

Lam-Williams Markov Chains on Symmetric Groups

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Abstract

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This paper reviews the current state of the Lam-Williams conjectures on a multivariate Markov chain on the symmetric group S_n . We start with Lam's work on random core partitions which led to a remarkable Markov chain on Weyl groups. Then we give a discussion on a multivariate generalization of this Markov chain in type A and state a series of related conjectures, due to Lam and Williams. Recent developments in proving the conjectures, due to Ayer and Linusson, are presented. Finally, a different perspective is offered, in which the stationary distributions are mappings from a certain simplex into a real projective space, and a formula is given for such stationary distributions.

Lam-Williams's Markov Chain on the Symmetric Groups

ANH HUYNH

This paper reviews the current state of the Lam-Williams conjectures on a multivariate Markov chain on the symmetric group S_n . We start with Lam's work on random core partitions which led to a remarkable Markov chain on Weyl groups. Then we give a discussion on a multivariate generalization of this Markov chain in type A and state a series of related conjectures, due to Lam and Williams. Recent developments in proving the conjectures, due to Ayyer and Linusson, are presented. Finally, a different perspective is offered, in which the stationary distributions are mappings from a certain simplex into a real projective space, and a formula is given for such stationary distributions.

1. MOTIVATION BY RANDOM CORE PARTITIONS

Given a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_m)$, one can attach to it $|\lambda| := \sum_i \lambda_i$ boxes arranged into rows from 1 to m so that row i contains λ_i boxes. In the English convention, the boxes are left-aligned. Such a collection of boxes is called the **Young diagram** for λ . This is an important device in combinatorial representation theory. For example, Young diagrams index irreducible representations of $GL(n)$, and a labeled version of Young diagrams can be used to write down the Schur polynomials.

DEFINITION 1.

- i. If a partition μ is contained in λ , i.e. $\mu_i \leq \lambda_i$ for all i , then one can also define a **skew Young diagram** λ/μ by removing the boxes that correspond to the Young diagram μ from the Young diagram λ .*
- ii. A skew Young diagram λ/μ is a **ribbon** if it is edge-connected and does not contain any 2×2 square.*
- iii. A Young diagram λ is called an **n -core** if one cannot take away from it any ribbons of size n from the lower-right boundary and still leave behind a valid Young diagram. The words **core partition** and **core** are also used, interchangeably.*
- iv. A box, or **cell**, in a Young diagram for λ is said to be **λ -addable**, or **addable** if the context is clear, if adding the box produces another valid Young diagram. A corner in the diagram where such a box can be added is called an **addable corner**.*

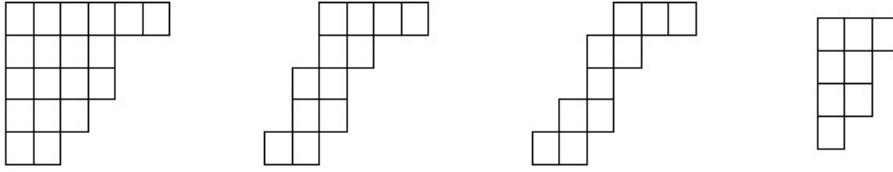
Example 2.

Figure 1. From left to right: the Young diagram for $\lambda = (6, 4, 4, 3, 2)$, the skew Young diagram λ/μ where $\mu = (2, 2, 1, 1)$, a ribbon that is also the Young diagram λ/ν for $\nu = (3, 2, 2, 1)$, the Young diagram for ν , which is also a 5-core.

The following algorithm produces all n -cores.

1. Take the empty n -core.
2. With each previously obtained n -core λ , repeat the following:
 - i. Suppose there is one addable corner of λ on diagonal d (do nothing if there isn't)
 - ii. Add all addable-corners on diagonals $d' \equiv d \pmod{n}$, this produces a new n -core λ' , (note that different addable-corners (mod n) may produce different diagrams).

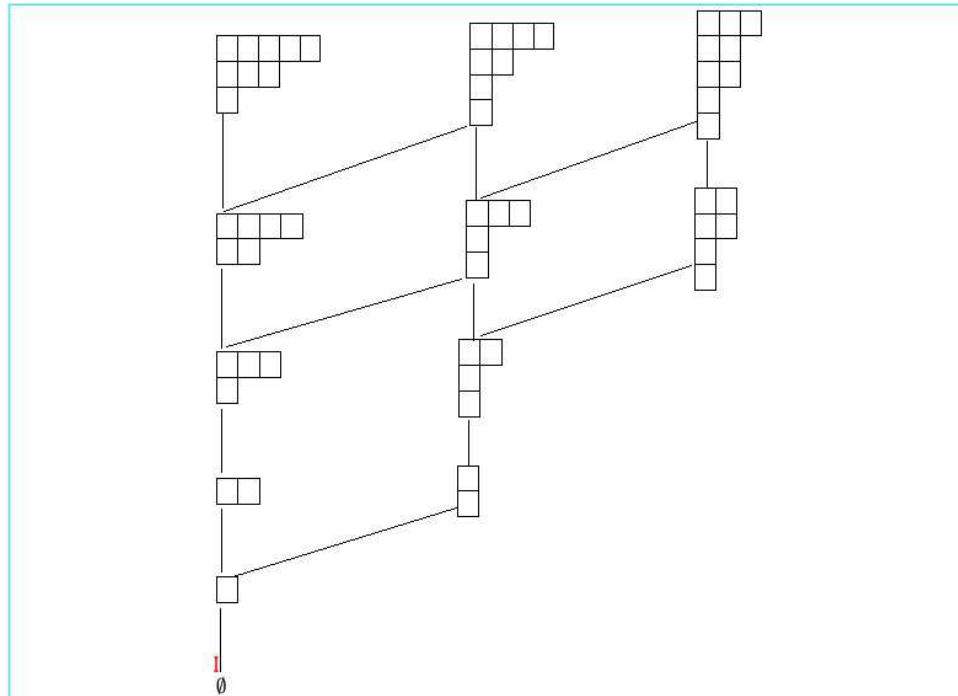
In fact, the algorithm is part of the following classical bijection. For the present purpose, the **affine Grassmannian elements** of \tilde{S}_n are those whose reduced expressions end with an s_0 on the right, see Section 2.1 for more details on \tilde{S}_n and reduced expressions.

PROPOSITION 3. (LAM) [15], [19] *There is a natural bijection between n -cores and the affine Grassmannian elements of \tilde{S}_n . The act of adding boxes along diagonals $k \pmod{n}$ corresponds to multiplying by s_k on the left.*

Example 4. The following 3-core (with the diagonals being labeled mod 3)

0	1	2	0	1	2
2	0	1	2		
1	2				

corresponds to the element $s_2s_1s_0s_2s_1s_0$ in \tilde{S}_3 .

Example 5.**Figure 2.** Growth of 3-cores

Suppose one grows n -cores according to the given algorithm, and furthermore, at each step one chooses the next n -core uniformly at random out of the possible moves, thus getting a random sequence $(\emptyset, \lambda^{(N_1)}, \lambda^{(N_2)}, \dots)$. Let $D(\lambda)$ be the curve drawing out the lower-right boundary of λ , and then scaled by the **degree** $\deg(\lambda)$, which is the length of the corresponding affine Grassmannian element, in both directions. Then Lam proved that, asymptotically, $D(\lambda^{(N)})$ has a definite shape of a piecewise-linear curve. To be precise, one needs to specify the distance between such curves D, D' by a norm $|D - D'|$, defined to be the supremum of the diagonal distance between D and D' .

THEOREM 6. (LAM) [12] *There exists a piecewise-linear curve $C = C_n$, so that for each $\varepsilon, \delta > 0$, there exists an M such that for every $N > M$, we have*

$$\text{Prob}(|D(\lambda^{(N)}) - C| > \delta) < \varepsilon$$

where $\lambda^{(N)}$ is a random core of degree N .

Example 7.

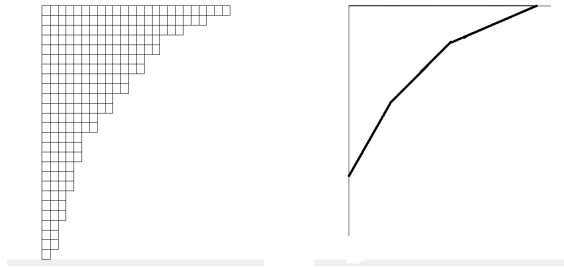


Figure 3. A random 4-core, and the curve C_4

In fact, via the bijection in Proposition 3, Theorem 6 came out as an application of Lam’s study of K -cohomology Schubert classes of the affine Grassmanian $\text{Gr}_{SL(n)}$. The precedent to this method is Okounkov’s work on random partitions using the Schur processes [21], in which Schur functions were used to construct a Plancherel measure on partitions. A Plancherel measure on n -cores was constructed by Lam using the “affine” analog of Schur functions, which are families of symmetric functions representing Schubert classes for the classical Grassmannians. One now turns to this part of Lam’s paper [12] before explaining the proof of Theorem 6.

2. A MARKOV CHAIN ON WEYL GROUPS

2.1. Affine Weyl groups and Coxeter arrangements

The purpose of this section is to fix notations. The reference for the concepts reviewed here can be found in [11]. A **Weyl group** W , associated with a crystallographic **root system** R , is a finite group of linear reflections on a real vector space V . The root system R , which is the orbit of a vector $v \in V$ under W can be decomposed into **positive** and **negative roots**:

$$R = R^+ \sqcup R^-.$$

Given R^+ , there is a set of **simple roots** that form a basis of V and from which all roots can be written as integer linear combinations. For each simple root α_i , $i \in I$, there is a **simple generator** s_i , and W is generated by these elements with braid relations imposed by the Dynkin diagram. Thus, each group element can be written as $w = s_{i_1} s_{i_2} \dots s_{i_l}$. If l is as small as can be, $s_{i_1} s_{i_2} \dots s_{i_l}$ is called a **reduced expression**, and $i_1 i_2 \dots i_l$ a **reduced word** for w . The number l is called **length** of w , denoted $l(w)$. There is a unique longest element, which is denoted by w_0 .

The **coroot** of a root α is denoted α^\vee and

$$\alpha^\vee = \frac{2}{\langle \alpha, \alpha \rangle} \alpha$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on V . One can take the \mathbb{Z} -span of the roots and obtain a **root lattice**, and similarly the \mathbb{Z} -span of the coroots is called the **coroot lattice**, denoted by Q^\vee . A coroot λ is said to be **anti-dominant** if $\langle \lambda, \alpha \rangle \leq 0$ for all positive roots $\alpha \in R^+$.

The **null root** δ_0 is the smallest nontrivial positive root that is orthogonal to all simple coroots. In particular, given labels $(a_i)_{i \in I}$ of vertices of a Dynkin diagram,

$$\delta_0 = \sum_{i \in I} a_i \alpha_i.$$

The corresponding **affine Weyl group** \widehat{W} acts on V as affine reflections. There is an **affine root system** R_{aff} consisting of elements $\{\alpha + n\delta_0 \mid \alpha \in R, n \in \mathbb{Z}\} \cup \{n\delta_0 \mid n \in \mathbb{Z} - \{0\}\}$ for \widehat{W} . The affine root $\hat{\alpha} \in R_{\text{aff}}$ is positive if and only if either $\alpha \in R^+$ and $n \geq 0$, or $\alpha \in R^-$ and $n > 0$. Thus, again, one has $R_{\text{aff}} = R_{\text{aff}}^+ \sqcup R_{\text{aff}}^-$.

The set of simple roots of the corresponding affine Weyl group \widehat{W} consist of the simple roots of R and $\alpha_0 = \delta_0 - \theta$ where θ is the highest root in R . One denotes by r_θ the reflection in the hyperplane perpendicular to θ . An affine Weyl group \widehat{W} that corresponds to W is the group that is generated by the $s_i, i \in I$ and s_0 , with similar relations as for W .

The **affine Coxeter arrangement** for \widehat{W} is the hyperplane arrangement in V consisting of hyperplanes $H_\alpha^k = \{v \in V \mid \langle v, \alpha \rangle = -k\}$, which corresponds to the affine root $\hat{\alpha} = \alpha + k\delta_0$. The connected components in this arrangement are called **alcoves**, and there is one alcove whose boundaries are hyperplanes that correspond to simple roots, which is called the **fundamental alcove**. The **Weyl chambers** are the connected components of the hyperplane arrangements consisting only of H_α^0 , and the chamber containing the fundamental alcove is called the **fundamental chamber**.

One will need the following facts. The elements of \widehat{W} are in bijection with the alcoves in the affine Coxeter arrangement for W , and thus one denotes an alcove by A_x where x is the corresponding affine Weyl group element. An element $x \in \widehat{W}$ can always be written as $x = wt_\lambda$ where $w \in W$, and t_λ is a translation by a coroot $\lambda \in Q^\vee$. Essentially, every alcove looks like an alcove that touches the origin, and can be gotten by translating this alcove to the desired position. An element $x \in \widehat{W}$ is **affine Grassmannian**, as mentioned in Section 1, if and only if its reduced expression ends with an s_0 on the right.

LEMMA 8. [13] *Let $\mu = (\mu_1, \dots, \mu_n) \in Q^\vee$ be an antidominant element of the coroot lattice. Then the n -core of the translation element $t_{(\mu_1, \dots, \mu_n)}$ has slope $(n - i)/i$ between diagonals $n\mu_i + i - 2$ and $n\mu_{i+1} + i - 2$ for $i = 1, 2, \dots, n - 1$. The slope is calculated between points of intersection of the boundary of the core and the diagonal.*

Example 9. The affine Weyl group $\widehat{W} = \tilde{S}_3$ acting on \mathbb{Z} is generated by simple transpositions s_0, s_1, s_2 , where s_i swaps $i + 3k$ and $i + 1 + 3k$ for all $k \in \mathbb{Z}$. So $s_1(7) = 8$ and $s_1(8) = 7$. The reflection r_θ is simply the transposition (13).

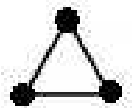


Figure 4. The Dynkin diagram for \tilde{S}_3 , all the roots are given the label 1.

The positive roots of S_3 are $e_1 - e_2, e_2 - e_3$, therefore the null vector is

$$\delta_0 = (e_1 - e_2) + (e_2 - e_3) = e_1 - e_3$$

The simple coroots coincide with the simple roots themselves.

The group \tilde{S}_3 acts naturally on the plane $x_1 + x_2 + x_3 = 0$ in \mathbb{R}^3 , thus one gets a Coxeter arrangement in the plane \mathbb{R}^2 , where hyperplanes are lines.

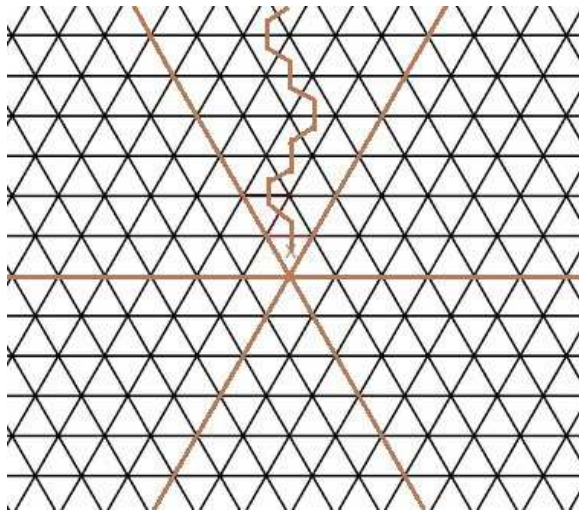


Figure 5. The Coxeter arrangement for \tilde{S}_3 , with a random walk from the fundamental alcove.

The affine Grassmannian elements are those whose reduced expression ends with s_0 on the right. This means from the fundamental alcove, one has to cross the north boundary first. Not recrossing hyperplanes means the same as having a reduced expression, and here one can see clearly that affine Grassmannian elements are confined to the fundamental chamber.

2.2. Three Markov chains

In order to use the bijection in Proposition 3 to prove Theorem 6, one defines a random walk in the affine Coxeter arrangement as follows.

DEFINITION 10. *A random walk Y in the affine Coxeter arrangement is a sequence of random variables $Y = (Y_0, Y_1, \dots, Y_l, \dots)$ where $Y_0 = A_{\text{id}}$, the fundamental alcove, and Y_{l+1} is picked uniformly at random amongst the alcoves adjacent to Y_l , with the constraint that the hyperplane separating Y_l and Y_{l+1} has not been crossed before, and that they all lie in the fundamental Weyl chamber.*

Notice that the constraint of not recrossing hyperplanes can be read from Y_l , because the hyperplanes that have been crossed are exactly those separating Y_l from Y_0 . Thus $\text{Prob}(Y_{l+1}|Y_l, \dots, Y_0) = \text{Prob}(Y_{l+1}|Y_l)$; in other words, Y is a Markov chain.

For simplicity, one modifies this Markov chain so that the transition probability is uniform. In other words, one assigns the following transition matrix

$$P_{A_{x_1}, A_{x_2}} = \begin{cases} 1/r & \text{if } x_1 \neq x_2, A_{x_1} \text{ and } A_{x_2} \text{ share an uncrossed hyperplane for boundary} \\ 1 - k/r & \text{if } x_1 = x_2 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

where $k + 1$ is the number of nonzero $P_{A_{x_1}, A_{x_2}}$ for each x_1 and $r = |I| + 1$. Notice that $k < r$ because each alcove has r hyperplanes on its boundary, and one has already been crossed. Denote by \tilde{Y} this **delayed random walk**.

The uniform transition probability allows one to simplify the problem by projecting to another Markov chain. The operation relies on Lemma 2.5 of [17].

LEMMA 11. [17] *Let Ω be the state space of a Markov chain X with transition matrix P . Let \sim be an equivalence relation on Ω with equivalence classes $\Omega^\# = \{[x] : x \in \Omega\}$ ($[x]$ denotes the equivalence class of x) and assume that P satisfies*

$$P(x, [y]) = P(x', [y])$$

whenever $x \sim x'$. Then X is a Markov chain with state space $\Omega^\#$ and transition matrix $P^\#$ defined by $P^\#([x], [y]) = P(x, [y])$.

Clearly one can project \tilde{Y} down to W . Then, one obtains the following Markov chain.

DEFINITION 12. *The process $Z = (Z_0, Z_1, \dots)$ is a Markov chain on W with transition matrix*

$$P_{w,v} = \begin{cases} 1/r & \text{if } v = s_i w < w \\ 1/r & \text{if } v = r_\theta w > w \\ 1 - k/r & \text{if } v = w \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

where $r = |I| + 1$ and $k + 1$ is the number of nonzero $P_{w,v}$ for each w .

The Markov chains defined so far are summarized in the following table.

X	Multivariate Markov chain on S_n , to be defined in Section 3
Y	Markov chain on the alcoves in the fundamental chamber
\tilde{Y}	Delayed version of Y , with positive probability of not moving
Z	Markov chain on S_n , which is the projection of \tilde{Y}

Table 1.

PROPOSITION 13. *The Markov chain Z is irreducible and aperiodic.*

That Z is aperiodic is due to the fact that from every state, the probability of staying in that state is positive. Irreducibility, i.e. strong-connectedness, is claimed by Lam to be proved in [10]. Thus Z has a unique limit stationary distribution $\zeta = (\zeta_w)_{w \in W}$.

Note 14. One uses ζ in general to denote the stationary distribution of any Markov chain on W .

2.3. Applications of the stationary distribution of Z

One now shows that asymptotically Y_N points in the direction of a certain vector.

THEOREM 15. (LAM) *The random walk Y asymptotes to one direction. Let $v(Y_i)$ be the unit vector pointing towards the mid point of Y_i , then*

$$\lim_{N \rightarrow \infty} v(Y_i) = \psi$$

where $\psi = \frac{1}{M} \sum_{w \in W: r_{\theta} w > w} \zeta_w w^{-1}(\theta^\vee)$, and M is a normalization factor.

Proof. It suffices to prove the analogous result for \tilde{Y}_N , since the difference is only on a set of measure 0 where \tilde{Y} eventually stops. Let $\kappa_{e,N}(Z)$ be the number of times the edge $e = (w \rightarrow u)$ is used in (Z_0, \dots, Z_N) . Then by the ergodic theorem for Markov chains¹, one has

$$\lim_{N \rightarrow \infty} \frac{1}{N} \kappa_{e,N}(Z) = \frac{\zeta_w}{r} \quad (3)$$

almost surely.

Denote $A_{x_i} = \tilde{Y}_i$ to be the alcove at step i in the delayed random walk, and let $w_i t_{\lambda^{(i)}} = x_i$, see Section 2.1. Recall from Proposition 3 that the next step in the walk is obtained by multiplying some s_j (or just id) to the left of x_i . Thus x_{i+1} is either x_i , or $s_j x_i$. In this form, $\lambda^{(i)}$ changes only if $x_{i+1} = s_0 x_i$. By using $s_0 = r_{\theta} t_{-\theta^\vee}$,

$$x_{i+1} = r_{\theta} t_{-\theta^\vee} w_i t_{\lambda^{(i)}} = r_{\theta} w_i t_{w_i^{-1}(-\theta^\vee)} t_{\lambda^{(i)}} = r_{\theta} w_i t_{\lambda^{(i)} + w_i^{-1}(-\theta^\vee)}.$$

Thus $\lambda^{(i+1)}$ is either the same as $\lambda^{(i)}$, or is equal to $\lambda^{(i)} + w_i^{-1}(-\theta^\vee)$. Together with (3), this shows that almost surely,

$$\lambda^{(N)} = \sum_{e \text{ where } \lambda^{(i)} \text{ changes}} \kappa_{e,N}(Z) w^{-1}(-\theta^\vee) \rightarrow \frac{1}{N} \sum_{w \text{ where } \lambda^{(i)} \text{ changes}} \frac{\zeta_w}{r} w^{-1}(-\theta^\vee).$$

1. **Ergodic theorem** (Theorem 4.16 [17]): Let f be a real-valued function defined on the state space Ω . If (X_t) is an irreducible Markov chain, then for any starting distribution μ ,

$$P_{\mu} \left(\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{s=0}^{t-1} f(X_s) = \sum_{x \in \Omega} f(x) \pi(x) \right) = 1$$

where π is the stationary distribution.

When projecting to W , the times when $x_{i+1} = s_0 x_i$ is at the transitions $w_i \rightarrow r_\theta w_i$. Thus, one concludes that eventually Y_N points in the direction of $\sum_{w \in W: r_\theta w > w} \zeta_w w^{-1}(-\theta^V)$. \square

The almost sure convergence implies convergence in probability, thus by picking a norm on V , one obtains the following corollary.

COROLLARY 16. *For each $\varepsilon > 0$ and $\delta > 0$, there is an $M = M(\varepsilon, \delta)$ so that*

$$\text{Prob}(|v(Y_N) - \psi| > \delta) < \varepsilon$$

for $N > M$.

Proof. (of Theorem 6)

By Proposition 3, one focuses on type A , where $W = S_n$. Let $\lambda^{(N)}$ denote the n -core that corresponds to Y_N . After picking the curve $C = C_n$, it suffices to show that for large enough N , $|D(\lambda^{(N)}) - C| > \delta$ implies $|v(Y_N) - \psi| > \delta'$ for some $\delta' > 0$. Suppose $Y_N = A_{x_N}$ where $x_N = w_N t_{\nu^{(N)}}$. Then for large N , $v(Y_N)$ is arbitrarily close to $v\left(A_{t_{\nu^{(N)}}}\right) = \nu^{(N)} / \|\nu^{(N)}\|$. Recall that ψ has rational coordinates, therefore there is also a translation element t_μ of $\widehat{W} = \tilde{S}_n$ which points in the direction of ψ . Then because one can pick the norm to be the absolute value of the maximum coordinate, it suffices to show that $|D(\lambda^{(N)}) - C| > \delta$ implies

$$\left| \nu_i^{(N)} / \|\nu^{(N)}\| - \mu_i / \|\mu\| \right| > \delta'$$

for some $\delta' > 0$ and some i .

Let $\lambda^{(N)}$ be the n -core that corresponds to $\nu^{(N)}$. Let C be the curve with prescribed slopes as in Lemma 8 for μ , it has $n - 1$ linear pieces. For $\nu^{(N)}$, there is a similar curve D_N and clearly $|D_N - D(\lambda^{(N)})|$ can be arbitrarily small as $N \rightarrow \infty$, because the diagram is being scaled down by $\deg(\lambda^{(N)}) > N$. Thus, by choosing N sufficiently large, $|D(\lambda^{(N)}) - C| > \delta$ implies that $|D_N - C| > 2\delta$. Since the slopes are determined by Lemma 8, the distance $|D_N - C|$ must be due to the difference between $(n\mu_i + i - 2)/\deg(\mu)$ and $(n\lambda_i^{(N)} + i - 2)/\deg(\lambda^{(N)})$, and thus $\left| \mu_i / \deg(\mu) - \lambda_i^{(N)} / \deg(\lambda^{(N)}) \right| > \delta'$ for some δ' which can be explicitly computed from δ for each n . By recalling the definition of the degree of an n -core, one completes the proof. \square

3. GENERALIZATION AND CONJECTURES

3.1. A multivariate Markov chain on the symmetric group

Lam states the following conjecture for the Markov chains Z as defined in Definition 12.

CONJECTURE 17. (LAM) *Let $W = S_n$. Then ζ_w/ζ_{w_0} is an integer for all $w \in W$, and $\zeta_{\text{id}}/\zeta_{w_0} = \prod_{k=0}^{n-1} \binom{n}{k} = \max_{w \in W} (\zeta_w/\zeta_{w_0})$.*

Building on this conjecture, especially on the integrality of ζ_w/ζ_{w_0} , Lam and Williams [14] made a series of conjectures on a generalization of this Markov chain on S_n for $n \geq 3$, which is presented in the following. The Lam-William multivariate Markov Chain $X = (X_0, X_1, \dots)$ on S_n is given by the transition matrix

$$P_{w,v} = \begin{cases} x_{w^{-1}(i+1)} & \text{if } v = (i, i+1)w < w. \\ x_{w^{-1}(1)} & \text{if } v = (1, n)w > w. \\ k_w & \text{if } v = w. \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

where k_w is chosen so that $\sum_v P_{w,v} = 1$ for each $w \in S_n$. Lam's Markov chain Z is the special case when $x_i = 1/n$ for all $i = 1, \dots, n-1$.

For Proposition 13 to be true, it only matters if an entry in the transition matrix is zero or not. Therefore, one can use the same argument and conclude that the multivariate Lam-Williams Markov chain has exactly one stationary weight (Proposition 1.1, [14]), which one scales so that

$$\zeta_{w_0} = \xi := x_1 \binom{n-1}{2} x_2 \binom{n-2}{2} \dots x_{n-2} \binom{2}{2}. \quad (5)$$

Notice that now even though one calls ζ the stationary distribution of X , in fact it is a normalized generator for the eigenspace of the eigenvector 1 of the transition matrix. From now on, ζ refers to this normalized generator.

CONJECTURE 18. (LAM-WILLIAMS) *In increasing strength:*

1. *Each ζ_w is a polynomial.*
2. *Each ζ_w is a polynomial with nonnegative integer coefficients.*
3. *Each ζ_w is a nonnegative integral sum of Schubert polynomials.*

Remark 19. Given a permutation $w = [w_1, w_2, \dots, w_n]$ in one-line notation. A **(letter) cyclic shift** $\chi(w)$ of w is one where w is transformed into $[w_1 + 1, w_2 + 1, \dots, w_n + 1] \pmod{n}$. It is clear from the definition by transition matrix that the stationary distributions at cyclic shifts of w are the same.

Remark 20. By virtue of the normalization, one can see from Gauss-Jordan elimination that all stationary distribution must thus be homogeneous polynomials of the same degree, which is $\prod_{i=2}^{n-1} \binom{i}{2}$.

Example 21. Let P be the transition matrix for the Lam-Williams Markov chain X on S_3 . Then if one orders S_3 by 123, 132, 213, 312, 231, 321, then

$$P^T - I = \begin{pmatrix} -x_1 & x_2 & x_1 & 0 & 0 & 0 \\ 0 & -x_1 - x_2 & 0 & 0 & x_1 & 0 \\ 0 & 0 & -x_1 - x_2 & x_1 & 0 & 0 \\ 0 & x_1 & 0 & -x_1 & 0 & x_2 \\ 0 & 0 & x_2 & 0 & -x_1 & x_1 \\ x_1 & 0 & 0 & 0 & 0 & -x_1 - x_2 \end{pmatrix} \quad (6)$$

and the stationary distribution is

$$\zeta_{123} = \zeta_{231} = \zeta_{312} = x_1 + x_2 = \mathfrak{S}_{s_2};$$

$$\zeta_{213} = \zeta_{132} = \zeta_{321} = x_1 = \mathfrak{S}_{s_1}.$$

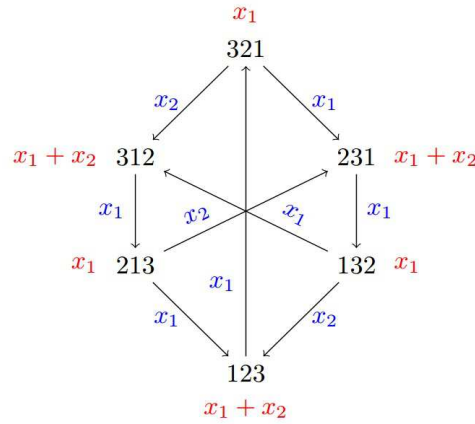


Figure 6. The Markov chain X on S_3

3.2. Schubert polynomials

Here is a brief review of Schubert polynomials, the main reference is [20]. One starts by defining the divided difference operator after Bernstein-Gelfand-Gelfand [3]

$$\partial_i := \partial_{s_i} = \frac{1 - t_i}{x_i - x_{i+1}}$$

where t_i is the operator that swaps the variable x_i and x_{i+1} . For example,

$$\partial_1(x_1^2 x_2 + x_1 x_3) = \frac{x_1^2 x_2 + x_1 x_3 - x_2^2 x_1 - x_2 x_3}{x_1 - x_2} = x_1 x_2 + x_3$$

The following identities are immediate:

$$\begin{aligned} \partial_i^2 &= 0; & \partial_i \partial_{i+1} \partial_i &= \partial_{i+1} \partial_i \partial_{i+1}; \\ \partial_i \partial_j &= \partial_j \partial_i & \text{if } |i - j| &\neq 1. \end{aligned} \quad (7)$$

Given a word $i_1 i_2 \dots i_r$, one abbreviates $\partial_{i_1} \partial_{i_2} \dots \partial_{i_r}$ by $\partial_{s_{i_1} s_{i_2} \dots s_{i_r}}$. If a, b are reduced words of the same permutation $w \in S_n$, then $\partial_a = \partial_b$. Thus, for each $w \in S_n$, the operator ∂_w can be defined unambiguously as ∂_a for any reduced word a of w . Furthermore, if a is not a reduced word, then $\partial_a = 0$. Thus,

$$\partial_u \partial_v = \begin{cases} \partial_{uv} & \text{if } l(uv) = l(u) + l(v) \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$

For each element $w \in S_n$, one defines a corresponding Schubert polynomial \mathfrak{S}_w after Lascoux-Schutzenberger [16].

$$\mathfrak{S}_{w_0} = \prod_{i=1}^n x_i^{n-i}, \quad \mathfrak{S}_w = \partial_{w^{-1}w_0} \mathfrak{S}_{w_0}.$$

Thus, one has $n!$ Schubert polynomials for each S_n .

PROPOSITION 22. *The Schubert polynomials have the following properties:*

- i. For each $w \in S_n$, \mathfrak{S}_w is a non-zero homogeneous polynomial in $\mathbb{Z}[x_1, \dots, x_{n-1}]$ of degree $l(w)$, of the form*

$$\mathfrak{S}_w = \sum_{\alpha} c_{\alpha} x^{\alpha} \quad (9)$$

summed over $\alpha \in \mathbb{N}^{n-1}$ such that $\alpha \subset \delta = (n-1, n-2, \dots, 1)$ as a partition (i.e. $\alpha_i \leq \delta_i$ for all i) and $|\alpha| = l(w)$.

- ii. \mathfrak{S}_w is symmetrical in x_i, x_{i+1} iff $w(i) < w(i+1)$.*

- iii. $\mathfrak{S}_{\text{id}} = 1$.*

- iv. Each \mathfrak{S}_w has a distinct largest term in reversed lexicographic order given by $x_1^{c_1} x_2^{c_2} \dots x_n^{c_n}$ where $c_i = \#\{j > i : w_i > w_j\}$.*

It is a well-known theorem by Borel (Proposition 8, [4]) that Schubert polynomials are the basis of the cohomology ring $H^*(GL(n)/B, \mathbb{Z})$ of the complete flag manifold. Here, B is a Borel subgroup, which one can take to be the group of invertible upper-triangular matrices. In fact, Schubert polynomials are the cohomology classes of the Schubert varieties in G/B .

It is also known that the Schubert polynomials form a basis for $\mathbb{Z}[x_1, x_2, \dots]$, therefore the essence in the strong Lam-Williams conjecture is that the coefficients are non-negative.

3.3. Other conjectures

Lam and Williams [14] also stated the following further conjectures for their Markov chain X .

CONJECTURE 23. (*Special value*)

$$\zeta_{\text{id}} = \mathfrak{S}_{123\dots n} \mathfrak{S}_{134\dots n2} \mathfrak{S}_{145\dots 23\dots} \mathfrak{S}_{1n23\dots(n-1)}. \quad (10)$$

Remark 24. One could use this conjecture for normalization, and prove instead that $\zeta_{w_0} = x_1^{\binom{n-1}{2}} x_2^{\binom{n-2}{2}} \cdots x_{n-2}$.

CONJECTURE 25. (*Monomial factor*) Assuming Conjecture 18, let $\eta(w)$ be the greatest monomial factor that can be factored out of ζ_w . Then η is an n -to-1 map from S_n to

$$\{x_1^{a_1+a_2+\dots+a_{n-2}} x_2^{a_2+\dots+a_{n-2}} \cdots x_{n-2}^{a_{n-2}} \mid (a_1, \dots, a_{n-2}) \in [n-2] \times [n-3] \times \dots \times [1]\}.$$

Moreover, $\eta(w) = x_1^{a_1+a_2+\dots+a_{n-2}} x_2^{a_2+\dots+a_{n-2}} \cdots x_{n-2}^{a_{n-2}}$ is given by

$$a_i = \#\{k \in [i+2, n] \mid w_k \in [w_i, w_{i+1}]\}$$

where $[w_i, w_{i+1}]$ denotes a cyclic subinterval of $[n]$. A cyclic subinterval means it is taken modulo n .

CONJECTURE 26. (*Special Schubert factor*) Consider the one-line notation of $w \in S_n$ with all its cyclic shifts. If there is an adjacent string of letters $1, 2$, then ζ_w is a multiple of $\mathfrak{S}_{134\dots n2}$. More generally, if there is an adjacent string of letters $1, 2, \dots, k$, then ζ_w is a multiple of $\mathfrak{S}_{1(k+1)(k+2)\dots n23\dots k}$.

These conjectures are supported with evidence up to $n=5$, see the Appendix. It should be noted that in order to write down the stationary distribution for S_n , sometimes one needs Schubert polynomials from S_m for some $m > n$. For example, in S_5 ,

$$\zeta_{51243} = \mathfrak{S}_{28513467} + \mathfrak{S}_{3751246} + \mathfrak{S}_{465123}.$$

4. THE WORK OF AYYER AND LINUSSON

This section summarizes some of the contributions by Ayer and Linusson to the Lam-Williams conjectures.

4.1. Lam-Williams Markov chain as a TASEP

In [2] Ayer and Linusson recast the Markov chain X in terms of **totally asymmetric exclusion processes** (TASEPs) on $\mathbb{Z}/N\mathbb{Z}$. Exclusion processes were first studied in biology as a prototype for the dynamics of ribosomes on RNA [18] and has since found applications in many fields, such as statistical physics [6]. In mathematics, Corteel and Williams [5] used TASEP's to give a combinatorial interpretation of the moments of Askey-Wilson polynomials.

The general exclusion process can be defined on any graph \mathcal{G} by assigning at most one particle at each vertex of \mathcal{G} and moving the particles according to a Markovian rule. In this paper one focuses on the exclusion process on $\mathbb{Z}/N\mathbb{Z}$, which is an exclusion process on a cycle of length N , with parameters $m = (m_1, \dots, m_n)$ of positive integers where $\sum_{i=1}^n m_i = N$. The state space Ω_m consists of configurations $\alpha \in \Omega_m$, each of which is an N -tuple with m_1 entries of 1's, m_2 entries of 2's, etc. Additionally, one considers the first and last entries adjacent, and this makes it similar to arranging the given numbers on a roundtable. It is easy to see that

$$|\Omega_m| = \binom{N}{m_1, \dots, m_n}.$$

The numbers $1, 2, \dots, n$ are designated classes of particles, and one denotes by α_i the class of the particle at site $i \pmod{N}$ on the ring.

Example 27. The configurations in $\Omega_{(2,1,1)}$ are $(3, 1, 1, 2), (3, 1, 2, 1), (3, 2, 1, 1)$ and all their cyclic shifts.

The evolution works as follows. Lower integers have higher priority than higher ones, and so if they are next to each other with the higher one to the left of the lower, then they can swap places, but not the other way around. To conform with Ayer-Linusson's convention, one uses the concept of exponential clock rate used in continuous Markov chain. Imagine putting alarm clocks at each of the states one can get to from the present state. The clock rings according to an exponentially distributed amount of time. Once a clock rings, one moves into the state that the clock occupies, and discards *all* clocks.

DEFINITION 28. (TASEP on $\mathbb{Z}/N\mathbb{Z}$) The **multispecies exclusion process** on Ω_m is defined by local transitions involving sites i and $i+1$ on the ring. If the current state is α , with $\alpha_i = a$ and $\alpha_{i+1} = b$, then

$$ab \rightarrow \begin{cases} ba & \text{with exponential rate 1 if } a > b, \\ ab & \text{otherwise} \end{cases}.$$

The process defined above is called totally asymmetric because there is only one possible direction of movement with larger integers going to the left. One can then define the multivariate, also called inhomogeneous by Ayer and Linusson, multispecies exclusion process on Ω_m by changing the rate 1 to be x_b whenever a swap between ab is possible. In this context, the Lam-Williams Markov chain is an inhomogeneous TASEP on $\Omega_{(1,1,\dots,1)}$, since there is exactly one element of each class in a permutation.

Remark 29. The TASEP on $\mathbb{Z}/N\mathbb{Z}$ in Definition 28 performs moves by multiplications on the right by a simple transition s_i , whereas the Lam-Williams Markov chain X uses multiplications on the left. The conjectures remain the same.

Remark 30. Let the **(position) cyclic shift** be $w = [w_1, w_2, \dots, w_n] \xrightarrow{c} c(w) = [w_2, w_3, \dots, w_n, w_1]$. Ayyer-Linusson's convention, which correspond to action on the positions, makes it clear that the stationary distribution at $c(w)$ is the same as that at w . Since Lam-Williams's convention correspond to permutation on the letters, the stationary distribution stays the same if the (letter) cyclic shift χ is as given in Remark 19.

4.2. Ferrari-Martin multiline processes and Conjecture 17

Even before Lam published his paper [12], Ferrari and Martin [9], inspired by ideas from their own work [8] and Ferrari, Fontes, Kohayakawa [7] and Angel [1], had come up with a method that can solve the first half of Conjecture 17. Essentially one analyses the moves in Ω_m by looking at elements of each class, one at a time. Formally, the state space Ω_m^{FM} of the **Ferrari-Martin multiline process** consists of cylinders of circumference N and height $n - 1$. Each site of a cylinder is either occupied (denoted by $*$) or not (denoted by \circ). The first row contains m_1 occupied sites, the second $m_1 + m_2$ occupied sites, and so on. One can think of the occupied sites on the first row as the origins of the first class particles, and the occupied sites on the second row as the origins of the first and second class particles, etc. It is clear that

$$|\Omega_m^{FM}| = \prod_{i=1}^{n-1} \binom{N}{\sum_{j=1}^i m_j}.$$

Then one defines evolution on this state space as follows. Given a configuration $\beta \in \Omega_m^{FM}$, a **ringing path** $P^{(i)}$ is a $n - 1$ -tuple of indices of sites on the rows of β , starting at site i on the bottom row. At every step, if the site is occupied, then the ringing path moves directly up to the next row, otherwise it goes diagonally once to the right. When the bell rings, for each site on the ringing path, if it is occupied and the site to its left is not, then one swaps these sites; otherwise, nothing happens. In other words, each row transforms by the two-class TASEP rule with the bell ringing at the site on the ringing path.

DEFINITION 31. *The **Ferrari-Martin multiline process** is a Markov chain on Ω_m^{FM} where the dynamics occur by ringing paths transition with exponential clock rate 1 on row $n - 1$.*

Example 32.

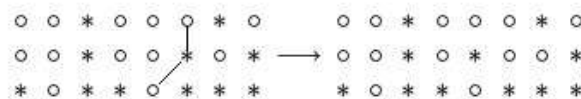


Figure 7. A ringing path and the transition after the bell rings, at site 5.

One can show that the moves are reversible, and thus by the detailed balance equation one gets a uniform stationary distribution, as proved in Theorem 3.1 of [9]. Furthermore, the Ferrari-Martin multiline process projects down (see Lemma 11) to the multispecies exclusion process Ω_m . The projection works as follows. A **bully path** is a path going through the multiline queue, always to the right or downward. Starting from an occupied site on the first row, it goes directly downward. It then moves rightward if necessary until it finds another occupied site on the second row not claimed by another bully path before it. Then it moves downward to the third row. And so on. This simply provides a rule to decide on each row of the multiline queue, which ones are the origins of first class particles, etc. Thus, one obtained a configuration in the multispecies exclusion process Ω_m in the end, since by the construction of Ω_m^{FM} there will be exactly m_1 first class particles, m_2 second class particles, etc. The fact that one can lump, i.e. that the hypothesis of Lemma 11 is satisfied is proved in Theorem 4.1 in [9].

Example 33.

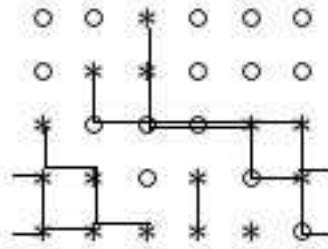
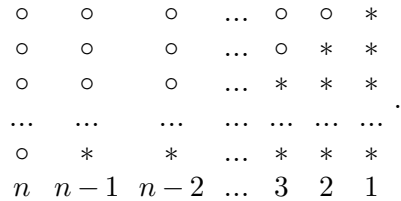


Figure 8. One configuration that corresponds to (1,2,3,4,5,6).

Proof. (First half of Conjecture 17, as implied by [9])

Since the stationary distribution of $\Omega_{(1,1,\dots,1)}^{FM}$ is uniform, each configuration has probability $p = 1/|\Omega_m^{FM}|$. Therefore, ζ_w is equal to the number of configurations that project down to w times p . There is only one configuration that projects down to w_0 , namely



Therefore ζ_w/ζ_{w_0} is always an integer. □

The latter half of the conjecture has not been proved, because it is hard in most cases to count the number of configurations that project down to the same element in S_n . Example 33 shows one possible configuration that projects down to id in S_6 , to illustrate how wild the configurations can be.

4.3. Generalization and the case $n=3$

Taking inspiration from Ferrari-Martin’s method, Ayer and Linusson gave the following generalization of Lam-Williams (middle strength) conjecture. One says that a vacancy is ***i*-covered** if it is on a bully path starting at row i .

CONJECTURE 34. (AYYER-LINUSSON) *There is a Markov chain on the state space of Ω_m^{FM} such that the stationary weight of any configuration β is given by*

$$v(C) = x_1^{C_1} x_2^{C_2} \dots x_{n-1}^{C_{n-2}} \prod_{1 \leq i < j \leq n} \left(\frac{x_j}{x_i} \right)^{z_{i,j}} \tag{11}$$

where C_j is the total number of vacancies below row j in β and $z_{i,j}$ are the number of vacancies on row j of β that are i -covered.

Remark 35. This conjecture implies the middle strength conjecture of Lam and Williams. All one needs to check is that $\sum_{i < j \leq n} z_{i,j} \leq C_i$ for all i , which is obvious, because the left hand side counts a select number of vacancies below row i , whereas the right hand side counts all of them.

Ayer and Linusson gave such a Markov chain for a three-species Ferrari-Martin model $\Omega_{(m_1, m_2, m_3)}^{FM}$. In the special case of one particle in each class, one recovers the Lam-William Markov chain for $n=3$. Observe that the configurations have two rows of size $m_1 + m_2 + m_3$ each. Suppose β is a configuration whose projection is α . If $\alpha_i = 3$, then site i on the second row of β is not 1-covered.

DEFINITION 36. *Ayer-Linusson Markov chain is a multivariate Ferrari-Martin process on $\Omega_{(m_1, m_2, m_3)}^{FM}$ where the rate of transition from β at column i is x_{α_i} if $\alpha_i = 1, 2$. If $\alpha_i = 3$, then the rate is x_1 if the 3 is 1-covered, and x_2 if it is not (this choice is inconsequential).*

Example 37.

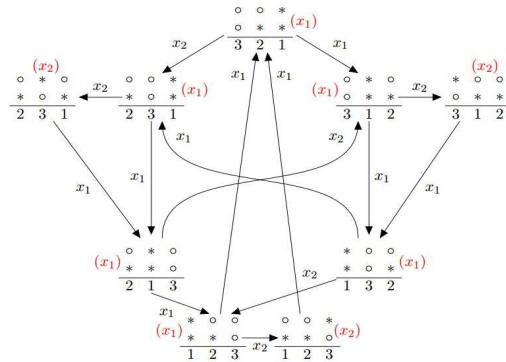


Figure 9. Ayer-Linusson Markov chain for $\Omega_{(1,1,1)}^{FM}$

Remark 38. It is hard to extend this method to the case of four classes of particles.

Remark 39. As it is, the Ferrari-Martin method seems to be suitable for the middle strength conjecture. However, it does not offer to explain the appearance of Schubert polynomials, and therefore might not be the appropriate tool for the strongest conjecture of Lam and Williams.

5. A FEW OBSERVATIONS

This section contains a few observations about the Lam-Williams conjectures by the author.

5.1. Stationary distributions as maps

As can be seen from the appendix, very few Schubert polynomials appear in the expansion of ζ_w . One is led to suspect that there might be a family of Markov chains that will yield Schubert positive stationary distribution. This section attempts to provide a framework where such a family might be realized.

Consider a Markov chain on S_n . Each row of the transition matrix corresponds to a point in an $(n! - 1)$ -dimensional vector space because the last entry is not needed. In fact, the coordinates of any such point are nonnegative and add up to at most 1, thus they are points in a $(n! - 1)$ -simplex, which is denoted by $\Delta^{n!-1}$.

DEFINITION 40. *The **Markovian space** \mathcal{M} of S_n is the space of Markov chains on S_n , considered as a subspace of $\mathbb{R}^{n!(n!-1)}$.*

Since the transition matrix has $n!$ rows, \mathcal{M} is a Cartesian product of $n! - 1$ -dimensional simplices

$$\mathcal{M} = \prod_{w \in S_n} \Delta^{n!-1}.$$

The Lam-Williams Markov chain actually lives on a smaller space. Since there are always states inaccessible to the current state at the very next move (there are only at most n arrows out of a state, and there are $n! - 1$ remaining states), there are always 0's on every row of the transition matrix. Therefore, each row in the transition matrix gives a point on the boundary of $\Delta^{n!-1}$. So let us define

$$\mathcal{H} = \prod_{w \in S_n} \partial \Delta^{n!-1}, \tag{12}$$

also as a subspace of $\mathbb{R}^{n!(n!-1)}$.

Since each $\partial\Delta^{n!-1}$ is homotopic to a $(n! - 2)$ -sphere, \mathcal{H} is homotopic to a product of spheres. The cohomology group of \mathcal{H} therefore is the exterior product of $\mathbb{Z}[x_w]/(x_w^2)$ for $w \in S_n$. Thus, one has

$$H^*(\mathcal{H}) \cong \mathbb{Z}[x_w: w \in S_n]/(x_w x_v + x_v x_w: v, w \in S_n).$$

The space \mathcal{H} , just like \mathcal{M} , acts on the vector space $\mathbb{R}^{n!}$ by performing one step of the Markov chain. For most points in \mathcal{H} , away from possibly some lower dimensional subspace, the transition probabilities from a point to any other point is positive. It easily follows that for such points, the corresponding Markov chain is aperiodic and irreducible. The unique stationary distribution thus generates a line through the origin in $\mathbb{R}^{n!}$, and one has a rational map γ from \mathcal{H} to $\mathbb{RP}^{n!}$.

Remark 41. The Lam-Williams Markov chain is simply a polynomial map Λ from Δ^n into \mathcal{H} , specified by the transition matrix. The composition of Λ and γ is given by ζ , which is expected to be Schubert positive by the Lam-Williams conjecture.

5.2. Formulas for γ and ζ

One can derive an exact formula for $\gamma: \mathcal{H} \rightarrow \mathbb{RP}^{n!}$. Let $\Pi = P^T - I$, then the stationary distribution ζ belongs to the nullspace of this matrix.

Remark 42. There is the issue of finding a convenient ordering of the elements of S_n so as to fit into the indices of the matrix Π . What one would like is to do it in such a way that when computing determinants and relevant quantities, the sign comes out naturally and clean. In the present absence of such an ordering, one designates $\text{sgn}(u)$ for the appropriate sign whenever needed, and attention must be paid when doing computations.

Convention 43. One denotes by $M|_w^v$ the matrix gotten by deleting row w and column v from the matrix M . One also uses M_w to denote row w of M and M^v to denote column v of M .

Because the stationary distributions are vectors in the one-dimensional nullspace of Π , it is perpendicular to all rows of Π . For a dense subset of \mathcal{H} , then, if one takes the first $n! - 1$ rows of Π (excluding the flows into id) and some vector not in the span of these rows, and apply the Gram-Schmidt process, the last basis vector will be in the nullspace of Π .

Define the matrix Γ by

$$\Gamma_{uw} := (\Pi_w \Pi_u^T) \tag{13}$$

Then let

$$\gamma(\xi) := \frac{1}{|\Gamma|_{\text{id}}^{\text{id}}} \begin{pmatrix} \Gamma|_{\text{id}}^{\text{id}} & \Pi|_{\text{id}} \xi \\ \Pi|_{\text{id}}^T & \xi \end{pmatrix} \tag{14}$$

be the determinantal formula of the Gram-Schmidt process for r_w , $w \in W \setminus \{\text{id}\}$ and ξ . Then for a good (generic) choice of ξ , $\gamma(\xi)$ is a basis for the nullspace of Π . Thus, for a dense

open subset of \mathcal{H} , one is safe to choose $\xi = \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}$. In short,

$$\zeta = \kappa \gamma \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix} \end{pmatrix}. \quad (15)$$

for some κ .

Even though this is an exact formula, it seems hopelessly complex and very slow computationally. However, one redeeming factor is that it is very symmetric in terms of Π . One is also free to choose from many possible ξ to tease out useful information. Furthermore, in the particular case of Lam-Williams's Markov chain, there are not many non-zero entries in the Gramian matrix Γ . For example, one can look at the entries corresponding to id , and find that

$$\begin{aligned} \zeta_{\text{id}} &= \kappa \frac{1}{|\Gamma|_{\text{id}}^{\text{id}}|} \begin{pmatrix} \Gamma|_{\text{id}}^{\text{id}} & \Pi^{\text{id}}|_{\text{id}} \\ (\Pi^{\text{id}}|_{\text{id}})^T & 1 \end{pmatrix} \\ \kappa &= \frac{\zeta_{\text{id}} |\Gamma|_{\text{id}}^{\text{id}}|}{|\Gamma|_{\text{id}}^{\text{id}}| + \text{sgn}((1, n)) x_1^2 |\Gamma|_{\text{id}, (1n)}^{\text{id}, (1n)}|} \end{aligned} \quad (16)$$

Thus, assuming Conjecture 23, or by Remark 24, one can also compute ζ_w as follow

$$\zeta_w = \frac{\mathfrak{S}_{12\dots n} \mathfrak{S}_{134\dots n2\dots} \mathfrak{S}_{1n2\dots(n-1)}}{|\Gamma|_{\text{id}}^{\text{id}}| + \text{sgn}((1, n)) x_1^2 |\Gamma|_{\text{id}, (1n)}^{\text{id}, (1n)}|} \sum_{\text{suitable } u} \text{sgn}(u, (1, n)) x_i x_1 |\Gamma|_{\text{id}, (1n)}^{\text{id}, u} \quad (17)$$

For instance,

$$\begin{aligned} x_1^{\binom{n-1}{2}} x_2^{\binom{n-2}{2}} \dots x_{n-2} &= \frac{\mathfrak{S}_{12\dots n} \mathfrak{S}_{134\dots n2\dots} \mathfrak{S}_{1n2\dots(n-1)}}{|\Gamma|_{\text{id}}^{\text{id}}| + \text{sgn}((1, n)) x_1^2 |\Gamma|_{\text{id}, (1n)}^{\text{id}, (1n)}|} \times \\ &\quad \left(-\sum_i x_i x_1 |\Gamma|_{\text{id}, (1n)}^{\text{id}, w_0} + \sum_i \text{sgn}(s_i w_0, (1, n)) x_i x_1 |\Gamma|_{\text{id}, (1n)}^{\text{id}, s_i w_0} \right) \end{aligned} \quad (18)$$

This can be verified directly by computer for small n . From here, one might hope to prove that the right hand side of (16) satisfy the Lam-Williams conjectures; one might also prove (17) instead of Conjecture 23. In general, without knowing the Lam-Williams conjectures, it is very surprising that one has the identity in (17).

5.3. Multi-Schur functions

For the available data, that is, up to $n=5$, the stationary distributions expand into Schubert polynomials that correspond to **vexillary permutations**. This is far from being true for all n and no proof in that direction is known, but if true it opens up many possibilities.

Recall that vexillary permutations are those that avoid the pattern 2143. The following formula is well-known.

THEOREM 44. (2.3 of [22]) *Let w be a vexillary permutation with shape $\lambda = (\lambda_1, \dots, \lambda_{l(w)})$ and flag $\phi = (\phi_1, \dots, \phi_{l(w)})$. Then \mathfrak{S}_w is a multi-Schur function, as given by*

$$\mathfrak{S}_w(x) = s_\lambda(x_1 + \dots + x_{\phi_1}, \dots, x_1 + \dots + x_{\phi_{l(w)}})$$

These multi-Schur functions can be defined using **Jacobi-Trudi determinant formula**, as follow

$$s_\lambda(x_1 + \dots + x_{\phi_1}, \dots, x_1 + \dots + x_{\phi_{l(w)}}) = \det (h_{\lambda_i - i + j}(x_1 + \dots + x_{\phi_i}))_{1 \leq i, j \leq n} \quad (19)$$

where h_m denote the homogeneous symmetric polynomials of degree m , see Chapter 3 [20]. This is a special case of formula 3.1 of [20]. In comparison with Section 5.2, one should note that, the matrices involved in (19) are $n \times n$ matrices, where most entries are nonzero polynomial of high degrees. In contrast, the matrices involved in section 5.2 are $n! \times n!$ matrices and most of their entries are 0 with the rest being quadratic polynomials.

6. APPENDIX

6.1. Lam-Williams Markov chain on S_4

Since one can lump states that are cyclic shifts of one another, it is enough to write down the Markov chain for permutations in S_4 whose one-line notations start with 4. With the ordering 4123, 4132, 4213, 4231, 4312, 4321, the matrix Π is given by

$$\begin{pmatrix} -x_1 & x_3 & x_2 & 0 & x_1 & 0 \\ 0 & -x_1 - x_3 & 0 & x_2 & 0 & x_1 \\ 0 & x_1 & -x_1 - x_2 & 0 & x_2 & 0 \\ x_1 & 0 & 0 & -x_1 - x_2 & 0 & x_2 \\ 0 & 0 & 0 & x_1 & -x_1 - x_2 & x_3 \\ 0 & 0 & x_1 & 0 & 0 & -x_1 - x_2 - x_3 \end{pmatrix}.$$

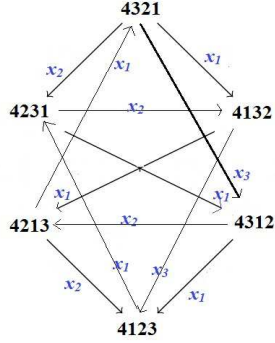


Figure 10. Lam-Williams Markov chain X on S_4/χ , where χ is the cyclic shift in Remark 19.

The stationary distribution is provided in the following table.

$$\begin{aligned}
 \zeta_{4123} &= \mathfrak{S}_{15324} + \mathfrak{S}_{2431} \\
 \zeta_{4132} &= \mathfrak{S}_{25134} \\
 \zeta_{4213} &= \mathfrak{S}_{3241} + \mathfrak{S}_{3412} + \mathfrak{S}_{4213} \\
 \zeta_{4231} &= \mathfrak{S}_{2431} + \mathfrak{S}_{3412} + \mathfrak{S}_{4132} \\
 \zeta_{4312} &= \mathfrak{S}_{3241} + \mathfrak{S}_{4132} \\
 \zeta_{4321} &= \mathfrak{S}_{4213}
 \end{aligned}$$

6.2. Lam-Williams Markov chain on S_5

It is not possible to include in this paper the 24×24 transition matrix, nor a sketch of the Markov chain on S_5 , even with lumping. Schubert expansion of stationary distribution in S_5 are included in the following table.

$$\begin{aligned}
 \zeta_{51234} &= \mathfrak{S}_{18532467} + 2\mathfrak{S}_{2753146} + \mathfrak{S}_{28431567} + 2\mathfrak{S}_{365214} + \mathfrak{S}_{3742156} + \mathfrak{S}_{3751246} + \mathfrak{S}_{456213} + \\
 &\quad \mathfrak{S}_{465123} \\
 \zeta_{51243} &= \mathfrak{S}_{28513467} + \mathfrak{S}_{3751246} + \mathfrak{S}_{465123} \\
 \zeta_{51324} &= \mathfrak{S}_{3742156} + \mathfrak{S}_{38241567} + \mathfrak{S}_{38412567} + \mathfrak{S}_{4723156} + \mathfrak{S}_{4731256} + \mathfrak{S}_{48213567} \\
 \zeta_{51342} &= \mathfrak{S}_{28413567} + \mathfrak{S}_{3741256} + \mathfrak{S}_{463125} + \mathfrak{S}_{4721356} + \mathfrak{S}_{48123567} \\
 \zeta_{51423} &= \mathfrak{S}_{38241567} + \mathfrak{S}_{4723156} + \mathfrak{S}_{48132567} \\
 \zeta_{51432} &= \mathfrak{S}_{48213567} \\
 \zeta_{52134} &= \mathfrak{S}_{456213} + \mathfrak{S}_{463215} + \mathfrak{S}_{4723156} + \mathfrak{S}_{536214} + \mathfrak{S}_{54321} + \mathfrak{S}_{546123} + 2\mathfrak{S}_{562314} + \\
 &\quad \mathfrak{S}_{563124} + \mathfrak{S}_{5713246} + \\
 &\quad + \mathfrak{S}_{634215} + 2\mathfrak{S}_{642315} + \mathfrak{S}_{643125} + \mathfrak{S}_{7324156} + \mathfrak{S}_{7413256} \\
 \zeta_{52143} &= \mathfrak{S}_{546123} + \mathfrak{S}_{563124} + \mathfrak{S}_{5721346} + \mathfrak{S}_{643125} + \mathfrak{S}_{7423156} \\
 \zeta_{52314} &= \mathfrak{S}_{365214} + \mathfrak{S}_{3751246} + \mathfrak{S}_{465123} + \mathfrak{S}_{562314} + \mathfrak{S}_{563124} + \mathfrak{S}_{5721346} + \mathfrak{S}_{625314} + \\
 &\quad \mathfrak{S}_{635124} + \mathfrak{S}_{7251346}
 \end{aligned}$$

$$\begin{aligned}
\zeta_{52341} &= \mathfrak{S}_{2753146} + \mathfrak{S}_{365214} + \mathfrak{S}_{3742156} + \mathfrak{S}_{3751246} + \mathfrak{S}_{3751246} + \mathfrak{S}_{456123} + \mathfrak{S}_{463215} + \\
&\quad \mathfrak{S}_{465123} + \\
&\quad + \mathfrak{S}_{4723156} + \mathfrak{S}_{536214} + \mathfrak{S}_{562314} + \mathfrak{S}_{563124} + \mathfrak{S}_{5713246} + \mathfrak{S}_{625314} + \mathfrak{S}_{634215} + \\
&\quad \mathfrak{S}_{635124} + \mathfrak{S}_{7153246} + \mathfrak{S}_{7243156} \\
\zeta_{52413} &= \mathfrak{S}_{3742156} + \mathfrak{S}_{463215} + \mathfrak{S}_{4723156} + \mathfrak{S}_{4731256} + \mathfrak{S}_{562314} + \mathfrak{S}_{5713246} + \mathfrak{S}_{634215} + \\
&\quad \mathfrak{S}_{642315} + \mathfrak{S}_{7243156} + \mathfrak{S}_{7341256} + \mathfrak{S}_{7413256} \\
\zeta_{52431} &= \mathfrak{S}_{4723156} + \mathfrak{S}_{4731256} + \mathfrak{S}_{5721346} + \mathfrak{S}_{7324156} + \mathfrak{S}_{7341256} + \mathfrak{S}_{7421356} \\
\zeta_{53124} &= \mathfrak{S}_{456213} + \mathfrak{S}_{463215} + \mathfrak{S}_{536214} + \mathfrak{S}_{54321} + \mathfrak{S}_{546123} + \mathfrak{S}_{562314} + \mathfrak{S}_{563124} + \mathfrak{S}_{625314} + \\
&\quad + 2\mathfrak{S}_{634215} + \mathfrak{S}_{635124} + 2\mathfrak{S}_{642315} + \mathfrak{S}_{643125} + \mathfrak{S}_{651324} + \mathfrak{S}_{7243156} + \mathfrak{S}_{7341256} + \\
&\quad \mathfrak{S}_{7413256} \\
\zeta_{53142} &= \mathfrak{S}_{536214} + \mathfrak{S}_{546123} + \mathfrak{S}_{562314} + \mathfrak{S}_{563124} + \mathfrak{S}_{634215} + \mathfrak{S}_{635124} + \mathfrak{S}_{642315} + \mathfrak{S}_{643125} + \\
&\quad + \mathfrak{S}_{652134} + \mathfrak{S}_{7324156} + \mathfrak{S}_{7341256} + \mathfrak{S}_{7421356} \\
\zeta_{53214} &= \mathfrak{S}_{642315} + \mathfrak{S}_{643125} + \mathfrak{S}_{652134} + \mathfrak{S}_{7421356} \\
\zeta_{53241} &= \mathfrak{S}_{54321} + \mathfrak{S}_{562314} + \mathfrak{S}_{634215} + 2\mathfrak{S}_{642315} + \mathfrak{S}_{643125} + \mathfrak{S}_{651324} + \mathfrak{S}_{7324156} + \mathfrak{S}_{7413256} \\
\zeta_{54123} &= \mathfrak{S}_{7243156} + \mathfrak{S}_{456213} + \mathfrak{S}_{536214} + \mathfrak{S}_{546123} + 2\mathfrak{S}_{625314} + \mathfrak{S}_{634215} + \mathfrak{S}_{635124} + \\
&\quad \mathfrak{S}_{7153246} \\
\zeta_{54132} &= \mathfrak{S}_{546123} + \mathfrak{S}_{635124} + \mathfrak{S}_{7251346} \\
\zeta_{54213} &= \mathfrak{S}_{634215} + \mathfrak{S}_{642315} + \mathfrak{S}_{643125} + \mathfrak{S}_{7324156} + \mathfrak{S}_{7341256} + \mathfrak{S}_{7421356} \\
\zeta_{54231} &= \mathfrak{S}_{54321} + \mathfrak{S}_{634215} + \mathfrak{S}_{642315} + \mathfrak{S}_{643125} + \mathfrak{S}_{7243156} + \mathfrak{S}_{7341256} + \mathfrak{S}_{7413256} \\
\zeta_{54312} &= \mathfrak{S}_{642315} + \mathfrak{S}_{7324156} + \mathfrak{S}_{7413256} \\
\zeta_{54321} &= \mathfrak{S}_{7421356}
\end{aligned}$$

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