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Deformation invariance of rational pairs

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A dissertation
submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy

University of Washington

2014

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Program Authorized to Offer Degree:
Mathematics

University of Washington

Abstract

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Rational pairs, recently introduced by Kollár and Kovács, generalize rational singularities to pairs (X, D) . Here X is a normal variety and D is a reduced divisor on X . Integral to the definition of a rational pair is the notion of a thrifty resolution, also defined by Kollár and Kovács, and in order to work with rational pairs it is often necessary to know whether a given resolution is thrifty. In this dissertation I present several foundational results that are helpful for identifying thrifty resolutions and analyzing their behavior. In 1978, Elkik proved that rational singularities are deformation invariant. The main result of this dissertation is an analogue of this theorem for rational pairs: given a flat family $X \rightarrow S$ and a Cartier divisor D on X , if the fibers over a smooth point $s \in S$ form a rational pair, then (X, D) is also rational near the fiber X_s .

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ACKNOWLEDGMENTS

First of all, I thank my advisor, Sándor Kovács, for nearly five years of guidance and support. I continue to be amazed that he managed to find a perfect niche for my mathematical tastes within the vast research area of algebraic geometry. The problem he suggested I work on magically incorporates all my favorite things about the subject.

A very special thank you goes to Jim Morrow, who admitted me into his REU program at UW in 2007 and, at the end of that summer, encouraged me to apply to graduate school. At the time I had never considered the possibility, and I am very grateful that he set me on this path.

Thank you to Steve Mitchell, Max Lieblich, Sara Billey, and Doug Lind, for wonderful advice and help about math and my career during my time at UW.

I wrote most of this thesis at the incomparable Milstead & Co. coffee shop in Fremont, so I thank everyone who works there for providing me delicious coffee and a perfect place to work.

I have been fortunate to meet many wonderful friends in Seattle, and I thank them for all the happy memories here. And most of all, thanks to my husband Toby, who has been a seemingly inexhaustible source of support and wisdom for years.

We were binding sheaves in the field, and lo, my sheaf arose, and also stood upright; and, behold, your sheaves stood round about, and bowed down to my sheaf!

—GENESIS 37:7, NEW ASV

Chapter 1

INTRODUCTION

Rational singularities were originally defined for surfaces by Michael Artin, and the notion was later extended to higher-dimensional varieties. Rational singularities are the singularities that are cohomologically smooth: if $\tilde{X} \rightarrow X$ is a resolution and X has rational singularities, then the cohomology of the structure sheaf of X is the same as that of the structure sheaf of \tilde{X} . In characteristic 0, rational singularities are characterized by one simple requirement: for a resolution $f: \tilde{X} \rightarrow X$, the natural map $\mathcal{O}_X \rightarrow \mathcal{R}f_*\mathcal{O}_{\tilde{X}}$, in the derived category of sheaves of \mathcal{O}_X -modules, is a quasi-isomorphism. To extend the definition to characteristic $p > 0$, we would add one more requirement: that the higher direct images of the canonical sheaf $\omega_{\tilde{X}}$ also vanish. In this dissertation, though, we assume characteristic 0.

There are many other well-studied classes of singularities for varieties, which we describe later in this dissertation. Many of them are also defined for pairs (X, D) , where X is a variety and D is a divisor on X . In [Kol13, Section 2.5], Kollár and Kovács introduced a generalization of rational singularities to reduced pairs. A reduced pair is a certain type of pair (X, D) : the variety X is normal, and the Weil divisor D has all of its coefficients equal to 1. This D is easy to visualize: it is just a union of codimension-1 subvarieties.

First we'll present two chapters of background results and definitions from the literature. Chapter 2 covers Cohen-Macaulay varieties, normal varieties, and rational singularities. Along with the statements of many well-known and important theorems—such as Serre's criterion for normal varieties, and Grothendieck duality for proper morphisms—we also provide proofs of some “folklore” theorems that are well known but for which we could not find appropriate references in the literature. We construct the dualizing complex of a projective variety, by putting together many pieces from [Har66]. We also show in (2.3.2) that on a normal variety, the dualizing sheaf may be identified with the canonical sheaf.

After defining rational singularities for varieties, we state the well-known and very useful characterization of rational singularities known as Kempf’s criterion: if $f : Y \rightarrow X$ is a resolution, then X has rational singularities if and only if X is CM and the natural map $\mathcal{R}f_*\omega_Y \rightarrow \omega_X$ is a quasi-isomorphism. Finally, we observe the famous fact that makes rational resolutions nice to work with: if X has one resolution that is rational, then every other resolution of X is also rational. As we will see below, this is *not* true for rational resolutions of pairs as defined in [Kol13]. This would present a big problem for the theory, but for the fact that restricting to a smaller class of resolutions—which we’ll describe below—yields an analogous result.

Chapter 3 includes the necessary background from [Kol13] about rational pairs. To define rational pairs, we first need to say what we mean by a resolution of pairs. Take a resolution $f : Y \rightarrow X$, such that the birational transform of D in Y , which we’ll call B , is an snc divisor: this is what we will call a resolution of pairs. All these notions are defined and explained in Chapter 3. The definition of a rational resolutions of a pair is formally very much like that of a rational resolution of a variety, but there is a problem, which we alluded to above: a pair can have both rational and non-rational resolutions. So taking the rational pairs to be the pairs for which rational resolutions exist would make the objects extremely hard to work with.

To fix this issue, Kollár and Kovács define rational pairs using only a special subclass of resolutions. These are called the thrifty resolutions, and for a resolution to be thrifty it must satisfy two conditions (called Condition 1 and Condition 2) which have to do with the intersections of the components of the divisors D in X and B in Y ; see (3.3.1) for the precise statement, and see the rest of Chapter 3 for the necessary definitions. A rational pair is a pair that has a thrifty rational resolution. This is a useful definition because a pair cannot have both rational and non-rational *thrifty* resolutions. Also, a nice analogue of the Grauert-Riemenschneider vanishing theorem holds for thrifty resolutions of pairs (X, D) : see (4.4.2) below. All of these results suggest that the theory of rational pairs might turn out to be similar to that of rational singularities of varieties.

In Chapter 4 we present several new results about thrifty resolutions. We mentioned above that there are two conditions for thriftiness (which we will define later in the disser-

tation, after we've built up the necessary vocabulary), but it's important to also note that the conditions aren't created equal. Condition 1 is often much easier to verify, because it is a property of the snc locus of the target pair (X, D) , and there are many nice properties one can exploit there. The other condition, Condition 2, is less straightforward to check because it's necessary to look at points in the source Y that don't lie over the snc locus of (X, D) . So alternative ways to verify Condition 2 would be very helpful for anyone studying rational pairs. There are two main results in Chapter 4. The first one gives a criterion for Condition 2, so that one only needs to check the easier Condition 1:

Proposition (4.3.1). *If $f: (Y, D_Y) \rightarrow (X, D)$ is a log resolution, then f satisfies Condition 2.*

In other words, a log resolution is thrifty if and only if it satisfies Condition 1.

The other main result from Chapter 4 shows that, given a birational morphism $Y \rightarrow X$ that satisfies Condition 1 for thriftiness, there is a thrifty log resolution that dominates it.

Theorem (4.3.2). *Suppose $f: (Y, D_Y) \rightarrow (X, D)$ satisfies Condition 1. Then there is a thrifty log resolution of (X, D) factoring through (Y, D_Y) .*

In Chapter 5, we investigate how the rationality property can pass from a pair to a subpair, or from a subpair to a neighborhood of the larger pair. To do this, we first need to verify that thrifty resolutions work well with certain pairs and subpairs. A subpair (X', D') of a pair (X, D) is just another pair with $X' \subset X$ and $D' \subset D$. We show that under certain nice conditions, a thrifty resolution of the larger pair (X, D) restricts to a thrifty resolution of the subpair (X', D') . This is an important prerequisite for checking that rationality of one implies rationality of the other, because rational pairs are defined by the existence of thrifty rational resolutions.

In Chapter 5, we present a Bertini-style result for rational pairs: general hyperplane sections of rational pairs are also rational. We also show that flat families of rational pairs are deformation invariant: given a flat family of rational pairs over a scheme S and a smooth point $s \in S$, then rationality of the fiber over s guarantees rationality of the larger pair in a neighborhood of the fiber. Elkik proved the analogous statement for rational singularities

in [Elk78], so this is a generalization of her result for pairs. This is the main result of the dissertation.

Theorem (5.2.8). *Let (X, D) be a pair, with D Cartier. Suppose $X \rightarrow S$ is a flat morphism, and $s \in S$ is a smooth point so that the fibers (X_s, D_s) form a reduced pair. If (X_s, D_s) is a rational pair, then (X, D) is a rational pair in a neighborhood of (X_s, D_s) .*

That is, if (X_s, D_s) is rational at x , then (X, D) is also rational at x .

Chapter 2

BACKGROUND: RATIONAL SINGULARITIES

We start with some relevant background from commutative algebra and algebraic geometry. We'll see that if a variety X has rational singularities, then it is both Cohen-Macaulay and normal, so we begin by discussing these properties.

2.1 Cohen-Macaulay varieties

If M is a module over a local ring A with maximal ideal \mathfrak{m} , then there is a notion of *depth* of M as an A -module. One can think of depth as a kind of dimension for a module: we start with a big thing—the whole module—and count how many times it's possible to kill off cyclic submodules one by one until there is nothing left. By definition, the depth of M is the maximum length of a regular sequence $\{x_1, \dots, x_r\} \subset \mathfrak{m}$. A set of elements $\{x_1, \dots, x_r\}$ is a regular sequence if x_1 is not a zero-divisor in M , and for all $i \geq 2$, $M/(x_1, \dots, x_{i-1})M$ is not zero and x_i is not a zero-divisor there.

Considering A as a module over itself, we say that A is Cohen-Macaulay if its depth equals its Krull dimension: $\text{depth } A = \dim A$. If depth and Krull dimension are viewed as two different ways to measure something like the dimension of a ring, then it would be nice if they yielded the same number. A ring is Cohen-Macaulay if this is actually true. A general discussion of depth, dimension, and the Cohen-Macaulay property can be found in [Mat89].

Definition 2.1.1 ([KM98, p. 153]). A variety X is Cohen-Macaulay or CM, or is said to have CM singularities, if all the local rings \mathcal{O}_x are Cohen-Macaulay.

A local complete intersection subscheme of a nonsingular variety is CM ([Har77, II.8.23]). In particular, complete intersections are CM, from which we immediately see that CM is a much weaker condition than smoothness.

We can say a few things about what CM schemes look like locally. If A is a local CM ring, all the minimal primes of A have the same dimension ([Eis95, 18.10–11]). Since the minimal primes of a local ring \mathcal{O}_x on X correspond to the irreducible components of X passing through the point x , this statement means that if \mathcal{O}_x is CM, then x cannot lie on two irreducible components of different dimensions. We can use this to easily construct a non-CM scheme: take the union of a plane and a line through it.

There is another local topological requirement for CM schemes. If a scheme is CM and connected, it must remain connected even if a subset of codimension 2 is removed. This is Hartshorne’s Connectedness Theorem ([Eis95, 18.12]), and it implies that a union of two planes meeting at one point in four-space is not CM.

This is a surprisingly useful property for a scheme to have. Mainly the Cohen-Macaulay property is helpful because there are many useful consequences for a special coherent sheaf called the dualizing sheaf.

Definition 2.1.2 ([Har77, p. 241]). Let X be a proper scheme of dimension n . A dualizing sheaf for X is a coherent sheaf ω_X that comes with a trace map $t: H^n(X, \omega_X) \rightarrow k$. The trace map has the property that if \mathcal{F} is coherent on X , the pairing

$$\mathrm{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \rightarrow H^n(X, \omega_X), \quad (\phi: \mathcal{F} \rightarrow \omega_X, \alpha) \mapsto (H^n \phi)(\alpha)$$

composed with t yields an isomorphism $\mathrm{Hom}(\mathcal{F}, \omega_X) \rightarrow H^n(X, \mathcal{F})^\vee$.

Every projective scheme over k has a dualizing sheaf ([Har77, III.7.5]): if X is projective of dimension n , embed X as a closed subscheme of \mathbf{P}^N for some N , and define the dualizing sheaf to be $\mathcal{E}xt_{\mathbf{P}^N}^{N-n}(\mathcal{O}_X, \omega_{\mathbf{P}^N})$. We will see that this sheaf is actually the $-n$ th cohomology of the dualizing complex ω_X^\bullet (constructed below).

If X is smooth over k , then X has a canonical sheaf, which is by definition the highest exterior power of the sheaf of differentials $\Omega_{X/k}$. Since the sheaf of differentials on X is a locally free sheaf of rank $\dim X$ if X is smooth, the canonical sheaf is invertible. This sheaf is also usually called ω_X , and for good reason: the canonical sheaf coincides with the dualizing sheaf in the smooth case ([Har77, III.7.12]). Actually, we will see below that there is a good way to define the canonical sheaf on a scheme that is merely normal, and not

necessarily smooth. In this case, since X is projective but not necessarily CM, we define the dualizing sheaf as the $\mathcal{E}xt$ sheaf above, and we use that it's equal to $h^{-n}(\omega_X^\bullet)$, the $-n$ th cohomology of the dualizing complex (which we'll construct below). Then, as we will see in (2.3.2), the dualizing sheaf and the canonical sheaf are again equal.

In the definition of the dualizing sheaf, it isn't required that X be CM. But if X is projective, CM, and equidimensional (which means that all its irreducible components have the same dimension—which, for CM schemes, is implied by connectedness), then its dualizing sheaf is very well-behaved. In this case, we have Serre duality, which greatly simplifies the computation of cohomology groups of sheaves on X :

Theorem 2.1.3 (Serre duality: see [Har77, III.7.7]). *Let \mathcal{F} be a locally free sheaf on X , where X is projective of dimension n . Then there are natural isomorphisms*

$$H^i(X, \mathcal{F}) \simeq H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X)^\vee.$$

Because X is assumed to be projective, a theorem of Serre applies: for every coherent sheaf \mathcal{F} , the cohomology groups $H^i(X, \mathcal{F})$ are all finite-dimensional k -vector spaces ([Har77, III.5.2]). So taking the dual of the right term in (2.1.3) gives equations for each i :

$$\dim_k H^i(X, \mathcal{F}) = \dim_k H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X).$$

What if a scheme X is not, or at least is not known to be, CM? Then there are still some useful duality results that hold on X , but they can only be stated in the derived category. The dualizing sheaf is not as useful in this case: we can't use it to get cohomology isomorphisms as we could with Serre duality. Instead, we'll use the dualizing complex ω_X^\bullet , which lies in the derived category of \mathcal{O}_X -modules. The dualizing complex encodes information about the dualizing sheaf and, in the non-CM case, whatever extra information is necessary to make a duality theory.

2.2 The dualizing complex

Here we will describe the dualizing complex of a projective scheme. The main reference for dualizing complexes and coherent duality theory is [Har66]. The construction below is just an expanded version of the one found there.

Let X be projective, with structure map $f: X \rightarrow \operatorname{Spec} k$, and factor f through $P = \mathbf{P}_k^N$:

$$\begin{array}{ccc} X & \xrightarrow{\sigma} & P \\ & \searrow f & \downarrow \pi \\ & & \operatorname{Spec} k \end{array}$$

Then σ is a closed immersion, and π is the smooth projection map. We define ω_X^\bullet to be the element of the derived category of \mathcal{O}_X -modules

$$\omega_X^\bullet = f^!k = \sigma^!\pi^!k,$$

where we view k as the structure sheaf of $\operatorname{Spec} k$.

Given any morphism $g: U \rightarrow V$, the functor $g^!$ takes an element of the derived category of \mathcal{O}_V -modules to the derived category of \mathcal{O}_U -modules; it is a sort of pullback. The complex $g^!\mathcal{F}^\bullet$ is defined for smooth morphisms g in [Har66, III.2 and III.8.7]. In the special case where π is the structure morphism and the complex is merely the structure sheaf on $\operatorname{Spec} k$, we have

$$\pi^!k = \pi^*k \otimes \omega_P[N].$$

Now $\pi^*k \simeq \mathcal{O}_P$, so this is just equal to the complex $\omega_P[N]$. The $[N]$ means “shift N places to the left”: this complex has the sheaf ω_P at the $-N$ th place, and zeroes everywhere else.

Now σ is a closed immersion, hence finite, and $g^!\mathcal{F}^\bullet$ for a finite morphism g is defined in [Har66, III.6 and III.8.7]. In the special case where σ is an embedding, $f^!k = \sigma^!\pi^!k = \sigma^!(\omega_P[N])$ is the complex

$$\mathcal{R}\mathcal{H}om_P(\mathcal{O}_X, \omega_P[N]) \simeq \mathcal{R}\mathcal{H}om_P(\mathcal{O}_X, \omega_P)[N].$$

Now all the sheaves in the above complex are supported on X , so we may view them as sheaves on X . This is the dualizing complex ω_X^\bullet . It has the property that the functor $D(-) = \mathcal{R}\mathcal{H}om(-, \omega_X^\bullet)$ is a dualizing functor in the derived category of \mathcal{O}_X -modules: $D \circ D$ is the identity functor. This is often useful in calculations involving ω_X^\bullet , such as those that come up in Grothendieck duality theory.

Now, if X has dimension n , take $-n$ th cohomology of the dualizing complex $\omega_X^\bullet = f^!k$:

$$\begin{aligned} h^{-n}(\omega_X^\bullet) &= h^{-n}(\mathcal{R}\mathcal{H}\text{om}_P(\mathcal{O}_X, \omega_P)[N]) \\ &= h^{N-n}(\mathcal{R}\mathcal{H}\text{om}_P(\mathcal{O}_X, \omega_P)) \\ &= \mathcal{E}\text{xt}_P^{N-n}(\mathcal{O}_X, \omega_P) \\ &= \omega_X. \end{aligned}$$

This is the dualizing sheaf of X , as defined above. This is what we meant when we claimed above that the dualizing complex contains information about the dualizing sheaf. In fact, the dualizing sheaf is the first nonzero cohomology sheaf of the dualizing complex. By [Har77, III.7.3], $\mathcal{E}\text{xt}_P^i(\mathcal{O}_X, \omega_P) = 0$ if $i < N - n$.

If we also require that all cohomologies above $-n$ vanish, so that there is only one nonzero cohomology sheaf of ω_X^\bullet , then we have a characterization of CM schemes:

Theorem 2.2.1 ([Con00, 3.5.1]). *A scheme X is CM and equidimensional if and only if there is an isomorphism in the derived category of \mathcal{O}_X -modules:*

$$\omega_X^\bullet \simeq \omega_X[n],$$

where $\omega_X[n]$ is the complex with the dualizing sheaf ω_X in the $-n$ th place, and zeroes everywhere else.

This follows from the proof of [Har77, III.7.6], which shows that if $X \hookrightarrow P = \mathbf{P}^N$ is a projective scheme of dimension n , then it is CM and equidimensional if and only if

$$\mathcal{E}\text{xt}_P^i(\mathcal{O}_X, \omega_P) = 0$$

for $i \neq \text{codim}(X, P)$. The upshot is that CM schemes are the ones where the dualizing complex is essentially the same object as the dualizing sheaf. The dualizing complex of a CM scheme does not need to include the extra information that would otherwise come in the form of nonzero cohomology groups.

Here is the main result about dualizing complexes.

Theorem 2.2.2 (Grothendieck duality: see [Har66, VII.3.4c]). *Let $f: Y \rightarrow X$ be a proper morphism, and let \mathcal{F}^\bullet be a bounded complex in the derived category of \mathcal{O}_Y -modules. Then*

there is an isomorphism in the derived category of \mathcal{O}_X -modules:

$$\mathcal{R}f_*\mathcal{R}\mathcal{H}om_Y(\mathcal{F}^\bullet, \omega_Y^\bullet) \simeq \mathcal{R}\mathcal{H}om_X(\mathcal{R}f_*\mathcal{F}^\bullet, \omega_X^\bullet).$$

This statement holds for any proper morphism. Of course, the theory is the most straightforward when the schemes X and Y are CM, because then the dualizing complex is very simple. In fact, if X is CM, then applying Grothendieck duality to the structure morphism $f: X \rightarrow \text{Spec } k$ and using that $\omega_X^\bullet \simeq \omega_X[n]$ reduces the statement to that of Serre duality (2.3).

2.3 Normal varieties

A scheme X is normal if all its local rings are integrally closed domains. Any scheme can be normalized: given any X , there is a normal scheme \tilde{X} and a normalization morphism $\tilde{X} \rightarrow X$, such that every other dominant morphism from a normal scheme to X factors uniquely through \tilde{X} . See [Har77, Ex. II.3.8]. The normalization morphism is finite if X is of finite type over k , which is true for the projective schemes we consider here.

Many theorems about the resolution of singularities include normality hypotheses. Because it is always possible to normalize a scheme, this isn't usually an impediment to using the theorems in more general situations. As we will see below, normalizing resolves the singularities in codimension 1. Once this is done, we can use theorems on resolution of singularities to smooth out whatever remains.

There are some useful ways to characterize normal varieties. We begin with Serre's criterion.

Theorem 2.3.1 (Serre's criterion: see [Har77, II.8.22A]). *A noetherian scheme X is normal if and only if it satisfies two conditions, called R_1 and S_2 :*

R_1 : *X is regular in codimension 1; that is, for every point $x \in X$ of codimension at least 1, the local ring \mathcal{O}_x is regular.*

S_2 : *For every point $x \in X$ of codimension at least 2, we have $\text{depth } \mathcal{O}_x$ at least 2.*

From this theorem we see that normality is a strong condition for curves: indeed, it is equivalent to being regular. Since the schemes in this dissertation are defined over a field of characteristic 0, this means smooth over k . A normal surface might have singularities, but they can only be isolated points. As the dimension increases, the normality condition reveals less about the scheme.

As an application of Serre's criterion, we can define the canonical sheaf on a normal scheme. We saw above how this is done on a smooth scheme: the canonical sheaf ω_X is highest exterior power of the sheaf of differentials $\Omega_{X/k}$, and it is invertible. Now assume that X is merely normal—not necessarily smooth over k , nor necessarily CM. Let $Z \subset X$ be the singular locus: by Serre's criterion, Z has codimension at least 2. So define the canonical sheaf $\omega_{X \setminus Z}$ on the smooth open subscheme $X \setminus Z$, using the sheaf of differentials. Take the corresponding Weil divisor K ; this is the canonical divisor of the scheme $X \setminus Z$. Now K corresponds to a Weil divisor on X , since the divisor class groups of X and $X \setminus Z$ are isomorphic ([Har77, II.6.5]). Call this divisor K_X : it is the canonical divisor of X , and we can take the canonical sheaf to be $\mathcal{O}_X(K_X)$. This sheaf is the pushforward via the inclusion $i: X \setminus Z \hookrightarrow X$ of $\omega_{X \setminus Z}$. It is not invertible on X in general, but it has rank 1.

The next well-known theorem shows that normal schemes share a nice property of smooth schemes: the dualizing sheaf coincides with the canonical sheaf.

Theorem 2.3.2. *Suppose X is normal and quasiprojective, let $\mathcal{F} = h^{-n}(\omega_X^\bullet)$ be the dualizing sheaf of X , and let $n = \dim X$. Write ω_X for the canonical sheaf of X , as defined above. Then $\mathcal{F} \simeq \omega_X$.*

Although this fact is widely known in the world of algebraic geometry, an adequate reference in the literature is missing. So we will include a proof here, for the reader's benefit.

Proof. Embed X as an open subscheme of a projective variety \bar{X} , which in turn may be viewed as a closed subscheme of \mathbf{P}^N . Now X is assumed to be normal but \bar{X} might not be, so let $\tilde{X} \rightarrow \bar{X}$ be the finite normalization morphism.

Consider the projection $\mathbf{P}^N \rightarrow L \simeq \mathbf{P}^n$ onto a linear subspace of dimension n , by repeatedly projecting from points not contained in the projective variety \bar{X} . Restricting

this projection map to \bar{X} yields a finite morphism $\bar{X} \rightarrow L$. By composing this with the normalization $\tilde{X} \rightarrow \bar{X}$, we obtain a finite morphism $p: \tilde{X} \rightarrow L$. That p is finite guarantees it is proper ([Har77, Ex. II.4.1]).

Now \tilde{X} is normal by assumption, so its singular set $\text{Sing } \tilde{X}$ has codimension at least 2. Let $V = L \setminus p(\text{Sing } \tilde{X})$, and let $U = p^{-1}(V)$. Then $L \setminus V$ also has codimension at least 2, because p is finite. We have a Cartesian diagram, with horizontal arrows the inclusions:

$$\begin{array}{ccc} U & \xrightarrow{i} & \tilde{X} \\ p \downarrow & & \downarrow p \\ V & \xrightarrow{j} & L \end{array}$$

The map $p: U \rightarrow V$ is proper and quasifinite, hence finite, and it is flat by [Har77, Ex. III.10.9]. The open set U is smooth, so we can define the canonical sheaf ω_U as the n th exterior power of the sheaf of differentials. The canonical sheaf $\omega_{\tilde{X}}$ is defined to be $i_*\omega_U$. We'll show that $\omega_{\tilde{X}} \simeq \mathcal{F}'$, where \mathcal{F}' is the dualizing sheaf $h^{-n}(\omega_{\tilde{X}}^\bullet)$.

By Grothendieck duality—see (2.2.2)—applied to the proper morphism p , we have

$$\mathcal{R}p_*\mathcal{R}\mathcal{H}\text{om}_{\tilde{X}}(\mathcal{O}_{\tilde{X}}, \omega_{\tilde{X}}^\bullet) \simeq \mathcal{R}\mathcal{H}\text{om}_L(\mathcal{R}p_*\mathcal{O}_{\tilde{X}}, \omega_L^\bullet).$$

Now we simplify. Since p is finite, p_* is exact ([Har77, Ex. III.8.2]), so the left side becomes $p_*\mathcal{R}\mathcal{H}\text{om}_{\tilde{X}}(\mathcal{O}_{\tilde{X}}, \omega_{\tilde{X}}^\bullet)$. Now take $-n$ th cohomology. Since p_* is exact, it commutes with h^{-n} . Also, $\mathcal{O}_{\tilde{X}}$ is its own locally free resolution, so we have

$$\begin{aligned} h^{-n}(p_*\mathcal{R}\mathcal{H}\text{om}_{\tilde{X}}(\mathcal{O}_{\tilde{X}}, \omega_{\tilde{X}}^\bullet)) &\simeq p_*(h^{-n}(\mathcal{R}\mathcal{H}\text{om}_{\tilde{X}}(\mathcal{O}_{\tilde{X}}, \omega_{\tilde{X}}^\bullet))) \\ &\simeq p_*(h^{-n}(\omega_{\tilde{X}}^\bullet)) \\ &= p_*\mathcal{F}'. \end{aligned}$$

Now we move over to the right side. First, L is smooth, so ω_L^\bullet is just the complex $\omega_L[n]$, with the canonical sheaf shifted n places to the left. Using this together with exactness of p_* , the right side becomes

$$\mathcal{R}\mathcal{H}\text{om}_L(p_*\mathcal{O}_{\tilde{X}}, \omega_L[n]) \simeq \mathcal{R}\mathcal{H}\text{om}_L(p_*\mathcal{O}_{\tilde{X}}, \omega_L)[n].$$

Now take $-n$ th cohomology of this complex:

$$\begin{aligned} h^{-n}(\mathcal{R}\mathcal{H}\text{om}_L(p_*\mathcal{O}_{\tilde{X}}, \omega_L)[n]) &\simeq h^{n-n}(\mathcal{R}\mathcal{H}\text{om}_L(p_*\mathcal{O}_{\tilde{X}}, \omega_L)) \\ &\simeq \mathcal{H}\text{om}_L(p_*\mathcal{O}_{\tilde{X}}, \omega_L). \end{aligned}$$

We conclude that $p_*\mathcal{F}' \simeq \mathcal{H}\text{om}_L(p_*\mathcal{O}_{\tilde{X}}, \omega_L)$. The same argument applied to $p: U \rightarrow V$, taking into account the smoothness of U , shows that $p_*\omega_U \simeq \mathcal{H}\text{om}(p_*\mathcal{O}_U, \omega_V)$. In other words, the sheaves $j_*p_*\omega_U$ and $\mathcal{H}\text{om}_L(p_*\mathcal{O}_{\tilde{X}}, \omega_L)$ are equal when restricted to V .

Now $p: U \rightarrow V$ is finite and flat, so $p_*\mathcal{O}_U$ is locally free of finite rank on V . Thus $p_*\mathcal{O}_U$ is a reflexive sheaf on V , and its pushforward $j_*p_*\mathcal{O}_U$ is a reflexive sheaf on L ([Har80, 1.6]). On the other hand, $\mathcal{H}\text{om}_L(p_*\mathcal{O}_{\tilde{X}}, \mathcal{O}_L)$ is reflexive by [Har80, 1.2], and this is preserved by tensoring with ω_L ([Har80, 1.1]). Thus $\mathcal{H}\text{om}_L(p_*\mathcal{O}_{\tilde{X}}, \omega_L)$ is reflexive. So we have two reflexive sheaves that are equal on V . Since the complement of V has codimension at least 2, these sheaves are actually equal on L ; see [Har80, 1.6].

Now we have a chain of isomorphisms:

$$p_*\omega_{\tilde{X}} \simeq j_*p_*\omega_U \simeq \mathcal{H}\text{om}_L(p_*\mathcal{O}_{\tilde{X}}, \omega_L) \simeq p_*\mathcal{F}'.$$

The first is from the commutative diagram and the definition of $\omega_{\tilde{X}}$, the second is from the reflexivity argument above, and the third is from the Grothendieck duality argument above. So $\omega_{\tilde{X}}$ and \mathcal{F}' have the same pushforward by p . We will show that this forces $\omega_{\tilde{X}} \simeq \mathcal{F}'$.

Now $\mathcal{F}'|_U$ is ω_U , so there is a morphism $\alpha: \mathcal{F}' \rightarrow i_*\omega_U = \omega_{\tilde{X}}$. Include this morphism in an exact sequence:

$$0 \longrightarrow \ker \alpha \longrightarrow \mathcal{F}' \longrightarrow \omega_{\tilde{X}} \longrightarrow \text{coker } \alpha \longrightarrow 0.$$

Push the whole thing forward by p . Because p is finite, and thus p_* is exact, we obtain a new exact sequence:

$$0 \longrightarrow p_*\ker \alpha \longrightarrow p_*\mathcal{F}' \longrightarrow p_*\omega_{\tilde{X}} \longrightarrow p_*\text{coker } \alpha \longrightarrow 0.$$

All the work above tells us that $p_*\mathcal{F}' \simeq p_*\omega_{\tilde{X}}$, so the pushforwards of the kernel and cokernel of α are zero. So just one statement remains to be justified: if g is affine, and $g_*\mathcal{G} = 0$ for

some sheaf \mathcal{G} , then $\mathcal{G} = 0$. To prove this, let W be some affine open set in the target of g , so that $g^{-1}W$ is affine. Then, since $g_*\mathcal{G} = 0$, we have that $\mathcal{G}(g^{-1}W) = 0$. Then $\mathcal{G}|_{g^{-1}W} = 0$ because $g^{-1}W$ is an affine scheme, and since this is true for an affine cover of X by sets of the form $g^{-1}W$, \mathcal{G} must be the zero sheaf.

Now we use the result from the previous paragraph: here p is the affine morphism—it is affine because it is finite—and $\ker \alpha$ and then $\operatorname{coker} \alpha$ play the role of \mathcal{G} . Then, by that result, $\ker \alpha$ and $\operatorname{coker} \alpha$ are both zero, so $\mathcal{F}' \simeq \omega_{\tilde{X}}$.

Our original goal was to show that the sheaves \mathcal{F} and ω_X are isomorphic, and the work above shows that $\mathcal{F}' \simeq \omega_{\tilde{X}}$. Note that in the case where X is actually projective—not merely quasiprojective—then $\tilde{X} = X$ and we are done. To finish the more general case, we use that fact that the open immersion $X \rightarrow \tilde{X}$ factors uniquely through the normalization \tilde{X} . Indeed, $\iota : X \rightarrow \tilde{X}$ is an open immersion, and therefore smooth. Then the canonical sheaf ω_X is just the pullback $\iota^*\omega_{\tilde{X}}$, and the dualizing complex ω_X^\bullet pulls back to the dualizing complex $\omega_{\tilde{X}}^\bullet$. So the isomorphism of sheaves on \tilde{X} implies that the desired isomorphism also holds on X : $\mathcal{F} \simeq \omega_X^\bullet$. \square

Thus when X is normal, it is perfectly reasonable to use the notation ω_X to refer to both the canonical sheaf and the dualizing sheaf.

Recall that for non-smooth X , ω_X need not be invertible. But it may be, and this is a nice situation: then the canonical divisor K_X is Cartier, and so is much easier to handle.

Theorem 2.3.3 ([Har77, 7.11]). *Let X be a closed subscheme of \mathbf{P}_k^N , and suppose that X is a local complete intersection. Then the dualizing sheaf of X is invertible.*

Because normal schemes are S_2 , they satisfy an algebraic analogue of the result from complex analysis called Hartog's Lemma. This analytic result says that holomorphic functions extend over subsets of codimension two. For the algebraic version of this theorem, first view a Noetherian, integrally closed domain A as a subring of its quotient field $\operatorname{Frac}(A)$. Then A is equal to the intersection of all the localizations $A_{\mathfrak{p}}$, where \mathfrak{p} is a prime of height 1 ([Eis95, 11.4]). Primes of height 1 correspond to points of codimension 1. So the geometric version of this theorem says that if a rational function on $\operatorname{Spec} A$ is regular at all codimen-

sion 1 points, then it is actually a regular function. Below is another formulation of the statement, discussed in [Vak13].

Theorem 2.3.4 (Algebraic Hartog’s Lemma: see [Vak13, 11.3.10]). *If A is a Noetherian, integrally closed domain, and f is a rational function that’s regular on $(\text{Spec } A) \setminus V$ where V has codimension 2, then f extends to be regular on V .*

A normal scheme is covered by affines $\text{Spec } A$, where A is an integrally closed domain, so this analogue of Hartog’s Lemma holds for normal schemes.

Normal schemes may have singularities, but Serre’s criterion tells us that the singular locus has codimension at least 2. If we resolve those singularities, the structure sheaf has a nice property:

Proposition 2.3.5. *Suppose Y is normal, $f: Y \rightarrow X$ is projective and birational, and X, Y are integral. Then X is normal if and only if the natural map $\mathcal{O}_X \rightarrow f_*\mathcal{O}_Y$ is an isomorphism.*

This statement is well-known, but a reference seems to be lacking in the literature, so we provide a quick proof below.

Proof. Under the given hypotheses, X normal implies $\mathcal{O}_X \rightarrow f_*\mathcal{O}_Y$ is an isomorphism by the proof of [Har77, III.11.4].

Now assume $\mathcal{O}_X \rightarrow f_*\mathcal{O}_Y$ is an isomorphism. Since Y is normal, the map f factors through the normalization \tilde{X} , by the universal property of normalization. We have a diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ & \searrow \tilde{f} & \nearrow \sigma \\ & \tilde{X} & \end{array}$$

where the normalization morphism σ is finite, and hence separated. By [Har77, Ex. II.4.9], \tilde{f} is also projective. The normalization σ is birational, so we have $K(\tilde{X}) = K(X)$, since the schemes in question are integral. Thus \tilde{f} must also be birational.

So \tilde{f} is projective and birational and the target is normal. Then, by the beginning of this proof, $\tilde{f}_*\mathcal{O}_Y = \mathcal{O}_{\tilde{X}}$. Apply the functor σ_* to this isomorphism. This yields $\sigma_*\tilde{f}_*\mathcal{O}_Y = \sigma_*\mathcal{O}_{\tilde{X}}$, which becomes $f_*\mathcal{O}_Y = \sigma_*\mathcal{O}_{\tilde{X}}$ because $f = \sigma \circ \tilde{f}$.

Since $f_*\mathcal{O}_Y = \mathcal{O}_X$, we have $\mathcal{O}_X = \sigma_*\mathcal{O}_{\tilde{X}}$. So if $\text{Spec } A$ is any affine in X , we have $A = \tilde{A}$. Then X is equal to its normalization \tilde{X} , and we are done. \square

As a generalization of Serre's condition S_2 , define a condition S_m :

Definition 2.3.6. Let X be a scheme. Then X satisfies Serre's condition S_m if for each $x \in X$, $\text{depth } \mathcal{O}_x \geq \min(m, \text{codim}(x, X))$.

With this definition we can rephrase the CM condition: a scheme X is CM if it satisfies condition S_m for all m .

Recall that the properties CM and normal do not imply other in general. For curves and surfaces, normal does imply CM, but this fails in higher dimensions.

In practice, the varieties we will see here will usually be both CM and normal. These properties work nicely together. Normality allows us to define a canonical sheaf ω_X , which has rank 1, and the Cohen-Macaulay property lets us identify the dualizing complex with the canonical sheaf: $\omega_X^\bullet \simeq \omega_X[n]$.

2.4 Rational singularities

Loosely speaking, rational singularities are the ones that cohomology doesn't see. In this section we collect several definitions and well-known results about rational singularities.

We will see below that a variety X with rational singularities is cohomologically smooth, in the sense that resolving the singularities on X does not affect the cohomology of locally free sheaves on X , nor the cohomology of the dualizing (canonical) sheaf ω_X .

Definition 2.4.1 ([KM98, 5.8]). Let X be a variety, and let $f: Y \rightarrow X$ be a resolution of singularities. Then f is a rational resolution if the natural morphism $\mathcal{O}_X \rightarrow \mathcal{R}f_*\mathcal{O}_Y$ is a quasi-isomorphism.

In characteristic $p > 0$, we would need to include a bit more in the definition: in addition to the quasi-isomorphism, we would require that the higher direct images $R^i f_*\omega_Y$ vanish for

$i > 0$ ([KM98, 5.9]). Requiring these sheaves to vanish along with the higher direct images of the structure sheaf lets us use Grothendieck duality in a particularly simple way, for then we have

$$\mathcal{R}f_*\mathcal{O}_Y \simeq f_*\mathcal{O}_Y \simeq \mathcal{O}_X \quad \text{and} \quad \mathcal{R}f_*\omega_Y \simeq f_*\omega_Y \simeq \omega_X \quad (\text{see (2.4.5) below})$$

and the statement of duality degenerates nicely. But in characteristic 0, we don't need to separately assume this third condition. It is always true, by a result first proved in [GR70]:

Theorem 2.4.2 (Grauert-Riemenschneider vanishing: see [KM98, 2.68]). *Suppose $f: Y \rightarrow X$ is a proper, birational morphism of integral schemes of finite type over a field of characteristic 0, with Y smooth. (For example, let $f: Y \rightarrow X$ be a resolution of singularities.) Then $R^i f_*\omega_Y = 0$ for all $i > 0$.*

Because of the Grauert-Riemenschneider vanishing theorem, rational resolutions in characteristic 0 are defined using only the requirement in (2.4.1). In characteristic $p > 0$, we would tack on the additional condition.

The next theorem shows that having rational singularities doesn't depend on the choice of resolution. A desire for rational pairs to satisfy an analogous property featured heavily in the development of their definition in [Kol13], as we will see below.

Theorem 2.4.3 ([KM98, 5.10]). *If X has a rational resolution, then every resolution of X is rational.*

Definition 2.4.4 ([KM98, p. 155]). *If some—equivalently, every—resolution $f: Y \rightarrow X$ is rational, then X has rational singularities.*

2.4.A Characterizing rational singularities in characteristic zero

Next we'll discuss two ways to characterize rational singularities over a field of characteristic zero. The first is called Kempf's criterion. Its proof relies on Grauert-Riemenschneider vanishing, and so it holds in characteristic 0. But it is not generally true in characteristic $p > 0$.

Theorem 2.4.5 (Kempf’s criterion: see [KM98, 5.12]). *Let X be a variety with dualizing complex ω_X^\bullet , and let $f: Y \rightarrow X$ be a resolution of singularities. Then the following are equivalent:*

1. X has rational singularities;
2. There is an isomorphism in the derived category: $\omega_X^\bullet \simeq f_*\omega_Y[n]$;
3. X is CM and the trace map $f_*\omega_Y \rightarrow \omega_X$ is an isomorphism.

The trace morphism $f_*\omega_Y \rightarrow \omega_X$ is a natural, non-zero morphism of sheaves on X that exists as long as X and Y are equidimensional of dimension n , projective over k , and f is generically finite. If f is also birational, as in the statement of Kempf’s criterion, then the trace morphism is injective ([KM98, 5.77]).

Here is a version of Kempf’s criterion that is a little less general, but has the feature of building in the assumption of GR vanishing so it works in any characteristic. We present it here because a very similarly-phrased result will come up later for rational pairs.

Theorem 2.4.6 (Kempf’s criterion (alternate version): see [Kol13, 2.77]). *Let X be a variety over a field of any characteristic, and $f: Y \rightarrow X$ a resolution. Then f is a rational resolution if and only if:*

1. X is CM, and
2. The natural map $\mathcal{R}f_*\omega_Y \rightarrow \omega_X$ is a quasi-isomorphism.

Kempf’s criterion says, in particular, that rational singularities are CM. The definition of rational singularities together with (2.3.5) tells us they’re normal. So all the background results on CM and normal schemes from the beginning of this dissertation are relevant here.

Another characterization of rational singularities in characteristic zero is found in [Kov00]. Recall that the definition of rational singularities essentially requires that the natural map $\mathcal{O}_X \rightarrow \mathcal{R}f_*\mathcal{O}_Y$ is a quasi-isomorphism, when f is a resolution of singularities. It turns out that with a little less, we can still ensure that X has rational singularities.

Theorem 2.4.7 ([Kov00, Theorem 1]). *Suppose Y has rational singularities, and let $f: Y \rightarrow X$ be a morphism. Let $\rho: \mathcal{O}_X \rightarrow \mathcal{R}f_*\mathcal{O}_Y$ be the natural morphism induced by f . If there is a morphism $\rho': \mathcal{R}f_*\mathcal{O}_Y \rightarrow \mathcal{O}_X$ so that the composition $\rho' \circ \rho$ is a quasi-isomorphism of \mathcal{O}_X with itself, then X has rational singularities.*

Here the map f need not be a resolution of singularities, as in (2.4.1), nor even birational, and Y need not be smooth. Also, we do not require that $\mathcal{O}_X \rightarrow \mathcal{R}f_*\mathcal{O}_Y$ is an isomorphism in the derived category, but merely that it has a left inverse.

2.4.B Cohomological smoothness

If X has rational singularities, then in many cases computing sheaf cohomology on X is the same as computing sheaf cohomology on a resolution Y of X . The fact that rational singularities do not affect the cohomology of the structure sheaf or the canonical sheaf comes straight out of (2.4.1). Using the Leray spectral sequence ([Har77, Ex. III.8.1]), together with the conditions that $R^i f_*\mathcal{O}_Y = 0$ and $R^i f_*\omega_Y = 0$ for $i > 0$ and the isomorphisms $f_*\mathcal{O}_Y \simeq \mathcal{O}_X$ and $f_*\omega_Y \simeq \omega_X$, we have isomorphisms for all i :

$$H^i(Y, \mathcal{O}_Y) \simeq H^i(X, \mathcal{O}_X)$$

$$H^i(Y, \omega_Y) \simeq H^i(X, \omega_X).$$

In fact, this type of cohomology isomorphism holds not only for the structure sheaf and the canonical sheaf, but also for locally free sheaves of finite rank. If X has rational singularities, then the cohomology of locally free sheaves on X is also not affected by a resolution of singularities:

Proposition 2.4.8. *Suppose X has rational singularities, and $f: Y \rightarrow X$ is a resolution. Let \mathcal{E} be locally free of finite rank on X . Then for every i there is an isomorphism*

$$H^i(X, \mathcal{E}) \simeq H^i(Y, f^*\mathcal{E}).$$

This result is widely known, but we include a sketch of its proof here.

Proof. The claim follows from the projection formula ([Har77, Ex. III.8.3]) and the Leray spectral sequence. The projection formula shows that the natural morphism in the derived category

$$\mathcal{E} \rightarrow \mathcal{R}f_*(f^*\mathcal{E})$$

is an isomorphism, and the Leray spectral sequence gives the cohomology isomorphism ([Har77, Ex. III.8.1]). \square

2.4.C Vanishing theorems from the literature

In this section we state a very useful vanishing theorem for smooth projective varieties, and its extension to varieties with rational singularities. First, we'll state a few preliminary definitions.

Definition 2.4.9 ([KM98, 0.4.3]). A \mathbf{Q} -Cartier divisor D on X is nef (“numerically effective”) if $(D.C) \geq 0$ for every irreducible curve C on X .

A \mathbf{Q} -Cartier divisor is a divisor D such that one of its multiples mD is Cartier. \mathbf{Q} -Cartier divisors are nice because we can pull them back via morphisms. The pullback f^* is already defined for Cartier divisor classes: it coincides with the pullback on invertible sheaves, since the Picard group and the Cartier divisor class group are isomorphic (as long as we assume integrality—see [Har77, II.6.15]). Now, if we just know that D is \mathbf{Q} -Cartier, let mD be Cartier. Then define f^*D to be the \mathbf{Q} -divisor $\frac{1}{m}f^*(mD)$.

Definition 2.4.10 ([KM98, 2.59]). A Cartier divisor D on X is big if for all m large enough, the global sections of $\mathcal{O}_X(mD)$ define a rational map $\phi: X \rightarrow \mathbf{P}^N$ such that X is birational to $\phi(X)$.

Thus “big” is a birational version of “ample”. If $f: Y \rightarrow X$ is birational, then a Cartier divisor D on X is big if and only if the pullback f^*D is big ([KM98, p. 67]).

Theorem 2.4.11 (Kawamata-Viehweg vanishing: [Kaw82], [Vie82], or see [KM98, 2.64]). *Let $f: Y \rightarrow X$ be a birational morphism over k (characteristic 0), with Y smooth, and let \mathcal{L} be a big and nef line bundle on X . (For example, let \mathcal{L} be ample.) Then*

1. $H^i(Y, \omega_Y \otimes f^* \mathcal{L}) = 0$ for $i > 0$
2. $H^i(Y, f^* \mathcal{L}^{-1}) = 0$ for $i < \dim Y$.

Statements (1) and (2) are equivalent by Serre duality. Now combine Kawamata-Viehweg vanishing with (2.4.8). The result is a Kodaira-type vanishing theorem on X .

Corollary 2.4.12 (Kodaira-type vanishing theorem). *Suppose X has rational singularities, and let \mathcal{L} be an ample line bundle on X . Then*

1. $H^i(X, \mathcal{L}^{-1}) = 0$ for $i > 0$
2. $H^i(X, \omega_X \otimes \mathcal{L}) = 0$ for $i < \dim X$.

Again, (1) and (2) are equivalent by Serre duality, which holds in this case because rational singularities are Cohen-Macaulay.

2.5 Singularities of pairs

Now let X be normal, let $f: Y \rightarrow X$ be a resolution of singularities, and let $D = \sum n_i D_i$ be a \mathbf{Q} -divisor on X with $0 \leq n_i \leq 1$ for all i . Let $D_Y \subset Y$ be the birational transform of D under f ; that is, $D_Y = (f^{-1})_* D$ (usually just written “ $f_*^{-1} D$ ”). Here the notation “ f^{-1} ” means the morphism representing the birational transformation f^{-1} on the largest open set where it’s defined.

Also, write K_X, K_Y for the canonical divisors on X and Y , so that $\omega_X = \mathcal{O}_X(K_X)$, and similarly for Y .

2.5.A Normal crossings conditions

The analogue of “smooth” for pairs (X, D) is simple normal crossings, or snc. A weaker notion is normal crossings, or nc.

Definition 2.5.1 ([Kol13, 1.7]). A pair $(X, D = \sum n_i D_i)$ has simple normal crossings (or “is snc”) if X is a smooth variety, each D_i is smooth, and all the intersections are transverse. If X is not smooth or if D doesn’t satisfy those hypotheses, then we can still discuss the

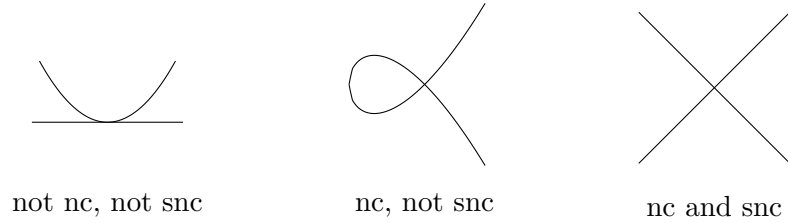


Figure 2.1: Normal crossings vs. simple normal crossings.

largest open set $U \subset X$ so that the pair $(U, D|_U)$ is snc. This pair $(U, D|_U)$ is called the snc locus of (X, D) , and is written $\text{snc}(X, D)$. We also may refer to the non-snc locus, $\text{non-snc}(X, D)$, which is just $X \setminus \text{snc}(X, D)$.

Definition 2.5.2 ([Kol13, 1.7]). Let (X, D) be as above. Then (X, D) has normal crossings if every point has an étale neighborhood $\pi: U \rightarrow X$ so that $(U, \pi^{-1}D)$ is snc.

Being snc is a local property in the Zariski topology, but it cannot be checked étale-locally. It's clear from the definition that being nc is local in the étale topology. For an example of something that's nc but not snc, take the nodal cubic curve $y^2 = x^2(x + 1)$ (Figure 2.1). If (X, D) is nc, it is also snc if and only if each of the irreducible components of D is smooth.

2.5.B Resolutions of pairs

A resolution of a pair (X, D) is a resolution of singularities $f: Y \rightarrow X$ so that the pair $(Y, B = f_*^{-1}D)$ is snc. We write $f: (Y, B) \rightarrow (X, D)$ for a resolution of pairs.

An especially nice kind of resolution of (X, D) is one where the entire preimage of D , together with any other exceptional components, is an snc divisor in Y .

Definition 2.5.3 ([Kol13, 1.12]). Let $f: Y \rightarrow X$ be a resolution, with D a divisor on X . If the exceptional locus $E := \text{Ex}(f) \subset Y$ is a divisor and the pair $(Y, B + E)$ is snc, then f is called a log resolution.

Log resolutions exist by [Hir64]. They are used to define several well-understood classes of singularities for pairs, which we'll describe now. Let f be a log resolution of a pair (X, D) , and suppose $K_X + D$ is \mathbf{Q} -Cartier. Note that this might be true even if K_X is not \mathbf{Q} -Cartier, which is one reason it's useful to study pairs (X, D) .

Now compare $(K_Y + B)$ with $f^*(K_X + D)$:

$$K_Y + B \sim f^*(K_X + D) + \sum a_i E_i.$$

Here the a_i are rational numbers and the E_i are exceptional prime divisors. The numbers a_i are called discrepancies. We can use discrepancies to define several levels of singularities, in increasing order of badness.

Definition 2.5.4 ([Kol13, 2.8]). A pair (X, D) has

terminal singularities	if $a_i > 0, \forall a_i$
canonical singularities	if $a_i \geq 0, \forall a_i$
Kawamata log terminal (klt) singularities	if $a_i > -1, \forall a_i$, and $[D] = 0$
purely log terminal (plt) singularities	if $a_i > -1, \forall a_i$
divisorial log terminal (dlt) singularities	if $a_i > -1, \forall a_i$ such that $f(E_i) \subset \text{non-snc}(X, D)$
log canonical (lc) singularities	if $a_i \geq -1, \forall a_i$

These definitions appear to depend on the choice of resolution $f: (Y, B) \rightarrow (X, D)$, but in fact they do not. Also, log canonical singularities are the worst ones that are discernible by calculating discrepancies: if there is any resolution $f: (Y, B) \rightarrow (X, D)$ with some $a_i < -1$, then for any n we can find a resolution of (X, D) with some $a_i < n$. In other words, once a pair is worse than log canonical, it has arbitrarily large discrepancies.

A general principle is that as we add components to the divisor D , the singularities of (X, D) get worse. This is because more exceptional divisors are required to make up the difference between $K_Y + B$ and $f^*(K_X + D)$. On the other hand, if we write $D = C + C'$, with C and C' effective, and if $K_X + C$ is also \mathbf{Q} -Cartier, then the singularities of (X, C) are at most as severe as those of (X, D)

We say “ X has terminal (canonical, lc) singularities” if the pair $(X, 0)$ does. To avoid confusion, when we refer to a pair (X, D) where $D \neq 0$ we merely say “ (X, D) is terminal (canonical, lc, ...)”.

In particular, if (X, D) is terminal (canonical, lc), then the variety X has the same type of singularities. We will see below that the analogous statement for rational pairs (X, D) is not true. That is, if (X, D) is rational, then it is *not* guaranteed that the variety X has rational singularities.

The next theorem from the literature shows how rational singularities are related to the singularities defined in terms of discrepancies

Theorem 2.5.5 ([Elk81, Theorem 1], or see [Kov00, Theorem 4]). *Log terminal singularities are rational.*

The converse is true with some extra hypotheses. If X is normal, with the canonical divisor K_X Cartier, then X has rational singularities if and only if X has canonical singularities ([KM98, 5.24]).

Chapter 3

BACKGROUND: RATIONAL PAIRS

In this chapter we describe a recent generalization of the notion of rational singularities to pairs (X, D) . This generalization is introduced and developed in [Kol13, Section 2.5]. We'll restrict our attention to certain types of pairs (X, D) for this section. We'll take X to be a projective variety and D a reduced (Weil) divisor on X . A reduced divisor is one that can be written as $\sum n_i D_i$, where the D_i are distinct and each $n_i = 1$. In particular, reduced divisors are effective, and we may think of them as mere unions of codimension-one subvarieties.

The divisors are reduced, so if $f: Y \rightarrow X$ is a resolution, then the birational transform B is the closure in Y of the image of D under the morphism f^{-1} .

Definition 3.0.6 ([Kol13, 4.15]). If $f: (Y, B) \rightarrow (X, D)$ is a resolution of a pair (X, D) , and we write

$$(K_Y + B) \sim f^*(K_X + D) + \sum a_i E_i$$

then each E_i with $a_i = -1$ is an lc (log canonical) place. The image of an lc place in X is an lc center.

The lc centers of a pair (X, D) are well defined: they depend only on the divisor, not on the choice of resolution $f: (Y, B) \rightarrow (X, D)$.

Definition 3.0.7 ([Kol13, 1.7]). If (X, D) is an snc pair, with $D = \sum D_i$, then a stratum of (X, D) is an irreducible component of any intersection of the D_i .

If (X, D) is snc, then its strata are exactly its lc centers: see [Kol13, 4.15]. In other words, they are exactly the closed subvarieties that, if blown up, would produce a discrepancy of -1 . When we discuss the definition of rational pairs in [Kol13] in the next section, we'll see that the “right” resolutions of pairs are the ones that satisfy certain conditions over the lc centers.

3.1 Rational resolutions

The first step towards extending rational singularities to pairs is determining what should be a rational resolution. The definition in [Koll13] is a natural extension of (2.4.1).

Definition 3.1.1 ([Koll13, 2.78]). Let X be a normal variety and let D be a reduced divisor on X . Let $f: (Y, B) \rightarrow (X, D)$ be a resolution so that (Y, B) is snc. Then $f: (Y, B) \rightarrow (X, D)$ is a rational resolution if

1. The natural morphism $\mathcal{O}_X(-D) \rightarrow \mathcal{R}f_*\mathcal{O}_Y(-B)$ is a quasi-isomorphism, and
2. The higher direct images $R^i f_*\omega_Y(B)$ vanish for $i > 0$.

Here the situation is a bit different from what we saw before in (2.4.1): there we did not need to include an additional condition about the higher direct images of the canonical sheaf, but here we do need to include the analogous condition (numbered 2). The analogue of the Grauert-Riemenschneider vanishing theorem that we would need for this to hold automatically is not true. However, we'll see in (4.4.2) below that the third condition is not a problem in practice, and usually can in fact be ignored as in (2.4.1). That is, an analogue of GR vanishing does hold for our restricted class of resolutions.

3.2 Characterizing rational resolutions

Here we state some analogues to Kempf's criterion (2.4.5) for rational resolutions. The next two results hold in any characteristic.

Theorem 3.2.1 ([Koll13, 2.83]). *Let (X, D) be a reduced pair, and let $(Y, B) \rightarrow (X, D)$ be a resolution. Suppose $n = \dim X = \dim Y$. Then f is a rational resolution if and only if two conditions hold:*

1. $\mathcal{R}\mathcal{H}om(\mathcal{O}_X(-D), \omega_Y^\bullet) \simeq f_*\omega_Y(-B)[n]$ (in the derived category), and
2. $R^i f_*\omega_Y(B) = 0$ for $i > 0$.

Now (2) here is the same as condition (3) in (3.1.1), so the content of this theorem is that (1) above is equivalent to the first two conditions in (3.1.1). This is a consequence of Grothendieck duality; see (2.2.2).

Remember that we only required D to be a reduced Weil divisor. In the case where D is Cartier, then we get a statement that looks very much like the statement of Kempf's criterion in (2.4.6):

Theorem 3.2.2 ([Kol13, 2.84]). *Let (X, D) be a reduced pair, and let $f: (Y, B) \rightarrow (X, D)$ be a resolution. Then f is a rational resolution of (X, D) if and only if:*

1. X is CM, and
2. The natural morphism $\mathcal{R}f_*\omega_Y(B) \rightarrow \omega_X(D)$ is a quasi-isomorphism.

We'll call the morphism $f_*\omega_Y(B) \rightarrow \omega_X(D)$ that results from taking the 0th cohomology of $\mathcal{R}f_*\omega_Y(B) \rightarrow \omega_X(D)$ the logarithmic trace. It will come up again in the last section of this dissertation.

Now a nice result analogous to (2.4.3) would be this: if a pair (X, D) has a rational resolution, then every resolution of pairs $(Y, B) \rightarrow (X, D)$ is rational. Then we could say that a pair (X, D) is rational if it has a rational resolution, analogously to how we define rational singularities for varieties. But this doesn't work, because there are some pairs that have rational resolutions and also non-rational resolutions. Here is a simple example.

Example 3.2.3. This is an expanded version of the discussion following [Kol13, 2.78]. Let $X = \mathbf{P}^2$, and let D be a union of two nonsingular, irreducible curves in X , meeting transversally at one point x . Then the identity map $(X, D) \rightarrow (X, D)$ is a rational resolution: the three conditions follow trivially, and (X, D) is snc. On the other hand, let $f: (Y, B) \rightarrow (X, D)$ be the resolution given by blowing up x . Then we have the beginning of a long exact sequence of sheaves on X :

$$0 \longrightarrow f_*\mathcal{O}_Y(-B) \longrightarrow f_*\mathcal{O}_Y \longrightarrow f_*\mathcal{O}_B \longrightarrow R^1f_*\mathcal{O}_Y(-B) \longrightarrow 0$$

The zero at the end of the sequence above is $R^1f_*\mathcal{O}_Y$, which vanishes because X is smooth and hence has rational singularities. Taking stalks at x , we see that $(R^1f_*\mathcal{O}_Y(-B))_x = 0$ if

and only if the map $(f_*\mathcal{O}_Y)_x \rightarrow (f_*\mathcal{O}_B)_x$ is surjective. But it cannot be surjective: the term on the left is just the local ring \mathcal{O}_x , and the term on the right requires two generators over \mathcal{O}_x since x has two preimages in B . So $f: (Y, B) \rightarrow (X, D)$ is not a rational resolution. In fact, it is noted in [Kol13, 2.78] that a resolution f of an snc pair (X, D) is rational if and only if f is an isomorphism over the generic point of every stratum of (X, D) . Put another way, $f(\text{Ex}(f))$ doesn't contain any stratum of (X, D) , and this fails here.

This example shows that rational resolutions of pairs (X, D) don't work as nicely as rational resolutions of X do: that one rational resolution exists is not enough to guarantee that all resolutions will be rational.

The problem with this analogy to (2.4.3) is not the way rational resolutions are defined, but rather that "resolution" is too general. In [Kol13], Kollár and Kovács found that by restricting attention to a certain subclass of resolutions of pairs (X, D) , it's possible to get a well-behaved notion.

3.3 Introduction to thrifty resolutions

Here we briefly describe this subclass of resolutions, and show how they are used to define rational pairs in [Kol13]. In the next section, we prove several foundational new results about thrifty resolutions.

Our definition comes from [Kol13, 2.79]. Note that it is phrased differently (but equivalently!) there.

Definition 3.3.1 ([Kol13, 2.79]). Suppose X is a normal variety and D is a reduced divisor on X . Let $f: (Y, B) \rightarrow (X, D)$ be a resolution, where B , the birational transform of D , is snc. Then f is a thrifty resolution if the following two conditions hold:

Condition 1. f is an isomorphism over the generic point of every stratum of $\text{snc}(X, D)$; equivalently, $f(\text{Ex}(f))$ does not contain any stratum of $\text{snc}(X, D)$

Condition 2. f is an isomorphism at the generic point of every stratum of (Y, D) ; equivalently, the exceptional locus $\text{Ex}(f)$ does not contain any stratum of (Y, B) .

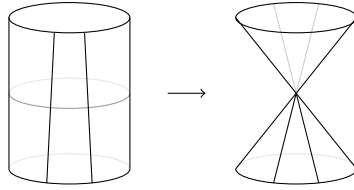


Figure 3.1: A thrifty, non-rational resolution.

In the next chapter, we will frequently refer to these conditions as Condition 1 and Condition 2.

In other words, this definition requires that no stratum of (Y, B) is contracted, and no stratum of the snc locus of (X, D) is in the image of what's contracted.

Observe that in the degenerate case $D = 0$, all resolutions are thrifty resolutions.

Theorem 3.3.2 ([Koll13, 10.45.2]). *In characteristic 0, thrifty resolutions exist. Any two of them can be dominated by a third thrifty resolution.*

The definition (3.3.1) takes care of what went wrong in (3.2.3). There, we took a pair (X, D) that was already snc, and we blew up one of its strata, the point x . But this was unnecessary: (X, D) was already snc at x . The thrifty resolutions are the ones that avoid altering the strata where the pair is already snc.

Thrifty resolutions needn't be rational. For example, take C to be a conic in \mathbf{P}^2 , and X to be the affine cone over C in \mathbf{A}^3 . Let D be the divisor on X given by the union of two lines through the cone point x , and let $f: (Y, B) \rightarrow (X, D)$ be the resolution given by blowing up x (Figure 3.1). This is not a rational resolution, by the same argument as we saw in (3.2.3): the blown-up point in D has two preimages in B , and so the map of stalks at x is not an isomorphism. But f is a thrifty resolution: Condition 1 holds because $\text{snc}(X, D)$ doesn't contain x , and Condition 2 is automatically satisfied because the resolution is log. (Also, clearly the components of B do not intersect.)

The question of whether rational resolutions are necessarily thrifty is currently open. Later, in (4.3.4), we give a partial answer: if a rational resolution is also a log resolution,

then it is thrifty.

Definition 3.3.3 ([Kol13, 2.80]). If X is a normal variety, with D a reduced divisor on X , then the pair (X, D) is rational if it has a thrifty rational resolution.

Since smooth varieties have rational singularities, it is reasonable that snc pairs (which are analogous to smooth varieties) should be rational. Indeed, it's clear from (3.3.3) that snc pairs are rational.

Now we have the analogue of (2.4.3) for pairs in characteristic 0.

Theorem 3.3.4 ([Kol13, 2.86]). *If (Y, D) has a thrifty rational resolution, then every other thrifty resolution is rational.*

In this way, the thrifty rational resolutions of pairs (X, D) are the ones that *do* behave as nicely as rational resolutions of the variety X do.

3.4 An alternate generalization

The definition of rational pairs in (3.1.1) is a natural analogue of the definition of rational singularities of varieties in (2.4.1). Why, then, was this theory not developed until 2013? Presumably this definition did not appear in the literature until very recently because it would be desirable to have a result like (2.4.3), where it is only required that the variety X has one rational resolution for every resolution to have the desired property. But the most obvious analogue of that theorem would be “if (Y, D) has a rational resolution, then every resolution of (Y, D) is rational”, and we have seen in (3.2.3) that this fails. But when we restrict to thrifty resolutions, we see that having a thrifty rational resolution implies that every other thrifty resolution is rational (as in (3.3.4)), so now (3.1.1) is a reasonable definition.

But there are other ways one might decide to extend rational singularities to pairs. In [ST08], Schwede and Takagi defined rational pairs so that they would satisfy a property of the form “if (Y, D) is rational, and C is a smaller divisor contained in D , then (Y, C) is also rational.” This is analogous to how terminal, canonical, klt, dlt, and lc singularities of pairs behave: adding components to D can only worsen those singularities, but removing components of D will not.

This is a big difference between the two new definitions. Under the definition in [Kol13], it is possible for a pair (Y, D) to be rational but for Y not to be rational ([Kol13, 2.81.2]). Arguably, this is a feature: it could allow us to treat some varieties X with worse singularities as if they had rational singularities.

In the rest of this work we will only be using the thrifty-resolutions definition of rational pairs in [Kol13]; the definition from [ST08] won't come up again here.

3.5 When is there a D so that (X, D) is rational?

We just saw that the definition of rational pairs from [Kol13] has a curious feature. The singularities for pairs defined in terms of discrepancies (terminal, canonical, log terminal, log canonical) have the property that adding pieces to the divisor D makes the singularities more severe. But this isn't true for rational pairs: see [Kol13, 2.81.2]. An interesting question is, does there exist a criterion for a variety X to have a divisor D so that (X, D) is rational? Such an X could be called “potentially rational”. Certainly X having rational singularities is sufficient, but it is not necessary. We do not settle this question here, but we will now describe one situation where the existence of a D is impossible.

Before we give the result, we'll recall a well-known fact about resolutions of singularities that will be useful. For lack of an adequate reference, a proof is included.

Theorem 3.5.1. *Suppose $f: X \rightarrow Y$ and $f': X' \rightarrow Y$ are resolutions of Y . Then $\mathcal{R}f_*\mathcal{O}_X \simeq \mathcal{R}f'_*\mathcal{O}_{X'}$ in the derived category. In particular, for each i , $R^i f_*\mathcal{O}_X \simeq R^i f'_*\mathcal{O}_{X'}$.*

Proof. Let $h: Z \rightarrow Y$ be a resolution dominating f and f' , so that there are morphisms $g: Z \rightarrow X$ and $g': Z \rightarrow X'$ with

$$h = f \circ g = f' \circ g'.$$

Then we have

$$\mathcal{R}f_*\mathcal{R}g_*\mathcal{O}_Z \simeq \mathcal{R}h_*\mathcal{O}_Z \simeq \mathcal{R}f'_*\mathcal{R}g'_*\mathcal{O}_Z,$$

by the Grothendieck spectral sequence.

Now X and X' are smooth, so they have rational singularities. So $\mathcal{R}g_*\mathcal{O}_Z \simeq \mathcal{O}_X$ and $\mathcal{R}g'_*\mathcal{O}_{X'} \simeq \mathcal{O}_{X'}$, from which we conclude that $\mathcal{R}f_*\mathcal{O}_X \simeq \mathcal{R}f'_*\mathcal{O}_{X'}$. \square

Now we describe a class of varieties X that cannot be part of a rational pair (X, D) .

Proposition 3.5.2. *Suppose A is an n -dimensional projective variety with nonzero top cohomology: that is, $H^n(A, \mathcal{O}_A) \neq 0$. Let X be the projective cone over A , so that X has dimension $n + 1$. Then there does not exist a divisor $D \subset X$ so that the pair (X, D) has a rational resolution.*

Proof. Let $f: Y \rightarrow X$ be any resolution of singularities. We'll show that $R^n f_* \mathcal{O}_Y \neq 0$.

By (3.5.1), it suffices to show that $R^n f'_* \mathcal{O}_{Y'} \neq 0$, where $f': Y' \rightarrow X$ is the resolution of Y given by blowing up the cone point x . Let $E \simeq A$ be the exceptional divisor of f' in Y' , and consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_{Y'}(-E) \longrightarrow \mathcal{O}_{Y'} \longrightarrow \mathcal{O}_E \longrightarrow 0.$$

Push it forward by f' , obtaining a long exact sequence. Since f' has no fibers of dimension greater than n , all the R^{n+1} terms vanish, and the tail of the sequence is

$$\longrightarrow R^n f'_* \mathcal{O}_{Y'} \longrightarrow R^n f'_* \mathcal{O}_E \longrightarrow 0.$$

Now f' maps E to a point, x . In particular, $f': E \rightarrow \{x\}$ is a map to an affine scheme. Thus $R^n f'_* \mathcal{O}_E \simeq H^n(E, \mathcal{O}_E)$. This cohomology group is nonzero by assumption, since $E \simeq A$. So $R^n f'_* \mathcal{O}_{Y'}$ cannot be zero either, or it would sandwich the nonzero group between zeroes. By (3.5.1), this means that $R^n f_* \mathcal{O}_Y \neq 0$.

Now let D be any reduced divisor on X , and let $f: (Y, B) \rightarrow (X, D)$ be a resolution of the pair. In particular, f is a resolution of X . We'll show this resolution of (X, D) cannot possibly be rational.

Consider the short exact sequence for the inclusion $B \subset Y$:

$$0 \longrightarrow \mathcal{O}_Y(-B) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_B \longrightarrow 0.$$

Push this forward by f , and observe that again all the R^{n+1} terms vanish (by looking at fiber dimensions). The tail of the sequence is then

$$\longrightarrow R^n f_* \mathcal{O}_Y(-B) \longrightarrow R^n f_* \mathcal{O}_Y \longrightarrow R^n f_* \mathcal{O}_B \longrightarrow 0.$$

Now $f: B \rightarrow D$ has fibers of dimension at most $n - 1$ (in order to be birational), so the group on the right is 0. The map $R^n f_* \mathcal{O}_Y(-B) \rightarrow R^n f_* \mathcal{O}_Y$ is then surjective. This means that $R^n f_* \mathcal{O}_Y(-B)$ is nonzero, since it maps onto a nonzero sheaf. According to (3.1.1), the higher direct images of $\mathcal{O}_Y(-B)$ must vanish in order for a resolution to be rational. So this resolution of (X, D) is not rational, and no rational resolution of the pair exists. \square

This result shows, in particular, that the cone over an abelian variety or a K3 surface cannot be the X in a rational pair (X, D) .

Chapter 4

THRIFTY RESOLUTIONS

In this section we introduce several foundational results about thrifty resolutions. There are two main types of results. Some are concerned with making it easier to tell whether a given resolution is thrifty, by pointing out some situations where one of the two conditions for thriftiness is automatically true. Others have to do with showing that thrifty resolutions of pairs behave nicely with restriction to subpairs.

4.1 Results from the literature

Two theorems from the literature will be very helpful in the proofs that follow. The first is from [Sza94]; see also [Kol13, 10.45].

Theorem 4.1.1 ([Sza94]). *If X is a variety over a field of characteristic 0, and D is a Weil divisor on X , then (X, D) has a log resolution that is an isomorphism over $\text{snc}(X, D)$.*

The second result is a very useful fact about dlt pairs. See also [Kol13, 4.16.2].

Theorem 4.1.2 ([Fuj07, 3.9.2]). *If (X, Δ) is a dlt pair, if D_1, \dots, D_r are the irreducible divisors that appear in Δ with coefficient 1, and if $\{i_1, \dots, i_s\} \subset \{1, \dots, r\}$, then every irreducible component of $D_{i_1} \cap \dots \cap D_{i_s}$ has pure codimension s .*

4.2 Examples and non-examples

Some pictures of thrifty and non-thrifty resolutions of the plane appear in Figure 4.1.

4.3 Finding thrifty resolutions

As defined in [Kol13], a resolution $(Y, B) \rightarrow (X, D)$ is thrifty if it satisfies the following two conditions:

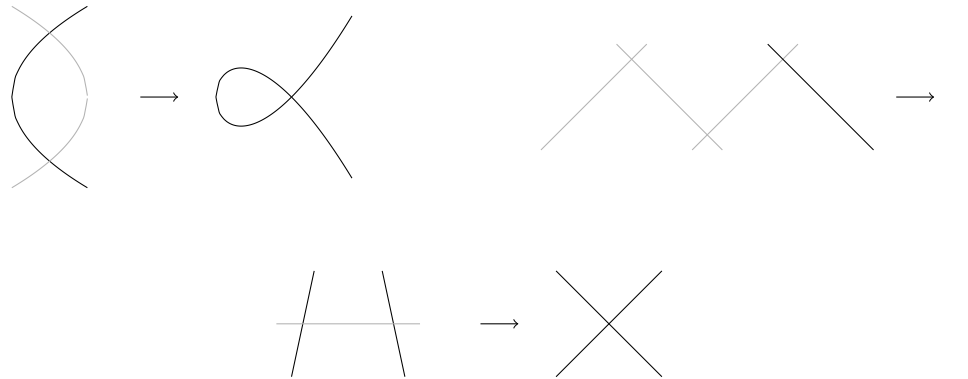


Figure 4.1: Two thrifty resolutions (above), and a non-thrifty resolution (below). The ambient space X is a plane.

1. f is an isomorphism over the generic point of every stratum of $\text{snc}(X, D)$; equivalently, no stratum of $\text{snc}(X, D)$ is contained in $f(\text{Ex}(f))$
2. f is an isomorphism at the generic point of every stratum of (Y, B) ; equivalently, no stratum of (Y, B) is contained in $\text{Ex}(f)$.

As we noted in the last chapter, it will be convenient for the rest of this dissertation to be able to refer to these conditions by name. We will therefore call them Condition 1 and Condition 2 from now on.

Observe that Condition 1 is a property of $\text{snc}(X, D)$: this makes it relatively easy to deal with, because X , the D_i , and all the intersections of the D_i are then smooth. Condition 2, on the other hand, is *not* a property of $\text{snc}(X, D)$, so we must examine points of (Y, B) that lie over the non-snc locus of (X, D) to check for it. Points outside of $\text{snc}(X, D)$ are trickier to deal with, so alternative ways to verify Condition 2 would be welcome.

We'll show below that Condition 2 is automatic if f is a log resolution. So if f is log, then it is thrifty if and only if it satisfies Condition 1. Recall that a resolution $(Y, B) \rightarrow (X, D)$ is log if the exceptional locus $\text{Ex}(f)$ is a divisor and the pair $(Y, B + \text{Ex}(f))$ is snc. This is a much stronger requirement than that (Y, B) be snc, as in the definition of a resolution of pairs.

Proposition 4.3.1. *If $f: (Y, B) \rightarrow (X, D)$ is a log resolution, then f satisfies Condition 2.*

Proof. Let E be the reduced divisor supported on $\text{Ex}(f)$, so that the pair $(Y, B + E)$ is snc and $B + E$ is reduced. Condition 2 fails exactly when a stratum of (Y, B) is contained in E . Let Z be some stratum of (Y, B) , so that Z is a component of some intersection $\bigcap_{i \in I} D'_i$, where $I = i_1, \dots, i_s$. Then by (4.1.2), which applies because (Y, B) is snc and therefore dlt, Z has codimension s .

Now if $Z \subset E$, then Z is contained in some component E_j of E . But then $Z \subset E_j \cap (\bigcap_{i \in I} D'_i)$. Now $(Y, B + E)$ is also dlt, and so this intersection has codimension $s + 1$ by (4.1.2). So this is impossible: Z cannot be contained in E . Thus Condition 2 always holds for a log resolution. \square

Now we list a few results about when a pair has a thrifty resolution, and when a resolution is guaranteed to be thrifty.

Theorem 4.3.2. *Suppose $f: Y \rightarrow X$ is a proper birational morphism between normal varieties, and $D \subset X$ is a reduced divisor. If f is an isomorphism over every stratum of $\text{snc}(X, D)$, then there is a thrifty log resolution of (X, D) factoring through (Y, B) .*

Proof. Let $B = f_*^{-1}D$ be the birational transform of D in Y . We'll construct a thrifty log resolution g of (X, D) starting with the pair (Y, B) .

First, blow up the exceptional components in Y of the birational morphism $f: Y \rightarrow X$. From this we obtain a new proper birational morphism $f': Y' \rightarrow X$, and a new birational transform B' of D in Y' . Now the exceptional locus of $Y' \rightarrow X$ is now a divisor in Y' ; call it E' .

Let h be a log resolution of the pair $(Y', B' + E')$ that is an isomorphism over $\text{snc}(Y', B' + E')$. To do this, we appeal to Szabó's theorem from [Sza94], which says that choosing such a resolution is possible; see [Kol13, 10.45.2].

Now write $g = f \circ f' \circ h$. It is a log resolution of X and the birational transform of D by g is an snc divisor. We'll show now that g is also an isomorphism over the generic point of every stratum of $\text{snc}(X, D)$.

Let $U \subset X$ be the open set over which f is an isomorphism. Let x be the generic point of any stratum of $\text{snc}(X, D)$. Then by assumption $x \in U$, so $x \in U \cap \text{snc}(X, D)$. The

pair (Y, B) is snc at the preimage of x in Y , because f is an isomorphism there. Similarly, (Y', B') is snc at the preimage of x in Y' : again, the blowups $f': Y' \rightarrow Y$ do not affect points outside the exceptional locus of f , so the entire preimage of $U \cap \text{snc}(X, D)$ in Y' is inside the locus where f' is an isomorphism. The exceptional locus E' is disjoint from the preimage of x in Y' , so the pair $(Y', B' + E')$ is also snc there. We chose the resolution h of $(Y', B' + E')$ to be an isomorphism over $\text{snc}(Y', B' + E')$, so the composition $g = f \circ f' \circ h$ is an isomorphism over x .

In other words, g satisfies Condition 1 of thriftiness from (3.3.1). Since g is a log resolution of (X, D) , it follows from (4.3.1) that g is thrifty. So there is a thrifty log resolution of (X, D) factoring through f . \square

Corollary 4.3.3. *Every thrifty resolution is dominated by a thrifty log resolution.*

Corollary 4.3.4. *If a log resolution of a pair is rational, then it is thrifty.*

Proof. Let $f: (Y, B) \rightarrow (X, D)$ be a rational resolution. We just need to check that f satisfies Condition 1 and Condition 2. Since f is log, Condition 2 is automatic by (4.3.1). Condition 1, on the other hand, is a property of $\text{snc}(X, D)$. Rational resolutions are defined in terms of sheaves, and $U = \text{snc}(X, D)$ is an open set, so the restriction of f to $f^{-1}(U)$ is still a rational resolution of $(U, D \cap U)$. Now snc pairs are dlt, so by [Kol13, 2.87] f is thrifty. \square

4.4 Properties of thrifty resolutions

Next we'll show that thrifty resolutions satisfy an analogue of the Grauert-Riemenschneider vanishing theorem; see (2.4.2). Thus the third condition in (3.1.1) is automatically true, at least in situations where we know the resolution in question is thrifty.

To prove the vanishing theorem, we'll start with a result from [Kol13].

Proposition 4.4.1 (Special case of [Kol13, 10.34]). *Let (Y, B) be snc, and $f: Y \rightarrow X$ a projective morphism. For any lc center Z of (Y, B) , write $F_Z \subset Z$ for the generic fiber of $f|_Z: Z \rightarrow f(Z)$. Set*

$$c = \max\{\dim F_Z : Z \text{ is an lc center}\}.$$

Then $R^i f_* \omega_Y(B) = 0$ for $i > c$.

The statement of [Kol13, 10.34] is more general than what we stated above, but we only need this version here.

Proposition 4.4.2 (GR-type vanishing for thrifty resolutions). *Let $f: (Y, B) \rightarrow (X, D)$ be a thrifty resolution. Then $R^i f_* \omega_Y(B) = 0$ for all $i > 0$.*

Proof. We'll appeal to (4.4.1), since (Y, B) is an snc pair and the resolution $f: Y \rightarrow X$ is projective. Let Z be a stratum of (Y, B) , and let F_Z be the generic fiber of the map $f|_Z: Z \rightarrow f(Z)$. The image $f(Z)$ is closed, since f is projective, and it is also irreducible because Z is, so it has a generic point.

By (4.4.1), $R^i f_* \omega_Y(B) = 0$ for all $i > c$, where c is defined to be

$$c := \max(\dim F_Z : Z \text{ is a stratum}).$$

Since f is thrifty, it is birational on every stratum of (Y, B) . In particular, it is dominant when restricted to each Z , so the dimension of each generic fiber is 0: see [Eis95, p. 290]. Thus $c = 0$, so $R^i f_* \omega_Y(B) = 0$ for all $i > 0$. \square

Next we check that birational transform works well under composition of birational maps.

Lemma 4.4.3. *Let (X, D) be a reduced pair. Suppose we have a commutative diagram*

$$\begin{array}{ccc} Z & \xrightarrow{h} & Y \\ & \searrow g & \downarrow f \\ & & X \end{array}$$

such that f, g, h are birational. Also, suppose that the birational transforms $D_Z := g_^{-1}D$, $B := f_*^{-1}D$ are defined. Then $h_*^{-1}B = D_Z$.*

Proof. First assume D is irreducible: that is, it has just one component. If we prove it in this case, we are done, because we can just repeat the argument for each component of D .

Now let $U_{fg} \subset X$ be the open set on which the morphisms f^{-1} and g^{-1} are defined. Then f^{-1}, g^{-1} are actually isomorphisms on U_{fg} . (Find this set by taking the open sets for

f and g individually, and then intersecting them.) Then $g^{-1}(U_{fg}) \subset Z$ maps into $f^{-1}(U_{fg})$ by h , by commutativity of the original diagram. Thus we have a new commutative diagram

$$\begin{array}{ccc} g^{-1}(U_{fg}) & \xrightarrow{h} & f^{-1}(U_{fg}) \\ & \searrow \simeq & \downarrow \simeq \\ & & U_{fg} \end{array}$$

Since two of the maps are isomorphisms, h must be too.

Now we need to check that the closure of $g^{-1}(D \cap U_{fg})$ is equal to the closure of $h^{-1}(B \cap f^{-1}(U_{fg}))$. But $B \cap f^{-1}(U_{fg}) = f^{-1}(D \cap U_{fg})$, by definition of the birational transform. So the diagram tells us that this is true. Taking closures in Z , then, we get the same thing. Thus we can use D_Z to denote the birational transform of D and also the birational transform of B . \square

Next we verify that thrifty resolutions are closed under composition.

Proposition 4.4.4. *If $g : (Z, D_Z) \rightarrow (Y, B)$ and $f : (Y, B) \rightarrow (X, D)$ be thrifty resolutions, then the composition $h = f \circ g$ is also a thrifty resolution.*

Proof. We need to check Conditions 1 and 2 for $h : (Z, D_Z) \rightarrow (X, D)$.

We begin with Condition 2: we'll check that that no stratum of (Z, D_Z) is in $\text{Ex}(h)$. Now we know that $g : Z \rightarrow Y$ is birational on all strata of (Z, D_Z) , because g satisfies Condition 2. So the image of a stratum S has the same dimension as S , and it is also contained in the intersection of the relevant components of B . A stratum is irreducible, so its image is irreducible, and since (Y, B) is snc the intersections have the right dimension. So the image is a stratum of (Y, B) . But then it cannot be contracted by f , since f also satisfies Condition 2 by hypothesis. Thus Condition 2 is true for h .

Now we move to Condition 1: we'll verify that no stratum of $\text{snc}(X, D)$ is in $h(\text{Ex}(h))$. Because f is thrifty, no stratum of $\text{snc}(X, D)$ is in the image of $\text{Ex}(f)$. Let V be some stratum of $\text{snc}(X, D)$. It is necessary to rule out the possibility that V is in $h(\text{Ex}(g))$. Now we are assuming f is thrifty, so $f_*^{-1}V$ exists, and it has the right dimension to be a stratum of (Y, B) (again because (Y, B) is snc). Then if V were in $h(\text{Ex}(g))$, we would have $f_*^{-1}V$

in the image of $\text{Ex}(g)$, but this is not possible because it's a stratum. So Condition 1 holds as well. \square

4.5 Thrifty resolutions of pairs and subpairs

The main results in this dissertation have to do with how the rationality property passes between a pair and a subpair. A subpair of (X, D) is just a pair (X', D') with $X' \subset X$ and $D' \subset D$. Here our subpairs will be of strictly smaller dimension: for example, one obtains a lower-dimensional subpair by cutting (X, D) with a hyperplane. Since we wish to study when rationality passes between a pair and a subpair, it will be useful to examine how thriftiness of resolutions is preserved when we restrict a resolution to certain nice subpairs.

First we'll verify that thrifty resolutions restrict well to hyperplane sections of pairs in \mathbf{P}^N . For the next two results, we'll use the following notation.

Notation 4.5.1. Let X be a projective variety, let (X, D) be a reduced pair, and let $f: (Y, C) \rightarrow (X, D)$ be a resolution. If $X_H = X \cap H$ is a general hyperplane section of X in \mathbf{P}^N , then let $D_H = D \cap H$; by generality of H , D_H is a divisor in X_H . Also, let $Y_H = f_*^{-1}X_H = f^*X_H$ —it is a pullback, by generality of H —and let $f_H = f|_{Y_H}$. Finally, let $C_H = (f_H)_*^{-1}D_H$.

Lemma 4.5.2. *Using the notation of (4.5.1), $f_H: (Y_H, C_H) \rightarrow (X_H, D_H)$ is a resolution.*

Proof. We need to verify that Y_H is smooth and (Y_H, C_H) is an snc pair. Smoothness of Y_H comes from the fact that a resolution of a projective variety restricts to a resolution of a general one of its hyperplane sections. The same is true for each component of C_H . As for their intersections, since H is general they are transverse. \square

Lemma 4.5.3. *With the notation of (4.5.1), if f is a thrifty log resolution, then f_H is also thrifty.*

Proof. Since f is log, $\text{Ex}(f) + C$ is an snc divisor in Y . Also, for general H , by the same reasoning as above we have that $C + \text{Ex}(f) + Y_H$ is snc in Y . Thus the exceptional components of f_H have the right codimension—that is, 1—and $\text{Ex}(f_H)$ is a divisor in Y_H . Also, $\text{Ex}(f_H) + C_H$ is snc.

So f_H is also a log resolution, which means that Condition 2 for thriftiness is automatic. We just need to verify Condition 1: no stratum of (X_H, D_H) is in $f_H(\text{Ex}(f_H))$. But for almost all H this is true: the only H for which this fails are those such that a component of $\text{Ex}(f)$ maps into, but not onto, a stratum of (X, D) , and then H cuts out exactly that image when it is intersected with D . But by generality this does not happen.

So f_H is thrifty as well. □

Now we'll move to a more general situation, where the divisor in question is not necessarily a hyperplane section of the variety X .

Theorem 4.5.4 (Existence of compatible thrifty resolutions). *Let A be a reduced, irreducible, normal, and Cartier divisor on X , and let D be another divisor on X such that $(X, A + D)$ is a reduced pair and $(A, A \cap D)$ is also a reduced pair. Then there is a thrifty resolution $\phi: (Y, B) \rightarrow (X, A + D)$ so that if we let $\phi_{\tilde{A}}: \phi_*^{-1}A \rightarrow A$, then*

$$(\phi_*^{-1}A, (\phi_{\tilde{A}}^{-1})_*(A \cap D)) \rightarrow (A, A \cap D)$$

is also a thrifty log resolution.

Proof. Let $\phi: (Y, B) \rightarrow (X, A + D)$ be a thrifty log resolution. Then if $\tilde{A} = \phi_*^{-1}A$, we have that $\phi_{\tilde{A}}: (\tilde{A}, (\phi_{\tilde{A}}^{-1})_*(A \cap D)) \rightarrow (A, A \cap D)$ is also a log resolution. This is just because restricting an snc pair gives another snc pair. We only need to show that $\phi_{\tilde{A}}$ is thrifty.

Since $\phi_{\tilde{A}}$ is log, it suffices to check for Condition 1 of thriftiness: Condition 2 is automatically satisfied by (4.3.1). So our task is to prove that no stratum of $\text{snc}(A, A \cap D)$ is in $\phi_{\tilde{A}}(\text{Ex}(\phi_{\tilde{A}}))$.

The first thing to show is that $\text{snc}(A, A \cap D) \subset \text{snc}(X, A + D)$. Let $x \in \text{snc}(A, A \cap D)$. Then x is a smooth point of A and also of $A \cap D_i$ for each component $A \cap D_i$ of $A \cap D$. Since A is Cartier (and $A \cap D_i$ is Cartier in each D_i) this means that x is a smooth point in X and in each D_i as well. Finally we need to show that A and any D_i 's passing through x intersect transversally at x . Because $x \in \text{snc}(A, A \cap D)$, we know that the local equations of the $A \cap D_i$ that pass through x form part of a regular system of parameters in the local ring $\mathcal{O}_{A,x}$. I claim that the local equations for those D_i , together with the local equation for A , form part of a regular sequence in $\mathcal{O}_{X,x}$. Modding out first by the local equation for

A , we get the ring $\mathcal{O}_{A,x}$, and then we already know the images of the other local equations work well. So $x \in \text{snc}(X, A + D)$.

Now any stratum of $\text{snc}(A \cap D)$ is again a stratum of $\text{snc}(X, A + D)$: take the D_i 's that already appeared in the intersection, and add A . So if ϕ is thrifty, then $\phi_{\bar{A}}$ is also thrifty. \square

Chapter 5

RATIONAL PAIRS AND SUBPAIRS

This section contains several new results about when rationality of pairs is preserved under various operations.

Given a reduced pair (X, D) , let (X', D') be a reduced subpair of (X, D) , in the sense that $X' \subset X$ and $D' \subset D$ and (X', D') is itself a reduced pair. In this section, we investigate when rationality passes from (X, D) to (X', D') , or from (X', D') to (X, D) (at least in a neighborhood of X').

5.1 Hyperplane sections of rational pairs

It is well known that rational singularities satisfy a Bertini-like theorem: if $X \subset \mathbf{P}^N$ is a projective variety with rational singularities, then a general hyperplane section of X also has rational singularities. We'll show next that the analogous statement holds for rational pairs.

Theorem 5.1.1 (Bertini-type result for rational pairs). *Let $X \subset \mathbf{P}^N$ be a normal projective variety, with $D \subset X$ a reduced divisor. If H is any hyperplane in \mathbf{P}^N , write X_H for the hyperplane section $X \cap H$, and similarly for D_H , as in the notation from (4.5.1) above. For a general hyperplane H passing through X , X_H is normal and $D_H \subset X_H$ is a reduced divisor, so (X_H, D_H) is also a reduced pair.*

If (X, D) is a rational pair, then (X_H, D_H) is also rational.

Proof. Let $f: (Y, C) \rightarrow (X, D)$ be a thrifty log resolution, which is also rational because (X, D) is rational. Then $f_H: (Y_H, C_H) \rightarrow (X_H, D_H)$ is also a thrifty resolution, by (4.5.2) and (4.5.3). We'll show that f_H is rational, and hence that (X_H, D_H) is a rational pair.

Start with the short exact sequence

$$0 \longrightarrow \mathcal{O}_Y(-Y_H) \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_{Y_H} \longrightarrow 0$$

Twist by $-C$, which is Cartier (by smoothness of Y), and get a new short exact sequence. Then push forward to X , obtaining the long exact sequence

$$\begin{aligned} 0 \longrightarrow f_*\mathcal{O}_Y(-Y_H - C) \longrightarrow f_*\mathcal{O}_Y(-C) \longrightarrow f_*\mathcal{O}_{Y_H}(-C_H) \longrightarrow \\ \longrightarrow R^1f_*\mathcal{O}_Y(-Y_H - C) \longrightarrow \dots \end{aligned}$$

Now Y_H is a pullback. By the projection formula, together with the rationality assumption on (X, D) ,

$$\begin{aligned} R^i f_*\mathcal{O}_Y(-Y_H - C) &\simeq R^i f_*\mathcal{O}_Y(-C) \otimes \mathcal{O}_X(-X_H) \\ &\simeq \begin{cases} \mathcal{O}_X(-D - X_H) & i = 0 \\ 0 & \text{else} \end{cases} \end{aligned}$$

So $R^i f_*\mathcal{O}_{Y_H}(-C_H) = 0$ for $i > 0$ by the long exact sequence. Also, there is now have a short exact sequence of pushforwards by f , yielding isomorphisms

$$\begin{aligned} f_*\mathcal{O}_{Y_H}(-C_H) &\simeq \mathcal{O}_X(-D)/\mathcal{O}_X(-D - X_H) \\ &\simeq \mathcal{O}_X(-D) \otimes \mathcal{O}_{X_H} \\ &\simeq \mathcal{O}_{X_H}(-D_H). \end{aligned}$$

So f_H is a thrifty rational resolution, and therefore (X_H, D_H) is a rational pair. \square

5.2 Deformation invariance of rational pairs

We are now ready to develop the main theorem: that rationality of pairs is preserved by deforming a flat family that is defined over a base scheme S . First we'll reduce to an especially simple case, where S is the spectrum of a regular local ring, and then we'll prove the claim in that situation with an induction argument. In order to reduce to the case where the base S is $\text{Spec } R$ for a regular local ring R , we need several preliminary results about how rational pairs and thrifty resolutions behave with respect to the base change by a morphism $\text{Spec } \mathcal{O}_{S,s} \rightarrow S$, where $s \in S$ is a smooth point.

5.2.A Flat families and base change by a local scheme near a point

The results that follow are all closely related, and they share a common notation. For convenience, we collect all the notation here and will refer back to it throughout the rest of the paper.

Notation 5.2.1. Let $f: X \rightarrow S$ be a morphism, and $s \in S$ a point. Let (X, D) be a reduced pair such that the fibers (X_s, D_s) form a reduced pair. Also, let $f: (Y, B) \rightarrow (X, D)$ be a resolution. Let $\text{Spec } \mathcal{O}_{S,s} \rightarrow S$ be the inclusion of the local scheme near s , and base change by this morphism: let $X' = X \times_S \text{Spec } \mathcal{O}_{S,s}$, and similarly for D', Y', B' . Write $\pi: X' \rightarrow X$ for the natural projection onto the first factor.

As we will see, all the salient aspects of this situation are preserved by the base change by $\text{Spec } \mathcal{O}_{S,s} \rightarrow S$. The next few results are standard commutative algebra, but for lack of a reference we include the proofs here.

Lemma 5.2.2. *With the notation of (5.2.1), let $x' \in X'$. If $\pi(x') = x$, and \mathcal{F} is a sheaf of \mathcal{O}_X -modules on X , then $\mathcal{F}_x \simeq (\pi^*\mathcal{F})_{x'}$. That is, stalks of \mathcal{O}_X -modules are preserved by the base change. In particular, $\mathcal{O}_{X,x} \simeq \mathcal{O}_{X',x'}$.*

Proof. We may assume the schemes are affine, so the base change corresponds to a pushout diagram of rings:

$$\begin{array}{ccc} B & \longrightarrow & B_{\mathfrak{p}} \\ \uparrow & & \uparrow \\ A & \longrightarrow & A_{\mathfrak{p}} \end{array}$$

Here A and B are rings, $\phi: A \rightarrow B$ is the homomorphism corresponding to the affine morphism $\text{Spec } B \rightarrow \text{Spec } A$, and $\mathfrak{p} \triangleleft A$ is a prime ideal. Then the \mathcal{O}_X -module \tilde{F} has the form \tilde{M} for a B -module M , and $\pi^*\mathcal{F} = (M \otimes_B B_{\mathfrak{p}})^{\sim} \simeq (M \otimes_A A_{\mathfrak{p}})^{\sim}$. Let \mathfrak{q}' be a prime ideal in the ring $B_{\mathfrak{p}}$. It corresponds to an ideal \mathfrak{q} in B that is disjoint from $f(A - \mathfrak{p})$. Here \mathfrak{q}' corresponds to $x' \in X'$; \mathfrak{q} to $x \in X$. It suffices to verify that $(M \otimes_A A_{\mathfrak{p}})_{\mathfrak{q}'} \simeq M_{\mathfrak{q}}$. By basic properties of localization (see [Sta14, Tag 02C7]) and the fact that $\phi(A - \mathfrak{p}) \subset B - \mathfrak{q}$, this is true.

Note in particular the case where $\mathcal{F} \simeq \mathcal{O}_X$: if $\pi(x') = x$, then $\mathcal{O}_{X,x} \simeq \mathcal{O}_{X',x'}$. □

Corollary 5.2.3. *With the notation of (5.2.1), let \mathcal{P} be a property of X that may be checked locally (i.e., at stalks). Then \mathcal{P} holds on the image of the projection $\pi(X') \subset X$ if and only if \mathcal{P} holds on X' .*

Example 5.2.4. We continue to use the notation of (5.2.1). Before moving on, we'll note a few specific applications of (5.2.3) to the varieties X and Y , and their counterparts X' and Y' under the base change. Here are a few useful choices for the property \mathcal{P} in (5.2.3). We'll refer back to these in the rest of the paper.

1. **Nonsingularity.** From (5.2.3) we immediately see that Y' is nonsingular (because Y is), and each component of $B' \subset Y'$ is nonsingular (because each component of B is).
2. **Codimension.** if $x' \in X'$ maps to $x \in X$, then x' and x have the same codimension. This is immediately clear from (5.2.2), because $\text{codim}(x, X) = \dim \mathcal{O}_{X,x}$ and similarly for $\text{codim}(x', X')$.
3. **Reduced divisors.** Moreover, since we are assuming that $D \subset X$ is a reduced divisor, the base change D' is also reduced. The coefficient of a component of a divisor is checked at the stalk at the generic point of each component, and we have just seen that codimension-1 points in X' map to codimension-1 points in X . Note that the function field also is involved in checking the coefficient of a divisor. The function field is also preserved—it is the stalk of the structure sheaf of any generic point of the base change—so reducedness is preserved.
4. **The snc locus.** The property of being in the snc locus of a pair is checked using stalks of the structure sheaf and of quotients of the structure sheaf. Nonsingularity of the variety and of each component of the divisor is checked on stalks of the structure sheaf. The other requirement for a point to be in the snc locus of a pair is that any components of the divisor that pass through the point must meet transversally. That is, the local equations that cut out the components form part of a regular sequence in the stalk of the structure sheaf at each point of the variety (see [SGA77, 3.1.5]). Now the stalks of the structure sheaf are preserved by the base change, and a component

of D' passes through x' in X' if and only if its image passes through $x = \pi(x')$ in X . So x' is in $\text{snc}(X', D')$ if and only if $x = \pi(x')$ is in $\text{snc}(X, D)$. Since (Y, B) is assumed to be snc in (5.2.1), we also have that (Y', B') is snc .

Corollary 5.2.5. *Using (5.2.1), let $\mathcal{F} \rightarrow \mathcal{G}$ be a morphism of coherent sheaves of \mathcal{O}_X -modules on X , and assume that the induced morphism $\pi^*\mathcal{F} \rightarrow \pi^*\mathcal{G}$ on X' is an isomorphism. Then $\mathcal{F} \rightarrow \mathcal{G}$ is an isomorphism in a neighborhood of the fiber X_s .*

Proof. First, note that the natural inclusion morphism $X_s \rightarrow X$ factors through the projection $\pi: X' \rightarrow X$. This is just because the local ring $\mathcal{O}_{S,s}$ maps to the residue field $\mathcal{O}_{S,s}/\mathfrak{m}_s = k(s)$, so the fiber $X_s = X \times_S \text{Spec } k(s)$ maps to $X' = X \times_S \text{Spec } \mathcal{O}_{S,s}$. So the fiber X_s is contained in the image of π . By (5.2.2), for any $x' \in X'$ with $\pi(x') = x$, there are isomorphisms

$$\mathcal{F}_x \simeq (\pi^*\mathcal{F})_{x'}, \quad \mathcal{G}_x \simeq (\pi^*\mathcal{G})_{x'}.$$

So $\mathcal{F}_x \rightarrow \mathcal{G}_x$ is an isomorphism for all $x \in X_s$. Now the stalk of the kernel sheaf is the kernel of the stalk morphism, and there is a natural isomorphism between the stalk of the cokernel sheaf and the cokernel of the morphism on stalks ([Vak13, 2.5.A–B]). So $\ker(\mathcal{F} \rightarrow \mathcal{G})_x = 0$ and $\text{coker}(\mathcal{F} \rightarrow \mathcal{G})_x = 0$ for all $x \in X_s$.

Now $\ker(\mathcal{F} \rightarrow \mathcal{G})$ and $\text{coker}(\mathcal{F} \rightarrow \mathcal{G})$ are coherent because \mathcal{F} and \mathcal{G} are. For any coherent sheaf \mathcal{H} , if $\mathcal{H}_x = 0$, then $\mathcal{H}|_U = 0$ for some neighborhood U of x . So the sheaves $\ker(\mathcal{F} \rightarrow \mathcal{G})$ and $\text{coker}(\mathcal{F} \rightarrow \mathcal{G})$ are zero in a neighborhood of x . Since this is true for all $x \in X_s$, it follows that $\mathcal{F} \rightarrow \mathcal{G}$ is an isomorphism in a neighborhood of the fiber X_s . \square

Lemma 5.2.6. *With the notation of (5.2.1), the morphism $f': Y' \rightarrow X'$ is a resolution of singularities. Moreover, B' is the birational transform of D' and (Y', B') is snc ; thus, $(Y', B') \rightarrow (X', D')$ is a resolution of pairs. If f is a log resolution, then so is f' , and if f is thrifty, so is f' .*

Proof. First of all, f' is proper and Y' is smooth: properness is always preserved by base change, and Y' is smooth by (5.2.4).

We'll now verify that f' is birational. Let η' be some codimension-0 point in X' . Irreducibility is not preserved by the base change, so there might be multiple such points; we may choose any one. This η' maps to some point η in X , which also has codimension 0 by (5.2.4). Now this η is the generic point of X , because X is irreducible. Because $Y \rightarrow X$ is surjective, this point η has a preimage ξ in Y , and this ξ is the generic point of Y . Then there is a point in $Y' \simeq Y \times_X X'$ mapping to ξ ; call this point ξ' . By (5.2.4), ξ' has codimension 0, and the map on stalks $\mathcal{O}_{X',\eta'} \rightarrow \mathcal{O}_{Y',\xi'}$ is exactly the same map as $\mathcal{O}_{X,\eta} \rightarrow \mathcal{O}_{Y,\xi}$, which is an isomorphism because $Y \rightarrow X$ is birational. Then f' is an isomorphism over every generic point of X' . So it is birational over each component of X' .

Now (Y', B') is snc, by (5.2.4). The same argument as above shows that $B' \rightarrow D'$ is birational over every component of D' , so $(Y', B') \rightarrow (X', D')$ is a resolution of pairs.

We'll show next that the exceptional locus of f in Y base changes to the exceptional locus of f' in Y' , and the image of the exceptional locus of f in X base changes to the image of the exceptional locus of f' in X' .

First, taking the base change of all the exceptional components in Y gives us exactly the exceptional components in Y' . If a point on X is in the image of $\text{Ex}(f)$ and also of the projection $X' \rightarrow X$, then its preimages in X' are in the image of $\text{Ex}(f')$. Indeed, given an exceptional component of the resolution $Y \rightarrow X$, the map $\mathcal{O}_{X',f'(y')} \rightarrow \mathcal{O}_{Y',y'}$ at its generic point y' is not an isomorphism, and this persists in the base change to the resolution $Y' \rightarrow X'$ by (5.2.2): it is precisely the same map as $\mathcal{O}_{X,f(y)} \rightarrow \mathcal{O}_{Y,y}$, if $\pi(y') = y$. From this the converse follows too: an exceptional component in Y' maps to an exceptional component in Y , and a point in $f'(\text{Ex}(f')) \subset X'$ maps to a point in $f(\text{Ex}(f)) \subset X$. So the base change of the exceptional locus is the exceptional locus, and the base change of the image of the exceptional locus is the image of the exceptional locus.

Now suppose f is a log resolution. We'll show the same is true of f' . We already know that the base change of a divisor is a divisor, because codimension of points is preserved. Consider an irreducible component F of the exceptional locus in Y' . We'll show F has codimension 1 in Y' . Assume the opposite: that F has codimension $k \geq 2$ in Y' . If ν is the

generic point of F , then the map on stalks

$$\mathcal{O}_{X',f'(\nu)} \rightarrow \mathcal{O}_{Y',\nu}$$

is *not* an isomorphism, because Z is exceptional, but there is a closed, codimension-1 subset V of Y' containing F so that the map on stalks at the generic point of V is an isomorphism. Then the map on stalks at the corresponding codimension- k point in Y is not an isomorphism, but at the codimension-1 point it is. In other words, the resolution $Y \rightarrow X$ also has an exceptional component that is not a divisor—but this is impossible, because we assume $Y \rightarrow X$ is a log resolution, so its exceptional components must be divisors.

So the exceptional locus is a divisor in Y' . Taking the snc locus commutes with this base change by (5.2.4), so f' is a log resolution.

Finally we show that if f is thrifty, then so is f' . To verify Conditions 1 and 2 of (3.3.1), we must examine the resolution f' at generic points of strata of $\text{snc}(X', D')$ and of (Y', B') .

We'll start with Condition 1. Let Z be a stratum of $\text{snc}(X', D')$, so that Z is an irreducible component of some intersection $D'_{i_1} \cap \cdots \cap D'_{i_r}$. Then, with $\pi: X' \rightarrow X$ the projection, $\pi(Z)$ is irreducible and contained in a component of the intersection $D_{i_1} \cap \cdots \cap D_{i_r}$ in X . Now, by applying (5.2.4), we see that $\pi(Z)$ has the same codimension in X as Z does in X' , and $\pi(Z) \subset \text{snc}(X, D)$. By [Kol13, 4.16.2], which says that the intersection of r components of a dlt divisor has pure codimension r , $\pi(Z)$ is actually equal to a stratum of $\text{snc}(X, D)$ (and is not merely contained in one). Now the assumption that f is thrifty means that $\pi(Z)$ is not contained in $f(\text{Ex}(f))$. By the argument above, which showed that taking the image of the exceptional locus commutes with the base change $X' \rightarrow X$, Z is not in $f'(\text{Ex}(f'))$, and so f' satisfies Condition 1.

Condition 2 is almost the same. If a stratum of (Y', B') lies in $\text{Ex}(f')$, then its image is a stratum of (Y, B) and lies in $\text{Ex}(f)$, but this is impossible because f is thrifty. So f' also satisfies Condition 2, and hence is thrifty. \square

5.2.B Passing from a member of a flat family to a neighborhood

In this section we prove the main theorem: given a pair (X, D) with D Cartier and a flat morphism $X \rightarrow S$, then if the fibers (X_s, D_s) over a smooth point $s \in S$ are a rational

(reduced) pair, (X, D) is also rational near X_s .

Because we assume D is Cartier, we may use (3.2.2), the analogue of Kempf's criterion for rational pairs, to conclude that (X, D) is rational near X_s . In order to check the second part of (3.2.2), we will need to exhibit a thrifty resolution f of X so that that $f_*\omega_Y(B) \simeq \omega_X(D)$, at least near X_s . The next lemma shows that actually we only need to verify that $f_*\omega_Y(B) \rightarrow \omega_X(D)$ is surjective: injectivity is automatic.

Lemma 5.2.7. *Suppose $f: Y \rightarrow X$ is a birational and proper morphism between normal varieties, D is a divisor in X , $B = f_*^{-1}D$ is the birational transform of D in Y , and ω_Y is torsion free. (For example, let $f: (Y, B) \rightarrow (X, D)$ be a resolution, so that ω_Y is invertible.) Then there is a logarithmic trace morphism $f_*\omega_Y(B) \rightarrow \omega_X(D)$, and it is injective.*

Proof. Let $U \subset X$ be the largest open set over which f is an isomorphism. The complement of U has codimension at least 2, because X is normal. Let $i: U \hookrightarrow X$ be the inclusion. On U , we have

$$f_*\omega_Y(B)|_U \simeq \omega_X(D)|_U,$$

because the restricted map $f: f^{-1}(U) \rightarrow U$ is an isomorphism. Now i^* and i_* are adjoint functors, so there is a natural morphism

$$f_*\omega_Y(B) \rightarrow i_*i^*f_*\omega_Y(B),$$

and the sheaf on the right can also be written as $i_*(f_*\omega_Y(B)|_U)$. Putting these maps together, we obtain a composition

$$f_*\omega_Y(B) \rightarrow i_*(f_*\omega_Y(B)|_U) \rightarrow i_*(\omega_X(D)|_U).$$

On the open set U , $\omega_X(D)$ and $i_*(\omega_X(D)|_U)$ are equal. The complement $X \setminus U$ has codimension at least 2 and the sheaves are reflexive, so they are equal on X by [Har80, 1.6]. From this we have the desired map $f_*\omega_Y(B) \rightarrow \omega_X(D)$.

Now we use the assumption that ω_Y is torsion free. Then $\omega_Y(B)$ is too, as is its push-forward $f_*\omega_Y(B)$: for any open set $V \subset X$, the sections of $f_*\omega_Y(B)$ on V are by definition the same as those of $\omega_Y(B)$ on $f^{-1}(V)$. Now $f_*\omega_Y(B) \rightarrow \omega_X(D)$ is an isomorphism at the

generic point of X , so the kernel of the morphism is a torsion sheaf. But $f_*\omega_Y(B)$ is torsion free, so the logarithmic trace map is injective. \square

Theorem 5.2.8 (Deformation invariance for rational pairs). *Let (X, D) be a pair, with D Cartier. Suppose $X \rightarrow S$ is a flat morphism, and $s \in S$ is a smooth point so that the fibers (X_s, D_s) form a reduced pair. If (X_s, D_s) is a rational pair, then (X, D) is a rational pair in a neighborhood of (X_s, D_s) .*

That is, if (X_s, D_s) is rational at x , then (X, D) is also rational at x .

Proof. The first step is to show that we may assume the base S is the spectrum of a regular local ring R . To see this, first base change the morphism $X \rightarrow S$ by the flat morphism $\text{Spec } \mathcal{O}_{S,s} \rightarrow S$, and let $X' = X \times_S \text{Spec } \mathcal{O}_{S,s}$ as in the notation of (5.2.1). Similarly, let $D' = D \times_S \text{Spec } \mathcal{O}_{S,s}$. Then $X' \rightarrow \text{Spec } \mathcal{O}_{S,s}$ is again flat.

We'll show that it suffices to prove the result for the new pair (X', D') : if (X', D') is rational near X_s , then (X, D) is also rational near X_s . Let $f: (Y, B) \rightarrow (X, D)$ be a thrifty log resolution, not necessarily rational. Then there is a Cartesian diagram:

$$\begin{array}{ccc} Y & \xleftarrow{\rho} & Y' \\ f \downarrow & & \downarrow f' \\ X & \xleftarrow{\pi} & X' \\ \downarrow & & \downarrow \\ S & \xleftarrow{\quad} & \text{Spec } \mathcal{O}_{S,s} \end{array}$$

By (5.2.3), if X' is normal and CM, then so is X at every point in the image of the projection $X' \rightarrow X$, and if D' is reduced, then so is D at every point in the image of $D' \rightarrow D$. Also, by (5.2.6), $f': (Y', B') \rightarrow (X', D')$ is a thrifty log resolution.

Suppose for now that we have shown (X', D') is a rational pair in a neighborhood U of X_s , so that every thrifty resolution of the pair is rational over U . Then f' is a rational resolution; that is, on U we have

$$\mathcal{R}(f')_* \mathcal{O}_{Y'}(-B') \simeq \mathcal{O}_{X'}(-D').$$

The map $\text{Spec } \mathcal{O}_{S,s} \rightarrow S$ is flat, so by cohomology and base change for flat morphisms

([Har77, III.9.3]), there is an isomorphism on U :

$$\pi^* \mathcal{R}f_* \mathcal{O}_Y(-B) \simeq \mathcal{O}_{X'}(-D').$$

To prove the original thrifty resolution f is then rational in a neighborhood of X_s , we need to verify that $\mathcal{R}f_* \mathcal{O}_Y(-B) \simeq \mathcal{O}_X(-D)$ near X_s . Now $\mathcal{O}_{X'}(-D') = \pi^* \mathcal{O}_X(-D)$, and the pushforwards $R^i f_* \mathcal{O}_Y(-B)$ are coherent. By (5.2.5) it follows that $\mathcal{R}f_* \mathcal{O}_Y(-B) \simeq \mathcal{O}_X(-D)$ in a neighborhood of X_s .

So it suffices to prove the result in the case where the base is $\text{Spec } \mathcal{O}_{S,s}$. We may then assume that $S = \text{Spec } R$, where R is a regular local ring of dimension n .

We'll prove that X is normal in a neighborhood of the fiber X_s . Then, by (3.2.2) and (4.4.2), we will only need to prove that X is CM near X_s and for some thrifty resolution $f: (Y, B) \rightarrow (X, D)$, the logarithmic trace $f_* \omega_Y(B) \rightarrow \omega_X(D)$ is an isomorphism in a neighborhood of X_s . By (5.2.7) the logarithmic trace is injective, so we will show that X is normal and CM and the map of sheaves is surjective in a neighborhood of X_s .

Following the idea of the proof of [Elk78, Théorème 2], we'll prove this using induction on n . Our base case is $n = 0$. If $n = 0$, then $S = \text{Spec } K = \{s\}$ for some field K . In this case $(X, D) = (X_s, D_s)$, so the conclusion is trivially true.

Now let $\dim R = n$. For the inductive hypothesis, assume the result is true for $\dim R < n$: if the base scheme is the spectrum of a regular local ring of dimension less than n , and if (X_s, D_s) is rational, then (X, D) is rational in a neighborhood of X_s .

Let t be a regular parameter in R . Then let $X_t = X \times_S \text{Spec}(R/tR)$, so that X_t is the pullback of the divisor defined by t in $S = \text{Spec } R$, and define D_t similarly. Then X_t is Cartier in X , and D_t in D . Note that (X_s, D_s) is a subpair of (X_t, D_t) : $X_s \subset X_t$ and $D_s \subset D_t$.

Now the regular local ring R/tR has dimension $n - 1$, so (X_t, D_t) is rational near X_s by the inductive hypothesis. In particular, X_t is normal and CM. Then X is also normal and CM in a neighborhood of X_t : these are properties that pass from a Cartier divisor to an open set in the whole space. Any neighborhood of X_t is also a neighborhood of X_s , so X is CM and normal near X_s . Working in this neighborhood, normality allows us to use (3.2.2), and since we also have in this neighborhood that X is CM, f is thrifty, and

$f_*\omega_Y(B) \rightarrow \omega_X(D)$ is injective, it just remains to show that $f_*\omega_Y(B) \rightarrow \omega_X(D)$ is surjective near X_s .

Let $(Y, B) \rightarrow (X, D)$ be a thrifty log resolution. There is then a proper morphism $Y_t \rightarrow X_t$ and a component Y_1 of Y_t mapping birationally to X_t . Write $f_1 = f|_{Y_1} : Y_1 \rightarrow X_t$, and let $B_1 = (f_1)_*^{-1}(D_t)$ be the birational transform of D_t . Now (Y_1, B_1) is not snc — Y_1 need not even be smooth—but the morphism $(Y_1, B_1) \rightarrow (X_t, D_t)$ is birational and satisfies Condition 1 from (3.3.1). By (4.3.2), there is a thrifty log resolution $f_2 : (Y_2, B_2) \rightarrow (X_t, D_t)$ factoring through (Y_1, B_1) . Write \tilde{f} for the intermediate birational morphism $Y_2 \rightarrow Y_1$.

By assumption, (X_t, D_t) is a rational pair, so f_2 is a rational resolution. Now D_t is Cartier in X_t , so by (3.2.2), the logarithmic trace map is an isomorphism:

$$(f_2)_*\omega_{Y_1}(B_1) \simeq \omega_{X_t}(D_t).$$

Because t is a regular parameter, we have an exact sequence

$$0 \longrightarrow \omega_Y \xrightarrow{\cdot t} \omega_Y \longrightarrow \omega_{Y_t} \longrightarrow 0$$

and similarly for ω_X, ω_{X_t} . Twist the sequences by B and D , respectively. Both operations are exact because B, D are Cartier. Push forward the sequence on Y by f . The result is a commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & f_*\omega_Y(B) & \xrightarrow{\cdot t} & f_*\omega_Y(B) & \longrightarrow & f_*\omega_{Y_t}(B_t) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \omega_X(D) & \xrightarrow{\cdot t} & \omega_X(D) & \longrightarrow & \omega_{X_t}(D_t) \longrightarrow 0 \end{array}$$

Both rows are exact: the top is exact by (4.4.2), the analogue of the Grauert-Riemenschneider vanishing theorem for thrifty resolutions.

By Grothendieck duality, $\omega_{Y_1}(B_1)$ is a subsheaf of $\omega_{Y_t}(B_t)$. Moreover, by (5.2.7), the logarithmic trace map $\tilde{f}_*\omega_{Y_2}(B_2) \rightarrow \omega_{Y_1}(B_1)$ is injective. Composing these injective maps, we get an injection $\tilde{f}_*\omega_{Y_2}(B_2) \rightarrow \omega_{Y_t}(B_t)$. Pushing forward by f and using that $f_2 = f \circ \tilde{f}$, we have an injective map

$$f_*\tilde{f}_*\omega_{Y_2}(B_2) = (f_2)_*\omega_{Y_2}(B_2) \hookrightarrow f_*\omega_{Y_t}(B_t).$$

The isomorphism $(f_2)_*\omega_{Y_2}(B_2) \rightarrow \omega_{X_t}(D_t)$ then factors through $f_*\omega_{Y_t}(B_t)$, so the composition

$$f_*\omega_{Y_2}(B_2) \rightarrow f_*\omega_{Y_t}(B_t) \rightarrow \omega_{X_t}(D_t)$$

is injective and surjective. Thus the right vertical arrow in the diagram is surjective.

To prove our desired result—that f is a rational resolution—we just need to verify that the middle vertical arrow is also surjective in a neighborhood of X_s . Let $x \in X_s$ be any point, not necessarily closed. Then x maps to the single closed point—the maximal ideal \mathfrak{m} —of $\text{Spec } R$. This is clear from the definition of R : it is $\mathcal{O}_{S,s}$ for a smooth point $s \in S$, and X_s is just the fiber over s .

Working in an affine neighborhood of our point x , we may assume $X = \text{Spec } A, Y = \text{Spec } C$, and $\omega_X(D), \omega_Y(B)$ correspond to finite modules M, N over A, C respectively. We have ring maps $R \rightarrow A \rightarrow C$. Considering N as an A -module via the map $A \rightarrow C$, and then thinking of $N \rightarrow M$ as a morphism of A -modules, we have the local version of the morphism $f_*\omega_Y(B) \rightarrow \omega_X(D)$.

The logarithmic trace map $f_*\omega_Y(B) \rightarrow \omega_X(D)$ is injective by (5.2.7), so $N \rightarrow M$ is injective and N is a submodule of M . The morphism of sheaves $f_*\omega_{Y_t}(B_t) \rightarrow \omega_{X_t}(D_t)$ corresponds locally to the map of modules $N/tN \rightarrow M/tM$. Let \mathfrak{p} be the ideal in A corresponding to the point x . Since $t \in \mathfrak{m}$ and \mathfrak{p} pulls back to \mathfrak{m} , t is also in \mathfrak{p} when we view t as an element of A .

Next we localize the entire diagram of A -modules at \mathfrak{p} , to get a diagram of maps between $A_{\mathfrak{p}}$ -modules:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & K \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & N_{\mathfrak{p}} & \xrightarrow{\cdot t} & N_{\mathfrak{p}} & \longrightarrow & N_{\mathfrak{p}}/tN_{\mathfrak{p}} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & M_{\mathfrak{p}} & \xrightarrow{\cdot t} & M_{\mathfrak{p}} & \longrightarrow & M_{\mathfrak{p}}/tM_{\mathfrak{p}} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
& & (M/N)_{\mathfrak{p}} & & (M/N)_{\mathfrak{p}} & & 0
\end{array}$$

Localization is exact, so the rows of the diagram are short exact, the first two arrows are injective, and the right arrow is surjective. Also, localization commutes with taking quotients, so $(M/tM)_{\mathfrak{p}} \simeq M_{\mathfrak{p}}/tM_{\mathfrak{p}}$, $(N/tN)_{\mathfrak{p}} \simeq N_{\mathfrak{p}}/tN_{\mathfrak{p}}$, and $(M/N)_{\mathfrak{p}} \simeq M_{\mathfrak{p}}/N_{\mathfrak{p}}$. By the snake lemma, there is a short exact sequence:

$$0 \longrightarrow K \longrightarrow (M/N)_{\mathfrak{p}} \xrightarrow{\cdot t} (M/N)_{\mathfrak{p}} \longrightarrow 0$$

In particular, notice that the map $(M/N)_{\mathfrak{p}} \rightarrow (M/N)_{\mathfrak{p}}$ given by multiplication by t is surjective, so $(M/N)_{\mathfrak{p}} = t(M/N)_{\mathfrak{p}}$.

All the modules we had before localizing were finite over A , so all the localized modules in the new diagram are finite over $A_{\mathfrak{p}}$. Now t is in the single maximal ideal of $A_{\mathfrak{p}}$ and $(M/N)_{\mathfrak{p}} = t(M/N)_{\mathfrak{p}}$, so $(M/N)_{\mathfrak{p}} = 0$ by Nakayama's lemma.

This argument holds for every point $x \in X_s$, so the cokernel sheaf $\omega_X(D)/f_*\omega_Y(B)$ is zero at every $x \in X_s$. Then, because the cokernel sheaf is coherent, it is actually zero on an open set of X containing X_s .

Now we've shown that $\omega_X(D) \simeq f_*\omega_Y(B)$ in a neighborhood of the fiber X_s . By the reductions above, the induction is complete and (X, D) is rational in a neighborhood of X_s . \square

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