New Approaches to Institutional Portfolio Performance Attribution and Private Equity Risk

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Abstract

New Approaches to Institutional Portfolio Performance Attribution and Private Equity Risk

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Economics

This work is comprised of three separate works which are summarized as follows:

1.) We develop a new method to estimate private equity funds' market beta from cash flows. Our methodology extends the widely known public market equivalent calculation to a cross-sectional regression. By simply regressing funds' internal rates of return on their paired market internal rates of return, we are able to estimate private equity market betas. For venture funds, we find a high market beta. For buyout funds, we find a low beta. Though we have a small sample, our results fall in line with those recently reported in the literature.

2.) We extend the Brinson et al. (1986) performance attribution framework to support institutional-specific requirements, including a hierarchical structure and multiple benchmark styles. By attributing

performance to four statistics (e.g. *manager alpha*, *portfolio construction*, *tactical* and *strategic*), we are able to remove the *interaction* term, which is a commonly referred to shortcoming of Brinson attribution. We subsequently modify Frongello (2002) linking to produce pro-rated multiperiod attributes which sum to meaningful statistics using notional portfolios of multitier excess returns.

3.) A number of methods have been developed to link single-period arithmetic attribution results. We present the first institutional portfolio empirical study comparing the most referenced methods for producing additive multiperiod attributes from their single-period counterparts. While our findings suggest the methods typically produce similar results, we find a pattern in the way the methods' results relate to one another. We find the Modified Frongello Method and Cariño Method to produce nearly identical results, the Frongello and Cariño methods to cluster and the Naïve and Menchero methods to be outliers.

Dedicated to my Grandma

Estimating Private Equity Market Beta Using Cash Flows: A Cross-Sectional Regression of Fund-Market Paired Internal Rates of Return

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Abstract: We develop a new method to estimate private equity funds' market beta from cash flows. Our methodology extends the widely known public market equivalent calculation to a cross-sectional regression. By simply regressing funds' internal rates of return on their paired market internal rates of return, we are able to estimate private equity market betas. For venture funds, we find a high market beta. For buyout funds, we find a low beta. Though we have a small sample, our results fall in line with those recently reported in the literature.

Introduction

Private equity (PE) is a major institutional asset class, representing a significant investment by colleges, foundations, pension funds and sovereign wealth funds. Optimal portfolio allocations are illusive as a lack of transaction-based performance measures make risk, return and asset class covariances difficult to estimate. The statistical analysis of PE time series based upon general partner (GP) reported returns are subject to asymmetric information and stale pricing biases.

PE investment is typically structured as a limited partnership where the PE firm acts as the GP. Limited partners (LPs) commit to provide a certain total amount of capital to the fund. Over the lifetime of the fund, the GP agrees to invest the committed capital and distribute returns. The lifetime of the fund is usually between 10 and 12 years, during which in the first half the GP draws down the capital and then begins to distribute returns as investments mature.

As the individual assets within PE funds are not publicly traded, pricing is not optimal. Infrequent and appraisal-based pricing makes times series stale and subject to biases such as autocorrelation. Additionally, information asymmetry between the LP and GP allows for the misreporting of performance.

Ljungqvist and Richardson (2003) were the first to analyze PE returns-based on cash flows. Using a single institutional investor's portfolio, their data is not subject to survivorship bias. Perhaps of greater importance, cash flows are not subject to gaming as there is little to no information asymmetry. By calculating internal rates of return (IIRs) for the funds, one can reduce noise of fund performance by NAV manipulation. Many methods, including the seminal work of Kaplan and Schoar (2005), rely upon cash flow reliability to provide the foundation to their PE performance measurement techniques.

We use PE cash flows from a single institutional investor between 1990 and 2012 to get more precise estimates of market beta. Our methodology uses cash flows to 1.) determine the IRR of a fund, 2.) calculate a benchmark IRR by capital weighting its returns with the fund's cash flows, and then 3.) cross-sectionally regress the fund IRR on the market IRR to estimate equity market beta.

We apply our method to venture capital (VC) and buyout (BO) cross-sections. We find that VC funds have a market beta of 2.38. For BOs, we find a market beta of 0.76. Our estimate for VC is more statistically significant than our estimate for BO. We argue that the factors specific to VC investments are correlated with market movements while BO factors are not.

We proceed as follows: Section 2 discusses the data, the issues inherent of PE performance and our treatment thereof. Section 3 describes our approach. Section 4 presents our empirical results. Section 5 concludes.

2 Data

In this section, we describe the data and our treatment of funds which are not completely liquidated.

2.1 Data Source

Detailed performance data was supplied by a single institutional investor. Cash flows (net of fees) and manager reported NAVs were provided. The cash flows and general performance data are highly detailed. The institutional investor's portfolio has a meaningful investment in PE. The sample size is small in comparison to much of the related literature. However, our method is not limited to small samples. We are currently in the process of attaining a larger dataset to further confirm the method's efficacy.

Comprehensive performance and transactional data was available as far back as late 1988. At the point-in-time of the authoring of this paper, the valuation cutoff date for performance data was December 31, 2012. In addition, any fund with an inception date[1] less than or equal to December 31, 2010 is considered.[2]

Alexander and Takahashi (2002) stated that capital contributions are typically concentrated in the first few years of a fund. Additionally, Ljungqvist and Richardson (2003) note that only 10% of funds started in 1999 had called more than 70% of their committed capital by 2001. We believe these observations justify a vintage year cutoff of two years less than the valuation cutoff. Finally, we also employ a *fund inclusion condition* (described in Section 2.3), which is much more robust than a heuristic vintage year cutoff.

Before the aforementioned conditions are met, our dataset consists of 50 VC funds and 62 BO funds spanning vintage years of 1990 to 2012. After all conditions are met, 17 VC funds and 20 BO funds remain with vintage years from 1990 to 2001 and 2002, respectively. The resulting maximum vintage years generally support the average ten years to maturity of PE funds reported across the literature.

We refer to our data as fund-level data. This means our least common denominator of cash flow detail throughout the portfolio is at the fund level. We do not have deal-level flows. We have detailed transfers for the fund, but we do not know what companies within the fund to which they are associated. This level of data is similar to Driessen et al. (2012), Brown et al. (2012), Harris et al. (2013),

[1] Inception date refers to the date of the institutional investor's initial commitment to a fund
[2] We refer to this condition as the *vintage year cutoff date*.

Ang et al. (2013) and most similarly to Ljungqvist and Richardson (2003), as their data came from a single institutional investor.[3]

The primary advantage of our dataset is its integrity. As the sample size is small, we have the ability to be more thoughtful in its treatment. This includes not only exclusion conditions, but also cash flow types. For example, our calculations account for recallable return of capital, stock distributions and commitment adjustments. We believe accounting for these types of specialized transfers makes our dataset quite accurate with respect to cash flows.

Secondarily, because our dataset is from a single institutional investor similar to Ljungqvist and Richardson (2003), it is not subject to selection or survivorship bias. However, this advantage may be mitigated by the size of the sample. The cross-section of funds may or may not be representative of the PE space.

2.2 Manager Reported NAV

At a high level, there are two main issues in PE with manager (GP) reported NAVs ($rNAV$s). The first issue is that because the companies that the GPs invest in are not publicly traded, their pricing is subject to inefficiencies. This is typically referred to as stale pricing in the literature. The second is an asymmetric information principle-agent problem.

The bulk of assets in PE funds are not traded in public markets. This makes determining their value difficult and costly. For VCs, an event such as negotiating new rounds of financing, an initial public offering (IPO), an acquisition or a shutdown triggers a new valuation. For BOs, GPs appraise companies

[3] Although their sample size is much larger than ours

intermittently as transactions between purchase and sale are atypical. Based on these triggers, Woodward (2009) reports that VCs are valued every year or two, while BOs are appraised no more frequently than quarterly. As such, $rNAV$s are a mix of current and stale values, introducing noise such as autocorrelation.

Further, many institutional investors mark their portfolios more frequently than quarterly. This creates a mixed-frequency problem. There is a vast literature dealing with this problem, which is outside the scope of this paper. That being said, if reporting monthly because the vast majority of the portfolio has monthly marks, the investments with quarterly marks will appear less volatile if no frequency-driven corrections are made.

With regard to the asymmetric information principle-agent problem, Brown et al. (2013) notes that GPs face a trade-off in deciding whether or not to game $rNAV$s. On one hand, GPs have an incentive to overstate recent performance to increase the probability of future fundraising.[4] On the other, they must weigh this incentive against the probability of LPs discovering NAV manipulation.

When calculating GP lifetime income, Metrick and Yasuda (2010) show that expected subsequent funds account for a larger portion of the total than the current fund. And as the scalability of GP management skills increase, so does the incentive to overstate recent fund performance. Ljungqvist and Richardson (2003) find that returns increase with the log of fund size and decrease with its level.[5] This illustrates a highly tested question as to how easily GPs' skills can be scaled. According to Gompers and Lerner

[4] Brown et al. (2013) refers to this incentive to manipulate returns in an attempt to assist in fundraising as *Fund Timing.*

[5] Ljungqvist and Richardson (2003) find no significant size effect for BOs. Further, they note that excess IRRs (their dependent variable) are maximized at a total commitment size between $1.1B and $1.2B.

(1996), GPs do not proportionately add partners as they increase capital under management. Each investment requires extensive due diligence and regular interactions. As such, finding portfolio managers with the appropriate level of talent may not be feasible.

Assuming away GP skill scalability concerns, a GP's reputation counterweighs the incentive to overstate NAVs. If maximizing the number of subsequent funds plays a role in GP utility maximization, credibility is a central concern. If LPs can determine that a GP is misreporting NAVs, this would jeopardize future fundraising, representing a credible threat. Further, if some amount of gaming is expected by LPs, higher performing GPs may actually be more conservative in stating their $rNAV$s to protect against idiosyncratic shocks being mistaken for gaming.

Kaplan and Schoar (2005) and Phalippou (2010) use tercile transition probabilities as a way to evaluate future performance based on past performance. Using a similar framework, Brown et al. (2013) finds that for both BOs and VCs, top-tercile funds are less likely to move to another tercile than mid-tercile funds. This suggests that mid-tercile funds are inflating NAVs toward the end of their fundraising efforts, whereas top-tercile funds NAVs are quite stable. These results support the aforementioned gaming trade-off intuition.[6]

As asymmetric information is the principle reason LPs hire GPs to invest in companies, the issues inherent of $rNAV$s are expected to persist. However, the *ex post* punishment that LPs can levy against GPs by cutting back subsequent funding modifies incentive compatibility. The equilibrium of NAV gaming and how it affects risk and excess returns continues to be a major topic in the literature.

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[6] Similar to Brown et al. (2013), I do not consider GPs gaming fees as another path of utility maximization. If interested, Phalippou (2009) discusses GP fee gaming.

2.3 Fund Inclusion Condition

The *fund inclusion condition* states, at time T, the $rNAV$ plus the remaining unfunded commitment must be less than or equal to 20% of the total commitment to the fund, in order for that fund to be included in the study:

$$rNAV_T + C_0 + \sum_{t=1}^{T} aC_t - \sum_{t=0}^{T} Call_t + \sum_{t=1}^{T} rDist_t \leq \left(C_0 + \sum_{t=1}^{T} aC_t\right) \cdot RT,$$

$$s.t. Vintage\ Year \leq 2010\ and\ T \leq 2012$$

where the periodicity of t is monthly, $rNAV_T$ is the NAV reported by the GP at time T, C_0 is the initial capital committed to the fund, aC_t is a commitment adjustment at time t, $rDist_t$ is a recallable return of capital at time t and RT is the residual threshold, which we set heuristically to 20%.[7] In most cases, $\sum_{t=1}^{T} aC_t$=0, but occasionally the LP adjusts the commitment to the fund mid-term. We allow for this corner case, as opposed to assuming the GP always starts a new fund to increase callable capital through means other than recallable returns of capital. Perhaps more likely, accounting for commitment adjustments allows for commitment reductions, which can increase the number of funds in our sample as a $-aC_t$ reduces the left hand side of the *fund inclusion condition* at a rate ($1/RT = 5$) faster than the right hand side.

Given this reasonably strict condition, the *vintage year cutoff* is not binding in our dataset. This makes intuitive sense, as a fund whose initial commitment was only two years prior would typically still be in the commitment-calling phase. This is widely supported in the literature. For example, Driessen et al. (2012) states that GPs fundraise about every two to five years. This implies that capital calls for current

[7] Brown et al. (2013) defines a fund as "resolved" once it has a $rNAV$ less than 2% of the fund's initial commitment. However, the size of their dataset is at least one order of magnitude larger than ours. Additionally, we reduce the 20% by the distributions to date. Based on our analysis, we believe our condition to be more flexible while maintaining a reasonable level of restriction.

funds are not exhausted sooner than two years. Supporting this assumption, Ljungqvist and Richardson (2003) mention it takes six years for the funds in their dataset to call 90% of their committed capital.

As the pertinent funds are still in their commitment phase, they are most likely not distributing any returns. As Ljungqvist and Richardson (2003) note for their institutional investor's portfolio, by the end of year three, only 12.9% of the total invested capital is returned. On average, they observed it took a little under seven years for committed capital to be returned. The timing of distributions with respect to calls provides a second check on the intuition behind our *vintage year cutoff* being non-binding. Nonetheless, we leave the *vintage year cutoff* condition in place to systematically protect against corner cases that could bias our results.

2.4 Estimation of Final Market Values

Similar to Kaplan and Schoar (2005), we assume $rNAV_T$ is the final cash flow. According to Driessen et al. (2012), converting the $rNAV$ into a market value (MV) is essentially the type of implicit systematic risk assumption we are trying to avoid by calculating fund IRRs. But with a small dataset, a compromise had to be struck regarding non-liquidated fund inclusion and sample size.

With a strong understanding of the PE portfolio, and a *fund inclusion condition* calibrated for our sample set, we are implicitly assuming this effect to be negligible. However, given the discussion of issues with $rNAV$s and our *fund inclusion condition* as it compares to other authors, we believe this assumption to be neither controversial nor restrictive. Simply put, by implementing conditions on our data, the $rNAV_T$s are so small that their distance from the truth is theoretically negligible. As a result, it should not materially affect the results.

3 Beta Estimation

In this section, we explain our method of estimating the market beta of PE.

3.1 Calculation of Paired Fund & Market IRRs

3.1.1 Fund IRR

The fund IRR (IRR_f) is calculated as

$$NPV = \sum_{t=0}^{T} \frac{Call_t - Dist_t}{(1 + IRR_f)^t} = 0$$

where $Dist_t$ is a distribution at time t inclusive of recallable return of capital ($rDist_t$) and $Dist_T = rNAV_T$ when $rNAV_T \neq 0$. As can be seen from the equation, we are assuming a constant rate of return.

Solving for when the net present value (NPV) equals zero produces the IRR_f. Finding a solution to this equation requires numerical optimization as the return cannot typically be found analytically.

There are known issues associated with using IRR_f. For example, the implicit assumption is that $Call_t \; \forall \; t$ have equal rates of return. Also, an IRR does not consider the cost of capital. However, these issues are immaterial to our analysis as we are not using this equation as an investment decision tool.

3.1.2 Benchmark IRR (market IRR)[8]

The Benchmark IRR (IRR_b) is calculated the same way as IRR_f:

$$NPV = \sum\nolimits_{t=0}^{T} \frac{Call_t - Dist_t}{(1 + IRR_b)^t} = 0$$

where $Dist_T = bNAV_T$ when $bNAV_T > 0$.

The difference is that we need to manually calculate the benchmark NAV at time t ($bNAV_t$). Doing so is relatively straightforward:

$$bNAV_t = bNAV_{t-1} \cdot (1 + r_{bt}) + Call_t - Dist_t$$

$$s.t. bNAV_t \geq 0 \; and \; bNAV_0 = 0$$

where r_{bt} is the benchmark return at time t.

At time T, as defined by the fund associated with benchmark b, $bNAV_T$ is used as the final distribution in calculating IRR_b. However, if at any time $bNAV_t < 0$, the cash flows are truncated as of time $t-1$ and IRR_b is calculated by setting $T = t - 1$. Intuitively, we are setting a "no short position" condition for the benchmark. This condition is only binding if the fund outperforms the benchmark by a wide margin. In our sample this particular condition is not binding.

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[8] Original credit for combining PE cash flows with the returns of a reference benchmark to determine the benchmark IRR (originally called the index comparison method (ICM) and later the public market equivalent (PME)) that would have been obtained had the PE cash flows been made into the benchmarks is originally attributed to Long and Nickels (1996). Later refinements were applied when Rouvinez (2003) introduced PME+, then when Cambridge Associates (2013) developed mPME and the often used KS-PME method of Kaplan and Schoar (2005) was subsequently introduced.

3.2 Regression

3.2.1 Linear Cross-Sectional Regression

Once we have calculated the pair of IRRs for each of the funds in our sample, we run a linear cross-sectional regression

$$\boldsymbol{IRR_f} = \alpha + \beta \cdot \boldsymbol{IRR_b} + \boldsymbol{\epsilon}$$

$$\boldsymbol{\epsilon} \sim \mathcal{N}(0, \sigma^2)$$

where $\boldsymbol{IRR_f}$ is a vector of all the IRR_fs in our sample, $\boldsymbol{IRR_b}$ is a vector of all IRR_bs and $\boldsymbol{\epsilon}$ is a vector of error terms. By implementing this method, we are implicitly assuming a linear relationship between the dependent variable and regressor, no multicollinearity, exogeneity and spherical error variance. While we realize these are rather idealized assumptions, we argue the simplicity of the method and utility of its empirical results outweigh its lack of theoretical rigor.[9]

3.2.2 Exogeneity Assumption

By assuming exogeneity, we are assuming the regressor is orthogonal to the error term. The consequences of this assumption in the context of our regression is a possible omitted variable bias. This bias would violate the Gauss-Markov theorem, and consequently the estimator would not be consistent. The obvious solution to this issue is to include additional regressors. If the identified additional regressors are returns series similar to the benchmark, adding them to the regression is straightforward. If they are of a different type (e.g. log change in the total dollar volume of VC investments), then an extension of the method might be required.

[9] Much of the literature leverages these assumptions. For example, Ljungqvist and Richardson (2003) assume fund returns are cross-sectionally i.i.d. Kaplan and Schoar (2005) assume homoscedasticity, but later apply corrections for heteroscedasticity. Finally, many authors including Driessen et at. (2012) and Korteweg and Sorensen (2010) assume exogeneity in their main specification, but then subsequently attempt to refine by adding regressors such as firm size and book-to-market factors to reduce omitted variable biases.

3.2.3 Spherical Error Variance Assumption

If the error terms are heteroscedastic or correlated, the estimator will still be unbiased[10] but it may not be efficient.

Given we have handled the omitted variable problem as discussed above, assuming the error terms are uncorrelated seems reasonable in our regression. Additionally, calculating heteroscedasticity-consistent standard errors such as White's standard errors would alleviate any further concerns regarding spherical error variance.

4 Empirical Results

In this section, we report market beta estimates of PE funds. For this study, our primary focus was the S&P 500 as a benchmark. The benchmark is investible, investment professionals are familiar with its mechanics and the VC funds in our sample are predominantly exposed to the United States. Additionally, this benchmark facilitates sanity checks against contemporary literature, as the S&P 500 is the often-used index. That being said, we also benchmark IRR_f to the NASDAQ, Russell 2000 and the Fama-French (1993) excess return (RMRF).

4.1 Venture Funds

Panel A of Table 1 presents the market beta estimates of VC funds ($\hat{\beta}_{VC}$) for four different benchmarks. Additionally, the regression results are illustrated in Figures 1-4. The confidence bands look to be quite tight at first glance, with exception to when the Russell 2000 IRR_b is the regressor.

[10] Given all other assumptions are not violated.

We find a statistically significant $\hat{\beta}_{VC}$ to the S&P 500 ($\hat{\beta}_{VC,S\&P}$) of 2.38. Our sample appears to be highly correlated with the benchmark. In fact, across all benchmarks tested, $\hat{\beta}_{VC}$ is notably larger than 1. And with exception to Russell 2000, the fit of the regression seems to be quite strong with an average R^2 of about 0.8. Further, being that the cash flows of our sample are net of fees, as noted by Driessen et al. (2012) and supported by the general PE fee structure articulated in Metrick and Yasuda (2010), $\hat{\beta}_{VC}$ would be expected to adjust upward by roughly 0.15 in a gross of fees estimate.

To justify their high $\hat{\beta}_{VC}$, Driessen et al. (2012) investigate a time series of dividend yields[11] against a five year moving average of S&P 500 returns. Supporting their $\hat{\beta}$, they find that if the stock market moving average does well (five years being the average duration of an investment), a large spike follows a year later in the dividend yield. As further support, Korteweg and Sorensen (2010) note there may be substantial VC-specific risk incorporated into $\hat{\beta}_{VC}$. When adding a VC-specific risk factor[12], their $\hat{\beta}_{VC}$ drops to ~ 1. This would seem to support Gompers and Lerner's (2000) thesis that money chasing deals affects pricing, which increases VC volatility. The correlation of dividend yields and market returns, coupled with momentum chasing in PE, could explain $\hat{\beta}_{VC} > 1$.

Comparing our results to the literature in Table 2: Panel 1, our $\hat{\beta}_{VC,S\&P}$ is in line with estimates of Korteweg and Sorensen (2010) and Driessen et al. (2012). When taking the standard error into account, $\hat{\beta}_{VC,S\&P}$ is also near Ang et al. (2013). The inclusion of vintage years 2002-2008 could explain the

[11] Summed over all funds, the dividend yield in year t for fund i is defined as the sum of all dividends paid over the year, divided by the size of fund i.

[12] Motivated by Gompers and Lerner (2000) and Kaplan and Schoar (2005), who suggest capital inflows into VC funds lead to higher valuations and subsequent lower performance, their VC-specific risk factor is defined as the log-change in the total dollar volume of VC investments in the target period.

downward deviation in their $\hat{\beta}_{VC}$ from ours. Additionally, Ang et al. (2013) note their estimates are sensitive to priors, although that by itself does not completely explain the difference.

As further evidence in favor of the hypothesis that VC-specific factors are correlated with our market beta, our alpha result of $\hat{\alpha}_{VC,S\&P} = -0.0724$ is directionally similar to the literature. Driessen et al. (2012), Ewens et al. (2013), Cochrane (2005), Hwang et al. (2005), etc., find negative VC fund alphas net of fees. While we are not suggesting our $\hat{\alpha}_{VC}$ results are actionable, due to possible omitted variable bias, the fact that our $\hat{\alpha}_{VC}$ is negative would suggest our specification to be reasonable and that the bulk of omitted variables are correlated with market beta.

Considering our sample size and model simplicity, our $\hat{\beta}_{VC}$ looks to perform quite well when comparing the results to the existing literature. All other studies we referenced have more observations, sometimes thousands more. Clearly, applying this technique to a larger sample would be a valuable next step.

4.2 Buyout Funds

Panel B of Table 1 presents the market beta estimates of BO funds ($\hat{\beta}_{BO}$s). Next, Panel B of Table 2 presents the $\hat{\beta}_{BO}$s in the literature. Additionally, Figures 5-8 present our regression results graphically.

We find a $\hat{\beta}_{BO}$ to the S&P 500 ($\hat{\beta}_{BO,S\&P}$) of 0.76. Though we have more observations for estimating our BO beta, the results are less statistically significant. Across all benchmarks, the size of the standard errors are such that the results are not actionable for practitioners. The best R^2 we have is 0.25 for Russell 2000. It is unclear if this is because of our sample set or because of issues regarding BO fundamentals. But as this issue is across the board, regardless of benchmark, we hypothesize this is because the market benchmarks are not accounting for fundamental drivers of BO performance.

Driessen et al. (2012) suggest that leverage is a key factor in low $\hat{\beta}_{BO}$. The seminal work by Modiglaini and Miller (1958) states that expected stock returns should increase in financial leverage. However, according to Kaplan and Stein (1990), in highly leveraged transactions, a large increase in debt has only a small effect on equity beta. Further, Korteweg (2004) finds no positive relation between leverage and expected stock returns.[13] As leverage is an important aspect to BO performance, this could explain the small $\hat{\beta}_{BO}$.

Ang et al. (2013) tests the Kaplan and Strömberg (2009) hypothesis that relative yields on corporate assets to high-yield debt explain BO returns. They find the asset-debt yield to be statistically significant when regressing against BO IRR_fs in support of this hypothesis. Additionally, they also find the default spread and VIX to be statistically significant. Perhaps incorporating these factors would better fit the regression. However, doing so is outside the scope of this paper.

Assuming a larger sample set would resolve our statistical significance issue, our estimates are reasonably in line with the literature estimates presented in Table 2: Panel B. Taking into account our standard errors, all of our regression results fall within the band of literature estimates. Ewens et al. (2013) has the best fit of those we tabulated when adjusting for statistical significance and vintage years in their sample set. Interestingly, our $\hat{\beta}_{BO,S\&P}$ and $\hat{\beta}_{BO,RMRF}$ are quite close numerically. Again, applying this technique to a larger sample would be a valuable next step.

[13] Korteweg (2004) implements a simple trading strategy based upon the assumption that highly levered companies are overvalued, yielding some preliminary evidence that factor loadings are too low for the companies. This could partially explain the *leverage puzzle*.

5 Application of Beta Results

Once $\hat{\beta}_{PE}$ is calculated for a portfolio, the question becomes how to apply it. Our principle intention, when developing an estimated β for PE, was to begin to resolve issues with volatility estimates due to the inherent aforementioned pricing issues. While we recognize that $\hat{\beta}_{PE}$ can be utilized in a number of different applications, we generally focus on two. The first use for our estimate is in the asset allocation process. The second is in evaluating performance on a risk-adjusted basis.

5.1 Asset Allocation

In terms of Markowitz (1952) mean-variance optimization (MVO), our $\hat{\beta}_{PE}$ is a useful input in concert with heuristic adjustments. MVO requires expected returns, standard deviations and a correlation matrix. The basic technique for estimating β_{PE} requires further explanation to satisfy these requirements.

Calculating the expected return for PE is relatively straightforward. After calculating the expected return of the benchmark (e.g. $E[R_{S\&P}]$):

$$E[R_{PE}] = \hat{\alpha} + \hat{\beta}_{PE} \cdot E[R_{S\&P}]$$

Then adjustments to this expected return can be made based on the conviction of investment management. For example, if the portfolio manager feels he can add value through fund selection, he can adjust the expected return by increasing $\hat{\alpha}$.

Next, estimating the volatility of PE requires a heuristic approach since the standard estimates of standard errors from our regression are probably biased. Our $\hat{\beta}_{PE}$ still helps to set a floor for the PE volatility relative the target benchmark volatility. For example, if $\hat{\beta}_{PE,S\&P}$ is 2.0 and the S&P 500 volatility is 10%, the prescribed minimum volatility of PE is 20%. Once this minimum volatility has been calculated, it is up to investment management to adjust it upward heuristically.

Finally, our method can also be used to calculate the covariances between PE and the other strategies. As an example, to calculate the covariance between PE and fixed income (FI) we would treat a FI proxy such as Barclays US Government Index as the benchmark and apply the same regression. Then multiplying $\hat{\beta}_{PE,FI}$ by σ_{FI}^2 would produce $Cov(PE, FI)$.

5.2 Risk-Adjusted Performance Evaluation

With regard to risk-adjusted performance evaluation, our $\hat{\beta}_{PE}$ is essentially a proxy. Typically, one would estimate β for a particular fund or portfolio by using its performance as inputs in the Capital Asset Pricing Model (CAPM). Following this estimation, $\hat{\beta}$ would be an input into any number of risk measures to facilitate an apples-to-apples comparative analysis.

Utilizing our $\hat{\beta}_{PE}$ is essentially a substitution with some caveats. Given the estimate is not fund-specific, some applications will have to be at an aggregated level. If using $\hat{\beta}_{PE}$ as an estimate of volatility relative to the market, the results would be an estimate for the PE portfolio, not the funds individually. However, there are a number of risk measures where $\hat{\beta}_{PE}$ can be used at the individual fund-level.

Jensen's Alpha is a good example of a risk-adjusted performance statistic that could be applied to individual funds. By substituting $\hat{\beta}_{PE}$ for the traditional $\hat{\beta}$, a more rigorous abnormal return could be calculated for fund f over a particular trailing period

$$\alpha_{J,f} = R_f - \hat{\beta}_{f,b} \cdot \bar{R}_{f,b} \rightarrow IRR_f - \hat{\beta}_{PE,b} \cdot IRR_{f,b}$$

where, for the trailing period, IRR_f is calculated as stated in Section 3.1.1, $IRR_{f,b}$ is the IRR as calculated in Section 3.1.2 using cash flows from fund f and returns from benchmark b, and $\hat{\beta}_{PE,b}$ is the estimated PE beta to benchmark b as calculated in Section 3.2.1.

Investment professionals use beta estimates in a number of ways. Sometimes it can be as simple as comparing the beta of one sub-portfolio to the beta of another. Other times, it may be used as an input into an equation to answer a more specific risk question. By using $\hat{\beta}_{PE}$ as a proxy for a fund or PE portfolio traditionally estimated $\hat{\beta}$, the results will be free from biases introduced by the issues inherent of PE performance reporting.

6 Conclusion

We developed a new econometric methodology to estimate the market beta to PE using cash flow data. Our method utilizes the public market equivalent literature to pair standard IRR calculations for individual funds to equity market IRRs. Using ordinary least squares cross-sectional regressions for VC and BO funds, we are able to estimate PE market betas.

We apply our technique to one institutional investor's PE portfolio. We find that VC funds have a high market beta, while that of BO funds is much lower. We theorize that the high VC beta can be at least partially explained by a factor that accounts for VC-specific market momentum. We also suggest that the low BO beta is not spurious, but a results of the equity markets not accounting for leverage.

A previously noted next step for this technique would be to apply it to a larger sample set. The results from those larger cross-sectional regressions would prove quite informative in the technique's broader applicability. Additionally, expanding the regressions to include VC and BO-specific factors could help explain the results.

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Table 1: IRR Market Equivalent Cross-Sectional Regression Results

Panel A: Venture Funds

Vintage	Obs.	Benchmark	Beta	Std. Error	R-Squared	p-Value
1990-01	17	S&P 500	2.37934	0.301367	0.806035	< 0.0001
1990-01	17	NASDAQ	1.38416	0.185421	0.787913	< 0.0001
1990-01	17	Russell 2000	3.31594	0.864989	0.494877	0.001628
1990-01	17	RMRF	2.86819	0.384154	0.78797	< 0.0001

Panel B: Buyout Funds

Vintage	Obs.	Benchmark	Beta	Std. Error	R-Squared	p-Value
1990-02	20	S&P 500	0.759268	0.373168	0.186985	0.056888
1990-02	20	NASDAQ	0.558739	0.244308	0.225157	0.034522
1990-02	20	Russell 2000	1.42942	0.575244	0.25542	0.023019
1990-02	20	RMRF	0.829202	0.435654	0.167543	0.073112

Table 2: Literature Beta Estimates of Private Equity Funds

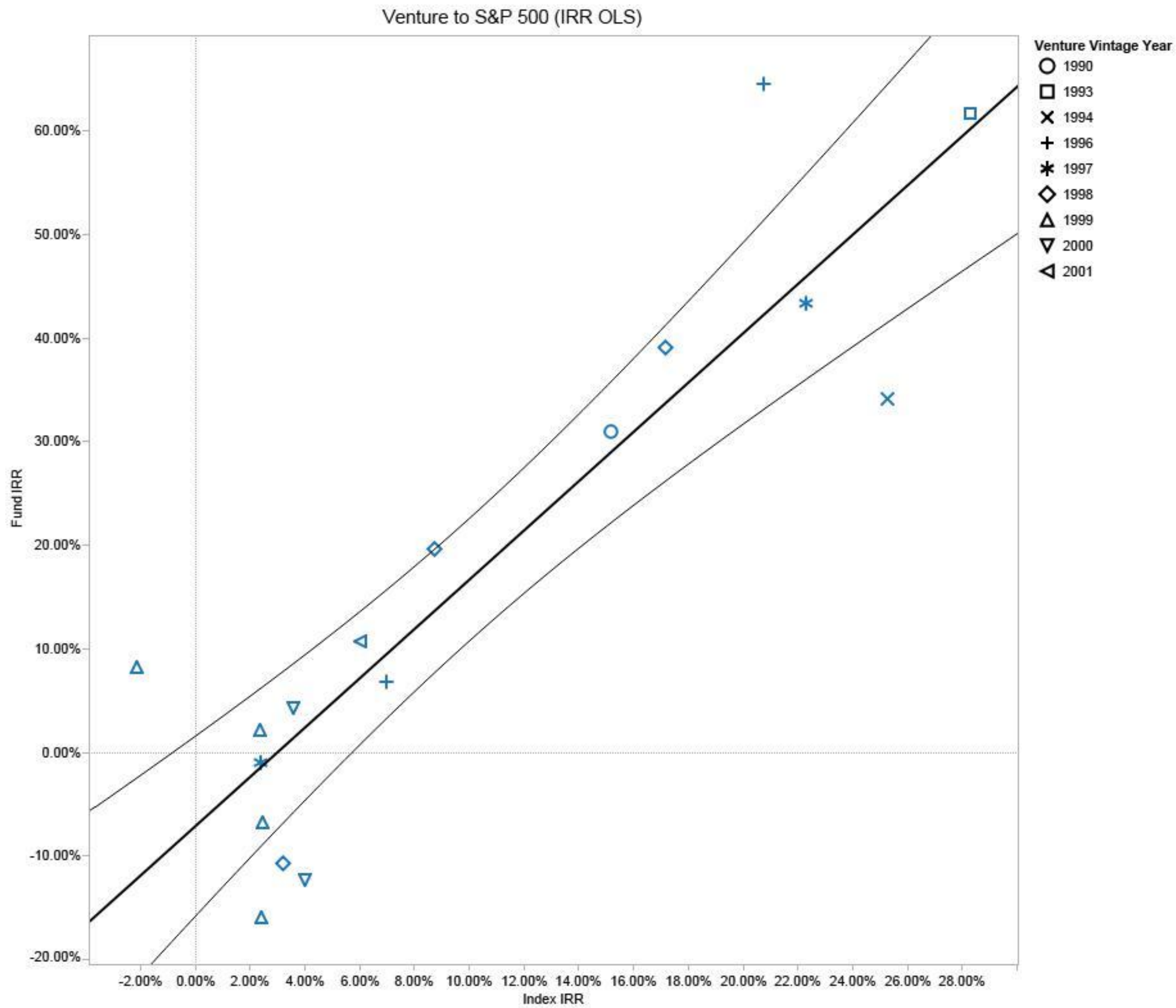
Panel A: Venture Funds

Article	Vintage	Obs.	Benchmark	Beta	Std. Dev.
Ljungqvist-Richardson (2003)	1981-93	19	Exposure-Based	1.12	0.06
Kaplan-Schoar (2005)	1980-97	577	S&P 500	1.23	0.42
Korteweg-Sorensen (2010)	N/A	5501	RMRF	2.6624	0.117
Driessen et al. (2012)	1980-03	686	S&P 500	2.73	0.55
Ewens et al. (2013)	1980-07	1040	RMRF	1.234	0.179
Ang et al. (2013)	1992-08	272	S&P 500	1.67	0.27

Panel B: Buyout Funds

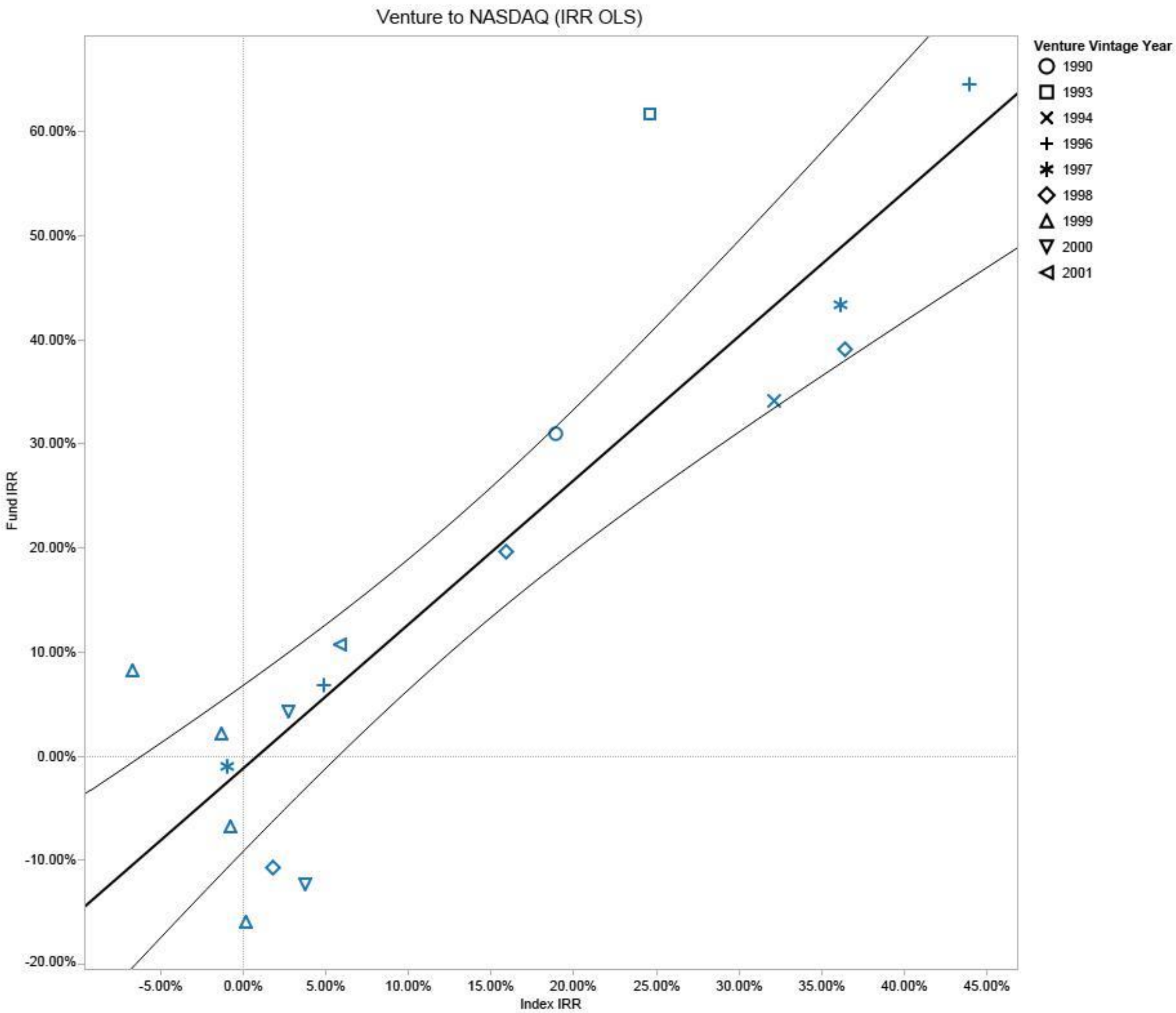
Article	Vintage	Obs.	Benchmark	Beta	Std. Dev.
Ljungqvist-Richardson (2003)	1981-93	54	Exposure-Based	1.08	0.11
Kaplan-Schoar (2005)	1980-97	169	S&P 500	0.41	0.29
Cao and Lerner (2008)	1981-03	307	RMRF	1.3	
Jegadeesh et al. (2010)	1994-08	180	S&P 500	0.7083	0.2227
Franzoni et al. (2012)	1975-07	139	RMRF	0.948	
Driessen et al. (2012)	1984-03	272	S&P 500	1.31	0.66

Figure 1:



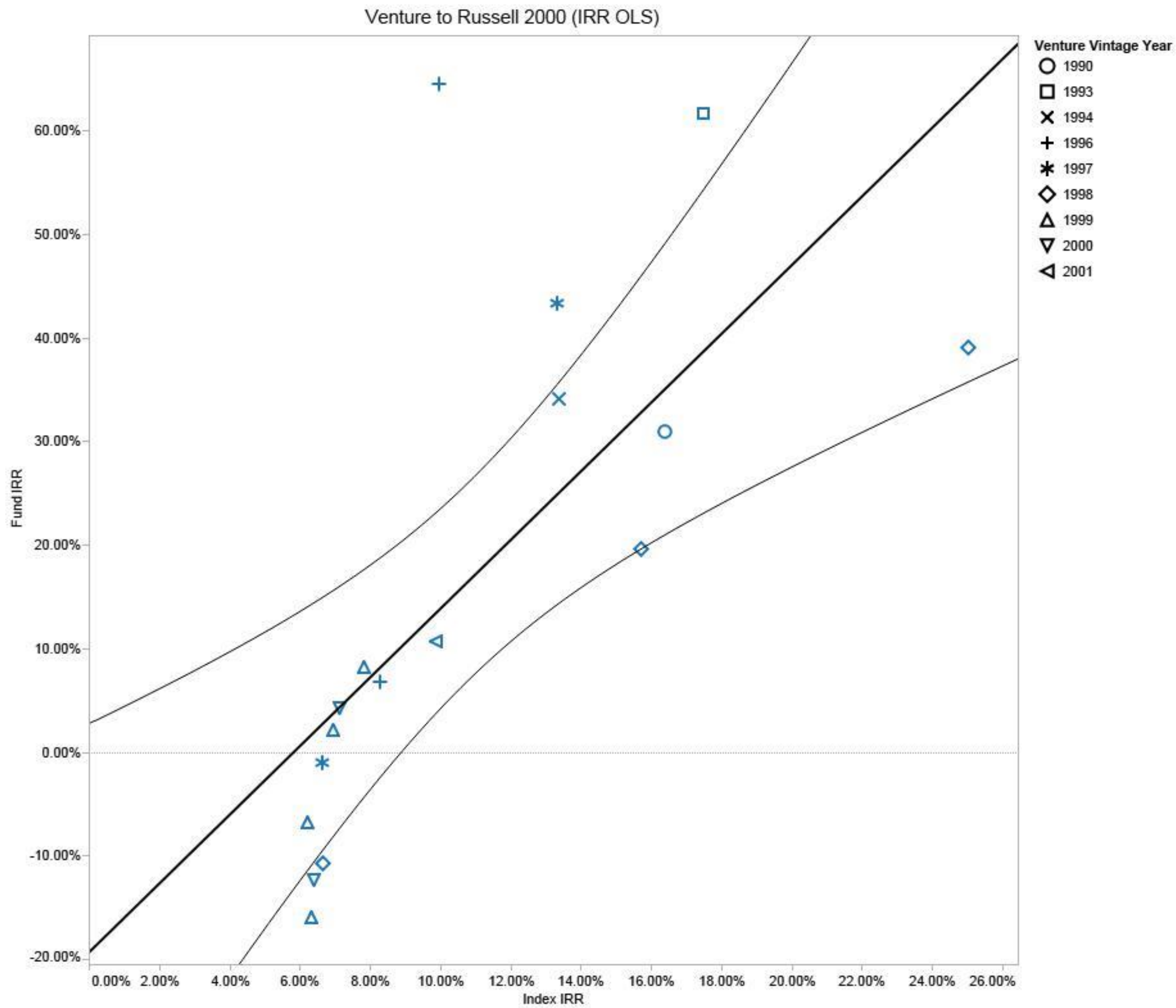
Venture to S&P 500 (IRR OLS)
Venture Vintage Year
1990
1993
1994
1996
1997
1998
1999
2000
2001
Fund IRR
60.00%
50.00%
40.00%
30.00%
20.00%
10.00%
0.00%
-10.00%
-20.00%
-2.00% 0.00% 2.00% 4.00% 6.00% 8.00% 10.00% 12.00% 14.00% 16.00% 18.00% 20.00% 22.00% 24.00% 26.00% 28.00%
Index IRR

Figure 2:



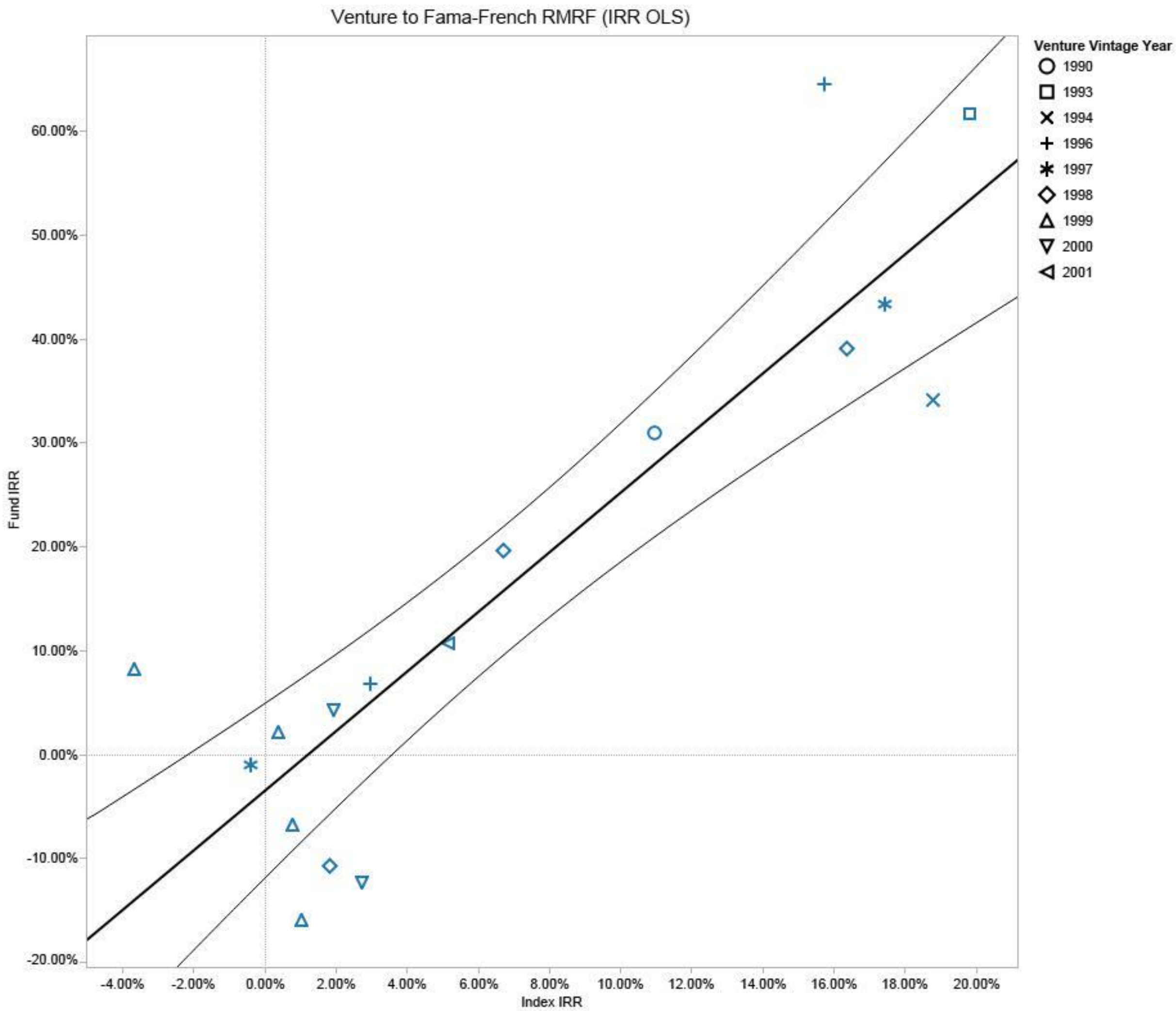
Venture to NASDAQ (IRR OLS)
Venture Vintage Year
1990
1993
1994
1996
1997
1998
1999
2000
2001
60.00%
50.00%
40.00%
30.00%
20.00%
10.00%
0.00%
-10.00%
-20.00%
Fund IRR
-5.00%
0.00%
5.00%
10.00%
15.00%
20.00%
25.00%
30.00%
35.00%
40.00%
45.00%
Index IRR

Figure 3:



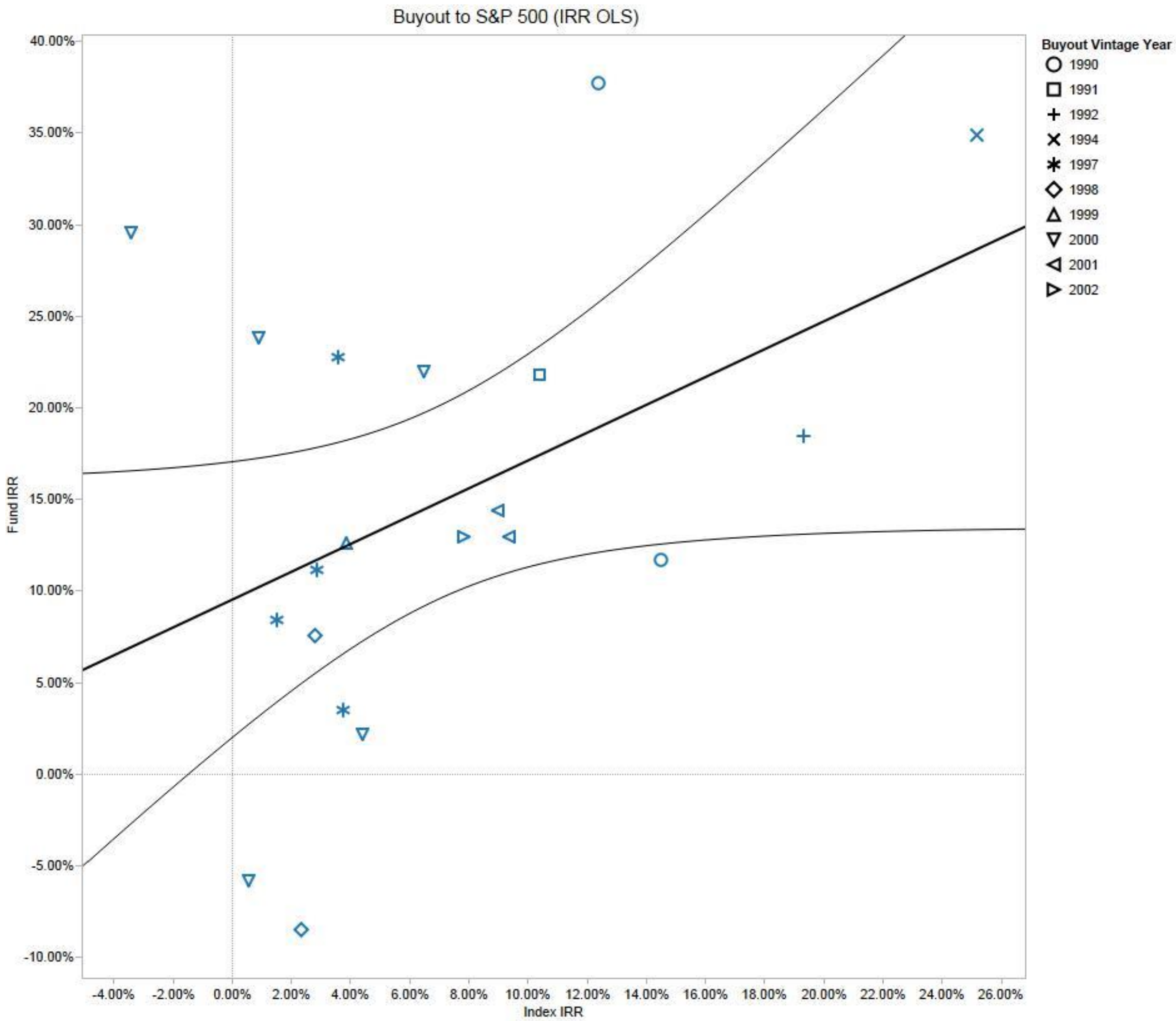
Venture to Russell 2000 (IRR OLS)
Venture Vintage Year
1990
1993
1994
1996
1997
1998
1999
2000
2001
Fund IRR
60.00%
50.00%
40.00%
30.00%
20.00%
10.00%
0.00%
-10.00%
-20.00%
0.00%
2.00%
4.00%
6.00%
8.00%
10.00%
12.00%
14.00%
16.00%
18.00%
20.00%
22.00%
24.00%
26.00%
Index IRR

Figure 4:



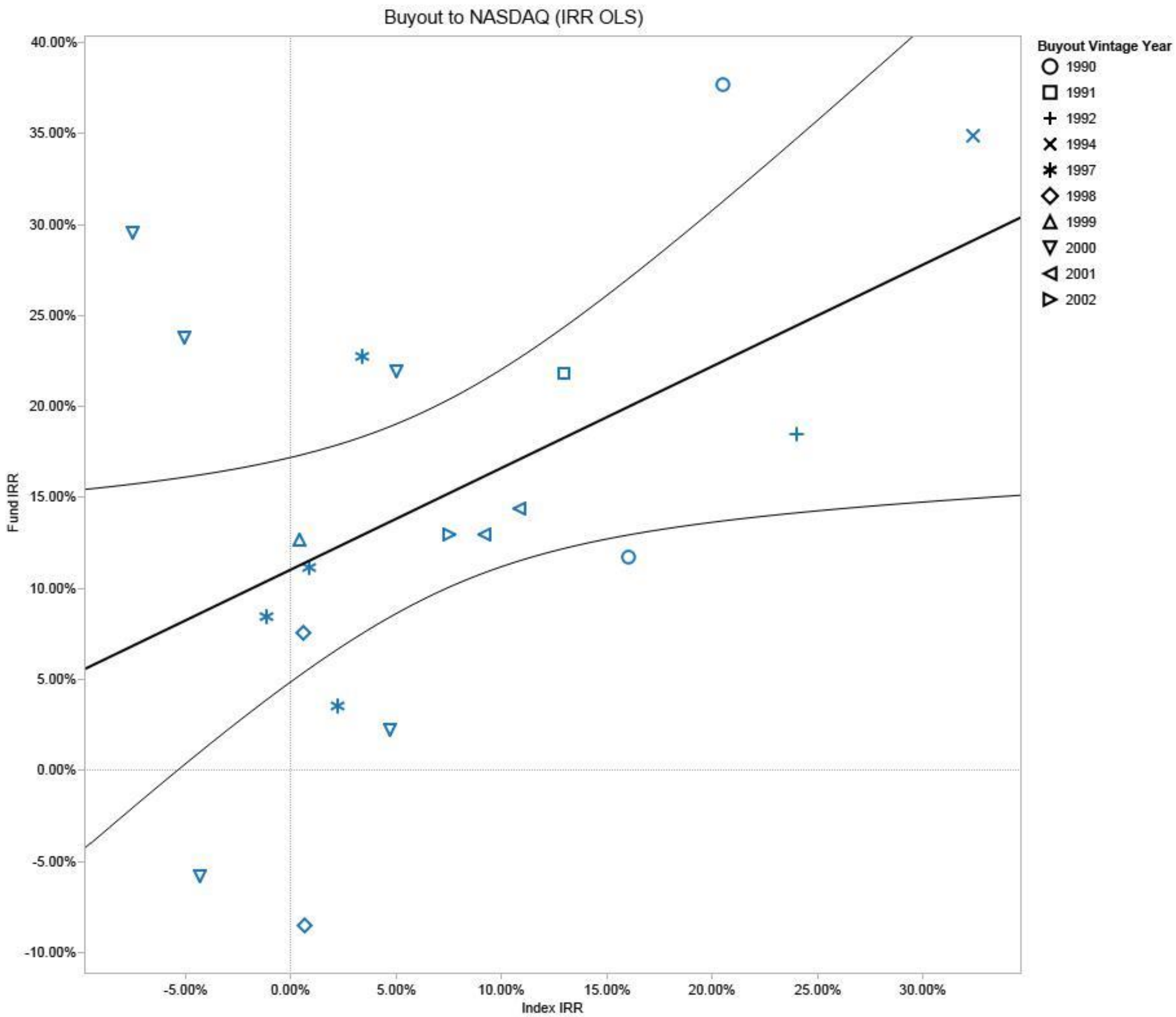
Venture to Fama-French RMRF (IRR OLS)
Venture Vintage Year
1990
1993
1994
1996
1997
1998
1999
2000
2001
Fund IRR
60.00%
50.00%
40.00%
30.00%
20.00%
10.00%
0.00%
-10.00%
-20.00%
-4.00%
-2.00%
0.00%
2.00%
4.00%
6.00%
8.00%
10.00%
12.00%
14.00%
16.00%
18.00%
20.00%
Index IRR

Figure 5:



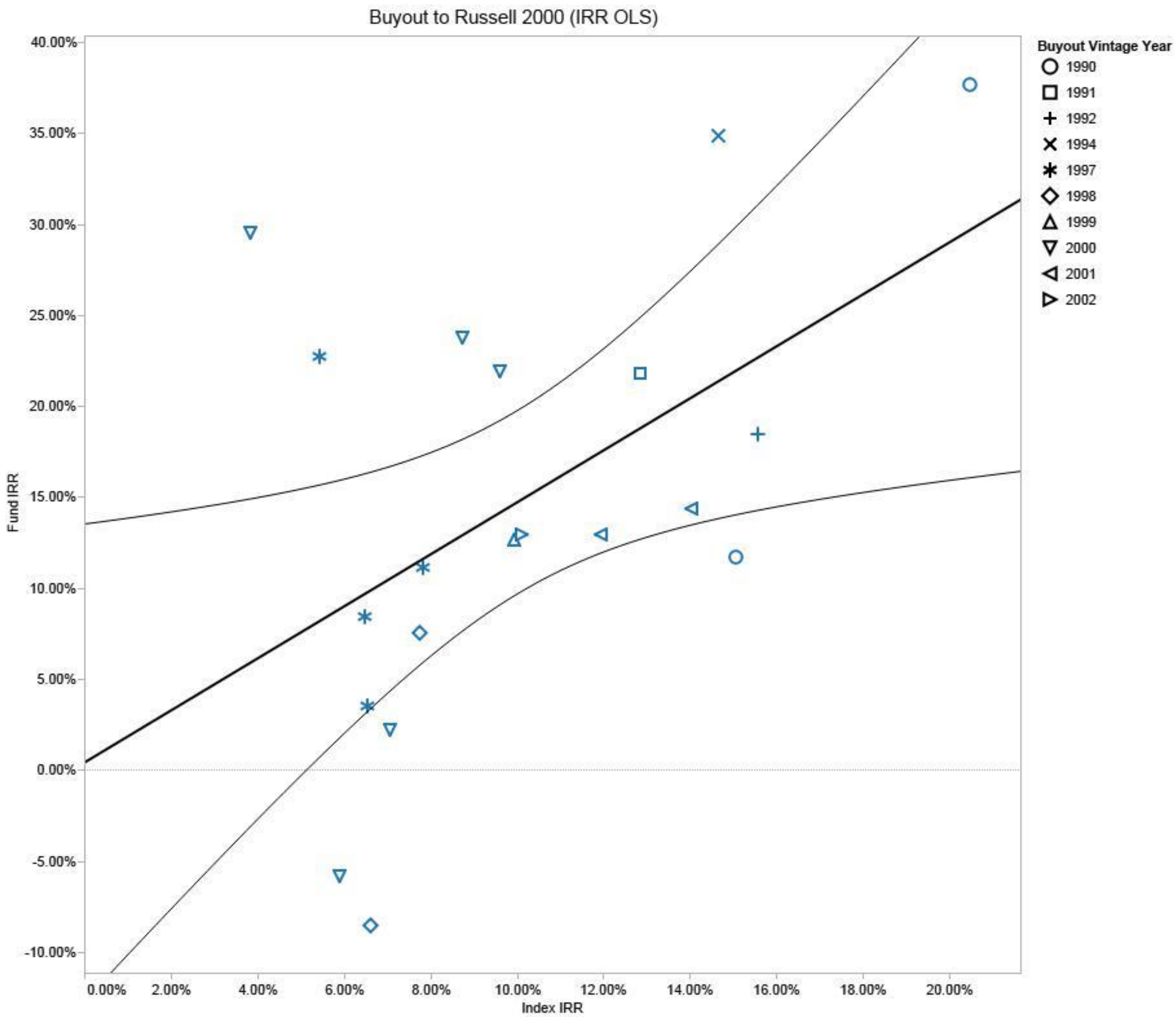
Buyout to S&P 500 (IRR OLS)
Buyout Vintage Year
1990
1991
1992
1994
1997
1998
1999
2000
2001
2002
Fund IRR
Index IRR
40.00%
35.00%
30.00%
25.00%
20.00%
15.00%
10.00%
5.00%
0.00%
-5.00%
-10.00%
-4.00%
-2.00%
0.00%
2.00%
4.00%
6.00%
8.00%
10.00%
12.00%
14.00%
16.00%
18.00%
20.00%
22.00%
24.00%
26.00%

Figure 6:



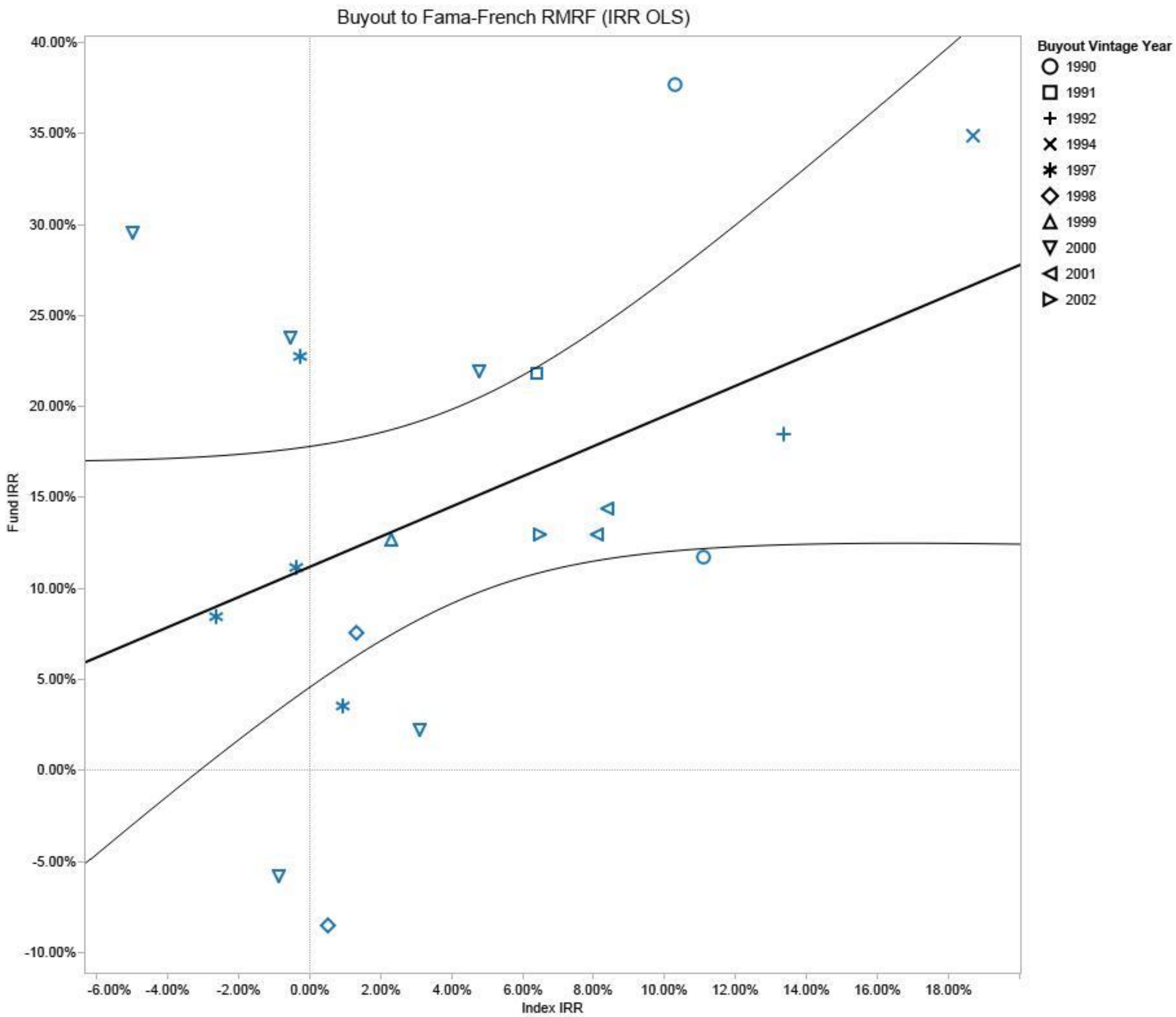
Buyout to NASDAQ (IRR OLS)
40.00%
35.00%
30.00%
25.00%
20.00%
15.00%
10.00%
5.00%
0.00%
-5.00%
-10.00%
Fund IRR
-5.00%
0.00%
5.00%
10.00%
15.00%
20.00%
25.00%
30.00%
Index IRR
Buyout Vintage Year
1990
1991
1992
1994
1997
1998
1999
2000
2001
2002

Figure 7:



Buyout to Russell 2000 (IRR OLS)
Buyout Vintage Year
1990
1991
1992
1994
1997
1998
1999
2000
2001
2002
Fund IRR
40.00%
35.00%
30.00%
25.00%
20.00%
15.00%
10.00%
5.00%
0.00%
-5.00%
-10.00%
Index IRR
0.00%
2.00%
4.00%
6.00%
8.00%
10.00%
12.00%
14.00%
16.00%
18.00%
20.00%

Figure 8:



Buyout to Fama-French RMRF (IRR OLS)
Buyout Vintage Year
1990
1991
1992
1994
1997
1998
1999
2000
2001
2002
Fund IRR
40.00%
35.00%
30.00%
25.00%
20.00%
15.00%
10.00%
5.00%
0.00%
-5.00%
-10.00%
-6.00%
-4.00%
-2.00%
0.00%
2.00%
4.00%
6.00%
8.00%
10.00%
12.00%
14.00%
16.00%
18.00%
Index IRR

Institutional Portfolio Attribution:

A Brinson Attribution Extension

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Working Paper

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Abstract: We extend the Brinson et al. (1986) performance attribution framework to support institutional-specific requirements, including a hierarchical structure and multiple benchmark styles. By attributing performance to four statistics (e.g. *manager alpha*, *portfolio construction*, *tactical* and *strategic*), we are able to remove the *interaction* term, which is a commonly referred to shortcoming of Brinson attribution. We subsequently modify Frongello (2002) linking to produce pro-rated multiperiod attributes which sum to meaningful statistics using notional portfolios of multitier excess returns.

Introduction

The contribution of each aspect of the investment management process is of fundamental importance to investors. In order to learn from mistakes and understand core competencies, a framework is needed to intuitively present historical performance in terms of the investment process. Performance attribution is such a framework and relates the impact of active investment by dissecting total plan returns into various meaningful components.

Contemporarily, approaches to performance attribution are founded on the seminal work of Brinson, Hood and Beebower (1986) (BHB). Their model presents a process of attributing an actual portfolio's return to investment management activities that contribute to said return (e.g. active *asset allocation*, *security selection*). And BHB does so without requiring complex calculations, while offering reasonably intuitive results.

This article focuses on an extension of BHB developed for the institutional space. With multiple layers of oversight, and a focus on hiring investment managers as opposed to direct security investment, institutional portfolio attribution has a specialized set of requirements. Our model endeavors to fulfill these requirements by extending BHB to have a more detailed set of attributes and more extensive benchmarking. Further, our framework can support a commonly implemented hierarchical structure without having different equations for the total portfolio versus a specific segment as noted in Christopherson et al. (2009) as the *Brinson-Fachler allocation effect*.

BHB illustrated an accounting decomposition of a manager's sector allocation and individual security selection activities. By breaking up the excess return of the portfolio over its benchmark into investment manager processes, plan sponsors can view the source(s) of manager performance. While quite useful and arguably derived from first principles, BHB is not without application issues.

The first and most regularly stated shortcoming of the BHB model is the *interaction*[14] term. While *asset allocation* and *security selection* may be independent decisions, one's contribution to the plan level active return depends upon the other. The degree of this dependence is described by the *interaction* term.

The second issue is that extending the model to a more complicated portfolio of managers and securities is not straightforward. More specifically, if benchmarking is done at both the strategy level and the manager level, it is unclear how to apply the framework. BHB assumes that the only allocation decision is its weight, while all benchmarking is done at the selection level. This only works for strategy-level benchmarking if the strategy benchmark is the weighted sum of the members' benchmarks. This is highly unlikely save the circumstance in which all strategy members share the same benchmark, as weights change over time.

An institutional portfolio necessitates solutions to these issues. Heavy scrutiny at a high level, coupled with performance goals, requires a simple and comprehensive set of attributes which support a hierarchical portfolio structure. For reference, figure 1 illustrates one such hierarchical structure.

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[14] Brinson and Fachler (1985) referred to this effect as the "cross product," and Brinson, Hood and Beebower (1986) called it "other."

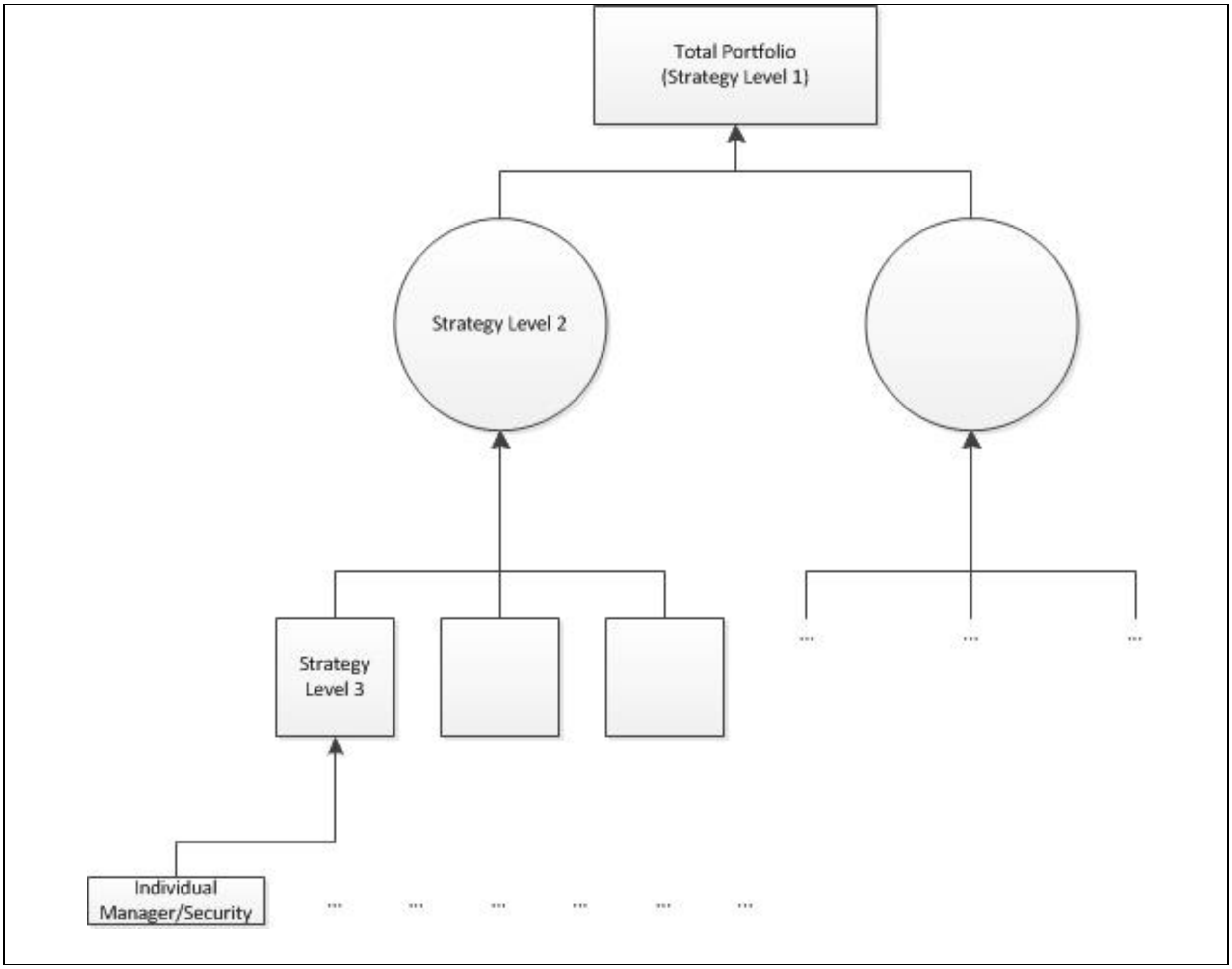

Figure 1: Example institutional portfolio hierarchical structure

Our attribution model poses solutions to both these issues. We introduce new intuitive terms and also handle portfolios with hierarchical benchmarking. In the institutional space, this facilitates reporting in environments where broader investable benchmarks are required at the strategy level, while more specific manager benchmarks are needed to minimize tracking error. In addition, moving away from the hard-to-explain *interaction* term is advantageous when presenting attribution results to executive management.

Following presentation of the model framework, we demonstrate an extension of the Frongello (2002) linking methodology. We use notional portfolios similar in definition to Brinson (1986) to pro-rate the attribution results. Doing so, we are able to report the resulting attributes in terms that sum up to meaningful statistics.

Framework

Ultimately, as in BHB, attribution boils down to splitting the active return into actual and intuitive investment management processes. The *active return*, realized portfolio performance less benchmark policy performance, is:[15]

$$R - \bar{R} = \sum_i (W_{a,i} \cdot R_{a,i}) - \sum_s (W_{t,s} \cdot \bar{R}_{pol,s})$$

where $W_{a,i}$ is the actual weight of asset i, $R_{a,i}$ is the actual return of asset i, $W_{t,s}$ is the weight of strategy s set during asset allocation, and $\bar{R}_{pol,s}$ is the return of the policy benchmark for strategy s. As an example of a total portfolio's policy benchmark, the often referred to benchmark in the literature is the 70/30 equities/treasuries percentage split. Continuing the 70/30 benchmark example, the attribution strategies below the total portfolio would be organized such that their target weights and asset strategies summed accordingly.[16] This means that attribution strategies cannot be overlapping.[17]

While the total portfolio policy benchmark is an important performance measure, it is broad and non-specific by design. Therefore, in order to have the flexibility to benchmark strategies and assets with exposures that are more indicative of the underlying investments, reference benchmarks are employed. These reference benchmarks can be the same as the policy benchmarks but do not have the policy benchmark summation restrictions. And they can be more meaningful when viewed in isolation.

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[15] If subscripts are omitted, the return is assumed to be at the total portfolio level

[16] i.e. $\bar{R} = 0.7 \cdot \bar{R}_{equities} + 0.3 \cdot \bar{R}_{treasuries} = \sum_s (W_{t,s} \cdot \bar{R}_{pol,s})$

[17] No attribution strategy can have an ancestor or descendant which is an attribution strategy

Armed with this style of benchmarking, we are able to create a set of attributes such that we can satisfy governance requirements, track excess returns with respect to both broad and specific market exposures and avoid the inclusion of the *interaction* term. Essentially, it's by adding this addition level of benchmarking that we are able to augment BHB to support a hierarchical structure. This allows for a systematic approach to attribution, while not deviating too far from the broadly accepted and understood Brinson-style attribution.

The first and most familiar attribute is *manager alpha*[18]. Defined as the excess return attributable to manager selection, *manager alpha* for asset i is:

$$\propto_i \equiv W_{a,i} \cdot \left(R_{a,i} - \bar{R}_{ref,i}\right)$$

where $\bar{R}_{ref,i}$ is the return of the reference benchmark for asset i. This attribute is similar to Brinson's *selection effect*, but differs in that the actual weight is used as opposed to the target weight. In the institutional portfolio space, target weights are not often set at the manager level. Further, the excess return in terms of its contribution to the total portfolio is a meaningful and easy to digest statistic.

The second attribute is also in terms of asset i. *Portfolio construction*, defined as the excess return attributable to bottom-up construction of the strategy is:

$$PC_i \equiv W_{a,i} \cdot \left(\bar{R}_{ref,i} - \bar{R}^{i}_{ref,s}\right)$$

where $\bar{R}^{i}_{ref,s}$ is the reference benchmark for strategy s of which asset i is a member.

During the asset allocation process, strategies are defined along with their respective reference benchmarks. Clearly, associating managers to a strategy is a function of choosing what expected

[18] In this paper, we refer to individual investments as assets for generality. But given institutional investments are predominantly with managers as opposed to individual assets, we opted to name it *manager alpha*.

manager exposures are elements in that strategy. At the time of assignment, the manager reference benchmark is theoretically the best proxy for these exposures. *Portfolio construction* reflects how the assignment of asset i to strategy s directionally affected the performance of that strategy, in general exposure terms. Evident by the name, this attribute describes how effective adding asset i to strategy s was ex post.

Manager alpha and *portfolio construction* are with respect to asset i. The remain two attributes are with respect to strategy s. Congruent to Brinson attribution, there are measure(s) associated with bottom-up decisions as well as measure(s) associated with top-down processes. In reference to institutional portfolios, asset allocation is very much a top-down process, while the day-to-day focus is often a bottom-up approach. For that reason, attributing performance in terms of both is important.

Tactical is such an attribute associated with strategy s. *Tactical* is measured as:

$$T_s \equiv \left(W_{a,s} - W_{t,s}\right) \cdot \left(\bar{R}_{ref,s} - \sum_s W_{t,s} \cdot \bar{R}_{ref,s}\right)$$

where $W_{a,s}$ is the strategy's actual weight as a percentage of the total portfolio and $W_{t,s}$ is the strategy's target weight set during asset allocation. Based on the above, *tactical* is the excess return attributable to the divergence of the actual strategy weight from its target. Similar to Brinson attribution's *allocation effect*, *tactical* can be interpreted as a strategy's tilt. Plainly, the *tactical* attribute measures the effect on the portfolio of adjusting the actual weight of strategy s after setting the target weight.

The final attribute is *strategic*. *Strategic* measures how the specific ex ante exposures of strategy s performed ex post, with respect to the policy benchmark. That is:

$$S_s \equiv W_{t,s} \cdot \left(\bar{R}_{ref,s} - \bar{R}_{pol,s}\right)$$

Essentially, the *strategic* attribute is the excess return due to the adjustment of the focus of strategy s. It represents investment management's decision to deviate from the broad policy benchmark by adjusting the intended ex ante exposures of strategy s.

Together, these four attributes encompass the *active return* in terms of intuitive investment management decisions[19]:

$$R - \bar{R} = \sum_i (\propto_i + PC_i) + \sum_s (T_s + S_s)$$

Note the absence of an *interaction* term.[20] There is no cross-product generated from the simultaneity of top-down and bottom-up decisions. Instead, interaction is scattered within α and PC.

Linking and Notional Portfolios

As we have clearly chosen arithmetic attribution, linking single period attributes to calculate multiperiod attribution effects is not a simple compounding. If we had chosen geometric attribution, teasing out trailing effects would be trivial. But as end-user reporting[21] is of primary importance, we opted for arithmetic excess returns.

In arithmetic attribution, the sum of the single period attributes is $R - \bar{R}$. Ideally, the natural multiperiod decomposition would be calculated in a similar fashion. Unfortunately, doing so would leave a residual which would grow as the number of periods increased. For this reason, a multiperiod linking

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[19] See Appendix A for a proof that the active return is equal to the sum of the portfolio attributes

[20] See Appendix B for an alternative to the *manager alpha* and *portfolio construction* attributes that includes the *interaction* term, which is useful for those still interested in how *interaction* fits into the institutional portfolio attribution framework

[21] As noted in Menchero (2004), reporting excess returns in arithmetic terms retains widespread popularity because it is regarded as more intuitive

algorithm is necessary. We opted for Frongello (2002) Linking as it was a standard amongst our peers as well as reasons that will become clear subsequently.[22]

Cariño (1999) explained it as the compound return of a portfolio over n periods,

$$R = (1 + R_1)(1 + R_2) \dots (1 + R_n) - 1$$

and the compound return of the benchmark equals,

$$\bar{R} = (1 + \bar{R}_1)(1 + \bar{R}_2) \dots (1 + \bar{R}_n) - 1$$

But the sum of the differences (or alternatively, sum of relative attribution effects) does not equal the difference in the compounded total return,

$$R - \bar{R} \neq (1 + R_1 - \bar{R}_1)(1 + R_2 - \bar{R}_2) \dots (1 + R_n - \bar{R}_n) - 1$$

As per Frongello (2002), defining the variable $G_{b,t}$ as the effect due to attribute b in time period t, similar to the above, the sum of all the variables $G_{b,t}$ does not equal the cumulative excess return,

$$\sum_t \sum_b G_{b,t} \neq R - \bar{R}$$

Therefore, a scaling term (scaled $G_{b,t}$ denoted as $F_{b,t}$) must be applied to all $G_{b,t}$ so that,

$$\sum_t \sum_b F_{b,t} = R - \bar{R}$$

As defined in Frongello (2002), the cumulative outperformance due to attribute b for periods 1 through n is:

$$b_{1,n} = \sum_{t=1}^{n} F_{b,t}$$

where:

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[22] Frongello linking is sometimes referred to as "dollar attribution" and credited to Wilshire as per Bonefede et al. (2002)

$$F_{b,t} = G_{b,t}\left(\prod_{j=1}^{t-1}(1+\acute{R}_j)\right) + \tilde{R}_t(\sum_{j=1}^{t-1} F_{j,b})$$

Typically, $\acute{R}_j$ is the actual return of the total portfolio and $\tilde{R}_t$ is the total portfolio policy benchmark return. However, in our institutional portfolio attribution framework, we vary $\acute{R}_j$ and $\tilde{R}_t$ depending on to what attribute we are referring. We do so such that the grand total still equals the multiperiod portfolio active return, $R - \bar{R}$.

The concept of a notional portfolio was introduced in BHB in reference to hypothetical portfolios composed of combinations of actual returns, benchmark returns, actual weights and target weights. These notional portfolios were differenced to produce the *asset allocation* and *security selection* attributes.[23]

As there are four attribute types, we introduce four notional portfolios. In doing so, we define the identity:

$$\sum_b b_{1,n}^{adj} = nP_b = \acute{R} - \tilde{R}$$

Where $b_{1,n}^{adj}$ is the pro-rated Frongello term and nP_b is the excess return defined by the notional portfolio applied to attribute b. Using this adjusted multiperiod attribute, we can use simple sums to present performance attribution in an intuitive and meaningful table. Figure 2 provides an abstract example of such a report.

[23] In this paper, we abstracted away from this concept as we felt this was implied. Instead, we opted to introduce notional portfolios where the application of hypothetical portfolios may not be so straightforward.

Attribution Strategy	Alpha[24]	Portfolio Construction[25]	Tactical[26]	Strategic[27]	**Total[28]**
	Attribution Decomposition				
S_1	$\alpha_{s,1}$	$PC_{s,1}$	$T_{s,1}$	$S_{s,1}$	$Tot_{s,1}$
S_2	$\alpha_{s,2}$	$PC_{s,2}$	$T_{s,2}$	$S_{s,2}$	$Tot_{s,2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_n	$\alpha_{s,n}$	$PC_{s,n}$	$T_{s,n}$	$S_{s,n}$	$Tot_{s,n}$
Grand Total	$\sum_{i=1}^{n} \alpha_{s,i} = R - \bar{R}_I^A$	$\sum_{i=1}^{n} PC_{s,i} = \bar{R}_I^A - \bar{R}_S^A$	$\sum_{i=1}^{n} T_{s,i} = \bar{R}_S^A - \bar{R}_S^T$	$\sum_{i=1}^{n} S_{s,i} = \bar{R}_S^T - \bar{R}$	$\sum_{i=1}^{n} Tot_{s,i} = R - \bar{R}$

Figure 2: Tabular Multiperiod (or single period) attribution report[29]

Each of the notional portfolios presents the summation of an attribute over the plan as a meaningful statistic. For *manager alpha*, the notional portfolio is the total plan excess return of the weighted[30] sum of the assets' returns over the assets' reference benchmarks:

$$nP_\alpha = R - \bar{R}_I^A$$

where $R = \sum_i w_i R_{a,i}$ is the weighted actual return for all assets in total plan, and $\bar{R}_I^A = \sum_i w_i \bar{R}_{ref,i}$ is the weighted reference asset benchmark return for those same assets. This describes how the assets performed versus their benchmark at the total plan level.

For *portfolio construction*, the notional portfolio is the total plan excess return of the asset benchmark return over the actual strategy reference benchmark return:

$$nP_{PC} = \bar{R}_I^A - \bar{R}_S^A$$

[24] $\alpha_{s,x} = \sum_i \alpha_i \ \ \forall i \in S_x$
[25] $PC_{s,x} = \sum_i PC_i \ \ \forall i \in S_x$
[26] $T_{s,x} = \sum_i T_i \ \ \forall i \in S_x$
[27] $S_{s,x} = \sum_i S_i \ \ \forall i \in S_x$
[28] $Tot_{s,x} = \alpha_{s,x} + PC_{s,x} + T_{s,x} + S_{s,x}$
[29] For an alternative decomposition using *strategy alpha* and *interaction* terms, refer to Appendix C
[30] Weights are calculated as the net asset value (NAV) percentage of total plan NAV

where $\bar{R}_S^A = \sum_S w_S^A \bar{R}_{ref,s}$ is the actual weighted sum of the strategies' reference benchmark returns. This notional portfolio characterizes how the asset benchmarks preformed relative to the strategy reference benchmarks.

Next, *tactical*'s notional portfolio is:

$$nP_T = \bar{R}_S^A - \bar{R}_S^T$$

where $\bar{R}_S^T = \sum_S w_S^T \bar{R}_{ref,s}$ is the target weighted sum of the strategies' reference benchmark returns. This can be interpreted as the excess return of the total plan sum of the strategy reference benchmarks when comparing the actual weight to the target weight. A positive number would imply that strategy tilts added value to the portfolio during the reference period.

Finally, the *strategic* notional portfolio is:

$$nP_S = \bar{R}_S^T - \bar{R}$$

This difference describes how the outcome of the decision to hone in on specific exposures within strategies performed relative to the broad policy benchmark.

Clearly, these notional portfolios are congruent to their individual asset and strategy counterparts. The interpretations of the notional portfolios only differ from the individual attributes in that they are calculated at the total plan level. But pro-rating the attributes in these terms scales them in terms easily interpretable regardless of number of periods or at what level in the hierarchy.

Numerical Example

To illustrate usage and directional interpretation of results, numbers were generated for the explicit purpose of providing an abstract example. Figure 3 presents hypothetical institutional multiperiod attributes and summations.

	Manager Alpha	Portfolio Construction	Tactical	Strategic	Total
Public Equity	0.85%	-0.25%	0.36%	-0.42%	0.54%
Real Estate	-0.18%	-0.11%	-0.01%	0.00%	-0.30%
Private Equity	-0.21%	0.02%	0.00%	0.00%	-0.19%
Absolute Return	0.29%	-0.02%	0.15%	0.00%	0.42%
Fixed Income	0.11%	-0.31%	-0.03%	-0.55%	-0.78%
Total Plan	**0.86%**	**-0.67%**	**0.46%**	**-0.97%**	**-0.32%**

Figure 3: Hypothetical multiperiod institutional portfolio attribution

As is evident in Figure 3, the individual asset attributes are aggregated into their respective attribution strategies. In doing so, five rows are generated for the five attribution strategies. These strategies can be in any level of the hierarchy, and there can be any number of them. For example, because attributes are additive, we could sum Private Equity and Public Equity to present a generalized Equity strategy. The only restriction is that they cannot be overlapping, as noted earlier.

Interpretation of Figure 1 is dependent on a few items. As this is an abstract example, exacting process implications would be too granular. Given performance attribution is in terms of excess returns, how the portfolio is benchmarked strongly influences the results. Further, interpreting the outcomes in a vacuum may lead to detrimental repositioning. Systematic factors and the number of periods analyzed must be taken into account as results could shift once a particular event enters or leaves the active window. For this reason, looking at multiple period lengths side-by-side is highly beneficial. More cohesive conclusions can be drawn when observing both contemporary and longer term horizons. That being said, the following analysis assumes a thorough benchmarking process.

Looking at *manager alpha*, we can interpret the attribute at both the strategy and total plan levels. For instance, at the total plan level the excess return of all the assets against their individual benchmarks is 86 bps. Drilling down to each individual strategy, we can see how each contributed to that outperformance. According to Figure 3, Public Equity stands out as the top performer followed by Absolute Return and then Fixed Income. Detracting from this outperformance is Real Estate and Private Equity.

The interpretation is twofold. At a total plan level, bottom-up asset choice and sizing added value. At the strategy level, time spent choosing assets in Public Equity was well spent while Real Estate was detrimental. Investment management could use this information to decide where to focus their efforts. For example, they could decide to passively invest in Real Estate's benchmark and focus more on their core competencies such as Public Equity and Absolute Return. Further, if Public Equity was bucketed into small, mid and large cap categories, investment management could evaluate how each capsize performed relative its benchmark and reallocate Public Equity accordingly. If small cap underperformed relative its benchmark but large cap outperformed, shifting NAV from small cap to large cap could increase Public Equity *manager alpha* going forward.

Next, *portfolio construction* at the total plan level negatively influenced the active return. This states that ex ante decisions on individual asset exposures underperformed their respective strategy exposures.[31] Given the size and direction, investment management could choose to rethink the

[31] Assuming the benchmarks are good representations of their respective asset exposures

allocations to certain exposures within strategies. Continuing the Public Equity capsize example, a negative *portfolio construction* attribute implies that the individual capsize categories underperformed the Public Equity reference benchmark. In that case, resizing the underperforming sectors and possibly adding unrepresented sectors within Public Equity could be a viable solution.

Taking a step back, a point we want to emphasize is that the flexibility to sum at any level in the hierarchy, coupled with the ability to sum across attributes, allows for targeted portfolio management. Some analysts may want to look at *manager alpha* in isolation for performance compensation, while higher level management needs to look at the sum of *manager alpha* and *portfolio construction* to decide how much NAV to allocate to specific sectors. This framework was specifically built to support aggregation across these dimensions.

Next, looking at *tactical* gleans some useful information. At the total plan level, tactical tilts added value. This could be due to directed decisions or because of rebalancing periodicity. The interpretation could be that investment management observed changes in the market and adjusted asset allocation fortuitously. Alternatively, the rebalancing periodicity may be too infrequent to correct for a range-bound market.

By comparing the strategy level *tactical* attributes, relative observations are quite clear. Tilts in Public Equity and Absolute Return were strong contributors to the direction of total plan *tactical*. Absolute Return is reasonably positive and Real Estate is of negligible consequence. Without knowing the direction of the tilts, it is difficult to tell a complete story. As portfolio managers know the actual and target weights of their strategies, interpretation is much simpler. For example, if we assume Public Equity had a positive tilt, the positive *tactical* attribute reveals that the Public Equity reference benchmark outperformed the weighted average strategy reference benchmark return. Thus, the 36 bps *tactical* attribute states the effect of leaning into the Public Equity strategy on the total plan in terms of its contribution to the active return.[32]

Next, plan level *strategic* is strongly negative and the largest attribute in absolute terms. This suggests that strategy level exposure decisions materially detracted from the active return. That being said, the only contributors to this aggregate attribute are Public Equity and Fixed Income.[33] As both strategy level attributes are negative, investment management should revisit the decision to adjust exposures away from policy benchmarks.

Finally, looking at the total plan attributes and their individual contributions to the active return, we can make some general observations. Specifically, at the total plan level, manager alpha and tactical are the only positive contributors to the active return. One interpretation of this result is that this particular team excels at picking assets and reading the market in the short-to-moderate term. Assuming this to be the case, a recommendation could be that the team focus on picking assets that fit within the constraints of the policy benchmark exposures, and subsequently resize these investments to take advantage of market timing opportunities.

As stated before, context is fundamentally tied to the interpretation of this institutional portfolio attribution framework. Portfolio liquidity and market conditions play major roles in translating the results into investment decisions. Attribution is intrinsically backward-looking. Portfolio management also requires forward-looking considerations. For example, market expectations may override suggestions to rebalance in spite of attribute indicators. The model translates performance into

[32] Typically, as in Private Equity, a zero value for tactical suggests the target and actual weights are the same.
[33] A zero value for strategic with other non-zero attributes implies $\bar{R}_{ref,s} = \bar{R}_{pol,s}$

attributes scaled in terms of active return and portfolio decisions. It is up to the user to frame the results in the appropriate context. It is up to the user to frame the results in the appropriate context.

Conclusion

We feel that this model lends a more intuitive and nuanced look into drivers of historical performance. With this information, decisions can be made on what aspects of the investment process to focus on, and whether to modify current exposures or introduce new ones based in various aggregations.

As the model lends itself naturally to programmatic generation, we were able to build a solution that systematically populates a database with historical and contemporaneous results for a number of trailing period lengths. This saves a substantial amount of the analysts' time, as typical attribution reporting is intensive from both a computation and data collection standpoint. Further, by leveraging object-oriented programming, we can minimize computational and human error, effectively raising the statistics' credibility.

Currently, we are working on pairing these historical attributes with forward-looking risk measures. We have made a fair amount of progress but, as is typical with risk, new challenges introduce themselves. Material to attribution, risk benchmarks should theoretically have even less tracking error than a somewhat broad strategy-level reference benchmark. But using a different benchmark decouples the paired models from one another. Further, heterogeneous asset pricing methods and reporting periodicities pose challenges typical to risk and create a mismatch with attribution's actual reported performance measurement style.

Appendix A

$$Active\ Management = R - \bar{R} = \sum_i (\propto_i + PC_i) + \sum_s (T_s + S_s)$$

Proof:

$$\propto_i + PC_i = W_{a,i} \cdot \left(R_{a,i} - \bar{R}_{ref,i}\right) + W_{a,i} \cdot \left(\bar{R}_{ref,i} - \bar{R}_{ref,s}\right)$$

$$= W_{a,i} \cdot R_{a,i} - W_{a,i} \cdot \bar{R}_{ref,i} + W_{a,i} \cdot \bar{R}_{ref,i} - W_{a,i} \cdot \bar{R}_{ref,s} = W_{a,i} \cdot R_{a,i} - W_{a,i} \cdot \bar{R}_{ref,s}$$

$$\therefore \sum_i (\propto_i + PC_i) = \sum_i \left(W_{a,i} \cdot R_{a,i} - W_{a,i} \cdot \bar{R}^i_{ref,s}\right) = \sum_i \left(W_{a,i} \cdot R_{a,i}\right) - \sum_i \left(W_{a,i} \cdot \bar{R}^i_{ref,s}\right)$$

$$= \sum_i \left(W_{a,i} \cdot R_{a,i}\right) - \sum_s \left(W_{a,s} \cdot \bar{R}_{ref,s}\right)$$

$$T_s + S_s = \left(W_{a,s} - W_{t,s}\right) \cdot \left(\bar{R}_{ref,s} - \sum_s W_{t,s} \cdot \bar{R}_{ref,s}\right) + W_{t,s} \cdot \left(\bar{R}_{ref,s} - \bar{R}_{pol,s}\right)$$

$$= \bar{R}_{ref,s} \cdot W_{a,s} - W_{a,s} \cdot \sum_s W_{t,s} \cdot \bar{R}_{ref,s} + W_{t,s} \cdot \sum_s W_{t,s} \cdot \bar{R}_{ref,s} - W_{t,s} \cdot \bar{R}_{pol,s}$$

$$\therefore \sum_s (T_s + S_s) = \sum_s \left(\bar{R}_{ref,s} \cdot W_{a,s} - W_{a,s} \cdot \sum_s W_{t,s} \cdot \bar{R}_{ref,s} + W_{t,s} \cdot \sum_s W_{t,s} \cdot \bar{R}_{ref,s} - W_{t,s} \cdot \bar{R}_{pol,s}\right)$$

$$= \sum_s \left(W_{a,s} \cdot \bar{R}_{ref,s}\right) - \sum_s \left(W_{t,s} \cdot \bar{R}_{pol,s}\right)$$

$$+ \left(\sum_s W_{t,s} \cdot \sum_s W_{t,s} \cdot \bar{R}_{ref,s} - \sum_s W_{a,s} \cdot \sum_s W_{t,s} \cdot \bar{R}_{ref,s}\right)$$

$$= \sum_s \left(W_{a,s} \cdot \bar{R}_{ref,s}\right) - \sum_s \left(W_{t,s} \cdot \bar{R}_{pol,s}\right)$$

$$\Rightarrow \sum_i (\propto_i + PC_i) + \sum_s (T_s + S_s)$$

$$= \sum_i \left(W_{a,i} \cdot R_{a,i}\right) - \sum_s \left(W_{a,s} \cdot \bar{R}_{ref,s}\right) + \sum_s \left(W_{a,s} \cdot \bar{R}_{ref,s}\right) - \sum_s \left(W_{t,s} \cdot \bar{R}_{pol,s}\right)$$

$$= \sum_i \left(W_{a,i} \cdot R_{a,i}\right) - \sum_s \left(W_{t,s} \cdot \bar{R}_{pol,s}\right)$$

$$\therefore \; Active\ Management = \sum_i (\propto_i + PC_i) + \sum_s (T_s + S_s) \blacksquare$$

Appendix B

As institutional portfolio attribution is driven just as much by reporting requirements as it is self-evaluation, an alternative bolt-on methodology may be necessary. For those familiar with and interested in the traditional *interaction* term, the *strategy alpha* attribute completes the model such that the sum of the attributes is still the active return.

Similar to *manager alpha*, *strategy alpha* is the excess return of strategy s over its reference benchmark. That is:

$$\alpha S_s \equiv W_{t,s} \cdot \left(R_{a,s} - \bar{R}_{ref,s}\right)$$

And the *interaction* term is calculated as,

$$I_s \equiv \left(W_{a,s} - W_{t,s}\right) \cdot \left(\sum\nolimits_{i \in s} \left(W_{a,i} \cdot R_{a,i}\right) - \bar{R}_{ref,s}\right)$$

and can be interpreted as a measure of investment management' skill in tactically adjusting the asset allocation in the same direction of the underlying assets excess return. Clearly, this is the same cross product as in BHB.

Conveniently,

$$\sum\nolimits_{s} (\alpha S_s + I_s) = \sum\nolimits_{i} (\propto_i + PC_i)$$

Proof:

$$\propto_i + PC_i = W_{a,i} \cdot \left(R_{a,i} - \bar{R}_{ref,i}\right) + W_{a,i} \cdot \left(\bar{R}_{ref,i} - \bar{R}_{ref,s}\right)$$

$$= W_{a,i} \cdot R_{a,i} - W_{a,i} \cdot \bar{R}_{ref,i} + W_{a,i} \cdot \bar{R}_{ref,i} - W_{a,i} \cdot \bar{R}_{ref,s} = W_{a,i} \cdot R_{a,i} - W_{a,i} \cdot \bar{R}_{ref,s}$$

$$\therefore \sum\nolimits_{i} (\propto_i + PC_i) = \sum\nolimits_{i} \left(W_{a,i} \cdot R_{a,i} - W_{a,i} \cdot \bar{R}^{i}_{ref,s}\right) = \sum\nolimits_{i} \left(W_{a,i} \cdot R_{a,i}\right) - \sum\nolimits_{i} \left(W_{a,i} \cdot \bar{R}^{i}_{ref,s}\right)$$

$$= \sum\nolimits_{i} \left(W_{a,i} \cdot R_{a,i}\right) - \sum\nolimits_{s} \left(W_{a,s} \cdot R_{ref,s}\right)$$

$$\alpha S_s + I_s = W_{t,s} \cdot \left(R_{a,s} - \bar{R}_{ref,s}\right) + \left(W_{a,s} - W_{t,s}\right) \cdot \left(\sum_{i \in s} \left(W_{a,i} \cdot R_{a,i}\right) - \bar{R}_{ref,s}\right)$$

$$= W_{t,s} \cdot R_{a,s} - W_{t,s} \cdot \bar{R}_{ref,s} + W_{a,s} \cdot \sum_{i \in s} \left(W_{a,i} \cdot R_{a,i}\right) - W_{t,s} \cdot \sum_{i \in s} \left(W_{a,i} \cdot R_{a,i}\right) - W_{a,s} \cdot \bar{R}_{ref,s} + W_{t,s} \cdot \bar{R}_{ref,s}$$

$$= W_{t,s} \cdot R_{a,s} - W_{a,s} \cdot \bar{R}_{ref,s} + W_{a,s} \cdot \sum_{i \in s} \left(W_{a,i} \cdot R_{a,i}\right) - W_{t,s} \cdot \sum_{i \in s} \left(W_{a,i} \cdot R_{a,i}\right)$$

$$\therefore \sum_{s} \left(\alpha S_s + I_s\right)$$

$$= \sum_{s} \left(W_{t,s} \cdot R_{a,s} - W_{a,s} \cdot \bar{R}_{ref,s} + W_{a,s} \cdot \sum_{i \in s} \left(W_{a,i} \cdot R_{a,i}\right) - W_{t,s} \cdot \sum_{i \in s} \left(W_{a,i} \cdot R_{a,i}\right)\right)$$

$$= \sum_{s} \left(W_{t,s} \cdot R_{a,s} - W_{a,s} \cdot \bar{R}_{ref,s}\right) = \sum_{s} \left(W_{t,s} \cdot R_{a,s}\right) - \sum_{s} \left(W_{a,s} \cdot \bar{R}_{ref,s}\right)$$

$$= \sum_{i} \left(W_{a,i} \cdot R_{a,i}\right) - \sum_{s} \left(W_{a,s} \cdot \bar{R}_{ref,s}\right) = \sum_{i} \left(\propto_i + PC_i\right) \blacksquare$$

Appendix C

Using the alternative attributes described in Appendix B, *alpha* and *portfolio construction* can be replaced in Figure 2 with:

	Alternative Decomposition		
Attribution Strategy	Strategy Alpha	Interaction	**Total**[34]
S_1	$\alpha S_{s,1}$	$I_{s,1}$	$aTot_{s,1}$
S_2	$\alpha S_{s,2}$	$I_{s,2}$	$aTot_{s,2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_n	$\alpha S_{s,n}$	$I_{s,n}$	$aTot_{s,n}$
Grand Total	$\sum_{i=1}^{n} \alpha S_{s,i}$	$\sum_{i=1}^{n} I_{s,i}$	$\sum_{i=1}^{n} aTot_{s,i} = R - \bar{R}_S^A$

As *strategy alpha* and *interaction* are strategy-level attributes, there is no need to pro-rate them more granularly than the sum of the *alpha* and *portfolio construction* notional portfolios. The alternative attribute's contribution to the total plan excess return of actual weighted return versus the actual strategy reference benchmark return is meaningful in our hierarchical framework.

[34] $aTot_{s,x} = \alpha S_{s,x} + I_{s,x} = \alpha_{s,x} + PC_{s,x}$

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Comparing Performance Attribution Linking Methods:

An Empirical Study

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Abstract: A number of methods have been developed to link single-period arithmetic attribution results. We present the first institutional portfolio empirical study comparing the most referenced methods for producing additive multiperiod attributes from their single-period counterparts. While our findings suggest the methods typically produce similar results, we find a pattern in the way the methods' results relate to one another. We find the Modified Frongello Method and Cariño Method to produce nearly identical results, the Frongello and Cariño methods to cluster and the Naïve and Menchero methods to be outliers.

INTRODUCTION

Attribution analysis is the exercise of comparing a portfolio's performance relative to a benchmark in terms of a predetermined set of effects. While single-period performance attribution methodologies are more or less standardized, there is no clear consensus on how to link the single-period results to produce multiperiod attributes.

In this paper, we compare the most referenced methods in an empirical study. We apply each of the methods to a single portfolio over time to produce a set of apples-to-apples results. This is a rarely attempted exercise as the implementation of each of these methods is quite labor intensive. However, utilizing proprietary software we developed, the application of different methods was relatively simple.

We compare Frongello, Modified Frongello, Reverse Frongello, Cariño, Menchero and Naïve methods. Once the linked attributes are calculated, how to compare the results is not as straightforward as it may seem. For a decades old portfolio, the number of multiperiod attributes for six different methods is staggering. Furthermore, some nontrivial criteria need to be established to decide whether the numerical results are similar to each other. Therefore, we implement a dispersion function to compare the matched linking method results for different trailing periods. We then use a threshold parameter as an input into a condition to filter the results. After filtering, we are left with a set of results that we can analyze.

Our discoveries are simultaneously benign and compelling. We find that at a reasonable threshold, the methods produce similar results. However, as the threshold is reduced patterns emerge. Naively compounding the attributes and the Menchero methods tend to be outliers, while the Frongello methods and the Cariño Method tend to cluster together. These results are proved to be quite robust to time and trailing period.

We proceed as follows: Section 2 reviews the attribution linking methods most referred to in the literature. Section 3 describes our approach to comparing these methods to one another. Section 4 discusses the dataset used for the analysis. Section 5 presents the results. Section 6 concludes.

2 ATTRIBUTION LINKING METHODS

$G_{tb} = original\ arithmetic\ attribute\ b\ in\ period\ t$

$R = cumulative\ portfolio\ return$

$\bar{R} = cumulative\ benchmark\ return$

$R_t = portfolio\ return\ in\ period\ t$

$\bar{R}_t = benchmark\ return\ in\ period\ t$

$$\sum_b G_{tb} = R_t - \bar{R}_t \nRightarrow \left(\sum_t \sum_b G_{tb} = R - \bar{R}\ or\ \left[\prod_t \prod_b (1 + G_{tb})\right] - 1 = R - \bar{R}\right)$$

2.1 Naïve[35] Method (N)

$F_b = adjusted\ trailing\ attribute\ b$

$$\sum_b F_b = R - \bar{R}$$

$$F_b^{COMP} = \frac{\prod_t (1 + G_{tb}) - 1}{\sum_b [\prod_t (1 + G_{tb}) - 1]} \cdot \left(\prod_t (1 + R_t) - \prod_t (1 + \bar{R}_t)\right)$$

$$F_b^{ADD} = \frac{\sum_t G_{tb}}{\sum_b \sum_t G_{tb}} \cdot \left(\prod_t (1 + R_t) - \prod_t (1 + \bar{R}_t)\right)$$

---

[35] Naming the method “Naïve” is attributed to Laker (2002)

Simply adding or compounding single-period attributes is what the literature refers to as the *naïve method*. The issue with this method put mathematically as in Cariño (2012):

The compound return of a portfolio is

$$R = (1 + R_1)(1 + R_2) \dots (1 + R_n) - 1$$

And the compound return of the benchmark equals

$$\bar{R} = (1 + \bar{R}_1)(1 + \bar{R}_2) \dots (1 + \bar{R}_n) - 1$$

But the sum of the differences (or alternatively, the sum of relative attribution effects) does not equal the difference in the compounded total return,

$$R - \bar{R} \neq (1 + R_1 - \bar{R}_1)(1 + R_2 - \bar{R}_2) \dots (1 + R_n - \bar{R}_n) - 1$$

Therefore, as Christopherson et al. (2009) puts it, the issue is that market values are additive within periods, while returns are multiplicative across periods. This mismatch leaves a residual over time if one attempts to simply add single period attributes or multiply equally weighted returns.

When using the Naïve Method, the individual attributes must be rescaled if they are to sum up to the cumulative excess return. To do so, we weight the linked attribute by its percentage of the total of all linked attributes in terms of the cumulative excess return. In terms of F_b^{COMP}, we compound attribute b over the trailing period, divide by the sum of all attributes compounded over the trailing period, and then multiply that by the cumulative excess return. Doing so, $\sum_b F_b^{COMP} = R - \bar{R}$.[36]

[36] For the results section of this paper, we focus on naïve compounding (i.e. F_b^{COMP}).

2.2 Scaling Coefficient Methods

Single period attributes are additive. However, attributes cannot be summed or compounded to explain the total excess return over multiple periods. Using a scaling coefficient, single-period attributes can be transformed so that they sum to the total active return over the periods in question.

As an argument in favor of coefficient methods, they are not order dependent. Meaning, whether calculating scaled attributes from $0 \rightarrow T$ or $T \rightarrow 0$, the scaled attributes will be the same. Arguing against, Mirabelli (2001) noted the coefficient algorithms require information about the returns of the entire period in order to calculate the coefficients of any individual period. Additionally but related, single-period results will be readjusted every time a new period is added, as the total returns change. This is considered a negative aspect of coefficient methods as the absolute contribution of a single period attribute to a trailing statistic is dynamic, which makes writing attribution narratives more complex.

$F_{tb} = adjusted\ arithmetic\ attribute\ b\ in\ time\ t$

$C_t = scaling\ factor\ in\ period\ t$

$F_{tb} = G_{tb}(C_t)$

$\sum_t \sum_b F_{tb} = R - \bar{R}$

2.2.1 Cariño Scaling Coefficient (C) (logarithmic linking coefficient)

$$C_t^{CAR} = \frac{[\ln(1 + R_t) - \ln(1 + \bar{R}_t)]/(R_t - \bar{R}_t)}{[\ln(1 + R) - \ln(1 + \bar{R})]/(R - \bar{R})} = \frac{f(R_t, \bar{R}_t)}{f(R, \bar{R})}$$

The Cariño method scales returns through the relationship between nominal returns and the natural log of the nominal returns. Christopherson et al. (2009) states that discrete returns and their effects already constitute an approximation to the truth. As such, it seems appropriate to distribute the residual proportionally.

The criticism of this method is that it tends to overweight periods of lower relative returns, and underweight periods of higher than average returns. According to Menchero (2004), this can be seen more clearly by doing a Taylor series expansion of $f(R_t, \bar{R}_t)$ for small returns:

$$f(R_t, \bar{R}_t) \approx 1 - \frac{1}{2}(R_t + \bar{R}_t)$$

Menchero (2004) refers to this behavior as not metric preserving. Meaning, that two periods with identical returns will not necessarily have the same linking coefficient. Cariño (2012) argues that the difference in the coefficients is a natural consequence of the compounding of attributes.

2.2.2 Menchero Scaling Coefficient (M)

$T = total\ number\ of\ periods$

$$C_T^{MEN} = (1/T) \cdot \left[(R - \bar{R}) / \left((1+R)^{1/T} - (1+\bar{R})^{1/T}\right)\right] + \frac{\left(R - \bar{R} - (1/T) \cdot \left[(R - \bar{R}) / \left((1+R)^{1/T} - (1+\bar{R})^{1/T}\right)\right] \cdot \sum_t (R_t - \bar{R}_t)\right) \cdot (R_t - \bar{R}_t)}{\sum_t (R_t - \bar{R}_t)^2}$$

Menchero (2004) states that the aim of his scaling coefficient method is to minimize the variance of attribution effects across periods. Referring to Cariño's method, he argues that there is no defensible basis for the scaling an attribute based on the magnitude of the returns of the period. By uniformly distributing the arithmetic attribution effects, Menchero suggests that his method is more stable in volatile markets.

2.3 Frongello Methods (recursive)

Cariño (2002) labeled the portions of excess return that arise when current attributes compound over attributes earned in prior periods as “cross products”. For recursive linking methods, he notes it is unclear how to allocate cross products of different attributes in different periods. It can be thought of as a multi-period interaction term. There is not a good answer, for example, on whether to assign a multi-period cross product to allocation or selection when they compound over one another. The Frongello methods are designed such that each one assigns the cross product to the prior effect, the latter effect or a combination of the two. It is up to the investment professional to decide which assignment to make by choosing the appropriate algorithm.

Bonafede et al. (2002) remarks that while the order of periods has no bearing on the resulting cumulative total return, the order of periods does have a bearing on the cumulative attribution results. Contrarily, Menchero (2004) opines that attribution linking should be independent of ordering of the periods, as relative performance is independent. He argues that if one assumes performance attribution should be order dependent, he is implicitly assuming that geometric attribution is incorrect. Whether or not order dependence is correct or desirable remains an argument of intuition. At this point, there is no universally accepted answer.

Each of the three Frongello algorithms is composed of two components. The first is the contribution to excess return due to active decisions in the current period. The second is the contribution to excess return due to past active decisions compounding into the present, based on some current rate of return.

2.3.1 (Simplified) Frongello (F) (latter effect)

$$F_{tb} = G_{tb} \cdot \prod_{j=1}^{t-1} (1 + R_j) + \bar{R}_t \cdot \sum_{j=1}^{t-1} F_{jb}$$

Generally, the Frongello method takes the current attribute G_{tb} and scales it by a total return through the prior period (in this case, the total portfolio return $\prod_{j=1}^{t-1}(1 + R_j)$). It then compounds this with the total return due to G_{tb} through the prior period ($\sum_{j=1}^{t-1} F_{jb}$), scaled by the current period return (in this case, the return of the benchmark $\bar{R}_t$). In this method, current attributes are earned based on the cumulative total performance of the portfolio. Thus, the cross product is assigned to the latter effect as opposed to the prior.

This algorithm is referred to as the Simplified Frongello Method or just the Frongello Method. Frongello (2006) states this algorithm's deviation from the Modified Frongello Method (Section 2.3.3) is a function of the portfolio's tracking error. As tracking error gets smaller, the two methods converge. And for most portfolio's, the differences although noticeable are typically immaterial.

2.3.2 Reverse Frongello (rF) (prior effect)

$$F_{tb} = G_{tb} \cdot \prod_{j=1}^{t-1} (1 + \bar{R}_j) + R_t \cdot \sum_{j=1}^{t-1} F_{jb}$$

In this method, current attributes are earned based on the cumulative benchmark return. Thus, the cross product is assigned to the prior effect.

2.3.3 Modified Frongello (mF) (half latter effect, half prior effect)

$$F_{tb} = G_{tb} \cdot 0.5 \cdot \left[\prod_{j=1}^{t-1} (1 + R_j) + \prod_{j=1}^{t-1} (1 + \bar{R}_j)\right] + 0.5 \cdot (R_t + \bar{R}_t) \cdot \sum_{j=1}^{t-1} F_{jb}$$

The equation states that attributes earn and are carried forward by the benchmark return plus half the portfolio outperformance. Essentially, the Modified Frongello algorithm assigns half of the cross product term to the prior effect and half to the latter. This approach attributes an equal-weighted cross product in all situations, giving top-down and bottom-up management equal credit.

According to Frongello (2006), the Modified Frongello Method's order dependence is typically immaterial. In most cases he tested, the Cariño method and Modified Frongello method returned identical results to "…many decimal places." Unlike Simplified Frongello, Modified Frongello compounds current attributes into the future based on an average rate of the total return of the portfolio and benchmark. This makes its effect on the trailing attribute similar to Cariño's coefficient method.

2.4 Method Characteristics

2.4.1 Minimum Required Characteristics (Cariño 2012)

Generality {mF,F,rF,C,M}

The method supports any additive single-period scheme.

Familiarity {mF,F,rF,C,M}

The interpretation of the multi-period and single period results are the same.

No Residuals/Distortions {mF,F,rF,C,M}

The method explains the excess return exactly without introducing unnecessary distortion.

2.4.2 Additional Differentiating Characteristics

Non A-Causal {F,rF,mF,N}

The linking methodology should not be dependent upon future events when scaling single-period attributes (Mirabelli 2001).

The Frongello methods are not dependent on future returns. Unlike the Cariño and Menchero methods, the prior period scaled coefficients do not change when additional periods are added.

Sincerity {mF,F,rF}

Explains the mechanical difference between single-period coefficient scaling versus recursive scaling (Frongello 2006).

Coefficient algorithms are functions of the cumulative active return. As such, every time a new period is added, the coefficients on past attributes change as the cumulative active return is updated. Frongello (2006) argues that this scaling process alters the attribute's contribution to the excess return of its respective period. However, the principle reason to look at a single-period scaled attribute for a particular trailing period is to assess its contribution to that particular trailing period. Admittedly, Frongello (2006) states this aspect of the mechanics of scaling methods is immaterial when assessing the trailing attribute itself. That being said, the Frongello methods (recursive scaling) does not reassess historical scaling when new periods are introduced.

Intuitive {mF,F,rF,N}

The method uses mathematics that are usable by a wide, non-academic audience. In other words, the methods are practitioner-friendly.

The Naïve Method is obviously the most intuitive. However, of the methods which seek to adjust attributes to sum properly, the Frongello Methods are typically sited as the most intuitive. All methods are challenging to implement, but Frongello Methods do not require rescaling single-period attributes, and their recursive design is more easily explainable and implemented by practitioners.

Order Independence {(mF),C,M,N}

The ordering of periods will not affect the cumulative attribution results.

Though the Modified Frongello Method is mathematically order dependent, according to Frongello (2006), the order dependence is not noticeable to many decimal places in almost all cases. As stated above, it is debatable as to whether or not order dependence is desirable. In the methods examined, order independence comes packaged with the undesirable restatement of adjusted attributes each period.

Return Sensitive {mF,F,rF,C}

Scaling terms are a function of the relative size of returns. Periods of low (high) returns will require higher (lower) scaling than periods of high (low) returns.

As stated above, the aim of Menchero's method is to minimize the variance of attribute effects across periods. The result is that the Menchero Method's scaling coefficients do not vary with the level of return. However, Frongello and Cariño both argue that as attributes are defined as components of excess return and compound with other periods, an attribute in a lower (higher) return period should have a higher (lower) scaling than a comparable attribute in a higher (lower) return period.

Associative {mF,F,rF,C}

The order in which the execution of the method is performed does not matter as long as the sequence of the periods is not altered (See Appendix A for Associative Property proofs for each method).

While the commutative property has been widely referenced in the literature, the associative property has not been explored. We thought this property to be a strong addition to the linking characteristics commonly mentioned when introducing a new method or comparing existing methods. Further, its relationship with the dispersion of our results is compelling.

3 COMPARISON APPROACH

Subsequent to calculating trailing attributes using all the methods described in Section 2, comparison of the results is not straightforward. The results vary by linking method, trailing period, attribute type, portfolio type and target entity (the particular fund or strategy). For our dataset, the permutations result in individual statistics in the hundreds of millions. Therefore, we have to decide how to group and then compare the results systematically.

We define:

$I = \{F, mF, rF, C, M, N\}$

$L_{i,T} = linking\ algorithm\ function\ i \in I\ for\ period\ length\ T$

$$MaxDist(J,T) = abs\left(\max_{j\in J} L_{j,T} - \min_{j\in J} L_{j,T}\right)$$

$$where\ abs(x) = absolute\ value\ of\ x, and\ \ J \subseteq I$$

3.1 Threshold Break Condition (TBC)

The TBC is a function of the threshold parameter. The threshold parameter is a user-supplied decimal which relates to the relative and absolute thresholds in the following manner:

$$TH = Threshold\ Parameter$$

$$rTH = Relative\ Threshold = TH$$

$$aTH = Absolute\ Threshold = rTH/100$$

The TBC states the maximum distance between any two linking algorithm results in a particular subset of I must be greater than $(TH \cdot 100)$ % of the absolute value of the average of the subset's linking algorithm results, and the absolute value of the maximum distance between any two linking algorithm results must be greater than $(TH \cdot 100)\ basis\ points$. Put mathematically:

$$MaxDist(J,T) > rTH \cdot \left(\sum_j abs\left(L_{j,T}\right)/n\right)\ and\ abs\left(MaxDist(J,T)\right) > aTH$$

$$where\ n = number\ of\ elements\ in\ J$$

This condition being met represents a material divergence within a subset J of linking algorithm results. By employing absolute and relative comparisons simultaneously, we make sure the spread of the employed attribution linking methods for a particular attribute is of interest. However, we realize that many divergences smaller than aTH may be material at the total portfolio level. Understanding this, we use both comparisons to highlight the patterns and then relax the absolute threshold comparison to verify our observations.

3.2 Portfolio Types

3.2.1 Notional Portfolios

The concept of a notional portfolio was introduced in Brinson et al. (1986) (BHB) in reference to hypothetical portfolios composed of combinations of actual returns, benchmark returns, actual weights and target weights. These notional portfolios were differenced to produce the *asset allocation* and *security selection* attributes.[37]

We utilize four notional portfolios to match the institutional portfolio attributes as laid out in Jiang et al. (2013). In doing so, we use the identity:

$$\sum_b b_{1,n}^{adj} = nP_b = \acute{R} - \tilde{R}$$

Where $b_{1,n}^{adj}$ is the pro-rated term and nP_b is the excess return defined by the notional portfolio applied to attribute b. Using this adjusted multiperiod attribute, we can use simple sums to present performance attribution in an intuitive and meaningful table. Figure 1 provides an abstract example of such a report.

Each of the notional portfolios presents the summation of an attribute over the plan as a meaningful statistic. For *manager alpha*, the notional portfolio is the total plan excess return of the weighted[38] sum of the assets' returns over the assets' reference benchmarks:

$$nP_\alpha = R - \bar{R}_I^A$$

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[37] In this paper, we abstracted away from this concept as we felt this was implied. Instead, we opted to introduce notional portfolios where the application of hypothetical portfolios may not be so straightforward.
[38] Weights are calculated as the net asset value (NAV) percentage of total plan NAV

where $R = \sum_i w_i R_{a,i}$ is the weighted actual return for all assets in total plan, and $\bar{R}_I^A = \sum_i w_i \bar{R}_{ref,i}$ is the weighted reference asset benchmark return for those same assets. This describes how the assets performed versus their benchmark at the total plan level.

For *portfolio construction*, the notional portfolio is the total plan excess return of the asset benchmark return over the actual strategy reference benchmark return:

$$nP_{PC} = \bar{R}_I^A - \bar{R}_S^A$$

where $\bar{R}_S^A = \sum_S w_S^A \bar{R}_{ref,s}$ is the actual weighted sum of the strategies' reference benchmark returns. This notional portfolio characterizes how the asset benchmarks preformed relative to the strategy reference benchmarks.

Next, the *tactical* notional portfolio is:

$$nP_T = \bar{R}_S^A - \bar{R}_S^T$$

where $\bar{R}_S^T$=$\sum_S w_S^T \bar{R}_{ref,s}$ is the target weighted sum of the strategies' reference benchmark returns. This can be interpreted as the excess return of the total plan sum of the strategy reference benchmarks when comparing the actual weight to the target weight. A positive number would imply that strategy tilts added value to the portfolio during the reference period.

Finally, the *strategic* notional portfolio is:

$$nP_S = \bar{R}_S^T - \bar{R}$$

This difference describes how the outcome of the decision to hone in on specific exposures within strategies performed relative to the broad policy benchmark.

Clearly, these notional portfolios are congruent to their individual asset and strategy attribute counterparts. The interpretations of the notional portfolios only differ from the individual attributes in that they are calculated at the total plan level. However, pro-rating the attributes in this manner scales them in terms easily interpretable regardless of number of periods or hierarchical level.

3.2.2 Control Portfolio

The control portfolio is a simplified version of the notional portfolios. Instead of having a notional portfolio for each of the attribute types, there is only one for all types. In this paper, we call this portfolio the control portfolio as it is the notional portfolio used for linking in all the literature save Jiang et al. (2013). The concept of notional portfolios as defined by section 3.2.1 was introduced along with a style of institutional attribution which applies to the dataset used in this paper, but not necessarily to the reader's as the concept is relatively new. As such, this traditional portfolio serves as a control group to the notional portfolios of section 3.2.1.

For all the attributes, the notional portfolio is the total plan excess return of the weighted sum of the assets' returns over the broad policy benchmark:

$$nP_C = R - \bar{R}$$

Figure 2 illustrates the use of the control portfolio. When using the notional portfolios in a multiperiod setting, the $Tot_{s,x}$ for strategy S_x is not meaningful as the individual attributes are scaled differently for each type. But in the control portfolio setting, the $Tot_{s,x}$ is the S_x strategy's contribution to the total plan excess return, $R - \bar{R}$. Simply put, the multiperiod column totals are meaningful using our notional portfolios, while the row totals are meaningful in the control portfolio setting.

3.2.3 Why Use Notional Portfolios

Laker (2005) describes a method using the Brinson (1986) quadrants, commonly referred to as "notional portfolios," to calculate residual-free multi-period attributes at the total portfolio level. As he notes himself, this method is not a solution for fund-level attributes. His suggestion for the fund-level attributes is to simply use the existing linking methods such as Frongello (2002), Cariño (1999), Menchero (2004), to name a few.

Extending what Laker (2005) refers to as the "exact method," we use notional portfolio trailing returns defined in Section 3.2.1 as inputs for each of the linking methods described in Section 2. Typically, the total portfolio trailing active returns would be used for this purpose (the Control Portfolio). Using our more granular notional portfolios, we are able to group attributes by their type. By doing so, we hope the residuals within attribute types will generally reduce in size, and the linking methods results will begin to converge as a result.

4 DATA

Detailed performance data was supplied by a single institutional investor. Using proprietary performance analytics software developed by the authors, single period[39] attribution results were calculated for the entire portfolio using the Institutional Attribution framework as described in Jiang et al. (2013), for dates 12/31/1988 through 12/31/2013.

[39] The periodicity of the data is monthly. As such, the single period attributes are calculated monthly. Private equity returns are only supplied to the institutional investor on a quarterly basis. Zero returns are entered for private equity off quarter months and market values are adjusted for cash flows on a monthly basis.

Following the calculation of these individual monthly attributes, trailing period attributes were calculated systematically for all methods described in Section 2. Using object oriented programming, human error inherent to spreadsheet attribution approaches was substantially reduced. Further, the data was scrubbed rigorously prior to implementation.

Between the dates of 12/31/1988 and 12/31/2013, there are more than 230,000 individual single period attributes. The attribution types include *alpha*, *portfolio construction*, *tactical* and *strategic*. These attribute types may be considered exotic compared to the BHB-style *selection*, *interaction* and *allocation* attributes.

The results are not attribution style dependent, with exception to the notional portfolio definitions. The concept of notional portfolios as we have laid out in Section 3.2.1 can be applied to any style of attribution including BHB. In any case, the interpretation of the results would be the same. The Institutional Attribution framework simply allowed us to further segregate the linking residuals as contrast against the Control Portfolio.

5 RESULTS

$$J_{ALL} = I = \{F, mF, rF, C, M, N\}$$

$$J_{no\ N} = \{F, mF, rF, C, M\}$$

$$J_{F\&C} = \{F, mF, rF, C\}$$

$$J_F = \{F, mF, rF\}$$

$$J_{mF\&C} = \{mF, C\}$$

$$J_{M\&N} = \{M, N\}$$

Using the TBC, we tabulate the results as percentage of threshold breaks, broken down into rows of subsets of $I\ \{J_{ALL}, J_{no\ N}, J_{F\&C}, J_F, J_{mF\&C}, J_{M\&N}\}$ and columns of different trailing period lengths $\{1\ Year, 3\ Year, 5\ Year, 10\ Year\}$. The results can be seen in Tables 1 and 2.

Many different threshold parameter values were employed, although only two (0.05 and 0.0006484) are reported. The 0.05 threshold parameter was chosen as a heuristic baseline. As practitioners, given the structure of the threshold condition, 5% of the average and 0.05 basis points seemed reasonable when choosing from a materiality standpoint. The seemingly random 0.0006484 threshold parameter was chosen as it was the point where $(J_{mF\&C}, 1\ Year)$ broke once. All other threshold parameters led to similar conclusions and no contradictory results. As such, they are not specifically reported herein.

5.1 No Overlapping Periods

As stated in the Section 4, there are more than 230,000 individual single period attributes. However, you will notice our number of observations (denoted as n in the sub-tables) is substantially smaller and dependent upon the trailing period length. In an effort to assure our results were not biased by serial correlation, we selected records with no overlap. For example, our $10\ Year$ statistics only include linked attributes from the months of 12/1993, 12/2003 and 12/2014. This reduces our number of observations, but we feel the number of remaining observations are reasonable with regard to statistical inference.

5.2 Table Structure Description

For both Table 1 and Table 2, the first row of sub-tables $\{(1,1), (1,2)\}$ are results when the threshold parameter is set to 0.05. The second row of sub-tables $\{(2,1), (2,2)\}$ are results when the threshold parameter is set to 0.0006484. The first column of sub-tables $\{(1,1), (2,1)\}$ are results using our notional portfolios previously defined in Section 3.2.1. The second column of sub-tables $\{(1,2),(2,2)\}$ are results using the control portfolio.

5.3 Observations

5.3.1 Modified Frongello ≈ Cariño

Table 1 describes the percentage of both relative and absolute threshold breaks by linking method subset and trailing period. Looking at the second row of sub-tables, there are some significant patterns. The most notable of which is that $J_{mF\&C}$ threshold breaks are negligible even with a diminutive threshold parameter of 0.0006484. This supports the footnoted claim in Christopherson et al. (2009) that the Modified Frongello Method and the Cariño Method are numerically equivalent to many decimal places.

Valtonen (2002) goes as far as to present a recursive version of the Cariño Method. He suggests the Cariño Method is almost identical to the Modified Frongello Method. He demonstrates this proposition with a heuristic proof using a Taylor series expansion. Our empirical analysis supports his findings.

5.3.2 Frongello and Cariño Methods Cluster Around the Mean

$J_{F\&C} = J_F$ across rows and columns for all trailing periods. The intuition is that the Modified Frongello Method falls in between Frongello and Reverse Frongello by design, and Cariño ≈ Modified Frongello. Additionally, $J_{F\&C}$ and J_F (along with $J_{mF\&C}$) threshold breaks are insignificant relative to the other subsets of I.[40]

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[40] The first row of sub-tables, $(1,1)$ and $(1,2)$ are particularly notable in that they support the Christopherson et al. (2009) claim that the differences between the linking methods are relatively immaterial.

5.3.3 Using Notional Portfolios Reduces Residuals and Divergence

Generally all break percentages in sub-table $(2,1)$ are smaller than in sub-table $(2,2)$.[41] This reduction in linking method divergence numerically illustrates how using our notional portfolios better distribute the residuals prior to applying the linking methodologies. This partially supports the "exact method" argument of Laker (2005) as described in Section 3.2.3.

5.3.4 Divergence of Methods Increases with Longer Periods

Of significance and expectedly, the percentage breaks in the second row of sub-tables gradually increase in size as the trailing periods lengths grow. Intuitively, this is due to compounding over longer periods. The only exception is $(J_{M\&N}, 1\ Year)$. We believe that given the trailing period length and number of methods employed, this is an artifact of the dataset.

5.3.5 Menchero and Naïve Methods Compete for Position Furthest from the Mean

With regard to $J_{M\&N}$, there is a notable spread between the Naïve and Menchero methods. Given that $J_{All} \approx J_{no\ N}$ for the lower threshold, one might assume that the Menchero method is the outlier the lion's share of the time. However, the unreported $J_{no\ M}$ is roughly equivalent to $J_{no\ N}$. Further, the difference between the two varies in sign from statistic to statistic. Resultantly, the singular conclusion we can draw with regard to the Naïve and Menchero methods is that they are outliers to the Frongello and Cariño methods, but neither method is consistently the furthest from the average.

[41] The exception being the $10\ Year$ trailing period percentages for J_{All} and $J_{no\ N}$. This, as all other $10\ Year$ break percentages are lower in the Notional Portfolios sub-table and both J_{All} and $J_{no\ N}$ are larger, is due to the divergence of the Menchero Method from the others. This is either an artifact of the dataset or the Menchero Method structure.

5.3.6 Associative Property Helps Differentiate the Linking Methods

As mentioned in sections 5.4 and 5.7, the Frongello methods and Cariño Method cluster about the mean while the Menchero and Naïve methods are consistent outliers. Interestingly, the Frongello methods and Cariño Method are associative while the Menchero and Naïve methods are not. This provides empirical evidence to support the importance of the associative property of linking methods.

5.3.7 Relaxing TBC Helps Validate the Observations of Sections 5.3.1–5.3.6

Table 2 relaxes the TBC by removing the absolute threshold. This is done to verify that the results drawn from Table 1 are not data dependent. We do not want to draw conclusions that are tied to the magnitude of the underlying individual attributes.

Relaxing the TBC, Table 2 percentages are all larger as is expected when relaxing constraints. More importantly, the growth of the percentages is proportional across subsets of I for each trailing period. This alleviates concerns that our results are skewed by the exclusion of smaller attributes whose sum may be meaningful at the total portfolio level.

6 CONCLUSION

In this paper we explored the often referenced arithmetic attribution linking methods and compared their results when applied to a single institutional investor’s portfolio over time. Using different subsets of the various methods we were able to analyze the dispersion of the results. And varying the threshold parameter, we were able to tease out patterns in the data.

On the onset we observed that the results of the methods were reasonably close to each other. However, as we reduced the threshold parameter, dependable patterns revealed themselves. We found that the Naïve Method and Menchero Methods were consistently outliers, while the Cariño and Frongello methods clustered about the mean of the methods' results. In addition, we verified that Modified Frongello and Cariño methods produce nearly identical results to many decimal places.

We also introduced the new linking method Associative characteristic. Interestingly, the Associative Property was satisfied for all methods save Naïve and Menchero. This leads us to believe that associativity is a necessary condition of an attribution linking method.

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Table 1: Percentage of Conditional Breaks (using relative and absolute thresholds)

(1,1) Notional Portfolios — *TH = 0.05*

	1 Year	3 Year	5 Year	10 Year
All	3.09%	1.49%	7.94%	2.40%
no N	2.71%	1.36%	2.17%	2.20%
F & C	0.03%	0.02%	0.03%	0.14%
F	0.03%	0.02%	0.03%	0.14%
mF & C	0.00%	0.00%	0.00%	0.00%
M & N	1.21%	0.65%	1.32%	0.21%
n =	18200	5824	3640	1456

(1,2) Control Portfolio — *TH = 0.05*

	1 Year	3 Year	5 Year	10 Year
All	3.94%	0.64%	1.07%	3.85%
no N	3.80%	0.62%	0.93%	3.85%
F & C	0.08%	0.15%	0.11%	1.85%
F	0.08%	0.15%	0.11%	1.85%
mF & C	0.00%	0.00%	0.00%	0.00%
M & N	2.43%	0.02%	0.14%	0.76%
n =	18200	5824	3640	1456

(2,1) Notional Portfolios — *TH = 0.0006484*

	1 Year	3 Year	5 Year	10 Year
All	18.83%	19.56%	26.65%	44.37%
no N	18.60%	19.25%	26.37%	44.23%
F & C	8.01%	10.37%	11.92%	21.02%
F	8.01%	10.37%	11.92%	21.02%
mF & C	0.01%	0.05%	0.08%	0.34%
M & N	6.89%	6.73%	11.65%	18.82%
n =	18200	5824	3640	1456

(2,2) Control Portfolio — *TH = 0.0006484*

	1 Year	3 Year	5 Year	10 Year
All	20.71%	21.39%	30.41%	35.03%
no N	20.46%	21.00%	30.03%	33.65%
F & C	12.27%	15.59%	16.29%	27.61%
F	12.27%	15.59%	16.26%	27.61%
mF & C	0.01%	0.14%	0.19%	2.27%
M & N	9.13%	7.14%	17.20%	25.07%
n =	18200	5824	3640	1456

The above tables describe the percentage of threshold breaks for combinations of trailing periods and sets of attribution linking methods. Row 1 is uses relative and absolute thresholds of 5% and 5 basis points, respectively. Row 2 uses relative and absolute thresholds of 0.06% and 0.06 basis points, respectively. Additionally, Column 1 uses the Notional Portfolios as defined in section 3.2.1, while Column 2 uses the Control Portfolio of section 3.2.2.

Table 2: Percentage of Conditional Breaks (using only relative thresholds)

(1,1)				*rTH = 0.05*
	Notional Portfolios			
	1 Year	**3 Year**	**5 Year**	**10 Year**
All	20.23%	28.73%	35.05%	57.83%
no N	19.65%	28.42%	34.42%	57.69%
F & C	3.50%	9.39%	13.49%	26.79%
F	3.50%	9.39%	13.49%	26.79%
mF & C	0.00%	0.03%	0.05%	0.07%
M & N	2.50%	4.95%	10.66%	28.09%
n =	18200	5824	3640	1456

(1,2)				*rTH = 0.05*
	Control Portfolio			
	1 Year	**3 Year**	**5 Year**	**10 Year**
All	20.34%	29.22%	38.74%	54.60%
no N	19.96%	26.58%	38.43%	50.96%
F & C	6.36%	14.71%	22.01%	38.74%
F	6.36%	14.71%	22.01%	38.74%
mF & C	0.01%	0.09%	0.03%	0.21%
M & N	5.25%	8.89%	17.97%	34.48%
n =	18200	5824	3640	1456

(2,1)				*rTH = 0.0006484*
	Notional Portfolios			
	1 Year	**3 Year**	**5 Year**	**10 Year**
All	29.02%	35.37%	39.09%	58.38%
no N	29.01%	35.37%	39.09%	58.38%
F & C	27.75%	35.06%	38.79%	58.24%
F	27.75%	35.06%	38.79%	58.24%
mF & C	0.43%	2.44%	4.81%	15.80%
M & N	25.94%	33.59%	38.02%	57.69%
n =	18200	5824	3640	1456

(2,2)				*rTH = 0.0006484*
	Control Portfolio			
	1 Year	**3 Year**	**5 Year**	**10 Year**
All	29.10%	35.47%	39.15%	58.38%
no N	29.10%	35.47%	39.15%	58.38%
F & C	28.66%	35.16%	38.90%	58.38%
F	28.66%	35.16%	38.87%	58.38%
mF & C	1.35%	5.79%	18.21%	44.99%
M & N	25.87%	33.83%	38.98%	58.38%
n =	18200	5824	3640	1456

The above tables describe the percentage of threshold breaks for combinations of trailing periods and sets of attribution linking methods. Row 1 is uses a relative threshold of 5%. Row 2 uses a relative threshold of 0.06%. Additionally, Column 1 uses the Notional Portfolios as defined in section 3.2.1, while Column 2 uses the Control Portfolio of section 3.2.2.

Figure 1: Tabular Multiperiod (or single period) attribution report using Notional Portfolios

	Attribution Decomposition[42]				
Attribution Strategy	Alpha[43]	Portfolio Construction[44]	Tactical[45]	Strategic[46]	**Total**[47]
S_1	$\alpha_{s,1}$	$PC_{s,1}$	$T_{s,1}$	$S_{s,1}$	$Tot_{s,1}$
S_2	$\alpha_{s,2}$	$PC_{s,2}$	$T_{s,2}$	$S_{s,2}$	$Tot_{s,2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_n	$\alpha_{s,n}$	$PC_{s,n}$	$T_{s,n}$	$S_{s,n}$	$Tot_{s,n}$
Grand Total	$\sum_{i=1}^{n} \alpha_{s,i} = R - \bar{R}_I^A$	$\sum_{i=1}^{n} PC_{s,i} = \bar{R}_I^A - \bar{R}_S^A$	$\sum_{i=1}^{n} T_{s,i} = \bar{R}_S^A - \bar{R}_S^T$	$\sum_{i=1}^{n} S_{s,i} = \bar{R}_S^T - \bar{R}$	$\sum_{i=1}^{n} Tot_{s,i} = R - \bar{R}$

Each of the column totals, when using notional portfolios, presents the summation of a particular attribute over the plan as a meaningful active management statistic, as detailed in Jiang et al. (2013).

Figure 2: Tabular Multiperiod (or single period) attribution report using Control Portfolio

	Attribution Decomposition				
Attribution Strategy	Alpha	Portfolio Construction	Tactical	Strategic	**Total**
S_1	$\alpha_{s,1}$	$PC_{s,1}$	$T_{s,1}$	$S_{s,1}$	$Tot_{s,1} = C_{S1}(R - \bar{R})$
S_2	$\alpha_{s,2}$	$PC_{s,2}$	$T_{s,2}$	$S_{s,2}$	$Tot_{s,2} = C_{S2}(R - \bar{R})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
S_n	$\alpha_{s,n}$	$PC_{s,n}$	$T_{s,n}$	$S_{s,n}$	$Tot_{s,n} = C_{Sn}(R - \bar{R})$
Grand Total	$\sum_{i=1}^{n} \alpha_{s,i}$	$\sum_{i=1}^{n} PC_{s,i}$	$\sum_{i=1}^{n} T_{s,i}$	$\sum_{i=1}^{n} S_{s,i}$	$\sum_{i=1}^{n} Tot_{s,i} = R - \bar{R}$

C_{Sx} is a scaling term such that, as is clear in Figure 2, $\sum_{i=1}^{n} C_{Si} = 1$. In a multiperiod setting, each individual $Tot_{s,x}$ can be defined as the S_x strategy's contribution to the total plan active return when applying the Control Portfolio to a particular linking algorithm.

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42 As defined in Jiang et al. (2013)

43 $\alpha_{s,x} = \sum_i \alpha_i \;\; \forall i \in S_x$

44 $PC_{s,x} = \sum_i PC_i \;\; \forall i \in S_x$

45 $T_{s,x} = \sum_i T_i \;\; \forall i \in S_x$

46 $S_{s,x} = \sum_i S_i \;\; \forall i \in S_x$

47 $Tot_{s,x} = \alpha_{s,x} + PC_{s,x} + T_{s,x} + S_{s,x}$

APPENDIX A

The associative property for each method has not been mentioned in the literature. This appendix addresses whether the associative property is satisfied for each method individually. When a method violates the associative property, it is proved through a numerical counterexample.

In the abstract examples, we prove associativity for the case of 3 periods. By the General Associative Law, it follows that the associativity holds for any number of periods.

Set: $\left(1+R_{x,y}\right)-1 \stackrel{\text{def}}{=} cumulative\ return\ between\ period\ x\ and\ period\ y = (1+R_x)\cdot \ldots \cdot \left(1+R_y\right)$

<u>Simplified Frongello</u> (true):[48]

$\mathcal{L}_{ib}^{FRONG}\left(G_{i,x,b}, G_{i,y,b}\right)$

$\stackrel{\text{def}}{=} for\ attribute\ b\ of\ entity\ i, the\ effect\ in\ period\ x\ linked\ to\ the\ effect\ in\ period\ y$

$= G_{iyb}\cdot(1+R_x)+\left(1+\bar{R}_y\right)\cdot G_{ixb}$

$$\mathcal{L}_{ib}^{FRONG}(G_{i1b}, G_{i2b}) = G_{i2b}\cdot(1+R_1)+(1+\bar{R}_2)\cdot G_{i1b}$$

$$\mathcal{L}_{ib}^{FRONG}\left(\mathcal{L}_{ib}^{FRONG}(G_{i1b}, G_{i2b}), G_{i3b}\right) = G_{i3b}\cdot\left(1+R_{1,2}\right)+(1+\bar{R}_3)\cdot \mathcal{L}_{ib}^{FRONG}(G_{i1b}, G_{i2b})$$

$$= G_{i3b}\cdot(1+R_1)\cdot(1+R_2)+(1+\bar{R}_3)\cdot(G_{i2b}\cdot(1+R_1)+(1+\bar{R}_2)\cdot G_{i1b})$$

$$\mathcal{L}_{ib}^{FRONG}(G_{i2b}, G_{i3b}) = G_{i3b}\cdot(1+R_2)+(1+\bar{R}_3)\cdot G_{i2b}$$

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[48] As Modified and Reverse Frongello methods share the same basic functional form as the Simplified Frongello Method, we omit these proofs, but state the Associative Property holds.

$$\mathcal{L}_{ib}^{FRONG}\left(G_{i1b}, \mathcal{L}_{b}^{FRONG}(G_{i2b}, G_{i3b})\right) = \mathcal{L}_{b}^{FRONG}(G_{i2b}, G_{i3b}) \cdot (1 + R_1) + \left(1 + \bar{R}_{2,3}\right) \cdot G_{i1b}$$

$$= (G_{i3b} \cdot (1 + R_2) + (1 + \bar{R}_3) \cdot G_{i2b}) \cdot (1 + R_1) + (1 + \bar{R}_2) \cdot (1 + \bar{R}_3) \cdot G_{i1b}$$

$$= G_{i3b} \cdot (1 + R_1) \cdot (1 + R_2) + (1 + \bar{R}_3) \cdot (G_{i2b} \cdot (1 + R_1) + (1 + \bar{R}_2) \cdot G_{i1b})$$

$$\therefore \boldsymbol{\mathcal{L}_{ib}^{FRONG}\left(\mathcal{L}_{ib}^{FRONG}(G_{i1b}, G_{i2b}), G_{i3b}\right) = \mathcal{L}_{ib}^{FRONG}\left(G_{i1b}, \mathcal{L}_{b}^{FRONG}(G_{i2b}, G_{i3b})\right)} \blacksquare$$

<u>Cariño Linking Method</u> (true):

Set:

$$f(R_x, \bar{R}_x) = [\ln(1 + R_x) - \ln(1 + \bar{R}_x)]/(R_x - \bar{R}_x)$$

$$f(R_{x,y}, \bar{R}_{x,y}) = [\ln(1 + R_{x,y}) - \ln(1 + \bar{R}_{x,y})]/(R_{x,y} - \bar{R}_{x,y})$$

$$C_x^{CAR} = \frac{[\ln(1 + R_x) - \ln(1 + \bar{R}_x)]/(R_x - \bar{R}_x)}{[\ln(1 + R) - \ln(1 + \bar{R})]/(R - \bar{R})} = \frac{f(R_x, \bar{R}_x)}{f(R_{x,y}, \bar{R}_{x,y})},$$

$$where \sum_i \sum_{j=x}^{y} \sum_b G_{i,j,b}(C_j^{CAR}) = R_{x,y} - \bar{R}_{x,y} = R - \bar{R}$$

$\mathcal{L}_{ib}^{CAR}(G_{i,x,b}, G_{i,y,b})$

$\stackrel{\text{def}}{=} for\ attribute\ b\ of\ entity\ i, the\ effect\ in\ peroid\ x\ linked\ to\ the\ effect\ in\ period\ y$

$$= \frac{G_{ixb} \cdot f(R_x, \bar{R}_x) + G_{iyb} \cdot f(R_y, \bar{R}_y)}{f(R_{x,y}, \bar{R}_{x,y})}$$

$$\mathcal{L}_{ib}^{CAR}(G_{i1b}, G_{i2b}) = \frac{G_{i1b} \cdot f(R_1, \bar{R}_1) + G_{i2b} \cdot f(R_2, \bar{R}_2)}{f(R_{1,2}, \bar{R}_{1,2})}$$

$$\mathcal{L}_{ib}^{CAR}\left(\mathcal{L}_{ib}^{CAR}(G_{i1b}, G_{i2b}), G_{i3b}\right) = \frac{\mathcal{L}_{ib}^{CAR}(G_{i1b}, G_{i2b}) \cdot f(R_{1,2}, \bar{R}_{1,2}) + G_{i3b} \cdot f(R_3, \bar{R}_3)}{f(R_{1,3}, \bar{R}_{1,3})}$$

$$= \boldsymbol{\frac{G_{i1b} \cdot f(R_1, \bar{R}_1) + G_{i2b} \cdot f(R_2, \bar{R}_2) + G_{i3b} \cdot f(R_3, \bar{R}_3)}{f(R_{1,3}, \bar{R}_{1,3})}}$$

$$\mathcal{L}_{ib}^{CAR}(G_{i2b}, G_{i3b}) = \frac{G_{i2b} \cdot f(R_2, \bar{R}_2) + G_{i3bb} \cdot f(R_3, \bar{R}_3)}{f(R_{2,3}, \bar{R}_{2,3})}$$

$$\mathcal{L}_{ib}^{CAR}\left(G_{i1b}, \mathcal{L}_{ib}^{CAR}(G_{i2b}, G_{i3b})\right) = \frac{G_{i1b} \cdot f(R_1, \bar{R}_1) + \mathcal{L}_{ib}^{CAR}(G_{i2b}, G_{i3b}) \cdot f\left(R_{2,3}, \bar{R}_{2,3}\right)}{f\left(R_{1,3}, \bar{R}_{1,3}\right)}$$

$$= \frac{\boldsymbol{G_{i1b} \cdot f(R_1, \bar{R}_1) + G_{i2b} \cdot f(R_2, \bar{R}_2) + G_{i3bb} \cdot f(R_3, \bar{R}_3)}}{\boldsymbol{f\left(R_{1,3}, \bar{R}_{1,3}\right)}}$$

$$\therefore \boldsymbol{\mathcal{L}_{ib}^{CAR}\left(\mathcal{L}_{ib}^{CAR}(G_{i1b}, G_{i2b}), G_{i3b}\right) = \mathcal{L}_{ib}^{CAR}\left(G_{i1b}, \mathcal{L}_{b}^{CAR}(G_{i2b}, G_{i3b})\right)} \blacksquare$$

Table A.1: Simplified Portfolio Performance Attribution (Menchero Method)

	Portfolio Return	Benchmark Return	Attribute
Period A	30.00%	25.00%	0.05
Period B	5.00%	6.00%	-0.01
Period C	20.00%	30.00%	-0.10

<u>Menchero Linking Method</u> (false) (using values from Table A.1)**:**

Disproving the *Associative Property* for the Menchero Linking Method is a simple exercise of finding one numerical counterexample. Using the table provided above, the three period linked attributes were calculated.

$A + B = 0.04737$

$(A + B) + C = -0.07126$

$B + C = -0.11726$

$A + (B + C) = -0.08542$

$\therefore (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} \neq \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C})$ ■

Table A.2: Simplified Portfolio Performance Attribution (Naïve Method)

	Portfolio Return	Benchmark Return	Security Selection	Asset Allocation
Period A	4.00%	3.00%	-0.010	0.020
Period B	5.00%	2.00%	0.001	0.029
Period C	2.00%	1.00%	-0.002	0.012

<u>Naïve Method</u> (false) (Using values from Table A.2)**:**

Disproving the *Associative Property* for the Naïve Linking Method is a simple exercise of finding one numerical counterexample (examples for both Addition and Compounding). Using the table provided above, the three period linked attributes were calculated.

Addition Method (Selection)[49]

$A + B = -0.00932$

$(A + B) + C = -0.01193$

$B + C = -0.00102$

$A + (B + C) = -0.01162$

$\therefore (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} \neq \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) \blacksquare$

Compounding Method (Selection)

$A + B = -0.00919$

$(A + B) + C = -0.01151$

49 Although the Allocation effects are not reported, they can also be shown to violate the Associative Property as well.

$B + C = -0.00101$

$A + (B + C) = -0.01162$

$\therefore (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} \neq \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C})$ ■