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Strichartz estimates for the wave equation on
Riemannian manifolds of bounded curvature

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Abstract

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Mathematics

Wave packet methods have proven to be a useful tool for the study of dispersive effects of the wave equation with coefficients of limited differentiability. In this thesis, we use scaled wave packet methods to prove Strichartz estimates on compact Riemannian manifolds under the condition that the Riemannian curvature tensor is uniformly bounded. This improves upon prior results for the case of metrics with two bounded derivatives.

TABLE OF CONTENTS

	Page
Glossary	iii
Chapter 1: Introduction	1
1.1 Wave Equation and Strichartz estimates	1
1.2 $C^{1,1}$ case	3
1.3 Main results	5
Chapter 2: Preliminaries and Reduction to the Model Operator	12
2.1 Harmonic coordinates on (M, g)	12
2.2 The wave equation on (M, g)	16
2.3 The model operator P	17
2.4 Reduction to a first order equation	22
Chapter 3: Regularity of the geodesic and Hamiltonian flows	24
Chapter 4: Small-time estimates for the phase functions	29
Chapter 5: Parametrix for the dyadically localized equation	32
Chapter 6: Energy flow estimates	36
Chapter 7: Wave packets and dispersive estimates	44
7.1 The wave packet frame	44
7.2 Weighted norm estimates for the parametrix	46
7.3 Proof of the dispersive estimate	58

Bibliography 61

GLOSSARY

- : $B_r = \{x \in \mathbb{R}^d : |x| < r\}$, $B_r(x_0) = \{x \in \mathbb{R}^d : |x - x_0| < r\}$
- : $W^{n,p}(\Omega) = \{f \in L^p(\Omega) : \partial^\alpha f \in L^p(\Omega) \ \forall |\alpha| \leq n\}$.
- : $\|f\|_{W^{n,p}(\Omega)} = \sum_{|\alpha| \leq n} \|\partial^\alpha f\|_{L^p(\Omega)}$
- : $\Pi_\omega^\perp =$ projection onto the subspace perpendicular to ω .
- : $a \lesssim b$ means $a \leq C_d b$, $C_d > 0$ a constant depending only on the dimension d .
- : $a \approx b$ means $a \lesssim b$ and $b \lesssim a$.

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Special thanks go to my parents, for their patience and love.

DEDICATION

To my parents

Chapter 1

INTRODUCTION

1.1 Wave Equation and Strichartz estimates

Consider the Cauchy problem for the wave equation on a manifold (M, g)

$$\begin{aligned} (\partial_t^2 - \Delta_g)u &= F(t, x) \in L^1([-1, 1]; H^{s-1}(M)) \\ u(0, x) &= f(x) \in H^s(M) \\ \partial_t u(0, x) &= g(x) \in H^{s-1}(M) \end{aligned} \tag{1.1}$$

where Δ_g is the Laplace-Beltrami operator, which equals $(\sqrt{|g|})^{-1} \sum_{i,j} \partial_i (\sqrt{|g|} g^{ij} \partial_j)$ in a local coordinate chart. An important family of estimates for the solutions of the Cauchy problem, which control mixed type L^p norms over space and time in terms of a Sobolev norm of the initial data, are the Strichartz estimates. More specifically, if $s \geq 0$, $q, r \in (2, \infty)$, and the admissibility condition is satisfied:

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - s, \quad \frac{1}{q} \leq \frac{d-1}{2} \left(\frac{1}{2} - \frac{1}{r} \right),$$

then for a smooth compact manifold, we have

$$\|u\|_{L_t^q([-1,1]; L^r(M))} \leq C(\|f\|_{H^s(M)} + \|g\|_{H^{s-1}(M)} + \|F\|_{L_t^1([-1,1]; H^{s-1}(M))}). \tag{1.2}$$

The first version of such Strichartz estimates was obtained globally on \mathbb{R}^{d+1} by Strichartz in [18], [19], where he proved (1.2) for $s = \frac{1}{2}$, and $q = r = \frac{2(d+1)}{d-1}$. The results were then extended to general smooth manifolds (M, g) . More details can be found in [15]. There are two crucial endpoint cases: one is when $r = 2$, $q = \infty$, which is the energy estimate; the other one is when equality holds in the second admissibility condition. We refer to [8], [9], [10] for more details.

The dispersive nature of solutions to the wave equation plays a crucial role in proving Strichartz estimates. Such dispersive properties are closely related to the geometry of the Hamiltonian flow induced by the wave operator, which, on the cotangent bundle $T^*(M)$, is given by

$$\frac{dx}{dt} = H_\xi(x, \xi), \quad \frac{d\xi}{dt} = -H_x(x, \xi) \quad (1.3)$$

where $H(x, \xi) = \sqrt{\sum_{i,j} g^{ij}(x) \xi_i \xi_j}$ is the square root of the principal symbol of the wave operator. From such a point of view, the energy contained in the initial data, roughly speaking, propagates along the bicharacteristic curves determined by (1.3). Hence, in order to study the dispersive properties of the solutions to the wave equation, it's important to understand the geometry of the geodesic flow determined by (1.3).

To clarify this, we study the model case in which \mathbb{R}^d is endowed with the flat metric, following the presentation of [5]. Fix a nonzero, radial Schwartz function on \mathbb{R}^d , $\psi(x)$ with $\text{supp}(\hat{\psi}(\xi)) \subset \{\xi : 1 \leq |\xi| \leq 2\}$ and define $\psi_\lambda(x) = \lambda^d \psi(\lambda x)$. Write

$$\hat{\psi}_\lambda(\xi) = \sum_\nu \hat{\psi}_\lambda^\nu(\xi),$$

where $\hat{\psi}_\lambda^\nu(\xi)$ is supported in a cone of angle $\lambda^{-\frac{1}{2}}$ about the direction ν , and ν is a $\lambda^{-\frac{1}{2}}$ scaled family of directions in the unit sphere. For a specific ν_0 , $\hat{\psi}_\lambda^{\nu_0}(\xi)$ is then supported in a rectangle centered at $\xi_0 = \lambda \nu_0$ of dimension λ in ν_0 direction and $\lambda^{\frac{1}{2}}$ in ν_0^\perp directions and $\psi_\lambda^{\nu_0}(x)$ is highly concentrated in a rectangle centered at origin, with λ^{-1} scale in the ν_0 direction and $\lambda^{-\frac{1}{2}}$ scale in ν_0^\perp directions. The solution of (1.3) with initial data $(0, \xi_0)$ is

$$x(t) = t\nu_0 \quad \xi(t) = \xi_0$$

and the coherent wave packet at scale λ

$$\psi_\lambda^{\nu_0}(t, x) = (e^{it\sqrt{-\Delta}} \psi_\lambda^{\nu_0})(x) = \int e^{i\langle x, \xi \rangle + it|\xi|} \hat{\psi}_\lambda^{\nu_0}(\xi) d\xi,$$

roughly speaking, will rapidly decay outside of the region

$$\{x : |\langle \nu_0, x + t\nu_0 \rangle| \leq \lambda^{-1}, |\langle \nu_0^\perp, x + t\nu_0 \rangle| \leq \lambda^{-\frac{1}{2}}\}.$$

This can be seen by noting that $|\xi|$ in the phase function is well approximated by $\langle \nu_0, \xi \rangle$ on the support of $\hat{\psi}_\lambda^{\nu_0}(\xi)$. The above regions are disjoint for different ν in the sum, and summing over ν shows that $\psi_\lambda(t, x) = (e^{it\sqrt{-\Delta}}\psi_\lambda)(x) = \sum_\nu \psi_\lambda^\nu(t, x)$ satisfies, for all N

$$|\psi_\lambda(t, x)| \leq C_N \lambda^{\frac{d+1}{2}} (1 + \lambda||x| - t|)^{-N},$$

when $t \approx 1$. For $0 \leq t \lesssim 1$, scaling then shows that

$$|\psi_\lambda(t, x)| \leq C_N t^{-\frac{d-1}{2}} \lambda^{\frac{d+1}{2}} (1 + \lambda||x| - t|)^{-N}. \quad (1.4)$$

Similar estimates can then be shown for $\cos(t\sqrt{-\Delta})\psi_\lambda$ and $\sin(t\sqrt{-\Delta})\psi_\lambda$. This leads to bounds for Schwartz kernel of the wave group localized to frequency scale λ , and Littlewood-Paley theory together with an interpolation argument leads to the proof of Strichartz estimates. More details can be found in [9], [10].

For a smooth manifold (M, g) , one can use a Fourier integral representation of $e^{it\sqrt{-\Delta_g}}$ to prove (1.4) for t less than the injectivity radius, which leads to the proof of Strichartz estimates on smooth manifolds. We note that the group property of $e^{it\sqrt{-\Delta_g}}$ plays a crucial role in the proof of Strichartz estimates described in [9], [10].

1.2 $C^{1,1}$ case

For a non-smooth metric g , the Fourier integral representation of $e^{it\sqrt{-\Delta_g}}$ fails. However, in [12], Smith developed new techniques to prove (1.2) under the condition that the metric g is $C^{1,1}$. There are two key ingredients in the proof:

The first one is to approximate $\sqrt{-\Delta_g}$ using a paradifferential approximation. For $k \geq 0$, take g_k^{ij} to be mollification of g^{ij} of scale $2^{-\frac{k}{2}}$ in space. Then since $g^{ij} \in C^{1,1}$, we have the estimates

$$\begin{aligned} \|g^{ij}(x) - g_k^{ij}(x)\|_{L^\infty} &\leq C 2^{-k}, \\ \|\nabla_x(g^{ij}(x) - g_k^{ij}(x))\|_{L^\infty} &\leq C 2^{-\frac{k}{2}}. \end{aligned}$$

We then pose

$$P = \sum_{k=1}^{\infty} p_k(x, D)\psi_k(D),$$

where $p_k(x, \xi) = \sqrt{\sum_{i,j} g_k^{ij}(x)\xi_i\xi_j}$ and ψ_k is the Littlewood-Paley cutoff function at frequency scale 2^k . The key estimate for P is that $P^2 + \Delta_g$ is an operator of order 1, in the sense that it maps $H^s \rightarrow H^{s-1}$ for $0 \leq s \leq 2$. One can then reduce the $L_t^q L_x^r$ type estimates for $\partial_t^2 - \Delta_g$ to the same estimates for $\partial_t^2 - P^2$ by absorbing the error term into the driving force F , provided that $0 \leq s \leq 2$.

The second ingredient is to construct a good parametrix for $\partial_t - iP$, meaning, we need to approximate the evolution operator e^{itP} . The key idea in [12] is to construct an approximation solution using an appropriate wave packet frame based on ψ'_λ described above. The wave packets constructed there are a family of L^2 functions suitably localized in both physical and frequency space at dyadic scale, and the key property of such wave packets is that the action of the wave group on each element of the frame can be well approximated by a function which is essentially a rigid translation of the original wave packet along the Hamiltonian flow induced by the wave operator. Therefore the solutions to the wave equation can be approximated by the superposition of the translated wave packet frame, up to an error term that involves a gain of one derivative. The exact solution to the Cauchy problem can then be produced by iteration. We note that such wave packet techniques only work for $d = 2, 3$, since the kernel $K_k(t, x, y)$ of the parametrix constructed in [12] for e^{itP} at each dyadic scale does not satisfies the bound

$$|K_k(t, x, y)| \leq C 2^{kd}(1 + 2^k|t|)^{-\frac{d-1}{2}}(1 + 2^k d(x, S_t(y)))^{-N}, \quad (1.5)$$

except for $t \approx 1$. However, for $d = 2, 3$, it does satisfy

$$|K_k(t, x, y)| \leq C 2^{kd}(1 + 2^k|t|)^{-\frac{d-1}{2}},$$

which implies the Strichartz estimates. The Strichartz estimates for general dimensions were established by Tataru in [20], [21], [22], where the FBI transform was used instead of the

wave packet technique. Here we also mention the work in Smith [13], where a modified FBI-transform was applied to approximate e^{itP} by the Hamiltonian flow map on the transform side. The parametrix constructed there is then a unitary group. We need the group property since for dispersive estimates we want $W(t)W(s)^*$ to satisfy the same estimates (1.5) for $t \rightarrow t - s$. We can then derive Strichartz estimates for all dimensions.

We remark here that the assumption of $C^{1,1}$ is the minimal regularity requirement among Hölder spaces for the validity of the Strichartz estimates. In fact, in [14], Smith and Sogge produced explicit examples of wave operators with Lipschitz or $C^{1,\alpha}$ metric for each $0 < \alpha < 1$, for which the Strichartz estimates fail.

1.3 Main results

In this thesis, we show that the regularity requirement of the metric can be slightly relaxed by replacing the $C^{1,1}$ condition by the condition of bounded Riemannian curvature tensor. This means that we require only certain combinations of the second order derivatives of g to be in L^∞ , rather than all the second order derivatives of g . Additionally, the condition $R \in L^\infty$ is geometrically invariant, and better behaved under a low regularity change of coordinates. Specifically, we prove the following theorem:

Theorem 1.1. *Suppose that (M, g) is a compact Riemannian manifold of C^1 structure that can be covered by a family of coordinate charts in which the metric g satisfies $g_{ij}(0) = \delta_{ij}$, $\sup_{i,j} \|g_{ij}\|_{W^{1,p}} \leq C_0$, some $p > d$. If the Riemannian curvature tensor satisfies $R \in L^\infty$ in each local coordinate chart, then the Strichartz estimates (1.2) hold.*

We want to adapt the ideas from [12], for which, as mentioned in the previous section, we need two ingredients.

First, under the bounded curvature condition, we need

$$\|g^{ij}(x) - g_k^{ij}(x)\|_{L^\infty} \leq C 2^{-k}, \quad (1.6)$$

where g_k^{ij} is a mollification of g^{ij} at physical scale $2^{-\frac{k}{2}}$. This will hold in local harmonic coordinates on (M, g) . This uses the fact that, in harmonic coordinates (see e.g. [4]),

$$\sum \partial_{x_m} (g^{mn} \partial_{x_n} g_{ij}) + Q(g, \nabla g) = \text{Ric}_{ij} \in L^\infty,$$

where $Q(g, \nabla g)$ is a quadratic form in first order derivatives of g_{ij} with coefficients given by combination of coefficients of g . By elliptic regularity, it follows that $\nabla^2 g \in BMO$, hence $g \in C^{2,*}$, which implies (1.6).

In Chapter 2, we use this idea to reduce matters to working with a compact perturbation of the Euclidean metric on \mathbb{R}^d , such that $\nabla^2 g \in BMO$ and $R \in L^\infty$. The procedure is similar to that in Taylor [23], Chapter 3 §9. Chapter 2 also reduces estimates for $\partial_t^2 - \Delta_g$ to the same estimates for $\partial_t - iP$, where P is a self-adjoint operator defined as

$$P = \beta_0(D)^2 + \frac{1}{2} \sum_{k=1}^{\infty} \beta_k(D) (p_k(x, D) + p_k(x, D)^*) \beta_k(D),$$

where $\beta_k^2(\xi)$ is a Littlewood-Paley partition of unity and $p_k(x, \xi) = \left(\sum_{i,j=1}^d g_k^{ij}(x) \xi_i \xi_j \right)^{\frac{1}{2}}$. We show that $P^2 + \Delta_g$ maps $H^s \rightarrow H^{s-1}$ for $0 \leq s \leq 2$.

The second ingredient we need is the construction of the exact evolution group $E(t) = e^{itP}$. We require $E(t)$ to satisfy the following properties:

- $E(t)$ is a strongly continuous 1-parameter unitary group on $L^2(\mathbb{R}^d)$
- The kernel $K_k(t, x, y)$ of $E(t)$ at each dyadic scale 2^k satisfies the dispersive estimates

$$|K_k(t, x, y)| \leq C 2^{kd} (1 + 2^k |t|)^{-\frac{d-1}{2}} (1 + 2^k d(x, S_t(y)))^{-N},$$

where $S_t(y)$ denotes the geodesic sphere with radius t centered at y , and $d(x, S_t(y))$ denotes the Euclidean distance of x to $S_t(y)$. Here we require $0 \leq t \leq 1$, which will guarantee that there are no conjugate points.

We start by constructing an approximation $W(t)$ to e^{itP} , in that

$$\begin{aligned}(\partial_t - iP)W(t) &= B(t) \\ W(0) &= I\end{aligned}$$

where $B(t)$ is an operator of order 0. The exact evolution group $E(t)$ then can be produced by iteration of $W(t)$. We note that neither the rigid translation technique in [12] nor the construction using the flow along P of the FBI transform in [13] give a good approximation of $W(t)$ to e^{itP} in the bounded curvature setting. In both of those approaches, one needs $\nabla^2 g \in L^\infty$ to show that $(\partial_t - iP)W(t)$ is of order 0. The key idea in our situation is to base $W(t)$ on the Lax parametrix construction. It works well since the solution to the eikonal equation

$$\partial_t \varphi_k(t, x, \eta) = p_k(x, \nabla_x \varphi_k(t, x, \eta)), \quad \varphi_k(0, x, \eta) = \langle x, \eta \rangle$$

can be expressed in terms of the geodesic/Hamiltonian flow induced by p_k . The bounded curvature condition plus Jacobi variation formula shows that the geodesic flow in this setting is a C^1 diffeomorphism, which is just as good as the $C^{1,1}$ case. Letting $(x(t, y, \eta), \xi(t, y, \eta))$ be the Hamiltonian flow of (y, η) along p_k at time t , then $\varphi_k(t, x, \eta) = \sum_j \eta_j y_j(t, x, \eta)$, where $y(t, x, \eta)$ is the inverse of the map $y \rightarrow x(t, y, \eta)$. The derivative estimates of $y(t, x, \eta)$ then can be obtained by studying the Hamiltonian flow induced by p_k .

In Chapter 3, we use the Jacobi variation formula and ODE theory to show the regularity of the geodesic flow and derivative estimates on $(y, \eta) \rightarrow (x(t, y, \eta), \xi(t, y, \eta))$. We also prove the invertibility of the map $y \rightarrow x(t, y, \eta)$ and the derivative estimates of $y(t, x, \eta)$, all of which are as regular as the $C^{1,1}$ case.

In Chapter 4, we use the results derived in Chapter 3 and a dilation argument to prove the desired estimates for $\varphi_k(t, x, \eta)$. The key point is that we get better estimates for small t , which is crucial to proving the dispersive estimates of the kernel $K_k(t, x, y)$ when t is small.

In Chapter 5, we introduce

$$(W_k(t)f)(x) = \frac{1}{(2\pi)^d} \int e^{i\varphi_k(t, x, \eta)} \psi_k(\eta) \hat{f}(\eta) d\eta,$$

and show that

$$(\partial_t - iP_k)(W_k(t)f) := B_k(t)f = \int e^{i\varphi_k(t,x,\eta)} b_k(t,x,\eta) \psi_k(\eta) \hat{f}(\eta) d\eta,$$

where, for $|\eta| \approx 2^k$, b_k satisfies

$$|\langle \eta, \partial_\eta \rangle^j \partial_\eta^\alpha \partial_x^\beta b_k(t,x,\eta)| \leq C_{j,\alpha,\beta} 2^{-kj} (t^{\frac{1}{2}} 2^{-\frac{k}{2}})^{|\alpha|} 2^{\frac{k}{2}|\beta|}.$$

This can be proved by using the estimates of φ_k in Chapter 4 and Fourier integral calculus arguments. We note that for t small b_k has better estimates, in particular $b_k \in S_{1,\frac{1}{2}}^0$ when $t \approx 2^{-k}$. This behavior will be crucial for the dispersive estimates in Chapter 7 for small t . We remark that we cannot use the transport equation as in the Lax parametrix construction to make $(\partial_t - iP_k)W_k(t)$ to be order less than 0, since the transport equations do not lead to a gain for general $p_k \in S_{1,\frac{1}{2}}^1$.

In Chapter 6, we introduce

$$\begin{aligned} W(t) &= \sum_k W_k(t), \\ B(t) &= \sum_k B_k(t). \end{aligned}$$

The exact evolution wave group $E(t) = e^{itP}$ then can be constructed explicitly as

$$E(t) = \sum_{m=0}^{\infty} t^m \int_{\Lambda^m} W(tr_{m+1}) B(tr_m) \cdots B(tr_1) d\mathbf{r},$$

where $\Lambda^m \subset \mathbb{R}_+^{m+1}$ is the m -simplex, consisting of $\mathbf{r} = (r_1, \dots, r_{m+1})$ with $r_j > 0$ for all j , and with $r_1 + \dots + r_{m+1} = 1$, and $d\mathbf{r}$ is the measure on Λ^m induced by projection onto (r_1, \dots, r_m) . We want to show that $E(t)$ satisfies a micro-locality property, that is, if f is micro-locally concentrated in a region U in phase space, then $E(t)f$ is micro-locally concentrated in the flowout of U under the Hamiltonian flow at time t .

One can show this for any finite product of terms using appropriate Fourier integral arguments. The difficulty in our setting is that we need to control products of terms of

arbitrary length. We start by showing that the dyadic localization is preserved for long products of B . Precisely, we show that

$$E(t) = \sum_{k=0}^{\infty} \sum_{m=0}^{2^{\frac{k}{4}}} t^m \int_{\Lambda^m} \tilde{W}_k(tr_{m+1}) \tilde{B}_k(tr_m) \cdots \tilde{B}_k(tr_2) \tilde{\psi}_k(D) B_k(tr_1) d\mathbf{r} + R(t),$$

where $R(t)$ is a smoothing operator, and

$$\begin{aligned} \tilde{W}_k(t) &= \tilde{\psi}_k(D)(W_{k-1} + W_k + W_{k+1})(t), \\ \tilde{B}_k(t) &= \tilde{\psi}_k(D)(B_{k-1} + B_k + B_{k+1})(t). \end{aligned}$$

Here, $\tilde{\psi}_k = \psi_{k-1} + \psi_k + \psi_{k+1}$. The terms for $m \geq 2^{\frac{k}{4}}$ give a smoothing operator $R(t)$ due to the fact that the norm of the m -th iteration can be bounded by $\frac{1}{m!}$. Littlewood-Paley theory then reduces the proof of Strichartz estimates to the estimates for the term

$$\tilde{E}_k(t) = \sum_{m=0}^{2^{\frac{k}{4}}} t^m \int_{\Lambda^m} \tilde{W}_k(tr_{m+1}) \tilde{B}_k(tr_m) \cdots \tilde{B}_k(tr_2) \tilde{\psi}_k(D) B_k(tr_1) d\mathbf{r}. \quad (1.7)$$

The goal of Chapter 7 is to prove that the kernel $K_k(t, x, y) = (\tilde{E}_k(t) \delta_y(\cdot))(x)$ satisfies the same dispersive estimates as on smooth manifolds; that is,

$$|K_k(t, x, y)| \leq C 2^{kd} (1 + 2^k |t|)^{-\frac{d-1}{2}} (1 + 2^k d(x, S_t(y)))^{-N}, \quad 2^{-k} \leq t \leq 1. \quad (1.8)$$

To motivate our proof of (1.8), let's take a closer look at the model case we described in Section 1.1. In that setting, for any ψ_k with Fourier transform $\hat{\psi}_k(\xi)$ supported in $\{\xi : 2^k \leq |\xi| \leq 2^{k+1}\}$, we make a conic decomposition $\{\psi_k^\nu\}$ such that $\hat{\psi}_k(\xi) = \sum_\nu \hat{\psi}_k^\nu(\xi)$, where $\hat{\psi}_k^\nu(\xi)$ is supported in a $2^k \times (2^{\frac{k}{2}})^{d-1}$ rectangle along the ν direction. The energy of each wave packet ψ_k^ν is then highly concentrated in a dual rectangle in physical space. Also, we note that the phase function of the Fourier integral representation of $e^{it\sqrt{-\Delta}}$ can be well approximated by $\langle \nu, \xi \rangle$ in the support of $\hat{\psi}_k^\nu$. The dispersion at $t \approx 1$ then comes from the fact that the wave packet ψ_k^ν propagates along the projected bicharacteristic curve in physical space, and different wave packets, essentially, do not overlap when $t \approx 1$. The scaling argument then shows that the dispersive estimates hold for any $t < 1$.

We want to adapt the same idea to prove the dispersive estimates in the bounded curvature scenario. One difficulty here is that the scaling argument does not work, since the paradifferential operator P does not scale properly. In order to handle this issue, we note that we have better estimates for the phase function $\varphi_k(t, x, \eta)$ when t is small. Thus, instead of scaling the operator, we work with a scaled wave packet frame with a finer localization in space. We construct a wave packet frame $\{\phi_\gamma\}$ that essentially is a spatial dilation by t^{-1} of the dyadic-parabolic wave packets at the scale $t2^k$ constructed in Smith [12]. We want to show that the operator $\tilde{E}_k(t)$ maps such a wave packet to a similar wave packet along the bicharacteristic flow. The dispersive estimates are then a consequence of such a mapping property. The difficulty is that the expression for $\tilde{E}_k(t)$ in (1.7) contains long products of operators, hence it is difficult to directly use Fourier integral calculus arguments. Instead of working with the operator $\tilde{E}_k(t)$ directly, we will prove an estimate for each single term $\tilde{B}_k(s)$ in the product that we can iterate. Specifically, for any function f with its Fourier transform supported in the dyadic region at the scale 2^k , we expand f as

$$f = \sum_{\gamma} c_{\gamma} \phi_{\gamma}$$

where

$$c_{\gamma} = \int \overline{\phi_{\gamma}(y)} f(y) dy.$$

For any given integer $M \geq 0$ and fixed t, x, ν , we then define a weighted norm space

$$\|f\|_{M,x,\nu}^2 = \sum_{\gamma'} (1 + 2^k d_t(x, \nu; x', \nu'))^{2M} |c_{\gamma'}|^2$$

where d_t is the pseudodistance function defined on the cosphere bundle $S^*(\mathbb{R}^d)$ as

$$d_t(x, \nu; x', \nu') = |\langle \nu, x - x' \rangle| + |\langle \nu', x - x' \rangle| + t|\nu - \nu'|^2 + t^{-1}|x - x'|^2.$$

For each single term $\tilde{B}_k(s)$ in $\tilde{E}_k(t)$, we prove the following weighted norm estimates

$$\|\tilde{B}_k(s)f\|_{M,\chi_s(x,\nu)} \leq C_M \|f\|_{M,x,\nu}, \tag{1.9}$$

where χ_s is the projected Hamiltonian flow on $S^*(\mathbb{R}^n)$ at time s . Iteration then shows that

$$\|\tilde{E}_k(t)f\|_{M,\chi_t(x_0,\nu_0)} \leq C_M e^{tC_M} \|f\|_{M,x_0,\nu_0},$$

which will lead to the dispersive estimates using arguments as for smooth manifolds.

To prove (1.9), we study the matrix representation $\{a(\gamma', \gamma)\}$ of $\tilde{B}_k(s)$ in terms of the wave packet frame $\{\phi_\gamma\}$, where

$$a(\gamma', \gamma) = \int \overline{\phi_{\gamma'}(y)} (\tilde{B}_k(s)\phi_\gamma)(y) dy.$$

The weighted norm estimates (1.9) then can be proven by using Schur's lemma for the matrix representation $\{a(\gamma', \gamma)\}$, which is a consequence of the following estimates we prove in Section 7.2,

$$|a(\gamma', \gamma)| \leq C_N (1 + 2^k d_t(\gamma'; \chi_s(\gamma)))^{-N} \quad (1.10)$$

Estimate (1.10) tells us that $\tilde{B}_k(s)$ essentially maps the wave packet ϕ_γ to $\phi_{\chi_s(\gamma)}$ for every γ . A key step in the proof of (1.10) is to show that the phase function of the Fourier integral representation of $\tilde{B}_k(s)$ can be linearized over the support of $\hat{\phi}_\gamma$, which closely follows the proof in Seeger-Sogge-Stein [11], where they essentially handle the case when $t = 1$. In our situation, better estimates for the phase function $\varphi_k(t, x, \eta)$ and the symbol $b_k(t, x, \eta)$ for small t are vital to the proof.

Chapter 2

PRELIMINARIES AND REDUCTION TO THE MODEL OPERATOR

In this chapter we study the regularity of g in harmonic coordinates. We then consider Sobolev spaces on M , and define the wave group for $\sqrt{-\Delta_g}$ using the orthonormal basis for $L^2(M)$ consisting of eigenfunctions of Δ_g . We conclude the chapter by reducing the proof of Theorem 1.1 to estimates for the evolution group e^{itP} of a pseudodifferential operator P on \mathbb{R}^d that is a paradifferential approximation to $\sqrt{-\Delta_g}$ in some harmonic coordinate chart.

2.1 Harmonic coordinates on (M, g)

We start with the assumption that (M, g) is a Riemannian manifold of C^1 structure with the following condition: there exists $r_0 > 0$, $C_0 < \infty$, and $p \in (d, \infty]$, and for each $z \in M$ a coordinate chart $F_z : B_{r_0} \rightarrow M$ with $F_z(0) = z$, so that the induced metric g on $B_{r_0} \subset \mathbb{R}^d$ satisfies

$$g_{ij}(0) = \delta_{ij}, \quad \sup_{ij} \|g_{ij}\|_{W^{1,p}} \leq C_0.$$

Since $W^{1,p}$ functions are of Hölder regularity $1 - \frac{d}{p} > 0$, by shrinking r_0 if needed we may additionally assume that, given $c_0 > 0$ to be determined,

$$\sup_{x \in B_{r_0}} |g_{ij}(x) - \delta_{ij}| \leq c_0.$$

Following Taylor [23], Chapter 3 §9, in particular [23, Prop. 9.1] and the comments following [23, (9.39)], after replacing r_0 by $\rho_0 = \rho_0(d, p, C_0, c_0) > 0$, we may assume that the induced coordinate functions, $f_z^i : F_z(B_{\rho_0}) \rightarrow \mathbb{R}^d$, are harmonic functions with respect to the Laplace-Beltrami operator of g , and that overlapping harmonic coordinate charts are

of regularity $W^{2,p}$ on their overlaps. The harmonic coordinates are related to the original F_z by a $W^{2,p}$ change of coordinates over B_{r_0} , and it follows that the original coordinates were also necessarily of regularity $W^{2,p} \subset C^{1,1-\frac{d}{p}}$ on their overlaps. Consequently, M is a manifold with $W^{2,p}$ structure. This is consistent with the fact that a metric g maintains its $W^{1,p}$ regularity under a $W^{2,p}$ change of coordinates, which can be seen by (2.1) below.

The space $W^{m,p}(B_\rho)$ admits a continuous linear extension operator to $W^{m,p}(\mathbb{R}^d)$ for all integers $m \geq 0$; see e.g. Stein [16], Chapter VI §3 Theorem 5. We may thus apply [23] Chapter 2 Proposition 1.1, together with the inclusions $W^{1,p}(\mathbb{R}^d) \subset L^\infty(\mathbb{R}^d)$, $W^{1,2}(\mathbb{R}^d) \subset L^{\frac{2p}{p-2}}(\mathbb{R}^d)$, to see that, on either \mathbb{R}^d or B_{ρ_0} ,

$$\|fg\|_{W^{1,p}} \leq C \|f\|_{W^{1,p}} \|g\|_{W^{1,p}}, \quad \|fg\|_{H^1} \leq C \|f\|_{W^{1,p}} \|g\|_{H^1}, \quad (2.1)$$

The Riemannian curvature tensor R for g in coordinates is given by

$$R_{ijkl} = \frac{1}{2} \left[\frac{\partial^2 g_{ik}}{\partial x_j \partial x_\ell} + \frac{\partial^2 g_{j\ell}}{\partial x_i \partial x_k} - \frac{\partial^2 g_{i\ell}}{\partial x_j \partial x_k} - \frac{\partial^2 g_{jk}}{\partial x_i \partial x_\ell} \right] + Q(g, \nabla g),$$

where $Q(g, \nabla g)$ is a quadratic form in first order derivatives of g_{ij} with coefficients given by combination of coefficients of g , hence $Q(g, \nabla g) \in L^{\frac{p}{2}}$ when $g \in W^{1,p}$ with $p > d$. Then R is defined as a distribution, and our key assumption is that R_{ijkl} is a bounded measurable function, and that uniformly in the local coordinates F_z ,

$$\|R_{ijkl}\|_{L^\infty(B_{\rho_0})} \leq C_0.$$

This is implied by assuming that R is a measurable function, together with the geometric condition that for all continuous vector fields v_j ,

$$\|\langle R(v_1, v_2)v_3, v_4 \rangle\|_{L^\infty(M)} \leq C_0 \quad \text{if} \quad \|g(v_j)\|_{L^\infty(M)} \leq 1.$$

In harmonic coordinates, the Ricci tensor Ric can be written, see e.g. [4], in the form

$$\text{Ric}_{ij} = \sum_{mn} \partial_{x_m} (g^{mn} \partial_{x_n} g_{ij}) + Q(g, \nabla g).$$

We have $\text{Ric}_{ij} \in L^\infty(B_{\rho_0})$, and following [23], Chapter 3 §10, we conclude that $g_{ij} \in W^{2,q}(B_\rho)$, for all $\rho < \rho_0$ and all $q < \infty$. In particular, $g_{ij} \in \text{Lip}(B_{.9\rho})$.

Next take $\phi \in C_c^\infty(B_{.8\rho_0}, [0, 1])$ with $\phi = 1$ on $B_{.7\rho_0}$, and $\chi \in C_c^\infty(B_{.9\rho_0}, [0, 1])$ with $\phi = 1$ on $B_{.8\rho_0}$.

We form a Riemannian metric $\tilde{g}_{ij} = \phi g_{ij} + (1 - \phi)\delta_{ij}$ on \mathbb{R}^d , and uniformly elliptic coefficients $a^{ij} = \chi g^{ij} + (1 - \chi)\delta^{ij}$ on \mathbb{R}^d . Then the following holds globally on \mathbb{R}^d ,

$$\sum_{m,n=1}^d \partial_{x_m} (a^{mn} \partial_{x_n} \tilde{g}_{ij}) \in L^\infty.$$

Since the a^{mn} are globally Lipschitz, we conclude from [23, Chapter 3] Proposition 10.3 that $\nabla^2 \tilde{g}_{ij} \in BMO_c(\mathbb{R}^d)$.

Note also that the Riemannian curvature tensor $\tilde{\mathbf{R}}$ of \tilde{g} belongs to $L^\infty(\mathbb{R}^d)$, where we use that \tilde{g} is Lipschitz, hence $\tilde{\mathbf{R}} = \phi \mathbf{R}$ modulo products of g and ∇g . Collecting this, and shrinking ρ_0 by a factor of 2, we have

Lemma 2.1. *Given $c_0 > 0$, there exists $\rho_0 > 0$ and $C_0 < \infty$ so that for each $z \in M$ there exists a coordinate chart $F_z : B_{\rho_0} \rightarrow M$, with $F_z(0) = z$, such that the induced metric on B_{ρ_0} agrees with the restriction of a metric g defined on \mathbb{R}^d , with*

$$\|\nabla^2 g_{ij}\|_{BMO} + \|g\|_{\text{Lip}} + \|\mathbf{R}_{ijkl}\|_{L^\infty} \leq C_0, \quad \|g - \mathbf{I}\|_{L^\infty} \leq c_0, \quad g = \mathbf{I} \text{ if } |x| > 2\rho_0.$$

In particular, g_{ij} belongs to $W^{2,q}(\mathbb{R}^d)$ for all $Q < \infty$.

We now fix a cover of M by a finite collection of harmonic coordinate charts $F_{z_j} : B_{\rho_0} \rightarrow M$. We may then take a partition of unity χ_j on M , with $\text{supp}(\chi_j) \subset F_{z_j}(B_{\rho_0})$, and such that $\chi_j \circ F_i$ belongs to $W^{2,p}(B_{\rho_0})$ for each i ; in particular, $\chi_j \circ F_j \in W_c^{2,p}(B_{\rho_0})$.

It follows from (2.1) that multiplication by χ_j maps $H_{\text{loc}}^s(B_{\rho_0})$ into $H_c^s(B_{\rho_0})$ for $s = 1, 2$. By interpolation this holds for $0 \leq s \leq 2$. We may then introduce Sobolev spaces $H^s(M) \subset L^2(M)$ for $0 \leq s \leq 2$ by

$$f \in H^s(M) \Leftrightarrow f \circ F_z \in H_{\text{loc}}^s(B_{\rho_0}) \quad \forall z \in M, \quad \|f\|_{H^s(M)} = \sum_j \|(\chi_j f) \circ F_j\|_{H^s(\mathbb{R}^d)}. \quad (2.2)$$

If f is supported in $F_i(B_{\rho_0}) \cap F_j(B_{\rho_0})$, then $\|f \circ F_i\|_{H^s} \leq C\|f \circ F_j\|_{H^s}$. This holds for $s = 0, 1$ since $F_j^{-1} \circ F_i$ is a C^1 diffeomorphism. It holds for $s = 2$ since $D(F_j^{-1} \circ F_i)$ is a multiplier on $W^{1,p}$ by (2.1). It then holds by interpolation for $0 \leq s \leq 2$. Consequently, there are natural continuous inclusions $H_c^s(B_{\rho_0}) \rightarrow H^s(M)$ for $0 \leq s \leq 2$ given by $g \rightarrow g \circ F_j^{-1}$, and one may identify $H^s(M)$ with a closed subspace of the finite direct sum over j of $H^s(B_{\rho_0})$.

An element of $(H^s)^*$ thus induces an element of $H_{\text{loc}}^{-s}(B_{\rho_0})$, and if we identify $H^{-s}(M)$ with $(H^s)^*$ for $0 \leq s \leq 2$, then the condition (2.2) holds for $-2 \leq s \leq 2$.

We observe here the following regularity property for Δ_g in harmonic coordinates, which follows, for example, from [7, Theorem 8.9]. Suppose that $u \in H^1(B_{\rho_0})$ is a weak solution to $\Delta_g u = f$, where $f \in L^2(B_{\rho_0})$. Then $u \in H^2(B_\rho)$ for all $\rho < \rho_0$, and

$$\|u\|_{H^2(B_\rho)} \leq C_\rho (\|u\|_{H^1(B_{\rho_0})} + \|f\|_{L^2(B_{\rho_0})}). \quad (2.3)$$

The Sobolev spaces for $|s| \leq 2$ can also be characterized using the spectral decomposition of Δ_g on $L^2(M)$. Consider the quadratic form on $H^1(M)$ given by

$$Q(u, v) = - \int \bar{u} (\Delta_g v) dm_g = \int g(d\bar{u}, dv) dm_g.$$

Then Q is symmetric, nonnegative, and coercive. By Rellich's Lemma, there exists a complete orthonormal basis $\{v_j\}$ of $L^2(M, dm_g)$ that diagonalizes Q , in that for $f, g \in H^1(M)$

$$Q(f, g) = \sum_j \lambda_j^2 \overline{c_j(f)} c_j(g),$$

where $c_j(f) = \int_M \bar{v}_j f dm_g$, and $0 = \lambda_0 \leq \lambda_1 \leq \dots$ is a sequence of real numbers converging to ∞ . The v_j are weak solutions in $H^1(M)$ to $-\Delta_g v_j = \lambda_j^2 v_j$, hence (2.3) gives $\|v_j\|_{H^2(M)} \leq C \lambda_j^2$. It follows that $c_j(f)$ can be defined for $f \in H^s(M)$ when $-2 \leq s \leq 0$ as the action of f on \bar{v}_j .

The operator $(1 - \Delta_g)$ is equivalent to multiplication by $(1 + \lambda_j^2)$ in the basis $\{v_j\}$, and the following theorem then gives a more natural definition of $H^s(M)$.

Theorem 2.2. For $-2 \leq s \leq 2$, the mapping $f \rightarrow \{c_j(f)\}_{j=0}^\infty$ is a continuous bijection, with continuous inverse, of $H^s(M)$ onto the space $\ell^2(\mathbb{N}, (1 + \lambda_j^2)^s)$. In particular, uniformly over $-2 \leq s \leq 2$ we have

$$\|f\|_{H^s(M)}^2 \approx \sum_{j=0}^\infty (1 + \lambda_j^2)^s |c_j(f)|^2, \quad c_j(f) = \int_M f \bar{v}_j dm_g,$$

and $\sum_{j=0}^\infty c_j(f) v_j$ converges to f in the topology of $H^s(M)$.

Proof. The theorem holds for $s = 0$ and $s = 1$ by construction of the v_j . For $s = 2$, we note that

$$\sum_{j=0}^N c_j((1 - \Delta_g)f) v_j = \sum_{j=0}^N (1 + \lambda_j^2) c_j(f) v_j = (1 - \Delta_g) \sum_{j=0}^N c_j(f) v_j$$

converges in $L^2(M)$ to $(1 - \Delta_g)f$ if $f \in H^2(M)$. It follows by elliptic regularity that $\sum_j c_j(f) v_j$ converges in $H^2(M)$ to f . Surjectivity onto $\ell^2(\mathbb{N}, (1 + \lambda_j^2)^2)$ follows similarly. The theorem follows for $0 \leq s \leq 2$ by interpolation, and for $-2 \leq s \leq 0$ by duality. \square

We note that the proof also shows that $-\Delta_g$ conjugates to multiplication by $\{\lambda_j^2\}$ in the basis $\{v_j\}$, as a map from $H^s(M) \rightarrow H^{s-2}(M)$, provided $0 \leq s \leq 2$.

2.2 The wave equation on (M, g)

Given data $(f, g) \in H^s(M) \oplus H^{s-1}(M)$, with $0 \leq s \leq 2$, we define the solution of the Cauchy problem to be

$$u(t) = \sum_{j=0}^\infty (\cos(t\lambda_j) c_j(f) + \lambda_j^{-1} \sin(t\lambda_j) c_j(g)) v_j$$

where we set $0^{-1} \sin(0t) = t$. It follows from Theorem 2.2 that

$$u \in C^0(H^s(M)) \cap C^1(H^{s-1}(M)) \cap C^2(H^{s-2}(M)), \quad u(0) = f, \quad \partial_t u(0) = g.$$

and additionally that $\partial_t^2 u = \Delta_g u$. In particular, this holds in the weak sense on B_{ρ_0} in each local coordinate chart F_z .

If the data (f, g) is localized in $F_z(B_{\rho_0/3})$ for some harmonic coordinate chart, then $W^{2,p}$ regularity of g for all $p < \infty$, together with Lemma 2.1 for c_0 small, shows that $u(t)$ is compactly supported in B_{ρ_0} if $|t| \leq \rho_0$.

By a partition of unity, we may reduce the proof of Theorem 1.1 to the case that the Cauchy data is supported in $F_z(B_{\rho_0/3})$ for some z and harmonic coordinates F_z , and thus work on \mathbb{R}^d with a metric satisfying the conditions of Lemma 2.1. After rescaling space and time by a factor $R \geq 1$, where $R^{-1}C_0 \leq c_0$, we can reduce Theorem 1.1 to the following

Theorem 2.3. *Suppose g is a Riemannian metric on \mathbb{R}^d , with $g_{ij} = \delta_{ij}$ if $|x| > R_0$, and such that, for c_d a small constant to be chosen depending on d ,*

$$\|R_{ijkl}\|_{L^\infty} + \|g - I\|_{\text{Lip}} + \|\nabla_x^2 g_{ij}\|_{BMO} \leq c_d,$$

Suppose $u \in C_t^0([-1, 1]; H^s(M)) \cap C_t^1([-1, 1]; H^{s-1}(M))$ is a weak solution to

$$(\partial_t^2 - \Delta_g)u = F, \quad u(0, \cdot) = f, \quad \partial_t u(0, \cdot) = g.$$

Then if $s \in [0, 2]$, $q, r \in (2, \infty)$, and the admissibility condition is satisfied:

$$\frac{1}{q} + \frac{d}{r} = \frac{d}{2} - s, \quad \frac{1}{q} \leq \frac{d-1}{2} \left(\frac{1}{2} - \frac{1}{r} \right),$$

the following estimate holds

$$\|u\|_{L_t^q([-1, 1]; L^r(M))} \leq C \left(\|f\|_{H^s(M)} + \|g\|_{H^{s-1}(M)} + \|F\|_{L_t^1([-1, 1], H^{s-1}(M))} \right).$$

2.3 The model operator P

We consider Δ_g for a metric on \mathbb{R}^d satisfying the conditions of Theorem 2.3.

Introduce $\beta_k(\xi) = \beta(2^{-k}\xi)$ if $k \geq 1$, and $\psi_k(\xi) = \beta_k(\xi)^2$, such that ψ_k is a Littlewood-Paley partition of unity. That is,

$$\text{supp}(\beta) \subset \left\{ \frac{9}{10} \leq |\xi| \leq \frac{20}{9} \right\}, \quad \sum_{k=0}^{\infty} \beta_k(\xi)^2 = 1.$$

We introduce a family of metrics $g_k(x)$ that are mollifications of $g(x)$ on spatial scale $2^{-\frac{k}{2}}$. Precisely, fix a radial function $\chi \in C_c^\infty(B_1)$ with $\int \chi(x) dx = 1$, and $\int x^\alpha \chi(x) dx = 0$ for $1 \leq |\alpha| \leq 4$, and define a smooth metric g_k on \mathbb{R}^d by

$$(g_k)_{ij}(x) = 2^{\frac{k}{2}d} \int \chi(2^{\frac{k}{2}}(x-y)) g_{ij}(y) dy.$$

By the conditions on g in Theorem 2.3, $\|g_k - I\|_{L^\infty} \leq c_0$. Furthermore,

$$\|\partial_x^\beta g_k^{ij}\|_{L^\infty} \leq C_\alpha \begin{cases} 1, & |\beta| \leq 1, \\ \log(k), & |\beta| = 2, \\ 2^{\frac{k}{2}(|\beta|-2)}, & |\beta| \geq 3. \end{cases} \quad (2.4)$$

The estimate for $|\beta| = 2$ follows by the fact that $|\text{avg}_Q(\nabla^2 g)| \lesssim 1 + |\log(\text{diam}(Q))|$, which follows from $\nabla^2 g \in BMO$ with support in $\{|x| \leq R_0\}$, and the last since $\partial_x^\theta \chi$ is an atom when $|\theta| \geq 1$.

In much of what follows we will need only the weaker estimates

$$\|\partial_x^\beta g_k^{ij}\|_{L^\infty} \leq C_\alpha \begin{cases} 1, & |\beta| \leq 1, \\ 2^{\frac{k}{2}(|\beta|-1)}, & |\beta| \geq 2. \end{cases} \quad (2.5)$$

Let $p_k(x, \xi) = \left(\sum_{i,j=1}^d g_k^{ij}(x) \xi_i \xi_j\right)^{\frac{1}{2}}$, so $p_k(x, \xi)$ is homogeneous of degree 1, and by (2.5) and the conditions of Theorem 2.3

$$\begin{aligned} |p_k(x, \xi) - |\xi|| + |\partial_x p_k(x, \xi)| &\leq c_0 |\xi|, \\ |\partial_\xi^\alpha \partial_x^\beta p_k(x, \xi)| &\leq C_{\alpha,\beta} 2^{\frac{k}{2} \max(0, |\beta|-1)} |\xi|^{1-|\alpha|}. \end{aligned} \quad (2.6)$$

Hence, $\partial_x^\beta p_k(x, \xi) \psi_k(\xi) \in S_{1, \frac{1}{2}}^1$, uniformly over k , if $|\beta| \leq 1$.

Define

$$P = \beta_0(D)^2 + \frac{1}{2} \sum_{k=1}^{\infty} \beta_k(D) (p_k(x, D) + p_k(x, D)^*) \beta_k(D),$$

and let $p(x, \xi)$ be the symbol of P . Then P is self-adjoint, and the $S_{1, \frac{1}{2}}^m$ calculus shows that

$$p(x, \xi) - \sum_{k=1}^{\infty} p_k(x, \xi) \psi_k(\xi) \in S_{1, \frac{1}{2}}^0.$$

In particular,

$$\partial_x^\alpha p \in S_{1, \frac{1}{2}}^1 \text{ for } |\beta| \leq 1.$$

We note for future use that the Garding inequality for P follows easily. Indeed, letting

$$b(x, \xi) = \left(\psi_0(\xi) + \sum_{k=1}^{\infty} p_k(x, \xi) \psi_k(\xi) \right)^{\frac{1}{2}},$$

then $b(x, D)^* b(x, D) - P \in \text{Op}(S_{1, \frac{1}{2}}^0)$, hence for $f \in H^{\frac{1}{2}}$, and some real C

$$\langle Pf, f \rangle \geq -C \|f\|_{L^2}^2. \quad (2.7)$$

Lemma 2.4.

$$\|P^2 u + \Delta_g u\|_{H^{s-1}(\mathbb{R}^d)} \leq C \|u\|_{H^s(\mathbb{R}^d)}, \quad 0 \leq s \leq 2. \quad (2.8)$$

Proof. From the fact that g^{ij} is Lipschitz, we have the following estimates:

$$|\partial_x^\beta \partial_\xi^\alpha p_k(x, \xi)| \leq C_{\alpha, \beta} \begin{cases} 2^{\frac{k}{2}(|\beta|-1)} (1 + |\xi|)^{1-|\alpha|}, & |\beta| \geq 1 \\ (1 + |\xi|)^{1-|\alpha|}, & \beta = 0. \end{cases} \quad (2.9)$$

Thus $\partial_x^\beta p_k(x, \xi) \beta_k(\xi) \in S_{1, \frac{1}{2}}^1$ for $|\beta| \leq 1$, with uniform bounds over k . Furthermore, β_k has disjoint support from β_j if $|j - k| > 1$. The asymptotic calculus then yields

$$P^2 = \sum_{k=0}^{\infty} \left(\sum_{i, j=1}^d g_k^{ij}(x) D_i D_j \right) \psi_k(D) + r(x, D), \quad r(x, \xi) \in S_{1, \frac{1}{2}}^1,$$

so that $r(x, D) : H^s \rightarrow H^{s-1}$ for all s . We next write

$$-\Delta_g = \sum_{i, j=1}^d g^{ij}(x) D_i D_j + |g|^{-\frac{1}{2}} (D_i (|g|^{\frac{1}{2}} g^{ij})) D_j.$$

Since $|g|^{-\frac{1}{2}} (D_i (|g|^{\frac{1}{2}} g^{ij})) \in W^{1,p}$ is a multiplier on H^s for $|s| \leq 1$, the second term maps $H^s \rightarrow H^{s-1}$ for $0 \leq s \leq 2$.

We thus need establish that, for each i, j , the operator

$$R(x, D) = \sum_{k=0}^{\infty} (g^{ij}(x) - g_k^{ij}(x)) \psi_k(D) D_i$$

maps $H^s(\mathbb{R}^d) \rightarrow H^s(\mathbb{R}^d)$ for $-1 \leq s \leq 1$.

By the vanishing moment conditions on the radial function χ , we can write

$$1 - \hat{\chi}(\xi) = |\xi|^2 h(\xi), \quad \text{where} \quad |\partial^\alpha h(\xi)| \leq C_\alpha \begin{cases} \min(1, |\xi|^{2-|\alpha|}), & |\xi| \leq 1, \\ |\xi|^{-2-|\alpha|}, & |\xi| \geq 1. \end{cases}$$

For $j, k \geq 0$, if we let $h_{j,k}(\xi) = \psi_j(\xi) h(2^{-\frac{k}{2}} \xi)$ we then have

$$|\partial_\xi^\alpha h_{j,k}(\xi)| \leq C_\alpha 2^{-|2j-k|} 2^{-j|\alpha|}. \quad (2.10)$$

That is, $\{2^{|2j-k|} h_{j,k}\}_{j=0}^\infty$ satisfies the same derivative estimates and localization properties of a Littlewood-Paley partition of unity, uniformly over k . We can write

$$g - g_k = 2^{-k} \sum_{j=0}^\infty g_{j,k}, \quad \text{where} \quad g_{j,k} = -(2\pi)^{-n} \widehat{h_{j,k}} * (\Delta g).$$

We then have

$$\text{supp}(\widehat{g_{j,k}}) \subset \{2^{j-1} \leq |\xi| \leq 2^{j+2}\}, \quad \|g_{j,k}\|_{L^\infty} \leq 2^{-|2j-k|},$$

For the second estimate we use that $\|\widehat{h_{j,k}} * (\Delta g)\|_{L^\infty} \leq C 2^{-|2j-k|} \|\Delta g\|_{BMO}$. This follows for $j \neq 0$ since $\widehat{h_{j,k}}$ is a molecule. For $j = 0$, we use also that $\text{supp}(\Delta g) \subseteq B_{R_0}$.

If $j < k - 1$, then $g_{j,k} \psi_k(D)u$ has spectrum supported in $\{2^{k-1} \leq |\xi| \leq 2^{k+2}\}$, so we can use orthogonality to estimate the sum over $j < k - 1$:

$$\begin{aligned} \left\| \sum_{k=0}^\infty \sum_{j=0}^{k-2} 2^{-k} g_{j,k} \psi_k(D) Du \right\|_{H^s}^2 &\leq C \sum_{k=0}^\infty \left\| \sum_{j=0}^{k-2} 2^{-k} g_{j,k} \psi_k(D) Du \right\|_{H^s}^2 \\ &\leq C \sum_{k=0}^\infty \left(\sum_{j=0}^{k-2} 2^{-|2j-k|} \|\psi_k(D)u\|_{H^s} \right)^2 \\ &\leq C \sum_{k=0}^\infty \|\psi_k(D)u\|_{H^s}^2 \leq C \|u\|_{H^s}^2. \end{aligned}$$

If $j > k + 1$, then $g_{j,k} \psi_k(D)u$ has spectrum supported in $\{2^{j-1} \leq |\xi| \leq 2^{j+2}\}$, and we estimate

the sum by

$$\begin{aligned}
\left\| \sum_{k=0}^{\infty} \sum_{j=k+2}^{\infty} 2^{-k} g_{j,k} \psi_k(D) Du \right\|_{H^s} &\leq \sum_{k=0}^{\infty} \sum_{j=k+2}^{\infty} 2^{-k} \|g_{j,k} \psi_k(D) Du\|_{H^s} \\
&\leq C \sum_{k=0}^{\infty} \sum_{j=k+2}^{\infty} 2^{-k+j^s} \|g_{j,k} \psi_k(D) Du\|_{L^2} \\
&\leq C \sum_{k=0}^{\infty} \sum_{j=k+2}^{\infty} 2^{k(1-s)+j(s-2)} \|\psi_k(D) u\|_{H^s} \\
&\leq C \sum_{k=0}^{\infty} 2^{-k} \|\psi_k(D) u\|_{H^s} \leq C \|u\|_{H^s}.
\end{aligned}$$

It remains to handle the case $|j - k| \leq 1$. For this, we note that, by (2.10), the function $a_k(\xi) := 2^k \sum_{|j-k| \leq 1} h_{j,k}(\xi)$ satisfies the properties of a Littlewood-Paley partition of unity, as does $2^{-k} \psi_k(D) D := \tilde{\psi}_k(D)$. We rewrite the remaining term as

$$\left\| \sum_{k=0}^{\infty} 2^{-k} (a_k(D) \Delta g) (\tilde{\psi}_k(D) u) \right\|_{H^s}$$

For $-1 \leq s \leq 0$, we dominate this by

$$\left\| \sum_{k=0}^{\infty} (a_k(D) \Delta g) (2^{-k} \tilde{\psi}_k(D) u) \right\|_{L^2} \leq C \|\Delta g\|_{BMO} \|u\|_{H^{-1}}$$

where we use the paraproduct estimate of Carleson [2] and Fefferman-Stein [6]; for a proof, see Stein [17, II.2.4, IV.4.3].

For $0 \leq s \leq 1$, we use that $(a_k(D) \Delta g) (\tilde{\psi}_k(D) u)$ is frequency supported in $|\xi| \leq 2^{k+3}$, and bound

$$\begin{aligned}
\left\| \sum_{k=0}^{\infty} 2^{-k} (a_k(D) \Delta g) (\tilde{\psi}_k(D) u) \right\|_{H^s} &\leq \sum_{k=0}^{\infty} 2^{k(s-1)} \left\| (a_k(D) \Delta g) (\tilde{\psi}_k(D) u) \right\|_{L^2} \\
&\leq C \sum_{k=0}^{\infty} 2^{k(s-1)} \|\Delta g\|_{BMO} \|\tilde{\psi}_k(D) u\|_{L^2} \\
&\leq C \sum_{k=0}^{\infty} 2^{-k} \|\Delta g\|_{BMO} \|\tilde{\psi}_k(D) u\|_{H^s} \\
&\leq C \|\Delta g\|_{BMO} \|u\|_{H^s}.
\end{aligned}$$

□

We also note here the following bounds:

$$\left\| \partial_x^\beta (\mathbf{g}_k - \mathbf{g}_{k-1}) \right\|_{L^\infty} \leq C_\beta 2^{-k + \frac{1}{2}|\beta|}. \quad (2.11)$$

For this, write

$$\chi(\xi) - \chi(2^{\frac{1}{2}}\xi) = |\xi|^2 \rho(\xi), \quad \rho \in \mathcal{S}(\mathbb{R}^d), \quad \rho(0) = 0.$$

Then, setting $\rho_k(\xi) = \rho(2^{-\frac{k}{2}}\xi)$, we have

$$\mathbf{g}_k - \mathbf{g}_{k-1} = 2^{-k} \widehat{\rho}_k * (\Delta \mathbf{g}).$$

The bound follows since $2^{-\frac{1}{2}|\beta|} \partial^\beta \widehat{\rho}_k$ is a molecule, uniformly over k , where by a molecule we mean a L^1 -norm preserving dilation of a Schwartz function of integral 0.

2.4 Reduction to a first order equation

By the above, we need to prove that

$$\|u\|_{L_t^q L_x^r} \leq C \left(\|u\|_{L_t^\infty H_x^s} + \|\partial_t u\|_{L_t^\infty H_x^{s-1}} + \|(D_t^2 - P^2)u\|_{L_t^1 H_x^{s-1}} \right),$$

If we pose $u = \langle D \rangle^{-s} v$, where $\langle D \rangle = (1 - \Delta)^{\frac{1}{2}}$, then v solves

$$(D_t^2 - P^2)v = [P^2, \langle D \rangle^{-s}]u + \langle D \rangle^{-s} (D_t^2 - P^2)u.$$

The $S_{1, \frac{1}{2}}$ calculus shows that $[P^2, \langle D \rangle^{-s}] \in S_{1, \frac{1}{2}}^{1-s}$, where we use that $\partial_x p(x, \xi) \in S_{1, \frac{1}{2}}^1$. Consequently, the above is equivalent to showing that

$$\|\langle D \rangle^{-s} u\|_{L_t^q L_x^r} \leq C \left(\|u\|_{L_t^\infty L_x^2} + \|\partial_t u\|_{L_t^\infty H_x^{-1}} + \|(D_t^2 - P^2)u\|_{L_t^1 H_x^{-1}} \right). \quad (2.12)$$

By (2.7), with $\mu = 1 + C_1$ we have

$$\langle (P + \mu)f, f \rangle \geq \|f\|_{L^2}^2 \quad \Rightarrow \quad \|(P + \mu)f\|_{L^2} \geq \|f\|_{L^2} \quad \text{when } f \in H^1.$$

By elliptic estimates we have $\|(P + \mu)f\|_{L^2(\mathbb{R}^d)} \geq \|f\|_{H^1(\mathbb{R}^d)}$, consequently $(P + \mu)^{-1}$ exists as a map from $L^2(\mathbb{R}^d) \rightarrow H^1(\mathbb{R}^d)$. One can show that $(P + \mu)^{-1} \in \text{Op}(S_{1, \frac{1}{2}}^{-1})$, for example by [1].

Note that since $(P + \mu)^2 - P^2 \in \text{Op}(S_{1, \frac{1}{2}}^1)$, the estimate (2.12) remains unchanged if we replace P by $P + \mu$. For notational simplicity we then assume P is invertible, with $P^{-1} \in \text{Op}(S_{1, \frac{1}{2}}^{-1})$.

The remainder of this paper is devoted to constructing the exact evolution group $E(t) = \exp(itP)$ for the self-adjoint operator P , and proving dispersive estimates for its kernel. The group $E(t)$ will satisfy following properties:

- $E(t)$ is a strongly continuous 1-parameter unitary group on $L^2(\mathbb{R}^d)$
- $E(t)$ is strongly continuous on $H^s(\mathbb{R}^d)$ for all $s \in \mathbb{R}$.
- $\partial_t E(t)$ is strongly continuous from $H^s(\mathbb{R}^d)$ into $H^{s-1}(\mathbb{R}^d)$ for all $s \in \mathbb{R}$.
- $E(0)f = f$, and $\partial_t E(t)f = iPE(t)f = iE(t)Pf$ when $f \in H^s$, for all $s \in \mathbb{R}$.

The second and third condition mean that, if $f \in H^s$, then $E(t)f \in C^0(H^s) \cap C^1(H^{s-1})$. For $s < 0$, we understand that $E(t)$ extends continuously to such an operator from $L^2(\mathbb{R}^d)$. It follows from the third and fourth conditions that $E(t)f \in C^j(H^{s-j})$ for all $s \in \mathbb{R}$ and all $j \in \mathbb{N}$.

Let

$$C(t) = \frac{1}{2}(E(t) + E(-t)), \quad S(t) = \frac{1}{2}(E(t) - E(-t))P^{-1}.$$

The solution $u \in C^0(L^2) \cap C^1(H^{-1})$ to the Cauchy problem

$$(\partial_t^2 - P^2)u = F, \quad u(0) = f, \quad \partial_t u(0) = g,$$

is then given by

$$u(t) = C(t)f + S(t)g + \int_0^t S(t-s)F(s) ds.$$

So we are reduced to showing that, for all $f \in H^2$, and q, r, s as in Theorem 1.1,

$$\|\langle D \rangle^{-s} E(t)f\|_{L_t^q L_x^r([-1, 1] \times \mathbb{R}^d)} \leq C \|f\|_{L^2(\mathbb{R}^d)}. \quad (2.13)$$

Chapter 3

REGULARITY OF THE GEODESIC AND HAMILTONIAN FLOWS

In this chapter we establish estimates for derivatives of all order on the geodesic and Hamiltonian flows of the metrics g_k , as well as spatial dilates of g_k . More generally, we will consider a metric g_M on \mathbb{R}^d , with estimates that depend on the parameter $M \in [1, \infty)$. We assume that for constants independent of M , the metric g_M satisfies the following conditions. First, for a sufficiently small constant c_d to be chosen depending only on the dimension d ,

$$\|\mathbf{R}_{ijkl}\|_{L^\infty} + \|(g_M)_{ij} - \delta_{ij}\|_{\text{Lip}} + \|\nabla_x^2 (g_M)_{ij}\|_{BMO} \leq c_d. \quad (3.1)$$

Next, for constants C_β independent of M , we assume that

$$\|\partial_x^\beta g_M^{ij}\|_{L^\infty} \leq C_\beta M^{|\beta|-1}, \quad |\beta| \geq 1, \quad (3.2)$$

and that the associated Riemann curvature tensor, \mathbf{R}_{ijkl} , satisfies

$$\|\partial_x^\beta \mathbf{R}_{ijkl}\|_{L^\infty} \leq C_\beta M^{|\beta|}, \quad |\beta| \geq 0. \quad (3.3)$$

We suppress the M in \mathbf{R}_{ijkl} , as well the Christoffel symbols Γ_{ij}^n , for simplicity of notation.

We let $\gamma(t, y, w)$ be the geodesic for g_M with initial conditions (y, w) :

$$\partial_t^2 \gamma^n = \sum_{ij} \Gamma_{ij}^n(\gamma) \dot{\gamma}^i \dot{\gamma}^j, \quad \gamma(0, y, w) = y, \quad \dot{\gamma}(0, y, w) = w,$$

where $\dot{\gamma} \equiv \partial_t \gamma$. Note that by (3.2) we have

$$\|\Gamma_{ij}^n\|_{L^\infty} \lesssim c_d, \quad (3.4)$$

where for this section $a \lesssim b$ means that $a \leq C_d b$ with C_d depending only on the dimension d .

Theorem 3.1. *Suppose that g_M satisfies (3.1)-(3.3), for a suitably small constant c_d . Then there are constants $C_{\alpha,\beta} < \infty$, depending only on the constants C_β in (3.2)-(3.3), so that over the set $\frac{1}{2} \leq |w| \leq 2$ and $0 \leq t \leq 1$,*

$$|\partial_y^\beta \partial_w^\alpha \gamma(t, y, w)| + |\partial_y^\beta \partial_w^\alpha \partial_t \gamma(t, y, w)| \leq C_{\alpha,\beta} M^{|\alpha|+|\beta|-1}, \quad |\alpha| + |\beta| \geq 1,$$

and

$$|D_y \gamma - \mathbf{I}| + |D_y \dot{\gamma}| + |D_w \dot{\gamma} - \mathbf{I}| + |\dot{\gamma} - w| \lesssim c_d, \quad |D_w \gamma - t \mathbf{I}| \lesssim c_d t. \quad (3.5)$$

Additionally, if either $|\alpha| \geq 1$ or $|\beta| \geq 2$,

$$|\partial_y^\beta \partial_w^\alpha \gamma(t, y, w)| \leq C_{\alpha,\beta} t M^{|\alpha|+|\beta|-1}, \quad |\alpha| \geq 1 \text{ or } |\beta| \geq 2.$$

Proof. We produce a (not orthonormal) frame $\{V_m\}_{m=1}^d$ along $\gamma(t, y, w)$ by parallel translating the standard frame $\{\partial_m\}_{m=1}^d$, yielding $V_m(t, y, w) = \sum_n v_m^n(t, y, w) \partial_n$. The dual frame $\{V^n\}_{n=1}^d$ under g_M is obtained by parallel translating $\sum_m g_M^{nm}(y) \partial_m$ along γ , so $v^{n,l}(t, y, w) = \sum_m g_M^{nm}(y) v_m^l(t, y, w)$, and regularity of the functions $v^{n,l}$ will follow directly from those for v_m^l . We have

$$\partial_t v_m^n = -\Gamma_{ij}^n(\gamma) \dot{\gamma}^i v_m^j, \quad v_m^n(0, t, w) = \delta_m^n. \quad (3.6)$$

Note that by (3.4) we have $|v_m^n - \delta_m^n| \lesssim c_d$ if $|t| \leq 1$. We then expand

$$\partial_{y^k} \gamma = \sum_m f_k^m(t, y, w) V_m = \sum_{mj} f_k^m(t, y, w) v_m^j(t, y, w) \partial_j. \quad (3.7)$$

Then, using (3.6),

$$\partial_{y^k} \dot{\gamma}^n = \partial_t \partial_{y^k} \gamma^n = \sum_m (\partial_t f_k^m) v_m^n - \sum_{ij} \Gamma_{ij}^n(\gamma) \dot{\gamma}^i f_k^m v_m^j. \quad (3.8)$$

Since $D_t^2 \partial_{y^k} \gamma = \sum_m (\partial_t^2 f_k^m) V_m$, the Jacobi variation formula yields

$$\partial_t^2 f_k^m = \sum_n \left(\sum_{ijlp} R_{ijlp}(\gamma) \dot{\gamma}^i v_n^j \dot{\gamma}^l v^{m,p} \right) f_k^n, \quad (3.9)$$

with the following initial conditions, the latter by (3.6) and (3.7),

$$f_k^m(0, y, w) = \delta_k^m, \quad \partial_t f_k^m(0, y, w) = \sum_i \Gamma_{ik}^m(y) w^i.$$

The bound $|\dot{\gamma}| \lesssim 1$ and $|v| \lesssim 1$, together with (3.4) yield, for $|t| \leq 1$,

$$|v_j^n - \delta_j^n| + |f_k^m - \delta_k^m| + |\partial_t f_k^m| + |\partial_{y_k} \gamma^n - \delta_k^n| + |\partial_{y_k} \dot{\gamma}^n| \lesssim c_d. \quad (3.10)$$

This yields the bound of (3.5) for the first two terms. The bounds for the terms in D_w follow by a similar consideration of the Jacobi field $\partial_{w^j} \gamma$.

Assume now that we have shown the following for $|\alpha| + |\beta| < N$, where $N \geq 1$,

$$|\partial_y^\beta \partial_w^\alpha v_j^n| + |\partial_y^\beta \partial_w^\alpha f_k^m| + |\partial_y^\beta \partial_w^\alpha \partial_t f_k^m| + |\partial_y^\beta \partial_w^\alpha \partial_{y_k} \gamma^n| + |\partial_y^\beta \partial_w^\alpha \partial_{y_k} \dot{\gamma}^n| \lesssim C_{\alpha, \beta} M^{|\alpha| + |\beta| - 1}. \quad (3.11)$$

By (3.6) and the Leibniz rule, for $|\alpha| + |\beta| = N$ we can write

$$\partial_t \partial_y^\beta \partial_w^\alpha v_m^n = -\Gamma_{ij}^n(\gamma) \dot{\gamma}^i \partial_y^\beta \partial_w^\alpha v_m^j + \mathcal{O}(M^{|\alpha| + |\beta|}).$$

Here we used that $|\partial_x^\beta \Gamma_{ij}^n(x)| \leq C_\beta M^{|\beta|}$. Similarly, by (3.9),

$$\partial_t^2 \partial_y^\beta \partial_w^\alpha f_k^m = \sum_n \left(\sum_{ijklp} R_{ijklp}(\gamma) \dot{\gamma}^i v_n^j \dot{\gamma}^l v^{m,p} \right) \partial_y^\beta \partial_w^\alpha f_k^n + \mathcal{O}(M^{|\beta|}),$$

By the initial conditions, we have

$$\partial_y^\beta v_n^m(0, y, w) = 0, \quad \partial_y^\beta f_k^m(0, y, w) = 0, \quad |\partial_t \partial_y^\beta f_k^m(0, y, w)| \leq C_\beta M^{|\beta|}.$$

An application of Gronwall's lemma then yields the following, for $|\beta| = N$,

$$|\partial_y^\beta \partial_w^\alpha v_j^n| + |\partial_y^\beta \partial_w^\alpha f_k^m| + |\partial_y^\beta \partial_w^\alpha \partial_t f_k^m| \leq C_{\alpha, \beta} M^{|\alpha| + |\beta|},$$

and (3.11) follows for $|\alpha| + |\beta| = N$ by (3.7) and (3.8), hence all α, β by induction.

The last estimate of the theorem now follows since $|\partial_t \partial_y^\beta \partial_w^\alpha \gamma| \leq C_{\alpha, \beta} M^{|\alpha| + |\beta| - 1}$, and $\partial_y^\beta \partial_w^\alpha \gamma(0, y, w) = 0$ if either $|\alpha| \geq 1$ or $|\beta| \geq 2$. \square

We now consider the related Hamiltonian flow. Let

$$p_M(x, \eta) = \left(\sum_{ij} g_M^{ij}(x) \eta_i \eta_j \right)^{\frac{1}{2}},$$

and consider the solution $(x(t, y, \eta), \xi(t, y, \eta))$ to the Hamiltonian flow

$$\dot{x} = (\nabla_{\xi} p_M)(x, \xi), \quad \dot{\xi} = -(\nabla_x p_M)(x, \xi), \quad x(0, y, \eta) = y, \quad \xi(0, y, \eta) = \eta.$$

These are related to the geodesic flow by the following,

$$x^i(t, y, \eta) = \gamma^i(t, y, w(y, \eta)), \quad \xi_j(t, y, \eta) = p_M(y, \eta) \sum_j g_{M,ij}(\gamma) \dot{\gamma}^j(t, y, w(y, \eta)),$$

where

$$w^i(y, \eta) = \frac{1}{p_M(y, \eta)} \sum_j g_M^{ij}(y) \eta_j.$$

It follows from (3.1) that

$$|D_y w| + |w - |\eta|^{-1} \eta| + |D_{\eta} w - (I - |\eta|^{-2} \eta \otimes \eta)| \lesssim c_d, \quad (3.12)$$

and from (3.2) and homogeneity that

$$|\partial_y^{\beta} \partial_{\eta}^{\alpha} w(y, \eta)| \leq C_{\alpha, \beta} M^{|\beta|-1} |\eta|^{-|\alpha|}.$$

Observe that $I - |\eta|^{-2} \eta \otimes \eta = \Pi_{\eta}^{\perp}$, the projection onto the plane perpendicular to η . We thus deduce the following corollary of Theorem 3.1.

Corollary 3.2. *Suppose that g_M satisfies (3.1)-(3.3), for a suitably small constant c_d . Then there are constants $C_{\alpha, \beta} < \infty$, depending only on the constants C_{β} in (3.2)-(3.3), so that over the interval $0 \leq t \leq 1$,*

$$|\eta| |\partial_y^{\beta} \partial_{\eta}^{\alpha} x(t, y, \eta)| + |\partial_y^{\beta} \partial_{\eta}^{\alpha} \xi(t, y, \eta)| \leq C_{\alpha, \beta} M^{|\alpha|+|\beta|-1} |\eta|^{1-|\alpha|}, \quad |\alpha| + |\beta| \geq 1,$$

and

$$|D_y x - I| + |D_{\eta} \xi - I| \lesssim c_d, \quad |D_y \xi| + |\xi - \eta| \lesssim c_d |\eta|, \quad |D_{\eta} x - t \Pi_{\eta}^{\perp}| \lesssim c_d t.$$

Additionally, if either $|\alpha| \geq 1$ or $|\beta| \geq 2$,

$$|\partial_y^{\beta} \partial_{\eta}^{\alpha} x(t, y, \eta)| \leq C_{\alpha, \beta} t M^{|\alpha|+|\beta|-1} |\eta|^{-|\alpha|}, \quad |\alpha| \geq 1 \text{ or } |\beta| \geq 2.$$

For the generating function $\varphi_k(t, x, \eta)$, we need consider the function $y(t, x, \eta)$ that is the inverse of the map $y \rightarrow x(t, y, \eta)$.

Theorem 3.3. *Suppose that g_M satisfies (3.1)-(3.3), for a suitably small constant c_d . Then there are constants $C_{\alpha, \beta} < \infty$, depending only on the constants C_β in (3.2)-(3.3), so that if $0 \leq t \leq 1$ and $\eta \neq 0$, the map $y \rightarrow x(t, y, \eta)$ is invertible, and the inverse map $y(t, x, \eta)$ satisfies $|D_x y - I| \lesssim c_d$, and*

$$|\partial_x^\beta \partial_\eta^\alpha y(t, x, \eta)| \leq C_{\alpha, \beta} M^{|\alpha|+|\beta|-1} |\eta|^{-|\alpha|}, \quad |\alpha| + |\beta| \geq 1,$$

Additionally, if either $|\alpha| \geq 1$ or $|\beta| \geq 2$,

$$|\partial_x^\beta \partial_\eta^\alpha y(t, x, \eta)| \leq C_{\alpha, \beta} t M^{|\alpha|+|\beta|-1} |\eta|^{-|\alpha|}, \quad |\alpha| \geq 1 \text{ or } |\beta| \geq 2.$$

Also, for the function $\xi(t, x, \eta) := \xi(t, y(t, x, \eta), \eta)$,

$$|\partial_x^\beta \partial_\eta^\alpha \xi(t, x, \eta)| \leq C_{\alpha, \beta} M^{|\alpha|+|\beta|-1} |\eta|^{1-|\alpha|}, \quad |\alpha| + |\beta| \geq 1.$$

Proof. For each $\eta \neq 0$ and $0 \leq t \leq 1$, the map $y \rightarrow x$ is proper, hence a closed mapping. It is open since $|D_y x - I| \lesssim c_d$, hence onto by connectivity of \mathbb{R}^d , and one-to-one by simple connectivity of \mathbb{R}^d . Thus $y \rightarrow x(t, y, \eta)$ is a diffeomorphism of \mathbb{R}^d , with inverse satisfying $|D_x y - I| \lesssim c_d$. The estimates of the theorem then are a consequence of the inverse function theorem and Corollary 3.2. \square

Chapter 4

SMALL-TIME ESTIMATES FOR THE PHASE FUNCTIONS

In this chapter we prove estimates on derivatives of the solution to the eikonal equation for g_k . Let g_k be the mollification of g at spatial scale $2^{-\frac{k}{2}}$ from Chapter 2, and let φ_k be the solution to the eikonal equation

$$\partial_t \varphi_k(t, x, \eta) = p_k(x, \nabla_x \varphi_k(t, x, \eta)), \quad \varphi_k(0, x, \eta) = \langle x, \eta \rangle.$$

Then $\varphi_k(t, x, \eta) = \sum_j \eta_j y_j(t, x, \eta)$, where $y(t, x, \eta)$ is as in Theorem 3.3, and the estimates of that theorem hold with $M = 2^k$. Furthermore,

$$\partial_{\eta_j} \varphi_k(t, x, \eta) = y_j(t, x, \eta), \quad \partial_{x_j} \varphi_k(t, x, \eta) = \xi_j(t, x, \eta).$$

We then easily read off the following from Theorem 3.3,

$$|\partial_x^\beta \varphi_k(t, x, \eta)| \leq C_\beta 2^{\frac{k}{2}(|\beta|-2)} |\eta|, \quad |\beta| \geq 2 \tag{4.1}$$

$$|\partial_x^\beta \partial_\eta \varphi_k(t, x, \eta)| \leq C_\beta t 2^{\frac{k}{2}(|\beta|-1)}, \quad |\beta| \geq 2, \tag{4.2}$$

$$|\partial_x^\beta \partial_\eta^\alpha \varphi_k(t, x, \eta)| \leq C_\beta t 2^{\frac{k}{2}(|\alpha|+|\beta|-2)} |\eta|^{1-|\alpha|}, \quad |\alpha| \geq 2. \tag{4.3}$$

Additionally,

$$|\partial_x \partial_\eta \varphi_k(t, x, \eta)| \leq C. \tag{4.4}$$

The following shows that some estimates can be improved with respect to derivatives in η , which is key to controlling the evolution operators for small t .

Theorem 4.1. *Assume that $|\alpha| \geq 2$ or $|\beta| \geq 2$. Then when $2^{-k} \leq t \leq 1$,*

$$|\partial_x^\beta \partial_\eta^\alpha \varphi_k(t, x, \eta)| \leq C_{\alpha, \beta} t^{\frac{|\alpha|}{2}} 2^{\frac{k}{2}(|\alpha|+|\beta|-2)} |\eta|^{1-|\alpha|},$$

and when $0 \leq t \leq 2^{-k}$,

$$|\partial_x^\beta \partial_\eta^\alpha \varphi_k(t, x, \eta)| \leq C_{\alpha, \beta} 2^{\frac{k}{2}(|\beta|-2)} |\eta|^{1-|\alpha|}.$$

Proof. If $|\alpha| \leq 1$, the estimates for all $0 \leq t \leq 1$ follow from (4.1)-(4.2). To handle $|\alpha| \geq 2$, we take a parameter ε with $2^{-k/2} \leq \varepsilon \leq 1$. Let $g_{\varepsilon, k}(x) = g_k(\varepsilon x)$, where g_k is the localization of g to frequency $2^{k/2}$. Similarly, let $p_{\varepsilon, k}(x, \xi) = p_k(\varepsilon x, \xi)$. Let $\varphi_{\varepsilon, k}$ be the solution to

$$\partial_t \varphi_{\varepsilon, k}(t, x, \eta) = p_{\varepsilon, k}(x, \nabla_x \varphi_{\varepsilon, k}(t, x, \eta)), \quad \varphi_{\varepsilon, k}(0, x, \eta) = \langle x, \eta \rangle.$$

Then by homogeneity we have

$$\varphi_k(t, x, \eta) = \varepsilon \varphi_{\varepsilon, k}(\varepsilon^{-1} t, \varepsilon^{-1} x, \eta). \quad (4.5)$$

The metric $g_{\varepsilon, k}(x)$ is the localization of $g(\varepsilon x)$ to frequency $\varepsilon 2^{k/2}$. Since $g(\varepsilon x)$ is Lipschitz with bounded curvature (uniformly over the range of ε), we can apply estimates (4.1)-(4.3) with $2^{\frac{k}{2}}$ replaced by $M = \varepsilon 2^{\frac{k}{2}}$.

For $2^{-\frac{k}{2}} \leq t \leq 1$ we take $\varepsilon = t$ in (4.5), and apply (4.3) with $M = t 2^{-\frac{k}{2}}$ to get

$$|\partial_x^\beta \partial_\eta^\alpha \varphi_k(t, x, \eta)| \leq C_{\alpha, \beta} t^{|\alpha|-1} 2^{\frac{k}{2}(|\alpha|+|\beta|-2)} |\eta|^{1-|\alpha|}.$$

For $|\alpha| \geq 2$ this implies the desired estimate.

For $0 \leq t \leq 2^{-\frac{k}{2}}$ we take $\varepsilon = 2^{-\frac{k}{2}}$ in (4.5), and apply (4.3) with $2^{\frac{k}{2}}$ replaced by 1 to get

$$|\partial_x^\beta \partial_\eta^\alpha \varphi_k(t, x, \eta)| \leq C_{\alpha, \beta} t 2^{\frac{k}{2}|\beta|} |\eta|^{1-|\alpha|}.$$

Since $t \leq t^{\frac{|\alpha|}{2}} 2^{\frac{k}{2}(|\alpha|-2)}$ for $t \geq 2^{-k}$ and $|\alpha| \geq 2$, and $t 2^{\frac{k}{2}|\beta|} \leq 2^{\frac{k}{2}(|\beta|-2)}$ for $0 \leq t \leq 2^{-k}$, this concludes the theorem for $0 \leq t \leq 2^{-\frac{k}{2}}$. \square

As a corollary we obtain the estimates we need for linearizing the phase function, and showing the symbols are slowly varying, for η in an appropriate conical region. Given a unit vector ν , and $2^{-k} \leq t \leq 1$, we define the dyadic/conic region

$$\Omega_{k, t}^\nu = \left\{ \eta : \frac{2}{3} 2^{k-1} \leq |\eta| \leq \frac{3}{2} 2^{k+2}, |\nu - |\eta|^{-1} \eta| \leq \frac{1}{16} t^{-\frac{1}{2}} 2^{-\frac{k}{2}} \right\}. \quad (4.6)$$

Note that on this region, since $t^{-\frac{1}{2}}2^{-\frac{k}{2}} \leq 1$,

$$|\eta| \geq \langle \nu, \eta \rangle \geq \frac{3}{4}|\eta|, \quad |\Pi_{\nu^\perp} \eta| \leq t^{-\frac{1}{2}}2^{\frac{k}{2}},$$

where Π_{ν^\perp} is projection onto the hyperplane perpendicular to ν .

Corollary 4.2. *The following estimates hold for $\eta \in \Omega_{k,t}^\nu$, where ∂_η and ∂_x denote general terms of the form ∂_{η_i} and ∂_{x_i} ,*

$$|\langle \nu, \partial_\eta \rangle^j \partial_\eta^\alpha \partial_x^\beta (\partial_\eta^2 \varphi_k)(t, x, \eta)| \leq C_{j,\alpha,\beta} t 2^{-k} 2^{-kj} (t^{\frac{1}{2}} 2^{-\frac{k}{2}})^{|\alpha|} 2^{\frac{k}{2}|\beta|}, \quad (4.7)$$

and

$$\begin{aligned} |\langle \nu, \partial_\eta \rangle^j \partial_\eta^\alpha \partial_x^\beta (\partial_\eta \partial_x \varphi_k)(t, x, \eta)| + 2^{-k} |\langle \nu, \partial_\eta \rangle^j \partial_\eta^\alpha \partial_x^\beta (\partial_x^2 \varphi_k)(t, x, \eta)| \\ \leq C_{j,\alpha,\beta} 2^{-kj} (t^{\frac{1}{2}} 2^{-\frac{k}{2}})^{|\alpha|} 2^{\frac{k}{2}|\beta|}. \end{aligned} \quad (4.8)$$

Proof. We observe that, by homogeneity, the estimates of Theorem 4.1 imply, for $|\eta| \approx 2^k$ and all j ,

$$|\partial_\eta^\alpha \partial_x^\beta \langle \eta, \partial_\eta \rangle^j (\partial_\eta^2 \varphi_k)(t, x, \eta)| \leq C_{j,\alpha,\beta} t 2^{-k} (t^{\frac{1}{2}} 2^{-\frac{k}{2}})^{|\alpha|} 2^{\frac{k}{2}|\beta|},$$

which, since $[\partial_\eta^\alpha, \langle \eta, \partial_\eta \rangle] = |\alpha| \partial_\eta^\alpha$, imply in turn that

$$|\langle \eta, \partial_\eta \rangle^j \partial_\eta^\alpha \partial_x^\beta (\partial_\eta^2 \varphi_k)(t, x, \eta)| \leq C_{j,\alpha,\beta} t 2^{-k} (t^{\frac{1}{2}} 2^{-\frac{k}{2}})^{|\alpha|} 2^{\frac{k}{2}|\beta|}.$$

Suppose (4.7) holds for $j < j_0$, and after rotation assume that $\eta = (1, 0, \dots, 0)$. We expand

$$\langle \eta, \partial_\eta \rangle^{j_0} = \eta_1^{j_0} \partial_{\eta_1}^{j_0} + \sum_{\substack{j+|\alpha| \leq j_0 \\ j < j_0}} c_{j_0,j,\alpha} \eta_1^j \eta'^\alpha \partial_{\eta_1}^j \partial_{\eta'}^\alpha$$

Since $\eta_1 \leq 2^{k+2}$ and $|\eta'| \leq t^{-\frac{1}{2}} 2^{\frac{k}{2}}$ on $\Omega_{k,t}^\nu$, the induction hypothesis of (4.7) for $j < j_0$ yields that

$$|\eta_1^{j_0} \partial_{\eta_1}^{j_0} \partial_\eta^\alpha \partial_x^\beta (\partial_\eta^2 \varphi_k)(t, x, \eta)| \leq C_{j,\alpha,\beta} t 2^{-k} (t^{\frac{1}{2}} 2^{-\frac{k}{2}})^{|\alpha|} 2^{\frac{k}{2}|\beta|},$$

which yields (4.7) for $j = j_0$ since $\eta_1 \geq 2^{k-2}$ on $\Omega_{k,t}^\nu$. Similar steps prove (4.8). \square

Chapter 5

PARAMETRIX FOR THE DYADICALLY LOCALIZED EQUATION

In this chapter we produce the exact wave group $E(t)$ for P by iteration of an approximate wave group $W(t)$, which we construct as a sum of approximate solutions at each dyadic scale.

Define

$$\tilde{p}_k(x, \eta) = \sum_{j=k-1}^{k+1} p_j(x, \eta) \psi_j(\eta),$$

and set

$$(D_t - \tilde{p}_k(x, D)) e^{i\varphi_k(t, x, \eta)} \psi_k(\eta) = e^{i\varphi_k(t, x, \eta)} b_k(t, x, \eta) \psi_k(\eta),$$

where D acts on x . Then $b_k(t, x, \eta) \psi_k(\eta)$ is given by the oscillatory integral

$$\left(\partial_t \varphi_k(t, x, \eta) - \frac{1}{(2\pi)^n} \int e^{i\langle x-y, \zeta \rangle + i\varphi_k(t, y, \eta) - i\varphi_k(t, x, \eta)} \tilde{p}_k(x, \zeta) dy d\xi \right) \psi_k(\eta).$$

We then write

$$\varphi_k(t, y, \eta) - \varphi_k(t, x, \eta) = (y - x) \cdot V(t, x, y - x, \eta),$$

where

$$V(t, x, h, \eta) = \int_0^1 (\nabla_x \varphi_k)(t, x + sh, \eta) ds = \int_0^1 \xi(t, x + sh, \eta) ds,$$

where $\xi(t, x, \eta) = \nabla_x \varphi_k(t, x, \eta)$ is as in Theorem 3.3. Then

$$V(t, x, 0, \eta) = \nabla_x \varphi(t, x, \eta), \quad \partial_{h_i} V_j(t, x, 0, \eta) = \frac{1}{2} \partial_{x_i} \partial_{x_j} \varphi_k(t, x, \eta). \quad (5.1)$$

By (3.1), we have $|V(t, x, h, \eta) - \eta| \leq \frac{1}{8} |\eta|$, and by Theorem 3.3,

$$|\partial_x^\beta \partial_h^\gamma \partial_\eta^\alpha V(t, x, h, \eta)| \leq C_{\alpha, \beta, \gamma} 2^{\frac{k}{2}(|\alpha| + |\beta| + |\gamma| - 1)} |\eta|^{1 - |\alpha|}, \quad |\alpha| + |\beta| + |\gamma| \geq 1. \quad (5.2)$$

We then make a change of variables: $y \rightarrow y + h$, followed by $\zeta \rightarrow V(t, x, h, \eta) + \zeta$, to write the oscillatory integral term as

$$\frac{1}{(2\pi)^n} \int e^{-i\langle h, \zeta \rangle} \tilde{p}_k(x, V(t, x, h, \eta) + \zeta) \psi_k(\eta) dh d\zeta.$$

We split the oscillatory integral up using a smooth cutoff χ , supported in $|\zeta| \leq 2$, with $\chi(\zeta) = 1$ for $|\zeta| \leq 1$. Specifically, we write

$$1 = \chi(2^{-k+4}\zeta)(1 - \chi(h)) + (1 - \chi(2^{-k+4}\zeta)) + \chi(h)\chi(2^{-k+4}\zeta).$$

The estimates (5.2), together with (2.9) ($C^1 S_{1, \frac{1}{2}}^1$ estimates on p_k), imply that

$$\begin{aligned} & |\partial_x^\gamma \partial_\eta^\theta \partial_\zeta^\alpha \partial_h^\beta \tilde{p}_k(x, V(t, x, h, \eta) + \zeta) \psi_k(\eta) \chi(2^{-k+4}\zeta)| \\ & \leq C_{\alpha, \beta, \gamma, \theta} \begin{cases} 2^{k(1-|\alpha|-|\theta|)} 2^{\frac{k}{2}(|\beta|+|\gamma|+|\theta|-1)}, & |\beta| + |\gamma| + |\theta| \geq 1 \\ 2^{k(1-|\alpha|)}, & |\beta| + |\gamma| + |\theta| = 0. \end{cases} \end{aligned} \quad (5.3)$$

Furthermore, it is supported where $\frac{9}{10}2^k \leq |\eta| \leq \frac{10}{9}2^{k+1}$.

Consider first

$$\begin{aligned} r_1(t, x, \eta) &= \int e^{-i\langle h, \zeta \rangle} \tilde{p}_k(x, V(t, x, h, \eta) + \zeta) \psi_k(\eta) \chi(2^{-k+4}\zeta) (1 - \chi(h)) dh d\zeta \\ &= \int e^{-i\langle h, \zeta \rangle} \Delta_\zeta^N \left(\tilde{p}_k(x, V(t, x, h, \eta) + \zeta) \psi_k(\eta) \chi(2^{-k+4}\zeta) \right) (1 - \chi(h)) |h|^{-2N} dh d\zeta. \end{aligned}$$

The estimates (5.3) show that the integrand is bounded by $2^{k(1-2N)} |h|^{-2N}$, and supported where $|\zeta| \leq 2^{k+2}$ and $|h| > 1$, with similar estimates on derivatives in (x, η) . Hence, for all N ,

$$|\partial_x^\beta \partial_\eta^\alpha r_1(t, x, \eta)| \leq C_{N, \alpha, \beta} 2^{-kN}, \quad (5.4)$$

and r_1 is supported where $2^{k-1} \leq |\eta| \leq 2^{k+2}$.

Next consider

$$\begin{aligned} r_2(t, x, \eta) &= \int e^{-i\langle h, \zeta \rangle} \tilde{p}_k(x, V(t, x, h, \eta) + \zeta) \psi_k(\eta) (1 - \chi(2^{-k+4}\zeta)) dh d\zeta \\ &= \int e^{-i\langle h, \zeta \rangle} (1 - \Delta_\zeta)^n \Delta_h^N \left(\tilde{p}_k(x, V(t, x, h, \eta) + \zeta) \psi_k(\eta) (1 - \chi(2^{-k+4}\zeta)) |\zeta|^{-2N} \right) \\ & \quad \times (1 + |h|^2)^{-n} dh d\zeta \end{aligned}$$

In this case the derivative estimates (5.3) show that the integrand is bounded by a constant times $2^{k(N+\frac{1}{2})}|\zeta|^{-2N}(1+|h^2|)^{-n}$ and is supported where $|\zeta| \geq 2^k$. It follows that $r_2(t, x, \eta)$ also satisfies the estimates (5.4).

We are thus left, modulo smoothing terms, with the symbol

$$\int e^{-i\langle h, \zeta \rangle} \tilde{p}_k(x, V(t, x, h, \eta) + \zeta) \psi_k(\eta) \chi(2^{-k+4}\zeta) \chi(h) dh d\zeta.$$

We take a Taylor expansion in ζ of \tilde{p}_k about $\zeta = 0$ to write this as

$$\sum_{|\gamma| < 2N} \frac{1}{\gamma!} \int e^{-i\langle h, \zeta \rangle} D_h^\gamma \left((\partial_\xi^\gamma \tilde{p}_k)(x, V(t, x, h, \eta)) \chi(h) \right) \psi_k(\eta) \chi(2^{-k+4}\zeta) dh d\zeta + r(t, x, \eta),$$

where $r(t, x, \eta)$ is given by

$$\sum_{|\gamma|=2N} \int_0^1 (1-s)^{N-1} \int e^{-i\langle h, \zeta \rangle} D_h^\gamma \left((\partial_\xi^\gamma \tilde{p}_k)(x, V(t, x, h, \eta) + s\zeta) \chi(h) \right) \times \psi_k(\eta) \chi(2^{-k+4}\zeta) dh d\zeta ds.$$

The estimates (5.3) show that $|\partial_x^\beta \partial_\eta^\alpha r(t, x, \eta)| \leq C_{N, \alpha, \beta} 2^{k(n+1-\frac{1}{2}|\alpha|+\frac{1}{2}|\beta|-N)}$.

For the terms with $|\gamma| < 2N$, let $\phi(h) = \widehat{\chi(2^k \cdot)}(h)$, which has integral $(2\pi)^n$ and vanishing moments of all non-zero order, and write the γ term as

$$\int e^{-i\langle h, \zeta \rangle} D_h^\gamma \left((\partial_\xi^\gamma \tilde{p}_k)(x, V(t, x, h, \eta)) \psi_k(\eta) \chi(h) \right) 2^{nk} \phi(2^k h) dh.$$

We Taylor expand $p_k(x, V(t, x, h, \eta))\chi(h)$ to order N about $h = 0$. The N -th order remainder term will lead to a term bounded by $2^{k(1-\frac{1}{2}|\gamma|-N)}$, with similar estimates on derivatives in (x, η) . All terms with h^θ with $\theta \neq 0$ integrate to 0 by the moment condition. Therefore, since $\partial_t \varphi_k(t, x, \eta) = p_k(t, x, \xi(t, x, \eta))$, we can write

$$b_k(t, x, \eta) = \left(p_k(t, x, \xi(t, x, \eta)) - \sum_{|\gamma| < 2N} \frac{1}{\gamma!} D_h^\gamma (\partial_\xi^\gamma \tilde{p}_k)(x, V(t, x, h, \eta)) \Big|_{h=0} \right) \psi_k(\eta) \quad (5.5)$$

plus a term $r(t, x, \eta)$, where $\text{supp}(r) \subseteq \text{supp}(\psi_k)$ and

$$|\partial_x^\beta \partial_\eta^\alpha r(t, x, \eta)| \leq C_{N, \alpha, \beta} 2^{k(\frac{1}{2}|\beta|-\frac{1}{2}|\alpha|+n+1-N)}.$$

The term $p_k(t, x, \nabla_x \varphi_k(t, x, \eta))$ combines with the $\gamma = 0$ term to give

$$\sum_{j=k-1}^{k+1} (p_k - p_j)(t, x, \xi(t, x, \eta)) \psi_j(\eta) \psi_k(\eta). \quad (5.6)$$

We will estimate this term similar to the term $|\gamma| = 1$, using the following estimate, which is a consequence of (2.11).

$$|\partial_x^\beta \partial_\xi^\alpha (p_k - p_j)(t, x, \xi)| \leq C_{\alpha, \beta} 2^{\frac{|\beta|}{2}-1} |\xi|^{1-|\alpha|}. \quad (5.7)$$

We now examine the terms in the sum when $|\gamma| \geq 1$. Observe that

$$\partial_h^\theta V(t, x, h, \eta)|_{h=0} = \frac{1}{1 + |\theta|} \partial_x^\theta \xi(t, x, \eta).$$

The γ term in (5.5) is then a finite linear combination of terms of the form

$$(\partial_\xi^{\gamma+\sigma} \tilde{p}_k)(x, \xi(t, x, \eta)) (\partial_x^{\theta_1} \xi(t, x, \eta)) \cdots (\partial_x^{\theta_l} \xi(t, x, \eta)),$$

where $\theta_1 + \cdots + \theta_l = \gamma$, each $\theta_i \neq 0$, and $l = |\sigma| \geq 1$.

We now consider estimates to show that $b_k(t, x, \eta)$ is slowly varying on the region $\Omega_{k,t}^\nu$ defined in (4.6). First note that, by (4.8), when $\theta \neq 0$, and $\eta \in \Omega_{k,t}^\nu$,

$$|\langle \nu, \partial_\eta \rangle^j \partial_x^\beta \partial_\eta^\alpha (\partial_x^\theta \xi(t, x, \eta))| \leq C_{j, \alpha, \beta, \theta} 2^{\frac{k}{2}(|\theta|+1)} 2^{-kj} (t^{\frac{1}{2}} 2^{-\frac{k}{2}})^{|\alpha|} 2^{\frac{k}{2}|\beta|}.$$

A recursion argument and (2.9) then show that, for $\eta \in \Omega_{k,t}^\nu$,

$$\begin{aligned} & \left| \langle \nu, \partial_\eta \rangle^j \partial_x^\beta \partial_\eta^\alpha \left((\partial_\xi^{\gamma+\sigma} \tilde{p}_k)(x, \xi(t, x, \eta)) \partial_x^{\theta_1} \xi(t, x, \eta) \cdots \partial_x^{\theta_l} \xi(t, x, \eta) \right) \right| \\ & \leq C_{j, \alpha, \beta, \gamma, \sigma} 2^{\frac{k}{2}(2-|\gamma|-|\sigma|)} 2^{-kj} (t^{\frac{1}{2}} 2^{-\frac{k}{2}})^{|\alpha|} 2^{\frac{k}{2}|\beta|}. \end{aligned}$$

The expression for $b_k(t, x, \eta)$ involves an asymptotic sum over $|\gamma| \geq 1$, where also $|\sigma| \geq 1$ in all terms. Similarly, using (5.7), we get the following bounds for (5.6),

$$\left| \langle \nu, \partial_\eta \rangle^j \partial_x^\beta \partial_\eta^\alpha \left(\sum_{j=k-1}^{k+1} (p_k - p_j)(t, x, \xi(t, x, \eta)) \psi_j(\eta) \psi_k(\eta) \right) \right| \leq C_{j, \alpha, \beta} 2^{-kj} (t^{\frac{1}{2}} 2^{-\frac{k}{2}})^{|\alpha|} 2^{\frac{k}{2}|\beta|}.$$

We consequently get the following.

Corollary 5.1. *The following estimates hold for $\eta \in \Omega_{k,t}^\nu$,*

$$|\langle \nu, \partial_\eta \rangle^j \partial_\eta^\alpha \partial_x^\beta b_k(t, x, \eta)| \leq C_{j, \alpha, \beta} 2^{-kj} (t^{\frac{1}{2}} 2^{-\frac{k}{2}})^{|\alpha|} 2^{\frac{k}{2}|\beta|}.$$

Chapter 6

ENERGY FLOW ESTIMATES

Define the operator $W_k(t)$ and $B_k(t)$ by

$$\begin{aligned} (W_k(t)f)(x) &= \frac{1}{(2\pi)^d} \int e^{i\varphi_k(t,x,\eta)} \psi_k(\eta) \hat{f}(\eta) d\eta, \\ (B_k(t)f)(x) &= \frac{1}{(2\pi)^d} \int e^{i\varphi_k(t,x,\eta)} b_k(t,x,\eta) \psi_k(\eta) \hat{f}(\eta) d\eta, \end{aligned}$$

In this chapter we show that the Fourier transforms of $W_k(t)f$ and $B_k(t)f$ are concentrated in the image of the support of \hat{f} under the Hamiltonian flow for g_k of time t . Since we will consider iterates of B_k , we need a finer localization than that given by standard S_1^0 symbols. We could work with S_ρ^0 cutoffs for any $\rho \in (\frac{1}{2}, 1)$, but fix $\rho = \frac{3}{4}$ for convenience. We thus consider cutoff functions $a(\eta)$ that satisfy

$$|\partial_\eta^\alpha a(\eta)| \leq C_\alpha 2^{-\frac{3}{4}k|\alpha|}, \quad \text{supp}(a) \subset \{\eta : \frac{4}{5}2^{k-1} < |\eta| < \frac{5}{4}2^{k+2}\}. \quad (6.1)$$

Given any compact set $K \subset \{\eta : \frac{7}{8}2^{k-1} < |\eta| < \frac{8}{7}2^{k+2}\}$, and $\delta > 0$, there exists such a cutoff a such that $a = 1$ on the $\frac{1}{3}\delta 2^{\frac{3k}{4}}$ neighborhood of K , $\text{supp}(a)$ is contained in the $2^{\frac{3k}{4}}\delta$ neighborhood of K , and such that the constants C_α depend only on δ and the dimension d . In particular they are independent of K . For example,

$$a(\eta) = (\delta^{-1}2^{-\frac{3k}{4}})^d \int_{K^*} \phi(\delta^{-1}2^{-\frac{3k}{4}}(\eta - \zeta)) d\zeta, \quad K^* = \{\zeta : \text{dist}(\zeta, K) \leq \frac{2}{3}\delta 2^{\frac{3k}{4}}\},$$

where $\int \phi = 1$ and $\text{supp}(\phi) \subset B_{1/3}$.

Lemma 6.1. *Suppose that a_1 and a_2 are cutoffs satisfying (6.1), and that $a_2 = 1$ on the $\delta 2^{\frac{3k}{4}}$ neighborhood of the projection onto η of the image of $\mathbb{R}^d \times \text{supp}(a_1)$ under the Hamiltonian flow of p_k at time t . Then for all N ,*

$$\|(1 - a_2(D))B_k(t)a_1(D)f\|_{H^N} \leq C_N 2^{-kN} \|f\|_{H^{-N}},$$

where the constant C_N depends only on N , the constants C_α in (6.1), and δ . The same holds with $B_k(t)$ replaced by $W_k(t)$.

Proof. We prove this using a modification of the Córdoba-Fefferman wave packet transform introduced in [3]. We use the version from [12], which is based on a Schwartz function with Fourier transform of compact support, instead of a Gaussian. Precisely, fix g a radial, real Schwartz function supported in $B_{1/4}$, with $\|g\|_{L^2} = (2\pi)^{\frac{d}{2}}$, and set

$$g_{x,\xi}(y) = 2^{\frac{kd}{4}} e^{i\langle \xi, y-x \rangle} g\left(2^{\frac{k}{2}}(y-x)\right)$$

For $f \in L^2(\mathbb{R}^d)$ define

$$(T_k f)(x, \xi) = \int f(z) \overline{g_{x,\xi}(z)} dz.$$

Then T_k is an isometry, with adjoint given by

$$(T_k^* F)(y) = \int F(x, \xi) g_{x,\xi}(y) dx d\xi,$$

and since the frequency localization of the operator it suffices to show that, for all N ,

$$\|T_k \langle D \rangle^N (1 - a_2(D)) B_k(t) a_1(D) T_k^* F\|_{L^2(\mathbb{R}^{2d})} \leq C_N 2^{-2kN} \|F\|_{L^2(\mathbb{R}^{2d})}.$$

This operator on $L^2(\mathbb{R}^{2d})$ is given by the following integral kernel,

$$K(x', \xi'; x, \xi) = \int_{\mathbb{R}^d} (B_k(t) a_1(D) g_{x,\xi})(y) \overline{\langle D \rangle^N (1 - a_2(D)) g_{x',\xi'}(y)} dy.$$

Letting $(x_t, \xi_t) = \chi_t(x, \xi)$, with χ_t the Hamiltonian flow, the kernel vanishes unless $|\xi' - \xi_t| \geq \delta 2^{\frac{3k}{4}} - \frac{1}{2}$. Thus it suffices to show that, for all N ,

$$|K(x', \xi'; x, \xi)| \leq C_N 2^{kN} (1 + 2^{\frac{k}{2}} |x' - x_t| + 2^{-\frac{k}{2}} |\xi' - \xi_t|)^{-N}. \quad (6.2)$$

For the purpose of this proof, we use $\tilde{g}_{x,\xi}(y)$ to denote a generic function of the form $2^{\frac{kd}{4}} e^{i\langle \xi, y-x \rangle} \tilde{g}\left(2^{\frac{k}{2}}(y-x)\right)$, where \tilde{g} is a Schwartz function that may differ in each occurrence, but all instances lie in a bounded family in $\mathcal{S}(\mathbb{R}^d)$ with bounds depending only on Schwartz seminorms of g , the constants C_α in (6.1), and the constants $C_{j,\alpha,\beta}$ in Corollaries 4.2 and 5.1.

For instance, we can write $\langle D \rangle^N (1 - a_2(D)) g_{x', \xi'} = 2^{kN} \tilde{g}_{x', \xi'}$. We prove the estimate (6.2) by showing that we can write $B_k(t) g_{x, \xi} = \tilde{g}_{x_t, \xi_t}$ provided $|\xi| \in [2^{k-1}, 2^{k+2}]$, where we ignore the factor $a_1(D)$ since it preserves the bounds and compact support of $\widehat{g}_{x, \xi}(\eta)$.

The analysis of the function $B_k(t) g_{x, \xi}$ is essentially from [3], with simplifications since \hat{g} is compact support. It is also a simple case of the proof of Lemma 7.2. \square

Lemma 6.2. *Let $W(t)f = \sum_{k=0}^{\infty} W_k(t)f$. Then*

$$(D_t - P)W(t) = \sum_{k=0}^{\infty} B_k(t) + R(t),$$

where $\|R(t)f\|_{H^N} \leq C_N \|f\|_{H^{-N}}$ for all N , uniformly over $0 \leq t \leq 1$.

Proof. Let $a_k(\eta)$ be supported in $\{\frac{4}{5}2^k < |\eta| < \frac{5}{4}2^{k+1}\}$, and equal 1 where $\{\frac{7}{8}2^k < |\eta| < \frac{8}{7}2^{k+1}\}$. Then for c_d small enough, the condition of Lemma 6.1 with $a_1 = \psi_k$ is satisfied for all t with $|t| \leq 1$. Therefore

$$\sum_{k=0}^{\infty} \sum_{|j-k|>1} p_j(x, D) \psi_j(D) W_k(t) = \sum_{k=0}^{\infty} \sum_{|j-k|>1} p_j(x, D) \psi_j(D) (1 - a_k(D)) W_k(t)$$

can be seen to satisfy the conditions of $R(t)$. Thus we can write

$$\begin{aligned} (D_t - P)W(t) &= \sum_{k=0}^{\infty} \left(D_t - \sum_{j=k-1}^{k+1} p_j(x, D) \psi_j(D) \right) W_k(t) + R(t) \\ &= \sum_{k=0}^{\infty} B_k(t) + R(t) \end{aligned}$$

for $R(t)$ as in the statement. \square

The analysis of the kernel in Lemma 6.1 shows that we can write

$$\sum_{|j-k|>1} p_j(x, D) \psi_j(D) (1 - a_k(D)) W_k(t) f(x) = \int K_k(t, x, y) (\psi_k(D)f)(y) dy,$$

where

$$|\partial_x^\beta \partial_y^\gamma K_k(t, x, y)| \leq C_{\alpha, \beta, N} 2^{-kN} (1 + |x - y|)^{-N}.$$

We can then write

$$\int K_k(t, x, y) (\psi_k(D)f)(y) dy = \frac{1}{(2\pi)^n} \int e^{i\varphi_k(t, x, \eta)} r_k(t, x, \eta) \psi_k(\eta) \hat{f}(\eta) d\eta,$$

with

$$r_k(t, x, \eta) = e^{-i\varphi_k(t, x, \eta)} \int K_k(t, x, y) e^{-i\langle y, \eta \rangle} dy,$$

so that for all N

$$|\partial_x^\beta \partial_\eta^\alpha r_k(t, x, \eta)| \leq C_{\alpha, \beta, N} 2^{-kN}, \quad 2^{k-1} \leq |\eta| \leq 2^{k+2}.$$

We can then incorporate r_k into b_k , and hence $R(t)$ into the sum of the $B_k(t)$. Thus, we can write

$$(D_t - P)W(t) = \sum_{k=1}^{\infty} B_k(t) \equiv B(t).$$

The proof of Lemma 6.1 also shows the following.

Lemma 6.3. *For all $s \in \mathbb{R}$ we have $\|B(t)f\|_{H^s} \leq C_s \|f\|_{H^s}$, uniformly over $0 \leq t \leq 1$.*

We can thus produce the exact wave group $E(t)$ for $D_t - P$ via iteration

$$E(t) = W(t) + \int_0^t W(t-s)B(s) ds + \int_0^t \int_0^s W(t-s)B(s-r)B(r) dr ds + \dots$$

To write the iteration more concisely, let $\Lambda^m \subset \mathbb{R}_+^{m+1}$ be the m -simplex, consisting of $\mathbf{r} = (r_1, \dots, r_{m+1})$ with $r_j > 0$ for all j , and with $r_1 + \dots + r_{m+1} = 1$. Let $d\mathbf{r}$ be the measure on Λ^m induced by projection onto (r_1, \dots, r_m) . Then we can write

$$E(t) = \sum_{m=0}^{\infty} t^m \int_{\Lambda^m} W(tr_{m+1})B(tr_m) \cdots B(tr_1) d\mathbf{r}. \quad (6.3)$$

If C_s is a uniform bound for the $H^s(\mathbb{R}^d)$ norm of both $W(t)$ and $B(t)$, then the m -th term has H^s operator norm at most $C_s^{m+1} t^m / m!$.

Theorem 6.4. *The expansion (6.3) converges uniformly over $0 \leq t \leq 1$, in the operator norm topology on each $H^s(\mathbb{R}^d)$. The limit $E(t)$ is a one parameter group of L^2 -unitary operators, and for $f \in H^s$, $F \in L^1([-1, 1], H^s)$, the solution to $(\partial_t - iP)u = F$, $u(0, \cdot) = f$ is given by $u(t, \cdot) = E(t)f + \int_0^t E(t-s)F(s, \cdot) ds$.*

Recall that $\tilde{\psi}_k = \psi_{k-1} + \psi_k + \psi_{k+1}$. We define

$$\begin{aligned}\tilde{W}_k(t) &= \tilde{\psi}_k(D)(W_{k-1} + W_k + W_{k+1})(t), \\ \tilde{B}_k(t) &= \tilde{\psi}_k(D)(B_{k-1} + B_k + B_{k+1})(t).\end{aligned}\tag{6.4}$$

Lemma 6.5. *If $m + 1 \leq 2^{\frac{k}{4}}$, then for all N , uniformly over t and \mathbf{r} ,*

$$\begin{aligned}\|W(tr_{m+1})B(tr_m) \cdots B_k(tr_1)f - \tilde{W}_k(tr_{m+1})\tilde{B}_k(tr_m) \cdots \tilde{\psi}_k(D)B_k(tr_1)f\|_{H^N} \\ \leq C_N 2^{-kN} \|f\|_{H^{-N}}.\end{aligned}$$

Proof. Fix t and \mathbf{r} , and without loss of generality assume $t \geq 0$. We introduce a family of intermediate cutoffs $\psi_{k,j}(D)$ for $1 \leq j \leq m$, which depend on $t\mathbf{r}$. Define points $\frac{10}{9} \leq p'_{j-1} < p_j < p'_j \leq \frac{5}{4}$ as follows. Fix c_0 and c_1 so that $e^{c_0} = \frac{10}{9}$, and $e^{c_0+2c_1} = \frac{5}{4}$. Let

$$p_j = e^{c_0+c_1(r_1+\cdots+r_j)t+c_1j}2^{-\frac{k}{4}}, \quad p'_j = e^{c_0+c_1(r_1+\cdots+r_j)t+c_1(j+\frac{1}{2})}2^{-\frac{k}{4}}$$

Thus ψ_k is supported where $|\eta| \in [e^{-c_0}2^k, e^{c_0}2^{k+1}]$, and $\tilde{\psi}_k(\eta) = 1$ on the set $\{\eta : |\eta| \in [p'_m{}^{-1}2^k, p'_m 2^{k+1}]\}$. Also,

$$|p'_j - p_j| \geq \frac{1}{2} c_1 2^{-\frac{k}{4}}, \quad |p_{j+1} - p'_j| \geq c_1 r_{j+1} t + \frac{1}{2} c_1 2^{-\frac{k}{4}}.$$

By the above comments we can construct a family of cutoffs $\psi_{k,j}(\xi)$, satisfying (6.1) with constants C_α depending only on the dimension d , such that

$$\psi_{k,j}(\eta) = 1 \quad \text{if } |\eta| \in [p_j^{-1}2^k, p_j 2^{k+1}], \quad \text{supp}(\psi_{k,j}) \subset \{\eta : |\eta| \in [p'_j{}^{-1}2^k, p'_j 2^{k+1}]\}.$$

Let $c'_d = \sup_{x,\xi} (|\xi|^{-1} |\nabla_x p_k(x, \xi)|) \lesssim c_d$. Then for solutions to the Hamiltonian flow,

$$\exp(-c'_d tr_j) |\xi(s)| \leq |\xi(s + tr_j)| \leq \exp(c'_d tr_j) |\xi(s)|.$$

Then if $c'_d \leq c_1$, the condition of Lemma 6.1 is satisfied for $a_2 = \psi_{k,j}$ and $a_1 = \psi_{k,j-1}$ with $\delta = \frac{1}{4}c_1$. Thus Lemma 6.1 yields

$$\|(1 - \psi_{k,j}(D))B(tr_j)\psi_{k,j-1}(D)\|_{H^s \rightarrow H^s} \leq C_{s,N} 2^{-kN}, \quad \forall s, N,$$

where if $j = 1$ this holds with $B(tr_j)\psi_{k,j-1}(D)$ replaced by $B_k(tr_1)$.

Since $B(t)\psi_{k,j}(D) = \tilde{B}_k(t)\psi_{k,j}(D)$, and $m < 2^{\frac{k}{4}}$, we can apply this repeatedly to write

$$\begin{aligned} W(tr_{m+1})B(tr_m) \cdots B(tr_2)B_k(tr_1) \\ = \tilde{W}_k(tr_{m+1})\psi_{k,m}(D)\tilde{B}_k(tr_m) \cdots \tilde{B}_k(tr_2)\psi_{k,1}(D)B_k(tr_1) + R_{\mathbf{r}}(t), \end{aligned}$$

where $\|R_{\mathbf{r}}(t)\|_{H^s \rightarrow H^s} \leq C_{s,N} 2^{-Nk}$ for all s, N . We then prove Lemma 6.5 by observing that the same steps let us write

$$\begin{aligned} \tilde{W}_k(tr_{m+1})\psi_{k,m}(D)\tilde{B}_k(tr_m) \cdots \tilde{B}_2(tr_2)\psi_{k,1}(D)B_k(tr_1) \\ = \tilde{W}_k(tr_{m+1})\tilde{B}_k(tr_m) \cdots \tilde{B}_k(tr_2)\tilde{\psi}_k(D)B_k(tr_1) + R_{\mathbf{r}}(t), \end{aligned}$$

for a similar $R_{\mathbf{r}}(t)$. Since $R_{\mathbf{r}}(t)$ is localized on the right at frequency 2^k , it follows that $\|R_{\mathbf{r}}(t)\|_{H^{-N} \rightarrow H^N} \leq C_N 2^{-kN}$ for all N . \square

Corollary 6.6. *One can write*

$$E(t) = \sum_{k=0}^{\infty} \sum_{m=0}^{2^{\frac{k}{4}}} t^m \int_{\Lambda^m} \tilde{W}_k(tr_{m+1})\tilde{B}_k(tr_m) \cdots \tilde{B}_k(tr_2)\tilde{\psi}_k(D)B_k(tr_1) d\mathbf{r} + R(t),$$

where for all N we have $\|R(t)f\|_{H^N} \leq C_N \|f\|_{H^{-N}}$, uniformly over $0 \leq t \leq 1$.

Proof. Consider

$$\sum_{m=2^{\frac{k}{4}}}^{\infty} t^m \int_{\Lambda^m} W(tr_{m+1})B(tr_m) \cdots B(tr_2)B_k(tr_1) d\mathbf{r}$$

For $0 \leq t \leq 1$, and all N , the $H^N \rightarrow H^N$ operator norm of this sum is bounded by the sum $\sum_{m \geq 2^{\frac{k}{4}}} C_N^{m+1}/m! \leq C_N 2^{-3kN}$. Since it is localized on the right at frequency 2^k it thus has $H^{-N} \rightarrow H^N$ operator norm bounded by 2^{-kN} . \square

We now define the operator

$$\tilde{E}_k(t) = \sum_{m=0}^{2^{\frac{k}{4}}} t^m \int_{\Lambda^m} \tilde{W}_k(tr_{m+1})\tilde{B}_k(tr_m) \cdots \tilde{B}_k(tr_2)\tilde{\psi}_k(D)B_k(tr_1) d\mathbf{r}.$$

The arguments leading to Lemma 6.5 apply equally well to conic localization. Precisely, suppose that $a_\omega(\eta)$ is supported in the set $|\omega - |\eta|^{-1}\eta| \leq \frac{1}{32}$, for some $\omega \in \mathbb{S}^{d-1}$, and let $\tilde{a}_\omega(\eta)$ be a smooth, homogeneous cutoff such that

$$\tilde{a}_\omega(\eta) = 1 \text{ if } |\omega - |\eta|^{-1}\eta| \leq \frac{1}{24}, \quad \text{supp}(\tilde{a}_\omega) \subset |\omega - |\eta|^{-1}\eta| \leq \frac{1}{16}.$$

Then, modulo $R_k(t)$ where $\|R_k(t)\|_{H^{-N} \rightarrow H^N} \leq C_N 2^{-kN}$ uniformly over $0 \leq t \leq 1$, one may write

$$\tilde{E}_k(t)a_\omega(D) = \sum_{m=0}^{2^{\frac{k}{4}}} t^m \int_{\Lambda^m} \tilde{a}_\omega(D) \tilde{W}_k(tr_{m+1}) \tilde{a}_\omega(D) \tilde{B}_k(tr_m) \cdots \tilde{a}_\omega(D) \tilde{\psi}_k(D) B_k(tr_1) a_\omega(D).$$

By taking a finite conic partition of unity, we may thus write

$$\tilde{E}_k(t) = \sum_{\omega} \tilde{E}_k^\omega(t) + R(t),$$

where $R(t)$ is as in Corollary 6.6, and for some a and \tilde{a} as above,

$$\tilde{E}_k^\omega(t) = \sum_{m=0}^{2^{\frac{k}{4}}} t^m \int_{\Lambda^m} \tilde{a}_\omega(D) \tilde{W}_k(tr_{m+1}) \tilde{a}_\omega(D) \tilde{B}_k(tr_m) \cdots \tilde{a}_\omega(D) \tilde{\psi}_k(D) B_k(tr_1) a_\omega(D). \quad (6.5)$$

We now turn to the proof of (2.13) for the operator $E(t)$, that is

$$\|\langle D \rangle^{-s} E(t) f\|_{L_t^q L_x^r([-1,1] \times \mathbb{R}^d)} \leq C \|f\|_{L^2(\mathbb{R}^d)}$$

for (q, r, s) satisfying the conditions of Theorem 1.1. A consequence of Corollary 6.6 is that $\psi_k(D)E(t) = \psi_k(D)E(t)\tilde{\psi}_k(D) + R(t)$, with $R(t)$ a smoothing operator, hence by Littlewood-Paley theory it suffices to prove

$$\|\psi_k(D)E(t)f\|_{L_t^q L_x^r([-1,1] \times \mathbb{R}^d)} \leq C 2^{ks} \|f\|_{L^2(\mathbb{R}^d)}.$$

By [9, Theorem 1.2], this is implied by the estimate

$$\|\psi_k(D)E(t-s)\psi_k(D)f\|_{L^\infty(\mathbb{R}^d)} \leq C 2^{kd} (1 + 2^k |t-s|)^{-\frac{d-1}{2}} \|f\|_{L^1(\mathbb{R}^d)}.$$

To apply [9, Theorem 1.2], we have used that $E(t)E^*(s) = E(t-s)$, and applied a scaling of (t, x) by a factor of 2^k .

By Corollary 6.6 and the comments following it, this is implied by proving the same estimate with $\psi_k(D)E(t-s)\psi_k(D)$ replaced by $\tilde{E}_k^\omega(t-s)$, that is

$$\|\tilde{E}_k^\omega(t-s)f\|_{L^\infty(\mathbb{R}^d)} \leq C 2^{kd} (1 + 2^k |t-s|)^{-\frac{d-1}{2}} \|f\|_{L^1(\mathbb{R}^d)}. \quad (6.6)$$

Chapter 7

WAVE PACKETS AND DISPERSIVE ESTIMATES

In this chapter we give the proof of the estimate (6.6), which is equivalent to the following pointwise bound on the integral kernel $K_k(t, x, y)$ of $\tilde{E}_k^\omega(t)$,

$$|K_k(t, x, y)| \leq C_d 2^{kd} (1 + 2^k |t|)^{-\frac{d-1}{2}}, \quad |t| \leq 1.$$

We in fact prove a stronger estimate, which captures the decay of the fundamental solution away from the light cone. Without loss of generality, we consider $0 \leq t \leq 1$ in the remainder of this section. Then we will show that, for all N , with $S_t(y)$ the geodesic sphere of radius t centered at y , and $\text{dist}(x, S_t(y))$ the Euclidean distance of x to the set $S_t(y)$,

$$|K_k(t, x, y)| \leq C_N 2^{kd} (1 + 2^k t)^{-\frac{d-1}{2}} (1 + 2^k |\text{dist}(x, S_t(y))|)^{-N}. \quad (7.1)$$

7.1 The wave packet frame

We will establish (7.1) for $2^{-k} \leq t \leq 1$; the proof for $0 \leq t \leq 2^{-k}$ follows by using the same proof as for $t = 2^{-k}$. We prove the estimate by studying the behavior of $\tilde{E}_k^\omega(t)$ in an appropriate frame of wave packets. The wave packet frame that we use at scale 2^k is essentially a spatial dilation by t^{-1} of the dyadic-parabolic wave packets at the scale $t2^k$ constructed in Smith [12]. The key difference is that our frame covers three dyadic regions instead of just one. We provide the details here for completeness.

We will be working with functions whose Fourier transform is supported in the set

$$A_k = \left\{ \eta : \frac{4}{5} 2^{k-1} \leq |\eta| \leq \frac{5}{4} 2^{k+2} \right\}$$

Let $A'_k = \left\{ \eta : \frac{2}{3} 2^{k-1} \leq |\eta| \leq \frac{3}{2} 2^{k+2} \right\}$. We construct a partition of unity on A_k , supported in

A'_k , of the form

$$1 = \sum_{\nu \in \Upsilon_{k,t}} \beta_{k,t}^\nu(\eta)^2 \text{ when } \eta \in A_k, \quad \text{supp}(\beta_{k,t}^\nu) \subset \Omega_{k,t}^\nu,$$

where $\Upsilon_{k,t}$ is a collection of unit vectors separated by $t^{-\frac{1}{2}}2^{-\frac{k}{2}}$, and $\beta_{k,t}^\nu(\eta)$ satisfies the following estimates

$$|\langle \nu, \partial_\eta \rangle^j \partial_\eta^\alpha \beta_{k,t}^\nu(\eta)| \leq C_{j,\alpha} (2^k)^{-j} (t^{-\frac{1}{2}}2^{\frac{k}{2}})^{-|\alpha|}.$$

Observe that $\Omega_{k,t}^\nu$, defined in (4.6), is contained in a rectangle of dimension 2^{k+3} along the direction ν and $t^{-\frac{1}{2}}2^{\frac{k}{2}}$ along the directions orthogonal to ν . For each ν , let $\Xi_{k,t}^\nu$ be a rectangular lattice in \mathbb{R}^n with spacing $2\pi \cdot 2^{-k-3}$ along the ν direction and spacing $2\pi \cdot t^{\frac{1}{2}}2^{-\frac{k}{2}}$ in directions orthogonal to ν . Let $\Gamma_{k,t} = \{\gamma = (x, \nu) : x \in \Xi_{k,t}^\nu, \nu \in \Upsilon_{k,t}\}$, and set

$$\hat{\phi}_\gamma(\eta) = 2^{-\frac{k-3}{2}} (t^{\frac{1}{2}}2^{-\frac{k}{2}})^{\frac{d-1}{2}} e^{-i\langle x, \eta \rangle} \beta_{k,t}^\nu(\eta).$$

Then

$$\begin{aligned} & |\langle \nu^\perp, \partial_y \rangle^\alpha \partial_y^\beta \phi_\gamma(y)| \\ & \leq C_{N,\alpha,\beta} 2^{\frac{k}{2}} (t^{-\frac{1}{2}}2^{\frac{k}{2}})^{\frac{d-1}{2}} 2^{k|\beta|} (t^{-\frac{1}{2}}2^{\frac{k}{2}})^{|\alpha|} (1 + 2^k |\langle \nu, y - x \rangle| + t^{-1}2^k |y - x|^2)^{-N}. \end{aligned} \quad (7.2)$$

For any function $f \in L^2(\mathbb{R}^n)$ with $\text{supp}(\hat{f}) \subset A_k$, we can expand f in terms of $\{\phi_\gamma\}_{\gamma \in \Gamma_{k,t}}$ as

$$f = \sum_{\gamma} c_\gamma \phi_\gamma$$

where

$$c_\gamma = \int \overline{\phi_\gamma(y)} f(y) dy.$$

For any given integer $M \geq 0$ and fixed t, x, ν , we then define a weighted norm space

$$\|f\|_{M,x,\nu}^2 = \sum_{\gamma'} (1 + 2^k d_t(x, \nu; x', \nu'))^{2M} |c_{\gamma'}|^2$$

where d_t is the pseudodistance function defined on the cosphere bundle $S^*(\mathbb{R}^d)$ as

$$d_t(x, \nu; x', \nu') = |\langle \nu, x - x' \rangle| + |\langle \nu', x - x' \rangle| + t|\nu - \nu'|^2 + t^{-1}|x - x'|^2.$$

These norms capture when a function is “close” to being a wave packet at (x, ν) , and will be used to proving localization properties of the composition of operators occurring in $\tilde{E}_k^\omega(t)$. The pointwise estimates on K will be a result of the following.

Theorem 7.1. *Let χ_s be the projected Hamiltonian flow on $S^*(\mathbb{R}^n)$ at time s . Then for $0 \leq s \leq t$, and all $M \geq 0$, all $(x, \nu) \in \mathbb{R}^d \times \mathbb{S}^{d-1}$,*

$$\|\tilde{E}_k^\omega(t)f\|_{M, \chi_t(x_0, \nu_0)} \leq C_M \|f\|_{M, x_0, \nu_0}.$$

The advantage of using normed spaces to establish localization of $\tilde{E}_k^\omega(t)$ is that it suffices to prove estimates for the individual factors in the products that make up $\tilde{E}_k^\omega(t)$. Precisely, we will show that in the next section that, for all ω, x, ν , and $0 \leq s \leq t$,

$$\|\tilde{a}_\omega(D)\tilde{\psi}_k(D)B_k(s)f\|_{M, \chi_s(x, \nu)} \leq C_M \|f\|_{M, x, \nu}, \quad (7.3)$$

and the same for B_k replaced by B_j or W_j with $|j - k| \leq 1$. The formula (6.5) for $\tilde{E}_k^\omega(t)$ then shows that

$$\begin{aligned} \|\tilde{E}_k^\omega(t)f\|_{M, \chi_t(x_0, \nu_0)} &\leq \sum_{m=0}^{\infty} \frac{t^m C_M^{m+1}}{m!} \|f\|_{M, x_0, \nu_0} \\ &= C_M e^{tC_M} \|f\|_{M, x_0, \nu_0}. \end{aligned}$$

7.2 Weighted norm estimates for the parametrix

In order to prove (7.3), we study the matrix representation of the operator $\tilde{a}_\omega(D)\tilde{\psi}_k(D)B_k(s)$ in the frame $\{\phi_\gamma\}_{\gamma \in \Gamma_{k,t}}$. The same proof works equally well when B_k is replaced by B_j or W_j with $|j - k| \leq 1$. The factor $\tilde{a}_\omega(D)\tilde{\psi}_k(D)$ ensures that the range of this operator can be expanded in the frame $\{\phi_\gamma\}_{\gamma \in \Gamma_{k,t}}$ with coefficients that vanish unless $|\nu - \gamma| \leq \frac{1}{8}$. Consequently, in the estimates that follow we assume $|\nu - \omega| \leq \frac{1}{8}$.

Then, for $\text{supp}(\hat{f}) \subset A_k \cap \{|\omega - |\eta|^{-1}\eta| \leq \frac{1}{16}\}$, we have

$$\tilde{a}_\omega(D)\tilde{\psi}_k(D)B_k(s)f = \tilde{a}_\omega(D)\tilde{\psi}_k(D)B_k(s) \left(\sum_{\gamma} c_\gamma \phi_\gamma \right) = \sum_{\gamma'} c_{\gamma'} \phi_{\gamma'}$$

where

$$c_{\gamma'} = \sum_{\gamma} a(\gamma', \gamma) c_{\gamma}$$

and

$$a(\gamma', \gamma) = \int \overline{\phi_{\gamma'}(y)} (\tilde{a}_{\omega}(D) \tilde{\psi}_k(D) B_k(s) \phi_{\gamma})(y) dy.$$

Hence we may regard $a(\gamma', \gamma)$ as the matrix representation of operator $P_k(s)$ mapping l^2 to l^2 . We will prove (7.3) by showing that $a(\gamma', \gamma_0)$ satisfies the following estimates

$$\begin{aligned} |a(\gamma', \gamma_0)| &\leq C_N (1 + t2^k |\nu_s - \nu'|^2)^{-N} \\ &\quad \times (1 + 2^k |\langle \nu_s, x_s - x' \rangle| + 2^k |\langle \nu', x_s - x' \rangle| + t^{-1} 2^k |x_s - x'|^2)^{-N} \end{aligned} \quad (7.4)$$

where $\gamma_0 = (x_0, \nu_0)$, $\gamma' = (x', \nu')$, and $(x_s, \nu_s) = \chi_s(x_0, \nu_0)$. Note that the matrix $a(\gamma', \gamma)$ vanishes unless $|\nu' - \omega| \leq \frac{1}{8}$, hence unless $|\nu' - \nu_s| \leq \frac{1}{4}$ for all $0 \leq s \leq 1$, where we use that $|\nu_s - \nu_0| \lesssim c_d$ since $|\nabla_x p_k(x, \eta)| \lesssim c_d |\eta|$.

Estimates (7.4) is then an immediate consequence of the following lemma.

Lemma 7.2. *For any l, β and N , the function $g(x) = B_k(s) \phi_{\gamma}$ satisfies the estimates*

$$\begin{aligned} |\langle \nu_s, \partial_x \rangle^l \langle \nu_s^{\perp}, \partial_x \rangle^{\beta} g(x)| &\leq C_{N,l,\beta} t^{-(\frac{d-1}{4})} 2^{k(\frac{d+1}{4})} \\ &\quad \times 2^{kl} (t^{-\frac{1}{2}} 2^{\frac{k}{2}})^{|\beta|} (1 + 2^k |\langle \nu_s, x - x_s \rangle| + t^{-1} 2^k |x - x_s|^2)^{-N}. \end{aligned} \quad (7.5)$$

If $g(x)$ satisfies (7.5), then for $|\nu' - \nu_s| \leq \frac{1}{4}$ we have

$$\begin{aligned} \left| \int g(x) \overline{\tilde{a}_{\omega}(D) \tilde{\psi}_k(D) \phi_{\gamma'}(x)} dx \right| &\leq C_N (1 + t2^k |\nu_s - \nu'|^2)^{-N} \\ &\quad \times (1 + 2^k |\langle \nu_s, x_s - x' \rangle| + 2^k |\langle \nu', x_s - x' \rangle| + t^{-1} 2^k |x_s - x'|^2)^{-N}, \end{aligned} \quad (7.6)$$

where C_N depends on only a finite number of the constants $C_{N,l,\beta}$.

Proof. We first prove (7.5). We can assume $\nu_0 = e_1$, where $\gamma = (x_0, \nu_0)$. Then

$$B_k(s) \phi_{\gamma} = 2^{-\frac{3}{2}} 2^{-k(\frac{d+1}{4})} t^{\frac{d-1}{4}} \int e^{i(\varphi_k(s,x,\eta) - \langle x_0, \eta \rangle)} b_k(s, x, \eta) \beta_{k,t}^{e_1}(\eta) d\eta.$$

We simplify the phase function by writing

$$\varphi_k(s, x, \eta) - \langle x_0, \eta \rangle = [\langle \partial_\eta \varphi_k(s, x, e_1) - x_0, \eta \rangle] + [\varphi_k(s, x, \eta) - \partial_\eta \varphi_k(s, x, e_1) \cdot \eta].$$

Then

$$B_k(s) \phi_\gamma = 2^{-\frac{3}{2}} 2^{-k(\frac{d+1}{4})} t^{\frac{d-1}{4}} \int e^{i\langle y(s, x, e_1) - x_0, \eta \rangle} e^{ih(s, x, \eta)} b_k(s, x, \eta) \beta_{k,t}^{e_1}(\eta) d\eta$$

where $y(s, x, e_1) = \partial_\eta \varphi_k(s, x, e_1)$ and $h(s, x, \eta) = \varphi_k(s, x, \eta) - \partial_\eta \varphi_k(s, x, e_1) \cdot \eta$. Write $\eta = (\eta_1, \eta')$ where $e_1 = (1, 0)$. We claim that for any β and $N_1, |\alpha| \geq 0$:

$$|\partial_{\eta_1}^{N_1} \partial_{\eta'}^\alpha \partial_x^\beta h(s, x, \eta)| \leq C_{N_1, |\alpha|, \beta} 2^{-kN_1} (t^{-\frac{1}{2}} 2^{\frac{k}{2}})^{-|\alpha|} 2^{\frac{k}{2}|\beta|} \quad \text{if } \eta \in \text{supp}(\beta_{k,t}^{e_1}). \quad (7.7)$$

The case $\beta \neq 0$ follows exactly the same as the case $\beta = 0$, so we consider $\beta = 0$ and suppress s, x in what follows. Then with $\varphi_k(\eta) = \varphi_k(s, x, \eta)$ where $0 \leq s \leq t$, then from (4.7) and the fact that $|\eta_1| \approx 2^k$ we see that for integer j and index α

$$|\partial_{\eta_1}^j \partial_{\eta'}^\alpha (\partial_{\eta'}^2 \varphi_k(\eta))| \leq C_{j, |\alpha|} t 2^{-k} 2^{-kj} (t^{-\frac{1}{2}} 2^{\frac{k}{2}})^{-|\alpha|} \quad \text{if } \eta \in \text{supp}(\beta_{k,t}^{e_1}). \quad (7.8)$$

We now prove (7.7) in different scenarios. First we note

$$\begin{aligned} h(\eta) &= \varphi_k(\eta_1, \eta') - \partial_{\eta_1} \varphi_k(\eta_1, 0) \cdot \eta_1 - \partial_{\eta'} \varphi_k(\eta_1, 0) \cdot \eta' \\ &= \varphi_k(\eta_1, \eta') - \varphi_k(\eta_1, 0) - \partial_{\eta'} \varphi_k(\eta_1, 0) \cdot \eta' \end{aligned}$$

since φ_k is homogeneous of degree 1 in η .

In case of $N_1 = 0$ and $|\alpha| = 0$, using the second order Taylor expansion of $\varphi_k(\eta)$ and (7.8), we get

$$|h(\eta)| \leq C_{0,0} t 2^{-k} (t^{-\frac{1}{2}} 2^{\frac{k}{2}})^2 = C_{0,0}.$$

In the case $N_1 \geq 1$ and $|\alpha| = 0$, we have

$$\partial_{\eta_1}^{N_1} h(\eta) = \partial_{\eta_1}^{N_1} \varphi_k(\eta_1, \eta') - \partial_{\eta_1}^{N_1} \varphi_k(\eta_1, 0) - \partial_{\eta_1}^{N_1} (\partial_{\eta'} \varphi_k(\eta_1, 0)) \cdot \eta',$$

hence

$$|\partial_{\eta_1}^{N_1} h(\eta)| \leq C_{N_1, 0} (t 2^{-k} 2^{-kN_1}) \cdot (t^{-\frac{1}{2}} 2^{\frac{k}{2}})^2 \leq C_{N_1, 0} 2^{-kN_1}.$$

In the case $N_1 \geq 0$ and $|\alpha| = 1$, we have

$$\begin{aligned} |\partial_{\eta_1}^{N_1} \partial_{\eta'} h(\eta)| &= |\partial_{\eta_1}^{N_1} \partial_{\eta'} \varphi_k(\eta_1, \eta') - \partial_{\eta_1}^{N_1} \partial_{\eta'} \varphi_k(\eta_1, 0)| \\ &\leq C_{N_1,1} (t 2^{-k} 2^{-kN_1}) \cdot (t^{-\frac{1}{2}} 2^{\frac{k}{2}}) \\ &= C_{N_1,1} 2^{-kN_1} (t^{-\frac{1}{2}} 2^{\frac{k}{2}})^{-1}. \end{aligned}$$

In the case $N_1 \geq 0$ and $|\alpha| \geq 2$, we have

$$\begin{aligned} |\partial_{\eta_1}^{N_1} \partial_{\eta'}^\alpha h(\eta)| &= |\partial_{\eta_1}^{N_1} \partial_{\eta'}^\alpha \varphi_k(\eta_1, \eta')| \\ &\leq C_{N_1,|\alpha|} t 2^{-k} 2^{-kN_1} (t^{-\frac{1}{2}} 2^{\frac{k}{2}})^{-(|\alpha|-2)} \\ &= C_{N_1,|\alpha|} 2^{-kN_1} (t^{-\frac{1}{2}} 2^{\frac{k}{2}})^{-|\alpha|}. \end{aligned}$$

This completes the proof of (7.7). It then follows that, on the support of $\beta_{k,t}^{e_1}$,

$$|\partial_{\eta_1}^{N_1} \partial_{\eta'}^\alpha \partial_x^\beta (e^{ih(s,x,\eta)})| \leq C_{N_1,|\alpha|,\beta} 2^{-kN_1} (t^{-\frac{1}{2}} 2^{\frac{k}{2}})^{-|\alpha|} 2^{\frac{k}{2}|\beta|}.$$

Therefore we may rewrite

$$B_k(s) \phi_\gamma = 2^{-\frac{3}{2}} 2^{-k(\frac{d+1}{4})} t^{\frac{d-1}{4}} \int e^{i(y(s,x,e_1)-x_0,\eta)} \tilde{b}_k(s, x, \eta) \beta_{k,t}^{e_1}(\eta) d\eta$$

where $\tilde{b}_k(s, x, \eta) = e^{ih(s,x,\eta)} b_k(s, x, \eta)$ satisfies the estimates in Corollary 5.1 for $\nu = e_1$.

Let $T = D_x y(s, x_s, e_1)$. We pose $x = x_s + {}^t T^{-1} \tilde{x}$, and define

$$f(s, \tilde{x}) = t^{\frac{d-1}{4}} 2^{-k(\frac{d+1}{4})} (B_k(s) \phi_\gamma)(x_s + {}^t T^{-1} \tilde{x}).$$

The chain rule shows that estimates (7.5) are equivalent to the following

$$\left| \partial_{\tilde{x}_1}^l \partial_{\tilde{x}'}^\beta f(s, \tilde{x}) \right| \leq C_{N,l,\beta} 2^{kl} (t^{-\frac{1}{2}} 2^{\frac{k}{2}})^{|\beta|} (1 + 2^k |\tilde{x}_1| + t^{-1} 2^k |\tilde{x}|^2)^{-N}. \quad (7.9)$$

This uses the fact that $|T - I| \lesssim c_d$, which holds by Corollary 3.3, and $\nu_s = c T e_1$, where for a prescribed ϵ we have $(1 + \epsilon)^{-1} \leq c \leq 1 + \epsilon$ by taking c_d small enough, since

$$\nu_s = c D_x \varphi_k(s, x_s, e_1) = c D_x (D_\eta \varphi_k(s, x_s, e_1) \cdot e_1) = c D_x y_1(s, x_s, e_1).$$

Set $\tilde{y}(s, \tilde{x}) = y(s, x_s + {}^t T^{-1} \tilde{x}, e_1) - x_0$. Immediately we see $\tilde{y}(0) = 0$ and $\partial_{\tilde{x}} \tilde{y}(0) = I$. We also observe that, by (4.8) and the fact that $y_1(s, x, e_1) = \varphi_k(s, x, e_1)$,

$$|\partial_{\tilde{x}}^\beta \tilde{y}_1| \leq \begin{cases} C_\beta 2^{\frac{k}{2}(|\beta|-2)}, & |\beta| \geq 2, \\ C_\beta, & |\beta| = 1, \end{cases} \quad (7.10)$$

and

$$|\partial_{\tilde{x}}^\beta \tilde{y}'(\tilde{x})| \leq C_\beta 2^{\frac{k}{2}(|\beta|-1)}, \quad |\beta| \geq 1. \quad (7.11)$$

We then can write $f(s, \tilde{x}) = F(s, \tilde{x}, \tilde{y}(s, \tilde{x}))$, where

$$F(s, \tilde{x}, \tilde{y}) = \int e^{i\tilde{y}\cdot\eta} \tilde{b}'_k(s, \tilde{x}, \eta) \rho_{k,t}^{e_1}(\eta) d\eta,$$

where $\rho_{k,t}^{e_1}(\eta) = 2^{-\frac{3}{2}} 2^{-k(\frac{d+1}{2})} t^{\frac{d-1}{2}} \beta_{k,t}^{e_1}(\eta)$ and $\tilde{b}_k(s, \tilde{x}, \eta) = \tilde{b}_k(s, x_s + {}^t T^{-1} \tilde{x}, \eta)$. We observe that $\tilde{b}_k(s, \tilde{x}, \eta)$ satisfies the estimates in Corollary 5.1 for $\nu = e_1$ since $|T - I| \lesssim c_d$.

In order to show (7.9), we first prove two intermediate results: For any N, β, l , we have

$$\left| \partial_{\tilde{x}_1}^l \partial_{\tilde{x}}^\beta f(s, \tilde{x}) \right| \leq C_{N,l,\beta} 2^{kl} (t^{-\frac{1}{2}} 2^{\frac{k}{2}})^{|\beta|} (1 + 2^k |\tilde{y}_1| + t^{-1} 2^k |\tilde{y}|^2)^{-N} (1 + t^{-\frac{1}{2}} 2^{\frac{k}{2}} |\tilde{x}'|)^{|\beta|}, \quad (7.12)$$

and

$$|\tilde{x}_1| + t^{-1} |\tilde{x}|^2 \approx |\tilde{y}_1| + t^{-1} |\tilde{y}|^2. \quad (7.13)$$

To prove (7.12), we first note that

$$\partial_{\tilde{x}}^\beta F(s, \tilde{x}, \tilde{y}(\tilde{x})) = \sum_{\beta' \leq \beta} \binom{\beta}{\beta'} \left(\partial_z^{\beta-\beta'} \partial_{\tilde{x}}^{\beta'} F(s, z, \tilde{y}(\tilde{x})) \right) \Big|_{z=\tilde{x}}.$$

The operator ∂_z acting on F only applies to the symbol \tilde{b}'_k , which has the effect of replacing the symbol by $2^{\frac{k}{2}|\beta-\beta'|}$ times \tilde{b}''_k where \tilde{b}''_k satisfies the estimates in Corollary 5.1 with $\nu = e_1$.

Thus, one is reduced to proving that, for all indices N, l, β :

$$\left| \partial_{\tilde{x}_1}^l \partial_{\tilde{x}}^\beta F(t, \tilde{y}(\tilde{x})) \right| \leq C_{N,l,\beta} 2^{k(l+\frac{|\beta|}{2})} (1 + 2^k |\tilde{y}_1(\tilde{x})| + t^{-1} 2^k |\tilde{y}(\tilde{x})|^2)^{-N} (1 + t^{-\frac{1}{2}} 2^{\frac{k}{2}} |\tilde{x}'|)^{|\beta|}. \quad (7.14)$$

In the expression above, we suppress the argument z of $F(t, z, \tilde{y}(\tilde{x}))$, and note that our proof will show that the estimate holds uniformly in z . In order to prove (7.14), first note that

$$\begin{aligned} & \partial_{\tilde{x}_1}^l \partial_{\tilde{x}'}^\beta F(s, \tilde{y}(\tilde{x})) \\ &= \sum_{\Omega} C_{\Omega} \cdot (\partial_{\tilde{y}_1}^m \partial_{\tilde{y}'}^\gamma F) \cdot (\partial_{\tilde{x}_1}^{l_1} \partial_{\tilde{x}'}^{\beta_1} \tilde{y}_1) \cdots (\partial_{\tilde{x}_1}^{l_m} \partial_{\tilde{x}'}^{\beta_m} \tilde{y}_1) \cdot (\partial_{\tilde{x}_1}^{\tilde{l}_1} \partial_{\tilde{x}'}^{\tilde{\beta}_1} \tilde{y}') \cdots (\partial_{\tilde{x}_1}^{\tilde{l}_{|\gamma|}} \partial_{\tilde{x}'}^{\tilde{\beta}_{|\gamma|}} \tilde{y}'), \end{aligned}$$

where the sum runs over $\Omega = \{m, \gamma, l_1, \dots, l_m, \tilde{l}_1, \dots, \tilde{l}_{|\gamma|}, \beta_1, \dots, \beta_m, \tilde{\beta}_1, \dots, \tilde{\beta}_{|\gamma|}\}$, where

$$l_1 + \dots + l_m + \tilde{l}_1 + \dots + \tilde{l}_{|\gamma|} = l, \quad \beta_1 + \dots + \beta_m + \tilde{\beta}_1 + \dots + \tilde{\beta}_{|\gamma|} = \beta.$$

Since $l_i + |\beta_i| > 0$, $\tilde{l}_i + |\tilde{\beta}_i| > 0$, it follows that $m + |\gamma| \leq l + |\beta|$. For each generic term in the sum, we will establish the following estimates:

$$|\partial_{\tilde{y}_1}^m \partial_{\tilde{y}'}^\gamma F| \leq C_{m,\gamma} 2^{km} (t^{-\frac{1}{2}} 2^{\frac{k}{2}})^{|\gamma|} (1 + 2^k |\tilde{y}_1| + t^{-1} 2^k |\tilde{y}'|^2)^{-N} \quad (7.15)$$

$$|\partial_{\tilde{x}_1}^{\tilde{l}_i} \partial_{\tilde{x}'}^{\tilde{\beta}_i} \tilde{y}'| \leq C_{\tilde{l}_i, \tilde{\beta}_i} 2^{k(\tilde{l}_i + \frac{|\tilde{\beta}_i|}{2} - \frac{1}{2})} t^{-\frac{1}{2}(|\tilde{\beta}_i| - 1)} \quad (7.16)$$

$$|\partial_{\tilde{x}_1}^{l_i} \partial_{\tilde{x}'}^{\beta_i} \tilde{y}_1| \leq C_{l_i, \beta_i} 2^{k(l_i + \frac{|\beta_i|}{2} - 1)} (1 + t^{-\frac{1}{2}} 2^{\frac{k}{2}} |\tilde{x}'|) \quad (7.17)$$

Proof of (7.15). We prove for all indices m, γ, l, β

$$|\tilde{y}_1^l \tilde{y}'^\beta \partial_{\tilde{y}_1}^m \partial_{\tilde{y}'}^\gamma F(s, \tilde{y})| \leq C_{m,\gamma,l,\beta} 2^{-kl} (t^{-\frac{1}{2}} 2^{\frac{k}{2}})^{-|\beta|} 2^{km} (t^{-\frac{1}{2}} 2^{\frac{k}{2}})^{|\gamma|}.$$

This can be seen by the following calculation:

$$\begin{aligned} |\tilde{y}_1^l \tilde{y}'^\beta \partial_{\tilde{y}_1}^m \partial_{\tilde{y}'}^\gamma F(\tilde{y})| &= \left| \tilde{y}_1^l \tilde{y}'^\beta \int \partial_{\tilde{y}_1}^m \partial_{\tilde{y}'}^\gamma (e^{i\tilde{y}\cdot\eta}) \tilde{b}'_k(s, x, \eta) \rho_{k,t}^\nu(\eta) d\eta \right| \\ &= \left| \tilde{y}_1^l \tilde{y}'^\beta \int e^{i\tilde{y}\cdot\eta} \eta_1^m \eta'^\gamma \tilde{b}'_k(s, \tilde{x}, \eta) \rho_{k,t}^\nu(\eta) d\eta \right| \\ &= \left| \int \partial_{\eta_1}^l \partial_{\eta'}^\beta (e^{i\tilde{y}\cdot\eta}) \eta_1^m \eta'^\gamma \tilde{b}'_k(s, \tilde{x}, \eta) \rho_{k,t}^\nu(\eta) d\eta \right| \\ &= 2^{km} (t^{-\frac{1}{2}} 2^{\frac{k}{2}})^{|\gamma|} \left| \int e^{i\tilde{y}\cdot\eta} \partial_{\eta_1}^l \partial_{\eta'}^\beta \tilde{b}''_k(s, \tilde{x}, \eta) \rho_{k,t}^\nu(\eta) d\eta \right| \end{aligned}$$

where $\tilde{b}'_k(s, \tilde{x}, \eta) = 2^{-km} (t^{-\frac{1}{2}} 2^{\frac{k}{2}})^{-|\gamma|} \tilde{b}_k(s, \tilde{x}, \eta)$ satisfies the estimates in Corollary 5.1. It then follows that

$$\begin{aligned} |\partial_{\tilde{y}_1}^m \partial_{\tilde{y}'}^\gamma F| &\leq C'_{m,\gamma,N} 2^{km} (t^{-\frac{1}{2}} 2^{\frac{k}{2}})^{|\gamma|} (1 + 2^k |\tilde{y}_1| + t^{-\frac{1}{2}} 2^{\frac{k}{2}} |\tilde{y}'|)^{-2N} \\ &\leq C_{m,\gamma,N} 2^{km} (t^{-\frac{1}{2}} 2^{\frac{k}{2}})^{|\gamma|} (1 + 2^k |\tilde{y}_1| + t^{-1} 2^k |\tilde{y}'|^2)^{-N}. \end{aligned}$$

In the estimate above, we use the fact that $|\Omega_{k,t}^{e_1}| \approx 2^k(t^{-\frac{1}{2}}2^{\frac{k}{2}})^{d-1}$, so $\int \rho_{k,t}^\nu(\eta) d\eta \lesssim 1$. \square

Proof of (7.16). If $|\tilde{\beta}_i| \geq 1$, we have $t^{-\frac{1}{2}(|\beta_i|-1)} \geq 1$, hence

$$|\partial_{\tilde{x}_1}^{\tilde{l}_i} \partial_{\tilde{x}'}^{\tilde{\beta}_i} \tilde{y}'| \leq C_{\tilde{l}_i, \tilde{\beta}_i} 2^{\frac{k}{2}(\tilde{l}_i + |\tilde{\beta}_i| - 1)} \leq C_{\tilde{l}_i, \tilde{\beta}_i} 2^{k(\tilde{l}_i + \frac{|\tilde{\beta}_i|}{2} - \frac{1}{2})} t^{-\frac{1}{2}(|\tilde{\beta}_i| - 1)}.$$

If $|\tilde{\beta}_i| = 0$, $\tilde{l}_i \geq 1$, then

$$|\partial_{\tilde{x}_1}^{\tilde{l}_i} \partial_{\tilde{x}'}^{\tilde{\beta}_i} \tilde{y}'| = |\partial_{\tilde{x}_1}^{\tilde{l}_i} \tilde{y}'| \leq C_{\tilde{l}_i, \tilde{\beta}_i} 2^{\frac{k}{2}(\tilde{l}_i - 1)} = C_{\tilde{l}_i, \tilde{\beta}_i} 2^{k(\tilde{l}_i - \frac{1}{2})} 2^{-\frac{k\tilde{l}_i}{2}} \leq C_{\tilde{l}_i, \tilde{\beta}_i} 2^{k(\tilde{l}_i - \frac{1}{2})} t^{\frac{1}{2}},$$

where we use the fact that $2^{-k} \leq t$. \square

Proof of (7.17). We need to deal with three different scenarios. If $l_i + |\beta_i| \geq 2$, then (7.10) tells us

$$|\partial_{\tilde{x}_1}^{l_i} \partial_{\tilde{x}'}^{\beta_i} \tilde{y}_1| \leq C_{l_i, \beta_i} 2^{\frac{k}{2}(l_i + |\beta_i| - 2)} \leq C_{l_i, \beta_i} 2^{k(l_i + \frac{|\beta_i|}{2} - 1)} (1 + t^{-\frac{1}{2}} 2^{\frac{k}{2}} |\tilde{x}'|).$$

If $|\beta_i| = 1$, $l_i = 0$, then

$$|\partial_{\tilde{x}_1}^{l_i} \partial_{\tilde{x}'}^{\beta_i} \tilde{y}_1| = |\partial_{\tilde{x}'}^{\beta_i} \tilde{y}_1| \leq C |\tilde{x}'| \leq C 2^{-\frac{k}{2}} (1 + t^{-\frac{1}{2}} 2^{\frac{k}{2}} |\tilde{x}'|) = C 2^{k(l_i + \frac{|\beta_i|}{2} - 1)} (1 + t^{-\frac{1}{2}} 2^{\frac{k}{2}} |\tilde{x}'|).$$

If $|\beta_i| = 0$, $l_i = 1$, then

$$|\partial_{\tilde{x}_1}^{l_i} \partial_{\tilde{x}'}^{\beta_i} \tilde{y}_1| = |\partial_{\tilde{x}_1} \tilde{y}_1| \leq C \leq C 2^{k(l_i + \frac{|\beta_i|}{2} - 1)} (1 + t^{-\frac{1}{2}} 2^{\frac{k}{2}} |\tilde{x}'|).$$

\square

Combining estimates (7.15)-(7.17), we get

$$\begin{aligned} & |(\partial_{\tilde{y}_1}^m \partial_{\tilde{y}'}^\gamma F) \cdot (\partial_{\tilde{x}_1}^{l_1} \partial_{\tilde{x}'}^{\beta_1} \tilde{y}_1) \cdots (\partial_{\tilde{x}_1}^{l_m} \partial_{\tilde{x}'}^{\beta_m} \tilde{y}_1) \cdot (\partial_{\tilde{x}_1}^{\tilde{l}_1} \partial_{\tilde{x}'}^{\tilde{\beta}_1} y') \cdots (\partial_{\tilde{x}_1}^{\tilde{l}_m} \partial_{\tilde{x}'}^{\tilde{\beta}_m} \tilde{y}')| \\ & \leq C'_\Omega (1 + 2^k |\tilde{y}_1| + t^{-1} 2^k |\tilde{y}'|^2)^{-N} 2^{k \cdot \Sigma(\Omega)} t^{\Sigma_1(\Omega)} (1 + t^{-\frac{1}{2}} 2^{\frac{k}{2}} |\tilde{x}'|)^{|\beta|} \end{aligned}$$

where

$$\begin{aligned} \Sigma(\Omega) &= m + \frac{1}{2}|\gamma| + \sum_{i=1}^{|\gamma|} \left(\tilde{l}_i + \frac{1}{2}|\tilde{\beta}_i| - \frac{1}{2} \right) + \sum_{i=1}^m \left(l_i + \frac{1}{2}|\beta_i| - 1 \right) \\ &= \sum_{i=1}^{|\gamma|} \left(\tilde{l}_i + \frac{1}{2}|\tilde{\beta}_i| \right) + \sum_{i=1}^m \left(l_i + \frac{1}{2}|\beta_i| \right) \\ &= l + \frac{1}{2}|\beta| \end{aligned}$$

and

$$\Sigma_1(\Omega) = -\frac{1}{2}|\gamma| - \frac{1}{2} \sum_{i=1}^{|\gamma|} (|\tilde{\beta}_i| - 1) = -\frac{1}{2} \sum_1^{|\gamma|} |\tilde{\beta}_i| \geq -\frac{1}{2}|\beta|.$$

This proves (7.12).

To prove (7.13), we first note $|\tilde{x}| \approx |\tilde{y}|$. By symmetry, the result is then reduced to proving

$$|\tilde{y}_1 - \tilde{x}_1| \lesssim t^{-1}|\tilde{x}|^2.$$

We consider the Taylor expansion of \tilde{y}_1

$$\tilde{y}_1(x) = \tilde{y}_1(0) + \partial_{\tilde{x}_1}\tilde{y}_1(0) \cdot \tilde{x}_1 + \partial_{\tilde{x}'}\tilde{y}_1(0) \cdot \tilde{x}' + R(\tilde{x})$$

where $|R(\tilde{x})| \lesssim |\tilde{x}|^2$, since $|\partial_{\tilde{x}_1}\tilde{y}_1| \lesssim 1$. Since $\tilde{y}_1(0) = 0$, $\partial_{\tilde{x}_1}\tilde{y}_1(0) = 1$, and $\partial_{\tilde{x}'}\tilde{y}_1(0) = 0$, we have

$$|\tilde{y}_1 - \tilde{x}_1| = |R(\tilde{x})| \lesssim |\tilde{x}|^2 \leq t^{-1}|\tilde{x}|^2.$$

Combine all the results above, we know for any N, l, β :

$$\begin{aligned} \left| \partial_{\tilde{x}_1}^l \partial_{\tilde{x}'}^\beta f(s, \tilde{x}) \right| &\leq C_{N,l,\beta} 2^{kl} (t^{-\frac{1}{2}} 2^{\frac{k}{2}})^{|\beta|} (1 + 2^k |\tilde{y}_1| + t^{-1} 2^k |\tilde{y}|^2)^{-N} (1 + t^{-\frac{1}{2}} 2^{\frac{k}{2}} |\tilde{x}'|)^{|\beta|} \\ &\leq C'_{N,l,\beta} 2^{kl} (t^{-\frac{1}{2}} 2^{\frac{k}{2}})^{|\beta|} (1 + 2^k |\tilde{y}_1| + t^{-1} 2^k |\tilde{y}|^2)^{-(N-\frac{|\beta|}{2})} \\ &\leq C''_{N',l,\beta} 2^{kl} (t^{-\frac{1}{2}} 2^{\frac{k}{2}})^{|\beta|} (1 + 2^k |\tilde{x}_1| + t^{-1} 2^k |\tilde{x}|^2)^{-N'}. \end{aligned}$$

This completes the proof of (7.9), hence (7.5).

We now turn to the proof of (7.6). We will now assume $\nu_s = e_1, x_s = 0$, and recall that $\gamma' = (x', \nu')$. The multiplier $\tilde{a}_\omega(D)\tilde{\psi}_k(D)$ is of class S_1^0 , and is seen to preserve the support properties and estimates (7.2) for $\phi_{\gamma'}$. Thus we ignore this factor, and just assume $|\nu' - e_1| \leq \frac{1}{4}$. Proving (7.6) is then equivalent to proving

$$\begin{aligned} \left| \int g(x) \overline{\phi_{\gamma'}(x)} dx \right| &\leq C_N (1 + t2^k |\nu' - e_1|^2)^{-N} \\ &\quad \times (1 + 2^k |\langle e_1, x' \rangle| + 2^k |\langle \nu', x' \rangle| + t^{-1} 2^k |x'|^2)^{-N}. \end{aligned} \quad (7.18)$$

First we show that for any N

$$\left| \int g(x) \overline{\phi_{\gamma'}(x)} dx \right| \leq C_N (1 + t^{\frac{1}{2}} 2^{\frac{k}{2}} |\nu' - e_1|)^{-N}. \quad (7.19)$$

Observe that (7.5) implies that, for any $N > 0$, we have

$$|\hat{g}(\eta)| \leq C'_N t^{\frac{d-1}{4}} 2^{-k(\frac{d+1}{4})} (1 + t^{\frac{1}{2}} 2^{-\frac{k}{2}} |\langle e_1^\perp, \eta \rangle|)^{-N}.$$

Suppose the angle between $|\eta|^{-1}\eta$ and e_1 is θ , then

$$|\langle e_1^\perp, \eta \rangle| = |\eta| |\langle e_1^\perp, |\eta|^{-1}\eta \rangle| = |\eta| |\sin \theta|.$$

We note that

$$|e_1 - |\eta|^{-1}\eta| = 2 \left| \sin \frac{1}{2}\theta \right|,$$

hence

$$\frac{|\langle e_1^\perp, \eta \rangle|}{|e_1 - |\eta|^{-1}\eta|} = |\eta| \frac{|\sin \theta|}{2 \left| \sin \frac{1}{2}\theta \right|} = |\eta| \left| \cos \frac{1}{2}\theta \right|.$$

Since $|\nu' - |\eta|^{-1}\eta| \leq \frac{1}{4} t^{-\frac{1}{2}} 2^{-\frac{k}{2}}$ and $|\nu' - e_1| \leq \frac{1}{4}$, we know $|2 \sin \frac{1}{2}\theta| \leq \frac{1}{2}$, so $|\cos \frac{1}{2}\theta| \geq \sqrt{\frac{15}{16}}$,

hence

$$\sqrt{\frac{15}{16}} |\eta| |e_1 - |\eta|^{-1}\eta| \leq |\langle e_1^\perp, \eta \rangle| \leq |\eta| |e_1 - |\eta|^{-1}\eta|.$$

Hence we can see that

$$\begin{aligned} 1 + t^{\frac{1}{2}} 2^{-\frac{k}{2}} |\langle e_1^\perp, \eta \rangle| &\approx 1 + t^{\frac{1}{2}} 2^{-\frac{k}{2}} |\eta| |e_1 - |\eta|^{-1}\eta| \\ &\approx 1 + t^{\frac{1}{2}} 2^{\frac{k}{2}} |e_1 - |\eta|^{-1}\eta| \\ &\approx 1 + t^{\frac{1}{2}} 2^{\frac{k}{2}} |e_1 - \nu'|. \end{aligned}$$

Here we use the fact that $|\eta| \approx 2^k$ and $|\nu' - |\eta|^{-1}\eta| \leq \frac{1}{4} t^{-\frac{1}{2}} 2^{-\frac{k}{2}}$. This proves (7.19).

We next prove

$$\begin{aligned} &\left| \int g(x) \overline{\phi_{\gamma'}(x)} dx \right| \\ &\leq C_N (1 + 2^k |\langle e_1, x' \rangle| + 2^k |\langle \nu', x' \rangle| + t^{-1} 2^k |x'|^2)^{-N} (1 + t 2^k |\nu' - e_1|^2)^N. \end{aligned} \quad (7.20)$$

We first prove

$$\begin{aligned} \sup_x [(1 + 2^k |\langle e_1, x \rangle| + t^{-1} 2^k |x|^2)^{-1} (1 + 2^k |\langle \nu', x - x' \rangle| + t^{-1} 2^k |x - x'|^2)^{-1}] \\ \lesssim (1 + 2^k |\langle \nu', x' \rangle| + t^{-1} 2^k |x - x'|^2)^{-1} (1 + t 2^k |\nu' - e_1|^2). \end{aligned} \quad (7.21)$$

It suffices to show that, for all x ,

$$|\langle \nu', x' \rangle| + t^{-1} |x'|^2 \lesssim t |\nu - e_1|^2 + |\langle e_1, x \rangle| + |\langle \nu', x - x' \rangle| + t^{-1} |x|^2 + t^{-1} |x - x'|^2. \quad (7.22)$$

Substituting $x' \rightarrow tx'$, $x \rightarrow tx$ reduces this estimate to the case $t = 1$. To prove (7.22) for $t = 1$, we first note $|x'|^2 \leq 2|x - x'|^2 + 2|x|^2$. We also have

$$\begin{aligned} |\langle \nu', x' \rangle| &\leq |\langle \nu', x - x' \rangle| + |\langle \nu', x \rangle| \\ &\leq |\langle \nu', x - x' \rangle| + |\langle \nu' - e_1, x \rangle| + |\langle e_1, x \rangle| \\ &\leq |\langle \nu', x - x' \rangle| + |\nu' - e_1|^2 + |x|^2 + |\langle e_1, x \rangle|. \end{aligned}$$

This proves (7.22). The bound (7.21) then implies that

$$\left| \int g(x) \overline{\phi_{\gamma'}(x)} dx \right| \leq C_N (1 + 2^k |\langle \nu', x' \rangle| + t^{-1} 2^k |x'|^2)^{-N} (1 + t 2^k |\nu' - e_1|^2)^N. \quad (7.23)$$

By symmetry, we can prove the following estimate similar to (7.21),

$$\begin{aligned} \sup_x [(1 + 2^k |\langle e_1, x \rangle| + t^{-1} 2^k |x|^2)^{-1} (1 + 2^k |\langle \nu', x - x' \rangle| + t^{-1} 2^k |x - x'|^2)^{-1}] \\ \lesssim (1 + 2^k |\langle e_1, x' \rangle| + t^{-1} 2^k |x - x'|^2)^{-1} (1 + t 2^k |\nu' - e_1|^2). \end{aligned} \quad (7.24)$$

By the same argument, we then see that

$$\left| \int g(x) \overline{\phi_{\gamma'}(x)} dx \right| \leq C_N (1 + 2^k |\langle e_1, x' \rangle| + t^{-1} 2^k |x'|^2)^{-N} (1 + t 2^k |\nu - e_1|^2)^N. \quad (7.25)$$

(7.23) and (7.25) together imply (7.20). This, together with (7.19), completes the proof of Lemma 7.2. \square

Proof of the bound (7.3). We first prove the result for $M = 0$. From Lemma 7.2, we can see that

$$|a(\gamma', \gamma)| \leq C_N (1 + 2^k d_t(x_s, \nu_s; x', \nu'))^{-N}, \quad (x_s, \nu_s) = \chi_s(\gamma).$$

Next we note that, for any $\delta > 0$,

$$\sup_{\gamma} \sum_{\gamma'} (1 + 2^k d_t(\gamma, \gamma'))^{-(d+\delta)} \leq C_\delta,$$

and

$$\sup_{\gamma'} \sum_{\gamma} (1 + 2^k d_t(\gamma, \gamma'))^{-(d+\delta)} \leq C_\delta,$$

where C_δ is a constant independent of s, t and k . This can be proved by using the estimate (2.3) in Smith [12]; in our case we just need to consider $k = k'$. It then follows that

$$\sup_{\gamma} \sum_{\gamma'} |a(\gamma', \gamma)| \leq C,$$

and

$$\sup_{\gamma'} \sum_{\gamma} |a(\gamma', \gamma)| \leq C,$$

where C is independent of s, t and k . By Schur's Lemma, we conclude

$$\|\tilde{a}_\omega(D) \tilde{\psi}_k(D) B_k(s) f\|_{0, x_s, \nu_s} \leq C \|f\|_{0, x_0, \nu_0}.$$

The weighted case $M \geq 1$ follows by showing that

$$a_M(\gamma', \gamma) = (1 + 2^k d_t(x', \nu'; \chi_s(x_0, \nu_0)))^M a(\gamma', \gamma) (1 + 2^k d_t(x, \nu; x_0, \nu_0))^{-M}$$

satisfies the conditions of Schur's Lemma. Taking $N = d + \delta + M$, it then suffices to show that

$$1 + 2^k d_t(x', \nu'; \chi_s(x_0, \nu_0)) \lesssim \left(1 + 2^k d_t(x', \nu'; \chi_s(x, \nu))\right) \left(1 + 2^k d_t(x, \nu; x_0, \nu_0)\right),$$

which in turn follows from

$$d_t(x', \nu'; \chi_s(x_0, \nu_0)) \leq d_t(x', \nu'; \chi_s(x, \nu)) + d_t(\chi_s(x, \nu); \chi_s(x_0, \nu_0)),$$

and the fact that $d_t(\chi_s(x, \nu); \chi_s(x_0, \nu_0)) \approx d_t(x, \nu; x_0, \nu_0)$. This fact was proven in [12] for $C^{1,1}$ metrics and $t = 1$. We give the proof for bounded curvature metrics in the following lemma. \square

Lemma 7.3. *Suppose that χ_s is the projected Hamiltonian flow on $S^*(\mathbb{R}^d)$ induced by $p_k(x, \xi) = \left(\sum_{ij} g_k^{ij}(x) \xi_i \xi_j\right)^{\frac{1}{2}}$, where g satisfies the conditions of Theorem 2.3. Then if $0 \leq s \leq t \leq 1$,*

$$d_t(\chi_s(y, \nu); \chi_s(y', \nu')) \approx d_t(y, \nu; y', \nu').$$

Proof. Let $\eta = \nu$ and $\eta' = \nu'$. If (x_s, ξ_s) is the (non-projected) Hamiltonian flow of (y, η) , then $|\xi_s - \eta| \lesssim c_d$, so we can replace ν_s by ξ_s in the distance function. From Corollary 3.2, when $|\eta| = 1$ we have the bound $|\partial_\eta x_s| \lesssim s$, $|\partial_y x_s| + |\partial_y \xi_s| + |\partial_\eta \xi_s| \lesssim 1$, and we deduce

$$|x'_s - x_s| + t|\xi'_s - \xi_s| \lesssim |y' - y| + t|\eta' - \eta|,$$

so applying this also to χ_{-s} we obtain

$$t^{-1}|x'_s - x_s|^2 + t|\xi'_s - \xi_s|^2 \lesssim t^{-1}|y' - y|^2 + t|\eta' - \eta|^2.$$

By symmetry it now suffices to show that

$$|\langle \eta, y' - y \rangle| \leq |\langle \xi_s, x'_s - x_s \rangle| + C t^{-1}|x'_s - x_s|^2 + C t|\eta' - \eta|^2.$$

Let $\varphi(s, x, \eta)$ be the generating function, so $y = \nabla_\eta \varphi(s, x_s, \eta)$ and $\xi_s = \nabla_x \varphi(s, x_s, \eta)$. Then

$$\begin{aligned} \langle \eta, y' - y \rangle - \langle \xi_s, x'_s - x_s \rangle &= \langle \eta, \nabla_\eta \varphi(x, x'_s, \eta') - \nabla_\eta \varphi(s, x_s, \eta) \rangle - \langle \nabla_x \varphi(s, x_s, \eta), x_s - x'_s \rangle \\ &= \varphi(s, x'_s, \eta') - \varphi(s, x_s, \eta) - (x'_s - x_s) \cdot \nabla_x \varphi(s, x_s, \eta) - (\eta' - \eta) \cdot \nabla_\eta \varphi(s, x'_s, \eta'). \end{aligned}$$

Next, we note that, by Theorem 3.3,

$$\begin{aligned} |(\eta' - \eta) \cdot (\nabla_\eta \varphi(s, x'_s, \eta') - \nabla_\eta \varphi(s, x_s, \eta))| &\leq C |\eta' - \eta| (|x'_s - x_s| + t|\eta' - \eta|) \\ &\leq C t^{-1}|x'_s - x_s|^2 + C t|\eta' - \eta|^2. \end{aligned}$$

Consequently, up to an error of desired size the above expression is the remainder for the second order Taylor expansion of $\varphi(s, x'_s, \eta') - \varphi(s, x_s, \eta)$. The estimates (4.1)–(4.4) give $|\partial_x^2 \varphi_k| \lesssim 1$, $|\partial_x \partial_\eta \varphi_k| \lesssim 1$, $|\partial_\eta^2 \varphi_k| \lesssim |s|$, and hence the remainder is dominated by

$$|x'_s - x_s|^2 + |x'_s - x_s| |\eta' - \eta| + t |\eta' - \eta|^2 \leq 2t^{-1} |x'_s - x_s|^2 + 2t |\eta' - \eta|^2.$$

□

7.3 Proof of the dispersive estimate

In this section we prove the inequality (7.1). We write $K_k(t, x, x_0) = (\tilde{E}_k^\omega(t) \delta_{x_0})(x)$. Since $\tilde{E}_k^\omega(t)$ has the factor $\psi_k(D)$ on the right, we may write

$$(\tilde{E}_k^\omega(t) \delta_{x_0})(x) = \sum_{\nu} (\tilde{E}_k^\omega(t) \beta_{k,t}^\nu(D)^2 \delta_{x_0})(x).$$

The function $\beta_{k,t}^\nu(D)^2 \delta_{x_0}$ has Fourier transform $e^{-i\langle x_0, \eta \rangle} \beta_{k,t}^\nu(\eta)^2$. Up to a normalization factor of $2^k \frac{d+1}{4} t^{-\frac{d-1}{4}}$, this has the same support and derivative estimates as the frame element ϕ_γ where $\gamma = (x_0, \nu)$; precisely, it is easy to verify that, for any M ,

$$\|\beta_{k,t}^\nu(D)^2 \delta_{x_0}\|_{M, x_0, \nu} \leq C_M 2^k \frac{d+1}{4} t^{-\frac{d-1}{4}},$$

and hence by Theorem 7.1 we have

$$\|\tilde{E}_k^\omega(t) \beta_{k,t}^\nu(D)^2 \delta_{x_0}\|_{M, x_t, \nu_t} \leq C_M 2^k \frac{d+1}{4} t^{-\frac{d-1}{4}}.$$

We observe that this implies, for all N , where $(x_t, \nu_t) = \chi_t(x_0, \nu)$,

$$|(\tilde{E}_k^\omega(t) \beta_{k,t}^\nu(D)^2 \delta_{x_0})(x)| \leq C_M 2^k \frac{d+1}{2} t^{-\frac{d-1}{2}} (1 + 2^k |\langle \nu_t, x - x_t \rangle| + 2^k t^{-1} |x - x_t|^2)^{-N}.$$

To see this, we use that the frame coefficients $\{c_{\gamma'}\}$ of $\tilde{E}_k^\omega(t) \beta_{k,t}^\nu(D)^2 \delta_{x_0}$, where $\gamma' = (x', \nu')$, satisfy

$$|c_{\gamma'}| \leq C_M 2^k \frac{d+1}{4} t^{-\frac{d-1}{4}} (1 + 2^k d_t(x', \nu', x_t, \nu_t))^{-M}$$

for all M . From estimates (7.2) on $|\phi_\gamma(x)|$, we follow the proof of [12, Lemma 2.5] to bound $|(\tilde{E}_k^\omega(t)\beta_{k,t}^\nu(D)^2\delta_{x_0})(x)|$ by

$$\begin{aligned} C_M 2^k t^{\frac{d+1}{2}} t^{-\frac{d-1}{2}} \sum_{\nu'} \sum_{x \in \Xi_{k,t}^{\nu'}} (1 + 2^k d_t(x', \nu'; x_t, \nu_t))^{-M} (1 + 2^k d_t(x, \nu'; x', \nu'))^{-M} \\ \leq C_M 2^k t^{\frac{d+1}{2}} t^{-\frac{d-1}{2}} \sum_{\nu'} (1 + 2^k d_t(x, \nu'; x_t, \nu_t))^{-M} \\ \leq C_M 2^k t^{\frac{d+1}{2}} t^{-\frac{d-1}{2}} (1 + 2^k |\langle \nu_t, x - x_t \rangle| + 2^k t^{-1} |x - x_t|^2)^{-M + \frac{d}{2}}. \end{aligned}$$

Consequently, it remains to show that

$$\sum_{\nu \in \Gamma_{t2^k}} (1 + 2^k |\langle \nu_t, x - x_t \rangle| + 2^k t^{-1} |x - x_t|^2)^{-N - \frac{d}{2}} \leq C_{d,N,r} (1 + 2^k \text{dist}(x, S_t(x_0)))^{-N}.$$

Here, $x_t \in S_t(x_0)$ for each ν , and ν_t is the unit normal to $S_t(x_0)$ at the point x_t , in that $\nu_t \cdot \partial_{\eta_j} x_t(\eta) = 0$, which follows by invariance of the form $\eta \cdot dx$ under the Hamiltonian flow.

By Corollary 3.2, if $(\tilde{x}_t, \tilde{\nu}_t) = \chi_t(x_0, \tilde{\nu}_0)$ and $(x_t, \nu_t) = \chi_t(x_0, \nu_0)$, then for c_d small

$$\frac{4}{5} t \leq \frac{|\tilde{x}_t - x_t|}{|\tilde{\nu}_0 - \nu_0|} \leq \frac{5}{4} t. \quad (7.26)$$

Consequently the points x_t are separated by $t^{\frac{1}{2}} 2^{-\frac{k}{2}}$ for $\nu \in \Upsilon_{k,t}$, and hence

$$\sum_{\nu \in \Gamma_{t2^k}} (1 + 2^k t^{-1} |x - x_t|^2)^{-\frac{d}{2}} \leq C_d.$$

It thus suffices to show that, for c_d small enough,

$$|\langle \nu_t, x - x_t \rangle| + t^{-1} |x - x_t|^2 \geq \frac{1}{64} \text{dist}(x, S_t(x_0)). \quad (7.27)$$

This is immediate if $|x - x_t| \geq \frac{1}{64} t$, so we may assume $|x - x_t| \leq \frac{1}{64} t$, and then $\text{dist}(x, S_t(x_0)) = \text{dist}(x, S_t(x_0) \cap B_{t/32}(x_t))$.

We observe that, by scaling, it suffices to prove this in the case $t = 1$. Precisely, $(t^{-1}x_t, \nu_t)$ is the flowout for time 1 of (x_0, ν_0) under the metric $g_k(t \cdot)$, and $t^{-1}S_t(x_0)$ is the corresponding unit geodesic sphere centered at x_0 , hence the two sides of (7.27) dilate by the same factor t . Furthermore, the metric $g_k(t \cdot)$ satisfies conditions (3.1)-(3.3) with $M = t2^{\frac{k}{2}} \leq 2^{\frac{k}{2}}$.

Without loss of generality, we assume $\nu_0 = e_1$ and $x_0 = 0$. We introduce the notation $\chi_1(0, \nu) = (x(\nu), n(\nu))$ to denote the mapping of the unit sphere \mathbb{S}^{d-1} onto the unit conormal bundle of $S_1(0)$. Then, by (3.5) and (3.12), assuming c_d small,

$$|x(\nu) - \nu| + |n(\nu) - \nu| \lesssim c_d, \quad (1 + c_d)^{-1} \lesssim \frac{|x(\nu) - x(\tilde{\nu})|}{|\nu - \tilde{\nu}|} \lesssim 1 + c_d.$$

It follows that $\nu \rightarrow x(\nu)$ is a smooth embedding of \mathbb{S}^{d-1} into \mathbb{R}^d , with image $S_1(x_0)$, and with uniform C^1 bounds on the map and its inverse. Additionally, the map from \mathbb{S}^{d-1} to the outer unit normal $n(\nu)$ of $S_1(0)$ at $x(\nu)$ is smooth, with uniform bounds on the C^1 norm by Corollary 3.2.

By the above, $S_1(0) \cap B_{1/32}(x(e_1)) \subset \{x(\nu) : |\Pi_{e_1}^\perp \nu| \leq \frac{1}{16}\}$. We parameterize this neighborhood of e_1 in \mathbb{S}^{d-1} by $\nu = (\sqrt{1 - |\eta'|^2}, \eta')$ with $|\eta'| \leq \frac{1}{16}$. Then, writing $x = (x_1, x')$, the map $x'(\eta') := \Pi_{e_1}^\perp x(\sqrt{1 - |\eta'|^2}, \eta')$ satisfies

$$|D_{\eta'} x'(\eta') - \mathbf{I}| \lesssim c_d, \quad |\eta'| \leq \frac{1}{2}.$$

Consequently, we may parameterize $S_1(0) \cap B_{1/16}(e_1)$ by a graph $(F(x'), x')$, with uniform bounds on the C^1 norm of $F(x')$ over the set $|x'| < \frac{1}{4}$. On the other hand, since the unit co-normal $(1, -\nabla_{x'} F) / \sqrt{1 + |\nabla_{x'} F|^2}$ is C^1 in x' over $|x'| < \frac{1}{4}$, it follows that F is uniformly bounded in $C^2(|x'| < \frac{1}{4})$. To summarize, for a constant C_d depending only on the dimension, provided c_d is sufficiently small,

$$S_1(0) \cap B_{1/16}(e_1) = \{(F(x'), x') : |x'| < \frac{1}{4}\} \cap B_{1/16}(e_1), \quad \|F\|_{C^2(|x'| < \frac{1}{4})} \leq C_d.$$

If $x = (x_1, x') \in B_{1/16}(e_1)$, then $\text{dist}(x, S_1(0)) \leq C_d |x_1 - F(x')|$. Consequently, it suffices to observe that

$$\begin{aligned} |x_1 - F(x')| &\leq (1 + |\nabla_{x'} F|^2)^{\frac{1}{2}} |\langle n(e_1), (x_1 - F(0), x') \rangle| + C_d |x'|^2 \\ &= |x_1 - F(0) - x' \cdot \nabla_{x'} F(0)| + C_d |x'|^2 \end{aligned}$$

which follows by Taylor's theorem, where C_d depends only on $\|F\|_{C^2(|x'| < \frac{1}{16})}$. \square

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