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Quadratic Points on Intersections of Two Quadrics over Local Fields

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Abstract

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By a recent result of Creutz and Viray, a smooth complete intersection of two quadrics in 4-dimensional projective space over a nonarchimedean local field k always has a quadratic point. We extend this result by showing that such an intersection of quadrics must in fact obtain points over all quadratic extensions of k except possibly one. Our approach uses the Theorems of Springer and Amer-Brumer, as well as Tian's theory of semistable models for intersections of two quadrics.

TABLE OF CONTENTS

	Page
Chapter 1: Introduction	1
Chapter 2: Intersections of two quadrics over local fields	4
Chapter 3: Semistable models	10
3.1 The rank 3 case	12
3.2 The rank 2 isotropic case	14
3.3 The rank 2 anisotropic case	15
3.4 The rank 1 case	25
Chapter 4: Proof of the main theorem	31

Chapter 1

INTRODUCTION

The intersection of two quadrics in \mathbb{P}^4 has long been a fertile ground of study in arithmetic geometry. This is largely because it is one of the simplest varieties from a geometric standpoint that nonetheless exhibits some rich arithmetic features, for example the failure of the local-global principle.

In one direction, one might ask whether we may pinpoint this failure as due to one of several named obstructions. Conjecturally, for X an intersection of two quadrics in \mathbb{P}^4 over a number field, the failure of the local-global principle is fully explained by the *Brauer-Manin obstruction*, though the story is still ongoing as of the time of this writing.

In another direction, given a field k and supposing $X(k) = \emptyset$ one may ask: over what extensions L/k does X obtain an L -point? For L/k of *odd* degree, the answer is satisfyingly regular. Indeed, the theorems of Springer and Amer-Brumer imply that $X(k) \neq \emptyset$ as soon as $X(L) \neq \emptyset$. In addition, X always has degree 4 points, obtained by cutting the degree 4 surface X with hyperplanes. However, the question of whether X has a degree 2 point turns out to be rather more subtle.

Recently, in [\[CV23\]](#), the authors Brendan Creutz and Bianca Viray proved that

a smooth complete intersection of two quadrics in \mathbb{P}^4 over a number field has index dividing 2. Their proof crucially uses the fact that an intersection of two quadrics in \mathbb{P}^4 over a local field k always obtains a point over *some* quadratic extension, a result which Creutz and Viray first proved in the same paper. Their proof of this striking fact used the following strategy: assuming X fails to have points over the unique unramified quadratic extension of k , they show that X must then obtain points over *all* of the ramified quadratic extensions.

The chief aim of the following exposition is to show that the phenomenon found by Creutz and Viray holds in general, namely if a smooth complete intersection of quadrics in \mathbb{P}_k^4 fails to have points over any one particular quadratic extension, then it must obtain points over all the others. In particular the following statement holds:

Theorem 1.1. *Let $X \subset \mathbb{P}_k^4$ be a smooth complete intersection of two quadrics over a nonarchimedean local field k of odd residue characteristic. Then there is at most one quadratic extension K/k such that $X(K) = \emptyset$.*

The organization is as follows. In Chapter 2 we first recall some important preliminary results on the intersection of two quadrics over nonarchimedean local fields. The centerpiece of this chapter is Theorem [2.7](#), which will be used to deduce the specific form of the defining equations for X . Chapter 3 contains much of the technical work of the proof. The main tool we put to use here is Tian’s theory of semistable models for intersections of two quadrics developed in [Tia17](#), an analog of Kollár’s theory for hypersurfaces.

Finally, the relatively shorter Chapter 4 is devoted to the proof of the main theorem for which much of the work has already been done in the previous two chapters. We conclude with an example showing that it is possible for a smooth complete intersection of two quadrics in \mathbb{P}_k^4 over a local field k to fail to obtain points over exactly one quadratic extension. In some sense this shows that the result of Theorem [1.1](#) is the best we can expect.

Chapter 2

INTERSECTIONS OF TWO QUADRICS OVER LOCAL FIELDS

We begin this section by recalling two fascinating and fundamental results from the study of quadratic forms. The first deals with zeros of a single quadratic form, while the second deals with simultaneous zeros of two quadratic forms.

Theorem 2.1 (Springer's Theorem). *Let Q be a quadratic form over a field k with $\text{char}(k) \neq 2$. Then Q has a nontrivial zero over k if and only if it has a nontrivial zero over an odd degree extension of k .*

Proof. See [Spr56]. □

Theorem 2.2 (Amer-Brumer Theorem). *Let Q_1, Q_2 be two quadratic forms over a field k with $\text{char}(k) \neq 2$. Then Q_1 and Q_2 have a nontrivial common zero over k if and only if the single quadratic form $Q_1 + tQ_2$ has a nontrivial zero over $k(t)$.*

Proof. See [Ame76] and [Bru78]. □

We now recall a key definition:

Definition 2.3 (Skorobogatov, 1996). *A scheme X over a perfect field k is said to be **split** if it contains a geometrically integral open subscheme.*

Example 2.4. *The variety defined by $x^2 - ay^2 = 0$ is split over k if and only if $a \in k^{\times 2}$. This variety is always split over $k[\sqrt{a}]$.*

The notion of a scheme X/k being split is an important arithmetic property. In the case that k is a nonarchimedean local field, the existence of an integral model \mathfrak{X} for X with split special fiber serves as an important criterion for X to have k -points. In the case of intersections of two quadrics we have the following theorem due to Creutz and Viray [CV23, Theorem 2.3]. Due to its fundamental importance, we'll include a sketch of the proof here.

Theorem 2.5. *Let k be a nonarchimedean local field with valuation ring \mathcal{O} and let X be a smooth complete intersection of two quadrics in \mathbb{P}_k^4 . If there exists an integral model $\mathfrak{X} \subset \mathbb{P}_{\mathcal{O}}^4$ with split special fiber, then $X(k) \neq \emptyset$.*

Proof. Let U be a geometrically integral open subscheme contained in the special fiber, which exists since the special fiber is assumed split. Let \mathbb{F} denote the residue field of \mathcal{O} . By the Lang-Weil bounds, there is an $N > 0$ such that U contains a smooth \mathbb{F}' -point for any extension \mathbb{F}'/\mathbb{F} with $\text{card}(\mathbb{F}') > N$. Choose such an extension \mathbb{F}'/\mathbb{F} such that $[\mathbb{F}' : \mathbb{F}]$ is odd.

By Hensel's Lemma, $X(L) \neq \emptyset$ where L/k is the odd degree unramified extension with residue field extension \mathbb{F}'/\mathbb{F} . Suppose Q_1, Q_2 are the quadratic forms defining X , i.e. $X = V(Q_1, Q_2)$. By Theorem 2.2, the form $Q_1 + tQ_2$ has a zero over $L[t]$. But this form also has a zero over $k[t]$ as a result of Theorem 2.1. Finally, by another

application of Theorem [2.2](#) we have $X(k) \neq \emptyset$. □

Theorem 2.6. *Let $X \subset \mathbb{P}_{\mathbb{F}}^4$ be a geometrically reduced complete intersection of two quadrics. If X is nonsplit, then X is contained in a rank 2 quadric.*

Proof. See [\[CV23, Corollary 2.7\]](#). □

The preceding two theorems will be used in the following typical way: if $X = V(Q_1, Q_2)$ is an intersection of two quadrics over a field k , then the quadrics containing X are precisely the forms in the associated *pencil of quadrics* $\lambda Q_1 + \mu Q_2$ for $[\lambda : \mu] \in \mathbb{P}_{\bar{k}}^1$. In the case that k is a nonarchimedean local field, if it can be shown that X has an integral model \mathfrak{X}/\mathcal{O} such that every form in the pencil has rank at least 3 modulo π , then the special fiber of \mathfrak{X} will not be contained in any rank 2 quadric as a result of Theorem [2.6](#). In view of Theorem [2.5](#), this is enough to guarantee $X(k) \neq \emptyset$.

Note that in Theorem [2.6](#), the rank 2 quadric containing X is *not* assumed to be defined over the ground field. However, in the relevant setup for the proof of Theorem [1.1](#) we may improve on this result and deduce that the special fiber of \mathfrak{X} is in fact contained in a rank 2 quadric with \mathbb{F} -coefficients. This key intermediate result will allow us to pin down the form of the defining equations for X for the work to follow.

Theorem 2.7. *Let $X \subset \mathbb{P}_k^4$ be a smooth complete intersection of two quadrics over a nonarchimedean local field k with integral model \mathfrak{X} . Let \mathcal{O} denote the valuation ring of k and \mathbb{F} the residue field, with absolute Galois group $G_{\mathbb{F}}$. Suppose the special fiber $\mathfrak{X}_{\mathbb{F}}$ is reduced, and suppose further that $\mathfrak{X}_{\mathbb{F}}$ is nonsplit but becomes split over the unique*

quadratic extension \mathbb{F}'/\mathbb{F} .

Then the special fiber of \mathfrak{X} is contained in a rank 2 quadric defined over \mathbb{F} .

Proof. Since $\mathfrak{X}_{\mathbb{F}}$ is nonsplit and reduced, it must be geometrically reducible. Consider the possible degrees of irreducible components of $\mathfrak{X}_{\mathbb{F}}$:

$$3 + 1, \quad 2 + 2, \quad 2 + 1 + 1, \quad 1 + 1 + 1 + 1$$

Suppose $\mathfrak{X}_{\mathbb{F}}$ contains an irreducible quadric surface S_1 . Since $\mathfrak{X}_{\mathbb{F}}$ is nonsplit, there is a $\sigma \in G_{\mathbb{F}}$ such that $\sigma(S_1) = S_2$, where S_2 is an irreducible quadric surface distinct from S_1 . This rules out the decomposition $2 + 1 + 1$.

Hence if $\mathfrak{X}_{\mathbb{F}}$ contains an irreducible quadric surface S_1 , it must be the union of two distinct quadric surfaces. By [CTSSD87, Lemma 1.6], $\mathfrak{X}_{\mathbb{F}}$ is contained in a quadric surface defined over \mathbb{F} .

Next suppose $\mathfrak{X}_{\mathbb{F}}$ contains a 2-plane L_1 . Since $\mathfrak{X}_{\mathbb{F}}$ is nonsplit, there is a $\sigma \in G_{\mathbb{F}}$ such that $\sigma(L_1) = L_2$, where L_2 is a 2-plane distinct from L_1 . This rules out the decomposition $3 + 1$.

So we have reduced to the case where $\mathfrak{X}_{\mathbb{F}}$ is a union of four 2-planes. By [CV23, Lemma 2.6], $\mathfrak{X}_{\mathbb{F}}$ is a cone. In addition, the underlying assumption that $\mathfrak{X}_{\mathbb{F}}$ is nonsplit but becomes split over \mathbb{F}' implies that the Galois group $G_{\mathbb{F}}$ acts nontransitively on the 4 planes but also does not fix any plane.

If the cone locus is 1-dimensional, then $\mathfrak{X}_{\mathbb{F}}$ is a cone over an intersection of two quadrics in \mathbb{P}^2 , i.e. a union of 4 points. If we denote these points by P_1, P_2, P_3, P_4

then there are 3 pairs of lines through them, which we denote (L_{12}, L_{34}) , (L_{13}, L_{24}) , and (L_{14}, L_{23}) . Since $G_{\mathbb{F}}$ does not act transitively on the points, one pair of lines in \mathbb{P}^2 corresponding to a pair of hyperplanes in \mathbb{P}^4 must be fixed. If l_1 and l_2 are the linear forms cutting out these planes, then $Q = V(l_1 l_2)$ is a rank 2 *reducible* quadric hypersurface containing $\mathfrak{X}_{\mathbb{F}}$.

Next consider the case where the cone locus is 0-dimensional. Then $\mathfrak{X}_{\mathbb{F}}$ is a cone over an intersection of two quadrics in \mathbb{P}^3 . Since $\mathfrak{X}_{\mathbb{F}}$ is a union of 4 planes, the residual intersection in \mathbb{P}^3 is a union of 4 lines. Denote the lines L_1, L_2, L_3 , and L_4 . Since an intersection of two quadrics in \mathbb{P}^3 is a curve of arithmetic genus 1, the only admissible configuration of the lines is the one in which each line intersects exactly two of the others. Without loss of generality we may assume the following values of the intersection pairing:

$$L_1 \cdot L_3 = L_2 \cdot L_4 = 0, \quad L_1 \cdot L_2 = L_1 \cdot L_4 = L_2 \cdot L_3 = L_3 \cdot L_4 = 1.$$

Suppose there is a $\sigma \in G_{\mathbb{F}}$ such that $\sigma(L_1) = L_2$. Then we may also assume $\sigma(L_3) = L_4$, since $\mathfrak{X}_{\mathbb{F}}$ is nonsplit. Since $G_{\mathbb{F}}$ does not act transitively, the hyperplane H_{12} spanned by L_1 and L_2 is fixed by Galois, as is the hyperplane H_{34} spanned by L_3 and L_4 . If l_{12} and l_{34} are the linear forms such that $H_{12} = V(l_{12})$ and $H_{34} = V(l_{34})$, then $V(l_{12} l_{34})$ is a rank 2 *reducible* quadric surface containing $\mathfrak{X}_{\mathbb{F}}$.

Suppose instead is a $\sigma \in G_{\mathbb{F}}$ such that $\sigma(L_1) = L_3$. Then we may also assume $\sigma(L_2) = L_4$, since $\mathfrak{X}_{\mathbb{F}}$ is nonsplit. Since $G_{\mathbb{F}}$ does not act transitively, the hyperplane H_{12} spanned by L_1 and L_2 and the hyperplane H_{34} spanned by L_3 and L_4 are in-

terchanged by Galois, so their union is Galois stable. If l_{12} and l_{34} are the linear forms such that $H_{12} = V(l_{12})$ and $H_{34} = V(l_{34})$, then $V(l_{12}l_{34})$ is a rank 2 irreducible quadric surface defined over \mathbb{F} and containing $\mathfrak{X}_{\mathbb{F}}$. This completes the proof. \square

Chapter 3

SEMISTABLE MODELS

This section contains much of the technical work of the proof of Theorem [1.1](#). We begin by reviewing Tian's generalization of Kollár's theory of semistable models for hypersurfaces in [Kol97](#) to the complete intersections of two quadrics. For more information, see [Tia17](#).

Let k be a nonarchimedean field with valuation ring \mathcal{O} , and let $\mathfrak{X} \in \mathbb{P}_{\mathcal{O}}^4$ be a complete intersection of 2 quadrics. Let $\mathbf{q}_1, \mathbf{q}_2 \in k^{15}$ denote the coefficient vectors of Q_1 and Q_2 , where $\mathfrak{X} = V(Q_1, Q_2)$, and consider the 2×15 matrix

$$A = \begin{pmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{pmatrix}.$$

Given a weight vector $\mathbf{w} \in \mathbb{N}^5$, define $f_{\mathbf{w}}$ to be the change of variables

$$f_{\mathbf{w}} : \mathbb{P}_{\mathcal{O}}^4 \rightarrow \mathbb{P}_{\mathcal{O}}^4$$

$$x_i \mapsto \pi_i^w x_i$$

where w_i indicates the i th component of \mathbf{w} . Next, we define the *multiplicity of \mathfrak{X} with respect to the weight vector \mathbf{w}* to be

$$\text{mult}_{\mathbf{w}}(\mathfrak{X}) := \min_m \{v(m)\}$$

where m ranges over all the 2×2 minors of $A_{f_{\mathbf{w}}}(\mathfrak{X})$. We say \mathfrak{X} is *semistable* if

$$\text{mult}_{\mathbf{w}}(g(\mathfrak{X})) \leq \frac{4}{5} \sum_{i=0}^4 w_i$$

for all $\mathbf{w} \in \mathbb{N}^5$ and all $g \in \text{Aut}(\mathbb{P}_{\mathcal{O}}^4)$.

Theorem 3.1 (Tian). *Let $X \subset \mathbb{P}_k^4$ be a smooth complete intersection of two quadrics.*

Then X has a semistable integral model.

Proof. See [Tia17, Theorem 2.7]. □

We will use the preceding theorem to exercise finer control over the special fiber of \mathfrak{X} . For example, one may immediately glean the following basic properties of a semistable model \mathfrak{X} :

Theorem 3.2. *Suppose X has a semistable integral model \mathfrak{X} . Then the special fiber of \mathfrak{X} is reduced.*

Proof. See [Tia17, Lemma 2.22(4)] □

Theorem 3.3. *Suppose X has an integral model \mathfrak{X} for which one of the defining quadric equations has the form*

$$x_0x_1 + \pi F(x_0, x_1, x_2, x_3, x_4).$$

Then \mathfrak{X} is not semistable.

Proof. Without loss of generality, suppose $Q_1 = x_0x_1 + \pi F(x_0, x_1, x_2, x_3, x_4)$ and let $X = V(Q_1, Q_2)$, with Q_2 arbitrary. If \mathfrak{X} were semistable, we would require $\text{mult}_{\mathbf{w}}(\mathfrak{X}) < \frac{4}{5} \sum_{i=0}^4 w_i$ for all weight vectors \mathbf{w} .

Consider the weight vector $\mathbf{w} = (1, 0, 0, 0, 0)$, and the corresponding morphism $f_{\mathbf{w}}^*$. Every element in the first row of the matrix $A_{f_{\mathbf{w}}^*(\mathfrak{X})}$ is divisible by π (see the beginning of the section for the definition of $A_{f_{\mathbf{w}}^*(\mathfrak{X})}$). It then follows that every minor of $A_{f_{\mathbf{w}}^*(\mathfrak{X})}$ has valuation at least 1, a contradiction. \square

In view of Theorems [2.7](#) and [3.3](#), we will deduce that X has a model \mathfrak{X} given as the vanishing of quadrics of the form:

$$\begin{aligned} Q_1 &: x_0 l_0(x_1, x_2, x_3, x_4) + x_1 l_1(x_1, x_2, x_3, x_4) + G(x_2, x_3, x_4) \\ Q_2 &: x_0^2 - ax_1^2 + \pi F(x_2, x_3, x_4) \end{aligned} \tag{3.1}$$

The primary goal of the remainder of this section will be to prove Theorem [1.1](#) under the assumption that X has a semistable model \mathfrak{X} given by equations [3.1](#). We will divide the labor along the possible values of the rank of G modulo π .

3.1 The rank 3 case

We will begin by considering the possibility that G is a rank 3 quadratic form modulo π . In this case it will turn out that X has points over any ramified extension. We begin with the following lemma:

Lemma 3.4. *Assume X has a semistable model \mathfrak{X} given by equations [3.1](#). Then F and G are linearly independent modulo π .*

Proof. Let $\mathbf{w} = (1, 1, 0, 0, 0)$. Since \mathfrak{X} is semistable we require in particular that $\text{mult}_{\mathbf{w}}(\mathfrak{X}) < \frac{8}{5}$. To satisfy this inequality, the matrix $A_{f_{\mathbf{w}}^*(\mathfrak{X})}$ must have a 2×2 submatrix with valuation at most 1. Every term in the second row of $A_{f_{\mathbf{w}}^*(\mathfrak{X})}$ has valuation

at least 1, as do the entries in the first row that are associated with coefficients of terms involving x_0 and x_1 . Thus, every minor has valuation at least 1, and any minor with valuation exactly 1 must involve entries from the 2×6 submatrix associated with coefficients of terms involving only the variables x_2, x_3 , and x_4 .

Let \mathbf{g} and \mathbf{h} be the coefficient vectors of G and F respectively, which comprise the terms in Q_1 and Q_2 involving x_2, x_3 , and x_4 . Since \mathfrak{X} is semistable, the 2×6 submatrix $\begin{pmatrix} \mathbf{g} \\ \pi\mathbf{h} \end{pmatrix}$ has a minor with valuation 1. Hence $\begin{pmatrix} \mathbf{g} \\ \mathbf{h} \end{pmatrix}$ has a minor with valuation 0. Alternatively put, G and F are linearly independent mod π . \square

We are now ready to prove Theorem [1.1](#) in the case that G is rank 3 modulo π .

Theorem 3.5. *Assume X has a semistable model \mathfrak{X} given by equations [3.1](#). If G has rank 3 modulo π , then $X(K) \neq \emptyset$ for any ramified quadratic extension K/k .*

Proof. Let K/k be a ramified quadratic extension and let $\varpi \in k$ be a uniformizer of K with $\varpi^2 = u\pi$ for some $u \in \mathcal{O}^\times$. Upon making the change of variables $(x_0, x_1, x_2, x_3, x_4) \mapsto (\varpi x_0, \varpi x_1, x_2, x_3, x_4)$ over K and canceling common factors of ϖ , we obtain a model \mathfrak{X}' of X over the valuation ring of K of the following shape modulo ϖ :

$$\begin{aligned} Q_1 &: G(x_2, x_3, x_4) \\ Q_2 &: x_0^2 - ax_1^2 + uF(x_2, x_3, x_4) \end{aligned} \tag{3.2}$$

By Lemma [3.4](#), every $\overline{\mathbb{F}}$ -linear combination of Q_1 and Q_2 has rank at least 3, so

the special fiber of \mathfrak{X}' is split by Theorem [2.6](#). Therefore $X(K) \neq \emptyset$ by Theorem [2.5](#). □

Next we will consider the case where $G(x_2, x_3, x_4)$ is a rank 2 quadratic form modulo π . Within this case there are two subcases to consider, which are amenable to distinctly different approaches.

3.2 The rank 2 isotropic case

In this section, we will consider the special case where $G(x_2, x_3, x_4)$ reduces to an isotropic rank 2 quadratic form modulo π . After a suitable change of variables and renaming l_0, l_1, F , and G we may suppose that G reduces to the hyperbolic plane x_2x_3 modulo π . Thus we are led to consider that \mathfrak{X} is given by the vanishing of quadrics of the following form:

$$Q_1 : x_0l_0(x_1, x_2, x_3, x_4) + x_1l_1(x_1, x_2, x_3, x_4) + x_2x_3 + \pi H(x_2, x_3, x_4) \tag{3.3}$$

$$Q_2 : x_0^2 - ax_1^2 + \pi F(x_2, x_3, x_4)$$

In this case a familiar conclusion will follow: it will turn out that an X has points over *any* ramified quadratic extension. We begin with the following lemma:

Lemma 3.6. *Suppose that X has an integral model \mathfrak{X} given by equations of type [3.3](#).*

If \mathfrak{X} is semistable, then $F(0, x_3, x_4)$ is not identically zero modulo π .

Proof. Since \mathfrak{X} is semistable we require in particular that $\text{mult}_{(1,1,1,0,0)}(\mathfrak{X}) < \frac{12}{5}$. If $F(0, x_3, x_4)$ were identically zero mod π then every minor of $A_{f^*_{(1,1,1,0,0)}\mathfrak{X}}$ would have

valuation at least 3, contradicting the semistability of \mathfrak{X} . \square

Theorem 3.7. *Assume X has a semistable model \mathfrak{X} with defining equations given by*

3.3. *Then $X(K) \neq \emptyset$ for every ramified quadratic extension K/k .*

Proof. Let K/k be a ramified quadratic extension and let $\varpi \in k$ be a uniformizer of K with $\varpi^2 = u\pi$ for some $u \in \mathcal{O}^\times$. Upon making the change of variables $(x_0, x_1, x_2, x_3, x_4) \mapsto (\varpi x_0, \varpi x_1, x_2, x_3, x_4)$ over K and canceling common factors of ϖ , we obtain a model \mathfrak{X}' of X of the following shape modulo ϖ :

$$\begin{aligned} Q_1 &: x_2 x_3 \\ Q_2 &: x_0^2 - a x_1^2 + u F(x_2, x_3, x_4) \end{aligned} \tag{3.4}$$

This surface is a union of 2 components given by $x_2 = 0$ and $x_3 = 0$. The restriction to the component given by $x_2 = 0$ is a quadric surface defined by the vanishing of a single quadratic form Q_2 , which has rank at least 3 by Theorem **3.6**. A quadric surface defined by a form of rank 4 is smooth, while a quadric surface defined by a form of rank 3 is a cone over a conic, both of which are geometrically integral. It follows that \mathfrak{X}' modulo ϖ is split, and so $X(K) \neq \emptyset$ by Theorem **2.5**. \square

3.3 The rank 2 anisotropic case

In this section we consider the case where G is an *anisotropic* quadratic form of rank 2 modulo π in equation **3.1**. In particular, we may suppose after a suitable change of

variables that our defining equations are of the following form:

$$\begin{aligned} & x_0 l_0(x_1, x_2, x_3) + x_1 l_1(x_1, x_2, x_3) + x_4 l_4(x_0, x_1) + x_2^2 - ax_3^2 + \pi cx_4^2 \\ & x_0^2 - ax_1^2 + \pi F(x_2, x_3, x_4) \end{aligned} \tag{3.5}$$

To facilitate the following arguments, we will sometimes write F in the form

$$F(x_2, x_3, x_4) = Q(x_2, x_3) + x_4 L(x_2, x_3, x_4)$$

for a quadratic form Q and linear form L .

Unlike in the previous two cases, it is *not* possible to prove that X defined by the vanishing of quadrics of the above form has points over every ramified extension. Indeed, there are examples where X does fail to have points over a single ramified extension; see for instance Example [4.4](#) below.

We begin with the following lemmas:

Lemma 3.8. *Suppose X has an integral model \mathfrak{X} given by equations of type [3.5](#), and further suppose that $L(0, 0, x_4) \equiv 0$ modulo π . If \mathfrak{X} is semistable, then $l_4(x_0, x_1)$, and c are not both identically zero modulo π . If furthermore $L \equiv 0$ modulo π , then $x_2^2 - ax_3^2$ and $Q(x_2, x_3)$ are linearly independent modulo π .*

Proof. For the first part, consider the weight vector $(1, 1, 1, 1, 0)$. Since \mathfrak{X} is semistable we require $\text{mult}_{(1,1,1,1,0)}(\mathfrak{X}) < \frac{16}{5}$. If $L(0, 0, x_4)$, $l_4(x_0, x_1)$, and c were all identically 0 modulo π then every minor of $A_{f^*_{(1,1,1,1,0)}} \mathfrak{X}$ would have valuation at least 4, a contradiction.

For the second part, we instead consider the weight vector $(1, 1, 0, 0, 0)$. Since \mathfrak{X} is semistable we require $\text{mult}_{(1,1,0,0,0)}(\mathfrak{X}) < \frac{8}{5}$. Let $\mathbf{q} \in \mathcal{O}^3$ denote the coefficient vector of $Q(x_2, x_3)$. Then by semistability, the 2×3 submatrix of coefficients $\begin{pmatrix} 1 & 0 & -a \\ \pi \mathbf{q} \end{pmatrix}$ must have a minor with valuation 1. Hence $\begin{pmatrix} 1 & 0 & -a \\ \mathbf{q} \end{pmatrix}$ has a minor with valuation 0. Alternatively put, $x_2^2 - ax_3^2$ and $Q(x_2, x_3)$ are linearly independent mod π . \square

Lemma 3.9. *Suppose X has an integral model \mathfrak{X} given by equations of the following form, where Q and F are quadratic forms and l_4 is a linear form:*

$$\begin{aligned} x_2^2 - ax_3^2 + \pi Q(x_0, x_1, x_2, x_3) + \pi x_4 l_4(x_0, x_1, x_2, x_3) \\ x_0^2 - ax_1^2 + \pi F(x_2, x_3, x_4) \end{aligned} \tag{3.6}$$

Further suppose that $l_4(x_0, x_1, 0, 0) \equiv 0$ modulo π . If \mathfrak{X} is semistable, then $x_0^2 - ax_1^2$ and $Q(x_0, x_1, 0, 0)$ are linearly independent modulo π .

Proof. Since \mathfrak{X} is semistable we require $\text{mult}_{(0,0,1,1,0)}(\mathfrak{X}) < \frac{8}{5}$. Let $\mathbf{q} \in \mathcal{O}^3$ denote the coefficient vector of $Q(x_0, x_1, 0, 0)$. Then by semistability, the 2×3 submatrix of coefficients $\begin{pmatrix} \pi \mathbf{q} \\ 1 & 0 & -a \end{pmatrix}$ must have a minor with valuation 1. Hence $\begin{pmatrix} \mathbf{q} \\ 1 & 0 & -a \end{pmatrix}$ has a minor with valuation 0. Alternatively put, $x_0^2 - ax_1^2$ and $Q(x_0, x_1, 0, 0)$ are linearly independent mod π . \square

For the following two theorems, we will restrict to the case where $F(0, 0, x_4) \not\equiv 0$ modulo π in the equations [3.5](#), and make the following observation. By adding an

appropriate multiple of Q_2 to Q_1 we may arrange so that Q_1 has no x_4^2 term. Thus we may assume that \mathfrak{X} is given by the vanishing of quadrics of the form

$$\begin{aligned} x_0 l_0(x_0, x_1, x_2, x_3) + x_1 l_1(x_1, x_2, x_3) + x_4 l_4(x_0, x_1) + x_2^2 - ax_3^2 + \pi H(x_2, x_3, x_4) \\ x_0^2 - ax_1^2 + \pi F(x_2, x_3, x_4), \end{aligned} \quad (3.7)$$

for linear forms l_0 , l_1 , and l_4 and some quadratic form H for which we may assume $H(0, 0, x_4) = 0$.

Theorem 3.10. *Assume X has a semistable model \mathfrak{X} with defining equations given by [3.7](#) and further suppose $F(0, 0, x_4)$ is not identically 0 modulo π . If l_0 and l_1 are identically zero modulo π and in addition $l_4(x_0, x_1) \equiv 0$ modulo π^2 , then $X(K) \neq \emptyset$ for every ramified quadratic extension K/k .*

Proof. Writing $l_0 = \pi \tilde{l}_0$, $l_1 = \pi \tilde{l}_1$, and $l_4 = \pi \tilde{l}_4$, the equations [3.7](#) take the following form:

$$\begin{aligned} x_2^2 - ax_3^2 + \pi x_0 \tilde{l}_0(x_0, x_1, x_2, x_3) + \pi x_1 \tilde{l}_1(x_1, x_2, x_3) + \pi x_4 \tilde{l}_4(x_0, x_1) + \pi H(x_2, x_3, x_4) \\ x_0^2 - ax_1^2 + \pi F(x_2, x_3, x_4), \end{aligned} \quad (3.8)$$

Let K/k be a ramified quadratic extension and let $\varpi \in k$ be a uniformizer of K with $\varpi^2 = u\pi$ for some $u \in \mathcal{O}^\times$. Since $H(0, 0, x_4) = 0$ we may rewrite $H(x_2, x_3, x_4) = H_{23}(x_2, x_3) + x_4 H_4(x_2, x_3)$.

Upon making the change of variables

$$(x_0, x_1, x_2, x_3, x_4) \mapsto (\varpi x_0, \varpi x_1, \varpi^2 x_2, \varpi^2 x_3, x_4)$$

over K and canceling common factors of ϖ , we obtain a model \mathfrak{X}' of X over the valuation ring of K of the following shape modulo ϖ :

$$\begin{aligned} Q_1 &: x_2^2 - ax_3^2 + ux_0\tilde{l}_0(x_0, x_1, 0, 0) + ux_1\tilde{l}_1(x_1, 0, 0) + uH_4(x_2, x_3) \\ Q_2 &: x_0^2 - ax_1^2 + uF(0, 0, x_4) \end{aligned} \tag{3.9}$$

Every nonzero $\overline{\mathbb{F}}$ -linear combination of forms Q_1 and Q_2 modulo ϖ is an orthogonal sum of a $\overline{\mathbb{F}}$ -linear combination of $x_0\tilde{l}_0(x_0, x_1, 0, 0) + x_1\tilde{l}_1(x_1, 0, 0)$ and $x_0^2 - ax_1^2$, which has rank at least 1 by Lemma 3.9, and a $\overline{\mathbb{F}}$ -linear combination of $x_2^2 - ax_3^2 + uH_4(x_2, x_3)$ and $F(0, 0, x_4)$, which has rank at least 2. Note also that Q_1 and Q_2 themselves have rank at least 3 modulo ϖ (by assumption in the case of Q_2 , and by Lemma 3.9 in the case of Q_1). Hence every form in the pencil $\lambda Q_1 + \mu Q_2$ has rank at least 3 modulo ϖ .

By Theorem 2.6, the special fiber of \mathfrak{X}' is split. Therefore $X(K) \neq \emptyset$ by Theorem 2.5. □

Theorem 3.11. *Assume X has a semistable model \mathfrak{X} with defining equations given by 3.7 and further suppose $F(0, 0, x_4)$ is not identically 0 modulo π . If $l_4(x_0, x_1) \equiv 0$ modulo π but l_0 and l_1 are not both identically zero modulo π , then $X(K) \neq \emptyset$ for every ramified quadratic extension K/k .*

Proof. Let K/k be a ramified quadratic extension and let $\varpi \in k$ be a uniformizer of K with $\varpi^2 = u\pi$ for some $u \in \mathcal{O}^\times$. Upon making the change of variables

$$(x_0, x_1, x_2, x_3, x_4) \mapsto (\varpi x_0, \varpi x_1, \varpi x_2, \varpi x_3, x_4)$$

over K and canceling common factors of ϖ , we obtain a model \mathfrak{X}' of X over the valuation ring of K of the following shape modulo ϖ :

$$\begin{aligned} Q_1 &: x_0 l_0(x_0, x_1, x_2, x_3) + x_1 l_1(x_1, x_2, x_3) + x_2^2 - ax_3^2 \\ Q_2 &: x_0^2 - ax_1^2 + uF(0, 0, x_4) \end{aligned} \tag{3.10}$$

It follows from our assumptions that every $\overline{\mathbb{F}}$ -linear combination of forms Q_1 and Q_2 modulo ϖ has rank at least 3. By Theorem [2.6](#), the special fiber of \mathfrak{X}' is split. Therefore $X(K) \neq \emptyset$ by Theorem [2.5](#). \square

Theorem 3.12. *Assume X has a semistable model \mathfrak{X} with defining equations given by [3.5](#) and further suppose $L \equiv 0$ modulo π . Then $X(K) \neq \emptyset$ for every ramified quadratic extension K/k .*

Proof. Let K/k be a ramified quadratic extension and let $\varpi \in k$ be a uniformizer of K with $\varpi^2 = u\pi$ for some $u \in \mathcal{O}^\times$. Since $L \equiv 0$ modulo π we may write

$$L(x_2, x_3, x_4) = \pi x_4 \tilde{L}(x_2, x_3, x_4)$$

for a suitable linear form \tilde{L} . Upon making the change of variables

$$(x_0, x_1, x_2, x_3, x_4) \mapsto (\varpi^2 x_0, \varpi^2 x_1, \varpi x_2, \varpi x_3, x_4)$$

over K and canceling common factors of ϖ , we obtain a model \mathfrak{X}' of X over the valuation ring of K of the following shape modulo ϖ :

$$\begin{aligned}
Q_1 &: x_4 l_4(x_0, x_1) + x_2^2 - ax_3^2 + ucx_4^2 \\
Q_2 &: x_0^2 - ax_1^2 + uQ(x_2, x_3) + u^2 x_4 \tilde{L}(0, 0, x_4)
\end{aligned}
\tag{3.11}$$

Every $\overline{\mathbb{F}}$ -linear combination of forms Q_1 and Q_2 modulo ϖ has rank at least 3 by Lemma 3.8. By Theorem 2.6, the special fiber of \mathfrak{X}' is split. Therefore $X(K) \neq \emptyset$ by Theorem 2.5. \square

We now come to the main theorem of the section.

Theorem 3.13. *Suppose X has a semistable model \mathfrak{X} with defining equations given by 3.5. Then there is a quadratic ramified extension K/k such that $X(K) \neq \emptyset$.*

Proof. We will divide the argument into two cases, depending on whether the x_4^2 coefficient in L is a multiple of a uniformizer.

Case 1: $L(0, 0, x_4) \not\equiv 0$ modulo π .

First we will consider the case where $L(0, 0, x_4)$ is not identically 0 modulo π . Let us first consider the subcase where $l_4(x_0, x_1)$ is also not identically 0 modulo π . Let K/k be a ramified quadratic extension and let $\varpi \in k$ be a uniformizer of K with $\varpi^2 = u\pi$ for some $u \in \mathcal{O}$. Upon making the change of variables

$$(x_0, x_1, x_2, x_3, x_4) \mapsto (\varpi x_0, \varpi x_1, \varpi x_2, \varpi x_3, x_4)$$

over K and canceling common factors of ϖ , we obtain a model \mathfrak{X}' of X over the valuation ring of K of the following shape modulo ϖ :

$$\begin{aligned}
Q_1 &: x_4 l_4(x_0, x_1) \\
Q_2 &: x_0^2 - ax_1^2 + uL(0, 0, x_4)
\end{aligned}
\tag{3.12}$$

Note that \mathfrak{X}' contains $Q_2|_{l_4(x_0, x_1)=0}$ as a component, which is a cone over a rank 2 quadric. If $l_4(0, x_1)$ is not identically 0 modulo π , this quadric has the form $Ax_0^2 + uL(0, 0, x_4)$, where A is a constant that depends only on the coefficients of $l_4(x_0, x_1)$ [Note: if $l_4(0, x_1) \equiv 0$ modulo π then we use the form $Ax_1^2 + uL(0, 0, x_4)$ instead].

By the classification of quadratic forms over local fields [Lam05, Theorem VI.2.2], the component $Q_2|_{l_4(x_0, x_1)=0}$ is split over the special fiber of \mathfrak{X}' if and only if u and $-A/L(0, 0, 1)$ belong to the same square class. This will be the case over *one* of the quadratic ramified extensions K/k . Since the special fiber of \mathfrak{X}' contains a component that is split over the residue field of K , we conclude that \mathfrak{X}' is split as well. By Theorem 2.5, $X(K) \neq \emptyset$.

Suppose instead we have $l_4(x_0, x_1) \equiv 0$ modulo π . By adding to Q_1 an appropriate multiple of Q_2 , we may write our defining equations for \mathfrak{X} in the form

$$\begin{aligned}
Q_1 &: x_0 l_0(x_0, x_1, x_2, x_3) + x_1 l_1(x_1, x_2, x_3) + x_4 l_4(x_0, x_1) + x_2^2 - ax_3^2 + \pi H(x_2, x_3, x_4) \\
Q_2 &: x_0^2 - ax_1^2 + \pi F(x_2, x_3, x_4)
\end{aligned}
\tag{3.13}$$

where we may assume $H(0, 0, x_4) = 0$. If at least one of l_0 and l_1 is not identically 0 modulo π , then X has points over every ramified quadratic extension by Theorem 3.11. So we may assume l_0 and l_1 are both identically zero modulo π . Writing $l_0 = \pi \tilde{l}_0$,

$l_1 = \pi \tilde{l}_1$, and $l_4 = \pi \tilde{l}_4$, our defining equations take the form

$$\begin{aligned} Q_1 &: x_2^2 - ax_3^2 + \pi x_0 \tilde{l}_0(x_0, x_1, x_2, x_3) + \pi x_0 \tilde{l}_1(x_1, x_2, x_3) + \pi x_4 \tilde{l}_4(x_0, x_1) + \pi H(x_2, x_3, x_4) \\ Q_2 &: x_0^2 - ax_1^2 + \pi F(x_2, x_3, x_4) \end{aligned} \tag{3.14}$$

If $\tilde{l}_4 \equiv 0$ modulo π then X has points over every ramified quadratic extension by Theorem [3.10](#), so we may assume \tilde{l}_4 is not identically zero modulo π .

Let K/k be a ramified quadratic extension and let $\varpi \in k$ be a uniformizer of K with $\varpi^2 = u\pi$ for some $u \in \mathcal{O}$. Upon making the change of variables

$$(x_0, x_1, x_2, x_3, x_4) \mapsto (\varpi x_0, \varpi x_1, \varpi^2 x_2, \varpi^2 x_3, x_4)$$

over K and canceling common factors of ϖ , we obtain a model \mathfrak{X}' of X over the valuation ring of K of the following shape modulo ϖ :

$$\begin{aligned} Q_1 &: x_4 \tilde{l}_4(x_0, x_1) \\ Q_2 &: x_0^2 - ax_1^2 + uL(0, 0, x_4) \end{aligned} \tag{3.15}$$

Note that \mathfrak{X}' contains $Q_2|_{\tilde{l}_4(x_0, x_1)=0}$ as a component, which is a cone over a rank 2 quadric. If $\tilde{l}_4(0, x_1)$ is not identically 0 modulo π , this quadric has the form $Ax_0^2 + uL(0, 0, x_4)$, where A is a constant that depends only on the coefficients of $\tilde{l}_4(x_0, x_1)$ [Note: if $\tilde{l}_4(0, x_1) \equiv 0$ modulo π then we use the form $Ax_1^2 + uL(0, 0, x_4)$ instead].

By the classification of quadratic forms over local fields, the component $Q_2|_{\tilde{l}_4(x_0, x_1)=0}$ is split over the special fiber of \mathfrak{X}' if and only if u and $-A/L(0, 0, 1)$ belong to the same square class. This will be the case over *one* of the quadratic ramified extensions

K/k . Since the special fiber of \mathfrak{X}' contains a component that is split over the residue field of K , we conclude that \mathfrak{X}' is split as well. By Theorem [2.5](#), $X(K) \neq \emptyset$.

Case 2: $L(0, 0, x_4) \equiv 0$ modulo π

Next we consider the case where $L(0, 0, x_4) \equiv 0$ modulo π . If $L \equiv 0$ modulo π then X has points over every ramified quadratic extension by Theorem [3.12](#), so we may assume $L(x_2, x_3, 0)$ is not identically zero modulo π .

Let K/k be a ramified quadratic extension and let $\varpi \in k$ be a uniformizer of K with $\varpi^2 = u\pi$ for some $u \in \mathcal{O}$. Upon making the change of variables

$$(x_0, x_1, x_2, x_3, x_4) \mapsto (\varpi^2 x_0, \varpi^2 x_1, \varpi x_2, \varpi x_3, x_4)$$

over K and canceling common factors of ϖ , we obtain a model \mathfrak{X}' of X over the valuation ring of K of the following shape modulo ϖ :

$$\begin{aligned} Q_1 &: x_4 l_4(x_0, x_1) + x_2^2 - ax_3^2 + ucx_4^2 \\ Q_2 &: x_4 L(x_2, x_3) \end{aligned} \tag{3.16}$$

Note that \mathfrak{X}' contains $Q_2|_{l_4(x_0, x_1)=0}$ as a component, which we will denote by \mathcal{Y} . If $l_4(x_0, x_1)$ is not identically zero modulo ϖ then \mathcal{Y} is a cone over a conic, which is split. Hence \mathfrak{X}' is split, and by Theorem [2.5](#) we have $X(K) \neq \emptyset$ regardless of the choice of ramified quadratic extension K .

If on the other hand $l_4(x_0, x_1) \equiv 0$ modulo π then c is nonzero modulo π by Lemma [3.8](#) and \mathcal{Y} is a cone over a rank 2 quadric. If $L(0, x_3)$ is not identically 0 modulo π , this quadric has the form $Ax_2^2 + ucx_4^2$, where A is a constant that depends only on

the coefficients of $L(x_2, x_3)$ [Note: if $L(0, x_3) \equiv 0$ modulo π then we use the form $Ax_3^2 + ucx_4^2$ instead].

By the classification of quadratic forms over local fields, \mathcal{Y} is split over the special fiber of \mathfrak{X}' if and only if u and $-A/c$ belong to the same square class. This will be the case over *one* of the quadratic ramified extensions K/k . Since the special fiber of \mathfrak{X}' contains a component that is split over the residue field of K , we conclude that \mathfrak{X}' is split as well. By Theorem [2.5](#), $X(K) \neq \emptyset$. \square

This concludes the case where G is a rank 2 form modulo π . We remark in passing that the proofs in this section may be used to provide a rich source of examples of a smooth complete intersection of two quadrics in \mathbb{P}^4 over a local field failing to have points over every quadratic extension. For instance, Example [4.4](#) below was discovered during the proof of Theorem [3.13](#) above.

3.4 The rank 1 case

Finally we will consider the case where G reduces to a rank 1 quadratic form modulo π in the equations [3.1](#), which we may assume is x_2^2 . By completing the square in x_2 and renaming l_0 , l_1 , and F we may further assume that \mathfrak{X} is given by the vanishing of quadrics of the form:

$$\begin{aligned} x_0 l_0(x_1, x_2, x_3, x_4) + x_1 l_1(x_1, x_2, x_3, x_4) + x_2^2 + \pi H(x_3, x_4) \\ x_0^2 - ax_1^2 + \pi F(x_2, x_3, x_4) \end{aligned} \tag{3.17}$$

Lemma 3.14. *Suppose that X has an integral model \mathfrak{X} given by equations of type*

3.17. *If \mathfrak{X} is semistable, then $F(0, x_3, x_4)$ is not identically zero modulo π .*

Proof. The proof is essentially identical to that of Lemma 3.6, but due to its brevity we'll repeat it here. Since \mathfrak{X} is semistable we require $\text{mult}_{(1,1,1,0,0)}(\mathfrak{X}) < \frac{12}{5}$. If $F(0, x_3, x_4)$ were identically zero mod π then every minor of $A_{f_{(1,1,1,0,0)}^* \mathfrak{X}}$ would have valuation at least 3, contradicting the semistability of \mathfrak{X} . \square

Lemma 3.15. *Again suppose X has an integral model \mathfrak{X} given by equations of type*

3.17, *and further suppose that $l_0(0, 0, x_3, x_4)$ and $l_1(0, 0, x_3, x_4)$ are identically zero modulo π . If \mathfrak{X} is semistable, then $l_0(x_1, x_2, 0, 0)$ and $l_1(x_1, x_2, 0, 0)$ are not both identically zero modulo π . Furthermore, $H(x_3, x_4)$ and $F(0, x_3, x_4)$ are linearly independent modulo π .*

Proof. Since \mathfrak{X} is semistable, we require $\text{mult}_{(0,0,1,0,0)}(\mathfrak{X}) < \frac{4}{5}$. If both $l_0(x_1, x_2, 0, 0)$ and $l_1(x_1, x_2, 0, 0)$ were identically zero modulo π , the hypothesis would imply that l_0, l_1 are identically zero modulo π . But then every minor of $A_{f_{(0,0,1,0,0)}^* \mathfrak{X}}$ would have valuation at least 1, a contradiction.

For the second part, let \mathbf{h} and \mathbf{f} denote the coefficient vectors of $H(x_3, x_4)$ and $F(0, x_3, x_4)$, respectively. Then the semistability of \mathfrak{X} with respect to the weight

vector $w = (1, 1, 1, 0, 0)$ combined with the given hypothesis forces the 2×3 submatrix of coefficients $\begin{pmatrix} \pi \mathbf{h} \\ \pi \mathbf{f} \end{pmatrix}$ to have a minor with valuation 2, since we require

$\text{mult}_{(1,1,1,0,0)}(\mathfrak{X}) < \frac{12}{5}$. Hence the coefficient matrix $\begin{pmatrix} \mathbf{h} \\ \mathbf{f} \end{pmatrix}$ has a minor with valuation 0, i.e. $H(x_3, x_4)$ and $F(0, x_3, x_4)$ are linearly independent modulo π . \square

Theorem 3.16. *Assume X has a semistable model \mathfrak{X} with defining equations given by [3.17](#) and further suppose $l_0(0, 0, x_3, x_4) \equiv l_1(0, 0, x_3, x_4) \equiv 0$ modulo π . Then $X(K) \neq \emptyset$ for every ramified quadratic extension K/k .*

Proof. Let K/k be a ramified quadratic extension and let $\varpi \in k$ be a uniformizer of K with $\varpi^2 = u\pi$ for some $u \in \mathcal{O}^\times$. Upon making the change of variables

$$(x_0, x_1, x_2, x_3, x_4) \mapsto (\varpi x_0, \varpi x_1, \varpi x_2, x_3, x_4)$$

over K and canceling common factors of ϖ , we obtain a model \mathfrak{X}' of X over the valuation ring of K of the following shape modulo ϖ :

$$Q_1 : x_0 l_0(x_1, x_2, 0, 0) + x_1 l_1(x_1, x_2, 0, 0) + x_2^2 + uH(x_3, x_4) \tag{3.18}$$

$$Q_2 : x_0^2 - ax_1^2 + uF(0, x_3, x_4)$$

Every $\overline{\mathbb{F}}$ -linear combination of forms Q_1 and Q_2 modulo ϖ is an orthogonal sum of a $\overline{\mathbb{F}}$ -linear combination of $x_0 l_0(x_1, x_2, 0, 0) + x_1 l_1(x_1, x_2, 0, 0) + x_2^2$ and $x_0^2 - ax_1^2$, which has rank 2 by Lemma [3.15](#), and a $\overline{\mathbb{F}}$ -linear combination of $H(x_3, x_4)$ and $F(0, x_3, x_4)$, which has rank at least 1 by Lemma [3.15](#) again. Hence every form in the pencil $\lambda Q_1 + \mu Q_2$ has rank at least 3 modulo ϖ . By Theorem [2.6](#), the special fiber of \mathfrak{X}' is split. Therefore $X(K) \neq \emptyset$ by Theorem [2.5](#). \square

Theorem 3.17. *Assume X has a semistable model \mathfrak{X} with defining equations given by [3.17](#). Then there is a quadratic ramified extension K/k such that $X(K) \neq \emptyset$.*

Proof. If $l_0(0, 0, x_3, x_4) \equiv l_1(0, 0, x_3, x_4) \equiv 0$ modulo π then the desired conclusion follows from [Theorem 3.16](#), so we may assume $l_0(0, 0, x_3, x_4)$ and $l_1(0, 0, x_3, x_4)$ are not both identically zero modulo π .

By [Lemma 3.14](#), $F(0, x_3, x_4)$ is not identically zero modulo π . We may in fact assume one of $F(0, x_3, 0)$ or $F(0, 0, x_4)$ is not identically 0 modulo π after completing the square with respect to x_3 and x_4 and renaming F , l_0 , l_1 , and H as necessary.

Let K/k be a ramified quadratic extension and let $\varpi \in k$ be a uniformizer of K with $\varpi^2 = u\pi$ for some $u \in \mathcal{O}^\times$. Upon making the change of variables

$$(x_0, x_1, x_2, x_3, x_4) \mapsto (\varpi x_0, \varpi x_1, \varpi x_2, x_3, x_4)$$

over K and canceling common factors of ϖ , we obtain a model \mathfrak{X}' of X over the valuation ring of K of the following shape modulo ϖ :

$$\begin{aligned} \tilde{Q}_1 &: x_0 l_0(0, 0, x_3, x_4) + x_1 l_1(0, 0, x_3, x_4) \\ \tilde{Q}_2 &: x_0^2 - ax_1^2 + uF(0, x_3, x_4) \end{aligned} \tag{3.19}$$

Since one of $F(0, x_3, 0)$ or $F(0, 0, x_4)$ is not identically zero modulo π , at least one of $\tilde{Q}_2|_{x_3=0}$ or $\tilde{Q}_2|_{x_4=0}$ is rank 3 modulo π . By symmetry, it will suffice to consider the case where $\tilde{Q}_2|_{x_3=0}$ is rank 3 modulo π (i.e. $F(0, 0, x_4)$ is not identically zero modulo π).

Next, let $r = \text{rank}(\tilde{Q}_1|_{x_3=0})$ modulo π . If $r = 0$, then $\tilde{Q}_1 \equiv x_3L(x_0, x_1)$ modulo π for some linear form L . Hence \mathfrak{X}' has the following form modulo π :

$$\begin{aligned}\tilde{Q}_1 &: x_3L(x_0, x_1) \\ \tilde{Q}_2 &: x_0^2 - ax_1^2 + uF(0, x_3, x_4)\end{aligned}\tag{3.20}$$

The component $\mathfrak{X}'|_{x_3=0}$ modulo ϖ is a cone over a rank 3 quadric, i.e. a smooth conic, which is geometrically integral. Hence \mathfrak{X}' is split modulo ϖ , so $X(K) \neq \emptyset$ by Theorem [2.5](#).

If instead $r \neq 0$ then we can write $\tilde{Q}_1 = x_3l_3(x_0, x_1) + x_4l_4(x_0, x_1)$ for linear forms l_3 and l_4 we must have l_4 not identically 0 modulo π . In the remainder of the argument we'll use the notation l_{ij} to conveniently refer to the coefficient of x_ix_j in l_i .

Let $\mathcal{Y} = \mathfrak{X}'|_{x_3=0}$, which has the following shape modulo ϖ :

$$\begin{aligned}\tilde{Q}_1 &: x_4l_4(x_0, x_1) \\ \tilde{Q}_2 &: x_0^2 - ax_1^2 + uF(0, 0, x_4)\end{aligned}\tag{3.21}$$

Note that \mathcal{Y} modulo ϖ contains $Q_2|_{l_4(x_0, x_1)=0}$ as a component, which is a cone over a rank 2 quadric. If $l_4(0, x_1)$ is not identically 0 modulo π , this quadric has the form $Ax_0^2 + uF(0, 0, x_4)$, where A is a constant that depends only on the coefficients of $l_4(x_0, x_1)$ [Note: if $l_4(0, x_1) \equiv 0$ modulo π then we use the form $Ax_1^2 + uL(0, 0, x_4)$ instead].

By the classification of quadratic forms over local fields, the rank 2 quadric $Ax_0^2 + uF(0, 0, x_4)$ is isotropic over K if and only if u and $-A/F(0, 0, x_4)$ belong to the same

square class. This will be the case over *one* of the quadratic ramified extensions K/k .

Choose a nontrivial zero $(\tilde{x}_0, \tilde{x}_4)$ of $Ax_0^2 + uF(0, 0, x_4)$ over the residue field of K .

Setting $\tilde{x}_1 = -l_{04}\tilde{x}_0/l_{41}$ we obtain a point $P = (\tilde{x}_0, \tilde{x}_1, 0, 0, \tilde{x}_4)$ on the special fiber of \mathfrak{X}' .

By the Jacobian criterion, a point $(x_0, x_1, x_2, x_3, x_4)$ on the special fiber of \mathfrak{X}' is smooth if and only if the matrix $J|(x_0, x_1, x_2, x_3, x_4)$ has rank 2, where

$$J = \begin{pmatrix} l_{03}x_3 + l_{04}x_4 & l_{13}x_3 + l_{14}x_4 & 0 & l_3(x_0, x_1) & l_4(x_0, x_1) \\ 2x_0 & -2ax_1 & 0 & 2uF(0, 1, 0)x_3 & 2uF(0, 0, 1)x_4 \end{pmatrix}.$$

Applying this criterion to the point P , we have

$$J|_P = \begin{pmatrix} l_{04}\tilde{x}_4 & l_{14}\tilde{x}_4 & 0 & l_3(\tilde{x}_0, \tilde{x}_1) & 0 \\ 2\tilde{x}_0 & -2a\tilde{x}_1 & 0 & 0 & 2uF(0, 0, 1)\tilde{x}_4 \end{pmatrix}.$$

Note that both the entries $l_{14}\tilde{x}_4$ and $2uF(0, 0, 1)\tilde{x}_4$ are nonzero, so $J|_P$ is rank 2. Hence

P is a smooth point over the residue field of K . By Hensel's Lemma, $X(K) \neq \emptyset$. \square

Chapter 4

PROOF OF THE MAIN THEOREM

The primary goal of this chapter is to outline the proof of Theorem [1.1](#). This section is relatively shorter owing to much of the technical work having already been done in the previous chapter.

Our starting point in the proof of the main theorem is the following theorem of Creutz and Viray:

Theorem 4.1 (Creutz, Viray). *Let $X \subset \mathbb{P}_k^4$ be a smooth complete intersection of two quadrics over a local field k . There is a quadratic extension K/k such that $X(K) \neq \emptyset$.*

Proof. See [\[CV23\]](#), Theorem 2.1]. □

In fact Creutz and Viray give two proofs of this theorem. One of these proofs actually gives a slightly stronger result that we will use in the proof of our main theorem below. We record this result as the following Theorem:

Theorem 4.2 (Creutz, Viray). *Let $X \subset \mathbb{P}_k^4$ be a smooth complete intersection of two quadrics over a local field k , with a semistable integral model $\mathfrak{X} \subset \mathbb{P}_{\mathcal{O}}^4$. Suppose that the special fiber of \mathfrak{X} is nonsplit and remains nonsplit over the unique quadratic extension \mathbb{F}'/\mathbb{F} . Then $X(K) \neq \emptyset$ for any ramified quadratic extension K/k .*

Proof. This follows from the proof of [CV23, Theorem 2.1]. □

We are now ready to prove the main theorem.

Theorem 4.3 (Main Theorem). *Let $X \subset \mathbb{P}_k^4$ be a smooth complete intersection of two quadrics over a nonarchimedean local field k of odd residue characteristic. Then there is at most one quadratic extension K/k such that $X(K) = \emptyset$.*

Proof. Let \mathcal{O} denote the valuation ring of k and \mathbb{F} its residue field. By Theorem [3.1], X has a semistable model $\mathfrak{X} \subset \mathbb{P}_k^4$. If the special fiber of \mathfrak{X} is split, then $X(k) \neq \emptyset$ by Theorem [2.5]. If on the other hand the special fiber of \mathfrak{X} is nonsplit and remains nonsplit over the unique quadratic extension \mathbb{F}'/\mathbb{F} , then Theorem [4.2] applies and $X(K) \neq \emptyset$ for every ramified extension K/k .

We may henceforth assume that the special fiber of \mathfrak{X} is nonsplit over \mathbb{F} but becomes split over the over the quadratic extension \mathbb{F}'/\mathbb{F} . By Theorem [3.2], the special fiber of \mathfrak{X} is reduced. Hence Theorem [2.7] applies, and the special fiber of \mathfrak{X} is contained in a rank 2 quadric Q defined over \mathbb{F} .

If Q is reducible modulo π then we may assume it takes the form $Q \equiv x_0x_1$ modulo π , which is not semistable by Theorem [3.3]. So Q is irreducible modulo π , in which case we may assume $Q \equiv x_0^2 - ax_1^2$ modulo π for some nonsquare unit $a \in \mathcal{O}^\times$. After completing the square in x_0 and x_1 , we may assume that \mathfrak{X} is given by the vanishing

of quadrics of the following form:

$$\begin{aligned} Q_1 &: x_0 l_0(x_1, x_2, x_3, x_4) + x_1 l_1(x_1, x_2, x_3, x_4) + G(x_2, x_3, x_4) \\ Q_2 &: x_0^2 - ax_1^2 + \pi F(x_2, x_3, x_4) \end{aligned} \tag{4.1}$$

Note that we can ensure Q_1 has no x_0^2 term by adding an appropriate multiple of Q_2 if necessary. In view of Theorems [3.4](#) and [3.5](#), we may reduce to the case where G is rank 1 or 2 modulo π . By changing variables in x_2, x_3 , and x_4 and renaming l_0, l_1 , and F if necessary, we may assume G has one of the following three forms modulo π :

(i) $G \equiv x_2 x_3 \pmod{\pi}$

(ii) $G \equiv x_2^2 \pmod{\pi}$

(iii) $G \equiv x_2^2 - ax_3^2 \pmod{\pi}$

For form (i), we may write $G = x_2 x_3 + \pi H(x_2, x_3, x_4)$ and then the desired result follows from Theorem [3.7](#). For form (ii), after completing the square and renaming l_0, l_1 , and F as necessary, we may write $G = x_2^2 + \pi H(x_3, x_4)$. The desired result then follows from Theorem [3.17](#). Similarly, for form (iii) after completing the square and renaming l_0, l_1 , and F as necessary, we may write $G = x_2^2 - ax_3^2 + \pi c x_4^2$, for some $c \in \mathcal{O}$. The desired result then follows from Theorem [3.13](#). \square

The next example shows that the result of Theorem [4.3](#) is the best possible in the sense that a smooth complete intersection of two quadrics $X \subset \mathbb{P}_k^4$ can fail to have points over exactly one quadratic extension of k .

Example 4.4. Let k be a nonarchimedean local field of odd characteristic with valuation ring \mathcal{O} . Fix a uniformizer π and a nonsquare unit $a \in \mathcal{O}^\times$, and let $X \subset \mathbb{P}_k^4$ be the intersection of the two quadrics

$$\begin{aligned} x_0x_4 + x_2^2 - ax_3^2 + \pi x_4^2 \\ x_0^2 - ax_1^2 + \pi Q(x_2, x_3) + \pi x_4^2 \end{aligned} \tag{4.2}$$

Consider the quadratic extension $K = k(\varpi)$ where $\varpi^2 = \pi$. Then $X(K) = \emptyset$.

Proof. Suppose there is a solution $(x_0, x_1, x_2, x_3, x_4)$ with $x_i \in K$. In such a solution we may in fact assume $x_i \in \mathcal{O}'$ where \mathcal{O}' is the valuation ring of K , and we may also assume not all x_i are divisible by ϖ .

By the second equation x_0 and x_1 must be divisible by ϖ , and it then follows from the first equation that x_2 and x_3 must also be divisible by ϖ . Writing $x_i = \varpi \tilde{x}_i$ for $i = 0$ to 3 for $\tilde{x}_i \in \mathcal{O}'$ and cancelling common factors of ϖ that appear in the equations, we have:

$$\begin{aligned} \tilde{x}_0x_4 + \varpi(\tilde{x}_2^2 - a\tilde{x}_3^2) + \varpi x_4^2 \\ \tilde{x}_0^2 - a\tilde{x}_1^2 + \varpi^2 Q(\tilde{x}_2, \tilde{x}_3) + x_4^2 \end{aligned} \tag{4.3}$$

Since x_4 is not divisible by ϖ , it follows from the first equation that \tilde{x}_0 is divisible by ϖ . Then by the second equation we have that $-a\tilde{x}_1^2 + x_4^2$ is divisible by ϖ . Again using the fact that x_4 is not divisible by ϖ , we have $(\tilde{x}_1/x_4)^2 \equiv 1/a$ modulo ϖ , so $1/a$ is a square modulo ϖ . But this is a contradiction, since 1 and a belong to different square classes in $K^\times/K^{\times 2}$. Therefore $X(K) = \emptyset$. \square

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