Faces of Polytopes

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Abstract

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The first two chapters of this thesis are about face numbers of polytopes. In Chapter 1, we settle a question of Bárány from the 1990s: do all convex d-polytopes satisfy $f_0, \dots, f_{d-1} \geq \min\{f_0, f_{d-1}\}$? We answer "yes" and prove a stronger statement: for all convex d-polytopes and $k = 0, \dots, d-1$,

$$f_k \geq \frac{1}{2}\left[\binom{\lceil \frac{d}{2} \rceil}{k} + \binom{\lfloor \frac{d}{2} \rfloor}{k}\right] f_0, \qquad f_k \geq \frac{1}{2}\left[\binom{\lceil \frac{d}{2} \rceil}{d-k-1} + \binom{\lfloor \frac{d}{2} \rfloor}{d-k-1}\right] f_{d-1}.$$

The former holds with equality precisely when $k = 0$ or for simple polytopes when $k = 1$. The latter holds with equality precisely when $k = d-1$ or for simplicial polytopes when $k = d-2$. These inequalities are asymptotically tight for dual cyclic and cyclic polytopes, respectively. We generalize the latter inequality to shellable, strongly regular CW spheres and balls, proving that every shellable, dual shellable, strongly regular CW $(d-1)$-sphere satisfies $f_0, \dots, f_{d-1} \geq \min\{f_0, f_{d-1}\}$.

Chapter 2 concerns the *fatness* parameter, defined for 4-polytopes and 3-spheres as

$$\frac{f_1 + f_2 - 20}{f_0 + f_3 - 10}.$$

We construct a family of shellable, dual shellable, strongly regular CW 3-spheres with arbitrarily high fatness, solving a problem of Ziegler. These spheres have f-vectors $(\Theta(n),$ $\Theta(n\alpha(n)), \Theta(n\alpha(n)), \Theta(n))$, where α is the inverse Ackermann function. We conjecture

that these spheres are realizable as convex 4-polytopes; if true, this would answer Eppstein, Kuperberg, and Ziegler's famous question of whether 4-polytopes may be arbitrarily fat.

Chapter 3 is about reconstruction problems. We prove that every 4-polytope can be reconstructed up to combinatorial isomorphism from its edge-polygon incidences, solving a problem of Grünbaum. For each $d \geq 3$, we prove that *not* every d-polytope can be reconstructed up to combinatorial isomorphism from its $(d-3)$-skeleton and dual $(d-3)$-skeleton together, answering a question of Samper. As a corollary, we fully characterize the subsets $K \subseteq \{0, \ldots, d-1\}$ such that every d-polytope can be reconstructed from the incidences of its faces of dimension in K. We further prove positive and negative reconstruction results on simplicial homology manifolds and pseudomanifolds.

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INTRODUCTION

This thesis is about polytopes! A *polytope* is a convex shape with flat sides, or more precisely, the convex hull of finitely many points in Euclidean space. Polytopes include convex polygons, cubes, pyramids, and many other beautiful geometric shapes in three dimensions and higher. They have numerous applications across commutative algebra, probability, and optimization.

The *faces* of a polytope are the lower-dimensional polytopes making up its boundary. For example, the faces of a three-dimensional cube are its 8 vertices, 12 edges, and 6 polygons. It is natural to ask, if a cube has *face numbers* $8, 12, 6$, then *which other lists of integers are the face numbers of some polytope?* This simple question has fascinated combinatorialists for centuries and is the primary focus of our work.

This thesis is based on five research papers by the author, all published or under consideration. Chapter 1 corresponds with [33], [35], and [34]; Chapter 2 corresponds with [31]; and Chapter 3 corresponds with [32].

Bárány's question on face numbers of polytopes

We recount three iconic results on the face numbers of polytopes. Given a d-polytope, we define f_k as the number of k-dimensional faces for $k = 0, \ldots, d-1$.

1. *Euler's equation*: for all d-polytopes, $\sum_{k=0}^{d-1} (-1)^k f_k = 1 - (-1)^d$. Euler discovered the equation $f_0 - f_1 + f_2 = 2$ for 3-polytopes circa 1752 [22, 21]. Various authors are credited with the general version; see Chapter 8 of Grünbaum [29] for further discussion.

2. The *upper bound theorem*: among d-polytopes with a fixed number of vertices, the *cyclic polytope*—a simplicial polytope with vertices on the curve $(t, t^2, \ldots, t^d)$—maximizes $f_1, \ldots, f_{d-1}$. McMullen proved the upper bound theorem in 1970 [44].

3. The *g-theorem*: a set of necessary and sufficient conditions for any list of integers $f_0, \dots, f_{d-1}$ to be the face numbers of a simplicial polytope. Billera and Lee proved sufficiency [6, 7] and Stanley proved necessity [52] in 1980.

Beyond these landmarks, the face numbers of polytopes are poorly understood. For example, we have no *lower* bound theorem for d-polytopes in general. Bárány asked the following question in the 1990s [2]: *does every d-polytope satisfy $f_0, \dots, f_{d-1} \geq \min\{f_0, f_{d-1}\}$?*

In Chapter 1, we answer Bárány's question in the affirmative and prove a stronger statement. For nonnegative integers d and k, define

$$\rho(d,k) := \frac{1}{2}\left[\binom{\lceil \frac{d}{2} \rceil}{k} + \binom{\lfloor \frac{d}{2} \rfloor}{k}\right].$$

Our main result is that for $i = 1, \dots, \lceil d/2 \rceil + 1$,

$$f_{i-1} \geq \rho(d, i-1) f_0, \qquad f_{d-i} \geq \rho(d, i-1) f_{d-1}.$$

We prove these inequalities tight for dual cyclic and cyclic polytopes, respectively, and characterize the cases where equality holds. Our primary tool is *solid angles*, a generalization of plane angles to higher dimensions. We partially extend this result on polytopes to the larger class of shellable, strongly regular CW complexes homeomorphic to a sphere or ball.

A related problem is Kalai's generalized upper bound conjecture [39]: *if C is a cyclic d-polytope and P another d-polytope with $f_{d-1}(P) \geq f_{d-1}(C)$, then $f_k(P) \geq f_k(C)$ for $k = 0, \dots, d-1$.* Kalai's conjecture is equivalent to a set of sharp, nonlinear lower bounds on $f_k(P)$ in terms of $f_{d-1}(P)$. Our main result above represents the "best linear approximation" of the nonlinear bounds conjectured by Kalai. At the end of Chapter 1, we explore a possible avenue to the generalized upper bound conjecture via *Grassmann angles*, an even broader notion of angles for polytopes.

Fat polytopes and spheres

The *f-vector* of any d-polytope is defined as $(f_0, \dots, f_{d-1})$. To rephrase our central question in f-vector language: *which integer vectors of length d are the f-vector of some d-polytope?*

Steinitz answered this question for $d = 3$ in 1906 [53], proving that the f-vectors of 3-polytopes are exactly the lattice points in the cone given by

- $f_0 - f_1 + f_2 = 2$ (Euler's equation),
- $2f_1 - 3f_0 \geq 0$ (with equality for simple polytopes), and
- $2f_1 - 3f_2 \geq 0$ (with equality for simplicial polytopes).

This is the so-called "f-vector cone" for 3-polytopes. Its apex is $(4, 6, 4)$, or the f-vector of the 3-dimensional simplex. For general d, the *f-vector cone* is the maximal set of linear inequalities satisfied by the f-vectors of all d-polytopes with equality at the f-vector $(\binom{d+1}{1}, \dots, \binom{d+1}{d})$ of the d-dimensional simplex. For $d > 3$, the set of lattice points in the f-vector cone is strictly larger than the set of f-vectors (see Ziegler [63]).

Chapter 2 is dedicated to the f-vector cone of 4-polytopes. Bayer gave a list of f-vector inequalities for 4-polytopes in [3] which might fully describe the f-vector cone. Only one gap remains in our understanding: the *fatness* parameter, defined as $(f_1 + f_2 - 20)/(f_0 + f_3 - 10)$. Eppstein, Kuperberg, and Ziegler asked the following in [19]: *do there exist 4-polytopes with arbitrarily high fatness?*

If we can construct arbitrarily fat 4-polytopes, then we will have a full picture of the f-vector cone. Ziegler constructed in [62] the fattest 4-polytopes known to date, having fatness $9 - \varepsilon$ for arbitrarily small ε. In contrast, we already know of arbitrarily fat strongly regular 3-spheres, for which Eppstein et al. gave a probabilistic construction in [19].

Bruggesser and Mani proved in [16] to much acclaim that all polytopes are shellable. Thus, fat shellable spheres are a natural step between fat spheres and fat polytopes. In personal communication [20], Ziegler posed the following problem: *do there exist shellable 3-spheres with arbitrarily high fatness?*

In Chapter 2, we answer in the affirmative, constructing a two-parameter family of shellable, dual shellable spheres with unbounded fatness (the first explicit construction for fat 3-spheres in general). We use double recursion, building each sphere by gluing together copies of two previous spheres. Our construction yields spheres with f-vectors $(\Theta(n), \Theta(n\alpha(n)), \Theta(n\alpha(n)), \Theta(n))$, where α is the inverse Ackermann function. We conjecture that these spheres may be realized as arbitrarily fat polytopes.

Reconstructing polytopes

Face numbers are exciting and fascinating, but they do not tell us everything. The strongest combinatorial information about a polytope is its *face lattice*, defined as the set of faces partially ordered by containment. Chapter 3 is about reconstruction problems: *how much information is needed to determine a polytope's face lattice up to isomorphism?*

The *k-skeleton* of a polytope is the set of faces of dimensions $0, \dots, k$ partially ordered by containment. Whitney proved in 1932 [55] that *the face lattice of a 3-polytope is determined by its 1-skeleton.* Generalizing Whitney's theorem, Grünbaum proved in [29] that the face lattice of any d-polytope is determined by its $(d-2)$-skeleton.

Grünbaum [29, p. 234] and Samper [49] posed the following reconstruction problems on 4-polytopes.

- (Grünbaum) *Is the face lattice of a 4-polytope determined by its edge-polygon incidences?*

- (Samper) *Is the face lattice of a 4-polytope determined by the combination of its 1-skeleton and dual 1-skeleton?*

We solve both problems in Chapter 3, the latter in higher generality. We first prove that the face lattice of every polytope is determined by its edge-polygon incidences. We then prove that for all $d \geq 3$, there exist combinatorially distinct d-polytopes P and Q with isomorphic $(d-3)$-skeleta and isomorphic dual $(d-3)$-skeleta, meaning the answer to Samper's question is "no". From these results and previous ones of Grünbaum, for any fixed dimension d, we characterize the subsets $K \subseteq [0, d-1]$ such that the face poset of every d-polytope can be reconstructed from the incidences of its faces of dimension in K.

A second type of reconstruction problem deals with simplicial polytopes and more general simplicial complexes. Perles famously discovered in 1970 [47] that every simplicial d-polytope is determined by its $\lfloor d/2 \rfloor$-skeleton. Dancis extended this result in [17] to homology $(d-1)$-manifolds for even d and to homology $(d-1)$-manifolds with vanishing middle homology (e.g. simplicial spheres) for odd d. Later in Chapter 3, we strengthen Dancis's result, showing that a simplicial homology manifold can be reconstructed from the incidences of its k- and $(k-1)$-faces for any $\lceil d/2 \rceil \leq k \leq d-2$. We also prove a generalization for normal, simplicial pseudomanifolds satisfying a certain link condition.

At the end of Chapter 3, we prove a negative reconstruction result: for any fixed $d \geq 3$, there exist normal, simplicial pseudomanifolds of dimension $d-1$ that cannot even be reconstructed from their $(d-2)$-skeleta.

Chapter 1

A POSITIVE ANSWER TO BÁRÁNY'S QUESTION

This chapter is about the face numbers of polytopes. If P is a d-polytope and $0 \leq k \leq d-1$, we define the k^{th} *face number* $f_k(P)$ as the number of k-dimensional faces of P, and the *face vector* or *f-vector* as $(f_0(P), \dots, f_{d-1}(P))$. While easily defined and extensively studied, face numbers still remain highly mysterious.

Face numbers are best understood in the case of simple and simplicial polytopes. In 1971, McMullen published the famous *g-conjecture*, a proposed set of necessary and sufficient conditions for a vector $(f_0, \dots, f_{d-1})$ to be the f-vector of some simplicial polytope [45]. Billera and Lee proved the sufficiency of these conditions in 1979 [6][7]; Stanley proved their necessity the same year [52]. The resulting *g-theorem* gives a full characterization of the f-vectors of simplicial, and consequently simple, polytopes.

Results on general polytopes are far more elusive. McMullen proved in 1970 that for a fixed dimension d and number of vertices $n > d$, the *cyclic polytope* $C(d,n)$ simultaneously maximizes all face numbers [44]. Something is also known about *flag f-numbers*, which count the chains of faces $\emptyset \subset G_1 \subset \cdots \subset G_s \subset P$ in a fixed sequence of dimensions. Significant results include Kalai's rigidity inequality for flag f-vectors [37, Theorem 1.4], as well as Bayer's inequalities for flag f-vectors of 4-polytopes [3].

The greatest hole in our understanding of face numbers is the question of lower bounds on f_k. It is well known that in a fixed dimension d, the d-simplex Δ^d simultaneously minimizes all face numbers. The most general lower bound previously was a theorem of Xue [56], first conjectured by Grünbaum in 1967 [29]: if P is a d-polytope with $f_0 = d + s \leq 2d$, then

$$f_k \geq \binom{d+1}{k+1} + \binom{d}{k+1} - \binom{d+1-s}{k+1}.$$

Xue later proved a similar inequality in the case $f_0 = 2d+1$ [57].

What if we do not know that $f_0 \leq 2d+1$? Bárány asked (see [2, (3.4)][5, Problem 15.6.5]) whether all d-polytopes satisfy $f_0, \dots, f_{d-1} \geq \min\{f_0, f_{d-1}\}$. We will settle Bárány's question with our main result, a general lower bound on f_k given f_0 or f_{d-1}. For all integers $0 \leq k < d$, we define

$$\rho(d,k) := \frac{1}{2}\left[\binom{\lceil \frac{d}{2} \rceil}{k} + \binom{\lfloor \frac{d}{2} \rfloor}{k}\right].$$

Our result is as follows: for all d-polytopes P and $k = 0, \dots, d-1$,

$$f_k \geq \rho(d,k) f_0, \tag{1.1}$$

$$f_k \geq \rho(d, d-k-1) f_{d-1}. \tag{1.2}$$

Equivalently, for $i = 1, \dots, d$,

$$f_{i-1} \geq \rho(d, i-1) f_0, \qquad f_{d-i} \geq \rho(d, i-1) f_{d-1}.$$

Note that (1.1) is nontrivial when $k \leq \lceil d/2 \rceil$, while (1.2) is nontrivial when $k \geq \lfloor d/2 \rfloor - 1$.

Our result answers Bárány's question and does better. We show that if P is a d-polytope and $0 \leq k \leq \lfloor d/2 \rfloor$, then $f_k \geq f_0$; likewise, if $\lceil d/2 \rceil - 1 \leq k \leq d-1$, then $f_k \geq f_{d-1}$. Notably, if d is odd and $k = (d-1)/2$, then $f_k \geq (k/2+1) f_0, (k/2+1) f_{d-1}$.

Bounds (1.1) and (1.2) are strongest when $k \approx d/4$ and $k \approx 3d/4$, respectively. These values evoke Björner's results on the partial unimodality of f-vectors [10]. Björner proved that for simplicial d-polytopes, $f_0 < \cdots < f_{\lfloor d/2 \rfloor - 1} \leq f_{\lfloor d/2 \rfloor}$ and $f_{\lfloor 3(d-1)/4 \rfloor} > \cdots > f_{d-1}$. Dually, for simple d-polytopes, $f_0 < \cdots < f_{\lceil (d-1)/4 \rceil}$ and $f_{\lceil d/2 \rceil - 1} \geq f_{\lceil d/2 \rceil} > \cdots > f_{d-1}$. In fact, Björner proved that these are the strongest unimodality results possible for simplicial and simple polytopes.

Our proof relies on *solid angles*, a generalization of angles in arbitrary finite dimension. Our primary information on solid angles is due to Perles and Shephard [48][51]. For us, the key feature distinguishing the boundary of a d-polytope from an arbitrary $(d-1)$-complex is that of angle deficiencies where several facets meet. Two results of Perles and Shephard (see Theorems 1.1.12–1.1.13) allow us to use this notion in proving (1.1) and (1.2).

The structure of this chapter is as follows. Section 1.1 provides the background for our work on polytopes and solid angles. Section 1.2 is dedicated to our main result: we prove a key proposition on angle sums, prove (1.1) and (1.2), and answer Bárány's question. In Section 1.3, we show that (1.1) and (1.2) are tight linear bounds. In section 1.4, we prove tighter, nonlinear bounds on face numbers in terms of Grassmann angles. Finally, in Section 1.5, we partially extend our results to the face numbers of shellable CW spheres and balls.

1.1 Preliminaries

In this section, we introduce the concepts and first results needed for our proof of (1.1) and (1.2). Most of these preliminaries revolve around polytopes, polytopal complexes, and solid angles. We also include a lemma about binomial coefficients.

1.1.1 Polytopes and polytopal complexes

We begin by introducing some key notions about polytopes and polytopal complexes. All polytopes are assumed to be convex. For any undefined terminology, we refer the reader to Grünbaum [29] and Ziegler [60].

Definition 1.1.1. A *polytope* is the convex hull of a finite set of points in a real vector space. A polytope with affine hull of dimension d is called a d-*polytope*. By convention, the empty set is a (-1)-polytope.

Definition 1.1.2. Let P be a d-polytope. A *face* of P is either P itself or a polytope $G = H \cap P$, where H is a codimension one hyperplane that does not intersect the interior of P. Faces of dimension 0, 1, $d-2$, and $d-1$ are called *vertices*, *edges*, *ridges*, and *facets*, respectively. For $k = 0, \dots, d-1$, we define $f_k(P)$—or f_k, when not ambiguous—as the number of k-dimensional faces of P.

Definition 1.1.3. Let P be a d-polytope and G a k-dimensional face of P. The *quotient polytope* P/G is defined as $H \cap P$, where H is an affine $(d-k-1)$-dimensional subspace intersecting exactly those faces of P which properly contain G.

Definition 1.1.4. A polytope is *simplicial* if every proper face is a simplex. A polytope is *simple* if its quotient by every nonempty face is a simplex.

Definition 1.1.5. A *polytopal complex* $\mathcal{P}$ is a finite collection of polytopes with the following properties:

- $\emptyset \in \mathcal{P}$.
- If $P \in \mathcal{P}$ and G is a face of $\mathcal{P}$, then $G \in \mathcal{P}$.
- If $P, Q \in \mathcal{P}$, then $P \cap Q$ is a face of both P and Q.

If $U \subset \mathcal{P}$, we define $\langle U \rangle$ as the subcomplex of $\mathcal{P}$ generated by U.

Definition 1.1.6. If P is a polytope, we define the *boundary complex* ∂P as the polytopal complex consisting of all proper faces of P.

Definition 1.1.7. Let P be a d-polytope. A *polytopal subdivision* of P is a polytopal complex with underlying space P. A d-polytope in this complex is called a *cell.*

Definition 1.1.8. For nonnegative integers d, a *d-diagram* is a polytopal subdivision $\mathcal{D}$ of some d-polytope P such that for all $G \in \mathcal{D}$, $G \cap \partial P$ is a face of P. An *interior vertex* of $\mathcal{D}$ is a vertex $x \in \mathcal{D}$ which is not a vertex of P.

We now give an elementary, but useful, result about d-diagrams.

Lemma 1.1.9. *Any d-diagram with $d \geq 1$ contains an interior vertex.*

Proof. Let $\mathcal{D}$ be a d-diagram with $d \geq 1$, and let P be the d-polytope of which $\mathcal{D}$ is a subdivision. Let D be a cell of $\mathcal{D}$, so $D \cap \partial P$ is a face of P. Note that $D \neq P$, because $P \cap \partial P = \partial P$ is not a face of P.

Since $\dim D = \dim P = d$, we know D is not a face of P, so $D \cap \partial P \neq D$. It follows that D contains some vertex $x \notin D \cap \partial P$. By definition, x is an interior vertex of $\mathcal{D}$. □

Definition 1.1.10. Let P be a d-polytope with vertex set S, and let $v \in \mathbb{R}^d$ be a vector. We say v is in *general position with respect to* P if for all subsets $U \subset S$ such that $|U| > 1$ and $\dim \operatorname{aff}(U) < d$, v is not parallel to $\operatorname{aff}(U)$.

1.1.2 Solid Angles

Next we discuss solid angles, a higher-dimensional analogue for the familiar angles of polygons. Solid angles are a crucial ingredient in proving our main result.

Definition 1.1.11. Let P be a d-polytope and G a nonempty face of P. We define the *solid angle of P at G*, denoted $\varphi(P, G)$, as follows. Let B be an open ball in $\operatorname{aff} P$ centered on G, intersecting only those faces of P which contain G. Then

$$\varphi(P, G) := \frac{m(B \cap P)}{m(B)},$$

where m is the volume function. For $0 \leq k \leq d$, we denote by $\varphi_k(P)$ the sum of solid angles of P at all k-dimensional faces. That is, if $\mathcal{G}_k$ is the set of all k-dimensional faces of P, then

$$\varphi_k(P) := \sum_{G \in \mathcal{G}_k} \varphi(P, G).$$

We give two essential results on solid angles by Perles and Shephard. The first places a lower bound on the k^{th} angle sum using orthogonal projections [48, (23)]. The second formalizes our intuition about the positive curvature of polytopes [51, Theorem 3].

Theorem 1.1.12 (Perles and Shephard)**.** *Let P be a d-polytope and $0 \leq k \leq d-1$. For all vectors $v \in \mathbb{R}^d$ in general position with respect to P, let $H_v \subset \mathbb{R}^d$ be a codimension one hyperplane orthogonal to v, and let $\pi_v : \mathbb{R}^d \to H_v$ be the orthogonal projection map. Then*

$$\varphi_k(P) \geq \frac{1}{2}\left[f_k(P) - \max_v f_k(\pi_v(P))\right].$$

Perles and Shephard stated the above inequality for $0 \leq k \leq d-2$; it easily extends to $k = d-1$, as the solid angle of a polytope at any facet is $1/2$.

Theorem 1.1.13 (Shephard)**.** *Let P be a d-polytope and G a k-dimensional face of P, where $0 \leq k \leq d-2$. Let $\mathcal{F}_G$ be the set of facets of P containing G. Then*

$$\sum_{F \in \mathcal{F}_G} \varphi(F, G) \leq 1,$$

with equality if and only if $k = d-2$.

1.1.3 A Combinatorial Lemma

Finally, we prove a lemma about the convexity of binomial coefficients.

Lemma 1.1.14. *For all nonnegative integers a, b, c,*

$$\binom{a}{c} + \binom{b}{c} \geq \binom{\lceil \frac{a+b}{2} \rceil}{c} + \binom{\lfloor \frac{a+b}{2} \rfloor}{c}.$$

Proof. Without loss of generality, suppose $a \geq b$. If $a - b > 1$,

$$\left[\binom{a}{c} + \binom{b}{c}\right] - \left[\binom{a-1}{c} + \binom{b+1}{c}\right] = \binom{a-1}{c-1} - \binom{b}{c-1} \geq 0.$$

Thus,

$$\binom{a}{c} + \binom{b}{c} \geq \binom{a-1}{c} + \binom{b+1}{c} \geq \binom{a-2}{c} + \binom{b+2}{c} \geq \cdots \geq \binom{\lceil \frac{a+b}{2} \rceil}{c} + \binom{\lfloor \frac{a+b}{2} \rfloor}{c}.$$

□

1.2 Main result

In this section, we prove our main theorem on face numbers and consequently answer Bárány's question. The key to our proof is the following proposition about angle sums. Recall that for all integers $0 \leq k < d$, we define

$$\rho(d,k) := \frac{1}{2}\left[\binom{\lceil \frac{d}{2} \rceil}{k} + \binom{\lfloor \frac{d}{2} \rfloor}{k}\right].$$

Proposition 1.2.1. *For all $(d-1)$-polytopes Q and all $k = 0, \dots, d-2$,*

$$\varphi_k(Q) \geq \rho(d, d-k-1).$$

Proof. Let Q be a $(d-1)$-polytope and $0 \leq k \leq d-2$. If $d = 2$, then $k = 0$, so $\varphi_k(Q) = \rho(d, d-k-1) = 1$. For the remainder of this proof, we will assume $d \geq 3$.

For each vector $v \in \mathbb{R}^{d-1}$ in general position with respect to Q, let H_v be a codimension one hyperplane orthogonal to v. Define $\pi_v : \mathbb{R}^{d-1} \to H_v$ to be the orthogonal projection map.

Let $\mathcal{F}$ be the set of facets of Q. For each $F \in \mathcal{F}$, let u_F be the outer unit normal of F. Define subcomplexes $Q_v^+, Q_v^- \subset \partial Q$ as follows:

$$Q_v^+ = \langle F \in \mathcal{F} \mid v \cdot u_F > 0 \rangle,$$
$$Q_v^- = \langle F \in \mathcal{F} \mid v \cdot u_F < 0 \rangle.$$

Then $Q_v^+ \cup Q_v^- = \partial Q$. Furthermore, $\pi_v|_{Q_v^+}$ and $\pi_v|_{Q_v^-}$ are homeomorphisms onto $\pi_v(Q)$. Since $\partial Q_v^+ = \partial Q_v^- = Q_v^+ \cap Q_v^-$, it follows that $\pi_v|_{Q_v^+ \cap Q_v^-}$ is a homeomorphism onto $\partial \pi_v(Q)$.

Both $\{\pi_v(G) \mid G \in Q_v^+\}$ and $\{\pi_v(G) \mid G \in Q_v^-\}$ are regular subdivisions of $\pi_v(Q)$ (see [60, Definition 5.3]). Let $\mathcal{Q}_v$ be the smallest common refinement of these two subdivisions:

$$\mathcal{Q}_v = \{\pi_v(G^+) \cap \pi_v(G^-) \mid G^+ \in Q_v^+, G^- \in Q_v^-\}.$$

We know that $\mathcal{Q}_v$ is itself a polytopal subdivision of $\pi_v(Q)$, specifically the meet of $\{\pi_v(G) \mid G \in Q_v^+\}$ and $\{\pi_v(G) \mid G \in Q_v^-\}$ in the set of polytopal subdivisions of $\pi_v(Q)$ partially ordered under refinement (see [18, Lemma 2.3.9]). We claim that $\mathcal{Q}_v$ is a $(d-2)$-diagram.

Suppose $G = \pi_v(G^+) \cap \pi_v(G^-) \in \mathcal{Q}_v$, where $G^+ \in Q_v^+$ and $G^- \in Q_v^-$. Since π_v maps Q_v^+, Q_v^- homeomorphically onto $\pi_v(Q)$ and maps $Q_v^+ \cap Q_v^-$ homeomorphically onto $\partial \pi_v(Q)$, we can observe that

$$\pi_v(G^+ \cap G^-) = \pi_v((G^+ \cap Q_v^-) \cap (G^- \cap Q_v^+)) = \pi_v(G^+ \cap Q_v^-) \cap \pi_v(G^- \cap Q_v^+),$$
$$\pi_v(G^+ \cap Q_v^-) = \pi_v(G^+ \cap (Q_v^+ \cap Q_v^-)) = \pi_v(G^+) \cap \partial \pi_v(Q),$$
$$\pi_v(G^- \cap Q_v^+) = \pi_v(G^- \cap (Q_v^+ \cap Q_v^-)) = \pi_v(G^-) \cap \partial \pi_v(Q).$$

Thus,

$$G \cap \partial \pi_v(Q) = \pi_v(G^+) \cap \pi_v(G^-) \cap \partial \pi_v(Q) = \pi_v(G^+ \cap Q_v^-) \cap \pi_v(G^- \cap Q_v^+) = \pi_v(G^+ \cap G^-).$$

Since $G^+ \cap G^- \in Q_v^+ \cap Q_v^-$ is a face of Q, it follows that $G \cap \partial \pi_v(Q) = \pi_v(G^+ \cap G^-)$ is a face of $\pi_v(Q)$. This verifies our claim that $\mathcal{Q}_v$ is a $(d-2)$-diagram.

By Lemma 1.1.9, $\mathcal{Q}_v$ contains an interior vertex x. Let $x = \pi_v(X^+) \cap \pi_v(X^-)$, where $X^+ \in Q_v^+$ and $X^- \in Q_v^-$ (Figure 1.1). We know $x \notin \partial \pi_v(Q)$, so $X^+, X^- \notin Q_v^+ \cap Q_v^-$. It

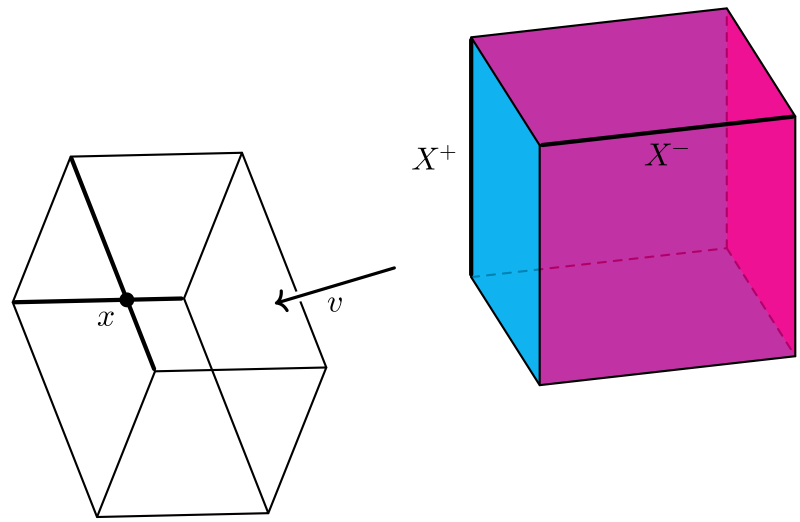

Figure 1.1: A $(d-1)$-polytope Q and the resulting $(d-2)$-diagram $\mathcal{Q}_v$. The back, bottom, and left facets of Q generate Q_v^+ (blue); the front, top, and right generate Q_v^- (pink).

follows that for any face G of Q, if $X^+ \subset G$, then G is a face of Q_v^+ but not a face of Q_v^-. Similarly, if $X^- \subset G$, then G is a face of Q_v^- but not a face of Q_v^+.

Let $\ell^+ = \dim X^+$ and $\ell^- = \dim X^-$. Since v is in general position with respect to Q, we know $\ell^+ = \dim \pi_v(X^+)$, $\ell^- = \dim \pi_v(X^-)$, and $x = \operatorname{aff} \pi_v(X^+) \cap \operatorname{aff} \pi_v(X^-)$. Because $\operatorname{aff} \pi_v(X^+)$ and $\operatorname{aff} \pi_v(X^-)$ intersect only at a point, it follows that $\ell^+ + \ell^- \leq \dim H_v = d-2$.

On the other hand, because $\pi_v(X^+)$ and $\pi_v(X^-)$ intersect nontrivially, we know v is parallel to $\operatorname{aff}(X^+ \cup X^-)$, so $\operatorname{aff}(X^+ \cup X^-) = \mathbb{R}^{d-1}$. Thus, $\ell^+ + \ell^- = d-2$. It follows that

$$\dim Q/X^+ = d - \ell^+ - 2 = \ell^-,$$
$$\dim Q/X^- = d - \ell^- - 2 = \ell^+.$$

As a result,

$$f_{k-\ell^+-1}(Q/X^+) \geq \binom{\ell^- + 1}{k - \ell^+} = \binom{\ell^- + 1}{d-k-1},$$
$$f_{k-\ell^--1}(Q/X^-) \geq \binom{\ell^+ + 1}{k - \ell^-} = \binom{\ell^+ + 1}{d-k-1}.$$

The above inequalities indicate that Q has at least $\binom{\ell^-+1}{d-k-1}$ many k-dimensional faces containing X^+ and at least $\binom{\ell^++1}{d-k-1}$ many k-dimensional faces containing X^-. We have established that these two sets of faces are disjoint, so Q has at least $\binom{\ell^-+1}{d-k-1}+\binom{\ell^++1}{d-k-1}$ many k-dimensional faces not contained in $Q^+ \cap Q^-$. In other words,

$$f_k(Q) - f_k(\pi_v(Q)) \geq \binom{\ell^- + 1}{d-k-1} + \binom{\ell^+ + 1}{d-k-1}.$$

By Lemma 1.1.14, it follows that for all $v \in \mathbb{R}^{d-1}$ in general position with respect to Q,

$$f_k(Q) - f_k(\pi_v(Q)) \geq \binom{\lceil \frac{d}{2} \rceil}{d-k-1} + \binom{\lfloor \frac{d}{2} \rfloor}{d-k-1} = 2\rho(d, d-k-1).$$

Thus,

$$f_k(Q) - \max_v f_k(\pi_v(Q)) \geq 2\rho(d, d-k-1).$$

Finally, we can use Theorem 1.1.12 to conclude that

$$\varphi_k(Q) \geq \frac{1}{2}\left[f_k(Q) - \max_v f_k(\pi_v(Q))\right] \geq \rho(d, d-k-1). \qquad \square$$

Our main result follows.

Theorem 1.2.2. *Let P be a convex d-polytope and suppose $k = 0, \dots, d-1$. Then*

$$f_k \geq \rho(d,k) f_0, \qquad f_k \geq \rho(d, d-k-1) f_{d-1}.$$

In the former, equality holds precisely when $k = 0$ or when $k = 1$ and P is simple. In the latter, equality holds precisely when $k = d-1$ or when $k = d-2$ and P is simplicial.

Proof. Let P be a d-polytope and $0 \leq k \leq d-1$. Let $\mathcal{F}$ be the set of facets of P, and let $\mathcal{G}$ be the set of k-dimensional faces of P. For all $F \in \mathcal{F}$ and $G \in \mathcal{G}$, if G is not a face of F, we set $\varphi(F, G) = 0$.

By Theorem 1.1.13, for all $G \in \mathcal{G}$, $\sum_{F \in \mathcal{F}} \varphi(F, G) \leq 1$. Equality is achieved if and only if $k \geq d-2$ (trivially if $k = d-1$). Thus,

$$\sum_{\substack{F \in \mathcal{F} \\ G \in \mathcal{G}}} \varphi(F, G) \leq |\mathcal{G}| = f_k(P),$$

with equality if and only if $k \geq d-2$.

By Proposition 1.2.1, for all $F \in \mathcal{F}$, $\sum_{G \in \mathcal{G}} \varphi(F, G) \geq \rho(d, d-k-1)$. Thus,

$$\sum_{\substack{F \in \mathcal{F} \\ G \in \mathcal{G}}} \varphi(F, G) \geq \rho(d, d-k-1)|\mathcal{F}| = \rho(d, d-k-1) f_{d-1}(P).$$

It follows that

$$f_k(P) \geq \rho(d, d-k-1) f_{d-1}(P).$$

This inequality is guaranteed to be strict if $k < d-2$.

It remains to determine when equality is achieved for $k \geq d-2$. We can observe that $\rho(d, 0) = 1$ and $\rho(d, 1) = d/2$. Thus, equality is trivial for $k = d-1$. If $k = d-2$, then equality holds if and only if $f_{d-2}(P)/f_{d-1}(P) = d/2$; equivalently, if and only if each facet of P contains exactly d ridges of P. This occurs precisely when P is simplicial.

By duality, for all $0 \leq k \leq d-1$,

$$f_k(P) \geq \rho(d, k) f_0(P).$$

Equality holds precisely when $k = 0$ or when $k = 1$ and P is simple. □

Note that if $k \leq \lfloor d/2 \rfloor$, then $\rho(d, k) \geq 1$, while if $k \geq \lceil d/2 \rceil - 1$, then $\rho(d, d-k-1) \geq 1$. We can therefore answer Bárány's question:

Corollary 1.2.3. *All d-polytopes satisfy $f_0, \ldots, f_{\lfloor d/2 \rfloor} \geq f_0$ and $f_{\lceil d/2 \rceil - 1}, \ldots, f_{d-1} \geq f_{d-1}$. Consequently,*

$$f_0, \ldots, f_{d-1} \geq \min\{f_0, f_{d-1}\}.$$

1.3 Proof of tightness

In this section, we will prove that the linear bounds in Theorem 1.2.2 are tight. That is, for any fixed dimension d and $k = 0, \ldots, d-1$, Theorem 1.2.2 gives the greatest constant lower bounds on $f_k(P)/f_0(P), f_k(P)/f_{d-1}(P)$ for d-polytopes P. We will prove this using the cyclic polytope $C(d, n)$; specifically, we will show that as n grows arbitrarily large, $f_k(C(d, n))/f_{d-1}(C(d, n))$ asymptotically approaches $\rho(d, d-k-1)$.

Definition 1.3.1. Let $n > d \geq 2$, and let $\gamma : \mathbb{R} \to \mathbb{R}^d$ be the moment curve, defined as

$$\gamma(t) = (t, t^2, \dots, t^d).$$

The *cyclic polytope* $C(d, n) \subset \mathbb{R}^d$ is the convex hull of n arbitrary, distinct points in the image of γ.

Definition 1.3.2. Let P be a simplicial d-polytope with vertex set S. We say P is *neighborly* if for each subset $U \subset S$ with $|U| \leq \lfloor d/2 \rfloor$, there exists a face of P with vertex set U.

The cyclic polytope is simplicial and neighborly, and its facets are determined by the *Gale evenness condition* (see Gale [26]). As a result, the combinatorial type of $C(d, n)$ does not depend on our choice of points on the moment curve.

Since $C(d, n)$ is neighborly, we know that for $k = 0, \dots, \lfloor d/2 \rfloor$,

$$f_k(C(d, n)) = \binom{n}{k+1}.$$

All remaining face numbers of $C(d, n)$ are determined by the Dehn–Sommerville equations. Billera and Björner provide the explicit formula in [5].

Theorem 1.3.3 (Billera and Björner)**.** *Let $C(d, n)$ be the cyclic d-polytope on n vertices. Then for $k = 0, \dots, d-1$,*

$$f_k(C(d, n)) = \frac{n - \delta(n-k-2)}{n-k-1} \sum_{j=0}^{\lfloor d/2 \rfloor} \binom{n-1-j}{k+1-j} \binom{n-k-1}{2j-k-1+\delta},$$

where $\delta = \lceil d/2 \rceil - \lfloor d/2 \rfloor$. In particular,

$$f_{d-1}(C(d, n)) = \binom{n - \lfloor \frac{d+1}{2} \rfloor}{n-d} + \binom{n - \lfloor \frac{d+2}{2} \rfloor}{n-d}.$$

For convenience, we will write $f_k(C(d, n))$ to denote the value given by Theorem 1.3.3 even when $d < 2$ or $k > d - 1$.

Lemma 1.3.4. *For all $n > d \geq 2$ and $0 \leq k \leq d-1$,*

$$f_k(C(d, n)) = \rho(d, d-k-1) f_{d-1}(C(d, n)) + f_k(C(d-2, n)).$$

Proof. Let $n > d \geq 2$ and $0 \leq k \leq d-1$. If $k < \lfloor d/2 \rfloor - 1$, then $\rho(d, d-k-1) = 0$ and $f_k(C(d,n)) = f_k(C(d-2,n)) = \binom{n}{k+1}$, so we are done.

Suppose $k \geq \lfloor d/2 \rfloor - 1$. By Theorem 1.3.3,

$$\begin{aligned}
f_k(C(d,n)) - f_k(C(d-2,n)) &= \frac{n - \delta(n-k-2)}{n-k-1}\binom{n - \lfloor \frac{d}{2} \rfloor - 1}{n-k-2}\binom{n-k-1}{n-d} \\
&= \frac{n - \delta(n-k-2)}{n - \lfloor \frac{d}{2} \rfloor}\binom{n - \lfloor \frac{d}{2} \rfloor}{n-k-1}\binom{n-k-1}{n-d} \\
&= \frac{n - \delta(n-k-2)}{n - \lfloor \frac{d}{2} \rfloor}\binom{n - \lfloor \frac{d}{2} \rfloor}{n-d}\binom{\lceil \frac{d}{2} \rceil}{d-k-1} \\
&= \frac{n - \delta(n-k-2)}{\lceil \frac{d}{2} \rceil}\binom{n - \lfloor \frac{d}{2} \rfloor - 1}{n-d}\binom{\lceil \frac{d}{2} \rceil}{d-k-1}.
\end{aligned}$$

If d is even, then

$$\begin{aligned}
\rho(d, d-k-1) &= \binom{\frac{d}{2}}{d-k-1}, \\
f_{d-1}(C(d,n)) &= \frac{2n}{d}\binom{n - \frac{d}{2} - 1}{n-d}.
\end{aligned}$$

If d is odd, then

$$\begin{aligned}
\rho(d, d-k-1) &= \frac{k+2}{d+1}\binom{\frac{d+1}{2}}{d-k-1}, \\
f_{d-1}(C(d,n)) &= 2\binom{n - \frac{d-1}{2} - 1}{n-d}.
\end{aligned}$$

Thus, in either case,

$$f_k(C(d,n)) - f_k(C(d-2,n)) = \rho(d, d-k-1) f_{d-1}(C(d,n)).$$

$\square$

Using Lemma 1.3.4, we can prove our desired tightness result.

Theorem 1.3.5. *Let d be a positive integer and $0 \leq k \leq d-1$. There exist infinite families of d-polytopes $\{P_n\}, \{Q_n\}$ such that for all $\varepsilon > 0$ and n sufficiently large,*

$$\frac{f_k(P_n)}{f_0(P_n)} < \rho(d,k) + \varepsilon, \qquad \frac{f_k(Q_n)}{f_{d-1}(Q_n)} < \rho(d, d-k-1) + \varepsilon.$$

Proof. The proof is trivial for $d = 1$. Fix $d \geq 2$ and $0 \leq k \leq d-1$, and observe that for all $n > d$,

$$f_{d-1}(C(d,n)) > \frac{(n-d)^{\lceil (d-1)/2 \rceil}}{\lceil \frac{d-1}{2} \rceil !},$$

$$f_k(C(d-2,n)) < \frac{k+2}{n-k-1} \left\lfloor \frac{d-2}{2} \right\rfloor n^{\lceil (d-1)/2 \rceil}.$$

Thus,

$$\lim_{n\to\infty} \frac{f_k(C(d-2,n))}{f_{d-1}(C(d,n))} = 0.$$

By Lemma 1.3.4, it follows that

$$\lim_{n\to\infty} \frac{f_k(C(d,n))}{f_{d-1}(C(d,n))} = \rho(d, d-k-1).$$

Accordingly, if $C(d,n)^*$ is the polar dual of $C(d,n)$, then

$$\lim_{n\to\infty} \frac{f_k(C(d,n)^*)}{f_0(C(d,n)^*)} = \rho(d,k).$$

□

1.4 The search for stronger bounds

While Theorem 1.2.2 gives the tightest linear bounds on f_k in terms of f_0 or f_{d-1}, there is still room for improvement via nonlinear bounds. In this section, we will use Grassmann angles to prove a stronger version of Theorem 1.2.2 (§1.4.1). We will then prove a connected result relating the face numbers of P to a projection in codimension two (§1.4.2).

1.4.1 Grassmann angles and nonlinear bounds

We can view solid angles as probabilities. Let $P \subset \mathbb{R}^d$ be a d-polytope and G a face of P. Then $2\varphi(P,G)$ is the probability that a random line through the relative interior of G intersects the interior of P. Equivalently, if $H \subset \mathbb{R}^d$ is a random codimension one hyperplane and $\pi : \mathbb{R}^d \to H$ the orthogonal projection map, then $1 - 2\varphi(P,G)$ is the probability that $\pi(G)$ is a proper face of $\pi(P)$. Summing over all k-dimensional faces yields the following identity, previously given by Feldman and Klain [23, (5)].

Theorem 1.4.1 (Feldman and Klain)**.** *Let $P \subset \mathbb{R}^d$ be a d-polytope and $0 \le k \le d-1$. Suppose $H \subset \mathbb{R}^d$ is a codimension one hyperplane chosen uniformly at random, and let $\pi : \mathbb{R}^d \to H$ be the orthogonal projection map. Then*

$$f_k(P) - 2\varphi_k(P) = \mathbb{E} f_k(\pi(P)),$$

where $\mathbb{E}$ denotes expected value.

If we replace our random line or hyperplane with a subspace of another dimension, we arrive at a more general notion of angles, first introduced by Grünbaum in 1968 [28].

Definition 1.4.2. Let P be a d-polytope, G a face of P, z a point in the relative interior of G, and $1 \le m \le d-1$. The *Grassmann angle* $\gamma^m(P, G)$ is the probability that $(S+z) \cap P = \{z\}$ for an m-dimensional linear subspace $S \subset \mathbb{R}^d$ chosen uniformly at random. For $k = 0, \dots, d-1$, if $\mathcal{G}_k$ is the set of k-dimensional faces of P, we define

$$\gamma_k^m(P) := \sum_{G \in \mathcal{G}_k} \gamma^m(P, G).$$

Note that $\gamma^m(P, G)$ is independent of our choice of z, and $\gamma^m(P, G) = 0$ if $\dim G > d - m - 1$.

Like solid angles, Grassmann angles can be understood in terms of projections. Let P be a d-polytope and $1 \le m \le d-1$. Let $H \subset \mathbb{R}^d$ be a codimension m linear subspace chosen uniformly at random, and let $\pi : \mathbb{R}^d \to H$ be the orthogonal projection map. Then for each face G of P, $\pi(G)$ is a proper face of $\pi(P)$ with probability $\gamma^m(P, G)$. In particular, $\gamma^1(P, G)$ is $1 - 2\varphi(P, G)$; see Theorem 1.4.1 and our surrounding discussion.

More generally, linearity of expectation gives us the following analogue of Theorem 1.4.1 for Grassmann angles.

Lemma 1.4.3. *Let $P \subset \mathbb{R}^d$ be a d-polytope, $1 \le m \le d-1$, and $0 \le k \le d-1$. Let $H \subset \mathbb{R}^d$ be a codimension m linear subspace chosen uniformly at random, and let $\pi : \mathbb{R}^d \to H$ be the orthogonal projection map. Then*

$$\gamma_k^m(P) = \mathbb{E} f_k(\pi(P)).$$

Of particular interest is the Grassmann angle $\gamma^2(P,G)$. The following result of Grünbaum [28] identifies $\gamma^2(P,G)$ with the *angle deficiency* of P at G.

Theorem 1.4.4 (Grünbaum). *Let P be a d-polytope with $d \geq 3$, and let G be a face of P. Let $\mathcal{F}_G$ be the set of facets of P containing G. Then*

$$\gamma^2(P,G) = 1 - \sum_{F \in \mathcal{F}_G} \varphi(F,G).$$

Following our proof of Theorem 1.2.2 and using the identity above, we can prove a stronger bound on f_k in terms of both f_{d-1} and γ_k^2.

Proposition 1.4.5. *For all d-polytopes P and $0 \leq k \leq d-1$,*

$$f_k(P) \geq \rho(d, d-k-1) f_{d-1}(P) + \gamma_k^2(P).$$

Proof. Let P be a d-polytope and $0 \leq k \leq d-1$. Let $\mathcal{F}$ be the set of facets of P and $\mathcal{G}$ the set of k-dimensional faces. Summing Theorem 1.4.4 over $\mathcal{G}$, we find

$$\gamma_k^2(P) = f_k(P) - \sum_{\substack{F \in \mathcal{F} \\ G \in \mathcal{G}}} \varphi(F,G) = f_k(P) - \sum_{F \in \mathcal{F}} \varphi_k(F).$$

By Proposition 1.2.1, it follows that

$$\gamma_k^2(P) \leq f_k(P) - \rho(d, d-k-1) f_{d-1}(P). \qquad \square$$

By Lemma 1.4.3, we know $\gamma_k^2(P) \geq f_k(\Delta^{d-2}) = \binom{d-1}{k+1}$. Combining this with Proposition 1.4.5 yields the following bounds on $f_k(P)$ in terms of $f_0(P)$ and $f_{d-1}(P)$, a slight improvement on Theorem 1.2.2.

Corollary 1.4.6. *Let P be a d-polytope and $k = 0, \dots, d-1$. Then*

$$f_k \geq \rho(d,k) f_0 + \binom{d-1}{k-1},$$

$$f_k \geq \rho(d, d-k-1) f_{d-1} + \binom{d-1}{k+1}.$$

1.4.2 Face number relations from a fixed projection

There is a deeper, combinatorial identity hidden behind Proposition 1.4.5. This identity describes the face numbers of a polytope P by its behavior under an arbitrary orthogonal projection of codimension two. We will require a notion of general position for subspaces, but it can be slightly weaker than our previous notion for vectors.

Definition 1.4.7. Let P be a d-polytope and S an m-dimensional subspace of $\mathbb{R}^d$. We say S is in *general position with respect to* P if for all $0 \leq k \leq d-1$, all k-dimensional faces G of P, and all points z in the relative interior of G,

$$\dim((S+z) \cap G) = \max\{0, m+k-d\},$$

where $S+z$ is the translation of S by the vector z.

Proposition 1.4.8. *Let P be a d-polytope with $d \geq 3$, $S \subset \mathbb{R}^d$ a two-dimensional subspace in general position with respect to P, and $\pi : \mathbb{R}^d \to S^\perp$ the orthogonal projection map. Let $\mathcal{F}$ be the set of facets of P. Then for all $0 \leq k \leq d-3$,*

$$f_k(P) - f_k(\pi(P)) = \frac{1}{2} \sum_{F \in \mathcal{F}} \left[f_k(F) - f_k(\pi(F)) \right].$$

Proof. Let $\mathcal{G}$ be the set of k-dimensional faces G of P such that $\pi(G)$ is not a face of $\pi(P)$. Let $\mathcal{E}$ be the set of pairs (F, G) such that F is a facet of P, G is a k-dimensional face of F, and $\pi(G)$ is not a face of $\pi(F)$. Then

$$\begin{aligned} |\mathcal{G}| &= f_k(P) - f_k(\pi(P)), \\ |\mathcal{E}| &= \sum_{F \in \mathcal{F}} \left[f_k(F) - f_k(\pi(F)) \right]. \end{aligned}$$

Suppose $G \in \mathcal{G}$, and let z be an arbitrary point in the relative interior of G. Since S is in general position with respect to P, each facet F of P containing G has $\dim((S+z) \cap F) \leq 1$. If $(F, G) \in \mathcal{E}$, then $\dim((S+z) \cap F) = 1$; if not, then $(S+z) \cap F = \{z\}$. Meanwhile, for all faces K of P containing G which are not facets, $(S+z) \cap K = \{z\}$.

Since $\pi(G)$ is not a face of $\pi(P)$, $S+z$ must intersect the relative interior of P. Thus, $(S+z)\cap P$ is a polygon. For all facets F of P containing G, $\dim((S+z)\cap F)=1$ if and only if $(S+z)\cap F$ is an edge of $(S+z)\cap P$ containing z. Thus, there are exactly two facets F, F' of P with $(F,G),(F',G)\in\mathcal{E}$.

Now consider an arbitrary $(F,G)\in\mathcal{E}$. We know $\dim\pi(F)=\dim\pi(P)=d-2$, so $\pi(F)$ cannot be a proper face of $\pi(P)$. Thus, $\mathrm{relint}(\pi(G))\subset\mathrm{relint}(\pi(F))\subseteq\mathrm{relint}(\pi(P))$. It follows that $G\in\mathcal{G}$.

We have shown that $(F,G)\mapsto G$ is a well-defined map from $\mathcal{E}$ to $\mathcal{G}$, under which the preimage of each $G\in\mathcal{G}$ has size exactly two. Thus,

$$f_k(P)-f_k(\pi(P))=\frac{1}{2}\sum_{F\in\mathcal{F}}\left[f_k(F)-f_k(\pi(F))\right]. \qquad \square$$

Let P be a polytope and F a facet of P. Let S,π be as defined in Proposition 1.4.8, and let T be the image of the orthogonal projection of $S^\perp$ onto $\mathrm{aff}(F)$, so T has codimension one in $\mathrm{aff}(F)$. Then $\pi(F)$ is combinatorially equivalent to the orthogonal projection of F onto T. If S is chosen uniformly at random, then the resulting $T\subset\mathrm{aff}(F)$ obeys a uniform distribution as well. It follows by Theorem 1.4.1 that

$$\mathbb{E}\left[f_k(F)-f_k(\pi(F))\right]=2\varphi_k(F).$$

Furthermore, by Lemma 1.4.3,

$$\mathbb{E}f_k(\pi(P))=\gamma_k^2(P).$$

Thus, we can arrive at Proposition 1.4.5 by applying linearity of expectation to Proposition 1.4.8. The argument goes:

$$\begin{aligned}
f_k(P)-f_k(\pi(P)) &= \frac{1}{2}\sum_{F\in\mathcal{F}}\left[f_k(F)-f_k(\pi(F))\right]\\
\Rightarrow f_k(P)-\mathbb{E}f_k(\pi(P)) &= \frac{1}{2}\sum_{F\in\mathcal{F}}\mathbb{E}\left[f_k(F)-f_k(\pi(F))\right]\\
\Rightarrow \qquad f_k(P)-\gamma_k^2(P) &= \sum_{F\in\mathcal{F}}\varphi_k(F)\\
\Rightarrow \qquad f_k(P)-\gamma_k^2(P) &\le \rho(d,d-k-1)f_{d-1}(P).
\end{aligned}$$

Remark 1.4.9. Proposition 1.4.8 illustrates a beautiful property of the cyclic polytope $C(d,n)$. Let $\pi : \mathbb{R}^d \to \mathbb{R}^{d-2}$ be the projection onto the first $d-2$ coordinates, so $\pi(C(d,n)) = C(d-2,n)$. Then by Proposition 1.4.8 and Lemma 1.3.4,

$$\sum_{F\in\mathcal{F}} [f_k(F) - f_k(\pi(F))] = 2\rho(d, d-k-1) f_{d-1}(C(d,n)),$$

where $\mathcal{F}$ is the set of facets of $C(d,n)$. Thus, all facets $F \in \mathcal{F}$ simultaneously attain

$$f_k(F) - f_k(\pi(F)) = 2\rho(d, d-k-1),$$

the minimum possible value for any $(d-1)$-polytope F and orthogonal projection π onto a codimension one subspace in general position. This property helps explain why cyclic polytopes have among the lowest face numbers for polytopes with a fixed number of facets, and optimistically, it may be a clue toward proving the generalized upper bound conjecture.

1.5 Face numbers of shellable spheres and balls

As some results about face numbers of polytopes continue to hold for larger classes of CW complexes, it is natural to ask a generalized version of Bárány's question: does every strongly regular CW $(d-1)$-sphere satisfy $f_0, \dots, f_{d-1} \geq \min\{f_0, f_{d-1}\}$? Furthermore, does it satisfy the inequalities of Theorem 1.2.2? In this section, we partially answer these questions.

Our main result on CW complexes is a generalization of Theorem 1.2.2: if X is a shellable, strongly regular CW sphere or ball of dimension $d-1$, then for all $k = \lfloor d/2 \rfloor - 1, \dots, d-1$,

$$f_k(X) \geq \rho(d, d-k-1) f_{d-1}(X) + \tfrac{1}{2} f_k(\partial X), \tag{1.3}$$

with equality precisely when $k = d-1$ or when $k = d-2$ and X is simplicial. Inequality (1.3) implies the following: if S is a strongly regular CW $(d-1)$-sphere whose face poset is both CL-shellable and dual CL-shellable, then for all $k = 0, \dots, d-1$,

$$f_k(S) \geq \rho(d,k) f_0(S), \qquad f_k(S) \geq \rho(d, d-k-1) f_{d-1}(S).$$

Consequently, $f_k(S) \geq \min\{f_0(S), f_d(S)\}$.

Björner and Wachs proved that a graded poset is CL-shellable if and only if it admits a recursive atom ordering [13]. Thus, when we require the face poset of S to be both CL-shellable and dual CL-shellable, it is equivalent to saying that the face poset admits both a recursive atom ordering and a recursive coatom ordering.

We would like to prove (1.3) in the same way as Theorem 1.2.2—but since CW spheres do not have solid angles or nice orthogonal projections, we will need to modify our approach. Let S be a shellable, strongly regular CW $(d-2)$-sphere and $(F_1, \dots, F_n)$ a shelling of its $(d-2)$-faces. In place of an "upper" and "lower" subcomplex, we will consider a "beginning" subcomplex $\mathcal{C} = \langle F_1, \dots, F_j \rangle$ and an "ending" subcomplex $\mathcal{D} = \langle F_{j+1}, \dots, F_n \rangle$ for some $1 \leq j \leq n$. This approach is motivated by the line shellings of polytopes constructed by Bruggesser and Mani in 1971 [16]; if S is the boundary complex of a d-polytope, then the "upper" and "lower" subcomplexes induced by an orthogonal projection coincide with "beginning" and "ending" subcomplexes induced by some line shelling. We will mimic the proof of Theorem 1.2.2, but in place of Proposition 1.2.1 about solid angles, we will show that

$$f_k(\operatorname{int} \mathcal{C}) + f_k(\operatorname{int} \mathcal{D}) \geq 2\rho(d, d-k-1). \tag{1.4}$$

Ultimately, we will prove inequality (1.3) by summing (1.4) over all d-faces of a CW d-sphere or CW d-ball X.

1.5.1 CW complexes

In this and the following subsection, we will introduce the basic notions of regular, strongly regular, and shellable CW complexes. We will then discuss face posets, the combinatorial tools for describing a CW complex's structure. We refer the reader to Björner [9, 12, 13] for any undefined terminology.

Definition 1.5.1. A *CW complex* of dimension $d-1 \geq 0$, or *CW* $(d-1)$*-complex*, is a set X of open balls of dimension at most $d-1$, together with a topological space $|X|$ and a set of attaching maps defined recursively as follows.

- If $d = 1$, then X is a finite set of points, and $|X|$ is the set X under the discrete topology.

- If $d > 1$, we construct X from a CW $(d-2)$-complex X'. We let $X = X' \cup \{F_1, \dots, F_n\}$, where $n \geq 0$ and $F_1, \dots, F_n$ are open $(d-1)$-balls. We then build the topological space $|X|$ by attaching $F_1, \dots, F_n$ to $|X'|$ with attaching maps $\partial F_i \to |X'|$, $i = 1, \dots, n$.

Definition 1.5.2. If X is a CW $(d-1)$-complex, we call the elements of $X \cup \{\emptyset\}$ the *faces* of X. For $k = 0, \dots, d-1$, we define $f_k(X)$ as the number of k-dimensional faces of X; similarly, if W is a subset of X, we define $f_k(W)$ as the number of k-dimensional faces belonging to W.

Definition 1.5.3. Let X be a CW complex. For any face $G \in X$, we define the *geometric realization* $|G|$ as the closure of the image of G in $|X|$ under the inclusion map. For a set of faces $W \subseteq X$, we define

$$|W| := \bigcup_{G \in W} |G|.$$

For faces $G, G' \in X$, we will often say "G is contained in G'" or "G' contains G" as shorthand to mean $|G| \subseteq |G'|$.

Definition 1.5.4. A CW $(d-1)$-complex is *pure* if every face is contained in a $(d-1)$-face.

Definition 1.5.5. Let X be a pure, strongly regular CW $(d-1)$-complex. If every $(d-2)$-face belongs to at most two $(d-1)$-faces, we call X a *pseudomanifold with boundary.* If every $(d-2)$-face belongs to exactly two $(d-1)$-faces, we call X a *pseudomanifold.*

Definition 1.5.6. A *CW $(d-1)$-sphere* (resp. *$(d-1)$-ball*) is a strongly regular CW $(d-1)$-complex X such that $|X| \cong \mathbb{S}^{d-1}$ (resp. $\mathbb{B}^{d-1}$).

Note that any CW sphere or ball is a pseudomanifold with boundary.

The following two definitions deal with certain subsets of CW complexes.

Definition 1.5.7. Let X be a CW complex and $W \subseteq X$ a set of faces. The *closure* of W, denoted $\langle W \rangle$, is the CW complex consisting of all faces of X which are contained in some member of W.

Definition 1.5.8. Let X be a $(d-1)$-dimensional pseudomanifold with boundary. The *boundary complex* ∂X is the closure of the set of $(d-2)$-faces of X contained in exactly one $(d-1)$-face. The *interior* $\operatorname{int} X$ is the set of faces of X not contained in ∂X.

Note that $\operatorname{int} X$ is not, in general, a CW complex.

If G is a face of some strongly regular CW complex, we will often write ∂G as shorthand for $\partial\langle G\rangle$. Henceforth, this is what ∂G will mean unless stated otherwise.

We are ready to define a shelling of a CW complex. Our definition is taken from [8].

Definition 1.5.9. Let X be a pure, regular CW $(d-1)$-complex with $(d-1)$-faces $F_1, \dots, F_n$. We call $(F_1, \dots, F_n)$ a *shelling* of X if either $d = 1$, or $d > 1$ and

- ∂F_1 admits a shelling.
- For all $1 < j \leq n$, $\partial F_j \cap \bigcup_{i=1}^{j-1} \partial F_i$ is a pure $(d-2)$-complex, and there exists a shelling of ∂F_j beginning with the $(d-2)$-faces of $\partial F_j \cap \bigcup_{i=1}^{j-1} \partial F_i$.

A pure, regular CW complex is *shellable* if it admits a shelling.

1.5.2 Face posets and shellings

When working with regular CW complexes, it is often useful to consider the face poset. The face poset describes a CW complex's combinatorial structure; that is, which faces are contained in which. This section will introduce some basic properties of these posets.

Of particular interest are diamond lattices, a special type of lattices which include the face posets of CW spheres.

Definition 1.5.10. Let $\mathcal{L}$ be a finite, graded poset. We say $\mathcal{L}$ is a *lattice* if any two elements of $\mathcal{L}$ have a unique least upper bound and a unique greatest lower bound. We say $\mathcal{L}$ is a *diamond lattice* if, in addition, each interval of $\mathcal{L}$ with length two has size exactly four.

Some sources [8] also refer to a diamond lattice as a "thin" lattice.

The following is a direct consequence of Proposition 3.1 in [58].

Lemma 1.5.11. *Let $\mathcal{L}$ be a diamond lattice of rank at least two. For every coatom F of $\mathcal{L}$, there exists an atom v of $\mathcal{L}$ such that $v \nleq F$.*

Lemma 1.5.12 (Xue [58, Proposition 3.2])**.** *Let $\mathcal{L} = [\hat{0}, \hat{1}]$ be a diamond lattice of rank d and G an element of rank $r+1$. Then for all $r \leq k \leq d-1$, the upper interval $[G, \hat{1}]$ has at least $\binom{d-r-1}{d-k-1}$ many elements of rank $k+1$.*

Definition 1.5.13. Let X be a CW complex. The *face poset* $\mathcal{F}(X)$ is the set of faces of X ordered by containment, with a unique minimal element $\hat{0}$ representing the empty face, and with a unique maximal element $\hat{1}$ added.

Note that for all pseudomanifolds X, including all spheres, $\mathcal{F}(X)$ is a diamond lattice.

The next three theorems, due to Björner, show that all shelling information about a regular CW complex is encoded in its face poset. These theorems involve *CL-shellability*, a version of shellability for lattices. The precise definition of a CL-shellable lattice will never be used and is hence omitted, but can be found in [9].

A lattice is *dual CL-shellable* if its dual lattice is CL-shellable.

Theorem 1.5.14 (Björner [8, Proposition 4.2])**.** *Let X be a pure, regular CW complex. Then X is shellable if and only if $\mathcal{F}(X)$ is dual CL-shellable.*

Theorem 1.5.15 (Björner [8, Proposition 4.3])**.** *Let X be a pure, strongly regular CW $(d-1)$-complex with a shelling $(F_1, \dots, F_n)$. For all $1 < j \leq n$, $\partial F_j \cap \bigcup_{i=1}^{j-1} \partial F_i$ is a PL-ball or PL-sphere of dimension $d-2$. Furthermore, X is a CW ball if and only if X is a pseudomanifold and each $\partial F_j \cap \bigcup_{i=1}^{j-1} \partial F_i$ is a PL-ball.*

Theorem 1.5.16 (Björner [8, Proposition 4.5])**.** *Let $\mathcal{L}$ be a finite, graded poset of rank $d+1$ with a unique least element $\hat{0}$ and greatest element $\hat{1}$. The following are equivalent:*

- *$\mathcal{L}$ is a dual CL-shellable diamond lattice.*

- *$\mathcal{L} \cong \mathcal{F}(X)$, X a shellable $(d-1)$-pseudomanifold.*

- $\mathcal{L} \cong \mathcal{F}(X)$, *$X$ a shellable CW $(d-1)$-sphere.*

Björner's arguments for Theorem 1.5.14 imply a stronger statement: if X is a pure, regular CW $(d-1)$-complex with $(d-1)$-faces $F_1, \dots, F_n$, then $\mathcal{F}(X)$ fully determines whether $(F_1, \dots, F_n)$ is a shelling. Thus, when discussing shellings of a CW complex X, we need never worry about the precise attaching maps used to construct $|X|$.

1.5.3 Main results on CW complexes

This section is dedicated to our main results on face numbers of CW complexes. We will begin with a topological lemma, used to relate shellings of a CW $(d-1)$-complex with shellings of an individual $(d-1)$-face's boundary (Lemma 1.5.17). Next, we will prove a $(d-1)$-sphere analogue of Proposition 1.2.1 by induction on d (Lemmas 1.5.18, 1.5.19). With these pieces in place, we will prove our lower bound on face numbers for shellable, strongly regular CW spheres and CW balls (Theorem 1.5.20). Finally, we will discuss conditions under which a CW $(d-1)$-sphere S must satisfy Bárány's property:

$$f_0, \dots, f_{d-1} \geq \min\{f_0, f_{d-1}\}.$$

Lemma 1.5.17. *Let X be a strongly regular CW $(d-1)$-ball with a shelling $(F_1, \dots, F_n)$, where $n > 1$. Let $\mathcal{C} = \partial F_n \cap \bigcup_{i=1}^{n-1} \partial F_i$. Then $\mathcal{C}$ is a CW $(d-2)$-ball and $\operatorname{int} \mathcal{C} \subseteq \operatorname{int} X$.*

Proof. By Theorem 1.5.15, $\langle F_i \rangle_{i=1}^{n-1}$ is a CW $(d-1)$-ball and $\mathcal{C}$ is a CW $(d-2)$-ball. Let

$$A = \bigcup_{i=1}^{n-1} |F_i| \cong \mathbb{B}^{d-1} \quad \text{and} \quad B = |F_n| \cong \mathbb{B}^{d-1}.$$

Then

$$A \cup B = |X| \cong \mathbb{B}^{d-1} \quad \text{and} \quad A \cap B = |\mathcal{C}| \cong \mathbb{B}^{d-2}.$$

Consider a point $x \in |\mathcal{C}| \backslash |\partial \mathcal{C}|$. By Mayer–Vietoris, the following is an exact sequence:

$$\tilde{H}_{d-1}(A \backslash x) \oplus \tilde{H}_{d-1}(B \backslash x) \to \tilde{H}_{d-1}(|X| \backslash x) \to \tilde{H}_{d-2}(|\mathcal{C}| \backslash x) \to \tilde{H}_{d-2}(A \backslash x) \oplus \tilde{H}_{d-2}(B \backslash x).$$

We know $x \in \partial A, \partial B$, so $A \backslash x \simeq B \backslash x \simeq \mathbb{B}^{d-1}$ (where $\simeq$ denotes homotopy equivalence). Thus, our exact sequence becomes

$$0 \longrightarrow \tilde{H}_{d-1}(|X| \backslash x) \longrightarrow \tilde{H}_{d-2}(|\mathcal{C}| \backslash x) \longrightarrow 0.$$

It follows that $\tilde{H}_{d-1}(|X| \backslash x) \cong \tilde{H}_{d-2}(|\mathcal{C}| \backslash x) \cong \mathbb{Z}$. Hence, $x \notin \partial |X| = |\partial X|$. We may conclude that $|\mathcal{C}| \backslash |\partial \mathcal{C}|$ and $|\partial X|$ are disjoint.

Let $G \in \operatorname{int} \mathcal{C}$. Then

$$|G| \cap (|\mathcal{C}| \backslash |\partial \mathcal{C}|) \neq \emptyset,$$

so $|G| \nsubseteq |\partial X|$. Thus, $G \in \operatorname{int} X$. This completes our proof that $\operatorname{int} \mathcal{C} \subseteq \operatorname{int} X$. □

Lemma 1.5.18. *Let S be a strongly regular CW $(d-1)$-sphere with a shelling $(F_1, \dots, F_n)$. Let $1 \leq j < n$, let $\mathcal{C} = \langle F_i \rangle_{i=1}^{j}$, and let $\mathcal{D} = \langle F_i \rangle_{i=j+1}^{n}$. Then there exist faces $C \in \operatorname{int} \mathcal{C}$ and $D \in \operatorname{int} \mathcal{D}$ such that $\dim C + \dim D \leq d - 1$.*

Proof. We will use induction on d. For our base case, suppose $d = 1$. Then $n = 2$ and $j = 1$, so $\mathcal{C} = \{C\}$ and $\mathcal{D} = \{D\}$ for the two vertices C, D of S. We can see that $\dim C + \dim D = d - 1 = 0$.

For our inductive step, fix $d > 1$ and suppose the statement holds for all strongly regular $(d-2)$-spheres. Let S be a strongly regular $(d-1)$-sphere with a shelling $(F_1, \dots, F_n)$. Let $1 \leq j < n$, let $\mathcal{C} = \langle F_i \rangle_{i=1}^{j}$, and let $\mathcal{D} = \langle F_i \rangle_{i=j+1}^{n}$. Since $\mathcal{C}, \mathcal{D}$ are pure, full-dimensional subcomplexes of a strongly regular CW sphere, $\mathcal{C}, \mathcal{D}$ must be pseudomanifolds with boundary, so $\partial \mathcal{C}, \partial \mathcal{D}$ and $\operatorname{int} \mathcal{C}, \operatorname{int} \mathcal{D}$ are well-defined. Furthermore, $\operatorname{int} \mathcal{C} = S \backslash \mathcal{D}$ and $\operatorname{int} \mathcal{D} = S \backslash \mathcal{C}$.

If $j = 1$, then $\operatorname{int} \mathcal{D}$ must include at least one vertex by Lemma 1.5.11. Let $C = F_1$ and let D be a vertex in $\operatorname{int} \mathcal{D}$. Then $\dim C + \dim D = d - 1$, as desired.

Now, suppose $j > 1$. Let Q be the set of $(d-2)$-faces of ∂F_j, and define subcomplexes $\mathcal{C}', \mathcal{D}'$ of ∂F_j as follows:

$$\mathcal{C}' = \langle R \in Q \mid R \in \partial F_1, \dots, \text{or } \partial F_{j-1} \rangle,$$

$$\mathcal{D}' = \langle R \in Q \mid R \in \partial F_{j+1}, \dots, \text{or } \partial F_n \rangle.$$

Since $F_1, \ldots, F_n$ is a shelling of S, there exists a shelling of ∂F_j beginning with the $(d-2)$-faces of $\mathcal{C}'$. Thus, by our inductive hypothesis, there exist faces $C \in \operatorname{int} \mathcal{C}'$ and $D' \in \operatorname{int} \mathcal{D}'$ such that $\dim C + \dim D' \leq d-2$.

By Theorem 1.5.15, $\mathcal{C}$ is a CW $(d-1)$-ball and $\partial F_j \cap \bigcup_{i=1}^{j-1} \partial F_i \supseteq \mathcal{C}'$ is a CW $(d-2)$-ball. Thus, $\mathcal{C}' = \partial F_j \cap \bigcup_{i=1}^{j-1} \partial F_i$. It follows by Lemma 1.5.17 that $C \in \operatorname{int} \mathcal{C}$.

We can see that $\mathcal{C}' \cap \mathcal{D}' = \partial \mathcal{C}' = \partial \mathcal{D}'$, so $D' \notin \mathcal{C}'$; thus, F_j is the only $(d-1)$-face of $\mathcal{C}$ containing D'. Consider the upper interval $[D', \hat{1}]$ in $\mathcal{F}(S)$. Since $\mathcal{F}(S)$ is a diamond lattice (Theorem 1.5.16), $[D', \hat{1}]$ must be a diamond lattice as well. Thus, by Lemma 1.5.11, there exists an atom D of $[D', \hat{1}]$ (i.e. a face D containing D' with $\dim D = \dim D' + 1$) such that $D \notin \langle F_j \rangle$. This implies that $D \notin \mathcal{C}$, so $D \in \operatorname{int} \mathcal{D}$.

Since $\dim C + \dim D' \leq d-2$, we know $\dim C + \dim D \leq d-1$. This completes our inductive step. ☐

Lemma 1.5.19. *Let S be a strongly regular CW $(d-2)$-sphere with a shelling $(F_1, \ldots, F_n)$. Let $0 \leq j \leq n$, let $\mathcal{C} = \langle F_i \rangle_{i=1}^{j}$, and let $\mathcal{D} = \langle F_i \rangle_{i=j+1}^{n}$. Then for $\lfloor d/2 \rfloor - 1 \leq k \leq d-2$,*

$$f_k(\operatorname{int} \mathcal{C}) + f_k(\operatorname{int} \mathcal{D}) \geq 2\rho(d, d-k-1).$$

Proof. First, suppose $j = 0$ or $j = n$. If $j = 0$, then $\mathcal{C} = \emptyset$ and $\mathcal{D} = S$; likewise, if $j = n$, then $\mathcal{C} = S$ and $\mathcal{D} = \emptyset$. Thus,

$$f_k(\operatorname{int} \mathcal{C}) + f_k(\operatorname{int} \mathcal{D}) = f_k(S).$$

By Lemma 1.5.12, we know

$$f_k(S) \geq \binom{\lceil \frac{d}{2} \rceil}{d-k-1} + \binom{\lfloor \frac{d}{2} \rfloor}{d-k-1},$$

so we are done.

Suppose instead that $0 < j < n$. Then by Lemma 1.5.18, there exist faces $C \in \operatorname{int} \mathcal{C}$ and $D \in \operatorname{int} \mathcal{D}$ such that $\dim C + \dim D \leq d-2$. We know C has rank $\dim C + 1$ in the face poset $L(S)$, so by Lemma 1.5.12, there are at least $\binom{d - \dim C - 1}{d-k-1}$ many k-faces of S containing

C. Thus,

$$f_k(\operatorname{int} \mathcal{C}) \geq \binom{d - \dim C - 1}{d-k-1}.$$

By a similar argument,

$$f_k(\operatorname{int} \mathcal{D}) \geq \binom{d - \dim D - 1}{d-k-1}.$$

We know that $(d - \dim C - 1) + (d - \dim D - 1) \geq d$. Hence, by Lemma 1.1.14,

$$f_k(\operatorname{int} \mathcal{C}) + f_k(\operatorname{int} \mathcal{D}) \geq \binom{\lceil \frac{d}{2} \rceil}{d-k-1} + \binom{\lfloor \frac{d}{2} \rfloor}{d-k-1}. \qquad \square$$

Theorem 1.5.20. *Let X be a shellable, strongly regular CW sphere or ball of dimension $d-1$. Then for all $\lfloor d/2 \rfloor - 1 \leq k \leq d-1$,*

$$f_k(X) \geq \rho(d, d-k-1) f_{d-1}(X) + \tfrac{1}{2} f_k(\partial X),$$

with equality precisely when $k = d-1$ or when $k = d-2$ and X is simplicial.

Proof. First, observe that $\rho(d, 0) = 1$ and $f_{d-1}(\partial X) = 0$, so the desired inequality holds for $k = d-1$. For the remainder of this proof, we will assume that $k < d-1$.

Let $n = f_{d-1}(X)$, and let $(F_1, \dots, F_n)$ be a shelling of X. Consider an arbitrary $1 \leq j \leq n$, and let Q be the set of $(d-2)$-faces of ∂F_j. Define pure subcomplexes $\mathcal{C}_j, \mathcal{D}_j$ of ∂F_j as follows:

$$\mathcal{C}_j = \langle R \in Q \mid R \in \partial F_1, \dots, \text{or } \partial F_{j-1} \rangle,$$
$$\mathcal{D}_j = \langle R \in Q \mid R \in \partial F_{j+1}, \dots, \partial F_n, \text{or } \partial X \rangle.$$

We can see that $\mathcal{C}_j \cup \mathcal{D}_j = \partial F_j$. Furthermore, since X is a pseudomanifold with boundary, the sets of $(d-2)$-faces of $\mathcal{C}_j$ and $\mathcal{D}_j$ are disjoint. In other words,

$$\mathcal{C}_j \cap \mathcal{D}_j = \partial \mathcal{C}_j = \partial \mathcal{D}_j.$$

If $j = 1$, then $\mathcal{C}_j = \emptyset = \partial F_j \cap \bigcup_{i=1}^{j-1} \partial F_{j-1}$. If $j > 1$, then by Theorem 1.5.15, $\partial F_j \cap \bigcup_{i=1}^{j-1} \partial F_{j-1}$ is a PL-ball or PL-sphere of dimension $d-2$. Since $\mathcal{C}_j$ contains all of the

$(d-2)$-faces of $\partial F_j \cap \bigcup_{i=1}^{j-1} \partial F_{j-1}$, it follows that $\mathcal{C}_j = \partial F_j \cap \bigcup_{i=1}^{j-1} \partial F_{j-1}$. Thus,

$$\mathcal{C}_j = \partial F_j \cap \bigcup_{i=1}^{j-1} \partial F_{j-1}, \qquad j = 1, \dots, n.$$

We now turn our attention to $\mathcal{D}_j$. By definition, if $j = 1$, then $\mathcal{D}_j = \partial F_j = \partial F_j \cap (\partial X \cup \bigcup_{i=j+1}^{n} \partial F_i)$. Likewise, if $j = n$, then $\mathcal{D}_j = \partial F_j \cap \partial X = \partial F_j \cap (\partial X \cup \bigcup_{i=j+1}^{n} \partial F_i)$. We claim that the equality $\mathcal{D}_j = \partial F_j \cap (\partial X \cup \bigcup_{i=j+1}^{n} \partial F_i)$ also holds for all $1 < j < n$.

Suppose $1 < j < n$. By definition, $\mathcal{D}_j \subseteq \partial F_j \cap (\partial X \cup \bigcup_{i=j+1}^{n} \partial F_i)$. Furthermore, by Lemma 1.5.17, $\partial F_j \backslash \mathcal{D}_j = \operatorname{int} \mathcal{C}_j \subseteq \operatorname{int}\langle F_i \rangle_{i=1}^{j}$. It follows that $\partial F_j \backslash \mathcal{D}_j$ is disjoint from $\partial F_{j+1}, \dots, \partial F_n, \partial X$, so $\partial F_j \backslash \mathcal{D}_j \subseteq \partial F_j \backslash (\partial X \cup \bigcup_{i=j+1}^{n} \partial F_i)$. Thus, $\mathcal{D}_j = \partial F_j \cap (\partial X \cup \bigcup_{i=j+1}^{n} \partial F_i)$. This proves our claim that

$$\mathcal{D}_j = \partial F_j \cap \left(\partial X \cup \bigcup_{i=j+1}^{n} \partial F_i \right), \qquad j = 1, \dots, n.$$

We are ready to prove our main inequality. Let $G \in X$. If $G \in \operatorname{int} \mathcal{C}_j$ for any $1 \le j \le n$, then $G \notin \mathcal{D}_j$, so $G \notin \partial F_{j+1}, \dots, \partial F_n, \partial X$. Likewise, if $G \in \operatorname{int} \mathcal{D}_j$ for any $1 \le j \le n$, then $G \notin \mathcal{C}_j$, so $G \notin \partial F_1, \dots, \partial F_{j-1}$. That is, G belongs to at most one of $\operatorname{int} \mathcal{C}_1, \dots, \operatorname{int} \mathcal{C}_n, \partial X$ and at most one of $\operatorname{int} \mathcal{D}_1, \dots, \operatorname{int} \mathcal{D}_n$. Thus, for $k = 0, \dots, d-1$,

$$\sum_{j=1}^{n} \left[f_k(\operatorname{int} \mathcal{C}_j) + f_k(\operatorname{int} \mathcal{D}_j) \right] \le 2 f_k(X) - f_k(\partial X).$$

Meanwhile, if $\lfloor d/2 \rfloor - 1 \le k < d-1$, then by Lemma 1.5.19,

$$\sum_{j=1}^{n} [f_k(\operatorname{int} \mathcal{C}_j) + f_k(\operatorname{int} \mathcal{D}_j)] \ge 2\rho(d, d-k-1)n.$$

We may conclude that for all $\lfloor d/2 \rfloor - 1 \le k < d-1$,

$$f_k(X) \ge \rho(d, d-k-1)n + \tfrac{1}{2} f_k(\partial X). \tag{1.5}$$

Finally, we will prove that equality holds in (1.5) if and only if $k = d-2$ and X is simplicial. Suppose $k = d-2$ and X is simplicial. Then

$$\rho(d, d-k-1)n + \tfrac{1}{2} f_k(\partial X) = \tfrac{1}{2} dn + \tfrac{1}{2} f_{d-2}(\partial X).$$

Each $(d-1)$-face of X is a simplex with exactly d many faces of dimension $d-2$. Meanwhile, each interior $(d-2)$-face of X is contained in two $(d-1)$-faces, and each boundary $(d-2)$-face is contained in one $(d-1)$-face. Thus, the sum $dn + f_{d-2}(\partial X)$ double-counts each $(d-2)$-face of X. It follows that

$$f_{d-2}(X) = \tfrac{1}{2}dn + \tfrac{1}{2}f_{d-2}(\partial X),$$

so we are done.

Conversely, suppose equality holds in (1.5). Then for all $1 \le j \le n$,

$$f_k(\operatorname{int} \mathcal{C}_j) + f_k(\operatorname{int} \mathcal{D}_j) = 2\rho(d, d-k-1), \tag{1.6}$$

and in particular,

$$f_k(\partial F_1) = 2\rho(d, d-k-1).$$

Furthermore, by Lemma 1.5.12 and Vandermonde's identity[1],

$$f_k(\partial F_1) \ge \binom{d}{d-k-1} = \binom{\lceil \frac{d}{2} \rceil}{d-k-1} + \binom{\lfloor \frac{d}{2} \rfloor}{d-k-1} + \sum_{i=1}^{d-k-2} \binom{\lceil \frac{d}{2} \rceil}{i} \binom{\lfloor \frac{d}{2} \rfloor}{d-k-i}.$$

It follows that

$$\sum_{i=1}^{d-k-2} \binom{\lceil \frac{d}{2} \rceil}{i} \binom{\lfloor \frac{d}{2} \rfloor}{d-k-i} = 0,$$

which can only occur if $k = d-2$.

For each $1 \le j \le n$, equation (1.6) now reduces to

$$\begin{aligned} & f_{d-2}(\operatorname{int} \mathcal{C}_j) + f_{d-2}(\operatorname{int} \mathcal{D}_j) = d \\ \Rightarrow \qquad & f_{d-2}(\partial F_j) = d. \end{aligned}$$

Thus, by [58, Proposition 3.3], each F_j is a simplex. This completes our treatment of equality and our proof. $\square$

For any graded poset $\mathcal{L}$ of rank $d+1$ and $k = 0, \dots, d-1$, let $f_k(\mathcal{L})$ be the number of elements of $\mathcal{L}$ with rank $k+1$ (so for a pure, regular CW d-complex X, $f_k(\mathcal{F}(X)) = f_k(X)$). The following are immediate consequences of Theorems 1.5.20 and 1.5.16.

[1] a misnomer: Zhu Shijie stated the identity in 1303, roughly 470 years before Vandermonde. See [1, pp. 59–60].

Corollary 1.5.21. *Let $\mathcal{L}$ be a diamond lattice of rank $d+1$. If $\mathcal{L}$ is CL-shellable, then for all $0 \leq k \leq \lceil d/2 \rceil$,*

$$f_k(\mathcal{L}) \geq \rho(d,k) f_0(\mathcal{L}).$$

If $\mathcal{L}$ is dual CL-shellable, then for all $\lfloor d/2 \rfloor - 1 \leq k \leq d-1$,

$$f_k(\mathcal{L}) \geq \rho(d, d-k-1) f_{d-1}(\mathcal{L}).$$

Corollary 1.5.22. *Let $\mathcal{L}$ be a diamond lattice of rank $d+1$ which is both CL-shellable and dual CL-shellable. Then*

$$f_0(\mathcal{L}), \dots, f_{d-1}(\mathcal{L}) \geq \min\{f_0(\mathcal{L}), f_{d-1}(\mathcal{L})\}.$$

1.6 The generalized upper bound conjecture

Recall the guiding question of this chapter: what are the smallest possible face numbers of a d-polytope with a given number of facets? We have obtained the tightest possible linear bounds, but the question of nonlinear bounds remains wide open.

A notable nonlinear bound on face numbers is Kalai's Generalized Upper Bound Theorem for simplicial polytopes [39]. Recall that $C(d,n)$ is the cyclic d-polytope on n vertices.

Theorem 1.6.1 (Generalized Upper Bound Theorem)**.** *Let P be a simplicial d-polytope satisfying $f_{d-1}(P) \geq f_{d-1}(C(d,n))$ for some fixed n. Then $f_k(P) \geq f_k(C(d,n))$ for $k = 0, \dots, d-1$.*

Kalai proved Theorem 1.6.1 using the g-theorem and the theory of algebraic shifting. He then made further conjectures at varying levels of generality. The following two remain open:

Conjecture 1.6.2 (Kalai)**.** *The Generalized Upper Bound Theorem applies to arbitrary polytopes.*

Conjecture 1.6.3 (Kalai)**.** *The Generalized Upper Bound Theorem applies to arbitrary Eulerian lattices.*

Theorem 1.2.2 is essentially the "best linear approximation" of Kalai's generalized upper bound conjecture for polytopes. Further progress could involve placing a lower bound on the γ_k^2 term of Proposition 1.4.5 in terms of f_{d-1}.

Finally, in light of our CW complex results, we will squeeze our own conjecture in between 1.6.2 and 1.6.3:

Conjecture 1.6.4. *The Generalized Upper Bound Theorem applies to arbitrary shellable, strongly regular CW spheres.*

Chapter 2

FAT POLYTOPES AND SPHERES

Ziegler writes that "it would be nice to have arbitrarily fat shellable 3-spheres" [20]. In this chapter, we construct arbitrarily fat shellable 3-spheres.

To explain why the above would be nice, we first consider f-vectors of polytopes. If P is a d-polytope, then the f-vector of P is defined as $(f_0, \dots, f_{d-1})$, where f_k represents the number of k-dimensional faces of P for $k = 0, \dots, d-1$. Herefrom arises the guiding question of much polyhedral combinatorics: which integer vectors are the f-vector of a polytope?

Steinitz answered this question for 3-polytopes in 1906 [53]. An integer vector (f_0, f_1, f_2) is the f-vector of a 3-polytope if and only if the following hold:

- $f_0 - f_1 + f_2 = 2$ (Euler's relation),
- $2f_1 - 3f_0 \geq 0$ (tight for simple polytopes),
- $2f_1 - 3f_2 \geq 0$ (tight for simplicial polytopes).

The above define a two-dimensional cone in $\mathbb{R}^3$ with apex $(4, 6, 4)$, the f-vector of the 3-simplex. Note that the d-simplex in general has f-vector $(d+1, \binom{d+1}{2}, \dots, \binom{d+1}{d})$.

A century later, we are nowhere near characterizing the f-vectors of 4-polytopes. We therefore ask a simpler question: what is the *f-vector cone*, or the closure of the convex cone spanned by all f-vectors in $\mathbb{Z}^4$ with apex the f-vector $(5, 10, 10, 5)$ of the 4-simplex? All 4-polytopes must satisfy the following (see Bayer [3]):

- $f_0 - f_1 + f_2 - f_3 = 0$ (Euler's relation),
- $f_0 \geq 5$ (tight for simplex),

- $f_3 \geq 5$ (tight for simplex),
- $f_1 - 2f_0 \geq 0$ (tight for simple polytopes),
- $f_2 - 2f_3 \geq 0$ (tight for simplicial polytopes),
- $5f_0 - 2f_1 - 2f_2 + 5f_3 \leq 10$ (proven by Bayer; tight for stacked polytopes and their duals).

The above define a three-dimensional cone in $\mathbb{R}^4$ with apex $(5, 10, 10, 5)$. However, we do not know if the set of f-vectors spans its whole interior. Here the *fatness* parameter, defined as $(f_1 + f_2 - 20)/(f_0 + f_3 - 10)$, comes into play. If there exist arbitrarily fat 4-polytopes, then this cone is exactly the f-vector cone; if fatness of 4-polytopes is bounded, then the f-vector cone is smaller. See Ziegler [63][61] for further discussion.

Eppstein, Kuperberg, and Ziegler introduced the fatness parameter in 2003 and constructed a 4-polytope with fatness ~ 5.048, the highest then known. Eppstein et al. further constructed a family of strongly regular CW 3-spheres with unbounded fatness, having f-vectors $(\Theta(n^{12}), \Theta(n^{13}), \Theta(n^{13}), \Theta(n^{12}))$. The latter construction was probabilistic and involved subdividing a high-genus surface into a CW complex with many edges, then thickening it to obtain a three-dimensional complex with many edges and ridges. [19]

In 2006, Paffenholz constructed a family of polytopes with fatness $6 - \varepsilon$ for arbitrarily small ε. Paffenholz's "E-construction" involved stellating a simple polytope P and merging any two of the resulting facets that share a ridge of P [46]. Around the same time, Ziegler constructed a family of polytopes with fatness $9 - \varepsilon$, the highest to date. Ziegler's polytopes arise from taking a "deformed product" of many large polygons, then projecting the product into $\mathbb{R}^4$ while preserving all edges and all large polygon faces [62].

A natural approach to constructing fat polytopes is to construct fat 3-spheres first, then attempt to find polytopes that realize them. However, for a sphere to have any hope of polytopality, it must be both shellable and dual shellable. It is unknown whether the probabilistic construction of Eppstein et al. can be specified to yield shellable spheres [20]. We construct an explicit family of arbitrarily fat, strongly regular CW 3-spheres that are both shellable and dual shellable. Our spheres have f-vectors $(\Theta(n), \Theta(n\alpha(n)), \Theta(n\alpha(n)), \Theta(n))$, where α is the inverse Ackermann function.

We generate our family of spheres recursively by gluing together small CW complexes into large ones. Our method is based on recursive constructions of Davenport-Schinzel sequences (see Sharir and Agarwal [50]) and a construction of pattern-avoiding binary matrices due to Füredi and Hajnal [25]. We ensure shellability by building CW complexes in a "bird's-eye view" where each facet has a well-defined top and bottom, making the list of facets from lowest to highest a shelling order.

The structure of this chapter is as follows. In Section 2.1, we introduce the main concepts of this chapter related to polytopes and CW complexes. Section 2.2 is a warm-up: we show a modified version of Füredi and Hajnal's matrix construction and use it to generate polytopal complexes with high *complexity*, a related parameter to fatness. In Section 2.3, we give our main construction of fat, shellable, dual shellable 3-spheres. We conjecture that these spheres are polytopal.

2.1 Preliminaries

In this section, we discuss important operations on CW complexes (§2.1.1) and the notion of a dual shelling (§2.1.2). We will not draw a distinction in this chapter between a CW complex and its geometric realization.

2.1.1 Operations on CW complexes

We will sometimes want to discuss the local structure of a CW complex around a specific face. It will then be useful to consider the relative interior, star, and vertex figure, which we define below.

Definition 2.1.1. Let Z be a face of some CW complex. The *relative interior* of Z, denoted relint Z, is the set $Z \backslash \partial Z$.

Definition 2.1.2. Let X be a pure, regular CW complex and Z a face of X. The *star* of Z, denoted $\operatorname{st} Z$, is the union of the relative interiors of all faces of X containing Z.

Definition 2.1.3. Let X be a pure, regular CW d-complex and v a vertex of X. Let U be an open neighborhood of v in X with $\operatorname{cl} U \subset \operatorname{st} v$ and with $Z \cap \operatorname{cl} U$ homeomorphic to a closed k-ball for all k-faces $Z \subseteq X$ containing v. The *vertex figure* of v, denoted X/v, is the CW $(d-1)$-complex with $(k-1)$-faces $Z/v := Z \cap \operatorname{cl} U \backslash U$ for $k = 1, \ldots, d$ and for k-faces $Z \subseteq X$ containing v.

Observation 2.1.4. Suppose X is a pure, regular CW complex and v a vertex of X. If X is strongly regular, then X/v is strongly regular. If X is a CW 3-sphere, then X/v is a CW 2-sphere. If X is a CW 3-ball and $v \in \operatorname{int} X$ or $v \in \partial X$, then X/v is respectively a CW 2-sphere or 2-ball (see [11, 278]).

Our main work in this chapter is on obtaining new CW complexes from old. The tools for this include subcomplexes and coning, defined next.

Definition 2.1.5. Let X be a CW complex. A *subcomplex* of X is a CW complex $W \subseteq X$ such that every face of W is a face of X, and the set of faces of W is closed under containment in X.

Definition 2.1.6. Let X be a regular CW complex, W a subcomplex of X, and v a point outside X. To *cone* over W at v is an operation wherein we add v as a vertex of X, then attach to each nonempty k-face $Z \subseteq W$ a new $(k+1)$-face isomorphic to a pyramid with base Z and apex v.

Observation 2.1.7. If we cone a strongly regular CW complex over any subcomplex, we obtain another strongly regular CW complex. If we cone a CW d-ball over its boundary, we obtain a CW d-sphere.

Hereafter, when we refer to any CW complex, "strongly regular" will be implied unless otherwise indicated. We will typically also omit the word "CW" and write *d-sphere* to mean "strongly regular CW d-sphere"; likewise *d-ball*.

2.1.2 Dual shellings

We recall the notion of shellings for CW complexes, and we introduce the notion of dual shellings. As we no longer distinguish between a complex and its geometric realization, our definition of a shelling can be simplified slightly.

Definition 2.1.8. Let X be a pure CW d-complex with facets $Y_1, \dots, Y_n$. The ordered set $(Y_1, \dots, Y_n)$ is a *shelling* of X if either $d = 0$, or $d > 0$ and

- ∂Y_1 admits a shelling,
- for $i = 2, \dots, n$, the intersection $Y_i \cap \bigcup_{j=1}^{i-1} Y_j$ is a pure $(d-1)$-complex, and there exists a shelling of ∂Y_i beginning with the $(d-1)$-faces of $Y_i \cap \bigcup_{j=1}^{i-1} Y_j$.

We call X *shellable* if it admits a shelling.

Definition 2.1.9. Let X be a pure CW d-complex with vertices $v_1, \dots, v_n$. The ordered set $(v_1, \dots, v_n)$ is a *dual shelling* of X if either $d = 0$, or $d > 0$ and

- X/v_1 admits a dual shelling,
- for $i = 2, \dots, n$, the set $X/v_i \cap \bigcup_{j=1}^{i-1} \operatorname{st} v_j$ is a nonempty union of vertices of X/v_i and their stars, and there exists a dual shelling of X/v_i beginning with the vertices in $X/v_i \cap \bigcup_{j=1}^{i-1} \operatorname{st} v_j$.

We call X *dual shellable* if it admits a dual shelling.

Björner and Wachs proved that a CW complex is shellable (respectively, dual shellable) if and only if its face poset is *dual CL-shellable* (respectively, *CL-shellable*) [13, Corollary 4.4].

We proceed to discuss the shellings and dual shellings of polytopes. These notions are well-defined because the boundary complex of any d-polytope is a $(d-1)$-sphere. The converse does not hold in general but holds when $d = 3$ due to the famous 1922 theorem of Steinitz [54] (see also [60, §4]).

Theorem 2.1.10 (Steinitz)**.** *Any 2-sphere is isomorphic to the boundary complex of a 3-polytope.*

For the most famous result on shellings, we turn to Bruggesser and Mani [16]:

Theorem 2.1.11 (Bruggesser and Mani)**.** *The boundary complex of any polytope is shellable.*

Bruggesser and Mani proved Theorem 2.1.11 using the technique of *line shellings*. To obtain a line shelling, we move away from our polytope in a straight line and begin listing facets in the order they become visible. We then approach along the same line, from the opposite direction, and list the remaining facets in the order they become obscured.

Under polarization, a shelling of any polytope becomes a dual shelling of the dual polytope. A line shelling becomes a *sweep*, a dual shelling induced by a linear functional [29, p. 142a].

Corollary 2.1.12. *Let P be a polytope and v a vertex of P. Then there exists a shelling of ∂P beginning (or ending) with the facets of P that contain v.*

The above follows from Bruggesser and Mani's line shellings; see [16, Proposition 2] and [60, Corollary 8.13].

Corollary 2.1.13. *The boundary complex of any d-polytope is a shellable, dual shellable $(d-1)$-sphere.*

The next three lemmas concern the shelling properties of 2- and 3-spheres. We include their proofs for completeness.

Lemma 2.1.14. *Let S be a 2-sphere, and let W be the union of a proper, nonempty subset of the facets of S. Then there exists a shelling of S beginning with the facets of W if and only if W is homeomorphic to a closed disc.*

Proof. If there exists a shelling of S beginning with the facets of W, then W is homeomorphic to a closed disc by Theorem 1.5.15.

Conversely, suppose W is homeomorphic to a closed disc. Let Q be the 2-sphere obtained from W by adding a new vertex v and coning over the boundary of W at v. Let Q' be the 2-sphere obtained from $\mathrm{cl}(S\backslash W)$ by adding a new vertex v' and coning over the boundary of $\mathrm{cl}(S\backslash W)$ at v'. By Theorem 2.1.10, Q and Q' are each isomorphic to the boundary complex of some 3-polytope.

By Corollary 2.1.12, there exists a shelling $(T_1, \dots, T_m, \dots, T_n)$ of Q such that $T_1, \dots, T_m$ are the facets of W. Also by Corollary 2.1.12, there exists a shelling $(T'_1, \dots, T'_{m'}, \dots, T'_{n'})$ of Q' such that $T'_{m'+1}, \dots, T'_{n'}$ are the facets of $\mathrm{cl}(S\backslash W)$.

Since $(T_1, \dots, T_n)$ is a shelling of Q, for $i = 2, \dots, m$, there exists a shelling of ∂T_i beginning with the edges of $T_i \cap \bigcup_{j=1}^{i-1} T_j$. Meanwhile, since $(T'_1, \dots, T'_{n'})$ is a shelling of Q', for $i = m' + 1, \dots, n'$, there exists a shelling of $\partial T'_i$ beginning with the edges of

$$T'_i \cap \bigcup_{j=1}^{i-1} T'_j = T'_i \cap \left(\bigcup_{j=1}^{m} T_j \cup \bigcup_{j=m'+1}^{i-1} T'_j \right).$$

Thus, by definition, $(T_1, \dots, T_m, T'_{m'+1}, \dots, T'_{n'})$ is a shelling of S beginning with the facets of W. ∎

The next lemma follows by duality.

Lemma 2.1.15. *Let S be a 2-sphere, and let U be the union of a proper, nonempty subset of the vertices of S and their stars. Then there exists a dual shelling of S beginning with the vertices in U if and only if U is homeomorphic to an open disc.*

From Lemmas 2.1.14–2.1.15, we obtain the following.

Lemma 2.1.16. *Let S be a 3-sphere with vertices $v_1, \dots, v_n$ and facets $T_1, \dots, T_m$.*

i. $(T_1, \dots, T_m)$ is a shelling of S if and only $T_i \cap \bigcup_{j=1}^{i-1} T_j$ is homeomorphic to a closed disc for $i = 2, \dots, m-1$.

ii. $(v_1, \dots, v_n)$ *is a dual shelling of* S *if and only if* $X/v_i \cap \bigcup_{j=1}^{i-1} \operatorname{st} v_j$ *is a union of vertices of* X/v_i *and their stars, and is homeomorphic to an open disc, for* $i = 2, \dots, n-1$.

2.1.3 Fatness

Lastly, we introduce *f*-vectors and the fatness and complexity parameters for polytopes and complexes.

Definition 2.1.17. Let X be a $(d+1)$-polytope, polytopal d-complex, or CW d-complex. We define the *f-vector* of X as $f(X) := (f_0(X), \dots, f_d(X))$.

Definition 2.1.18. Let X be a 4-polytope, polytopal 3-complex, or CW 3-complex with $f(X) = (f_0, f_1, f_2, f_3)$. The *fatness* of X is defined as

$$\frac{f_1 + f_2 - 20}{f_0 + f_3 - 10}.$$

Definition 2.1.19. Let X be a 4-polytope, polytopal 3-complex, or CW 3-complex with $f(X) = (f_0, f_1, f_2, f_3)$. Let f_{03} be the number of vertex-facet incidences in X. The *complexity* of X is defined as

$$\frac{f_{03} - 20}{f_0 + f_3 - 10}.$$

The fatness and complexity of spheres are asymptotically equivalent. Ziegler gives the following in [61].

Proposition 2.1.20 (Ziegler)**.** *For any 3-sphere not isomorphic to the boundary of the 4-simplex, its fatness* F *and complexity* C *satisfy*

$$C \leq 2F - 2, \qquad F \leq 2C - 2.$$

The next two theorems represent the state of the art on fat polytopes and spheres.

Theorem 2.1.21 (Ziegler)**.** *There exist 4-polytopes with fatness* $9 - \varepsilon$ *for arbitrarily small* ε.

Ziegler constructs these polytopes in [62] by taking a "deformed product" of many large, even-sided polygons and projecting it into $\mathbb{R}^4$. These polytopes have fatness approaching 9 and complexity approaching 16.

Theorem 2.1.22 (Eppstein, Kuperberg, and Ziegler)**.** *There exist arbitrarily fat 3-spheres.*

Eppstein et al. construct these spheres in [19] from randomly chosen, finite covering spaces of high-genus surfaces. It is unknown whether these spheres can be specified to be shellable [20].

2.2 Warm-up: binary matrices

In this section, we motivate our main construction with an analogue in the setting of binary matrices. We construct a family of $n \times n$ binary matrices avoiding the pattern

$$\begin{bmatrix} & 1 & & 1 \\ 1 & & 1 & \end{bmatrix},$$

whose total number of ones is superlinear in n. Füredi and Hajnal gave the first such construction in [25]; ours is broadly the same, but we make some small adjustments for accuracy, and we include for completeness a proof of the superlinear bound.

We first introduce the Ackermann function (§2.2.1), which we will use to compute asymptotic bounds both here and in our main construction of spheres. We then give the aforementioned construction of pattern-avoiding matrices (§2.2.2). Finally, we use these matrices to generate a family of polytopal 3-complexes with unbounded complexity (§2.2.3).

2.2.1 The Ackermann function

The objects we will construct are huge, and we cannot describe their asymptotic behavior with polynomials, exponentials, or similar. Thus, we introduce the extremely fast-growing Ackermann function $A(s,t)$ and a related function $C(s,t)$. Several distinct versions of both functions appear in the literature; we define $A(s,t)$ and $C(s,t)$ as in [50] and [25], respectively. Our discussion in this subsection parallels [50, 21–22].

Definition 2.2.1. For positive integers s and t, the *Ackermann function* $A(s,t)$ is defined recursively as follows.

- $A(1,t) = 2t$ for all t.

$A(s,t)$

$s \backslash t$	1	2	3	4
1	2	4	6	8
2	2	4	8	16
3	2	4	16	2^{16}
4	2	4	2^{16}	$\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{2^{16}\text{ twos}}$

$C(s,t)$

$s \backslash t$	1	2	3	4
1	1	1	1	1
2	2	2	2	2
3	2	4	8	16
4	2	8	2^{11}	$2^{11+2^{11}}$

Table 2.1: Values of $A(s,t)$ and $C(s,t)$ for $s,t < 5$.

- $A(s,1) = 2$ for all s.
- $A(s,t) = A(s-1, A(s,t-1))$ for $s,t > 1$.

The *inverse Ackermann function* is defined as $\alpha(n) = \min\{s \mid A(s,s) \geq n\}$.

Definition 2.2.2. For positive integers s and t, the function $C(s,t)$ is defined recursively as follows.

- $C(1,t) = 1$ for all t.
- $C(s,1) = 2$ for all $s > 1$.
- $C(s,t) = C(s,t-1)C(s-1, C(s,t-1))$ for $s,t > 1$.

Values of $A(s,t)$ and $C(s,t)$ for small s and t are given in Table 2.1.

Observation 2.2.3. The following are easy to check.

$$A(2,t) = 2^t, \qquad A(s,2) = 4, \qquad C(2,t) = 2, \qquad C(3,t) = 2^t, \qquad C(s,2) = 2^{s-1}.$$

Lemma 2.2.4. *Let s and t be positive integers.*

i. $A(s,t+1) > A(s,t)$.

ii. $A(s,t) \geq t+2$ *for* $t > 1$.

iii. $A(s+1,t) \geq A(s,t+1)$ *for* $t > 2$.

iv. $A(s,3) \geq 2^{s+1}$.

Proof. We will prove i-iv in order.

i. By definition, $A(1,t+1) = 2(t+1) > 2t = A(1,t)$. We also know that $A(s,2) = 4 > 2 = A(s,1)$. The cases $s,t > 1$ follow by induction:

$$A(s,t+1) = A(s-1,A(s,t)) > A(s-1,A(s,t-1)) = A(s,t).$$

ii. If $t = 2$, we have $A(s,2) = 4 = t+2$. The cases $t > 2$ then follow from i.

iii. Let $t > 2$. Then by ii and the recursive definition of $A(s,t)$,

$$A(s+1,t) = A(s,A(s+1,t-1)) \geq A(s,t+1).$$

iv. If $s = 1$, we have $A(1,3) = 6 > 4 = 2^{s+1}$. If $s > 1$, then by iii,

$$A(s,3) \geq A(s-1,4) \geq A(s-2,5) \geq \cdots \geq A(2,s+1) = 2^{s+1}.$$ □

Lemma 2.2.5. *Let* s *and* t *be positive integers.*

i. $C(s,t+1) \geq C(s,t)$.

ii. $C(s,t+1) \geq 2C(s,t)$ *for* $s > 2$.

Proof. We will prove i and then ii.

i. By definition, $C(1,t+1) = 1 = C(1,t)$. If $s > 1$, then

$$C(s,t+1) = C(s,t)C(s-1,C(s,t)) \geq C(s,t).$$

ii. Let $s > 2$. It follows from i that $C(s-1,t) \geq C(s-1,1) = 2$ for all t. Thus,

$$C(s,t+1) = C(s,t)C(s-1,C(s,t)) \geq 2C(s,t). \qquad \square$$

When working with $A(s,t)$ and $C(s,t)$, we will often use their monotonicity in t without explicit reference to Lemmas 2.2.4i and 2.2.5i.

Lemma 2.2.6. *For all positive integers $s > 1$ and t,*

$$A(s-1,t) \leq tC(s,t) < A(s,t+1).$$

Proof. First, we will show that $tC(s,t) \geq A(s-1,t)$. For $s = 2$, this is immediate, as $tC(2,t) = 2t = A(1,t)$.

For $s > 2$, we will prove the stronger statement that $C(s,t) \geq A(s-1,t)$. We will do so by double induction on s and t, with base cases $s = 3$ and $t = 1$. The base case $s = 3$ is immediate, as $C(3,t) = 2^t = A(2,t)$. The base case $t = 1$ is also immediate, as $C(s,1) = 2 = A(s-1,1)$.

For the inductive step, fix $s > 3$, and suppose $C(s-1,t) \geq A(s-2,t)$ for all t. Fix $t > 1$ as well, and suppose $C(s,t-1) \geq A(s-1,t-1)$. Then

$$\begin{aligned} C(s,t) &= C(s,t-1)C(s-1,C(s,t-1)) \\ &\geq C(s-1,C(s,t-1)) \\ &\geq A(s-2,A(s-1,t-1)) \\ &= A(s-1,t). \end{aligned}$$

This completes the inductive step. We may conclude that $C(s,t) \geq A(s-1,t)$ for all $s > 2$, so $tC(s,t) \geq A(s-1,t)$ for all $s > 1$ and all t.

Second, we will show that $tC(s,t) \leq A(s,t+1) - 2$. We will again use double induction on s and t. The base cases will be $s = 2, 3$ and $t = 1$, which we check in order.

$s = 2$:

$$tC(2,t) = 2t \leq 2^{t+1} - 2 = A(2,t+1) - 2.$$

$s = 3$:

$$tC(3,t) = t2^t \leq 2^{2t} - 2 \leq 2^{A(3,t)} - 2 = A(3, t+1) - 2.$$

$t = 1$:

$$C(s,1) = 2 = A(s,2) - 2.$$

For the inductive step, fix $s > 3$, and assume $tC(s-1,t) \leq A(s-1,t+1) - 2$ for all t. Fix $t > 1$ as well, and assume $(t-1)C(s,t-1) \leq A(s,t) - 2$.

By the inductive hypothesis,

$$tC(s,t-1) \leq 2(t-1)C(s,t-1) < 2A(s,t) - 2.$$

Using the above, Lemma 2.2.5ii, and finally the inductive hypothesis again,

$$\begin{aligned} tC(s,t) &= tC(s,t-1)C(s-1, C(s,t-1)) \\ &< (2A(s,t) - 2)C(s-1, A(s,t) - 2) \\ &\leq (A(s,t) - 1)C(s-1, A(s,t) - 1) \\ &\leq A(s-1, A(s,t)) - 2 \\ &= A(s, t+1) - 2. \end{aligned}$$

This completes the inductive step. We may conclude that $tC(s,t) \leq A(s,t+1) - 2$ for all $s > 1$ and all t. □

2.2.2 Pattern-avoiding matrices

We formally introduce the notion of pattern-avoiding matrices, then proceed with our construction. As mentioned, our construction is adapted from [25], and we have also consulted [41, §2.3] as a reference. Recall that a binary matrix is a matrix of ones and zeros.

Definition 2.2.7. Let M be a binary matrix. The *weight* of M, denoted $|M|$, is the number of ones in M.

Definition 2.2.8. Let M and N be binary matrices. We say that M *contains* N if we can obtain N from M by deleting a set of rows, deleting a set of columns, and changing a set of ones to zeros. We say that M *avoids* N if we cannot obtain N from M in this way.

Let
$$N = \begin{bmatrix} & 1 & & 1 \\ 1 & & 1 & \end{bmatrix}, \qquad N' = \begin{bmatrix} 1 & & 1 \\ 1 & 1 & \end{bmatrix}$$
(zeros are left blank). We will construct arbitrarily large $n \times n$ binary matrices M, avoiding both N and N', such that $|M| = \Theta(n\alpha(n))$. Avoiding N' is not really the goal, but it is counterintuitively easier to prove that our matrices avoid both N and N' than that they just avoid N.

For positive integers s and t, we will define a $tC(s,t) \times tC(s,t)$ binary matrix $M(s,t)$. For any fixed $M(s,t)$, our definition will ensure that there exist $1 = c_1 < \cdots < c_{C(s,t)} \leq tC(s,t)$ such that the first t rows have leading ones in column c_1, the next t rows in column c_2, and so on. Following [25], we will divide $M(s,t)$ into *horizontal blocks* of consecutive rows, ending at rows $t, 2t, \ldots, tC(s,t)$. We will similarly divide $M(s,t)$ into *vertical blocks* of consecutive columns, beginning at columns $c_1, \ldots, c_{C(s,t)}$.

Construction 2.2.9. We define $M(s,t)$ recursively as follows.

- $M(1,t)$ is the $t \times t$ matrix with all ones in the first column and all zeros elsewhere.
- $M(s,1) = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$ for $s > 1$.
- $M(s,t)$ for $s,t > 1$ is obtained from $M(s,t-1)$ and $M(s-1, C(s,t-1))$ as shown in Figure 2.1. $H_1, \ldots, H_{C(s,t-1)}$ are the horizontal blocks of $M(s,t-1)$, which appears $C(s-1, C(s,t-1))$ many times. $V_1, \ldots, V_{C(s-1,C(s,t-1))}$ are the vertical blocks of $M(s-1, C(s,t-1))$, which appears once. $C(s,t)$ many extra ones are introduced, one per row of $M(s-1, C(s,t-1))$.

We refer the reader to [25, Definition 7.2] for a narrative description of the recursive step above; our recursive step (Figure 2.1) is identical. However, we use different base cases

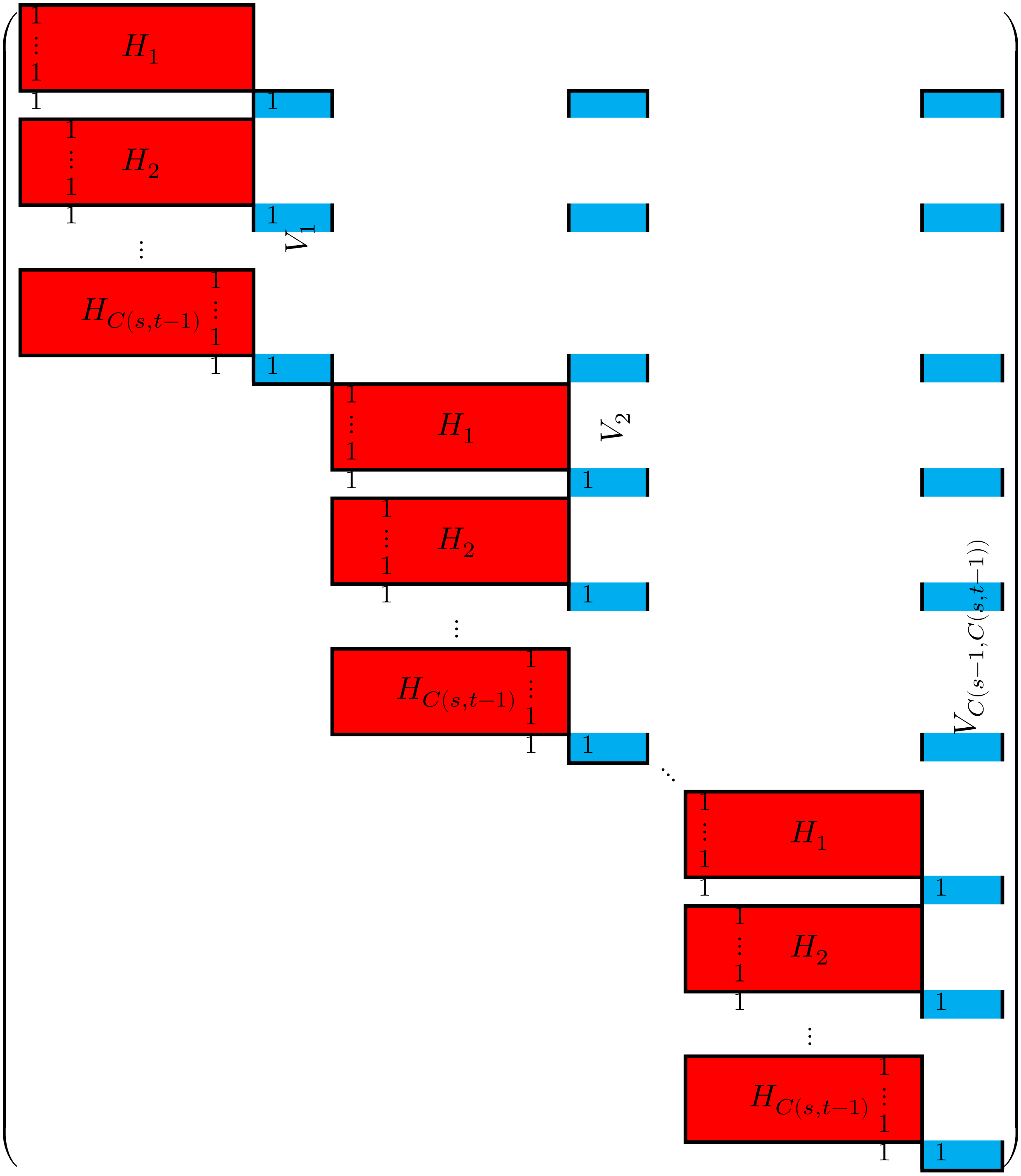

Figure 2.1: Recursive construction of $M(s,t)$ from $M(s,t-1)$ and $M(s-1,C(s,t-1))$.

$M(s,t)$

s \ t	1	2	3	4
1	$\begin{bmatrix}1\end{bmatrix}$	$\begin{bmatrix}1&\\1&\end{bmatrix}$	$\begin{bmatrix}1&&\\1&&\\1&&\end{bmatrix}$	$\begin{bmatrix}1&&&\\1&&&\\1&&&\\1&&&\end{bmatrix}$
2	$\begin{bmatrix}1&\\&1\end{bmatrix}$	$\begin{bmatrix}\mathbf{1}&&&\\1&&\underline{1}&\\&\mathbf{1}&&\\&1&\underline{1}&\end{bmatrix}$	$\begin{bmatrix}\mathbf{1}&&&&&\\\mathbf{1}&&\mathbf{1}&&&\\1&&&&\underline{1}&\\&\mathbf{1}&&&&\\&\mathbf{1}&\mathbf{1}&&&\\&1&&&\underline{1}&\end{bmatrix}$	$\begin{bmatrix}\mathbf{1}&&&&&&&\\\mathbf{1}&&\mathbf{1}&&&&&\\\mathbf{1}&&&&\mathbf{1}&&&\\1&&&&&&\underline{1}&\\&\mathbf{1}&&&&&&\\&\mathbf{1}&\mathbf{1}&&&&&\\&\mathbf{1}&&&\mathbf{1}&&&\\&1&&&&&\underline{1}&\end{bmatrix}$
3	$\begin{bmatrix}1&\\&1\end{bmatrix}$	$\begin{bmatrix}\mathbf{1}&&&&&&&\\1&&\underline{1}&&&&&\\&\mathbf{1}&&&&&&\\&1&\underline{1}&&&&\underline{1}&\\&&&\mathbf{1}&&&&\\&&&1&&\underline{1}&&\\&&&&\mathbf{1}&&&\\&&&&1&\underline{1}&\underline{1}&\end{bmatrix}$	$\begin{bmatrix}\mathbf{1}&&&&&&&&&&&&&&&&&&&&&&&\\\mathbf{1}&&\mathbf{1}&&&&&&&&&&&&&&&&&&&&&\\1&&&&&&&&\underline{1}&&&&&&&&&&&&&&&\\&\mathbf{1}&&&&&&&&&&&&&&&&&&&&&&\\&\mathbf{1}&\mathbf{1}&&&&\mathbf{1}&&&&&&&&&&&&&&&&&\\&1&&&&&&&\underline{1}&&&&&&&&&&\underline{1}&&&&&\\&&&\mathbf{1}&&&&&&&&&&&&&&&&&&&&\\&&&\mathbf{1}&&\mathbf{1}&&&&&&&&&&&&&&&&&&\\&&&1&&&&&\underline{1}&&&&&&&&&&&&\underline{1}&&&\\&&&&\mathbf{1}&&&&&&&&&&&&&&&&&&&\\&&&&\mathbf{1}&\mathbf{1}&\mathbf{1}&&&&&&&&&&&&&&&&&\\&&&&1&&&&\underline{1}&&&&&&&&&&&&&&\underline{1}&\\&&&&&&&&&\mathbf{1}&&&&&&&&&&&&&&\\&&&&&&&&&\mathbf{1}&&\mathbf{1}&&&&&&&&&&&&\\&&&&&&&&&1&&&&&&&&\underline{1}&&&&&&\\&&&&&&&&&&\mathbf{1}&&&&&&&&&&&&&\\&&&&&&&&&&\mathbf{1}&\mathbf{1}&&&&\mathbf{1}&&&&&&&&\\&&&&&&&&&&1&&&&&&&\underline{1}&\underline{1}&&&&&\\&&&&&&&&&&&&\mathbf{1}&&&&&&&&&&&\\&&&&&&&&&&&&\mathbf{1}&&\mathbf{1}&&&&&&&&&\\&&&&&&&&&&&&1&&&&&\underline{1}&&&\underline{1}&&&\\&&&&&&&&&&&&&\mathbf{1}&&&&&&&&&&\\&&&&&&&&&&&&&\mathbf{1}&\mathbf{1}&\mathbf{1}&&&&&&&&\\&&&&&&&&&&&&&1&&&&\underline{1}&&&&&\underline{1}&\end{bmatrix}$	

Table 2.2: Selected matrices $M(s,t)$, colored as in the proof of Lemma 2.2.10.

$M(1,t)$ and $M(s,1)$ to ensure that each has the claimed number of horizontal and vertical blocks. Small examples of $M(s,t)$ are given in Table 2.2.

Lemma 2.2.10. *$M(s,t)$ avoids both N and N' for all positive integers s and t.*

Proof. We argue by double induction on s and t. For the base cases, observe that each matrix $M(1,t)$ or $M(s,1)$ avoids both N and N'.

For the inductive step, fix $s,t>1$, and suppose $M(s,t-1)$ and $M(s-1,C(s,t-1))$ each avoid N,N'. In $M(s,t)$, color each "1" red, blue, or black: red if inherited from a copy of $M(s,t-1)$, blue if inherited from $M(s-1,C(s,t-1))$, or black otherwise. Our construction of $M(s,t)$ implies (a)–(d) below, while the inductive hypothesis implies (e). We **bold** red ones and <u>underline</u> blue ones to assist colorblind readers.

(a) A red "1" cannot be in the same row as a blue or black "1".

(b) A blue "1" cannot be in the same column as a red or black "1".

(c) Given a black "1", any other "1" must be in a higher row or a further-right column.

(d) $M(s,t)$ avoids the colored matrices

$$\begin{bmatrix} & \mathbf{1} \\ \underline{1} & \end{bmatrix}, \qquad \begin{bmatrix} & \underline{1} & \\ \mathbf{1} & & \mathbf{1} \end{bmatrix}.$$

(e) $M(s,t)$ avoids the colored matrices

$$\begin{bmatrix} & \mathbf{1} & & \mathbf{1} \\ \mathbf{1} & & \mathbf{1} & \end{bmatrix}, \qquad \begin{bmatrix} \mathbf{1} & & \mathbf{1} \\ \mathbf{1} & \mathbf{1} & \end{bmatrix}, \qquad \begin{bmatrix} & \underline{1} & & \underline{1} \\ \underline{1} & & \underline{1} & \end{bmatrix}, \qquad \begin{bmatrix} \underline{1} & & \underline{1} \\ \underline{1} & \underline{1} & \end{bmatrix}.$$

No coloring of N' is consistent with (a)-(e), so $M(s,t)$ must avoid N'.

It remains to prove that $M(s,t)$ avoids N. By way of contradiction, suppose $M(s,t)$ contains N. Let $1 \leq r_1 < r_2 \leq tC(s,t)$ be the rows and $1 \leq k_1 < k_2 < k_3 < k_4 \leq tC(s,t)$ the columns of $M(s,t)$ in which N appears. Then the only coloring of N satisfying (a)–(e) is

$$\begin{array}{c} \\ r_1 \\ r_2 \end{array} \begin{array}{c} \begin{array}{cccc} k_1 & k_2 & k_3 & k_4 \end{array} \\ \begin{bmatrix} & \underline{1} & & \underline{1} \\ 1 & & \underline{1} & \end{bmatrix} \end{array}.$$

Let V_i be the first vertical block of $M(s-1, C(s,t-1))$ that appears in $M(s,t)$ to the right of column k_1. Let $k > k_1$ be the column containing the first blue "1" in row r_2. By construction, k is the first column of V_i. It follows that $k \leq k_2$; otherwise, the blue "1" in position (r_1, k_2) must have come from a vertical block V_j of $M(s-1, C(s,t-1))$ with $j < i$, contradicting the minimality of i.

$M(s,t)$ therefore contains one of the colored matrices

$$\begin{array}{c} \\ r_1 \\ r_2 \end{array} \begin{array}{c} \begin{array}{ccccc} k_1 & & k_2 & k_3 & k_4 \end{array} \\ \begin{bmatrix} & & \underline{1} & & \underline{1} \\ 1 & \underline{1} & & \underline{1} & \end{bmatrix} \end{array}, \qquad \begin{array}{c} \\ r_1 \\ r_2 \end{array} \begin{array}{c} \begin{array}{cccc} k_1 & k_2 & k_3 & k_4 \end{array} \\ \begin{bmatrix} & \underline{1} & & \underline{1} \\ 1 & \underline{1} & \underline{1} & \end{bmatrix} \end{array}.$$

This contradicts (e). We may conclude that $M(s,t)$ avoids N, and the inductive step is complete. $\square$

For binary matrices M, recall that $|M|$ denotes the weight of M. By construction, $|M(s,t)|$ satisfies the following for all s and t.

- $|M(1,t)| = t$ for all t.
- $|M(s,1)| = 2$ for all $s > 1$.
- $|M(s,t)| = C(s,t) + C(s-1, C(s,t-1))|M(s,t-1)| + |M(s-1, C(s,t-1))|$ for $s,t > 1$.

For $s,t > 1$, the recursive formula for $|M(s,t)|$ may be rearranged to obtain

$$\frac{|M(s,t)|}{C(s,t)} - \frac{|M(s,t-1)|}{C(s,t-1)} = 1 + \frac{|M(s-1,C(s,t-1))|}{C(s,t-1)C(s-1,C(s,t-1))}.$$

If we substitute the dummy variable u for $t-1$ in the above, then sum the resulting equation from $u=1$ to $u=t-1$, we find

$$\begin{aligned}
\frac{|M(s,t)|}{C(s,t)} - \frac{|M(s,1)|}{C(s,1)} &= (t-1) + \sum_{u=1}^{t-1} \frac{|M(s-1,C(s,u))|}{C(s,u)C(s-1,C(s,u))} \\
\Rightarrow \qquad \frac{|M(s,t)|}{C(s,t)} &= t + \sum_{u=1}^{t-1} \frac{|M(s-1,C(s,u))|}{C(s,u)C(s-1,C(s,u))} \\
\Rightarrow \qquad \frac{|M(s,t)|}{tC(s,t)} &= 1 + \frac{1}{t}\sum_{u=1}^{t-1} \frac{|M(s-1,C(s,u))|}{C(s,u)C(s-1,C(s,u))}. \qquad (2.1)
\end{aligned}$$

Lemma 2.2.11. *For all positive integers s and t with $t \geq s$,*

$$\frac{s}{3} < \frac{|M(s,t)|}{tC(s,t)} \leq s.$$

Proof. First, we will prove that $|M(s,t)|/(tC(s,t)) > s/3$ for $t \geq s$. We argue by induction on s. The base case $s = 1$ is immediate, as $|M(1,t)|/(tC(1,t)) = 1 > 1/3$ for all t.

For the inductive step, fix $s > 1$, and suppose that $|M(s-1,t)|/(tC(s-1,t)) > (s-1)/3$ for all $t \geq s-1$. For all $u > 1$, we can observe that $C(s,u) \geq C(s,2) = 2^{s-1} > s-1$. Thus, by the inductive hypothesis, for all $u > 1$,

$$\frac{|M(s-1,C(s,u))|}{C(s,u)C(s-1,C(s,u))} > \frac{s-1}{3}.$$

Hence, by (2.1), for all $t \geq s$,

$$\frac{|M(s,t)|}{tC(s,t)} \geq 1 + \frac{(s-1)(t-2)}{3t} > \frac{s}{3}.$$

This completes the inductive step. We may conclude that $|M(s,t)|/(tC(s,t)) > s/3$ for $t \geq s$.

Second, we will prove that $|M(s,t)|/(tC(s,t)) \leq s$ for all s and t. We argue by induction on s. The base case $s = 1$ is again immediate, as $|M(1,t)|/(tC(1,t)) = 1$ for all t.

For the inductive step, fix $s > 1$, and suppose that $|M(s-1,t)|/(tC(s-1,t)) \leq s-1$ for all t. If $t = 1$, we have $|M(s,1)|/C(s,1) = 1 \leq s$, so we are done. Suppose $t > 1$. By the inductive hypothesis, for all u,

$$\frac{|M(s-1,C(s,u))|}{C(s,u)C(s-1,C(s,u))} \leq s-1.$$

Thus, by (2.1),

$$\frac{|M(s,t)|}{tC(s,t)} \leq 1 + \frac{(s-1)(t-1)}{t} < s.$$

This completes the inductive step. We may conclude that $|M(s,t)|/(tC(s,t)) \leq s$ for all s and t. Hence, $s/3 < |M(s,t)|/(tC(s,t)) \leq s$ for all s and $t \geq s$. □

Theorem 2.2.12 (Füredi and Hajnal)**.** *For arbitrarily large n, there exist $n \times n$ binary matrices M avoiding N such that $|M| = \Theta(n\alpha(n))$.*

Proof. Let $n = sC(s,s)$ and $M = M(s,s)$. Then M is an $n \times n$ binary matrix avoiding N (Lemma 2.2.10). Furthermore, by Lemmas 2.2.6 and 2.2.11, we have $|M| = \Theta(s^2C(s,s)) = \Theta(n\alpha(n))$. □

Remark 2.2.13. The growth rate $\Theta(n\alpha(n))$ in Theorem 2.2.12 is the highest possible (see [25, Theorem 8.1]).

2.2.3 Polytopal 3-complexes

To conclude this section, we use our pattern-avoiding matrices to construct polytopal 3-complexes with unbounded complexity. Recall that complexity is defined as $(f_{03} - 20)/(f_0 + f_3 - 10)$.

Definition 2.2.14. The *moment curve* $x : \mathbb{R} \to \mathbb{R}^d$ is the parametric curve defined by

$$x(t) = (t, t^2 \dots, t^d).$$

Remark 2.2.15. It is well known that any finite set of points on the moment curve is in general position (i.e. a subset of size k has affine hull of dimension $\min\{k-1, d\}$).

The most famous combinatorial property of the moment curve is *Gale's evenness criterion*, which characterizes the facets of any polytope with vertices on the moment curve. We will use a related property derived from Gale's criterion and Radon's theorem by Lee and Nevo [42, Propositions 2.2&2.4].

Proposition 2.2.16 (Lee and Nevo)**.** *Let $A, B \subset \mathbb{R}$ be finite sets with $|A \cap B| \le d$, and let $x : \mathbb{R} \to \mathbb{R}^d$ be the moment curve. Then $\operatorname{conv} x(A) \cap \operatorname{conv} x(B) \supsetneq \operatorname{conv} x(A \cap B)$ if and only if there exist $t_1 < \dots < t_{d+2}$ such that either*

- $\{t_i \mid i \text{ odd}\} \subseteq A$ *and* $\{t_i \mid i \text{ even}\} \subseteq B$, *or*

- $\{t_i \mid i \text{ odd}\} \subseteq B$ *and* $\{t_i \mid i \text{ even}\} \subseteq A$.

Corollary 2.2.17. *Let M be an $m \times n$ binary matrix with pairwise distinct rows, and let $x : \mathbb{R} \to \mathbb{R}^3$ be the moment curve. For $i = 1, \dots, m$, let $P_i = \operatorname{conv}\{x(j) \mid M_{ij} = 1\}$. Then the collection of polytopes $P_1, \dots, P_m$ and their faces is a polytopal 3-complex if and only if M avoids the matrices*

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & & 1 & & 1 \\ & 1 & & 1 & \end{bmatrix}, \quad \begin{bmatrix} & 1 & & 1 & \\ 1 & & 1 & & 1 \end{bmatrix}.$$

Theorem 2.2.18. *For arbitrarily large n, there exist polytopal 3-complexes Q such that $f_0(Q), f_3(Q) \leq n$ and $f_{03}(Q) = \Theta(n\alpha(n))$.*

Proof. Let M be an $n \times n$ matrix as in Theorem 2.2.12. Delete any row of M containing fewer than four ones, and let $m \leq n$ be the number of remaining rows. Since we have deleted at most $3n$ many ones, we still have $|M| = \Theta(n\alpha(n))$.

Let $P_i = \operatorname{conv}\{x(j) \mid M_{ij} = 1\}$ for $i = 1, \dots, m$. Then each P_i is a 3-polytope, since it is the convex hull of at least four points in general position. Let Q be the collection of 3-polytopes $P_1, \dots, P_m$ and their faces. By Corollary 2.2.17, Q is a polytopal 3-complex.

We have $f_3(Q) = m$, and $f_0(Q)$ is the number of nonempty columns of M. Thus, $f_0(Q), f_3(Q) \leq n$. Finally, by construction, we have $f_{03}(Q) = |M| = \Theta(n\alpha(n))$. □

2.3 Shellable spheres

This section is dedicated to our main construction: a family of shellable, dual shellable 3-spheres with unbounded fatness. We obtain these spheres by gluing together balls. Where we previously constructed a matrix from two smaller matrices, entering ones where columns of the first matrix intersect rows of the second, we now construct a ball from two smaller balls, creating incidences between vertices of the first ball and facets of the second.

We begin by introducing new recursive functions that will describe combinatorial properties of our spheres (§2.3.1). We then give our main construction (§2.3.2). Next, we prove that our spheres are shellable and dual shellable (§2.3.3) and give asymptotic bounds on their face numbers (§2.3.4). We use these bounds to prove our main result: the existence of arbitrarily fat, shellable, dual shellable 3-spheres (§2.3.5). Finally, we conjecture that a subfamily of our spheres can be realized as arbitrarily fat 4-polytopes (§2.3.6).

2.3.1 Auxiliary functions

We aim to construct 3-spheres using a similar double recursion as in Section 2.2. In place of $C(s,t)$, we will use two new functions $K(s,t)$ and $K'(s,t)$.

$K(s,t)$

$s \backslash t$	1	2	3
1	2	2	2
2	4	8	16
3	8	2^{15}	$2^{7\cdot 2^{13}+11}$

$K'(s,t)$

$s \backslash t$	1	2	3
1	5	5	5
2	7	19	43
3	11	$7 \cdot 2^{13} - 5$	$(7 \cdot 2^{12} - 1)2^{7\cdot 2^{13}-3} - 5$

Table 2.3: Values of $K(s,t)$ and $K'(s,t)$ for $s, t < 4$.

Definition 2.3.1. For positive integers s and t, the functions $K(s,t)$ and $K'(s,t)$ are defined recursively as follows.

- $K(1,t) = 2,\ K'(1,t) = 5$ for all t.
- $K(s,1) = 2^s,\ K'(s,1) = 2^s + 3$ for all s.
- For $s, t > 1$,

$$K(s,t) = K(s,t-1)K(s-1, K'(s,t-1)),$$
$$K'(s,t) = K'(s,t-1)K(s-1,K'(s,t-1)) + K'(s-1,K'(s,t-1)).$$

Values of $K(s,t)$ and $K'(s,t)$ for small s and t are given in Table 2.3. We note that there is a more concise way to define $K(s,t)$ and $K'(s,t)$ for $t \geq 0$ by setting $K(s,0) = 2$ and $K'(s,0) = 1$ for $s > 1$, but we avoid this to remain consistent in our use of positive indices.

Observation 2.3.2. It is easy to check that for all positive integers t,

$$K(2,t) = 2^{t+1}, \qquad K'(2,t) = 3 \cdot 2^{t+1} - 5.$$

Lemma 2.3.3. *Let s and t be positive integers.*

i. $K(s,t+1) \geq K(s,t)$.

ii. $K'(s, t+1) \geq K'(s, t)$.

iii. $K'(s, t+1) > 2K'(s, t)$ *for* $s > 1$.

iv. $K'(s, t) > K(s, t)$.

v. $K(s, t) \geq 2^{st-t+1}$.

Proof. We prove i-v in order.

i. By definition, $K(1, t+1) = 2 = K(1, t)$ for all t. If $s > 1$, then

$$K(s, t+1) = K(s, t)K(s-1, K'(s, t)) \geq K(s, t).$$

ii. By definition, $K'(1, t+1) = 5 = K'(1, t)$ for all t. If $s > 1$, then

$$K'(s, t+1) = K'(s, t)K(s-1, K'(s, t)) + K'(s-1, K'(s, t)) > K'(s, t).$$

iii. Let $s > 1$. It follows from i that $K(s-1, t) \geq K(s-1, 1) \geq 2$ for all t. Thus,

$$K'(s, t+1) = K'(s, t)K(s-1, K'(s, t)) + K'(s-1, K'(s, t)) > 2K'(s, t).$$

iv. We know $K'(1, t) = 5 > 2 = K(1, t)$ for all t, and $K'(s, 1) = 2^s + 3 > 2^s = K(s, 1)$ for all s. The cases $s, t > 1$ follow by induction:

$$\begin{aligned} K'(s, t) &= K'(s, t-1)K(s-1, K'(s, t-1)) + K'(s-1, K'(s, t-1)) \\ &> K(s, t-1)K(s-1, K'(s, t-1)) \\ &= K(s, t). \end{aligned}$$

v. For $s = 1$ and any t, we have $K(1, t) = 2 = 2^{st-t+1}$. For $t = 1$ and any s, we have $K(s, 1) = 2^s = 2^{st-t+1}$. The cases $s, t > 1$ follow by induction:

$$K(s, t) = K(s, t-1)K(s-1, K'(s, t-1)) \geq 2^{s(t-1)-t+2}2^{s-1} = 2^{st-t+1}. \qquad \square$$

For convenience, we will generally use Lemma 2.3.3i-ii without explicit reference.

Lemma 2.3.4. *For all positive integers s and t,*

$$1 < \frac{K'(s,t)}{K(s,t)} < 1 + 2^{3-s}.$$

Proof. The inequality $K'(s,t)/K(s,t) > 1$ follows immediately from Lemma 2.3.3iv. It remains to prove that $K'(s,t)/K(s,t) < 1 + 2^{3-s}$.

For $s, t > 1$, the recursive formulae for $K(s,t)$ and $K'(s,t)$ imply

$$\frac{K'(s,t)}{K(s,t)} - \frac{K'(s,t-1)}{K(s,t-1)} = \frac{1}{K(s,t-1)} \cdot \frac{K'(s-1, K'(s,t-1))}{K(s-1, K'(s,t-1))}.$$

Substituting the dummy variable u for $t-1$ and summing from $u = 1$ to $u = t-1$, we find

$$\begin{aligned} \frac{K'(s,t)}{K(s,t)} - \frac{K'(s,1)}{K(s,1)} &= \sum_{u=1}^{t-1} \frac{1}{K(s,u)} \cdot \frac{K'(s-1, K'(s,u))}{K(s-1, K'(s,u))} \\ \Rightarrow \qquad \frac{K'(s,t)}{K(s,t)} &= 1 + 3 \cdot 2^{-s} + \sum_{u=1}^{t-1} \frac{1}{K(s,u)} \cdot \frac{K'(s-1, K'(s,u))}{K(s-1, K'(s,u))}. \qquad (2.2) \end{aligned}$$

We proceed by induction on s. Our base cases are $s = 1, 2$. For $s = 1$, we have $K'(1,t)/K(1,t) = 5/2 < 5 = 1 + 2^{3-s}$. For $s = 2$, we have $K'(2,t)/K(2,t) < 3 = 1 + 2^{3-s}$ (see Observation 2.3.2).

For the inductive step, fix $s > 2$, and suppose that for all t,

$$\frac{K'(s-1,t)}{K(s-1,t)} < 1 + 2^{4-s}.$$

Note that this implies $K'(s-1,t)/K(s-1,t) < 3$.

If $t = 1$, we have $K'(s,1)/K(s,1) = 1 + 3 \cdot 2^{-s} < 1 + 2^{3-s}$, so we are done. If $t > 1$, then by (2.2), the inductive hypothesis, and Lemma 2.3.3v,

$$\begin{aligned} \frac{K'(s,t)}{K(s,t)} &< 1 + 3 \cdot 2^{-s} + 3 \sum_{u=1}^{t-1} \frac{1}{K(s,u)} \\ &\le 1 + 3 \cdot 2^{-s} + 3 \sum_{u=1}^{t-1} 2^{u-su-1} \\ &< 1 + 3 \cdot 2^{-s} + \frac{3}{2^s - 2} \\ &< 1 + 2^{3-s}. \end{aligned}$$

This completes the inductive step. We may conclude that $1 < K'(s,t)/K(s,t) < 1 + 2^{3-s}$ for all s and t. □

Lemma 2.3.5. *For all positive integers s and t,*

$$A(s,t) \leq tK(s,t) < tK'(s,t) < A(s+1,t+2).$$

Proof. The inequality $tK(s,t) < tK'(s,t)$ follows immediately from Lemma 2.3.3iv. Thus, it will suffice to prove that $A(s,t) \leq tK(s,t)$ and $tK'(s,t) < A(s+1,t+2)$.

First, we will prove that $tK(s,t) \geq A(s,t)$. For $s = 1$, this is immediate, as $tK(1,t) = 2t = A(1,t)$.

For $s > 1$, we will prove the stronger statement that $K(s,t) \geq A(s,t)$. We will do so by double induction on s and t, with base cases $s = 2$ and $t = 1$. The base case $s = 2$ is immediate, as $K(2,t) = 2^{t+1} > 2^t = A(2,t)$. The base case $t = 1$ is also immediate, as $K(s,1) = 2 = A(s,1)$.

For the inductive step, fix $s > 2$, and suppose $K(s-1,t) \geq A(s-1,t)$ for all t. Fix $t > 1$ as well, and suppose $K(s,t-1) \geq A(s,t-1)$. Then

$$\begin{aligned} K(s,t) &= K(s,t-1)K(s-1,K'(s,t-1)) \\ &\geq K(s-1,K'(s,t-1)) \\ &\geq A(s-1,A(s,t-1)) \\ &= A(s,t). \end{aligned}$$

This completes the inductive step. We may conclude that $K(s,t) \geq A(s,t)$ for all $s > 1$, so $tK(s,t) \geq A(s,t)$ for all s and t.

Second, we will prove that $tK'(s,t) \leq A(s+1,t+2) - 3$. We will again use double induction on s and t. The base cases are $s = 1,2$ and $t = 1$, which we check in order.

$s = 1$:

$$tK'(1,t) = 5t \leq 2^{t+2} - 3 = A(2,t+2) - 3.$$

$s = 2$:

$$tK'(2,t) = t(3 \cdot 2^{t+1} - 5) < 2^{2(t+1)} - 3 \leq 2^{A(3,t+1)} - 3 = A(3,t+2) - 3.$$

$t = 1$, using Lemma 2.2.4iv:

$$K'(s, 1) = 2^s + 3 \leq 2^{s+2} - 3 \leq A(s + 1, 3) - 3.$$

For the inductive step, fix $s > 2$, and suppose $tK'(s-1, t) \leq A(s, t+2) - 3$ for all t. Fix $t > 1$ as well, and suppose $(t - 1)K'(s, t - 1) \leq A(s + 1, t + 1) - 3$.

By the inductive hypothesis,

$$\begin{aligned} t(K'(s, t-1) + 1) &= tK'(s, t-1) + (t-2) + 2 \\ &\leq (2t-2)K'(s, t-1) + 2 \\ &\leq 2A(s+1, t+1) - 4. \end{aligned}$$

Using the above, Lemma 2.3.3iii, and the inductive hypothesis,

$$\begin{aligned} tK'(s, t) &= t(K'(s, t-1)K(s-1, K'(s, t-1)) + K'(s-1, K'(s, t-1))) \\ &< t(K'(s, t-1) + 1)K'(s-1, K'(s, t-1)) \\ &\leq (2A(s+1, t+1) - 4)K'(s-1, A(s+1, t+1) - 3) \\ &< (A(s+1, t+1) - 2)K'(s-1, A(s+1, t+1) - 2) \\ &\leq A(s, A(s+1, t+1)) - 3 \\ &= A(s+1, t+2) - 3. \end{aligned}$$

This completes the inductive step. We may conclude that $tK'(s, t) \leq A(s + 1, t + 2) - 3$ for all s and t. □

2.3.2 Construction

We will construct a family of CW 3-complexes $X(s, t)$, where s and t range over the positive integers. The complex $X(s, t)$ will be a 3-ball for $t > 1$. Subsequently, we will use $X(s, t)$ to construct our desired family of 3-spheres $S(s, t)$.

We will draw $X(s, t)$ from a "bird's-eye view" in $\mathbb{R}^3$, so the boundary of each facet will have a well-defined top and bottom, as will the boundary of $X(s, t)$ itself for $t > 1$. In our drawing, each ridge and facet will have the shape of a convex polygon, giving $X(s,t)$ the appearance of a polytopal complex. This is an optical illusion; we do not necessarily claim that $X(s,t)$ can be realized with convex polytopes in $\mathbb{R}^3$.

Our construction of $X(s,t)$ will ensure the following (see Figure 2.2).

(a) $X(s,t)$ is a 3-ball for $t > 1$.

(b) Projected onto the page, all vertices of $X(s,t)$ map to points on the unit circle, $X(s,t)$ maps to their convex hull, and the image of each face is the convex hull of the images of its vertices. The projection map acts homeomorphically on the top and bottom of $\partial X(s,t)$, as well as the top and bottom of the boundary of each facet. We call the boundary of the image of $X(s,t)$ the *profile* of $X(s,t)$.

(c) The bottom of $\partial X(s,t)$ contains a fan of triangular faces $\triangle a_1a_2a_3, \triangle a_1a_3a_4, \dots,$ $\triangle a_1 a_{K'(s,t)-1} a_{K'(s,t)}$, where vertices $a_1, \dots, a_{K'(s,t)}$ appear counterclockwise along the profile of $X(s,t)$, and $a_{K'(s,t)}, a_1, a_2$ are consecutive. We call vertex a_1 the *hub*, vertices $a_1, \dots, a_{K'(s,t)}$ the *roots*, and these triangles the *root triangles* of $X(s,t)$.

(d) For each index $1 < i < K'(s,t)$, one of the following holds:

- a_i, a_{i+1} are adjacent on the profile of $X(s,t)$,
- the bottom of $\partial X(s,t)$ contains a triangle $\triangle a_i b_i a_{i+1}$ such that a_i, b_i, a_{i+1} appear consecutively counterclockwise on the profile of $X(s,t)$, or
- the bottom of $\partial X(s,t)$ contains triangles $\triangle a_i b_i a_{i+1}$ and $\triangle b_i d_i a_{i+1}$ belonging to a common facet, vertices a_i, b_i, d_i, a_{i+1} appear counterclockwise on the profile of $X(s,t)$, vertices a_i, b_i are consecutive, vertices d_i, a_{i+1} are consecutive, and vertices a_i, d_i do not belong to a common ridge.

In the latter two cases, we call $\triangle a_i b_i a_{i+1}$ a *tooth* and b_i its *tip*.

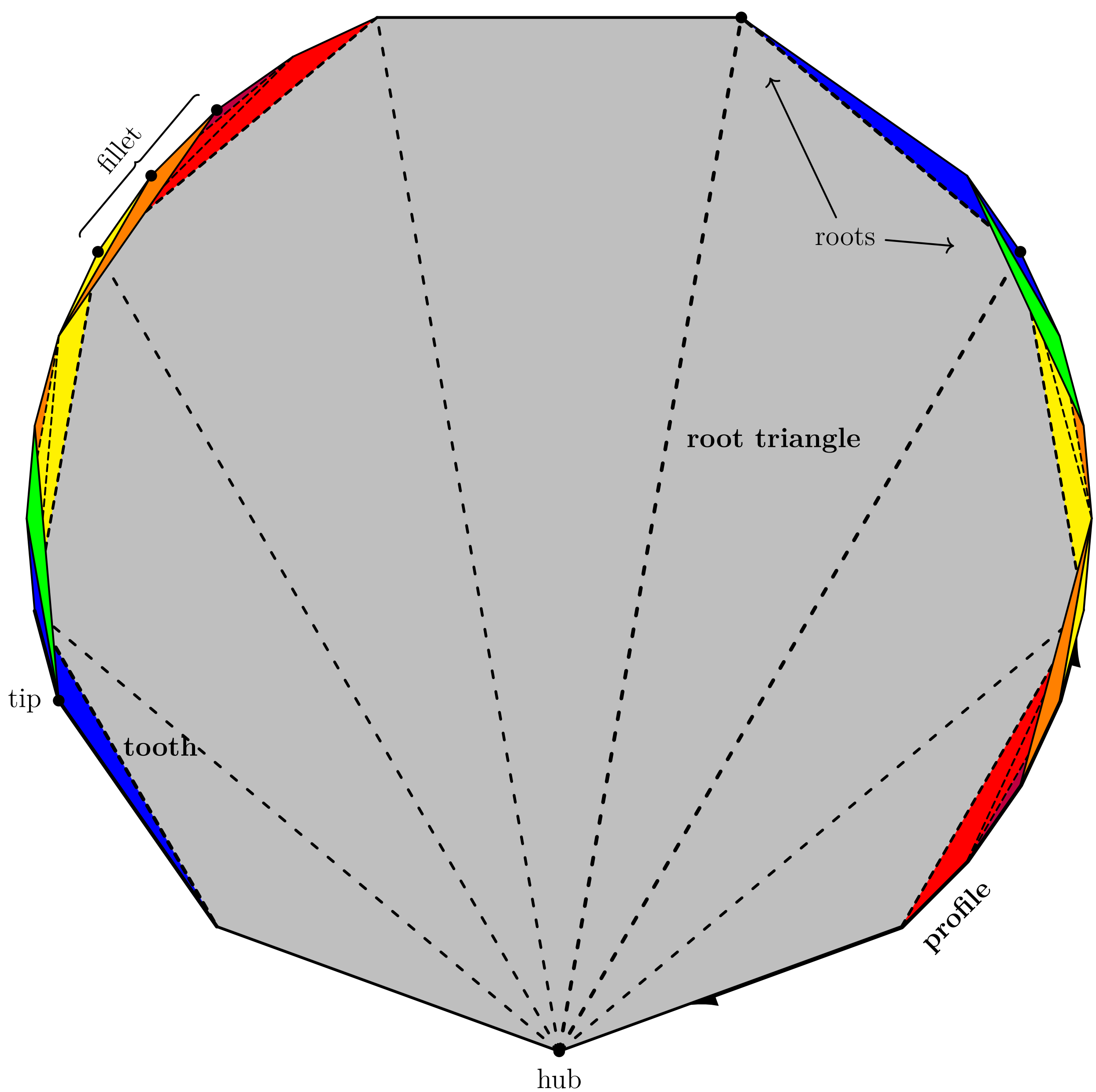

Figure 2.2: Part of the boundary of a hypothetical $X(s,t)$ with nine roots, six teeth, and four fillets of three vertices each. Adjacent ridges drawn in different colors (i.e. any pair of adjacent ridges pictured that includes a root triangle and a tooth, or includes neither) necessarily belong to distinct facets.

(e) For exactly $K(s,t)$ many indices $2 < i < K'(s,t)$, there is a list of consecutive vertices $b_{i-1}, c_1, \dots, c_{t-1}, c_t = a_i, b_i$ ordered counterclockwise along the profile of $X(s,t)$ such that

- c_t is a root, and b_{i-1}, b_i are tips of consecutive teeth that both contain c_t,
- triangles $\triangle b_{i-1}c_1c_2, \triangle b_{i-1}c_2c_3, \dots, \triangle b_{i-1}c_{t-1}c_t$ are on the bottom of $\partial X(s,t)$,
- triangles $\triangle b_ic_1c_2, \triangle b_ic_2c_3, \dots, \triangle b_ic_{t-1}c_t$ are on the top of $\partial X(s,t)$,
- each edge $b_ic_1, \dots, b_ic_t$ belongs to exactly one ridge outside $\{\triangle b_ic_1c_2, \triangle b_ic_2c_3, \dots, \triangle b_ic_{t-1}c_t\}$, which is a triangle whose additional vertex appears counterclockwise from b_i, and
- if we pick two nonconsecutive vertices from the list $(c_1, \dots, c_t)$, these two vertices will never belong to a common facet.

We call the list of vertices $(c_1, \dots, c_t)$ a *fillet* of $X(s,t)$.

(f) The hub a_1 does not belong to a common facet with any non-root vertex.

Observation 2.3.6. $X(s,t)$ has t many vertices in each fillet, $K(s,t)$ many fillets, and $K'(s,t)$ many roots including the hub.

Construction 2.3.7. For all positive integers s and t, we define $X(s,t)$ recursively as follows.

- $X(s,1)$ is a triangulated disc consisting of $2^s + 1$ many root triangles and $2^s + 1$ many teeth (Figure 2.3). We regard this as a "degenerate" 3-complex with no facets, so $\partial X(s,1) = X(s,1)$, and the "top" and "bottom" of $X(s,1)$ both equal $X(s,1)$ in its entirety. We let each root contained in two teeth be a fillet. (The second, third, and fifth statements in (e) are here vacuously true.)

- For $t > 1$, $X(1,t)$ is a 3-ball with hub a_1, roots $a_1, \dots, a_5$ in counterclockwise order, teeth $\triangle a_2b_2a_3, \triangle a_3b_3a_4, \triangle a_4b_4a_5$, and fillets $(c_1, \dots, c_{t-1}, a_3), (c'_1, \dots, c'_{t-1}, a_4)$ (Figure 2.4). It has the following facets:

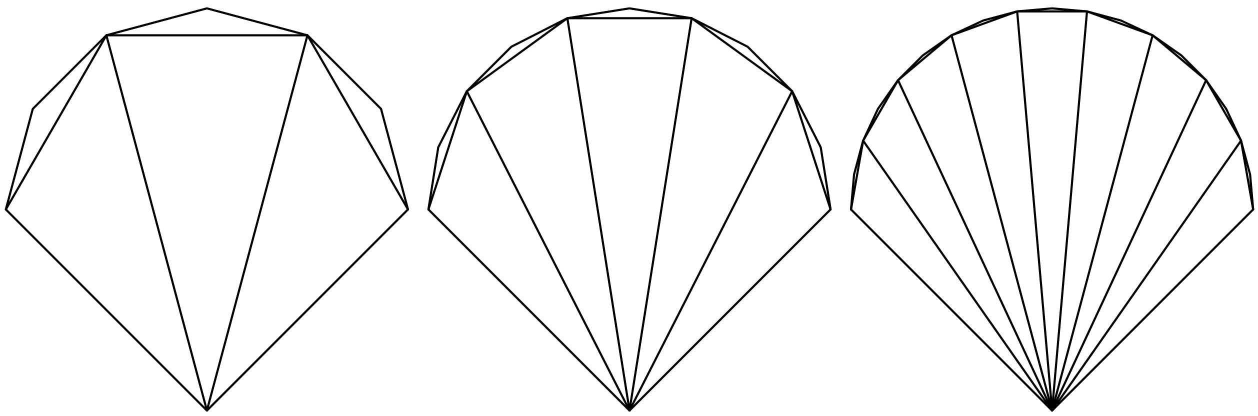

Figure 2.3: From left to right, CW complexes $X(s,1)$ for $s = 1,2,3$.

- a facet on vertices $a_1a_2a_3a_4a_5$ (gray), isomorphic to a connected sum of simplices $a_1a_2a_3a_5$ and $a_1a_3a_4a_5$,
- a facet on vertices $a_2b_2c_{t-1}a_3a_5$ (blue), isomorphic to a connected sum of simplices $a_2b_2a_3a_5$ and $b_2c_{t-1}a_3a_5$,
- a facet on vertices $c_{t-1}a_3b_3c'_{t-1}a_4a_5$ (pink), isomorphic to a connected sum of simplices $c_{t-1}a_3b_3a_5$, $a_3b_3a_4a_5$, and $b_3c'_{t-1}a_4a_5$,
- a facet isomorphic to a simplex on vertices $c'_{t-1}a_4b_4a_5$ (yellow), and
- for each $i = 1, \dots, t-2$, four facets isomorphic to simplices on vertices $b_2c_ic_{i+1}a_5$, $c_ic_{i+1}b_3a_5$, $b_3c'_ic'_{i+1}a_5$, and $c'_ic'_{i+1}b_4a_5$ (not colored).

- For $s, t > 1$, $X(s,t)$ is constructed from $X(s-1, K'(s,t-1))$ and $K(s-1, K'(s,t-1))$ many copies of $X(s,t-1)$. Let o be the hub of $X(s-1, K'(s,t-1))$. First, we reflect $X(s-1, K'(s,t-1))$ horizontally. Next, for each fillet of $X(s-1, K'(s,t-1))$, we do the following (see Figure 2.5).

  1. Label the vertices in the fillet $c_1, \dots, c_{K'(s,t-1)}$ going *clockwise* (as we have reflected $X(s-1, K'(s,t-1))$, this is consistent with our notation in (e)). Let b be

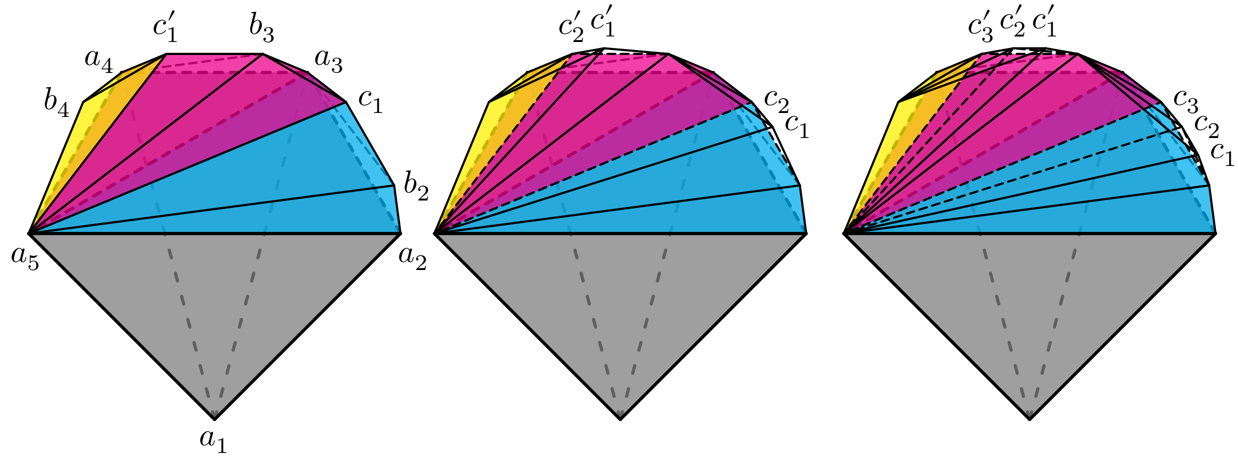

Figure 2.4: From left to right, CW complexes $X(1,t)$ for $t = 2, 3, 4$.

the tip clockwise of our fillet, b' the tip counterclockwise of our fillet, and a the root counterclockwise of b', so $\triangle c_{K'(s,t-1)} b' a$ is a tooth and $\triangle o c_{K'(s,t-1)} a$ a root triangle. By (e), triangles $\triangle bc_2c_1, \triangle bc_3c_2, \dots, \triangle bc_{K'(s,t-1)}c_{K'(s,t-1)-1}$ are on the top of $\partial X(s-1, K'(s,t-1))$, while triangles $\triangle b'c_2c_1, \triangle b'c_3c_2, \dots,$ $\triangle b'c_{K'(s,t-1)}c_{K'(s,t-1)-1}$ are on the bottom of $\partial X(s-1, K'(s,t-1))$.

For $i = 1, \dots, K'(s,t-1) - 1$, let Y_i be the facet containing $\triangle bc_{i+1}c_i$, and let Z_i be the unique ridge such that $bc_i \subset Z_i \subset Y_i$ and $Z_i \neq \triangle bc_{i+1}c_i$. By (e), each Z_i is a triangle lying to the left of $\overrightarrow{bc_i}$, and $Y_{i-1} \cap Y_i = Z_i$ for $i > 1$.

2. On the bottom, attach a new facet isomorphic to a simplex on vertices o, a, b', $c_{K'(s,t-1)}$, introducing ridges $\triangle ob'a$ and $\triangle oc_{K'(s,t-1)}b'$. By (f), vertices o and b' were not previously on a common facet, so we maintain strong regularity.

3. On the bottom, attach a new facet on vertices $o, b', c_1, \dots, c_{K'(s,t-1)}$, isomorphic to a connected sum of simplices $oc_2c_1b', oc_3c_2b', \dots, oc_{K'(s,t-1)}c_{K'(s,t-1)-1}b'$, introducing ridges $\triangle oc_1b', \triangle oc_2c_1, \triangle oc_3c_2, \dots, \triangle oc_{K'(s,t-1)}c_{K'(s,t-1)-1}$. By (e)–(f), no two nonconsecutive vertices in the list $(c_1, \dots, c_{K'(s,t-1)}, o)$ were originally on a common facet, so we maintain strong regularity. Vertices $b', c_1, \dots, c_{K'(s,t-1)-1}$

will become roots of $X(s,t)$. The triangles $\triangle ob'a, \triangle oc_1b', \triangle oc_2c_1, \triangle oc_3c_2, \dots,$ $\triangle oc_{K'(s,t-1)}c_{K'(s,t-1)-1}$ will replace $\triangle oc_{K'(s,t-1)}a$ as root triangles.

4. Draw new vertices $\alpha_1, \dots, \alpha_{K'(s,t-1)-1}$ and $\beta_1, \dots, \beta_{K'(s,t-1)-2}$ so that projected onto the page, the following vertices lie in clockwise order on the unit circle:

$$\begin{aligned}&c_1,\ \alpha_1,\ \beta_1,\\ &\dots,\\ &c_{K'(s,t-1)-2},\ \alpha_{K'(s,t-1)-2},\ \beta_{K'(s,t-1)-2},\\ &c_{K'(s,t-1)-1},\ \alpha_{K'(s,t-1)-1},\ c_{K'(s,t-1)}.\end{aligned}$$

For each $i = 1, \dots, K'(s,t-1)-1$, extend ridge Z_i to include α_i, replacing edge bc_i with edges $b\alpha_i, \alpha_i c_i$. For $i < K'(s,t-1)-1$, extend facet Y_i to include α_i, β_i, and α_{i+1}, keeping the extended Z_i and Z_{i+1} as faces, and replacing $\triangle bc_{i+1}c_i$ with triangles $\triangle b\alpha_{i+1}\alpha_i, \triangle\alpha_{i+1}\beta_i\alpha_i, \triangle\alpha_{i+1}c_{i+1}\beta_i$ on the top and $\triangle c_{i+1}\beta_i c_i, \triangle\beta_i\alpha_i c_i$ on the bottom of ∂Y_i. For $i = K'(s,t-1)-1$, extend facet Y_i to include α_i, keeping the extended Z_i as a face, and replacing $\triangle bc_{i+1}c_i$ with triangles $\triangle bc_{i+1}\alpha_i$ on the top and $\triangle c_{i+1}\alpha_i c_i$ on the bottom of ∂Y_i. Triangles

$$\begin{aligned}&\triangle c_2\beta_1c_1,\\ &\dots,\\ &\triangle c_{K'(s,t-1)-1}\beta_{K'(s,t-1)-2}c_{K'(s,t-1)-2},\\ &\triangle c_{K'(s,t-1)}\alpha_{K'(s,t-1)-1}c_{K'(s,t-1)-1}\end{aligned}$$

will replace $\triangle c_{K'(s,t-1)}b'a$ as teeth of $X(s,t)$.

5. Attach a copy of $X(s,t-1)$ to our complex by identifying the hub of $X(s,t-1)$ with vertex b, the non-hub roots of $X(s,t-1)$ with vertices $\alpha_1, \dots, \alpha_{K'(s,t-1)-1}$, and the teeth of $X(s,t-1)$ with a subset of the triangles

$$\triangle\alpha_2\beta_1\alpha_1,\ \dots,\ \triangle\alpha_{K'(s,t-1)-1}\beta_{K'(s,t-1)-2}\alpha_{K'(s,t-1)-2}.$$

By construction, no two nonconsecutive vertices in the list $(\alpha_1, \dots, \alpha_{K'(s,t-1)-1})$ were previously on a common facet, so our hub and root identifications preserve strong regularity. By (f), no tip of $X(s, t-1)$ was previously on a common facet with the hub of $X(s, t-1)$, so our tooth identifications preserve strong regularity as well.

In each fillet of $X(s, t-1)$, we have identified the last vertex with α_i for some $1 < i < K'(s, t-1) - 1$. Append to each fillet of $X(s, t-1)$ the corresponding vertex c_i to obtain a fillet of $X(s, t)$.

We repeat these steps $K(s-1, K'(s, t-1))$ many times, once for each fillet of $X(s-1, K'(s, t-1))$. The original complex $X(s-1, K'(s, t-1))$ contributes $K'(s-1, K'(s, t-1))$ many roots, and each repetition adds $K(s, t-1)$ many fillets and $K'(s, t-1)$ many roots. The resulting complex therefore has $K(s, t-1)K(s-1, K'(s, t-1)) = K(s, t)$ many fillets and $K'(s, t-1)K(s-1, K'(s, t-1)) + K'(s-1, K'(s, t-1)) = K'(s, t)$ many roots, as desired. Each fillet contains $t-1$ many vertices from $X(s, t-1)$ and one new vertex, for the desired total of t.

Finally, consider any tooth $\triangle aba'$ of $X(s-1, K'(s, t-1))$ remaining on the boundary of our complex, where a, b, a' appear counterclockwise along the profile. If a, b, a' are consecutive, we simply let $\triangle aba'$ be a tooth of $X(s, t)$. Otherwise, by (d), the bottom of our complex contains another triangle $\triangle adb$ belonging to the same facet, vertices a, d are consecutive, vertices b, a' are consecutive, and vertices a', d do not belong to a common ridge. In this case, we delete edge ab and the two triangles, draw a new edge $a'd$ and triangles $\triangle ada', \triangle dba'$, and let $\triangle ada'$ be a tooth of $X(s, t)$. We repeat this for every tooth of $X(s-1, K'(s, t-1))$ remaining on the boundary of our complex.

The proofs of (a)–(f) are routine double induction. We sketch the proof of (a) below and omit (b)–(f).

We can easily check that $X(s, 1)$ is a disc for all s and that $X(1, t)$ is a 3-ball for all $t > 1$. In the cases $s, t > 1$, recall that we obtain $X(s, t)$ from $X(s-1, K'(s, t-1))$ by repeatedly

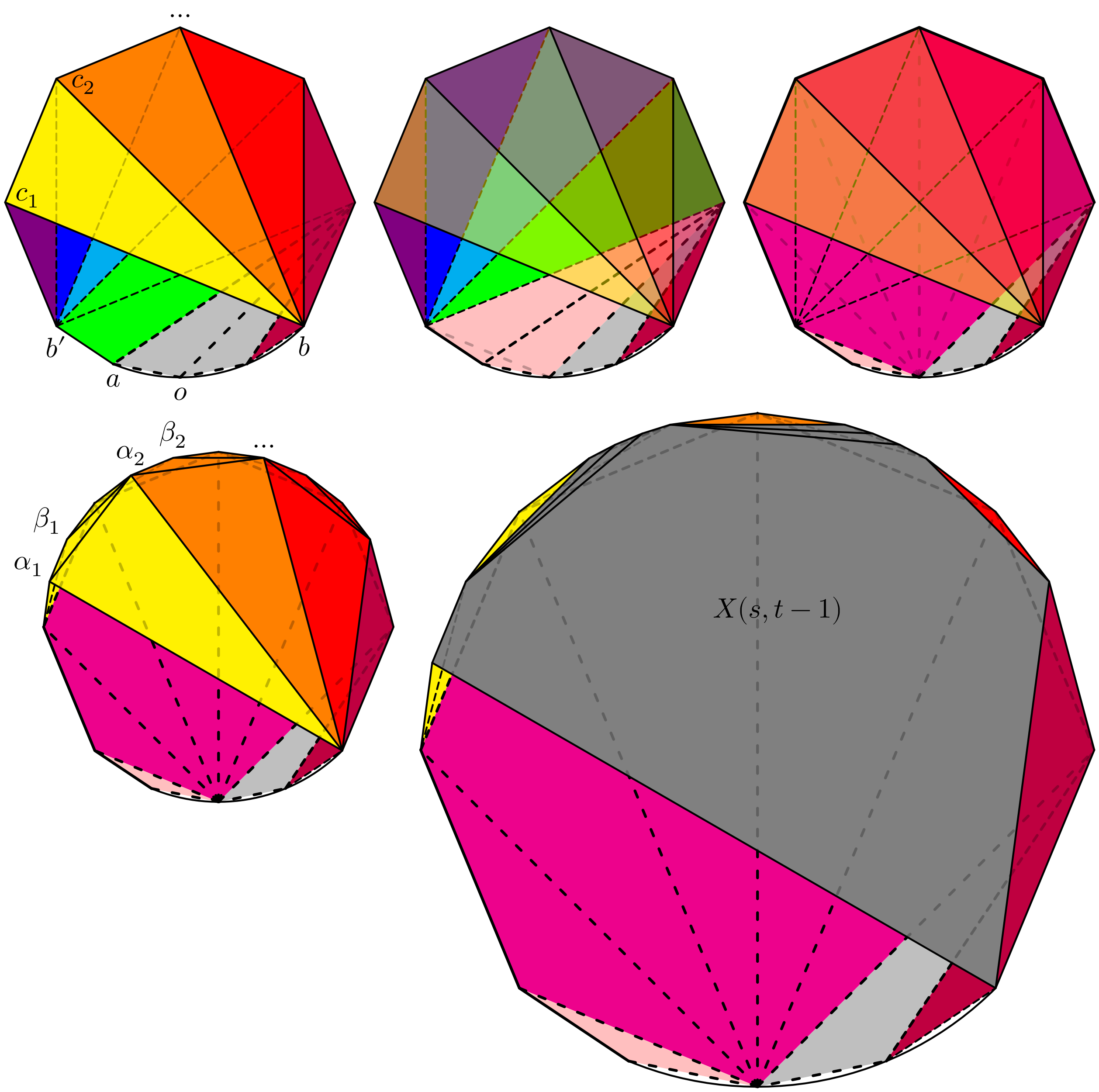

Figure 2.5: From left to right and then down, attaching a copy of $X(s, t-1)$ to $X(s-1, K'(s, t-1))$. We have "zoomed in" on one fillet of $X(s-1, K'(s, t-1))$ and the teeth neighboring it. The numerous other vertices of $X(s-1, K'(s, t-1))$, if pictured, would lie along the bottom arc.

subdividing faces, attaching a facet along a disc, or attaching a copy of $X(s, t-1)$ along a disc.

If we subdivide faces of a 3-ball, or if we attach two 3-balls along a disc, the result is a 3-ball. Thus, if $X(s, t-1)$ and $X(s-1, K'(s, t-1))$ are both 3-balls, then $X(s, t)$ is a 3-ball and we are done. This leaves only the case $t = 2$, in which $X(s-1, K'(s, t-1))$ is a 3-ball and $X(s, t-1)$ is a disc. However, if $t = 2$, then the disc along which we attach our two complexes is equal to $X(s, t-1)$ itself; thus, the result is still a 3-ball. It follows that $X(s, t)$ is a 3-ball for all $t > 1$.

At last, we use $X(s, t)$ to construct our family of 3-spheres $S(s, t)$. Note that all vertices of $X(s, t)$ lie on the boundary of $X(s, t)$.

Definition 2.3.8. For all s and all $t > 1$, we define $S(s, t)$ as the 3-sphere obtained by coning over the boundary of $X(s, t)$.

2.3.3 Proofs of shellability

We will now prove that $S(s, t)$ is shellable and dual shellable.

Proposition 2.3.9. *$S(s, t)$ is shellable for all s and all $t > 1$.*

Proof. First, we will show that for all s and t, the facets of $X(s, t)$ can be listed in some order $Y_1, \dots, Y_n$ such that $Y_i \cap \bigcup_{j=1}^{i-1} Y_j$ is contained in the bottom of ∂Y_i for $i = 1, \dots, n$. Intuitively, this order represents a way of "stacking" facets from lowest to highest.

We proceed by double induction on s and t. For the base case $s = 1, t > 1$, we may order the facets of $X(1, t)$ as listed in Construction 2.3.7. The base case $t = 1$ is trivial, as $X(s, 1)$ has no facets.

For the inductive step, fix $s, t > 1$, and assume that both $X(s, t-1)$ and $X(s-1, K'(s, t-1))$ admit such an order. Recall Steps 1-5 of Construction 2.3.7. We begin with the facets of $X(s-1, K'(s, t-1))$ listed in the given order. On Step 2, we prepend the new facet to our list. On Step 3, we do the same. After Step 4, our list still satisfies the required property, as the intersection of any two facets either stays the same or grows from a triangle into a quadrilateral (not changing which facet is on top). Finally, on Step 5, we list the facets of our new copy of $X(s, t-1)$ in the given order and append them to our current list. We do this process for each repetition of Steps 1-5 to obtain the desired list of facets of $X(s,t)$. This completes the inductive step.

Now, fix s and $t > 1$. Let $Y_1, \dots, Y_n$ be the list of facets of $X(s,t)$ described above. Let v be the new vertex introduced when constructing $S(s,t)$ from $X(s,t)$. Then each facet of $S(s,t)$ containing v is isomorphic to a pyramid with apex v and base contained in $\partial X(s,t)$.

Let W be the union of all pyramidal facets having apex v and base contained in the bottom of $\partial X(s,t)$. Then $S(s,t)/v$ is a 2-sphere, and $W/v \subset S(s,t)/v$ is a closed disc. Thus, by Lemma 2.1.14, there exists a shelling $(T_1/v, \dots, T_k/v, \dots, T_m/v)$ of $S(s,t)/v$ such that $T_1, \dots, T_k$ are the facets of W.

We will use Lemma 2.1.16i to prove that

$$(T_1,\ \dots,\ T_k,\ Y_1,\ \dots,\ Y_n,\ T_{k+1},\ \dots,\ T_m)$$

is a shelling of $S(s,t)$. For $i = 1, \dots, n$, we can see that

$$Y_i \cap \left(\bigcup_{j=1}^{k} T_j \cup \bigcup_{j=1}^{i-1} Y_j \right)$$

is exactly the bottom of ∂Y_i, which is homeomorphic to a disc. Meanwhile, since $(T_1/v, \dots, T_m/v)$ is a shelling of $S(s,t)/v$, for $i = 2, \dots, m-1$, we know that $T_i/v \cap \bigcup_{j=1}^{i-1} T_j/v$ is a union of consecutive edges of T_i/v homeomorphic to an interval. Thus, $T_i \cap \bigcup_{j=1}^{i-1} T_j$ is a union of consecutive lateral triangles of T_i homeomorphic to a disc.

Consequently, for $i = k+1, \dots, m-1$,

$$T_i \cap \left(\bigcup_{j=1}^{k} T_j \cup \bigcup_{j=1}^{n} Y_j \cup \bigcup_{j=k+1}^{i-1} T_j \right)$$

is a union of consecutive lateral triangles of T_i homeomorphic to a disc, plus the base of T_i. This union is itself homeomorphic to a disc.

By Lemma 2.1.16i, we may conclude that the above is a shelling of $S(s,t)$. ☐

Proposition 2.3.10. *$S(s,t)$ is dual shellable for all s and all $t > 1$.*

Proof. Suppose $t > 1$. Let $v_1, \dots, v_{n-1}$ be the vertices of $X(s,t)$ in counterclockwise order, where v_1 is the hub of $X(s,t)$. Let v_n be the new vertex introduced when constructing $S(s,t)$ from $X(s,t)$. We will use Lemma 2.1.16ii to show that $(v_1, \dots, v_n)$ is a dual shelling of $S(s,t)$.

Fix an arbitrary $1 < i < n$. We will first prove that $S(s,t)/v_i \cap \bigcup_{j=1}^{i-1} \operatorname{st} v_j$ is a union of vertices of $S(s,t)/v_i$ and their stars. Let Z be a face of $S(s,t)$ such that $v_i, v_j \in Z$ for some $j < i$. List the vertices of Z in index order, and let v_k be the vertex of Z immediately preceding v_i, so $j \leq k < i$. If $Z \subseteq X(s,t)$, then by (b), the image of Z projected onto the page is the convex hull of the images of its vertices. Thus, $v_i v_k$ is an edge of Z. If instead $Z \nsubseteq X(s,t)$, then Z is isomorphic to a pyramid with apex v_n and base contained in $X(s,t)$. By applying our previous argument to the base of Z, we see again that $v_i v_k$ is an edge of Z.

We have shown that for all $j < i$ and all faces $Z \subseteq S(s,t)$, if $\operatorname{relint} Z \subseteq \operatorname{st} v_i \cap \operatorname{st} v_j$, then $\operatorname{relint} Z \subset \operatorname{st}(v_i v_k)$ for some edge $v_i v_k$ with $j \leq k < i$. Thus, $\operatorname{st} v_i \cap \bigcup_{j=1}^{i-1} \operatorname{st} v_j$ is a union of stars of edges incident to v_i. It follows that $S(s,t)/v_i \cap \bigcup_{j=1}^{i-1} \operatorname{st} v_j$ is a union of stars of vertices of $S(s,t)/v_i$.

Second, we will prove that $S(s,t)/v_i \cap \bigcup_{j=1}^{i-1} \operatorname{st} v_j$ is homeomorphic to an open disc. Let w be the vertex of $S(s,t)/v_i$ induced by edge $v_i v_n$. If $i = n-1$, then $S(s,t)/v_i \cap \bigcup_{j=1}^{i-1} \operatorname{st} v_j = S(s,t)/v_i \backslash w$, so we are done.

Suppose $i < n-1$. In our picture of $X(s,t)$, let H be a plane perpendicular to the page and intersecting the relative interiors of edges $v_{i-1}v_i, v_i v_{i+1}$. Let H^+ be the resulting open half-space containing v_i. As a consequence of (b), any k-face of $X(s,t)$ containing v_i must intersect $\operatorname{cl} H^+$ on a closed k-ball. Thus, by definition, the subdivision of $H \cap X(s,t)$ induced by $X(s,t)$ is exactly the vertex figure $X(s,t)/v_i$. Let $U = X(s,t)/v_i \cap \bigcup_{j=1}^{i-1} (\operatorname{st} v_j \cap X(s,t))$.

Let J be a second plane perpendicular to the page, intersecting v_i and the relative interior of edge $v_{n-1}v_1$. Let J^+ be the resulting open half-space containing $v_1, \dots, v_{i-1}$, and let $U' = X(s,t)/v_i \cap J^+$. Then U' is homeomorphic to a half-disc with $U' \cap \partial U' = \partial(X(s,t)/v_i) \cap J^+$. For any edge $v_i v_j$ with $j < i$, we have $\operatorname{relint} v_i v_j \subset J^+$, so the vertex $v_i v_j / v_i$ of $X(s,t)/v_i$ is in U'.

Let Z/v_i be a face of $X(s,t)/v_i$. If $\operatorname{relint}(Z/v_i) \subseteq U$, then as previously shown, $v_i v_j / v_i \in$

Z_i/v_i for some edge v_iv_j with $j < i$. Conversely, if $\operatorname{relint}(Z/v_i) \not\subseteq U$, then $Z \cap \{v_1, \dots, v_{i-1}\} = \emptyset$. In this case, the image of Z on the page is a convex polygon with vertices outside the image of J^+; thus, $Z \cap J^+ = \emptyset$, so $Z/v_i \cap U' = \emptyset$. We may conclude that the faces of $X(s,t)/v_i$ with relative interiors contained in U are exactly the faces that nontrivially intersect U'. Consequently, U deformation retracts onto U', so U is simply connected.

We obtain the vertex figure $S(s,t)/v_i$ by coning over the boundary of $X(s,t)/v_i$ at w. For each nonempty k-face (i.e. edge or vertex) e of $\partial(X(s,t)/v_i)$, let $\hat{e}$ be the $(k+1)$-face of $S(s,t)/v_i$ containing e and w. Then

$$S(s,t)/v_i \cap \bigcup_{j=1}^{i-1} \operatorname{st} v_j = U \cup \bigcup \{\operatorname{relint} \hat{e} \mid \operatorname{relint} e \subseteq U \cap \partial U\}.$$

It follows that $S(s,t)/v_i \cap \bigcup_{j=1}^{i-1} \operatorname{st} v_j$ deformation retracts onto U. Thus, $S(s,t)/v_i \cap \bigcup_{j=1}^{i-1} \operatorname{st} v_j$ is simply connected. We may conclude that $S(s,t)/v_i \cap \bigcup_{j=1}^{i-1} \operatorname{st} v_j$ is homeomorphic to an open disc by the Riemann mapping theorem (see [43]).

For each $1 < i < n$, we have shown that $S(s,t)/v_i \cap \bigcup_{j=1}^{i-1} \operatorname{st} v_j$ is a union of vertices of $S(s,t)/v_i$ and their stars and is homeomorphic to an open disc. By Lemma 2.1.16ii, $(v_1, \dots, v_n)$ is therefore a dual shelling of $S(s,t)$. $\square$

2.3.4 Face asymptotics

In this subsection, we will derive recursive formulae for the face numbers of $S(s,t)$ and determine their asymptotic rates of growth. Our formulae will depend on $K(s,t)$, $K'(s,t)$, and one more function that we now define.

Recall that when constructing $X(s,t)$ from $X(s-1, K'(s,t-1))$, each time we attach a copy of $X(s,t-1)$, we introduce exactly two new root triangles that do not correspond with a tooth.

Definition 2.3.11. For all positive integers s and t, we define $R(s,t)$ as the number of root triangles minus the number of teeth in $X(s,t)$. Equivalently,

- $R(1,t) = 0$ for all t.

- $R(s, 1) = 0$ for all s.

- $R(s, t) = 2K(s-1, K'(s, t-1)) + R(s-1, K'(s, t-1))$ for $s, t > 1$.

Lemma 2.3.12. *For all $s, t > 1$,*

$$\frac{2K(s,t)}{K(s,t-1)} \leq R(s,t) < \frac{3K(s,t)}{K(s,t-1)}.$$

Proof. First, we will show that $R(s, t) < K(s, t)$ for all s and t. This is clearly true if $s = 1$ or if $t = 1$, as $R(s, t)$ then equals zero. The cases $s, t > 1$ follow by induction:

$$\begin{aligned} R(s,t) &= 2K(s-1, K'(s,t-1)) + R(s-1, K'(s,t-1)) \\ &< 3K(s-1, K'(s,t-1)) \\ &< K(s,t-1)K(s-1, K'(s,t-1)) \\ &= K(s,t). \end{aligned}$$

The inequality $3 < K(s, t-1)$ used above is a consequence of Lemma 2.3.3v.

We have shown that $R(s, t) < K(s, t)$ for all s and t. For $s, t > 1$, it follows that

$$\frac{2K(s,t)}{K(s,t-1)} = 2K(s-1, K'(s,t-1)) \leq R(s,t) < 3K(s-1, K'(s,t-1)) = \frac{3K(s,t)}{K(s,t-1)}. \qquad \square$$

Definition 2.3.13. For $i = 0, 1, 2, 3$ and positive integers s and t, we define functions $F_i(s, t)$ recursively as follows. We denote by $F(s, t)$ the vector $(F_0(s, t), F_1(s, t), F_2(s, t), F_3(s, t))$.

- $F(1, t) = (2t + 7,\ 10t + 15,\ 16t + 12,\ 8t + 4)$ for all $t > 1$.

- $F(s, 1) = (2^{s+1} + 5,\ 3 \cdot 2^{s+1} + 9,\ 2^{s+3} + 8,\ 2^{s+2} + 4)$ for all s.

- For $s, t > 1$,

$$\begin{aligned} F_0(s,t) = K(s-1, K'(s,t-1))[F_0(s,t-1) + R(s,t-1) - 2] \\ + F_0(s-1, K'(s,t-1)), \end{aligned}$$

$$
\begin{aligned}
F_1(s,t) = K(s-1, K'(s,t-1))[F_1(s,t-1) + 3K'(s,t-1) + 3R(s,t-1) - 4] \\
+ F_1(s-1, K'(s,t-1)), \\
F_2(s,t) = K(s-1, K'(s,t-1))[F_2(s,t-1) + 3K'(s,t-1) + 4R(s,t-1) + 1] \\
+ F_2(s-1, K'(s,t-1)), \\
F_3(s,t) = K(s-1, K'(s,t-1))[F_3(s,t-1) + 2R(s,t-1) + 3] \\
+ F_3(s-1, K'(s,t-1)).
\end{aligned}
$$

Proposition 2.3.14. *The f-vector of $S(s,t)$ agrees with $F(s,t)$ for all positive integers s and $t > 1$.*

Proof. Let $s, t > 1$. Recall Steps 1–5 of Construction 2.3.7, which we repeat for each fillet of $X(s-1, K'(s,t-1))$ when constructing $X(s,t)$. In Steps 1–4, we add the following numbers of faces to $X(s-1, K'(s,t-1))$ and $\partial X(s-1, K'(s,t-1))$.

i	total i-faces added (Steps 1–4)	boundary i-faces added (Steps 1–4)
0	$2K'(s,t-1) - 3$	$2K'(s,t-1) - 3$
1	$7K'(s,t-1) - 10$	$6K'(s,t-1) - 10$
2	$5K'(s,t-1) - 5$	$4K'(s,t-1) - 7$
3	2	

In Step 5, we attach a copy of $X(s,t-1)$ along a disc D with

$$
\begin{aligned}
f(D) = (2K'(s,t-1) - R(s,t-1) - 2,\ 4K'(s,t-1) - 2R(s,t-1) - 7, \\
2K'(s,t-1) - R(s,t-1) - 4), \\
f(\partial D) = (2K'(s,t-1) - R(s,t-1) - 2,\ 2K'(s,t-1) - R(s,t-1) - 2).
\end{aligned}
$$

If $t > 2$, we thus add the following numbers of faces to $X(s-1, K'(s,t-1))$ and $\partial X(s-1, K'(s,t-1))$.

i	total i-faces added (Step 5)	boundary i-faces added (Step 5)
0	$f_0(X(s,t-1)) - 2K'(s,t-1) + R(s,t-1) + 2$	$f_0(\partial X(s,t-1)) - 2K'(s,t-1) + R(s,t-1) + 2$
1	$f_1(X(s,t-1)) - 4K'(s,t-1) + 2R(s,t-1) + 7$	$f_1(\partial X(s,t-1)) - 6K'(s,t-1) + 3R(s,t-1) + 12$
2	$f_2(X(s,t-1)) - 2K'(s,t-1) + R(s,t-1) + 4$	$f_2(\partial X(s,t-1)) - 4K'(s,t-1) + 2R(s,t-1) + 8$
3	$f_3(X(s,t-1))$	

The case $t = 2$ is slightly different: every face of $X(s,1)$ is identified with an existing face, so we do not add any faces in Step 5. The middle column above is still consistent, as

$$f(X(s,1)) = (2^{s+1}{+}4,\ 2^{s+2}{+}5,\ 2^{s+1}{+}2,\ 0) = (2K'(s,1){-}2,\ 4K'(s,1){-}7,\ 2K'(s,1){-}4,\ 0),$$

making all values in the middle column zero. However, the right column assumes that interior faces of D will be in the interior of our 3-complex after identification, which is only true for $t > 2$. Nevertheless, we can make the right column consistent with $t = 2$ by redefining $f(\partial X(s,1))$ as

$$f(\partial X(s,1)) = (2^{s+1}{+}4,\ 3{\cdot}2^{s+1}{+}6,\ 2^{s+2}{+}4) = (2K'(s,1){-}2,\ 6K'(s,1){-}12,\ 4K'(s,1){-}8),$$

making all values in the right column zero. This is what $f(\partial X(s,1))$ will mean for the remainder of our proof; the abuse of notation is harmless, as $S(s,t)$ is only defined for $t > 1$.

For all $s,t > 1$, we have added the following numbers of faces to $X(s-1, K'(s,t-1))$ and $\partial X(s-1, K'(s,t-1))$ in Steps 1–5.

i	total i-faces added (Steps 1–5)	boundary i-faces added (Steps 1–5)
0	$f_0(X(s,t-1)) + R(s,t-1) - 1$	$f_0(\partial X(s,t-1)) + R(s,t-1) - 1$
1	$f_1(X(s,t-1)) + 3K'(s,t-1) + 2R(s,t-1) - 3$	$f_1(\partial X(s,t-1)) + 3R(s,t-1) + 2$
2	$f_2(X(s,t-1)) + 3K'(s,t-1) + R(s,t-1) - 1$	$f_2(\partial X(s,t-1)) + 2R(s,t-1) + 1$
3	$f_3(X(s,t-1)) + 2$	

For all s and t, let

$$\begin{aligned}
F_0'(s,t) &= f_0(X(s,t)) + 1,\\
F_1'(s,t) &= f_1(X(s,t)) + f_0(\partial X(s,t)),\\
F_2'(s,t) &= f_2(X(s,t)) + f_1(\partial X(s,t)),\\
F_3'(s,t) &= f_3(X(s,t)) + f_2(\partial X(s,t)),
\end{aligned}$$

and $F'(s,t) = (F_0'(s,t), F_1'(s,t), F_2'(s,t), F_3'(s,t))$. Recall that for $t > 1$, we obtain the sphere $S(s,t)$ by coning over the boundary of $X(s,t)$, so $f(S(s,t)) = F'(s,t)$.

We can observe that $f(X(1,t)) = (2t+6, 8t+9, 10t, 4t-4)$ and $f(\partial X(1,t)) = (2t+6, 6t+12, 4t+8)$ for all $t > 1$, so

$$F'(1,t) = (2t+7,\ 10t+15,\ 16t+12,\ 8t+4) = F(1,t).$$

As previously discussed, for all s, we have $f(X(s,1)) = (2^{s+1}+4, 2^{s+2}+5, 2^{s+1}+2, 0)$ and $f(\partial X(s,1)) = (2^{s+1}+4, 3\cdot 2^{s+1}+6, 2^{s+2}+4)$. Thus,

$$F'(s,1) = (2^{s+1}+5,\ 3\cdot 2^{s+1}+9,\ 2^{s+3}+8,\ 2^{s+2}+4) = F(s,1).$$

For $s, t > 1$, our preceding computations yield

$$\begin{aligned}
F_0'(s,t) &= K(s-1, K'(s,t-1))[F_0'(s,t-1) + R(s,t-1) - 2]\\
&\qquad + F_0'(s-1, K'(s,t-1)),\\
F_1'(s,t) &= K(s-1, K'(s,t-1))[F_1'(s,t-1) + 3K'(s,t-1) + 3R(s,t-1) - 4]\\
&\qquad + F_1'(s-1, K'(s,t-1)),\\
F_2'(s,t) &= K(s-1, K'(s,t-1))[F_2'(s,t-1) + 3K'(s,t-1) + 4R(s,t-1) + 1]\\
&\qquad + F_2'(s-1, K'(s,t-1)),\\
F_3'(s,t) &= K(s-1, K'(s,t-1))[F_3'(s,t-1) + 2R(s,t-1) + 3]\\
&\qquad + F_3'(s-1, K'(s,t-1)).
\end{aligned}$$

These match the recursive formulae for $F(s,t)$. Thus, $F'(s,t) = F(s,t)$ for all s and t. We may conclude that the f-vector of $S(s,t)$ agrees with $F(s,t)$ for all s and $t > 1$. $\square$

Proposition 2.3.15. *For all s and t with $t > s$,*

$$1 < \frac{F_0(s,t)}{tK(s,t)} < \frac{F_3(s,t)}{tK(s,t)} < 72, \tag{2.3}$$

$$\frac{3s}{2} < \frac{F_1(s,t)}{tK(s,t)} < \frac{F_2(s,t)}{tK(s,t)} < 158s. \tag{2.4}$$

Proof. Observe that for all $t > 1$ and for all s, respectively,

$$\frac{F(1,t)}{tK(1,t)} = \left(1 + \frac{7}{2t},\ 5 + \frac{15}{2t},\ 8 + \frac{6}{t},\ 4 + \frac{2}{t}\right),$$

$$\frac{F(s,1)}{K(s,1)} = (2 + 5 \cdot 2^{-s},\ 6 + 9 \cdot 2^{-s},\ 8 + 2^{3-s},\ 4 + 2^{2-s}).$$

For $s, t > 1$, we may rearrange the formula for $F_0(s,t)$ to obtain

$$\frac{F_0(s,t)}{K(s,t)} - \frac{F_0(s,t-1)}{K(s,t-1)} = \frac{R(s,t-1)-2}{K(s,t-1)} + \frac{F_0(s-1,K'(s,t-1))}{K(s,t)}.$$

Substituting the dummy variable u for $t-1$ and summing from $u = 1$ to $u = t-1$, we get

$$\frac{F_0(s,t)}{K(s,t)} - \frac{F_0(s,1)}{K(s,1)} = \sum_{u=1}^{t-1} \frac{R(s,u)-2}{K(s,u)} + \sum_{u=1}^{t-1} \frac{F_0(s-1,K'(s,u))}{K(s,u+1)}.$$

Equivalently,

$$\frac{F_0(s,t)}{K(s,t)} = 2 + 5 \cdot 2^{-s} + \sum_{u=1}^{t-1} \frac{R(s,u)-2}{K(s,u)} + \sum_{u=1}^{t-1} \frac{K'(s,u)}{K(s,u)} \cdot \frac{F_0(s-1,K'(s,u))}{K'(s,u)K(s-1,K'(s,u))}. \tag{2.5}$$

Through similar manipulations, we find that for all $s, t > 1$,

$$\frac{F_1(s,t)}{K(s,t)} = 6 + 9 \cdot 2^{-s} + \sum_{u=1}^{t-1} \frac{3R(s,u)-4}{K(s,u)} + \sum_{u=1}^{t-1} \frac{K'(s,u)}{K(s,u)} \left[3 + \frac{F_1(s-1,K'(s,u))}{K'(s,u)K(s-1,K'(s,u))}\right], \tag{2.6}$$

$$\frac{F_2(s,t)}{K(s,t)} = 8 + 2^{3-s} + \sum_{u=1}^{t-1} \frac{4R(s,u)+1}{K(s,u)} + \sum_{u=1}^{t-1} \frac{K'(s,u)}{K(s,u)} \left[3 + \frac{F_2(s-1,K'(s,u))}{K'(s,u)K(s-1,K'(s,u))}\right], \tag{2.7}$$

$$\frac{F_3(s,t)}{K(s,t)} = 4 + 2^{2-s} + \sum_{u=1}^{t-1} \frac{2R(s,u)+3}{K(s,u)} + \sum_{u=1}^{t-1} \frac{K'(s,u)}{K(s,u)} \cdot \frac{F_3(s-1,K'(s,u))}{K'(s,u)K(s-1,K'(s,u))}. \tag{2.8}$$

Note by Lemma 2.3.3v that $K'(s,u) > s-1$ for all u. For the remainder of our proof, when working with expressions of the form $F_i(s-1,K'(s,u))$, we will use this inequality without explicit reference.

We begin our proof of (2.3). It is clear that $F_3(s,t) > F_0(s,t)$ for all s and t, so we need only prove that $F_0(s,t)/(tK(s,t)) > 1$ and $F_3(s,t)/(tK(s,t)) < 72$ for $t > s$. We will prove these in order.

We prove that $F_0(s,t)/(tK(s,t)) > 1$ by induction on s. The base case $s = 1$ holds, as $F_0(1,t)/(tK(1,t)) = 1 + 7/(2t) > 1$ for all $t > 1$.

For the inductive step, let $s > 1$, and suppose $F_0(s-1,t)/(tK(s-1,t)) > 1$ for all $t > s-1$. Recall that $K'(s,u)/K(s,u) > 1$ for all u by Lemma 2.3.3iv. Thus, by (2.5) and the inductive hypothesis, for all $t > 1$,

$$\frac{F_0(s,t)}{K(s,t)} > 5 \cdot 2^{-s} + \sum_{u=1}^{t-1} \frac{R(s,u)-2}{K(s,u)} + t + 1.$$

It is easy to verify by Lemma 2.3.3v that $\sum_{u=1}^{t-1} 2/K(s,u) < 1$. Thus, $F_0(s,t)/K(s,t) > t$. It follows that $F_0(s,t)/(tK(s,t)) > 1$. This completes our proof that $F_0(s,t)/(tK(s,t)) > 1$ for $t > s$.

Next, we prove by induction on s that

$$\frac{F_3(s,t)}{tK(s,t)} \leq \prod_{i=1}^{s} (1 + 2^{3-i}).$$

Note that $\prod_{i=1}^{\infty}(1+2^{3-i}) \approx 71.527$, so the above inequality implies that $F_3(s,t)/(tK(s,t)) < 72$. The base case $s = 1$ is immediate, as $F_3(1,t)/(tK(1,t)) = 4 + 2/t \leq 5$ for all $t > 1$.

For the inductive step, let $s > 1$, and suppose $F_3(s-1,t)/(tK(s-1,t)) \leq \prod_{i=1}^{s-1}(1+2^{3-i})$ for all $t > s-1$. By Lemmas 2.3.3v and 2.3.12, we have

$$\sum_{u=1}^{t-1} \frac{2R(s,u)+3}{K(s,u)} < \sum_{u=1}^{t-2} \frac{6}{K(s,u)} + \sum_{u=1}^{t-1} \frac{3}{K(s,u)} < \frac{9}{2}.$$

Meanwhile, by Lemma 2.3.4 and the inductive hypothesis, for all u,

$$\frac{K'(s,u)}{K(s,u)} \cdot \frac{F_3(s-1,K'(s,u))}{K'(s,u)K(s-1,K'(s,u))} < \prod_{i=1}^{s} (1+2^{3-i}).$$

Hence, by (2.8),

$$\begin{aligned}\frac{F_3(s,t)}{K(s,t)} &< \frac{19}{2} + (t-1)\prod_{i=1}^{s}(1+2^{3-i}) \\ &< t\prod_{i=1}^{s}(1+2^{3-i}).\end{aligned}$$

Thus, $F_3(s,t)/(tK(s,t)) < \prod_{i=1}^{s}(1+2^{3-i})$. This completes our proof that $F_3(s,t)/(tK(s,t)) \leq \prod_{i=1}^{s}(1+2^{3-i})$ for $t > s$, which in turn completes our proof of (2.3).

We now begin our proof of (2.4). It is clear that $F_2(s,t) > F_1(s,t)$ for all s and t, so we need only prove that $F_1(s,t)/(tK(s,t)) > 3s/2$ and $F_2(s,t)/(tK(s,t)) < 158s$ for $t > s$. We will prove these in order.

We prove that $F_1(s,t)/(tK(s,t)) > 3s/2$ by induction on s. The base case $s = 1$ holds, as $F_1(1,t)/(tK(1,t)) = 5 + 15/(2t) > 3/2$ for all $t > 1$.

For the inductive step, let $s > 1$, and suppose $F_1(s-1,t)/(tK(s-1,t)) > 3(s-1)/2$ for all $t > s-1$. We can verify by Lemma 2.3.3v that $\sum_{u=1}^{t-1} 4/K(s,u) < 2$, so in (2.6), the first three terms of the right-hand side must sum to a positive number. Recall again that $K'(s,u)/K(s,u) > 1$ for all u. Thus, by (2.6) and the inductive hypothesis, for all $t > s$, we have

$$\begin{aligned}
\frac{F_1(s,t)}{K(s,t)} &> \sum_{u=1}^{t-1}\left[3 + \frac{F_1(s-1,K'(s,u))}{K'(s,u)K(s-1,K'(s,u))}\right] \\
&> (t-1)\left[3 + \frac{3(s-1)}{2}\right] \\
&= \frac{3(s+1)(t-1)}{2} \\
&\geq \frac{3st}{2}.
\end{aligned}$$

It follows that $F_1(s,t)/(tK(s,t)) > 3s/2$. This completes our proof that $F_1(s,t)/(tK(s,t)) > 3s/2$ for $t > s$.

Next, we prove by induction on s that

$$\frac{F_2(s,t)}{tK(s,t)} \leq 11s\prod_{i=2}^{s}(1+2^{3-i}).$$

Note that $11\prod_{i=2}^{\infty}(1+2^{3-i}) \approx 157.359$, so the above implies that $F_2(s,t)/(tK(s,t)) < 158s$. The base case $s = 1$ is immediate, as $F_2(1,t)/(tK(1,t)) = 8 + 6/t \leq 11$ for all $t > 1$.

For the inductive step, let $s > 1$, and suppose $F_2(s-1,t)/(tK(s-1,t)) \leq 11(s-1)\prod_{i=2}^{s-1}(1+2^{3-i})$ for all $t > s-1$. By Lemmas 2.3.3v and 2.3.12, we have

$$\sum_{u=1}^{t-1}\frac{4R(s,u)+1}{K(s,u)} < \sum_{u=1}^{t-2}\frac{12}{K(s,u)} + \sum_{u=1}^{t-1}\frac{1}{K(s,u)} < \frac{13}{2}.$$

Meanwhile, by Lemma 2.3.4 and the inductive hypothesis, for all u,

$$\begin{aligned}\frac{K'(s,u)}{K(s,u)}\left[3+\frac{F_2(s-1,K'(s,u))}{K'(s,u)K(s-1,K'(s,u))}\right] &< (1+2^{3-s})\left[3+11(s-1)\prod_{i=2}^{s-1}(1+2^{3-i})\right]\\ &< 11s(1+2^{3-s})\prod_{i=2}^{s-1}(1+2^{3-i})\\ &= 11s\prod_{i=2}^{s}(1+2^{3-i}).\end{aligned}$$

Thus, by (2.7),

$$\begin{aligned}\frac{F_2(s,t)}{K(s,t)} &< \frac{33}{2}+11s(t-1)\prod_{i=2}^{s}(1+2^{3-i})\\ &< 11st\prod_{i=2}^{s}(1+2^{3-i}).\end{aligned}$$

It follows that $F_2(s,t)/(tK(s,t)) < 11s\prod_{i=2}^{s}(1+2^{3-i})$. This completes our proof that $F_2(s,t)/(tK(s,t)) \leq 11s\prod_{i=2}^{s}(1+2^{3-i})$ for $t > s$, which in turn completes our proof of (2.4). □

2.3.5 Main result

We now prove our main result.

Theorem 2.3.16. *For arbitrarily large n, there exist shellable, dual shellable 3-spheres S such that*

$$f(S) = (\Theta(n), \Theta(n\alpha(n)), \Theta(n\alpha(n)), \Theta(n)).$$

Proof. For arbitrary $s > 1$, let $n = sK(s-1,s)$, and let $S = S(s-1,s)$. By Propositions 2.3.9-2.3.10, S is shellable and dual shellable.

By Propositions 2.3.14-2.3.15,

$$\begin{aligned}f_0(S), f_3(S) &= \Theta(sK(s-1,s)),\\ f_1(S), f_2(S) &= \Theta(s^2K(s-1,s)).\end{aligned}$$

By Lemma 2.3.5, we have $s = \Theta(\alpha(n))$. Thus,

$$f(S) = (\Theta(n), \Theta(n\alpha(n)), \Theta(n\alpha(n)), \Theta(n)).$$ □

```
sage: P = Polyhedron(vertices=[[0,-1,0,0],[1,0,0,0],[12/13,5/13,3/13,6/13],
[4/5,3/5,1/3,3/5],[3/5,4/5,0,0],[5/13,12/13,1/3,15/13],[-5/13,12/13,8/13,
16/13],[-3/5,4/5,2/5,0],[-4/5,3/5,9/10,8/5],[-1,0,4/3,0],[-4/5,-3/5,1,12]])
```

Figure 2.6: SageMath code for a 4-polytope on eleven vertices with boundary isomorphic to $S(1,2)$. All vertices project onto the unit circle in the first two coordinates.

2.3.6 Further questions

We have constructed arbitrarily fat, shellable, dual shellable 3-spheres. This leaves us with a key question: are they polytopes?

Observation 2.3.17. $S(1,t)$ is isomorphic to the boundary complex of a 4-polytope for all $t > 1$.

Figure 2.6 gives SageMath code for a 4-polytope realizing $S(1,2)$. The ten vertices of $X(1,2)$ are arranged roughly as in Figure 2.4 (left), taking the first three coordinates respectively as east, north, and "up" out of the page, and ignoring the fourth coordinate. Label vertices as follows:

$$v = \left(-\tfrac{4}{5}, -\tfrac{3}{5}, 1, 12\right), \quad a = \left(-1, 0, \tfrac{4}{3}, 0\right), \quad w_2 = \left(\tfrac{4}{5}, \tfrac{3}{5}, \tfrac{1}{3}, \tfrac{3}{5}\right), \quad x_2 = \left(-\tfrac{5}{13}, \tfrac{12}{13}, \tfrac{8}{13}, \tfrac{16}{13}\right).$$

Here v is the unique vertex disjoint from $X(1,2)$, a is the root of $X(1,2)$ clockwise-adjacent to the hub, and w_2 and x_2 are the initial vertices of each fillet. For each successive $t > 2$, we may obtain a polytope realizing $S(1,t)$ by placing new vertices w_t and x_t beyond triangles $\triangle avw_{t-1}$ and $\triangle avx_{t-1}$, respectively, and taking the convex hull.

Conjecture 2.3.18. *For infinitely many s, there exists $t > s$ such that $S(s,t)$ is isomorphic to the boundary complex of a 4-polytope.*

If one believes in arbitrarily fat 4-polytopes, then we propose $S(s,t)$ as plausible candidates. Intuitively, we imagine a version of Construction 2.3.7 where every step yields a subcomplex of the boundary of a 4-polytope. At each recursion, we have many degrees of freedom when adding vertices in Step 4, especially if we relax the requirement that all vertices project onto the unit circle. Steps 3 and 5 are tricky—3 requires many vertices to lie on a common hyperplane, and 5 requires two complexes to fit together exactly—but we suspect that suitable choices in Step 4 make the rest of the construction possible.

There has been progress on finding polytopal realizations of 3-spheres algorithmically [15][24][27]. However, while we conjecture that realizations of many $S(s,t)$ exist, we are not optimistic that they can be found by computer search. The sphere $S(3,4)$ already has more than $2^{2^{2^{15}}}$ vertices, putting it far beyond the scope of relevant algorithms.

Finally, we remark that ours is one of many related constructions for fat 3-spheres. One could modify Construction 2.3.7 by subdividing some facets or splitting some vertices without changing the f-vector asymptotically. Where we add one vertex to obtain $S(s,t)$ from $X(s,t)$, one could add many more vertices and still keep fatness roughly the same. Such modifications could yield further candidates for realizability as fat 4-polytopes.

Chapter 3

RECONSTRUCTING POLYTOPES AND PSEUDOMANIFOLDS

This chapter is dedicated to reconstruction problems on polytopes and simplicial complexes. Reconstruction problems formalize the following question: if P is an unknown d-polytope, how much of the face poset of P must we know to reconstruct the rest?

The *face poset* $\mathcal{F}(P)$ of a d-polytope P is the set of faces of P, partially ordered by containment. Many reconstruction problems begin with the restriction of $\mathcal{F}(P)$ to faces of certain dimensions. One such problem is to reconstruct P from its *k-skeleton*, defined for $0 \leq k < d$ as the restriction of $\mathcal{F}(P)$ to faces of dimensions $0, \ldots, k$.

Reconstruction problems are important to the computational study of polytopes. Suppose we wish to encode d-polytopes efficiently (up to combinatorial type) in a computer database. If every d-polytope can be reconstructed from its k-skeleton, then it suffices to record only the k-skeleton, rather than the entire face poset.

These problems have long been of interest, even beyond their computational relevance. In 1932, Whitney proved that any 3-polytope can be recovered from its vertex-edge graph. Specifically, Whitney showed that if G is the vertex-edge graph of a 3-polytope, then the boundaries of facets are exactly the induced, non-separating cycles in G. [55]

Grünbaum proved in 1967 that every d-polytope is determined by its $(d-2)$-skeleton. Grünbaum's method paralleled Whitney's: if $\mathcal{C}$ is the $(d-2)$-skeleton of a d-polytope, Grünbaum showed that the boundaries of facets are exactly the induced, non-separating subcomplexes of $\mathcal{C}$ homeomorphic to S^{d-2}. Grünbaum noted that *not* every d-polytope is determined by its $(d-3)$-skeleton, giving the counterexample of two distinct 4-polytopes with vertex-edge graph K_8. [29, pp. 228-229]

Two other results of Grünbaum are as follows. First, if P is a d-polytope and $0 \leq a < b < d$, then the incidences between a- and b-faces of P uniquely determine the restriction of $\mathcal{F}(P)$ to dimensions $a, \dots, b$ [29, p. 233]. Effectively, given any two "levels" of a face poset, we can recover everything in between.

Second, if $d \geq 5$, the restriction of $\mathcal{F}(P)$ to dimensions $1, \dots, d-2$ is enough to reconstruct P [29, p. 234]. Grünbaum states that the $d = 4$ analogue is an open problem:

Problem 3.0.1 (Grünbaum)**.** Are all 4-polytopes determined by their edge-ridge incidences?

Problem 3.0.1 is the final piece in a large puzzle. If P is an arbitrary d-polytope, and we are given the restriction of $\mathcal{F}(P)$ to certain dimensions, then which dimensions must be included to guarantee we can reconstruct P? The answer is trivial for $d = 2$: either the vertex set or the edge set is enough. Whitney's theorem covers $d = 3$: we can reconstruct P from its vertex-edge graph or, dually, its facet-ridge graph. For $d \geq 5$, Grünbaum's results imply the following:

- We can reconstruct P if our subposet includes vertices or edges *and* includes facets or ridges.

- We cannot necessarily reconstruct P otherwise, as neither the $(d-3)$-skeleton nor the dual $(d-3)$-skeleton is sufficient in general.

Only for $d = 4$ has the answer remained unknown.

A more general question arises naturally: if we know *several* subposets of $\mathcal{F}(P)$, each being the restriction of $\mathcal{F}(P)$ to a distinct set of dimensions, when are these enough to reconstruct P? Samper recently asked a question of this type [49]:

Problem 3.0.2 (Samper)**.** Is every 4-polytope determined by the combination of its vertex-edge graph and its facet-ridge graph?

Another genre of reconstruction problem deals with simplicial polytopes. If we know that P is a simplicial d-polytope, then we can reconstruct P from relatively little information, as the possible face posets of simplicial polytopes are far more restricted than those of general polytopes.

One celebrated theorem is attributed to Perles from 1970: any simplicial d-polytope is determined by its $\lfloor \frac{d}{2} \rfloor$-skeleton [47]. In 1984, Dancis extended this theorem to homology $(d-1)$-manifolds for even d and to homology $(d-1)$-manifolds with trivial $\frac{d-1}{2}$-homology for odd d. Dancis further proved that all homology $(d-1)$-manifolds are determined by their $\lceil \frac{d}{2} \rceil$-skeleta [17].

In 1987, Blind and Mani proved that any simplicial polytope is determined by its facet-ridge graph [14]. Kalai published a simpler proof the following year. Kalai worked in the dual setting, giving an algorithm to reconstruct a simple polytope from its vertex-edge graph [38] (see also Joswig [36]). By an additional result of Kalai, for any simplicial d-polytope P and $1 \leq k < d$, the k-skeleton of P is determined by the incidences of its k- and $(k-1)$-faces. Putting this all together, for any $\lfloor \frac{d}{2} \rfloor \leq k < d$, we can reconstruct P from its k-, $(k-1)$-face incidences (see [40, Theorem 19.5.30]).

Yang recently proved that any shellable, simplicial sphere is determined by its facet-ridge incidences, extending Blind and Mani's 1987 result. Yang used the graph theoretic tools of good acyclic orientations and k-systems. [59]

This chapter builds on the results above, both for polytopes and for simplicial complexes. We answer Problem 3.0.1 in the affirmative, proving that all 4-polytopes are determined by their edge-ridge incidences (Theorem 3.2.3). Our method involves Alexander duality, which we use to prove a "non-separating" condition on the boundaries of facets akin to the conditions of Whitney and Grünbaum. We then answer Problem 3.0.2 in the negative by constructing distinct 4-polytopes with the same vertex-edge graph and the same facet-ridge graph. More generally, for each $d \geq 3$, we construct distinct d-polytopes with identical $(d-3)$-skeleta and identical dual $(d-3)$-skeleta (Theorem 3.3.4).

Next, we partially generalize Kalai and Perles's results on k-, $(k-1)$-face incidences from simplicial polytopes to pseudomanifolds. We prove that any simplicial homology $(d-1)$-manifold with $d \geq 4$ is determined by its k-, $(k-1)$-face incidences for arbitrary

$\lceil \frac{d}{2} \rceil \leq k \leq d-2$ (Theorem 3.5.3). We further extend this to normal pseudomanifolds in which each $(2d-2k-1)$-dimensional link is a homology manifold (Theorem 3.5.5). Finally, we show that the reconstruction results for simplicial polytopes and manifolds fail for normal pseudomanifolds in general, as for each $d \geq 3$ we construct distinct, normal $(d-1)$-pseudomanifolds with identical $(d-2)$-skeleta (Theorem 3.5.9).

The structure of this chapter is as follows. In Section 3.1, we define our objects of study and recall existing reconstruction results. In Section 3.2, we discuss 4-polytopes and solve Problem 3.0.1. Section 3.3 is home to our counterexamples for Problem 3.0.2 and its higher-dimensional analogues. In Section 3.4, we combine our preceding results with those of Grünbaum into a single reconstruction theorem, characterizing the incidence information needed to identify arbitrary d-polytopes for all $d \geq 4$. We conclude in Section 3.5 with our results for reconstructing simplicial complexes.

3.1 Preliminaries

In this section, we introduce the objects and tools involved in reconstruction problems. The reader may consult Grünbaum [29] or Ziegler [60] for any undefined terminology. The reader may also refer to Bayer [4] for a broader survey of existing reconstruction results.

3.1.1 Skeleta of polytopes

Definition 3.1.1. Let P be a d-polytope. The *face poset* $\mathcal{F}(P)$ is the set of faces of P, partially ordered by containment. For $k = 0, \dots, d-1$, we define $\mathcal{F}^k(P)$ as the set of k-dimensional faces of P.

We may now state our reconstruction problem more precisely. Let P be an unknown d-polytope. Suppose we know, up to isomorphism, some induced subposets of $\mathcal{F}(P)$; each subposet has the form $\bigcup_{k \in K} \mathcal{F}^k(P)$ for some $K \subseteq \{0, \dots, d-1\}$. From which subposets of this kind is it guaranteed that we can reconstruct $\mathcal{F}(P)$?

This is a broad question, but we can simplify it using the notion of an (a, b)-*skeleton* and a theorem of Grünbaum.

Definition 3.1.2. Let P be a d-polytope and $0 \leq a \leq b < d$. We define the (a,b)-*skeleton* of P as the set

$$\mathcal{F}_a^b(P) := \bigcup_{k=a}^{b} \mathcal{F}^k(P),$$

with partial order induced by $\mathcal{F}(P)$. In particular, we call $\mathcal{F}_0^k(P)$ the *k-skeleton* of P, and we call $\mathcal{F}_{d-k-1}^{d-1}(P)$ the *dual k-skeleton* of P.

Theorem 3.1.3 (Grünbaum [29, p. 233])**.** *Let P, Q be d-polytopes and $0 \leq a < b < d$. Then any poset isomorphism $\varphi : \mathcal{F}^a(P) \cup \mathcal{F}^b(P) \to \mathcal{F}^a(Q) \cup \mathcal{F}^b(Q)$ extends to an isomorphism $\mathcal{F}_a^b(P) \to \mathcal{F}_a^b(Q)$.*

In other words, if P is a d-polytope and $0 \leq a < b < d$, then the (a,b)-skeleton of P is determined by $\mathcal{F}^a(P) \cup \mathcal{F}^b(P)$. Thus, our problem reduces to that of reconstructing a d-polytope from a set of (a,b)-skeleta.

Problem 3.1.4 (Reconstruction)**.** Let $d \geq 2$. Characterize all

$$0 \leq a_1 \leq b_1 < d,$$
$$\vdots$$
$$0 \leq a_n \leq b_n < d$$

such that the face poset of every d-polytope P is determined by the list of isomorphism types of $\mathcal{F}_{a_1}^{b_1}(P), \dots, \mathcal{F}_{a_n}^{b_n}(P)$.

Let us take this opportunity to answer Problem 3.1.4 for $d = 2, 3$.

Observation 3.1.5. The face poset of any 2-polytope P is determined by $\mathcal{F}^0(P)$, as well as by $\mathcal{F}^1(P)$. In other words, the number of vertices or edges in a polygon determines its face poset.

Observation 3.1.6. By Whitney's theorem [55], the face poset of any 3-polytope P is determined by $\mathcal{F}_0^1(P)$, as well as by $\mathcal{F}_1^2(P)$. Nothing else will suffice; for example, let P be the cross-polytope and Q the stacked polytope on six vertices. Then $f(P) = f(Q) = (6, 12, 8)$, so $\mathcal{F}^k(P) \cong \mathcal{F}^k(Q)$ for $k = 0, 1, 2$. However, $\mathcal{F}(P) \not\cong \mathcal{F}(Q)$.

It remains to answer Problem 3.1.4 for $d \geq 4$. Two additional theorems of Grünbaum take us much of the distance. The first says that we can reconstruct a d-polytope from its $(d-2)$-skeleton; the second says that if $d \geq 5$, then the $(1, d-2)$-skeleton is sufficient.

Theorem 3.1.7 (Grünbaum [29, p. 228])**.** *Let P, Q be d-polytopes with $d \geq 2$. Any isomorphism $\varphi : \mathcal{F}_0^{d-2}(P) \to \mathcal{F}_0^{d-2}(Q)$ extends to an isomorphism $\mathcal{F}(P) \to \mathcal{F}(Q)$. Dually, any isomorphism $\varphi : \mathcal{F}_1^{d-1}(P) \to \mathcal{F}_1^{d-1}(Q)$ extends to an isomorphism $\mathcal{F}(P) \to \mathcal{F}(Q)$.*

Theorem 3.1.8 (Grünbaum [29, p. 234])**.** *Let P, Q be d-polytopes with $d \geq 5$. Any isomorphism $\varphi : \mathcal{F}_1^{d-2}(P) \to \mathcal{F}_1^{d-2}(Q)$ extends to an isomorphism $\mathcal{F}(P) \to \mathcal{F}(Q)$.*

Recall that for each $d \geq 3$, there exist distinct d-polytopes with identical $(d-3)$-skeleta. Dually, there exist distinct d-polytopes with identical dual $(d-3)$-skeleta. Thus, for $d \geq 5$, Theorem 3.1.8 answers Problem 3.1.4 in the $n = 1$ case: we can reconstruct every d-polytope P from $\mathcal{F}_a^b(P)$ if and only if $a \leq 1$ and $b \geq d-2$.

3.1.2 Manifolds and simplicial complexes

We next recall the definition of a pseudomanifold, and we introduce CW manifolds.

Definition 3.1.9. A *pseudomanifold* is a pure, strongly regular CW complex in which every ridge belongs to exactly two facets.

Definition 3.1.10. A *CW manifold* is a pseudomanifold homeomorphic to a manifold.

Definition 3.1.11. Let X be a CW pseudomanifold. A *singularity* of X is a nonempty face $G \subseteq X$ such that for an arbitrary point g in the relative interior of G, no open neighborhood of g in X is homeomorphic to a ball.

A CW manifold is therefore a pseudomanifold with no singularities.

Next, we discuss simplicial complexes, which will be the setting for our reconstruction results in Section 3.5.

Definition 3.1.12. A *simplicial complex* is a strongly regular CW complex in which every face is a simplex.

Suppose X is a simplicial complex on a vertex set V. Sometimes, we will not regard the faces of X as topological balls, but rather as subsets of V. So if G is a k-simplex in X with vertices $v_1, \ldots, v_{k+1} \in V$, we may write $G = \{v_1, \ldots, v_{k+1}\}$. If B is the $(k-1)$-face of G disjoint from v_1, we may write $G = B \cup v_1$ or $B = G \backslash v_1$.

We define the face poset and (a, b)-skeleton of any CW complex (including any simplicial complex) the same way as for a polytope.

Definition 3.1.13. Let G be a face of some simplicial complex X. The *link* of G, denoted $\mathrm{Lk}(G, X)$, is the subcomplex of X with faces $\{A \backslash G : A \in \mathcal{F}(X), G \subseteq A\}$.

Definition 3.1.14. Let X be a connected, simplicial $(d-1)$-pseudomanifold. Then X is *normal* if for all k-faces G with $0 \leq k \leq d-3$, $\mathrm{Lk}(G, X)$ is connected.

The final complexes we introduce are *homology manifolds*, which lie between normal pseudomanifolds and simplicial manifolds in generality. Homology manifolds may be defined with coefficients in $\mathbb{Z}$ or in any fixed field; for the homology manifolds in this chapter, our field of choice is $\mathbb{Z}_2$.

Definition 3.1.15. A simplicial *homology* $(d-1)$*-manifold* is a connected, simplicial $(d-1)$-pseudomanifold X such that for each $0 \leq k \leq d-2$ and k-face $G \subseteq X$, $\mathrm{Lk}(G, X)$ has the homology of a $(d-k-2)$-sphere. That is,

$$H_i(\mathrm{Lk}(G, X), \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & i = 0, \\ 0, & i = 1, \ldots, d-k-3, \\ \mathbb{Z}_2, & i = d-k-2. \end{cases}$$

We will use the following theorem of Dancis to prove our positive reconstruction results for homology manifolds and pseudomanifolds.

Theorem 3.1.16 (Dancis [17]). *Let X and Y be simplicial homology $(d-1)$-manifolds. Then any isomorphism $\varphi : \mathcal{F}_0^{\lceil d/2 \rceil}(X) \to \mathcal{F}_0^{\lceil d/2 \rceil}(Y)$ extends to an isomorphism $\mathcal{F}(X) \to \mathcal{F}(Y)$. If d is odd and $H_{(d-1)/2}(X, \mathbb{Z}_2) = H_{(d-1)/2}(Y, \mathbb{Z}_2) = 0$, then any isomorphism $\varphi : \mathcal{F}_0^{(d-1)/2}(X) \to \mathcal{F}_0^{(d-1)/2}(Y)$ likewise extends to an isomorphism $\mathcal{F}(X) \to \mathcal{F}(Y)$.*

3.2 Reconstructing a 4-polytope from edges and ridges

In this section, we will prove that the face poset of a 4-polytope is uniquely determined by its edge-ridge incidences. We begin by showing how to reconstruct the facets of a CW 3-sphere from its $(1, 2)$-skeleton.

Lemma 3.2.1. *Let X be a strongly regular CW 3-sphere and $\mathcal{E} \subseteq \mathcal{F}^1(X), \mathcal{R} \subseteq \mathcal{F}^2(X)$ nonempty sets. Then there exists a facet F with $\mathcal{F}_1^2(F) = \mathcal{E} \cup \mathcal{R}$ if and only if the following hold:*

1. *For each ridge $R \in \mathcal{R}$, every edge of R is in $\mathcal{E}$.*

2. *Each edge $E \in \mathcal{E}$ belongs to exactly two ridges in $\mathcal{R}$.*

3. *The incidence graphs of $\mathcal{E} \cup \mathcal{R}$ and $\mathcal{F}_1^2(X) \backslash (\mathcal{E} \cup \mathcal{R})$ are connected.*

Proof. One direction is immediate: if there is such a facet, then $\mathcal{E}$ and $\mathcal{R}$ must satisfy conditions 1-3. To prove the other direction, assume $\mathcal{E}$ and $\mathcal{R}$ satisfy these conditions.

Let $M \subset \mathcal{F}(X)$ be the set of all ridges in $\mathcal{R}$ and all their faces; that is, $M := \bigcup_{R \in \mathcal{R}} \mathcal{F}(R)$. Observe that $|M|$ is a connected 2-pseudomanifold embedded in S^3. We will show that the complement of $|M|$ in S^3 has two connected components.

Since $|M|$ is a 2-pseudomanifold, any singularity of $|M|$ is located at a vertex. Let us construct a new manifold from $|M|$ embedded in S^3 by removing these singularities.

Let v be a vertex of X which is a singularity in $|M|$. Draw an open ball $B \subset X$ around v, sufficiently small that B only intersects the faces of X containing v. Then $|M| \cap \partial B$ is a disjoint union of circles $C_1, \ldots, C_n$. Delete $|M| \cap B$ from $|M|$, and glue a disc D_i to each circle C_i so that $D_1, \dots, D_n$ are disjoint and embedded in B. Repeat this operation for each singularity of $|M|$, and call the resulting manifold N. Since N is a 2-manifold embedded in S^3, N must be orientable (see [30, p. 256]).

If we delete all singularities of $|M|$, the resulting space embeds in N and is therefore orientable. Since removing vertices of a 2-pseudomanifold does not affect orientability, $|M|$ must be orientable as well. It follows that $\tilde{H}^2(|M|, \mathbb{Z}) = \mathbb{Z}$, so by Alexander duality, $\tilde{H}_0(X \backslash |M|, \mathbb{Z}) = \mathbb{Z}$. We may conclude that $X \backslash |M|$ has two connected components.

Let A, B be the connected components of $X \backslash |M|$. Then the closures of A and B in X must each be the union of a nonempty set of facets. We claim that either A or B is the interior of a single facet.

By way of contradiction, suppose A and B each contain the interiors of at least two facets. Let F be a facet with $\operatorname{int} F \subset A$. Since $\operatorname{int} F \neq A$, we know $\partial F \nsubseteq M$, so there exists a ridge $R \in \partial F \backslash M$. Thus, $\operatorname{relint} R \subset A$. By a similar argument, there exists a ridge R' such that $\operatorname{relint} R' \subset B$. However, this places R and R' in different connected components of the incidence graph of $\mathcal{F}_1^2(X) \backslash (\mathcal{E} \cup \mathcal{R})$, contradicting our connectedness assumption.

We have shown that either A or B is the interior of some facet F. Thus, $\partial F \subseteq M$. Any edge E in ∂F belongs to two ridges in ∂F, and by assumption, these must be the only ridges in M containing E. In other words, there can be no incidence between a face in $(\mathcal{E} \cup \mathcal{R}) \cap \partial F$ and a face in $(\mathcal{E} \cup \mathcal{R}) \backslash \partial F$. Since the incidence graph of $\mathcal{E} \cup \mathcal{R}$ is connected, it follows that $\partial F = M$ and $\mathcal{F}_1^2(F) = \mathcal{E} \cup \mathcal{R}$. □

Lemma 3.2.2. *Let X, Y be strongly regular CW 3-spheres and $\varphi : \mathcal{F}_1^2(X) \to \mathcal{F}_1^2(Y)$ an isomorphism. Then φ extends to an isomorphism $\mathcal{F}_1^3(X) \to \mathcal{F}_1^3(Y)$.*

Proof. Let $\mathcal{E} \subseteq \mathcal{F}^1(X)$, $\mathcal{R} \subseteq \mathcal{F}^2(X)$. We can see that $\mathcal{E}, \mathcal{R}$ satisfy the conditions of Lemma 3.2.1 if and only if $\varphi(\mathcal{E}) \subseteq \mathcal{F}^1(Y), \varphi(\mathcal{R}) \subseteq \mathcal{F}^2(Y)$ satisfy those conditions. Thus, for each facet F of X, we may define $\varphi(F) := F'$, where F' is the unique facet of Y such that

$$\varphi\left(\mathcal{F}_1^2(F)\right) = \mathcal{F}_1^2(F').$$

This gives us the desired isomorphism $\varphi : \mathcal{F}_1^3(X) \to \mathcal{F}_1^3(Y)$. □

Our theorem follows from Lemma 3.2.2 by duality.

Theorem 3.2.3. *Let P, Q be 4-polytopes and $\varphi : \mathcal{F}_1^2(P) \to \mathcal{F}_1^2(Q)$ an isomorphism. Then φ extends to an isomorphism $\mathcal{F}(P) \to \mathcal{F}(Q)$.*

From an algorithmic standpoint, given the $(1, 2)$-skeleton of a 4-polytope, we can enumerate the subsets satisfying conditions 1–3 of Lemma 3.2.1 and therefore corresponding to facets. We can similarly enumerate the subsets satisfying dual conditions and therefore corresponding to vertex figures.

Remark 3.2.4. Theorem 3.2.3 can be generalized to (strongly regular) CW 3-spheres, as CW 3-spheres are piecewise linear and therefore have well-defined duals.

3.3 The $(d-3)$-skeleton and dual $(d-3)$-skeleton are not enough

In this section, we will show that a d-polytope cannot generally be identified from the combined information of its $(d-3)$-skeleton and dual $(d-3)$-skeleton. For each $d \geq 3$, we will construct combinatorially distinct d-polytopes P, Q whose $(d-3)$-skeleta are isomorphic and whose dual $(d-3)$-skeleta are also isomorphic.

Fix $d \geq 3$, and let $e_1, \dots, e_d$ be the standard basis vectors of $\mathbb{R}^d$. Consider the d-polytope

$$W := \operatorname{conv}\{\pm e_1, \dots, \pm e_{d-1}, 2e_d - e_1, e_d \pm e_2, \dots, e_d \pm e_{d-1}\}.$$

Here, W is a wedge of two $(d-1)$-dimensional cross-polytopes F and F' sharing a single vertex at e_1. Specifically, F is the $(d-1)$-dimensional cross-polytope with vertices $\{\pm e_1, \dots, \pm e_{d-1}\}$, and F' is the $(d-1)$-dimensional cross-polytope with vertices $\{e_1, 2e_d - e_1, e_d \pm e_2, \dots, e_d \pm e_{d-1}\}$. The proper, nonempty faces of W are as follows:

- facets F and F',
- the vertex at e_1,
- for each k-face G of F, $0 \leq k \leq d-2$, excluding the vertex at e_1:

– G itself,

– a corresponding k-face G' of F', directly "above" G in the e_d direction,

– a "lateral" $(k+1)$-face equal to the convex hull of G and G'.

We construct new d-polytopes P and Q from W, each by shifting a vertex of F' in the $-e_d$ direction. To construct P, we move the vertex at $2e_d - e_1$ to $e_d - e_1$. To construct Q, we leave the vertex at $2e_d - e_1$ put, and instead move the vertex at $e_d + e_2$ to $\frac{1}{2}e_d + e_2$. So

$$\begin{aligned}
P &:= \operatorname{conv}\{\pm e_1, \dots, \pm e_{d-1}, e_d - e_1, e_d \pm e_2, \dots, e_d \pm e_{d-1}\},\\
Q &:= \operatorname{conv}\left\{\pm e_1, \dots, \pm e_{d-1}, 2e_d - e_1, \frac{1}{2}e_d + e_2, e_d - e_2, e_d \pm e_3, \dots, e_d \pm e_{d-1}\right\}.
\end{aligned}$$

When we construct P from W, we introduce a new ridge with vertices $\{e_d \pm e_2, \dots, e_d \pm e_{d-1}\}$, subdividing F' into two pyramidal facets. One of these pyramidal facets contains the vertex at e_1; the other contains the vertex at $e_d - e_1$. All faces of W besides F' remain intact.

Similarly, when we construct Q from W, we introduce a new ridge with vertices $\{e_1, 2e_d - e_1, e_d \pm e_3, \dots, e_d \pm e_{d-1}\}$, again subdividing F' into two pyramidal facets. One of these pyramidal facets contains the vertex at $\frac{1}{2}e_d + e_2$; the other contains the vertex at $e_d - e_2$. As before, all faces of W besides F' remain intact. See Figures 3.1–3.2 for a comparison of P and Q when $d = 3, 4$.

Claim 3.3.1. *$\mathcal{F}(P)$ is not isomorphic to $\mathcal{F}(Q)$.*

Proof. Consider the vertex of Q at e_1. This vertex is contained in 2^{d-2} facets of F, meaning e_1 is contained in 2^{d-2} lateral facets of Q. Additionally, e_1 is contained in F itself, as well as the pyramidal facets $\operatorname{conv}\{e_1, 2e_d - e_1, \frac{1}{2}e_d + e_2, e_d \pm e_3, \dots, e_d \pm e_{d-1}\}$ and $\operatorname{conv}\{e_1, 2e_d - e_1, e_d - e_2, e_d \pm e_3, \dots, e_d \pm e_{d-1}\}$. Thus, e_1 belongs to a total of $2^{d-2} + 3$ facets of Q.

We will show that no vertex of P belongs to this many facets. Let v be a vertex of P contained in F. Then v belongs to 2^{d-2} facets of F, so v is contained in 2^{d-2} lateral facets of P. Furthermore, if $v \neq e_1$, then the vertex $v + e_d$ belongs to the same 2^{d-2} lateral facets of P. Thus, each vertex of P is contained in exactly 2^{d-2} lateral facets.

There are only three non-lateral facets of P: the cross-polytope F and the pyramids $\operatorname{conv}\{e_1, e_d \pm e_2, \ldots, e_d \pm e_{d-1}\}$ and $\operatorname{conv}\{e_d - e_1, e_d \pm e_2, \ldots, e_d \pm e_{d-1}\}$. There is no vertex shared by all three non-lateral facets, so each vertex of P belongs to at most $2^{d-2} + 2$ facets in total.

Since Q has a vertex contained in $2^{d-2} + 3$ facets, and P does not, we may conclude that P and Q are combinatorially distinct. $\square$

Claim 3.3.2. *$\mathcal{F}_0^{d-3}(P)$ is isomorphic to $\mathcal{F}_0^{d-3}(Q)$.*

Proof. Recall that when we constructed each of P, Q from W, we introduced one new ridge and subdivided the facet F' into two pyramidal facets, but we preserved all faces of dimension $d-3$ or lower. Thus, $\mathcal{F}_0^{d-3}(P) \cong \mathcal{F}_0^{d-3}(Q) \cong \mathcal{F}_0^{d-3}(W)$. $\square$

Claim 3.3.3. *$\mathcal{F}_2^{d-1}(P)$ is isomorphic to $\mathcal{F}_2^{d-1}(Q)$.*

Proof. Let A, A' be the two facets of P resulting from our subdivision of F':

$$A := \operatorname{conv}\{e_1, e_d \pm e_2, \ldots, e_d \pm e_{d-1}\},$$
$$A' := \operatorname{conv}\{e_d - e_1, e_d \pm e_2, \ldots, e_d \pm e_{d-1}\}.$$

Let B, B' be the analogous facets of Q:

$$B := \operatorname{conv}\left\{e_1, 2e_d - e_1, \frac{1}{2}e_d + e_2, e_d \pm e_3, \ldots, e_d \pm e_{d-1}\right\},$$
$$B' := \operatorname{conv}\{e_1, 2e_d - e_1, e_d - e_2, e_d \pm e_3, \ldots, e_d \pm e_{d-1}\}.$$

Then A, A' are pyramids over a common $(d-2)$-cross-polytope, as are B, B'. Thus, there is an isomorphism $\psi : \mathcal{F}(A) \cup \mathcal{F}(A') \to \mathcal{F}(B) \cup \mathcal{F}(B')$ such that $\psi(\partial(A \cup A')) = \partial(B \cup B')$.

We may construct $\mathcal{F}_2^{d-1}(P)$ from $\mathcal{F}(A) \cup \mathcal{F}(A')$ in the following steps:

1. For each k-face $G \in \partial(A \cup A')$ with $k \geq 1$, add to the poset a “lateral” $(k+1)$-face $L(G)$ containing G. Let $G, L(G) \subseteq L(G')$ if and only if $G \subseteq G'$.

2. For each k-face $G \in \partial(A \cup A')$ with $k \geq 2$, add to the poset an "opposite" k-face $O(G)$ contained in $L(G)$ and disjoint from $A \cup A'$. Let $O(G) \subseteq O(G'), L(G')$ if and only if $G \subseteq G'$.

3. Add one facet F containing exactly the faces $O(G)$ added in step 2.

Note that in step 1, we must specify $k \geq 1$ because the vertex at e_1 does not belong to a "lateral" edge.

We may construct $\mathcal{F}_2^{d-1}(Q)$ from $\mathcal{F}(B) \cup \mathcal{F}(B')$ in the same steps, simply replacing $A \cup A'$ with $B \cup B'$ at each occurrence. It follows that $\mathcal{F}_2^{d-1}(P) \cong \mathcal{F}_2^{d-1}(Q)$. $\square$

In conclusion,

Theorem 3.3.4. *For all $d \geq 3$, there exist d-polytopes P and Q such that*

$$\mathcal{F}_0^{d-3}(P) \cong \mathcal{F}_0^{d-3}(Q),$$
$$\mathcal{F}_2^{d-1}(P) \cong \mathcal{F}_2^{d-1}(Q),$$
$$\mathcal{F}(P) \ncong \mathcal{F}(Q).$$

Note that when $d = 3$, polytopes satisfying Theorem 3.3.4 are unremarkable. It simply means that $f_0(P) = f_0(Q)$ and $f_2(P) = f_2(Q)$, a property shared by many pairs of distinct 3-polytopes. Theorem 3.3.4 is nontrivial for $d \geq 4$.

Remark 3.3.5. When $d = 4$, Theorem 3.3.4 gives combinatorially distinct 4-polytopes P and Q with isomorphic graphs and isomorphic dual graphs.

3.4 The $(1, d-2)$*-skeleton is necessary and sufficient*

We are now ready to answer Problem 3.1.4 for $d \geq 4$.

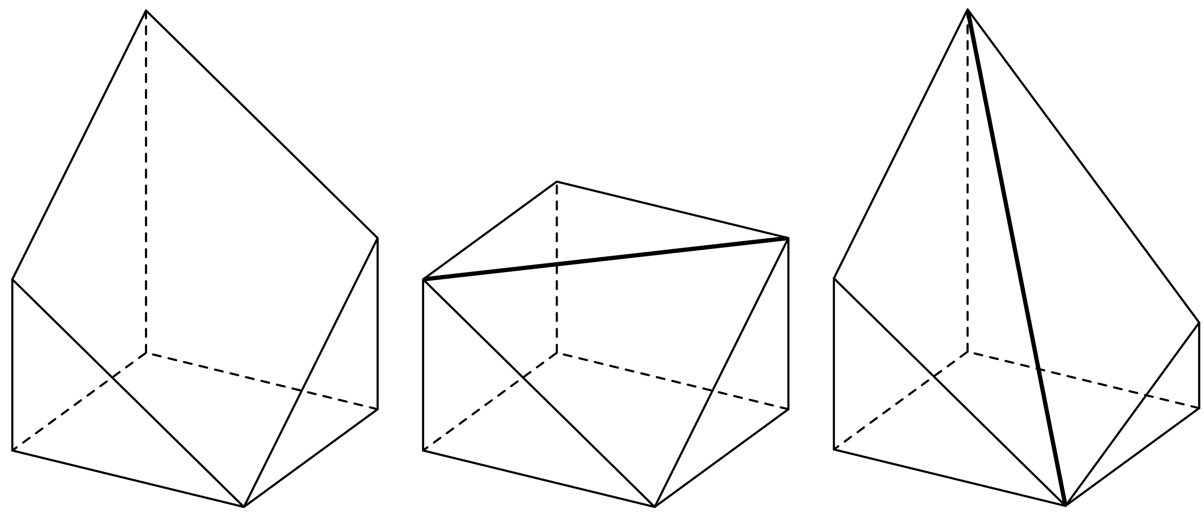

Figure 3.1: From left to right, polytopes W, P, and Q for $d = 3$. The square base is F; the bold edge is introduced when we construct P or Q from W.

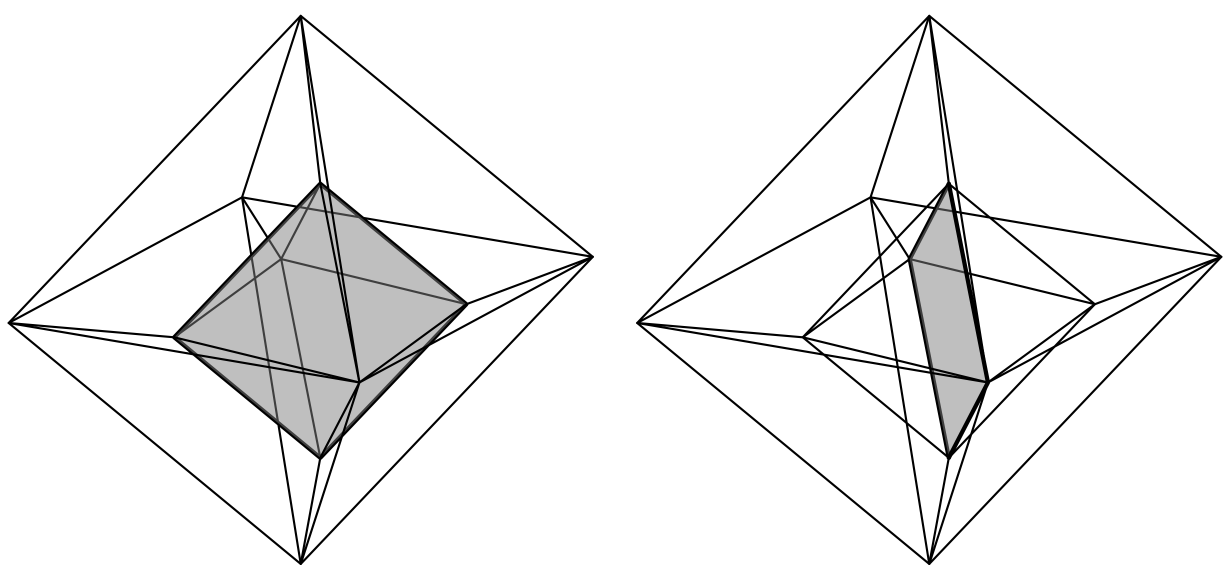

Figure 3.2: Schlegel diagrams of P (left) and Q (right) for $d = 4$. The outer octahedron is F; the shaded square is the ridge introduced when we construct P or Q from W (not pictured).

Theorem 3.4.1. *Fix $d \geq 4$, and let*

$$0 \leq a_1 \leq b_1 < d,$$
$$\vdots$$
$$0 \leq a_n \leq b_n < d.$$

The following are equivalent:

i. *For all d-polytopes P and Q, if $\mathcal{F}_{a_i}^{b_i}(P) \cong \mathcal{F}_{a_i}^{b_i}(Q)$ for $i = 1, \dots, n$, then $\mathcal{F}(P) \cong \mathcal{F}(Q)$.*

ii. *There exists $1 \leq i \leq n$ such that $a_i \leq 1$ and $b_i \geq d-2$.*

Proof. By Theorems 3.1.8 and 3.2.3, ii implies i.

Suppose i holds, and let P, Q be d-polytopes as in Theorem 3.3.4. Then $\mathcal{F}(P) \ncong \mathcal{F}(Q)$, so there exists $1 \leq i \leq n$ such that $\mathcal{F}_{a_i}^{b_i}(P) \ncong \mathcal{F}_{a_i}^{b_i}(Q)$. Since $\mathcal{F}_2^{d-1}(P) \cong \mathcal{F}_2^{d-1}(Q)$, we know $a_i \leq 1$. Similarly, since $\mathcal{F}_0^{d-3}(P) \cong \mathcal{F}_0^{d-3}(Q)$, we know $b_i \geq d-2$. We may conclude that i implies ii. □

If we wish to identify all d-polytopes from a set of (a,b)-skeleta, then knowing more than one skeleton does not help. Theorem 3.4.1 shows this for $d \geq 4$: either we know the $(1, d-2)$-skeleton, which is sufficient on its own, or we do not have enough information. We can make analogous statements for $d = 2, 3$ (Observations 3.1.5–3.1.6).

Of course, there are *specific* d-polytopes which can be identified from a set of a (a,b)-skeleta, but not from any one of those skeleta alone. For the simplest example, let $d = 3$ and P be the square pyramid. Then we can identify P from $\mathcal{F}^0(P)$ and $\mathcal{F}^2(P)$, as P is the only 3-polytope with five vertices and five facets. However, we cannot identify P from $\mathcal{F}^0(P)$ or $\mathcal{F}^2(P)$ alone, as P is neither the only 3-polytope with five vertices nor the only 3-polytope with five facets.

We conclude this section with a still-open problem of Grünbaum [29, p. 234].

Problem 3.4.2 (Grünbaum)**.** For $d \geq 5$, do there exist d-polytopes P, Q with an isomorphism $\varphi : \mathcal{F}_{d-3}^{d-2}(P) \to \mathcal{F}_{d-3}^{d-2}(Q)$ that does not extend to an isomorphism $\mathcal{F}_{d-3}^{d-1}(P) \to \mathcal{F}_{d-3}^{d-1}(Q)$?

3.5 Simplicial complexes

Our focus so far has been on reconstructing polytopes, but we can ask the same questions about any class of objects with graded face posets: CW spheres, CW manifolds, simplicial manifolds, pseudomanifolds, and so on. In this section, we prove positive and negative reconstruction results for homology manifolds and normal, simplicial pseudomanifolds.

Recall that when we discuss homology manifolds, the homologies in question have $\mathbb{Z}_2$-coefficients.

Lemma 3.5.1. *Let X, Y be normal, simplicial $(d-1)$-pseudomanifolds with $d \geq 4$, and let $2 \leq k \leq d-2$. Then any isomorphism $\varphi : \mathcal{F}^k_{k-1}(X) \to \mathcal{F}^k_{k-1}(Y)$ extends to an isomorphism $\mathcal{F}^k_{k-2}(X) \to \mathcal{F}^k_{k-2}(Y)$.*

Proof. We will call a subset T of $\mathcal{F}^k_{k-1}(X)$ *triangular* if $T = \{A_1, A_2, A_3, B_1, B_2, B_3\}$ for some distinct k-faces A_1, A_2, A_3 and distinct $(k-1)$-faces B_1, B_2, B_3, and

$$B_1 \subset A_2, A_3,$$
$$B_2 \subset A_1, A_3,$$
$$B_3 \subset A_1, A_2.$$

Let $\sim$ be the minimal equivalence relation on triangular subsets of $\mathcal{F}^k_{k-1}(X)$ such that if T, T' are triangular subsets sharing two $(k-1)$-faces, then $T \sim T'$. For any triangular subset T, we denote by $[T]$ the equivalence class of T under $\sim$.

For all triangular subsets $T = \{A_1, A_2, A_3, B_1, B_2, B_3\}$ as above, define

$$I(T) := A_1 \cap A_2 \cap A_3.$$

Since A_1, A_2, A_3 are distinct, we can see that $A_1 \cap A_2 = B_3$ and $A_1 \cap A_3 = B_2$. Thus, $I(T) = B_2 \cap B_3$. Since B_2 and B_3 are $(k-1)$ simplices belonging to the k-simplex A_1, it follows that $I(T)$ is a $(k-2)$-face of X. Furthermore, if $B_2, B_3 \in T'$ for some other triangular subset T', then $I(T') = B_2 \cap B_3 = I(T)$. As a result, for all triangular subsets T, T' with

$T \sim T'$, we have $I(T) = I(T')$. We may therefore define $I[T] \coloneqq I(T)$ as a function on equivalence classes.

For example, suppose X is the boundary complex of the 4-simplex on vertex set $\{1, 2, 3, 4, 5\}$ and $k = 2$. Then $T \coloneqq \{123, 124, 134, 14, 13, 12\}$ and $T' \coloneqq \{123, 125, 135, 15, 13, 12\}$ are triangular subsets. Since T and T' share the $(k-1)$-faces 12 and 13, we have $T \sim T'$, and $I(T) = I(T') = 1$.

Returning to the general case, let C be a $(k-2)$-face of X. We will prove that $C = I[T]$ for exactly one equivalence class $[T]$. Since $k \leq d-2$, we know $C \subseteq G$ for some $(k+1)$-face G of X, and the closed interval $[C, G]$ in $\mathcal{F}(X)$ is a Boolean lattice of length three. Thus, the open interval (C, G) in $\mathcal{F}(X)$ is a triangular subset, and $C = I[(C, G)]$.

We have shown that $C = I[T]$ for some triangular subset T; it remains to prove that $[T]$ is unique. Suppose $C = I[T] = I[T']$ for some triangular subsets T, T'. Let $A \in T$ be a k-face and let $B_1, B_2 \in T$ be $(k-1)$-faces with $B_1, B_2 \subset A$. Let G be a $(k+1)$-face of X with $A \subset G$. As before, the open interval (C, G) in $\mathcal{F}(X)$ is a triangular subset. Furthermore, $B_1, B_2 \in (C, G)$, so $(C, G) \sim T$. By the same reasoning, we can find a $(k+1)$-face G' of X such that $(C, G') \sim T'$.

Since X is a normal pseudomanifold, we know that $\mathrm{Lk}(C, X)$ is a normal pseudomanifold of dimension at least 2, and $G \backslash C, G' \backslash C$ are 2-faces of $\mathrm{Lk}(C, X)$. Thus, there is a sequence of 2-faces $G \backslash C = U_1, \dots, U_n = G' \backslash C \subseteq \mathrm{Lk}(C, X)$ such that U_m, U_{m+1} share an edge for each $1 \leq m < n$. For $1 \leq m \leq n$, let $U_m = G_m \backslash C$ with G_m a $(k+1)$-face of X, so $G_1 = G$ and $G_n = G'$. Then $(C, G_1), \dots, (C, G_n)$ are triangular subsets, and

$$(C, G) = (C, G_1) \sim \cdots \sim (C, G_n) = (C, G').$$

Thus, $T \sim T'$. It follows that $C = I[T]$ for a unique equivalence class $[T]$ of triangular subsets.

Define triangular subsets of $\mathcal{F}_{k-1}^{k}(Y)$, equivalence classes, and the function I identically as for X. Then the isomorphism $\varphi : \mathcal{F}_{k-1}^{k}(X) \to \mathcal{F}_{k-1}^{k}(Y)$ takes triangular subsets to triangular subsets and preserves equivalence classes. Thus, we can extend φ to an isomorphism

$\mathcal{F}^k_{k-2}(X) \to \mathcal{F}^k_{k-2}(Y)$ by letting

$$\varphi(I[T]) := I[\varphi(T)]$$

for all triangular subsets T of $\mathcal{F}^k_{k-1}(X)$. □

By applying Lemma 3.5.1 repeatedly, we can use the $(k-1,k)$-skeleton of a normal, simplicial pseudomanifold to reconstruct its entire k-skeleton.

Lemma 3.5.2. *Let X, Y be normal, simplicial $(d-1)$-pseudomanifolds with $d \geq 3$, and let $1 \leq k \leq d-2$. Then any isomorphism $\varphi : \mathcal{F}^k_{k-1}(X) \to \mathcal{F}^k_{k-1}(Y)$ extends to an isomorphism $\mathcal{F}^k_0(X) \to \mathcal{F}^k_0(Y)$.*

Our theorem on homology manifolds now follows from Theorem 3.1.16 and Lemma 3.5.2.

Theorem 3.5.3. *Let X, Y be simplicial homology $(d-1)$-manifolds with $d \geq 4$, and let $\lceil \frac{d}{2} \rceil \leq k \leq d-2$. Then any isomorphism $\varphi : \mathcal{F}^k_{k-1}(X) \to \mathcal{F}^k_{k-1}(Y)$ extends to an isomorphism $\mathcal{F}(X) \to \mathcal{F}(Y)$. If d is odd and $H_{(d-1)/2}(X, \mathbb{Z}_2) = H_{(d-1)/2}(Y, \mathbb{Z}_2) = 0$, then the same holds for $d \geq 3$ and $\frac{d-1}{2} \leq k \leq d-2$.*

Proof. By Lemma 3.5.2, we may extend φ to an isomorphism $\mathcal{F}^k_0(X) \to \mathcal{F}^k_0(Y)$. By Theorem 3.1.16, we may then extend φ to an isomorphism $\mathcal{F}(X) \to \mathcal{F}(Y)$. □

Yang proved that any shellable, simplicial sphere is determined by its facet-ridge incidences [59]. Theorem 3.5.3 and Yang's result imply the following.

Corollary 3.5.4. *Let X, Y be shellable, simplicial $(d-1)$-spheres with $d \geq 2$, and let $\lfloor \frac{d}{2} \rfloor \leq k < d$. Then any isomorphism $\varphi : \mathcal{F}^k_{k-1}(X) \to \mathcal{F}^k_{k-1}(Y)$ extends to an isomorphism $\mathcal{F}(X) \to \mathcal{F}(Y)$.*

Comparing the above results on simplicial manifolds to Theorem 3.4.1 on polytopes, we can see that the face posets of simplicial manifolds are generally easier to reconstruct. This makes sense: knowing that every face is a simplex, and therefore that every lower interval is a Boolean lattice, is an enormous hint. The "triangular subsets" in the proof of Lemma 3.5.1 are how we take advantage of this information.

The situation changes for pseudomanifolds, as Theorem 3.1.16 no longer applies directly. Nevertheless, we can generalize Theorem 3.5.3 to normal pseudomanifolds whose face links meet certain criteria.

Theorem 3.5.5. *Let $d \geq 5$ and $\lceil \frac{d+1}{2} \rceil \leq k \leq d-2$. Let X and Y be normal, simplicial $(d-1)$-pseudomanifolds in which the link of each $(2k-d-1)$-face is a homology $(2d-2k-1)$-manifold. Then any isomorphism $\varphi : \mathcal{F}^k_{k-1}(X) \to \mathcal{F}^k_{k-1}(Y)$ extends to an isomorphism $\mathcal{F}(X) \to \mathcal{F}(Y)$.*

Proof. By Lemma 3.5.2, we can extend φ to an isomorphism $\mathcal{F}^k_0(X) \to \mathcal{F}^k_0(Y)$. It remains to prove that we can extend the isomorphism to higher-dimensional faces.

Let G be a $(2k-d-1)$-face of X. Then the map $\varphi|_{\mathcal{F}^{d-k}_0(\mathrm{Lk}(G,X))} : \mathcal{F}^{d-k}_0(\mathrm{Lk}(G,X)) \to \mathcal{F}^{d-k}_0(\mathrm{Lk}(\varphi(G),Y))$ is an isomorphism. By assumption, $\mathrm{Lk}(G,X)$ and $\mathrm{Lk}(\varphi(G),Y)$ are homology $(2d-2k-1)$-manifolds. Thus, by Theorem 3.1.16, $\varphi|_{\mathcal{F}^{d-k}_0(\mathrm{Lk}(G,X))}$ extends to an isomorphism $\mathcal{F}(\mathrm{Lk}(G,X)) \to \mathcal{F}(\mathrm{Lk}(\varphi(G),Y))$.

Let $k < r < d$, and let $v_1, \dots, v_{r+1}$ be vertices in X. Then the following are equivalent:

- $\{v_1, \dots, v_{r+1}\}$ is an r-face of X.
- $\{v_1, \dots, v_{2k-d}\}$ is a face $G \in \mathcal{F}^{2k-d-1}(X)$, and $\{v_{2k-d+1}, \dots, v_{r+1}\}$ is a $(d-2k+r)$-simplex in $\mathrm{Lk}(G,X)$.
- $\varphi\{v_1, \dots, v_{2k-d}\}$ is a face $\varphi(G) \in \mathcal{F}^{2k-d-1}(Y)$, and $\varphi\{v_{2k-d+1}, \dots, v_{r+1}\}$ is a $(d-2k+r)$-simplex in $\mathrm{Lk}(\varphi(G),Y)$.
- $\varphi\{v_1, \dots, v_{r+1}\}$ is an r-face of Y.

For each r-face G in X with $k < r < d$, we may now define $\varphi(G) := G'$, where G' is the unique r-face of Y such that $\varphi(\mathcal{F}^0(G)) = \mathcal{F}^0(G')$. This gives our desired isomorphism $\varphi : \mathcal{F}(X) \to \mathcal{F}(Y)$. ☐

Remark 3.5.6. Theorem 3.5.5 holds for $d \geq 4$ and $\lceil \frac{d}{2} \rceil \leq k \leq d-2$ if we instead require the link of each $(2k-d)$-face to be a homology $(2d-2k-2)$-manifold having trivial $(d-k-1)$-homology with $\mathbb{Z}_2$-coefficients.

Let us illustrate why the conditions of Theorem 3.5.5 are important. For arbitrary $d \geq 3$, we will construct a pair of normal, simplicial $(d-1)$-pseudomanifolds X and Y which are not isomorphic, but whose $(d-2)$-skeleta *are* isomorphic.

For each $d \geq 2$, let

$$V_{d-1} := \{1, \dots, d\} \times \{0, 1, 2\}.$$

We define M^{d-1} as the pure $(d-1)$-complex on V_{d-1} whose facets are exactly the sets $\{(1, a_1), \dots, (d, a_d)\} \subset V_{d-1}$ with

$$\sum_{i=1}^{d} a_i \not\equiv 0 \pmod 3.$$

For $0 \leq k \leq d-2$, the k-faces of M^{d-1} are the sets $\{(m_1, a_1), \dots, (m_{k+1}, a_{k+1})\} \subset V$ with $m_1 < \dots < m_{k+1}$. To see why, observe that regardless of $a_1, \dots, a_{k+1}$, we can always choose $a_{k+2}, \dots, a_d \in \{0, 1, 2\}$ such that $\sum_{i=1}^{d} a_i \not\equiv 0 \pmod 3$.

Each $(d-2)$-face

$$G = \{(1, a_1), \dots, (n-1, a_{n-1}), (n+1, a_{n+1}), \dots, (d, a_d)\}$$

of M^{d-1} is contained in exactly two facets. These facets are $G \cup \{(n, a_n)\}$ and $G \cup \{(n, a'_n)\}$, where

$$a_n \equiv 1 - \sum_{i=1}^{n-1} a_i - \sum_{i=n+1}^{d} a_i \pmod 3,$$
$$a'_n \equiv 2 - \sum_{i=1}^{n-1} a_i - \sum_{i=n+1}^{d} a_i \pmod 3.$$

Thus, M^{d-1} is a pseudomanifold for each $d \geq 2$.

Observe that M^1 is a six-vertex circle and therefore a 1-manifold. Additionally, for each $d \geq 3$ and $G \in \mathcal{F}^k(M^{d-1})$ with $0 \leq k \leq d-3$, we have $\mathrm{Lk}(G, M^{d-1}) \cong M^{d-k-2}$. By strong induction on d, we can see that M^{d-1} is a normal $(d-1)$-pseudomanifold for all $d \geq 2$.

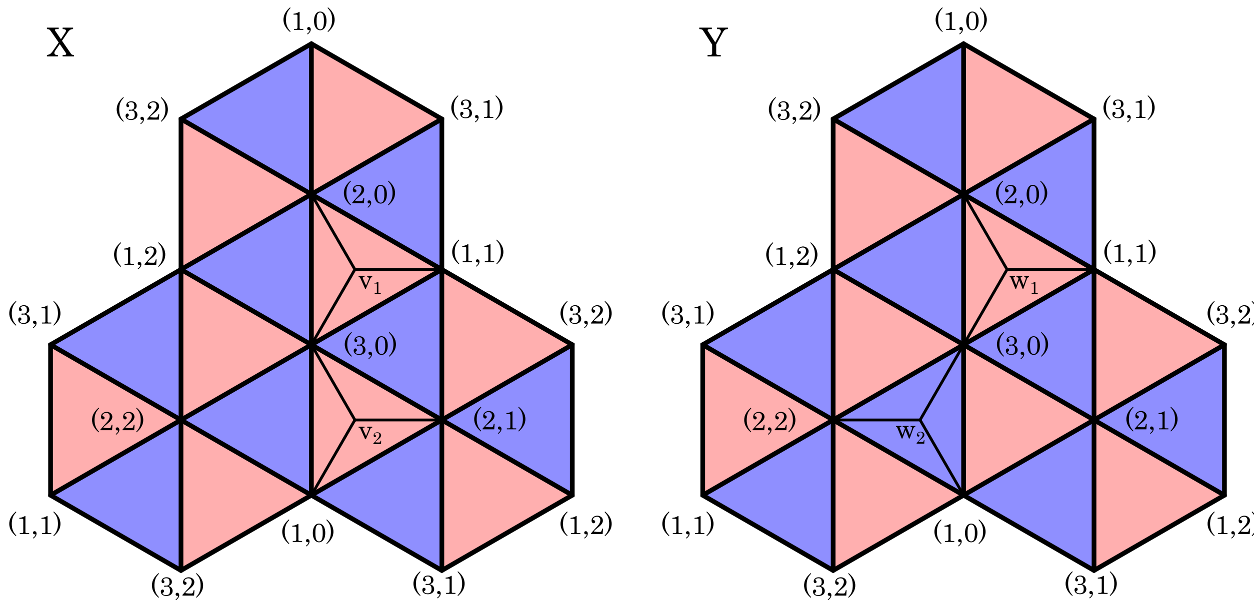

Figure 3.3: Normal $(d-1)$-pseudomanifolds X and Y for $d = 3$. In this case, X and Y are manifolds homeomorphic to the 2-torus.

Further, M^{d-1} is orientable. For each facet $\{(1, a_1), \dots, (d, a_d)\}$, we may set $[(1, a_1), \dots, (d, a_d)]$ as the positive orientation if $\sum_{i=1}^{d} a_i \equiv 1 \pmod 3$; negative if $\sum_{i=1}^{d} a_i \equiv 2 \pmod 3$.

Now, fix $d \geq 3$. Beginning with M^{d-1}, we construct a normal $(d-1)$-pseudomanifold X by a stellar subdivision of the facets $\{(1,1), (2,0), (3,0), \dots, (d,0)\}$ and $\{(1,0), (2,1), (3,0), \dots, (d,0)\}$. We call the new vertices v_1 and v_2, respectively.

Beginning again with M^{d-1}, we similarly construct a normal $(d-1)$-pseudomanifold Y by a stellar subdivision of the facets $\{(1,1), (2,0), (3,0), \dots, (d,0)\}$ and $\{(1,0), (2,2), (3,0), \dots, (d,0)\}$. We call the new vertices w_1 and w_2, respectively. Figure 3.3 shows the structures of X and Y for $d = 3$.

Claim 3.5.7. *$\mathcal{F}(X)$ is not isomorphic to $\mathcal{F}(Y)$.*

Proof. Let $E_1, E_2 \in \mathcal{F}^1(X)$ be the edges

$$E_1 = \mathrm{Lk}(v_1, X) \setminus \mathrm{Lk}(v_2, X) = \{(1,1), (2,0)\},$$

$$E_2 = \operatorname{Lk}(v_2, X) \setminus \operatorname{Lk}(v_1, X) = \{(1,0), (2,1)\}.$$

Similarly, let $D_1, D_2 \in \mathcal{F}^1(Y)$ be the edges

$$D_1 = \operatorname{Lk}(w_1, Y) \setminus \operatorname{Lk}(w_2, Y) = \{(1,1), (2,0)\},$$
$$D_2 = \operatorname{Lk}(w_2, Y) \setminus \operatorname{Lk}(w_1, Y) = \{(1,0), (2,2)\}.$$

(We write $S\backslash T$, as above, to denote the subcomplex of S consisting of all faces disjoint from T.)

By way of contradiction, suppose there is an isomorphism $\varphi : \mathcal{F}(X) \to \mathcal{F}(Y)$. Observe that v_1, v_2 are the only vertices of X contained in just d facets, while w_1, w_2 are the only such vertices of Y. Thus, $\varphi\{v_1, v_2\} = \{w_1, w_2\}$. It follows that

$$\begin{aligned}
\varphi\{E_1, E_2\} &= \varphi\{\operatorname{Lk}(v_1, X) \setminus \operatorname{Lk}(v_2, X), \operatorname{Lk}(v_2, X) \setminus \operatorname{Lk}(v_1, X)\} \\
&= \{\operatorname{Lk}(w_1, Y) \setminus \operatorname{Lk}(w_2, Y), \operatorname{Lk}(w_2, Y) \setminus \operatorname{Lk}(w_1, Y)\} \\
&= \{D_1, D_2\}.
\end{aligned}$$

This is impossible, because

$$\operatorname{Lk}(E_1, X)\backslash\{v_1, v_2\} = \operatorname{Lk}(E_2, X)\backslash\{v_1, v_2\},$$

while

$$\operatorname{Lk}(D_1, Y)\backslash\{w_1, w_2\} \neq \operatorname{Lk}(D_2, Y)\backslash\{w_1, w_2\}.$$

Hence, by contradiction, $\mathcal{F}(X)$ is not isomorphic to $\mathcal{F}(Y)$. □

Claim 3.5.8. *$\mathcal{F}_0^{d-2}(X)$ is isomorphic to $\mathcal{F}_0^{d-2}(Y)$.*

Proof. Let $\varphi : \mathcal{F}_0^{d-2}(X) \to \mathcal{F}_0^{d-2}(Y)$ be the map taking $v_1, v_2, (2,1), (2,2)$ to $w_1, w_2, (2,2)$, $(2,1)$, respectively, and fixing all other vertices. Then φ permutes the faces of $\mathcal{F}_0^{d-2}(M^{d-1}) \cong \mathcal{F}_0^{d-2}(X\backslash\{v_1, v_2\}) \cong \mathcal{F}_0^{d-2}(Y\backslash\{w_1, w_2\})$. We can also see that $\varphi(\operatorname{Lk}(v_1, X)) = \operatorname{Lk}(w_1, Y)$ and $\varphi(\operatorname{Lk}(v_2, X)) = \operatorname{Lk}(w_2, Y)$. Thus, $\varphi : \mathcal{F}_0^{d-2}(X) \to \mathcal{F}_0^{d-2}(Y)$ is an isomorphism. □

In conclusion,

Theorem 3.5.9. *For all $d \geq 3$, there exist normal, orientable $(d-1)$-pseudomanifolds X and Y such that*

$$\mathcal{F}_0^{d-2}(X) \cong \mathcal{F}_0^{d-2}(Y),$$
$$\mathcal{F}(X) \ncong \mathcal{F}(Y).$$

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