned}
\alpha \frac{1 - e^{r_1(t-T)}}{r_1} &< 0 \\
1 - e^{r_1(t-T)} &< 0 \\
1 &< e^{r_1(t-T)} \\
0 &< t - T \\
T &< t,
\end{aligned}$$

which is never the case in our problem since T is the terminal time.

A.1.3 Case 3: $\phi_i = 0$

Both spouses head to their natural dispositions, and there is no interaction.

A.2 Singular Solution

Now, we must determine whether or not there is a singular solution. For ϕ_1 to be singular, $\lambda_1 x_2 \equiv 0$ for an interval of time. This means that $\lambda_1 = 0$ or $x_2 = 0$. First, we will consider the case for $\lambda_1 = 0$.

If $\lambda_1 = 0$, but $\dot{\lambda}_1 \neq 0$, then ϕ_1 cannot be singular. Therefore, $\dot{\lambda}_1 = 0$ means

$$0 = \dot{\lambda}_1 = -\alpha - \lambda_2\phi_2, \tag{A.7}$$

$$\dot{\lambda}_2 = -(1-\alpha) + \lambda_2 r_2, \tag{A.8}$$

$$\text{implies } \lambda_2(t) = \frac{1-\alpha}{r_2} - \frac{1-\alpha}{r_2}e^{r_2(t-T)}, \tag{A.9}$$

$$0 = -\alpha - \phi_2[\frac{1-\alpha}{r_2}, -\frac{1-\alpha}{r_2}e^{r_2(t-T)}]. \tag{A.10}$$

If $\alpha = 0$,

$$0 = -\phi_2[\frac{1}{r_2} - \frac{1}{r_2}e^{r_2(t-T)}], \tag{A.11}$$

$$\frac{\phi_2}{r_2} = \frac{\phi_2}{r_2}e^{r_2(t-T)}, \tag{A.12}$$

$$1 = e^{r_2(t-T)}, \tag{A.13}$$

$$t = T. \tag{A.14}$$

This is not an interval.

If $\alpha = 1$,

$$0 = -1. \tag{A.15}$$

$$\tag{A.16}$$

This is a contradiction. If $\alpha \in (0,1)$,

$$0 = -\alpha - \phi_2[\frac{1-\alpha}{r_2} - \frac{1-\alpha}{r_2}e^{r_2(t-T)}], \tag{A.17}$$

$$\phi_2 = \frac{\alpha}{-\frac{1-\alpha}{r_2} + \frac{1-\alpha}{r_2}e^{r_2(t-T)}}. \tag{A.18}$$

$$\text{We need} 0 < \frac{\alpha}{-\frac{1-\alpha}{r_2} + \frac{1-\alpha}{r_2}e^{r_2(t-T)}} < M = u_{2,max} \tag{A.19}$$

$$-\frac{1-\alpha}{r_2} + \frac{1-\alpha}{r_2}e^{r_2(t-T)} > 0 \tag{A.20}$$

$$e^{r_2(t-T)} > 1 \tag{A.21}$$

$$r_2(t-T) > 0 \tag{A.22}$$

$$t > T. \tag{A.23}$$

$$\tag{A.24}$$

This is a contradiction because $t \in [0, T]$. Therefore, $\lambda_1 \neq 0$ for an interval of time.

For ϕ_2 to be singular, $\lambda_2 x_1 \equiv 0$. The analysis for λ_2 is similar to the analysis for λ_1, so we come to the same conclusion that $\lambda_2 \neq 0$ for an interval of time. The remaining possible cases are

1. $x_2 = 0, x_1 = 0$ and ϕ_1, ϕ_2 both singular,

2. $x_2 = 0$, ϕ_1 singular, and $\phi_2 = 0$ or $\phi_2 = u_{2,max} = M_2$, and

3. $x_1 = 0$, ϕ_2 singular, and $\phi_1 = 0$ or $\phi_1 = u_{1,max} = M_1$.

We will examine these cases one at a time.

1. Case: $x_2 = 0, x_1 = 0$ and ϕ_1, ϕ_2 both singular

   The state equations are reduced to

$$0 = r_1 \bar{x}_1 \tag{A.25a}$$

$$0 = r_2 \bar{x}_2. \tag{A.25b}$$

   The parameters r_1 and r_2 are never equal to zero. We are not interested in the cases where $\bar{x}_1$ or $\bar{x}_2$ are equal to zero.

2. Case: $x_2 = 0$, ϕ_1 singular, and $\phi_2 = 0$ or $\phi_2 = u_{2,max} = M_2$ This means:

$$\dot{x}_1 = r_1(\bar{x}_1 - x_1) \tag{A.26}$$

$$\dot{x}_2 = r_2 \bar{x}_2 + \phi_2 x_1. \tag{A.27}$$

   Thus, we can solve for x_1.

$$x_1(t) = \bar{x}_1 + (x_1^0 - \bar{x}_1)e^{-r_1 t} \tag{A.28}$$

   Plugging this into the equation for x_2,

$$\dot{x}_2 = r_2 \bar{x}_2 + \phi_2[\bar{x}_1 + (x_1^0 - \bar{x}_1)e^{-r_1 t}]. \tag{A.29}$$

   This is not true for an interval if $\dot{x}_2 \neq 0$ at $x_2 = 0$. Therefore, $\dot{x}_2 = 0$

$$0 = r_2\bar{x}_2 + u_2[\bar{x}_1 + (x_1^0 - \bar{x}_1)e^{-r_1 t}] \tag{A.30}$$

$$\phi_2 = \frac{-r_2\bar{x}_2}{\bar{x}_1 + (x_1^0 - \bar{x}_1)e^{-r_1 t}} = M_2. \tag{A.31}$$

For most parameters, equation A.30 only applies to specific parameters. However, if $x_1^0 = \bar{x}_1$ and

$$-r_2\bar{x}_2 = M_2\bar{x}_1, \tag{A.32}$$

ϕ_1 will be a singular solution while ϕ_2 is not.

3. Case: $x_1 = 0$, ϕ_2 singular, and $\phi_1 = 0$ or $\phi_1 = u_{1,max} = M_1$

   This is similar to the previous case. If $x_2^0 = \bar{x}_1$ and

$$-r_1\bar{x}_1 = M_1\bar{x}_2, \tag{A.33}$$

   ϕ_2 will be singular while ϕ_1 is not.

A.3 Code

```
function g = mp4(r1,x0,r2,y0,x_i,y_i,u1_max, u2_max, alpha1,
    alpha2, e1, e2,u0, T)
test = -1;
delta = 0.0001;
N = round(T*1000);



t = linspace(0,T,N+1);
h = T/N;
h2 = h/2;

u1 = zeros(1,N+1);
u2= zeros(1,N+1);
u1_temp = zeros(1,N+1);
u2_temp= zeros(1,N+1);
switchfn1 = zeros(1,N+1);
switchfn2 = zeros(1,N+1);

x = zeros(1,N+1);
y = zeros(1,N+1);
x(1) = x_i;
y(1) = y_i;



lambda1 = zeros(1,N+1);
lambda2 = zeros(1,N+1);
mu1 = zeros(1,N+1);
mu2 = zeros(1,N+1);
```

```
while test < 0
    oldu1 = u1;
    oldu2 = u2;
    oldx = x;
    oldy = y;
    oldlambda1 = lambda1;
    oldlambda2 = lambda2;
    oldmu1 = mu1;
    oldmu2 = mu2;
%runge kunge 4 solving x and y forward in time
    for i = 1:N
        k1x = r1*(x0-x(i)) + u1(i)*y(i);
        k1y = r2*(y0-y(i)) + u2(i)*x(i);

        k2x = r1*(x0-(x(i)+h2*k1x)) + 0.5*(u1(i)+u1(i+1))*(y(i)+h2
           *k1y);
        k2y = r2*(y0-(y(i)+h2*k1y)) + 0.5*(u2(i)+u2(i+1))*(x(i)+h2
           *k1x);

        k3x = r1*(x0-(x(i)+h2*k2x)) + 0.5*(u1(i)+u1(i+1))*(y(i)+h2
           *k2y);
        k3y = r2*(y0-(y(i)+h2*k2y)) + 0.5*(u2(i)+u2(i+1))*(x(i)+h2
           *k2x);

        k4x = r1*(x0-(x(i)+h*k3x)) + u1(i+1)*(y(i)+h*k3y);
        k4y = r2*(y0-(y(i)+h*k3y)) + u2(i+1)*(x(i)+h*k3x);

        x(i+1) = x(i) + (h/6)*(k1x + 2*k2x + 2*k3x + k4x);
```

```
    y(i+1) = y(i) + (h/6)*(k1y + 2*k2y + 2*k3y + k4y);
end

% runge kutta solving lambda and mu backward in time
for i = 1:N
    j = N + 2 - i;
    k1l1 = -alpha1 + lambda1(j)*r1 - u2(j)*lambda2(j);
    k1l2 = -(1-alpha1)-lambda1(j)*u1(j) + lambda2(j)*r2;
    k1m1 = -(1-alpha2) + mu1(j)*r1 - mu2(j)*u2(j);
    k1m2 = -alpha2 -u1(j)*mu1(j)+mu2(j)*r2;



    k2l1 = -alpha1 + (lambda1(j)-h2*k1l1)*r1 - (lambda2(j)-h2*
        k1l2)*0.5*(u2(j)+u2(j-1));
    k2l2 = -(1-alpha1)-(lambda1(j)-h2*k1l1)*0.5*(u1(j)+u1(j-1)
        ) + (lambda2(j)-h2*k1l2)*r2;
    k2m1 = -(1-alpha2) + (mu1(j)-h2*k1m1)*r1 - (mu2(j)-h2*k1m2
        )*0.5*(u2(j)+u2(j-1));
    k2m2 = -alpha2 -0.5*(u1(j)+u1(j-1))*(mu1(j)-h2*k1m1) +(mu2
        (j)-h2*k1m2)*r2;

    k3l1 = -alpha1 + (lambda1(j)-h2*k2l1)*r1 - (lambda2(j)-h2*
        k2l2)*0.5*(u2(j)+u2(j-1));
    k3l2 = -(1-alpha1)-(lambda1(j)-h2*k2l1)*0.5*(u1(j)+u1(j-1)
        ) + (lambda2(j)-h2*k2l2)*r2;
    k3m1 = -(1-alpha2) + (mu1(j)-h2*k2m1)*r1 - (mu2(j)-h2*k2m2
        )*0.5*(u2(j)+u2(j-1));
    k3m2 = -alpha2-0.5*(u1(j)+u1(j-1))*(mu1(j)-h2*k2m1) +(mu2(
        j)-h2*k2m2)*r2;
```

```
        k4l1 = -alpha1 + (lambda1(j)-h*k3l1)*r1 - (lambda2(j)-h*
            k3l2)*u2(j-1);
        k4l2 = -(1-alpha1)-(lambda1(j)-h*k3l1)*u1(j-1) + (lambda2(
            j)-h*k3l2)*r2;
        k4m1 = -(1-alpha2) + (mu1(j)-h*k3m1)*r1 - (mu2(j)-h*k3m2)*
            u2(j-1);
        k4m2 = -alpha2-(mu1(j)-h*k3m1)*u1(j)+(mu2(j)-h*k3m2)*r2;

        lambda1(j-1) = lambda1(j)-(h/6)*(k1l1 + 2*k2l1 + 2*k3l1 +
            k4l1);
        lambda2(j-1) = lambda2(j)-(h/6)*(k1l2 + 2*k2l2 + 2*k3l2 +
            k4l2);
        mu1(j-1) = mu1(j)-(h/6)*(k1m1 + 2*k2m1 + 2*k3m1 + k4m1);
        mu2(j-1) = mu2(j)-(h/6)*(k1m2 + 2*k2m2 + 2*k3m2 + k4m2);


    end

    %update controls
    for i = 1:N+1
        if alpha1 == 0
            switchfn1(i) = lambda1(i)*y(i);
            if (switchfn1(i) < 0)
                u1_temp(i) = 0;
            else
                u1_temp(i) = u1_max;
            end
        else
```

```
        u1_temp(i) = max(0,e1*lambda1(i)*y(i)+u0);
    end

    if alpha2 == 0
        switchfn2(i) = mu2(i)*x(i);
        if (switchfn2(i) < 0)
            u2_temp(i) = 0;
        else
            u2_temp(i) = u2_max;
        end
    else
        u2_temp(i) = max(0,e2*mu2(i)*x(i)+u0);
    end
end

u1 = 0.5*(u1_temp + oldu1);
u2 = 0.5*(u2_temp + oldu2);



%check tolerance
temp1 = delta*sum(abs(u1))-sum(abs(oldu1 - u1));
temp2 = delta*sum(abs(x))-sum(abs(oldx - x));
temp3 = delta*sum(abs(lambda1))-sum(abs(oldlambda1 - lambda1))
    ;
temp4 = delta*sum(abs(y))-sum(abs(oldy - y));
temp5 = delta*sum(abs(lambda2))-sum(abs(oldlambda2 - lambda2))
    ;
temp6 = delta*sum(abs(mu1))-sum(abs(oldmu1 - mu1));
```

```
    temp7 = delta*sum(abs(mu2))-sum(abs(oldmu2 - mu2));
    temp8 = delta*sum(abs(u2))-sum(abs(oldu2 - u2));
    test = min(temp1, min(temp2,min(temp3,min(temp4,min(temp5,min(
        temp6,min(temp7,temp8)))))));
end

    g(1,:) = t;
    g(2,:) = x;
    g(3,:) = y;
    g(4,:) = lambda1;
    g(5,:) = lambda2;
    g(6,:) = mu1;
    g(7,:) = mu2;
    g(8,:) = u1;
    g(9,:) = u2;

end

function f = marriageRepair(alpha1,alpha2, r1, r2,x0,y0, flag)
close all



eps =0.3;
e1 = eps;
e2= eps;
T = 5;
u0= 0.5;
u1_max = u0;
u2_max = u0;
```

```
I = 2;
%J = 0.2;
n = 5;
%xlist = -I*x0 + 2*I*x0*rand(100,1);
%ylist = -I*y0 + 2*I*y0*rand(100,1);
xlist = linspace(I*x0, -I*x0,n);
ylist = linspace(I*y0, -y0*I,n);
% xlist = linspace(x0-J, x0+J,n);
% ylist = linspace(y0-J, y0+J,n);

x1 =[];
x2 = [];
u1 = [];
u2 = [];

for i = 1:n
    for j = 1:n
        x_i = xlist(i);
        y_i = ylist(j);
        g = mp4(r1,x0,r2,y0,x_i,y_i,u1_max, u2_max, alpha1,alpha2,
            e1, e2,u0, T);
        ind = find(g(1,:)== 3);
        x1_i = g(2,1:ind);
        x2_i = g(3,1:ind);
        u1_i = g(8,1:ind);
        u2_i = g(9,1:ind);

    if flag
        figure(1)
```

```
            subplot(2,1,2)
            plot(x1_i, x1_i.*u2_i, 'b')
            hold on

            subplot(2,1,1)
            plot(x2_i, x2_i.*u1_i, 'b')
            hold on
        end



%save all x and y values from each iteration
        x1 = [x1, x1_i];
        x2 = [x2, x2_i];
        u1 = [u1,u1_i];
        u2 = [u2, u2_i];
        end
end

%split data into two regions
x1_neg = x1(x1<0);
x2_neg = x2(x2<0);
u1_neg = u1(x2<0);
u2_neg = u2(x1<0);

x1_pos = x1(x1>0);
x2_pos = x2(x2>0);
u1_pos = u1(x2>0);
u2_pos = u2(x1>0);
```

```
%fit data of each region
coeff1_neg = polyfit(x2_neg, x2_neg.*u1_neg,1);
coeff2_neg = polyfit(x1_neg, x1_neg.*u2_neg,1);

coeff1_pos = polyfit(x2_pos, x2_pos.*u1_pos,1);
coeff2_pos = polyfit(x1_pos, x1_pos.*u2_pos,1);

x1_fit_neg = linspace(min(x1_neg), max(x1_neg), 100);
x2_fit_neg = linspace(min(x2_neg), max(x2_neg), 100);

u1_fit_neg = polyval(coeff1_neg, x2_fit_neg);
u2_fit_neg = polyval(coeff2_neg, x1_fit_neg);

x1_fit_pos = linspace(min(x1_pos), max(x1_pos), 100);
x2_fit_pos = linspace(min(x2_pos), max(x2_pos), 100);

u1_fit_pos = polyval(coeff1_pos, x2_fit_pos);
u2_fit_pos = polyval(coeff2_pos, x1_fit_pos);


    if flag
        figure(1)
        subplot(2,1,2)
        plot(x1_fit_pos, u2_fit_pos, 'r', 'LineWidth', 1.25)
        plot(x1_fit_neg, u2_fit_neg, 'r',  'LineWidth', 1.25)
        xlabel('x_1')
        ylabel('\phi_2*x_1')
```

```
% Create text strings for the slopes
slope1text = sprintf('b_2 = %.2f', coeff2_neg(1));
slope2text = sprintf('a_2 = %.2f', coeff2_pos(1));

% Display the first slope text at (-0.2, 0)
text(x0, 0.1, slope1text, 'Color', 'r');

% Display the second slope text at (0.2, 0)
text(-x0, 0.1, slope2text, 'Color', 'r');



subplot(2,1,1)
plot(x2_fit_pos, u1_fit_pos, 'r','LineWidth', 1.25)
plot(x2_fit_neg, u1_fit_neg, 'r', 'LineWidth', 1.25)
xlabel('x_2')
ylabel('\phi_1*x_2')

% Create text strings for the slopes
slope3text = sprintf('b_1 = %.2f', coeff1_neg(1));
slope4text = sprintf('a_1 = %.2f', coeff1_pos(1));

% Display the first slope text at (-0.2, 0)
text(-y0-.1, 0, slope3text, 'Color', 'r');

% Display the second slope text at (0.2, 0)
text(y0, 0, slope4text, 'Color', 'r');
```

```
        hold off;
        sgtitle(['\alpha_1 = ', num2str(alpha1), ', \alpha_2 = ',
            num2str(alpha2)])
    end

f(1) = coeff1_pos(1); %a1
f(2) = coeff1_neg(1); %b1
f(3) = coeff2_pos(1); %a2
f(4) = coeff2_neg(1); %b2

function f = eqPts(r1, x0,r2, y0,a1,b1,a2,b2)
    num1 = x0 +(a1/r1)*y0;
    num2 = x0 +(b1/r1)*y0;
    num3 = y0+(a2/r2)*x0;
    num4 = y0+(b2/r2)*x0;

    %1
    if num1 >0 && num3 >0
        f(1) = num1/(1 - (a1*a2)/(r1*r2));
        f(2) = num3/(1 - (a1*a2)/(r1*r2));

        %2
    elseif num1 < 0 && num4 >0
            f(1) = num1/(1 - (a1*b2)/(r1*r2));
            f(2) = num4/(1 - (a1*b2)/(r1*r2));

        %3
    elseif num2 < 0 && num4 <0
            f(1) = num2/(1 - (b1*b2)/(r1*r2));
```

```
            f(2) = num4/(1 - (b1*b2)/(r1*r2));

        %4
    elseif num2>0 && num3<0
            f(1) = num2/(1 - (b1*a2)/(r1*r2));
            f(2) = num3/(1 - (b1*a2)/(r1*r2));
    else
        f(1) = NaN;
        f(2) = NaN;
    end

%generate table of equilibrium points
clear all
close all
clc
a = linspace(0,1,11);
r1 = 0.8;
x0 = -0.26;
r2 = 0.8;
y0 = 0.12;
%T = 1;

columnNames = {'alpha_1', 'alpha_2', 'x_1^*', 'x_2^*'};
% columnNames = {'alpha_1', 'alpha_2', 'x_1^* neg', 'x_2^* neg', '
    neg stability',...
%       'x_1^* pos', 'x_2^* posNeg', '  stability'};
% Initialize an empty table with variable names
dataTable = table([], [], [], [], 'VariableNames', columnNames);
```

```
for i = 1:length(a)
    for j = 1:length(a)
        alpha1 = a(i);
        alpha2 = a(j);
        g = marriageRepair(alpha1,alpha2, r1, r2,x0,y0, false);
        a1 = g(1);
        b1 = g(2);
        a2 = g(3);
        b2 = g(4);
        h= eqPts(r1, x0,r2, y0,a1,b1,a2,b2);

        dataRow = {alpha1, alpha2, h(1), h(2)};
        dataTable = [dataTable; dataRow];
         i,j
    end
end

disp(dataTable);
```