On the γ_2-positivity of Smooth Toric Threefolds

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Abstract

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In this thesis we consider the classification of smooth toric varieties with positive second chern character. We give a complete proof, without using the classification of smooth toric Fano threefolds, that the only such threefold with positive second chern character is $\mathbb{P}^3$. We also provide some initial steps towards proving that the only smooth toric Fano varieties in any dimension with positive second chern character are the projective spaces $\mathbb{P}^n$.

Contents

Chapter 0.

Introduction

The classification of the 18 toric Fano threefolds was completed independently by Watanabe and Watanabe [19] and Batyrev [3] in 1982. In 1999, Batyrev [2] classified 123 toric Fano fourfolds and Sato [16] found discovered one missing from the list, completing the classification. In 2009, Kreuzer and Nill [11] classified the 866 toric Fano fivefolds, and in 2007 Øbro [13] devised an algorithm to compute the number of smooth toric Fano varieties in any dimension and computed the number of smooth toric Fano varieties up to dimension 8.

2-Fano varieties were defined by de Jong and Starr in [8].

Definition 0.1. *A smooth projective variety is 2-Fano if it is Fano and the second chern character*

$$ch_2(X) = ch_2(T_X) = \frac{1}{2}\left(c_1(T_X)^2 - 2c_2(T_X)\right),$$

has $ch_2(X) \cdot S > 0$ *for all* $S \in \overline{NE_2(X)}$*, where* c_i *is the* i*-th graded piece of* $c(T_X)$*, the total chern class of* X*.*

They prove that under certain hypotheses on spaces of rational curves on a 2-Fano variety X, that X has a rational surface through a general point of X, and in [9], provide some first steps towards the classification of 2-Fano varieties.

Araujo and Castravet in [1] classify 2-Fano varieties of dimensions 2 and 3. The only 2-Fano surface is $\mathbb{P}^2$ and there are two 2-Fano threefolds, namely $\mathbb{P}^3$ and the smooth quadric hypersurface in $\mathbb{P}^4$.

The focus of this thesis is on the intersection of toric Fano varieties and 2-Fano varieties. Namely, we address the question of which toric Fano varieties are also 2-Fano. The only known 2-Fano toric varieties are the projective spaces $\mathbb{P}^n$. In [12], Nobili uses the classification of smooth toric Fano fourfolds, due to Batyrev in [3], to show that the 2-Fano toric Fano fourfold is $\mathbb{P}^4$.

Sato and Sumiyoshi make the following definition [17, Definition 3.1].

Definition 0.2. *Let X be a smooth toric variety with torus-invariant divisiors $D_1, \ldots, D_k$. Define $\gamma_2(X) \in A^2(X)$ to be*

$$\gamma_2(X) = \sum D_i^2.$$

We say that X is γ_2-positive if for all non-zero $S \in \overline{NE_2(X)}$,

$$\gamma_2(X) \cdot S > 0.$$

They show that for a smooth toric variety, $2\gamma_2(X) = \mathrm{ch}_2(X)$. Therefore X is γ_2-positive if and only if $\mathrm{ch}_2(X) \cdot S > 0$ for all $S \in \overline{\mathrm{NE}_2(X)}$.

Sato and Suyama [18] prove that every smooth projective toric variety X with $\rho(X) \leq 3$ is not γ_2-positive. In [15], Sato proves that every toric Fano variety of dimension n with $5 \leq n \leq 7$ is not γ_2-positive, using the classification and a computer algorithm.

This thesis explores the concept of γ_2-positivity on smooth toric Fano varieties, with a focus on smooth toric Fano threefolds. Notably, we have the following conjecture, known for dimensions $n \leq 7$.

Conjecture 0.3. *Let X be a smooth toric Fano variety of dimension n. Then X is γ_2-positive if and only if $X \cong \mathbb{P}^n$.*

The main subject of this thesis is the following proposition.

Proposition 0.4. *Let X be a smooth toric Fano threefold. Then X is γ_2-positive if and only if $X \cong \mathbb{P}^3$.*

This can be deduced from the results of Araujo and Castravet on Fano threefolds which in turn use the classification of smooth Fano threefolds. It can also be proved using the classification of the 18 smooth toric Fano threefolds due independently to Batyrev [3] and Watanabe and Watanabe [19].

By a theorem of Batyrev [3, Proposition 3.2], every projective toric variety X has a *centrally symmetric relation*, *i.e* there exists a primitive collection $\{x_1, \ldots x_k\}$ with

$$\sum_{i=1}^{k} x_i = 0.$$

We define the *degree* of the centrally symmetric relation to be k. If X has a centrally symmetric relation of maximal degree $k = n + 1$, then $X \cong \mathbb{P}^n$. The primary result of this thesis is a proof, without using any classification of Fano or toric Fano varieties, that if X is a smooth Fano toric threefold with a centrally symmetric relation of degree 2 or 3, then X is not γ_2-positive. We treat the cases of degree 2 and 3 separately.

The outline of this thesis is as follows.

Chapter 1 contains the relevant background for smooth toric varieties. In Chapter 2, we define $\gamma_2(X)$ and provide results on the intersection theory of toric varieties. In Chapter 3, we prove some fundamental results about smooth toric surfaces. In Chapter 4, we prove several results on the structure of smooth toric threefolds and on the behavior of γ_2. Additionally, we provide a complete description of the behavior of γ_2 on smooth toric threefolds X with $\rho(X) = 2$. In Chapter 5, we describe $\gamma_2(X)$ for X the blowup of a smooth toric threefold at a torus-invariant subvariety. We prove the main result of this thesis in Chapter 6, namely, that every smooth toric Fano threefold X with $\rho(X) > 1$ is not γ_2-positive. Finally, in Chapter 7, we show that under certain hypotheses, the methods of our main result can be applied to smooth toric Fano varieties of dimension n with a centrally symmetric of degree 2 or of degree n, to show that they are not $\gamma_2(X)$-positive.

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Chapter 1.

Toric Varieties

1.1 First Definitions

The following definitions and results are the standard treatment of toric varieties. We refer the reader to Fulton's excellent book [7] for details and proofs.

Definition 1.1. *A normal variety X over an algebraically closed field k is a* toric variety *if X admits an open embedding $T \hookrightarrow X$, with $T \cong \mathbb{G}_m^n$, an algebraic torus, such that the action $T \times T \to T$ extends to an action on X, $T \times X \to X$.*

We work over $k = \mathbb{C}$, but note that the results hold over any algebraically closed field. A *lattice* N is a free abelian group of finite rank, *i.e.* $N \cong \mathbb{Z}^n$. We let M denote the dual lattice, $M = N^\vee = \mathrm{Hom}(N, \mathbb{Z})$.

An *algebraic torus* over $\mathbb{C}$ is a product $\mathbb{G}_m^n$, where $\mathbb{G}_m$ is the group variety $\mathbb{C}^*$ with multiplication coming from $\mathbb{C}$. As an affine variety

$$\mathbb{G}_m \cong \operatorname{Spec} \mathbb{C}[X, X^{-1}].$$

For every $k \in \mathbb{Z}$, we have a homomorphism of algebraic groups:

$$\mathbb{G}_m \to \mathbb{G}_m, z \mapsto z^k,$$

and every such homomorphism is of this form.

We let $N = \mathrm{Hom}(\mathbb{G}_m, T) \cong \mathbb{Z}^n$ be the lattice of *one-parameter subgroups* of T. Given a lattice N, we let T_N denote the associated torus, and we let λ_v denote the image of $v \in N$ under the identification $N = \mathrm{Hom}(\mathbb{G}_m, T_N)$:

$$\lambda_v : \mathbb{G}_m \to T_N.$$

We denote the dual lattice, $M = \mathrm{Hom}(N, \mathbb{Z}) \cong \mathbb{Z}^n$. Every *character* χ : $T_N \to \mathbb{G}_m$ is given by a unique $u \in M$. To $u = (a_1, \ldots, a_n) \in M$, we associate a monomial

$$\chi^u = X_1^{a_1} \ldots X_n^{a_n} \in \mathbb{C}[M] = \mathbb{C}[X_1, X_1^{-1}, \ldots, X_n, X_n^{-1}].$$

Let $M_\mathbb{R} = M \otimes \mathbb{R}$, and $N_\mathbb{R} = N \otimes \mathbb{R}$, the associated real vector spaces.

Definition 1.2. *A* convex rational polyhedral cone *is a set of the form*

$$\sigma = \{\lambda_1 x_1 + \cdots + \lambda_k v_k \mid \lambda_i \in \mathbb{R}_{\geq 0}\} \subseteq N_{\mathbb{R}},$$

where the x_i are non-zero elements of the lattice N. Given non-zero lattice elements $x_1, \ldots, x_r$ we denote

$$\langle x_1, \ldots, x_r \rangle = \{\lambda_1 x_1 + \cdots + \lambda_k v_k \mid \lambda_i \in \mathbb{R}_{\geq 0}\},$$

the cone generated by the set $\{x_1, \ldots, x_r\}$.

Definition 1.3. *Let σ be a one-dimensional cone in $N_{\mathbb{R}}$. Then there is a unique minimal generator $x \in N$ so that $\sigma = \mathbb{R}_{\geq 0} x$. We call such a lattice element a* primitive vector*. Any σ has a unique associated set of primitive vectors, which we refer to as the* generators *of σ.*

Definition 1.4. *A cone $\sigma \subseteq N_{\mathbb{R}}$ is to be* strongly convex *if there are no non-trivial linear subspaces of $N_{\mathbb{R}}$ contained in σ,* i.e.

$$\sigma \cap (-\sigma) = \{0\}.$$

Definition 1.5. *The dimension of a cone σ, $dim(\sigma)$ is defined to be the dimensional of the linear subspace of $N_{\mathbb{R}}$ of the vectors in σ. A 1-dimensional cone is called a* ray*. If σ is simplicial, generated by primitive vectors $x_1, \ldots, x_k$, then $dim(\sigma) = k$.*

Definition 1.6. *A cone σ is called* simplicial *if its ray generators are linearly independent, and is called* smooth *if additionally its generators can be extended to a $\mathbb{Z}$-basis of the lattice N.*

For us a cone $\sigma \subseteq N_{\mathbb{R}}$ will always mean a *smooth convex rational polyhedral* cone.

Definition 1.7. *A* semigroup *is a set S with a commutative and associative operation*

$$+ : S \times S \to S,$$

and with an identity element $0 \in S$, so $0 + a = a$ for all $a \in S$.

A map $\varphi : S \to S'$ is called a *semigroup morphism* if $\varphi(u_1 + u_2) = \varphi(u_1) + \varphi(u_2)$ for all u_1 and u_2. There is a covariant functor from semigroups to $\mathbb{C}$-algebras which sends a semigroup S to the $\mathbb{C}$-algebra $\mathbb{C}[S]$, generated by χ^u for $u \in S$. Multiplication in $\mathbb{C}[S]$ is defined by

$$\chi^u \cdot \chi^v = \chi^{u+v}, \text{ and } \chi^0 = 1.$$

A morphism of semigroups $\varphi : S \to S'$ induces a morphism of $\mathbb{C}$-algebras:

$$f : \mathbb{C}[S] \to \mathbb{C}[S'], \text{ where } f(\chi^u) = \chi^{\varphi(u)} \in \mathbb{C}[S'].$$

For any cone $\sigma \subseteq N_{\mathbb{R}}$, the set of lattice points $\sigma \cap N$ has the natural structure of a semigroup.

Proposition 1.8 (Gordan's Lemma)**.** *If σ is a cone, then $S_\sigma = \sigma^\vee \cap M$ is a finitely generated semigroup.*

Definition 1.9. *Suppose we have a cone σ, then we define the* corresponding affine toric variety

$$U_\sigma = Spec(\mathbb{C}[S_\sigma]) = Spec(A_\sigma).$$

Definition 1.10. *A* **fan** *Σ is a finite collection of cones σ in $N_\mathbb{R}$ such that*

1. *If σ and τ are in Σ, then $\sigma \cap \tau$ is Σ and is a face of both σ and τ.*
2. *If σ is in Σ and τ is a face of σ, then τ is in Σ.*
3. *Every σ in Σ does not contain any linear subspace of $N_\mathbb{R}$,* i.e. $\sigma \cap -\sigma = \{0\}$.

Proposition 1.11. *Let σ be a cone in N. Then every face τ of σ is of the the form $\tau = \sigma \cap u^\perp$ for some $u \in \sigma^\vee \cap M$. Given $\tau = \sigma \cap u^\perp$,*

$$S_\tau = S_\sigma + \mathbb{Z}_{\geq 0} \cdot (-u).$$

This proposition implies the following, which allows us to glue together affine sets U_σ as σ varies over the cones in a fan Σ.

Lemma 1.12. *If τ is a face of σ, then the induced map $U_\tau \to U_\sigma$ embeds U_τ as a principal open subset of U_σ.*

We describe how to construct a toric variety $X = X_\Sigma$ from a fan Σ in N. From the definition of a fan, any two cones σ and τ in Σ meet at a common face $\sigma \cap \delta$. The lemma gives us open immersions $\phi_\tau : U_{\sigma\cap\tau} \to U_\tau$ and $\phi_\sigma : U_{\sigma\cap\tau} \to U_\sigma$.

Lemma 1.13 (Separation Lemma)**.** *For any two cones σ and σ' whose intersection is a face of each, there exists $u \in \sigma^\vee \cap \sigma'^\vee$ such that*

$$\tau = \sigma \cap u^\perp = \sigma' \cap u^\perp.$$

Definition 1.14. *Suppose Δ is a fan in N. The* support *of Δ is the union in $N_\mathbb{R}$ over all cones in Δ:*

$$|\Delta| := \bigcup_{\sigma\in\Delta} \sigma.$$

Definition 1.15. *Given a fan Σ in N, we define the* toric variety associated to, *X_Σ, to be the following. Let $\tilde{X}$ be the disjoint union over all $\sigma \in \Sigma$. Then for any σ and σ' in Σ, $\sigma \cap \sigma'$ is a face of each, by definition. Then $U_{\sigma\cap\sigma'}$ can be embedded in both U_σ and U_σ as principle open subvarieties. Finally we glue U_σ to $U_{\sigma'}$ along the open subvarieties. We do this for all σ, $\sigma' \in \Sigma$ to obtain $X = X_\Sigma$.*

Definition 1.16. *We say that a fan Σ is complete if $|\Sigma| = N_\mathbb{R}$,* i.e., *for all $v \in N_\mathbb{R}$, there exists $\sigma \in \Sigma$ with $v \in \sigma$.*

Proposition 1.17. *A toric variety $X = X_\Sigma$ is complete if and only if Σ is complete.*

Every toric variety we consider in this thesis is complete.

Definition 1.18. *For any $\sigma \in \Sigma$, we have a corresponding closed torus-invariant subvariety of X, denoted $V(\sigma)$*

Notation 1.19. *We let $\Sigma(k)$ denote the set of k-dimensional cones of Σ, and, by abuse of notation, we write $x \in \Sigma(1)$ to denote the primitive vector associated to $\mathbb{R}_{\geq 0}x \in \Sigma(1)$.*

Note that every torus-invariant subvariety Y of a toric variety X is a complete intersection. Indeed if $Y \subseteq X$ is any torus-invariant variety $V(\sigma)$, with $\sigma = \langle x_1, \ldots, x_k \rangle$, Then $Y = D_1 \ldots D_k$.

Proposition 1.20 (Orbit-Cone Correspondence)**.** *There is an order-reversing correspondence between cones $\sigma \in \Sigma$ and torus-invariant subvarieties of X. In otherwords, if $\sigma \subseteq \sigma'$, Then $V(\sigma') \subseteq V(\sigma)$. If $dim(\sigma) = k$, then $dim(V(\sigma)) = n - k$.*

1.2 Morphisms

Proposition 1.21. *[7] Suppose we have toric varieties X and X' with Σ and Σ' in N and N'. Then for any map $N \to N'$, we have an associated birational map $\varphi : X \to X'$. If, additionally, the image of every cone of Σ is contained in a cone of Σ', then φ is a morphism.*

Note that *any* map of lattices $N \to N'$ induces a rational map $X \to X'$. Such a map will always be defined on the open torus.

If X and X' are of the same dimension, then the map is necessarily birational.

Proposition 1.22. *Suppose $f : X' \to X$ is a birational toric morphism between smooth toric varieties. Then either f is an isomorphism or $\rho(X') > \rho(X)$.*

Proof. Let Σ' and Σ be the fans associated to X' and X, respectively, and let $\varphi : N' \to N$ be the associated map of lattices. Then for all $\sigma' \in \Sigma'$, there exists $\sigma \in \Sigma$ such that $\varphi(\sigma') \subseteq \sigma$. Suppose f is not an isomorphism. Then there exists $\varphi(\sigma') \neq \sigma$, and there must be some y, a primitive vector of σ', such that $y \notin \Sigma(1)$. Therefore, $\rho(X') > \rho(X)$. □

Proposition 1.23 (Blowups)**.** *Suppose X is a smooth toric variety and ζ a torus-invariant subvariety.*

Then for some ordering of the torus-invariant divisors, $\zeta = \prod_{i=1}^{k} D_i$ where k is the codimension of ζ. Let $Y = Bl_\zeta X$, then the fan associated to Y is obtained from the fan associated to X, by removing the cone $\sigma = \langle x_1, \ldots, x_k \rangle$, adding the primitive vector $y = \sum_{i=1}^{k} x_i$ and adding every cone of the form

$$\sigma_i = \langle x_1, \ldots, \hat{x}_i, \ldots, x_k, y \rangle,$$

along with all of its proper faces.

Theorem 1.24 (Fiber Bundles, [4])**.** *Let X be a smooth toric variety. Suppose there is a r-dimensional linear supspace $H \subseteq N_{\mathbb{Q}}$ such that for every $\sigma \in \Sigma$, there exists an $\eta, \tau \in \Sigma$ such that*

1. $\sigma = \eta + \tau$.
2. $\eta \subseteq H$.
3. $dim(\eta) = r$.
4. $\tau \cap H = 0$.

Fix an $(n-r)$-dimensional subspace $H^{\perp} \subseteq N_{\mathbb{Q}}$ complementary to H, i.e. *so that $N_{\mathbb{Q}} = H \oplus H^{\perp}$. Then the following hold*

1. *The fan $\Sigma_F = \{\sigma \in \Sigma | \sigma \subseteq H\}$ is the fan of a smooth toric variety F.*
2. *Consider the projection $\pi : N_{\mathbb{Q}} \to H^{\perp}$. The fan $\Sigma_Z = \{\pi(\sigma) \mid \sigma \in \Sigma\}$ is the fan of a smooth toric variety Z.*
3. *The projection π induces an equivariant morphism $\bar{\pi} : X \to Z$ such that for every affine open $U \subseteq X$, $\bar{\pi}^{-1}(U) \cong U \times F$.*

That is, X is a toric vector bundle over Z with fiber F.

Theorem 1.25. *[10] Let $X = X_{\Sigma}$. If $dim(X) = 2$ or 3, then X is projective. If $dim(X) = 2$, then Σ is a splitting fan and X is a projective space bundle.*

Proposition 1.26. *In that case we have $x_1 + \ldots x_m = 0$ and*

$$y_1 + \cdot y_n = \sum a_i x_i$$

and

$$X = \mathbb{P}_{\mathbb{P}^{n-1}}(\mathcal{O} \oplus \bigoplus \mathcal{O}(a_i))$$

The following is due to Kleinschmidt [10].

Proposition 1.27. *Any smooth toric variety X with $\rho(X) = 2$ is a projective bundle over a projective space.*

1.3 Toric Fano Varieties

Proposition 1.28. *If X is a smooth toric variety, with torus-invariant divisors D_i, then*

$$\Omega_X^n \cong \mathcal{O}_X\left(-\sum D_i\right).$$

Corollary 1.29. *For any smooth toric variety X with torus-invariant divisors $\{D_i\}$, we have $-K_X = \sum D_i$.*

Theorem 1.30 (Cone Theorem). *Suppose X is a complete toric variety. Then*

$$NE(X) = \sum_{\sigma \in \Sigma(n-1)} \mathbb{R}_{\geq 0}\dot{[}V(\sigma)].$$

In particular NE(X) is a closed rational polyhedral cone, and it is strictly convex if and only if X is projective.

Theorem 1.31 (Toric Kleiman criterion). *Suppose X is a smooth toric variety, a combination of the torus-invariant divisors of X, $D = \sum a_i D_i$, is ample if and only if $D \cdot V(\sigma) > 0$ for all cones of dimension $n-1$.*

In other words, a Cartier divisor on a toric variety is ample if and only if its intersection with any torus invariant curve is positive.

Definition 1.32. *A variety X is* Fano *if X is projective, normal, and $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor.*

Proposition 1.33. *[19] X is Fano if and only if for every top-dimensional cone σ, with spanning primitive vectors $x_1, \ldots, x_n$, there exists a linear functional $f : N \to \mathbb{Z}$ such that $f(x_1) = f(x_2) = \cdots = f(x_n) = -1$ and for all other primitive vectors x, $f(x) \geq 0$.*

Remark 1.34. *This is the same thing as saying that the primitive vectors minimally span a convex polytope with the origin as the only interior lattice point.*

Casagrande [5] proves the following bound on the Picard number of a smooth toric Fano variety X.

Proposition 1.35. *Any smooth toric Fano variety X of dimension n satisfies $\rho(X) \leq 2n$.*

Furthermore, if $\rho(X) = 2n$, then n is even and $X \cong S^{\frac{n}{2}}$, where S is the blowup of $\mathbb{P}^2$ at the three torus-invariant points.

Proposition 1.36. *Suppose X is a smooth toric Fano variety of dimension $n \geq 3$, with torus-invariant D and E, so $D \cdot E \equiv 0$, and the corresponding primitve relation is of the form $x + y = w$. Then if F is the torus-invariant divisor corresponding to w, then $D \cdot F \not\equiv 0$ and $E \cdot F \not\equiv 0$.*

Proof. Suppose $D \cdot E \equiv 0$, then the corresponding primitive vectors x and y are not adjacent to on another, so

$$x + y = z,$$

for some primitive vector z, since X is Fano, the degree of the relation must be positive and X does not have a centrally symmetric relation of degree 2. Then if z is not adjacent to x,

$$z + x = w,$$

for some primitive vector w. But then, w does not meet y, so

$$\begin{aligned} y + w &= (z - x) + (z + x) \\ &= 2z, \end{aligned}$$

a contradiction. So z meets both x and y, and letting F be the torus-invariant divisor correspdonding to w, we have the result. □

Lemma 1.37. *Suppose X is a smooth toric Fano variety with torus-invariant divisors D and E, and corresponding primitiive vectors x and y, respectively, such that*

$$x + y = 0.$$

Then, if there exists D_1 adjacent to D but not E, then there exists E_1 adjacent to both D_1 and E, and letting x_1 and y_1 be the primitive vectors corresponding to D_1 and E_1 respectively,

$$\begin{aligned} x + y_1 &= x_1, \text{ and} \\ y + x_1 &= y_1. \end{aligned}$$

Additionally, x_1 does not meet y, and y_1 does not meet x.

Proof. Suppose we have D_1 adjacent to D, but not adjacent to E, Then

$$x_1 + y = y_1$$

for some primitive vector y_1, since X is Fano, hence the degree of the relation must be strictly positive, and we cannot have $x_1 + y = 0$, since $x + y = 0$. Then, y_1 is not adjacent to x,

$$\begin{aligned} y_1 + x &= x_1 + y + x \\ &= x_1. \end{aligned}$$

If y_1 does not meet x_1, then we have

$$x_1 + y_1 = w,$$

where w meets neither x nor y. So,

$$\begin{aligned} w + y &= (x_1 + y_1) + y \\ &= (x_1 + y) + y_1 \\ &= 2y_1. \end{aligned}$$

A contradiction, so y_1 meets x_1. Finally, if y_1 does not meet y, then we have $y_1 + y = w$, but then w does not meet x_1, so

$$\begin{aligned} w + x_1 &= (y_1 + y) + x_1 \\ &= y_1 + (y + x_1) \\ &= 2y_1. \end{aligned}$$

A contradiction. So, we get the result. □

Proposition 1.38. *If X is a smooth toric Fano variety of dimension $n \geq 3$, and X has a centrally symmetric relation of degree 2,*

$$x + y = 0.$$

Then for any maximal cone of the form $\{x, x_1, \ldots, x_{n-1}\}$, there exists $k \in \{1, \ldots, n-1\}$ such that for all $i \neq k$, x_i meets y.

Proof. Suppose we have a maximal cone spanned by $\{x, x_1, \ldots, x_{n-1}\}$. Then we have for each i, $x_i = y_i$, or $x_i = y_i + x$. Supposing there is some i for which, $x_i \neq y_i$, we have some $j \in \{1, \ldots, n-1\}$, so that $y_j \neq x_j$. Then suppose $x_k \neq y_i$ for $k \neq j$. Then

$$\begin{aligned} y_k &= x_k + y \\ &= x_k - x \\ &= x_k - (x_j - y_j), \end{aligned}$$

so y_k is in the affine linear subspace of dimension $n - 1$ spanned by

$$\{x_1, \ldots, x_{n-1}, y_j\}.$$

But that's impossible since y_j meets all of the x_i, and the polytope of X has as a face the convex hull of $\{x_1, \ldots, x_{n-1}, y_j\}$.

So, for *any* $n - 1$ primitive vectors x_i meeting x, there is *at most* one which does not meet y, and in that case $y_i + x = x_i$. □

Corollary 1.39. *If X is Fano and $\dim(X) \geq 3$, then for any torus-invariant divisors D and E, either D is adjacent to E, or there exists F adjacent to both D and E.*

Proof. Suppose $D \cdot E \equiv 0$. Then if the corresponding primitive relation is of the form $x + y = z$, then we have the result by Proposition 1.36. If $x + y = 0$, then we have the result by Proposition 1.38. □

Chapter 2.

Intersection Theory

In this chapter we outline the intersection theory of a smooth toric variety X and define the primary object of study, $\gamma_2(X)$. We begin with some preliminary definitions. Suppose X is an arbitrary variety, we let the group of cycles $Z(X)$ denote the free abelian group generated by subvarieties of X. We denote $Z_k(X)$ to be k-th graded piece of $Z(X)$, *i.e.* the group of k-cycles generated by subvarieties of dimension k. A cycle $\sum a_i Y_i$, is called effective if $a_i \geq 0$ for all i.

The Chow group $A(X)$ is the the group of cycles $Z(X)$ modulo rational equivalence, and is graded by dimension, *i.e.*,

$$A(X) = \bigoplus_{i=1}^{n} A_i(X).$$

If X is smooth, we denote $A^k(X) \cong A_{n-k}(X)$, the group of codimension-k cycles on X modulo rational equivalence. We define $N^k(X)$ to be the $\mathbb{R}$-vector space $A^k(X) \otimes \mathbb{R}$, and likewise for $N_k(X)$.

Given $u \in M$ we get a rational function χ^u on X, and the following lemma shows us how to compute the corresponding principal divisor.

Lemma 2.1. *[7] Let $u \in M$. Then*

$$div(\chi^u) = \sum_{x_i \in \Sigma(1)} u(x_i) D_i.$$

Proposition 2.2. *[7, Section 5.1] Let X be a toric variety. Then for $1 \leq k \leq n$, the Chow group $A_k(X)$ is generated by the classes $[V(\sigma)]$ where σ varies over all $n-k$ dimensional cones in Σ.*

We define $\mathrm{NE}_k(X) \subseteq N_k(X)$ to be the cone spanned by classes of effective k-cycles, and define $\mathrm{NE}(X)$ to be $\mathrm{NE}_1(X)$. We let $\overline{\mathrm{NE}_k(X)}$ be the closure of $\mathrm{NE}_k(X)$ in $N_k(X)$.

Proposition 2.3. *We have the following short exact sequence giving a presentation for the Chow group $A^1(X)$.*

$$0 \to M \to \mathbb{Z}^k \to A^1(X) \to 0$$

Definition 2.4. *Suppose X is a smooth toric variety, we say D_1 is* adjacent *to D_2 if $D_1 \cap D_2 \neq \emptyset$.*

Dualizing the exact sequence

$$0 \to M \to \mathbb{Z}^k \to A^1(X) \to 0,$$

we obtain

$$0 \to A_1(X) \to \mathbb{Z}^k \to N \to 0.$$

Therefore we get the following proposition.

Proposition 2.5. *[14] Let X be a smooth toric variety, then*

$$A_1(X) \cong \{(a_1, \ldots, a_k) \in \mathbb{Z}^k \mid \sum a_i x_i = 0\},$$

the lattice of linear relations among the primitive vectors of X.

Proposition 2.6. *[14] Suppose $C \subseteq X$ is a torus-invariant curve. Then $C = D_1 \cdots D_{n-1}$, for some torus-invariant divisors D_i. $C \cong \mathbb{P}^1$ and meets exactly two torus-invariant divisors E_1 and E_2 at a torus-invariant point. Let x_i and y_i be the primitive vectors corresponding to the D_i and the E_i respectively. Then, the linear relation associated to C is*

$$y_1 + y_2 + \sum a_i x_i = 0,$$

where $a_i = D_i \cdot C = deg(D_i|_C)$.

Corollary 2.7. *Suppose $C = D_1 \cdot D_2$ is a torus invariant curve on a smooth toric threefold X. Then C meets exactly two divisors E_1 and E_2, at a point. Let x_1, x_2, y_1, y_2 be the associated primitve divisors, then the linear relation assoicated to C is*

$$y_1 + y_2 + (D_1^2 \cdot D_2)x_1 + (D_1 \cdot D_2^2)x_2 = 0.$$

Reid proves the following proposition, which shows that, NE(X) is a closed, polyhedral cone.

Proposition 2.8. *[14] NE(X) is generated by torus-invariant curves.*

2.1 Primitive Vectors and Relations

Definition 2.9. *A nonempty subset $\mathcal{P} = \{x_1, \ldots, x_r\}$ of $\Sigma(1)$ is called a* primitive collection *if*

i. $\langle x_1, \ldots, x_r \rangle \notin \Sigma$

ii. For each i, $\langle x_1, \ldots, \hat{x_i}, \ldots, x_r \rangle \in \Sigma$.

Definition 2.10. *[14] Let $\mathcal{P} = \{x_1, \ldots, x_r\}$ be a primitive relation. and define the focus of $\mathcal{P}$*

$$\sigma(\mathcal{P}) := \langle y_1, \ldots, y_k \rangle$$

to be the cone of minimal dimension with $\sum x_i \in \sigma(\mathcal{P})$. Then, there is a unique linear relation

$$\sum x_i = \sum a_i y_i$$

with $a_i \in \mathbb{Z}_{>0}$. Then define the primitive relation associated to $\mathcal{P}$,

$$\mathcal{R} = \mathcal{R}(\mathcal{P}) := \sum_1^r x_i - \sum_1^k a_i y_i$$

Lemma 2.11. *[3, Section 3.2] If*

$$\sum_{i=1}^r x_i = \sum_{j=1}^s a_j y_j$$

is a primitive relation, then

$$\{x_1, \ldots, x_r\} \cap \{y_1, \ldots, y_s\} = \emptyset.$$

Proof. For all i $\langle x_1, \ldots, \hat{x_i}, \ldots x_l \rangle$ is a cone for all i, and $\mathrm{conv}(y_1, \ldots, y_k)$ is a cone. Suppose $y_1 = x_1$. Then $x_1 + \sum_2^l x_i = a_1 x_1 + \sum_2^k a_i y_i$, and

$$\sum_2^l x_i = (a_1 - 1)y_1 + \sum_2^k a_i y_i.$$

Observe that $\sum_2^l x_i \in \mathrm{int}(\mathrm{cone}(x_2, \ldots, x_l)$. For two cones $\sigma, \sigma' \in \Sigma$, if $\mathrm{int}(\sigma) \cap \sigma' \neq \emptyset$ then $\sigma \subseteq \sigma'$. so $\{x_2, \ldots, x_l\}$ is contained in $\{y_1, \ldots, y_k\}$. But then $\{x_1, \ldots x_l\}$ is contained in $\{y_1, \ldots, y_k\}$, but that's impossible because $\langle x_1, \ldots, x_l \rangle \notin \Sigma$. So $\{x_i\} \cap \{y_i\} = \emptyset$. □

Lemma 2.12. *Suppose $\mathcal{P} = \{x_1, \ldots x_k\}$ is a primitive collection of a smooth toric variety X, then*

$$2 \leq |\mathcal{P}| = k \leq n + 1.$$

We say that $\mathcal{P}$ has size k.

Definition 2.13. *The degree of a primitive relation $\mathcal{R}$ (or of the corresponding primitive collection) is defined to be*

$$deg(\mathcal{R}) := r - c_1 - \cdots - c_k.$$

Definition 2.14. *A primitive relation $\mathcal{R}$ with focus 0,* i.e.*,*

$$\sum_{i=1}^k x_i = 0$$

is called a centrally symmetric relation. Note that $k = deg(\mathcal{R})$.

Given a primitive relation $\mathcal{R}$, with corresponding 1-cycle $\gamma \in A_1(X)$, $-K_X \cdot \gamma = \deg(\mathcal{R})$, and we get the following fundamental lemma.

Lemma 2.15. *A smooth toric variety is Fano if and only if deg($\mathcal{P}$) > 0 for every primitive relation $\mathcal{P}$ of Σ.*

Lemma 2.16. *If $\mathcal{P} = \{x_1, \dots, x_k\}$ is a primitive collection on X, then*

$$\mathcal{P}' = \{x_1, \dots, \hat{x}_i, \dots, x_k\}$$

is a primitive collection on D_i.

Proof. First, observe that

$$0 \equiv D_1 \cdot D_2 \cdots D_k = (D_2 \cdots D_k)|_{D_1}$$

So the $D_i|_{D_1}$ do not meet at a point, thus the images of $x_2, \dots, x_k$ do not span a maximal cone. Then for any i, since $\mathcal{P}$ is a primitive collection,

$$(D_2 \cdots \hat{D}_i \cdot D_k)|_{D_1} = D_1 \cdot D_2 \cdots \hat{D}_i \cdots D_k$$

is an effective cycle on D_1. So $\mathcal{P}'$ is a primitive collection on D_i. □

Corollary 2.17. *If $\mathcal{P}$ is a centrally symmetric relation on X, then $\mathcal{P}'$ is a centrally symmetric collection on D_i.*

Proof. If $\sum x_i = 0$ in N, then $\sum x_i' = 0$ in N' where x_i' are the images of x_i. □

Proposition 2.18. *[3, Proposition 3.2] For any smooth projective toric variety there exists a centrally symmetric relation.*

Corollary 2.19. *If S is a smooth toric surface with $\rho(S) \geq 2$, then S has a centrally symmetric relation of degree 2.*

Proof. If S has a centrally symmetric relation of degree 3, $S \cong \mathbb{P}^2$ by Proposition 2.20. Otherwise, by Proposition 2.18, S has a centrally symmetric relation of degree 2. □

Proposition 2.20. *If X has a primitive collection of maximal degree, then $X \cong \mathbb{P}^n$.*

Proof. We proceed by induction. If $\dim(X) = 2$ and $\{x_1, x_2, x_3\}$ is a primitive collection, then $X \cong \mathbb{P}^2$.

Suppose the result holds for $\dim(X) = n - 1$. Then on X, we have

$$\{x_1, \dots, x_{n+1}\}$$

is a primitive collection. For any i,

$$\{x_1, \dots, \hat{x}_i, \dots, x_{n+1}\}$$

is a primitive collectiion on D_i. Therefore, by induction, $D_i \cong \mathbb{P}^{n-1}$ for all i, and each D_i meets exactly $n-1$ other $\mathbb{P}^{n-1}$. There can be no other torus-invariant divisors since one would have to meet on of the D_i, but we have already accounted for all such divisors. Therefore, every torus-invariant divisor $D \cong \mathbb{P}^{n-1}$, Therefore $X \cong \mathbb{P}^n$. □

Proposition 2.21. *[6] Suppose X is a smooth Fano toric variety. Then if D_1 and D_2 are two divisors such that $D_1 \cap D_2 = \emptyset$, then there exists D_3 such that $D_1 \cdot D_3$ and $D_2 \cdot D_3$ are effective classes.*

2.2 γ_2

In [9], de Jong and Starr explore the behavior of the second graded piece of the total chern character of a Fano variety X,

$$ch_2(T_X) = \frac{1}{2}\left(c_1(T_X)^2 - 2c_2(T_X)\right),$$

where c_i is the i-th graded piece of $c(T_X)$. We say that X is ch_2-positive if for all $S \in \overline{\mathrm{NE}_2(X)}$,

$$\mathrm{ch}_2(X) \cdot S > 0.$$

They show that every projective space is $\mathrm{ch}_2(X)$ positive, as well as all weighted-projective spaces. Total complete intersections X in a projective space $\mathbb{P}^n$ with multi-degree $(d_1, \ldots, d_k)$ is ch_2-positive if and only if $\sum d_i^2 < n+1$.

Proposition 2.22. *[12] If X is a smooth toric variety with torus-invariant divisors D_i, then*

$$ch_2(\mathcal{T}_X) \equiv \frac{1}{2}\left(\sum D_i^2\right).$$

Sato and Suyama define the related γ_2 in [18].

Definition 2.23. *For any smooth toric variety X with $dim(X) \geq 2$, let D_i be the torus-invariant divisors of X and define*

$$\gamma_2(X) = \sum D_i^2$$

Proposition 2.24.

$$2ch_2(X) = \gamma_2(X).$$

Proof.

$$c(T_X) = 1 + \sum D_i + \sum_{i<j} D_i \cdot D_j + \cdots$$

So $c_1(T_X) = \sum D_i = -K_X$, and $c_2(T_X) = \sum_{i<j} D_i \cdot D_j$.

$$c_1(T_X)^2 = \sum D_i^2 + 2\sum_{i<j} D_i \cdot D_j,$$

so

$$\begin{aligned}
ch_2(T_X) &= \frac{1}{2}\left(c_1(T_X)^2 - 2c_2(T_X)\right) \\
&= \frac{1}{2}\left(\sum D_i^2 + 2\sum_{i<j} D_i \cdot D_j - 2\sum_{i<j} D_i \cdot D_j\right) \\
&= \frac{1}{2}\sum D_i^2 \\
&= \frac{\gamma_2(X)}{2}.
\end{aligned}$$

□

Proposition 2.25. *Let $X = \mathbb{P}^n$. Then X is γ_2-positive.*

Proof. Let $D_1, \ldots, D_{n+1}$ be the torus-invariant divisors of X. Any effective surface $S \in \mathrm{NE}_2(X)$ is numerically equivalent to a positive multiple of a linear subspace of dimension 2. In particular,

$$S \equiv D_3 \cdots D_n.$$

For torus-invariant D_i, D_i^2 is linearly equivalent to any affine linear subspace of dimension $n-2$, in particular, for all i, $D_i \equiv D_1 \cdot D_2$, thus

$$\begin{aligned}
\gamma_2(X) &\equiv \sum D_i^2 \\
&\equiv \sum D_1 \cdot D_2 \\
&\equiv (n+1) D_1 \cdot D_2.
\end{aligned}$$

So for any $S \in \mathrm{NE}_2(X)$,

$$\begin{aligned}
\gamma_2(X) \cdot S &= (n+1) D_1 \cdot D_2 \cdot S \\
&= (n+1) D_1 \cdot D_2 \cdot a(D_3 \cdots D_{n+1}) \\
&= n+1.
\end{aligned}$$

So X is γ_2-positive. □

Chapter 3.

Surfaces

Fulton [7, Section 2.5] demonstrates that a smooth toric surface is given by a sequence of lattice points $x_i \subseteq N \cong \mathbb{P}^2$, in counter-clockwise order, such that succesive pairs generate the lattice. In that case, for all i, we must have

$$x_{i-1} + x_{i+1} + a_i x_i = 0 \tag{3.1}$$

for some $a_i \in \mathbb{Z}$. If x_{i-1} does not meet x_{i+1} in a cone, and $a_i \leq 0$, then Equation 3.1 is the primitive relation corresponding to the primitive collection $\{x_{i-1}, x_{i+1}\}$.

Proposition 3.1. *For every primitive vector x_i, we have a corresponding torus-invariant curve C_i, and $C_i^2 = a_i$.*

Proof. Since x_i is adjacent to x_{i-1}, we have a basis for the lattice $\{x_{i-1}, x_i\}$. Since $x_{i-1} + x_{i+1} + ax_i = 0$, the dual vector $\hat{x}_i$ gives us

$$\begin{aligned} D_i - aD_{i+1} &\equiv 0, \text{ so} \\ D_i^2 - aD_i \cdot D_{i+1} = D_i^2 - a &= 0. \end{aligned}$$

□

Example 3.2. *Any smooth toric surface S with $\rho(S) = 1$ has three primitive vectors. Then we must have*

$$x_1 + x_2 + x_3 = 0,$$

where all three torus-invariant curves C_i, have $C_i^2 = 1$, and $S \cong \mathbb{P}^2$.

Example 3.3. *If $\rho(S) = 2$, we have four primitive vectors, x_1, x_2, x_3, x_4, and, without loss of generality,*

$$\begin{aligned} x_1 + x_3 &= 0, \text{ and} \\ x_2 + x_4 &= ax_3 \\ &= -ax_1. \end{aligned}$$

We may assume $a \geq 0$.

If $a = 0$, then for all C_i, we have $C_i^2 = 0$, and $S \cong \mathbb{P}^1 \times \mathbb{P}^1$.

If $a > 0$, then we have two torus-invariant curves, C_1 and C_3 with $C_1^2 = -C_3^2 = a$, and $S \cong \mathbb{F}_a$ is a Hirzebruch surface of degree a.

Lemma 3.4. *Any smooth toric surface S with $\rho(S) \geq 3$ has at most one D_i with $D_i^2 > 0$ and at most two D_i with $D_i^2 = 0$.*

Proof. Every such surface is a blowup of a Hirzebruch. If we start with $\mathbb{P}^1 \times \mathbb{P}^1$, then we have four. If we blowup somewhere, we end up with only two. Any further blowups can only decrease the self-intersections. Otherwise suppose we start with a Hirzbruch $\mathbb{F}_a$. Then we start with 2, and blowing-up anywhere removes one. It's possible we could create a new one eventually with the original curve that had positive self-intersection. But that can only happen once in a sequence of blowups, and in any case we only get one more. So, any smooth surface with picard number greater than 2, has at most two curves with zero self-intersection. ☐

Lemma 3.5. *[7] Blowing up a surface with self-intersection numbers $a_1, \dots, a_k$ at the torus-invariant curve $D_j \cdot D_{j+1}$ results in a surface with self-intersection numbers*

$$a_1, \dots, a_j - 1, -1, a_{j+1} - 1, \dots, a_k$$

Proposition 3.6. *[7] Every smooth toric surface can be obtained by a sequence of toric blowups starting from either $\mathbb{P}^2$ or $\mathbb{F}^a$.*

Proposition 3.7. *Let X be a smooth toric surface, and let $\{C_i\}$ be the torus-invariant divisors, then*

$$\sum C_i^2 = 3(2 - \rho(X)).$$

Proof. If $X \cong \mathbb{P}^2$, there are exactly three torus-invariant curves, each with self intersection 1. Then, $\gamma_2(X) = 3$. If $\rho(X) = 2$, then $X \cong \mathbb{F}_a$, and we have $D_1^2 = D_3^2 = 0$, and $D_2^2 = -D_4^2 = a$, so $\gamma_2(X) = \sum D_i = 0$. By 3.6, every smooth toric surface X with $\rho(X) \geq 3$ can be obtained by a sequence of blowups at torus-invariant points starting at $\mathbb{F}_a$. So we need only prove that the relation is preserved under a blowup. Suppose X has $\gamma_2(X) = 3(2 - \rho(X))$, and let $\tilde{X}$ be the blowup of X and some torus-invariant point p. Without loss of generality, let $p = D_1 \cdot D_2$. Then, we blowup at p, to obtain exceptional divisor $E \subseteq \tilde{X}$, with $E^2 = -1$. By, 3.5, $\tilde{D_1}^2 = D_1^2 - 1$, and $\tilde{D_2}^2 = D_2^2 - 1$. All other self-intersections are unaltered. Therefore, we have

$$\begin{aligned}
\gamma_2(\tilde{X}) &= \gamma_2(X) - 3 \\
&= 3(2 - \rho(X)) - 3 \\
&= 6 - 3(\rho(\tilde{X}) - 1) - 3 \\
&= 6 - 3\rho(\tilde{X}) + 3 - 3 \\
&= 3(2 - \rho(\tilde{X}).
\end{aligned}$$

□

Lemma 3.8. *If X is a smooth toric Fano surface, then for all i, $a_i \geq -1$.*

Proof. If X is Fano, then $-K_X$ is ample, so by Kleiman's criterion, $-K_X$ intersects positively with every effective curve. In particular $-K_X \cdot D_i > 0$, where

$$-K_X \cdot D_i = 2 + a_i.$$

Therefore $a_i \geq -1$. □

Lemma 3.9. *A smooth toric Fano surface X must have $\rho(X) \leq 4$*

Proof. $\sum a_i = 3(2 - \rho(X))$, but all $a_i \geq -1$, so $\sum a_i \geq -(\rho(X) + 2)$. So

$$\begin{aligned} 3(2 - \rho(X)) &\geq -(\rho(X) + 2), \\ 6 - 3\rho(X) &\geq -\rho(X) - 2, \\ 8 &\geq 2\rho(X), \\ 4 &\geq \rho(X). \end{aligned}$$

□

Chapter 4.

Threefolds

In this section we describe the general behavior of $\gamma_2(X)$ for a smooth toric threefold X.

We begin with the following fundamental proposition which describes the relationship between the top self-intersection D^3 and $\gamma_2(X) \cdot D$.

Proposition 4.1. *Suppose X is a smooth toric threefold, and D a torus-invariant divisor, then*

$$\begin{aligned} \gamma_2(X) \cdot D &= D^3 + \gamma_2(D) \\ &= D^3 + 3(2 - \rho(D)). \end{aligned}$$

Proof. By Proposition 3.7, we have for each i,

$$D_i|_D^2 = 3(2 - \rho(D)),$$

so

$$\begin{aligned} \gamma_2(X) \cdot D &= (D^2 + \sum_{D_i \neq D} D_i^2) \cdot D \\ &= D^3 + \sum_{D_i \neq D} D_i^2 \cdot D \\ &= D^3 + \sum_{D_i \neq D} D_i|_D^2 \\ &= D^3 + 3(2 - \rho(D)). \end{aligned}$$

□

Corollary 4.2. *If $\rho(D) \geq 2$,*

$$\gamma_2(X) \cdot D \leq D^3,$$

and if $D^3 \leq 0$ and $\rho(D) \geq 2$, then $\gamma_2(X) \cdot D \leq 0$, and X is not γ_2-positive.

In general, it is fairly straight-forward to compute D^3 for a particular torus-invariant divisor D on a smooth toric threefold X. Let D be such a divisor, and let D_i be the divisors adjacent to D. Let x_i and x be the primitive vectors corresponding to the D_i and D, let $\hat{x}$ be any element of $M = N^\vee$ with $\hat{x}(x) = 1$, and $a_i = \hat{x}(x_i)$. Then, we get, locally near D,

$$D \equiv \sum a_i D_i.$$

so

$$\begin{aligned} D^3 &= \sum a_i D_i \cdot D^2. \\ &= \sum a_i D|^2_{D_i}. \end{aligned}$$

Note that the values $D|^2_{D_i}$ are not intrinsic to D, indeed they are the self-intersection numbers of D restricted to each D_i, and the a_i can be any integers. In fact, for any set of a_i, we have a smooth toric threefold with the above set-up and for each i $\hat{x}(x_i) = a_i$. In particular this means the knowledge of the picard number of a torus-invariant divisor D only gives us information about D^3 if $\rho(D) = 1$, in which case, $\gamma_2(X)\cdot \geq 3$.

Proposition 4.3 ($\rho(D) = 1$)**.** *Suppose X is a smooth, toric threefold, and suppose D is a torus-invariant divisor with $\rho(D) = 1$. Let D_i be the divisors adjacent to D, and x_i, x, be the primitive vectors corresponding to D_i and D, respectively. Then then exists $a \in \mathbb{Z}$, with $D_i^2 \cdot D = a$ for all i and*

$$D^3 = a^2.$$

Proof. We have

$$x_1 + x_2 + x_3 + ax = 0.$$

In particular all $D_i \cdot D^2 = a$ for all i. Using the basis $\{x, x_1, x_2\}$, and the dual functional $\hat{x}$, we obtain the numerical relation, locally near D,

$$D - aD_3 \equiv 0,$$

$$D^3 - a(D_3 \cdot D^2) = 0$$

and

$$D^3 = a^2 \geq 0.$$

□

Corollary 4.4. *Let X be a smooth toric threefold. If there exists a torus-invariant divisor $D \subseteq X$ with $\rho(D) = 1$, then*

$$\gamma_2(X) \cdot D = 3 + a^2 > 0.$$

In particular, $\gamma_2(X) \cdot D > 0$ for all $\rho(D) = 1$.

Corollary 4.5. *Let X be a smooth toric threefold. If there exists a torus-invariant divisor $D \subseteq X$, with $\rho(D) = 1$, and $D^3 = 0$, then X has a centrally symmetric relation of degree 3.*

Proposition 4.6 ($\rho(D) = 2$). *Suppose X is a smooth toric threefold, and D a torus-invariant divisor with $\rho(D) = 2$. Then $D \cong \mathbb{F}_a$ for some $a \geq 0$.*

i. If $a = 0$, $D \cong \mathbb{P}^1 \times \mathbb{P}^1$, and for some b_1 and b_2,

$$\begin{aligned} b_1 &= D^2 \cdot D_1 = D^2 \cdot D_3, \\ b_2 &= D^2 \cdot D_2 = D^2 \cdot D_4, \text{ and} \\ D^3 &= 2b_1 b_2. \end{aligned}$$

ii. If $a > 0$, then

$$\begin{aligned} b_1 &= D^2 \cdot D_1 = D^2 \cdot D_3, \\ b_2 &= D^2 \cdot D_2, \\ b_4 &= D^2 \cdot D_4, \\ ab_1 &= b_2 - b_4, \text{ and} \\ D^3 &= b_1(b_2 + b_4). \end{aligned}$$

Proof. Let D_i be the surrounding divisors, in order.

Without loss of generality assume

$$\begin{aligned} 0 &= D_1^2 \cdot D = D_3^2 \cdot D, \text{ and} \\ a &= D_2^2 \cdot D = -D_4^2 \cdot D. \end{aligned}$$

If $a = 0$, we have

$$\begin{aligned} & x_1 + x_3 + (D_2 \cdot D^2)x = 0. \\ & = x_1 + x_3 + (D_4 \cdot D^2)x = 0. \\ & x_2 + x_4 + (D_1 \cdot D^2)x = 0. \\ & = x_2 + x_4 + (D_3 \cdot D^2)x = 0. \end{aligned}$$

Let $b_1 = D_3 \cdot D^2 = D_1 \cdot D^2$ and $b_2 = D_2 \cdot D^2 = D_4 \cdot D^2$.

Then, using the basis $\{x_1, x_2, x\}$ and the dual functional $\hat{x}$, we obtain

$$D - b_2 D_3 - b_1 D_4 \equiv 0$$

locally near D. So

$$D^2 \equiv b_2^2 D_3^2 + b_1^2 D_4^2 + 2b_1 b_2 D_3 \cdot D_4,$$

and

$$\begin{aligned} D^3 &= b_2^2 D_3^2 \cdot D + b_1^2 D_4^2 \cdot D + 2b_1 b_2 D_3 \cdot D_3 \cdot D \\ &= 2b_1 b_2. \end{aligned}$$

If $a > 0$, then let $b_i = D_i \cdot D^2 = D|^2_{D_i}$. Then we have the set of equations defining the relations associated to the curves $D_i \cdot D$.

$$\begin{aligned} 0 &= x_1 + x_3 + (D_2^2 \cdot D)x_2 + (D_2 \cdot D^2)x \\ &= x_1 + x_3 + ax_2 + b_2 x \\ 0 &= x_1 + x_3 + (D_4^2 \cdot D)x_4 + (D_4 \cdot D^2)x \\ &= x_1 + x_3 - ax_4 + b_4 x. \\ 0 &= x_2 + x_4 + (D_1^2 \cdot D)x_1 + (D_1 \cdot D^2)x \\ &= x_2 + x_4 + b_1 x \\ 0 &= x_2 + x_4 + (D_3^2 \cdot D)x_3 + (D_3 \cdot D^2)x \\ &= x_2 + x_4 + b_3 x. \end{aligned}$$

So

$$\begin{aligned} a(x_2 + x_4) + (b_2 - b_4)x = 0, \text{ and} \\ b_1 = b_3. \end{aligned}$$

Since

$$\begin{aligned} 0 &= x_2 + x_4 + b_3 x, \text{ then} \\ 0 &= a(x_2 + x_4) + ab_3 x, \end{aligned}$$

and we have $ab_3 = ab_1 = b_2 - b_4$. Note that

$$\begin{aligned} x_4 &= -b_3 x - x_2, \text{ and} \\ x_3 &= -x_1 - ax_2 - b_2 x, \end{aligned}$$

so using the basis $\{x, x_1, x_2\}$ and the dual functional $\hat{x}$, we get, that near D,

$$D - b_2 D_3 - b_3 D_4 \equiv 0.$$

and

$$\begin{aligned} 0 &= D^3 - b_2 D_3 \cdot D^2 - b_3 D_4 \cdot D^2 \\ &= D^3 - b_2 b_3 - b_3 b_4 \\ &= D^3 - b_3(b_2 + b_4). \end{aligned}$$

So

$$D^3 = b_3(b_2 + b_4).$$

□

4.1 Smooth Threefolds with Picard Number 2

In this section we classify all smooth toric threefolds X with Picard number 2 and show that none are γ_2-positive. There are two types, according to Batyrev's classification, depending on the maximal degree d of a centrally symmetric relation on X.

First, we show that there is only one combinatorial type of a toric threefold with picard number 2.

Proposition 4.7. *Suppose $\rho(X) = 2$. Then the torus-invariant divisors of X consist of three Hirzebruch surfaces D_1, D_2, and D_3, and two $\mathbb{P}^2$'s, E_1 and E_2. The D_i each meet both E_1 and E_2 and $E_1 \cdot E_2 \equiv 0$.*

Proof. We must have some D with $\rho(D) \geq 2$, and since there are only four other torus-invariant divisors, we assume $\rho(D) = 2$. Then D does meet all four other divisors $D_1, \dots, D_4$, and without loss of generality D_1 meets D_3. Then D_2 and D_4 each have $\rho(D_i) = 1$, while D_1 and D_3 each have $\rho(D_i) = 2$.

Thus we have determined the combinatorial type of X. □

Corollary 4.8. *If $\rho(X) = 2$, then X has a primitive collection $\mathcal{P}$ with $|\mathcal{P}| = 3$.*

Proof. By the proof of 4.7, we must have three Hirzebruch surfaces pair-wise meeting each other, and, by the proof, they do not all three meet at a point. So we have a primitive collection $\mathcal{P}$ with $|\mathcal{P}| = 3$. □

Proposition 4.9. *There are two types of smooth toric threefolds X with $\rho(X) = 2$.*

Type I. *The fibers of the Hirzebruch surfaces D_i are of the form $D_i \cdot D_j$, $D_i^2 \cdot D_j = 0$ for all i, j, and we have a centrally symmetric relation of degree 2:*

$$y_1 + y_2 = 0.$$

Type II. *The fibers of the D_i are of the form $D_i \cdot E_j$, $D_i^2 \cdot E_j = 0$, for all i, j, and we have a centrally symmetric relation of degree 3:*

$$x_1 + x_2 + x_3 = 0.$$

In other words the fibers l of the D_i (torus-invariant curves with $l^2 = 0$ on D_i) are either all of the form $D_i \cdot E_j$ or all of the form $D_i \cdot D_j$. There is only one smooth toric threefold satisfying both conditions, namely $\mathbb{P}^2 \times \mathbb{P}^1$, where $D_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ for all i.

Proposition 4.10. *If X is a smooth toric threefold of picard number 2, with a centrally symmetric relation of degree 2. Then $D_i^3 = 0$ for all i, and $E_1^3 = E_2^3 = a^2$ where $x_1 + x_2 + x_3 = ay_1$.*

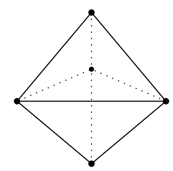

Figure 4.1: A threefold with picard number 2.

Proof. Let y_1 be the first $\mathbb{P}^1$, and let x_1, x_2, and x_3 the adjacent Hirzebruch surfaces. Then $ay_1 = x_1 + x_2 + x_3 = by_2$.

Suppose the fibers of D_1 attach to D_2 and D_3. Then, as we saw earlier that if one of the D_i is such that its fiber are attached to E_1, then they all must be. Then

$$D_i|_{D_j}^2 = 0,$$

for all $i \neq j$. Therefore, at D_i, we must have

$$y_1 + y_2 = a_i x_i.$$

But this holds for all i. So $a_i = 0$ for all i, and

$$y_1 + y_2 = 0.$$

Then we have

$$x_1 + x_2 + x_3 = ay_1 = -ay_2.$$

So,

$$\begin{aligned} D_1 - D_3 &\equiv 0 \\ D_2 - D_3 &\equiv 0 \\ E_1 + aD_3 - E_2 &\equiv 0. \end{aligned}$$

So,

$$\begin{aligned} E_1^3 &\equiv E_1^2(E_2 - aD_3) \\ &\equiv -aE_1|_{D_3}^2. \\ &= a^2 \geq 0. \end{aligned}$$

$$\begin{aligned} E_2^3 &\equiv E_2^2(E_1 + aD_3) \\ &\equiv aE_2|_{D_3}^2. \\ &= a^2 \geq 0. \end{aligned}$$

$$D_1^3 \equiv D_1 \cdot D_2 \cdot D_3 = 0.$$

□

Proposition 4.11. *Suppose X is a smooth toric threefold with $\rho(X) = 2$ and with a centrally symmetric relation of degree 3. Then,*

$$\sum D_i^3 = 0,$$

and $D_i^3 \leq 0$ for some i.

Proof. By assumption, X has a centrally symmetric relation of degree 3

$$x_1 + x_2 + x_3 = 0,$$

and $E_i|_{D_j}^2 = 0$ for all i, j.

Now, write

$$y_1 + y_2 = dx_1 + cx_2$$

where we get

$$D_2|_{D_1}^2 = -c, \text{ and}$$
$$D_3|_{D_1}^2 = c.$$

Similarly,

$$D_1|_{D_2}^2 = -d, \text{ and}$$
$$D_3|_{D_2}^2 = d.$$

Then, write

$$\begin{aligned} y_1 + y_2 &= d(-x_2 - x_3) + cx_2 \\ &= (c - d)x_2 - dx_3 \end{aligned}$$

to get

$$D_2|_{D_3}^2 = d - c, \text{ and}$$
$$D_1|_{D_3}^2 = c - d.$$

Rewrite the above equations in terms of the basis $\{x_1, x_2, y_1\}$

$$y_2 = dx_1 + cx_2 - y_1,$$
$$x_3 = -x_1 - x_2.$$

So, we have

$$\begin{aligned} D_1 - D_3 + dE_2 &\equiv 0 \\ D_2 - D_3 + cE_2 &\equiv 0 \\ E_1 - E_2 &\equiv 0. \end{aligned}$$

Then,

$$E_1^3 = E_2^2 = E_1 \cdot E_2^2 = 0,$$

and

$$\begin{aligned} D_1^3 &\equiv D_1^2(D_3 - dE_2) \\ &\equiv D_1|_{D_3}^2 - dD_1|_{E_2}^2 \\ &= (c-d) - d \\ &= c - 2d. \\ D_2^3 &\equiv D_2^2(D_3 - cE_2) \\ &\equiv D_2|_{D_3}^2 - cD_2|_{E_2}^2 \\ &= (d-c) - c \\ &= d - 2c, \text{ and} \\ D_3^3 &\equiv D_3^2(D_2 + cE_2) \\ &\equiv D_3|_{D_2}^2 + cD_3|_{E_2}^2 \\ &= d + c. \end{aligned}$$

□

Note that c and d can take any value in $\mathbb{Z}^2$. In particular any one of the D_i^3 can be arbitrarily large.

Then observe that

$$\begin{aligned} D_1^3 + D_2^3 + D_3^3 &= (c - 2d) + (d - 2c) + (d + c) \\ &= 0. \end{aligned}$$

So,

$$\gamma_2(X) \cdot \sum D_i = \sum D_i^3 = 0,$$

where $\sum D_i \in \mathrm{NE}_2(X)$ is an effective combination of torus-invariant surfaces. Therefore, X is not γ_2-positive. Additionally, there must be some i with $D_i^3 \leq 0$.

Proposition 4.12. *If a smooth toric threefold X has $\rho(X) = 2$ there exists a torus-invariant divisor D such that $\gamma_2(X) \cdot D \leq 0$, and X is not γ_2-positive.*

Proof. If X has a centrally symmmetric relation of degree 2, then by Proposition 4.10, there exists a torus-invariant divisor D with $D^3 = 0$ and $\rho(D) = 2$. So

$$\gamma_2(X) \cdot D = D^3 = 0.$$

If X has a centrally symmetric relation of degree 3, then by Proposition 4.11, there exist three divisor D_1, D_2, and D_3, each of Picard number 2, and

$$\sum D_i^3 = 0.$$

But for each D_i, we have $\gamma_2(X) \cdot D = D^3$, so

$$\gamma_2(X) \cdot \sum D_i = \sum D_i^3 \leq 0.$$

So for some i, $\gamma_2(X) \cdot D_i \leq 0$. ☐

Chapter 5.

Blowups of Threefolds

In this chapter, we investigate how the two types of toric blowups of smooth toric threefolds affect the behavior of $\gamma_2(X)$. Suppose we blowup X at a torus-invariant subvariety Z to obtain

$$\tilde{X} = \mathrm{Bl}_Z(X) \to X.$$

We have two possibilities.

1. $Z = p$ is a torus-invariant point of X.
2. $Z = l$ is a torus-invariant curve of X.

5.1 Blowups of Points

If $Z = p$, then we may write $p = D_1 \cdot D_2 \cdot D_3$, for torus-invariant divisors D_i. Let x_1, x_2, x_3 be the corresponding primitive vectors, let E be the exceptional divisor of $\tilde{X} \to X$, with corresponding primitive vector y, and let $\tilde{D}_i$ be the strict transform of D_i with corresponding primitive vector x_i'. Then

$$x_1' + x_2' + x_3' = y,$$

and $E \cong \mathbb{P}^2$. We have $E^2 \cdot D_i = -1$, and $E \cdot D_i^2 = 1$, for all i. $\rho(\tilde{D}_i) = \rho(D_i) + 1$.

Proposition 5.1. *Let X is a smooth toric threefold and $\pi : \tilde{X} \to X$ the blowup of X at a torus-invariant point p. For $D \subseteq X$ a torus-invariant divisor, let $\tilde{D}$ denote the proper transform of D, and let E be the exceptional divisor of π. Then,*

$$\gamma_2(\tilde{X}) \cdot \tilde{D} = \begin{cases} \gamma_2(X) \cdot D - 4, & \text{if } p \in D \\ \gamma_2(X) \cdot D, & \text{otherwise.} \end{cases}$$

and

$$\gamma_2(\tilde{X}) \cdot E = 4.$$

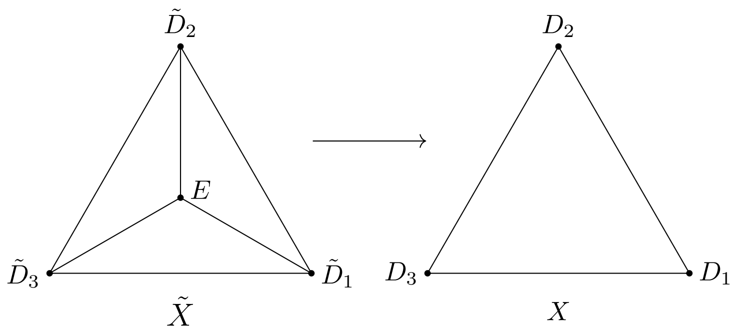

Figure 5.1: Blowup of a point on a toric threefold.

Proof. Let $\{D_i\}$ be the set of all torus-invariant divisors on X, with $D_1 \cdot D_2 \cdot D_3 = p$. Then

$$\pi^*(D_i) = \tilde{D}_i + \epsilon_i E$$

with $\epsilon_i = 1$ if $i \in \{1, 2, 3\}$, and $\epsilon_i = 0$ otherwise. Then, for $j \in \{1, 2, 3\}$,

$$\begin{aligned}
\gamma_2(\tilde{X}) \cdot \tilde{D}_j &= \left(\sum \tilde{D}_i{}^2 + E^2\right) \cdot \tilde{D}_j \\
&= \left(\sum (\pi^* D_i - \epsilon_i E)^2 + E^2\right) \cdot \tilde{D}_j \\
&= \left(\sum (\pi^* D_i^2 + 2\epsilon_i \pi^*(D_i) \cdot E + \epsilon_i E^2) + E^2\right) \cdot \tilde{D}_j \\
&= \sum \pi^*(D_i)^2 \cdot \tilde{D}_j + (1+3)E^2 \cdot \tilde{D}_j \\
&= \sum D_i^2 \cdot D_j + 4E^2 \cdot \tilde{D}_j \\
&= \gamma_2(X) \cdot D_j + 4(E|_{\tilde{D}})^2 \\
&= \gamma_2(X) \cdot D_j - 4.
\end{aligned}$$

If $j \notin \{1, 2, 3\}$, then $E \cdot \tilde{D}_j \equiv 0$, in which case

$$\gamma_2(\tilde{X}) \cdot \tilde{D}_j = \gamma_2(X) \cdot D_j.$$

Since

$$\begin{aligned}
\gamma_2(\tilde{X}) \cdot \tilde{D}_j &= \tilde{D}_j^3 + 3(2 - \rho(\tilde{D}_j)), \\
\gamma_2(X) \cdot D_j &= D_j^3 + 3(2 - \rho(D_j))
\end{aligned}$$

and $\rho(\bar{D}) = \rho(D) + 1$, we have for $j \in \{1, 2, 3\}$,

$$\begin{aligned}\gamma_2(\tilde{X}) \cdot \tilde{D}_j &= \tilde{D}_j^3 + 3(2 - \rho(\tilde{D}_j)) \\ &= \tilde{D}_j^3 + 3(2 - (\rho(D_j) + 1)) \\ &= \tilde{D}_j^3 + 3 - 3\rho(D_j) \\ &= \gamma_2(X) \cdot D_j - 4. \\ &= D_j^3 + 3(2 - \rho(D_j)) - 4. \\ &= D_j^3 + 2 - 3\rho(D_j).\end{aligned}$$

So $\bar{D}^3 = D^3 - 1$. If $j \notin \{1, 2, 3\}$,

$$\tilde{D}_j^3 = D_j^3.$$

Finally,

$$\begin{aligned}E^3 &= E \cdot (\pi^* D_1 - \tilde{D}_1)(\pi^* D_2 - \tilde{D}_2) \\ &= E \cdot \tilde{D}_1 \cdot \tilde{D}_2 \\ &= 1.\end{aligned}$$

Then, since $\rho(E) = 1$, $\gamma_2(\tilde{X}) \cdot E = E^3 + 3 = 4$. □

5.2 Blowups of Lines

Theorem 5.2. *Suppose Z is a line $l = D_1 \cdot D_2$ on X. There are two divisors meeting l properly at a point, which we denote F_1 and F_2. Then,*

$$\begin{aligned}\gamma_2(\tilde{X}) \cdot \tilde{D}_i &= \gamma_2(X) \cdot D_i - 2l_i|_E^2 - l|_{D_i}^2. \\ &= \gamma_2(X) \cdot D_i - 2(l|_{D_i'}^2 - l|_{D_1}^2) - l|_{D_i}^2. \\ &= \gamma_2(X) \cdot D_i + l|_{D_i}^2 - 2l|_{D_i'}^2 \\ \gamma_2(\tilde{X}) \cdot \tilde{F}_i &= \gamma_2(X) \cdot F_i - 3. \\ \gamma_2(\tilde{X}) \cdot E &= -D_1^2 \cdot D_2 - D_2^2 \cdot D_1 = -l|_{D_1}^2 - l|_{D_2}^2.\end{aligned}$$

Note that $\rho(\tilde{D}_i) = \rho(D_i)$, $\rho(\tilde{F}_i) = \rho(F_i) + 1$ for $i = 1, 2$, and $\rho(E) = 2$. So, in terms of top self-intersections, we have the following corollary.

Corollary 5.3. *Let Z be the blowup a line $l = D_1 \cdot D_2$ on X.*

$$\begin{aligned}\tilde{D}_1^3 &= D_1^3 + l|_{D_1}^2 - 2l|_{D_2}^2 \\ \tilde{D}_2^3 &= D_2^3 + l|_{D_2}^2 - 2l|_{D_1}^2 \\ \tilde{F}_i^3 &= F_i^3. \\ E^3 &= -l|_{D_1}^2 - l|_{D_2}^2.\end{aligned}$$

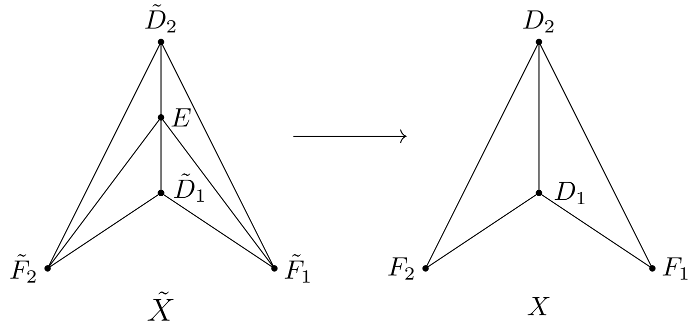

Figure 5.2: Blowup of a line.

$$\begin{aligned}
\gamma_2(\tilde{X}) &= \sum \tilde{G}_i^2 + \sum \tilde{D}_i^2 + \sum \tilde{F}_i^2 + E^2 \\
&= \sum \pi^*(G_i)^2 + \sum (\pi^*(D_i) - E)^2 + \sum \pi^*(F_i)^2 + E^2 \\
&= \sum \pi^*(G_i)^2 + \sum (\pi^*(D_i)^2 - 2\pi^*(D_i) \cdot E + E^2) + \sum \pi^*(F_i)^2 + E^2 \\
&= \sum \pi^*(G_i)^2 + \sum \pi^*(D_i)^2 + \sum \pi^*(F_i)^2 - 2\sum \pi^*(D_i) \cdot E + 3E^2 \\
&= \pi^*(\gamma_2(X)) - 2\pi^*(D_1 + D_2) \cdot E + 3E^2. \\
&= \sum \pi^*(G_i)^2 + \sum \pi^*(D_i)^2 + \sum \pi^*(F_i)^2 - 2\sum (\tilde{D}_i + E) \cdot E + 3E^2 \\
&= \sum \pi^*(G_i)^2 + \sum \pi^*(D_i)^2 + \sum \pi^*(F_i)^2 - 2\sum (\tilde{D}_i \cdot E) - E^2 \\
&= \pi^*(\gamma_2(X)) - 2\sum (\tilde{D}_i \cdot E) - E^2
\end{aligned}$$

So

$$\begin{aligned}
\gamma_2(\tilde{X}) \cdot \tilde{F}_i &= \gamma_2(X) \cdot F_i - 2\sum_j \tilde{D}_j \cdot E \cdot \tilde{F}_i - E^2 \cdot \tilde{F}_i \\
&= \gamma_2(X) \cdot F_i - 4 + 1 \\
&= \gamma_2(X) \cdot F_i - 3.
\end{aligned}$$

Since

$$\begin{aligned}
\pi_*(E \cdot \tilde{D}_i) &= \pi_*\left((\pi^* D_2 - \tilde{D}_2) \cdot \tilde{D}_1\right) \\
&= \pi_*(\pi^* D_2 \cdot \tilde{D}_1) - \pi_*(\tilde{D}_2 \cdot \tilde{D}_1) \\
&= D_2 \cdot \pi_*(\tilde{D}_1) - 0 \\
&= D_2 \cdot D_1 \\
&= l,
\end{aligned}$$

we have

$$\begin{aligned}
\gamma_2(\tilde{X}) \cdot \tilde{D}_i &= \pi^*(\gamma_2(X)) \cdot \tilde{D}_i - 2\pi^*(D_1 + D_2) \cdot E \cdot \tilde{D}_i + 3E^2 \cdot \tilde{D}_i. \\
&= \pi^*(\gamma_2(X)) \cdot \tilde{D}_i - 2(D_1 + D_2) \cdot \pi_*(E \cdot \tilde{D}_i) + 3E^2 \cdot \tilde{D}_i. \\
&= \gamma_2(X) \cdot D_i - 2 \sum \tilde{D}_j \cdot E \cdot \tilde{D}_i - E^2 \cdot \tilde{D}_i \\
&= \gamma_2(X) \cdot D_i - 2E \cdot \tilde{D}_i^{\,2} - 2\tilde{D}_{i+1} \cdot \tilde{D}_i \cdot E - E^2 \cdot \tilde{D}_i \\
&= \gamma_2(X) \cdot D_i - 2E \cdot \tilde{D}_i^{\,2} - E^2 \cdot \tilde{D}_i \\
&= \gamma_2(X) \cdot D_i - 2\tilde{D}_i|_E^2 - E|_{D_i}^2. \\
&= \gamma_2(X) \cdot D_i - 2l_i|_E^2 - l|_{D_i}^2.
\end{aligned}$$

Lemma 5.4.

$$\begin{aligned}
E^3 &\equiv -E^2 \cdot D_1 - E^2 \cdot D_2 \\
&= -l|_{D_1}^2 - l|_{D_2}^2.
\end{aligned}$$

Proof. $E^2 \cdot \tilde{D}_i = E|_{\tilde{D}_i}^2 = l|_{D_i}^2$, since

$$\pi|_{\tilde{D}_i} \tilde{D}_i \to D_i$$

is an isomorphism, and $E^2 \cdot \tilde{F}_i = 0$. Then we have

$$\begin{aligned}
0 &= E \cdot \pi^* D_1 \cdot \pi^* D_2 \\
&= E \cdot (\tilde{D}_1 + E) \cdot (\tilde{D}_2 + E) \\
&= E^2 \cdot \tilde{D}_1 + E^2 \cdot \tilde{D}_2 + E^3 \\
&= l|_{D_1}^2 + l|_{D_2}^2 + E^3.
\end{aligned}$$

□

Lemma 5.5. *The exceptional divisor E is a Hirzebruch surface. The curves $l_1|_E$ and $l_2|_E$ are the torus-invariant sections of*

$$\pi|_E : E \to l$$

with self-intersection numbers

$$l_1|_E^2 = l|_{D_2}^2 - l|_{D_1}^2, \tag{5.1}$$

and

$$l_2|_E^2 = -l_1|_E^2 = l|_{D_1}^2 - l|_{D_2}^2. \tag{5.2}$$

Additionally, for the other two torus-invariant curves f_1 and f_2 on E, f_i is a fiber of $\pi|_E$ and $f_i^2 = 0$.

Proof.

$$\begin{aligned} 0 &= E \cdot \pi^* D_1 \cdot \pi^* D_1 \\ &= E \cdot (\tilde{D_1} + E) \cdot (\tilde{D_1} + E) \\ &= E \cdot \tilde{D_1}^2 + 2E^2 \cdot \tilde{D_1} + E^3 \\ &= l_1|_E^2 + 2l|_{D_1}^2 + E^3. \end{aligned}$$

So by Lemma 5.4

$$\begin{aligned} 0 &= l_1|_E^2 + 2l|_{D_1}^2 - l|_{D_1}^2 - l|_{D_2}^2. \\ &= l_1|_E^2 + l|_{D_1}^2 - l|_{D_2}^2, \text{ or} \\ l_1|_E^2 &= l|_{D_2}^2 - l|_{D_1}^2. \end{aligned}$$

□

We give an alternate proof using linear relations:

Proof. Write

$$\begin{aligned} 0 &= y_1 + y_2 + c_1 x_1 + c_2 x_2. \\ &= y_1 + y_2 + (c_1 - c_2)x_1 + c_2(x_1 + x_2) \\ &= y_1 + y_2 + (c_1 - c_2)x_1 + c_2 e. \end{aligned}$$

These are the linear relations representing l and l_1 in $A_1(X)$ and $A_1(\tilde{X})$, respectively. Then $l = D_1 \cdot D_2$ and we have

$$\begin{aligned} l_2 \cdot E = l \cdot D_1 &= l|_{D_2}^2 = c_1 \\ l_1 \cdot E = l \cdot D_2 &= l|_{D_1}^2 = c_2 \\ l_1 \cdot D_1 = l_1|_E^2 &= c_1 - c_2 \\ l_2 \cdot D_2 = l_2|_E^2 &= c_2 - c_1. \end{aligned}$$

In particular,

$$l_1|_E^2 = -l_2|_E^2 = l|_{D_2}^2 - l|_{D_1}^2,$$

and, by Lemma 5.4

$$E^3 = -c_1 - c_2.$$

□

Chapter 6.

Fano Threefolds

In this section, we prove that every smooth toric Fano threefold X with $\rho(X) \geq 2$ is not γ_2-positive.

The key to finding a torus-invariant divisor D with $\gamma_2(X) \cdot D \leq 0$ comes from a proposition of Batyrev [3] which naturally separates toric threefolds into three categories.

Namely, we use the following corollary of Proposition 2.18.

Corollary 6.1. *Every smooth projective toric threefold X has a centrally symmetric relation of one of the following types.*

i. $x_1 + x_2 = 0$,

ii. $x_1 + x_2 + x_3 = 0$,

iii. $x_1 + x_2 + x_3 + x_4 = 0$.

By, Proposition 2.20, if X has a centrally symmetric relation of degree 4, then $X \cong \mathbb{P}^3$. Therefore if X is a smooth toric Fano threefold with $\rho(X) \geq 2$, X must have either a centrally symmetric relation of degree either 2 or 3.

Lemma 6.2. *Suppose X is a smooth toric threefold with a torus-invariant divisor D with $\rho(D) \geq 2$. Let x be the primitive vector corresponding to D. Then there exists D_1 and D_2 adjacent to D, with corresponding primitive vectors x_1 and x_2, such that*

$$x_1 + x_2 + ax = 0.$$

The remaining divisors meeting D are split into $\{E_i\}$ and $\{F_j\}$ with

$$E_i \cdot F_j \cdot D_1 \equiv 0$$

for all i, j.

Proof. Every torus-invariant curve on D is of the form $G|_D$ for some torus-invariant G. On D, we have the primitive relation $x_1' + x_2' = 0$, corresponding to $D_1|_D$ and $D_2|_D$. We must have at least one primitive vector on either side of the line spanned by x_1' and x_2'. The corresponding torus-invariant curves of D, other than $D_1|_D$ and $D_2|_D$ must be of the form $E_i|_D$ and $F_j|_D$, for some E_i and F_j on X, and $E_i|D_1 \cdot F_j|D_1 \equiv 0$ for all i, j, thus

$$E_i \cdot F_j \cdot D_1 \equiv 0.$$

□

Lemma 6.3. *Suppose X is a smooth toric threefold with a torus-invariant divisor D with $\rho(D) \geq 2$. Let $\{E_i\}$ and $\{F_j\}$ as in the lemma, If $|\{E_i\}|$ and $|\{F_j\}|$ are both at most 2, then, for all i and j,*

$$E_i^2 \cdot D \geq -1, \text{ and}$$
$$F_j^2 \cdot D \geq -1.$$

Proof. D is the blowup of a Hirzebruch surface $\mathbb{F}_a$ where $D_1|_D$ and $D_2|_D$ are the strict transforms of the positive and negative sections.

If $|\{E_i\}| = 1$, we have only one divisor on one side, so we did not blow up on that side, and $E_1|_D^2 = 0$. If we have two divisors, then we blew up once, yielding two divisors E_1 and E_2 with $E_1|_D^2 = E_2|_D^2 = -1$. Similarly for the F_j. □

Corollary 6.4. *Suppose X is a smooth toric threefold, with a torus-invariant divisor D with $\rho(D) \geq 2$. Then we have $x_1 + x_2 + ax = 0$, with corresponding torus-invariant divisors D_1 and D_2. If for all other torus-invariant divisors E, $E^2 \cdot D \geq -1$, then*

$$D_1^3 \leq 2aD_1^2 \cdot D \cdot D - a^2 D_1 \cdot D^2.$$

Proof. Suppose we have $x_1 + x_2 + ax = 0$. Let y_i and z_j be the primitive vectors corresponding to E_i and F_j, respectively. There exists a maximal cone spanned by $\{x, x_1, y_1\}$ and using the dual functional $\hat{y}_1$, we have $\hat{y}_1(x) = \hat{y}_1(x_1) = 0$. Then, we have $\hat{y_1}(y_i) > 0$ and $\hat{y}_1(z_j) < 0$. for all i, j. We obtain numerical relation, locally near D,

$$\sum a_i' E_i - \sum b_j' F_j \equiv 0.$$

with $a_i' > 0$ and $b_j' > 0$. The functional $\hat{x}_1$ yields the relation, locally near D,

$$D - aD_1 + \sum a_i'' E_i + \sum b_j'' F_j \equiv 0,$$

for some a_i'' and b_j''.

Multiplying the first equation by $n > 0$, and adding it to the second, we obtain

$$D - aD_1 + \sum (na_i' + a_i'')E_i - \sum (nb_i' - b_i'')F_i \equiv 0.$$

Choosing n large enough, and setting $a_i = na'_i + a''_i$ and $b_j = nb'_i - b''_i$, we can ensure that $a_i > 0$ and $b_j > 0$ for all i, j, and we get

$$D - aD_1 \equiv -\sum a_i E_i + \sum b_j F_j.$$

with all a_i, b_j positive. Squaring, we obtain

$$\begin{aligned} D^2 - 2aD \cdot D_1 + a^2 \cdot D_1^2 \equiv \sum a_i^2 E_i^2 + 2 \sum a_i a_{i+1} E_i E_{i+1} \\ + \sum b_i^2 F_i^2 + 2 \sum b_i b_{i+1} F_i F_{i+1}. \end{aligned}$$

Dotting with D,

$$\begin{aligned} D^3 - 2aD^2 \cdot D_1 + a^2 D \cdot D_1^2 = \sum a_i^2 E_i^2 \cdot D + 2 \sum a_i a_{i+1} \\ + \sum b_i^2 F_i^2 \cdot D + 2 \sum b_i b_{i+1}. \end{aligned}$$

By assumption, for all i, j, $E_i^2 \cdot D \geq -1$ and $F_j^2 \cdot D \geq -1$.

If, for any i, $E_i^2 \cdot D_1 = 0$, there is only one E_i, and there are no $a_i a_{i+1}$, so

$$\sum a_i^2 E_i^2 \cdot D = 0.$$

If not, there are exactly two E_i, both with self-intersection -1. Then,

$$\begin{aligned} \sum a_i^2 E_i^2 \cdot D + 2 \sum a_i a_{i+1} &= -a_1^2 + 2a_1 a_2 - a_2^2 \\ &= -(a_1 - a_2)^2 \\ &\leq 0. \end{aligned}$$

Similarly,

$$\sum b_i^2 F_i^2 \cdot D + 2 \sum b_i b_{i+1} \leq 0.$$

So,

$$\begin{aligned} D^3 - 2aD^2 \cdot D_1 + a^2 D \cdot D_1^2 \leq 0, \text{ and} \\ D^3 \leq 2aD^2 \cdot D_1 - a^2 D \cdot D_1^2. \end{aligned}$$

□

Remark 6.5. *The role of D and E in the corollary are completely symmetric, so*

$$D^3 \leq 2aD^2 \cdot D_2 - a^2 D \cdot D_2^2.$$

Proposition 6.6 (Centrally Symmetric Relation of Degree 2)**.** *If X is a smooth toric Fano threefold with centrally symmetric relation of degree 2, with corresponding divisors D and E, then there exists D_i meeting both D and E,*

$$\gamma_2(X) \cdot D_i \leq D_i^3 \leq 0,$$

and X is not γ_2-positive.

Proof. Suppose X is a smooth toric Fano threefold with a centrally symmetric relation of degree 2, $x + y = 0$. Let $\{x_i\}$ be the primitive vectors adjacent to x and $\{y_j\}$, the primitive vectors adjacent to y. Then, By Lemma 1.37, $|\{x_i\}| = |\{y_j\}|$, and we may reorder the indices so that for each i, either $x_i = y_i$, or $x_i + y = y_i$, and x_i is adjacent to y_i. By Proposition 1.38, we have that for at least one i, $x_i = y_i$. Then, by Lemma 6.3, if x_i meets x_{i+1}, but not y_{i+1}, $D_i \cdot D_{i+1}^2 = 0$, otherwise, $x_{i+1} \neq y_{i+1}$ and x_i meets both x_{i+1} and y_{i+1}, and

$$\begin{aligned} D_i \cdot D_{i+1}^2 &= D_i \cdot E_{i+1}^2 \\ &= -1. \end{aligned}$$

Thus, the hypotheses of Corollary 6.4 are satisfied with $a = 0$, thus $D_i^3 \leq 0$. Finally, $\rho(D_i) \geq 2$, so

$$\gamma_2(X) \cdot D_i \leq D_i^3 \leq 0.$$

□

Proposition 6.7. *Suppose X has a centrally symmetric relation of degree 3, but not a centrally symmetric relation of degree 2. If X is Fano, then for for each D_i, there are at most two torus-invariant divisors on either side of the relation restricted to D_i.*

Proof. We assume X does not have a centrally symmetric relation of degree 2. Otherwise, X is not γ_2-positive by Proposition 6.6.

Observe that x_1, x_2, x_3 lie in a plane containing the origin. Therefore, there must be at least one vector on either side of this plane.

Recall by Proposition 1.35, since $\dim(X)$ is odd, $\rho(X) \leq 2n - 1 = 5$. Then there are at most 4 vectors y_j on one side of the plane spanned by the x_i. In that case there is just one z on the other. But then no y_j meets z, so for each i, we have $y_j + z = 0$ or $y_j + z = w_j$. We assumed that there are no centrally symmetric relations of degree 2, so $y_j + z = w_j$ for each j. The only vectors meeting both w and one of the y_j, are the x_i. So each w_j is some x_i. But there are only three w_j and four y_j, so, without loss of generality,

$$y_1 + z = y_2 + z = x_i.$$

A contradiction, so there can be at most three vectors on each side. In that case, suppose we have x_1 meeting all three y_1, y_2, y_3.

If y_1 does not meet y_3, then $y_1 + y_3 = w$, where w must be y_2, otherwise we have 4 vectors on one side. Then, we have $y_1 + y_3 = y_2$, but, without loss of generality, y_3 does not meet x_2. So $y_3 + x_2 = w$ which again must be one of the y_j, but y_1 does not meet y_3, so $w = y_2$, but $y_3 + y_1 = y_2$, a contradiction.

If y_1 *does* meet y_3, then without loss of generality, y_1 meets x_2, y_3 meets x_3, and y_3 does not meet x_2, so we have $y_3 + x_2 = y_j$, where y_j meets both y_3 and x_2. y_2 cannot meet x_2, so $y_j = y_1$, and $y_3 + x_2 = y_1$. Then, since y_2 does not meet x_2, we have $y_2 + x_2 = y_j$, where y_j meets both y_2 and x_2. y_3 does not meet x_2, so we must have $y_2 + x_2 = y_1$, but $y_3 + x_2 = y_1$, a contradiction.

So no x_i can meet all three y_i; in other words, every x_i meets at most two of the y_i. ☐

Lemma 6.8. *Suppose X has a centrally symmetric relation of degree 3, with associated torus-invariant divisors D_1, D_2, and D_3, then*

$$\sum_{i \neq j} (D_i^2 \cdot D_j) \leq 0.$$

Proof. On D_i, we $x_{i-1} + x_{i+1} = 0$ so D_{i-1} and D_{i+1} are the strict transforms of the positive and negative curves of a Hirzebruch surface, $\mathbb{F}_a$. On $\mathbb{F}_a$, the sums of the self-intersections are 0. D_i is obtained via succesively blowing up $\mathbb{F}_a$, and at each step the sum of the self-intersections can only decrease. So for each i,

$$D_i \cdot D_{i+1}^2 + D_i \cdot D_{i-1}^2 \leq 0.$$

Summing them up, we get

$$\begin{aligned} 0 &\geq \sum D_i \cdot D_{i+1}^2 + D_i \cdot D_{i-1}^2 \\ &= \sum_{i \neq j} (D_i^2 \cdot D_j) \end{aligned}$$

☐

Proposition 6.9 (Centrally Symmetric Relation of Degree 2)**.** *Suppose X is a smooth Fano toric threefold with a centrally symmetric relation of degree 3. Then X is not γ_2-positive.*

Proof. If X has a centrally symmetric relation of degree 2, then by Proposition 6.6, X is not γ_2-positive. Then suppose X does not have a centrally symmetric relation of degree 2. By Proposition 6.7, each D_i has at most 2 divisors on either side of the relation, so by Lemma 6.3 the hypotheses of Corollary 6.4 are satisfied with $a = 1$. So, for each $i \neq j$, we have

$$D_i^3 - 2D_i^2 \cdot D_j + D_i \cdot D_j^2 \leq 0.$$

Summing these over all $i \neq j$, we obtain

$$\begin{aligned} 0 &\geq \sum_{i \neq j} D_i^3 - 2D_i^2 \cdot D_j + D_i \cdot D_j^2 \\ &= \sum_i 2D_i^3 + \sum_{i \neq j} D_i^2 \cdot D_j. \end{aligned}$$

And, by Lemma 6.8,

$$\sum_{i \neq j} D_i^2 \cdot D_j \leq 0.$$

So,

$$\sum_i D_i^3 \leq 0.$$

Each D_i has $\rho(D_i) \geq 2$, so

$$\gamma_2(X) \cdot \sum D_i \leq \sum D_i^3 \leq 0,$$

and X is not γ_2-positive.

□

Theorem 6.10. *Let X be a smooth toric Fano threefold, with $X \ncong \mathbb{P}^3$. Then X is not γ_2-positive.*

Proof. Any such X has a centrally symmetric relation of degree 2 or 3. If X has a centrally symmetric relation of degree 2, then the result follows from Proposition 6.6. Otherwise, if X only has a centrally symmetric relation of degree 3, then the result follows from Proposition 6.9. □

Chapter 7.

Higher Dimensions

In this chapter we show that for a smooth toric variety X of dimension at least 4, the behavior of $\gamma_2(X)$ depends on the torus-invariant threefolds of X, and are able to show that under certain assumptions, smooth toric Fano varieties with centrally symmetric relations of degrees 2 or n, are not γ_2-positive.

Lemma 7.1. *Suppose X is a smooth toric variety of dimension $n \geq 4$. Then any torus-invariant surface $S \subseteq X$ is of the form $S = D_1 \cdots D_{n-2}$ for torus-invariant D_i. For each i, let*

$$\zeta_i = D_1 \cdots \hat{D}_i \cdots D_{n-2}.$$

Each ζ_i is a torus-invariant threefold in X, and $D_i \cdot \zeta_i = S$ for all i, and, for each i,

$$\begin{aligned} \gamma_2(X) \cdot S &= D_i^2 \cdot S + \gamma_2(D_i) \cdot S|_{D_i} \\ &= D_i^3 \cdot \zeta_i + \gamma_2(D_i) \cdot S|_{D_i}. \end{aligned}$$

Proof. $\gamma_2(X) = \sum D_i^2$, so

$$\begin{aligned} \gamma_2(X) \cdot S &= (D_1^2 + \cdots + D_k^2) \cdot (D_1 \cdots D_{n-2}) \\ &= (D_1^3 \cdot D_2 \cdots D_{n-2}) + D_1 \cdot (D_2^2 + \cdots + D_k^2) \cdot (D_2 \cdots D_{n-2}) \\ &= D_1^2 \cdot S + \gamma_2(D_1) \cdot S|_{D_1} \\ &= D_1^3 \cdot \zeta_1 + \gamma_2(D_1) \cdot S|_{D_1}. \end{aligned}$$

□

Corollary 7.2. *Let X be a smooth toric threefold of dimesion $n \geq 4$, with torus-invariant threefold $\zeta \subseteq X$. Then for any torus-invariant divisor D meeting ζ properly,*

$$\gamma_2(X) \cdot D \cdot \zeta = S|_\zeta^3 + \gamma_2(D) \cdot S|_D.$$

Proof. Since D meets ζ properly, $\zeta = D_1 \cdots D_{n-3}$ for $D_i \neq D$ for all i. Then,

$$\begin{aligned}\gamma_2(X) \cdot D \cdot \zeta &= D^3 \cdot \zeta + \gamma_2(D) \cdot (D \cdot \zeta)|_D. \\ &= D|_\zeta^3 + \gamma_2(D) \cdot \zeta|_D \\ &= S|_\zeta^3 + \gamma_2(D) \cdot S|_D.\end{aligned}$$

□

Corollary 7.3. *Let S and ζ_i, as in the proof, then,*

$$\gamma_2(X) \cdot S = 3(2 - \rho(S)) + \sum S|_{\zeta_i}^3.$$

If $\rho(S) \geq 2$, then

$$\gamma_2(X) \cdot S \leq \sum S|_{\zeta_i}^3.$$

Proof. First note, that is X is a threefold, $\zeta_1 = X$, and

$$\gamma_2(X) \cdot S = 3(2 - \rho(S)) + S^3,$$

and if $\rho(S) \geq 2$, $\gamma_2(X) \cdot S \leq S^3$.

Then, suppose $\dim(X) \geq 4$. Then, as in the lemma, any $S = D_1 \cdot D_{n-2}$, and there are exactly $n-2$ threefolds ζ_i on X with $S \subseteq \zeta_i$. We have

$$\gamma_2(X) \cdot D_1 \cdot \zeta_1 = S|_{\zeta_1}^3 + \gamma_2(D_1) \cdot S|_{D_1}.$$

But, for all $i > 1$, $\zeta_i \subseteq D_1$. So the torus-invariant threefolds in D_1 containing $S|_{D_1}$ are exactly the $\zeta_i|_{D_1}$. Then, by induction, we have the result. □

Proposition 7.4. *Suppose we have a smooth toric variety X of dimension n with a torus-invariant surface $S \subseteq X$, with $S \cong \mathbb{P}^2$. Let $S = D_1 \cdots D_{n-2}$ and $\zeta_i = D_i \cdots \hat{D}_i \cdots D_{n-2}$. S meets exactly three divisors E_1, E_2, and E_3. Let y_i be the corresponding primitive vectors, then for each i, we have*

$$y_1' + y_2' + y_3' + ax_i' = 0.$$

Where the y_i' and x_i' correspond to the restrictions of E_i and ζ_i to D_i. Then,

$$\gamma_2(X) \cdot S = 3 + \sum a_i^2.$$

Proof. If $n = 3$, then by Proposition 4.3, $\gamma_2(X) \cdot S = 3 + a^2$.

Then suppose $n \geq 4$. By the corollary, since $\rho(S) = 1$, we have

$$\gamma_2(X) \cdot S = 3 + \sum S|_{\zeta_i}^3.$$

Then, where on each ζ_i we have the primitive relation

$$y_1' + y_2' + y_3' + a_i x_i' = 0.$$

Where the y_i' and x_i' correspond to the restrictions of E_i and S to D_i. Therefore, we have for all i, $S|_{\zeta_i}^3 = a_i^2$, again by Proposition 4.3. Then,

$$\gamma_2(X) \cdot S = 3 + \sum a_i^2.$$

□

Example 7.5. *If $X \cong \mathbb{P}^n$, then every $\zeta_i \cong \mathbb{P}^3$, so $a_i = 1$ for all i. Then, $\gamma_2(S) = n + 1$.*

We prove the following for the blowup at a point, in any dimension n.

Proposition 7.6. *Let S be a torus invariant subsurface of a smooth toric variety of dimension n. Let $\pi : Y \to X$ be the blowup of X at a torus-invariant point p. Let $\tilde{S} = \pi_*^{-1}(S)$. Then*

$$\gamma_2(Y) \cdot \tilde{S} = \begin{cases} \gamma_2(X) \cdot S - (n+1), & \text{if } p \in S, \\ \gamma_2(X) \cdot S, & \text{otherwise.} \end{cases}$$

Proof. Let $S = D_1 \dots D_{n-2}$, and let $p \in S$ *i.e.* $p = D_1 \dots D_n$, for some D_{n-1} and D_n. Then, let

$$\pi : \tilde{X} = \mathrm{Bl}_p X \to X,$$

and let $E \subseteq \tilde{X}$ be the exceptional divisor, and $\tilde{S}$ the strict transform of S. Denote all other divisors besides the D_i by F_j, where $\pi^* F_j = \tilde{F}_j$ for all j. We note that

$$\pi^*(D_i) = \tilde{D}_i + E$$

Then

$$\begin{aligned}
\gamma_2(\tilde{X}) \cdot \tilde{S} &= \left(\sum \tilde{D}_i^2 + \sum \tilde{F}_i^2 + E^2 \right) \cdot \tilde{S} \\
&= \left(\sum (\pi^*(D_i^2) - E) + \sum \tilde{F}_i^2 + E^2 \right) \cdot \tilde{S} \\
&= \sum \pi^*(D_i)^2 \cdot S + (n+1) E^2 \cdot \tilde{S} \\
&= \gamma_2(X) \cdot S + (n+1)(E|_{\tilde{S}})^2. \\
&= \gamma_2(X) \cdot S - (n+1).
\end{aligned}$$

If $S \subseteq E$, then without loss of generality $S = E \cdot D_1 \cdots D_{n-3}$. Let $\zeta = D_1 \cdots D_{n-3}$.

$$\gamma_2(\tilde{X}) \cdot S = E|_\zeta^3 + \gamma_2(E) \cdot \zeta|_E.$$

but, $E|_\zeta$ is the exceptional divisor of a blowup at a point, so by Proposition 5.1, $E^3 \cdot D_1 \cdots D_{n-3} = 1$. $E \cong \mathbb{P}^{n-1}$, so by Example 7.5, $\gamma_2(E) \cdot \zeta|_E = n$. Then,

$$\gamma_2(\tilde{X}) \cdot S = n + 1.$$

□

Proposition 7.7 (Centrally Symmetric Relations of Degree 2)**.** *Suppose X is a smooth toric variety of dimension $n \geq 3$ with a centrally symmetric relation of degree 2 with corresponding divisors D_1 and D_2. If there exists a torus-invariant surface S intersection both D_1 and D_2 properly, and if for any other torus-invariant divisor E, $E|_S^2 \geq -1$, Then*

$$\gamma_2(X) \cdot S \leq 0.$$

Proof. Since S intersects both D_1 and D_2 properly, for any ζ_i containing S, the relation descends to a centrally symmetric relation of degree 2 on ζ_i corresponding to $D_1|_{\zeta_i}$ and $D_2|_{\zeta_i}$, and $S_|\zeta_i$ intersects both. Then, since for all other torus-invariant divisors E, $E|_S^2 \geq -1$, by Proposition 6.6

$$(S|_{\zeta_i})^3 \leq 0$$

for all i. Then, by Corollary 7.3,

$$\gamma_2(X) \cdot S \leq (S|_{\zeta_i})^3 \leq 0.$$

□

Theorem 7.8. *Suppose X is a smooth toric variety of dimension $n \geq 3$ with a centrally symmetric relation of degree n with corresponding divisors $D_1, \ldots, D_n$. For every $\beta \subseteq \{1, \ldots, n\}$ with $|\beta| = n-2$, let*

$$S_\beta = \prod_{i \in \beta} D_i.$$

Suppose that for every such β and every torus-invariant divisor E not among the D_i, that $E|_{S_\beta}^2 \geq -1$, Then

$$\gamma_2(X) \cdot \sum_\beta S_\beta \leq 0.$$

Suppose X is a smooth toric variety of dimension n with a centrally symmetric relation

$$\sum_{i=1}^n x_i = 0,$$

Then for any $\alpha \subseteq \{1, \ldots, n\}$ with $|\alpha| = n - 3$, we get a torus-invariant threefold

$$\zeta_\alpha = \prod_{i \in \alpha} D_i.$$

For each α and $i \notin \alpha$, we get a torus-invariant surface

$$S_{\alpha,i} = S_\alpha \cdot D_i.$$

Then ζ_α has a centrally symmetric relation of degree three and each $S_{\alpha,i}$ has a centrally symmetric relation of degree 2, all coming from the restrictions of the remaining D_j. Suppose for all other torus-invariant E, and for all i, $E|^2_{S_{\alpha,i}} \geq -1$. Then, by Proposition 6.9

$$\sum_{i \notin \alpha} (S_{\alpha,i}|_{\zeta_i})^3 \leq 0.$$

Now, letting $\beta \subseteq \{1, \dots, n\}$, with $|\beta| = n-2$, we obtain a torus-invariant surface

$$S_\beta = \prod_{i \in \beta} D_i.$$

Then every ζ containing S_β, is of the form ζ_α, for some α, and each S_β is contained in exactly $n-2$ torus-invariant threefolds, and each

Then,

$$\begin{aligned} \gamma_2(X) \cdot \sum_\beta S_\beta &\leq \sum_\beta \sum_{\alpha \subseteq \beta} S_\beta|^3_{\zeta_\alpha} \\ &\leq 0. \end{aligned}$$

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