On weak quantum symmetry and Frobenius-Perron theory

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Abstract

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Chapter 1 describes the history and notion of a fusion category, as well as some natural settings where fusion categories appear. We discuss the fact that every fusion category is equivalent to the representation category of a weak Hopf algebra, and the importance of an invariant called the Frobenius-Perron (FP) dimension.

Chapter 2 explores algebraic structures in corepresentation categories over weak bialgebras, an investigation which was inspired by studying the symmetries of path algebras of quivers. We prove that for a weak bialgebra H, the formulaic notions of H-comodule algebra, H-comodule coalgebra, and H-comodule Frobenius algebra are equivalent via an isomorphism of categories to the categorical notions of algebra, coalgebra, and Frobenius algebra in the monoidal category of right H-comodules. We also provide a monoidal functor which constructs "weak quantum symmetries" from "quantum symmetries."

Chapter 3 takes a slightly different perspective, and seeks to generalize the categories where we can apply the notion of FP dimension. We generalize the definition of the FP dimension of an object in a fusion category to any $\mathbf{k}$-linear category with a chosen endofunctor. In particular, we define the FP dimension for abelian and derived categories, as well as provide examples and applications. This work has particular significance because it defines a new invariant for derived categories, and not many derived invariants are known. Chapter 4 takes this one step further by calculating the Frobenius-Perron dimension for the abelian

categories associated to a special generalization of A-D-E quiver algebras, and introducing a family of a abelian categories which produce arbitrarily large FP dimension. Along the way, we introduce a result that can be applied to calculate the FP dimension of a radical square zero bound quiver algebra.

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DEDICATION

To my parents.

Chapter 1

INTRODUCTION

1.1 Contributions of this thesis

This thesis focuses on two main, and perhaps seemingly unrelated, subjects: Frobenius-Perron theory of endofunctors ([CGW+19a], [Wic19], and [CGW+19b]) and the study of (co)representation categories of weak bialgebras ([WWW19]). However, both of these topics find their root in a single source: fusion categories. From a mathematical perspective, fusion categories can be seen as a generalization of finite groups (or rather, their representation categories). To explain how the work in this thesis fits into this framework, we will build the theory of fusion categories from the ground up, starting from the theory of groups.

Group theory was first developed in the 19th century as a way to understand the symmetries of mathematical and physical objects. A classical example of a group is the collection of all symmetries of a square, that is, all physical operations that we can perform on a rigid square that yield a square in the same position that we started with. We denote this collection of symmetries by D_8. We can see that some symmetries of the square include rotating the square by 90 degrees or flipping it over the diagonal. In fact, there are only 8 such symmetries (hence the name D_8).

Upon further study, we begin to notice some special properties of these symmetries. If we compose any of these operations, that is, perform one and then the other, we still end up with a symmetry of the square. We also have a symmetry-preserving operation that does nothing. Furthermore, every symmetry-preserving operation has an opposite operation that returns the square to its original state. For example, if we flip the square twice, the end result is equivalent to the "do nothing" symmetry-preserving operation. We can formalize these abstract properties using the definition of a group.

Definition 1.1.1. A *finite group* is a finite set G along with a set map $m : G \times G \to G$ satisfying the following properties. Here we write $gh \coloneqq m(g,h)$ for $(g,h) \in G \times G$.

- associativity: for all $g,h,k \in G$, $(gh)k = g(hk)$,
- unitality: there exists an element $1_G \in G$ such that for all $g \in G$, $1_G g = g = g 1_G$,
- existence of inverses: for all g in G, there exists an element $g^{-1} \in G$ such that $gg^{-1} = 1_G = g^{-1}g$.

Now that we have a formal definition of what a group of symmetries is, how can we study groups and understand their properties? Since groups formalize the symmetries of mathematical and physical objects, it makes sense to understand the collection of symmetries G by understanding the objects whose symmetries are described by G. This leads us to the definition of a representation of a group.

Notation 1.1.2. In the following, we will work over a field $\mathbf{k}$. When we write $\otimes$, we mean $\otimes_{\mathbf{k}}$.

Definition 1.1.3. A *representation of a finite group* G is a vector space V over $\mathbf{k}$ along with a group homomorphism $\rho : G \to \mathrm{Aut}(V)$, where $\mathrm{Aut}(V)$ is the collection of all automorphisms (or symmetries) of V. If V is a representation of G, we say that G *acts on* V.

The restriction that ρ is a group homomorphism yields two conditions. The first says that 1_G acts as the identity (or "do nothing" map) on V. The second condition says that composition of symmetry operations in G is compatible with the action on V. These are exactly the conditions that we would want for a vector space V to have symmetries described by G.

Remark 1.1.4. In our example of the symmetries of the square, we do not need to use vector spaces. One approach we could take is to label the corners of the square, and represent each symmetry-preserving operation by a permutation of the corners. The corners of the square form a set with four elements, so we could have simply defined a group action on a set instead. However, we do not lose any information by defining group actions on vector spaces, as we can always take a group action on a set and write it as a group action on a vector space. In the D_8 example, we could consider a 4-dimensional vector space V, with a basis in one-to-one correspondence with the corners of the square. Since each symmetry-preserving operation is a permutation of the corners of the square, we can extend this map $\mathbf{k}$-linearly to get an automorphism of V. It can be checked that this satisfies the definition of a representation of D_8.

The utility of defining group actions on vector spaces is that we get richer structure and more possible actions than defining group actions on sets. Furthermore, we can use the full power of linear algebra to make conclusions. Eventually, we will see that we can define group actions on mathematical objects with more structure than a vector space.

Now that we have defined a representation of a group, it is natural to ask: how can we classify the representations of a group? In other words, a group like D_8 represents the symmetries of the square. Can we find all mathematical objects (vector spaces) that have symmetries described by D_8?

In the special case where $|G|$ does not divide $\operatorname{char} \mathbf{k}$, we have an important theorem that helps us achieve our goal. First, we need a definition.

Definition 1.1.5. A *subrepresentation* of a representation $\rho : G \to \mathrm{Aut}(V)$ is defined to be a vector space $W \subset V$ such that $\rho(g)(w) \in W$ for all $g \in G, w \in W$.

A *simple representation* is a group representation that has no subrepresentations besides itself and the zero vector space. We remark that simple representations are often referred to as *irreducible representations* in the literature.

Theorem 1.1.6 (Maschke's Theorem [Mas98])**.** *Every representation of a finite group G over a field $\mathbf{k}$ with characteristic not dividing the order of G is a direct sum of simple representations.*

Maschke's theorem tells us that in this case, the simple representations are the building blocks of all representations. To understand any representation, it is enough to gain an understanding of the simple ones.

In the case where $\mathbf{k} = \mathbb{C}$, we have a theorem about the number of simple representations.

Theorem 1.1.7. *Let G be a finite group. There are finitely many simple representations ρ_i of G over $\mathbb{C}$, and we have*

$$\sum_i d_i^2 = |G|,$$

where d_i denotes the degree of $\rho_i : G \to \mathrm{Aut}(V_i)$ (the dimension of the vector space V_i).

The moral of the story is that we can get all of the representations of G by taking direct sums of the simple ones, and there are only finitely many simples. This reduces the original classification problem to finding the simple representations, which we can use techniques from group representation theory to solve. In the case of D_8, we can apply these techniques to show that there are only five simple representations. It seems then that by Maschke's theorem, the representation theory of D_8 is completely understood.

Until now, we have discussed studying the individual representations of G without any thought as to how they behave together. Nevertheless, the collection of representations of G has a rich structure. In particular, the collection of finite-dimensional representations of G forms a *category* which we call $\mathrm{Rep}_{f.d.}(G)$. The objects are the finite-dimensional representations of G, while the morphisms are vector space maps which are compatible with the G-action.

This category has many other special properties. If (V, ρ_V) and (W, ρ_W) are two representations of G, then we can define a representation of G on their $\mathbf{k}$-tensor product $(V \otimes W, \rho_{V\otimes W})$ via $\rho_{V\otimes W}(g) = \rho_V(g) \otimes \rho_W(g)$ for all $g \in G$. Since our representations are finite-dimensional, we can take the vector space dual of a representation to yield a new representation. Furthermore, $\mathrm{Rep}_{f.d.}(G)$ is an *abelian category* (see [Wei94, Sec. 1.2, Defn. 1.2.2] for a definition). Practically speaking, we can think of abelian categories as full subcategories of the category of modules over a ring via the following theorem.

Theorem 1.1.8. *[Wei94, Freyd-Mitchell Embedding Theorem 1.6.1] If $\mathcal{A}$ is a small abelian category, then there exists a ring R and an exact, fully faithful functor from $\mathcal{A}$ into R-*Mod*, which embeds $\mathcal{A}$ as a full subcategory in the sense that* $\mathrm{Hom}_{\mathcal{A}}(M,N) \cong \mathrm{Hom}_R(M,N)$.

In this case of $\mathrm{Rep}_{f.d.}(G)$, the ring under consideration would be the *group algebra* $\mathbf{k}G$. It is well known that $\mathrm{Rep}_{f.d.}(G) \cong \mathbf{k}G\text{-mod}$, where $\mathbf{k}G\text{-mod}$ denotes the category of finite-dimensional left $\mathbf{k}G$-modules. We recall the definition of the group algebra below.

Definition 1.1.9. Let G be a finite group. Then the *group algebra* $\mathbf{k}G$ is the $|G|$-dimensional $\mathbf{k}$-vector space defined as $\mathrm{Span}_{\mathbf{k}}\{g \in G\}$. We define the multiplication map $m : \mathbf{k}G \otimes \mathbf{k}G \to \mathbf{k}G$ as follows, where $\alpha_i, \beta_j \in \mathbf{k}$, $g_i, h_j \in G$:

$$\left(\sum_i \alpha_i g_i\right)\left(\sum_j \beta_j h_j\right) = \sum_{i,j} \alpha_i \beta_j (g_i h_j),$$

and $g_i h_j$ denotes multiplication in G. The unit is 1_G.

There are many more special properties of $\mathrm{Rep}_{f.d.}(G)$, which are captured in the definition of a fusion category.

Definition 1.1.10. [EGNO15] A *fusion category* $\mathcal{C}$ is a category that has the following properties:

(1) **k-linear:** for any two objects V, W, $\mathrm{Hom}_{\mathcal{C}}(V,W)$ has the structure of a $\mathbf{k}$-vector space that is compatible with composition of morphisms,

(2) **abelian,**

(3) **locally finite:** for any two objects V, W, $\dim \mathrm{Hom}_{\mathcal{C}}(V,W) < \infty$. Furthermore, every object has finite length; that is, for all $V \in \mathcal{C}$ there exists a filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

such that V_i/V_{i-1} is simple for all i,

(4) **monoidal:** essentially, there is a tensor product on objects in the category [EGNO15, Def. 2.1.1],

(5) **rigid:** essentially, the objects of the category have duals [EGNO15, Def. 2.10.11],

(6) **finite semisimple:** there are finitely many simple objects, and every object in the category can be written as a direct sum of simples,

(7) **the monoidal product is bilinear on morphisms when considered as a bifunctor,**

(8) **the endomorphism ring of the monoidal unit is isomorphic to k.**

If we remove requirement (6), then $\mathcal{C}$ is called a *tensor category.* If we remove requirement (8), then $\mathcal{C}$ is called a *multifusion category.* If we remove both (6) and (8), then $\mathcal{C}$ is called a *multitensor category.*

One might ask why we would want to generalize the properties of $\mathrm{Rep}_{f.d.}(G)$ in this way. One answer is that fusion and tensor categories arise naturally in physical settings, for example, in the study of topological phases of matter, which appear in condensed matter physics and topological quantum computing. For our purposes, we will define a topological phase of matter to be a physical system that can be described by an anyon model: a theory of quasiparticles on a 2-dimensional surface. A famous example of a topological phase of matter is given by the Ising model of ferromagnetism, where the quasiparticles correspond to elementary excitations of the ground state. The physics of an anyon model is described by the following properties:

(1) A finite list of particle *types*,

(2) Rules for *fusing* (two particles combine) and *splitting* particles (one particle splits into two),

(3) Rules for *braiding* (exchanging particles).

We will not focus on (3), as this involves the notion of a braided tensor category (see [EGNO15] for a definition). Though we omit some details, we remark that the full picture of an anyon model is given by a unitary modular tensor category (a fusion category with additional structure).

The particle types can be thought of as labels distinguishing different types of particles. For example, we could have an anyon model with 3 types of particles, labeled a, b, c. We always assume that we have a special label $\mathbb{1}$, called the vacuum, which corresponds to having no particles at all. For a particle of type a, we assume that there exists a corresponding particle of type $\overline{a}$, called the *charge conjugate* of a, and a charge conjugation map C on the set of labels such that $C^2 = \mathrm{Id}$ and $C(a) = \overline{a}$. We also assume that $\mathbb{1} = \overline{\mathbb{1}}$.

The fusion rules specify the possible particle types c that can result from fusing an a particle with a b particle. In this model, we assume that when we combine two particles, we get another particle. But the particle that we get might not necessarily be uniquely determined. Rather, we have a list of particle types that can be obtained, and the number of ways in which we can obtain them. The final particle we obtain is determined probabilistically, as in quantum mechanics. We use the notation:

$$a \times b = \sum_c N_{ab}^c c,$$

where each N_{ab}^c is a non-negative integer and we sum over all particle labels. Physically, N_{ab}^c represents the possible number of ways an a and a b particle can combine to result in a c particle. It also represents the number of possible ways a c particle can split into an a and a b particle. There are several assumptions that we need to make. For one, fusion should be associative: $(a \times b) \times c = a \times (b \times c)$. We also want $a \times b = b \times a$, and $\mathbb{1} \times a = a$.

The last assumption (along with some other physical assumptions, see [Pre, Sec. 9.12]) implies that $\overline{a}$ is the unique type of particle that fuses with an a particle to yield $\mathbb{1}$, and that $\mathbb{1}$ can only split into charge conjugate pairs $a, \overline{a}$. Therefore, $\overline{a}$ represents an antiparticle corresponding to a. If we combine a particle with its corresponding antiparticle, they should annihilate and yield the vacuum. If we want to create particles from the vacuum, they should come in particle-antiparticle pairs.

There are more physical assumptions that we want to make; for the sake of brevity, we will not mention them here. The point that we want to make is that we need the language of fusion categories to define an anyon model mathematically. We have the following dictionary between physical and mathematical concepts (see [Del19] for a more detailed dictionary).

Physics	Math
anyon model	unitary modular tensor category $\mathcal{C}$ (a fusion category with extra structure)
anyon types	isomorphism classes of simple objects in $\mathcal{C}$
quantum superposition (every anyon state can be expressed as a superposition of the anyon types)	$\mathcal{C}$ is a semisimple abelian category (every object can be expressed as a direct sum of the simple objects)
fusion rules: $a \times b = \sum_c N_{ab}^c c, \quad a \times (b \times c) = (a \times b) \times c, \quad \mathbb{1} \times a = a = a \times 1$	$\mathcal{C}$ is a monoidal category with bilinear bifunctor $\otimes$, with $a \otimes b = \bigoplus_c N_{ab}^c c$
conjugate charges $a, \overline{a}$	left duals a^* of simple objects a ($\mathcal{C}$ is rigid, which means that left and right duals exist)

For physicists, fusion and tensor categories provide the mathematical formalism to describe certain physical models. For mathematicians, fusion and tensor categories are interesting objects of study in and of themselves. Not every fusion category comes from representations of a group. For example, the category of representations of any finite-dimensional semisimple Lie algebra yields a fusion category. Because of this, there is an active program of research to classify fusion categories and where they come from. One tool that has been crucial to this effort is the *Frobenius-Perron (FP) dimension.* First, we need some definitions.

Definition 1.1.11. The *spectral radius* of a matrix A with entries in $\mathbb{C}$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ is defined to be

$$\rho(A) = \max\{|\lambda_1|, \ldots, |\lambda_n|\}.$$

Definition 1.1.12. [EGNO15, Def. 3.3.3, Prop. 4.5.4] Let $\mathcal{C}$ be a fusion category and let $\{X_1, \ldots, X_n\}$ be a complete list of non-isomorphic simples. By semisimplicity, for every object V of $\mathcal{C}$ and every j, we have a decomposition for some integers $a_{ij} \geq 0$:

$$V \otimes_{\mathcal{C}} X_j \cong \bigoplus_{i=1}^{n} X_i^{\oplus a_{ij}}.$$

The *Frobenius-Perron (FP) dimension of* V is defined to be

$$\mathrm{FPdim}(V) \coloneqq \rho((a_{ij})_{n \times n}).$$

Example 1.1.13. [EGNO15, Ex. 4.5.5] If $\mathcal{C} = \mathrm{Rep}_{f.d.}(G)$ for some finite group G, then if V is a representation of G of dimension d, $\mathrm{FPdim}(V) = d$.

This example justifies calling FPdim a "dimension." However, in a general fusion category, the FP dimension of an object does not have to be an integer. There are many theorems regarding the numerical properties of the FP dimensions of objects of a fusion category; we list one here.

Theorem 1.1.14. *[ENO05, Thm. 8.33] Let $\mathcal{C}$ be a fusion category. The following two conditions are equivalent.*

(i) For all $X \in \mathcal{C}$, the number $\mathrm{FPdim}(X)$ is an integer.

(ii) $\mathcal{C}$ is the representation category of a finite-dimensional quasi-Hopf algebra.

We also remark the the FP dimension has a physical meaning for a topological phase of matter. In a unitary modular tensor category, the FP dimension (also known as the *quantum dimension*, which is defined using the trace) of an anyon encodes some of its physical properties. Anyons with quantum dimension 1 are called abelian, and those with quantum dimension greater than 1 are non-abelian. We can use the FP dimension along with other properties of the anyon (for example, the twist, which represents exchange of identical particles via braiding) to define fermions, bosons, and semions (see [Del19]).

We can also define the FP dimension of a fusion category as follows.

Definition 1.1.15. [EGNO15, Def. 3.3.12, Rem. 6.1.8] Let $\mathcal{C}$ be a fusion category and let $\{X_1, \dots, X_n\}$ be a complete list of non-isomorphic simples. The *Frobenius-Perron (FP) dimension of* $\mathcal{C}$ is defined to be

$$\mathrm{FPdim}(\mathcal{C}) := \sum_{i=1}^{n} \mathrm{FPdim}(X_i)^2.$$

Example 1.1.16. If $\mathcal{C} = \mathrm{Rep}_{f.d.}(G)$ for some finite group G, then by Example 1.1.13, Definition 1.1.15, and Theorem 1.1.7,

$$\mathrm{FPdim}(\mathcal{C}) = |G|.$$

There are many useful theorems concerning the FP dimension of a category, but the main one that we focus on in this thesis is the following embedding theorem.

Theorem 1.1.17. *[EGNO15, Prop. 6.3.3] Let $\mathcal{C}$ and $\mathcal{D}$ be fusion categories. If $F : \mathcal{C} \to \mathcal{D}$ is an injective tensor functor, then* $\mathrm{FPdim}(\mathcal{C}) \leq \mathrm{FPdim}(\mathcal{D})$.

This theorem shows that the FP dimension is an invariant of a fusion category. We can apply this theorem in the following way. If we have two fusion categories $\mathcal{C}$ and $\mathcal{D}$, and we find that $\mathrm{FPdim}(\mathcal{C}) > \mathrm{FPdim}(\mathcal{D})$, we immediately know that $\mathcal{C} \ncong \mathcal{D}$ and that $\mathcal{C} \not\hookrightarrow \mathcal{D}$. Hence one fruitful direction in the classification of fusion categories has been classifiying all fusion categories of a particular FP dimension (for example, see [ENO05, EGO04, JL09]).

Since the FP dimension has been such an important tool in the classification of fusion categories, we would like to extend these ideas to other categories of importance to algebraists, namely abelian and derived categories. In Chapter 3, we generalize the definition of the FP dimension of an object to any **k**-linear category with a chosen endofunctor, specialize this definition to abelian and derived categories, and give some results. This gives us a new invariant for abelian and derived categories, and we show that the analogy of Theorem 1.1.17 holds. We build on this work in Chapter 4 by focusing on derived categories of certain abelian categories associated to A-D-E quivers.

In the prequel, we discussed how fusion categories generalize the special categorical properties of $\mathrm{Rep}_{f.d.}(G)$ and how the FP dimension is a powerful tool for studying and classifying fusion categories. But in fact, we already know something about where all fusion categories come from. There is a remarkable theorem that says that every fusion category is the representation category of another algebraic structure called a weak Hopf algebra.

Theorem 1.1.18. *([Hay99a, Szl01], see also [ENO05, Cor. 2.22]) Every fusion category is equivalent to the category of finite dimensional representations of a weak Hopf algebra.*

Therefore, fusion categories generalize groups from two perspectives: from the properties of the categories themselves, and from the properties of the algebraic structures that give rise to these categories through their representations. We now explore how this algebraic structure generalizes the definition of a group, or rather, that of a group algebra.

First we need to recall the definition of an algebra and a coalgebra. All of the algebraic structures that we describe here are $\mathbf{k}$-linear.

Definition 1.1.19. A $\mathbf{k}$-*algebra* is a $\mathbf{k}$-vector space A equipped with a multiplication map $m : A \otimes A \to A$ and unit map $u : \mathbf{k} \to A$ so that $m(m \otimes \mathrm{Id}) = m(\mathrm{Id} \otimes m)$ and $m(u \otimes \mathrm{Id}) = \mathrm{Id} = m(\mathrm{Id} \otimes u)$. We reserve the notation 1 to mean

$$1 := 1_A := u(1_{\mathbf{k}}).$$

A $\mathbf{k}$-*coalgebra* is a $\mathbf{k}$-vector space C equipped with a comultiplication map $\Delta : C \to C \otimes C$ and counit map $\varepsilon : C \to \mathbf{k}$ so that $(\Delta \otimes \mathrm{Id})\Delta = (\mathrm{Id} \otimes \Delta)\Delta$ and $(\varepsilon \otimes \mathrm{Id})\Delta = \mathrm{Id} = (\mathrm{Id} \otimes \varepsilon)\Delta$.

Definition 1.1.20. A *bialgebra* is a quintuple $(H, m, u, \Delta, \varepsilon)$ such that

(i) (H, m, u) is an algebra,

(ii) (H, Δ, ε) is a coalgebra,

(iii) $\Delta(ab) = \Delta(a)\Delta(b)$ for all $a, b \in H$,

(iv) $\Delta(1) = 1 \otimes 1$,

(v) $\varepsilon(ab) = \varepsilon(a)\varepsilon(b)$ for all $a, b, c \in H$,

(vi) $\varepsilon(1) = 1$.

We can summarize this definition by saying that a bialgebra is a vector space equipped with the structures of an algebra and a coalgebra that have good compatibility conditions. Items (iii)-(vi) simply say that Δ and ε are algebra maps. It turns out that this is equivalent to requiring that m and u are coalgebra maps.

Now let us try to understand how the group algebra $\mathbf{k}G$ fits into this picture. We defined the algebra structure on $\mathbf{k}G$ in Definition 1.1.9. Since the elements of G are a basis for $\mathbf{k}G$, we can define a coalgebra structure by extending the following relations $\mathbf{k}$-linearly:

$$\forall g \in G, \quad \Delta(g) = g \otimes g, \quad \varepsilon(g) = 1.$$

It suffices to check the relations on a basis of $\mathbf{k}G$, since all maps are $\mathbf{k}$-linear. For $g \in G$, we have

$$(\Delta \otimes \mathrm{Id})\Delta(g) = (\Delta \otimes \mathrm{Id})(g \otimes g) = g \otimes g \otimes g = (\mathrm{Id} \otimes \Delta)(g \otimes g) = (\mathrm{Id} \otimes \Delta)\Delta(g),$$

and

$$(\varepsilon \otimes \mathrm{Id})\Delta(g) = (\varepsilon \otimes \mathrm{Id})(g \otimes g) = g = (\mathrm{Id} \otimes \varepsilon)(g \otimes g) = (\mathrm{Id} \otimes \varepsilon)\Delta(g).$$

Finally, $\mathbf{k}G$ is a bialgebra, since for all $g, h \in G$, we have

$$\Delta(gh) = gh \otimes gh = (g \otimes g)(h \otimes h) = \Delta(g)\Delta(h), \qquad \Delta(1) = \Delta(1_G) = 1_G \otimes 1_G = 1 \otimes 1,$$

and

$$\varepsilon(gh) = 1 = \varepsilon(g)\varepsilon(h), \qquad \varepsilon(1) = \varepsilon(1_G) = 1.$$

Now, $\mathbf{k}G$ has an additional piece of structure: every element of G has an inverse. When we have a bialgebra structure in place, we can generalize the notion of inverse with the following definition.

Definition 1.1.21. A *Hopf algebra* is a sextuple $(H, m, u, \Delta, \varepsilon, S)$, where $(H, m, u, \Delta, \varepsilon)$ is a bialgebra and $S : H \to H$ is a $\mathbf{k}$-linear map called the *antipode* that satisfies the following property for all $h \in H$:

$$m(S \otimes \mathrm{Id})\Delta(h) = \varepsilon(h) = m(\mathrm{Id} \otimes S)\Delta(h).$$

If we define $S : \mathbf{k}G \to \mathbf{k}G$ by requiring that $S(g) = g^{-1}$ and extending $\mathbf{k}$-linearly, we can see that $\mathbf{k}G$ is a Hopf algebra. For $g \in G$, we have

$$m(S \otimes \mathrm{Id})\Delta(g) = m(S(g) \otimes g) = m(g^{-1} \otimes g) = 1 = \varepsilon(g),$$

and the other relation is proved similarly.

Remark 1.1.22. For our purposes in this chapter, it suffices to remark that a Hopf algebra is a special case of a weak Hopf algebra (see Definition 2.2.3 for a precise definition).

One might ask why we went to the trouble of generalizing the definition of a group algebra in this way. One motivation is Theorem 1.1.18, which says that weak Hopf algebras are precisely the algebraic structures that yield fusion categories. Yet another motivation comes from our study of symmetry. In the prequel, we discussed how groups formalize collections of symmetries of mathematical objects, and we examined group actions on vector spaces. It is natural to ask for groups to act on mathematical objects with more structure: in particular, on algebras and coalgebras. As before, we can define group actions via automorphisms:

Definition 1.1.23. A *group action* by a group G on a $\mathbf{k}$-algebra A is a group homomorphism $\rho : G \to \mathrm{Aut}(A)$.

The additional piece of structure that we obtain here is that ρ needs to respect the multiplication and unit of A. That is, for all $g \in G$ and $a, b \in A$, $\rho(g)(ab) = \rho(g)(a)\rho(g)(b)$, and $\rho(g)(1_A) = 1_A$.

The study of group actions on algebraic structures found its motivations in algebraic geometry. It started with group actions on commutative algebras, in particular with group actions on polynomial rings.

Definition 1.1.24. Given a vector space V of dimension n, we define $\mathbf{k}[V]$ to be the commutative $\mathbf{k}$-algebra generated by all $\mathbf{k}$-linear maps $V \to \mathbf{k}$. That is, if we fix a basis of V and let $\{x_i\}$ be the dual basis, then $\mathbf{k}[V] = \mathbf{k}[x_1, \dots, x_n]$.

Proposition 1.1.25. *Given a representation $\rho : G \to \mathrm{Aut}(V)$, we have an induced action ρ' on $\mathbf{k}[V]$:*

$$\forall v \in V, g \in G, f \in \mathbf{k}[V], \quad \rho'(g)(f)(x) \coloneqq f(\rho(g^{-1})(x)).$$

It is natural to ask: which polynomials are invariant (that is, fixed) under a group action on $\mathbf{k}[V]$? This simple question led to the development of invariant theory. Building off of this framework, we can consider group actions on coordinate rings of other varieties, and other graded commutative algebras. The logical next step (for noncommutative algebraists) would be to consider group actions on noncommutative algebras.

Unlike graded commutative algebras, a noncommutative algebra does not have an associated geometric space via the Proj or Spec constructions. Yet studying symmetries of noncommutative algebras still makes sense from a geometric perspective. Noncommutative projective schemes were first developed in [AZ94] (see [Rog14] for a comprehensive survey on this topic). In commutative algebraic geometry, we can construct a geometric space $\mathrm{Proj}(A)$ from a commutative graded ring A. When B is noncommutative, however, it might not be easy to find a geometric space that acts as the analogue of $\mathrm{Proj}(B)$. However, there are many properties of $\mathrm{Proj}(A)$ that are determined by its category of coherent sheaves. To this end, a generalization of the category of coherent sheaves on $\mathrm{Proj}(A)$ was developed for noncommutative algebras. By studying this category, noncommutative geometers can make geometric conclusions about B without having to construct an actual geometric space.

One of the requirements on B for this construction to work is that B should be connected $\mathbb{N}$-graded. In this case, *connected* simply means that $B_0 \cong \mathbf{k}$, where B_0 is the degree 0 part of B. (There are other requirements, but we will focus on the connected part here.) It seems then that if we want to study symmetries of noncommutative spaces, we should be studying group actions on connected graded $\mathbf{k}$-algebras.

However, the symmetries described by group actions on connected graded algebras do not capture the full collection of symmetries of these algebras, particularly in the noncommutative case. For example, consider the skew polynomial ring $\mathbf{k}_{-1}[x, y] := \mathbf{k}\langle x, y\rangle/(xy + yx)$ in the case where $\mathbf{k}$ is an algebraically closed field of characteristic 0. This is a noncommutative Artin-Schelter regular algebra of global dimension 2, which by definition is connected graded. In [CKWZ16], Chan, Kirkman, Walton, and Zhang classify all finite-dimensional inner faithful Hopf actions with trivial homological determinant on global dimension 2 Artin-Schelter regular algebras. In particular, they show that there are Hopf actions on $\mathbf{k}_{-1}[x, y]$ that do *not* come from a group algebra action or the action of its dual.

Examples like these led to the notion of quantum symmetry. For noncommutative connected graded algebras, the "right" kind of symmetries to consider are Hopf algebra actions (a generalization of group actions, which we will define in Chapter 2). As the previous example shows, if we restrict ourselves to group actions, we lose genuine symmetries of noncommutative algebras. A theorem that supports that Hopf algebra actions are the right kind of generalization of group actions is given below. This incredible result states that for commutative domains, there are no quantum symmetries, only group actions. That is, the notion of Hopf action reduces to the notion of group action in the commutative domain case.

Theorem 1.1.26. *[EW14][Thm. 1.3] If a cosemisimple Hopf algebra H over $\mathbf{k}$ acts inner faithfully on a commutative domain over $\mathbf{k}$, then H is a finite group algebra.*

The study of Hopf actions on connected graded algebras is an active field of research.

However, there are noncommutative algebras of interest that are graded, but not connected: for example, the path algebra of a quiver. To understand symmetries of nonconnected graded algebras, we posit that we must extend our definition of symmetries to "weak symmetry," that is, (co)actions of weak bialgebras and weak Hopf algebras on algebras, coalgebras, and Frobenius algebras. We explore this idea further in Chapter 2, where we give both categorical and formulaic definitions of a "weak symmetry," and show that we have isomorphisms of categories linking the two notions. We also provide a monoidal functor that creates weak symmetries from quantum symmetries. We hope that this work will be the first in a systematic study of weak symmetries.

The contribution of this thesis to the study of mathematical symmetries centers on two main topics: applying the techniques used in the study of fusion categories to other categories of interest to algebraists and geometers, and understanding weak symmetries of algebraic structures. Though seemingly different, these topics find their source in the desire to formalize and understand symmetries, which started in the 19th century and has been a focus of mathematical research to the present day.

1.2 List of publications

This thesis contains material pertaining to the first three papers on this list.

Publications

1. [CGW+19a] Jianmin Chen, Zhibin Gao, Elizabeth Wicks, James J. Zhang, Xiaohong Zhang, and Hong Zhu. Frobenius-Perron theory of endofunctors. *Algebra Number Theory*, 13(9):2005-2055, 2019.

2. [Wic19] Elizabeth Wicks. Frobenius-Perron theory of modified *ADE* bound quiver algebras. *J. Pure Appl. Algebra*, 223(6):2673-2708, 2019.

Preprints

3. [WWW19] Chelsea Walton, Elizabeth Wicks, and Robert Won. Algebraic structures in comodule categories over weak bialgebras. *ArXiv e-prints*, 2019.

4. [CGW+19b] Jianmin Chen, Zhibin Gao, E. Wicks, James J. Zhang, Xiaohong Zhang, and Hong Zhu. Frobenius-Perron theory for projective schemes. *ArXiv e-prints*, 2019.

Chapter 2

ALGEBRAIC STRUCTURES IN COMODULE CATEGORIES OVER WEAK BIALGEBRAS

2.1 Introduction

This chapter is a study of quantum symmetry in the context of weak bialgebra and weak Hopf algebra (co)actions on algebraic structures, based on joint work with Chelsea Walton and Robert Won [WWW19]. Previously, we discussed how Hopf algebras can be thought of as generalizations of groups and can be used to generalize group actions on connected graded noncommutative algebras. Our goal is to use weak Hopf algebras and bialgebras to generalize group actions on nonconnected graded algebras, coalgebras, and Frobenius algebras.

First, we need some definitions.

Definition 2.1.1. Let (A, m, u) be a $\mathbf{k}$-algebra and let V be a vector space. Then V is a left A-module if there exists a $\mathbf{k}$-linear map $l : A \otimes V \to V$ such that:

- $l(\mathrm{Id}_A \otimes l) = l(m \otimes \mathrm{Id}_V)$,
- $l(u \otimes \mathrm{Id}_V) = \mathrm{Id}_V$.

We denote $l(a \otimes v) := a.v$ for $a \in A, v \in V$.

Definition 2.1.2. [Mon93] Let H be a Hopf algebra and let A be a $\mathbf{k}$-algebra. We say that A is a *left H-module algebra* if A is a left H-module such that for all $h \in H, a, b \in A$, we have

- $h.(ab) = (h_1.a)(h_2.b)$,
- $h.1_A = \varepsilon(h)1_A$,

where 1_A is the unit of A, ε is the counit of H, and $\Delta(h) = h_1 \otimes h_2$ (here we use sumless Sweedler notation). We say that H *acts on* A and we call this a Hopf action.

Since a Hopf action should correspond to a symmetry of an algebra, the action should be compatible with the algebra structure of A. The first requirement shows that we have compatibility with multiplication, while the second shows that we have compatibility with the unit. However, this definition seems somewhat ad-hoc. How did we come up with these formulaic compatibility conditions? The answer is that these compatibility conditions appear in a natural way when considering Hopf actions from a categorical perspective.

The category of modules over a Hopf algebra H is always a monoidal category, denoted ${}_H\mathcal{M}$, where the tensor product is $\otimes := \otimes_{\mathbf{k}}$ [EGNO15]. Recall the definition of an algebra in a monoidal category (Definition 2.3.4). We have the following classical theorem, which connects the formulaic definition of a Hopf action to the categorical definition.

Theorem 2.1.3 (see [EGNO15, Ex. 7.8.3(6)])**.** *Let A be a $\mathbf{k}$-algebra and H be a Hopf algebra. Then A is a left H-module algebra if and only if A is an algebra in the monoidal category of left H-modules ${}_H\mathcal{M}$.*

With this theorem in mind, we can redefine our notion of symmetry categorically by consider algebras, coalgebras, and Frobenius algebras in ${}_H\mathcal{M}$. However, we don't have to only focus on modules. Since H is a coalgebra, we can also look at comodules. That is, we can consider algebras, coalgebras, and Frobenius algebras in $\mathcal{M}^H$ (the monoidal category of right comodules of H) in order to understand symmetries.

Remark 2.1.4. In our work, we focus on coactions rather than actions. The main reason for this is that we have many nice examples of weak bialgebra coactions. In particular, for any quiver Q, we have an example of an associated weak bialgebra $\mathfrak{H}(Q)$ that coacts on $\mathbf{k}Q$ (see Example 2.2.8). Since we wanted to extend our notion of symmetry to nonconnected graded algebras like $\mathbf{k}Q$, this example led us to start with coactions. In the finite-dimensional case, we do not lose anything from taking this perspective; a weak bialgebra action can be thought of as a coaction of the dual weak bialgebra.

As in the bialgebra case, we would like to define the notion of an action for a weak bialgebra categorically as well as formulaically. To do this, there are several issues that need to be taken into account. First, we need to understand the monoidal structure of $\mathcal{M}^H$ when H is a weak bialgebra. If H is a weak bialgebra that is not a bialgebra, the tensor product is not $\otimes_{\mathbf{k}}$ and the monoidal unit is not $\mathbf{k}$ (see Section 2.3.2). So, algebras in $\mathcal{M}^H$ are not canonically $\mathbf{k}$-algebras in this case. Yet we still want to have a formulaic definition of a weak bialgebra coacting on an algebra that corresponds to the categorical notion. We were able to achieve this goal, as we will show below.

Convention. We employ Sweedler notation freely here and throughout (see Section 2.2.3).

Definition 2.1.5 ([Böh00, Definition 2.1], see also [CDG00, Proposition 4.10])**.** Let H be a weak bialgebra and H_t be its target counital subalgebra [Definition 2.2.2]. We say that a $\mathbf{k}$-algebra $(A,\ m : A \otimes A \to A,\ u : \mathbf{k} \to A)$ is a *right H-comodule algebra* if A is a right H-comodule via $\rho : A \to A \otimes H$, $a \mapsto a_{[0]} \otimes a_{[1]}$ so that

$$(ab)_{[0]} \otimes (ab)_{[1]} = a_{[0]}b_{[0]} \otimes a_{[1]}b_{[1]}$$

and $\rho(1_A) \in A \otimes H_t$. Here, $m(a \otimes b) = ab$ and $1_A = u(1_{\mathbf{k}})$.

As desired, there is a correspondence between right H-comodule algebras and algebras in $\mathcal{M}^H$; this comprises our first result.

Notation 2.1.6. If $\mathcal{D}$ is a category whose objects have underlying $\mathbf{k}$-vector space structures, let $\mathcal{D}_{f.d.}$ denote the full subcategory of $\mathcal{D}$ whose objects have finite $\mathbf{k}$-dimension.

Theorem 2.1.7 (Theorem 2.4.3)**.** *Let H be a weak bialgebra. Consider the category $\mathcal{A}^H$ of right H-comodule algebras (see Definition 2.4.1) and the category $\mathsf{Alg}(\mathcal{M}^H)$ of algebras in the monoidal category $\mathcal{M}^H$ of right H-comodules (see Section 2.3.2 and Definition 2.3.4). Then there is an isomorphism of categories*

$$\mathcal{A}^H \cong \mathsf{Alg}(\mathcal{M}^H),$$

which restricts to an isomorphism $\mathcal{A}^H_{f.d.} \cong \mathsf{Alg}(\mathcal{M}^H_{f.d.})$.

The utility of this theorem is that it gives a correspondence between understanding weak symmetries of algebras formulaically and categorically. This result has appeared as a special case of [BCM02, Proposition 3.9], but we include a full proof here, both for the reader's benefit and to build on it for subsequent results on comodule coalgebras and comodule Frobenius algebras. The proofs of our results are all constructive, explicit, and written purely in the language of weak bialgebras.

To study comodule coalgebras over weak bialgebras, we note that a formulaic definition of an H-comodule coalgebra has appeared in the literature.

Definition 2.1.8 ([Jia09, Definition 1], [VZ10, Definition 2.1])**.** Let H be a weak bialgebra and ε_s be its source counital map [Definition 2.2.2]. A $\mathbf{k}$-coalgebra $(C,\ \Delta : C \to C \otimes C,\ \varepsilon : C \to \mathbf{k})$ is called a *right H-comodule coalgebra* if C is a right H-comodule via $\rho : C \to C \otimes H$, $c \mapsto c_{[0]} \otimes c_{[1]}$ so that

$$c_{1,[0]} \otimes c_{2,[0]} \otimes c_{1,[1]}c_{2,[1]} = c_{[0],1} \otimes c_{[0],2} \otimes c_{[1]}$$

and $\varepsilon(c_{[0]})c_{[1]} = \varepsilon(c_{[0]})\varepsilon_s(c_{[1]})$.

Moreover, we introduce the definition of an H-comodule Frobenius algebra.

Definition 2.1.9. (Definition 2.6.1) A Frobenius $\mathbf{k}$-algebra $(A, m, u, \Delta, \varepsilon)$ is called a *right H-comodule Frobenius algebra* if (A, m, u) is a right H-comodule algebra and (A, Δ, ε) is a right H-comodule coalgebra.

We establish results relating the formulaic definitions above with categorical notions.

Theorem 2.1.10 (Theorems 2.5.2 and 2.6.2)**.** *Let H be a weak bialgebra and let $\mathcal{M}^H$ denote the monoidal category of right H-comodules (see Section 2.3.2). Consider the categories below:*

- *the category $\mathcal{C}^H$ of right H-comodule coalgebras (Definition 2.5.1);*
- *the category $\mathcal{F}^H$ of right H-comodule Frobenius algebras (Definition 2.6.1);*

- *the category* $\mathsf{Coalg}(\mathcal{M}^H)$ *of coalgebras in* $\mathcal{M}^H$ *[Definition 2.3.4]; and*

- *the category* $\mathsf{FrobAlg}(\mathcal{M}^H)$ *of Frobenius algebras in* $\mathcal{M}^H$ *[Definition 2.3.4].*

Then, we have the following isomorphisms of categories:

$$\mathcal{C}^H \cong \mathsf{Coalg}(\mathcal{M}^H) \qquad \text{and} \qquad \mathcal{F}^H \cong \mathsf{FrobAlg}(\mathcal{M}^H),$$

which restrict to isomorphisms $\mathcal{C}^H_{f.d.} \cong \mathsf{Coalg}(\mathcal{M}^H_{f.d.})$ *and* $\mathcal{F}^H_{f.d.} \cong \mathsf{FrobAlg}(\mathcal{M}^H_{f.d.})$.

Remark 2.1.11. The restrictions of the category isomorphisms to their respective categories of finite-dimensional corepresentations in Theorems 2.1.7 and 2.1.10 is clear, and will be omitted in the statements and proofs in Sections 2.4–2.6. We mention the restriction here as one may be interested in category of representations ${}_H\mathcal{M}$ of a finite-dimensional weak bialgebra H, in which case one can use the monoidal equivalence: ${}_H\mathcal{M}_{f.d.} \cong \mathcal{M}^{H^*}_{f.d.}$.

One application for this study of weak quantum symmetries is to understand Frobenius algebras in tensor categories, which arise physically in quantum field theory. These include the Frobenius algebras in modular tensor categories that are used to examine rational conformal field theories [FRS02], those in C^*-tensor categories that comprise Q-systems in subfactor theory [BKLR15], the commutative Frobenius algebras that characterize 2-dimensional topological quantum field theories in symmetric tensor categories [Koc04], and more (see [RFFS07] for a survey of Frobenius algebras in topological and conformal field theory). The reason we can apply our results in this way is the following theorem.

Theorem 2.1.12. ([Hay99a, Theorem 4.1], see also [Szl01, Theorem 1.4] [ENO05, Corollary 2.22]). *Every fusion category is tensor equivalent to the category of finite-dimensional comodules over a face algebra (i.e., over a certain weak bialgebra).*

In light of Theorem 2.1.12, some of our contributions (Definition 2.1.9 and Theorem 2.1.10) provide an explicit characterization of Frobenius algebras in any fusion category.

As mentioned above, our interest in comodule algebras over weak bialgebras is motivated by coactions of Hayashi's face algebras [Hay93] (see Theorem 2.1.12), which include weak bialgebras $\mathfrak{H}(Q)$ attached to a finite quiver Q (see Example 2.2.8). In [Hay99b, Section 2], Hayashi showed that the corresponding path algebra $\mathbf{k}Q$ is a comodule over $\mathfrak{H}(Q)$, and we further establish that $\mathbf{k}Q$ is an $\mathfrak{H}(Q)$-comodule algebra in order to illustrate Theorem 2.1.7 (Example 2.4.8). The $\mathfrak{H}(Q)$-comodule structure on $\mathbf{k}Q$ is also used for our running examples of weak comodule coalgebras (Example 2.5.7) and weak comodule Frobenius algebras (Examples 2.6.6 and 2.6.7), which illustrate Theorem 2.1.10.

To produce another collection of examples for Theorem 2.1.7 above, we employ the quantum transformation groupoids presented in [NV02, Section 2.6]. These are weak Hopf algebras $H(L, B, \triangleleft)$ that depend on a Hopf algebra L, a strongly separable algebra B, and an action $B \otimes L \xrightarrow{\triangleleft} B$ making B a right L-module algebra (Definition 2.7.2). (Note that minor adjustments to the presentation of $H(L, B, \triangleleft)$ in [NV02] were needed.) Our last result is that we construct a monoidal functor from a category of L-bicomodules to the corepresentation category $\mathcal{M}^{H(L,B,\triangleleft)}$.

Theorem 2.1.13 (Theorem 2.7.5)**.** *Let L, B, $\triangleleft$ be as above, and let $H(L, B, \triangleleft)$ denote the corresponding quantum transformation groupoid. Let L-$\mathsf{Bicomod}$ be the monoidal category of L-bicomodules given in Definition 2.7.4. Then there is a monoidal functor*

$$\widehat{\Gamma} : L\text{-}\mathsf{Bicomod} \to \mathcal{M}^{H(L,B,\triangleleft)}.$$

This theorem provides a mechanism for creating "weak quantum symmetry" (weak Hopf algebra coactions) from "ordinary quantum symmetry" (Hopf algebra actions). Several examples of this construction are provided at the end of Section 2.7.

2.2 Preliminaries on weak bialgebras

In this section, we begin by providing the algebraic set-up material for working over a field in Section 2.2.1. We then define and recall properties of weak bialgebras and of weak Hopf algebras in Section 2.2.2. Next, we define (co)modules over weak bialgebras in Section 2.2.3. We end by providing examples of weak bialgebras in Section 2.2.4.

2.2.1 Algebraic structures over a field

Fix a base field $\mathbf{k}$, and reserve $\otimes$ to mean $\otimes_{\mathbf{k}}$. All algebraic structures in this work are $\mathbf{k}$-linear.

Recall that a $\mathbf{k}$-*algebra* is a $\mathbf{k}$-vector space A equipped with a multiplication map $m : A \otimes A \to A$ and unit map $u : \mathbf{k} \to A$ so that $m(m \otimes \mathrm{Id}) = m(\mathrm{Id} \otimes m)$ and $m(u \otimes \mathrm{Id}) = \mathrm{Id} = m(\mathrm{Id} \otimes u)$. We reserve the notation 1 to mean

$$1 := 1_A := u(1_{\mathbf{k}}).$$

A $\mathbf{k}$-*coalgebra* is a $\mathbf{k}$-vector space C equipped with a comultiplication map $\Delta : C \to C \otimes C$ and counit map $\varepsilon : C \to \mathbf{k}$ so that $(\Delta \otimes \mathrm{Id})\Delta = (\mathrm{Id} \otimes \Delta)\Delta$ and $(\varepsilon \otimes \mathrm{Id})\Delta = \mathrm{Id} = (\mathrm{Id} \otimes \varepsilon)\Delta$. When we refer to a (co)algebra, we mean a (co)associative (co)unital (co)algebra.

A *Frobenius algebra over* $\mathbf{k}$ is a $\mathbf{k}$-vector space A that is simultaneously a $\mathbf{k}$-algebra (A, m, u) and a $\mathbf{k}$-coalgebra (A, Δ, ε) so that $(m \otimes \mathrm{Id})(\mathrm{Id} \otimes \Delta) = \Delta m = (\mathrm{Id} \otimes m)(\Delta \otimes \mathrm{Id})$.

If (C, Δ, ε) is a coalgebra and $c \in C$, we use sumless Sweedler notation and write

$$\Delta(c) := c_1 \otimes c_2.$$

We use the compact notation $\Delta^2(c)$ to mean $(\Delta \otimes \mathrm{Id})\Delta(c)$. Since Δ is coassociative,

$$\Delta^2(c) := (\Delta \otimes \mathrm{Id})\Delta(c) = (\mathrm{Id} \otimes \Delta)\Delta(c).$$

Using Sweedler notation, we write $\Delta^2(c) = c_1 \otimes c_2 \otimes c_3$. Then we have

$$c_1 \otimes c_2 \otimes c_3 := c_{1,1} \otimes c_{1,2} \otimes c_2 = c_1 \otimes c_{2,1} \otimes c_{2,2},$$

where we denote $\Delta(c_1) = c_{1,1} \otimes c_{1,2}$ and $\Delta(c_2)$ similarly.

There may be multiple coproducts of 1 appearing in the same formula. We denote different copies of 1 using primed notation. For example,

$$\Delta(1) \otimes \Delta(1) = (1_1 \otimes 1_2) \otimes (1'_1 \otimes 1'_2) = 1_1 \otimes 1_2 \otimes 1'_1 \otimes 1'_2.$$

2.2.2 Terminology and properties

Definition 2.2.1. A *weak bialgebra* is a quintuple $(H, m, u, \Delta, \varepsilon)$ such that

(i) (H, m, u) is an algebra,

(ii) (H, Δ, ε) is a coalgebra,

(iii) $\Delta(ab) = \Delta(a)\Delta(b)$ for all $a, b \in H$,

(iv) $\varepsilon(abc) = \varepsilon(ab_1)\varepsilon(b_2c) = \varepsilon(ab_2)\varepsilon(b_1c)$ for all $a, b, c \in H$,

(v) $\Delta^2(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1)$.

We can rewrite (v) in Sweedler notation as follows:

$$1_1 \otimes 1_2 \otimes 1_3 = 1_1 \otimes 1_2 1'_1 \otimes 1'_2 = 1_1 \otimes 1'_1 1_2 \otimes 1'_2.$$

The difference between a bialgebra and a weak bialgebra can be understood as a weakening of the compatibility between the algebra and coalgebra structures. In a weak bialgebra, we still have that comultiplication is multiplicative (e.g., condition (iii)), but the counit is no longer multiplicative and we do not necessarily have $\Delta(1) = 1 \otimes 1$ or $\varepsilon(1) = 1$. Instead, we have weak multiplicativity of the counit (condition (iv)) and weak comultiplicativity of the unit (condition (v)).

Definition 2.2.2 (ε_s, ε_t, H_s, H_t)**.** Let $(H, m, u, \Delta, \varepsilon)$ be a weak bialgebra. We define the *source and target counital maps*, respectively as follows:

$$\varepsilon_s : H \to H, \quad x \mapsto 1_1 \varepsilon(x 1_2)$$
$$\varepsilon_t : H \to H, \quad x \mapsto \varepsilon(1_1 x) 1_2.$$

We denote the images of these maps as $H_s := \varepsilon_s(H)$ and $H_t := \varepsilon_t(H)$. We call H_s and H_t the *counital subalgebras* of H (see Proposition 2.2.4).

With these definitions in mind, we can define a weak Hopf algebra.

Definition 2.2.3. A *weak Hopf algebra* is a sextuple $(H, m, u, \Delta, \varepsilon, S)$, where $(H, m, u, \Delta, \varepsilon)$ is a weak bialgebra and $S : H \to H$ is a $\mathbf{k}$-linear map called the *antipode* that satisfies the following properties for all $h \in H$:

(i) $S(h_1)h_2 = \varepsilon_s(h)$

(ii) $h_1 S(h_2) = \varepsilon_t(h)$

(iii) $S(h_1)h_2 S(h_3) = S(h)$.

Note that if H is a weak bialgebra, then H is a bialgebra precisely when ε is multiplicative or when $\Delta(1) = 1 \otimes 1$. Similarly, if H is a weak Hopf algebra, the following are equivalent: H is a Hopf algebra; $\Delta(1) = 1 \otimes 1$; $\varepsilon(xy) = \varepsilon(x)\varepsilon(y)$ for all $x, y \in H$; $S(x_1)x_2 = \varepsilon(x)1$ for all $x \in H$; and $x_1 S(x_2) = \varepsilon(x)1$ for all $x \in H$ [BNS99, p.5].

We notice that the antipode axioms for a weak Hopf algebra use the maps $\varepsilon_s, \varepsilon_t$ instead of ε as for ordinary Hopf algebras. In fact, the counital subalgebras $H_s = \varepsilon_s(H)$ and $H_t = \varepsilon_t(H)$ are often used instead of $\mathbf{k} = \varepsilon(H)$ in the weak Hopf algebra setting. These subalgebras have special properties that we will need below.

Proposition 2.2.4 (Facts about counital subalgebras)**.** *Let H be a weak bialgebra.*

(i) *[NV02, Prop. 2.2.1(iii)] $h \in H_s \Leftrightarrow \Delta(h) = 1_1 \otimes h1_2$ and $h \in H_t \Leftrightarrow \Delta(h) = 1_1 h \otimes 1_2$.*

(ii) *[NV02, Prop. 2.2.2] H_s (resp., H_t) is a left (resp., right) coideal subalgebra of H.*

(iii) *[NV02, Prop. 2.2.2] If $y \in H_s$ and $z \in H_t$, then $yz = zy$.*

(iv) *[BCJ11, Prop. 1.17] H_s is a coalgebra with counit $\varepsilon|_{H_s}$ and comultiplication*

$$\begin{aligned} \Delta_s : H_s &\to H_s \otimes H_s \\ y &\mapsto 1_1 \otimes \varepsilon_s(y1_2) = y_1 \otimes \varepsilon_s(y_2). \end{aligned}$$

(v) *[BCJ11, Prop. 1.17] H_t is a coalgebra with counit $\varepsilon|_{H_t}$ and comultiplication*

$$\begin{aligned}\Delta_t : H_t &\to H_t \otimes H_t \\ z &\mapsto \varepsilon_t(1_1 z) \otimes 1_2 = \varepsilon_t(z_1) \otimes z_2.\end{aligned}$$

Proposition 2.2.5 (Useful weak bialgebra identities)**.** *[NV02, Prop. 2.2.1] Let H be a weak bialgebra and let $g, h \in H$. The following relations hold:*

$$\varepsilon_s(\varepsilon_s(h)) = \varepsilon_s(h), \qquad \varepsilon_t(\varepsilon_t(h)) = \varepsilon_t(h), \tag{2.2.5.1}$$

$$h_1 \varepsilon_s(h_2) = h = \varepsilon_t(h_1) h_2, \tag{2.2.5.2}$$

$$\Delta(1) = 1_1 \otimes \varepsilon_t(1_2) = \varepsilon_s(1_1) \otimes 1_2 \in H_s \otimes H_t. \tag{2.2.5.3}$$

2.2.3 Modules and comodules

We now direct our attention to the (co)representation theory of weak bialgebras. Let H be a weak bialgebra and let M be a $\mathbf{k}$-vector space. We call M a right H*-(co)module* if it is a right (co)module under the (co)algebra structure of H. We define left H*-(co)modules* similarly. We call M an (H, H)*-bi(co)module* if it is a bi(co)module under the (co)algebra structure of H.

Given an algebra A, we denote the actions for a right A-module M and for a left A-module N as follows:

$$\nu_M : M \otimes A \to M, \quad m \otimes a \mapsto m \triangleleft a,$$

$$\mu_N : A \otimes N \to N, \quad a \otimes n \mapsto a \triangleright n.$$

Now suppose that M and N are (A, A)-bimodules. Recall that the *tensor product* of M and N is an (A, A)-bimodule given as follows:

$$M \otimes_A N := (M \otimes N)/\operatorname{im}(\nu_M \otimes \mathrm{Id}_N - \mathrm{Id}_M \otimes \mu_N).$$

That is, $M \otimes_A N$ is a $\mathbf{k}$-quotient space of $M \otimes N$.

Given a coalgebra C, we denote the coactions for a right C-comodule M and for a left C-comodule N using sumless Sweedler notation:

$$\rho_M : M \to M \otimes C,\ \ m \mapsto m_{[0]} \otimes m_{[1]}, \qquad \lambda_N : N \to C \otimes N,\ \ n \mapsto n_{[-1]} \otimes n_{[0]}.$$

A C-comodule M must satisfy the coassociativity and unitality axioms. Namely, for right C-comodules we get

$$m_{[0],[0]} \otimes m_{[0],[1]} \otimes m_{[1]} = m_{[0]} \otimes m_{[1],1} \otimes m_{[1],2}, \tag{2.2.5.4}$$

$$m = \varepsilon(m_{[1]}) m_{[0]}. \tag{2.2.5.5}$$

for all $m \in M$. We let $\mathsf{Comod}\text{-}C$ denote the category of right C-comodules.

Now let M, N be (C, C)-bicomodules. Recall that the *cotensor product* of M and N is a (C, C)-bicomodule given as follows:

$$M \otimes^C N := \ker\left(\rho_M \otimes \mathrm{Id}_N - \mathrm{Id}_M \otimes \lambda_N\right).$$

By definition, $M \otimes^C N$ is a $\mathbf{k}$-subspace of $M \otimes N$.

2.2.4 Examples

Next, we present several examples of weak bialgebras.

Example 2.2.6 (Groupoid algebras)**.** [NV02, Example 2.5] Let G be a finite groupoid (a finite category in which every morphism is invertible). We define the groupoid algebra $\mathbf{k}G$, as follows. As a vector space, $\mathbf{k}G$ has basis given by the morphisms $g \in G$. The product of two morphisms is given by composition if it is defined and 0 otherwise. This extends linearly to define multiplication on $\mathbf{k}G$. The unit of $\mathbf{k}G$ is $1_{\mathbf{k}G} = \sum_{i=1}^{n} e_i$ where e_i denotes the identity morphism of the ith object of G. Then $\mathbf{k}G$ can be given the structure of a weak Hopf algebra via

$$\Delta(g) = g \otimes g, \qquad \varepsilon(g) = 1, \qquad S(g) = g^{-1}$$

for all $g \in G$. The groupoid algebra is a Hopf algebra if and only if G is a group. Moreover, for any $g \in G$, we have

$$\varepsilon_s(g) = 1_1\varepsilon(g1_2) = \sum_{i=1}^n e_i\varepsilon(ge_i) = e_{t(g)}$$

where $t(g)$ denotes the target of g and composition is written left-to-right. Hence, the counital subalgebra $(\mathbf{k}\,G)_s$ is equal to $\mathbf{k}\,e_1 \oplus \cdots \oplus \mathbf{k}\,e_n$. Similarly, $(\mathbf{k}\,G)_t = \bigoplus_{i=1}^n \mathbf{k}\,e_i$.

The following example is not provided in the literature, to the best of our knowledge, and its verification follows exactly as the example above.

Example 2.2.7 (Path algebras)**.** Let Q be a finite quiver with vertex set $Q_0 = \{1, 2, \dots, n\}$ and edge set Q_1. We let e_i denote the trivial path at vertex i. The path algebra $\mathbf{k}\,Q$ has a weak bialgebra structure given by $\Delta(e_i) = e_i \otimes e_i$ and $\varepsilon(e_i) = 1$ for each $1 \le i \le n$, and $\Delta(p) = p \otimes p$ and $\varepsilon(p) = 1$ for each $p \in Q_1$. With this structure, $\mathbf{k}\,Q$ is a bialgebra if and only if $|Q_0| = 1$. As above, $(\mathbf{k}\,Q)_s = \bigoplus_{i=1}^n \mathbf{k}\,e_i = (\mathbf{k}\,Q)_t$.

Example 2.2.8 (Hayashi's face algebras attached to quivers)**.** Let Q be a finite quiver. Denote by Q_ℓ the set of all paths of length ℓ in Q, and let $Q_0 = \{1, 2, \dots, n\}$. For a path p, let $s(p)$ and $t(p)$ denote the source and target of p, respectively. We define the weak bialgebra $\mathfrak{H}(Q)$ as follows. As a $\mathbf{k}$-vector space, $\mathfrak{H}(Q) = \bigoplus_{\ell \ge 0} \bigoplus_{p,q \in Q_\ell} \mathbf{k}\,x_{p,q}$, for indeterminates $x_{p,q}$. The algebra structure of $\mathfrak{H}(Q)$ is given by

$$x_{p,q} \cdot x_{p',q'} = \delta_{t(p),s(p')}\delta_{t(q),s(q')}x_{pp',qq'} = \begin{cases} x_{pp',qq'} & \text{if } t(p) = s(p') \text{ and } t(q) = s(q') \\ 0 & \text{otherwise} \end{cases}$$

with $1 = \sum_{i,j \in Q_0} x_{i,j}$. For $p, q \in Q_\ell$, the coalgebra structure is given by

$$\Delta\left(x_{p,q}\right) = \sum_{t \in Q_\ell} x_{p,t} \otimes x_{t,q} \qquad \text{and} \qquad \varepsilon\left(x_{p,q}\right) = \delta_{p,q}.$$

For each $i \in Q_0$, the *face idempotents* of $\mathfrak{H}(Q)$ are $a_j := \sum_{i=1}^n x_{i,j}$ and $a'_j := \sum_{i=1}^n x_{j,i}$. For $p, q \in Q_\ell$, we can compute $\varepsilon_s(x_{p,q}) = \delta_{p,q}\sum_{i=1}^n x_{i,t(q)}$ and $\varepsilon_t(x_{p,q}) = \delta_{p,q}\sum_{j=1}^n x_{s(q),j}$. Hence, as $\mathbf{k}$-vector spaces, $\mathfrak{H}(Q)_s = \bigoplus_{j=1}^n \mathbf{k}\,a_j$ and $\mathfrak{H}(Q)_t = \bigoplus_{j=1}^n \mathbf{k}\,a'_j$. We refer the reader to [Hay93] for more details.

2.3 Monoidal categories and corepresentation categories

In this section, we begin by recalling facts about monoidal categories, functors between them, and algebraic structures within them in Section 2.3.1. We then recall the definition of the corepresentation category of a weak bialgebra in Section 2.3.2, and discuss functors between this category and other monoidal categories in Section 2.3.3.

2.3.1 Monoidal categories

Definition 2.3.1. [EGNO15] A *monoidal category* $\mathcal{C}$ consists of the following data:

- a category $\mathcal{C}$,
- a bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,
- a natural isomorphism $\alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \xrightarrow{\sim} X \otimes (Y \otimes Z)$ for each $X, Y, Z \in \mathcal{C}$,
- an object $\mathbb{1} \in \mathcal{C}$,
- natural isomorphisms $l_X : \mathbb{1} \otimes X \xrightarrow{\sim} X$, $r_X : X \otimes \mathbb{1} \xrightarrow{\sim} X$ for each $X \in \mathcal{C}$,

such that the following two axioms are satisfied.

The pentagon axiom. For all objects $W, X, Y, Z \in \mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
 & ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "\alpha_{W\otimes X,Y,Z}"'] \arrow[dr, "\alpha_{W,X,Y}\otimes \mathrm{Id}_Z"] & \\
(W \otimes X) \otimes (Y \otimes Z) \arrow[d, "\alpha_{W,X,Y\otimes Z}"'] & & (W \otimes (X \otimes Y)) \otimes Z \arrow[d, "\alpha_{W,X\otimes Y,Z}"] \\
W \otimes (X \otimes (Y \otimes Z)) & & W \otimes ((X \otimes Y) \otimes Z) \arrow[ll, "\mathrm{Id}_W \otimes \alpha_{X,Y,Z}"]
\end{tikzcd}$$

The triangle axiom. For all objects $X, Y \in \mathcal{C}$, the following diagram commutes:

$$\begin{tikzcd}
(X \otimes \mathbb{1}) \otimes Y \arrow[rr, "\alpha_{X,\mathbb{1},Y}"] \arrow[dr, "r_X \otimes \mathrm{Id}_Y"'] & & X \otimes (\mathbb{1} \otimes Y) \arrow[dl, "\mathrm{Id}_X \otimes l_Y"] \\
 & X \otimes Y &
\end{tikzcd}$$

Example 2.3.2. Examples of monoidal categories include the following:

- $\mathsf{Vec}_{\mathbf{k}}$, the category of finite-dimensional $\mathbf{k}$-vector spaces, with $\otimes = \otimes_{\mathbf{k}}$, $\mathbb{1} = \mathbf{k}$, and with the canonical associativity and unit isomorphisms;

- ${}_A\mathcal{M}_A$, the category of (A, A)-bimodules with monoidal product being the tensor product $\otimes_A$, for an algebra A;

- ${}^C\mathcal{M}^C$, the category of (C, C)-bicomodules with monoidal product being the cotensor product $\otimes^C$, for a coalgebra C.

If C is a weak bialgebra, we can endow the category of right C-comodules with the structure of a monoidal category; see Section 2.3.2 below.

Definition 2.3.3. [Str07, p.85] [DP08] [Szl05, Eq.6.46, 6.47] Let $(\mathcal{C}, \otimes_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}, \alpha_{*,*,*}, l_*, r_*)$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, \mathbb{1}_{\mathcal{D}}, \alpha_{*,*,*}, l_*, r_*)$ be monoidal categories.

(i) A *monoidal functor* $(F, F_{*,*}, F_0) : \mathcal{C} \to \mathcal{D}$ consists of the following data:

- a functor $F : \mathcal{C} \to \mathcal{D}$,
- a natural transformation $F_{X,Y} : F(X) \otimes_{\mathcal{D}} F(Y) \to F(X \otimes_{\mathcal{C}} Y)$ for all $X, Y \in \mathcal{C}$,
- a morphism $F_0 : \mathbb{1}_{\mathcal{D}} \to F(\mathbb{1}_{\mathcal{C}})$ in $\mathcal{D}$,

that satisfy the associativity and unitality constraints: for all $X, Y, Z \in \mathcal{C}$,

$$F_{X,Y\otimes_{\mathcal{C}}Z}\,(\mathrm{Id}_{F(X)} \otimes_{\mathcal{D}} F_{Y,Z})\,\alpha_{F(X),F(Y),F(Z)} = F(\alpha_{X,Y,Z})\,F_{X\otimes_{\mathcal{C}}Y,Z}\,(F_{X,Y} \otimes_{\mathcal{D}} \mathrm{Id}_{F(Z)}),$$
$$F(l_X)^{-1}\,l_{F(X)} = F_{\mathbb{1}_{\mathcal{C}},X}\,(F_0 \otimes_{\mathcal{D}} \mathrm{Id}_{F(X)}),$$
$$F(r_X)^{-1}\,r_{F(X)} = F_{X,\mathbb{1}_{\mathcal{C}}}\,(\mathrm{Id}_{F(X)} \otimes_{\mathcal{D}} F_0).$$

A *strong monoidal functor* is a monoidal functor where $F_0, F_{X,Y}$ are isomorphisms for all $X, Y \in \mathcal{C}$.

(ii) A *comonoidal functor* $(F, F^{*,*}, F^0) : \mathcal{C} \to \mathcal{D}$ consists of the following data:

- a functor $F : \mathcal{C} \to \mathcal{D}$,
- a natural transformation $F^{X,Y} : F(X \otimes_{\mathcal{C}} Y) \to F(X) \otimes_{\mathcal{D}} F(Y)$ for all $X, Y \in \mathcal{C}$,
- a morphism $F^0 : F(\mathbb{1}_{\mathcal{C}}) \to \mathbb{1}_{\mathcal{D}}$ in $\mathcal{D}$,

that satisfy the coassociativity and counitality constraints dual to above. A *strong comonoidal functor* is a comonoidal functor where $F^0, F^{X,Y}$ are isomorphisms for all $X, Y \in \mathcal{C}$.

(iii) A *Frobenius monoidal functor* $(F, F_{*,*}, F_0, F^{*,*}, F^0) : \mathcal{C} \to \mathcal{D}$ is a functor where $(F, F_{*,*}, F_0)$ is monoidal, $(F, F^{*,*}, F^0)$ is comonoidal, such that for all $X, Y, Z \in \mathcal{C}$:

$$(F_{X,Y} \otimes_{\mathcal{D}} \mathrm{Id}_{F(Z)})\alpha^{-1}_{F(X),F(Y),F(Z)}(\mathrm{Id}_{F(X)} \otimes_{\mathcal{D}} F^{Y,Z}) = F^{X\otimes_{\mathcal{C}} Y,Z} F(\alpha^{-1}_{X,Y,Z}) F_{X,Y\otimes_{\mathcal{C}} Z},$$
$$(\mathrm{Id}_{F(X)} \otimes_{\mathcal{D}} F_{Y,Z})\alpha_{F(X),F(Y),F(Z)}(F^{X,Y} \otimes_{\mathcal{D}} \mathrm{Id}_{F(Z)}) = F^{X,Y\otimes_{\mathcal{C}} Z} F(\alpha_{X,Y,Z}) F_{X\otimes_{\mathcal{C}} Y,Z}.$$

Our definition of a (co)monoidal functor is also called a lax (co)monoidal functor in the literature, and our definition of a strong (co)monoidal functor is sometimes simply referred to as a (co)monoidal functor in other references. It is well known that a strong monoidal functor is the same as a strong comonoidal functor. Moreover, any strong (co)monoidal functor is Frobenius monoidal [DP08, Proposition 3].

Now we turn our attention to algebraic structures in monoidal categories.

Definition 2.3.4 ($\mathsf{Alg}(\mathcal{C})$, $\mathsf{Coalg}(\mathcal{C})$, $\mathsf{FrobAlg}(\mathcal{C})$)**.** Let $(\mathcal{C}, \otimes, \mathbb{1}, \alpha, l, r)$ be a monoidal category.

(i) An *algebra* in $\mathcal{C}$ is a triple (A, m, u), where A is an object in $\mathcal{C}$, and $m : A \otimes A \to A$, $u : \mathbb{1} \to A$ are morphisms in $\mathcal{C}$, satisfying unitality and associativity constraints:

$$m(m \otimes \mathrm{Id}) = m(\mathrm{Id} \otimes m)\alpha_{A,A,A}, \tag{2.3.4.1}$$
$$m(u \otimes \mathrm{Id}) = l_A, \qquad m(\mathrm{Id} \otimes u) = r_A. \tag{2.3.4.2}$$

A *morphism* of algebras (A, m_A, u_A) to (B, m_B, u_B) is a morphism $f : A \to B$ in $\mathcal{C}$ so that $fm_A = m_{B\otimes B}(f \otimes f)$ and $fu_A = u_B$. Algebras in $\mathcal{C}$ and their morphisms form a category, which we denote by $\mathsf{Alg}(\mathcal{C})$.

(ii) A *coalgebra* in $\mathcal{C}$ is a triple (C, Δ, ε), where C is an object in $\mathcal{C}$, and $\Delta : C \to C \otimes C$, $\varepsilon : C \to \mathbb{1}$ are morphisms in $\mathcal{C}$, satisfying counitality and coassociativity constraints:

$$\alpha_{C,C,C}(\Delta \otimes \mathrm{Id})\Delta = (\mathrm{Id} \otimes \Delta)\Delta, \tag{2.3.4.3}$$

$$(\varepsilon \otimes \mathrm{Id})\Delta = l_C^{-1}, \qquad (\mathrm{Id} \otimes \varepsilon)\Delta = r_C^{-1}. \tag{2.3.4.4}$$

A *morphism* of coalgebras $(C, \Delta_C, \varepsilon_C)$ to $(D, \Delta_D, \varepsilon_D)$ is a morphism $g : C \to D$ in $\mathcal{C}$ so that $\Delta_D g = (g \otimes g)\Delta_C$ and $\varepsilon_D g = \varepsilon_C$. Coalgebras in $\mathcal{C}$ and their morphisms form a category, which we denote by $\mathsf{Coalg}(\mathcal{C})$.

(iii) A *Frobenius algebra* in $\mathcal{C}$ is a quintuple $(A, m, u, \Delta, \varepsilon)$ such that $(A, m, u) \in \mathsf{Alg}(\mathcal{C})$ and $(A, \Delta, \varepsilon) \in \mathsf{Coalg}(\mathcal{C})$, so that

$$(m \otimes \mathrm{Id})\alpha_{A,A,A}^{-1}(\mathrm{Id} \otimes \Delta) = \Delta m = (\mathrm{Id} \otimes m)\alpha_{A,A,A}(\Delta \otimes \mathrm{Id}). \tag{2.3.4.5}$$

A *morphism* of Frobenius algebras in $\mathcal{C}$ is a morphism in $\mathcal{C}$ that lies in both $\mathsf{Alg}(\mathcal{C})$ and $\mathsf{Coalg}(\mathcal{C})$. Frobenius algebras in $\mathcal{C}$ and their morphisms form a category, which we denote by $\mathsf{FrobAlg}(\mathcal{C})$.

Algebras, coalgebras, and Frobenius algebras in $\mathsf{Vec}_{\mathbf{k}}$ are the same as $\mathbf{k}$-algebras, $\mathbf{k}$-coalgebras, and Frobenius algebras over $\mathbf{k}$, respectively (cf. Section 2.2.1).

Next, we see that monoidal, comonoidal, and Frobenius monoidal functors preserve, respectively, algebras, coalgebras, and Frobenius algebras in monoidal categories.

Proposition 2.3.5. *[Str07, p.100-101] [Szl05, Lem. 2.1] [DP08, Cor. 5] [KR09, Prop. 2.13]*

(i) *Let* $(F, F_{*,*}, F_0) : \mathcal{C} \to \mathcal{D}$ *be a monoidal functor. If* $(A, m, u) \in \mathsf{Alg}(\mathcal{C})$, *then*

$$(F(A),\ F(m)F_{A,A},\ F(u)F_0) \in \mathsf{Alg}(\mathcal{D}).$$

(ii) *Let* $(F, F^{*,*}, F^0) : \mathcal{C} \to \mathcal{D}$ *be a comonoidal functor. If* $(C, \Delta, \varepsilon) \in \mathsf{Coalg}(\mathcal{C})$*, then*

$$(F(C),\ F^{C,C}F(\Delta),\ F^0F(\varepsilon)) \in \mathsf{Coalg}(\mathcal{D}).$$

(iii) *Let* $(F, F_{*,*}, F_0, F^{*,*}, F^0) : \mathcal{C} \to \mathcal{D}$ *be a Frobenius monoidal functor. If* $(A, m, u, \Delta, \varepsilon)$ *is in* $\mathsf{FrobAlg}(\mathcal{C})$*, then*

$$(F(A),\ F(m)F_{A,A},\ F(u)F_0,\ F^{A,A}F(\Delta),\ F^0F(\varepsilon)) \in \mathsf{FrobAlg}(\mathcal{D}).$$

□

2.3.2 The corepresentation category $\mathcal{M}^H$

For a weak bialgebra $H = (H, m, u, \Delta, \varepsilon)$, the category $\mathcal{M}^H$ of right H-comodules can be given the structure of a monoidal category:

$$\mathcal{M}^H = (\mathsf{Comod}\text{-}H,\ \overline{\otimes},\ \mathbb{1} = H_s,\ \alpha = \alpha_{\mathsf{Vec}_{\mathbf{k}}},\ l,\ r).$$

We give further details about the tensor product $\overline{\otimes}$ and unitality constraints l and r below. We also recall some basic facts about $\mathcal{M}^H$ which were provided in [BCJ11], some of which appeared in the preprint [Nil98]. The connection between $\mathcal{M}^H$ and the bi(co)module categories ${}_{H_s}\mathcal{M}_{H_s}$ and ${}^{H_s}\mathcal{M}^{H_s}$ are also highlighted below.

Remark 2.3.6. By [NV02, Remark 2.4.1], H^{cop} is a weak Hopf algebra. Hence the category of left H-comodules is isomorphic to the category of right H^{cop}-comodules and we can identify ${}^H\mathcal{M}$ with $\mathcal{M}^{H^{cop}}$. Note that the category of left H-comodules can also be given the structure of a monoidal category, but one would need to use the counital subalgebra H_t in place of H_s, along with implementing other similar substitutions.

2.3.2.1 Tensor product $\overline{\otimes}$

Given $M, N \in \mathcal{M}^H$, the tensor product $M \otimes N$ has a coassociative, but not necessarily counital, right H-coaction given by $m \otimes n \mapsto m_{[0]} \otimes n_{[0]} \otimes m_{[1]}n_{[1]}$. Hence, in $\mathcal{M}^H$ we force counitality by defining the monoidal product of M and N to be

$$M \mathbin{\overline{\otimes}} N := \left\{ m \otimes n \in M \otimes N \mid m \otimes n = \varepsilon(m_{[1]}n_{[1]})m_{[0]} \otimes n_{[0]} \right\}. \tag{2.3.6.1}$$

We use the notation $m \mathbin{\overline{\otimes}} n$ for the element $m \otimes n$ when we view it as an element of $M \mathbin{\overline{\otimes}} N$. The right H-coaction on $M \mathbin{\overline{\otimes}} N$ is given by

$$\begin{aligned} \rho_{M \overline{\otimes} N} : M \mathbin{\overline{\otimes}} N &\to M \mathbin{\overline{\otimes}} N \otimes H \\ m \mathbin{\overline{\otimes}} n &\mapsto m_{[0]} \mathbin{\overline{\otimes}} n_{[0]} \otimes m_{[1]}n_{[1]}. \end{aligned} \tag{2.3.6.2}$$

The inclusion of $M \mathbin{\overline{\otimes}} N$ into $M \otimes N$ is given by

$$\begin{aligned} \iota_{M,N} : M \mathbin{\overline{\otimes}} N &\to M \otimes N \\ \varepsilon(m_{[1]}n_{[1]})m_{[0]} \mathbin{\overline{\otimes}} n_{[0]} &\mapsto \varepsilon(m_{[1]}n_{[1]})m_{[0]} \otimes n_{[0]}. \end{aligned}$$

There is also a projection

$$\begin{aligned} \eta_{M,N} : M \otimes N &\to M \mathbin{\overline{\otimes}} N \\ m \otimes n &\mapsto \varepsilon(m_{[1]}n_{[1]})m_{[0]} \mathbin{\overline{\otimes}} n_{[0]}. \end{aligned}$$

Next, we study $\overline{\otimes}$ of morphisms of H-comodules. If $f : M \to M'$ and $g : N \to N'$ are morphisms of right H-comodules, then we define

$$\begin{aligned} f \mathbin{\overline{\otimes}} g : M \mathbin{\overline{\otimes}} N &\to M' \mathbin{\overline{\otimes}} N' \\ \varepsilon(m_{[1]}n_{[1]})m_{[0]} \mathbin{\overline{\otimes}} n_{[0]} &\mapsto \varepsilon(m_{[1]}n_{[1]})f(m_{[0]}) \mathbin{\overline{\otimes}} g(n_{[0]}). \end{aligned}$$

Note that since f and g are right H-comodule morphisms, we have

$$\varepsilon(f(m)_{[1]}g(n)_{[1]})f(m)_{[0]} \mathbin{\overline{\otimes}} g(n)_{[0]} = \varepsilon(m_{[1]}n_{[1]})f(m_{[0]}) \mathbin{\overline{\otimes}} g(n_{[0]}),$$

and so $\varepsilon(m_{[1]}n_{[1]})f(m_{[0]}) \mathbin{\overline{\otimes}} g(n_{[0]}) \in M' \mathbin{\overline{\otimes}} N'$. This shows that for $m \mathbin{\overline{\otimes}} n \in M \mathbin{\overline{\otimes}} N$,

$$(f \mathbin{\overline{\otimes}} g)(m \mathbin{\overline{\otimes}} n) = f(m) \mathbin{\overline{\otimes}} g(n). \tag{2.3.6.3}$$

2.3.2.2 Unit object $\mathbb{1} = H_s$

The counital subalgebra H_s is naturally a right H-comodule since the image of $\Delta|_{H_s}$ is a subspace of $H_s \otimes H$ (see Proposition 2.2.4(iv)), and so $\Delta|_{H_s}$ can be viewed as a map $H_s \to H_s \otimes H$. By [BCJ11, Theorem 3.1], H_s is the unit object of the monoidal category $\mathcal{M}^H$.

2.3.2.3 The unit isomorphisms l and r

By [BCJ11, Section 3], the monoidal category $\mathcal{M}^H$ has unit isomorphisms:

$$l_M : H_s \overline{\otimes} M \to M, \qquad x \overline{\otimes} m = \varepsilon(x_{[1]}m_{[1]})x_{[0]} \overline{\otimes} m_{[0]} \;\mapsto\; \varepsilon(xm_{[1]})m_{[0]},$$

$$r_M : M \overline{\otimes} H_s \to M, \qquad m \overline{\otimes} x = \varepsilon(x_{[1]}m_{[1]})x_{[0]} \overline{\otimes} m_{[0]} \;\mapsto\; \varepsilon(m_{[1]}x)m_{[0]},$$

for all $M \in \mathcal{M}^H$. Moreover, the inverses of these maps are given as follows:

$$l_M^{-1} : M \to H_s \overline{\otimes} M, \qquad m \;\mapsto\; \varepsilon(1_2 m_{[1]})1_1 \overline{\otimes} m_{[0]},$$

$$r_M^{-1} : M \to M \overline{\otimes} H_s, \qquad m \;\mapsto\; m_{[0]} \overline{\otimes} \varepsilon(m_{[1]}1_2)1_1.$$

2.3.3 Functors between $\mathcal{M}^H$ and other monoidal categories

In this section, we discuss the functors $U : \mathcal{M}^H \to \mathsf{Vec}_{\mathbf{k}}$, $\Phi : \mathcal{M}^H \to {}_{H_s}\mathcal{M}_{H_s}$, and $\Psi : \mathcal{M}^H \to {}^{H_s}\mathcal{M}^{H_s}$. We will see below that U and Φ are monoidal, and also that U and Ψ are comonoidal.

2.3.3.1 The forgetful functor $U : \mathcal{M}^H \to \mathsf{Vec}_{\mathbf{k}}$

To begin, we have by [BCJ11, Theorem 3.2] that the forgetful functor

$$U : \mathcal{M}^H \to \mathsf{Vec}_{\mathbf{k}}$$

is both a monoidal functor via

$$\begin{aligned} U_{M,N} : M \otimes N &\xrightarrow{\eta_{M,N}} M \overline{\otimes} N &\quad \text{and} \quad U_0 : \mathbf{k} &\xrightarrow{u_{H_s}} H_s \\ m \otimes n &\mapsto \varepsilon(m_{[1]}n_{[1]})m_{[0]} \overline{\otimes} n_{[0]} & 1_{\mathbf{k}} &\mapsto 1_H, \end{aligned} \tag{2.3.6.4}$$

and a comonoidal functor via

$$\begin{array}{ccc} U^{M,N}: M \mathbin{\overline{\otimes}} N \xrightarrow{\iota_{M,N}} M \otimes N & \text{and} & U^0 : H_s \xrightarrow{\varepsilon|_{H_s}} \mathbf{k} \\ \varepsilon(m_{[1]}n_{[1]})m_{[0]} \mathbin{\overline{\otimes}} n_{[0]} \mapsto \varepsilon(m_{[1]}n_{[1]})m_{[0]} \otimes n_{[0]} & & x \mapsto \varepsilon(x). \end{array} \tag{2.3.6.5}$$

Here, $\eta_{M,N}$ and $\iota_{M,N}$ are defined in Section 2.3.2.1. In fact, by [Szl05, Section 6], U is a Frobenius monoidal functor. We also remark that

$$U_{M,N}U^{M,N} = \mathrm{Id}_{M \overline{\otimes} N}. \tag{2.3.6.6}$$

2.3.3.2 The functor $\Phi : \mathcal{M}^H \to {}_{H_s}\mathcal{M}_{H_s}$

The monoidal structure of U induces an H_s-bimodule action on any right H-comodule. Explicitly, if M is a right H-comodule, then the left H_s-action μ_M and right H_s-action ν_M are given by, respectively,

$$\mu_M(x \otimes m) =: x \triangleright m = \varepsilon(x m_{[1]})m_{[0]}, \qquad \nu_M(m \otimes x) =: m \triangleleft x = \varepsilon(m_{[1]}x)m_{[0]}, \tag{2.3.6.7}$$

for all $x \in H_s$, $m \in M$; this is consistent with Section 2.3.2.3. Moreover, these actions commute with right H-coaction in the following way: for all $x \in H_s, m \in M$,

$$(x \triangleright m)_{[0]} \otimes (x \triangleright m)_{[1]} = m_{[0]} \otimes x m_{[1]} \quad \text{and} \quad (m \triangleleft x)_{[0]} \otimes (m \triangleleft x)_{[1]} = m_{[0]} \otimes m_{[1]}x. \tag{2.3.6.8}$$

For a morphism $f : M \to N \in \mathcal{M}^H$, define $\Phi(f) = f$. Indeed, if $m \in M$ and $x \in H_s$, then

$$f(m \triangleleft x) = f(\varepsilon(m_{[1]}x)m_{[0]}) = \varepsilon(m_{[1]}x)f(m_{[0]}) = \varepsilon(f(m)_{[1]}x)f(m)_{[0]} = f(m) \triangleleft x.$$

Likewise, $f(x \triangleright m) = x \triangleright f(m)$, for all $x \in H_s$ and $m \in M$. Therefore, f is also an H_s-bimodule morphism.

Next, note that as a monoidal functor, U factors through the monoidal functor

$$\Phi : \mathcal{M}^H \to {}_{H_s}\mathcal{M}_{H_s}, \quad \text{where}$$

$$\Phi_{M,N} : M \otimes_{H_s} N \to M \mathbin{\overline{\otimes}} N$$

$$m \otimes_{H_s} n \mapsto \varepsilon(m_{[1]}n_{[1]})m_{[0]} \mathbin{\overline{\otimes}} n_{[0]}.$$

and $\Phi_0 : H_s \to H_s$ is the identity map. In fact, $\Phi_{M,N}$ is an isomorphism of H-comodules, with inverse

$$\Phi^{-1}_{M,N} : M \mathbin{\overline{\otimes}} N \to M \otimes_{H_s} N$$

$$\varepsilon(m_{[1]}n_{[1]})m_{[0]} \mathbin{\overline{\otimes}} n_{[0]} \mapsto \varepsilon(m_{[1]}n_{[1]})m_{[0]} \otimes_{H_s} n_{[0]} = m \otimes_{H_s} n.$$

Throughout, we use $\Phi_{M,N}$ and $\Phi^{-1}_{M,N}$ to identify $M \mathbin{\overline{\otimes}} N$ with $M \otimes_{H_s} N$.

2.3.3.3 The functor $\Psi : \mathcal{M}^H \to {}^{H_s}\mathcal{M}^{H_s}$

The comonoidal structure of U induces an H_s-bicomodule action on any right H-comodule. Explicitly, if M is a right H-comodule, then the left H_s-coaction and right H_s-coaction on M are given respectively by

$$\lambda^s_M(m) := \varepsilon\big(m_{[1]}1_1\big)1_2 \otimes m_{[0]}, \quad \rho^s_M(m) := m_{[0]} \otimes 1_1\varepsilon\big(m_{[1]}1_2\big) \qquad \forall m \in M. \tag{2.3.6.9}$$

As a comonoidal functor, U factors through the comonoidal functor

$$\Psi : \mathcal{M}^H \to {}^{H_s}\mathcal{M}^{H_s}, \quad \text{where}$$

$$\Psi^{M,N} : M \mathbin{\overline{\otimes}} N \to M \otimes^{H_s} N$$

$$\varepsilon(m_{[1]}n_{[1]})m_{[0]} \otimes n_{[0]} \mapsto \varepsilon(m_{[1]}n_{[1]})m_{[0]} \otimes^{H_s} n_{[0]}$$

is the inclusion and $\Psi^0 : H_s \to H_s$ is the identity map. Here, $M \otimes^{H_s} N$ is the cotensor product of M and N using the H_s-coactions given in (2.3.6.9) and Definition 2.2.2. In fact, $M \mathbin{\overline{\otimes}} N$ and $M \otimes^{H_s} N$ are equal as subspaces of $M \otimes N$.

2.4 Weak comodule algebras

The goal of this section is to reconcile the two notions of comodule algebras over weak bialgebras available in the literature, which is a special case of [BCM02, Proposition 3.9].

For the remainder of the paper, fix $H = (H, m, u, \Delta, \varepsilon)$ a weak bialgebra, and set the notation

$$1 \coloneqq 1_H = u(1_{\mathbf{k}}).$$

Definition 2.4.1 ($\mathcal{A}^H =: \mathcal{A}$)**.** and [Böh00, Definition 2.1] [CDG00, Proposition 4.10]. Consider the *category $\mathcal{A}^H$ of right H-comodule algebras* defined as follows. The objects of $\mathcal{A}^H$ are algebras in $\mathsf{Vec}_{\mathbf{k}}$,

$$(A,\ m_A : A \otimes A \to A,\ u_A : \mathbf{k} \to A),$$

with $1_A \coloneqq u_A(1_{\mathbf{k}})$, so that

- the $\mathbf{k}$-vector space A is a right H-comodule via

  $$\rho_A : A \to A \otimes H, \quad a \mapsto a_{[0]} \otimes a_{[1]},$$

  namely, the coassociativity and counit axioms hold (equations (2.2.5.4) and (2.2.5.5));

- the multiplication map m_A is compatible with ρ_A in the sense that $\forall a, b \in A$,

  $$(ab)_{[0]} \otimes (ab)_{[1]} = a_{[0]} b_{[0]} \otimes a_{[1]} b_{[1]}, \tag{2.4.1.1}$$

- the unit map u_A is compatible with ρ_A in the sense that

  $$\rho_A(1_A) \quad \in \quad A \otimes H_t. \tag{2.4.1.2}$$

The morphisms of $\mathcal{A}^H$ are $\mathbf{k}$-linear algebra maps that are also H-comodule maps. We write $\mathcal{A}$ for this category when H is understood.

Remark 2.4.2. It follows from [CDG00, Proposition 4.10] that (2.4.1.2) is equivalent to each of the conditions below:

$$\begin{aligned}
\rho_A(1_A) \quad &\in \quad (1_A)_{[0]} \otimes \varepsilon\big(1_{(1)}(1_A)_{[1]}\big)1_{(2)};\\
\rho_A(1_A) \quad &\in \quad (1_A)_{[0]} \otimes \varepsilon\big((1_A)_{[1]}1_{(1)}\big)1_{(2)};
\end{aligned}$$

$$\begin{aligned}
1_{[0]}a \otimes 1_{[1]} &= a_{[0]} \otimes \varepsilon\big(1_{(1)}a_{[1]}\big)1_{(2)} && \forall a \in A;\\
a1_{[0]} \otimes 1_{[1]} &= a_{[0]} \otimes \varepsilon\big(a_{[1]}1_{(1)}\big)1_{(2)} && \forall a \in A;\\
(\rho_A)^2(1_A) &= (1_A)_{[0]} \otimes (1_A)_{[1]}1_{(1)} \otimes 1_{(2)};\\
(\rho_A)^2(1_A) &= (1_A)_{[0]} \otimes 1_{(1)}(1_A)_{[1]} \otimes 1_{(2)}.
\end{aligned}$$

Recall from Section 2.3.2 the monoidal structure of the category $\mathcal{M}^H$ of right H-comodules. Our main result is the following.

Theorem 2.4.3. *The category $\mathcal{A}^H =: \mathcal{A}$ of right H-comodule algebras is isomorphic to the category $\mathsf{Alg}(\mathcal{M}^H)$ of algebras in the monoidal category $\mathcal{M}^H$.*

Proof. We achieve this result by constructing mutually inverse functors F and G below.

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, bend left, "F"] & & \mathsf{Alg}(\mathcal{M}^H) \arrow[ll, bend left, "G"]
\end{tikzcd}$$

First, take $(A,\ m_A,\ u_A) \in \mathcal{A}$, and consider the assignment

$$\begin{aligned}
F(A,\ m_A,\ u_A) &:= (A, \quad m_A\, U^{A,A} : A \,\overline{\otimes}\, A \to A, \quad \nu_A(u_A \otimes \mathrm{Id}_{H_s}) : H_s \to A)\\
F(f) &:= f
\end{aligned}$$

for $f : (A, m_A, u_A) \to (B, m_B, u_B) \in \mathcal{A}$. Here, $U^{A,A}$ and ν_A are given in (2.3.6.5) and (2.3.6.7), respectively.

Claim 2.4.4. The assignment F above yields a functor from $\mathcal{A}$ to $\mathsf{Alg}(\mathcal{M}^H)$.

Proof of Claim 2.4.4. First, A is a right H-comodule, by the definition of $\mathcal{A}$. Next, take $\overline{m}_A := m_A\, U^{A,A}$ and we get, by the definition of $A \,\overline{\otimes}\, A$ (§2.3.2.1), that

$$\overline{m}_A(a \,\overline{\otimes}\, b) = m_A U^{A,A}(a \,\overline{\otimes}\, b) = m_A(a \otimes b) = ab$$

for $a, b \in A$. Since

$$\begin{aligned}
\rho_A\, \overline{m}_A(a \,\overline{\otimes}\, b) \;=\; \rho_A(ab) \quad &= (ab)_{[0]} \otimes (ab)_{[1]} \\
&\overset{(2.4.1.1)}{=} a_{[0]}b_{[0]} \otimes a_{[1]}b_{[1]} \\
&= (\overline{m}_A \otimes \mathrm{Id}_H)(a_{[0]} \,\overline{\otimes}\, b_{[0]} \otimes a_{[1]}b_{[1]}) \\
&\overset{(2.3.6.2)}{=} (\overline{m}_A \otimes \mathrm{Id}_H)\rho_{A \overline{\otimes} A}(a \,\overline{\otimes}\, b),
\end{aligned}$$

we also get that $\overline{m}_A$ is a map in $\mathcal{M}^H$. Moreover, $\overline{m}_A$ is associative (that is, (2.3.4.1) holds) because m_A is associative.

Now, take $\overline{u}_A := \nu_A(u_A \otimes \mathrm{Id}_{H_s})$ and $1_{H_s} =: 1$. If $x \in H_s$, then

$$\begin{aligned}
\rho_A\, \overline{u}_A(x) \overset{(2.3.6.7)}{=} \rho_A(1_A \lhd x) &\overset{(2.3.6.8)}{=} (1_A)_{[0]} \otimes (1_A)_{[1]}x \\
&\overset{2.2.2,(2.4.1.2)}{=} (1_A)_{[0]} \otimes \varepsilon_t((1_A)_{[1]})x \\
&\overset{2.2.2,2.2.4(\mathrm{iii}),(2.2.5.3),(2.4.1.2)}{=} (1_A)_{[0]} \otimes \varepsilon((1_A)_{[1]}1_1)1_2x \\
&\overset{2.2.4(\mathrm{iii}),(2.2.5.3)}{=} \varepsilon((1_A)_{[1]}1_1)(1_A)_{[0]} \otimes x1_2 \\
&\overset{(2.3.6.7)}{=} (1_A \lhd 1_1) \otimes x1_2 \\
&= (\overline{u}_A \otimes \mathrm{Id}_H)(1_1 \otimes x1_2) \\
&\overset{2.2.4(\mathrm{i})}{=} (\overline{u}_A \otimes \mathrm{Id}_H)\Delta(x) \\
&\overset{\S 2.3.2.2}{=} (\overline{u}_A \otimes \mathrm{Id}_H)\rho_{H_s}(x).
\end{aligned}$$

So, $\overline{u}_A$ is a map in $\mathcal{M}^H$. Moreover, if $x \in H_s$ and $a \in A$, then

$$\begin{aligned}
\overline{m}_A(\overline{u}_A \overline{\otimes} \mathrm{Id}_A)(x \overline{\otimes} a) &\overset{(2.3.6.7)}{=} \overline{m}_A((1_A \triangleleft x) \overline{\otimes} a) \overset{(2.3.6.7)}{=} \varepsilon((1_A)_{[1]}x)\, \overline{m}_A((1_A)_{[0]} \overline{\otimes} a) \\
&\overset{(2.3.6.1)}{=} \varepsilon((1_A)_{[1]}x)\, m_A\left(\varepsilon((1_A)_{[0],[1]}a_{[1]})\,(1_A)_{[0],[0]} \otimes a_{[0]}\right) \\
&\overset{2.2.4\text{(iii)},(2.4.1.2)}{=} \varepsilon(x(1_A)_{[1]})\, \varepsilon((1_A)_{[0],[1]}a_{[1]})\,(1_A)_{[0],[0]}\, a_{[0]} \\
&\overset{(2.2.5.4)}{=} \varepsilon(x(1_A)_{[1],2})\, \varepsilon((1_A)_{[1],1}a_{[1]})(1_A)_{[0]}\, a_{[0]} \\
&\overset{2.2.1\text{(iv)}}{=} \varepsilon(x(1_A)_{[1]}a_{[1]})(1_A)_{[0]}\, a_{[0]} \\
&\overset{(2.4.1.1)}{=} \varepsilon(xa_{[1]})a_{[0]} \\
&\overset{\S 2.3.2.3}{=} l_A(x \overline{\otimes} a).
\end{aligned}$$

With a similar computation for the right unit constraint r_A in §2.3.2.3, we conclude that the unit axiom (2.3.4.2) holds for the map $\overline{u}_A$. Therefore, F sends objects to objects.

To see that F sends morphisms to morphisms, take $f : A \to B \in \mathcal{A}$, which means that $fm_A = m_B(f \otimes f)$ and $fu_A = u_B$. The map f must also be a morphism of H-comodules, so $(f \otimes f)U^{A,A} = U^{A,A}(f \overline{\otimes} f)$. Now we get:

$$\begin{aligned}
F(f)\, m_A\, U^{A,A} &= f\, m_A\, U^{A,A} &&= m_B\,(f \otimes f)\, U^{A,A} \\
&= m_B\, U^{A,A}\,(f \overline{\otimes} f) &&= m_B\, U^{A,A}\,(F(f) \overline{\otimes} F(f)).
\end{aligned}$$

We also get $F(f)(\nu_A(u_A \otimes \mathrm{Id}_{H_s})) = \nu_B(u_B \otimes \mathrm{Id}_{H_s})$ by the following computation. Recall by §2.3.3.2 that a right H-comodule map f must also be an H_s-bimodule map. So we get for $x \in H_s$,

$$\begin{aligned}
f\overline{u}_A(x) &= f(\nu_A(u_A \otimes \mathrm{Id}_{H_s})(x)) \overset{(2.3.6.7)}{=} f(1_A \triangleleft x) \\
&\overset{\S 2.3.3.2}{=} f(1_A) \triangleleft x \;=\; 1_B \triangleleft x \overset{(2.3.6.7)}{=} \nu_B(u_B \otimes \mathrm{Id}_{H_s})(x) \;=\; \overline{u}_B(x).
\end{aligned}$$

This ends the proof of Claim 2.4.4. □

Now, take $(A,\ \overline{m}_A : A \overline{\otimes} A \to A,\ \overline{u}_A : H_s \to A) \in \mathsf{Alg}(\mathcal{M}^H)$, and consider the assignment

$$\begin{aligned}
G(A,\ \overline{m}_A,\ \overline{u}_A) &:= (A,\quad \overline{m}_A\, U_{A,A} : A \otimes A \to A,\quad \overline{u}_A\, U_0 : \mathbf{k} \to A), \\
G(g) &:= g,
\end{aligned}$$

for $g : (A, \overline{m}_A, \overline{u}_A) \to (B, \overline{m}_B, \overline{u}_B) \in \mathsf{Alg}(\mathcal{M}^H)$. Here, $U_{A,A}$ and U_0 are given in (2.3.6.4).

Claim 2.4.5. The assignment G above yields a functor from $\mathsf{Alg}(\mathcal{M}^H)$ to $\mathcal{A}$.

Proof of Claim 2.4.5. Recall from Section 2.3.3.1 that the forgetful functor $U : \mathcal{M}^H \to \mathsf{Vec}_{\mathbf{k}}$ is monoidal, and that

$$G(A,\ \overline{m}_A,\ \overline{u}_A) := (A,\ \overline{m}_A\, U_{A,A},\ \overline{u}_A\, U_0) = (U(A),\ U(\overline{m}_A)U_{A,A},\ U(\overline{u}_A)U_0).$$

By Proposition 2.3.5(i), we have that $G(A,\ \overline{m}_A,\ \overline{u}_A) \in \mathsf{Alg}(\mathsf{Vec}_{\mathbf{k}})$.

It remains to show that A is an H-comodule algebra in the sense of Definition 2.4.1. First, we show that $\overline{m}_A\, U_{A,A}$ satisfies (2.4.1.1), i.e., that the following diagram commutes, where we use τ to denote the flip map in $\mathsf{Vec}_{\mathbf{k}}$ (that is, $\tau(a \otimes b) = b \otimes a$):

$$\begin{tikzcd}
A \otimes A \arrow[rr, "U_{A,A}"] \arrow[d, "\rho_A \otimes \rho_A"'] & & A \,\overline{\otimes}\, A \arrow[r, "\overline{m}_A"] \arrow[d, "\rho_{A \overline{\otimes} A}"'] & A \arrow[d, "\rho_A"] \\
A \otimes H \otimes A \otimes H \arrow[dr, "\mathrm{Id} \otimes \tau \otimes \mathrm{Id}"'] & & A \,\overline{\otimes}\, A \otimes H \arrow[r, "\overline{m}_A \otimes \mathrm{Id}"'] & A \otimes H \\
& A \otimes A \otimes H \otimes H \arrow[ur, "U_{A,A} \otimes m"'] & &
\end{tikzcd}$$

We know that the right square commutes since $\overline{m}_A$ is an H-comodule map. To show that the left pentagon commutes, let $a \otimes b \in A \otimes A$. Then

$$\begin{aligned}
\rho_{A \overline{\otimes} A}\, U_{A,A}(a \otimes b) &\overset{(2.3.6.4)}{=} \rho_{A \overline{\otimes} A}(\varepsilon(a_{[1]} b_{[1]}) a_{[0]} \,\overline{\otimes}\, b_{[0]}) \\
&\overset{(2.3.6.2)}{=} \varepsilon(a_{[1]} b_{[1]}) a_{[0],[0]} \,\overline{\otimes}\, b_{[0],[0]} \otimes a_{[0],[1]} b_{[0],[1]} \\
&\overset{(2.2.5.4)}{=} \varepsilon(a_{[1],2} b_{[1],2}) a_{[0]} \,\overline{\otimes}\, b_{[0]} \otimes a_{[1],1} b_{[1],1} \\
&\overset{2.2.1((\mathrm{iii}))}{=} a_{[0]} \,\overline{\otimes}\, b_{[0]} \otimes (a_{[1]} b_{[1]})_1 \varepsilon((a_{[1]} b_{[1]})_2) \\
&\overset{(2.3.4.4)}{=} a_{[0]} \,\overline{\otimes}\, b_{[0]} \otimes a_{[1]} b_{[1]}.
\end{aligned}$$

Similarly,

$$(U_{A,A} \otimes m)(\mathrm{Id} \otimes \tau \otimes \mathrm{Id})(\rho_A \otimes \rho_A)(a \otimes b)$$

$$
\begin{aligned}
&= (U_{A,A} \otimes m)(\mathrm{Id} \otimes \tau \otimes \mathrm{Id})(a_{[0]} \otimes a_{[1]} \otimes b_{[0]} \otimes b_{[1]}) \\
&= (U_{A,A} \otimes \mathrm{Id})(a_{[0]} \otimes b_{[0]} \otimes a_{[1]} b_{[1]}) \\
&\overset{(2.3.6.4)}{=} \varepsilon(a_{[0],[1]} b_{[0],[1]}) a_{[0],[0]} \,\overline{\otimes}\, b_{[0],[1]} \otimes a_{[1]} b_{[1]} \\
&\overset{(2.2.5.4)}{=} \varepsilon(a_{[1],1} b_{[1],1}) a_{[0]} \,\overline{\otimes}\, b_{[0]} \otimes a_{[1],2} b_{[1],2} \\
&\overset{2.2.1((iii))}{=} a_{[0]} \,\overline{\otimes}\, b_{[0]} \otimes \varepsilon((a_{[1]} b_{[1]})_1)(a_{[1]} b_{[1]})_2 \\
&\overset{(2.3.4.4)}{=} a_{[0]} \,\overline{\otimes}\, b_{[0]} \otimes a_{[1]} b_{[1]}.
\end{aligned}
$$

Moreover, (2.4.1.2) holds since

$$
\rho_A(1_A) = \rho_A \overline{u}_A U_0(1_{\mathbf{k}}) \overset{(2.3.6.4)}{=} \rho_A\, \overline{u}_A(1_{H_s}) \overset{\overline{u}_A \in \mathcal{M}^H}{=} (\overline{u}_A \otimes \mathrm{Id}_H)\, \rho_{H_s}(1) \overset{\S 2.3.2.2}{=} \overline{u}_A(1_1) \otimes 1_2,
$$

the latter of which belongs to $A \otimes H_t$ by (2.2.5.3). Therefore, G sends objects to objects.

Now take $g : A \to B \in \mathsf{Alg}(\mathcal{M}^H)$. Since g is a map in $\mathsf{Alg}(\mathcal{M}^H)$, by definition $G(g) = g$ is an H-comodule map. Thus a straightforward calculation yields $(g \,\overline{\otimes}\, g) U_{A,A} = U_{A,A}(g \otimes g)$. We also have that $g \overline{m}_A = \overline{m}_B (g \,\overline{\otimes}\, g)$ and $g \overline{u}_A = \overline{u}_B$. So, we get:

$$
\begin{aligned}
G(g)\, \overline{m}_A\, U_{A,A} \quad &= \quad g\, \overline{m}_A\, U_{A,A} &&= \quad \overline{m}_B\, (g \,\overline{\otimes}\, g)\, U_{A,A} \\
&= \quad \overline{m}_B\, U_{A,A}\, (g \otimes g) &&= \quad \overline{m}_B\, U_{A,A}\, (G(g) \otimes G(g)),
\end{aligned}
$$

and likewise $G(g)\, \overline{u}_A\, U_0 = \overline{u}_B\, U_0$. Therefore, $G(g)$ is an algebra map in $\mathsf{Vec}_{\mathbf{k}}$. Functoriality is clear by the definition of G. Hence, G also sends morphisms to morphisms, and this ends the proof of Claim 2.4.5. □

Claim 2.4.6. *F and G are mutually inverse functors.*

Proof. We will show that FG (resp., GF) is the identity functor on $\mathsf{Alg}(\mathcal{M}^H)$ (resp., $\mathcal{A}$) as follows. First, note that

$$
FG(A,\ \overline{m}_A,\ \overline{u}_A) \ = \ (A, \quad \overline{m}_A\, U_{A,A}\, U^{A,A}, \quad \nu_A\, (\overline{u}_A\, U_0 \otimes \mathrm{Id}_{H_s})).
$$

By (2.3.6.6), it is clear that $\overline{m}_A\, U_{A,A}\, U^{A,A} = \overline{m}_A$ as morphisms in $\mathcal{M}^H$. Recall from Section 2.3.3.2 that a morphism of right H-comodules is a morphism of H_s-bimodules. Hence, for $x \in H_s$

$$\nu_A(\overline{u}_A\, U_0 \otimes \mathrm{Id}_{H_s})(1_{\mathbf{k}} \otimes x) \overset{(2.3.6.4)}{=} \nu_A(\overline{u}_A(1) \otimes x) \overset{(2.3.6.7)}{=} \overline{u}_A(1) \triangleleft x = \overline{u}_A(1 \triangleleft x) = \overline{u}_A(x).$$

So, $\nu_A(\overline{u}_A\, U_0 \otimes \mathrm{Id}_{H_s}) = \overline{u}_A$ as morphisms in $\mathsf{Alg}(\mathcal{M}^H)$ as well. On the other hand,

$$GF(A,\ m_A,\ u_A) \ = \ (A, \quad m_A\, U^{A,A}\, U_{A,A}, \quad \nu_A(u_A \otimes \mathrm{Id}_{H_s})U_0).$$

We have that $m_A\, U^{A,A}\, U_{A,A} = m_A$ as $\mathbf{k}$-linear maps by the following computation for $a \otimes b \in A \otimes A$:

$$\begin{aligned} m_A\, U^{A,A}\, U_{A,A}(a \otimes b) &\overset{(2.3.6.5)}{=} m_A U_{A,A}(\varepsilon(a_{[1]}b_{[1]})a_{[0]} \mathbin{\overline{\otimes}} b_{[0]}) &&\overset{(2.3.6.4)}{=} m_A(\varepsilon(a_{[1]}b_{[1]})a_{[0]} \otimes b_{[0]}) \\ &= \varepsilon(a_{[1]}b_{[1]})a_{[0]}b_{[0]} &&\overset{(2.4.1.1)}{=} \varepsilon((ab)_{[1]})(ab)_{[0]} \\ &\overset{(2.2.5.5)}{=} m_A(a \otimes b). \end{aligned}$$

We also get $\nu_A(u_A \otimes \mathrm{Id}_{H_s})U_0 = u_A$ due to the calculation below:

$$\nu_A(u_A \otimes \mathrm{Id}_{H_s})U_0(1_{\mathbf{k}}) \ = \ \nu_A(u_A(1_{\mathbf{k}}) \otimes 1) \overset{(2.3.6.7)}{=} \varepsilon(u_A(1_{\mathbf{k}})_{[1]})u_A(1_{\mathbf{k}})_{[0]} \overset{(2.2.5.5)}{=} u_A(1_{\mathbf{k}}).$$

Moreover, $FG(g) = g$ for any $g \in \mathsf{Alg}(\mathcal{M}^H)$, and $GF(f) = f$ for any $f \in \mathcal{A}$. □

Therefore, the functors F and G (defined, respectively, in Claims 2.4.4 and 2.4.5) yield an isomorphism between the categories $\mathcal{A} \coloneqq \mathcal{A}^H$ and $\mathsf{Alg}(\mathcal{M}^H)$. □

Now consider the following examples.

Example 2.4.7. The counital subalgebra H_s is an object of $\mathcal{A}^H \cong \mathsf{Alg}(\mathcal{M}^H)$, as the unit object of a monoidal category is an algebra in the category; see Section 2.3.2.2.

Example 2.4.8. Let Q be a finite quiver. let Q_ℓ denote the paths of length ℓ in Q, write $Q_0 = \{1, \dots, n\}$, and let e_i denote the trivial path at vertex i. Let $\mathbf{k}Q$ be the path algebra of Q and let $\mathfrak{H}(Q)$ be Hayashi's face algebra as defined in Example 2.2.8. Recall that $\mathfrak{H}(Q)_t$ has $\mathbf{k}$-basis $\{\sum_{i=1}^n x_{j,i} \mid 1 \le j \le n\}$. In [Hay99b, Equation 2.15], Hayashi notes that $\mathbf{k}Q$ is a right $\mathfrak{H}(Q)$-comodule. For a path $p \in Q_\ell$, the right coaction is given by

$$\rho_{\mathbf{k}Q} : \mathbf{k}Q \to \mathbf{k}Q \otimes \mathfrak{H}(Q), \quad p \mapsto \sum_{q \in Q_\ell} q \otimes x_{q,p}.$$

Indeed, for a path $p \in Q_\ell$:

$$(\mathrm{Id} \otimes \Delta)\rho_{\mathbf{k}Q}(p) = (\mathrm{Id} \otimes \Delta)\left(\sum_{q \in Q_\ell} q \otimes x_{q,p}\right) = \sum_{q,r \in Q_\ell} q \otimes x_{q,r} \otimes x_{r,p}$$

while

$$(\rho_{\mathbf{k}Q} \otimes \mathrm{Id})\rho_{\mathbf{k}Q}(p) = (\rho_{\mathbf{k}Q} \otimes \mathrm{Id})\left(\sum_{q \in Q_\ell} q \otimes x_{q,p}\right) = \sum_{r,q \in Q_\ell} r \otimes x_{r,q} \otimes x_{q,p}.$$

So, $\rho_{\mathbf{k}Q}$ is coassociative. To see that $\rho_{\mathbf{k}Q}$ is counital, observe that

$$(\mathrm{Id} \otimes \varepsilon)\rho_{\mathbf{k}Q}(p) = (\mathrm{Id} \otimes \varepsilon)\left(\sum_{q \in Q_\ell} q \otimes x_{q,p}\right) = p.$$

In fact, $\mathbf{k}Q$ is a right $\mathfrak{H}(Q)$-comodule algebra. First note that

$$\rho_{\mathbf{k}Q}(1_{\mathbf{k}Q}) = \rho_{\mathbf{k}Q}\left(\sum_{i=1}^{n} e_i\right) = \sum_{i,j=1}^{n} e_j \otimes x_{j,i} = \sum_{j=1}^{n} e_j \otimes \left(\sum_{i=1}^{n} x_{j,i}\right)$$

and so $\rho_{\mathbf{k}Q}(1_{\mathbf{k}Q}) \in \mathbf{k}Q \otimes \mathfrak{H}(Q)_t$. Now let $p \in Q_\ell$ and $q \in Q_m$. Then if p and q are composable,

$$\rho_{\mathbf{k}Q}(pq) = \sum_{r \in Q_{\ell+m}} r \otimes x_{r,pq} = \left(\sum_{s \in Q_\ell} s \otimes x_{s,p}\right)\left(\sum_{t \in Q_m} t \otimes x_{t,q}\right) = \rho_{\mathbf{k}Q}(p)\rho_{\mathbf{k}Q}(q),$$

since each path $r \in Q_{\ell+m}$ can be written uniquely as a composition st where $s \in Q_\ell$ and $t \in Q_m$. If p and q are not composable, then

$$\rho_{\mathbf{k}Q}(p)\rho_{\mathbf{k}Q}(q) = \left(\sum_{s \in Q_\ell} s \otimes x_{s,p}\right)\left(\sum_{t \in Q_m} t \otimes x_{t,q}\right) = \sum_{s \in Q_\ell, t \in Q_m} st \otimes x_{s,p}x_{t,q} = 0 = \rho_{\mathbf{k}Q}(pq).$$

Therefore, $\mathbf{k}Q$ and $\mathfrak{H}(Q)$ satisfy Definition 2.4.1, so $\mathbf{k}Q$ is an $\mathfrak{H}(Q)$-comodule algebra.

2.5 Weak comodule coalgebras

The goal of this section is to dualize the main result of the previous section: Theorem 2.4.3. That result pertained to comodule algebras over weak bialgebras, and here we consider comodule coalgebras over weak bialgebras. We then prove the analogue of Theorem 2.4.3 for these structures.

Definition 2.5.1 ($\mathcal{C}^H =: \mathcal{C}$)**.** ([Jia09, Def. 1], [VZ10, Def. 2.1]; [AAGRFV+03, Prop 3.12].) Consider the *category* $\mathcal{C}^H$ *of right* H*-comodule coalgebras* defined as follows. The objects of $\mathcal{C}^H$ are coalgebras in $\mathsf{Vec}_{\mathbf{k}}$,

$$(C,\ \Delta_C : C \to C \otimes C,\ \varepsilon_C : C \to \mathbf{k}),$$

so that

- the $\mathbf{k}$-vector space C is a right H-comodule via

$$\rho_C : C \to C \otimes H, \quad c \mapsto c_{[0]} \otimes c_{[1]},$$

  namely, the coassociativity and counit axioms hold (Eqs. (2.2.5.4), (2.2.5.5));

- the comultiplication map Δ_C is compatible with ρ_C and $\rho_{C\otimes C}$ in the sense that $\forall c \in C$,

$$c_{1,[0]} \otimes c_{2,[0]} \otimes c_{1,[1]}c_{2,[1]} = c_{[0],1} \otimes c_{[0],2} \otimes c_{[1]}, \tag{2.5.1.1}$$

- the counit ε_C is compatible with ρ_C in the sense that $\forall c \in C$,

$$\varepsilon_C(c_{[0]})c_{[1]} = \varepsilon_C(c_{[0]})\varepsilon_s(c_{[1]}). \tag{2.5.1.2}$$

The morphisms of $\mathcal{C}^H$ are $\mathbf{k}$-linear coalgebra maps that are also H-comodule maps. In other words, a $\mathbf{k}$-linear map $f : C \to D$ between $C, D \in \mathcal{C}^H$ is a morphism in $\mathcal{C}^H$ if

$$f(c_{[0]}) \otimes c_{[1]} = f(c)_{[0]} \otimes f(c)_{[1]}, \qquad \forall c \in C, \tag{2.5.1.3}$$

$$f(c_1) \otimes f(c_2) = f(c)_1 \otimes f(c)_2, \qquad \forall c \in C, \tag{2.5.1.4}$$

$$\varepsilon_C = \varepsilon_D f. \tag{2.5.1.5}$$

We write $\mathcal{C}$ for this category when H is understood.

Theorem 2.5.2. *Fix a weak bialgebra* H*. Then the category* $\mathcal{C}^H =: \mathcal{C}$ *of right* H*-comodule coalgebras is isomorphic to the category* $\mathsf{Coalg}(\mathcal{M}^H)$ *of coalgebras in* $\mathcal{M}^H$*.*

Proof. We achieve this result by constructing mutually inverse functors F and G below.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, bend left, "F"] & & \mathsf{Coalg}(\mathcal{M}^H) \arrow[ll, bend left, "G"] \end{tikzcd}$$

First, take $(C,\ \Delta_C,\ \varepsilon_C) \in \mathcal{C}$, and consider the assignment

$$\begin{aligned} F(C,\ \Delta_C,\ \varepsilon_C) &\coloneqq (C, \quad U_{C,C}\,\Delta_C : C \to C\,\overline{\otimes}\,C, \quad (\varepsilon_C \otimes \mathrm{Id})\rho^s_C : C \to H_s) \\ F(f) &\coloneqq f \end{aligned}$$

for $f : (C, \Delta_C, \varepsilon_C) \to (D, \Delta_D, \varepsilon_D) \in \mathcal{C}$. Here, $U_{C,C}$ and ρ^s_C are given in Equations 2.3.6.4 and 2.3.6.9, respectively.

Claim 2.5.3. The assignment F above yields a functor from $\mathcal{C}$ to $\mathsf{Coalg}(\mathcal{M}^H)$.

Proof. First we will show that F takes objects of $\mathcal{C}$ to objects of $\mathsf{Coalg}(\mathcal{M}^H)$. Let $(C, \Delta_C, \varepsilon_C)$ be a right H-comodule coalgebra. Define

$$\overline{\Delta}_C \coloneqq U_{C,C}\,\Delta_C \qquad \text{and} \qquad \overline{\varepsilon}_C \coloneqq (\varepsilon_C \otimes \mathrm{Id})\rho^s_C.$$

We will show that $(C, \overline{\Delta}_C, \overline{\varepsilon}_C)$ is an object of $\mathsf{Coalg}(\mathcal{M}^H)$. To do this, it suffices to show that $\overline{\Delta}_C, \overline{\varepsilon}_C$ are H-comodule maps, and that $\overline{\Delta}_C$ is coassociative and counital (with respect to $\overline{\varepsilon}_C$).

First, we will show that $\overline{\varepsilon}_C$ is an H-comodule map; in other words, that the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "\overline{\varepsilon}_C"] \arrow[d, "\rho_C"'] & H_s \arrow[d, "\rho_{H_s}"] \\ C \otimes H \arrow[r, "\overline{\varepsilon}_C \otimes \mathrm{Id}"] & H_s \otimes H. \end{tikzcd}$$

By Section 2.3.2.2, we have $\rho_{H_s} \coloneqq \Delta|_{H_s}$. For $c \in C$ we have

$$\rho_{H_s}\overline{\varepsilon}_C(c) = \Delta|_{H_s}\overline{\varepsilon}_C(c)$$

$$
\begin{aligned}
&\overset{2.2.2,(2.3.6.9)}{=} \Delta|_{H_s}(\varepsilon_C(c_{[0]})\varepsilon_s(c_{[1]}))\\
&\overset{(2.5.1.2)}{=} \Delta|_{H_s}(\varepsilon_C(c_{[0]})c_{[1]})\\
&= \varepsilon_C(c_{[0]})c_{[1],1} \otimes c_{[1],2}\\
&\overset{(2.2.5.4)}{=} \varepsilon_C(c_{[0],[0]})c_{[0],[1]} \otimes c_{[1]}\\
&\overset{(2.5.1.2)}{=} \varepsilon_C(c_{[0],[0]})\varepsilon_s(c_{[0],[1]}) \otimes c_{[1]} \qquad \text{(applied to } c_{[0]}\text{)}\\
&\overset{2.2.2,(2.3.6.9)}{=} (\overline{\varepsilon}_C \otimes \mathrm{Id})(c_{[0]} \otimes c_{[1]})\\
&= (\overline{\varepsilon}_C \otimes \mathrm{Id})\rho_C(c).
\end{aligned}
$$

Next we will show that $\overline{\Delta}_C$ is an H-comodule map; in other words, that the following diagram commutes:

$$
\begin{tikzcd}
C \arrow[r, "\overline{\Delta}_C"] \arrow[d, "\rho_C"'] & C \mathbin{\overline{\otimes}} C \arrow[d, "\rho_{C\overline{\otimes}C}"] \\
C \otimes H \arrow[r, "\overline{\Delta}_C \otimes \mathrm{Id}"] & C \mathbin{\overline{\otimes}} C \otimes H.
\end{tikzcd}
$$

Let $c \in C$. Then we have

$$
\begin{aligned}
(\overline{\Delta}_C \otimes \mathrm{Id})\rho_C(c) &= (\overline{\Delta}_C \otimes \mathrm{Id})(c_{[0]} \otimes c_{[1]})\\
&= U_{C,C}\Delta_C(c_{[0]}) \otimes c_{[1]}\\
&= U_{C,C}(c_{[0],1} \otimes c_{[0],2}) \otimes c_{[1]}\\
&\overset{(2.3.6.4)}{=} \varepsilon(c_{[0],1,[1]}c_{[0],2,[1]})c_{[0],1,[0]} \mathbin{\overline{\otimes}} c_{[0],2,[0]} \otimes c_{[1]}\\
&\overset{(2.5.1.1)}{=} \varepsilon(c_{[0],[1]})c_{[0],[0],1} \mathbin{\overline{\otimes}} c_{[0],[0],2} \otimes c_{[1]} \qquad \text{(applied to } c_{[0]}\text{)}\\
&\overset{(2.2.5.4)}{=} \varepsilon(c_{[1],1})c_{[0],1} \mathbin{\overline{\otimes}} c_{[0],2} \otimes c_{[1],2}\\
&= c_{[0],1} \mathbin{\overline{\otimes}} c_{[0],2} \otimes \varepsilon(c_{[1],1})c_{[1],2}\\
&\overset{(2.3.4.4)}{=} c_{[0],1} \mathbin{\overline{\otimes}} c_{[0],2} \otimes c_{[1]}.
\end{aligned}
$$

In the last equation, (2.3.4.4) is applied to $c_{[1]} \in H$, for $H \in \mathsf{Coalg}(\mathsf{Vec}_{\mathbf{k}})$. Similarly,

$$
\begin{aligned}
\rho_{C\overline{\otimes}C}\overline{\Delta}_C(c) &= \rho_{C\overline{\otimes}C}U_{C,C}\Delta_C(c)\\
&= \rho_{C\overline{\otimes}C}U_{C,C}(c_1 \otimes c_2)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{(2.3.6.4)}{=} \rho_{C \overline{\otimes} C}\big(\varepsilon(c_{1,[1]}c_{2,[1]})c_{1,[0]} \mathbin{\overline{\otimes}} c_{2,[0]}\big) \\
&\overset{(2.3.6.2)}{=} \varepsilon(c_{1,[1]}c_{2,[1]})c_{1,[0],[0]} \mathbin{\overline{\otimes}} c_{2,[0],[0]} \otimes c_{1,[0],[1]}c_{2,[0],[1]} \\
&\overset{(2.2.5.4)}{=} \varepsilon(c_{1,[1],2}c_{2,[1]})c_{1,[0]} \mathbin{\overline{\otimes}} c_{2,[0],[0]} \otimes c_{1,[1],1}c_{2,[0],[1]} \qquad \text{(applied to } c_1\text{)} \\
&\overset{(2.2.5.4)}{=} \varepsilon(c_{1,[1],2}c_{2,[1],2})c_{1,[0]} \mathbin{\overline{\otimes}} c_{2,[0]} \otimes c_{1,[1],1}c_{2,[1],1} \qquad \text{(applied to } c_2\text{)} \\
&\overset{2.2.1((\mathrm{iii}))}{=} \varepsilon\big((c_{1,[1]}c_{2,[1]})_2\big)c_{1,[0]} \mathbin{\overline{\otimes}} c_{2,[0]} \otimes (c_{1,[1]}c_{2,[1]})_1 \\
&= c_{1,[0]} \mathbin{\overline{\otimes}} c_{2,[0]} \otimes \varepsilon\big((c_{1,[1]}c_{2,[1]})_2\big)(c_{1,[1]}c_{2,[1]})_1 \\
&\overset{(2.3.4.4)}{=} c_{1,[0]} \mathbin{\overline{\otimes}} c_{2,[0]} \otimes c_{1,[1]}c_{2,[1]} \\
&\overset{(2.5.1.1)}{=} c_{[0],1} \mathbin{\overline{\otimes}} c_{[0],2} \otimes c_{[1]},
\end{aligned}
$$

so the diagram commutes.

Now we will show that $\overline{\Delta}_C$ is counital. In particular, we will show that the following diagram commutes, where l_C, r_C are the left and right unit isomorphisms defined in Section 2.3.2.3:

$$
\begin{tikzcd}
H_s \overline{\otimes} C & C \arrow[l, "l_C^{-1}"'] \arrow[r, "r_C^{-1}"] \arrow[d, "\overline{\Delta}_C"] & C \overline{\otimes} H_s \\
& C \overline{\otimes} C \arrow[ul, "\overline{\varepsilon}_C \overline{\otimes} \mathrm{Id}"] \arrow[ur, "\mathrm{Id} \overline{\otimes} \overline{\varepsilon}_C"'] &
\end{tikzcd}
$$

We will give a proof that the right hand side of the diagram commutes; the left is similar. Let $c \in C$. We will show that $r_C(\mathrm{Id} \overline{\otimes} \overline{\varepsilon}_C)\overline{\Delta}_C(c) = c$.

$$
\begin{aligned}
r_C(\mathrm{Id} \overline{\otimes} \overline{\varepsilon}_C)\overline{\Delta}_C(c) &= r_C(\mathrm{Id} \overline{\otimes} \overline{\varepsilon}_C)U_{C,C}\Delta_C(c) \\
&= r_C(\mathrm{Id} \overline{\otimes} \overline{\varepsilon}_C)U_{C,C}(c_1 \otimes c_2) \\
&\overset{(2.3.6.4)}{=} r_C(\mathrm{Id} \overline{\otimes} \overline{\varepsilon}_C)\big(\varepsilon(c_{1,[1]}c_{2,[1]})c_{1,[0]} \mathbin{\overline{\otimes}} c_{2,[0]}\big) \\
&\overset{(2.3.6.9),2.2.2}{=} r_C\Big(\varepsilon(c_{1,[1]}c_{2,[1]})c_{1,[0]} \mathbin{\overline{\otimes}} \varepsilon_s(c_{2,[0],[1]})\varepsilon_C(c_{2,[0],[0]})\Big) \\
&\overset{(2.5.1.2)}{=} r_C\Big(\varepsilon(c_{1,[1]}c_{2,[1]})c_{1,[0]} \mathbin{\overline{\otimes}} c_{2,[0],[1]}\varepsilon_C(c_{2,[0],[0]})\Big) \qquad \text{(applied to } c_{2,[0]}\text{)} \\
&= r_C\Big(\varepsilon_C(c_{2,[0],[0]})\varepsilon(c_{1,[1]}c_{2,[1]})c_{1,[0]} \mathbin{\overline{\otimes}} c_{2,[0],[1]}\Big) \\
&\overset{(2.2.5.4)}{=} r_C\Big(\varepsilon_C(c_{2,[0]})\varepsilon(c_{1,[1]}c_{2,[1],2})c_{1,[0]} \mathbin{\overline{\otimes}} c_{2,[1],1}\Big) \qquad \text{(applied to } c_2\text{)} \\
&\overset{*}{=} r_C\Big(\varepsilon_C(c_{2,[0]})\varepsilon(c_{1,[1]}\varepsilon_s(c_{2,[1]})_2)c_{1,[0]} \mathbin{\overline{\otimes}} \varepsilon_s(c_{2,[1]})_1\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\overset{2.2.4\text{(i)}}{=} r_C\left(\varepsilon_C(c_{2,[0]})\varepsilon(c_{1,[1]}\varepsilon_s(c_{2,[1]})1_2)c_{1,[0]} \,\overline{\otimes}\, 1_1\right)\\
&= r_C\left(\varepsilon_C(c_{2,[0]})c_{1,[0]} \,\overline{\otimes}\, 1_1\varepsilon(c_{1,[1]}\varepsilon_s(c_{2,[1]})1_2)\right)\\
&\overset{2.2.2}{=} r_C\left(\varepsilon_C(c_{2,[0]})c_{1,[0]} \,\overline{\otimes}\, \varepsilon_s(c_{1,[1]}\varepsilon_s(c_{2,[1]}))\right)\\
&\overset{(2.5.1.2)}{=} r_C\left(\varepsilon_C(c_{2,[0]})c_{1,[0]} \,\overline{\otimes}\, \varepsilon_s(c_{1,[1]}c_{2,[1]})\right) \qquad \text{(applied to } c_2\text{)}\\
&\overset{(2.5.1.1)}{=} r_C\left(\varepsilon_C(c_{[0],2})c_{[0],1} \,\overline{\otimes}\, \varepsilon_s(c_{[1]})\right)\\
&\overset{(2.3.4.4)}{=} r_C(c_{[0]} \,\overline{\otimes}\, \varepsilon_s(c_{[1]}))\\
&\overset{\S 2.3.2.3}{=} \varepsilon(c_{[0],[1]}\varepsilon_s(c_{[1]}))c_{[0],[0]}\\
&\overset{(2.2.5.4)}{=} \varepsilon(c_{[1],1}\varepsilon_s(c_{[1],2}))c_{[0]}\\
&\overset{(2.2.5.2)}{=} \varepsilon(c_{[1]})c_{[0]}\\
&\overset{(2.2.5.5)}{=} c.
\end{aligned}
$$

To justify $*$:

$$
\begin{aligned}
c_1 \otimes \varepsilon_C(c_{2,[0]})c_{2,[1],1} \otimes c_{2,[1],2} &= c_1 \otimes \Delta(\varepsilon_C(c_{2,[0]})c_{2,[1]})\\
&\overset{(2.5.1.2)}{=} c_1 \otimes \Delta(\varepsilon_C(c_{2,[0]})\varepsilon_s(c_{2,[1]}))\\
&= c_1 \otimes \varepsilon_C(c_{2,[0]})\varepsilon_s(c_{2,[1]})_1 \otimes \varepsilon_s(c_{2,[1]})_2.
\end{aligned}
$$

Finally, we will show that $\overline{\Delta}_C$ is coassociative; that is, the following diagram commutes:

$$
\begin{tikzcd}
C \arrow[r, "\overline{\Delta}_C"] \arrow[d, "\overline{\Delta}_C"'] & C \,\overline{\otimes}\, C \arrow[d, "\mathrm{Id}\,\overline{\otimes}\overline{\Delta}_C"] \\
C \,\overline{\otimes}\, C \arrow[r, "\overline{\Delta}_C\overline{\otimes}\mathrm{Id}"] & C \,\overline{\otimes}\, C \,\overline{\otimes}\, C.
\end{tikzcd}
$$

Consider an arbitrary element $c \in C$. We have the following:

$$
\begin{aligned}
&(\mathrm{Id}\,\overline{\otimes}\overline{\Delta}_C)\overline{\Delta}_C(c)\\
&= (\mathrm{Id}\,\overline{\otimes}\overline{\Delta}_C)U_{C,C}\Delta_C(c)\\
&= (\mathrm{Id}\,\overline{\otimes}\overline{\Delta}_C)U_{C,C}(c_1 \otimes c_2)\\
&\overset{(2.3.6.4)}{=} (\mathrm{Id}\,\overline{\otimes}\overline{\Delta}_C)(\varepsilon(c_{1,[1]}c_{2,[1]})c_{1,[0]} \,\overline{\otimes}\, c_{2,[0]})
\end{aligned}
$$

$$
\begin{aligned}
&= (\mathrm{Id}\,\overline{\otimes} U_{C,C}\Delta_C)(\varepsilon(c_{1,[1]}c_{2,[1]})c_{1,[0]}\,\overline{\otimes}\,c_{2,[0]})\\
&= \varepsilon(c_{1,[1]}c_{2,[1]})c_{1,[0]}\,\overline{\otimes}\,U_{C,C}(c_{2,[0],1}\otimes c_{2,[0],2})\\
&\overset{(2.3.6.4)}{=} \varepsilon(c_{1,[1]}c_{2,[1]})c_{1,[0]}\,\overline{\otimes}\,\varepsilon(c_{2,[0],1,[1]}c_{2,[0],2,[1]})c_{2,[0],1,[0]}\,\overline{\otimes}\,c_{2,[0],2,[0]}\\
&= \varepsilon(c_{1,[1]}c_{2,[1]})\varepsilon(c_{2,[0],1,[1]}c_{2,[0],2,[1]})c_{1,[0]}\,\overline{\otimes}\,c_{2,[0],1,[0]}\,\overline{\otimes}\,c_{2,[0],2,[0]}\\
&\overset{(2.5.1.1)}{=} \varepsilon(c_{1,[1]}c_{2,1,[1]}c_{2,2,[1]})\varepsilon(c_{2,1,[0],[1]}c_{2,2,[0],[1]})c_{1,[0]}\,\overline{\otimes}\,c_{2,1,[0],[0]}\,\overline{\otimes}\,c_{2,2,[0],[0]} \quad \text{(applied to } c_2\text{)}\\
&\overset{(2.3.4.3)}{=} \varepsilon(c_{1,[1]}c_{2,[1]}c_{3,[1]})\varepsilon(c_{2,[0],[1]}c_{3,[0],[1]})c_{1,[0]}\,\overline{\otimes}\,c_{2,[0],[0]}\,\overline{\otimes}\,c_{3,[0],[0]}\\
&\overset{(2.2.5.4)}{=} \varepsilon(c_{1,[1]}c_{2,[1],2}c_{3,[1]})\varepsilon(c_{2,[1],1}c_{3,[0],[1]})c_{1,[0]}\,\overline{\otimes}\,c_{2,[0]}\,\overline{\otimes}\,c_{3,[0],[0]} \qquad \text{(applied to } c_2\text{)}\\
&\overset{(2.2.5.4)}{=} \varepsilon(c_{1,[1]}c_{2,[1],2}c_{3,[1],2})\varepsilon(c_{2,[1],1}c_{3,[1],1})c_{1,[0]}\,\overline{\otimes}\,c_{2,[0]}\,\overline{\otimes}\,c_{3,[0]} \qquad \text{(applied to } c_3\text{)}\\
&\overset{2.2.1((\mathrm{iii}))}{=} \varepsilon(c_{1,[1]}(c_{2,[1]}c_{3,[1]})_2)\varepsilon((c_{2,[1]}c_{3,[1]})_1)c_{1,[0]}\,\overline{\otimes}\,c_{2,[0]}\,\overline{\otimes}\,c_{3,[0]}\\
&\overset{2.2.1((\mathrm{iv}))}{=} \varepsilon(c_{1,[1]}c_{2,[1]}c_{3,[1]})c_{1,[0]}\,\overline{\otimes}\,c_{2,[0]}\,\overline{\otimes}\,c_{3,[0]}.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&(\overline{\Delta}_C\,\overline{\otimes}\,\mathrm{Id})\overline{\Delta}_C(c)\\
&= (\overline{\Delta}_C\,\overline{\otimes}\,\mathrm{Id})(\varepsilon(c_{1,[1]}c_{2,[1]})c_{1,[0]}\,\overline{\otimes}\,c_{2,[0]})\\
&= (U_{C,C}\Delta_C\,\overline{\otimes}\,\mathrm{Id})(\varepsilon(c_{1,[1]}c_{2,[1]})c_{1,[0]}\,\overline{\otimes}\,c_{2,[0]})\\
&= \varepsilon(c_{1,[1]}c_{2,[1]})U_{C,C}(c_{1,[0],1}\otimes c_{1,[0],2})\,\overline{\otimes}\,c_{2,[0]}\\
&\overset{(2.3.6.4)}{=} \varepsilon(c_{1,[1]}c_{2,[1]})\varepsilon(c_{1,[0],1,[1]}c_{1,[0],2,[1]})c_{1,[0],1,[0]}\,\overline{\otimes}\,c_{1,[0],2,[0]}\,\overline{\otimes}\,c_{2,[0]}\\
&\overset{(2.5.1.1)}{=} \varepsilon(c_{1,1,[1]}c_{1,2,[1]}c_{2,[1]})\varepsilon(c_{1,1,[0],[1]}c_{1,2,[0],[1]})c_{1,1,[0],[0]}\,\overline{\otimes}\,c_{1,2,[0],[0]}\,\overline{\otimes}\,c_{2,[0]} \quad \text{(applied to } c_1\text{)}\\
&\overset{(2.3.4.3)}{=} \varepsilon(c_{1,[1]}c_{2,[1]}c_{3,[1]})\varepsilon(c_{1,[0],[1]}c_{2,[0],[1]})c_{1,[0],[0]}\,\overline{\otimes}\,c_{2,[0],[0]}\,\overline{\otimes}\,c_{3,[0]}\\
&\overset{(2.2.5.4)}{=} \varepsilon(c_{1,[1],2}c_{2,[1]}c_{3,[1]})\varepsilon(c_{1,[1],1}c_{2,[0],[1]})c_{1,[0]}\,\overline{\otimes}\,c_{2,[0],[0]}\,\overline{\otimes}\,c_{3,[0]} \qquad \text{(applied to } c_1\text{)}\\
&\overset{(2.2.5.4)}{=} \varepsilon(c_{1,[1],2}c_{2,[1],2}c_{3,[1]})\varepsilon(c_{1,[1],1}c_{2,[1],1})c_{1,[0]}\,\overline{\otimes}\,c_{2,[0]}\,\overline{\otimes}\,c_{3,[0]} \qquad \text{(applied to } c_2\text{)}\\
&\overset{2.2.1((\mathrm{iii}))}{=} \varepsilon((c_{1,[1]}c_{2,[1]})_2c_{3,[1]})\varepsilon((c_{1,[1]}c_{2,[1]})_1)c_{1,[0]}\,\overline{\otimes}\,c_{2,[0]}\,\overline{\otimes}\,c_{3,[0]}\\
&\overset{2.2.1((\mathrm{iv}))}{=} \varepsilon(c_{1,[1]}c_{2,[1]}c_{3,[1]})c_{1,[0]}\,\overline{\otimes}\,c_{2,[0]}\,\overline{\otimes}\,c_{3,[0]}.
\end{aligned}
$$

Therefore, $\overline{\Delta}_C$ is coassociative.

Now suppose that $f : C \to D$ is a morphism in $\mathcal{C}$. We will show that $F(f) := f$ is also a morphism in $\mathsf{Coalg}(\mathcal{M}^H)$. In other words, the following two diagrams commute:

$$\begin{tikzcd} C \arrow[r, "\overline{\Delta}_C"] \arrow[d, "f"'] & C \mathbin{\overline{\otimes}} C \arrow[d, "f \overline{\otimes} f"] \\ D \arrow[r, "\overline{\Delta}_D"] & D \mathbin{\overline{\otimes}} D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\overline{\varepsilon}_C"] \arrow[d, "f"'] & H_s \\ D \arrow[ur, "\overline{\varepsilon}_D"'] & \end{tikzcd}$$

Recall from Section 2.3.2.1 that $(f \mathbin{\overline{\otimes}} f)(c \mathbin{\overline{\otimes}} c') \coloneqq f(c) \mathbin{\overline{\otimes}} f(c')$ for $c \mathbin{\overline{\otimes}} c' \in C \mathbin{\overline{\otimes}} C$. We start with the first diagram. For any $c \in C$, we have

$$\begin{aligned}
(f \mathbin{\overline{\otimes}} f)\overline{\Delta}_C(c) &= (f \mathbin{\overline{\otimes}} f)U_{C,C}\Delta_C(c) \\
&= (f \mathbin{\overline{\otimes}} f)U_{C,C}(c_1 \otimes c_2) \\
&\overset{(2.3.6.4)}{=} (f \mathbin{\overline{\otimes}} f)(\varepsilon(c_{1,[1]}c_{2,[1]})c_{1,[0]} \mathbin{\overline{\otimes}} c_{2,[0]}) \\
&= \varepsilon(c_{1,[1]}c_{2,[1]})f(c_{1,[0]}) \mathbin{\overline{\otimes}} f(c_{2,[0]}) \\
&\overset{(2.5.1.3)}{=} \varepsilon(f(c_1)_{[1]}f(c_2)_{[1]})f(c_1)_{[0]} \mathbin{\overline{\otimes}} f(c_2)_{[0]} \\
&\overset{(2.5.1.4)}{=} \varepsilon(f(c)_{1,[1]}f(c)_{2,[1]})f(c)_{1,[0]} \mathbin{\overline{\otimes}} f(c)_{2,[0]} \\
&= U_{D,D}(f(c)_1 \otimes f(c)_2) \\
&= U_{D,D}\Delta_D f(c) \\
&= \overline{\Delta}_D f(c).
\end{aligned}$$

We also have

$$\begin{aligned}
\overline{\varepsilon}_D f(c) &\overset{(2.3.6.9),2.2.2}{=} \varepsilon_s(f(c)_{[1]})\varepsilon_D(f(c)_{[0]}) \\
&\overset{(2.5.1.3)}{=} \varepsilon_s(c_{[1]})\varepsilon_D(f(c_{[0]})) \\
&\overset{(2.5.1.5)}{=} \varepsilon_s(c_{[1]})\varepsilon_C(c_{[0]}) \\
&= \overline{\varepsilon}_C(c).
\end{aligned}$$

We have shown that $F(f) \coloneqq f$ is a morphism in $\mathsf{Coalg}(\mathcal{M}^H)$. It is clear that F is functorial since $F(fg) = fg = F(f)F(g)$ and $F(\mathrm{Id}_C) = \mathrm{Id}_C = \mathrm{Id}_{F(C)}$ for any morphisms f, g in $\mathcal{C}$ and any $C \in \mathcal{C}$. $\square$

Now take $(C,\ \overline{\Delta}_C,\ \overline{\varepsilon}_C) \in \mathsf{Coalg}(\mathcal{M}^H)$, and consider the assignment

$$\begin{aligned} G(C,\ \overline{\Delta}_C,\ \overline{\varepsilon}_C) &\coloneqq (C, \quad U^{C,C}\,\overline{\Delta}_C : C \to C \otimes C, \quad U^0\,\overline{\varepsilon}_C : C \to \mathbf{k}) \\ G(\overline{f}) &\coloneqq \overline{f} \end{aligned}$$

for $\overline{f} : (C, \overline{\Delta}_C, \overline{\varepsilon}_C) \to (D, \overline{\Delta}_D, \overline{\varepsilon}_D) \in \mathsf{Coalg}(\mathcal{M}^H)$. Here, $U^{C,C}$ and U^0 are given in (2.3.6.4).

Claim 2.5.4. The assignment G given above yields a functor from $\mathsf{Coalg}(\mathcal{M}^H)$ to $\mathcal{C}$.

Proof. To begin, define

$$\Delta_C \coloneqq U^{C,C}\overline{\Delta}_C \qquad \text{and} \qquad \varepsilon_C \coloneqq U^0\overline{\varepsilon}_C.$$

Then by Proposition 2.3.5(ii) and the fact that the forgetful functor U is comonoidal (see Section 2.3.3.1), we have that $(C, \Delta_C, \varepsilon_C)$ is a coalgebra in $\mathsf{Vec}_{\mathbf{k}}$. It remains to check that C is a right H-comodule coalgebra; that is, C satisfies (2.5.1.1) and (2.5.1.2).

It is easily checked that (2.5.1.1) is equivalent to the commutativity of the following diagram, where τ is the flip map in $\mathsf{Vec}_{\mathbf{k}}$ and m is the multiplication in H:

$$\begin{tikzcd} C \arrow[rrr, "\rho_C"] \arrow[d, "\Delta_C"'] & & & C \otimes H \arrow[d, "\Delta_C \otimes \mathrm{Id}"] \\ C \otimes C \arrow[r, "\rho_C \otimes \rho_C"] & C \otimes H \otimes C \otimes H \arrow[r, "\mathrm{Id} \otimes \tau \otimes \mathrm{Id}"] & C \otimes C \otimes H \otimes H \arrow[r, "\mathrm{Id} \otimes \mathrm{Id} \otimes m"] & C \otimes C \otimes H. \end{tikzcd}$$

Using the definition of Δ_C, we can rewrite the diagram as follows:

$$\begin{tikzcd} C \arrow[rrr, "\rho_C"] \arrow[d, "\overline{\Delta}_C"'] & & & C \otimes H \arrow[d, "\overline{\Delta}_C \otimes \mathrm{Id}"] \\ C \overline{\otimes} C \arrow[rrr, "\rho_{C \overline{\otimes} C}"] \arrow[d, "U^{C,C}"'] & & & C \overline{\otimes} C \otimes H \arrow[d, "U^{C,C} \otimes \mathrm{Id}"] \\ C \otimes C \arrow[r, "\rho_C \otimes \rho_C"] & C \otimes H \otimes C \otimes H \arrow[r, "\mathrm{Id} \otimes \tau \otimes \mathrm{Id}"] & C \otimes C \otimes H \otimes H \arrow[r, "\mathrm{Id} \otimes \mathrm{Id} \otimes m"] & C \otimes C \otimes H. \end{tikzcd}$$

Since $\overline{\Delta}_C \in \mathsf{Coalg}(\mathcal{M}^H)$, it is an H-comodule morphism, which leads to commutativity of the top square. To show commutativity of the rest of the diagram, let $c \,\overline{\otimes}\, d \in C \,\overline{\otimes}\, C$, and consider the computation below:

$$\begin{aligned}(U^{C,C}\otimes \mathrm{Id})\rho_{C\overline{\otimes}C}(c\,\overline{\otimes}\,d) &= (U^{C,C}\otimes \mathrm{Id})(c_{[0]}\,\overline{\otimes}\,d_{[0]}\otimes c_{[1]}d_{[1]})\\ &\overset{(2.3.6.5)}{=} c_{[0]}\otimes d_{[0]}\otimes c_{[1]}d_{[1]}\\ &= (\mathrm{Id}\otimes \mathrm{Id}\otimes m)(c_{[0]}\otimes d_{[0]}\otimes c_{[1]}\otimes d_{[1]})\\ &= (\mathrm{Id}\otimes \mathrm{Id}\otimes m)(\mathrm{Id}\otimes \tau\otimes \mathrm{Id})(c_{[0]}\otimes c_{[1]}\otimes d_{[0]}\otimes d_{[1]})\\ &= (\mathrm{Id}\otimes \mathrm{Id}\otimes m)(\mathrm{Id}\otimes \tau\otimes \mathrm{Id})(\rho_C\otimes \rho_C)(c\otimes d)\\ &= (\mathrm{Id}\otimes \mathrm{Id}\otimes m)(\mathrm{Id}\otimes \tau\otimes \mathrm{Id})(\rho_C\otimes \rho_C)U^{C,C}(c\,\overline{\otimes}\,d).\end{aligned}$$

Therefore, (2.5.1.1) holds.

Now we will show that (2.5.1.2) holds. Recall from (2.3.6.9) and Definition 2.2.2 that for $c\in C$, we have $\rho^s_C(c) = c_{[0]}\otimes \varepsilon_s(c_{[1]})$. Then (2.5.1.2) holds if and only if

$$(\varepsilon_C\otimes \mathrm{Id})\rho(c) = (\varepsilon_C\otimes \mathrm{Id})\rho^s_C(c), \quad \forall c\in C.$$

Writing ε_C in terms of $\overline{\varepsilon}_C$ yields that (2.5.1.2) holds if and only if

$$(U^0\otimes \mathrm{Id})(\overline{\varepsilon}_C\otimes \mathrm{Id})\rho(c) = (U^0\otimes \mathrm{Id})(\overline{\varepsilon}_C\otimes \mathrm{Id})\rho^s_C(c), \quad \forall c\in C,$$

where U^0 is defined in (2.3.6.5). In fact, we will show that if $(C,\overline{\Delta}_C,\overline{\varepsilon}_C)\in \mathsf{Coalg}(\mathcal{M}^H)$, then

$$(U^0\otimes \mathrm{Id})(\overline{\varepsilon}_C\otimes \mathrm{Id})\rho(c) = \overline{\varepsilon}_C(c) = (U^0\otimes \mathrm{Id})(\overline{\varepsilon}_C\otimes \mathrm{Id})\rho^s_C(c), \quad \forall c\in C. \tag{2.5.4.1}$$

Recall that since $\overline{\varepsilon}_C\in \mathsf{Coalg}(\mathcal{M}^H)$, it is an H-comodule map. In other words,

$$\overline{\varepsilon}_C(c_{[0]})\otimes c_{[1]} = \rho_{H_s}\overline{\varepsilon}_C(c) = \overline{\varepsilon}_C(c)_1\otimes \overline{\varepsilon}_C(c)_2, \quad \forall c\in C, \tag{2.5.4.2}$$

where ρ_{H_s} is defined in Section 2.3.2.2. Therefore,

$$\begin{aligned}
(U^0 \otimes \mathrm{Id})(\overline{\varepsilon}_C \otimes \mathrm{Id})\rho(c) &= (U^0 \otimes \mathrm{Id})(\overline{\varepsilon}_C(c_{[0]}) \otimes c_{[1]}) && \overset{(2.3.6.5)}{=} \varepsilon(\overline{\varepsilon}_C(c_{[0]}))c_{[1]} \\
&\overset{(2.5.4.2)}{=} \varepsilon(\overline{\varepsilon}_C(c)_1)\overline{\varepsilon}_C(c)_2 && \overset{(2.3.4.4)}{=} \overline{\varepsilon}_C(c) \\
&\overset{(2.2.5.1)}{=} \varepsilon_s(\overline{\varepsilon}_C(c)) && \overset{(2.3.4.4)}{=} \varepsilon_s(\varepsilon(\overline{\varepsilon}_C(c)_1)\overline{\varepsilon}_C(c)_2) \\
&= \varepsilon(\overline{\varepsilon}_C(c)_1)\varepsilon_s(\overline{\varepsilon}_C(c)_2) && \overset{(2.5.4.2)}{=} \varepsilon(\overline{\varepsilon}_C(c_{[0]}))\varepsilon_s(c_{[1]}) \\
&= (U^0 \otimes \mathrm{Id})(\overline{\varepsilon}_C(c_{[0]}) \otimes \varepsilon_s(c_{[1]})) \\
&= (U^0 \otimes \mathrm{Id})(\overline{\varepsilon}_C \otimes \mathrm{Id})(c_{[0]} \otimes \varepsilon_s(c_{[1]})) \\
&= (U^0 \otimes \mathrm{Id})(\overline{\varepsilon}_C \otimes \mathrm{Id})\rho^s_C(c).
\end{aligned}$$

So, (2.5.4.1) holds, and we have shown that G takes objects of $\mathsf{Coalg}(\mathcal{M}^H)$ to objects of $\mathcal{C}$.

It remains to show that G takes morphisms of $\mathsf{Coalg}(\mathcal{M}^H)$ to morphisms of $\mathcal{C}$, since if so, it is clear that G is functorial. Let $\overline{f} : C \to D \in \mathsf{Coalg}(\mathcal{M}^H)$. Then $G(\overline{f}) = \overline{f}$ is an H-comodule map. It remains to check that it is a coalgebra map: conditions (2.5.1.4) and (2.5.1.5). Notice that $U^{C,C}(\overline{f} \,\overline{\otimes}\, \overline{f}) = (\overline{f} \otimes \overline{f})U^{C,C}$ by definition of $\overline{\otimes}$ and the fact that $U^{C,C}$ is inclusion; see (2.3.6.3) and (2.3.6.5). Then (2.5.1.4) holds as

$$\begin{aligned}
(\overline{f} \otimes \overline{f})\Delta_C &= (\overline{f} \otimes \overline{f})U^{C,C}\overline{\Delta}_C \\
&= U^{C,C}(\overline{f} \,\overline{\otimes}\, \overline{f})\overline{\Delta}_C \\
&= U^{C,C}\overline{\Delta}_D\overline{f} \qquad (\text{since } \overline{f} \in \mathsf{Coalg}(\mathcal{M}^H)) \\
&= \Delta_D\overline{f}.
\end{aligned}$$

On the other hand, (2.5.1.5), we have

$$\varepsilon_D\overline{f} \;= U^0\overline{\varepsilon}_D\overline{f} \;=\; U^0\overline{\varepsilon}_C \;=\; \varepsilon_C,$$

where the second equality holds because $\overline{f} \in \mathsf{Coalg}(\mathcal{M}^H)$. Therefore, G is well-defined on morphisms. □

Claim 2.5.5. The functors F and G are mutually inverse.

Proof. Clearly, FG and GF are the identity on morphisms. It remains to show that they act as the identity on objects. Suppose that $(C, \overline{\Delta}_C, \overline{\varepsilon}_C) \in \mathsf{Coalg}(\mathcal{M}^H)$. Then

$$\begin{aligned} FG(C, \overline{\Delta}_C, \overline{\varepsilon}_C) &= F(C,\ U^{C,C}\overline{\Delta}_C,\ U^0\overline{\varepsilon}_C) \\ &= (C,\ U_{C,C}U^{C,C}\overline{\Delta}_C,\ (U^0\overline{\varepsilon}_C \otimes \mathrm{Id})\rho^s_C) \\ &\overset{(2.3.6.6)}{=} (C,\ \overline{\Delta}_C,\ (U^0\overline{\varepsilon}_C \otimes \mathrm{Id})\rho^s_C) \\ &\overset{(2.5.4.1)}{=} (C, \overline{\Delta}_C, \overline{\varepsilon}_C). \end{aligned}$$

Now suppose that $(C, \Delta_C, \varepsilon_C) \in \mathcal{C}$. We have

$$\begin{aligned} GF(C, \Delta_C, \varepsilon_C) &= G(C,\ U_{C,C}\Delta_C,\ (\varepsilon_C \otimes \mathrm{Id})\rho^s_C) \\ &= (C,\ U^{C,C}U_{C,C}\Delta_C,\ U^0(\varepsilon_C \otimes \mathrm{Id})\rho^s_C)\ =\ (C, \Delta_C, \varepsilon_C), \end{aligned}$$

where the last equality is justified by the following calculations. For any $c \in C$, we have

$$\begin{aligned} U^{C,C}U_{C,C}\Delta_C(c) &= U^{C,C}U_{C,C}(c_1 \otimes c_2) && \overset{(2.3.6.4)}{=} U^{C,C}(\varepsilon(c_{1,[1]}c_{2,[1]})c_{1,[0]} \overline{\otimes} c_{2,[0]}) \\ &\overset{(2.3.6.5)}{=} \varepsilon(c_{1,[1]}c_{2,[1]})c_{1,[0]} \otimes c_{2,[0]} && \overset{(2.5.1.1)}{=} \varepsilon(c_{[1]})c_{[0],1} \otimes c_{[0],2} \\ &= \Delta_C(\varepsilon(c_{[1]})c_{[0]}) && \overset{(2.2.5.5)}{=} \Delta_C(c). \end{aligned}$$

Similarly,

$$\begin{aligned} U^0(\varepsilon_C \otimes \mathrm{Id})\rho^s_C(c) &= U^0(\varepsilon_C(c_{[0]})\varepsilon_s(c_{[1]})) && \overset{(2.5.1.2)}{=} U^0(\varepsilon_C(c_{[0]})c_{[1]}) \\ &\overset{(2.3.6.5)}{=} \varepsilon(\varepsilon_C(c_{[0]})c_{[1]}) && = \varepsilon_C(c_{[0]}\varepsilon(c_{[1]})) \\ &\overset{(2.2.5.5)}{=} \varepsilon_C(c). \end{aligned}$$

□

Therefore, the functors F and G (defined, respectively, in Claims 2.5.3 and 2.5.4) yield an isomorphism between the categories $\mathcal{C} := \mathcal{C}^H$ and $\mathsf{Coalg}(\mathcal{M}^H)$. □

Now consider the following examples.

Example 2.5.6. The counital subalgebra H_s is an object of $\mathcal{C}^H \cong \mathsf{Coalg}(\mathcal{M}^H)$, as the unit object of a monoidal category is a coalgebra in the category; see Section 2.3.2.2.

Example 2.5.7. Let Q be a finite quiver. Let Q_ℓ denote the paths of length ℓ in Q, let $Q_0 = \{1, 2, \ldots, n\}$, and write e_i for the trivial path at vertex i. Let $\mathfrak{H}(Q)$ denote Hayashi's face algebra as defined in Example 2.2.8. Recall from Example 2.4.8 that the path algebra $\mathbf{k}Q$ is an $\mathfrak{H}(Q)$-comodule algebra via the right coaction $\rho_{\mathbf{k}Q}(p) = \sum_{q \in Q_\ell} q \otimes x_{q,p}$ for $p \in Q_\ell$. We now show that this coaction makes the path coalgebra $\mathbf{k}Q$ an $\mathfrak{H}(Q)$-comodule coalgebra.

The coalgebra structure that we consider on $\mathbf{k}Q$ is given as follows. If $i \in Q_0$ then $\Delta_{\mathbf{k}Q}(e_i) = e_i \otimes e_i$ and $\varepsilon_{\mathbf{k}Q}(e_i) = 1$. For $\ell \geq 1$ and for a path $p \in Q_\ell$, write $p = p_1 p_2 \cdots p_\ell$ where $p_1, \ldots, p_\ell \in Q_1$. Then

$$\Delta_{\mathbf{k}Q}(p) = (e_{s(p)} \otimes p) + \left(\textstyle\sum_{i=1}^{\ell-1} p_1 \cdots p_i \otimes p_{i+1} \cdots p_\ell\right) + (p \otimes e_{t(p)})$$

and $\varepsilon_{\mathbf{k}Q}(p) = 0$.

First, we verify the compatibility of the comultiplication $\Delta_{\mathbf{k}Q}$ with the coactions $\rho_{\mathbf{k}Q}$ and $\rho_{\mathbf{k}Q \otimes \mathbf{k}Q}$. If $i \in Q_0$, then

$$\begin{aligned}
\rho_{\mathbf{k}Q\otimes\mathbf{k}Q}(\Delta_{\mathbf{k}Q}(e_i)) &= \rho_{\mathbf{k}Q\otimes\mathbf{k}Q}(e_i \otimes e_i) = \textstyle\sum_{j,k \in Q_0} e_j \otimes e_k \otimes x_{j,i} x_{k,i} = \sum_{j \in Q_0} e_j \otimes e_j \otimes x_{j,i} \\
&= (\Delta_{\mathbf{k}Q} \otimes \mathrm{Id}) \left(\textstyle\sum_{j \in Q_0} e_j \otimes x_{j,i}\right) = (\Delta_{\mathbf{k}Q} \otimes \mathrm{Id}) \rho_{\mathbf{k}Q}(e_i).
\end{aligned}$$

Now $p \in Q_\ell$ for some $\ell \geq 1$. Then we have

$$\begin{aligned}
\rho_{\mathbf{k}Q\otimes\mathbf{k}Q}(\Delta_{\mathbf{k}Q}(p)) &= \rho_{\mathbf{k}Q\otimes\mathbf{k}Q}\left(e_{s(p)} \otimes p + \left(\textstyle\sum_{i=1}^{\ell-1} p_1 \cdots p_i \otimes p_{i+1} \cdots p_\ell\right) + p \otimes e_{t(p)}\right) \\
&= \sum_{\substack{j \in Q_0 \\ q \in Q_\ell}} e_j \otimes q \otimes x_{j,s(p)} x_{q,p} + \sum_{i=1}^{\ell-1} \sum_{\substack{r \in Q_i \\ u \in Q_{\ell-i}}} r \otimes u \otimes x_{r, p_1 \cdots p_i} x_{u, p_{i+1} \cdots p_\ell} \\
&\quad + \sum_{\substack{v \in Q_\ell \\ k \in Q_0}} v \otimes e_k \otimes x_{v,p} x_{k,t(p)} \\
&= \sum_{q \in Q_\ell} e_{s(q)} \otimes q \otimes x_{q,p} + \sum_{i=1}^{\ell-1} \sum_{\substack{r \in Q_i, u \in Q_{\ell-i} \\ \text{with } t(r)=s(u)}} r \otimes u \otimes x_{ru,p} + \sum_{v \in Q_\ell} v \otimes e_{t(v)} \otimes x_{v,p}.
\end{aligned}$$

Further,

$$(\Delta_{\mathbf{k}Q} \otimes \mathrm{Id}) \rho_{\mathbf{k}Q}(p) = (\Delta_{\mathbf{k}Q} \otimes \mathrm{Id}) \left(\textstyle\sum_{q \in Q_\ell} q \otimes x_{q,p}\right)$$

$$= \sum_{q \in Q_\ell} \left(e_{s(q)} \otimes q \otimes x_{q,p} + \left(\sum_{i=1}^{\ell-1} q_1 \cdots q_i \otimes q_{i+1} \cdots q_\ell \otimes x_{q,p} \right) + q \otimes e_{t(q)} \otimes x_{q,p} \right).$$

So, for each $p \in Q_\ell$ we have $\rho_{\mathbf{k}Q \otimes \mathbf{k}Q}(\Delta_{\mathbf{k}Q}(p)) = (\Delta_{\mathbf{k}Q} \otimes \mathrm{Id})\rho_{\mathbf{k}Q}(p)$, as desired.

For compatibility of the counit $\varepsilon_{\mathbf{k}Q}$ with the coaction, first let $i \in Q_0$. Then

$$(\varepsilon_{\mathbf{k}Q} \otimes \mathrm{Id})(\rho_{\mathbf{k}Q}(e_i)) = (\varepsilon_{\mathbf{k}Q} \otimes \mathrm{Id}) \left(\textstyle\sum_{j \in Q_0} e_j \otimes x_{j,i} \right) = \textstyle\sum_{j \in Q_0} x_{j,i},$$

while

$$(\varepsilon_{\mathbf{k}Q} \otimes \varepsilon_s)(\rho_{\mathbf{k}Q}(e_i)) = (\varepsilon_{\mathbf{k}Q} \otimes \varepsilon_s) \left(\textstyle\sum_{j \in Q_0} e_j \otimes x_{j,i} \right) = \textstyle\sum_{j \in Q_0} \varepsilon_s(x_{j,i}) = \varepsilon_s(x_{i,i}) = \textstyle\sum_{j \in Q_0} x_{j,i}.$$

Now let $\ell \geq 1$ and let $p \in Q_\ell$. Then

$$(\varepsilon_{\mathbf{k}Q} \otimes \mathrm{Id})(\rho_{\mathbf{k}Q}(p)) = (\varepsilon_{\mathbf{k}Q} \otimes \mathrm{Id}) \left(\textstyle\sum_{q \in Q_\ell} q \otimes x_{q,p} \right) = 0 = (\varepsilon_{\mathbf{k}Q} \otimes \varepsilon_s)(\rho_{\mathbf{k}Q}(p)).$$

Hence, by Definition 2.5.1, the path coalgebra $\mathbf{k}Q$ is an $\mathfrak{H}(Q)$-comodule coalgebra.

2.6 Weak comodule Frobenius algebras

We now define comodule Frobenius algebras over weak bialgebras and prove a result analogous to Theorems 2.4.3 and 2.5.2. Recall the category $\mathcal{A}^H$ of right H-comodule algebras and the category $\mathcal{C}^H$ of right H-comodule coalgebras from Definitions 2.4.1 and 2.5.1, respectively.

Definition 2.6.1 ($\mathcal{F}^H =: \mathcal{F}$)**.** Consider the *category $\mathcal{F}^H$ of right H-comodule Frobenius algebras* defined as follows. The objects of $\mathcal{F}^H$ are Frobenius algebras in $\mathsf{Vec}_{\mathbf{k}}$,

$$(A,\ \ m_A : A \otimes A \to A,\ \ u_A : \mathbf{k} \to A,\ \ \Delta_A : A \to A \otimes A,\ \ \varepsilon_A : A \to \mathbf{k}),$$

so that $(A, m_A, u_A) \in \mathcal{A}^H$ and $(A, \Delta_A, \varepsilon_A) \in \mathcal{C}^H$. The morphisms of $\mathcal{F}^H$ are $\mathbf{k}$-linear maps that simultaneously belong to $\mathcal{A}^H$ and $\mathcal{C}^H$. When the weak bialgebra H is clear from context, we denote this category by $\mathcal{F}$.

Theorem 2.6.2. *The category $\mathcal{F}$ of right H-comodule Frobenius algebras is isomorphic to the category $\mathsf{FrobAlg}(\mathcal{M}^H)$ of Frobenius algebras in $\mathcal{M}^H$.*

Proof. Similar to the proofs of Theorems 2.4.3 and 2.5.2, we achieve this result by constructing mutually inverse functors F and G below.

$$\begin{tikzcd} \mathcal{F} \arrow[rr, bend left, "F"] & & \mathsf{FrobAlg}(\mathcal{M}^H) \arrow[ll, bend left, "G"] \end{tikzcd}$$

First, take $(A, m_A, u_A, \Delta_A, \varepsilon_A) \in \mathcal{F}$, and consider the assignment

$$\begin{aligned} F(A, m_A, u_A, \Delta_A, \varepsilon_A) \;&\coloneqq (A,\ m_A U^{A,A},\ \nu_A(u_A \otimes \mathrm{Id}_{H_s}),\ U_{A,A}\,\Delta_A,\ (\varepsilon_A \otimes \mathrm{Id})\rho^s_A) \\ F(f) \;&\coloneqq f \end{aligned}$$

for $f : (A, m_A, u_A, \Delta_A, \varepsilon_A) \to (B, m_B, u_B, \Delta_B, \varepsilon_B) \in \mathcal{F}$. Here, $U^{A,A}$, ν_A, $U_{A,A}$, and ρ^s_A are given in Eqs. (2.3.6.4), (2.3.6.5), (2.3.6.7) and (2.3.6.9), respectively.

Claim 2.6.3. The assignment F above yields a functor from $\mathcal{F}$ to $\mathsf{FrobAlg}(\mathcal{M}^H)$.

Proof. Define $(A, \overline{m}_A, \overline{u}_A, \overline{\Delta}_A, \overline{\varepsilon}_A) \coloneqq F(A, m_A, u_A, \Delta_A, \varepsilon_A)$. By Theorems 2.4.3 and 2.5.2, in particular by Claims 2.4.4 and 2.5.3, $(A, \overline{m}_A, \overline{u}_A) \in \mathsf{Alg}(\mathcal{M}^H)$ and $(A, \overline{\Delta}_A, \overline{\varepsilon}_A) \in \mathsf{Coalg}(\mathcal{M}^H)$. It remains to show that the Frobenius algebra condition (2.3.4.5) holds; namely, that

$$(\overline{m}_A \mathbin{\overline{\otimes}} \mathrm{Id})(\mathrm{Id} \mathbin{\overline{\otimes}} \overline{\Delta}_A) \;=\; \overline{\Delta}_A \overline{m}_A \;=\; (\mathrm{Id} \mathbin{\overline{\otimes}} \overline{m}_A)(\overline{\Delta}_A \mathbin{\overline{\otimes}} \mathrm{Id}).$$

(Recall from Section 2.3.2 that the associativity constraint of $\mathcal{M}^H$ is the identity.) We will show the left equality; the proof of the right equality is similar. Let $a \mathbin{\overline{\otimes}} b \in A \mathbin{\overline{\otimes}} A$. Then

$$\begin{aligned} (\overline{m}_A \mathbin{\overline{\otimes}} \mathrm{Id})(\mathrm{Id} \mathbin{\overline{\otimes}} \overline{\Delta}_A)(a \mathbin{\overline{\otimes}} b) &= (\overline{m}_A \mathbin{\overline{\otimes}} \mathrm{Id})(\mathrm{Id} \mathbin{\overline{\otimes}} U_{A,A}\Delta_A)(a \mathbin{\overline{\otimes}} b) \\ &= (\overline{m}_A \mathbin{\overline{\otimes}} \mathrm{Id})(\mathrm{Id} \mathbin{\overline{\otimes}} U_{A,A})(a \mathbin{\overline{\otimes}} b_1 \otimes b_2) \\ &\overset{(2.3.6.4)}{=} (\overline{m}_A \mathbin{\overline{\otimes}} \mathrm{Id})(a \mathbin{\overline{\otimes}} \varepsilon(b_{1,[1]} b_{2,[1]}) b_{1,[0]} \mathbin{\overline{\otimes}} b_{2,[0]}) \\ &= (m_A U^{A,A} \mathbin{\overline{\otimes}} \mathrm{Id})(a \mathbin{\overline{\otimes}} \varepsilon(b_{1,[1]} b_{2,[1]}) b_{1,[0]} \mathbin{\overline{\otimes}} b_{2,[0]}) \\ &\overset{(2.3.6.5)}{=} (m_A \mathbin{\overline{\otimes}} \mathrm{Id})(a \otimes \varepsilon(b_{1,[1]} b_{2,[1]}) b_{1,[0]} \mathbin{\overline{\otimes}} b_{2,[0]}) \end{aligned}$$

$$
\begin{aligned}
&= \varepsilon(b_{1,[1]}b_{2,[1]})ab_{1,[0]} \,\overline{\otimes}\, b_{2,[0]} \\
&\overset{(2.5.1.1)}{=} a\varepsilon(b_{[1]})b_{[0],1} \,\overline{\otimes}\, b_{[0],2} \\
&= ab_1 \,\overline{\otimes}\, b_2,
\end{aligned}
$$

where the last equality is justified by the following computation:

$$
b_1 \otimes b_2 \;=\; \Delta_C(b) \overset{(2.2.5.5)}{=} \Delta_C(\varepsilon(b_{[1]})b_{[0]}) \;=\; \varepsilon(b_{[1]})\Delta_C(b_{[0]}) \;=\; \varepsilon(b_{[1]})b_{[0],1} \otimes b_{[0],2}.
$$

On the other hand, we have

$$
\begin{aligned}
\overline{\Delta}_A\overline{m}_A(a \,\overline{\otimes}\, b) &= U_{A,A}\Delta_A m_A U^{A,A}(a \,\overline{\otimes}\, b) \\
&\overset{(2.3.4.5)}{=} U_{A,A}(m_A \otimes \mathrm{Id})(\mathrm{Id} \otimes \Delta_A)U^{A,A}(a \,\overline{\otimes}\, b) \\
&= U_{A,A}(m_A \otimes \mathrm{Id})(\mathrm{Id} \otimes \Delta_A)(a \otimes b) \\
&= U_{A,A}(ab_1 \otimes b_2) \\
&\overset{(2.3.6.5)}{=} U_{A,A}U^{A,A}(ab_1 \,\overline{\otimes}\, b_2) \\
&\overset{(2.3.6.6)}{=} ab_1 \,\overline{\otimes}\, b_2.
\end{aligned}
$$

We have shown that F takes objects to objects. By Theorems 2.4.3 and 2.5.2, F is functorial and F takes morphisms of $\mathcal{F}$ to morphisms of $\mathsf{FrobAlg}(\mathcal{M}^H)$. $\square$

Now take $(A,\ \overline{m}_A,\ \overline{u}_A,\ \overline{\Delta}_A,\ \overline{\varepsilon}_A) \in \mathsf{FrobAlg}(\mathcal{M}^H)$, and consider the assignment

$$
\begin{aligned}
G(A,\ \overline{m}_A,\ \overline{u}_A,\ \overline{\Delta}_A,\ \overline{\varepsilon}_A) &:= (A,\ \overline{m}_A U_{A,A},\ \overline{u}_A U_0,\ U^{A,A}\,\overline{\Delta}_A,\ U^0\,\overline{\varepsilon}_A) \\
G(\overline{f}) &:= \overline{f}
\end{aligned}
$$

for $\overline{f} : (A,\overline{m}_A,\overline{u}_A,\overline{\Delta}_A,\overline{\varepsilon}_A) \to (B,\overline{m}_B,\overline{u}_B,\overline{\Delta}_B,\overline{\varepsilon}_B) \in \mathsf{FrobAlg}(\mathcal{M}^H)$. Here, $U_{A,A}$, $U^{A,A}$, U_0, and U^0 are given in Section 2.3.3.1.

Claim 2.6.4. The assignment G given above yields a functor from $\mathsf{FrobAlg}(\mathcal{M}^H)$ to $\mathcal{F}$.

Proof. Recall from Section 2.3.3.1 that the forgetful functor $U : \mathcal{M}^H \to \mathsf{Vec}_{\mathbf{k}}$ is Frobenius monoidal. Then by 2.3.5(iii),

$$
G(A,\overline{m}_A,\overline{u}_A,\overline{\Delta}_A,\overline{\varepsilon}_A) = (U(A),\ U(\overline{m}_A)U_{A,A},\ U(\overline{u}_A)U_0,\ U^{A,A}U(\overline{\Delta}_A),\ U^0U(\overline{\varepsilon}_A))
$$

is a Frobenius algebra in $\mathsf{Vec}_{\mathbf{k}}$. Moreover, by Theorems 2.4.3 and 2.5.2, in particular by Claims 2.4.5 and 2.5.4, we know that

$$(A, m_A, u_A, \Delta_A, \varepsilon_A) := G(A, \overline{m}_A, \overline{u}_A, \overline{\Delta}_A, \overline{\varepsilon}_A)$$

is both a right H-comodule algebra and a right H-comodule coalgebra. Hence G takes objects to objects.

By Theorems 2.4.3 and 2.5.2, G is functorial and G takes morphisms of $\mathcal{F}$ to morphisms of $\mathsf{FrobAlg}(\mathcal{M}^H)$. □

Now by Claims 2.4.6 and 2.5.5, the functors F defined in Claim 2.6.3 and G defined in Claim 2.6.4 are mutual inverses. Therefore, these functors yield an isomorphism between the categories $\mathcal{F} := \mathcal{F}^H$ and $\mathsf{FrobAlg}(\mathcal{M}^H)$. □

Now consider the following examples.

Example 2.6.5. The counital subalgebra H_s is an object of $\mathcal{F}^H \cong \mathsf{FrobAlg}(\mathcal{M}^H)$, as the unit object of a monoidal category is a Frobenius algebra in the category; see Section 2.3.2.2.

Example 2.6.6. We showed in Examples 2.4.8 and 2.5.7 that $\mathbf{k}Q$ is an $\mathfrak{H}(Q)$-comodule algebra and an $\mathfrak{H}(Q)$-comodule coalgebra. By definition, $\mathbf{k}Q$ is an $\mathfrak{H}(Q)$-comodule Frobenius algebra if $\mathbf{k}Q$ is first a Frobenius $\mathbf{k}$-algebra. But it is straightforward to show that under the given algebra and coalgebra structure that $\mathbf{k}Q$ is a Frobenius $\mathbf{k}$-algebra if and only if $Q = Q_0$ (i.e., Q is arrowless). In this case, $\mathbf{k}Q \cong \mathfrak{H}(Q)_s$ in the category $\mathcal{F}$, so this example reduces to Example 2.6.5. Thus $\mathbf{k}Q = \mathbf{k}Q_0$ is a Frobenius algebra in $\mathcal{M}^{\mathfrak{H}(Q)}$ by Theorem 2.6.2.

On the other hand, we provide below an example of a Frobenius algebra over the weak bialgebra $\mathfrak{H}(Q)$ that is not the counital subalgebra $\mathfrak{H}(Q)_s$.

Example 2.6.7. Take Q to be the two cycle with vertices 1 and 2, with arrows $p : 1 \to 2$ and $p^* : 2 \to 1$.

$$Q = \begin{tikzcd} 1 \arrow[r, bend left, "p"] & 2 \arrow[l, bend left, "p^*"] \end{tikzcd}$$

Consider the $\mathbf{k}$-algebra $A = \mathbf{k}Q/(pp^* - e_1,\ p^*p - e_2)$. Then A admits the structure of a Frobenius $\mathbf{k}$-algebra via

$$\begin{aligned} \Delta(e_1) &= e_1 \otimes e_1 + p \otimes p^*, & \Delta(p) &= e_1 \otimes p + p \otimes e_2, \\ \Delta(p^*) &= p^* \otimes e_1 + e_2 \otimes p^*, & \Delta(e_2) &= p^* \otimes p + e_2 \otimes e_2, \end{aligned}$$

and $\varepsilon(e_1) = \varepsilon(e_2) = 1$, $\varepsilon(p) = \varepsilon(p^*) = 0$. Namely, A is isomorphic to the matrix algebra $\mathrm{Mat}_2(\mathbf{k})$ via $e_1, p, p^*, e_2 \mapsto E_{11}, E_{12}, E_{21}, E_{22}$, respectively. Moreover, we have a coaction $\rho : A \to A \otimes \mathfrak{H}(Q)$ given by

$$\begin{aligned} \rho(e_1) &= e_1 \otimes x_{1,1} + e_2 \otimes x_{2,1}, & \rho(e_2) &= e_1 \otimes x_{1,2} + e_2 \otimes x_{2,2} \\ \rho(p) &= p \otimes x_{p,p} + p^* \otimes x_{p^*,p}, & \rho(p^*) &= p \otimes x_{p,p^*} + p^* \otimes x_{p^*,p^*}, \end{aligned}$$

that yields that A belongs to the category $\mathcal{F}$. The coassociativity and counitality of ρ follows from a similar proof to Example 2.4.8. We already know that A is a Frobenius algebra, so it only remains to check that $\mathbf{k}Q$ is an $\mathfrak{H}(Q)$-comodule algebra and an $\mathfrak{H}(Q)$-comodule coalgebra. The first condition follows from the proof of Example 2.4.8 and by checking that $\rho(p)\rho(p^*) = \rho(e_1)$ and $\rho(p^*)\rho(p) = \rho(e_2)$. The second condition can be checked by hand. By Theorem 2.6.2, we can give A the structure of a Frobenius algebra in $\mathcal{M}^{\mathfrak{H}(Q)}$.

2.7 Weak quantum symmetry via quantum transformation groupoids

In the section, we consider a weak Hopf algebra that arises from a double-crossed product construction: the quantum transformation groupoid $H(L, B, \triangleleft)$ [Definition 2.7.2]. Here, L is a Hopf algebra, B a certain right L-module algebra via action $\triangleleft$ [Notation 2.7.1]. To achieve a supply of weak quantum symmetries, we produce a monoidal functor from a category of L-bicomodules to H-comodules [Theorem 2.7.5], thus illustrating the main result of Section 2.4. We present explicit examples of these new weak quantum symmetries afterward.

We assume in this section that $\mathbf{k}$ is an algebraically closed field. Moreover, consider the standing notation and hypotheses below.

Notation 2.7.1 (L, B, $e^{(i)}$, $\triangleleft$, $\triangleright$, ω)**.** Let L be a Hopf algebra over $\mathbf{k}$. Let B be a strongly separable $\mathbf{k}$-algebra with symmetric separability idempotent $e = e^{(1)} \otimes e^{(2)} \in B \otimes B^{op}$, where we omit summation notation. If we use multiple copies of the separability idempotent in a calculation, we will denote the additional copies with primes like so: $e'^{(1)} \otimes e'^{(2)}$. We have the following identities:

$$be^{(1)} \otimes e^{(2)} = e^{(1)} \otimes e^{(2)}b, \qquad \forall b \in B, \tag{2.7.1.1}$$

$$e^{(1)}e^{(2)} = 1_B. \tag{2.7.1.2}$$

Assume that B is a right L-module algebra with action

$$B \otimes L \to B, \qquad b \otimes h \mapsto b \triangleleft h, \qquad \forall b \in B,\ h \in L.$$

So, B^{op} is a left L-module algebra with action via the antipode S_L of L,

$$L \otimes B^{op} \to B^{op}, \qquad h \otimes a \mapsto h \triangleright a := a \triangleleft S_L(h), \qquad \forall a \in B^{op},\ h \in L. \tag{2.7.1.3}$$

Let $\omega : B \to \mathbf{k}$ be the nondegenerate trace form of B, which is characterized by

$$\omega(e^{(1)})e^{(2)} \ = \ e^{(1)}\omega(e^{(2)}) \ = \ 1_B. \tag{2.7.1.4}$$

We assume the following useful identities:

$$\omega((h \triangleright a)b) \ = \ \omega(a(b \triangleleft h)), \tag{2.7.1.5}$$

$$e^{(1)} \otimes (h \triangleright e^{(2)}) \ = \ (e^{(1)} \triangleleft h) \otimes e^{(2)}, \tag{2.7.1.6}$$

for $a \in B^{op}$, $b \in B$, and $h \in L$.

Now we consider the following weak Hopf algebra introduced in [NV02, Section 2.6], but modified slightly here; see also [Nil98, Example 3, Appendix D]. We verify that this is indeed a weak Hopf algebra in the appendix.

Definition 2.7.2 ($H := H(L, B, \triangleleft)$)**.** Let $H := H(L, B, \triangleleft)$ be the weak Hopf algebra, which is $B^{op} \otimes L \otimes B$ as a $\mathbf{k}$-vector space, with the following structure:

- multiplication $\quad m((a\otimes h\otimes b)\otimes(a'\otimes h'\otimes b')) = (h_1 \triangleright a')a\otimes h_2h'_1\otimes(b\triangleleft h'_2)b';$
- unit $\quad u(1_{\mathbf{k}}) = 1_B\otimes 1_L\otimes 1_B;$
- comultiplication $\quad \Delta(a\otimes h\otimes b) = (a\otimes h_1\otimes e^{(1)})\otimes((h_2\triangleright e^{(2)})\otimes h_3\otimes b);$
- counit $\quad \varepsilon(a\otimes h\otimes b) = \omega(a(b\triangleleft S_L^{-1}(h)));$
- antipode $\quad S(a\otimes h\otimes b) = b\otimes S_L(h)\otimes a;$

for all $a,a'\in B^{op}$, $b,b'\in B$, and $h,h'\in L$. We refer to H as a *quantum transformation groupoid* (QTG).

Lemma 2.7.3. *The source and target counital subalgebras of the QTG $H := H(L,B,\triangleleft)$ are given, respectively, as follows:*

$$H_s = 1_B\otimes 1_L\otimes B \qquad \textit{and} \qquad H_t = B^{op}\otimes 1_L\otimes 1_B.$$

Proof. This follows from the computations below: For $a\in B^{op}, h\in L, b\in B$, we get

$$\begin{aligned}
\varepsilon_s(a\otimes h\otimes b) &= (1_B\otimes 1_L\otimes e^{(1)})\varepsilon((a\otimes h\otimes b)(e^{(2)}\otimes 1_L\otimes 1_B))\\
&= (1_B\otimes 1_L\otimes e^{(1)})\varepsilon((h_1\triangleright e^{(2)})a\otimes h_2 1_L\otimes(b\triangleleft 1_L)1_B)\\
&= (1_B\otimes 1_L\otimes e^{(1)})\varepsilon((h_1\triangleright e^{(2)})a\otimes h_2\otimes b)\\
&= \omega((h_1\triangleright e^{(2)})a(b\triangleleft S_L^{-1}(h_2)))(1_B\otimes 1_L\otimes e^{(1)})\\
&\overset{(2.7.1.6)}{=} \omega(e^{(2)}a(b\triangleleft S_L^{-1}(h_2)))(1_B\otimes 1_L\otimes(e^{(1)}\triangleleft h_1))\\
&\overset{(2.7.1.1)}{=} \omega(e^{(2)})(1_B\otimes 1_L\otimes((a(b\triangleleft S_L^{-1}(h_2))e^{(1)})\triangleleft h_1))\\
&\overset{(2.7.1.4)}{=} 1_B\otimes 1_L\otimes(a(b\triangleleft S_L^{-1}(h_2)))\triangleleft h_1\\
&= 1_B\otimes 1_L\otimes(a\triangleleft h_{1,1})(b\triangleleft S_L^{-1}(h_2)h_{1,2})\\
&= 1_B\otimes 1_L\otimes(a\triangleleft h_1)(b\triangleleft \varepsilon_L(h_2))\\
&= 1_B\otimes 1_L\otimes(a\triangleleft h)b, \quad \text{and}
\end{aligned}$$

$$\varepsilon_t(a\otimes h\otimes b) = \varepsilon((1_B\otimes 1_L\otimes e^{(1)})(a\otimes h\otimes b))(e^{(2)}\otimes 1_L\otimes 1_B)$$

$$
\begin{aligned}
&= \varepsilon((1_L \triangleright a)1_B \otimes 1_L h_1 \otimes (e^{(1)} \triangleleft h_2)b)(e^{(2)} \otimes 1_L \otimes 1_B) \\
&= \varepsilon(a \otimes h_1 \otimes (e^{(1)} \triangleleft h_2)b)(e^{(2)} \otimes 1_L \otimes 1_B) \\
&= \omega(a((e^{(1)} \triangleleft h_2)b \triangleleft S_L^{-1}(h_1))(e^{(2)} \otimes 1_L \otimes 1_B) \\
&= \omega(a((e^{(1)} \triangleleft h_2) \triangleleft S_L^{-1}(h_1)_1)(b \triangleleft S_L^{-1}(h_1)_2))(e^{(2)} \otimes 1_L \otimes 1_B) \\
&= \omega(a(e^{(1)} \triangleleft h_2 S_L^{-1}(h_{1,2}))(b \triangleleft S_L^{-1}(h_{1,1})))(e^{(2)} \otimes 1_L \otimes 1_B) \\
&= \omega(ae^{(1)}\varepsilon_L(h_2)(b \triangleleft S_L^{-1}(h_1)))(e^{(2)} \otimes 1_L \otimes 1_B) \\
&= \omega(ae^{(1)}(b \triangleleft S_L^{-1}(h)))(e^{(2)} \otimes 1_L \otimes 1_B) \\
&\overset{(2.7.1.1)}{=} \omega(e^{(1)}(b \triangleleft S_L^{-1}(h)))(e^{(2)}a \otimes 1_L \otimes 1_B) \\
&\overset{(2.7.1.5)}{=} \omega((S_L^{-1}(h) \triangleright e^{(1)})b)(e^{(2)}a \otimes 1_L \otimes 1_B) \\
&\overset{(2.7.1.3)}{=} \omega((e^{(1)} \triangleleft h)b)(e^{(2)}a \otimes 1_L \otimes 1_B) \\
&\overset{(2.7.1.6)}{=} \omega(e^{(1)}b)((h \triangleright e^{(2)})a \otimes 1_L \otimes 1_B) \\
&\overset{\omega \text{ is a trace}}{=} \omega(be^{(1)})((h \triangleright e^{(2)})a \otimes 1_L \otimes 1_B) \\
&\overset{(2.7.1.1)}{=} \omega(e^{(1)})((h \triangleright (e^{(2)}b))a \otimes 1_L \otimes 1_B) \\
&\overset{(2.7.1.4)}{=} (h \triangleright b)a \otimes 1_L \otimes 1_B.
\end{aligned}
$$

□

One avenue of obtaining "weak quantum symmetries" is via the following construction. First, let us consider the following monoidal category.

Definition 2.7.4 (L-$\mathsf{Bicomod}$)**.** Let $(L\text{-}\mathsf{Bicomod}, \otimes, \mathbf{k})$ be the monoidal category consisting of L-bicomodules with the following monoidal structure. Observe that if $X \in L\text{-}\mathsf{Bicomod}$, then

$$x_{[0],[-1]} \otimes x_{[0],[0]} \otimes x_{[1]} = x_{[-1]} \otimes x_{[0],[0]} \otimes x_{[0],[1]}. \tag{2.7.4.1}$$

For $X, X' \in L\text{-}\mathsf{Bicomod}$, the bicomodule structure on $X \otimes X'$ is determined by

$$X \otimes X' \to L \otimes X \otimes X', \quad x \otimes x' \mapsto x_{[-1]}x'_{[-1]} \otimes x_{[0]} \otimes x'_{[0]}, \tag{2.7.4.2}$$

$$X \otimes X' \to X \otimes X' \otimes L, \quad x \otimes x' \mapsto x_{[0]} \otimes x'_{[0]} \otimes x_{[1]}x'_{[1]}. \tag{2.7.4.3}$$

This monoidal category has unit object $\mathbf{k}$ via the counit of L, and has the canonical isomorphisms for associativity and unitality.

We remark that, as an abelian category, L-$\mathsf{Bicomod}$ is the same as the category ${}^L\mathcal{M}^L$ defined in Example 2.3.2. However, they differ as monoidal categories, since the monoidal product in L-$\mathsf{Bicomod}$ is the tensor product $\otimes_{\mathbf{k}}$ while the monoidal product in ${}^L\mathcal{M}^L$ is the cotensor product $\otimes^L$.

Theorem 2.7.5. *Consider the assignment $\Gamma : \mathcal{M}^L \to \mathcal{M}^H$, where for an object $X \in \mathcal{M}^L$, take*

$$\Gamma(X) = B^{op} \otimes X \otimes B, \quad \textit{with}$$
$$\rho_{\Gamma(X)}(a \otimes x \otimes b) = (a \otimes x_{[0]} \otimes e^{(1)}) \otimes ((x_{[1],1} \triangleright e^{(2)}) \otimes x_{[1],2} \otimes b) \in \Gamma(X) \otimes H,$$

and for a morphism $f \in \mathcal{M}^L$, take $\Gamma(f) = \mathrm{Id}_{B^{op}} \otimes f \otimes \mathrm{Id}_B$. Then,

(i) *Γ is a functor;*

(ii) *Let $U' : L$-$\mathsf{Bicomod} \to \mathcal{M}^L$ denote the forgetful functor which forgets the left L-comodule structure. Then the precomposition $\widehat{\Gamma} = \Gamma \circ U' : L$-$\mathsf{Bicomod} \to \mathcal{M}^H$ admits a monoidal structure, where*

$$\widehat{\Gamma}_{X,X'} : \widehat{\Gamma}(X) \;\overline{\otimes}\; \widehat{\Gamma}(X') \longrightarrow \widehat{\Gamma}(X \otimes X')$$
$$(a \otimes x \otimes b) \,\overline{\otimes}\, (a' \otimes x' \otimes b') \mapsto (x_{[-1]} \triangleright a')a \otimes x_{[0]} \otimes x'_{[0]} \otimes (b \triangleleft x'_{[1]})b'$$

$$\widehat{\Gamma}_0 : \mathbb{1}_{\mathcal{M}^H} = H_s \longrightarrow \widehat{\Gamma}(\mathbf{k}) = \widehat{\Gamma}(\mathbb{1}_{L\text{-}\mathsf{Bicomod}})$$
$$1_B \otimes 1_L \otimes b \mapsto 1_B \otimes 1_{\mathbf{k}} \otimes b.$$

Proof. (i) Consider the following calculations:

$$(\rho_{\Gamma(X)} \otimes \mathrm{Id})\rho_{\Gamma(X)}(a \otimes x \otimes b)$$
$$= (\rho_{\Gamma(X)} \otimes \mathrm{Id})[(a \otimes x_{[0]} \otimes e^{(1)}) \otimes ((x_{[1],1} \triangleright e^{(2)}) \otimes x_{[1],2} \otimes b)]$$

$$= \left[a \otimes x_{[0],[0]} \otimes e'^{(1)} \otimes (x_{[0],[1],1} \triangleright e'^{(2)}) \otimes x_{[0],[1],2} \otimes e^{(1)}\right] \otimes ((x_{[1],1} \triangleright e^{(2)}) \otimes x_{[1],2} \otimes b)$$
$$\overset{(2.2.5.4)}{=} (a \otimes x_{[0]} \otimes e'^{(1)}) \otimes \left[(x_{[1],1} \triangleright e'^{(2)}) \otimes x_{[1],2,1} \otimes e^{(1)} \otimes ((x_{[1],2,2} \triangleright e^{(2)}) \otimes x_{[1],2,3} \otimes b\right]$$
$$= (\mathrm{Id} \otimes \Delta)\left[(a \otimes x_{[0]} \otimes e'^{(1)}) \otimes ((x_{[1],1} \triangleright e'^{(2)}) \otimes x_{[1],2} \otimes b)\right]$$
$$= (\mathrm{Id} \otimes \Delta)\rho_{\Gamma(X)}(a \otimes x \otimes b),$$

where the third equality holds by the comodule and the coassociativity axioms. Moreover,

$$(\mathrm{Id} \otimes \varepsilon)\rho_{\Gamma(X)}(a \otimes x \otimes b)$$
$$= (\mathrm{Id} \otimes \varepsilon)\left[(a \otimes x_{[0]} \otimes e^{(1)}) \otimes ((x_{[1],1} \triangleright e^{(2)}) \otimes x_{[1],2} \otimes b)\right]$$
$$= (a \otimes x_{[0]} \otimes e^{(1)})\, \omega((x_{[1],1} \triangleright e^{(2)})(b \triangleleft S_L^{-1}(x_{[1],2})))$$
$$\overset{(2.7.1.6)}{=} (a \otimes x_{[0]} \otimes (e^{(1)} \triangleleft x_{[1],1}))\, \omega(e^{(2)}(b \triangleleft S_L^{-1}(x_{[1],2})))$$
$$\overset{(2.7.1.1)}{=} (a \otimes x_{[0]} \otimes (((b \triangleleft S_L^{-1}(x_{[1],2}))e^{(1)}) \triangleleft x_{[1],1}))\, \omega(e^{(2)})$$
$$\overset{(2.7.1.4)}{=} a \otimes x_{[0]} \otimes ((b \triangleleft S_L^{-1}(x_{[1],2})) \triangleleft x_{[1],1})$$
$$= a \otimes x_{[0]} \otimes (b \triangleleft S_L^{-1}(x_{[1],2})x_{[1],1})$$
$$= a \otimes \varepsilon_L(x_{[1]})x_{[0]} \otimes b$$
$$= a \otimes x \otimes b.$$

So, $\Gamma(X)$ is an object of $\mathcal{M}^H$. It is clear that $\Gamma(f)$ is a morphism in $\mathcal{M}^H$, and with this, Γ is a functor.

(ii) First, we have that $\widehat{\Gamma}_{X,X'}$ is a morphism in $\mathcal{M}^H$ due to the following calculation:

$$(\widehat{\Gamma}_{X,X'} \otimes \mathrm{Id}_H)\rho_{\widehat{\Gamma}(X)\overline{\otimes}\widehat{\Gamma}(X')}[(a \otimes x \otimes b) \,\overline{\otimes}\, (a' \otimes x' \otimes b')]$$
$$\overset{(2.3.6.2)}{=} (\widehat{\Gamma}_{X,X'} \otimes \mathrm{Id}_H)\Big[(a \otimes x_{[0]} \otimes e^{(1)}) \,\overline{\otimes}\, (a' \otimes x'_{[0]} \otimes e'^{(1)})$$
$$\otimes \big((x_{[1],1} \triangleright e^{(2)}) \otimes x_{[1],2} \otimes b\big)\Big((x'_{[1],1} \triangleright e'^{(2)}) \otimes x'_{[1],2} \otimes b'\Big)\Big]$$
$$= \Big[(x_{[0],[-1]} \triangleright a')a \otimes x_{[0],[0]} \otimes x'_{[0],[0]} \otimes (e^{(1)} \triangleleft x'_{[0],[1]})e'^{(1)}\Big]$$
$$\otimes \big((x_{[1],1} \triangleright e^{(2)}) \otimes x_{[1],2} \otimes b\big)\Big((x'_{[1],1} \triangleright e'^{(2)}) \otimes x'_{[1],2} \otimes b'\Big)$$
$$\overset{(2.7.4.1)}{=} \Big[(x_{[-1]} \triangleright a')a \otimes x_{[0],[0]} \otimes x'_{[0],[0]} \otimes (e^{(1)} \triangleleft x'_{[0],[1]})e'^{(1)}\Big]$$

$$
\begin{aligned}
&\qquad\qquad \otimes \Big((x_{[0],[1],1} \triangleright e^{(2)}) \otimes x_{[0],[1],2} \otimes b\Big)\Big((x'_{[1],1} \triangleright e'^{(2)}) \otimes x'_{[1],2} \otimes b'\Big) \\
&\overset{(2.7.1.6)}{=} \Big[(x_{[-1]} \triangleright a')a \otimes x_{[0],[0]} \otimes x'_{[0],[0]} \otimes e^{(1)}e'^{(1)}\Big] \\
&\qquad\qquad \otimes \Big((x_{[0],[1],1}x'_{[0],[1]} \triangleright e^{(2)}) \otimes x_{[0],[1],2} \otimes b\Big)\Big((x'_{[1],1} \triangleright e'^{(2)}) \otimes x'_{[1],2} \otimes b'\Big) \\
&= \Big[(x_{[-1]} \triangleright a')a \otimes x_{[0]} \otimes x'_{[0]} \otimes e^{(1)}e'^{(1)}\Big] \\
&\qquad\qquad \otimes \Big((x_{[1]}x'_{[1]} \triangleright e^{(2)}) \otimes x_{[2]} \otimes b\Big)\Big((x'_{[2]} \triangleright e'^{(2)}) \otimes x'_{[3]} \otimes b'\Big) \\
&\overset{\text{mult in } H}{=} \Big[(x_{[-1]} \triangleright a')a \otimes x_{[0]} \otimes x'_{[0]} \otimes e^{(1)}e'^{(1)}\Big] \\
&\qquad\qquad \otimes \Big((x_{[2],1} \triangleright (x'_{[2]} \triangleright e'^{(2)}))(x_{[1]}x'_{[1]} \triangleright e^{(2)}) \otimes x_{[2],2}x'_{[3],1} \otimes (b \triangleleft x'_{[3],2})b'\Big) \\
&= \Big[(x_{[-1]} \triangleright a')a \otimes x_{[0]} \otimes x'_{[0]} \otimes e^{(1)}e'^{(1)}\Big] \\
&\qquad\qquad \otimes \Big((x_{[2]}x'_{[2]} \triangleright e'^{(2)})(x_{[1]}x'_{[1]} \triangleright e^{(2)}) \otimes x_{[3]}x'_{[3]} \otimes (b \triangleleft x'_{[4]})b'\Big) \\
&= \Big[(x_{[-1]} \triangleright a')a \otimes x_{[0]} \otimes x'_{[0]} \otimes e^{(1)}e'^{(1)}\Big] \otimes \Big((x_{[1]}x'_{[1]} \triangleright e^{(2)}e'^{(2)}) \otimes x_{[2]}x'_{[2]} \otimes (b \triangleleft x'_{[3]})b'\Big) \\
&\overset{e \text{ idemp.}}{=} \Big[(x_{[-1]} \otimes a')a \otimes x_{[0]} \otimes x'_{[0]} \otimes e^{(1)}\Big] \otimes \Big((x_{[1]}x'_{[1]} \triangleright e^{(2)}) \otimes x_{[2]}x'_{[2]} \otimes (b \triangleleft x'_{[3]})b'\Big) \\
&= \Big[(x_{[-1]} \triangleright a')a \otimes x_{[0],[0]} \otimes x'_{[0],[0]} \otimes e^{(1)}\Big] \\
&\qquad\qquad \otimes \Big((x_{[0],[1],1}x'_{[0],[1],1} \triangleright e^{(2)}) \otimes x_{[0],[1],2}x'_{[0],[1],2} \otimes (b \triangleleft x'_{[1]})b'\Big) \\
&\overset{(2.7.4.2),(2.7.4.3)}{=} \Big[(x_{[-1]} \triangleright a')a \otimes (x_{[0]} \otimes x'_{[0]})_{[0]} \otimes e^{(1)}\Big] \\
&\qquad\qquad \otimes \Big(((x_{[0]} \otimes x'_{[0]})_{[1],1} \triangleright e^{(2)}) \otimes (x_{[0]} \otimes x'_{[0]})_{[1],2} \otimes (b \triangleleft x'_{[1]})b'\Big) \\
&= \rho_{\widehat{\Gamma}(X \otimes X')}\Big[(x_{[-1]} \triangleright a')a \otimes x_{[0]} \otimes x'_{[0]} \otimes (b \triangleleft x'_{[1]})b'\Big] \\
&= \rho_{\widehat{\Gamma}(X \otimes X')}\widehat{\Gamma}_{X,X'}[(a \otimes x \otimes b) \,\overline{\otimes}\, (a' \otimes x' \otimes b')].
\end{aligned}
$$

Next, we have that $\widehat{\Gamma}_0$ is a morphism in $\mathcal{M}^H$ due to the calculation below. By Lemma 2.7.3, an arbitrary element of H_s is of the form $1_B \otimes 1_L \otimes b$. Now,

$$
\begin{aligned}
&(\widehat{\Gamma}_0 \otimes \mathrm{Id}_H)\rho_{H_s}[1_B \otimes 1_L \otimes b] \\
&\overset{\S 2.3.2.2}{=} (\widehat{\Gamma}_0 \otimes \mathrm{Id}_H)\Delta_H[1_B \otimes 1_L \otimes b] \\
&= (\widehat{\Gamma}_0 \otimes \mathrm{Id}_H)[(1_B \otimes 1_L \otimes e^{(1)}) \otimes ((1_L \triangleright e^{(2)}) \otimes 1_L \otimes b)] \\
&= (1_B \otimes 1_{\mathbf{k}} \otimes e^{(1)}) \otimes ((1_L \triangleright e^{(2)}) \otimes 1_L \otimes b)
\end{aligned}
$$

$$= \rho_{\widehat{\Gamma}(\mathbf{k})}[1_B \otimes 1_{\mathbf{k}} \otimes b]$$
$$= \rho_{\widehat{\Gamma}(\mathbf{k})}\widehat{\Gamma}_0[1_B \otimes 1_L \otimes b].$$

Moreover, we have that $\widehat{\Gamma}_{X,X'}$ satisfies the associativity constraint required for $\widehat{\Gamma}$ to be monoidal (see Definition 2.3.3):

$$\begin{aligned}
&\widehat{\Gamma}_{X,X'\otimes X''}(\mathrm{Id}\otimes\widehat{\Gamma}_{X',X''})[(a\otimes x\otimes b)\,\overline{\otimes}\,(a'\otimes x'\otimes b')\,\overline{\otimes}\,(a''\otimes x''\otimes b'')]\\
&= \widehat{\Gamma}_{X,X'\otimes X''}[(a\otimes x\otimes b)\,\overline{\otimes}\,((x'_{[-1]}\triangleright a'')a'\otimes x'_{[0]}\otimes x''_{[0]}\otimes(b'\triangleleft x''_{[1]})b'')]\\
&= (x_{[-1]}\triangleright((x'_{[-1]}\triangleright a'')a'))a\otimes x_{[0]}\otimes(x'_{[0]}\otimes x''_{[0]})_{[0]}\otimes(b\triangleleft(x'_{[0]}\otimes x''_{[0]})_{[1]})(b'\triangleleft x''_{[1]})b''\\
&\overset{(2.7.4.3)}{=} (x_{[-1],2}x'_{[-1]}\triangleright a'')(x_{[-1],1}\triangleright a')a\otimes x_{[0]}\otimes x'_{[0],[0]}\otimes x''_{[0],[0]}\otimes(b\triangleleft x'_{[0],[1]}x''_{[0],[1]})(b'\triangleleft x''_{[1]})b''\\
&= (x_{[-1]}x'_{[-1]}\triangleright a'')(x_{[-2]}\triangleright a')a\otimes x_{[0]}\otimes x'_{[0]}\otimes x''_{[0]}\otimes(b\triangleleft x'_{[1]}x''_{[1]})(b'\triangleleft x''_{[2]})b''\\
&= (x_{[0],[-1]}x'_{[0],[-1]}\triangleright a'')(x_{[-1]}\triangleright a')a\otimes x_{[0],[0]}\otimes x'_{[0],[0]}\otimes x''_{[0]}\otimes(b\triangleleft x'_{[1]}x''_{[1],1})(b'\triangleleft x''_{[1],2})b''\\
&\overset{(2.7.4.2)}{=} ((x_{[0]}\otimes x'_{[0]})_{[-1]}\triangleright a'')(x_{[-1]}\triangleright a')a\otimes(x_{[0]}\otimes x'_{[0]})_{[0]}\otimes x''_{[0]}\otimes(((b\triangleleft x'_{[1]})b')\triangleleft x''_{[1]})b''\\
&= \widehat{\Gamma}_{X\otimes X',X''}[(x_{[-1]}\triangleright a')a\otimes x_{[0]}\otimes x'_{[0]}\otimes((b\triangleleft x'_{[1]})b')\,\overline{\otimes}\,(a''\otimes x''\otimes b'')]\\
&= \widehat{\Gamma}_{X\otimes X',X''}(\widehat{\Gamma}_{X,X'}\,\overline{\otimes}\,\mathrm{Id})[(a\otimes x\otimes b)\,\overline{\otimes}\,(a'\otimes x'\otimes b')\,\overline{\otimes}\,(a''\otimes x''\otimes b'')].
\end{aligned}$$

Finally, $\widehat{\Gamma}_0$ satisfies the left unit constraint required for $\widehat{\Gamma}$ to be monoidal (see Definition 2.3.3):

$$\begin{aligned}
&\widehat{\Gamma}(l_X)\,\widehat{\Gamma}_{\mathbf{k},X}\,(\widehat{\Gamma}_0\,\overline{\otimes}\,\mathrm{Id})[(1_B\otimes 1_L\otimes b)\,\overline{\otimes}\,(a'\otimes x'\otimes b)]\\
&= \widehat{\Gamma}(l_X)\,\widehat{\Gamma}_{\mathbf{k},X}[(1_B\otimes 1_{\mathbf{k}}\otimes b)\,\overline{\otimes}\,(a'\otimes x'\otimes b)]\\
&= \widehat{\Gamma}(l_X)[(1_{\mathbf{k}}\triangleright a')1_B\otimes 1_{\mathbf{k}}\otimes x'_{[0]}\otimes(b\triangleleft x'_{[1]})b']\\
&= a'\otimes x'_{[0]}\otimes(b\triangleleft x'_{[1]})b'\\
&\overset{(2.7.1.4)}{=} \omega(e^{(2)})(a'\otimes x'_{[0]}\otimes(b\triangleleft x'_{[1]})b'e^{(1)})\\
&\overset{(2.7.1.1)}{=} \omega(e^{(2)}(b\triangleleft x'_{[1]})b')(a'\otimes x'_{[0]}\otimes e^{(1)})\\
&= \omega(\varepsilon_L(x'_{[1]})e^{(2)}((b\triangleleft x'_{[2]})b'))(a'\otimes x'_{[0]}\otimes e^{(1)})\\
&= \omega\Big((S_L^{-1}(x'_{[2]})x'_{[1]}\triangleright e^{(2)})((b\triangleleft x'_{[3]})b')\Big)(a'\otimes x'_{[0]}\otimes e^{(1)})\\
&\overset{(2.7.1.5)}{=} \omega\Big((x'_{[1]}\triangleright e^{(2)})\Big(((b\triangleleft x'_{[3]})b')\triangleleft S_L^{-1}(x'_{[2]})\Big)\Big)(a'\otimes x'_{[0]}\otimes e^{(1)})
\end{aligned}$$

$$= \varepsilon\Big((x'_{[1]} \triangleright e^{(2)}) \otimes x'_{[2]} \otimes (b \triangleleft x'_{[3]})b'\Big)\,(a' \otimes x'_{[0]} \otimes e^{(1)})$$
$$= \varepsilon\Big((x'_{[1],1} \triangleright e^{(2)}) \otimes x'_{[1],2,1} \otimes (b \triangleleft x'_{[1],2,2})b'\Big)\,(a' \otimes x'_{[0]} \otimes e^{(1)})$$
$$\overset{\text{mult. in } H}{=} \varepsilon\Big((1_B \otimes 1_L \otimes b)((x'_{[1],1} \triangleright e^{(2)}) \otimes x'_{[1],2} \otimes b')\Big)\,(a' \otimes x'_{[0]} \otimes e^{(1)})$$
$$= \varepsilon\big((1_B \otimes 1_L \otimes b)(a' \otimes x' \otimes b')_{[1]}\big)\,(a' \otimes x' \otimes b')_{[0]}$$
$$\overset{\S 2.3.2.3}{=} l_{\widehat{\Gamma}(X)}[(1_B \otimes 1_L \otimes b)\,\overline{\otimes}\,(a' \otimes x' \otimes b')],$$

and the right unit constraint holds similarly. □

Notice that the monoidal structure on $\widehat{\Gamma}$ is prompted by the multiplication of $H := H(L,B,\triangleleft)$; see Example 2.7.7 below. Indeed, H (with coaction defined by Δ_H) is an algebra in the monoidal category $\mathcal{M}^H$. However, H is not necessarily a coalgebra in $\mathcal{M}^H$, which leads to the question below.

Question 2.7.6. Under what conditions does the functor $\widehat{\Gamma}$ of Theorem 2.7.5 admit a comonoidal structure? Further, when does it admit a Frobenius monoidal structure?

Example 2.7.7 ($X = L$)**.** Take any Hopf algebra L with strongly separable module algebra B as in Notation 2.7.1, and let $H = H(L,B,\triangleleft)$ be the corresponding QTG.

By Theorem 2.7.5, $\widehat{\Gamma} : L\text{-}\mathsf{Bicomod} \to \mathcal{M}^H$ is monoidal and so by Proposition 2.3.5, we obtain an induced functor $\mathsf{Alg}(L\text{-}\mathsf{Bicomod}) \to \mathsf{Alg}(\mathcal{M}^H)$. By Theorem 2.4.3, there is an isomorphism of categories $G : \mathsf{Alg}(\mathcal{M}^H) \to \mathcal{A}^H$. Explicitly, we have (using Claim 2.4.5) that

$$\mathsf{Alg}(L\text{-}\mathsf{Bicomod}) \xrightarrow{\text{ind. by } \widehat{\Gamma}} \mathsf{Alg}(\mathcal{M}^H) \xrightarrow{G} \mathcal{A}^H \tag{2.7.7.1}$$

$$(L, m_L, u_L) \mapsto \big(\widehat{\Gamma}(L),\, \widehat{\Gamma}(m_L)\widehat{\Gamma}_{L,L},\, \widehat{\Gamma}(u_L)\widehat{\Gamma}_0\big) \mapsto \Big(\widehat{\Gamma}(L),\, \widehat{\Gamma}(m_L)\widehat{\Gamma}_{L,L}U_{\widehat{\Gamma}(L),\widehat{\Gamma}(L)},\, \widehat{\Gamma}(u_L)\widehat{\Gamma}_0 U_0\Big).$$

Note that $L \in \mathsf{Alg}(L\text{-}\mathsf{Bicomod})$ via Δ_L. Then $\widehat{\Gamma}(L) = B^{op} \otimes L \otimes B = H$ as a $\mathbf{k}$-vector space. By Theorem 2.7.5, the right H-coaction on $\widehat{\Gamma}(L)$ coincides with Δ_H.

We claim that

$$\left(\widehat{\Gamma}(L),\ \widehat{\Gamma}(m_L)\widehat{\Gamma}_{L,L}U_{\widehat{\Gamma}(L),\widehat{\Gamma}(L)},\ \widehat{\Gamma}(u_L)\widehat{\Gamma}_0U_0\right) = (H, m_H, u_H),$$

that is, under the functor (2.7.7.1), L maps to H. We verify this as follows. To verify that $\widehat{\Gamma}(u_L)\widehat{\Gamma}_0U_0 = u_H$, notice that

$$\begin{aligned}
\widehat{\Gamma}(u_L)\widehat{\Gamma}_0U_0(1_{\mathbf{k}}) &\overset{(2.3.6.4)}{=} \widehat{\Gamma}(u_L)\widehat{\Gamma}_0(1_H)\\
&= \widehat{\Gamma}(u_L)\widehat{\Gamma}_0(1_B \otimes 1_L \otimes 1_B)\\
&= \widehat{\Gamma}(u_L)(1_B \otimes 1_{\mathbf{k}} \otimes 1_B)\\
&= (\mathrm{Id} \otimes u_L \otimes \mathrm{Id})(1_B \otimes 1_{\mathbf{k}} \otimes 1_B)\\
&= 1_B \otimes 1_L \otimes 1_B\\
&= 1_H.
\end{aligned}$$

To show that $\widehat{\Gamma}(m_L)\widehat{\Gamma}_{L,L}U_{\widehat{\Gamma}(L),\widehat{\Gamma}(L)} = m_H$, we first note that for any $X \in L$-$\mathsf{Bicomod}$, a straightforward computation shows,

$$U_{\widehat{\Gamma}(X),\widehat{\Gamma}(X)}((a \otimes x \otimes b) \otimes (a' \otimes x' \otimes b')) = (a \otimes x \otimes be^{(1)}) \mathbin{\overline{\otimes}} (a' \otimes x'_{[0]} \otimes (e^{(2)} \triangleleft x'_{[1]})b').$$

Therefore,

$$\begin{aligned}
&\widehat{\Gamma}(m_L)\widehat{\Gamma}_{L,L}U_{\widehat{\Gamma}(L),\widehat{\Gamma}(L)}((a \otimes x \otimes b) \otimes (a' \otimes x' \otimes b'))\\
&= \widehat{\Gamma}(m_L)\widehat{\Gamma}_{L,L}((a \otimes x \otimes be^{(1)}) \mathbin{\overline{\otimes}} (a' \otimes x'_{[0]} \otimes (e^{(2)} \triangleleft x'_{[1]})b'))\\
&= \widehat{\Gamma}(m_L)((x_{[-1]} \triangleright a')a \otimes x_{[0]} \otimes x'_{[0],[0]} \otimes ((be^{(1)}) \triangleleft x'_{[0],[1]})(e^{(2)} \triangleleft x'_{[1]})b')\\
&= \widehat{\Gamma}(m_L)\left((x_{[-1]} \triangleright a')a \otimes x_{[0]} \otimes x'_{[0]} \otimes ((be^{(1)}) \triangleleft x'_{[1],1})(e^{(2)} \triangleleft x'_{[1],2})b'\right)\\
&= \widehat{\Gamma}(m_L)\left((x_{[-1]} \triangleright a')a \otimes x_{[0]} \otimes x'_{[0]} \otimes ((be^{(1)}e^{(2)}) \triangleleft x'_{[1]})b'\right)\\
&\overset{(2.7.1.2)}{=} \widehat{\Gamma}(m_L)\left((x_{[-1]} \triangleright a')a \otimes x_{[0]} \otimes x'_{[0]} \otimes (b \triangleleft x'_{[1]})b'\right)\\
&= (\mathrm{Id} \otimes m_L \otimes \mathrm{Id})\left((x_{[-1]} \triangleright a')a \otimes x_{[0]} \otimes x'_{[0]} \otimes (b \triangleleft x'_{[1]})b'\right)\\
&= (x_{[-1]} \triangleright a')a \otimes x_{[0]}x'_{[0]} \otimes (b \triangleleft x'_{[1]})b'\\
&= (x_1 \triangleright a')a \otimes x_2x'_1 \otimes (b \triangleleft x'_2)b' \qquad \text{(since the bicoaction is given by } \Delta_L\text{)}
\end{aligned}$$

$$= m_H((a \otimes x \otimes b) \otimes (a' \otimes x' \otimes b')).$$

Example 2.7.8 ($X = \mathbb{1}_{L\text{-}\mathsf{Bicomod}} = \mathbf{k}$)**.** Again, take any L and B as in Notation 2.7.1, and let $H(L, B, \triangleleft)$ be the corresponding QTG. Then, $\mathbf{k} \in \mathsf{Alg}(L\text{-}\mathsf{Bicomod})$ via u_L. By Theorem 2.7.5 we get that $\widehat{\Gamma}(\mathbf{k}) \in \mathsf{Alg}(\mathcal{M}^{H(L,B,\triangleleft)})$ with $H(L, B, \triangleleft)$-coaction given by

$$\widehat{\Gamma}(\mathbf{k}) \to \widehat{\Gamma}(\mathbf{k}) \otimes H(L, B, \triangleleft), \quad (a \otimes 1_{\mathbf{k}} \otimes b) \mapsto (a \otimes 1_{\mathbf{k}} \otimes e^{(1)}) \otimes (e^{(2)} \otimes 1_{\mathbf{k}} \otimes b)$$

and

$$\widehat{\Gamma}(\mathbf{k}) \cong B^{op} \otimes B \quad \text{as } \mathbf{k}\text{-algebras.}$$

Indeed, the identification is given by $a \otimes 1_{\mathbf{k}} \otimes b$ corresponding to $a \otimes b$. Here,

$$\begin{aligned}
&G(m_{\widehat{\Gamma}(\mathbf{k})})[(a \otimes 1_{\mathbf{k}} \otimes b) \otimes (a' \otimes 1_{\mathbf{k}} \otimes b')] \\
&= m_{\widehat{\Gamma}(\mathbf{k})}[(a \otimes 1_{\mathbf{k}} \otimes b) \,\overline{\otimes}\, (a' \otimes 1_{\mathbf{k}} \otimes b')] \\
&\overset{2.3.5(\mathrm{i})}{=} \widehat{\Gamma}(m_{\mathbf{k}})\, \widehat{\Gamma}_{\mathbf{k},\mathbf{k}}[(a \otimes 1_{\mathbf{k}} \otimes b) \,\overline{\otimes}\, (a' \otimes 1_{\mathbf{k}} \otimes b')] \\
&= \widehat{\Gamma}(m_{\mathbf{k}})[a'a \otimes 1_{\mathbf{k}} \otimes 1_{\mathbf{k}} \otimes bb'] \\
&= a'a \otimes 1_{\mathbf{k}} \otimes bb',
\end{aligned}$$

$$G(u_{\widehat{\Gamma}(\mathbf{k})})(1_{\mathbf{k}}) = u_{\widehat{\Gamma}(\mathbf{k})}(1_B \otimes 1_{\mathbf{k}} \otimes 1_B) \overset{2.3.5(\mathrm{i})}{=} \widehat{\Gamma}(u_{\mathbf{k}})\, \widehat{\Gamma}_0(1_B \otimes 1_{\mathbf{k}} \otimes 1_B) \qquad = 1_B \otimes 1_{\mathbf{k}} \otimes 1_B.$$

Remark 2.7.9. Note from the example above that $\widehat{\Gamma}(\mathbb{1}_{L\text{-}\mathsf{Bicomod}}) \cong B^{op} \otimes B$, and by Section 2.3.2.2 and Lemma 2.7.3 that $\mathbb{1}_{\mathcal{M}^{H(L,B,\triangleleft)}} \cong B$. So, if the functor $\widehat{\Gamma}$ is strong monoidal, then $B = \mathbf{k}$.

We end this section with a non-trivial example of a QTG and the accompanying functor $\widehat{\Gamma}$ in Theorem 2.7.5.

Example 2.7.10. Let G be a finite group, and assume that $\mathbf{k}$ is a field such that $|G| \in \mathbf{k}^{\times}$. Take $L = \mathbf{k}G$ as a Hopf algebra, and also take $B = \mathbf{k}G$ as a strongly separable right L-module algebra via the adjoint action $b \triangleleft h = h^{-1}bh$ for $h, b \in G$. Here, a (symmetric) separability idempotent of B is given by $|G|^{-1}\sum_{g\in G} g\otimes g^{-1}$, and $\omega = \varepsilon_{\mathbf{k}G}$; in particular, the identities (2.7.1.5) and (2.7.1.6) hold. In this case, the QTG

$$H \coloneqq H(\mathbf{k}G, \mathbf{k}G, \triangleleft_{\mathrm{adj}})$$

is a weak Hopf algebra of dimension $|G|^3$ with structure determined by

$$\begin{aligned}
m_H((a\otimes h\otimes b)\otimes(a'\otimes h'\otimes b')) &= ha'h^{-1}a\otimes hh'\otimes h'^{-1}bh'b',\\
u_H(1_{\mathbf{k}}) &= 1_{\mathbf{k}G}\otimes 1_{\mathbf{k}G}\otimes 1_{\mathbf{k}G},\\
\Delta_H(a\otimes h\otimes b) &= |G|^{-1}\textstyle\sum_{g\in G}(a\otimes h\otimes g)\otimes(hg^{-1}h^{-1}\otimes h\otimes b),\\
\varepsilon(a\otimes h\otimes b) &= 1,\\
S(a\otimes h\otimes b) &= b\otimes h^{-1}\otimes a,
\end{aligned}$$

for $a, a', h, h', b, b' \in G$. Take $1_H \coloneqq u_H(1_{\mathbf{k}})$. Note that when G is non-trivial, we get that H is not a Hopf algebra as $\Delta_H(1_H) \neq 1_H\otimes 1_H$ in this case.

Now given a right G-comodule V, we may extend linearly so that V is a right $\mathbf{k}G$-comodule. Using the antipode, we obtain a left $\mathbf{k}G$-comodule structure on V, and since $\mathbf{k}G$ is cocommutative, this makes V a $\mathbf{k}G$-bicomodule. Further, we can extend the $\mathbf{k}G$-coactions to $V^{\otimes k}$ for all $k \geq 1$ to obtain a $\mathbf{k}G$-bicomodule structure on the tensor algebra $T(V)$ so that $T(V) \in \mathsf{Alg}(\mathbf{k}G\text{-}\mathsf{Bicomod})$. By Theorem 2.7.5, $(\mathbf{k}G)^{op}\otimes T(V)\otimes \mathbf{k}G \in \mathsf{Alg}(\mathcal{M}^H)$, and so by Theorem 2.4.3, it is also a right H-comodule algebra (i.e., in the category $\mathcal{A}^H$). Similarly, we can take any ideal I of $T(V)$ that is stable under the $\mathbf{k}G$-coactions on $T(V)$ to get that $A \coloneqq T(V)/I$ belongs to $\mathsf{Alg}({}^{\mathbf{k}G}\mathcal{M}^{\mathbf{k}G})$ and $(\mathbf{k}G)^{op}\otimes A\otimes \mathbf{k}G \in \mathsf{Alg}(\mathcal{M}^H)$ $(\cong \mathcal{A}^H)$.

Chapter 3

FROBENIUS-PERRON THEORY OF ENDOFUNCTORS

This chapter is based on joint work with Jianmin Chen, Zhibin Gao, James J. Zhang, Xiaohong Zhang, and Hong Zhu [CGW+19a]. Throughout let $\mathbf{k}$ be an algebraically closed field, and let everything be over $\mathbf{k}$.

The Frobenius-Perron dimension has become a crucial concept in the study of fusion categories and representations of semisimple weak Hopf algebras since it was introduced by Etingof-Nikshych-Ostrik [ENO05] (also see [EGNO15, EGO04, Nik04]). In this chapter, we ask the question: can we extend the notion of Frobenius-Perron dimension to other categories of interest to algebraists? We answer this question in the affirmative for any $\mathbf{k}$-linear category with a chosen endofunctor. In particular, we develop Frobenius-Perron type invariants to study abelian and derived categories. Abelian categories are the bread and butter of algebraists, as the category of representations of a ring is always an abelian category. Derived categories were first developed by Grothendieck in the context of algebraic geometry, but have since found their way to other fields, including noncommutative algebraic geometry and physics. It is our hope that this new invariant will aid those trying to understand these categories.

3.0.1 Definitions

The first goal is to introduce the Frobenius-Perron dimension of an endofunctor of a category. This definition was inspired by the definition of the FP dimension of an object in a fusion category, which we recall below.

Definition 3.0.1. [EGNO15, Def. 3.3.3, Prop. 4.5.4] Let $\mathcal{C}$ be a fusion category and let $\{X_1, \ldots, X_n\}$ be a complete list of non-isomorphic simples. By semisimplicity, for every object V of $\mathcal{C}$ and every j, we have a decomposition for some integers $a_{ij} \geq 0$:

$$V \otimes_{\mathcal{C}} X_j \cong \bigoplus_{i=1}^{n} X_i^{\oplus a_{ij}}.$$

The *Frobenius-Perron (FP) dimension of* V is defined to be

$$\operatorname{FPdim}(V) := \rho((a_{ij})_{n\times n}),$$

where ρ is the spectral radius.

We can interpret this definition in another way. Instead of thinking about the object V, we can think about the endofunctor $V \otimes_{\mathcal{C}} - : \mathcal{C} \to \mathcal{C}$. For each simple object, we examine the object it is mapped to under this endofunctor. We then study its decomposition into simples, take the data from there, and put it into a matrix. We can see that this definition requires the existence of simple objects as well as semisimplicity. But what can we do when our category does not have a notion of subobject in the first place?

Well, notice that

$$\operatorname{Hom}_{\mathcal{C}}(X_j, V \otimes_{\mathcal{C}} X_k) \cong \operatorname{Hom}_{\mathcal{C}}\left(X_j, \bigoplus_{i=1}^{n} X_i^{\oplus a_{ik}}\right) \cong \bigoplus_{i=1}^{n} \operatorname{Hom}_{\mathcal{C}}(X_j, X_i)^{\oplus a_{ik}} \cong \mathbf{k}^{\oplus a_{jk}},$$

where the last isomorphism is due to the fact that X_j, X_i are simple objects, so $\operatorname{Hom}(X_j, X_i) \cong \delta_{ji}\,\mathbf{k}$. Therefore, we can write

$$a_{jk} = \dim \operatorname{Hom}_{\mathcal{C}}(X_j, V \otimes_{\mathcal{C}} X_k).$$

Therefore, instead of defining the FP dimension using the decomposition into simples, we could have defined it by taking dimensions of Hom spaces. In this context, the fact that we needed about these simple objects was that $\operatorname{Hom}(X_j, X_i) \cong \delta_{ji}\,\mathbf{k}$. These observations led to our definition of Frobenius-Perron dimension for a category with a chosen endofunctor (see 3.2).

If an object V in a fusion category $\mathcal{C}$ is considered as the associated tensor endofunctor $V \otimes_{\mathcal{C}} -$, then our definition of the Frobenius-Perron dimension agrees with the definition given in [ENO05] (see Example 3.2.12 for details). Our definition applies to the derived category of projective schemes and finite dimensional algebras, as well as other abelian and additive categories (Definitions 4.1.11 and 3.2.5). In Section 3.2, we define the following invariants of an endofunctor:

Frobenius-Perron dimension (denoted by fpd, and fpd^n for $n \geq 1$),

Frobenius-Perron growth (denoted by fpg),

Frobenius-Perron curvature (denoted by fpv), and

Frobenius-Perron series (denoted by FP).

One can further define the above invariants for an abelian or a triangulated category. Note that the FP dimension/growth/curvature of a category can be a non-integer, see Proposition 3.5.12(1), Example 3.8.7, and Remark 3.5.13(5) for non-integral values of fpd, fpg, and fpv respectively.

If $\mathfrak{A}$ is an abelian category, let $D^b(\mathfrak{A})$ denote the bounded derived category of $\mathfrak{A}$. On the one hand, one might say that it is reasonable to call fpd a dimension function since

$$\mathrm{fpd}(D^b(\mathrm{Mod}-\mathbf{k}[x_1, \cdots, x_n])) = n$$

(Proposition 3.4.3(1)), but on the other hand, one might argue that fpd should not be called a dimension function since

$$\mathrm{fpd}(D^b(coh(\mathbb{P}^n))) = \begin{cases} 1, & n = 1 \\ \infty, & n \geq 2 \end{cases}$$

(Propositions 3.6.5 and 3.6.7). In the latter case, fpd is an indicator of representation type of the category of $coh(\mathbb{P}^n)$, namely, $coh(\mathbb{P}^n)$ is tame if $n = 1$, and is of wild representation type for all $n \geq 2$. A similar statement holds for projective curves in terms of genus (Proposition 3.6.5).

We also define the Frobenius-Perron ("fp") version of global dimension (Definition 3.2.8(1)) for any triangulated category. We remark that in general, the fpgldim A does not agree with the classical global dimension of A (Theorem 3.7.8).

Our second goal is to provide several applications.

3.0.2 Embeddings

As the following theorems show, the Frobenius-Perron dimension increases when the "size" of the endofunctors and categories increase.

Theorem 3.0.2 (Theorem 3.3.2)**.** *Suppose $\mathcal{C}$ and $\mathcal{D}$ are* $\mathbf{k}$*-linear categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a fully faithful functor. Let $\sigma_{\mathcal{C}}$ and $\sigma_{\mathcal{D}}$ be endofunctors of $\mathcal{C}$ and $\mathcal{D}$ respectively. Suppose that $F \circ \sigma_{\mathcal{C}}$ is naturally isomorphic to $\sigma_{\mathcal{D}} \circ F$. Then* $\mathrm{FP}(u, t, \sigma_{\mathcal{C}}) \leq \mathrm{FP}(u, t, \sigma_{\mathcal{D}})$.

By taking σ to be the suspension functor of a pre-triangulated category, we have the following immediate consequence. (Note that the fp-dimension of a triangulated category $\mathcal{T}$ is defined to be $\mathrm{fpd}(\Sigma)$, where Σ is the suspension of $\mathcal{T}$.)

Corollary 3.0.3. *Let $\mathcal{T}_2$ be a pre-triangulated category and $\mathcal{T}_1$ a full pre-triangulated subcategory of $\mathcal{T}_2$. Then the following hold.*

(1) $\mathrm{fpd}\, \mathcal{T}_1 \leq \mathrm{fpd}\, \mathcal{T}_2$.

(2) $\mathrm{fpg}\, \mathcal{T}_1 \leq \mathrm{fpg}\, \mathcal{T}_2$.

(3) $\mathrm{fpv}\, \mathcal{T}_1 \leq \mathrm{fpv}\, \mathcal{T}_2$.

(4) If $\mathcal{T}_2$ has fp-subexponential growth, so does $\mathcal{T}_1$.

Fully faithful embeddings of derived categories of projective schemes have been investigated in the study of Fourier-Mukai transforms, birational geometry, and noncommutative crepant resolutions (NCCRs) by Bondal-Orlov [BO01, BO02], Van den Bergh [VdB04], Bridgeland [Bri02], Bridgeland-King-Reid [BKR01] and more.

Note that if $\operatorname{fpgldim}(\mathcal{T}) < \infty$, then $\operatorname{fpg}(\mathcal{T}) = 0$. If $\operatorname{fpg}(\mathcal{T}) < \infty$, then $\operatorname{fpv}(\mathcal{T}) \leq 1$. Hence, fpd, fpgldim, fpg and fpv measure the "size", "representation type", or "complexity" of a triangulated category $\mathcal{T}$ at different levels. Corollary 3.0.3 has many consequences concerning non-existence of fully faithful embeddings provided that we compute the invariants fpd, fpg and fpv of various categories efficiently.

3.0.3 Tame vs wild

Here we mention a couple of more applications. First we extend the classical trichotomy on the representation types of quivers to the fpd.

Theorem 3.0.4. *Let Q be a finite quiver and let $\mathcal{Q}$ be the bounded derived category of finite dimensional left $\mathbf{k}Q$-modules.*

(1) $\mathbf{k}Q$ is of finite representation type if and only if $\operatorname{fpd}\mathcal{Q} = 0$.

(2) $\mathbf{k}Q$ is of tame representation type if and only if $\operatorname{fpd}\mathcal{Q} = 1$.

(3) $\mathbf{k}Q$ is of wild representation type if and only if $\operatorname{fpd}\mathcal{Q} = \infty$.

By the classical theorems of Gabriel [Gab72] and Nazarova [Naz73], the quivers of finite and tame representation types correspond to the ADE and $\widetilde{A}\widetilde{D}\widetilde{E}$ diagrams respectively.

The above theorem fails for quiver algebras with relations (Proposition 3.5.12). As we have already seen, fpd is related to the "size" of a triangulated category, as well as, the representation types. We will see soon that fpg is also closely connected with the complexity of representations. When we focus on the representation types, we make some tentative definitions.

Let $\mathcal{T}$ be a triangulated category (such as $D^b(\operatorname{Mod}_{f.d.}-A)$).

(i) We call $\mathcal{T}$ *fp-trivial*, if $\operatorname{fpd}\mathcal{T} = 0$.

(ii) We call $\mathcal{T}$ *fp-tame*, if $\operatorname{fpd}\mathcal{T} = 1$.

(iii) We call $\mathcal{T}$ *fp-potentially-wild*, if $\operatorname{fpd}\mathcal{T} > 1$. Further,

(iiia) $\mathcal{T}$ is *fp-finitely-wild*, if $1 < \operatorname{fpd}\mathcal{T} < \infty$.

(iiib) $\mathcal{T}$ is *fp-locally-finitely-wild*, if $\operatorname{fpd}\mathcal{T} = \infty$ and $\operatorname{fpd}^n(\mathcal{T}) < \infty$ for all n.

(iiic) $\mathcal{T}$ is *fp-wild*, if $\operatorname{fpd}^1\mathcal{T} = \infty$.

There are other notions of tame/wildness in representation theory, see for example, [GK02, Dro04]. Following the above definition, fpd provides a quantification of the tame-wild dichotomy. By Theorem 3.0.4, finite/tame/wild representation types of the path algebra $\mathbf{k}Q$ are equivalent to the fp-version of these properties of $\mathcal{Q}$. Let A be a quiver algebra with relations and let $\mathcal{A}$ be the derived category $D^b(\mathrm{Mod}_{f.d.}-A)$. Then, in general, finite/tame/wild representation types of A are NOT equivalent to the fp-version of these properties of $\mathcal{A}$ (Example 3.5.5). It is natural to ask:

Question 3.0.5. For which classes of algebras A, is the fp-wildness of $\mathcal{A}$ equivalent to the classical and other wildness of A in representation theory literature?

3.0.4 Complexity

The complexity of a module or of an algebra is an important invariant in studying representations of finite dimensional algebras [AE81, Car96, CDW94, GLW09]. Let A be the quiver algebra $\mathbf{k}Q/(R)$ with relations R. The *complexity* of A is defined to be the complexity of the A-module $T := A/Jac(A)$, namely,

$$cx(A) = cx(T) := \limsup_{n\to\infty} \log_n(\dim \operatorname{Ext}^n_A(T,T)) + 1.$$

Let GKdim denote the Gelfand-Kirillov dimension of an algebra (see [KL85] and [MR87, Ch. 13]). Under some reasonable hypotheses, one can show

$$cx(A) = \operatorname{GKdim}\left(\bigoplus_{n=0}^{\infty} \operatorname{Ext}^n_A(T,T)\right).$$

It is easy to see that $cx(A)$ is an derived invariant. We extend the definition of the complexity to any triangulated category (Definition 3.8.2(4)).

Theorem 3.0.6. *Let A be the algebra $\mathbf{k}Q/(R)$ with relations R and let $\mathcal{A}$ be the bounded derived category of finite dimensional left A-modules. Then*

$$\operatorname{fpg}(\mathcal{A}) \leq cx(A) - 1.$$

The equality $\operatorname{fpg}(\mathcal{A}) = cx(A) - 1$ holds under some hypotheses (Theorem 3.8.4(2)).

3.0.5 Frobenius-Perron function

If $\mathcal{T}$ is a triangulated category with Serre functor S, we have a fp-function

$$fp : \mathbb{Z}^2 \longrightarrow \mathbb{R}_{\geq 0}$$

which is defined by

$$fp(a,b) := \operatorname{fpd}(\Sigma^a \circ S^b) \in \mathbb{R}_{\geq 0}.$$

Then $\operatorname{fpd}(\mathcal{T})$ is the value of the fp-function at $(1,0)$.

The fp-function for the projective line $\mathbb{P}^1$ and the quiver A_2 are given in the Examples 3.5.1 and 3.5.4 respectively.

The statements in Theorem 3.0.4, Questions 3.0.5 and 3.7.11 indicate that $fp(1,0)$ predicts the representation type of $\mathcal{T}$ for certain triangulated categories. It is expected that values of the fp-function at other points in $\mathbb{Z}^2$ are sensitive to other properties of $\mathcal{T}$.

3.0.6 Other properties

The paper contains some basic properties of fpd, one of which we mention here.

Proposition 3.0.7 (Serre duality)**.** *Let $\mathcal{C}$ be a* Hom*-finite category with Serre functor S. Let σ be an endofunctor of $\mathcal{C}$.*

(1) If σ has a right adjoint $\sigma^!$, then

$$\operatorname{fpd}(\sigma) = \operatorname{fpd}(\sigma^! \circ S).$$

(2) If σ is an equivalence with quasi-inverse σ^{-1}, then

$$\operatorname{fpd}(\sigma) = \operatorname{fpd}(\sigma^{-1} \circ S).$$

(3) If $\mathcal{C}$ is n-Calabi-Yau, then we have a duality, for all i,

$$\operatorname{fpd}(\Sigma^i) = \operatorname{fpd}(\Sigma^{n-i}).$$

3.0.7 Computations

Our third goal is to develop methods for computation. To use fp-invariants, we need to compute as many examples as possible. In general it is extremely difficult to calculate useful invariants for derived categories, as the definitions of these invariants are quite sophisticated. We develop some techniques for computing fp-invariants. In Sections 3.4 to 3.8, we compute the fp-dimension for some non-trivial examples.

3.0.8 Conventions

(1) Usually Q means a quiver.

(2) $\mathcal{T}$ is a (pre)-triangulated category with suspension functor $\Sigma = [1]$.

(3) If A is an algebra over the base field $\mathbf{k}$, then $\operatorname{Mod}_{f.d.}-A$ denotes the category of finite dimensional left A-modules.

(4) If A is an algebra, then we use $\mathfrak{A}$ for the abelian category $\operatorname{Mod}_{f.d.}-A$.

(5) When $\mathfrak{A}$ is an abelian category, we use $\mathcal{A}$ for the bounded derived category $D^b(\mathfrak{A})$.

3.1 Preliminaries

3.1.1 Classical Definitions

Let A be an $n \times n$-matrix over complex numbers $\mathbb{C}$. The *spectral radius* of A is defined to be

$$\rho(A) := \max\{|r_1|, |r_2|, \cdots, |r_n|\} \quad \in \mathbb{R}$$

where $\{r_1, r_2, \cdots, r_n\}$ is the complete set of eigenvalues of A. When each entry of A is a positive real number, $\rho(A)$ is also called the *Perron root* or the *Perron-Frobenius eigenvalue* of A. When applying ρ to the adjacency matrix of a graph (or a quiver), the spectral radius of the adjacency matrix is sometimes called the *Frobenius-Perron dimension* of the graph (or the quiver).

Let us mention a classical result concerning the spectral radius of simple graphs. A finite graph G is called *simple* if it has no loops and no multiple edges. Smith [Smi70] formulated the following result:

Theorem 3.1.1. *[DGF+13, Theorem 1.3] Let G be a finite, simple, and connected graph with adjacency matrix A.*

(1) $\rho(A) = 2$ if and only if G is one of the extended Dynkin diagrams of type $\widetilde{A}\widetilde{D}\widetilde{E}$.

(2) $\rho(A) < 2$ if and only if G is one of the Dynkin diagrams of type ADE.

We refer to [DGF+13] and [HPR80] for the diagrams of the ADE and $\widetilde{A}\widetilde{D}\widetilde{E}$ quivers/graphs.

In order to include some infinite-dimensional cases, we extend the definition of the spectral radius in the following way. Let $A := (a_{ij})_{n\times n}$ be an $n \times n$-matrix with entries a_{ij} in $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. Define $A' = (a'_{ij})_{n\times n}$ where

$$a'_{ij} = \begin{cases} a_{ij} & a_{ij} \neq \pm\infty, \\ x_{ij} & a_{ij} = \infty, \\ -x_{ij} & a_{ij} = -\infty. \end{cases}$$

In other words, we are replacing ∞ in the (i,j)-entry by a finite real number, called x_{ij}, in the (i,j)-entry. And every x_{ij} is considered as a variable or a function mapping $\mathbb{R} \to \mathbb{R}$.

Definition 3.1.2. Let A be an $n \times n$-matrix with entries in $\overline{\mathbb{R}}$. The *spectral radius* of A is defined to be

$$\rho(A) := \liminf_{\text{all } x_{ij} \to \infty} \rho(A') \quad \in \overline{\mathbb{R}}. \tag{3.1.2.1}$$

Remark 3.1.3. It also makes sense to use $\limsup$ instead of $\liminf$ in (3.1.2.1). We choose to take $\liminf$ in this paper.

Here is an easy example.

Example 3.1.4. Let $A = \begin{pmatrix} 1 & -\infty \\ 0 & 2 \end{pmatrix}$. Then $A' = \begin{pmatrix} 1 & -x_{12} \\ 0 & 2 \end{pmatrix}$. It is obvious that

$$\rho(A) = \lim_{x_{12} \to \infty} \rho(A') = \lim_{x_{12} \to \infty} 2 = 2.$$

3.1.2 **k**-*linear categories*

If $\mathcal{C}$ is a $\mathbf{k}$-linear category, then $\operatorname{Hom}_{\mathcal{C}}(M,N)$ is a $\mathbf{k}$-module for all objects M,N in $\mathcal{C}$. If $\mathcal{C}$ is also abelian, then $\operatorname{Ext}^i_{\mathcal{C}}(M,N)$ are $\mathbf{k}$-modules for all $i \geq 0$. Let dim be the $\mathbf{k}$-vector space dimension.

Remark 3.1.5. One can use a dimension function other than dim. Even when a category $\mathcal{C}$ is not $\mathbf{k}$-linear, it might still make sense to define some other dimension function on the Hom-sets of the category $\mathcal{C}$. The definition of Frobenius-Perron dimension given in the next section can be modified to fit this kind of dim.

3.1.3 Frobenius-Perron dimension of a quiver

In this subsection we recall some known elementary definitions and facts.

Definition 3.1.6. Let Q be a quiver.

(1) If Q has finitely many vertices, then the *Frobenius-Perron dimension* of Q is defined to be

$$\operatorname{fpd} Q := \rho(A(Q))$$

where $A(Q)$ is the adjacency matrix of Q.

(2) Let Q be any quiver. The *Frobenius-Perron dimension* of Q is defined to be

$$\operatorname{fpd} Q := \sup\{\operatorname{fpd} Q'\}$$

where Q' runs over all finite subquivers of Q.

See [ES11, Propositions 2.1 and 3.2] for connections between fpd of a quiver and its representation types, as well as its complexity.

We need the following well-known facts in linear algebra.

Lemma 3.1.7. *(1) Let B be a square matrix with nonnegative entries and let A be a principal minor of B. Then $\rho(A) \leq \rho(B)$.*

(2) Let $A := (a_{ij})_{n\times n}$ and $B := (b_{ij})_{n\times n}$ be two square matrices such that $0 \leq a_{ij} \leq b_{ij}$ for all i, j. Then $\rho(A) \leq \rho(B)$.

Let Q be a quiver with vertices $\{v_1, \cdots, v_n\}$. An oriented cycle based at a vertex v_i is called *indecomposable* if it is not a product of two oriented cycles based at v_i. For each vertex v_i let θ_i be the number of indecomposable oriented cycles based at v_i. Define the *cycle number* of a quiver Q to be

$$\Theta(Q) := \max\{\theta_i \mid \forall\ i\}.$$

The following result is well-known.

Theorem 3.1.8. *Let Q be a quiver and let $\Theta(Q)$ be the cycle number of Q.*

(1) $\operatorname{fpd}(Q) = 0$ *if and only if* $\Theta(Q) = 0$*, namely, Q is acyclic.*

(2) $\operatorname{fpd}(Q) = 1$ *if and only if* $\Theta(Q) = 1$.

(3) $\operatorname{fpd}(Q) > 1$ *if and only if* $\Theta(Q) \geq 2$.

3.1.4 Abelian and derived categories

We start with some preliminary definitions of additive, abelian, derived, and triangulated categories.

Definition 3.1.9. [Wei94, Defn. 1.2.2] Let $\mathcal{A}$ be a category. We call $\mathcal{A}$ an *abelian category* if the following conditions hold.

(i) For any two objects $A, B \in \mathcal{A}$, the set of morphisms $\mathrm{Hom}_{\mathcal{A}}(A, B)$ is given the structure of an abelian group such that composition distributes over addition. In particular, given a diagram in $\mathcal{A}$ of the form

$$A \xrightarrow{f} B \underset{g}{\overset{g'}{\rightrightarrows}} C \xrightarrow{h} D$$

we have $h(g + g')f = hgf + hg'f$ in $\mathrm{Hom}_{\mathcal{A}}(A, D)$.

(ii) For any $A, B \in \mathcal{A}$, the product $A \times B$ exists.

(iii) $\mathcal{A}$ has a zero object (one that is both initial and terminal).

(iv) Every map in $\mathcal{A}$ has a kernel and a cokernel.

(v) Every monic in $\mathcal{A}$ is the kernel of its cokernel.

(vi) Every epi in $\mathcal{A}$ is the cokernel of its kernel.

If only Items (i) and (ii) are satisfied, we call $\mathcal{C}$ an *additive category*.

Practically speaking, we can think of abelian categories as full subcategories of the category of modules over a ring via the following theorem.

Theorem 3.1.10. *[Wei94, Freyd-Mitchell Embedding Theorem 1.6.1] If $\mathcal{A}$ is a small abelian category, then there exists a ring R and an exact, fully faithful functor from $\mathcal{A}$ into R-*Mod*, which embeds $\mathcal{A}$ as a full subcategory in the sense that* $\mathrm{Hom}_{\mathcal{A}}(M, N) \cong \mathrm{Hom}_R(M, N)$.

Before defining the derived category, we need to define the category of chain complexes of an abelian category.

Definition 3.1.11. Let $\mathcal{A}$ be an abelian category. We will define $\mathrm{Ch}(\mathcal{A})$, the category of chain complexes of $\mathcal{A}$. The objects consist of a family $\{X_n\}_{n\in\mathbb{Z}}$ of objects in $\mathcal{A}$, together with a collection of maps $d_n : X_n \to X_{n-1}$ in $\mathcal{A}$ that have the property $d_n \circ d_{n+1} = 0$ for all $n \in \mathbb{Z}$. Such an object X can be written in the following manner:

$$X = \cdots \longrightarrow X_{n+2} \xrightarrow{d_{n+2}} X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \xrightarrow{d_{n-1}} X_{n-2} \longrightarrow \cdots$$

A morphism $f : X \to Y$ is a collection of maps $\{f_n : X_n \to Y_n\}_{n\in\mathbb{Z}}$ in $\mathcal{A}$ that commute with the d maps. That is, the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & X_{n+2} \arrow[r, "d_{n+2}"] \arrow[d, "f_{n+2}"] & X_{n+1} \arrow[r, "d_{n+1}"] \arrow[d, "f_{n+1}"] & X_n \arrow[r, "d_n"] \arrow[d, "f_n"] & X_{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "f_{n-1}"] & X_{n-2} \arrow[r] \arrow[d, "f_{n-2}"] & \cdots \\
\cdots \arrow[r] & Y_{n+2} \arrow[r, "d_{n+2}"] & Y_{n+1} \arrow[r, "d_{n+1}"] & Y_n \arrow[r, "d_n"] & Y_{n-1} \arrow[r, "d_{n-1}"] & Y_{n-2} \arrow[r] & \cdots
\end{tikzcd}$$

We use $\mathrm{Ch}^b(\mathcal{A})$ to denote the full subcategory of $\mathrm{Ch}(\mathcal{A})$ whose objects consist of bounded chain complexes (complexes X for which there exist $m \le n$ in $\mathbb{Z}$ such that $X_i = 0$ for $i > n, i < m$).

We have a special endofunctor $\Sigma : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})$ which we call the suspension. If $X \in \mathrm{Ch}(\mathcal{A})$, we define $\Sigma(X)_n = X_{n+1}$. We can think of this as shifting the chain complex X one unit to the right. If $f : X \to Y$ is a morphism in $\mathrm{Ch}(\mathcal{A})$, we can define $\Sigma(f)_n : \Sigma(X)_n \to \Sigma(X)_n$ to be $f_{n+1} : X_{n+1} \to X_{n+1}$. Clearly, Σ is an automorphism of $\mathrm{Ch}(\mathcal{A})$. The suspension of $\mathrm{Ch}^b(\mathcal{A})$ is defined analagously.

It can be checked that both $\mathrm{Ch}(\mathcal{A}), \mathrm{Ch}^b(\mathcal{A})$ are abelian categories.

Definition 3.1.12. Let C, D be objects of $\mathrm{Ch}(\mathcal{A})$. The nth *homology* of C is defined to be the quotient $H_n(C) := \ker d_n / \operatorname{Im} d_{n+1}$. It can be checked that a morphism $f : C \to D$ induces a morphism $H_n(C) \to H_n(D)$ for each n [Wei94, Ex. 1.1.2].

A morphism $C \to D$ of chain complexes is called a *quasi-isomorphism* if the induced maps $H_n(C) \to H_n(D)$ are all isomorphisms.

Definition 3.1.13. [Wei94, Ch. 10] Let $\mathcal{C}$ be an abelian category. Let S be a collection of morphisms in $\mathcal{C}$. Under certain conditions, we can define a category $S^{-1}\mathcal{C}$, called the *localization of* $\mathcal{C}$ *with respect to* S, as follows.

The objects of $S^{-1}\mathcal{C}$ are the objects of $\mathcal{C}$.

We define a left fraction $fs^{-1} : X \to Y$ to be the following data, where $s \in S$ and f is a morphism in $\mathcal{C}$:

$$X \xleftarrow{s} X_1 \xrightarrow{f} Y$$

We call $fs^{-1} : X \to Y$ equivalent to $gt^{-1} : X \to Y$ if there exists another left fraction $X \leftarrow X_3 \to Y$ that fits into a commutative diagram in $\mathcal{C}$:

$$\begin{tikzcd} & X_1 \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & X_3 \arrow[l] \arrow[r] \arrow[u] \arrow[d] & Y \\ & X_2 \arrow[ul, "t"] \arrow[ur, "g"'] & \end{tikzcd}$$

It can be shown that this is an equivalence relation. Then we define the morphisms of $S^{-1}\mathcal{C}$ to be equivalence classes of left fractions of morphisms of $\mathcal{C}$.

Definition 3.1.14. We define the *derived category* $D(\mathcal{A})$ to be the localization of $\mathrm{Ch}(\mathcal{A})$ with respect to the collection of quasi-isomorphisms.

We define the *bounded derived category* $D^b(\mathcal{A})$ to be the localization of $\mathrm{Ch}^b(\mathcal{A})$ with respect to the collection of quasi-isomorphisms.

Since $\Sigma : \mathrm{Ch}(\mathcal{A}) \to \mathrm{Ch}(\mathcal{A})$ is exact and preserves quasi-isomorphisms, by [Wei94, 10.5.2] there exists a functor extending Σ to the derived category called the *suspension* (which we will also denote $\Sigma : D(\mathcal{A}) \to D(\mathcal{A})$). The suspension of $D^b(\mathcal{A})$ is defined analagously.

Remark 3.1.15. The derived and bounded derived categories exist and satisfy the universal property of localization of a category by the Gabriel-Zisman Theorem [Wei94, Thm. 10.3.7].

The derived category of an abelian category is a special case of a triangulated category. First we need to define the notion of a triangle in a category. Essentially, triangles are replacements for short exact sequences. For example, the derived category is not abelian, so we cannot define kernels or cokernels and therefore we cannot define exact sequences. However, we can construct objects called distinguished triangles which yield similar homological data.

Definition 3.1.16. [Wei94, Ch. 10.2] Suppose $\mathcal{K}$ is a category equipped with an automorphism $T : \mathcal{K} \to \mathcal{K}$. A *triangle* on an ordered triple of (A, B, C) of objects of $\mathcal{K}$ is a triple $(u : A \to B, v : B \to C, w : C \to T(A))$ of morphisms. A triangle is usually displayed as follows:

$$\begin{tikzcd} & C \arrow[dl, "w"'] & \\ A \arrow[rr, "u"'] & & B \arrow[ul, "v"'] \end{tikzcd}$$

A morphism of triangles is a triple $(f : A \to A', g : B \to B', h : C \to C')$ forming a commutative diagram:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, "h"] & T(A) \arrow[d, "T(f)"] \\ A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & T(A') \end{tikzcd}$$

An isomorphism of triangles (f, g, h) is a morphism of triangles where f, g, h are isomorphisms in $\mathcal{K}$.

Definition 3.1.17. [Wei94, Defn. 10.2.1] An additive category $\mathcal{K}$ is called a *triangulated category* if it is equipped with an automorphism $T : \mathcal{K} \to \mathcal{K}$ (called the suspension or translation functor) and with a distinguished family of triangles (u, v, w) (called exact triangles) subject to the following axioms:

(TR1) Every morphism $u : A \to B$ is part of an exact triangle (u, v, w). If $A = B$ and $C = 0$, the triangle $(\mathrm{Id}_A, 0, 0)$ is exact. If (u, v, w) is a triangle on (A, B, C) isomorphic to an exact triangle (u', v', w') on (A', B', C'), then (u, v, w) is also an exact triangle.

(TR2) (Rotation). If (u, v, w) is an exact triangle on (A, B, C), then both of its rotates $(v, w, -T(u))$ and $(-T^{-1}(w), u, v)$ are exact triangles on $(B, C, T(A))$ and $(T^{-1}(C), A, B)$ respectively.

(TR3) (Morphisms). Given two exact triangles

$$\begin{tikzcd} & C \arrow[dl, "w"'] & \\ A \arrow[rr, "u"'] & & B \arrow[ul, "v"'] \end{tikzcd} \qquad \begin{tikzcd} & C' \arrow[dl, "w'"'] & \\ A' \arrow[rr, "u'"'] & & B' \arrow[ul, "v'"'] \end{tikzcd}$$

with morphisms $f : A \to A', g : B \to B'$ such that $gu = u'f$, there exists a morphism $h : C \to C'$ so that (f, g, h) is a morphism of triangles:

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[r, "v"] \arrow[d, "g"] & C \arrow[r, "w"] \arrow[d, dashed, "\exists h"] & T(A) \arrow[d, "T(f)"] \\ A' \arrow[r, "u'"] & B' \arrow[r, "v'"] & C' \arrow[r, "w'"] & T(A') \end{tikzcd}$$

(TR4) (The octahedral axiom). Given objects A, A', B, B', C, C' in $\mathcal{K}$, suppose there are three exact triangles: (u, j, ∂) on (A, B, C'); (v, x, i) on (B, C, A'); (vu, y, δ) on (A, C, B'). Then there is a fourth exact triangle $(f, g, T(j)i)$ on (C', B', A') such that in the following octahedron: (1) the four exact triangles form four of the faces; (2) the remaining four faces commute (that is, $\partial = \delta f$, $x = gy$); (3) $yv = fj$; and (4) $u\delta = ig$.

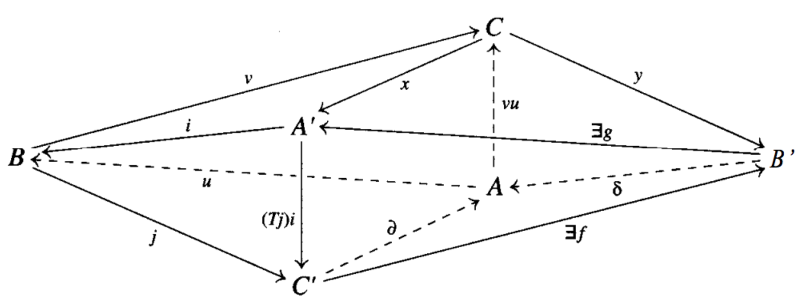

Figure 3.1: [Wei94, p. 375]

Definition 3.1.18. A *pre-triangulated category* is an additive category $\mathcal{K}$ equipped with an automorphism $T : \mathcal{K} \to \mathcal{K}$ (called the suspension or translation functor) and with a distinguished family of triangles (u, v, w) (called exact triangles) subject to (TR1), (TR2), and (TR3).

Clearly a triangulated category is pre-triangulated. It is an open problem as to whether pre-triangulated is equivalent to triangulated. That is, is the octahedral axiom (TR4) a consequence of the other three axioms?

Remark 3.1.19. The derived categories $D(\mathcal{A}), D^b(\mathcal{A})$ are triangulated (hence pre-triangulated) categories. In this case $T := \Sigma$, the suspension functor. The triangles in $D(\mathcal{A}), D^b(\mathcal{A})$ are related to short exact sequences in an intermediate category $\mathbf{K}$ involving mapping cones [Wei94, Defn. 10.1.3].

Definition 3.1.20. [EGNO15, Defn. 1.2.2, 1.2.3] A $\mathbf{k}$-linear category $\mathcal{C}$ is an additive category such that for any objects $X, Y \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ has the structure of a $\mathbf{k}$-vector space, and composition of morphisms respects this $\mathbf{k}$-linear structure. A $\mathbf{k}$-linear endofunctor of $\mathcal{C}$ is an additive functor from $\mathcal{C} \to \mathcal{C}$ that respects the vector space structure on Hom-sets.

3.2 Definitions

Notation 3.2.1. Throughout the rest of the paper, let $\mathcal{C}$ denote a $\mathbf{k}$-linear category. A functor between two $\mathbf{k}$-linear categories is assumed to preserve the $\mathbf{k}$-linear structure. For simplicity, $\dim(A, B)$ stands for $\dim \mathrm{Hom}_{\mathcal{C}}(A, B)$ for any objects A and B in $\mathcal{C}$.

The set of finite subsets of nonzero objects in $\mathcal{C}$ is denoted by Φ and the set of subsets of n nonzero objects in $\mathcal{C}$ is denoted by Φ_n for each $n \geq 1$. It is clear that $\Phi = \bigcup_{n \geq 1} \Phi_n$. We do not consider the empty set as an element of Φ.

Definition 3.2.2. Let $\phi := \{X_1, X_2, \cdots, X_n\}$ be a finite subset of nonzero objects in $\mathcal{C}$, namely, $\phi \in \Phi_n$. Let σ be an endofunctor of $\mathcal{C}$.

(1) The *adjacency matrix* of (ϕ, σ) is defined to be

$$A(\phi, \sigma) := (a_{ij})_{n \times n}, \quad \text{where } a_{ij} := \dim(X_i, \sigma(X_j)) \ \ \forall i, j.$$

(2) An object M in $\mathcal{C}$ is called a *brick* [ASS06, Definition 2.4, Ch. VII] if

$$\mathrm{Hom}_{\mathcal{C}}(M, M) = \mathbf{k}\,.$$

If $\mathcal{C}$ is a pre-triangulated category, an object M in $\mathcal{C}$ is called an *atomic* object if it is a brick and satisfies

$$\operatorname{Hom}_{\mathcal{C}}(M, \Sigma^{-i}(M)) = 0, \quad \forall\, i > 0. \tag{3.2.2.1}$$

(3) $\phi \in \Phi$ is called a *brick set* (respectively, an *atomic set*) if each X_i is a brick (respectively, atomic) and

$$\dim(X_i, X_j) = \delta_{ij}$$

for all $1 \le i, j \le n$. The set of brick (respectively, atomic) n-object subsets is denoted by $\Phi_{n,b}$ (respectively, $\Phi_{n,a}$). We write $\Phi_b = \bigcup_{n\ge1} \Phi_{n,b}$ (respectively, $\Phi_a = \bigcup_{n\ge1} \Phi_{n,a}$). Define the *b-height* of $\mathcal{C}$ to be

$$h_b(\mathcal{C}) = \sup\{n \mid \Phi_{n,b} \text{ is nonempty}\}$$

and the *a-height* of $\mathcal{C}$ (when $\mathcal{C}$ is pre-triangulated) to be

$$h_a(\mathcal{C}) = \sup\{n \mid \Phi_{n,a} \text{ is nonempty}\}.$$

Remark 3.2.3. (1) A brick may not be atomic. Let A be the algebra

$$\mathbf{k}\langle x, y\rangle/(x^2, y^2 - 1, xy + yx).$$

This is a 4-dimensional Frobenius algebra (of injective dimension zero). There are two simple left A-modules

$$S_0 := A/(x, y-1), \quad \text{and} \quad S_1 := A/(x, y+1).$$

Let M_i be the injective hull of S_i for $i = 0, 1$. (Since A is Frobenius, M_i is projective.) There are two short exact sequences

$$0 \longrightarrow S_0 \longrightarrow M_0 \xrightarrow{f} S_1 \longrightarrow 0$$

and

$$0 \longrightarrow S_1 \xrightarrow{g} M_1 \longrightarrow S_0 \longrightarrow 0.$$

It is easy to check that $\mathrm{Hom}_A(M_i, M_j) = \mathbf{k}$ for all $0 \le i, j \le 1$. Let $\mathcal{A}$ be the derived category $D^b(\mathrm{Mod}_{f.d.}\text{-}A)$ and let X be the complex

$$\cdots \longrightarrow 0 \longrightarrow M_0 \xrightarrow{g\circ f} M_1 \longrightarrow 0 \longrightarrow \cdots$$

An easy computation shows that $\mathrm{Hom}_{\mathcal{A}}(X, X) = \mathbf{k} = \mathrm{Hom}_{\mathcal{A}}(X, X[-1])$. So X is a brick, but not atomic.

(2) A brick object is called a *schur* object by several authors, see [CC15] and [CKW15]. It is also called *endo-simple* by others, see [vR08, vR16].

(3) An atomic object in a triangulated category is close to being a point-object defined by Bondal-Orlov [BO01, Definition 2.1]. A point-object was defined on a triangulated category with Serre functor. In this paper we do not automatically assume the existence of a Serre functor in general. When $\mathcal{C}$ is not a pre-triangulated category, we can not even ask for (3.2.2.1). In that case we can only talk about bricks.

Definition 3.2.4. Retain the notation as in Definition 4.1.10, and we use Φ_b as the testing objects. When $\mathcal{C}$ is a pre-triangulated category, Φ_b is automatically replaced by Φ_a unless otherwise stated.

(1) The *nth Frobenius-Perron dimension* of σ is defined to be

$$\mathrm{fpd}^n(\sigma) := \sup_{\phi\in\Phi_{n,b}} \{\rho(A(\phi,\sigma))\}.$$

If $\Phi_{n,b}$ is empty, then by convention, $\mathrm{fpd}^n(\sigma) = 0$.

(2) The *Frobenius-Perron dimension* of σ is defined to be

$$\mathrm{fpd}(\sigma) := \sup_n \{\mathrm{fpd}^n(\sigma)\} = \sup_{\phi\in\Phi_b}\{\rho(A(\phi,\sigma))\}.$$

(3) The *Frobenius-Perron growth* of σ is defined to be

$$\mathrm{fpg}(\sigma) := \sup_{\phi\in\Phi_b}\{\limsup_{n\to\infty} \log_n(\rho(A(\phi,\sigma^n)))\}.$$

By convention, $\log_n 0 = -\infty$.

(4) The *Frobenius-Perron curvature* of σ is defined to be

$$\operatorname{fpv}(\sigma) := \sup_{\phi \in \Phi_b} \{ \limsup_{n\to\infty} (\rho(A(\phi, \sigma^n)))^{1/n} \}.$$

This is motivated by the concept of the *curvature* of a module over an algebra due to Avramov [Avr98].

(5) We say σ has *fp-exponential growth* (respectively, *fp-subexponential growth*) if $\operatorname{fpv}(\sigma) > 1$ (respectively, $\operatorname{fpv}(\sigma) \le 1$).

Sometimes we prefer to have all information from the Frobenius-Perron dimension. We make the following definition.

Definition 3.2.5. Let $\mathcal{C}$ be a category and σ be an endofunctor of $\mathcal{C}$.

(1) The *Frobenius-Perron theory* (or fp-theory) of σ is defined to be the set

$$\{\operatorname{fpd}^n(\sigma^m)\}_{n\ge 1, m\ge 0}.$$

(2) The *Frobenius-Perron series* (or fp-series) of σ is defined to be

$$\operatorname{FP}(u,t,\sigma) := \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \operatorname{fpd}^n(\sigma^m) t^m u^n.$$

Remark 3.2.6. To define the Frobenius-Perron dimension, one only needs have an assignment $\tau : \Phi_n \to M_{n\times n}(\mathrm{Mod}-\mathbf{k})$, for every $n \ge 1$, satisfying the property that

if ϕ_1 is a subset of ϕ_2, then $\tau(\phi_1)$ is a principal submatrix of $\tau(\phi_2)$.

Then we define the adjacency matrix of $\phi \in \Phi_n$ to be

$$A(\phi, \tau) = (a_{ij})_{n\times n}$$

where

$$a_{ij} = \dim\,(\tau(\phi))_{ij} \quad \forall i, j.$$

Then the Frobenius-Perron dimension of τ is defined in the same way as in Definition 4.1.11. If there is a sequence of τ_m, the Frobenius-Perron series of $\{\tau_m\}$ is defined in the same way as in Definition 3.2.5 by replacing σ^m by τ_m. See Example 3.2.7 next.

Example 3.2.7. (1) Let $\mathfrak{A}$ be a $\mathbf{k}$-linear abelian category. For each $m \geq 1$ and $\phi = \{X_1, \cdots, X_n\}$, define

$$E^m : \phi \longrightarrow \left(\mathrm{Ext}^m_{\mathfrak{A}}(X_i, X_j)\right)_{n\times n}.$$

By convention, let $\mathrm{Ext}^0_{\mathfrak{A}}(X_i, X_j)$ denote $\mathrm{Hom}^0_{\mathfrak{A}}(X_i, X_j)$. Then, for each $m \geq 0$, one can define the Frobenius-Perron dimension of E^m as mentioned in Remark 3.2.6.

(2) Let $\mathfrak{A}$ be the $\mathbf{k}$-linear abelian category $\mathrm{Mod}_{f.d.}-A$ where A is a finite dimensional commutative algebra over a base field $\mathbf{k}$. For each $m \geq 1$ and $\phi = \{X_1, \cdots, X_n\}$, define

$$T_m : \phi \longrightarrow \left(\mathrm{Tor}^A_m(X_i, X_j)\right)_{n\times n}.$$

By convention, let $\mathrm{Tor}^A_0(X_i, X_j)$ denote $X_i \otimes_A X_j$. Then, for each $m \geq 0$, one can define the Frobenius-Perron dimension of T_m as mentioned in Remark 3.2.6.

Definition 3.2.8. (1) Let $\mathfrak{A}$ be an abelian category. The *Frobenius-Perron dimension* of $\mathfrak{A}$ is defined to be

$$\mathrm{fpd}\,\mathfrak{A} := \mathrm{fpd}(E^1)$$

where $E^1 := \mathrm{Ext}^1_{\mathfrak{A}}(-,-)$ is defined as in Example 3.2.7(1). The *Frobenius-Perron theory* of $\mathfrak{A}$ is the collection

$$\{\mathrm{fpd}^m(E^n)\}_{m\geq 1, n\geq 0}$$

where $E^n := \mathrm{Ext}^n_{\mathcal{A}}(-,-)$ is defined as in Example 3.2.7(1).

(2) Let $\mathcal{T}$ be a pre-triangulated category with suspension Σ. The *Frobenius-Perron dimension* of $\mathcal{T}$ is defined to be

$$\mathrm{fpd}\,\mathcal{T} := \mathrm{fpd}(\Sigma).$$

The *Frobenius-Perron theory* of $\mathcal{T}$ is the collection

$$\{\operatorname{fpd}^m(\Sigma^n)\}_{m\geq 1, n\in\mathbb{Z}}.$$

The *fp-global dimension* of $\mathcal{T}$ is defined to be

$$\operatorname{fpgldim}\mathcal{T} := \sup\{n \mid \operatorname{fpd}(\Sigma^n) \neq 0\}.$$

If $\mathcal{T}$ possesses a Serre functor S, the *Frobenius-Perron S-theory* of $\mathcal{T}$ is the collection

$$\{\operatorname{fpd}^m(\Sigma^n \circ S^w)\}_{m\geq 1, n,w\in\mathbb{Z}}.$$

Remark 3.2.9. (1) The Frobenius-Perron dimension (respectively, Frobenius-Perron theory, fp-global dimension) can be defined for suspended categories [KV87] and pre-n-angulated categories [GKO13] in the same way as Definition 3.2.8(2) since there is a suspension functor Σ.

(2) When $\mathfrak{A}$ is an abelian category, another way of defining the Frobenius-Perron dimension $\operatorname{fpd}\mathfrak{A}$ is as follows. We first embed $\mathfrak{A}$ into the derived category $D^b(\mathfrak{A})$. The suspension functor Σ of $D^b(\mathfrak{A})$ maps $\mathfrak{A}$ to $\mathfrak{A}[1]$ (so it is not an endofunctor of $\mathfrak{A}$). The adjacency matrix $A(\phi, \Sigma)$ is still defined as in Definition 4.1.10(1) for brick sets ϕ in $\mathfrak{A}$. Then we can define

$$\operatorname{fpd}(\Sigma\mid_{\mathfrak{A}}) := \sup_{\phi\in\Phi_b, \phi\subset\mathfrak{A}}\{\rho(A(\phi,\Sigma))\}$$

as in Definition 4.1.11(2) by considering only the brick sets in $\mathfrak{A}$. Now $\operatorname{fpd}(\mathfrak{A})$ agrees with $\operatorname{fpd}(\Sigma\mid_{\mathfrak{A}})$.

Lemma 3.2.10. *Let $\mathfrak{A}$ be an abelian category and $n \geq 1$. Then* $\operatorname{fpd}^n(D^b(\mathfrak{A})) \geq \operatorname{fpd}^n(\mathfrak{A})$. *A similar statement holds for* fpd, fpg *and* fpv.

Proof. This follows from the fact that there is a fully faithful embedding $\mathfrak{A} \to D^b(\mathfrak{A})$ and that E^1 on $\mathfrak{A}$ agrees with Σ on $D^b(\mathfrak{A})$. □

For any category $\mathcal{C}$ with an endofunctor σ, we define the *σ-quiver* of $\mathcal{C}$, denoted by $Q^{\sigma}_{\mathcal{C}}$, as follows:

(1) the vertex set of $Q^{\sigma}_{\mathcal{C}}$ consists of bricks in $\Phi_{1,b}$ in $\mathcal{C}$ (respectively, atomic objects in $\Phi_{1,a}$ when $\mathcal{C}$ is pre-triangulated), and

(2) the arrow set of $Q^{\sigma}_{\mathcal{C}}$ consists of $n_{X,Y}$-arrows from X to Y, for all $X, Y \in \Phi_{1,b}$ (respectively, in $\Phi_{1,a}$), where $n_{X,Y} = \dim(X, \sigma(Y))$.

If σ is E^1, this quiver is denoted by $Q^{E^1}_{\mathcal{C}}$, which will be used in later sections.

The following lemma follows from the definition.

Lemma 3.2.11. *Retain the above notation. Then* $\operatorname{fpd}\sigma \leq \operatorname{fpd} Q^{\sigma}_{\mathcal{C}}$.

The fp-theory was motivated by the Frobenius-Perron dimension of objects in tensor or fusion categories [EGNO15], as shown in the following example.

Example 3.2.12. First we recall the definition of the Frobenius-Perron dimension given in [EGNO15, Definitions 3.3.3 and 6.1.6]. Let $\mathcal{C}$ be a fusion category. Suppose that $\{X_1, \cdots, X_n\}$ is the complete list of non-isomorphic simple objects in $\mathcal{C}$. Since $\mathcal{C}$ is semisimple, every object X in $\mathcal{C}$ is a direct sum

$$X = \bigoplus_{i=1}^{n} X_i^{\oplus a_i}$$

for some integers $a_i \geq 0$. For every object V in $\mathcal{C}$ and every j, write

$$V \otimes_{\mathcal{C}} X_j \cong \bigoplus_{i=1}^{n} X_i^{\oplus a_{ij}} \tag{3.2.12.1}$$

for some integers $a_{ij} \geq 0$. Then, by [EGNO15, Definition 3.3.3], the **Frobenius-Perron dimension** of V is defined to be

$$\mathbf{FPdim}(V) := \rho((a_{ij})_{n\times n}). \tag{3.2.12.2}$$

Next we use Definition 4.1.11(2) to calculate the Frobenius-Perron dimension. Let σ be the tensor functor $V \otimes_{\mathcal{C}} -$ that is a $\mathbf{k}$-linear endofunctor of $\mathcal{C}$. If ϕ is a brick subset of $\mathcal{C}$, then ϕ is a subset of $\phi_n := \{X_1, \cdots, X_n\}$. For simplicity, assume that ϕ is $\{X_1, \cdots, X_s\}$ for some $s \leq n$. It follows from (3.2.12.1) that

$$\operatorname{Hom}_{\mathcal{C}}(X_i, \sigma(X_j)) = \operatorname{Hom}_{\mathcal{C}}(X_i, V \otimes_{\mathcal{C}} X_j) = \operatorname{Hom}_{\mathcal{C}}(X_i, \bigoplus_{k=1}^{n} X_k^{\oplus a_{kj}}) = \mathbf{k}^{\oplus a_{ij}}, \quad \forall\, i, j.$$

Hence the adjacency matrix of (ϕ_n, σ) is

$$A(\phi_n, \sigma) = (a_{ij})_{n\times n}$$

and the adjacency matrix of (ϕ, σ) is a principal minor of $A(\phi_n, \sigma)$. By Lemma 3.1.7(1), $\rho(A(\phi, \sigma)) \leq \rho(A(\phi_n, \sigma))$. By Definition 4.1.11(2), the *Frobenius-Perron dimension* of the functor $\sigma = V \otimes_{\mathcal{C}} -$ is

$$\operatorname{fpd}(V \otimes_{\mathcal{C}} -) = \sup_{\phi\in\Phi_b}\{\rho(A(\phi, \sigma))\} = \rho(A(\phi_n, \sigma)) = \rho((a_{ij})_{n\times n}),$$

which agrees with (3.2.12.2). This justifies calling $\operatorname{fpd}(V \otimes_{\mathcal{C}} -)$ the Frobenius-Perron dimension of V.

3.3 Basic properties

For simplicity, "Frobenius-Perron" is abbreviated to "fp".

3.3.1 Embeddings

It is clear that the fp-series and the fp-dimensions are invariant under equivalences of categories. We record this fact below. Recall that the Frobenius-Perron series $\operatorname{FP}(u, t, \sigma)$ of an endofunctor σ is defined in Definition 3.2.5(2).

Lemma 3.3.1. *Let $F : \mathcal{C} \to \mathcal{D}$ be an equivalence of categories. Let $\sigma_{\mathcal{C}}$ and $\sigma_{\mathcal{D}}$ be endofunctors of $\mathcal{C}$ and $\mathcal{D}$ respectively. Suppose that $F \circ \sigma_{\mathcal{C}}$ is naturally isomorphic to $\sigma_{\mathcal{D}} \circ F$. Then $\operatorname{FP}(u, t, \sigma_{\mathcal{C}}) = \operatorname{FP}(u, t, \sigma_{\mathcal{D}})$.*

Let $\mathbb{R}_+$ denote the set of non-negative real numbers union with $\{\infty\}$. Let

$$f(u,t) := \sum_{m,n=0}^{\infty} f_{m,n} t^m u^n \quad \text{and} \quad g(u,t) := \sum_{m,n=0}^{\infty} g_{m,n} t^m u^n$$

be two elements in $\mathbb{R}_+[[u, t]]$. We write $f \leq g$ if $f_{m,n} \leq g_{m,n}$ for all m, n.

Theorem 3.3.2. *Let $F : \mathcal{C} \to \mathcal{D}$ be a faithful functor that preserves brick subsets.*

(1) Let $\sigma_{\mathcal{C}}$ and $\sigma_{\mathcal{D}}$ be endofunctors of $\mathcal{C}$ and $\mathcal{D}$ respectively. Suppose that $F \circ \sigma_{\mathcal{C}}$ is naturally isomorphic to $\sigma_{\mathcal{D}} \circ F$. Then $\mathrm{FP}(u, t, \sigma_{\mathcal{C}}) \le \mathrm{FP}(u, t, \sigma_{\mathcal{D}})$.

(2) Let $\tau_{\mathcal{C}}$ and $\tau_{\mathcal{D}}$ be assignments of $\mathcal{C}$ and $\mathcal{D}$ respectively satisfying the property in Remark 3.2.6. Suppose that $\rho(A(\phi, \tau_{\mathcal{C}})) \le \rho(A(F(\phi), \tau_{\mathcal{D}}))$ for all $\phi \in \Phi_{n,b}(\mathcal{C})$ and all n. Then $\mathrm{FP}(u, t, \tau_{\mathcal{C}}) \le \mathrm{FP}(u, t, \tau_{\mathcal{D}})$.

Proof. (1) For every $\phi = \{X_1, \cdots, X_n\} \in \Phi_n(\mathcal{C})$, let $F(\phi)$ be $\{F(X_1), \cdots, F(X_n)\}$ in $\Phi_n(\mathcal{D})$. By hypothesis, if $\phi \in \Phi_{n,b}(\mathcal{C})$, then $F(\phi)$ is in $\Phi_{n,b}(\mathcal{D})$. Let $A = (a_{ij})$ (respectively, $B = (b_{ij})$) be the adjacency matrix of $(\phi, \sigma_{\mathcal{C}})$ (respectively, of $(F(\phi), \sigma_{\mathcal{D}})$). Then, by the faithfulness of F,

$$\begin{aligned} a_{ij} &= \dim(X_i, \sigma_{\mathcal{C}}(X_j)) \le \dim(F(X_i), F(\sigma_{\mathcal{C}}(X_j))) \\ &= \dim(F(X_i), \sigma_{\mathcal{D}}(F(X_j))) = b_{ij}. \end{aligned}$$

By Lemma 3.1.7(2),

$$\rho(A(\phi, \sigma_{\mathcal{C}})) =: \rho(A) \le \rho(B) := \rho(A(F(\phi), \sigma_{\mathcal{D}})). \tag{3.3.2.1}$$

By definition,

$$\mathrm{fpd}^n(\sigma_{\mathcal{C}}) \le \mathrm{fpd}^n(\sigma_{\mathcal{D}}). \tag{3.3.2.2}$$

Similarly, for all n, m, $\mathrm{fpd}^n(\sigma_{\mathcal{C}}^m) \le \mathrm{fpd}^n(\sigma_{\mathcal{D}}^m)$. The assertion follows.

(2) The proof of part (2) is similar. □

3.3.2 (a-)Hereditary algebras and categories

Recall that the global dimension of an abelian category $\mathfrak{A}$ is defined to be

$$\operatorname{gldim} \mathfrak{A} := \sup\{n \mid \mathrm{Ext}^n_{\mathfrak{A}}(X, Y) \ne 0, \text{ for some } X, Y \in \mathfrak{A}\}.$$

The global dimension of an algebra A is defined to be the global dimension of the category of left A-modules. An algebra (or an abelian category) is called *hereditary* if it has global dimension at most one.

There is a nice property concerning the indecomposable objects in the derived category of a hereditary abelian category (see [Kel07, Section 2.5]).

Lemma 3.3.3. *Let $\mathfrak{A}$ be a hereditary abelian category. Then every indecomposable object in the derived category $D(\mathfrak{A})$ is isomorphic to a shift of an object in $\mathfrak{A}$.*

Note that every brick (or atomic) object in an additive category is indecomposable. Based on the property in the above lemma, we make a definition.

Definition 3.3.4. An abelian category $\mathfrak{A}$ is called *a-hereditary* (respectively, *b-hereditary*) if every atomic (respectively, brick) object X in the bounded derived category $D^b(\mathfrak{A})$ is of the form $M[i]$ for some object M in $\mathfrak{A}$ and $i \in \mathbb{Z}$. The object M is automatically a brick object in $\mathfrak{A}$.

If α is an auto-equivalence of an abelian category $\mathfrak{A}$, then it extends naturally to an auto-equivalence, denoted by $\overline{\alpha}$, of the derived category $\mathcal{A} := D^b(\mathfrak{A})$. The main result in this subsection is the following. Recall that the b-height of $\mathfrak{A}$, denoted by $h_b(\mathfrak{A})$, is defined in Definition 4.1.10(3) and that the Frobenius-Perron global dimension of $\mathcal{A}$, denoted by $\operatorname{fpgldim} \mathcal{A}$, is defined in Definition 3.2.8(2).

Theorem 3.3.5. *Let $\mathfrak{A}$ be an a-hereditary abelian category with an auto-equivalence α. For each n, define $n' = \min\{n, h_b(\mathfrak{A})\}$. Let $\mathcal{A}$ be $D^b(\mathfrak{A})$.*

(1) If $m < 0$ or $m > \operatorname{gldim} \mathfrak{A}$, then

$$\operatorname{fpd}(\Sigma^m \circ \overline{\alpha}) = 0.$$

As a consequence, $\operatorname{fpgldim} \mathcal{A} \leq \operatorname{gldim} \mathfrak{A}$.

(2) For each n,

$$\operatorname{fpd}^n(\alpha) \leq \operatorname{fpd}^n(\overline{\alpha}) \leq \max_{1 \leq i \leq n'} \{\operatorname{fpd}^i(\alpha)\}. \tag{3.3.5.1}$$

If $\operatorname{gldim} \mathfrak{A} < \infty$, then

$$\operatorname{fpd}^n(\overline{\alpha}) = \max_{1 \leq i \leq n'} \{\operatorname{fpd}^i(\alpha)\}. \tag{3.3.5.2}$$

(3) Let $g := \operatorname{gldim} \mathfrak{A} < \infty$. Let β be the assignment $(X, Y) \to (\operatorname{Ext}^g_{\mathfrak{A}}(X, \alpha(Y)))$. Then

$$\operatorname{fpd}^n(\Sigma^g \circ \overline{\alpha}) = \max_{1 \leq i \leq n'} \{\operatorname{fpd}^i(\beta)\}. \tag{3.3.5.3}$$

(4) For every hereditary abelian category $\mathfrak{A}$*, we have* $\operatorname{fpd}(\mathcal{A}) = \operatorname{fpd}(\mathfrak{A})$.

Proof. (1) Since $\mathcal{A}$ is a-hereditary, every atomic object in $\mathcal{A}$ is of the form $M[i]$.

Case 1: $m < 0$. Write ϕ as $\{M_1[d_1], \cdots, M_n[d_n]\}$ where d_i is decreasing and M_i is in $\mathfrak{A}$. Then, for $i \le j$,

$$a_{ij} = \operatorname{Hom}_{\mathcal{A}}(M_i[d_i], (\Sigma^m \circ \overline{\alpha})M_j[d_j]) = \operatorname{Hom}_{\mathcal{A}}(M_i, \alpha(M_j)[d_j - d_i + m]) = 0$$

since $d_j - d_i + m < 0$. Thus the adjacency matrix $A := (a_{ij})_{n\times n}$ is strictly lower triangular. As a consequence, $\rho(A) = 0$. By definition, $\operatorname{fpd}(\Sigma^m \circ \overline{\alpha}) = 0$.

Case 2: $m > \operatorname{gldim}\mathfrak{A}$. Write ϕ as $\{M_1[d_1], \cdots M_n[d_n]\}$ where d_i is increasing and M_i is in $\mathfrak{A}$. Then, for $i \ge j$,

$$a_{ij} = \operatorname{Hom}_{\mathcal{A}}(M_i[d_i], (\Sigma^m \circ \overline{\alpha})M_j[d_j]) = \operatorname{Hom}_{\mathcal{A}}(M_i, \alpha(M_j)[d_j - d_i + m]) = 0$$

since $d_j - d_i + m > \operatorname{gldim}\mathfrak{A}$. Thus the adjacency matrix $A := (a_{ij})_{n\times n}$ is strictly upper triangular. As a consequence, $\rho(A) = 0$. By definition, $\operatorname{fpd}(\Sigma^m \circ \overline{\alpha}) = 0$.

(2) Let F be the canonical fully faithful embedding $\mathfrak{A} \to \mathcal{A}$. By Theorem 3.3.2 and (3.3.2.2),

$$\operatorname{fpd}^n(\alpha) \le \operatorname{fpd}^n(\overline{\alpha}).$$

For the other assertion, write ϕ as a disjoint union $\phi_{d_1} \cup \cdots \cup \phi_{d_s}$ where d_i is strictly decreasing and the subset ϕ_{d_i} consists of objects of the form $M[d_i]$ for $M \in \mathfrak{A}$. For any objects $X \in \phi_{d_i}$ and $Y \in \phi_{d_j}$ for $i < j$, $\operatorname{Hom}_{\mathcal{A}}(X, Y) = 0$. Thus the adjacency matrix of $(\phi, \overline{\alpha})$ is of the form

$$A(\phi, \overline{\alpha}) = \begin{pmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ * & A_{22} & 0 & \cdots & 0 \\ * & * & A_{33} & \cdots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ * & * & * & \cdots & A_{ss} \end{pmatrix} \tag{3.3.5.4}$$

where each A_{ii} is the adjacency matrix $A(\phi_{d_i}, \overline{\alpha})$. For each ϕ_{d_i}, we have

$$A(\phi_{d_i}, \overline{\alpha}) = A(\phi_{d_i}[-d_i], \overline{\alpha}) = A(\phi_{d_i}[-d_i], \alpha)$$

which implies that

$$\rho(A_{ii}) \le \mathrm{fpd}^{s_i}(\alpha) \le \max_{1\le j\le n'} \mathrm{fpd}^j(\alpha)$$

where s_i is the size of A_{ii} and $n' = \min\{n, h_b(\mathfrak{A})\}$. By using the matrix (3.3.5.4),

$$\rho(A(\phi,\overline{\alpha})) = \max_i\{\rho(A_{ii})\} \le \max_{1\le j\le n'} \mathrm{fpd}^j(\alpha).$$

Then (3.3.5.1) follows.

Suppose now that $g := \mathrm{gldim}\,\mathfrak{A} < \infty$. Let $\phi \in \Phi_{n,a}(\mathcal{A})$. Pick any $M \in \Phi_{1,b}(\mathfrak{A})$. Then, for $g' \gg g$, $\phi' := \phi \cup \{M[g']\} \in \Phi_{n+1,a}(\mathcal{A})$. By Lemma 3.1.7(1), $\rho(A(\phi',\overline{\alpha})) \ge \rho(A(\phi,\overline{\alpha}))$. Hence $\mathrm{fpd}^n(\overline{\alpha})$ is increasing as n increases. Therefore (3.3.5.2) follows from (3.3.5.1).

(3) The proof is similar to the proof of part (2). Let F be the canonical fully faithful embedding $\mathfrak{A} \to \mathcal{A}$. By Theorem 3.3.2(2) and (3.3.2.2),

$$\mathrm{fpd}^n(\beta) \le \mathrm{fpd}^n(\Sigma^g \circ \overline{\alpha}).$$

By the argument at the end of proof of part (2), $\mathrm{fpd}^n(\Sigma^g \circ \overline{\alpha})$ increases when n increases. Then

$$\max_{1\le j\le n'} \mathrm{fpd}^j(\beta) \le \mathrm{fpd}^n(\Sigma^g \circ \overline{\alpha}).$$

For the other direction, write ϕ as a disjoint union $\phi_{d_1} \cup \cdots \cup \phi_{d_s}$ where d_i is strictly increasing and ϕ_{d_i} consists of objects of the form $M[d_i]$ for $M \in \mathfrak{A}$. For objects $X \in \phi_{d_i}$ and $Y \in \phi_{d_j}$ for $i < j$, $\mathrm{Hom}_{\mathcal{A}}(X, \Sigma^g(\alpha(Y))) = 0$. Let $\gamma = \Sigma^g \circ \overline{\alpha}$. Then the adjacency matrix of (ϕ,γ) is of the form (3.3.5.4), namely,

$$A(\phi,\gamma) = \begin{pmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ * & A_{22} & 0 & \cdots & 0 \\ * & * & A_{33} & \cdots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ * & * & * & \cdots & A_{ss} \end{pmatrix}$$

where each A_{ii} is the adjacency matrix $A(\phi_{d_i},\gamma)$. For each ϕ_{d_i}, we have

$$A(\phi_{d_i},\gamma) = A(\phi_{d_i}[-d_i],\gamma) = A(\phi_{d_i}[-d_i],\beta)$$

which implies that

$$\rho(A_{ii}) \leq \mathrm{fpd}^{s_i}(\beta) \leq \max_{1\leq j\leq n'} \mathrm{fpd}^j(\beta)$$

where s_i is the size of A_{ii}. By using matrix (3.3.5.4),

$$\rho(A(\phi,\gamma)) = \max_i\{\rho(A_{ii})\} \leq \max_{1\leq j\leq n'} \mathrm{fpd}^j(\beta).$$

The assertion follows.

(4) Take α to be the identity functor of $\mathfrak{A}$ and $g = 1$ (since $\mathfrak{A}$ is hereditary). By (3.3.5.3), we have

$$\mathrm{fpd}^n(\Sigma) = \max_{1\leq i\leq n'}\{\mathrm{fpd}^i(E^1)\}.$$

By taking $\sup_n$, we obtain that $\mathrm{fpd}(E^1) = \mathrm{fpd}(\Sigma)$. The assertion follows. $\square$

3.3.3 Categories with Serre functor

Recall from [Kel08, Section 2.6] that if a Hom-finite category $\mathcal{C}$ has a Serre functor S, then there is a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(X,Y)^* \cong \mathrm{Hom}_{\mathcal{C}}(Y,S(X))$$

for all $X,Y \in \mathcal{C}$. A (pre-)triangulated Hom-finite category $\mathcal{C}$ with Serre functor S is called *n-Calabi-Yau* if there is a natural isomorphism

$$S \cong \Sigma^n.$$

(In [Kel08, Section 2.6] it is called *weakly n-Calabi-Yau.*) We now prove Proposition 3.0.7.

Proposition 3.3.6 (Serre duality). *Let $\mathcal{C}$ be a* Hom-*finite category with Serre functor S. Let σ be an endofunctor of $\mathcal{C}$.*

(1) If σ has a right adjoint $\sigma^!$, then

$$\mathrm{fpd}(\sigma) = \mathrm{fpd}(\sigma^! \circ S).$$

(2) If σ is an equivalence with quasi-inverse σ^{-1}, then

$$\operatorname{fpd}(\sigma) = \operatorname{fpd}(\sigma^{-1} \circ S).$$

(3) If $\mathcal{C}$ is (pre-)triangulated and n-Calabi-Yau, then we have a duality

$$\operatorname{fpd}(\Sigma^{i}) = \operatorname{fpd}(\Sigma^{n-i})$$

for all i.

Proof. (1) Let $\phi = \{X_1, \cdots, X_n\} \in \Phi_{n,b}$ and let $A(\phi, \sigma)$ be the adjacency matrix with (i,j)-entry $a_{ij} = \dim(X_i, \sigma(X_j))$. By Serre duality,

$$a_{ij} = \dim(X_i, \sigma(X_j)) = \dim(\sigma(X_j), S(X_i)) = \dim(X_j, (\sigma^! \circ S)(X_i)),$$

which is the (j,i)-entry of the adjacency matrix $A(\phi, \sigma^! \circ S)$. Then $\rho(A(\phi,\sigma)) = \rho(A(\phi, \sigma^! \circ S))$. It follows from the definition that $\operatorname{fpd}^n(\sigma) = \operatorname{fpd}^n(\sigma^! \circ S)$ for all $n \geq 1$. The assertion follows from the definition.

(2,3) These are consequences of part (1). □

3.3.4 Opposite categories

Lemma 3.3.7. *Let σ be an endofunctor of $\mathcal{C}$ and suppose that σ has a left adjoint σ^*. Consider σ^* as an endofunctor of the opposite category $\mathcal{C}^{op}$ of $\mathcal{C}$. Then*

$$\operatorname{fpd}^n(\sigma\mid_{\mathcal{C}}) = \operatorname{fpd}^n(\sigma^* \mid_{\mathcal{C}^{op}}).$$

for all n.

Proof. Let $\phi := \{X_1, \cdots, X_n\}$ be a brick subset of $\mathcal{C}$ (which is also a brick subset of $\mathcal{C}^{op}$). Then

$$\dim_{\mathcal{C}}(X_i, \sigma(X_j)) = \dim_{\mathcal{C}}(\sigma^*(X_i), X_j) = \dim_{\mathcal{C}^{op}}(X_j, \sigma^*(X_i)),$$

which implies that the adjacency matrix of σ^* as an endofunctor of $\mathcal{C}^{op}$ is the transpose of the adjacency matrix of σ. The assertion follows. □

Definition 3.3.8. (1) Two pre-triangulated categories $(\mathcal{T}_i, \Sigma_i)$, for $i = 1, 2$, are called *fp-equivalent* if

$$\mathrm{fpd}^n(\Sigma_1^m) = \mathrm{fpd}^n(\Sigma_2^m)$$

for all $n \geq 1, m \in \mathbb{Z}$.

(2) Two algebras are *fp-equivalent* if their bounded derived categories of finitely generated modules are fp-equivalent.

(3) Two pre-triangulated categories with Serre functors $(\mathcal{T}_i, \Sigma_i, S_i)$, for $i = 1, 2$, are called *fp-S-equivalent* if

$$\mathrm{fpd}^n(\Sigma_1^m \circ S_1^k) = \mathrm{fpd}^n(\Sigma_2^m \circ S_2^k)$$

for all $n \geq 1, m, k \in \mathbb{Z}$.

Proposition 3.3.9. *Let $\mathcal{T}$ be a pre-triangulated category.*

(1) $\mathcal{T}$ and $\mathcal{T}^{op}$ are fp-equivalent.

(2) Suppose S is a Serre functor of $\mathcal{T}$. Then $(\mathcal{T}, S)$ and $(\mathcal{T}^{op}, S^{op})$ are fp-S-equivalent.

Proof. (1) Let Σ be the suspension of $\mathcal{T}$. Then $\mathcal{T}^{op}$ is also pre-triangulated with suspension functor being $\Sigma^{-1} = \Sigma^*$ (restricted to $\mathcal{T}^{op}$). The assertion follows from Lemma 3.3.7.

(2) Note that the Serre functor of $\mathcal{T}^{op}$ is equal to $S^{-1} = S^*$ (restricted to $\mathcal{T}^{op}$). The assertion follows by Lemma 3.3.7. $\square$

Corollary 3.3.10. *Let A be a finite dimensional algebra.*

(1) A and A^{op} are fp-equivalent.

(2) Suppose A has finite global dimension. In this case, the bounded derived category of finite dimensional A-modules has a Serre functor. Then A and A^{op} are fp-S-equivalent.

Proof. (1) Since A is finite dimensional, the $\mathbf{k}$-linear dual induces an equivalence of triangulated categories between $D^b(\mathrm{Mod}_{f.d.}-A)^{op}$ and $D^b(\mathrm{Mod}_{f.d.}-A^{op})$. The assertion follows from Proposition 3.3.9(1).

The proof of (2) is similar by using Proposition 3.3.9(2) instead. □

There are examples where $\mathcal{T}$ and $\mathcal{T}^{op}$ are not triangulated equivalent, see Example 3.3.12. In this paper, a $\mathbf{k}$-algebra A is called *local* if A has a unique maximal ideal $\mathfrak{m}$ and $A/\mathfrak{m} \cong \mathbf{k}$. The following lemma is easy and well-known.

Lemma 3.3.11. *Let A be a finite dimensional local algebra over $\mathbf{k}$. Let $\mathfrak{A}$ be the category $\mathrm{Mod}_{f.d.}-A$ and $\mathcal{A}$ be $D^b(\mathfrak{A})$.*

(1) Let X be an object in $\mathcal{A}$ such that $\mathrm{Hom}_{\mathcal{A}}(X, X[-i]) = 0$ for all $i > 0$. Then X is of the form $M[n]$ where M is an object in $\mathfrak{A}$ and $n \in \mathbb{Z}$.

(2) Every atomic object in $\mathcal{A}$ is of the form $M[n]$ where M is a brick object in $\mathfrak{A}$ and $n \in \mathbb{Z}$. Namely, $\mathfrak{A}$ is a-hereditary.

Proof. (2) is an immediate consequence of part (1). We only prove part (1).

On the contrary we suppose that $H^m(X) \neq 0$ and $H^n(X) \neq 0$ for some $m < n$. Since X is a bounded complex, we can take m to be minimum of such integers and n to be the maximum of such integers. Since A is local, there is a nonzero map from $H^n(X) \to H^m(X)$, which induces a nonzero morphism in $\mathrm{Hom}_{\mathcal{A}}(X, X[m-n])$. This contradicts the hypothesis. □

Example 3.3.12. Let m, n be integers ≥ 2. Define $A_{m,n}$ to be the algebra

$$\mathbf{k}\langle x_1, x_2\rangle/(x_1^m, x_2^n, x_1x_2).$$

It is easy to see that $A_{m,n}$ is a finite dimensional local connected graded algebra generated in degree 1 (with $\deg x_1 = \deg x_2 = 1$). If $A_{m,n}$ is isomorphic to $A_{m',n'}$ as algebras, by [BZ17, Theorem 1], these are isomorphic as graded algebras. Suppose $f : A_{m,n} \to A_{m',n'}$ is an isomorphism of graded algebras and write

$$f(x_1) = ax_1 + bx_2, \quad f(x_2) = cx_1 + dx_2.$$

Then the relation $f(x_1)f(x_2) = 0$ forces $b = c = 0$. As a consequence, $m = m'$ and $n = n'$. So we have proven that

(1) $A_{m,n}$ is isomorphic to $A_{m',n'}$ if and only if $m = m'$ and $n = n'$.

Next we claim that

(2) if $m \neq n$, then the derived category $D^b(\mathrm{Mod}_{f.d.}-A_{m,n})$ is not triangulated equivalent to $D^b(\mathrm{Mod}_{f.d.}-A^{op}_{m,n})$.

Let m, n, m', n' be integers ≥ 2. Suppose that $D^b(\mathrm{Mod}_{f.d.}-A_{m,n})$ is triangulated equivalent to $D^b(\mathrm{Mod}_{f.d.}-A_{m',n'})$. Since $A_{m,n}$ is local, by [Yek99, Theorem 2.3], every tilting complex over $A_{m,n}$ is of the form $P[n]$ where P is a progenerator over $A_{m,n}$. As a consequence, $A_{m,n}$ is Morita equivalent to $A_{m',n'}$. Since both $A_{m,n}$ and $A_{m',n'}$ are local, Morita equivalence implies that $A_{m,n}$ is isomorphic to $A_{m',n'}$. By part (1), $m = m'$ and $n = n'$. In other words, if $(m,n) \neq (m',n')$, then $D^b(\mathrm{Mod}_{f.d.}-A_{m,n})$ is not triangulated equivalent to $D^b(\mathrm{Mod}_{f.d.}-A_{m',n'})$. As a consequence, if $m \neq n$, then $D^b(\mathrm{Mod}_{f.d.}-A_{m,n})$ is not triangulated equivalent to $D^b(\mathrm{Mod}_{f.d.}-A_{n,m})$. By definition, $A^{op}_{m,n} \cong A_{n,m}$. Therefore the claim (2) follows.

We can show that $D^b(\mathrm{Mod}_{f.d.}-A)$ is dual to $D^b(\mathrm{Mod}_{f.d.}-A^{op})$ by using the $\mathbf{k}$-linear dual. In other words, $D^b(\mathrm{Mod}_{f.d.}-A)^{op}$ is triangulated equivalent to $D^b(\mathrm{Mod}_{f.d.}-A^{op})$. Therefore the following is a consequence of part (2).

(3) Suppose $m \neq n$ and let $\mathcal{A}$ be $D^b(\mathrm{Mod}_{f.d.}-A_{m,n})$. Then $\mathcal{A}$ is not triangulated equivalent to $\mathcal{A}^{op}$. But by Proposition 3.3.9(1), $\mathcal{A}$ and $\mathcal{A}^{op}$ are fp-equivalent.

3.4 Derived category over a commutative ring

Throughout this section A is a commutative algebra and $\mathcal{A} = D^b(\mathrm{Mod}-A)$. (In other sections $\mathcal{A}$ usually denotes $D^b(\mathrm{Mod}_{f.d.}-A)$.)

Lemma 3.4.1. *Let A be a commutative algebra. Let X be an atomic object in $\mathcal{A}$. Then X is of the form $M[i]$ for some simple A-module M and some $i \in \mathbb{Z}$. As a consequence,* $\mathrm{Mod}-A$ *is a-hereditary.*

Proof. Consider X as a bounded above complex of projective A-modules. Since A is commutative, every $f \in A$ induces naturally a morphism of X by multiplication. For each i, $H^i(X)$ is an A-module. We have natural morphisms of A-algebras

$$A \to \mathrm{Hom}_{\mathcal{A}}(X, X) \to \mathrm{End}_A(H^i(X)).$$

By definition, $\mathrm{Hom}_{\mathcal{A}}(X, X) = \mathbf{k}$. Thus $\mathrm{Hom}_{\mathcal{A}}(X, X) = A/\mathfrak{m}$ for some ideal $\mathfrak{m}$ of A that has codimension 1. Hence the A-action on $H^i(X)$ factors through the map $A \to A/\mathfrak{m}$. This means that $H^i(X)$ is a direct sum of $A/\mathfrak{m}$.

Let $n = \sup X$ and $m = \inf X$. Then $H^m(X) = (A/\mathfrak{m})^{\oplus s}$ and $H^n(X) = (A/\mathfrak{m})^{\oplus t}$ for some $s, t > 0$. If $m < n$, then

$$\mathrm{Hom}_{\mathcal{A}}(X, X[m-n]) \cong \mathrm{Hom}_{\mathcal{A}}(X[n], X[m]) \cong \mathrm{Hom}_A(H^n(X), H^m(X)) \neq 0$$

which contradicts (3.2.2.1). Therefore $m = n$ and $X = M[n]$ for $M := H^n(X)$. Since X is atomic, M has only one copy of $A/\mathfrak{m}$. □

Lemma 3.4.2. *Let A be a commutative algebra. Let X and Y be two atomic objects in $\mathcal{A}$. Then $\mathrm{Hom}_{\mathcal{A}}(X, Y) \neq 0$ if and only if there is an ideal $\mathfrak{m}$ of A of codimension 1 such that $X \cong A/\mathfrak{m}[m]$ and $Y \cong A/\mathfrak{m}[n]$ for some $0 \leq n - m \leq \mathrm{projdim}\, A/\mathfrak{m}$.*

Proof. By Lemma 3.4.1, $X \cong A/\mathfrak{m}[m]$ for some ideal $\mathfrak{m}$ of codimension 1 and some integer m. Similarly, $Y \cong A/\mathfrak{n}[n]$ for ideal $\mathfrak{n}$ of codimension 1 and integer n.

Suppose $\mathrm{Hom}_{\mathcal{A}}(X, Y) \neq 0$. If $\mathfrak{m} \neq \mathfrak{n}$, then clearly $\mathrm{Hom}_{\mathcal{A}}(X, Y) = 0$. Hence $\mathfrak{m} = \mathfrak{n}$. Further, $\mathrm{Ext}_A^{n-m}(A/\mathfrak{m}, A/\mathfrak{m}) \cong \mathrm{Hom}_{\mathcal{A}}(X, Y) \neq 0$ implies that $0 \leq n - m \leq \mathrm{projdim}\, A/\mathfrak{m}$. The converse can be proved in a similar way. □

If A is an affine commutative ring over $\mathbf{k}$, then every simple A-module is 1-dimensional. Hence $(A/\mathfrak{m})[i]$ is a brick (and atomic) object in $\mathcal{A}$ for every $i \in \mathbb{Z}$ and every maximal ideal $\mathfrak{m}$ of A. The fp-global dimension $\mathrm{fpgldim}(\mathcal{A})$ is defined in Definition 3.2.8(2).

Proposition 3.4.3. *Let A be an affine commutative domain of global dimension $g < \infty$.*

(1) $\operatorname{fpd}(\mathcal{A}) = g$.

(2) $\operatorname{fpd}(\Sigma^i) = \binom{g}{i}$ *for all* i.

(3) $\operatorname{fpgldim}(\mathcal{A}) = g$.

Proof. (1) By Lemma 3.4.1, every atomic object is of the form $M[i]$ for some $M \cong A/\mathfrak{m}$ where $\mathfrak{m}$ is an ideal of codimension 1, and $i \in \mathbb{Z}$. It is well-known that

$$\dim \operatorname{Ext}^i_A(A/\mathfrak{m}, A/\mathfrak{m}) = \binom{g}{i}, \quad \forall \ i. \tag{3.4.3.1}$$

If $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are two different maximal ideals, then

$$\operatorname{Ext}^i_A(A/\mathfrak{m}_1, A/\mathfrak{m}_2) = 0 \tag{3.4.3.2}$$

for all i. Let ϕ be an atomic n-object subset. We can decompose ϕ into a disjoint union $\phi_{A/\mathfrak{m}_1} \cup \cdots \cup \phi_{A/\mathfrak{m}_s}$ where $\phi_{A/\mathfrak{m}}$ consists of objects of the form $A/\mathfrak{m}[i]$ for $i \in \mathbb{Z}$. It follows from (3.4.3.2) that the adjacency matrix is a block-diagonal matrix. Hence, we only need to consider the case when $\phi = \phi_{A/\mathfrak{m}}$ after we use the reduction similar to the one used in the proof of Theorem 3.3.5. Let $\phi = \phi_{A/\mathfrak{m}} = \{A/\mathfrak{m}[d_1], \cdots, A/\mathfrak{m}[d_m]\}$ where d_i is increasing. By Lemma 3.4.2, we have $d_{i+1} - d_i > g$, or $d_i + g < d_{i+1}$, for all $i = 1, \cdots, m-1$. Under these conditions, the adjacency matrix is lower triangular with each diagonal being g. Thus $\operatorname{fpd}(\Sigma) = g$.

The proof of (2) is similar. (3) is a consequence of (2). □

Suggested by Theorem 3.3.5, we could introduce some secondary invariants as follows. The *stabilization index* of a triangulated category $\mathcal{T}$ is defined to be

$$SI(\mathcal{T}) = \min\{n \mid \operatorname{fpd}^{n'} \mathcal{T} = \operatorname{fpd} \mathcal{T}, \ \forall \ n' \geq n\}.$$

The *global stabilization index* of $\mathcal{T}$ is defined to be

$$GSI(\mathcal{T}) = \max\{SI(\mathcal{T}') \mid \text{for all thick triangulated full subcategories } \mathcal{T}' \subseteq \mathcal{T}\}.$$

It is clear that both stabilization index and global stabilization index can be defined for an abelian category.

Similar to Proposition 3.4.3, one can show the following. Suppose that A is affine. For every i, let

$$d_i := \sup\{\dim \operatorname{Ext}^i_A(A/\mathfrak{m}, A/\mathfrak{m}) \mid \text{ maximal ideals } \mathfrak{m} \subseteq A\}.$$

Proposition 3.4.4. *Let A be an affine commutative algebra. Then, for each i,* $\operatorname{fpd}(\Sigma^i) = d_i < \infty$ *and* $\rho(A(\phi, \Sigma^i)) \leq d_i$ *for all* $\phi \in \Phi_{n,a}$. *As a consequence, for each integer i, the following hold.*

(1) $\operatorname{fpd}(\Sigma^i) = \operatorname{fpd}^1(\Sigma^i)$. *Hence the stabilization index of $\mathcal{A}$ is 1.*

(2) $\operatorname{fpd}(\Sigma^i)$ *is a finite integer.*

Theorem 3.4.5. *Let A be an affine commutative algebra and $\mathcal{A}$ be $D^b(\mathrm{Mod}-A)$. Let $\mathcal{T}$ be a triangulated full subcategory of $\mathcal{A}$ with suspension $\Sigma_{\mathcal{T}}$. Let i be an integer.*

(1) $\operatorname{fpd}(\Sigma^i_{\mathcal{T}}) = \operatorname{fpd}^1(\Sigma^i_{\mathcal{T}})$. *As a consequence, the global stabilization index of $\mathcal{A}$ is 1.*

(2) $\operatorname{fpd}(\Sigma^i_{\mathcal{T}})$ *is a finite integer.*

(3) If $\mathcal{T}$ is isomorphic to $D^b(\mathrm{Mod}_{f.d.}-B)$ for some finite dimensional algebra B, then B is Morita equivalent to a commutative algebra.

Proof. (1,2) These are similar to Proposition 3.4.4.

(3) Since B is finite dimensional, it is Morita equivalent to a basic algebra. So we can assume B is basic and show that B is commutative. Write B as a $\mathbf{k}Q/(R)$ where Q is a finite quiver with admissible ideal $R \subseteq (\mathbf{k}Q)_{\geq 2}$. We will show that B is commutative.

First we claim that each connected component of Q consists of only one vertex. Suppose not. Then Q contains distinct vertices v_1 and v_2 with an arrow $\alpha : v_1 \to v_2$. Let S_1 and S_2

be the simple modules corresponding to v_1 and v_2 respectively. Then $\{S_1, S_2\}$ is an atomic set in $\mathcal{T}$. The arrow represents a nonzero element in $\mathrm{Ext}^1_B(S_1, S_2)$. Hence

$$\mathrm{Hom}_{\mathcal{T}}(S_1, S_2[1]) \cong \mathrm{Ext}^1_B(S_1, S_2) \neq 0.$$

By Lemma 3.4.2, S_1 is isomorphic to a complex shift of S_2. But this is impossible. Therefore, the claim holds.

It follows from the claim in the last paragraph that $B = B_1 \oplus \cdots \oplus B_n$ where each B_i is a finite dimensional local ring corresponding to a vertex, say v_i. Next we claim that each B_i is commutative. Without loss of generality, we can assume $B_i = B$.

Now let ι be the fully faithful embedding from

$$\iota : \mathcal{T} := D^b(\mathrm{Mod}_{f.d.} - B) \longrightarrow \mathcal{A} := D^b(\mathrm{Mod} - A).$$

Let S be the unique simple left B-module. Then, by Lemma 3.4.1, there is a maximal ideal $\mathfrak{m}$ of A such that $\iota(S) = A/\mathfrak{m}[w]$ for some $w \in \mathbb{Z}$. After a shift, we might assume that $\iota(S) = A/\mathfrak{m}$. The left B-module B has a composition series such that each simple subquotient is isomorphic to S, which implies that, as a left A-module, $\iota(B)$ is generated by $A/\mathfrak{m}$ in $\mathcal{A}$. By induction on the length of B, one sees that, for every $n \in \mathbb{Z}$, $H^n(\iota(B))$ is a left $A/\mathfrak{m}^d$-module for some $d \gg 0$ (we can take $d = \mathrm{length}({}_B B)$). Since $\mathrm{Hom}_{\mathcal{A}}(\iota(B), \iota(B)[-i]) = \mathrm{Hom}_{\mathcal{T}}(B, B[-i]) = 0$ for all $i > 0$, the proof of Lemma 3.3.11(2) shows that $\iota(B) \cong M[m]$ for some left $A/\mathfrak{m}^d$-module M and $m \in \mathbb{Z}$. Since there are nonzero maps from S to B and from B to S, we have nonzero maps from $A/\mathfrak{m}$ to $\iota(B)$ and from $\iota(B)$ to $A/\mathfrak{m}$. This implies that $m = 0$. Since B is local (and then $B/\mathfrak{m}_B$ is 1-dimensional for the maximal ideal $\mathfrak{m}_B$), this forces that $M = A/I$ where I is an ideal of A containing $\mathfrak{m}^d$. Finally,

$$B^{op} = \mathrm{End}_B(B) \cong \mathrm{End}_{\mathcal{A}}(A/I, A/I) = \mathrm{End}_A(A/I, A/I) \cong A/I$$

which is commutative. Hence B is commutative. □

3.5 Examples

In this section we give three examples.

3.5.1 Frobenius-Perron theory of projective line $\mathbb{P}^1 := \operatorname{Proj} \mathbf{k}[t_0, t_1]$

Example 3.5.1. Let $coh(\mathbb{P}^1) =: \mathfrak{A}$ denote the category of coherent sheaves on $\mathbb{P}^1$. It is well-known (and follows from [BB07, Example 3.18]) that

Claim 5.1.1: Every brick object X in $\mathfrak{A}$ (namely, satisfying $\operatorname{Hom}_{\mathbb{P}^1}(X, X) = \mathbf{k}$) is either $\mathcal{O}(m)$ for some $m \in \mathbb{Z}$ or $\mathcal{O}_p$ for some $p \in \mathbb{P}^1$.

Let ϕ be in $\Phi_{n,b}(coh(\mathbb{P}^1))$. If $n = 1$ or ϕ is a singleton, then there are two cases: either $\phi = \{\mathcal{O}(m)\}$ or $\phi = \{\mathcal{O}_p\}$. Let E^1 be the functor $\operatorname{Ext}^1_{\mathbb{P}^1}(-,-)$. In the first case, $\rho(A(\phi, E^1)) = 0$ because $\operatorname{Ext}^1_{\mathbb{P}^1}(\mathcal{O}(m), \mathcal{O}(m)) = 0$, and in the second case, $\rho(A(\phi, E^1)) = 1$ because $\operatorname{Ext}^1_{\mathbb{P}^1}(\mathcal{O}_p, \mathcal{O}_p) = 1$.

If $|\phi| > 1$, then $\mathcal{O}(m)$ can not appear in ϕ as $\operatorname{Hom}_{\mathbb{P}^1}(\mathcal{O}(m), \mathcal{O}(m')) \neq 0$ and $\operatorname{Hom}_{\mathbb{P}^1}(\mathcal{O}(m), \mathcal{O}_p) \neq 0$ for all $m \leq m'$ and $p \in \mathbb{P}^1$. Hence, ϕ is a collection of $\mathcal{O}_p$ for finitely many distinct points p's. In this case, the adjacency matrix is the identity $n \times n$-matrix and $\rho(A(\phi, E^1)) = 1$. Therefore

$$\operatorname{fpd}^n(coh(\mathbb{P}^1)) = \operatorname{fpd}(coh(\mathbb{P}^1)) = 1 \tag{3.5.1.1}$$

for all $n \geq 1$. Since $coh(\mathbb{P}^1)$ is hereditary, by Theorem 3.3.5(3,4), we obtain that

$$\operatorname{fpd}^n(D^b(coh(\mathbb{P}^1))) = \operatorname{fpd}(D^b(coh(\mathbb{P}^1))) = 1 \tag{3.5.1.2}$$

for all $n \geq 1$.

Let K_2 be the Kronecker quiver

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \end{tikzcd} \tag{3.5.1.3}$$

By a result of Beilinson [Bei78], the derived category $D^b(\operatorname{Mod}_{f.d.} - \mathbf{k}\, K_2)$ is triangulated equivalent to $D^b(coh(\mathbb{P}^1))$. As a consequence,

$$\operatorname{fpd}(D^b(\operatorname{Mod}_{f.d.} - \mathbf{k}\, K_2)) = \operatorname{fpd}(D^b(coh(\mathbb{P}^1))) = 1. \tag{3.5.1.4}$$

It is easy to see, or by Theorem 3.1.8(1),

$$\operatorname{fpd} K_2 = 0$$

where fpd of a quiver is defined in Definition 3.1.6.

This implies that

$$\operatorname{fpd}(D^b(\operatorname{Mod}_{f.d.}-\mathbf{k}\,K_2)) > \operatorname{fpd} K_2. \tag{3.5.1.5}$$

Next we consider some general auto-equivalences of $D^b(coh(\mathbb{P}^1))$. Let

$$(m) : coh(\mathbb{P}^1) \to coh(\mathbb{P}^1)$$

be the auto-equivalence induced by the shift of degree m of the graded modules over $\mathbf{k}[t_0, t_1]$ and let Σ be the suspension functor of $D^b(coh(\mathbb{P}^1))$. Then the Serre functor S of $D^b(coh(\mathbb{P}^1))$ is $\Sigma \circ (-2)$. Let σ be the functor $\Sigma^a \circ (b)$ for some $a, b \in \mathbb{Z}$. By Theorem 3.3.5(1),

$$\operatorname{fpd}^n(\Sigma^a \circ (b)) = 0, \quad \forall\ a \neq 0, 1.$$

For the rest we consider $a = 0$ or 1. By Theorem 3.3.5(2,3), we only need to consider fpd on $coh(\mathbb{P}^1)$.

If ϕ is a singleton $\{\mathcal{O}(n)\}$, then the adjacency matrix is

$$A(\phi, \sigma) = \dim(\mathcal{O}, \Sigma^a \mathcal{O}(b)) = \begin{cases} 0 & a = 0, b < 0, \\ b+1 & a = 0, b \geq 0, \\ 0 & a = 1, b \geq -1, \\ -b-1 & a = 1, b < -1. \end{cases}$$

This follows from the well-known computation of $H^i_{\mathbb{P}^1}(\mathcal{O}(m))$ for $i = 0, 1$ and $m \in \mathbb{Z}$. (It also follows from a more general computation [AZ94, Theorem 8.1].) If $\phi = \{\mathcal{O}_p\}$ for some $p \in \mathbb{P}^1$, then the adjacency matrix is

$$A(\phi, \sigma) = \dim(\mathcal{O}_p, \Sigma^a(\mathcal{O}_p)) = 1, \quad \text{for}\ \ a = 0, 1.$$

It is easy to see from the above computation that

$$\mathrm{fpd}^1(\Sigma^a \circ (b)) = \begin{cases} 1 & a = 0, b < 0, \\ b+1 & a = 0, b \geq 0, \\ 1 & a = 1, b \geq -1, \\ -b-1 & a = 1, b < -1. \end{cases} \tag{3.5.1.6}$$

Now we consider the case when $n > 1$. If $\phi \in \Phi_{n,b}(coh(\mathbb{P}^1))$, ϕ is a collection of $\mathcal{O}_p$ for finitely many distinct p's. In this case, the adjacency matrix $A(\phi, \Sigma^a \circ (b))$ is the identity $n \times n$-matrix for $a = 0, 1$, and $\rho(A(\phi, \sigma)) = 1$. Therefore

$$\mathrm{fpd}^n(\Sigma^a \circ (b)) = 1 \tag{3.5.1.7}$$

for all $n > 1$, when restricted to the category $coh(\mathbb{P}^1)$.

It follows from Theorem 3.3.5(2,3) that

Claim 5.1.2: Consider $\Sigma^a \circ (b)$ as an endofunctor of $D^b(coh(\mathbb{P}^1))$. For $a, b \in \mathbb{Z}$ and $n \geq 1$, we have

$$\mathrm{fpd}^n(\Sigma^a \circ (b)) = \begin{cases} 0 & a \neq 0, 1, \\ 1 & a = 0, b < 0, \\ b+1 & a = 0, b \geq 0, \\ 1 & a = 1, b \geq -1, \\ -b-1 & a = 1, b < -1. \end{cases} \tag{3.5.1.8}$$

Since $S = \Sigma \circ (-2)$, we have the following (also see Figure 1)

$$\mathrm{fpd}^n(\Sigma^a \circ S^b) = \mathrm{fpd}^n(\Sigma^{a+b} \circ (-2b)) = \begin{cases} 0 & a+b \neq 0, 1, \\ 1 & a+b = 0, b > 0, \\ -(2b-1) & a+b = 0, b \leq 0, \\ 1 & a+b = 1, b \leq 0, \\ 2b-1 & a+b = 1, b > 0. \end{cases} \tag{3.5.1.9}$$

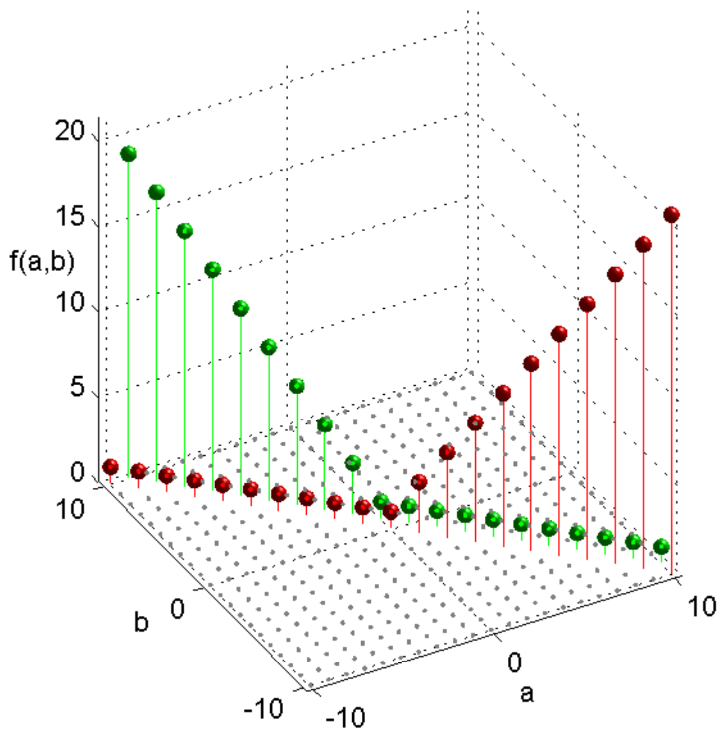

Figure 3.2: fp-S-theory for $\mathbb{P}^1$, where $f(a,b) = \operatorname{fpd}(\Sigma^a \circ S^b)$.

Claim 5.1.3: Since $D^b(coh(\mathbb{P}^1))$ and $D^b(\mathrm{Mod}_{f.d.} - \mathbf{k}\, K_2)$ are equivalent, the fp-theory of $D^b(\mathrm{Mod}_{f.d.} - \mathbf{k}\, K_2)$ agrees with (3.5.1.9) and Figure 1 (next page).

3.5.2 Frobenius-Perron theory of the quiver A_2

We start with the following example.

Example 3.5.2. Let A be the $\mathbb{Z}$-graded algebra $\mathbf{k}[x]/(x^2)$ with $\deg x = 1$. Let $\mathcal{C} := \mathrm{gr} - A$ be the category of finitely generated graded left A-modules. Let $\sigma := (-)$ be the degree shift functor of $\mathcal{C}$. It is clear that σ is an autoequivalence of $\mathcal{C}$. Let $\mathfrak{A}$ be the additive subcategory of $\mathcal{C}$ generated by $\sigma^n(A) = A(n)$ for all $n \in \mathbb{Z}$. Note that $\mathfrak{A}$ is not abelian and that every object in $\mathfrak{A}$ is of the form $\bigoplus_{n\in\mathbb{Z}} A(n)^{\oplus p_n}$ for some integers $p_n \geq 0$. Since the Hom-set in the graded module category consists of homomorphisms of degree zero, we have

$$\mathrm{Hom}_{\mathfrak{A}}(A, A(n)) = \begin{cases} \mathbf{k} & n = 0, 1 \\ 0 & \text{otherwise} \end{cases}.$$

In the following diagram each arrow represents 1-dimensional Hom for all possible Hom-set for different objects $A(n)$

$$\cdots \longrightarrow A(-2) \longrightarrow A(-1) \longrightarrow A(0) \longrightarrow A(1) \longrightarrow A(2) \longrightarrow \cdots \tag{3.5.2.1}$$

(where the number of arrows from $A(m)$ to $A(n)$ agrees with $\dim \mathrm{Hom}(A(m), A(n))$). It is easy to see that the set of indecomposable objects is $\{A(n)\}_{n\in\mathbb{Z}}$, which is also the set of bricks in $\mathfrak{A}$.

Lemma 3.5.3. *Retain the notation as in Example 3.5.2. When restricting σ onto the category $\mathfrak{A}$, we have, for every $m \geq 1$,*

$$\mathrm{fpd}^m(\sigma^n) = \begin{cases} 1 & n = 0, 1, \\ 0 & \text{otherwise} \end{cases}. \tag{3.5.3.1}$$

Proof. When $n = 0$, (3.5.3.1) is trivial. Let $n = 1$. For each set $\phi \in \Phi_{m,b}$, we can assume that $\phi = \{A(d_1), A(d_2,), \cdots, A(d_m)\}$ for a strictly increasing sequence $\{d_i \mid i = 1, 2, \cdots, m\}$. For any $i < j$, the (i, j)-entry of the adjacency matrix is

$$a_{ij} = \dim(A(d_i), A(d_j + 1)) = 0.$$

Thus $A(\phi, \sigma)$ is a lower triangular matrix with

$$a_{ii} = \dim(A(d_i), A(d_i + 1)) = 1.$$

Hence $\rho(A(\phi, \sigma)) = 1$. So $\mathrm{fpd}^m(\sigma) = 1$.

Similarly, $\mathrm{fpd}^m(\sigma^n) = 0$ when $n > 1$ as $\dim(A(d_i), A(d_i + 2)) = 0$ for all i.

Let $n < 0$. Let $\phi = \{A(d_1), A(d_2,), \cdots, A(d_m)\} \in \Phi_{m,b}$ where d_i are strictly decreasing. Then $a_{ij} = \dim(A(d_i), A(d_j + n)) = 0$ for all $i \leq j$. Thus $\rho(A(\phi, \sigma^n)) = 0$ and (3.5.3.1) follows in this case. $\square$

Example 3.5.4. Consider the quiver A_2

$$\bullet_2 \longleftarrow \bullet_1. \tag{3.5.4.1}$$

Let P_i (respectively, I_i) be the projective (respectively, injective) left $\mathbf{k}\,A_2$-modules corresponding to vertices i, for $i = 1, 2$, It is well-known that there are only three indecomposable left modules over A_2, with Auslander-Reiten quiver (or AR-quiver, for short)

$$P_2 \longrightarrow P_1 (= I_2) \longrightarrow I_1 \tag{3.5.4.2}$$

where each arrow represents a nonzero homomorphism (up to a scalar) [Sch14, Ex.1.13, pp.24-25]. The AR-translation (or translation, for short) τ is determined by $\tau(I_1) = P_2$. Let $\mathcal{T}$ be $D^b(\mathrm{Mod}_{f.d.} - \mathbf{k}\,A_2)$. The Auslander-Reiten theory can be extended from the module category to the derived category. It is direct that, in $\mathcal{T}$, we have the AR-quiver of all indecomposable objects

$$\begin{tikzcd}[column sep=small] & P_2[-1] \arrow[dr] & & I_1[-1] \arrow[dr] & & P_1 = I_2 \arrow[dr] & & P_2[1] \arrow[dr] & & I_1[1] & \\ \cdots & & P_1[-1] \arrow[ur] & & P_2 \arrow[ur] & & I_1 \arrow[ur] & & P_1[1] \arrow[ur] & & \cdots \end{tikzcd} \tag{3.5.4.3}$$

The above represents all possible nonzero morphisms (up to a scalar) between non-isomorphic indecomposable objects in $\mathcal{T}$. Note that $\mathcal{T}$ has a Serre functor S and that the AR-translation τ can be extended to a functor of $\mathcal{T}$ such that $S = \Sigma \circ \tau$ [RVdB02, Proposition I.2.3] or [CB, Remarks (2), p. 23]. After we identifying

$$P_2[i] \leftrightarrow A(3i), \quad P_1[i] \leftrightarrow A(3i+1), \quad I_1[i] \leftrightarrow A(3i+2),$$

(3.5.4.3) agrees with (3.5.2.1). Using the above identification, at least when restricted to objects, we have

$$\Sigma(A(i)) \cong A(i+3), \tag{3.5.4.4}$$

$$\tau(A(i)) \cong A(i-2), \tag{3.5.4.5}$$

$$S(A(i)) \cong A(i+1). \tag{3.5.4.6}$$

It follows from the definition of the AR-quiver [ARS97, VII] that the degree of τ is -2, see also [ASS06, Picture on p. 131]. Equation (3.5.4.5) just means that the degree of τ is -2.

By equation (3.5.4.6), the Serre functor S satisfies the property of σ defined in Example 3.5.2. By Lemma 3.5.3 or (3.5.3.1), we have

$$\operatorname{fpd}^n(\Sigma^a \circ S^b) = \operatorname{fpd}^n(\sigma^{3a+b}) = \begin{cases} 1 & 3a+b = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the fp-S-theory of $\mathcal{T}$ is given as above, and given as in Figure 2 (next page).

In particular, we have proven

$$\operatorname{fpgldim}(D^b(\mathrm{Mod}_{f.d.} - \mathbf{k}\, A_2)) = \operatorname{fpd}(D^b(\mathrm{Mod}_{f.d.} - \mathbf{k}\, A_2)) = \operatorname{fpd}(\Sigma) = 0,$$

which is less than $\operatorname{gldim} \mathbf{k}\, A_2 = 1$.

3.5.3 An example of non-integral Frobenius-Perron dimension

In the next example, we "glue" K_2 in (3.5.1.3) and A_2 in (3.5.4.1) together.

Example 3.5.5. Let G_2 be the quiver

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, bend left=40, "\beta"] \arrow[r, bend left=15, "\gamma"'] & \underset{2}{\bullet} \arrow[l, bend left=40, "\alpha"] \end{tikzcd} \tag{3.5.5.1}$$

consisting of two vertices 1 and 2, with arrow $\alpha : 2 \to 1$ and $\beta, \gamma : 1 \to 2$ satisfying relations

$$R: \quad \beta\alpha = \gamma\alpha = 0, \alpha\beta = \alpha\gamma = 0. \tag{3.5.5.2}$$

Note that (G_2, R) is a quiver with relations. The corresponding quiver algebra with relations is a 5-dimensional algebra

$$A = \mathbf{k}\, e_1 + \mathbf{k}\, e_2 + \mathbf{k}\, \alpha + \mathbf{k}\, \beta + \mathbf{k}\, \gamma$$

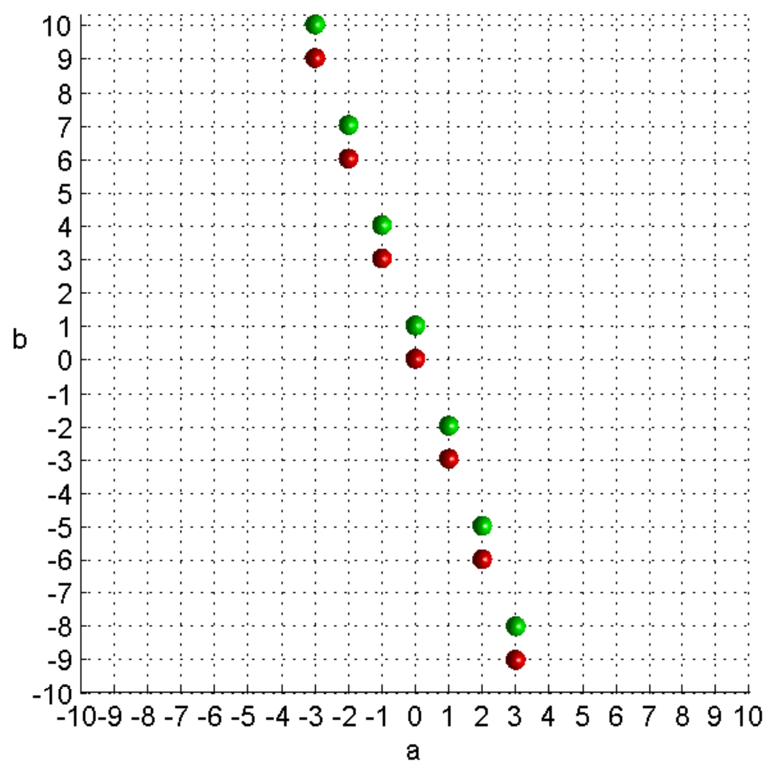

Figure 3.3: fp-S-theory for A_2

satisfying, in addition to (3.5.5.2),

$$e_1^2 = e_1, \quad e_2^2 = e_2, \quad 1 = e_1 + e_2, \tag{3.5.5.3}$$

$$e_2\alpha = 0, \quad \alpha e_1 = 0, \quad e_1\beta = e_1\gamma = 0, \quad \beta e_2 = \gamma e_2 = 0, \tag{3.5.5.4}$$

and

$$\alpha e_2 = \alpha = e_1\alpha, \quad \beta e_1 = \beta = e_2\beta, \quad \gamma e_1 = \gamma = e_2\gamma. \tag{3.5.5.5}$$

We can use the following matrix form to represent the algebra A

$$A = \begin{pmatrix} \mathbf{k}\, e_1 & \mathbf{k}\, \alpha \\ \mathbf{k}\, \beta + \mathbf{k}\, \gamma & \mathbf{k}\, e_2 \end{pmatrix}.$$

For each $i = 1, 2$, let S_i be the left simple A-module corresponding to the vertex i and P_i be the projective cover of S_i. Then $P_1 \cong Ae_1$ is isomorphic to the first column of A, namely, $\begin{pmatrix} \mathbf{k}\, e_1 \\ \mathbf{k}\, \beta + \mathbf{k}\, \gamma \end{pmatrix}$; and $P_2 \cong Ae_2$ is isomorphic to the second column of A, namely, $\begin{pmatrix} \mathbf{k}\, \alpha \\ \mathbf{k}\, e_2 \end{pmatrix}$.

We will show that the Frobenius-Perron dimension of the category of finite dimensional representations of (G_2, R) is $\sqrt{2}$, by using several lemmas below that contain some detailed computations.

Lemma 3.5.6. *Let $V = (V_1, V_2)$ be a representation of (G_2, R). Let $\overline{W} = \operatorname{im} \alpha$ and $K = \ker \alpha$. Take a $\mathbf{k}$-space decomposition $V_2 = W \oplus K$ where $W \cong \overline{W}$. Then there is a decomposition of (G_2, R)-representations $V \cong (\overline{W} \oplus T, W \oplus K) \cong (\overline{W}, W) \oplus (T, K)$ where α is the identity when restricted to W (and identifying W with $\overline{W}$) and is zero when restricted to K, where β and γ are zero when restricted to $\overline{W}$.*

Proof. Since $\overline{W} = \operatorname{im}\, \alpha$, $V_2 \cong W \oplus K$ where $K = \ker \alpha$ and $W \cong \overline{W}$. Write $V_1 = \overline{W} \oplus T$ for some $\mathbf{k}$-subspace $T \subseteq V_1$. The assertion follows by using the relations in (3.5.5.2). □

Recall that A_2 is the quiver given in (3.5.4.1) and K_2 is the Kronecker quiver given in (3.5.1.3). By the above lemma, the subrepresentation (W, W) (where we identify $\overline{W}$ with

W) is in fact a representation of $\begin{pmatrix} \mathbf{k}\,e_1 & \mathbf{k}\,\alpha \\ 0 & \mathbf{k}\,e_2 \end{pmatrix}$ $(\cong \mathbf{k}\,A_2)$ and the subrepresentation (T,K) is a representation of $\begin{pmatrix} \mathbf{k}\,e_1 & 0 \\ \mathbf{k}\,\beta + \mathbf{k}\,\gamma & \mathbf{k}\,e_2 \end{pmatrix}$ $(\cong \mathbf{k}\,K_2)$.

Let I_n be the $n \times n$-identity matrix. Let $Bl(\lambda)$ denote the block matrix

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}.$$

Lemma 3.5.7. *Suppose $\mathbf{k}$ is of characteristic zero. The following is a complete list of indecomposable representations of (G_2, R).*

(1) $P_2 \cong (\mathbf{k}, \mathbf{k})$, where $\alpha = I_1$ and $\beta = \gamma = 0$.

(2) $X_n(\lambda) = (K, K)$ with $\dim K = n$, where $\alpha = 0$, $\beta = I_n$ and $\gamma = Bl(\lambda)$ for some $\lambda \in \mathbf{k}$.

(3) $Y_n = (K, K)$ with $\dim K = n$, where $\alpha = 0$, $\beta = Bl(0)$ and $\gamma = I_n$.

(4) $S_{2,n} = (T, K)$ with $\dim T = n$ and $\dim K = n+1$, where $\alpha = 0$, $\beta = (I_n, 0)$ and $\gamma = (0, I_n)$.

(5) $S_{1,n} = (T, K)$ with $\dim T = n+1$ and $\dim K = n$, where $\alpha = 0$, $\beta = (I_n, 0)^\tau$ and $\gamma = (0, I_n)^\tau$.

As a consequence, $\mathbf{k}\,G_2/(R)$ is of tame representation type [Definition 3.7.1].

Proof. (1) By Lemma 3.5.6, this is the only case that could happen when $\alpha \neq 0$. Now we assume $\alpha = 0$.

(2,3,4,5) If $\alpha = 0$, then we are working with representations of Kronecker quiver K_2 (3.5.1.3). The classification follows from a classical result of Kronecker [Ben91, Theorem 4.3.2].

By (1-5), for each integer n, there are only finitely many 1-parameter families of indecomposable representations of dimension n. Therefore A is of tame representation type. $\square$

The following is a consequence of Lemma 3.5.7 and a direct computation.

Lemma 3.5.8. *Retain the hypotheses of Lemma 3.5.7. The following is a complete list of brick representations of* (G_2, R).

(1) $P_2 \cong (\mathbf{k}, \mathbf{k})$, *where* $\alpha = I_1$ *and* $\beta = \gamma = 0$.

(2) $X_1(\lambda) = (\mathbf{k}, \mathbf{k})$, *where* $\alpha = 0$, $\beta = I_1$ *and* $\gamma = \lambda I_1$ *for some* $\lambda \in \mathbf{k}$.

(3) $Y_1 = (\mathbf{k}, \mathbf{k})$, *where* $\alpha = 0$, $\beta = 0$ *and* $\gamma = I_1$.

(4) $S_{2,n}$ *for* $n \geq 0$.

(5) $S_{1,n}$ *for* $n \geq 0$.

The set $\Phi_{1,b}$ *consists of the above objects.*

Let $X_1(\infty)$ denote Y_1. We have the following short exact sequences of (G_2, R)-representations

$$0 \to S_1 \to P_2 \to S_2 \to 0,$$

$$0 \to S_2 \to X_1(\lambda) \to S_1 \to 0,$$

$$0 \to S_2^{n+1} \to S_{2,n} \to S_1^n \to 0,$$

$$0 \to S_2^n \to S_{1,n} \to S_1^{n+1} \to 0,$$

$$0 \to S_2^2 \to S_{2,n} \to S_{1,n-1} \to 0,$$

where $n \geq 1$ for the last exact sequence, and have the following nonzero homs, where $A = \mathbf{k}G_2/(R)$,

$$\mathrm{Hom}_A(X_1(\lambda), S_{1,n}) \neq 0, \quad \forall\, n \geq 1$$

$$\mathrm{Hom}_A(S_{2,n}, X_1(\lambda)) \neq 0, \quad \forall\, n \geq 1$$

$$\mathrm{Hom}_A(S_{2,m}, S_{2,n}) \neq 0, \quad \forall\, m \leq n$$

$$\mathrm{Hom}_A(S_{1,n}, S_{1,m}) \neq 0, \quad \forall\, m \leq n$$

$$\mathrm{Hom}_A(S_{2,n}, S_{1,m}) \neq 0, \quad \forall\, m + n \geq 1$$

Lemma 3.5.9. *Retain the hypotheses of Lemma 3.5.7. The following is the complete list of zero hom-sets between brick representations of G_2 in both directions.*

(1) $\mathrm{Hom}_A(X_1(\lambda), X_1(\lambda')) = \mathrm{Hom}_A(X_1(\lambda'), X_1(\lambda)) = 0$ *if* $\lambda \neq \lambda'$ *in* $\mathbf{k} \cup \{\infty\}$.

(2) $\mathrm{Hom}_A(S_1, S_2) = \mathrm{Hom}_A(S_2, S_1) = 0$.

As a consequence, if $\phi \in \Phi_{n,b}$ for some $n \geq 2$, then $\phi = \{S_1, S_2\}$ or $\phi = \{X_1(\lambda_i)\}_{i=1}^{n}$ for different parameters $\{\lambda_1, \cdots, \lambda_n\}$.

We also need to compute the Ext^1_A-groups.

Lemma 3.5.10. *Retain the hypotheses of Lemma 3.5.7. Let $\lambda \neq \lambda'$ be in $\mathbf{k} \cup \{\infty\}$.*

(1) $\mathrm{Ext}^1_A(X_1(\lambda), X_1(\lambda)) = \mathrm{Hom}_A(X_1(\lambda), X_1(\lambda)) = \mathbf{k}$.

(2) $\mathrm{Ext}^1_A(X_1(\lambda), X_1(\lambda')) = \mathrm{Hom}_A(X_1(\lambda), X_1(\lambda')) = 0$.

(3) $\begin{pmatrix} \mathrm{Ext}^1_A(S_1, S_1) & \mathrm{Ext}^1_A(S_1, S_2) \\ \mathrm{Ext}^1_A(S_2, S_1) & \mathrm{Ext}^1_A(S_2, S_2) \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{k}^{\oplus 2} \\ \mathbf{k} & 0 \end{pmatrix}$.

(4) $\mathrm{Ext}^1_A(P_2, P_2) = 0$.

(5) $\dim \mathrm{Ext}^1_A(S_{2,n}, S_{2,n}) \leq 1$ *for all* n.

(6) $\dim \mathrm{Ext}^1_A(S_{1,n}, S_{1,n}) \leq 1$ *for all* n.

Remark 3.5.11. In fact, one can show the following stronger version of Lemma 3.5.10(5,6).

(5') $\mathrm{Ext}^1_A(S_{2,n}, S_{2,n}) = 0$ for all n.

(6') $\mathrm{Ext}^1_A(S_{1,n}, S_{1,n}) = 0$ for all n.

Proof of Lemma 3.5.10. (1,2) Consider a minimal projective resolution of $X_1(\lambda)$

$$P_1 \to P_2 \xrightarrow{f_\lambda} P_1 \to X_1(\lambda) \to 0$$

where f_λ sends $e_2 \in P_2$ to $\gamma - \lambda\beta \in P_1$. More precisely, we have

$$\begin{pmatrix} \mathbf{k}\, e_1 \\ \mathbf{k}\, \beta + \mathbf{k}\, \gamma \end{pmatrix} \xrightarrow{e_1 \to \alpha} \begin{pmatrix} \mathbf{k}\, \alpha \\ \mathbf{k}\, e_2 \end{pmatrix} \xrightarrow{e_2 \to \gamma - \lambda\beta} \begin{pmatrix} \mathbf{k}\, e_1 \\ \mathbf{k}\, \beta + \mathbf{k}\, \gamma \end{pmatrix} \longrightarrow P_1/(\mathbf{k}(\gamma - \lambda\beta)) \longrightarrow 0.$$

Applying $\mathrm{Hom}_A(-, X_1(\lambda'))$ to the truncated projective resolution of the above, we obtain the following complex

$$\mathbf{k} \xleftarrow{\;0\;} \mathbf{k} \xleftarrow{\;g\;} \mathbf{k} \longleftarrow 0.$$

If g is zero, this is case (1). If $g \neq 0$, this is case (2).

(3) The proof is similar to the above by considering minimal projective resolutions of S_1 and S_2.

(4) This is clear since P_2 is a projective module.

(5,6) Let S be either $S_{2,n}$ or $S_{1,n}$. By Example 3.5.1, $\mathrm{fpd}(\mathrm{Mod}_{f.d} - \mathbf{k}\, K_2) = 1$. This implies that

$$\dim \mathrm{Ext}^1_{\mathbf{k}\, K_2}(S, S) \leq 1$$

where S is considered as an indecomposable K_2-module.

Let us make a comment before we continue the proof. Following a more careful analysis, one can actually show that

$$\mathrm{Ext}^1_{\mathbf{k}\, K_2}(S, S) = 0.$$

Using this fact, the rest of the proof would show the assertions (5',6') in Remark 3.5.11.

Now we continue the proof. There is a projective cover $P_1^b \xrightarrow{f} S$ so that $\ker f$ is a direct sum of finitely many copies of S_2. Since P_2 is the projective cover of S_2, we have a minimal projective resolution

$$\longrightarrow P_2^a \longrightarrow P_1^b \longrightarrow S \longrightarrow 0$$

for some a, b. In the category $\mathrm{Mod}_{f.d} - \mathbf{k}\, K_2$, we have a minimal projective resolution of S

$$0 \longrightarrow S_2^a \longrightarrow P_1^b \longrightarrow S \longrightarrow 0$$

where S_2 is a projective $\mathbf{k}\, K_2$-module. Hence we have a morphism of complexes

$$\begin{tikzcd} & {} \arrow[r] & P_2^a \arrow[r] \arrow[d] & P_1^b \arrow[r] \arrow[d, "="] & S \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & {} & S_2^a \arrow[r] & P_1^b \arrow[r] & S \arrow[r] & 0 \end{tikzcd}$$

Applying $\mathrm{Hom}_A(-, S)$ to above, we obtain that

$$\begin{tikzcd} \cdots & \mathrm{Hom}_A(P_2^a, S) \arrow[l] & \mathrm{Hom}_A(P_1^b, S) \arrow[l, "h"'] & \mathrm{Hom}_A(S, S) \arrow[l] & 0 \arrow[l] \\ \cdots & \mathrm{Hom}_A(S_2^a, S) \arrow[l] \arrow[u, "g"] & \mathrm{Hom}_A(P_1^b, S) \arrow[l, "f"'] \arrow[u, "="'] & \mathrm{Hom}_A(S, S) \arrow[l] \arrow[u, "="'] & 0 \arrow[l] \end{tikzcd}$$

Note that g is an isomorphism. Since $\dim \mathrm{Ext}^1_{\mathbf{k}\, K_2}(S, S) \leq 1$, the cokernel of f has dimension at most 1. Since g is an isomorphism, the cokernel of h has dimension at most 1. This implies that $\mathrm{Ext}^1_A(S, S)$ has dimension at most 1. □

Proposition 3.5.12. *Let $\mathfrak{A}$ be the category $\mathrm{Mod}_{f.d.} - A$ where A is in Example 3.5.5.*

(1) $\mathrm{fpd}^n\, \mathfrak{A} = \begin{cases} \sqrt{2} & n = 2, \\ 1 & n \neq 2. \end{cases}$ *As a consequence,* $\mathrm{fpd}\, \mathfrak{A} = \sqrt{2}$.

(2) $SI(\mathfrak{A}) = 2$.

(3) $\mathrm{fpd}\, \mathcal{A} \geq \sqrt{2}$.

Proof. (1) This is a consequence of Lemmas 3.5.9 and 3.5.10. Parts (2,3) follow from part (1). □

Remark 3.5.13. Let A be the algebra given in Example 3.5.5. We list some facts, comments and questions.

(1) The algebra A is non-connected $\mathbb{N}$-graded Koszul.

(2) The minimal projective resolutions of S_1 and S_2 are

$$\cdots \longrightarrow P_1^{\oplus 4} \longrightarrow P_2^{\oplus 4} \longrightarrow P_1^{\oplus 2} \longrightarrow P_2^{\oplus 2} \longrightarrow P_1 \longrightarrow S_1 \longrightarrow 0,$$

and

$$\cdots \longrightarrow P_2^{\oplus 4} \longrightarrow P_1^{\oplus 2} \longrightarrow P_2^{\oplus 2} \longrightarrow P_1 \longrightarrow P_2 \longrightarrow S_2 \longrightarrow 0.$$

(3) For $i \geq 0$, we have

$$\mathrm{Ext}_A^i(S_1, S_1) = \begin{cases} \mathbf{k}^{\oplus 2^{i/2}} & i \text{ is even} \\ 0 & \text{is odd} \end{cases}$$

$$\mathrm{Ext}_A^i(S_1, S_2) = \begin{cases} 0 & i \text{ is even} \\ \mathbf{k}^{\oplus 2^{(i+1)/2}} & \text{is odd} \end{cases}$$

$$\mathrm{Ext}_A^i(S_2, S_2) = \begin{cases} \mathbf{k}^{\oplus 2^{i/2}} & i \text{ is even} \\ 0 & \text{is odd} \end{cases}$$

$$\mathrm{Ext}_A^i(S_2, S_1) = \begin{cases} 0 & i \text{ is even} \\ \mathbf{k}^{\oplus 2^{(i-1)/2}} & \text{is odd.} \end{cases}$$

(4) One can check that every algebra of dimension 4 or less has either infinite or integral fpd. Hence, A is an algebra of smallest $\mathbf{k}$-dimension that has finite non-integral (or irrational) fpd. It is unknown if there is a finite dimensional algebra A such that $\mathrm{fpd}(\mathrm{Mod}_{f.d.} - A)$ is transcendental.

(5) Several authors have studied the connection between tame-wildness and complexity [BS10, ES11, Far07, FW11, KÏ3, Ric90]. The algebra A is probably the first explicit example of a tame algebra with infinite complexity.

(6) It follows from part (3) that the fp-curvature of $\mathcal{A} := D^b(\mathrm{Mod}_{f.d.} - A)$ is $\sqrt{2}$ (some details are omitted). As a consequence, $\mathrm{fpg}(\mathcal{A}) = \infty$. By Theorem 3.8.3, the complexity of A is ∞. We don't know what $\mathrm{fpd}\,\mathcal{A}$ is.

3.6 σ-decompositions

We fix a category $\mathcal{C}$ and an endofunctor σ. For a set of bricks B in $\mathcal{C}$ (or a set of atomic objects when $\mathcal{C}$ is triangulated), we define

$$\mathrm{fpd}^n\,|_B\,(\sigma) = \sup\{\rho(A(\phi,\sigma)) \mid \phi := \{X_1, \cdots, X_n\} \in \Phi_{n,b}, \text{ and } X_i \in B \ \ \forall i\}.$$

Let $\Lambda := \{\lambda\}$ be a totally ordered set. We say a set of bricks B in $\mathcal{C}$ has a σ-*decomposition* $\{B^\lambda\}_{\lambda\in\Lambda}$ (based on Λ) if the following holds.

(1) B is a disjoint union $\bigcup_{\lambda\in\Lambda} B^\lambda$.

(2) If $X \in B^\lambda$ and $Y \in B^\delta$ with $\lambda < \delta$, $\mathrm{Hom}_{\mathcal{C}}(X, \sigma(Y)) = 0$.

The following lemma is easy.

Lemma 3.6.1. *Let n be a positive integer. Suppose that B has a σ-decomposition $\{B^\lambda\}_{\lambda\in\Lambda}$. Then*

$$\mathrm{fpd}^n\,|_B(\sigma) \le \sup_{\lambda\in\Lambda, m\le n} \{\mathrm{fpd}^m\,|_{B^\lambda}(\sigma)\}.$$

Proof. Let ϕ be a brick set that is used in the computation of $\mathrm{fpd}^n\,|_B(\sigma)$. Write

$$\phi = \phi_{\lambda_1} \cup \cdots \cup \phi_{\lambda_s} \tag{3.6.1.1}$$

where λ_i is strictly increasing and $\phi_{\lambda_i} = \phi \cap B^{\lambda_i}$. For any objects $X \in \phi^{\lambda_i}$ and $Y \in \phi^{\lambda_j}$, where $\lambda_i < \lambda_j$, by definition, $\mathrm{Hom}_{\mathcal{C}}(X, \sigma(Y)) = 0$. Listing the objects in ϕ in the order that suggested by (3.6.1.1), then the adjacency matrix of (ϕ,σ) is of the form

$$A(\phi,\sigma)=\begin{pmatrix} A_{11} & 0 & 0 & \cdots & 0 \\ * & A_{22} & 0 & \cdots & 0 \\ * & * & A_{33} & \cdots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ * & * & * & \cdots & A_{ss} \end{pmatrix}$$

where each A_{ii} is the adjacency matrix $A(\phi_{\lambda_i},\sigma)$. By definition,

$$\rho(A_{ii}) \leq \operatorname{fpd}^{s_i}|_{B^{\lambda_i}}(\sigma)$$

where s_i is the size of A_{ii}, which is no more than n. Therefore

$$\rho(A(\phi,\sigma)) = \max_i\{\rho(A_{ii})\} \leq \sup_{\lambda\in\Lambda, m\leq n}\{\operatorname{fpd}^m|_{B^\lambda}(\sigma)\}.$$

The assertion follows. □

We give some examples of σ-decompositions.

Example 3.6.2. Let $\mathfrak{A}$ be an abelian category and $\mathcal{A}$ be the derived category $D^b(\mathfrak{A})$. Let $[n]$ be the n-fold suspension Σ^n.

(1) Suppose that α is an endofunctor of $\mathfrak{A}$ and $\overline{\alpha}$ is the induced endofunctor of $\mathcal{A}$. For each $n\in\mathbb{Z}$, let $B^n := \{M[-n] \mid M \text{ is a brick in } \mathfrak{A}\}$ and $B := \bigcup_{n\in\mathbb{Z}} B^n$. If $M_i[-n_i]\in B^{n_i}$, for $i=1,2$, such that $n_1<n_2$, then

$$\operatorname{Hom}_{\mathcal{A}}(M_1[-n_1],\overline{\alpha}(M_2[-n_2])) = \operatorname{Ext}_{\mathfrak{A}}^{n_1-n_2}(M_1,\alpha(M_2)) = 0.$$

Then B has a $\overline{\alpha}$-decomposition $\{B^n\}_{n\in\mathbb{Z}}$ based on $\mathbb{Z}$.

(2) Suppose $g := \operatorname{gldim}\mathfrak{A} < \infty$. Let σ be the functor $\Sigma^g\circ\overline{\alpha}$. For each $n\in\mathbb{Z}$, let $B^n := \{M[n] \mid M \text{ is a brick in } \mathfrak{A}\}$ and $B := \bigcup_{n\in\mathbb{Z}} B^n$. If $M_i[n_i]\in B^{n_i}$, for $i=1,2$, such that $n_1<n_2$, then

$$\operatorname{Hom}_{\mathcal{A}}(M_1[n_1],\sigma(M_2[n_2])) = \operatorname{Ext}_{\mathfrak{A}}^{n_2-n_1+g}(M_1,\alpha(M_2)) = 0.$$

Then B has a σ-decomposition $\{B^n\}_{n\in\mathbb{Z}}$ based on $\mathbb{Z}$.

Example 3.6.3. Let C be a smooth projective curve and let $\mathfrak{A}$ be the the category of coherent sheaves over C. Every coherent sheaf over C is a direct sum of a torsion subsheaf and a locally free subsheaf. Define

$$B^0 = \{T \text{ is a torsion brick object in } \mathfrak{A}\},$$

$$B^{-1} = \{F \text{ is a locally free brick object in } \mathfrak{A}\}$$

and

$$B = B^{-1} \cup B^0.$$

Let σ be the functor $E^1 := \mathrm{Ext}^1_{\mathfrak{A}}(-,-)$. If $F \in B^{-1}$ and $T \in B^0$, then

$$\mathrm{Ext}^1_{\mathfrak{A}}(F,T) = 0.$$

Hence, B has an E^1-decomposition based on the totally ordered set $\Lambda := \{-1, 0\}$.

The next example is given in [BB07].

Example 3.6.4. Let C be an elliptic curve. Let $\mathfrak{A}$ be the the category of coherent sheaves over C and $\mathcal{A}$ be the derived category $D^b(\mathfrak{A})$.

First we consider coherent sheaves. Let Λ be the totally ordered set $\mathbb{Q} \cup \{+\infty\}$. The slope of a coherent sheaf $X \neq 0$ [BB07, Definition 4.6] is defined to be

$$\mu(X) := \frac{\chi(X)}{\mathrm{rank}(X)} \in \Lambda$$

where $\chi(X)$ is the Euler characteristic of X and $\mathrm{rank}(X)$ is the rank of X. If X and Y are bricks such that $\mu(X) < \mu(Y)$, by [BB07, Corollary 4.11], X and Y are semistable, and thus by [BB07, Proposition 4.9(1)], $\mathrm{Hom}_{\mathfrak{A}}(Y, X) = 0$. By Serre duality (namely, Calabi-Yau property),

$$\mathrm{Hom}_{\mathcal{A}}(X, Y[1]) = \mathrm{Ext}^1_{\mathfrak{A}}(X, Y) = \mathrm{Hom}_{\mathfrak{A}}(Y, X)^* = 0. \tag{3.6.4.1}$$

Write $B = \Phi_{1,b}(\mathfrak{A})$ and B^λ be the set of (semistable) bricks with slope λ. Then $B = \bigcup_{\lambda \in \Lambda} B^\lambda$. By (3.6.4.1), $\mathrm{Ext}^1_{\mathfrak{A}}(X, Y) = 0$ when $X \in B^\lambda$ and $Y \in B^\nu$ with $\lambda < \nu$. Hence B has an E^1-decomposition. By Lemma 3.6.1, for every $n \geq 1$,

$$\mathrm{fpd}^n(E^1) = \mathrm{fpd}^n|_B(E^1) \leq \sup_{\lambda \in \Lambda, m \leq n} \{\mathrm{fpd}^m|_{B^\lambda}(E^1)\}.$$

Next we compute $\mathrm{fpd}^n\mid_{B^\lambda}(E^1)$. Let SS^λ be the full subcategory of $\mathfrak{A}$ consisting of semistable coherent sheaves of slope λ. By [BB07, Summary], SS^λ is an abelian category that is equivalent to SS^∞. Therefore one only needs to compute $\mathrm{fpd}^n\mid_{B^\infty}(E^1)$ in the category SS^∞. Note that SS^∞ is the abelian category of torsion sheaves and every brick object in SS^∞ is of the form $\mathcal{O}_p$ for some $p \in C$. In this case, $A(\phi, E^1)$ is the identity matrix. Consequently, $\rho(A(\phi, E^1)) = 1$. This shows that $\mathrm{fpd}^n\mid_{B^\lambda}(E^1) = \mathrm{fpd}^n\mid_{B^\infty}(E^1) = 1$ for all $n \geq 1$. It is clear that $\mathrm{fpd}^n(E^1) \geq \mathrm{fpd}^n\mid_{B^\infty}(E^1) = 1$. Combining with Lemma 3.6.1, we obtain that $\mathrm{fpd}^n(E^1) = 1$ for all n. (The above approach works for functors other than E^1.)

Finally we consider the fp-dimension for the derived category $\mathcal{A}$. It follows from Theorem 3.3.5(3) that

$$\mathrm{fpd}^n(\Sigma) = \mathrm{fpd}^n(E^1) = 1$$

for all $n \geq 1$. By definition,

$$\mathrm{fpd}(\mathcal{A}) = \mathrm{fpd}(\mathfrak{A}) = 1.$$

As we explained before fpd is an indicator of the representation types of categories.

Drozd-Greuel studied a tame-wild dichotomy for vector bundles on projective curves [DG01] and introduced the notion of VB-finite, VB-tame and VB-wild similar to the corresponding notion in the representation theory of finite dimensional algebras. In [DG01] Drozd-Greuel showed the following:

Let C be a connected smooth projective curve. Then

(a) C is VB-finite if and only if C is $\mathbb{P}^1$.

(b) C is VB-tame if and only if C is elliptic (that is, of genus 1).

(c) C is VB-wild if and only if C has genus $g \geq 2$.

We now prove a fp-version of Drozd-Greuel's result [DG01, Theorem 1.6]. We thank Max Lieblich for providing ideas in the proof of Proposition 3.6.5(3) next.

Proposition 3.6.5. *Suppose* $\mathbf{k} = \mathbb{C}$. *Let* $\mathbb{X}$ *be a connected smooth projective curve and let* g *be the genus of* $\mathbb{X}$.

(1) If $g = 0$ *or* $\mathbb{X} = \mathbb{P}^1$, *then* $\operatorname{fpd} D^b(coh(\mathbb{X})) = 1$.

(2) If $g = 1$ *or* $\mathbb{X}$ *is an elliptic curve, then* $\operatorname{fpd} D^b(coh(\mathbb{X})) = 1$.

(3) If $g \geq 2$, *then* $\operatorname{fpd} D^b(coh(\mathbb{X})) = \infty$.

Proof. (1) The assertion follows from (3.5.1.4).

(2) The assertion follows from Example 3.6.4.

(3) By Theorem 3.3.5(4), $\operatorname{fpd}(D^b(coh(\mathbb{X}))) = \operatorname{fpd}(coh(\mathbb{X}))$. Hence it suffices to show that $\operatorname{fpd}(coh(\mathbb{X})) = \infty$.

For each n, let $\{x_i\}_{i=1}^n$ be a set of n distinct points on $\mathbb{X}$. By [DG01, Lemma 1.7], we might further assume that $2x_i \nsim x_j + x_k$ for all $i \neq j$, as divisors on $\mathbb{X}$. Write $\mathcal{E}_i := \mathcal{O}(x_i)$ for all i. By [DG01, p.11], $\operatorname{Hom}_{\mathcal{O}_{\mathbb{X}}}(\mathcal{E}_i, \mathcal{E}_j) = 0$ for all $i \neq j$, which is also a consequence of a more general result [HL10, Proposition 1.2.7]. It is clear that $\operatorname{Hom}_{\mathcal{O}_{\mathbb{X}}}(\mathcal{E}_i, \mathcal{E}_i) = \mathbf{k}$ for all i. Let ϕ_n be the set $\{\mathcal{E}_1, \ldots, \mathcal{E}_n\}$. Then it is a brick set of non-isomorphic vector bundles on $\mathbb{X}$ (which are stable with $\operatorname{rank}(\mathcal{E}_i) = \deg(\mathcal{E}_i) = 1$ for all i).

Define the sheaf $\mathcal{H}_{ij} = \mathcal{H}om(\mathcal{E}_i, \mathcal{E}_j)$ for all i, j. Then $\deg(\mathcal{H}_{ij}) = 0$. By the Riemann-Roch Theorem, we have

$$\begin{aligned} 0 &= \deg(\mathcal{H}_{ij}) \\ &= \chi(\mathcal{H}_{ij}) - \operatorname{rank}(\mathcal{H}_{ij})\chi(\mathcal{O}_{\mathbb{X}}) \\ &= \dim \operatorname{Hom}_{\mathcal{O}_{\mathbb{X}}}(\mathcal{E}_i, \mathcal{E}_j) - \dim \operatorname{Ext}^1_{\mathcal{O}_{\mathbb{X}}}(\mathcal{E}_i, \mathcal{E}_j) - (1-g) \\ &= \delta_{ij} - \dim \operatorname{Ext}^1_{\mathcal{O}_{\mathbb{X}}}(\mathcal{E}_i, \mathcal{E}_j) + (g-1), \end{aligned}$$

which implies that $\dim \operatorname{Ext}^1_{\mathcal{O}_{\mathbb{X}}}(\mathcal{E}_i, \mathcal{E}_j) = g - 1 + \delta_{ij}$. This formula was also given in [DG01, p.11 before Lemma 1.7] when $i \neq j$.

Define the matrix A_n with entries $a_{ij} := \dim \operatorname{Ext}^1_{\mathcal{O}_{\mathbb{X}}}(\mathcal{E}_i, \mathcal{E}_j) = g - 1 + \delta_{ij}$, which is the adjacency matrix of (ϕ_n, E^1). This matrix has entries g along the diagonal and entries

$g-1$ everywhere else. Therefore the vector $(1,\ldots,1)$ is an eigenvector for this matrix with eigenvalue $n(g-1)+1$. So $\rho(A_n) \geq n(g-1)+1 \geq n+1$. Since we can define ϕ_n for arbitrarily large n, we must have $\mathrm{fpd}(coh(\mathbb{X})) = \infty$. $\square$

Question 3.6.6. Let $\mathbb{X}$ be a smooth irreducible projective curve of genus $g \geq 2$. Is $\mathrm{fpd}^n(\mathbb{X})$ finite for each n? If yes, do these invariants recover g?

Proposition 3.6.7. *Suppose* $\mathbf{k} = \mathbb{C}$*. Let* $\mathbb{Y}$ *be a smooth projective scheme of dimension at least 2. Then*

$$\mathrm{fpd}^1(coh(\mathbb{Y})) = \mathrm{fpd}(coh(\mathbb{Y})) = \mathrm{fpd}^1(D^b(coh(\mathbb{Y}))) = \mathrm{fpd}(D^b(coh(\mathbb{Y}))) = \infty.$$

Proof. It is clear that $\mathrm{fpd}^1(coh(\mathbb{Y}))$ is smallest among these four invariants. It suffices to show that $\mathrm{fpd}^1(coh(\mathbb{Y})) = \infty$.

It is well-known that $\mathbb{Y}$ contains an irreducible projective curve $\mathbb{X}$ of arbitrarily large (either geometric or arithmetic) genus, see, for example, [CFZ16, Theorem 0.1] or [Che97, Theorems 1 and 2]. Let $\mathcal{O}_{\mathbb{X}}$ be the coherent sheaf corresponding to the curve $\mathbb{X}$ and let g be the arithmetic genus of $\mathbb{X}$. In the abelian category $coh(\mathbb{X})$, we have

$$\dim \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{X}}}(\mathcal{O}_{\mathbb{X}}, \mathcal{O}_{\mathbb{X}}) = \dim H^1(\mathbb{X}, \mathcal{O}_{\mathbb{X}}) = g.$$

Since $coh(\mathbb{X})$ is a full subcategory of $coh(\mathbb{Y})$, we have

$$\dim \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{Y}}}(\mathcal{O}_{\mathbb{X}}, \mathcal{O}_{\mathbb{X}}) \geq \dim \mathrm{Ext}^1_{\mathcal{O}_{\mathbb{X}}}(\mathcal{O}_{\mathbb{X}}, \mathcal{O}_{\mathbb{X}}) = g.$$

By taking $\phi = \{\mathcal{O}_{\mathbb{X}}\}$, one sees that $\mathrm{fpd}^1(coh(\mathbb{Y})) \geq \mathrm{fpd}^1(coh(\mathbb{X})) \geq g$ for all such $\mathbb{X}$. Since g can be arbitrarily large, the assertion follows. $\square$

3.7 Representation types

3.7.1 Representation types

We first recall some known definitions and results.

Definition 3.7.1. Let A be a finite dimensional algebra.

(1) We say A is of *finite representation type* if there are only finitely many isomorphism classes of finite dimensional indecomposable left A-modules.

(2) We say A is *tame* or *of tame representation type* if it is not of finite representation type, and for every $n \in \mathbb{N}$, all but finitely many isomorphism classes of n-dimensional indecomposables occur in a finite number of one-parameter families.

(3) We say A is *wild* or *of wild representation type* if, for every finite dimensional $\mathbf{k}$-algebra B, the representation theory of B can be embedded into that of A.

The following is the famous trichotomy result due to Drozd [Dro80].

Theorem 3.7.2 (Drozd's trichotomy theorem)**.** *Every finite dimensional algebra is either of finite, tame, or wild representation type.*

Remark 3.7.3. (1) An equivalent and more precise definition of a wild algebra is the following. An algebra A is called *wild* if there is a faithful exact embedding of abelian categories

$$Emb: \quad \mathrm{Mod}_{f.d.} - \mathbf{k}\langle x, y\rangle \to \mathrm{Mod}_{f.d.} - A \tag{3.7.3.1}$$

that respects indecomposability and isomorphism classes.

(2) A stronger notion of wildness is the following. An algebra A is called *strictly wild*, also called *fully wild*, if Emb in (3.7.3.1) is a fully faithful embedding.

(3) It is clear that strictly wild is wild. The converse is not true.

We collect some celebrated results in terms of representation types of path algebras [Gab72] and [Naz73, DF73].

Theorem 3.7.4. *Let Q be a finite connected quiver.*

(1) *[Gab72] The path algebra* $\mathbf{k}Q$ *is of finite representation type if and only if the underlying graph of* Q *is a Dynkin diagram of type* ADE.

(2) *[Naz73, DF73] The path algebra* $\mathbf{k}Q$ *is of tame representation type if and only if the underlying graph of* Q *is an extended Dynkin diagram of type* $\widetilde{A}\widetilde{D}\widetilde{E}$.

Our main goal in this section is to prove Theorem 3.0.4. We thank Klaus Bongartz for suggesting the following lemma.

Lemma 3.7.5. *[Bon16a] Let* A *be a finite dimensional algebra that is strictly wild. Then, for each integer* $a > 0$, *there is a finite dimensional brick left* A*-module* N *such that* $\dim \operatorname{Ext}^1_A(N, N) \geq a$.

Proof. Let V be the vector space $\bigoplus_{i=1}^{a} \mathbf{k}\, x_i$ and let B be the finite dimensional algebra $\mathbf{k}\langle V\rangle/(V^{\otimes 2})$. By [Bon16b, Theorem 2(i)], there is a fully faithful exact embedding

$$\mathrm{Mod}_{f.d.} - B \longrightarrow \mathrm{Mod}_{f.d.} - \mathbf{k}\langle x, y\rangle.$$

Since A is strictly wild, there is a fully faithful exact embedding

$$\mathrm{Mod}_{f.d.} - \mathbf{k}\langle x, y\rangle \longrightarrow \mathrm{Mod}_{f.d.} - A.$$

Hence we have a fully faithful exact embedding

$$F: \quad \mathrm{Mod}_{f.d.} - B \longrightarrow \mathrm{Mod}_{f.d.} - A. \tag{3.7.5.1}$$

Let S be the trivial B-module $B/B_{\geq 1}$. It follows from an easy calculation that $\dim \operatorname{Ext}^1_B(S, S) = \dim(V)^* = a$. Since F is fully faithful exact, F induces an injection

$$F : \operatorname{Ext}^1_B(S, S) \longrightarrow \operatorname{Ext}^1_A(F(S), F(S)).$$

Thus $\dim \operatorname{Ext}^1_A(F(S), F(S)) \geq a$. Since S is simple, it is a brick. Hence, $F(S)$ is a brick. The assertion follows by taking $N = F(S)$. $\square$

Proposition 3.7.6. *(1) Let A be a finite dimensional algebra that is strictly wild, then*

$$\mathrm{fpd}^1(E^1) = \mathrm{fpd}(\mathfrak{A}) = \mathrm{fpd}(\mathcal{A}) = \infty.$$

(2) If $A := \mathbf{k}Q$ is wild, then

$$\mathrm{fpd}^1(E^1) = \mathrm{fpd}(\mathfrak{A}) = \mathrm{fpd}(\mathcal{A}) = \infty.$$

Proof. (1) For each integer a, by Lemma 3.7.5, there is a brick N in $\mathfrak{A}$ such that $\mathrm{Ext}^1_{\mathfrak{A}}(N,N) \geq a$. Hence $\mathrm{fpd}^1(E^1) \geq a$. Since a is arbitrary, $\mathrm{fpd}^1(E^1) = \infty$. Consequently, $\mathrm{fpd}(\mathfrak{A}) = \infty$. By Lemma 3.2.10, $\mathrm{fpd}(\mathcal{A}) = \infty$.

(2) It is well-known that a wild path algebra is strictly wild, see a comment of Gabriel [Gab75, p.149] or [Ari05, Proposition 7]. The assertion follows from part (1). □

The following lemma is based on a well-understood AR-quiver theory for acyclic quivers of finite representation type and the hammock theory introduced by Brenner [Bre86]. We refer to [RV87] if the reader is interested in a more abstract version of the hammock theory.

For a class of quivers including all ADE quivers, there is a convenient (though not essential) way of positioning the vertices as in [ASS06, Example IV.2.6]. A quiver Q is called *well-positioned* if the vertices of Q are located so that all arrows are strictly from the right to the left of the same horizontal distance. For example, the following quiver D_n is well-positioned:

$$\begin{tikzcd}
 & & & & n-1 \arrow[dl] \\
1 & 2 \arrow[l] & \cdots \arrow[l] & n-2 \arrow[l] & \\
 & & & & n \arrow[ul]
\end{tikzcd}$$

Lemma 3.7.7. *Let Q be a quiver such that*

(i) the underlying graph of Q is a Dynkin diagram of type A, or D, or E, and that

(ii) Q is well-positioned.

Let $A = \mathbf{k}Q$ and let M, N be two indecomposable left A-modules in the AR-quiver of A. Then the following hold.

(1) There is a standard way of defining the order or degree for indecomposable left A-modules M, denoted by $\deg M$, such that all arrows in the the AR-quiver have degree 1, or equivalently, all arrows are from the left to the right of the same horizontal distance. As in (3.5.4.2), when $Q = A_2$, $\deg P_2 = 0$, $\deg P_1 = 1$ and $\deg I_1 = 2$.

(2) If $\mathrm{Hom}_A(M, N) \neq 0$, then $\deg M \leq \deg N$.

(3) The degree of the AR-translation τ is -2.

(4) If $\mathrm{Ext}^1_A(M, N) \neq 0$, then $\deg M \geq \deg N + 2$.

(5) There is no oriented cycle in the E^1-quiver of $\mathfrak{A} := \mathrm{Mod}_{f.d.} - \mathbf{k}Q$, denoted by $Q^{E^1}_{\mathfrak{A}}$, defined before Lemma 3.2.11.

(6) $\mathrm{fpd}(\mathfrak{A}) = 0$.

Proof. (1) This is a well-known fact in AR-quiver theory. For each given quiver Q as described in (i,ii), one can build the AR-quiver by using the Auslander-Reiten translation τ and *the Knitting Algorithm*, see [Sch14, Ch. 3]. Some explicit examples are given in [Gab80, Ch. 6] and [Sch14, Ch. 3].

(2) This follows from (1). Note that the precise dimension of $\mathrm{Hom}_A(M, N)$ can be computed by using hammock theory [Bre86, RV87]. Some examples are given in [Sch14, Ch. 3].

(3) This follows from the definition of the translation τ in the AR-quiver theory [ARS97, VII]. See also, [CB, Remarks (2), p. 23].

(4) By Serre duality, $\mathrm{Ext}^1_R(M,N) = \mathrm{Hom}_A(N,\tau M)^*$ [RVdB02, Proposition I.2.3] or [CB, Lemma 1, p. 22]. If $\mathrm{Ext}^1_R(M,N) \neq 0$, then, by Serre duality and part (2), $\deg N \leq \deg \tau M = \deg M - 2$. Hence $\deg M \geq \deg N + 2$.

(5) In this case, every indecomposable module is a brick. Hence the E^1-quiver $Q^{E^1}_{\mathfrak{A}}$ has the same vertices as the AR-quiver. By part (4), if there is an arrow from M to N in the quiver $Q^{E^1}_{\mathfrak{A}}$, then $\deg M \geq \deg N + 2$. This means that all arrows in $Q^{E^1}_{\mathfrak{A}}$ are from the right to the left. Therefore there is no oriented cycle in $Q^{E^1}_{\mathfrak{A}}$.

(6) This follows from part (5), Theorem 3.1.8(1) and Lemma 3.2.11. □

Theorem 3.7.8. *Let Q be a finite quiver whose underlying graph is a Dynkin diagram of type ADE and let $A = \mathbf{k}Q$. Then* $\mathrm{fpd}(\mathfrak{A}) = \mathrm{fpd}(\mathcal{A}) = \mathrm{fpgldim}(\mathcal{A}) = 0$.

Proof. Since the path algebra A is hereditary, $\mathfrak{A}$ is a-hereditary of global dimension 1. By Theorem 3.3.5(3), $\mathrm{fpd}(\mathfrak{A}) = \mathrm{fpd}(\mathcal{A})$. If Q_1 and Q_2 are two quivers whose underlying graphs are the same, then, by Bernstein-Gelfand-Ponomarev (BGP) reflection functors [BGP73], $D^b(\mathrm{Mod}_{f.d.} - \mathbf{k}Q_1)$ and $D^b(\mathrm{Mod}_{f.d.} - \mathbf{k}Q_2)$ are triangulated equivalent. Hence we only need prove the statement for one representative. Now we can assume that Q satisfies the hypotheses (i,ii) of Lemma 3.7.7. By Lemma 3.7.7(6), $\mathrm{fpd}(\mathfrak{A}) = 0$. Therefore $\mathrm{fpd}(\mathcal{A}) = 0$, or equivalently, $\mathrm{fpd}(\Sigma) = 0$. By Theorem 3.3.5(1), $\mathrm{fpd}(\Sigma^i) = 0$ for all $i \neq 0, 1$. Therefore $\mathrm{fpgldim}(\mathcal{A}) = 0$. □

3.7.2 Weighted projective lines

To prove Theorem 3.0.4, it remains to show part (2) of the theorem. Our proof uses a result of [CGW+19b] about weighted projective lines, which we now review. Details can be found in [GL87, Section 1].

For $t \geq 1$, let $\mathbf{p} := (p_0, p_1, \cdots, p_t)$ be a $(t+1)$-tuple of positive integers, called the *weight sequence*. Let $\mathbf{D} := (\lambda_0, \lambda_1, \cdots, \lambda_t)$ be a sequence of distinct points of the projective line $\mathbb{P}^1$ over $\mathbf{k}$. We normalize $\mathbf{D}$ so that $\lambda_0 = \infty$, $\lambda_1 = 0$ and $\lambda_2 = 1$ (if $t \geq 2$). Let

$$S := \mathbf{k}[X_0, X_1, \cdots, X_t]/(X_i^{p_i} - X_1^{p_1} + \lambda_i X_0^{p_0}, i = 2, \cdots, t).$$

The image of X_i in S is denoted by x_i for all i. Let $\mathbb{L}$ be the abelian group of rank 1 generated by $\overrightarrow{x_i}$ for $i = 0, 1, \cdots, t$ and subject to the relations

$$p_0\overrightarrow{x_0} = \cdots = p_i\overrightarrow{x_i} = \cdots = p_t\overrightarrow{x_t} =: \overrightarrow{c}\,.$$

The algebra S is $\mathbb{L}$-graded by setting $\deg x_i = \overrightarrow{x_i}$. The corresponding *weighted projective line*, denoted by $\mathbb{X}(\mathbf{p}, \mathbf{D})$ or simply $\mathbb{X}$, is a noncommutative space whose category of coherent sheaves is given by the quotient category

$$coh(\mathbb{X}) \coloneqq \frac{\mathrm{gr}^{\mathbb{L}} - S}{\mathrm{gr}^{\mathbb{L}}_{f.d.} - S}$$

where $\mathrm{gr}^{\mathbb{L}} - S$ is the category of noetherian $\mathbb{L}$-graded left S-modules and $\mathrm{gr}^{\mathbb{L}}_{f.d.} - S$ is the full subcategory of $\mathrm{gr}^{\mathbb{L}} - S$ consisting of finite dimensional modules.

The weighted projective lines are classified into the following three classes:

$$\mathbb{X} \text{ is } \begin{cases} \textit{domestic} & \text{if } \mathbf{p} \text{ is } (p,q), (2,2,n), (2,3,3), (2,3,4), (2,3,5); \\ \textit{tubular} & \text{if } \mathbf{p} \text{ is } (2,3,6), (3,3,3), (2,4,4), (2,2,2,2); \\ \textit{wild} & \text{otherwise.} \end{cases}$$

Let $\mathbb{X}$ be a weighted projective curve. Let $Vect(\mathbb{X})$ be the full subcategory of $coh(\mathbb{X})$ consisting of all vector bundles. Similar to the elliptic curve case [Example 3.6.4], one can define the concepts of *degree*, *rank* and *slope* of a vector bundle on a weighted projective curve $\mathbb{X}$, see [LM93, Section 2] for details. For each $\mu \in \mathbb{Q} \cup \{\infty\}$, let $Vect_\mu(\mathbb{X})$ be the full subcategory of $Vect(\mathbb{X})$ consisting of all vector bundles of slope μ.

Lemma 3.7.9. *Let $\mathbb{X} = \mathbb{X}(\mathbf{p}, \mathbf{D})$ be a weighted projective line.*

(1) $coh(\mathbb{X})$ is noetherian and hereditary.

(2)

$$D^b(coh(\mathbb{X})) \cong \begin{cases} D^b(\mathrm{Mod}_{f.d.} - \mathbf{k}\,\widetilde{A}_{p,q}) & \text{if } \mathbf{p} = (p,q), \\ D^b(\mathrm{Mod}_{f.d.} - \mathbf{k}\,\widetilde{D}_n) & \text{if } \mathbf{p} = (2,2,n), \\ D^b(\mathrm{Mod}_{f.d.} - \mathbf{k}\,\widetilde{E}_6) & \text{if } \mathbf{p} = (2,3,3), \\ D^b(\mathrm{Mod}_{f.d.} - \mathbf{k}\,\widetilde{E}_7) & \text{if } \mathbf{p} = (2,3,4), \\ D^b(\mathrm{Mod}_{f.d.} - \mathbf{k}\,\widetilde{E}_8) & \text{if } \mathbf{p} = (2,3,5). \end{cases}$$

(3) Let M be a generic simple object in $coh(\mathbb{X})$. Then $\mathrm{Ext}^1_{\mathbb{X}}(M,M) = 1$.

(4) $\mathrm{fpd}^1(coh(\mathbb{X})) \geq 1$.

(5) If $\mathbb{X}$ is tubular or domestic, then $\mathrm{Ext}^1_{\mathbb{X}}(X,Y) = 0$ for all $X \in Vect_{\mu'}(\mathbb{X})$ and $Y \in Vect_{\mu}(\mathbb{X})$ with $\mu' < \mu$.

(6) If $\mathbb{X}$ is domestic, then $\mathrm{Ext}^1_{\mathbb{X}}(X,Y) = 0$ for all $X \in Vect_{\mu'}(\mathbb{X})$ and $Y \in Vect_{\mu}(\mathbb{X})$ with $\mu' \leq \mu$. As a consequence, $\mathrm{fpd}(\Sigma\,|_{Vect_{\mu'}(\mathbb{X})}) = 0$ for all $\mu < \infty$.

(7) Suppose $\mathbb{X}$ is tubular. Then every indecomposable vector bundle $\mathbb{X}$ is semi-stable.

(8) Suppose $\mathbb{X}$ is tubular and let $\mu \in \mathbb{Q}$. Then each $Vect_{\mu}(\mathbb{X})$ is a uniserial category. Accordingly indecomposables in $Vect_{\mu}(\mathbb{X})$ decomposes into Auslander-Reiten components, which all are tubes of finite rank.

Proof. (1) This is well-known.

(2) [GL87, 5.4.1].

(3) Let M be a generic simple object. Then M is a brick and $\mathrm{Ext}^1(M,M) = 1$.

(4) Follows from (3) by taking $\phi := \{M\}$.

(5) This is [Sch12, Corollary 4.34(i)] since tubular is also called elliptic in [Sch12].

(6) This is [Sch12, Comments after Corollary 4.34] since domestic is also called parabolic in [Sch12]. The consequence is clear.

(7) [GL87, Theorem 5.6(i)].

(8) [GL87, Theorem 5.6(iii)]. □

We will use the following result which is proved in [CGW+19b].

Theorem 3.7.10. *[CGW+19b] Let $\mathbb{X}$ be a weighted projective line.*

(1) If $\mathbb{X}$ is domestic, then $\operatorname{fpd} D^b(coh(\mathbb{X})) = 1$*.*

(2) If $\mathbb{X}$ is tubular, then $\operatorname{fpd} D^b(coh(\mathbb{X})) = 1$*.*

(3) If $\mathbb{X}$ is wild, then $\operatorname{fpd} D^b(coh(\mathbb{X})) \geq \dim \operatorname{Hom}_{\mathbb{X}}(\mathcal{O}_{\mathbb{X}}, \mathcal{O}_{\mathbb{X}}(\vec{\omega}))$ *where $\vec{\omega}$ is the dualizing element [GL87, Sec. 1.2].*

There is a similar statement for smooth complex projective curves (Proposition 3.6.5). The authors are interested in answering the following question.

Question 3.7.11. Let $\mathbb{X}$ be a wild weighted projective line. What is the exact value of $\operatorname{fpd}^n D^b(coh(\mathbb{X}))$?

3.7.3 Tubes

The following example is studied in [CGW+19b], which is dependent on direct linear algebra calculations.

Example 3.7.12. [CGW+19b] Let ξ be a primitive nth root of unity. Let T_n be the algebra

$$T_n := \frac{\mathbf{k}\langle g, x\rangle}{(g^n - 1, xg - \xi gx)}.$$

This algebra can be expressed by using a group action. Let G be the group

$$\{g \mid g^n = 1\} \cong \mathbb{Z}/(n)$$

acting on the polynomial ring $\mathbf{k}[x]$ by $g \cdot x = \xi x$. Then T_n is naturally isomorphic to the skew group ring $\mathbf{k}[x] * G$. Let $\overrightarrow{A_{n-1}}$ denote the cycle quiver with n vertices, namely, the quiver with one oriented cycle connecting n vertices. It is also known that T_n is isomorphic to the path algebra of the quiver $\overrightarrow{A_{n-1}}$. Then $\operatorname{fpd}(\operatorname{Mod}_{f.d.} - T_n) = 1$ by [CGW+19b].

3.7.4 Proof of Theorem 3.0.4

Proof of Theorem 3.0.4. Part (1) follows from Theorems 3.7.4(1) and 3.7.8. Part (3) follows from Proposition 3.7.6(2). It remains to deal with part (2).

(2) By Theorem 3.7.4(2), Q must be of type either $\overrightarrow{A_{n-1}}$, or $\widetilde{A}_{p,q}$, or $\widetilde{D}_n$, or $\widetilde{E}_{6,7,8}$. If Q is of type $\overrightarrow{A_{n-1}}$, the assertion follows from Example 3.7.12. If Q is of type $\widetilde{A}_{p,q}$, $\widetilde{D}_n$, or $\widetilde{E}_{6,7,8}$, the assertion follows from Lemma 3.7.9(2) and Theorem 3.7.10(1). □

3.8 Complexity

The concept of complexity was first introduced by Alperin-Evens in the study of group cohomology in 1981 [AE81]. Since then the study of complexity has been extended to finite dimensional algebras, Frobenius algebras, Hopf algebras and commutative algebras. First we recall the classical definition of the complexity for finite dimensional algebras and then give a definition of the complexity for triangulated categories. We give the following modified (but equivalent) version, which can be generalized.

Definition 3.8.1. Let A be a finite dimensional algebra and $T = A/J(A)$ where $J(A)$ is the Jacobson radical of A. Let M be a finite dimensional left A-module.

(1) The *complexity* of M is defined to be

$$cx(M) := \limsup_{n\to\infty} \log_n(\dim \operatorname{Ext}^n_A(M,T)) + 1.$$

(2) The *complexity* of the algebra A is defined to be

$$cx(A) := cx(T).$$

In the original definition of *complexity* by Alperin-Evens [AE81] and in most of other papers, the dimension of n-syzygies is used instead of the dimension of the Ext^n-groups, but it is easy to see that the asymptotic behavior of these two series are the same, therefore these give rise to the same complexity. It is well-known that $cx(M) \leq cx(A)$ for all finite dimensional left A-modules M. Next we introduce the notion of a complexity for a triangulated category which is partially motivated by the work in [BHZ19, Section 4].

Definition 3.8.2. Let $\mathcal{T}$ be a pre-triangulated category. Let d be a real number.

(1) The left subcategory of complexity less than d is defined to be

$$_d\mathcal{T} := \left\{ X \in \mathcal{T} \mid \lim_{n\to\infty} \frac{\dim \operatorname{Hom}_{\mathcal{T}}(X, \Sigma^n(Y))}{n^{d-1}} = 0, \forall\, Y \in \mathcal{T} \right\}.$$

(2) The right subcategory of complexity less than d is defined to be

$$\mathcal{T}_d := \left\{ X \in \mathcal{T} \mid \lim_{n\to\infty} \frac{\dim \operatorname{Hom}_{\mathcal{T}}(Y, \Sigma^n(X))}{n^{d-1}} = 0, \forall\, Y \in \mathcal{T} \right\}.$$

(3) The *complexity* of $\mathcal{T}$ is defined to be

$$cx(\mathcal{T}) := \inf \{ d \mid {_d\mathcal{T}} = \mathcal{T} \}$$

(4) The *Frobenius-Perron complexity* of $\mathcal{T}$ is defined to be

$$\operatorname{fpcx}(\mathcal{T}) := \operatorname{fpg}(\Sigma) + 1.$$

Note that it is not hard to show that $cx(\mathcal{T}) = \inf \{ d \mid \mathcal{T}_d = \mathcal{T} \}$.

Theorem 3.8.3. *Let $\mathcal{T}$ be a pre-triangulated category. Then* $\operatorname{fpcx}(\mathcal{T}) \leq cx(\mathcal{T})$.

Proof. Let d be any number strictly larger than $cx(\mathcal{T})$. We need to show that $\operatorname{fpcx}(\mathcal{T}) \leq d$.

Let $\phi \in \Phi_{m,a}$ be an atomic set and let $X := \bigoplus_{X_i \in \phi} X_i$. Then, by definition,

$$\lim_{n\to\infty} \frac{\dim \operatorname{Hom}_{\mathcal{T}}(X, \Sigma^n(X))}{n^{d-1}} = 0.$$

Then there is a constant C such that $\dim \operatorname{Hom}_{\mathcal{T}}(X, \Sigma^n(X)) < Cn^{d-1}$ for all $n > 0$. Since each X_i is a direct summand of X, we have

$$a_{ij}(n) := \dim \operatorname{Hom}_{\mathcal{T}}(X_i, \Sigma^n(X_j)) < Cn^{d-1}$$

for all i, j. This means that each entry $a_{ij}(n)$ in the adjacency matrix of $A(\phi, \Sigma^n)$ is less than Cn^{d-1}. Therefore $\rho(A(\phi, \Sigma^n)) < mCn^{d-1}$. By Definition 4.1.11(3), $\mathrm{fpg}(\Sigma) \le d-1$. Thus $\mathrm{fpcx}(\mathcal{T}) \le d$ as desired. □

We will prove that the equality $\mathrm{fpcx}(\mathcal{T}) = cx(\mathcal{T})$ holds under some extra hypotheses. Let A be a finite dimensional algebra with a complete list of simple left A-modules $\{S_1, \cdots, S_w\}$. We use n for any integer and i, j for integers between 1 and w. Define, for $i \le j$,

$$p_{ij}(n) \coloneqq \min\{\dim \mathrm{Ext}_A^n(S_i, S_j), \dim \mathrm{Ext}_A^n(S_j, S_i)\}$$

and

$$P_n \coloneqq \max\{p_{ij}(n) \mid i \le j\}.$$

We say A satisfies *averaging growth condition* (or *AGC* for short) if there are positive integers C and d, independent of the choices of n and (i, j), such that

$$\dim \mathrm{Ext}_A^n(S_i, S_j) \le C \max\{P_{n-d}, P_{n-d+1}, \cdots, P_{n+d}\} \tag{3.8.3.1}$$

for all n and all $1 \le i, j \le w$.

Theorem 3.8.4. *Let A be a finite dimensional algebra and $\mathcal{A} = D^b(\mathrm{Mod}_{f.d.}-A)$.*

(1) $cx(A) = cx(\mathcal{A})$. As a consequence, $cx(A)$ is a derived invariant.

(2) If A satisfies AGC, then $\mathrm{fpcx}(\mathcal{A}) = cx(\mathcal{A}) = cx(A)$. As a consequence, if A is local or commutative, then $\mathrm{fpcx}(\mathcal{A}) = cx(\mathcal{A}) = cx(A)$.

We will prove Theorem 3.8.4 after the next lemma.

Let $\mathcal{T}$ be a pre-triangulated category with suspension Σ. We use X, Y, Z for objects in $\mathcal{T}$. Fix a family ϕ of objects in $\mathcal{T}$ and a positive number d. Define

$${}_{d}(\phi) = \left\{X \in \mathcal{T} \mid \lim_{n\to\infty} \frac{\dim \mathrm{Hom}_{\mathcal{T}}(X, \Sigma^n(Y))}{n^{d-1}} = 0, \forall\ Y \in \phi\right\}, \tag{3.8.4.1}$$

$$(\phi)_d = \left\{ X \in \mathcal{T} \mid \lim_{n\to\infty} \frac{\dim \operatorname{Hom}_{\mathcal{T}}(Y, \Sigma^n(X))}{n^{d-1}} = 0, \forall\, Y \in \phi \right\}, \tag{3.8.4.2}$$

$$^d(\phi) = \left\{ X \in \mathcal{T} \mid \lim_{n\to\infty} \frac{\sum_{i\le n} \dim \operatorname{Hom}_{\mathcal{T}}(X, \Sigma^i(Y))}{n^d} = 0, \forall\, Y \in \phi \right\}, \tag{3.8.4.3}$$

$$(\phi)^d = \left\{ X \in \mathcal{T} \mid \lim_{n\to\infty} \frac{\sum_{i\le n} \dim \operatorname{Hom}_{\mathcal{T}}(Y, \Sigma^i(X))}{n^d} = 0, \forall\, Y \in \phi \right\}. \tag{3.8.4.4}$$

Lemma 3.8.5. *The following are full thick pre-triangulated subcategories of $\mathcal{T}$ closed under direct summands.*

(1) $_d(\phi)$.

(2) $(\phi)_d$.

(3) $^d(\phi)$.

(4) $(\phi)^d$.

Proof. We only prove (1). The proofs of other parts are similar. Suppose $X \in {}_d(\phi)$. Using the fact $\lim_{n\to\infty} \frac{n^{d-1}}{(n+1)^{d-1}} = 1$, we see that $X[1] = \Sigma(X)$ is in $_d(\phi)$. Similarly, $X[-1]$ is in $_d(\phi)$. If $f : X_1 \to X_2$ be a morphism of objects in $_d(\phi)$, and let X_3 be the mapping cone of f, then, for each $Y \in \phi$, we have an exact sequence

$$\to \operatorname{Hom}_{\mathcal{T}}(X_1, \Sigma^{n-1}(Y)) \to \operatorname{Hom}_{\mathcal{T}}(X_3, \Sigma^n(Y)) \to \operatorname{Hom}_{\mathcal{T}}(X_2, \Sigma^n(Y)) \to$$

which implies that $X_3 \in {}_d(\phi)$. Therefore $_d(\phi)$ is a thick pre-triangulated subcategory of $\mathcal{T}$. If $X \in {}_d(\phi)$ and $X = Y \oplus Z$, it is clear that $Y, Z \in {}_d(\phi)$. Therefore $_d(\phi)$ is closed under taking direct summands. □

Proof of Theorem 3.8.4. (1) Let $c = cx(A)$. For every $d < c$, we have that

$$\limsup_{n\to\infty} \frac{\dim \operatorname{Ext}^n_A(T,T)}{n^{d-1}} = \infty$$

which implies that $T \notin {}_d\mathcal{A}$. Therefore $d \le cx(\mathcal{A})$.

Conversely, let $d > c$. It follows from the definition that

$$\limsup_{n\to\infty} \frac{\dim \operatorname{Ext}_A^n(T,T)}{n^{d-1}} = 0.$$

This means that $T \in (\{T\})_d$. Since T generates $\mathcal{A}$, we have $\mathcal{A} = (\{T\})_d$. Again, since T generates $\mathcal{A}$, we have $\mathcal{A} = \mathcal{A}_d = {}_d\mathcal{A}$. By definition, $d \geq cx(\mathcal{A})$ as desired.

(2) Assume that A satisfies AGC. Let

$$\begin{aligned}
c_1 &= \operatorname{fpcx}(\mathcal{A}),\\
c_2 &= \limsup_{n\to\infty} \log_n(C \max\{P_{n-d}, P_{n-d+1}, \cdots, P_{n+d}\}) + 1,\\
c_3 &= \limsup_{n\to\infty} \log_n(P_n) + 1,\\
c_4 &= cx(A) = cx(\mathcal{A}).
\end{aligned}$$

By calculus, we have $c_2 = c_3$. Let ϕ be the atomic set of simple objects $\{S_i\}_{i=1}^w$. Then $\rho(\phi, \Sigma^n) \geq p_{ij}(n)$, for all i, j, by Lemma 3.1.7(2). So $\rho(\phi, \Sigma^n) \geq P_n$. As a consequence, $c_1 \geq c_3$. Let $T = A/J = \bigoplus_{i=1}^w S_i^{d_i}$ for some finite numbers $\{d_i\}_{i=1}^w$. Let D be $\max_i\{d_i\}$. By AGC, namely, (3.8.3.1),

$$\begin{aligned}
\dim \operatorname{Ext}_A^n(T,T) &= \sum_{i,j} d_i d_j \dim \operatorname{Ext}_A^n(S_i, S_j)\\
&\leq w^2 DC \max\{P_{n-d}, P_{n-d+1}, \cdots, P_{n+d}\}
\end{aligned}$$

which implies that $c_4 = cx(A) = cx(T) \leq c_2$. Combining with Theorem 3.8.3, we have $c_1 = c_2 = c_3 = c_4$ as desired.

If A is local, then there is only one simple module S_1. Then (3.8.3.1) is automatic. If A is commutative, then $\operatorname{Ext}_A^i(S_i, S_j) = 0$ for all n and all $i \neq j$. Again, in this case, (3.8.3.1) is obvious. The consequence follows from the main assertion. □

For all well-studied finite dimensional algebras A, (3.8.3.1) holds. For example, the algebra A in Example 3.5.5 satisfies AGC. This can be shown by using the computation given in Remark 3.5.13(3). It is natural to ask if every finite dimensional algebra satisfies AGC.

Lemma 3.8.6. *(1) Let $\mathfrak{A}$ be an abelian category and $\mathcal{A} = D^b(\mathfrak{A})$. If* $\operatorname{gldim} \mathfrak{A} < \infty$*, then* $\operatorname{fpcx}(\mathcal{A}) = 0$.

(2) Let $\mathcal{T}$ be a pre-triangulated category. If $\operatorname{fpgldim} \mathcal{T} < \infty$*, then* $\operatorname{fpcx}(\mathcal{T}) = 0$.

Proof. Both are easy and proofs are omitted. □

We conclude with examples of non-integral fpg of a triangulated category.

Example 3.8.7. (1) Let α be any real number in $\{0\} \cup \{1\} \cup [2, \infty)$. By [KL85, Theorem 1.8, or p. 14], there is a finitely generated algebra R with $\operatorname{GKdim} R = \alpha$. More precisely, [KL85, Theorem 1.8] implies that there is a 2-dimensional vector space $V \subset R$ that generates R such that, there are positive integers $a < b$, for every $n > 0$,

$$an^{\alpha} < \dim(\mathbf{k}\,1 + V)^n < bn^{\alpha}.$$

Define a filtration $\mathcal{F}$ on R by

$$F_i R = (\mathbf{k}\,1 + V)^i, \quad \forall\ i.$$

Let A be the associated graded algebra $\operatorname{gr} R$ with respect to this grading. Then A is connected graded and generated by two elements in degree 1 and satisfying, for every $n > 0$,

$$an^{\alpha} < \sum_{i=0}^{n} \dim A_i < bn^{\alpha}. \tag{3.8.7.1}$$

To match up with the definition of complexity, we further assume that there are $c < d$ such that, for every $n > 0$,

$$cn^{\alpha-1} < \dim A_n < dn^{\alpha-1}. \tag{3.8.7.2}$$

This can be achieved, for example, by replacing A by its polynomial extension $A[t]$ (with $\deg t = 1$) and replacing α by $\alpha + 1$.

Next we make A a differential graded (dg) algebra by setting elements in A_i to have cohomological degree i and $d_A = 0$. For this dg algebra, we denote the derived category of left dg A-modules by $\mathcal{A}$. Let $\mathcal{O}$ be the object ${}_AA$ in $\mathcal{A}$. By the definition of the cohomological degree of A, we have

$$\operatorname{Hom}_{\mathcal{A}}(\mathcal{O}, \Sigma^i\mathcal{O}) = A_i, \quad \forall\ i. \tag{3.8.7.3}$$

Let $\mathcal{T}$ be the full triangulated subcategory of $\mathcal{A}$ generated by $\mathcal{O}$. (3.8.7.3) implies that $\mathcal{O}$ is an atomic object. Now using (3.8.7.3) together with (3.8.7.2), we obtain that

$$\operatorname{fpcx}(\mathcal{T}) \geq \alpha. \tag{3.8.7.4}$$

By (3.8.7.2)-(3.8.7.3), we have that, for every $d > \alpha$, $\mathcal{O} \in {}_d(\{\mathcal{O}\})$. Since $\mathcal{O}$ generates $\mathcal{T}$, we have ${}_d(\{\mathcal{O}\}) = \mathcal{T}$. The last equation means that $\mathcal{O} \in (\mathcal{T})_d$. Since $\mathcal{O}$ generates $\mathcal{T}$, we have $(\mathcal{T})_d = \mathcal{T}$. By definition, $d > cx(\mathcal{T})$. Combining these with Theorem 3.8.3 and (3.8.7.4), we have, for every $d > \alpha$,

$$\alpha \leq \operatorname{fpcx}(\mathcal{T}) \leq cx(\mathcal{T}) < d$$

which implies that $\operatorname{fpcx}(\mathcal{T}) = cx(\mathcal{T}) = \alpha$. This construction implies that

$$\operatorname{GKdim}\left(\bigoplus_{i=0}^{\infty} \operatorname{Hom}_{\mathcal{T}}(\mathcal{O}, \Sigma^i(\mathcal{O}))\right) = \operatorname{GKdim} A = \alpha. \tag{3.8.7.5}$$

(2) We now consider an extreme case. Let $a := \{a_i\}_{i=0}^{\infty}$ be any sequence of non-negative integers with $a_0 = 1$. Define B to be the dg algebra $\bigoplus B_i$ such that

(i) $\dim B_i = a_i$ for all i. In particular, $B_0 = \mathbf{k}$. Elements in B_i have cohomological degree i.

(ii) $(\bigoplus_{i>0} B_i)^2 = 0$.

(iii) Differential $d_B = 0$.

In this case, $\operatorname{GKdim} B = 0$. Similar to part (1), the derived category of left dg B-modules is denoted by $\mathcal{B}$. Let $\mathcal{O}$ be the object ${}_BB$ in $\mathcal{B}$. Then

$$\operatorname{Hom}_{\mathcal{B}}(\mathcal{O}, \Sigma^i\mathcal{O}) = B_i, \quad \forall\ i,$$

and $\mathcal{O}$ is an atomic object. Let $\mathcal{T}$ be the full triangulated subcategory of $\mathcal{B}$ generated by $\mathcal{O}$. The argument in part (1) shows that

$$\operatorname{fpcx}(\mathcal{T}) = \limsup_{n\to\infty} \log_n(a_n) + 1.$$

Now let r be any real number ≥ 1 and let $a_i = \begin{cases} 1 & i = 0 \\ \lfloor i^{r-1} \rfloor & i \geq 1 \end{cases}$. Then we have $\operatorname{fpcx}(\mathcal{T}) = r$.

Let r be any real number ≥ 1 and $a_i = \lfloor r^i \rfloor$ for all $i \geq 0$. Then $\operatorname{fpcx}(\mathcal{T}) = \begin{cases} 1 & r = 1 \\ \infty & r > 1 \end{cases}$. Using a similar method (with details omitted), $\operatorname{fpv}(\mathcal{T}) = r$.

Chapter 4

FROBENIUS-PERRON THEORY OF MODIFIED A-D-E BOUND QUIVER ALGEBRAS

4.1 Introduction

This chapter is based on independent work [Wic19]. As discussed in the introduction and in Chapter 3, the Frobenius-Perron dimension of a finite tensor category has proven to be an important invariant since it was introduced by Etingof, Nikshych, and Ostrik [ENO05]. One can define the Frobenius-Perron dimension of an object in a tensor category as well as the dimension of the tensor category itself, and these dimensions encode many useful properties. These ideas have been essential to the classification of fusion categories and understanding the representation theory of semisimple Hopf algebras.

In Chapter 3, a new type of Frobenius-Perron dimension was introduced, which we will denote fpd [CGW+19a]. This extends the definition of Frobenius-Perron dimension of an object in a fusion category to a much more general setting [CGW+19a, Ex. 2.11]. In fact, this definition of fpd can be applied to any **k**-linear category along with a chosen endofunctor. Chapter 3 focuses on understanding the fpd of abelian categories (see Definition 4.1.10) and derived categories, where the endofunctor is the suspension. The results in [CGW+19a] suggest that the Frobenius-Perron dimension of an abelian or derived category is an important and useful invariant with connections to representation theory.

The intent of this chapter is to gain a better understanding of the fpd as it relates to the representation theory of finite-dimensional algebras. First, we review an important theorem from [CGW+19a], which is analogous to the finite tensor category result [EGNO15, Prop. 6.3.3].

Theorem 4.1.1. *[CGW+19a] If $F : \mathcal{A} \to \mathcal{B}$ is an embedding of abelian categories, then*

$\operatorname{fpd} \mathcal{A} \leq \operatorname{fpd} \mathcal{B}$.

The previous theorem is useful in that we can use the fpd to show non-existence of embeddings between abelian categories. However, calculating the fpd can be a difficult task. If we could develop strategies for calculating the fpd in certain abelian categories, we could compare those categories via their fpd. To that end, we prove the following theorem, which greatly simplifies the calculation of the fpd for radical squared zero finite-dimensional algebras. First we need some hypotheses.

Hypothesis 4.1.2. *Let $A = \mathbf{k}Q/(\geq 2)$ for some finite quiver Q, where (≥ 2) is the ideal generated by paths of length greater than 1, and let J be the ideal of A generated by the loops. Let Q' be the quiver formed from Q by removing all the loops. Define $C := \mathbf{k}Q' \cong \mathbf{k}Q/\tilde{J}$, where $\tilde{J}$ is the ideal of $\mathbf{k}Q$ generated by the loops, and $B := A/J \cong \mathbf{k}Q'/(\geq 2) = C/(\geq 2)$. Let $\tilde{P}_i$ (respectively $\tilde{I}_i, \tilde{S}_i$) denote the indecomposable projective (respectively injective, simple) B-module at each vertex i. Note that as A-modules, $\tilde{S}_i \cong S_i$ for each i.*

Calculating the fpd requires an understanding of the brick modules (Definition 4.1.10).

Theorem 4.1.3. *Assume Hypothesis 4.1.2. Then there is a one-to-one correspondence between the isomorphism classes below:*

$$\{\text{brick } A\text{-modules}\} \longleftrightarrow \{\text{brick } C\text{-modules annihilated by } (\geq 2)\}$$

Furthermore, if M, N are brick A-modules, we have a natural isomorphism

$$\operatorname{Hom}_A(M, N) \cong \operatorname{Hom}_B(M, N) \cong \operatorname{Hom}_C(M, N).$$

We apply this theorem to some examples. We hope that these results in conjuction with Theorem 4.1.1 will be of use to those who are trying to compare or understand these categories.

In particular, we focus on a family of bound quiver algebras related to the ADE quiver algebras (Definition 4.1.12). The ADE quivers and related quivers appear in settings as diverse as the classification of semisimple Lie algebras [Hum78], the McKay correspondence [McK80], the classification of minimal models of two-dimensional conformal field theory [Cap88], and more. In addition to their ubiquity, the ADE quivers are of finite representation type and their representations are well-known [Gab72], making variations of their representation categories ideal to study.

Theorem 4.1.4. *Let A be a modified ADE bound quiver algebra with N_i loops at each vertex i (Definition 4.1.12). Then*

$$\operatorname{fpd}(A\text{-}\mathrm{mod}) = \max\{N_1, \ldots, N_n\}.$$

Some consequences of this theorem include non-existence of embeddings. Suppose that A is a modified ADE bound quiver algebra with at least one loop on a single vertex. Then the category of representations of A does not embed into any of the categories of representations of the A_n, D_n, or E_n quiver algebras, since these have fpd 0 [CGW+19a]. Similarly, if A is a modified ADE bound quiver algebra with at least two loops on a single vertex, the category of representations of A does not embed into any of the categories of representations of the $\tilde{A}_n, \tilde{D}_n$, or $\tilde{E}_n$ quiver algebras, since these have fpd 1 [CGW+19a]. And if A, B are modified ADE bound quiver algebras where the maximum number of loops at a single vertex of A is N, and the maximum number of loops of at a single vertex of B is M, then if $N < M$, B-mod cannot embed into A-mod.

The following question would be interesting to investigate.

Question 4.1.5. What is the fpd for the $\tilde{A}\tilde{D}\tilde{E}$ modified bound quiver algebras?

Notice that Example 4.4.1 is of type $\tilde{A}(1)$. In this case we have a formula for the fpd which depends on the number of loops at each vertex, which is irrational in many cases. So we do not get the same result as Theorem 4.1.4. It would be interesting to see if there is a formula that describes the fpd of the $\tilde{A}\tilde{D}\tilde{E}$ modified bound quiver algebras.

It is natural to ask about the fpd of the derived category.

Question 4.1.6. Let A be a modified ADE bound quiver algebra. What is fpd $D^b(A\text{-}\mathrm{mod})$?

We defined the modified ADE bound quiver algebras with a certain arrow orientation (Definition 4.1.12). We ask if the fpd is invariant under changing directions of arrows, and show that the question has a positive answer up to $n = 3$ (Theorem 4.6.1).

Question 4.1.7. Let Q' be a quiver whose underlying graph is A_n, D_n, or E_n for some n. Let Q be the quiver formed from Q' by adding N_i loops to each vertex i, and let $A = kQ/(\geq 2)$. Is it true that

$$\operatorname{fpd}(A\text{-}\mathrm{mod}) = \max\{N_1, \ldots, N_n\}?$$

The fpd of an abelian category is an element of $\mathbb{R}^{\geq 0} \cup \{\pm\infty\}$. It is natural to ask

Question 4.1.8. For a finite-dimensional algebra A, what values of $\operatorname{fpd} A\text{-}\mathrm{mod}$ are possible?

As a partial answer to this question, we provide the following family of examples using Theorem 4.1.3. The general answer to this question is unknown.

Proposition 4.1.9. *There exists a family of finite-dimensional radical squared zero algebras A which can be used to construct arbitrarily large irrational values of* $\operatorname{fpd}(A\text{-}\mathrm{mod})$.

4.1.1 Conventions and definitions

We assume that the base field $\mathbf{k}$ is algebraically closed.

If A is a finite-dimensional $\mathbf{k}$-algebra, then $A\text{-}\mathrm{mod}$ (respectively $\mathrm{mod}\text{-}A$) denotes the category of finite-dimensional left (respectively right) A-modules. We will use A^{op} to denote the opposite algebra of A.

Let Q denote a finite quiver. The ideal generated by all paths of lengths greater than 1 in $\mathbf{k}Q$ is denoted by (≥ 2). If $A = \mathbf{k}Q/I$ is a bound quiver algebra, we denote the indecomposable projective (respectively injective, simple) module at vertex i by P_i (respectively I_i, S_i).

The spectral radius of a square matrix M with entries in $\mathbb{R}$ or $\mathbb{C}$ is denoted $\rho(M)$ [HJ13, Defn. 1.2.9].

We review the relevant definitions, which can be found in Chapter 3 as well as [CGW+19a].

Definition 4.1.10. Let $\mathcal{C}$ be a $\mathbf{k}$-linear abelian category, and let $\phi := \{X_1, X_2, \ldots, X_n\}$ be a finite subset of nonzero objects in $\mathcal{C}$.

(1) The *adjacency matrix* of ϕ is defined to be

$$A(\phi) := (a_{ij})_{n\times n}, \quad \text{where } a_{ij} := \dim \operatorname{Ext}^1_{\mathcal{C}}(X_i, X_j) \ \ \forall i, j.$$

(2) An object M in $\mathcal{C}$ is called a *brick* [ASS06, Definition 2.4, Ch. VII] if

$$\operatorname{Hom}_{\mathcal{C}}(M, M) = \mathbf{k}\,.$$

(3) $\phi \in \Phi$ is called a *brick set* if each X_i is a brick and

$$\dim \operatorname{Hom}_{\mathcal{C}}(X_i, X_j) = \delta_{ij}$$

for all $1 \le i, j \le n$. The set of brick n-object subsets is denoted by $\Phi_{n,b}$. We write $\Phi_b = \bigcup_{n\ge 1} \Phi_{n,b}$.

Definition 4.1.11. The *nth Frobenius-Perron dimension* of $\mathcal{C}$ is defined to be

$$\operatorname{fpd}^n(\mathcal{C}) := \sup_{\phi\in\Phi_{n,b}} \{\rho(A(\phi))\}.$$

If $\Phi_{n,b}$ is empty, then by convention, $\operatorname{fpd}^n(\mathcal{C}) = -\infty$.

The *Frobenius-Perron dimension* of $\mathcal{C}$ is defined to be

$$\operatorname{fpd}(\mathcal{C}) := \sup_n \{\operatorname{fpd}^n(\mathcal{C})\} = \sup_{\phi\in\Phi_b} \{\rho(A(\phi))\}.$$

Definition 4.1.12. We define a family of algebras closely related to the ADE path algebras.

Define $A(n, N_1, \ldots, N_n) = \mathbf{k}\, Q_A/(\ge 2)$ for $n \ge 1$, where Q_A is the following quiver with N_i loops (labeled a_i^l for $l = 1, \ldots, N_i$) at vertex i:

$$\begin{tikzcd}
1 \arrow[r, "x_2"] \arrow[loop below, "a_1^l"] & 2 \arrow[r, "x_3"] \arrow[loop below, "a_2^l"] & \cdots \arrow[r, "x_n"] & n \arrow[loop below, "a_n^l"]
\end{tikzcd}$$

Define $D(n, M_1, \ldots, M_n) = \mathbf{k}\, Q_D/(\geq 2)$ for $n \geq 4$, where Q_D is the following quiver with M_i loops (labeled b_i^l for $l = 1, \ldots, M_i$) at vertex i:

$$\begin{tikzcd}
 & & & n-1 \arrow[loop above, "b_{n-1}^l"] & \\
1 \arrow[r, "x_2"] \arrow[loop below, "b_1^l"] & 2 \arrow[r, "x_3"] \arrow[loop below, "b_2^l"] & \cdots \arrow[r, "x_{n-2}"] & n-2 \arrow[u, "x_{n-1}"] \arrow[r, "x_n"] \arrow[loop below, "b_{n-2}^l"] & n \arrow[loop below, "b_n^l"]
\end{tikzcd}$$

Define $E(n, K_1, \ldots, K_n) := \mathbf{k}\, Q_E/(\geq 2)$ for $n \in \{6, 7, 8\}$, where Q_E is the following quiver with K_i loops (labeled c_i^l for $l = 1, \ldots, K_i$) at vertex i:

$$\begin{tikzcd}
 & & 2 \arrow[loop above, "c_2^l"] & & \\
1 \arrow[r, "x_3"] \arrow[loop below, "c_1^l"] & 3 \arrow[r, "x_4"] \arrow[loop below, "c_3^l"] & 4 \arrow[u, "x_2"] \arrow[r, "x_5"] \arrow[loop below, "c_4^l"] & \cdots \arrow[r, "x_n"] & n \arrow[loop below, "c_n^l"]
\end{tikzcd}$$

We refer to the algebras above as the modified ADE bound quiver algebras. When the number of each loops at each vertex is given, we refer to these algebras as $A(n), D(n)$, and $E(n)$. These algebras appear in Theorems 4.5.1, 4.5.2 and 4.5.5.

4.2 Preliminaries

We compile a list of results which we use later in the paper. The reader is welcome to skip this section and come back to it for reference.

Definition 4.2.1. We define the path algebra $\mathbf{k}Q$ of a finite quiver Q (that is, a directed graph with finitely many edges and vertices) to be the associative $\mathbf{k}$-algebra determined by the generators e_i, α for all vertices i of Q and all edges α of Q, such that for all i, j, α,

$$e_i e_j = \delta_{ij}, \quad e_{t(\alpha)}\alpha = \alpha, \quad \alpha e_{s(\alpha)} = \alpha,$$

where $t(\alpha)$ is the vertex that is the target of α and $s(\alpha)$ is the vertex that is the source of α. Note that the definiton of path algebra in [ASS06] and in Chapter 3 is the opposite algebra of $\mathbf{k}Q$ as we have defined it here.

Fact 4.2.2. Let A be a finite-dimensional $\mathbf{k}$-algebra.

(a) [Sch14, Prob. 5.6] The following functors are mutually inverse contravariant equivalences

$$D(-) := \mathrm{Hom}_{\mathbf{k}}(-, \mathbf{k}) : \mathrm{mod}\text{-}A \to A\text{-}\mathrm{mod},$$
$$D(-) := \mathrm{Hom}_{\mathbf{k}}(-, \mathbf{k}) : A\text{-}\mathrm{mod} \to \mathrm{mod}\text{-}A.$$

(b) [Sch14, Prob. 5.6] D gives duality between injective left modules and projective right modules (and vice versa). This is a consequence of $\mathrm{Ext}^1_A(DX, DY) \cong \mathrm{Ext}^1_A(Y, X)$.

(c) [Sch14, Prob. 5.6, Prop. 5.7] Define the Nakayama functor $\nu(-) = D \circ \mathrm{Hom}_A(-, A)$. Now suppose that $A = \mathbf{k}Q/I$ is a bound quiver algebra. We define $I_i := \nu(Ae_i) = D(e_i A)$. Then I_i is an indecomposable injective left A-module.

(d) [Ben91, Lem. 1.7.5,1.7.7] Let $A = \mathbf{k}Q/I$ be a bound quiver algebra. Then

$$\dim \mathrm{Hom}_A(P_i, S_j) = \delta_{ij} = \dim \mathrm{Hom}_A(S_i, I_j).$$

(e) [Sch14, Prob. 2.8, Cor. 2.12] Let $A = \mathbf{k}Q$ be a path algebra. Then

$$\dim \mathrm{Hom}_A(I_j, I_i) = \text{ the number of paths from } i \to j = \dim \mathrm{Hom}_A(P_j, P_i).$$

(f) [ASS06, Lem. III.2.12(b)] Let $A = \mathbf{k}Q/I$ be a bound quiver algebra. Then

$$\dim \mathrm{Ext}^1_A(S_i, S_j) = \text{ the number of arrows from } i \to j.$$

(g) [ASS06, Defn. II.1.2, Thm. III.1.6] Let $A = \mathbf{k}Q/I$ be a bound quiver algebra. There exists an equivalence of categories $A\text{-}\mathrm{mod} \cong \mathrm{Rep}_{\mathbf{k}}(Q, I)$. We will freely identify these categories throughout the rest of the paper.

(h) [AF74, 4.3, Ex. 4.16] Let A be a $\mathbf{k}$-algebra and let I be an ideal. Then there is an equivalence of categories between $\mathcal{C} = A/I\text{-}\mathrm{mod}$ and $\mathcal{D}$, which is defined to be the full subcategory of $A\text{-}\mathrm{mod}$ whose objects are those A-modules which are annihilated by I.

(i) Let Q be a finite quiver and let $V = (V_i, f_\alpha) \in \mathrm{Rep}_{\mathbf{k}}(Q)$. Suppose $\alpha_1 \dots \alpha_n$ is a non-zero path, where each α_i is an edge of Q. Let I be an ideal of $\mathbf{k}Q$ containing $\alpha_1 \dots \alpha_n$. If $f_{\alpha_1} \circ \cdots \circ f_{\alpha_n}$ is non-zero, then V is not annihilated by I. This is a consequence of (7).

(j) [DK94, Cor. 1.7.3] Suppose that Q is a finite quiver with an indecomposable representation $V = (V_i, f_\beta) \in \mathrm{Rep}_{\mathbf{k}}(Q)$. If α is an edge of Q such that $Q - \alpha$ is disconnected, and $V_{t(\alpha)} \neq 0 \neq V_{s(\alpha)}$, then $f_\alpha \neq 0$.

(k) [ASS06, Lem. 2.4, 2.6] Let $A = \mathbf{k}Q/(\geq 2)$ be a radical squared zero algebra described by a finite quiver Q such that $e_j A e_i$ is one-dimensional for all j. Then $P_i = Ae_i$ corresponds to the representation (V_j, f_α), where

$$V_j = \begin{cases} \mathbf{k} \text{ if } j = i \\ \mathbf{k} \text{ if } j = t(\alpha) \text{ for some arrow } \alpha \text{ with source } i \\ 0 \text{ otherwise} \end{cases},$$

$$f_\alpha = \begin{cases} 1 \text{ if } V_{s(\alpha)} = \mathbf{k} = V_{t(\alpha)} \\ 0 \text{ otherwise} \end{cases}.$$

Similarly, $I_i = D(e_iA)$ corresponds to the representation (W_j, g_β), where

$$W_j = \begin{cases} \mathbf{k} \text{ if } j = i \\ \mathbf{k} \text{ if } j = s(\beta) \text{ for some arrow } \beta \text{ with target } i\,, \\ 0 \text{ otherwise} \end{cases}$$

$$g_\beta = \begin{cases} 1 \text{ if } W_{s(\beta)} = \mathbf{k} = W_{t(\beta)} \\ 0 \text{ otherwise} \end{cases}.$$

We need the following results for our calculations.

Lemma 4.2.3. *Assume Hypothesis 4.1.2.*

(a) If $\tilde{P}_i, \tilde{P}_j$ are in the same brick set in A-mod, then we have $\operatorname{Ext}^1_A(\tilde{P}_i, \tilde{P}_j) = 0$ if $i \neq j$. If $\tilde{P}_i \ncong S_i$, then $\operatorname{Ext}^1_A(\tilde{P}_i, \tilde{P}_i) = 0$.

(b) Dually, if $\tilde{I}_i, \tilde{I}_j$ are in the same brick set, then we have $\operatorname{Ext}^1_A(\tilde{I}_i, \tilde{I}_j) = 0$ if $i \neq j$. If $\tilde{I}_i \ncong S_i$, then $\operatorname{Ext}^1_A(\tilde{I}_i, \tilde{I}_i) = 0$.

(c) If $i \neq j$, then we have $\operatorname{Ext}^1_A(\tilde{P}_i, S_j) = 0$.

(d) Dually, if $i \neq j$, $\operatorname{Ext}^1_A(S_i, \tilde{I}_j) = 0$.

Proof. We will prove (1) and (3), since the other results are dual.

(1) Fix a vertex i. Let $P_i = Ae_i$ be the indecomposable projective A-module at vertex i. Notice that $\tilde{P}_i \cong P_i/JP_i$ as A-modules. Let $a_1, \ldots, a_N$ be a complete list of the loops in Q with source i, and let $x_1, \ldots, x_M$ be a complete list of arrows that are not loops with source i in Q. Note that for $1 \leq m \leq M, 1 \leq n \leq N$, $\mathbf{k}\, x_m \cong S_{t(x_m)}$ and $\mathbf{k}\, a_n \cong S_i$. Then the following is a projective resolution of $\tilde{P}_i$ in A.

$$\begin{tikzcd} \bigoplus_{n=1}^N \left(\bigoplus_{m=1}^M P_{t(x_m)} \oplus P_i^{\oplus N}\right) \arrow[r, "\psi"] \arrow[d] & P_i^{\oplus N} \arrow[r, "\varphi"] \arrow[d] & P_i \arrow[r] & \tilde{P}_i \arrow[r] & 0 \\ \bigoplus_{n=1}^N \left(\bigoplus_{m=1}^M \mathbf{k}\, x_m \bigoplus_{l=1}^N \mathbf{k}\, a_l\right) \arrow[ur, hook] & \bigoplus_{n=1}^N \mathbf{k}\, a_n \arrow[ur, hook] & & & \end{tikzcd}$$

We need to take first homology of the following sequence:

$$0 \to \mathrm{Hom}_A(P_i, \tilde{P}_j) \xrightarrow{\varphi^*} \bigoplus_{n=1}^{N} \mathrm{Hom}_A(P_i, \tilde{P}_j) \xrightarrow{\psi^*} \bigoplus_{n=1}^{N} \mathrm{Hom}_A\left(\bigoplus_{m=1}^{M} P_{t(x_m)} \bigoplus P_i^{\oplus N}, \tilde{P}_j \right).$$

Case 1: $i \neq j$.

We will show that $\mathrm{Hom}_A(P_i, \tilde{P}_j) = 0$; therefore, $\mathrm{Ext}^1_A(\tilde{P}_i, \tilde{P}_j) = 0$.

By Proposition 4.3.4, $\mathrm{Hom}_A(\tilde{P}_i, \tilde{P}_j) \cong \mathrm{Hom}_B(\tilde{P}_i, \tilde{P}_j)$. Then

$$\dim \mathrm{Hom}_A(P_i, \tilde{P}_j) = \dim e_i \tilde{P}_j = \dim e_i B e_j = \dim \mathrm{Hom}_B(\tilde{P}_i, \tilde{P}_j) = \dim \mathrm{Hom}_A(\tilde{P}_i, \tilde{P}_j) = 0,$$

where the last equality holds because $\tilde{P}_i, \tilde{P}_j$ are in the same brick set.

Case 2: $i = j$.

Notice that $\psi^* = \oplus_{n=1}^{N} \psi_n^*$, where $\psi_n^* : \mathrm{Hom}_A(P_i, \tilde{P}_j) \to \mathrm{Hom}_A\left(\bigoplus_{m=1}^{M} P_{t(x_m)} \oplus P_i^{\oplus N}, \tilde{P}_j \right)$. We will show that $\ker \psi_n^* = 0$ for each n. Therefore, $\mathrm{Ext}^1_A(\tilde{P}_i, \tilde{P}_j) = 0$.

$$\ker \psi_n^* = \left\{ g : P_i \to \tilde{P}_j : \quad \left[\bigoplus_{m=1}^{M} P_{t(x_m)} \bigoplus_{l=1}^{N} P_i \xrightarrow{\psi_n} P_i \xrightarrow{g} \tilde{P}_j \right] = 0 \right\}$$

Since $\mathrm{Im}\, \psi_n = \oplus_{m=1}^{M} \mathbf{k}\, x_m \oplus_{l=1}^{N} \mathbf{k}\, a_l$, we have

$$\ker \psi_n^* = \left\{ g : P_i \to \tilde{P}_j : \quad g(x_m) = 0 = g(a_l), \quad 1 \le m \le M, 1 \le l \le N \right\}$$

Let $g \in \ker \psi_n^*$. Since as a vector space, we have

$$\tilde{P}_i = \frac{\mathbf{k}\, e_i \oplus \mathbf{k}\, x_1 \oplus \cdots \oplus \mathbf{k}\, x_M \oplus \mathbf{k}\, a_1 \oplus \cdots \oplus \mathbf{k}\, a_N}{\mathbf{k}\, a_1 \oplus \cdots \oplus \mathbf{k}\, a_N},$$

we have that

$$g(e_i) = [\alpha e_i + \beta_1 x_1 + \cdots + \beta_N x_M]$$

for some $\alpha, \beta_1, \ldots, \beta_M \in \mathbf{k}$.

We can see that $[\beta_1 x_1 + \cdots + \beta_M x_M] = 0$ by noting that since none of the x_k have target i,

$$[\alpha e_i] = e_i[\alpha e_i + \beta_1 x_1 + \cdots + \beta_M x_M] = e_i g(e_i) = g(e_i) = [\alpha e_i + \beta_1 x_1 + \cdots + \beta_M x_M].$$

There exists at least one non-loop arrow x_j with source i because $\tilde{P}_i \ncong S_i$. Hence

$$0 = g(x_j) = x_j g(e_i) = x_j[\alpha e_i] = [\alpha x_j] \Rightarrow \alpha = 0.$$

Then $g(e_i) = 0$, so $g = 0$ and hence $\ker \psi_n^* = 0$.

(3) We will use the same projective resolution of $\tilde{P}_i$ as the proof of (1) and the same notation.

$$\begin{tikzcd}
\bigoplus_{n=1}^N \left(\bigoplus_{m=1}^M P_{t(x_m)} \oplus P_i^{\oplus N}\right) \arrow[r, "\psi"] \arrow[d] & \bigoplus_{n=1}^N P_i \arrow[r, "\varphi"] \arrow[d] & P_i \arrow[r] & \tilde{P}_i \arrow[r] & 0 \\
\bigoplus_{n=1}^N \left(\bigoplus_{m=1}^M \mathbf{k}\, x_m \bigoplus_{l=1}^N \mathbf{k}\, a_l\right) \arrow[ur, hook] & \bigoplus_{n=1}^N \mathbf{k}\, a_n \arrow[ur, hook] & & &
\end{tikzcd}$$

Apply $\mathrm{Hom}_A(-, S_j)$:

$$0 \to \mathrm{Hom}_A(P_i, S_j) \xrightarrow{\varphi^*} \bigoplus_{n=1}^N \mathrm{Hom}_A(P_i, S_j) \xrightarrow{\psi^*} \bigoplus_{n=1}^N \mathrm{Hom}_A\left(\bigoplus_{m=1}^M P_{t(x_m)} \bigoplus P_i^{\oplus N}, S_j\right).$$

Then $\mathrm{Ext}_A^1(\tilde{P}_i, S_j) = 0$ since $\mathrm{Hom}_A(P_i, S_j) = 0$ if $i \neq j$ (Fact 4.2.2). □

Lemma 4.2.4. *Assume Hypothesis 4.1.2. Let $\tilde{P}_i$ denote the indecomposable projective B-module at vertex i. Suppose that M is an A-module such that $\mathrm{Hom}_A(P_i, M) = 0$. Then*

$$\mathrm{Ext}_A^1(\tilde{P}_i, M) = 0.$$

Proof. Let $a_1, \ldots, a_N$ be a complete list of loops at i. First we construct a projective resolution of $\tilde{P}_i$. As a vector space,

$$\tilde{P}_i \cong \frac{P_i}{\mathbf{k}\, a_1 \oplus \cdots \oplus \mathbf{k}\, a_N}.$$

We have a projective resolution:

$$\begin{tikzcd}
\bigoplus_{i=1}^N P_i \arrow[r, "\varphi"] \arrow[d] & P_i \arrow[r, "\pi"] & \tilde{P}_i \arrow[r] & 0 \\
\mathbf{k}\, a_1 \oplus \cdots \oplus \mathbf{k}\, a_N \arrow[ur, hook] & & &
\end{tikzcd}$$

Now take $\mathrm{Hom}_A(-, M)$:

$$0 \longrightarrow \mathrm{Hom}_A(P_i, M) \xrightarrow{\varphi^*} \bigoplus_{l=1}^{L} \mathrm{Hom}_A(P_i, M)$$

Since $\mathrm{Hom}_A(P_i, M) = 0$,

$$\mathrm{Ext}_A^1\left(\tilde{P}_i, M\right) = 0. \qquad \square$$

Definition 4.2.5. A vertex i in a quiver is called a sink if there are no arrows α with $s(\alpha) = i$.

Lemma 4.2.6. *Assume Hypothesis 4.1.2. Let S_i denote the simple A-module at vertex i. If i is a sink in the quiver Q' describing B, and if M is an A-module such that* $\mathrm{Hom}_A(P_i, M) = 0$, *then*

$$\mathrm{Ext}_A^n(S_i, M) = 0, \ \forall n.$$

Proof. Let P_i denote the indecomposable projective at vertex i in A. Let $a_1, \ldots, a_N$ denote the loops with source i. If i is a sink in Q', then as a vector space, $\ker(P_i \to S_i) = \mathbf{k}\, a_1 \oplus \cdots \oplus \mathbf{k}\, a_N$. We have the following projective resolution of S_i:

$$\begin{tikzcd}
(P_i^{\oplus N})^{\oplus N^2} \arrow[r] \arrow[d] & (P_i^{\oplus N})^{\oplus N} \arrow[r] \arrow[d] & P_i^{\oplus N} \arrow[r] \arrow[d] & P_i \arrow[r, "\pi"] & S_i \arrow[r] & 0 \\
(\mathbf{k}\, a_1 \oplus \cdots \oplus \mathbf{k}\, a_N)^{\oplus N^2} \arrow[ur, hook] & (\mathbf{k}\, a_1 \oplus \cdots \oplus \mathbf{k}\, a_N)^{\oplus N} \arrow[ur, hook] & \mathbf{k}\, a_1 \oplus \cdots \oplus \mathbf{k}\, a_N \arrow[ur, hook] & & &
\end{tikzcd}$$

We must take $\mathrm{Hom}_A(-, M)$ of this resolution. But since $\mathrm{Hom}_A(P_i, M) = 0$, all the terms are 0. So $\mathrm{Ext}_A^n(S_i, M) = 0$ for any n. $\square$

The following lemma comes in handy when calculating Ext^1 groups.

Lemma 4.2.7. *Assume Hypothesis 4.1.2. Let M be an A-module which is annihilated by all of the loops in A, and let P_i be the indecomposable projective A-module at vertex i. Let $\{L_1, \ldots, L_n\}$ be a complete list of loops of A with source i. Then for any subset $\{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$,*

$$\{g : P_i \to M : g(L_{i_1}) = \cdots = g(L_{i_k}) = 0\} = \mathrm{Hom}_A(P_i, M).$$

Proof. For any $g : P_i \to M$, $g(L_j) = L_j g(e_i) = 0$, since L_j annihilates M. $\square$

We use the following proposition to show that certain indecomposable modules cannot be bricks.

Proposition 4.2.8. *Let Q be a finite quiver with vertices i, j, k with arrows $\alpha : i \to j$, $\beta : j \to k$. Suppose that for any arrow γ, $j \neq s(\gamma)$ except when $\gamma = \beta$, and $j \neq t(\gamma)$ except when $\gamma = \alpha$. If $V = (V_m, f_\gamma)$ is a representation of Q such that $V_i = \mathbf{k}^l, V_j = \mathbf{k}^m, V_k = \mathbf{k}^n$ for $l, n \geq 1$ and $m \geq 2$, $f_\alpha \neq 0 \neq f_\beta$, and $f_\beta \circ f_\alpha = 0$, then V is decomposable.*

Proof. Notice that removing the vertex j creates two disjoint quivers Q', Q'', where Q' contains vertex i and Q'' contains vertex k. Let $V(Q'), V(Q'')$ denote the set of vertices of Q', Q'' respectively.

First, here is a general picture of Q:

$$\begin{tikzcd}
\ddots \arrow[dr] & & & & \iddots \\
\cdots & i \arrow[l] \arrow[r, "\alpha"] & j \arrow[r, "\beta"] & k \arrow[r] \arrow[ur] & \cdots \\
\iddots \arrow[ur] & & & \vdots \arrow[u] & \ddots \arrow[ul]
\end{tikzcd}$$

Here is a general picture of $V = (V_a, f_\gamma)$ (for simplicity, denote f_α by f and f_β by g):

$$\begin{tikzcd}
\ddots \arrow[dr] & & & & \iddots \\
\cdots & \mathbf{k}^l \arrow[l] \arrow[r, "f"] & \mathbf{k}^m \arrow[r, "g"] & \mathbf{k}^n \arrow[r] \arrow[ur] & \cdots \\
\iddots \arrow[ur] & & & \vdots \arrow[u] & \ddots \arrow[ul]
\end{tikzcd}$$

Since $\operatorname{Im} f \neq 0$, $g \neq 0$, and $g \circ f = 0$, we may choose a basis $\{e_1, \ldots, e_p, e_{p+1}, \ldots, e_m\}$ for $\mathbf{k}^m$ such that $1 \leq p \leq m-1$ and

$$g(e_i) = 0 \text{ for } i = 1, \ldots, p$$

$$g(e_j) \neq 0 \text{ for } j = p+1, \ldots, m.$$

Since $g \circ f = 0, \operatorname{Im} f \subset \mathbf{k}\, e_1 \oplus \cdots \oplus \mathbf{k}\, e_p$. Define a map of representations $\varphi : V \to V$, where for any $a_1, \ldots, a_m \in \mathbf{k}$,

$$\begin{aligned}
&\varphi_a = \mathrm{Id}_{V_a} \;\text{ if } a \in V(Q') \\
&\varphi_a = 0 \text{ if } a \in V(Q'') \\
&\varphi_j : \mathbf{k}^m \to \mathbf{k}^m \\
&\quad (a_1, \ldots, a_m) \mapsto (a_1, \ldots, a_p, 0, \ldots, 0).
\end{aligned}$$

Since φ is a non-trivial idempotent in the endomorphism ring of V, V is decomposable by [DK94, Cor. 1.7.3]. □

4.3 Results for radical squared zero algebras

We will apply Theorem 4.1.3 to calculate the Frobenius-Perron dimension of the category of representations of a radical squared zero bound quiver algebra. To prove Theorem 4.1.3, we need some additional results.

Lemma 4.3.1. *Let A be a finite-dimensional algebra over a field $\mathbf{k}$. Then if V is a brick object in $A\text{-}\mathrm{mod}$ and $x \in Z(A)$, there exists $\lambda \in \mathbf{k}$ such that for all $v \in V$, $x.v = \lambda v$.*

Proof. We will prove the contrapositive. Notice that if x is a central element of A, then we have an induced endomorphism of any A-module V defined by:

$$\varphi_x : V \to V, \quad \varphi_x(v) = x.v.$$

Suppose that there exists a central element x of A and a representation V such that $\varphi_x \neq \lambda \mathrm{Id}_V$ for any $\lambda \in \mathbf{k}$. Then φ_x and Id_V are linearly independent elements of $\mathrm{Hom}_A(V, V)$, so V is not a brick. □

Proposition 4.3.2. *If x is a central nilpotent element of A and V is a brick module, then the ideal (x) annihilates V.*

Proof. By Lemma 4.3.1, for all $v \in V$ we must have $x.v = \lambda v$ for some $\lambda \in \mathbf{k}$. Suppose $x^n = 0$. Then $0 = x^n.v = \lambda^n v$ for every $v \in V$. Hence $\lambda = 0$ and x annihilates V. Since x is central, the two-sided ideal generated by x must also annihilate V. □

Corollary 4.3.3. *Let J be an ideal generated by central nilpotent elements of A. If V is a brick A-module, then J annihilates V.*

Proposition 4.3.4. *Let J be an ideal generated by central nilpotent elements. Then there is a one-to-one correspondence between the isomorphism classes below:*

$$\{\textit{brick } A\textit{-modules}\} \longleftrightarrow \{\textit{brick } A/J\textit{-modules}\}$$

Furthermore, if M, N are brick A/J-modules, then we have a natural isomorphism

$$\mathrm{Hom}_A(M,N) \cong \mathrm{Hom}_{A/J}(M,N).$$

Specifically, if we define J_{nc} to be the ideal generated by all central nilpotent elements, we get the one-to-one correspondence and natural isomorphism above.

Proof. Suppose that M, N are brick A-modules. Then J annihilates both M and N by Corollary 4.3.3. By Fact 4.2.2 (8) we have a natural isomorphism

$$\mathrm{Hom}_A(M,N) \cong \mathrm{Hom}_{A/J}(M,N).$$

Therefore M, N are also brick A/J-modules.

Now suppose that L, P are brick A/J-modules. Then L, P can be thought of as A-modules via Fact 4.2.2 (8), and we have

$$\mathrm{Hom}_{A/J}(L,P) \cong \mathrm{Hom}_A(L,P).$$

Therefore L, P are also brick A-modules. □

Proof of Theorem 4.1.3. First suppose that L, P are brick C-modules annihilated by (≥ 2). Then by Fact 4.2.2 (8) we have natural isomorphisms

$$\mathrm{Hom}_C(L,P) \cong \mathrm{Hom}_{C/(\geq 2)}(L,P) \cong \mathrm{Hom}_B(L,P) = \mathrm{Hom}_{A/J}(L,P) \cong \mathrm{Hom}_A(L,P).$$

Therefore L, P are brick A-modules.

Now suppose that M, N are brick A-modules. Then we have natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_A(M,N) &\cong \mathrm{Hom}_{A/J}(M,N) \quad \text{(Proposition 4.3.4)} \\ &= \mathrm{Hom}_B(M,N) \\ &\cong \mathrm{Hom}_{C/(\geq 2)}(M,N) \\ &\cong \mathrm{Hom}_C(M,N) \quad \text{(Fact 4.2.2 (8))}. \end{aligned}$$

Therefore, M, N are brick C-modules. □

4.4 An example of arbitrarily large irrational Frobenius-Perron dimension

Example 4.4.1. Consider the following quiver $Q(n,m)$ with n loops at vertex 1 and m loops at vertex 2:

$$\begin{tikzcd} 1 \arrow[loop left, "a_i"] \arrow[r, bend left, "x"] & 2 \arrow[l, bend left, "y"] \arrow[loop right, "b_j"] \end{tikzcd}$$

Let $A = \mathbf{k}\,Q(n,m)/(\geq 2)$. We want to calculate $\mathrm{fpd}\,(A\text{-}\mathrm{mod})$. We can see that $J = (a_1, \ldots, a_n, b_1, \ldots, b_m)$ is an ideal generated by central nilpotents. Let $B \coloneqq A/J$. Then B is described by the following quiver with relations $(\geq 2) = 0$:

$$\begin{tikzcd} 1 \arrow[r, bend left, "x"] & 2 \arrow[l, bend left, "y"] \end{tikzcd}$$

By Theorem 4.1.3, we have a one-to-one correspondence between bricks of A and bricks of B, as well as the following natural isomorphism for any A-modules M, N:

$$\mathrm{Hom}_A(M,N) \cong \mathrm{Hom}_B(M,N).$$

Hence it suffices to compute the dimensions of Hom_A-spaces in B.

The representation theory of B is well-known [CGW+19a]. There are 4 non-isomorphic indecomposable modules. First, we have the two simples S_1, S_2, which are described by $S_i =$

$Be_i/(\geq 1)$. Then we have the two indecomposable projectives M_1, M_2, where $M_i = Be_i$. S_1, S_2 are bricks since they are simples. We have $\mathrm{Hom}_B(M_i, M_i) = \mathrm{Hom}_B(Be_i, Be_i) \cong e_i B e_i \cong ke_i$, so each M_i is a brick as well. Under the one-to-one correspondence, these map to bricks in A where $S_i = Be_i/(\geq 1) = Ae_i/(\geq 1)$, which are simple in A, and $M_i = Be_i = Ae_i/Je_i$.

We know that $\mathrm{Hom}_A(M_i, M_j) \cong \mathrm{Hom}_B(M_i, M_j) = \mathrm{Hom}_B(Be_i, Be_j) \cong e_j B e_i$, which is non-zero for $i \neq j$. So M_1, M_2 cannot be in the same brick set. Furthermore, we have the following short exact sequences:

$$0 \to S_1 \to M_2 \to S_2 \to 0, \qquad 0 \to S_2 \to M_1 \to S_1 \to 0.$$

Hence we cannot have a brick set containing both S_i, M_j for any i, j.

Therefore there are only 5 possible brick sets: $\phi_1 = \{M_1\}$, $\phi_2 = \{M_2\}$, $\phi_3 = \{S_1, S_2\}$, $\phi_4 = \{S_1\}$, and $\phi_5 = \{S_2\}$. Since ϕ_4, ϕ_5 are subsets of ϕ_3, $\rho(A(\phi_{(4 \text{ or } 5)})) \leq \rho(A(\phi_3))$, so we do not need to consider them. We can compute the following.

$\rho(A(\phi_1)) = \mathrm{Ext}^1(M_1, M_1) = 0$ by Lemma 4.2.3.

$\rho(A(\phi_2)) = \mathrm{Ext}^1(M_2, M_2) = 0$ by Lemma 4.2.3.

$\rho(A(\phi_3)) = \rho\begin{pmatrix} n & 1 \\ 1 & m \end{pmatrix} = \frac{1}{2}\left(\sqrt{(m-n)^2 + 4} + m + n\right)$ by Fact 4.2.2 (6).

Then

$$\mathrm{fpd}\,(A\text{-}\mathrm{mod}) = \sup\{\rho(A(\phi_i)) : 1 \leq i \leq 3\} = \frac{1}{2}\left(\sqrt{(m-n)^2 + 4} + m + n\right).$$

Proof of Proposition 4.1.9. We can get arbitrarily large irrational numbers using this example. Let $r \in \mathbb{R}^{\geq 0}$. Let N be the smallest integer greater than or equal to r. Let $n = N, m = N + 1$. Then we have

$$\mathrm{fpd}\,(A\text{-}\mathrm{mod}) = \frac{1}{2}\left(\sqrt{(N+1-N)^2 + 4} + N + 1 + N\right) = \frac{1}{2}\left(2N + \sqrt{5} + 1\right) = N + \frac{1+\sqrt{5}}{2} > N \geq r.$$

□

Question 4.4.2. What is $\mathrm{fpd}\, D^b(A\text{-}\mathrm{mod})$? Is it still irrational in the cases where $\mathrm{fpd}\,(A\text{-}\mathrm{mod})$ is irrational?

4.5 Frobenius-Perron dimension of modified ADE bound quiver algebras

4.5.1 $A(n)$ algebras

Theorem 4.5.1. *Let $A(n)$ be the finite dimensional radical squared zero algebra described by the following quiver Q with N_i loops (labeled a_i^l for $l = 1, \ldots, N_i$) at vertex i:*

$$\begin{tikzcd} 1 \arrow[r, "x_2"] \arrow[loop below, "a_1^l"] & 2 \arrow[r, "x_3"] \arrow[loop below, "a_2^l"] & \cdots \arrow[r, "x_n"] & n \arrow[loop below, "a_n^l"] \end{tikzcd}$$

Then

$$\operatorname{fpd}(A(n)\text{-}\operatorname{mod}) = \max\{N_1, \ldots, N_n\}.$$

Proof. We have $A(n) = \mathbf{k}Q/(\geq 2)$. Let J be the ideal generated by the loops. Let $B := A(n)/J$. In other words, B is the radical squared zero algebra described by the following quiver Q':

$$\begin{tikzcd} 1 \arrow[r, "x_2"] & 2 \arrow[r, "x_3"] & \cdots \arrow[r, "x_n"] & n \end{tikzcd}$$

We can see that $B = C/(\geq 2)$, where $C := \mathbf{k}Q'$.

By Theorem 4.1.3, we have a one-to-one correspondence between bricks of $A(n)$ and bricks of C annihilated by (≥ 2), as well as the following natural isomorphism for any brick $A(n)$-modules M, N:

$$\operatorname{Hom}_{A(n)}(M, N) \cong \operatorname{Hom}_B(M, N) \cong \operatorname{Hom}_C(M, N).$$

Hence it suffices to compute the dimensions of $\operatorname{Hom}_{A(n)}$-spaces in B or C.

The set of brick modules of C is a subset of the set of indecomposable modules of C, which are well known by Gabriel's theorem [Gab72]. The dimension vectors of these modules correspond precisely to the positive roots for the A_n root system [Hum78, p. 64].

Let $E_i \coloneqq (0, \ldots, 1, \ldots, 0)$, the n-dimensional vector with a 1 in the ith position and 0 elsewhere. Then the positive roots of A_n are given by the following vectors:

$$\left\{ \sum_{l=i}^{j} E_l : 1 \leq i \leq j \leq n \right\},$$

or in different notation,

$$\begin{aligned}
&(1, 0, \ldots, 0)\\
&(0, 1, \ldots, 0)\\
&\vdots\\
&(0, \ldots, 0, 1)\\
&(1, 1, 0, \ldots, 0)\\
&(0, 1, 1, 0, \ldots, 0)\\
&\vdots\\
&(0, \ldots, 0, 1, 1)\\
&(1, 1, 1, 0, \ldots, 0)\\
&(0, 1, 1, 1, 0, \ldots, 0)\\
&\vdots\\
&(0, \ldots, 0, 1, 1, 1)\\
&\vdots\\
&(1, \ldots, 1).
\end{aligned}$$

However by Fact 4.2.2 (9), (10), any dimension vector with more than two 1's in a row gives a C-module that is not annihilated by (≥ 2). Therefore, the only indecomposables that could possibly be bricks of $A(n)$ are the following. Note that below we use Schiffler's notation for representations [Sch14, Remark 1.1].

$$(1) = (1, 0, \ldots, 0)$$

$$\begin{aligned}
(2) =&(0,1,\dots,0)\\
&\vdots\\
(n) =&(0,\dots,0,1)\\
\binom{1}{2} =&(1,1,0,\dots,0)\\
\binom{2}{3} =&(0,1,1,0,\dots,0)\\
&\vdots\\
\binom{n-1}{n} =&(0,\dots,0,1,1).
\end{aligned}$$

We will show that all of these are bricks of $A(n)$ and calculate the Hom matrix.

Because $(i),(j)$ are simple,

$$\dim \operatorname{Hom}_C((i),(j)) = \delta_{ij}, \tag{4.5.1.1}$$

so $(i),(j)$ are bricks.

If we have $\binom{i}{i+1}, \binom{j}{j+1}$ with $i \neq j, i \neq j+1$, and $i \neq j-1$, then we have no overlap between the non-zero vector spaces of the representations, and $\operatorname{Hom}_C\left(\binom{i}{i+1}, \binom{j}{j+1}\right) = 0$. So we only need to check the morphisms in the cases $i = j, i = j+1, i = j-1$.

Denote the indecomposable projective (respectively injective, respectively simple) of B at vertex i by $\tilde{P}_i$ (respectively $\tilde{I}_i$, respectively $\tilde{S}_i$). Then by Fact 4.2.2 (11),

$$\tilde{P}_i = \binom{i}{i+1} = \tilde{I}_{i+1}. \tag{4.5.1.2}$$

If $i = j$,

$$\dim \operatorname{Hom}_{A(n)}\left(\binom{i}{i+1}, \binom{i}{i+1}\right) = \dim \operatorname{Hom}_B\left(\tilde{P}_i, \tilde{P}_i\right) = \dim \operatorname{Hom}_B\left(Be_i, Be_i\right) = \dim e_i B e_i = 1,$$

so $\binom{i}{i+1}$ is a brick.

If $i = j+1$,

$$\dim \operatorname{Hom}_{A(n)}\left(\binom{i}{i+1}, \binom{i-1}{i}\right) = \dim \operatorname{Hom}_B\left(\tilde{P}_i, \tilde{P}_{i-1}\right) = \dim \operatorname{Hom}_B(Be_i, Be_{i-1}) = \dim e_i B e_{i-1} = 1.$$

If $i = j - 1$,

$$\operatorname{Hom}_{A(n)}\left(\begin{pmatrix} i \\ i+1 \end{pmatrix}, \begin{pmatrix} i+1 \\ i+2 \end{pmatrix}\right) = \operatorname{Hom}_B\left(\tilde{P}_i, \tilde{P_{i+1}}\right) = \operatorname{Hom}_B(Be_i, Be_{i+1}) = e_i B e_{i+1} = 0.$$

Therefore,

$$\dim \operatorname{Hom}_{A(n)}\left(\begin{pmatrix} i \\ i+1 \end{pmatrix}, \begin{pmatrix} j \\ j+1 \end{pmatrix}\right) = \begin{cases} 1 & \text{if } i = j+1 \\ 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}. \tag{4.5.1.3}$$

Recall that by Fact 4.2.2 (4), $\dim \operatorname{Hom}_B(\tilde{S}_i, \tilde{I}_j) = \delta_{ij}$. Therefore since $\tilde{S}_i = (i), \tilde{I}_{j+1} = \binom{j}{j+1}$,

$$\dim \operatorname{Hom}_{A(n)}\left((i), \begin{pmatrix} j \\ j+1 \end{pmatrix}\right) = \begin{cases} 1 & \text{if } i = j+1 \\ 0 & \text{otherwise} \end{cases}. \tag{4.5.1.4}$$

Now let us consider $\binom{i}{i+1}, (j)$. By Fact 4.2.2 (4),

$$\dim \operatorname{Hom}_{A(n)}\left(\begin{pmatrix} i \\ i+1 \end{pmatrix}, (j)\right) = \dim \operatorname{Hom}_B(\tilde{P}_i, \tilde{S}_j) = \delta_{ij}. \tag{4.5.1.5}$$

Compiling all this information, we have the following block matrix H. We define $H_{ij} = \dim \operatorname{Hom}_{A(n)}(X_i, X_j)$, where $X_i, X_j \in \left\{(1), \ldots, (n), \binom{1}{2}, \ldots, \binom{n-1}{n}\right\}$:

$$
H = \begin{array}{cc}
 & \begin{array}{cccccc|ccccc} (1) & (2) & (3) & \cdots & (n-1) & (n) & \binom{1}{2} & \binom{2}{3} & \cdots & \binom{n-2}{n-1} & \binom{n-1}{n} \end{array} \\
\begin{array}{c} (1) \\ (2) \\ (3) \\ \vdots \\ (n-1) \\ (n) \\ \hline \binom{1}{2} \\ \binom{2}{3} \\ \vdots \\ \binom{n-2}{n-1} \\ \binom{n-1}{n} \end{array} &
\left(\begin{array}{cccccc|ccccc}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 & 0 & 0 & 1 & \ddots & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \ddots & \ddots & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}\right)
\end{array}
$$

Now we can calculate the $\mathrm{Ext}^1_{A(n)}$ matrix. We already know what the Ext^1 groups for the simple objects should be. Recall from Lemma 4.2.3 that

$$
\begin{aligned}
\mathrm{Ext}^1_{A(n)}(S_i, \tilde{I}_j) &= 0, \ \forall i \neq j \\
\mathrm{Ext}^1_{A(n)}(\tilde{I}_i, \tilde{I}_j) &= 0, \ \text{if } i \neq j \text{ or } \tilde{I}_i \text{ is not simple} \\
\mathrm{Ext}^1_{A(n)}(\tilde{P}_i, S_j) &= 0, \ \forall i \neq j \\
\mathrm{Ext}^1_{A(n)}(\tilde{P}_i, \tilde{P}_j) &= 0, \ \text{if } i \neq j \text{ or } \tilde{P}_i \text{ is not simple.}
\end{aligned}
$$

Let $*$ denote a possible non-zero matrix entry. We have $(i) = S_i$ and $\binom{j}{j+1} = \tilde{P}_j = \tilde{I}_{j+1}$.

Using the same notation as before, if we define $E_{ij} = \mathrm{Ext}^1_{A(n)}(X_i, X_j)$, we have

$$
E = \begin{array}{cc}
 & \begin{array}{cccccc|ccccc} (1) & (2) & (3) & \cdots & (n-1) & (n) & \binom{1}{2} & \binom{2}{3} & \cdots & \binom{n-2}{n-1} & \binom{n-1}{n} \end{array} \\
\begin{array}{c} (1) \\ (2) \\ (3) \\ \vdots \\ (n-1) \\ (n) \\ \hline \binom{1}{2} \\ \binom{2}{3} \\ \vdots \\ \binom{n-2}{n-1} \\ \binom{n-1}{n} \end{array} &
\left(\begin{array}{cccccc|ccccc}
N_1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & N_2 & 1 & 0 & \cdots & 0 & * & 0 & \cdots & 0 & 0 \\
0 & 0 & N_3 & 1 & \cdots & 0 & 0 & * & \ddots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \ddots & \ddots & 0 & 0 \\
0 & 0 & 0 & 0 & N_{n-1} & 1 & 0 & 0 & 0 & * & 0 \\
0 & 0 & 0 & 0 & 0 & N_n & 0 & 0 & 0 & 0 & * \\
\hline
* & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\end{array}
$$

Now suppose that we have a brick set

$$
\phi = \left\{ (i_1), \ldots, (i_M), \begin{pmatrix} j_1 \\ j_1 + 1 \end{pmatrix}, \ldots, \begin{pmatrix} j_L \\ j_L + 1 \end{pmatrix} : i_1 < i_2 < \cdots < i_M, \quad j_1 < j_2 < \cdots < j_L \right\}.
$$

We will show that for $\ast \in \{0,1\}$,

$$A(\phi) = \begin{array}{c c}
 & \begin{array}{cccccc|ccccc} (i_1) & (i_2) & (i_3) & \cdots & (i_{M-1}) & (i_M) & \binom{j_1}{j_1+1} & \binom{j_2}{j_2+2} & \cdots & \binom{j_{L-1}}{j_{L-1}+1} & \binom{j_L}{j_L+1} \end{array} \\
\begin{array}{c} (i_1) \\ (i_2) \\ (i_3) \\ \vdots \\ (i_{M-1}) \\ (i_M) \\ \hline \binom{j_1}{j_1+1} \\ \binom{j_2}{j_2+2} \\ \vdots \\ \binom{j_{L-1}}{j_{L-1}+1} \\ \binom{j_L}{j_L+1} \end{array} &
\left(\begin{array}{cccccc|ccccc}
N_{i_1} & \ast & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & N_{i_2} & \ast & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & N_{i_3} & \ast & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & N_{i_{M-1}} & \ast & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & N_{i_M} & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\end{array}$$

Assuming the claim about $A(\phi)$, if there are no simples in ϕ, $\rho(A(\phi)) = 0 \le \max\{N_1, \ldots, N_n\}$. Else,

$$\rho(A(\phi)) = \max\{N_{i_1}, \ldots, N_{i_M}\} \le \max\{N_1, \ldots, N_n\}. \tag{4.5.1.6}$$

To compute $A(\phi)$, notice that the upper leftmost and bottom rightmost blocks are given by examination of the Ext^1 matrix. To get the upper rightmost and bottom leftmost blocks, it suffices to show that for all m, l,

$$\mathrm{Ext}^1_{A(n)}\left((i_m), \begin{pmatrix} j_l \\ j_l + 1 \end{pmatrix}\right) = 0 = \mathrm{Ext}^1_{A(n)}\left(\begin{pmatrix} j_l \\ j_l + 1 \end{pmatrix}, (i_m)\right).$$

By examination of the Hom and Ext^1 matrices, if $(i) \in \{(1), \ldots, (n)\}$ and $\binom{j}{k} \in \left\{\binom{1}{2}, \ldots, \binom{n-1}{n}\right\}$ are in the same brick set, then

$$\mathrm{Ext}^1_{A(n)}\left((i), \begin{pmatrix} j \\ k \end{pmatrix}\right) = 0 = \mathrm{Ext}^1_{A(n)}\left(\begin{pmatrix} j \\ k \end{pmatrix}, (i)\right).$$

Therefore,

$$\mathrm{Ext}^1_{A(n)}\left((i_m), \begin{pmatrix} j_l \\ j_l + 1 \end{pmatrix}\right) = 0 = \mathrm{Ext}^1_{A(n)}\left(\begin{pmatrix} j_l \\ j_l + 1 \end{pmatrix}, (i_m)\right).$$

To complete the proof, notice that

$$\tilde{\phi} := \{(1), (2), \ldots, (n)\}$$

is a brick set. By Eq. (4.5.1.6) we have

$$\rho(A(\tilde{\phi})) = \max\{N_1, N_2, \ldots, N_n\}.$$

Let Φ_b the set of brick sets of $A(n)$-modules. Again by Eq. (4.5.1.6),

$$\max\{N_1, N_2, \ldots, N_n\} = \rho(A(\tilde{\phi})) \leq \sup_{\phi \in \Phi_b} \rho(A(\phi)) \leq \max\{N_1, N_2, \ldots, N_n\}.$$

Therefore,

$$\operatorname{fpd}(A(n)\text{-}\mathrm{mod}) = \sup_{\phi \in \Phi_b} \rho(A(\phi)) = \max\{N_1, N_2, \ldots, N_n\}. \qquad \square$$

4.5.2 $D(n)$ algebras

Theorem 4.5.2. *Let $D(n)$ be the finite dimensional radical squared zero algebra described by the following quiver Q with N_i loops (labeled a_i^l for $l = 1, \ldots, N_i$) at vertex i:*

$$\begin{tikzcd}
 & & & n-1 \arrow[loop above, "a^l_{n-1}"] & \\
1 \arrow[r, "x_2"] \arrow[loop below, "a^l_1"] & 2 \arrow[r, "x_3"] \arrow[loop below, "a^l_2"] & \cdots \arrow[r, "x_{n-2}"] & n-2 \arrow[u, "x_{n-1}"] \arrow[r, "x_n"] \arrow[loop below, "a^l_{n-2}"] & n \arrow[loop below, "a^l_n"]
\end{tikzcd}$$

Then

$$\operatorname{fpd}(D(n)\text{-}\mathrm{mod}) = \max\{N_1, \ldots, N_n\}.$$

Proof. We will reuse some notation from the previous theorem. Let $D(n) = \mathbf{k}Q/(\geq 2)$. Let J be the ideal generated by the loops, and let $B := D(n)/J$. In other words, B is the radical squared zero algebra described by the following quiver Q':

$$\begin{tikzcd} & & & n-1 & \\ 1 \arrow[r, "x_2"] & 2 \arrow[r, "x_3"] & \cdots \arrow[r, "x_{n-2}"] & n-2 \arrow[u, "x_{n-1}"] \arrow[r, "x_n"] & n \end{tikzcd}$$

We can see that $B = C/(\geq 2)$, where $C := \mathbf{k}Q'$.

By Theorem 4.1.3, we have a one-to-one correspondence between bricks of $D(n)$ and bricks of C annihilated by (≥ 2), as well as the following natural isomorphism:

$$\mathrm{Hom}_{D(n)}(M,N) \cong \mathrm{Hom}_B(M,N) \cong \mathrm{Hom}_C(M,N).$$

Hence it suffices to compute the dimensions of $\mathrm{Hom}_{D(n)}$-spaces in B or C.

The indecomposable modules of C are well known by Gabriel's theorem [Gab72]. The dimension vectors of these modules correspond precisely to the positive roots for the D_n root system.

Let ε_i be the vector in $\mathbb{R}^n$ with a 1 in the ith component and 0 everywhere else. Recall from [Hum78, p. 64] that the root system for D_n consists of the vectors

$$\{\pm(\varepsilon_i \pm \varepsilon_j) : i \neq j\}.$$

Let

$$E_1 := \varepsilon_1 - \varepsilon_2, \ldots, E_{n-1} := \varepsilon_{n-1} - \varepsilon_n, E_n := \varepsilon_{n-1} + \varepsilon_n$$

be the standard base for D_n. Here we write out the other positive roots in terms of the standard base.

For $1 \leq i < j \leq n$,

$$\varepsilon_i - \varepsilon_j = E_i + E_{i+1} + \cdots + E_{j-1} = \sum_{l=i}^{j-1} E_l.$$

For $1 \leq i \leq n-2$,

$$\varepsilon_i + \varepsilon_n = E_i + E_{i+1} + \cdots + E_{n-2} + E_n = \left(\sum_{l=i}^{n-2} E_l\right) + E_n.$$

For $1 \le i \le n-2$,

$$\varepsilon_i + \varepsilon_{n-1} = E_i + E_{i+1} + \cdots + E_{n-2} + E_{n-1} + E_n = \sum_{l=i}^{n} E_l.$$

For $1 \le i < j \le n-2$,

$$\varepsilon_i + \varepsilon_j = (E_i + E_{i+1} + \cdots + E_{n-2}) + (E_j + E_{j+1} + \cdots + E_{n-2} + E_{n-1} + E_n) = \sum_{l=i}^{n-2} E_l + \sum_{k=j}^{n} E_k.$$

Let $\tilde{P}_i$ (respectively $\tilde{I}_i$) denote the indecomposable projective (respectively injective) of B at vertex i. By Fact 4.2.2 (11), we have

$$\tilde{P}_1 = \tilde{I}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = (1, 1, 0, \ldots, 0) \tag{4.5.2.1}$$

$$\tilde{P}_2 = \tilde{I}_3 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} = (0, 1, 1, 0, \ldots, 0) \tag{4.5.2.2}$$

$$\vdots \tag{4.5.2.3}$$

$$\tilde{P}_{n-3} = \tilde{I}_{n-2} = \begin{pmatrix} n-3 \\ n-2 \end{pmatrix} = (0, \ldots, 0, 1, 1, 0, 0) \tag{4.5.2.4}$$

$$\tilde{I}_{n-1} = \begin{pmatrix} n-2 \\ n-1 \end{pmatrix} = (0, \ldots, 0, 1, 1, 0) \tag{4.5.2.5}$$

$$\tilde{I}_n = \begin{pmatrix} n-2 \\ n \end{pmatrix} = (0, \ldots, 0, 1, 0, 1) \tag{4.5.2.6}$$

$$\tilde{P}_{n-2} = \begin{pmatrix} & n-2 & \\ n-1 & & n \end{pmatrix} = (0, \ldots, 0, 1, 1, 1). \tag{4.5.2.7}$$

Proposition 4.5.3. *The only indecomposable representations of C which are annihilated by (≥ 2) are the following:*

$$(1) = E_1 = (1, 0, \ldots, 0)$$

$$(2) = E_2 = (0, 1, 0, \ldots, 0)$$

$$\vdots$$

$$(n) = E_n = (0, \ldots, 0, 1)$$

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = E_1 + E_2 = (1, 1, 0, \ldots, 0)$$

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = E_2 + E_3 = (0, 1, 1, 0, \ldots, 0)$$

$$\vdots$$

$$\begin{pmatrix} n-3 \\ n-2 \end{pmatrix} = E_{n-3} + E_{n-2} = (0, \dots, 0, 1, 1, 0, 0)$$

$$\begin{pmatrix} n-2 \\ n-1 \end{pmatrix} = E_{n-2} + E_{n-1} = (0, \dots, 0, 1, 1, 0)$$

$$\begin{pmatrix} n-2 \\ n \end{pmatrix} = E_{n-2} + E_{n} = (0, \dots, 0, 1, 0, 1)$$

$$\begin{pmatrix} n-2 \\ n-1 \;\; n \end{pmatrix} = E_{n-2} + E_{n-1} + E_{n} = (0, \dots, 0, 1, 1, 1).$$

Proof. Case 0: $E_n = \varepsilon_{n-1} + \varepsilon_n$.

This representation is annihilated by (≥ 2) because E_n is the simple module at vertex n.

Case 1: Roots of the form $\varepsilon_i - \varepsilon_j$ *for* $1 \leq i < j \leq n$.

We have

$$\varepsilon_i - \varepsilon_j = E_i + E_{i+1} + \cdots + E_{j-1} = (0, \dots, 0, 1, \dots, 1, 0, \dots, 0),$$

where the first 1 is in the ith position and the last 1 is in the $(j-1)$th position. Notice that there must be a 0 in the nth position since $j \leq n$. By examination of the graph for C, we can see that if $j > i+2$ we have three 1's in a row, which corresponds to two non-zero paths of length 1 in a row by Fact 4.2.2 (10). Since the vector spaces involved are 1-dimensional, the composition of these non-zero paths must also be non-zero. By Fact 4.2.2 (9), if $j > i+2$ the representation is not annihilated by (≥ 2).

Therefore only dimension vectors of the following form are annihilated by (≥ 2):

$$\varepsilon_i - \varepsilon_{i+2} = E_i + E_{i+1} \text{ for } 1 \leq i \leq n-2$$

$$\varepsilon_i - \varepsilon_{i+1} = E_i \text{ for } 1 \leq i \leq n-1$$

Case 2: Roots of the form $\varepsilon_i + \varepsilon_n$ *for* $1 \leq i \leq n-2$.

We have

$$\varepsilon_i + \varepsilon_n = E_i + E_{i+1} + \cdots + E_{n-2} + E_n = (0, \dots, 0, 1, \dots, 1, 0, 1),$$

where the first 1 is in the ith position. If $i < n-2$, then we have three one-dimensional vector spaces which sit adjacent to each other in the graph of C, which corresponds to a non-zero path of length 2 by Fact 4.2.2 (10). By Fact 4.2.2 (9), such a representation is not annihilated by (≥ 2). If $i = n-2$, there is no such path.

Then the only representation annihilated by (≥ 2) is the one corresponding to $i = n-2$:

$$E_{n-2} + E_n = (0, \dots, 0, 1, 0, 1).$$

Case 3: Roots of the form $\varepsilon_i + \varepsilon_{n-1}$ for $1 \leq i \leq n-2$.

We have

$$\varepsilon_i + \varepsilon_{n-1} = E_i + E_{i+1} + \cdots + E_{n-2} + E_{n-1} + E_n = (0, \dots, 0, 1, \dots, 1, 1),$$

where the first 1 is in the ith position. If $i < n-2$, then we have three one-dimensional vector spaces which sit adjacent to each other in the graph of C, which corresponds to a non-zero path of length 2 by Fact 4.2.2 (10). By Fact 4.2.2 (9), such a representation is not annihilated by (≥ 2). If $i = n-2$, there is no such path.

By Fact 4.2.2 (9), the only representation annihilated by (≥ 2) is the one corresponding to $i = n-2$:

$$E_{n-2} + E_{n-1} + E_n = (0, \dots, 0, 1, 1, 1).$$

Case 4: Roots of the form $\varepsilon_i + \varepsilon_j$ for $1 \leq i < j \leq n-2$.

We have

$$\begin{aligned}\varepsilon_i + \varepsilon_j &= (E_i + E_{i+1} + \cdots E_{n-2}) + (E_j + E_{j+1} + \cdots + E_{n-2} + E_{n-1} + E_n) \\ &= E_i + E_{i+1} + \cdots + E_{j-1} + 2(E_j + \cdots + E_{n-2}) + E_{n-1} + E_n.\end{aligned}$$

If $j = n-2$, then

$$\begin{aligned}\varepsilon_i + \varepsilon_j &= E_i + E_{i+1} + \cdots + E_{j-1} + 2(E_j + \cdots + E_{n-2}) + E_{n-1} + E_n \\ &= E_i + \cdots + E_{n-3} + 2E_{n-2} + E_{n-1} + E_n \\ &= (0, \dots, 0, 1, \dots, 1, 2, 1, 1).\end{aligned}$$

If $j \leq n-3$, then

$$\begin{aligned}\varepsilon_i + \varepsilon_j &= E_i + E_{i+1} + \cdots + E_{j-1} + 2(E_j + \cdots + E_{n-2}) + E_{n-1} + E_n \\ &= (0, \ldots, 0, 1, \ldots, 1, 2, \ldots, 2, 1, 1).\end{aligned}$$

By Lemma 4.5.4 below, these representations are not annihilated by (≥ 2). Therefore, no representations of this type are allowed. □

Lemma 4.5.4. *Suppose that we have a representation V of C of the form*

$$\begin{tikzcd} & & & & \mathbf{k} & \\ V_1 \arrow[r] & \cdots \arrow[r] & V_{n-4} \arrow[r] & \mathbf{k}^2 \arrow[r, "f"] & \mathbf{k}^2 \arrow[r, "h"] \arrow[u, "g"] & \mathbf{k} \end{tikzcd} \tag{4.5.4.1}$$

or

$$\begin{tikzcd} & & & & \mathbf{k} & \\ V_1 \arrow[r] & \cdots \arrow[r] & V_{n-4} \arrow[r] & \mathbf{k} \arrow[r, "f"] & \mathbf{k}^2 \arrow[r, "h"] \arrow[u, "g"] & \mathbf{k} \end{tikzcd} \tag{4.5.4.2}$$

for some $\mathbf{k}$-vector spaces V_i such that f, g, h are non-zero and $g \circ f = 0 = h \circ f$. Then V is decomposable.

Proof. Define Q' to be the full subquiver containing the vertices $\{1, \ldots, n-3\}$, and let Q'' be the full subquiver containing the vertices $\{n-1, n\}$. Let $V(Q'), V(Q'')$ denote the set of vertices of Q', Q'' respectively.

We will start with Eq. (4.5.4.1).

Since $g \neq 0$, we can choose a basis e_1, e_2 for $V_{n-2} = \mathbf{k}^2$ such that $g = \pi_1$. Then

$$g \circ f = 0 \Rightarrow \operatorname{Im} f \subset \mathbf{k}\, e_2$$

$$h \circ f = 0, h \neq 0 \Rightarrow h = h_1 \pi_1 \text{ for some non-zero } h_1 \in \mathbf{k}\,.$$

Then we have a map of representations $\varphi : V \to V$ such that for any $a_1, a_2 \in \mathbf{k}$,

$$\varphi_a = 0 \text{ if } a \in V(Q')$$

$$
\begin{aligned}
\varphi_a &= \mathrm{Id}_{V_a} \text{ if } a \in V(Q'') \\
\varphi_{n-2} &: \mathbf{k}^2 \to \mathbf{k}^2 \\
&(a_1, a_2) \mapsto (a_1, 0).
\end{aligned}
$$

Since φ is a non-trivial idempotent in the endomorphism ring of V, V is decomposable by [DK94, Cor. 1.7.3].

Now let us prove the assertion for Eq. (4.5.4.2). By choice of basis for V_{n-3} and the fact that $f \neq 0$, we can assume that $f = \iota_2$. Since $g \circ f = 0 = h \circ f$ and g, h are non-zero, we must have $g = g_1 \pi_1, h = h_1 \pi_1$ for some non-zero $g_1, h_1 \in \mathbf{k}$.

Then we have a map of representations $\varphi : V \to V$ such that for any $a_1, a_2 \in \mathbf{k}$,

$$
\begin{aligned}
\varphi_a &= 0 \text{ if } a \in V(Q') \\
\varphi_a &= \mathrm{Id}_{V_a} \text{ if } a \in V(Q'') \\
\varphi_{n-2} &: \mathbf{k}^2 \to \mathbf{k}^2 \\
&(a_1, a_2) \mapsto (a_1, 0).
\end{aligned}
$$

Since φ is a non-trivial idempotent in the endomorphism ring of V, V is decomposable by [DK94, Cor. 1.7.3]. □

The list of indecomposables given in Proposition 4.5.3 are in fact bricks of $D(n)$, as can be seen by examination of the following Hom matrix, where $H_{ij} = \dim \mathrm{Hom}_{D(n)}(X_i, X_j)$ for brick objects X_i, X_j of $D(n)$-mod:

$$
\begin{array}{c|ccccccc|cccc|ccc}
 & (1) & (2) & \cdots & (n-3) & (n-2) & (n-1) & (n) & \binom{1}{2} & \binom{2}{3} & \cdots & \binom{n-3}{n-2} & \binom{n-2}{n-1} & \binom{n-2}{n} & \binom{n-2}{n-1\ n} \\
\hline
(1) & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
(2) & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & 0 & \ddots & \ddots & 0 & \vdots & \vdots & \vdots & 0 & \ddots & 0 & 0 & \vdots & \vdots & \vdots \\
(n-3) & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\
(n-2) & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
(n-1) & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
(n) & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
\hline
\binom{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 \\
\binom{2}{3} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & 0 & \vdots & \vdots & \vdots \\
\binom{n-3}{n-2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
\hline
\binom{n-2}{n-1} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
\binom{n-2}{n} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
\binom{n-2}{n-1\ n} & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1
\end{array}
$$

To calculate the entries of the Hom matrix, recall from Fact 4.2.2 (4), (5) that

$$\dim \mathrm{Hom}_C(\tilde{P}_i, \tilde{S}_j) = \delta_{ij},$$

$$\dim \mathrm{Hom}_C(\tilde{S}_i, \tilde{I}_j) = \delta_{ij},$$

$$\dim \mathrm{Hom}_C(\tilde{I}_j, \tilde{I}_i) = \text{ the number of paths from } i \to j \text{ in } C = \dim \mathrm{Hom}_C(\tilde{P}_j, \tilde{P}_i).$$

Using Eq. (4.5.2.1) through Eq. (4.5.2.7) in conjunction with these facts, we get most of the Hom matrix. We can calculate the remaining entries:

$$\dim \mathrm{Hom}_C\left((i), \begin{pmatrix} n-2 \\ n-1\ \ n \end{pmatrix}\right) = \begin{cases} 0 & i \in \{1, \ldots, n-2\} \\ 1 & i \in \{n-1, n\} \end{cases}$$

$$\dim \mathrm{Hom}_C\left(\begin{pmatrix} n-2 \\ n-1 \end{pmatrix}, (i)\right) = \begin{cases} 1 & i = n-2 \\ 0 & \text{otherwise} \end{cases}$$

$$\dim \operatorname{Hom}_C\left(\begin{pmatrix} n-2 \\ n \end{pmatrix}, (i)\right) = \begin{cases} 1 & i = n-2 \\ 0 & \text{otherwise} \end{cases}$$

$$\dim \operatorname{Hom}_C\left(\begin{pmatrix} n-2 \\ n-1 \end{pmatrix}, \begin{pmatrix} n-2 \\ n-1\ \ n \end{pmatrix}\right) = 0$$

$$\dim \operatorname{Hom}_C\left(\begin{pmatrix} n-2 \\ n \end{pmatrix}, \begin{pmatrix} n-2 \\ n-1\ \ n \end{pmatrix}\right) = 0$$

$$\dim \operatorname{Hom}_C\left(\begin{pmatrix} n-2 \\ n-1\ \ n \end{pmatrix}, \begin{pmatrix} n-2 \\ n-1 \end{pmatrix}\right) = 1$$

$$\dim \operatorname{Hom}_C\left(\begin{pmatrix} n-2 \\ n-1\ \ n \end{pmatrix}, \begin{pmatrix} n-2 \\ n \end{pmatrix}\right) = 1.$$

We have the following Ext^1 matrix using the same notation as before, where $E_{ij} = \dim \operatorname{Ext}^1_{D(n)}(X_i, X_j)$, and $*$ denotes a possible non-zero entry:

$$\begin{array}{c|ccccccc|cccc|ccc}
 & (1) & (2) & \cdots & (n-3) & (n-2) & (n-1) & (n) & \binom{1}{2} & \binom{2}{3} & \cdots & \binom{n-3}{n-2} & \binom{n-2}{n-1} & \binom{n-2}{n} & \binom{n-2}{n-1\ n} \\
\hline
(1) & N_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
(2) & 0 & N_2 & \ddots & 0 & 0 & 0 & 0 & * & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & 0 & 0 & \vdots & \vdots & \vdots \\
(n-3) & 0 & 0 & 0 & N_{n-3} & 1 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & * \\
(n-2) & 0 & 0 & 0 & 0 & N_{n-2} & 1 & 1 & 0 & 0 & 0 & * & 0 & 0 & * \\
(n-1) & 0 & 0 & 0 & 0 & 0 & N_{n-1} & 0 & 0 & 0 & 0 & 0 & * & 0 & * \\
(n) & 0 & 0 & 0 & 0 & 0 & 0 & N_n & 0 & 0 & 0 & 0 & 0 & * & * \\
\hline
\binom{1}{2} & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
\binom{2}{3} & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & 0 & \ddots & \ddots & 0 & \vdots & \vdots & \vdots \\
\binom{n-3}{n-2} & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\binom{n-2}{n-1} & 0 & 0 & 0 & 0 & * & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & * \\
\binom{n-2}{n} & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\
\binom{n-2}{n-1\ n} & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0
\end{array}$$

Recall the results of Lemmas 4.2.3, 4.2.4 and 4.2.6:

$$\operatorname{Ext}^1_{D(n)}(\tilde{P}_i, \tilde{S}_j) = 0 = \operatorname{Ext}^1_{D(n)}(\tilde{S}_i, \tilde{I}_j) \text{ if } i \neq j \tag{4.5.4.3}$$

$$\operatorname{Ext}^1_{D(n)}(\tilde{P}_i, \tilde{P}_j) = 0 = \operatorname{Ext}^1_{D(n)}(\tilde{I}_i, \tilde{I}_j) \text{ if they are not simple} \tag{4.5.4.4}$$

$$\operatorname{Ext}^1_{D(n)}(\tilde{P}_i, M) = 0 \text{ if } \operatorname{Hom}_{D(n)}(P_i, M) = 0 \tag{4.5.4.5}$$

$$\operatorname{Ext}^1_{D(n)}(\tilde{S}_i, M) = 0 \text{ if } i \text{ is a sink in } Q' \text{ and } \operatorname{Hom}_{D(n)}(P_i, M) = 0. \tag{4.5.4.6}$$

By Eq. (4.5.4.3) and Eq. (4.5.4.4), we get all of the entries in the four top left block matrices. We also get the entries in the middle rightmost block, the middle bottom block, and the bottom rightmost block. It remains to check the top rightmost block and the bottom leftmost block.

For the top rightmost block, note that Eq. (4.5.4.4) gives all of the entries in the $\left(\begin{smallmatrix} n-2 \\ n-1 \end{smallmatrix}\right), \left(\begin{smallmatrix} n-2 \\ n \end{smallmatrix}\right)$ columns.

Claim. $\operatorname{Ext}^1_{D(n)}\left((i), \left(\begin{smallmatrix} n-2 \\ n-1\ n \end{smallmatrix}\right)\right) = 0$ for $i \le n-4$.

We have the following projective resolution for $(i) = S_i$:

$$\begin{tikzcd}
\cdots \arrow[r] & P_{i+1} \oplus P_i^{N_i} \arrow[r] \arrow[dr] & P_i \arrow[r] & S_i \arrow[r] & 0 \\
& & \mathbf{k}\,x_{i+1} \oplus \left(\bigoplus_{l=1}^{N_i} \mathbf{k}\,a_i^l\right) \arrow[u, hook] & &
\end{tikzcd}$$

Now take $\operatorname{Hom}_{D(n)}\left(-, \left(\begin{smallmatrix} n-2 \\ n-1\ n \end{smallmatrix}\right)\right) = \operatorname{Hom}_{D(n)}(-, \tilde{P}_{n-2})$:

$$0 \longrightarrow \operatorname{Hom}_{D(n)}(P_i, \tilde{P}_{n-2}) \longrightarrow \operatorname{Hom}_{D(n)}(P_{i+1}, \tilde{P}_{n-2}) \oplus \operatorname{Hom}_{D(n)}(P_i, \tilde{P}_{n-2})^{N_i} \longrightarrow \cdots$$

Then for $j \neq n-2$,

$$\begin{aligned}
\dim \operatorname{Hom}_{D(n)}(P_j, \tilde{P}_{n-2}) &= \dim \operatorname{Hom}_{D(n)}\left(D(n)e_j, D(n)e_{n-2}/Je_{n-2}\right) \\
&= \dim e_j D(n) e_{n-2}/Je_{n-2} \\
&= \text{ the number of paths } n-2 \to j.
\end{aligned}$$

Since $i \le n-4$, there are no paths from $n-2$ to $j = i$ or to $j = i+1$. Therefore, $\operatorname{Hom}_{D(n)}(P_{i+1}, \tilde{P}_{n-2}) \oplus \operatorname{Hom}_{D(n)}(P_i, \tilde{P}_{n-2})^{N_i} = 0$.

Now let us examine the bottom left block matrix. We already know the $\left(\begin{smallmatrix} n-2 \\ n-1\ n \end{smallmatrix}\right)$ row of the matrix by Eq. (4.5.4.3).

Claim. $\mathrm{Ext}^1_{D(n)}\left(\binom{n-2}{n},(i)\right) = 0$ for $i \notin \{n-1, n-2\}$, and $\mathrm{Ext}^1_{D(n)}\left(\binom{n-2}{n-1},(i)\right) = 0$ for $i \notin \{n, n-2\}$.

We have the following projective resolution of $\binom{n-2}{n}$, which is equivalent as a $\mathbf{k}$-vector space to $\frac{P_{n-2}}{\mathbf{k}x_{n-1}\oplus\left(\bigoplus_{l=1}^{N_{n-2}}\mathbf{k}a^l_{n-2}\right)}$:

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n-1}\oplus P_{n-2}^{N_{n-2}} \arrow[rr] \arrow[dr] & & P_{n-2} \arrow[r] & \binom{n-2}{n} \arrow[r] & 0 \\
& & kx_{n-1}\oplus\left(\bigoplus_{l=1}^{N_{n-2}} ka^l_{n-2}\right) \arrow[ur, hook] & & &
\end{tikzcd}$$

Take $\mathrm{Hom}_{D(n)}(-,(i)) = \mathrm{Hom}_{D(n)}(-,S_i)$:

$$0 \longrightarrow \mathrm{Hom}_{D(n)}(P_{n-2},S_i) \longrightarrow \mathrm{Hom}_{D(n)}(P_{n-1},S_i)\oplus\mathrm{Hom}_{D(n)}(P_{n-2},S_i)^{N_{n-2}} \longrightarrow \cdots$$

Since $n-1 \neq i \neq n-2$, $\mathrm{Hom}_{D(n)}(P_{n-1},S_i)\oplus\mathrm{Hom}_{D(n)}(P_{n-2},S_i)^{N_{n-2}} = 0$ so $\mathrm{Ext}^1_{D(n)}\left(\binom{n-2}{n},(i)\right) = 0$.

The proof for $\binom{n-2}{n-1}$ is analogous.

Claim. Let ϕ be any brick set. Then $\rho(A(\varphi)) \leq \max\{N_1,\ldots,N_n\}$. Since $\phi = \{(1),\ldots,(n)\}$ is a brick set such that $\rho(A(\varphi)) = \max\{N_1,\ldots,N_n\}$, we must have $\mathrm{fpd}\,(D(n)\text{-}\mathrm{mod}) = \max\{N_1,\ldots,N_n\}$.

First notice that $\left\{\binom{n-2}{n},\binom{n-2}{n-1}\right\}$ and $\left\{\binom{\;n-2\;}{n-1\;n}\right\}$ are maximal brick subsets of $\left\{\binom{n-2}{n},\binom{n-2}{n-1},\binom{\;n-2\;}{n-1\;n}\right\}$. Let ϕ be a brick set. We can divide our proof of the claim into the following cases.

Case 1: $\binom{n-2}{n},\binom{n-2}{n-1},\binom{\;n-2\;}{n-1\;n} \notin \phi$.

Then for some $1 \leq j_l \leq n-3$, $1 \leq i_m \leq n$,

$$\phi = \left\{(i_1),\ldots,(i_M),\begin{pmatrix} j_1 \\ j_1+1\end{pmatrix},\ldots,\begin{pmatrix} j_L \\ j_L+1\end{pmatrix}\right\}.$$

Note that by examination of the Hom matrix, if $(i),\binom{j}{j+1}$ are in the same brick set, then $j \neq i \neq j+1$, so

$$\mathrm{Ext}^1_{D(n)}\left((i),\begin{pmatrix} j \\ j+1\end{pmatrix}\right) = 0 = \mathrm{Ext}^1_{D(n)}\left(\begin{pmatrix} j \\ j+1\end{pmatrix},(i)\right). \tag{4.5.4.7}$$

Then by examination of the Ext^1 matrix we have for some $* \in \{0,1\}$:

$$
A(\phi) = \begin{array}{c} \\ (i_1) \\ (i_2) \\ (i_3) \\ \vdots \\ (i_{M-1}) \\ (i_M) \\ \binom{j_1}{j_1+1} \\ \binom{j_2}{j_2+2} \\ \vdots \\ \binom{j_{L-1}}{j_{L-1}+1} \\ \binom{j_L}{j_L+1} \end{array}
\begin{array}{c}
\begin{array}{cccccc|ccccc} (i_1) & (i_2) & (i_3) & \cdots & (i_{M-1}) & (i_M) & \binom{j_1}{j_1+1} & \binom{j_2}{j_2+1} & \cdots & \binom{j_{L-1}}{j_{L-1}+1} & \binom{j_L}{j_L+1} \end{array} \\
\left(\begin{array}{cccccc|ccccc}
N_{i_1} & * & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & N_{i_2} & * & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & N_{i_3} & * & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & N_{i_{M-1}} & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & N_{i_M} & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\end{array}
$$

If there are no simples,

$$\rho(A(\phi)) = 0.$$

Else,

$$\rho(A(\phi)) = \max\{N_{i_1}, \dots, N_{i_M}\} \le \max\{N_1, \dots, N_n\}.$$

Case 2: $\binom{n-2}{n} \in \phi, \binom{n-2}{n-1} \notin \phi$.

Then for some $1 \le j_l \le n-3$, $1 \le i_m \le n$,

$$\phi = \left\{(i_1), \dots, (i_M), \binom{j_1}{j_1+1}, \dots, \binom{j_L}{j_L+1}, \binom{n-2}{n}\right\}.$$

Suppose that $i_1 \le \cdots \le i_M$. By examination of the Hom matrix, since $\binom{n-2}{n} \in \phi$ we have $(n-2), (n), \binom{n-2}{n-1\ n} \notin \phi$. Since $(n) \notin \phi$, the largest possible value of i_M is $n-1$.

Let $*$ denote a possible non-zero entry. We will show that

$$A(\phi) = \begin{array}{c|cccccc|c|ccccc}
 & (i_1) & (i_2) & (i_3) & \cdots & (i_{M-1}) & (i_M) & \binom{n-2}{n} & \binom{j_1}{j_1+1} & \binom{j_2}{j_2+1} & \cdots & \binom{j_{L-1}}{j_{L-1}+1} & \binom{j_L}{j_L+1} \\
\hline
(i_1) & N_{i_1} & * & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(i_2) & 0 & N_{i_2} & * & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(i_3) & 0 & 0 & N_{i_3} & * & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
(i_{M-1}) & 0 & 0 & 0 & 0 & N_{i_{M-1}} & * & 0 & 0 & 0 & 0 & 0 & 0 \\
(i_M) & 0 & 0 & 0 & 0 & 0 & N_{i_M} & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\binom{n-2}{n} & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\binom{j_1}{j_1+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\binom{j_2}{j_2+2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\binom{j_{L-1}}{j_{L-1}+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\binom{j_L}{j_L+1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array},$$

We have the top right block matrix because $(n-2) \notin \phi$. We have the middle row and column since $(n-2), (n) \notin \phi$. By Eq. (4.5.4.7), we have the rest of the $A(\phi)$ matrix.

If there are no simples,

$$\rho(A(\phi)) = 0.$$

Else,

$$\rho(A(\phi)) = \max\{N_{i_1}, \ldots, N_{i_M}\} \leq \max\{N_1, \ldots, N_n\}.$$

Case 3: $\binom{n-2}{n-1} \in \phi, \binom{n-2}{n} \notin \phi$.

By symmetry, we can use the proof for Case 2.

Case 4: $\binom{n-2}{n-1}, \binom{n-2}{n} \in \phi$.

Then for some $1 \leq j_l \leq n-3$, $1 \leq i_m \leq n$,

$$\phi = \left\{(i_1), \ldots, (i_M), \binom{j_1}{j_1+1}, \ldots, \binom{j_L}{j_L+1}, \binom{n-2}{n-1}, \binom{n-2}{n}\right\}.$$

Because $\binom{n-2}{n-1}, \binom{n-2}{n} \in \phi$, it is clear from the Hom matrix that $(n-2), (n-1), (n), \binom{n-2}{n-1\ n} \notin \phi$. Therefore we have the $A(\phi)$ matrix below:

$$
\begin{array}{cc}
 & \begin{array}{ccccccccccccc} (i_1) & (i_2) & (i_3) & \cdots & (i_{M-1}) & (i_M) & \binom{n-2}{n-1} & \binom{n-2}{n} & \binom{j_1}{j_1+1} & \binom{j_2}{j_2+1} & \cdots & \binom{j_{L-1}}{j_{L-1}+1} & \binom{j_L}{j_L+1} \end{array} \\
\begin{array}{c} (i_1) \\ (i_2) \\ (i_3) \\ \vdots \\ (i_{M-1}) \\ (i_M) \\ \hline \binom{n-2}{n-1} \\ \binom{n-2}{n} \\ \hline \binom{j_1}{j_1+1} \\ \binom{j_2}{j_2+2} \\ \vdots \\ \binom{j_{L-1}}{j_{L-1}+1} \\ \binom{j_L}{j_L+1} \end{array} &
\left(\begin{array}{cccccc|cc|ccccc}
N_{i_1} & * & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & N_{i_2} & * & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & N_{i_3} & * & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & N_{i_{M-1}} & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & N_{i_M} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\end{array}
$$

If there are no simples,

$$\rho(A(\phi)) = 0.$$

Else,

$$\rho(A(\phi)) = \max\{N_{i_1}, \ldots, N_{i_M}\} \leq \max\{N_1, \ldots, N_n\}.$$

Case 5: $\binom{n-2}{n-1\ n} \in \phi$.

Then for some $1 \leq j_l \leq n-3$, $1 \leq i_m \leq n$,

$$\phi = \left\{(i_1), \ldots, (i_M), \binom{j_1}{j_1+1}, \ldots, \binom{j_L}{j_L+1}, \binom{n-2}{n-1\ n}\right\}.$$

By examination of the Hom matrix, since $\binom{n-2}{n-1\ n} \in \phi$, we cannot have $(n-2)$,$(n-1)$,(n), $\binom{n-2}{n-1}$ or $\binom{n-2}{n}$ in ϕ.

$$
A(\phi) = \begin{array}{cc}
 & \begin{array}{cccccc|c|ccccc} (i_1) & (i_2) & (i_3) & \cdots & (i_{M-1}) & (i_M) & \binom{n-2}{n-1\ n} & \binom{j_1}{j_1+1} & \binom{j_2}{j_2+1} & \cdots & \binom{j_{L-1}}{j_{L-1}+1} & \binom{j_L}{j_L+1} \end{array} \\
\begin{array}{c} (i_1) \\ (i_2) \\ (i_3) \\ \vdots \\ (i_{M-1}) \\ (i_M) \\ \hline \binom{n-2}{n-1\ n} \\ \hline \binom{j_1}{j_1+1} \\ \binom{j_2}{j_2+2} \\ \vdots \\ \binom{j_{L-1}}{j_{L-1}+1} \\ \binom{j_L}{j_L+1} \end{array} &
\left(\begin{array}{cccccc|c|ccccc}
N_{i_1} & * & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & N_{i_2} & * & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & N_{i_3} & * & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & N_{i_{M-1}} & * & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & N_{i_M} & * & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\end{array}
$$

This can be checked by examination of the Ext^1 matrix. If there are no simples,

$$\rho(A(\phi)) = 0.$$

Else,

$$\rho(A(\phi)) = \max\{N_{i_1}, \dots, N_{i_M}\} \leq \max\{N_1, \dots, N_n\}.$$

We have exhausted all cases (and possibly the reader). □

4.5.3 $E(6), E(7), E(8)$ algebras

Theorem 4.5.5. *Suppose that $n \in \{6, 7, 8\}$, and let $E(n)$ be the finite dimensional radical squared zero algebra described by the following quiver Q with N_i loops (labeled a_i^l for $l = 1, \dots, N_i$) at vertex i:*

$$\begin{tikzcd}
 & & 2 \arrow[loop above, "a_2^l"] & & \\
1 \arrow[r, "x_3"] \arrow[loop below, "a_1^l"] & 3 \arrow[r, "x_4"] \arrow[loop below, "a_3^l"] & 4 \arrow[u, "x_2"] \arrow[r, "x_5"] \arrow[loop below, "a_4^l"] & \cdots \arrow[r, "x_n"] & n \arrow[loop below, "a_n^l"]
\end{tikzcd}$$

Then

$$\operatorname{fpd}(E(n)\text{-}\operatorname{mod}) = \max\{N_1, \dots, N_n\}.$$

Proof. We will reuse some notation from the previous theorem. We have $E(n) = \mathbf{k}Q/(\geq 2)$. Let J be the ideal generated by the loops, and define $B \coloneqq E(n)/J$. In other words, B is the radical squared zero algebra described by the following quiver Q':

$$\begin{tikzcd}
 & & 2 & & \\
1 \arrow[r, "x_3"] & 3 \arrow[r, "x_4"] & 4 \arrow[u, "x_2"] \arrow[r, "x_5"] & \cdots \arrow[r, "x_n"] & n
\end{tikzcd}$$

We can see that $B = C/(\geq 2)$, where $C \coloneqq \mathbf{k}Q'$.

By Theorem 4.1.3, we have a one-to-one correspondence between bricks of $E(n)$ and bricks of C annihilated by (≥ 2), as well as the following natural isomorphism:

$$\operatorname{Hom}_{E(n)}(M, N) \cong \operatorname{Hom}_B(M, N) \cong \operatorname{Hom}_C(M, N).$$

Hence it suffices to compute the dimensions of $\operatorname{Hom}_{E(n)}$-spaces in B or C.

The indecomposable modules of C are well known by Gabriel's theorem [Gab72]. The dimension vectors of these modules correspond precisely to the positive roots for the E_n root system, which are described in [Hum78, p. 64]. An explicit description of the positive roots for E_6, E_7, E_8 is given in [dPa, dPb, dPc].

We will show that the only indecomposable C-modules annihilated by (≥ 2) are:

$$\left\{(1), (2), \dots, (n), \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \end{pmatrix}, \dots, \begin{pmatrix} n-1 \\ n \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \end{pmatrix}, \begin{pmatrix} & 4 & \\ 2 & & 5 \end{pmatrix}\right\}. \tag{4.5.5.1}$$

This follows from examining the positive roots for E_n with respect to the next proposition.

Proposition 4.5.6. *The following indecomposable representations of C are not annihilated by (≥ 2) for any $l, m, k \geq 1$ for any entries $*$:*

$$(l, *, m, k, *, \dots, *) \tag{4.5.6.1}$$
$$(*, *, *, l, m, k, *, \dots, *) \tag{4.5.6.2}$$
$$(*, *, *, l, m, k) \text{ for } N = 6 \tag{4.5.6.3}$$
$$(*, *, *, *, l, m, k) \text{ for } N = 7 \tag{4.5.6.4}$$
$$(*, *, *, *, *, l, m, k) \text{ for } N = 8 \tag{4.5.6.5}$$
$$(*, *, *, *, l, m, k, *) \text{ for } N = 8 \tag{4.5.6.6}$$
$$(*, 1, 1, 2, 1, *, \dots, *) \tag{4.5.6.7}$$
$$(*, 1, 1, 1, *, \dots, *) \tag{4.5.6.8}$$
$$(*, *, 1, 1, 1, *, \dots, *). \tag{4.5.6.9}$$

Proof. Now in the case of $m = 1$, Eq. (4.5.6.1) through Eq. (4.5.6.6) follow from Fact 4.2.2 (9) and (10), since all of these representations have a path $\beta\alpha$ of length 2 such that $f_\beta \circ f_\alpha \neq 0$. We also have Eqs. (4.5.6.8) and (4.5.6.9) for the same reason. By Proposition 4.2.8, we have Eq. (4.5.6.1) through Eq. (4.5.6.6) for $m \geq 2$.

The proof of Eq. (4.5.6.7) is almost identical to the proof of Lemma 4.5.4 Eq. (4.5.4.2), so we omit it. $\square$

The indecomposables given in Eq. (4.5.5.1) are bricks of $E(n)$ by examination of the following Hom matrix, where $H_{ij} = \dim \mathrm{Hom}_{E(n)}(X_i, X_j)$ for some brick objects X_i, X_j of $E(n)$-mod:

$$
\begin{array}{cc}
 & \begin{array}{cccccccc|ccccc|cc|c}
(1) & (2) & (3) & (4) & (5) & \cdots & (n-1) & (n) & \binom{1}{3} & \binom{3}{4} & \binom{5}{6} & \cdots & \binom{n-1}{n} & \binom{4}{2} & \binom{4}{5} & \binom{4}{2\ 5}
\end{array} \\
\begin{array}{c}
(1) \\ (2) \\ (3) \\ (4) \\ (5) \\ \vdots \\ \vdots \\ (n) \\ \binom{1}{3} \\ \binom{3}{4} \\ \binom{5}{6} \\ \vdots \\ \binom{n-1}{n} \\ \binom{4}{2} \\ \binom{4}{5} \\ \binom{4}{2\ 5}
\end{array} &
\left(\begin{array}{cccccccc|ccccc|cc|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \ddots & \ddots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 1 & \ddots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1
\end{array}\right)
\end{array}
$$

To calculate the entries of the Hom matrix, recall from Fact 4.2.2 (4) and (5) that

$$\dim \mathrm{Hom}_C(\tilde{P}_i, \tilde{S}_j) = \delta_{ij},$$

$$\dim \mathrm{Hom}_C(\tilde{S}_i, \tilde{I}_j) = \delta_{ij},$$

$$\dim \mathrm{Hom}_C(\tilde{I}_j, \tilde{I}_i) = \text{ the number of paths from } i \to j \text{ in } C = \dim \mathrm{Hom}_C(\tilde{P}_j, \tilde{P}_i).$$

Furthermore, if there is no overlap between the two representations (that is, one representation has non-zero vector spaces only at vertices $i_1, \ldots, i_M$ and the other representation has non-zero vector spaces only at vertices $j_1, \ldots, j_L$ such that $i_m \neq j_l$ for any m, l), then there are no morphisms between them. The remaining Hom spaces can be calculated:

$$\dim \mathrm{Hom}_C\left((4), \begin{pmatrix} & 4 & \\ 2 & & 5 \end{pmatrix}\right) = 0$$

$$\dim \mathrm{Hom}_C\left((5), \begin{pmatrix} & 4 & \\ 2 & & 5 \end{pmatrix}\right) = 1$$
$$\dim \mathrm{Hom}_C\left(\begin{pmatrix} 4 \\ 2 \end{pmatrix}, (2)\right) = 0$$
$$\dim \mathrm{Hom}_C\left(\begin{pmatrix} 4 \\ 2 \end{pmatrix}, (4)\right) = 1$$
$$\dim \mathrm{Hom}_C\left(\begin{pmatrix} 4 \\ 5 \end{pmatrix}, (4)\right) = 1$$
$$\dim \mathrm{Hom}_C\left(\begin{pmatrix} 4 \\ 5 \end{pmatrix}, (5)\right) = 0$$
$$\dim \mathrm{Hom}_C\left(\begin{pmatrix} 4 \\ 2 \end{pmatrix}, \begin{pmatrix} & 4 & \\ 2 & & 5 \end{pmatrix}\right) = 0$$
$$\dim \mathrm{Hom}_C\left(\begin{pmatrix} 4 \\ 5 \end{pmatrix}, \begin{pmatrix} & 4 & \\ 2 & & 5 \end{pmatrix}\right) = 0$$
$$\dim \mathrm{Hom}_C\left(\begin{pmatrix} & 4 & \\ 2 & & 5 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix}\right) = 1$$
$$\dim \mathrm{Hom}_C\left(\begin{pmatrix} & 4 & \\ 2 & & 5 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \end{pmatrix}\right) = 1.$$

We have the following Ext^1 matrix, where using the same notation as before, $E_{ij} = \dim \mathrm{Ext}^1_{E(n)}(X_i, X_j)$, and $*$ denotes a possible non-zero entry:

$$
\begin{array}{c|cccccccc|ccccc|cc|c}
 & (1) & (2) & (3) & (4) & (5) & \cdots & (n-1) & (n) & \binom{1}{3} & \binom{3}{4} & \binom{5}{6} & \cdots & \binom{n-1}{n} & \binom{4}{2} & \binom{4}{5} & \binom{4}{2\ 5} \\
\hline
(1) & N_1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\
(2) & 0 & N_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & * \\
(3) & 0 & 0 & N_3 & 1 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & * \\
(4) & 0 & 1 & 0 & N_4 & 1 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & * \\
(5) & 0 & 0 & 0 & 0 & N_5 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * \\
\vdots & 0 & 0 & 0 & 0 & 0 & \ddots & \ddots & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & * \\
\vdots & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & * \\
(n) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & N_n & 0 & 0 & 0 & 0 & * & 0 & 0 & * \\
\hline
\binom{1}{3} & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\binom{3}{4} & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\binom{5}{6} & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & 0 & 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\binom{n-1}{n} & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
\binom{4}{2} & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\
\binom{4}{5} & 0 & * & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * \\
\hline
\binom{4}{2\ 5} & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0
\end{array}
$$

Recall the results of Lemmas 4.2.3, 4.2.4 and 4.2.6:

$$
\begin{aligned}
&\mathrm{Ext}^1_{E(n)}(\tilde{P}_i, \tilde{S}_j) = 0 = \mathrm{Ext}^1_{E(n)}(\tilde{S}_i, \tilde{I}_j) \text{ if } i \neq j \\
&\mathrm{Ext}^1_{E(n)}(\tilde{P}_i, \tilde{P}_j) = 0 = \mathrm{Ext}^1_{E(n)}(\tilde{I}_i, \tilde{I}_j) \text{ if they are not simple} \\
&\mathrm{Ext}^1_{E(n)}(\tilde{P}_i, M) = 0 \text{ if } \mathrm{Hom}_{E(n)}(P_i, M) = 0 \\
&\mathrm{Ext}^1_{E(n)}(\tilde{S}_i, M) = 0 \text{ if } i \text{ is a sink in } Q' \text{ and } \mathrm{Hom}_{E(n)}(P_i, M) = 0 \\
&\dim \mathrm{Ext}^1_{E(n)}((i),(j)) = \text{ the number of arrows } i \to j.
\end{aligned}
$$

These equations give us all the zero entries of the Ext^1 matrix, except

$$\mathrm{Ext}^1_{E(n)}\left(\binom{4}{2}, (i)\right) = 0, \quad i \neq 4, 5 \tag{4.5.6.10}$$

$$\mathrm{Ext}^1_{E(n)}\left(\binom{4}{5},(i)\right)=0,\quad i\neq 2,4. \tag{4.5.6.11}$$

Let us prove this. As a vector space,

$$\binom{4}{2}=\frac{\mathbf{k}\,e_4\oplus\mathbf{k}\,x_2\oplus\mathbf{k}\,x_5\oplus\mathbf{k}\,a_4^1\oplus\cdots\oplus\mathbf{k}\,a_4^{N_4}}{\mathbf{k}\,x_5\oplus\mathbf{k}\,a_4^1\oplus\cdots\oplus\mathbf{k}\,a_4^{N_4}}.$$

We have the following projective resolution of $\binom{4}{2}$:

$$\begin{tikzcd}
\cdots \arrow[r] & P_5\oplus P_4^{N_4} \arrow[r] \arrow[dr] & P_4 \arrow[r] & \binom{4}{2} \arrow[r] & 0 \\
& & \mathbf{k}\,x_5\oplus\mathbf{k}\,a_4^1\oplus\cdots\oplus\mathbf{k}\,a_4^{N_4} \arrow[u, hook] & &
\end{tikzcd}$$

Take $\mathrm{Hom}_{E(n)}(-,(i))=\mathrm{Hom}_{E(n)}(-,S_i)$:

$$0\longrightarrow \mathrm{Hom}_{E(n)}(P_4,S_i)\longrightarrow \mathrm{Hom}_{E(n)}(P_5,S_i)\oplus\mathrm{Hom}_{E(n)}(P_4,S_i)^{N_4}\longrightarrow\cdots$$

If $i\neq 4,5$, then $\mathrm{Hom}_{E(n)}(P_5,S_i)\oplus\mathrm{Hom}_{E(n)}(P_4,S_i)^{N_4}=0$, so $\mathrm{Ext}^1_{E(n)}\left(\binom{4}{2},(i)\right)=0$.

Similarly, as a vector space we have

$$\binom{4}{5}=\frac{\mathbf{k}\,e_4\oplus\mathbf{k}\,x_2\oplus\mathbf{k}\,x_5\oplus\mathbf{k}\,a_4^1\oplus\cdots\oplus\mathbf{k}\,a_4^{N_4}}{\mathbf{k}\,x_2\oplus\mathbf{k}\,a_4^1\oplus\cdots\oplus\mathbf{k}\,a_4^{N_4}}.$$

We have the following projective resolution of $\binom{4}{2}$:

$$\begin{tikzcd}
\cdots \arrow[r] & P_2\oplus P_4^{N_4} \arrow[r] \arrow[dr] & P_4 \arrow[r] & \binom{4}{5} \arrow[r] & 0 \\
& & \mathbf{k}\,x_2\oplus\mathbf{k}\,a_4^1\oplus\cdots\oplus\mathbf{k}\,a_4^{N_4} \arrow[u, hook] & &
\end{tikzcd}$$

Take $\mathrm{Hom}_{E(n)}(-,(i))=\mathrm{Hom}_{E(n)}(-,S_i)$:

$$0\longrightarrow \mathrm{Hom}_{E(n)}(P_4,S_i)\longrightarrow \mathrm{Hom}_{E(n)}(P_2,S_i)\oplus\mathrm{Hom}_{E(n)}(P_4,S_i)^{N_4}\longrightarrow\cdots$$

If $i \neq 2,4$, then $\operatorname{Hom}_{E(n)}(P_2, S_i) \oplus \operatorname{Hom}_{E(n)}(P_4, S_i)^{N_4} = 0$, so $\operatorname{Ext}^1_{E(n)}\left(\binom{4}{5}, (i)\right) = 0$.

Claim. For any brick set ϕ, $\rho(A(\phi)) \leq \max\{N_1, \ldots, N_n\}$. Because $\phi = \{(1), \ldots, (n)\}$ is a brick set and $\rho(A(\phi)) = \max\{N_1, \ldots, N_n\}$, we must have $\operatorname{fpd}(E(n)\text{-}\operatorname{mod}) = \max\{N_1, \ldots, N_n\}$.

Before we begin, note that by examination of the Hom and Ext^1 matrices, if $(i) \in \{(1), \ldots, (n)\}$ and $\binom{j}{k} \in \left\{\binom{1}{3}, \binom{3}{4}, \binom{5}{6}, \ldots, \binom{n-1}{n}\right\}$ are in the same brick set, then

$$\operatorname{Ext}^1_{E(n)}\left((i), \begin{pmatrix} j \\ k \end{pmatrix}\right) = 0 = \operatorname{Ext}^1_{E(n)}\left(\begin{pmatrix} j \\ k \end{pmatrix}, (i)\right).$$

Case 1: $\binom{4}{2}, \binom{4}{5}, \binom{\;4\;}{2\;5} \notin \phi$.

Then

$$\phi = \left\{(i_1), \ldots, (i_M), \begin{pmatrix} j_1 \\ k_1 \end{pmatrix}, \ldots, \begin{pmatrix} j_L \\ k_L \end{pmatrix}\right\},$$

where

$$(i_m) \in \{(1), \ldots, (n)\}, \quad \begin{pmatrix} j_l \\ k_l \end{pmatrix} \in \left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \end{pmatrix}, \ldots, \begin{pmatrix} n-1 \\ n \end{pmatrix}\right\}.$$

Then $A(\phi)$ is a principal submatrix of the following upper triangular matrix, whose eigenvalues are contained in the set $\{0, N_1, \ldots, N_n\}$. Therefore, $\rho(A(\phi)) \leq \max\{N_1, \ldots, N_n\}$.

$$\begin{array}{c|c}
 & \begin{array}{cccccccccc} (1) & (3) & (4) & (2) & (5) & \cdots & (n) & \binom{j_1}{k_1} & \cdots & \binom{j_L}{k_L} \end{array} \\
\begin{array}{c} (1) \\ (3) \\ (4) \\ (2) \\ (5) \\ \vdots \\ (n) \\ \binom{j_1}{k_1} \\ \vdots \\ \binom{j_L}{k_L} \end{array} &
\left(\begin{array}{ccccccc|ccc}
N_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & N_3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & N_4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & N_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & N_5 & * & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \ddots & * & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & N_n & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\end{array}$$

Case 2: $\begin{pmatrix} & 4 & \\ 2 & & 5 \end{pmatrix} \in \phi$.

Then by examination of the Hom matrix, $(2), (4), \binom{4}{2}, \binom{4}{5} \notin \phi$. So

$$\phi = \left\{(i_1), \ldots, (i_M), \begin{pmatrix} j_1 \\ k_1 \end{pmatrix}, \ldots, \begin{pmatrix} j_L \\ k_L \end{pmatrix}, \begin{pmatrix} & 4 & \\ 2 & & 5 \end{pmatrix}\right\},$$

where

$$(i_m) \in \{(1), \ldots, (n)\}, \quad \begin{pmatrix} j_l \\ k_l \end{pmatrix} \in \left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \end{pmatrix}, \ldots, \begin{pmatrix} n-1 \\ n \end{pmatrix}\right\}.$$

Then $A(\phi)$ is given by the following upper triangular matrix, whose eigenvalues are contained in the set $\{0, N_1, \ldots, N_n\}$. Notice that it is upper triangular because $(4) \notin \phi$. Therefore, $\rho(A(\phi)) \le \max\{N_1, \ldots, N_n\}$.

$$
\begin{array}{cc}
 & \begin{array}{ccc|ccc|c} (i_1) & \cdots & (i_M) & \binom{j_1}{k_1} & \cdots & \binom{j_L}{k_L} & \binom{4}{2\ 5} \end{array} \\
\begin{array}{c} (i_1) \\ \vdots \\ (i_M) \\ \binom{j_1}{k_1} \\ \vdots \\ \binom{j_L}{k_L} \\ \binom{4}{2\ 5} \end{array} &
\left(\begin{array}{ccc|ccc|c}
N_{i_1} & * & * & 0 & 0 & 0 & * \\
0 & \ddots & * & 0 & 0 & 0 & * \\
0 & 0 & N_{i_M} & 0 & 0 & 0 & * \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\end{array}
$$

Case 3: $\binom{4}{2}, \binom{4}{5} \in \phi$.

Then $(2), (4), (5), \binom{4}{2\ 5} \notin \phi$. So

$$\phi = \left\{(i_1), \ldots, (i_M), \binom{j_1}{k_1}, \ldots, \binom{j_L}{k_L}, \binom{4}{2}, \binom{4}{5}\right\},$$

where

$$(i_m) \in \{(1), \ldots, (n)\}, \quad \binom{j_l}{k_l} \in \left\{\binom{1}{3}, \binom{3}{4}, \binom{5}{6}, \ldots, \binom{n-1}{n}\right\}.$$

Then $A(\phi)$ is given by the following upper triangular matrix, whose eigenvalues are contained in the set $\{0, N_1, \ldots, N_n\}$. Notice that it is upper triangular because $(4) \notin \phi$. Therefore, $\rho(A(\phi)) \le \max\{N_1, \ldots, N_n\}$.

$$
\begin{array}{cc}
 & \begin{array}{ccc|ccc|cc} (i_1) & \cdots & (i_M) & \binom{j_1}{k_1} & \cdots & \binom{j_L}{k_L} & \binom{4}{2} & \binom{4}{5} \end{array} \\
\begin{array}{c} (i_1) \\ \vdots \\ (i_M) \\ \hline \binom{j_1}{k_1} \\ \vdots \\ \binom{j_L}{k_L} \\ \hline \binom{4}{2} \\ \binom{4}{5} \end{array} &
\left(\begin{array}{ccc|ccc|cc}
N_{i_1} & * & * & 0 & 0 & 0 & 0 & 0 \\
0 & \ddots & * & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & N_{i_M} & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\end{array}
$$

Case 4: $\binom{4}{2} \in \phi, \binom{4}{5} \notin \phi$.

Then $(2), (4), \binom{4}{2\ 5} \notin \phi$. So

$$\phi = \left\{ \binom{4}{2}, (i_1), \ldots, (i_M), \binom{j_1}{k_1}, \ldots, \binom{j_L}{k_L} \right\},$$

where

$$(i_m) \in \{(1), \ldots, (n)\}, \quad \binom{j_l}{k_l} \in \left\{ \binom{1}{3}, \binom{3}{4}, \binom{5}{6}, \ldots, \binom{n-1}{n} \right\}.$$

Then $A(\phi)$ is given by the following upper triangular matrix, whose eigenvalues are contained in the set $\{0, N_1, \ldots, N_n\}$. Notice that it is upper triangular because $(4) \notin \phi$. Therefore, $\rho(A(\phi)) \le \max\{N_1, \ldots, N_n\}$.

$$
\begin{array}{cc}
 & \begin{array}{c|ccc|ccc} \binom{4}{2} & (i_1) & \cdots & (i_M) & \binom{j_1}{k_1} & \cdots & \binom{j_L}{k_L} \end{array} \\
\begin{array}{c} \binom{4}{2} \\ \hline (i_1) \\ \vdots \\ (i_M) \\ \hline \binom{j_1}{k_1} \\ \vdots \\ \binom{j_L}{k_L} \end{array} &
\left(\begin{array}{c|ccc|ccc}
0 & * & * & * & 0 & 0 & 0 \\ \hline
0 & N_{i_1} & * & * & 0 & 0 & 0 \\
0 & 0 & \ddots & * & 0 & 0 & 0 \\
0 & 0 & 0 & N_{i_M} & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)
\end{array}
$$

Case 5: $\binom{4}{2} \notin \phi, \binom{4}{5} \in \phi$.

Then $(4), (5), \binom{4}{2\ 5} \notin \phi$. So

$$\phi = \left\{ \begin{pmatrix} 4 \\ 5 \end{pmatrix}, (i_1), \ldots, (i_M), \begin{pmatrix} j_1 \\ k_1 \end{pmatrix}, \ldots, \begin{pmatrix} j_L \\ k_L \end{pmatrix} \right\},$$

where

$$(i_m) \in \{(1), \ldots, (n)\}, \quad \begin{pmatrix} j_l \\ k_l \end{pmatrix} \in \left\{ \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 6 \end{pmatrix}, \ldots, \begin{pmatrix} n-1 \\ n \end{pmatrix} \right\}.$$

Then $A(\phi)$ is given by the following upper triangular matrix, whose eigenvalues are contained in the set $\{0, N_1, \ldots, N_n\}$. Notice that it is upper triangular because $(4) \notin \phi$. Therefore, $\rho(A(\phi)) \leq \max\{N_1, \ldots, N_n\}$.

$$\begin{array}{c|ccc|ccc}
\binom{4}{5} & (i_1) & \cdots & (i_M) & \binom{j_1}{k_1} & \cdots & \binom{j_L}{k_L}
\end{array}$$

$$\begin{array}{c}
\binom{4}{5} \\ \hline (i_1) \\ \vdots \\ (i_M) \\ \hline \binom{j_1}{k_1} \\ \vdots \\ \binom{j_L}{k_L}
\end{array}
\left(\begin{array}{c|ccc|ccc}
0 & * & * & * & 0 & 0 & 0 \\ \hline
0 & N_{i_1} & * & * & 0 & 0 & 0 \\
0 & 0 & \ddots & * & 0 & 0 & 0 \\
0 & 0 & 0 & N_{i_M} & 0 & 0 & 0 \\ \hline
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

We have exhausted all possible cases. Therefore for $n \in \{6, 7, 8\}$,

$$\operatorname{fpd}(E(n)\text{-}\operatorname{mod}) = \max\{N_1, \ldots, N_n\}. \qquad \square$$

4.6 Modified ADE quiver algebras with different arrow orientations

We have calculated the fpd of modified ADE quiver algebras with arrows in a certain direction (Theorem 4.1.4). It is natural to ask what happens if the directions of the arrows change. It is clear that the fpd remains the same upon taking the opposite algebra, which is the radical squared zero bound quiver algebra of the opposite quiver [CGW+19a]. However, we do not know what happens in other cases.

It is possible that the directions of the arrows do not matter (Question 4.1.7). We will show that the directions of the arrows do not matter for $A(n)$ if $n \leq 3$. Preliminary calculations for $n = 4$ also suggest a positive answer to this question, but the question requires further analysis.

Theorem 4.6.1. *For the modified ADE bound quiver algebras defined in Definition 4.1.12, if $A \in \{A(1), A(2), A(3)\}$, then $\operatorname{fpd}(A\text{-}\operatorname{mod})$ is invariant under change of direction of the arrows x_i.*

Proof. The proof for $n = 1, 2$ is clear because there is only one possible direction for the arrows up to isomorphism.

For $n = 3$, let A be the finite-dimensional algebra described by the following quiver, with relations such that paths of length greater than or equal to 2 are 0:

$$\begin{tikzcd} 1 \arrow[loop left, "a_n"] \arrow[r, "x"] & 2 \arrow[loop below, "b_m"] & 3 \arrow[l, "y"'] \arrow[loop right, "c_l"] \end{tikzcd}$$

Label the loops $a_1, \dots, a_N, b_1, \dots, b_M, c_1, \dots, c_L$. The brick objects of A are in one-to-one correspondence with the brick objects in B, where B is the finite dimensional algebra described by the following quiver:

$$\begin{tikzcd} 1 \arrow[r, "x"] & 2 & 3 \arrow[l, "y"'] \end{tikzcd}$$

The six non-isomorphic indecomposable representations of B are well known [Sch14, Ex. 1.14]. We have the three simples $S_1 = I_1, S_2 = P_2, S_3 = I_3$, and the additional representations are P_1, P_3, I_2. All of these are brick objects.

$$S_1 = (1) = (1, 0, 0)$$
$$S_2 = (2) = (0, 1, 0)$$
$$S_3 = (3) = (0, 0, 1)$$
$$P_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = (1, 1, 0)$$
$$P_3 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} = (0, 1, 1)$$
$$I_2 = \begin{pmatrix} 1 & & 3 \\ & 2 & \end{pmatrix} = (1, 1, 1)$$

We can determine the brick sets by examining the Hom matrix below, where $H_{ij} =$

$\dim \mathrm{Hom}(i,j)$. The $*$ represents a non-zero entry.

$$H = \begin{array}{c} \\ S_1 \\ S_2 \\ S_3 \\ P_1 \\ I_2 \\ P_3 \end{array} \begin{array}{c} \begin{array}{cccccc} S_1 & S_2 & S_3 & P_1 & I_2 & P_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & * & 1 & * \\ 0 & 0 & 1 & 0 & 0 & 0 \\ * & 0 & 0 & 1 & * & 0 \\ * & 0 & * & 0 & 1 & 0 \\ 0 & 0 & * & 0 & * & 1 \end{pmatrix} \end{array}$$

The maximal brick sets are:

$$\phi_1 = \{I_2\},$$
$$\phi_2 = \{S_1, S_2, S_3\},$$
$$\phi_3 = \{P_3, S_1\},$$
$$\phi_4 = \{P_1, P_3\},$$
$$\phi_5 = \{P_1, S_3\}.$$

We have $\rho(A(\phi_1)) = |\mathrm{Ext}^1_A(I_2, I_2)| = 0$ by Lemma 4.2.3 (2) since $I_2 \ncong S_2$. Since $|\mathrm{Ext}^1_A(S_i, S_j)|$ is the number of arrows from i to j, we have

$$\rho(A(\phi_2)) = \rho \begin{pmatrix} N & 1 & 0 \\ 0 & M & 0 \\ 0 & 1 & L \end{pmatrix} = \max\{N, M, L\}.$$

We have by Lemma 4.2.3 (1), (3) that

$$\rho(A(\phi_3)) = \rho \begin{pmatrix} |\mathrm{Ext}^1(P_3, P_3)| & |\mathrm{Ext}^1(P_3, S_1)| \\ |\mathrm{Ext}^1(S_1, P_3)| & |\mathrm{Ext}^1(S_1, S_1)| \end{pmatrix}$$
$$= \rho \begin{pmatrix} 0 & 0 \\ |\mathrm{Ext}^1(S_1, P_3)| & N \end{pmatrix}$$

$$= N.$$

By Lemma 4.2.3 we have

$$\rho(A(\phi_4)) = \rho\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

By Lemma 4.2.3 (1), (3) we have

$$\begin{aligned} \rho(A(\phi_5)) &= \rho\begin{pmatrix} |\operatorname{Ext}^1(P_1, P_1)| & |\operatorname{Ext}^1(P_1, S_3)| \\ |\operatorname{Ext}^1(S_3, P_1)| & |\operatorname{Ext}^1(S_3, S_3)| \end{pmatrix} \\ &= \rho\begin{pmatrix} 0 & 0 \\ |\operatorname{Ext}^1(S_3, P_1)| & L \end{pmatrix} \\ &= L. \end{aligned}$$

Therefore,

$$\operatorname{fpd}(A\text{-}\operatorname{mod}) = \max\{N, M, L\}.$$

Every modified ADE bound quiver algebra with arrows in an arbitrary orientation is isomorphic to one of type A, A^{op}, or $A(3)$. Since fpd is invariant under taking the opposite algebra by [CGW+19a, Cor. 3.10], the theorem is true for $n = 3$. □

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