Operations Management of On-Demand and Crowdsourced Systems

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Abstract

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Over the last few years, crowdsourced and on-demand service platforms have provided unconventional accessibility to services and products, and hence, studying their operations is of great importance. Research in this thesis focuses on studying operations and pricing of these service platforms and providing policies and guidelines on how such systems should be optimally implemented in practice.

Some crowdsourced platforms do not provide on-demand services and they merely match crowd providers with customers. An example of these platforms is a crowdfunding platform which is the focus of the second chapter in this thesis. Additionally, there are some crowdsourced platforms that provide on-demand services, such as crowdsourced last-mile delivery platforms that provide fast 1-hour and 2-hour delivery services to customers. Studying these platforms is the focus of the third chapter in this thesis. Finally, some on-demand service platforms do not base their operations on the crowd, but they revolve around matching independent and third-party logistics service providers with retailers; an example is an on-demand warehousing platform, which is the focus of my research in the fourth chapter.

In this thesis, we first study the application of revenue-sharing contracts in crowdfunding. In particular, we study an emergent model of crowdfunding that allows firms to raise capital

from a pool of investors and repay them a multiple of their investment by sharing a percentage of its future revenue, under a revenue-sharing contract. This means that investors will receive $M > 1$ dollars for every dollar that they have invested over an investment horizon of uncertain duration. This contract, as a flexible repayment agreement, is linked with the financial performance of the firm, allowing variable payments and investment horizons and, thus, reducing financial stress on the firm. If the firm's business does well, the firm is obligated to increase the payments, thus reducing the investment horizon, which results in a higher effective interest rate for the investors. Therefore, revenue-sharing contracts intuitively align firms' and investors' incentives in a way that was not possible with traditional fixed-rate loans. We also compare revenue-sharing contracts with equity crowdfunding and fixed-rate loans and observe that revenue-sharing contracts result in higher net present values and lower bankruptcy probabilities due to their flexible nature.

We next study labor planning and pricing for crowdsourced last-mile delivery systems that are utilized to deliver on-demand orders. We develop our optimization model by combining crowdsourcing, robust queuing, and robust routing theory, which allows for capturing uncertainties, trend and seasonality in: customer demands, crowd availability, service times, and traffic patterns. For a given delivery time window and a guarantee level to deliver on-time, we analytically derive the optimal delivery assignments to available independent crowd drivers and compute their optimal hourly wages. We evaluate the performance of our robust model against a stochastic counterpart and a modification (to accommodate crowdsourcing) of the well-known Savings Algorithm via a realistic simulation study, based on the Seattle transportation network, that allows for nonstationary customer purchasing patterns, heterogeneous crowd drivers, as well as time-varying traffic patterns. We show that our robust solution performs significantly better than the two benchmarks, both in terms of the percentage of on-time deliveries and cost savings. We show that crowdsourcing last-mile deliveries can significantly reduce logistical costs for retailers.

Finally, we study on-demand warehousing platforms that match independent warehouse providers who possess excess capacity with retailers who seek on-demand warehouse capacity without imposing large fixed costs or long-term leases. Although traditional warehousing may be cheaper, it must be acquired based only on demand forecasts and not the realized demand. However, on-demand warehousing can be acquired during the selling season after observing the demand. We study a firm's capacity decision in the presence of these on-demand platforms. Our results indicate that companies can rely on on-demand warehousing to supplement their traditional warehousing, with hybrid strategies constituting the optimal outcome in many cases. We find that, as firms' demand variability increases, firms benefit more from on-demand warehousing. Our results further show that as there is less uncertainty in the capacity provided by independent warehouse providers, firms' benefit from on-demand warehousing increases as they can rely on on-demand capacity to cope with their demand uncertainty.

TABLE OF CONTENTS

Page

LIST OF TABLES

LIST OF FIGURES

Figure Number Page

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DEDICATION

To my mom and dad,

Afsaneh and Ali

&

To my husband,

Siavash

Chapter 1

INTRODUCTION

The emergence and development of e-commerce has resulted in many platforms that revolve around matching independent service providers with customers. This is called the sharing economy. The focus of this thesis is studying applications of the sharing economy and deriving optimal policies for their operations. In the sharing economy, there is uncertainty in supply as well as demand, and therefore, matching supply and demand is one of the main challenges in the operations of the sharing economy. Matching supply and demand is even more significant for platforms that offer fast and on-demand services, such as on-demand crowd delivery platforms or ride-sharing platforms. However, not all applications of the sharing economy entail on-demand services (e.g., crowdfunding platforms).

Chapter 2 of this thesis is dedicated to studying crowdfunding platforms. Crowdfunding is a relatively new approach for firms to raise capital from a crowd of individuals, rather than traditional sources of capital (e.g., banks, venture capitalists, etc.). In particular, in Chapter 2, we study an application of revenue-sharing contracts in crowdfunding.

Chapter 2 is motivated by an emergent model of crowdfunding, pioneered by Bolstr (www.bolstr.com), Localstake (www.localstake.com), and Startwise (www.startwise.com), which targets small and medium-sized firms needing capital; however, this model is not intrinsically limited by firm size, and could be applied to any sized firm. These platforms match a firm needing funding with investors from the crowd, and the firm pays back the investors via *revenue-sharing contracts*. This more flexible repayment agreement is linked with the financial performance of the firm, allowing variable payments and investment horizons, thus reducing financial stress on the borrower. In contrast, if business goes well, the borrower is obligated to increase the payments, thus reducing the investment horizon, which results in a higher effective interest rate for the investors. Therefore, revenue-sharing contracts intuitively align a firm's and investors' incentives, in a way not possible with traditional fixed-rate loans.

The proposed revenue-sharing contract bears some superficial similarity to performance-sensitive debt (e.g., step-up bonds and performance-pricing loans) studied by [150], where the debt payments depend on the borrower's performance: the borrower pays higher interest rates during low performance and lower interest rates during high performance. However, this approach has the opposite behaviour of our proposed revenue-sharing contract, since, under the latter, a high performing firm will have high debt payments and will pay off the fixed loan amount early, resulting in a higher effective interest rate. Performance-sensitive debt has been shown to harm both the borrower and investors via earlier borrower default [150], whereas our new model can result in positive outcomes for all parties.

We next, in Chapters 3–4, study two applications of the sharing economy that provide on-demand services. In one of these applications, which is the focus of my research in Chapter 3, the on-demand services are offered by the crowd.

Chapter 3 is motivated by the current practice of retailers such as Amazon and Walmart. Amazon is using independent crowdsourced drivers under a program called Amazon Flex (flex.amazon.com), and pays the drivers an hourly wage of $18-$25. In addition to its warehouses, Amazon is using vacant lands, such as outdoor parking lots, as delivery stations, and is using crowdsourced drivers via the Flex program for making its Amazon Now and Amazon Fresh deliveries [172]. Walmart similarly offers fast deliveries in all of its major markets and product categories under the crowdsourced Spark delivery program (www.drive4spark.com). In addition, Target has acquired Shipt, an on-demand delivery service that utilizes a crowdsourced network of personal shoppers [202].

The emergence and development of e-commerce has resulted in customers that demand faster and cheaper delivery services. According to [108], 25% of customers do not proceed with their online orders if same-day delivery is not available. Nevertheless, only 20% of retailers offer same-day delivery, which is indicative of the challenges that are inherent in providing on-demand delivery services. To overcome these challenges, an increasing number of retailers have been considering crowdsourcing last-mile deliveries. This allows retailers to deal with demand uncertainty better, as they can flexibly utilize independent crowdsourced drivers after their demand realization without long-term capital commitments.

In Chapter 3, we study last-mile deliveries via independent crowd drivers that decide whether to work and when to work. Specifically, we study crowdsourced last-mile delivery systems for on-demand orders, with guaranteed delivery time windows, where the time between placing an order and receiving it ranges from 2 hours to same-day. In this chapter, we propose a novel robust crowdsourcing optimization model to study labor planning and pricing for crowdsourced last-mile delivery systems that are utilized for satisfying on-demand orders with guaranteed delivery time windows.

In Chapter 4, we study capacity flexibility via on-demand warehousing. On-demand warehousing is an application of the sharing economy that provide on-demand warehouse capacity by matching independent and third-party logistics (3PL) warehouse providers with retailers.

In recent years, due to a growing number of uncertainties, small or large businesses increasingly demand for more flexible warehousing capacity that is compatible with their fast-changing needs and their fluctuating demands. Flexe (www.flexe.com), Flowspace (www.flow.space), and Ware2go (www.ware2go.co) are platforms that address this demand. Also known as "the Airbnb of warehousing," Flexe matches independent warehouse providers who possess excess capacity with retailers who seek on-demand warehouse capacity without imposing large fixed costs or long-term leases. Although traditional warehousing may be cheaper, it must be acquired based only on demand forecasts, whereas on-demand warehousing may be acquired after observing the actual demand. In Chapter 4, we study capacity decisions in the presence of on-demand warehousing and derive optimal policies for firms given uncertainty in their demand as well as uncertainty in the supply of on-demand capacity provided by independent warehouse providers.

In Chapter 5, we present a summary of the research in this thesis and provide some extensions for future research in this area. Finally, all the proofs in this thesis can be found in the Appendix section.

1.1 Literature Review

1.1.1 Crowdfunding & Revenue-Sharing

There are many papers that study different crowdfunding models, papers that study various aspects of the interface of operations and finance, as well as a vast literature on revenue-sharing contracts, primarily in the areas of supply chain management and educational financing. In this section, we survey the most relevant results in these three streams. However, to the best of our knowledge, there are no studies investigating crowdfunding via revenue-sharing contracts, except [77], which is a book chapter summary of a preliminary version of this research. Thus, my research in Chapter 2 uniquely lies at the intersection of these two literature streams.

[10] provided a detailed study on how crowdfunding impacts the financing decisions of entrepreneurs, banks, and venture capital investors. [55] considered fundraising success in crowdfunding and studied the optimal referring policies by entrepreneurs, empirically and analytically. [52] studied how entrepreneurs can signal their product's quality to contributors

via campaign design in reward-based crowdfunding. [216] studied a data set from Prosper, focusing on a study of herding behavior in microloan markets. [141] also analyzed data from Prosper and studied the "home bias" in crowdfunding markets. [111] also studied a dataset from Prosper and showed that lenders use standard information along with nonstandard, or soft, information to evaluate borrower creditworthiness. [23] compared two forms of crowdfunding, profit sharing and pre-ordering of products, and showed that if the firm's investment goal is relatively high with respect to the market size, then the firm prefers profit-sharing crowdfunding. Finally, [208] study, both theoretically and empirically, the difference between auctions and posted prices on Prosper.com.

My research in Chapter 2 is also relevant to the literature on the interface of operations and finance. In particular, my research is relevant to operational financing. There can be many sources of financing, including traditional bank financing, as well as more novel arrangements that have been attracting attention in the OM literature, such as buyer financing [68] or supplier trade credits [135] or both [127]. We identify crowdfunding as an additional source of financing, and while we focus on revenue-sharing crowdfunding, we also consider equity crowdfunding.

The structure of our crowdfunding contracts, revenue-sharing, has also been extensively studied in the supply chain management literature. However, the combination with financial aspects is limited. [126] studied contract design for a supply chain with one supplier and one retailer in the presence of financial constraints and bankruptcy costs, and they show that a revenue-sharing contract can still coordinate the supply chain. [125] showed that revenue-sharing contracts can achieve high efficiency in the presence of cost uncertainty and working capital constraints. Similarly, we show that a firm can benefit significantly from revenue-sharing contracts under stochastic cash flows.

Revenue sharing contracts have also appeared in the education field, as a novel approach to funding students. [159] studied an income-contingent loan program for the financing of education, which was originally proposed by [85] for professional education and by [84] for vocational education. One example of such a loan program is the Yale Tuition Postponement Option, which started in 1971, but was discontinued in 1978 [131]. Another example is the Pay-it-Forward plan in Oregon: in 2013, state legislators proposed a program in which students could attend public colleges without paying tuition, and in return they would pay 3% percent of their future income to the state for several years after graduation [171]. Similarly, the Back-a-Boiler program at Purdue University, which started in Fall 2016 and has already raised $2.2 million, provides funds to undergraduate students to finance their education through an Income Share Agreement in which students agree to pay a percentage of their future income over a standard payment term [177]. The results and insights of my research in Chapter 2 of this thesis can also be applied to these student loan agreements.

1.1.2 Crowdsourcing Last-Mile Deliveries

In this stream, [139] develop a heuristic algorithm for a problem where a set of taxi drivers are serving both people and parcels. [7] develop algorithms for the problem of crowdsourcing deliveries using a rolling horizon approach which repeatedly matches drivers and deliveries every time a driver or a delivery arrives. [48] study the use of ride-sharing platforms such as Uber and Lyft, along with an in-house van delivery system, for making last-mile deliveries, and derive exact and asymptotic results for the expected number of packages that can be delivered during a time horizon. However, these papers focus on algorithmic solutions, whereas we focus on deriving *analytical* pricing and labor planning policies for on-demand crowdsourced delivery systems with guaranteed delivery time windows. The paper most relevant to ours is [178], which studies crowdsourcing deliveries. They show that crowdsourcing is not as scalable as truck deliveries, in terms of operating cost. These results are in contrast to our results in Chapter 3 of this thesis, where we show crowdsourcing last-mile deliveries can minimize the firm's delivery cost significantly. Additionally, they conclude that crowdsourcing results in cost savings in areas with low demand densities. However, we show that crowd delivery should be utilized in populated areas with higher demands; similarly, [176] study user acceptance in crowd-shipping systems and show that urban areas are more inclined to use crowd-shipping. We conjecture that these different conclusions are due to different choices in modeling: 1) [178] do not consider crowdsourcing deliveries with time windows and only focus on making deliveries without deadlines. We, in contrast, consider a firm that uses crowdsourcing last-mile deliveries for orders that should be delivered within a pre-specified time window (e.g., fast same-day and 2-hour deliveries). 2) [178] consider a firm that either utilizes crowd-delivery or its own truck-delivery system, for all orders, and no hybrid solution is possible. However, we consider the case where the firm can satisfy some customers' orders via the crowdsourcing system, and others using the alternative 3PL provider; i.e., in our model, a hybrid solution is feasible. 3) [178] model a firm that pays a base fare, per-minute fare, and a per-mile fare to crowd drivers, and a fixed hourly wage to truck drivers. In contrast, we consider a firm that pays crowd drivers an hourly wage, motivated by, for example, Amazon's Flex program, whereas 3PL delivery options (such as FedEx, UPS, and USPS) charge the firm per package delivery, after any applicable volume discounts, as a function of distance, as is current practice.

1.1.3 The Sharing Economy and On-Demand Platforms

There are a number of related papers that study peer-to-peer sharing. [100] develop a distributionally robust optimization model for studying service region design for electric vehicle sharing providers that offer one-way trips to customers. [170] study a matching problem in a ride-sharing platform (such as Uber and Lyft), and find that parameter-based policies can achieve better performances than the policy that matches customers with the closest drivers. [25] characterize the equilibrium of peer-to-peer product sharing where owners rent their assets to non-owner consumers on an as-needed basis to generate income from renting. However, these papers do not consider the design of a shared logistics service where independent workers are utilized for making fast on-demand last-mile deliveries, which is the focus of my research in Chapter 3 of this thesis. There are also several papers studying the operation and pricing of on-demand service platforms. [181] show that a dynamic pricing approach, in which a ride-sharing platform offers higher prices when the number of available drivers drops below a threshold, is more robust with respect to system parameters than the optimal static policy. [96] analyze an on-demand service provider that is using self-scheduling agents and wants to maximize its profit; they show that under self-scheduling, the firm offers lower service levels to its customers, which lowers the firm's profit and is costly to customers. [34] study spatial pricing solutions in ride-sharing platforms and show that the platform's profit and consumer surplus are higher as the demand pattern across the network is more "balanced". [43] show that the use of surge pricing on service platforms with self-scheduling providers benefits customers, providers, and the platform. [102] study fixed-commission contracts in on-demand matching platforms that match an independent and uncertain supply side with an uncertain demand side; they show that, if the supply curve is concave in wage, under the optimal commission contract the platform can earn at least 75% of its optimal profit. [33] analyze surge pricing and its corresponding supply response, within a geographic region, for a ride-sharing platform. [200] examines pricing in on-demand service platforms where agents are independent and customers are delay sensitive, and shows that uncertainties in agent opportunity costs and customer valuations impact the platform's per service price. Similarly, [11] study the optimal price and wage rate on on-demand service platforms with price- and time-sensitive customers and earning-sensitive service providers. [94] examine surge pricing for an on-demand platform with uncertain supply and demand, and show that the platform can signal market conditions to independent workers using surge pricing, which increases the platform's profit. [24] analyze labor welfare in on-demand service platforms with independent workers. [29] study competition between two ride-sharing platforms and show that if all independent service agents provide ride-sharing services through both platforms, the drivers, the platforms, and the customers are weakly worse off. In contrast to the above papers, in my research in Chapter 3 crowd drivers are utilized to deliver packages to customers within guaranteed delivery time windows. Utilizing crowd drivers, whose presence is uncertain, to deliver packages to customers, whose orders are uncertain and nonstationary, all within the promised delivery time window, distinguishes our work.

1.1.4 Queueing and Routing via Robust Optimization

In this stream, [16] study the worst-case performances of single-server and multi-server queueing systems using robust optimization. [197] is one of the first papers that studies the capacitated vehicle routing problem with demand uncertainty using a robust optimization approach. Similarly, [90] consider the capacitated vehicle routing problem under the case where demand is uncertain and belongs to a polyhedron uncertainty set, and the objective is to minimize total routing costs. [50] utilize a robust optimization framework to develop branch-and-bound algorithms for the stochastic multiple-vehicle routing problem, where the location of demand points and their distribution is unknown; their objective is to minimize the worst-case work load over all subregions, where each subregion is served by a single vehicle. [143] utilize robust optimization for order assignments in last-mile delivery faced by food service providers. However, these papers apply robust optimization to classic routing problems; in contrast, in this thesis we apply robust optimization to a novel application of on-demand deliveries via crowd drivers. Furthermore, no paper has combined robust queueing with robust routing, as we do.

1.1.5 Capacity Flexibility via On-Demand Warehousing

To the best of our knowledge, we are the first to study capacity flexibility via on-demand warehousing. However, there are papers that study capacity investments under uncertainty and resource flexibility in supply chain. [140] empirically study pricing and competition between fixed-capacity firms (i.e. hotels) and flexible-capacity firms (i.e. Airbnb). Perhaps the most relevant work to ours, is by [199] who study the timing of capacity investments under stochastic demand. They show that in a monopoly market, start-ups with the goal of maximizing their survival probability prefer early investments than established firms seeking to maximize their expected profit. [95] study capacity between competing firms and show that when firms' capacity are asymmetric, capacity sharing can lower the firms profitability in equilibrium. [56] study the role of resource flexibility in learning demand and alleviating censored sales information, which results in better decisions.

1.2 Outline and Contributions

1.2.1 Chapter 2: Crowdfunding via Revenue-Sharing Contracts

In this chapter, we provide an analysis of the firm's multi-period problem of maximizing its expected net present value, subject to platform fees and investor participation constraints, for stochastic revenues and costs.

Since the stochastic model is difficult to analyze, we derive a deterministic approximation model for it, in which we use a cash buffer to cope with uncertainties. We then solve the approximation problem analytically, which provides generalizable insights. We also solve our stochastic model numerically using Monte Carlo simulation and a grid-based optimization framework, for serially correlated random cash-flows that are parameterized using real data from Bolstr. We then compare the NPVs of the approximate and optimal solutions in the true stochastic model. We conclude that our approximation provides high quality solutions: the worst-case average error of the approximation solution's NPV, over all feasible Bolstr campaigns for all levels of cash-flow uncertainty, is approximately 0.2%.

Finally, we compare the performance of the proposed revenue-sharing contract with equity crowdfunding and fixed-rate loans, and we identify which type of financing results in a higher NPV or a lower chance of bankruptcy for a firm; we find that the revenue-sharing contract is superior in most cases.

The contributions of this chapter are as follows:

1. We are the first, to our knowledge, to study a new emergent model of crowdfunding pioneered by Bolstr, Localstake, and Startwise. Furthermore, our models are parameterized using real data from 56 Bolstr campaigns.

2. We study a firm's expected NPV maximization problem under a revenue-sharing contract, which is an intractable stochastic model. To overcome the technical difficulties, we design a tractable deterministic approximation model, in which we use a cash buffer to cope with cash-flow uncertainties. We solve the approximation model analytically, which provides qualitative insights. Numerical experiments, calibrated on real Bolstr data, indicate that the approximation solutions, when inserted into the true stochastic model, result in an expected NPV that is within 0.2% of the true optimal NPV, on average. Furthermore, the approximation solutions result in almost the same bankruptcy probabilities as the optimal stochastic solutions.

3. Our results provide managerial guidelines for the firm; for instance:

    (a) A firm can attain a higher NPV and a comparable probability of bankruptcy under a revenue-sharing contract than under equity crowdfunding. We show that the NPV benefit of revenue-sharing increases as the firm's cashflow volatility increases.

(b) A firm can attain a higher NPV and a lower probability of bankruptcy under a revenue-sharing contract than under a more traditional fixed-rate loan. We also show that these benefits are more significant for firms with higher levels of cash-flow uncertainty. Intuitively, these benefits are due to the more flexible nature of the revenue-sharing contract.

(c) As cash-flow uncertainty increases, the optimal investment amount increases, the revenue-sharing percentage decreases, resulting in a stochastically larger investment horizon (e.g., larger mean and variance). However, the firm's maximized NPV is rather insensitive to cash-flow uncertainty. Thus, while the details of the optimal revenue-sharing contract can change considerably with cash-flow uncertainty, the bottom line NPV is rather robust to this uncertainty.

Research in this chapter is a joint work with Michael R Wagner, and, has resulted in a publication in Manufacturing & Service Operations Management, volume 21, number 4, pp. 875-893, 2019.

1.2.2 Chapter 3: Crowdsourcing Last-Mile Deliveries

In this chapter, we study labor planning and pricing for crowdsourced last-mile delivery systems that are utilized to deliver on-demand orders. We develop our optimization model by combining crowdsourcing, robust queueing, and robust routing theory, which allows for capturing uncertainties, trend and seasonality in: customer demands, crowd availability, service times, and traffic patterns. For a given delivery time window and a guarantee level to deliver on-time, we analytically derive the optimal delivery assignments to available independent crowd drivers and compute their optimal hourly wages.

This chapter makes the following contributions:

1. Research in this chapter is the first, to our knowledge, that analytically studies crowdsourcing last-mile deliveries, with guaranteed delivery time windows (e.g., same-day or 2-hour deliveries), under non-stationary uncertainties. We develop our optimization model by combining crowdsourcing, robust queuing, and robust routing theory. We derive closed-form expressions for the delivery cost and system times of all customers in the crowd-delivery system, which allows us to carefully analyze crowd compensation and labor planning. Our results can help firms in designing crowdsourced delivery systems, for their on-demand orders with guaranteed delivery times, to minimize their last-mile delivery costs. Additionally, our results help firms to decide on the on-time delivery guarantee level depending on their customers' sensitivities to delay and crowd drivers' opportunity costs.

2. Our results show that crowdsourcing can help firms significantly decrease their delivery costs, while keeping the promise of fast on-time deliveries to customers. Our analysis shows that, under low levels of uncertainty or for low on-time guarantee levels, firms should assign fewer packages in each crowd delivery tour to create a stream of available delivery work, which encourages crowd drivers' participation, without having to offer them high wages. In contrast, under high levels of uncertainty or for high on-time guarantee levels, firms should assign more packages in each delivery tour, but at the same time offer higher hourly wages.

3. We evaluate the performance of our proposed robust optimal policy against two benchmarks, a stochastic counterpart and a non-randomized heuristic. Our results show that

the proposed robust optimal policy is superior to both benchmarks in terms of the firm's cost savings and the number of on-time deliveries for customers' on-demand orders.

4. We design realistic numerical experiments that cover 17 Seattle ZIP codes that are served by an Amazon Flex depot, using real travel distances on road networks. Our experimental results show that the performance of our proposed robust solution, which is faster and easier to implement, performs significantly better than the benchmarks. Furthermore, our results show that a crowd-delivery system, if designed optimally, is significantly cheaper than the exclusive use of a 3PL firm, especially for customers' fast same-day and 2-hour delivery orders.

This chapter is a joint work with Michael R Wagner.

1.2.3 Chapter 4: Capacity Flexibility via On-Demand Warehousing

In this chapter, we analyze firms' capacity investment decisions in the presence of on-demand warehousing. In the absence of on-demand warehousing, firms are committed to long-term leases (e.g., 5-year leases) based on their expected demand; variability in demand does not come into play and there is no flexibility. However, on-demand warehousing allows firms to adjust their capacity based on their realized demand. In this chapter, we develop our optimization model given uncertainties in the firm's demand, availability of independent warehouse providers, and providers' excess capacity.

This chapter makes the following contributions:

1. We are the first, to our knowledge, to study an emergent model of on-demand warehousing pioneered by Flexe (www.flexe.com), Flowspace (www.flow.space), and Ware2go (www.ware2go.co).

2. We study a firm's optimal capacity investment decisions in the presence of on-demand warehousing. We derive closed-form solutions for the firm's optimal policies under uncertainties in customers' demand, availability of independent warehouse providers, and providers' excess capacity.

3. Our results provide managerial guidelines for firms, as follows:

   (a) Surge pricing results in all providers providing their excess capacity to firms under high demand states, due to higher prices from the surge, which could benefit firms at optimality.

   (b) If a firm's demand realization is high, the firm chooses to acquire on-demand warehousing to complement its primary traditional capacity. Therefore, a hybrid strategy where the firm operates both traditional and on-demand warehousing capacity may be optimal. This finding is consistent with practice where established big firms utilize on-demand capacity as a "filler" for managing variability [189]. Our analysis can help firms decide what capacity mix should be chosen in order to benefit from the flexibility provided via on-demand warehousing.

   (c) Our results show that the benefit is higher for firms with moderate levels of traditional capacity.

   (d) Our results indicate that as variability in firms' demand increases, firms' benefit from capacity flexibility via on-demand warehousing increases, as it allows firms to absorb demand variabilities and benefit from high demand realizations.

   (e) Our results indicate that as firms' expected demand increases, firms should increase their traditional capacity investment without changing their on-demand capacity utilization.

This chapter is a joint work with Leela Nageswaran and Michael R Wagner.

1.2.4 Chapter 5: Conclusion and Future Work

This chapter provides a summary of our research findings in this thesis. We also discuss future research directions in the area of the sharing economy and on-demand service platforms.

Chapter 2

CROWDFUNDING VIA REVENUE-SHARING CONTRACTS

2.1 Introduction

The crowdfunding industry is already large, and growing fast. In 2013, the industry was estimated to have raised over $5.1 billion worldwide [162]. Looking to the future, PricewaterhouseCoopers has approximated that by 2025, crowdfunding will be a $150 billion industry [175]. A recent Wall Street Journal article [119] provides some insight for this growth: 1) an individual crowdfunding investment can be risky, offering high returns, 2) lenders can diversify their risk by spreading their investments over different crowdfunding campaigns, and 3) due to recent financial crises, investors have decreased trust in traditional financial investments (e.g., banks, stocks, etc.), so nontraditional lending markets have increased appeal. A recent regulatory change has also reduced the barriers to entry: starting from May 2016, ordinary people are permitted to invest in small firms [185]; before May 2016, only accredited investors (i.e., those with an annual income of at least $200,000 or a net worth of at least $1 million) could invest [61]. This new crowdfunding regulation, which allows firms to raise up to $1 million over a 12-month period, provides a great opportunity for firms to raise their investment targets more easily.

A number of different firms have become rather well known: Kickstarter, Indiegogo, GoFundMe, Kiva, Prosper, Lending Club. These firms have vastly different crowdfunding models driving their businesses. For example, Kickstarter and Indiegogo, perhaps the most well known crowdfunding firms, essentially solicit donations for individuals and firms needing capital. Kiva deals with microloans, targeting low-income entrepreneurs worldwide, that must be repaid to the lender. Prosper and Lending Club operate in the peer-to-peer lending marketplace, where loans have fixed repayment terms. The peer-to-peer lending market was the most dominant alternative finance market in 2015 in the United States, with approximately \$25.7 billion raised [2].

Notably, many of the loans administered by, for example, Bolstr are repaid well before their estimated investment horizon: according to the Bolstr website, the estimated investment horizons usually range from 2-5 years for loan sizes of \$25,000 - \$500,000, yet [53] report on the example of a lobster roll restaurant in Chicago that paid back its loan of \$70,000 in *seven months*. Furthermore, in 2016, Bolstr announced that "An investor who participated at the minimum investment level in every deal would have a portfolio tracking to a 19.18% net Internal Rate of Return."

The speed of funding in crowdfunding can be very fast, with respect to traditional funding sources. Traditional loans, such as from banks and the U.S. Small Business Administration (SBA), have low approval rates and typically take 2-3 months to fund. However, revenue-sharing contracts can fund very fast in practice. The following quotes are real subject lines from marketing emails from Bolstr:

"Paddy Wagon Raised \$20,000 in Less than 10 Minutes,"

"Dubina Brewing Co. Raised \$30,000 in 20 minutes,"

"The Bacon Jams Raised \$40,000 in Less than 30 Minutes,"

"Underground Butcher Raised \$75,000 in Less Than 1 Hour."

Firms can alternatively raise investments from other online lenders such as Lending Club, Prosper, and Kabbage, but according to the Bolstr website, loans from these lenders have annual percentage rates as high as 80%, whereas revenue-sharing loans have annual percentage rates of $8-25\%$. Therefore, revenue-sharing contracts have advantages over traditional and alternative funding sources such as having flexible payments, fast funding time, and lower equivalent interest rates. In this thesis we show that, the firm's NPV under a revenue-sharing contract is larger than that of equity crowdfunding, and the firm's probabilities of bankruptcy are comparable. Additionally, in most cases considered, a firm's Net Present Value (NPV) under a revenue-sharing contract is larger than the NPV of a fixed-rate loan, even when the latter has low annual interest rates. Similarly, a firm's probability of bankruptcy is lower under a revenue-sharing contract than under a fixed-rate loan. These benefits stem from the flexible nature of the revenue-sharing contract.

Under a crowdfunding revenue-sharing contract, the firm decides how much investment $Y \geq 0$ is needed, an investment multiple $M \geq 1$ (where investors are guaranteed to be paid back M times their initial investment), and a revenue-sharing proportion $\gamma \geq 0$ (in each period the firm pays all investors a proportion γ of its revenues). These variables induce a stochastic investment horizon T and a stochastic bankruptcy time B (possibly infinite). The platform fees consist of an origination percentage $\alpha \in [0, 1]$ (where the firm pays a fee αY to the platform at time zero) and a servicing percentage $\beta \in [0, 1]$ (where in each period the firm pays a β percent of all investor revenue payments to the platform). We design a stochastic programming formulation of the firm's problem, which is a rather intractable model. Thus, we derive a deterministic approximation model for the main stochastic model.

2.2 Stochastic Model of Firm

In this section we detail our basic stochastic model of a firm, whose revenue and cost in time period t, $R_t \geq 0$ and $C_t \geq 0$, are random variables; we model these cashflows using random walks with drift, which is explained in detail in Section 2.2.2. Many firms on Bolstr, Localstake, and Starwise have raised money to upgrade their existing stores, build a store in a new location, buy new equipment, and create/upgrade their websites. Due to this expansion, the firm has cash-flow shortages for a limited time and therefore needs to raise capital, in the amount of $Y \geq 0$ (e.g. dollars), via an intermediary platform that pairs interested individual investors that are willing to invest. Therefore, the investment Y is a buffer to avoid a negative cash flow.

The investors do not receive equity in the firm, but are paid back, with interest, via a revenue-sharing contract: at the end of each time period t, the firm is contractually obligated to pay out to all investors a proportion $\gamma \geq 0$ of its revenues for that time period. These payments continue until each investor receives a multiple $M \geq 1$ of his/her initial investment, which occurs at time $t = T$ (for all investors); this definition of M is motivated by practical implementations (e.g., Bolstr, Localstake, and Startwise). Note that the investment payments are not fixed, since they depend on firm revenues, which can vary. The contract's duration is therefore the stochastic stopping time

$$T = \min \left\{ \hat{T} \geq 1 : \sum_{t=1}^{\hat{T}} \gamma R_t \geq MY \right\}, \tag{2.1}$$

which captures the firm's contractual obligation to pay γ percent of its revenue to investors until a total nominal amount of MY has been paid. Time period $t = 0$ is the initialization of the revenue-sharing contract when the firm receives total investment Y. The firm must also pay the platform 1) an origination fee of $\alpha \in [0, 1]$ percent of the total amount raised Y at time $t = 0$ and 2) a servicing fee of $\beta \in [0, 1]$ percent of all revenue payments made to investors at times $t > 0$.

We next discuss cash flows, and, for simplicity, we assume the risk-free interest rate is zero (i.e., cash does not earn interest). If, in period t, $R_t - C_t < 0$, there is a cash shortfall; however, this can potentially be addressed using an excess of cash from previous periods. Thus, we focus on cumulative cash flows. If there exists $\tau \geq 1$ such that $\sum_{t=1}^{\tau}(R_t - C_t) < 0$, then the firm needs cash in month τ. The firm's cumulative cash flow at the end of month

$\tau = 1, \ldots, T$, during the revenue sharing contract, is $\sum_{t=1}^{\tau}(R_t - C_t) + (1-\alpha)Y - (\beta + 1)\gamma \sum_{t=1}^{\tau} R_t$; if $\tau > T$, after the completion of the contract, the cumulative cash flow is $\sum_{t=1}^{\tau}(R_t - C_t) + (1-\alpha)Y - (\beta+1)\gamma \sum_{t=1}^{T} R_t$. These two scenarios can be combined into one expression for the cash flow in period $\tau \geq 1$: $\sum_{t=1}^{\tau}(R_t - C_t) + (1-\alpha)Y - (\beta+1)\gamma \sum_{t=1}^{\min\{\tau,T\}} R_t$.

Due to the stochasticity of the firm's cash flows, the firm can potentially go bankrupt, for any combination of contractual parameters (Y, M, γ). [9] and [203] model bankruptcy as occurring during the first period where the firm is cash-flow negative; we adopt this approach. Letting B denote the time period where the firm goes bankrupt, we model B as a stochastic stopping time (that depends on the stochastic stopping time T):

$$B = \min\left\{ \hat{B} \geq 1 : \sum_{t=1}^{\hat{B}}(R_t - C_t) + (1-\alpha)Y - (\beta+1)\gamma \sum_{t=1}^{\min\{\hat{B},T\}} R_t < 0 \right\}. \tag{2.2}$$

The platform and investors receive their payments at the end of each month if the borrower is not bankrupt; thus, the revenue sharing contract is in effect for $\tau = 1, \ldots, \min\{B, T\}$. As costs and revenues are stochastic, a risk-neutral firm wants to maximize the firm's expected NPV

$$E\left[\sum_{t=1}^{B} \frac{R_t - C_t}{(1+r_t)^t}\right] - (\beta+1)\gamma E\left[\sum_{t=1}^{\min\{B,T\}} \frac{R_t}{(1+r_t)^t}\right] + (1-\alpha)Y, \tag{2.3}$$

where $\gamma E\left[\sum_{t=1}^{\min\{B,T\}} \frac{R_t}{(1+r_t)^t}\right]$ is the expected NPV of all payments made to investors and $\beta\gamma E\left[\sum_{t=1}^{\min\{B,T\}} \frac{R_t}{(1+r_t)^t}\right]$ is the expected NPV of all servicing fees paid to the platform, αY is the origination fee paid to the platform at time $t = 0$, and, in period t, risk is quantified via imputed discount rates r_t [6]. We assume these discount rates are known. According to [174], small-sized firms can estimate their NPV by using the cost of capital as the discount rate. Alternatively, [208] assume that firms determine their discount rate according to the lowest interest rate offered to them from other financial institutions.

Finally, we consider the investors. We assume that there is a large pool of investors and the target investment is raised by n investors, where investor i invests an amount $y_i \in (0, Y]$, and $\sum_{i=1}^{n} y_i = Y$. We model the investor participation constraints as

$$E\left[\sum_{t=1}^{\min\{B,T\}} \frac{y_i}{Y} \frac{\gamma R_t}{(1+\delta_t)^t} - y_i\right] \geq A_i, \;\; i = 1, ..., n, \tag{2.4}$$

where $\frac{y_i}{Y}$ is the fraction of the total payment γR_t investor i receives in period t, δ_t is the discount rate of investors in period t, the left-hand-side is the investor's expected NPV, and $A_i \geq 0$ is the target rate of return, in NPV terms, for investor i. In other words, A_i captures investor i's opportunity cost of alternative investments. We simplify the constraints in Expression (2.4) to the following single constraint

$$E\left[\sum_{t=1}^{\min\{B,T\}} \frac{\gamma}{Y} \frac{R_t}{(1+\delta_t)^t}\right] \geq \max_{1 \leq i \leq n} \left\{\frac{A_i}{y_i}\right\} + 1. \tag{2.5}$$

The firm is able to select the investment amount Y, the multiple M, and the revenue-sharing proportion γ in order to maximize its expected NPV. The above analysis results in the firm's problem:

$$
\begin{aligned}
z_F = \quad \max_{Y,M,\gamma} \quad & E\left[\sum_{t=1}^{B} \frac{R_t - C_t}{(1+r_t)^t}\right] - (\beta+1)\gamma E\left[\sum_{t=1}^{\min\{B,T\}} \frac{R_t}{(1+r_t)^t}\right] + (1-\alpha)Y \\
\text{s.t.} \quad & T = \min\left\{\hat{T} \geq 1 : \sum_{t=1}^{\hat{T}} \gamma R_t \geq MY\right\} \\
& B = \min\left\{\hat{B} \geq 1 : \sum_{t=1}^{\hat{B}} (R_t - C_t) + (1-\alpha)Y - (\beta+1)\gamma \sum_{t=1}^{\min\{\hat{B},T\}} R_t < 0\right\} \\
& E\left[\sum_{t=1}^{\min\{B,T\}} \frac{\gamma}{Y}\frac{R_t}{(1+\delta_t)^t}\right] \geq \max_{1\leq i\leq n}\left\{\frac{A_i}{y_i}\right\} + 1 \\
& Y, \gamma \geq 0, M \geq 1.
\end{aligned}
\tag{2.6}
$$

2.2.1 Model Parameterization Using Data from Bolstr.com

We have collected cost and revenue projections from 56 campaigns on the Bolstr platform, and performed regression analyses on them. The R^2 values for the cost regressions ranged from 0.595 to 0.995, with a mean of 0.905 and a standard deviation of 0.079. The R^2 values for the revenue regressions ranged from 0.725 to 0.995, with a mean of 0.922 and a standard deviation of 0.065. Therefore, the Bolstr data suggest that linear models of cost and revenue projections, in expectation, are reasonable assumptions.

We denote a and b as the intercept and slope of a generic revenue regression line, respectively; similarly, we denote c and d as the intercept and slope of an arbitrary cost regression line, respectively. Letting $E[R_t]$ and $E[C_t]$ denote the expected revenue and cost in period t, respectively, we assign

$$
E[R_t] = a + bt \qquad \text{and} \qquad E[C_t] = c + dt. \tag{2.7}
$$

Many, but not all, of our results will utilize the linearity of cash flows.

2.2.2 Modeling Cashflows

Motivated by the cash flow models in [63], we generate revenues $(R_1, R_2, \ldots)$ and costs $(C_1, C_2, \ldots)$ using random walk processes:

$$R_t = R_{t-1} + Z_t^r \qquad \text{and} \qquad C_t = C_{t-1} + Z_t^c, \qquad t \geq 1, \tag{2.8}$$

where $R_0 = a$ and $C_0 = c$ are given in Equation (2.7). The Z_t^r are independent normal random variables with common mean $\mu^r = b$, where b is given by Equation (2.7), and standard deviation $\sigma^r = \mu^r/k$, where k is a tunable parameter. Similarly, Z_t^c are independent normal random variables with common mean $\mu^c = d$, where d is given by Equation (2.7), and standard deviation $\sigma^c = \mu^c/k$. The means can be easily calculated: $E[R_t] = a + bt$ and $E[C_t] = c + dt$, in agreement with Equation (2.7). Furthermore, the random walk model exhibits serial correlation: it is straightforward to show that, for $s < t$, $cov(R_s, R_t) = s(\sigma^r)^2$ and $cov(C_s, C_t) = s(\sigma^c)^2$. Note that Brownian motion is a limit of a random walk process [117]. Therefore, our proposed random walk models for revenues and costs can be considered as approximate Brownian motion processes with drifts μ^r and μ^c and volatilities σ^r and σ^c, respectively [183, 190].

2.2.3 Analysis Roadmap

We found an analytical solution to Problem (2.6) to be intractable. In the next section, we derive an approximation for the stochastic problem. The approximation problem is a deterministic relaxation where $\sigma^r = \sigma^c = 0$, but we add a cash-flow buffer to the bankruptcy definition to deal with cash-flow uncertainties in the stochastic model; in this case, we are able to derive analytical solutions that provide generalizable insights. These results are provided in Section 2.3. To evaluate the quality of our approximation, we also solve Problem (2.6) for real Bolstr data, using Monte Carlo simulation, to determine the stochastic stopping times (B, T) and expectations, combined with numerical optimization on a fine grid of (Y, M, γ) space. We utilize random walks with drift for the cash flows, where $\sigma^r = \mu^r/k$ and $\sigma^c = \mu^c/k$ for various values of k; this analysis can be found in Section 2.4. Finally, in Sections 2.5–2.6, we show that the revenue-sharing contract compares favorably with equity crowdfunding and fixed-rate loans, respectively.

2.3 Deterministic Approximation to Stochastic Model

In this section, we consider a deterministic approximation to Problem (2.6) where the cash flows R_t and C_t are known exactly. This simplification results in the conversion of the bankruptcy time B and investment duration T into parameters, rather than random variables. We assume, given that cash flows are known exactly, the firm desires to avoid bankruptcy. Thus, reversing the condition for bankruptcy in Equation (2.2), we introduce constraints that require the firm to be cash-flow positive above $\theta \geq 0$, for all time periods $\tau \geq 1$, where θ is a cash buffer to account for cash-flow uncertainties in the stochastic problem:

$$\sum_{t=1}^{\tau}(R_t - C_t) + (1-\alpha)Y - (\beta+1)\gamma \sum_{t=1}^{\min\{\tau,T\}} R_t \geq \theta, \quad \tau \geq 1. \tag{2.9}$$

Similarly, in project management, [145] developed a deterministic schedule and used a project buffer for dealing with resource uncertainty. They determined the size of the buffer numerically; similarly, in Section 2.4.2 we determine the size of the cash-buffer θ numerically as a function of problem data. We show that as the uncertainty of revenues and costs increases, θ should increase to make the deterministic approximation solution feasible for the stochastic problem and provide a high quality approximation for the stochastic problem.

These constraints imply that $B = \infty$ as the firm will never go bankrupt. In Section

2.4 we show, via computational experiments, that the optimal solution to the stochastic model in Problem (2.6) induces a low probability of firm bankruptcy over feasible Bolstr campaigns, which suggests that the constraints in (2.9) are unlikely to eliminate the optimal solution to the stochastic model; in Section 2.4.2 we evaluate the approximation quality of the deterministic model developed in this section for Problem (2.6), with very encouraging results.

Next, since the variables Y, M, and γ are continuous, we assume that the definition of T in Equation (2.1) holds exactly and deterministically:

$$\sum_{t=1}^{T} \gamma R_t = MY. \tag{2.10}$$

These simplifications result in a deterministic approximation to Problem (2.6), parameterized by the investment duration T:

$$\begin{aligned}
\hat{z}_F(T) = \quad & \max_{Y,M,\gamma} \quad \sum_{t=1}^{\infty} \frac{R_t - C_t}{(1+r_t)^t} - (\beta+1)\gamma \sum_{t=1}^{T} \frac{R_t}{(1+r_t)^t} + (1-\alpha)Y \\
& \text{s.t.} \quad \sum_{t=1}^{T} \gamma R_t = MY && \text{(contractual obligation)} \\
& \qquad \sum_{t=1}^{\tau} (R_t - C_t) + (1-\alpha)Y - (\beta+1)\gamma \sum_{t=1}^{\min\{\tau,T\}} R_t \geq \theta, \ \tau \geq 1 && \text{(cash-flow constraints)} \\
& \qquad \sum_{t=1}^{T} \frac{\gamma}{Y} \frac{R_t}{(1+\delta_t)^t} \geq \max_{1\leq i\leq n} \left\{ \frac{A_i}{y_i} \right\} + 1 && \text{(investor participation)} \\
& \qquad Y, \gamma \geq 0, M \geq 1.
\end{aligned} \tag{2.11}$$

2.3.1 Analysis for fixed $T \in \mathbb{N}$

In this subsection, we solve Model (2.11) for a fixed $T \in \mathbb{N}$. To begin our analysis, we point out some simplifications. First, we let $\hat{A} = \max_{1\leq i\leq n} \left\{ \frac{A_i}{y_i} \right\}$ to simplify the exposition. Second, the contractual obligation constraint can be used to solve for $\gamma = \frac{MY}{\sum_{t=1}^{T} R_t}$, and γ is eliminated as a variable. In the subsequent analysis, it will be convenient to define the set

$$X \triangleq \left\{ \tau \in \mathbb{N} : \sum_{t=1}^{\tau} (R_t - C_t) < \theta \right\}, \tag{2.12}$$

which indexes all the time periods where the firm, without any investment, has cash flow below θ. It is also convenient to define the parameters $Z_\tau(T)$, which depend only on firm problem data and T:

$$Z_\tau(T) \triangleq (\hat{A}+1)(\beta+1)\frac{\sum_{t=1}^{\min\{\tau,T\}} R_t}{\sum_{t=1}^{T} \frac{R_t}{(1+\delta_t)^t}} - (1-\alpha). \tag{2.13}$$

The resulting model has the following solution.

Proposition 2.1. *Problem* (2.11)*, with $T \in \mathbb{N}$ fixed and $X \neq \emptyset$, is feasible if and only if $Z_\tau(T) < 0,\ \forall \tau \in X$ and $\max_{\tau \in X} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\} \le \min_{\substack{\tau \notin X \\ Z_\tau(T) > 0}} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\}$, and has the optimal solution $M^*(T) = \frac{(\hat{A}+1)\sum_{t=1}^{T} R_t}{\sum_{t=1}^{T} \frac{R_t}{(1+\delta_t)^t}}$ and $\gamma^*(T) = \frac{(\hat{A}+1)Y^*(T)}{\sum_{t=1}^{T} \frac{R_t}{(1+\delta_t)^t}}$,*

where

- *if $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T} \frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T} \frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) \ge 0$, then $Y^*(T) = \max_{\tau \in X} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\}$; alternatively, if $X = \emptyset$, then $Y^* = 0$, $\gamma^* = 0$, and $M^* = 1$ for all $T \in \mathbb{N}$.*

- *if $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T} \frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T} \frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) < 0$, then $Y^*(T) = \min_{\substack{\tau \notin X \\ Z_\tau(T) > 0}} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\}$; alternatively, if $X = \emptyset$, then $Y^*(T) = \min_{\substack{\tau \in \mathbb{N} \\ Z_\tau(T) > 0}} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\}$ for all $T \in \mathbb{N}$.*

The maximized NPV is $\hat{z}_F(T) = \sum_{t=1}^{\infty} \frac{R_t - C_t}{(1+r_t)^t} - \left((\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T} \frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T} \frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) \right) Y^*(T)$.

Note that Proposition 2.1 only requires that the cash flows R_t and C_t be deterministic, but does not require them to be linear. The feasibility constraints in Proposition 2.1 can be interpreted as financial conditions where the firm can eventually survive on its own; as an example where this is not possible, consider the case where $C_t > R_t$ for all t. In particular, $Z_\tau(T) < 0,\ \forall \tau \in X$ ensures that the firm is cash-flow positive for all periods $\tau \in X$, due to the investment $Y^*(T)$, and the condition $Y^*(T) \leq \min_{\substack{\tau \notin X \\ Z_\tau(T)>0}} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\}$ ensures that the firm can afford to pay back $M^*(T)Y^*(T)$ to investors and $\beta M^*(T)Y^*(T)$ to the platform.

If the firm's discount rates r_t are not too large, relative to the investors' discount rates δ_t, and $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) \geq 0$ holds, then we see that, intuitively, if $X = \emptyset$, then the firm does not need any investment and $Y^*(T) = 0$. Alternatively, if $X \neq \emptyset$ and the feasibility conditions are satisfied, then $\left((\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha)\right) Y^*(T)$ can be interpreted as the firm's cost for avoiding bankruptcy.

Alternatively, if $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) < 0$, then the firm's discount rates r_t are relatively larger than the investors' discount rates δ_t. Therefore the firm's gain from the $(1-\alpha)Y$ investment at time zero is greater than the firm's NPV of future payments to the investors and the platform. As a result, the firm benefits by raising a larger investment.

In Figure 2.1 we plot the average of the objective function value $\hat{z}_F(T)$ and the average of the optimal variables $(Y^*(T), M^*(T), \gamma^*(T))$ from Proposition 2.1 as a function of T, over feasible Bolstr campaigns, for the following parameter set: For each campaign we let linear revenues $R_t = E[R_t]$ and costs $C_t = E[C_t]$, where $E[R_t]$ and $E[C_t]$ are given in Equation (2.7) and only use the given campaign's data, $\theta = 0$, $\alpha = 0.05$ and $\beta = 0.01$ (per a Bolstr memorandum), $\hat{A} = 0.1$ (i.e., a 10% NPV return for investors), and $r_t = \delta_t = 0.01,\ \forall t$ (the discount rate per period, typically a month). These results are useful in the sequel for interpreting the results for the stochastic problem where the level of variability in costs and revenues is small.

The plot on the top left of Figure 2.1 suggests that $\hat{z}_F(T)$ converges rather quickly to an asymptote. In the next section, for linear cash-flows, we derive conditions for which

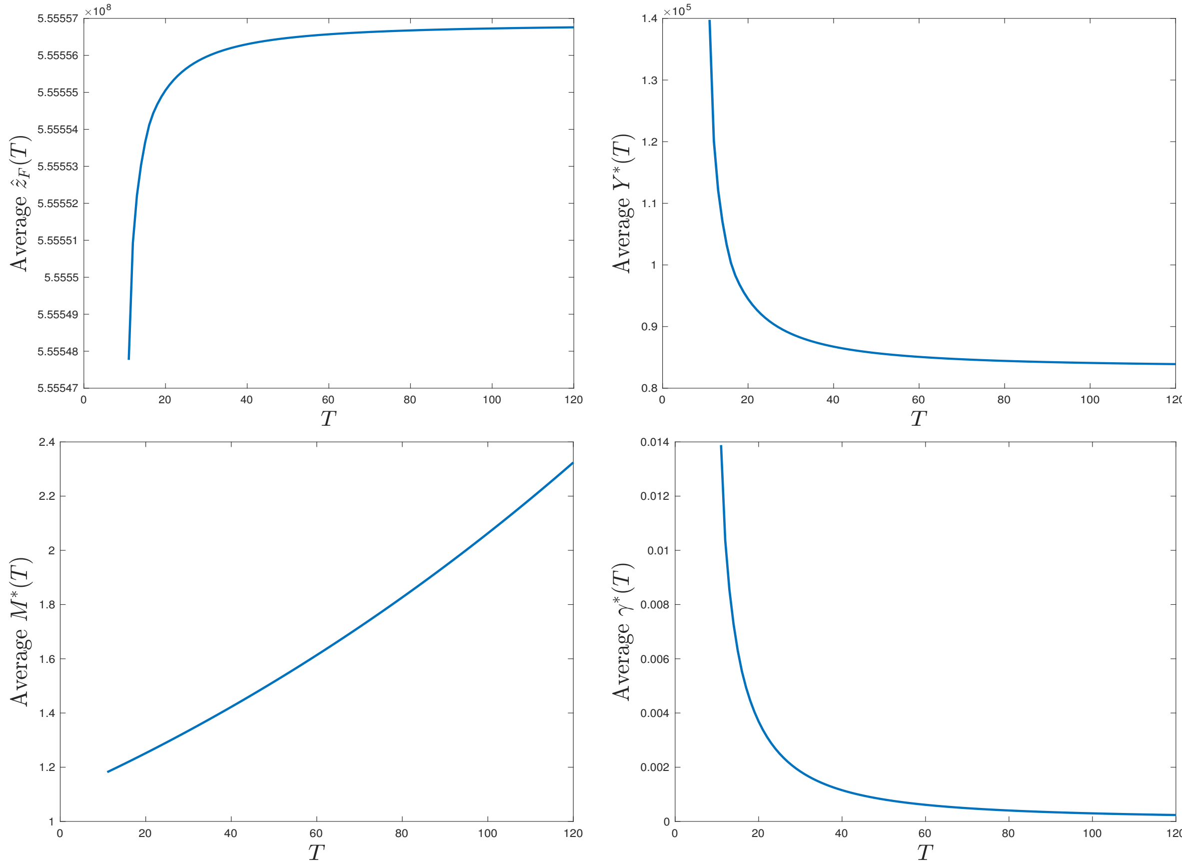

Figure 2.1: Illustration of Proposition 2.1 for feasible Bolstr campaigns with linear costs and revenues.

$\max_T \hat{z}_F(T)$ is attained when $T \to \infty$. However, our numerical results suggest that relatively small values of T, say $T \in \{18, \ldots, 30\}$, suffice to attain almost all the potential value of $\max_T \hat{z}_F(T)$. In the top right plot, we observe that $Y^*(T)$ decreases rather quickly to an asymptote as well. In the bottom left plot we see that $M^*(T)$ is increasing in T and in the bottom right plot we see that $\gamma^*(T)$ is decreasing in T, which intuitively align with the increased investment duration T. We point out that our model was only feasible for $T \geq 12$ for all feasible campaigns for $\theta = 0$; in the next section, for linear cash flows, we provide a rigorous proof that Problem (2.11) can only be feasible for large enough T.

2.3.2 The Optimal Investment Horizon T^*

In the previous section, we fixed $T \in \mathbb{N}$ and solved Problem (2.11); we provided closed-form expressions in Proposition 2.1 for $\gamma^*(T)$, $M^*(T)$, and $Y^*(T)$ for feasible problems. In this section, we find the optimal investment horizon T^*. Proposition 2.1 indicates that Model (2.11) becomes

$$\begin{aligned} \hat{z}_F(T) = \quad & \max_{T\in\mathbb{N}} \quad \sum_{t=1}^{\infty} \frac{R_t - C_t}{(1+r_t)^t} - \left((\beta+1)(\hat{A}+1) \frac{\sum_{t=1}^{T} \frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T} \frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) \right) Y^*(T) \\ & \text{s.t.} \quad Z_\tau(T) < 0, \ \tau \in X \\ & \qquad \max_{\tau \in X} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\} \leq \min_{\substack{\tau \notin X \\ Z_\tau(T)>0}} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\}, \end{aligned} \tag{2.14}$$

where $Y^*(T)$ is defined in Proposition 2.1. It is convenient to define the term $R^\infty \triangleq \sum_{t=1}^{\infty} \frac{R_t}{(1+\delta_t)^t}$, which we assume is finite, in order to present the next set of results, which are valid for large T and characterize the structure of Problem (2.14).

Lemma 2.1. *If* $(\hat{A}+1)(\beta+1)\sum_{t=1}^{\tau} R_t/R^\infty < (1-\alpha)$ *for all* $\tau \in X$*, then* $Z_\tau(T) < 0$ *for all* $\tau \in X$*.*

Lemma 2.2. *For* $\tau \in X$ *and* $Z_\tau(T) < 0$, $\max_{\tau\in X} \left\{ \frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)} \right\}$ *is strictly decreasing in* T*.*

Lemma 2.3. *For* $\tau \notin X$ *and* $Z_\tau(T) > 0$, $\min_{\substack{\tau\notin X \\ Z_\tau(T)>0}} \left\{ \frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)} \right\}$ *is strictly increasing in* T*.*

The next lemma builds upon Lemmas 2.2–2.3 to characterize the objective function of Problem (2.14).

Lemma 2.4. *If* $r_t = r \geq \delta = \delta_t, \forall t$ *or* $r_t = \delta_t, \forall t$*, then the objective function of Problem* (2.14) *is increasing in* T*, for large* T*.*

Lemmas 2.2 and 2.3 suggest that $\max_{\tau \in X} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\} \leq \min_{\substack{\tau \notin X \\ Z_\tau(T) > 0}} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\}$ is attained asymptotically. However, this is not always possible. Consider the following example.

Example 2.1. *Let the revenues* $R_t = 10$ *for* $t \geq 1$ *and the costs are* $C_1 = 15$, $C_2 = 5$, *and* $C_t = 10$ *for* $t \geq 3$, *and* $\theta = 0$. *The set* $X = \{1\}$ *and, for feasible values of* $(\hat{A}, \alpha, \beta, \delta_t)$, $\max_{\tau \in X} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\} > 0$ *for all feasible* T. *In contrast,* $\sum_{t=1}^{\tau}(R_t - C_t) = 0$ *for all* $\tau \notin X$, *implying* $\min_{\substack{\tau \notin X \\ Z_\tau(T) > 0}} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\} = 0.$

In the next subsection, we see that an assumption of linear cash flows resolves the issue encountered in the above counterexample, and Problem (2.14) is solvable analytically. Recall that the linearity of revenues and costs are supported by real data from Bolstr, as explained in Section 2.2.1.

2.3.2.1 Linear Firm Costs and Revenues

In this subsection, we consider Problem (2.14) when the costs and revenues are linear: $R_t = a + bt$ and $C_t = c + dt$. We could, for instance, parameterize (a, b, c, d) using the regression analysis on Bolstr data. The set X, for linear costs and revenues, simplifies to

$$X = \left\{ \tau \in \mathbb{N} : (a - c)\tau + (b - d)\frac{\tau(\tau + 1)}{2} < \theta \right\}. \tag{2.15}$$

The following function is useful for the subsequent analysis and results:

$$f(\tau) \triangleq \frac{\displaystyle\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} = \frac{(a - c)\tau + (b - d)\dfrac{\tau(\tau + 1)}{2} - \theta}{\eta(T)\left(a\tau + b\frac{\tau(\tau+1)}{2}\right) - (1 - \alpha)}, \tag{2.16}$$

where $\eta(T) \triangleq \frac{(\hat{A}+1)(\beta+1)}{\sum_{t=1}^{T} \frac{R_t}{(1+\delta_t)^t}} > 0$. This function allows us to write $\max_{\tau \in X} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\} =$

$\max_{\tau \in X} f(\tau)$ and $\min_{\substack{\tau \notin X \\ Z_\tau(T)>0}} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\} = \min_{\substack{\tau \notin X \\ Z_\tau(T)>0}} f(\tau)$, where $\max_{\tau \in X} f(\tau)$ is a lower bound on $Y^*(T)$ to allow cash-flow positivity at or above θ and $\min_{\substack{\tau \notin X \\ Z_\tau(T)>0}} f(\tau)$ is an upper bound on $Y^*(T)$ to ensure that the firm can afford to pay back $M^*(T)Y^*(T)$ to investors and $\beta M^*(T)Y^*(T)$ to the platform. Our next two results characterize the cases where the set X

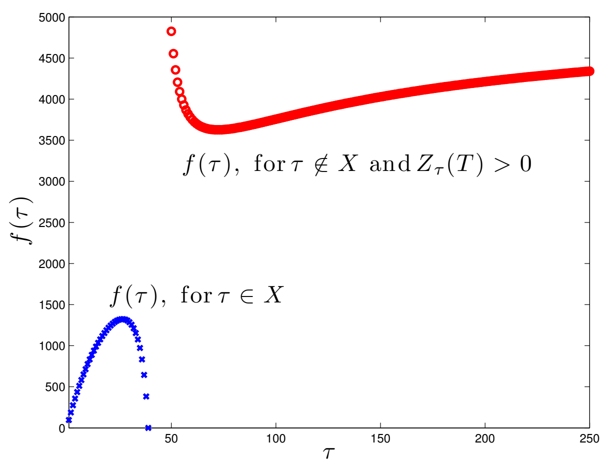

Figure 2.2: The form of the function $f(\tau)$.

is unbounded or empty, which results in infeasible and trivial firm optimization problems, respectively.

Lemma 2.5. *If 1) $b < d$ or 2) $b = d$ and $a < c + \theta$, then Problem* (2.14) *is infeasible.*

Under the conditions of Lemma 2.5, the firm will eventually go bankrupt for all values of (Y, M, γ).

Lemma 2.6. *If 1) $b > d$ and $(b-d) \geq (c-a)+\theta$ and $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) \geq 0$ or 2) $b = d$ and $a \geq c + \theta$, then $Y^*(T) = 0$ for all $T \in \mathbb{N}$ in Problem* (2.11).

Under either of the conditions of Lemma 2.6, the firm does not need any outside funding and is cash-flow positive at or above θ for all periods $\tau \geq 1$.

Another set of conditions where the firm is cash-flow positive at or above θ for all periods $\tau \geq 1$ is $b > d$ and $b - d \geq (c-a) + \theta$ and $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) < 0$. In the next lemma, we show that under these conditions, if feasible, the firm borrows money from investors although it is already cash-flow positive at or above θ. The reason is that, under these conditions, the firm's gain from raising an investment at time $t = 0$ is larger than the NPV of its future monthly payments to the investors and the platform, due to its high discount rates.

Lemma 2.7. *If $b > d$ and $b - d \geq (c-a) + \theta$ and $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) < 0$, then for $\theta \leq \theta_L(T)$, where $\theta_L(T)$ is a function of T and problem parameters, we have for all $T \in \mathbb{N}$:*

- *if $cb - da \geq 0$, then $Y^*(T) = \min\{f(\lfloor\tau^{**}\rfloor), f(\lceil\tau^{**}\rceil)\}$, where τ^{**} is real and equal to*

$$\begin{aligned}\tau^{**} &= \frac{(1-\alpha)(b-d) - \eta(T)\theta b}{(cb-da)\eta(T)} \\ &+ \frac{\sqrt{((1-\alpha)(b-d) - \eta(T)\theta b)^2 - \eta(T)(cb-da)\left((2(c-a)-(b-d))(1-\alpha) + \eta(T)\theta(2a+b)\right)}}{(cb-da)\eta(T)}.\end{aligned}$$

- *if $cb - da < 0$, then $Y^*(T) = (b-d)/\eta(T)b$.*

We have characterized all combinations of (a, b, c, d), except the case where $b > d$ and $b - d < (c-a) + \theta$. We break this case into two sub-cases: i) $b > d$ and $b - d < (c-a) + \theta$ and $cb > da$, and ii) $b > d$ and $b - d < (c-a) + \theta$ and $cb \leq da$. These cases result in Problem (2.14) being feasible with a non-trivial solution. In particular, Case i includes firms that have cash-flow shortages for a limited time, but have expectations of positive cash flows in the future. Case ii includes firms that have positive cash flows, but they are below θ, and the firm receives investment to increase its cash flow to at least θ. Recall that θ is a cash buffer in the deterministic approximation problem to account for cash-flow uncertainties in the stochastic problem. We begin by characterizing the set X for both these cases.

Lemma 2.8. *If $b > d$ and $b - d < (c - a) + \theta$, then*

$$X = \left\{ \tau \in \mathbb{N} : 1 \leq \tau < \frac{\left(c - a - \frac{b-d}{2}\right) + \sqrt{\left(c - a - \frac{b-d}{2}\right)^2 + 2\theta(b - d)}}{b - d} \right\}.$$

From the set X it is clear that as θ increases, the number of periods where the firm has a cash flow below θ increases. The following lemma indicates that the problem is not feasible for θ larger than a threshold.

Lemma 2.9. *The first constraint in Problem* (2.14)*, $Z_\tau(T) < 0$ for all $\tau \in X$, is equivalent to the conditions $\theta \leq \hat{\theta}(T)$ and $(ad - bc) + (b - d)\sqrt{(a + b/2)^2 + 2(1 - \alpha)b/\eta(T)} \geq 0$, where $\hat{\theta}(T)$ is a function of T and problem parameters.*

Note that the left-hand side of the second condition in Lemma 2.9 is increasing in T, albeit asymptotically, due to $\eta(T)$. Therefore, if it is possible for the first constraint of Problem (2.14) to hold, it will be feasible for a large enough T and a small enough θ. We next analyze Cases i and ii, building upon the condition in Lemma 2.9, to address the second constraint in Problem (2.14).

Case i: $b > d$ and $b - d < (c - a) + \theta$ and $cb > da$. The next three lemmas characterize the second constraint in Problem (2.14) for Case i: $b > d$ and $b - d < (c - a) + \theta$ and $cb > da$, assuming the feasibility conditions of the first constraint in Lemma 2.9 hold.

Lemma 2.10. *If $b > d$, $cb > da$, and $(ad - bc) + (b - d)\sqrt{(a + b/2)^2 + 2(1 - \alpha)b/\eta(T)} \geq 0$, then for $\theta \leq \tilde{\theta}(T)$ and $b - d < (c - a) + \theta$, where $\tilde{\theta}(T)$ is a function of T and problem* *parameters:*

a) If $\theta \geq \frac{((b-d)-2(c-a))(1-\alpha)}{\eta(T)(2a+b)}$, *then* $\max_{\tau \in X} f(\tau) = \max\{f(\lfloor \tau^* \rfloor), f(\lceil \tau^* \rceil)\}$, *where* τ^* *is real and equal to*

$$\begin{aligned}\tau^* &= \frac{(1-\alpha)(b-d)-\eta(T)\theta b}{(cb-da)\eta(T)} - \\ &\quad \frac{\sqrt{((1-\alpha)(b-d)-\eta(T)\theta b)^2 - \eta(T)(cb-da)\left((2(c-a)-(b-d))(1-\alpha)+\eta(T)\theta(2a+b)\right)}}{(cb-da)\eta(T)}.\end{aligned}$$

b) Otherwise, $\max_{\tau \in X} f(\tau) = \dfrac{a+b-c-d-\theta}{\eta(T)(a+b)-(1-\alpha)}$.

Lemma 2.11. *If* $b > d$, $cb > da$, *and* $(ad-bc)+(b-d)\sqrt{(a+b/2)^2+2(1-\alpha)b/\eta(T)} \geq 0$, *then for* $\theta \leq \tilde{\theta}(T)$ *and* $b-d < (c-a)+\theta$, *where* $\tilde{\theta}(T)$ *is a function of* T *and problem parameters, we have* $\min_{\substack{\tau \notin X \\ Z_\tau(T)>0}} \{f(\tau)\} = \min\{f(\lfloor \tau^{**} \rfloor), f(\lceil \tau^{**} \rceil)\}$, *where* τ^{**} *is real and equal to*

$$\begin{aligned}\tau^{**} &= \frac{(1-\alpha)(b-d)-\eta(T)\theta b}{(cb-da)\eta(T)} + \\ &\quad \frac{\sqrt{((1-\alpha)(b-d)-\eta(T)\theta b)^2 - \eta(T)(cb-da)\left((2(c-a)-(b-d))(1-\alpha)+\eta(T)\theta(2a+b)\right)}}{(cb-da)\eta(T)}.\end{aligned}$$

Lemma 2.12. *If* $b > d$, $cb > da$, *and* $(ad-bc)+(b-d)\sqrt{(a+b/2)^2+2(1-\alpha)b/\eta(T)} \geq 0$, *then* $\exists\, \bar{\theta}(T)$ *such that for* $\theta \leq \bar{\theta}(T)$ *and* $b-d < (c-a)+\theta$, *where* $\bar{\theta}(T)$ *is a function of* T *and problem parameters, the inequality* $\max_{\tau \in X}\{f(\tau)\} \leq \min_{\substack{\tau \notin X \\ Z_\tau(T)>0}} \{f(\tau)\}$ *holds for* T *large enough.*

Lemmas 2.10 – 2.12 prove that, for linear cash flows and Case i, if θ is small enough and T is large enough, Problem (2.14) is feasible. We next accomplish the same task for Case ii.

Case ii: $b > d$ **and** $b-d < (c-a)+\theta$ **and** $cb \leq da$**.** The next three lemmas characterize the second constraint in Problem (2.14) for Case ii: $b > d$ and $b-d < (c-a)+\theta$ and $cb \leq da$, assuming the feasibility conditions of the first constraint in Lemma 2.9 hold.

Lemma 2.13. *If $b > d$ and $cb \leq da$, and $(ad-bc)+(b-d)\sqrt{(a+b/2)^2+2(1-\alpha)b/\eta(T)} \geq 0$, then for $\theta \leq \tilde{\theta}(T)$ and $b-d < (c-a)+\theta$, where $\tilde{\theta}(T)$ is a function of T and problem parameters, we have $\max\limits_{\tau\in X} f(\tau) = \dfrac{a+b-c-d-\theta}{\eta(T)(a+b)-(1-\alpha)}$.*

Lemma 2.14. *If $b > d$ and $cb \leq da$, and $(ad-bc)+(b-d)\sqrt{(a+b/2)^2+2(1-\alpha)b/\eta(T)} \geq 0$, then for $\theta \leq \tilde{\theta}(T)$ and $b-d < (c-a)+\theta$, where $\tilde{\theta}(T)$ is a function of T and problem parameters, we have $\min\limits_{\substack{\tau\notin X\\ Z_\tau(T)>0}} \{f(\tau)\} = (b-d)/\eta(T)b$.*

Note that from $b > d$ and $cb \leq da$, we conclude $a \geq c$. Conditions $b > d$ and $a \geq c$ represent a firm that has higher revenues than costs in all periods, with the revenue growth larger than that of cost; however, the firm's cash flow is not above θ in all periods, which drives the need for investment.

Lemma 2.15. *If $b > d$ and $cb \leq da$, and $(ad-bc)+(b-d)\sqrt{(a+b/2)^2+2(1-\alpha)b/\eta(T)} \geq 0$, then $\exists\ \bar{\theta}(T)$ such that for $\theta \leq \bar{\theta}(T)$ and $b-d < (c-a)+\theta$, where $\bar{\theta}(T)$ is a function of T and problem parameters, the inequality $\max\limits_{\tau\in X}\{f(\tau)\} \leq \min\limits_{\substack{\tau\notin X\\ Z_\tau(T)>0}} \{f(\tau)\}$ holds.*

The conclusion of Lemmas 2.12 and 2.15, $\max\limits_{\tau\in X}\{f(\tau)\} \leq \min\limits_{\substack{\tau\notin X\\ Z_\tau(T)>0}} \{f(\tau)\}$, guarantees that the firm is cash-flow positive at or above θ in all months for Cases i and ii, respectively, which is only possible if θ is small enough. Lemmas 2.2 – 2.3 prove that the inequality $\max\limits_{\tau\in X}\{f(\tau)\} \leq \min\limits_{\substack{\tau\notin X\\ Z_\tau(T)>0}} \{f(\tau)\}$, if feasible, is feasible for T large enough. Thus, Problem (2.14), under linear cash flows and Cases i and ii, is feasible for T large enough and θ small enough. The results in Lemmas 2.7, 2.10, and 2.13 provide closed-form solutions for $Y^*(T)$ under all non-trivial cases and Proposition 2.1 provides closed-form solutions for $M^*(T)$, and $\gamma^*(T)$ as a function of $Y^*(T)$. We found that $T = 120$ worked well to generate high-quality approximations for problems parameterized by real Bolstr data. These optimal variables for Problem (2.14) are used as approximate solutions for the stochastic model in Problem (2.6), whose quality we explore in the next section.

To complete the analysis of the deterministic problem, we now collect all these results to solve Problem (2.14) for the cases where $r_t = r \geq \delta = \delta_t, \forall t$ or $r_t = \delta_t, \forall t$. Proposition 2.1, and Lemmas 2.7, 2.10, and 2.13 for linear cash flows, provide closed-form solutions for the optimal $\gamma^*(T)$, $M^*(T)$, and $Y^*(T)$, assuming model feasibility and a fixed $T \in \mathbb{N}$. Lemma 2.4 indicates that the objective of Problem (2.14) is strictly increasing in T. Lemmas 2.1 – 2.3 prove that Problem (2.14), if feasible, is feasible for large enough T. We also note that $\lim\limits_{T\to\infty} \eta(T) = \lim\limits_{T\to\infty} \dfrac{(\hat{A}+1)(\beta+1)}{\sum_{t=1}^{T} \frac{(a+bt)}{(1+\delta_t)^t}} = (\hat{A}+1)(\beta+1)/R^\infty$. Lemmas 2.11 – 2.12 and 2.14 – 2.15 prove, for Cases i and ii, respectively, that for linear cash flows and large enough T, the model is feasible. Together, these results solve Problem (2.14) for linear costs and revenues, which we summarize in the next propositions. Specifically, in Proposition 2.2 we characterize the optimal solutions for $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) \geq 0$ and in Proposition 2.3 we do the same for $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) < 0$.

Proposition 2.2.

For $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) \geq 0$ *and either* $r_t = r \geq \delta = \delta_t, \forall t$ *or* $r_t = \delta_t, \forall t$, *if* $b > d$, $b-d < (c-a)+\theta$, $\theta \leq \lim\limits_{T\to\infty}\bar{\theta}(T)$, *and* $(ad-bc)+(b-d)\sqrt{(a+b/2)^2 + 2(1-\alpha)b\frac{R^\infty}{(\hat{A}+1)(\beta+1)}} \geq 0$, *then*

- *if* $cb > da$ *and* $\theta \geq \frac{((b-d)-2(c-a))(1-\alpha)(\hat{A}+1)(\beta+1)}{R^\infty(2a+b)}$, *then* $T^* = \infty$, $M^* = \infty$, $Y^* = \max\{f(\lfloor\tau^*\rfloor), f(\lceil\tau^*\rceil)\}$, $\gamma^* = \frac{(\hat{A}+1)Y^*}{R^\infty}$, *where*

$$\tau^* = \frac{(1-\alpha)(b-d) - \theta b(\hat{A}+1)(\beta+1)/R^\infty}{(cb-da)(\hat{A}+1)(\beta+1)/R^\infty} -$$
$$\frac{\sqrt{\left((1-\alpha)(b-d) - \theta b\frac{(\hat{A}+1)(\beta+1)}{R^\infty}\right)^2 - \frac{(\hat{A}+1)(\beta+1)}{R^\infty}(cb-da)\left((2(c-a)-(b-d))(1-\alpha) + \frac{(\hat{A}+1)(\beta+1)}{R^\infty}\theta(2a+b)\right)}}{(cb-da)(\hat{A}+1)(\beta+1)/R^\infty}.$$

- *Otherwise,* $T^* = \infty$, $M^* = \infty$, $Y^* = \frac{a+b-c-d-\theta}{(\hat{A}+1)(\beta+1)(a+b)/R^\infty - (1-\alpha)}$, $\gamma^* = \frac{(\hat{A}+1)\frac{a+b-c-d-\theta}{\eta(T)(a+b)-(1-\alpha)}}{R^\infty}$.

The firm's maximized NPV is $\hat{z}_F = \sum_{t=1}^{\infty} \frac{a-c+(b-d)t}{(1+r_t)^t} - \left((\beta+1)(\hat{A}+1) \frac{\sum_{t=1}^{T} \frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T} \frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) \right) Y^*.$

Proposition 2.3.

For $(\beta+1)(\hat{A}+1) \frac{\sum_{t=1}^{T} \frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T} \frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) < 0$ *and* $r_t = r \geq \delta = \delta_t, \forall t,$ *if* $b > d$, $b - d \geq (c-a) + \theta$, *and* $\theta \leq \lim_{T\to\infty} \theta_L(T)$ *or if* $b > d$, $b - d < (c-a) + \theta$, $\theta \leq \lim_{T\to\infty} \bar{\theta}(T)$, *and* $(ad - bc) + (b-d)\sqrt{(a+b/2)^2 + 2(1-\alpha)b\frac{R^{\infty}}{(\hat{A}+1)(\beta+1)}} \geq 0$, *then*

- *if* $cb - da \geq 0$, *then* $Y^*(T) = \min\{f(\lfloor \tau^{**} \rfloor), f(\lceil \tau^{**} \rceil)\}$
- *if* $cb - da < 0$, *then* $Y^*(T) = (b-d)/\eta(T)b$,

where τ^{**} *is real and equal to*

$$\tau^{**} = \frac{(1-\alpha)(b-d) - \theta b(\hat{A}+1)(\beta+1)/R^{\infty}}{(cb-da)(\hat{A}+1)(\beta+1)/R^{\infty}} +$$
$$\frac{\sqrt{\left((1-\alpha)(b-d) - \theta b\frac{(\hat{A}+1)(\beta+1)}{R^{\infty}}\right)^2 - \frac{(\hat{A}+1)(\beta+1)}{R^{\infty}}(cb-da)\left((2(c-a)-(b-d))(1-\alpha) + \frac{(\hat{A}+1)(\beta+1)}{R^{\infty}}\theta(2a+b)\right)}}{(cb-da)(\hat{A}+1)(\beta+1)/R^{\infty}},$$

and $T^* = \infty$, $M^* = \infty$, $\gamma^* = \frac{(\hat{A}+1)Y^*}{R^{\infty}}$. *The firm's maximized NPV is* $\hat{z}_F = \sum_{t=1}^{\infty} \frac{a-c+(b-d)t}{(1+r_t)^t} - \left((\beta+1)(\hat{A}+1) \frac{\sum_{t=1}^{T} \frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T} \frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) \right) Y^*.$

From Propositions 2.2 and 2.3, we see that if $r_t = r \geq \delta = \delta_t, \forall t$ or $r_t = \delta_t, \forall t$, then Y^* and γ^* are finite, but M^* and T^* are infinite. This solution corresponds to a financial perpetuity (an annuity with no termination) with non-fixed payments that are a function of firm revenues. Note that perpetuities are common in modern business (e.g., anyone can purchase a perpetuity through Bank of America's Merrill Edge brokerage). More prominent examples include LeBron James, a four time NBA MVP, and soccer star David Beckham having lifetime contracts with Nike and Adidas, respectively [136]. Furthermore, [88] provides a study of three crowdfunding platforms and showed that many individuals and firms use crowdfunding to make direct and long term connections with investors; a perpetuity precisely achieves a long-term connection.

Our perpetuity solution can also be interpreted as a pseudo type of equity crowdfunding. [182] explains that the present value of stock is equivalent to the discounted present value of all future dividends, which are typically a share of profits (page 198 of [169]). If we replace profits with revenues, we obtain our perpetuity contract. In Section 2.5 we formally study the difference between our model and an equity crowdfunding model that shares profits, rather than revenues, and we show that the revenue-sharing contract is superior.

2.4 Analysis of Stochastic Model

2.4.1 Simulation-based Numerical Optimization of Stochastic Model

In order to solve Problem (2.6), we let revenues R_t and costs C_t of each of the 56 Bolstr campaigns follow the random walk processes in (2.8). We set the highest allowable values of σ^r and σ^c as $\mu^r/3$ and $\mu^c/3$, respectively, so that revenues and costs are non-negative with high probability. More specifically, we consider $\sigma^r = \mu^r/k$ and $\sigma^c = \mu^c/k$ for $k \in \{3, 4, 5, 6, 7, 8, 9, 10, 15\}$, along with the additional deterministic case of $k \to \infty$.

In our base parameter set, we let $r_t = \delta_t = 0.01$, for all t, $\alpha = 0.05$, $\beta = 0.01$, and $\hat{A} = 0.1$ (i.e., a 10% NPV return for investors); note that r is the discount rate per period, which is typically a month and that the choice of $\alpha = 0.05$ and $\beta = 0.01$ is supported by a Bolstr memorandum. Many online platforms such as Bolstr, LendingClub, and Prosper charge 1% servicing fee for collecting and processing payments. These online lenders usually charge borrowers origination fees of typically 1% − 8%.

We approximate an infinite horizon by considering $t \in \{1, ..., N\}$ where $N = 1000$; we selected this value of N so that $N > \max\{B, T\}$ holds with high probability; furthermore, there is no evidence that revenue-sharing contracts at Bolstr and Localstake last longer than 1000 months, and as pointed out earlier most Bolstr campaigns last 2-5 years. For analyzing Problem (2.6) numerically, we discretized the (Y, M, γ) space. In particular, we considered values of $Y \in \{0, \Delta_Y, 2\Delta_Y, \ldots, \overline{Y}\}$, $M \in \{1, 1+\Delta_M, 1+2\Delta_M, \ldots, \overline{M}\}$, and $\gamma \in \{0, \Delta_\gamma, 2\Delta_\gamma, \ldots, \overline{\gamma}\}$, where $(\Delta_Y, \Delta_M, \Delta_\gamma, \overline{Y}, \overline{M}, \overline{\gamma}) = (5000, 0.25, 0.01, 2000000, 3, 1)$ were chosen to balance computational time and solution quality.

For each campaign, we used Monte Carlo simulation to generate $m = 1000$ realizations of the $(R_1, \ldots, R_N)$ and $(C_1, \ldots, C_N)$ vectors, which allowed us to generate m realizations of the T and B random variables for each (Y, M, γ) tuple in the discretized set. Then, for each variable tuple, averaging over the m trials, we estimate $E\left(\sum_{t=1}^{B} \frac{R_t - C_t}{(1+r)^t}\right)$, $E\left(\sum_{t=1}^{\min\{B,T\}} \frac{R_t}{(1+r)^t}\right)$, and $E\left(\sum_{t=1}^{\min\{B,T\}} \frac{R_t}{(1+\delta)^t}\right)$, which allowed us to evaluate the feasibility of the variable tuple. Finally, we evaluated the objective function for each feasible (Y, M, γ) tuple and chose the one that maximizes the objective function as the optimal solution. Depending on the standard deviations, 90-95% of the 56 Bolstr campaigns were feasible.

2.4.2 Evaluation of Deterministic Approximation in Section 2.3

In this subsection, we evaluate the value of the approximate Problem (2.14) with respect to that of the stochastic Problem (2.6), for T large enough. For the numerical results presented below, we consider $T = 120$, though the performance is insensitive for larger T. In Section 2.4.3 we show that if we use the approximation solutions in the true stochastic model, the expected investment horizon T is close to the choice of $T = 120$. The main parameters that we vary for each campaign are the standard deviations σ^r and σ^c; all other problem parameters are listed above. As mentioned previously, we consider $\sigma^r \in [0, \mu^r/3]$ and $\sigma^c \in [0, \mu^c/3]$, so that revenues and costs are non-negative with high probability.

For each level of variability, we find the optimal $\theta \in [0, 2M]$ which makes the approximation solutions $(Y^*(\theta), M^*(\theta), \gamma^*(\theta))$ feasible for Problem (2.6) and results in the minimum approximation error for the stochastic problem. We denote this value of θ by θ^* in the sequel. In the left panel of Figure 2.3 we show that as cash-flow variability increases, θ^* increases to provide a larger cash buffer to absorb the additional variability.

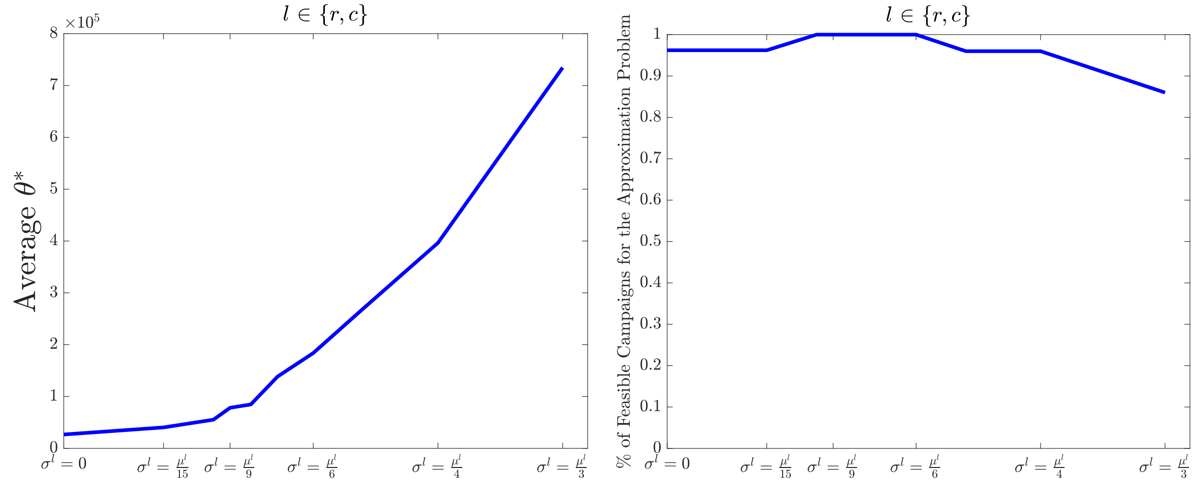

Figure 2.3: Left: The average of θ^* over feasible campaigns. Right: Percentage of feasible campaigns for Problem (2.6) which are feasible for Problem (2.14).

The right panel of Figure 2.3 shows the percentage of feasible campaigns for Problem (2.6) that are also feasible for Problem (2.11), for θ^*, and can therefore be approximated via the approximation Problem (2.14). As the level of cash-flow uncertainty decreases, the approximation Problem (2.14) provides an approximate solution for Problem (2.6) for almost all campaigns.

For evaluating the quality of the approximation, we input the approximation solutions $(Y(\theta^*), M(\theta^*), \gamma(\theta^*))$ into Model (2.6) and compare the firm's expected NPV under this solution with the true optimal expected NPV z_F, calculated numerically. In the left panel of Figure 2.4, we present the average error, over all feasible Bolstr campaigns, as a function of the standard deviations σ^r and σ^c. The length of each bar above and below the average value is equal to the standard deviation of the error over feasible campaigns. We observe that for the highest level of variability, $\sigma^r = \mu^r/3$ and $\sigma^c = \mu^c/3$, the average error is 0.2% with a standard deviation of 0.4%. As variability decreases, the mean and standard deviation of the errors decrease such that for $k \to \infty$ the mean and standard deviation of the errors are 0.003% and 0.01%, respectively. For $k = 6$ ($\sigma^r = \mu^r/6$ and $\sigma^c = \mu^c/6$) the average error is within 0.03%. Consequently, we expect that the approximation quality is acceptable for reasonable levels of variability encountered in practice, and we conclude that the approximation problem provides tractable solutions for the intractable stochastic problem.

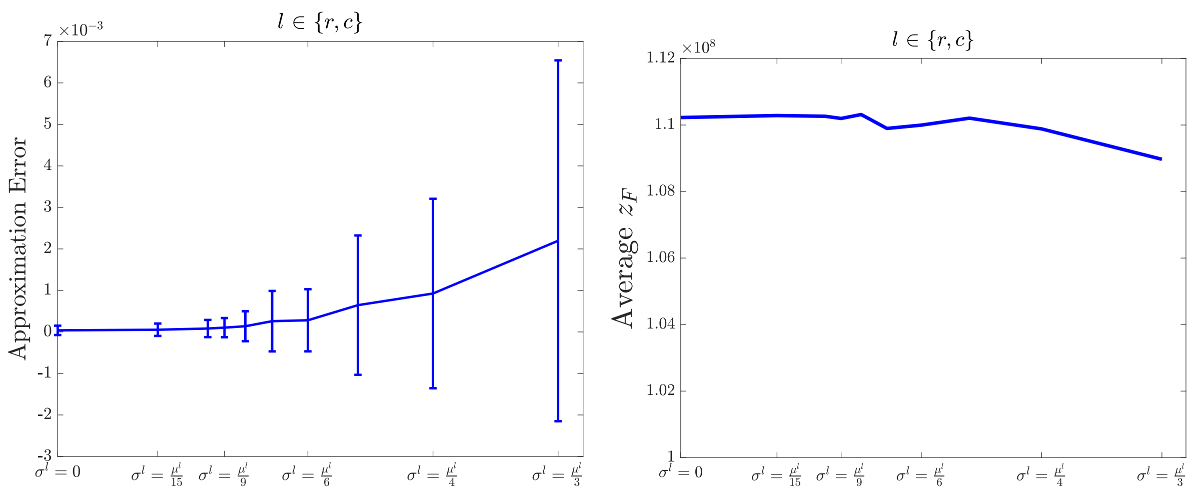

Figure 2.4: Left: Evaluation of the approximation quality of Problem (2.14) for stochastic Problem (2.6). Right: z_F over feasible campaigns.

Finally, the average values of z_F, as a function of cash-flow variability, are shown in the right panel of Figure 2.4. As variability decreases, z_F is increasing rather significantly for higher values of cash-flow variability and very slowly for smaller values of variability. In other words, firms can increase their maximized expected NPV significantly by slightly decreasing their cash-flow uncertainty in very uncertain environments. However, when cash-flow uncertainty is not significant, the maximized expected NPV is slowly decreasing in the standard deviations σ^r and σ^c, and is rather robust to their precise values.

2.4.3 Estimations of the T and B Distributions

In this subsection, we analyze the distributions of B, the firm's stochastic bankruptcy time, and T, the stochastic duration of the contract, for feasible campaigns under both the stochastic and approximation solutions. We present results for two qualitatively different campaigns that are feasible for all $k \geq 3$. One campaign, Campaign 1, has a relatively high bankruptcy probability for $k = 3$ (as representative of a risky firm) and the other campaign, Campaign 2, has a zero bankruptcy probability for $k = 3$ (as representative of a risk-less firm). The estimated probabilities $P(B < \infty)$ are given in Table 2.1, for the optimal and approximate solutions, respectively, as a function of k. We also evaluated the estimated probabilities $P(B < T)$, and they are identical to those in Table 2.1. We see that, as cash-flow variability decreases, the firm's bankruptcy probability decreases for Campaign 1 (and Campaign 2's probability remains at zero) for both optimal and approximate variables; furthermore, for a given k, the probabilities are mostly identical across the two sets of variables, providing further evidence of the quality of the approximation.

k ($\sigma^r = \mu^r/k$ & $\sigma^c = \mu^c/k$)	$P(B < \infty)$ for Campaign 1 (Y^*, M^*, γ^*)	$P(B < \infty)$ for Campaign 2 (Y^*, M^*, γ^*)	$P(B < \infty)$ for Campaign 1 $(Y(\theta^*), M(\theta^*), \gamma(\theta^*))$	$P(B < \infty)$ for Campaign 2 $(Y(\theta^*), M(\theta^*), \gamma(\theta^*))$
3	0.061	0	0.059	0
4	0.008	0	0.005	0
5	0.001	0	0.002	0
6	0.001	0	0.001	0
∞	0	0	0	0

Table 2.1: Estimated Probabilities $P(B < \infty)$.

Figure 2.5 shows the average of $P(B < \infty)$ over feasible Bolstr campaigns under both stochastic and approximation solutions. The approximation solutions result in almost the same bankruptcy probabilities as the stochastic optimal solutions, for feasible Bolstr campaigns over different levels of cash-flow variability.

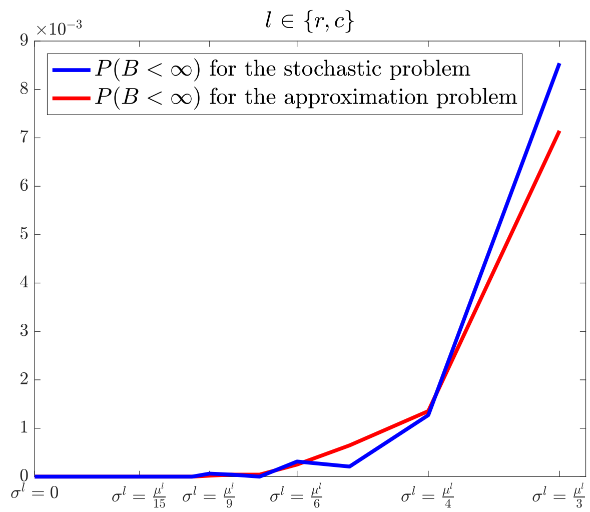

Figure 2.5: The average of $P(B < \infty)$ over feasible campaigns.

In the left plots of Figures 2.6 and 2.7, we provide the estimated distributions of T for $k \in \{3, 4, 5, \infty\}$ for Campaigns 1–2, respectively, for Problem (2.6) under the stochastic optimal solutions. Visually, we see that the mean and standard deviation of T for the stochastic optimal solutions are decreasing in k (we confirmed this numerically). In other words, as k increases, the cash-flow standard deviations σ^r and σ^c decrease, resulting in a problem with less uncertainty, which requires less investment Y^*, and can be paid back earlier with more certainty about the investment duration. Furthermore, when $k \to \infty$, resulting in $\sigma^r = \sigma^c = 0$, our stochastic model returns a deterministic investment duration.

In the right plots of Figures 2.6 and 2.7, we see the estimated distributions of T for $k \in \{3, 4, 5, \infty\}$ for Campaigns 1–2, respectively, for Problem (2.6) under the approximation solutions. Similar to the left plots, the standard deviation of T is decreasing in k. However, the mean of the investment horizon is approximately the same for different levels of variability, and close to $T = 120$, which we set earlier for the approximation problem. Therefore if we use the approximation solution for setting the contract terms (Y, M, γ), for T large enough, then the expected investment horizon is close to T for all firms with different levels of variability.

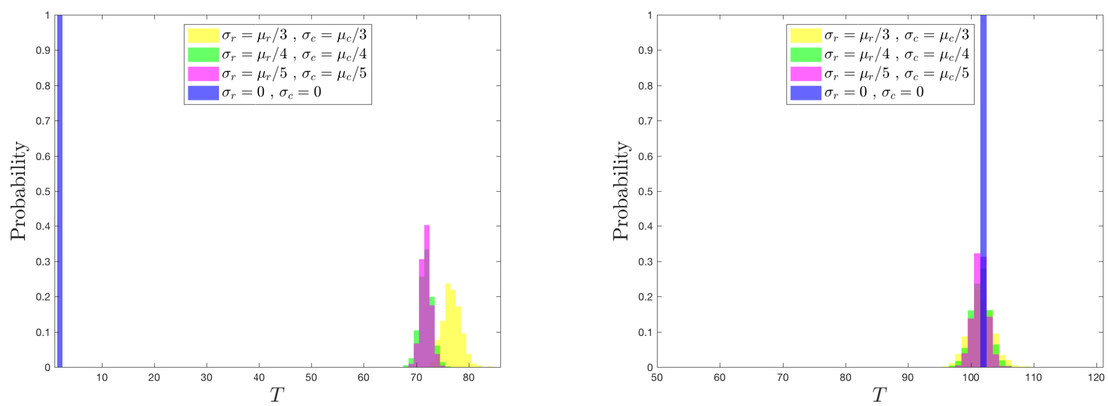

Figure 2.6: Estimated distributions of T in the stochastic problem for (Y^*, M^*, γ^*) (left) and for $(Y(\theta^*), M(\theta^*), \gamma(\theta^*))$ (right), for Campaign 1.

Finally, since we observed that the firm can complete the repayment of the revenue-sharing contract within our horizon of $N = 1000$ months, this value of N is unlikely a limiting factor of our numerical analysis. We also expect that it is unlikely that the firm will go bankrupt at some time $t > N$, under the less restrictive financial conditions due to the absence of repayment requirements, and we believe our estimates of the probability of firm bankruptcy are not limited by the value of $N = 1000$.

2.4.4 Sensitivity Analysis for Investors' Opportunity Cost

In this subsection, we perform sensitivity analysis for Problem (2.6) with respect to investors' opportunity cost $\hat{A}$. Figure 2.8 shows the average of z_F over the feasible Bolstr campaigns, for

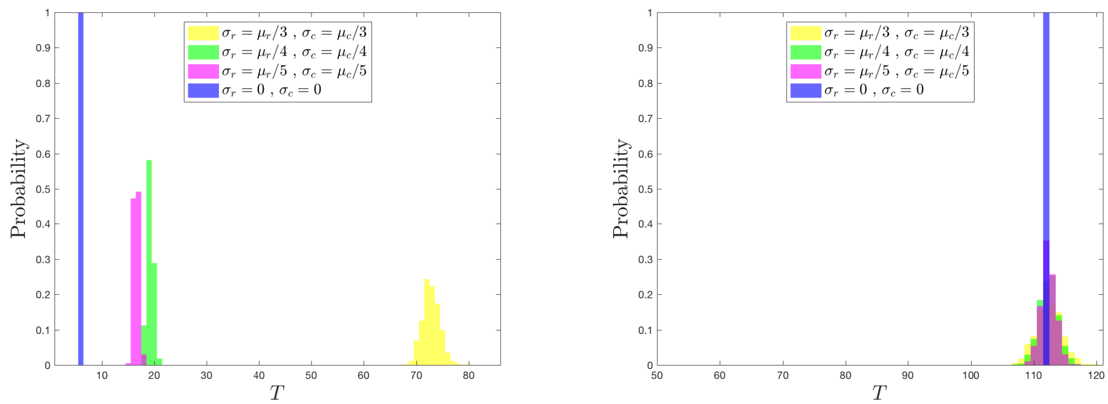

Figure 2.7: Estimated distributions of T in the stochastic problem for (Y^*, M^*, γ^*) (left) and for $(Y(\theta^*), M(\theta^*), \gamma(\theta^*))$ (right), for Campaign 2.

different values of $\hat{A}$, as a function of k. In this subsection we are analyzing the firm's problem for the cases where investors' opportunity cost is high, and specifically we are analyzing the optimal solutions under a revenue-sharing contract with $\hat{A} \in \{0.1, 0.2, 0.3, 0.4\}$.

As shown in Figure 2.8, as investors' opportunity cost increases, the firm's maximized NPV decreases. However, the firm's maximized NPV is rather robust with respect to $\hat{A}$ under a flexible revenue-sharing contract. Therefore, firms can benefit from flexible revenue-sharing contracts even with investors with moderately high opportunity costs, and the benefit increases as cash-flow variability decreases.

2.5 Equity Crowdfunding

In this section, we analyze an equity crowdfunding investment, and compare it with our revenue-sharing contract. According to pages 33-34 of [36], investors purchase equity from the firm and receive returns in the form of profit-sharing. Similarly, [182] explains that the present value of equity is equal to the discounted present value of all future dividends, which are typically shares of profit [169].

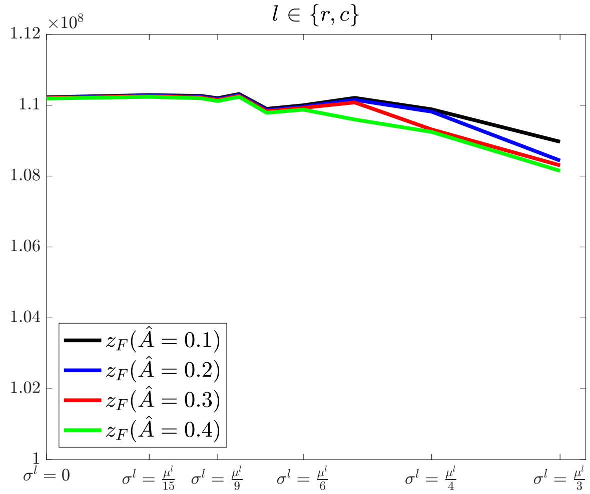

Figure 2.8: The average of z_F over feasible campaigns.

As in the analyses of previous sections, we assume that the firm raises Y from n investors such that $Y = \sum_{i=1}^{n} y_i$. Investors, in exchange for their equity investments, collectively receive η percent of the firm's monthly profit indefinitely, divided among the investors proportionally to their individual investment amounts.

As in our previous analyses, the investors' discount rate in period t is δ_t, and investor i's expected NPV is given by $E\left[\sum_{t=1}^{B} \frac{y_i}{Y} \frac{\eta \max\{R_t - C_t, 0\}}{(1+\delta_t)^t}\right]$, where $\max\{R_t - C_t, 0\}$ captures profit sharing. Thus, the investors' participation constraints are $E\left[\sum_{t=1}^{B} \frac{y_i}{Y} \frac{\eta \max\{R_t - C_t, 0\}}{(1+\delta_t)^t}\right] - y_i \geq A_i,\ i = 1, \ldots, n \quad \Leftrightarrow \quad E\left[\sum_{t=1}^{B} \frac{\eta}{Y} \frac{\max\{R_t - C_t, 0\}}{(1+\delta_t)^t}\right] \geq \hat{A} + 1$. As in our previous analyses, we assume that the platform charges an origination fee α and a servicing fee β. The firm's discount rate in period t is again r_t. The expected NPV of the firm after the crowdfunding investment (from the firm's perspective) is $V \triangleq E\left[\sum_{t=1}^{B} \frac{R_t - C_t}{(1+r_t)^t}\right] + (1-\alpha)Y - (\beta + 1)\eta E\left[\sum_{t=1}^{B} \frac{\max\{R_t - C_t, 0\}}{(1+r_t)^t}\right]$. As a result of their investment, investors collectively own a $\frac{Y}{V}$ proportion of the company's value, and the remaining $\left(1 - \frac{Y}{V}\right)$ proportion belongs to the firm. The expected NPV of the firm's remaining ownership is given by $\left(1 - \frac{Y}{V}\right)V$, which results in the firm's maximization problem being defined as

$$\begin{aligned} z_E = \quad \max_{Y,\eta} \quad & E\left[\sum_{t=1}^{B} \frac{R_t - C_t}{(1+r_t)^t}\right] - \alpha Y - (\beta+1)\eta E\left[\sum_{t=1}^{B} \frac{\max\{R_t - C_t, 0\}}{(1+r_t)^t}\right] \\ \text{s.t.} \quad & B = \min\left\{\hat{B} \geq 1 : \sum_{t=1}^{\hat{B}}(R_t - C_t) + (1-\alpha)Y - (\beta+1)\eta\sum_{t=1}^{\hat{B}}\max\{R_t - C_t, 0\} < 0\right\} \\ & E\left[\sum_{t=1}^{B} \frac{\eta}{Y}\frac{\max\{R_t - C_t, 0\}}{(1+\delta_t)^t}\right] \geq \hat{A} + 1 \\ & Y, \eta \geq 0; \end{aligned} \tag{2.17}$$

for simplicity, we assume that investors never sell their equity and only benefit from the dividends of profit sharing.

Figure 2.9 presents the results of numerical comparisons between revenue-sharing and equity crowdfunding contracts, where investors' opportunity cost $\hat{A} = 0.1$ and $r_t = \delta_t = 0.01, \forall t$. The left plot of Figure 2.9 presents the ratios z_E/z_F, where z_F (z_E) is the maximized NPV of revenue-sharing (equity) crowdfunding, averaged over feasible Bolstr campaigns, as a function of the volatility of cash flows. Since these ratios are less than one, they can be interpreted as the percentages of the revenue-sharing NPV that are attained by the equity crowdfunding contract. We conclude that, if feasible, a firm can attain a higher NPV under a revenue-sharing contract than an equity crowdfunding contract. However, we note that 4 out of 56 campaigns were feasible for high levels of uncertainty under equity crowdfunding, but not under revenue-sharing crowdfunding. The right plot of Figure 2.9 presents the firm's bankruptcy probabilities for both revenue-sharing and equity crowdfunding, again averaged over feasible Bolstr campaigns, as a function of the volatility of cash flows. We observe that the firm's bankruptcy probabilities under revenue-sharing contracts are within 0.1% of its bankruptcy probabilities under equity crowdfunding contracts. Furthermore, for low and medium cash-flow volatility levels, revenue-sharing contracts have lower bankruptcy probabilities. In summary, similar bankruptcy probabilities and higher NPVs highlight the superiority of our proposed revenue-sharing contract over equity crowdfunding contracts.

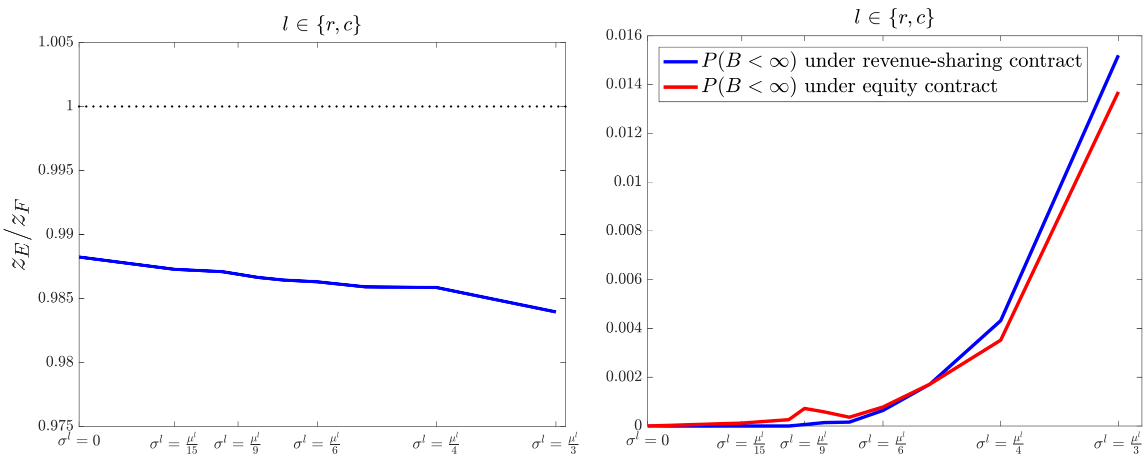

Figure 2.9: Comparisons of revenue-sharing and equity contracts.

2.6 Fixed-Rate Loans

In this section we analyze the performance of a standard fixed-rate loan, offered by other traditional and alternative lenders, to serve as a benchmark for the revenue-sharing contract. We assume these lenders offer fixed-rate loans with flexible payment terms: The firm can decide on the investment amount Y (e.g. dollars) and the duration D (e.g. months) for a loan with a fixed interest rate of s per period (e.g. month). For simplicity in our analysis, we consider s to be constant with respect to D, and we show that, even for this conservative interest rate structure, revenue-sharing contracts result in higher NPVs for firms.

The loan payment per period is a standard amortization and is equal to $\dfrac{sY}{1-(1+s)^{-D}}$ [196]. We assume the lender charges the borrower an origination fee of w percent of Y. The firm's maximization problem can be written as:

$$
\begin{aligned}
z_L = \quad & \max_{Y,D} \quad E\left[\sum_{t=1}^{B} \frac{R_t - C_t}{(1+r_t)^t}\right] - E\left[\sum_{t=1}^{\min\{B,D\}} \frac{sY}{\left(1-(1+s)^{-D}\right)(1+r_t)^t}\right] + (1-w)Y \\
& \text{s.t.} \quad B = \min\left\{\hat{B} \geq 1 : \sum_{t=1}^{\hat{B}}(R_t - C_t) + (1-w)Y - \sum_{t=1}^{\min\{\hat{B},D\}} \frac{sY}{1-(1+s)^{-D}} < 0\right\} \\
& \qquad\quad Y, D \geq 0.
\end{aligned}
\tag{2.18}
$$

Next, we solve the above problem numerically for Y and D. For this purpose, we need a base parameter set for the monthly interest rate s. For a conservative comparison, we consider $w = 0$. Interest rates and eligibility requirements vary across lenders. Banks usually have more strict eligibility requirements but they offer relatively lower interest rates than some online lenders who have less stringent eligibility requirements. We consider monthly interest rates $s \in \{0.03/12, 0.07/12, 0.14/12, 0.21/12\}$ for the numerical analysis.

We performed the numerical analysis for $D \in [1, 120]$ months and $Y \in [0, 2000000]$. We selected the upper bound of D to be 120 to account for long term loans offered by the US Small Business Administration (SBA) and banks that give more flexibility to firms. The upper bound of Y is chosen so that it is consistent with the numerical analysis for Problem (2.6). Figure 2.10 presents the firm's maximized NPV under loans with monthly fixed-rates $s \in \{0.03/12, 0.07/12, 0.14/12, 0.21/12\}$ as a ratio to the firm's maximized NPV under revenue-sharing contracts, with investors' opportunity costs $\hat{A} \in \{0.1, 0.2\}$. The ratios in Figure 2.10 are interpreted as percentages of the revenue-sharing NPV that are attained by the fixed-rate loans. Therefore, a ratio less than one indicates that the firm's maximized NPV is higher under a revenue-sharing contract and a ratio greater than one indicates that the firm's maximized NPV is higher under a fixed-rate loan.

The firm's maximized NPV under a revenue-sharing contract with $\hat{A} \in \{0.1, 0.2\}$ is

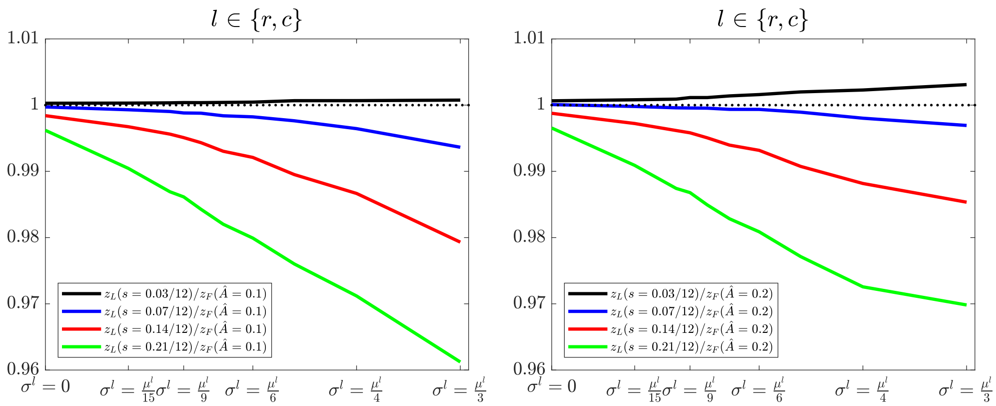

Figure 2.10: Comparisons of maximized NPV under revenue-sharing contracts and fixed-rate loans.

higher than loans with monthly fixed-rates as small as $s = 0.07/12$. The firm's benefit from a flexible revenue-sharing contract is more significant when cash-flow variability is high. For a revenue-sharing contract with $\hat{A} = \{0.1, 0.2\}$, the firm benefits from loans with a monthly fixed-rate $s = 0.03/12$, but not loans with higher rates, especially for higher levels of cash-flow uncertainty. However, most loans offered to firms, especially small to medium-sized firms, have relatively high interest rates. This underscores the importance of flexible payments that are possible under revenue-sharing contracts. Additionally, most low-rate loans, such as those offered by the SBA, have a processing time of 2–3 months. In contrast, according to marketing emails from Bolstr, firms can raise investments under revenue-sharing contracts within hours.

Figure 2.11 shows the average bankruptcy probabilities over feasible Bolstr campaigns under loans with monthly fixed-rates $s \in \{0.03/12, 0.07/12, 0.14/12, 0.21/12\}$ and under revenue-sharing contracts with investors' opportunity costs $\hat{A} \in \{0.1, 0.2\}$.

The bankruptcy probability for a firm under a revenue-sharing contract with $\hat{A} \leq 0.2$ is smaller than that of loans with monthly rates as small as $s = 0.14/12$. The highest

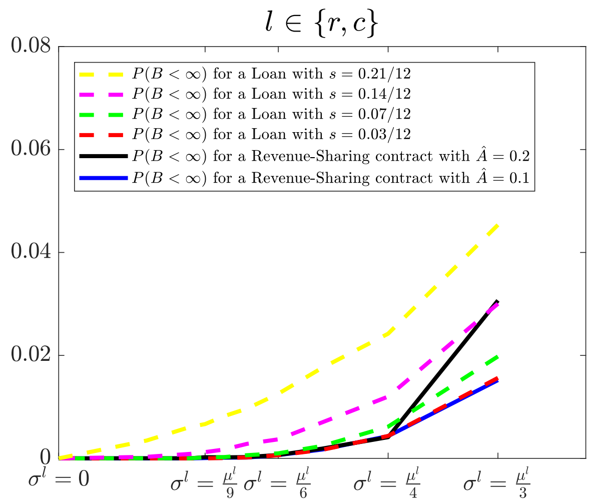

Figure 2.11: Comparison of average bankruptcy probabilities under revenue-sharing contracts and fixed-rate loans.

bankruptcy probability is for a loan with monthly rate $s = 0.21/12$ and this again underscores the importance of flexible revenue-sharing contracts, especially for firms with higher levels of cash-flow variability.

We conclude that firms benefit significantly more from revenue-sharing contracts than traditional fixed-rate loans, if investors' opportunity costs are not too large or if loans have non-negligible interest rates. These results are of great importance to small firms which usually have higher levels of cash-flow uncertainty and need investment; according to a report by the SBA [1], 73% of small firms used some type of financing in $2015 - 2016$.

2.7 Conclusion

In this chapter we analyzed an emergent model of crowdfunding in which a firm borrows capital and then pays back investors via revenue sharing contracts. Specifically, the firm pays the investors a percentage of its revenues monthly until a predetermined investment multiple is paid, over an uncertain investment horizon.

Research in this chapter is the first, to our knowledge, that studies this new model of crowdfunding. This model is facilitated by a platform (e.g., Bolstr, Localstake, or Startwise) that matches investors with a firm needing capital. This new model helps firms in need of investment to survive and thrive with a flexible contract whose terms depend on the firm's performance. Indeed, when these contracts are used optimally, we provide evidence that the likelihood of firm bankruptcy is small, even for highly variable cash flows, due to the flexible monthly payments facilitated by the contract. If the revenue performance of the firm goes well, then the monthly payments to investors increase, which results in higher effective interest rates for investors. If revenue performance is poor, payments are lowered to reduce financial stress on the firm. We use real data from 56 Bolstr campaigns to motivate and calibrate our analytical models, and to parameterize our numerical studies.

The first part of the research in this chapter formulates a stochastic programming model of the firm's expected NPV maximization problem with the contract parameters as variables, for given values of the platform's origination and servicing fees, and investors' opportunity costs; unfortunately, this model is difficult to analyze. Therefore, in the second part of this chapter, we formulate a deterministic approximation that we solve analytically. In the approximation problem, we use a cash buffer for dealing with cash-flow uncertainties. In the third part, we evaluate the quality of the approximation solutions for the main stochastic model, over feasible Bolstr campaigns, for different levels of cash-flow uncertainty. We see that the worst-case average error over the campaigns is approximately 0.2%. Therefore, we conclude that the approximation problem provides high quality solutions for the intractable stochastic problem.

In the final part of this chapter, we compare revenue-sharing contracts with equity crowdfunding and fixed-rate loans. We find that, for most cases considered, revenue-sharing contracts provide a higher NPV and a lower probability of bankruptcy than equity crowdfunding or a fixed-rate loan. We also show that these benefits are more significant for firms with higher levels of cash-flow uncertainty.

Chapter 3

CROWDSOURCING LAST-MILE DELIVERIES

3.1 Introduction

McKinsey & Company approximates that last-mile delivery costs can be more than half of the total delivery cost of online orders [179]. Large online retailers and start-ups have credited their success to efficient last-mile delivery services that they can offer to their customers [113]. According to [114], a 40% decrease in shipping cost is equivalent to a 15-20% increase in profit or, equivalently, a 15-20% decrease in price. Therefore, firms can compete with their rivals if they can offer cost efficient and fast last-mile deliveries [152].

Last-mile delivery should be scalable as it is dynamic and growing fast with, in 2015, 7-10% growth rate in the United States and Germany and over 100% in developing markets; e-commerce is the leading factor for this growth [113]. During the 2013 holiday season, due to limited resources, UPS and other third-party logistics (3PL) providers could not deliver Amazon packages on time, which resulted in millions of dollars of refunds for Amazon customers [28].

Crowdsourcing last-mile deliveries allows retailers to deal with demand uncertainty better as they can flexibly utilize independent crowdsourced drivers after their demand realization without long-term capital commitments [114]. Moreover, crowdsourcing provides retailers more control over deliveries, and it allows the retailer to offer delivery services when a 3PL provider is not available.

In this chapter, we study last-mile deliveries via independent crowd drivers that decide whether to work and when to work. Specifically, we study crowdsourced last-mile delivery systems for on-demand orders, with guaranteed delivery time windows, where the time between placing an order and receiving it ranges from 2 hours to same-day. According to [200], crowd drivers' independence and customer delay sensitivity are integral to crowdsourced on-demand service platforms. Firms should intelligently decide on the crowd compensation and labor planning to have enough crowd drivers willing to participate, such that they are not too busy, to ensure on-time deliveries, nor too idle, where their effective earning rate is too low for participation. It is challenging for retailers to make pricing and labor planning decisions since there is uncertainty in customer demands (*when* and *where* will a customer place an order), crowd availability (*whether* and *when* independent crowd drivers are available for delivery services), customer delivery service times (delivering to a house differs from delivering to a gated community), as well as road traffic conditions.

Research in this chapter is related to three streams of research: crowdsourcing last-mile deliveries, the sharing economy and on-demand platforms, and queuing and routing via robust optimization. In particular, we are not aware of any paper that combines aspects of all three streams, as we do in this chapter.

3.2 A Robust Crowdsourcing Model

Challenges in crowdsourced delivery platforms come from uncertainties in both the supply of crowd drivers and the demand of customers. In these systems, the availability of independent crowd drivers is not guaranteed, as each driver has their own individual uncertain opportunity cost, which results in an uncertain supply of crowd drivers that depends on the offered wage rate. On the other hand, customer orders must be delivered within guaranteed delivery time windows, under uncertainty in the timing of customer orders, on-location service times, and delivery locations, possibly with trend and seasonality. These uncertainties make it challenging to deliver customer orders on-time for a given service level guarantee. We study how a firm should design an efficient crowdsourced last-mile delivery system, to ensure participation of a sufficient number of crowd drivers, for on-time delivery of all customer orders, while still minimizing delivery costs.

In this chapter, we develop our main model utilizing robust optimization, a modern approach to decision making under uncertainty, where uncertainty is captured using deterministic set membership (whose structures are motivated by various limit theorems of probability), rather than stochastic distributions. There are advantages in using robust optimization in studying crowdsourcing last-mile deliveries with guaranteed delivery time windows. First, unlike stochastic optimization, robust models revolve around worst-case scenarios or, in other words, they "under-promise but over-deliver" [129]. This is important for our application since customers are sensitive to delay and, hence, firms need to make guaranteed on-time package deliveries; robust optimization allows a firm to promise on-time deliveries with high probability.

Furthermore, the robust approach allows us to capture trend and seasonality in customer arrivals, service times, and traffic patterns, as well as heterogeneity in crowd drivers, over multiple service subregions with varying areas and populations; capturing all these patterns is typically not tractable in a stochastic modeling approach. Even if a stochastic approach is tractable, the robust approach is appealing since there is ample evidence in the literature that misspecified distributions in stochastic programming can result in sub-optimal solutions, and robust optimization results in better outcomes; for example, [31] provides this evidence in an inventory management context, and [18] in a resource allocation context. Additionally, in practice, there are not always enough data available (e.g., the problem of "small data"); even if there are, using this historical data for choosing a distribution assumes that the past data are of high quality and are representative of future, which is not always true. As a result, developing models that do not make any distributional assumption is important [149].

3.2.1 Problem Definition

In crowdsourced delivery platforms, as the firm should make sure a sufficient number of crowd drivers are available at the distribution center/pick-up location for on-time delivery of customers' fast (1-hour, 2-hour, or same-day) delivery orders, the firm asks for availability of crowd drivers beforehand. In particular, the firm announces the delivery work that it needs, along with associated details, including the start date and time. Each crowd driver, who sees this announcement, can choose to sign up for it and provide the requested delivery services, or not, depending on what compensation the firm has offered compared to their opportunity costs. Hence, in our problem, the firm aims to make labor planning and crowd compensation decisions for a time horizon (e.g., half-a-day or a day) in advance of demand realization.

In this study, a firm has two options for last-mile delivery: 1) the firm can outsource all deliveries to a third-party logistics (3PL) firm, such as Deliv or FedEx, or 2) assign all packages to crowd drivers to deliver. The main monetary difference between using 3PL companies for delivery and utilizing crowd drivers is the payment structure: 3PL companies typically charge firms per package delivery, whereas the crowd is paid hourly wages for value-added time. In our model, we also allow for hybrid solutions, where both a 3PL firm and the crowd are utilized, to add a layer of flexibility to a firm's delivery system. Hybrid solutions are motivated by practice: Amazon, in addition to using crowdsourced delivery services under the Flex program out of its warehouse stations and vacant lots [172], utilizes the 3PL companies UPS and FedEx for making its fast same-day deliveries [120, 138]. Similarly, Walmart has started using a crowd delivery program (Spark), while still using 3PL firms (such as FedEx). Furthermore, some retailers, such as Macy's and Best Buy, utilize both 3PL carriers and crowdsourced delivery firms for satisfying their fast same-day and two-day deliveries [147, 148, 41]. Finally, [48] study a hybrid solution where the firm can use ride-sharing platforms such as Uber and Lyft along with in-house van delivery systems for making last-mile deliveries. However, this approach has not been successful for all firms: Walmart retracted from using ride-sharing platforms [180].

3PL firms typically charge a fee per package for delivery, which is a function of the distance between the origin and the destination, and the delivery time window; we let c_T denote this per package cost (the distance and delivery time window dependence is implicit), where the T subscript indicates a (delivery) truck. In contrast, in crowdsourced delivery applications drivers are typically paid an hourly wage w. As is the case in practice, we assume the driver is paid only for travel and servicing time (i.e., value-adding time), but not idle time.

Firms typically offer customers guarantees on delivery times. For instance, Amazon offers next-day, same-day, and even 2-hour guaranteed delivery windows. We let α denote the desired delivery duration requested by the customer, and we assume that all customers have the same target delivery duration. We also refer to α as the *service level* that all customers are guaranteed to receive.

As a firm needs to make crowd compensation and labor planning decisions before demand is realized, without using the package characteristics (e.g., location) of the realized demand, we develop a static optimization model that allows us to derive generic optimal policies for the crowdsourced last-mile delivery system. We adopt a simple randomized allocation policy, where a $P \in [0, 1]$ proportion of deliveries are randomly assigned to be delivered by the crowd, and the remaining $(1 - P)$ proportion is delivered using a 3PL firm. This randomized allocation policy allows us to derive generic analytical results and managerial insights under demand and supply uncertainties, without using the package characteristic of the realized demand. In Section 3.4, we propose a dynamic benchmark model that uses package characteristics (e.g., location) to determine which packages should be crowdsourced and which should be assigned to the 3PL firm, and compare its performance with the randomized allocation policy.

In the next section, we develop a robust optimization model of crowdsourcing last-mile deliveries for on-demand orders with guaranteed delivery time windows. We aim to derive optimal policies for crowdsourced last-mile deliveries under supply and demand uncertainties, and under the service-level guarantee that the firm offers its customers.

3.2.2 Models of Uncertainty

Customer Interarrival Times. Let the interarrival time of customer i be A_i, with mean and standard deviation of μ_i^a and σ_i^a, respectively. If customers arrive according to a Poisson process with rate λ, the random allocation policy splits this process into a Poisson process served by the crowd, of rate $P\lambda$, and another Poisson process of rate $(1-P)\lambda$ that is served by a 3PL firm; in this analysis, we assume that $P > 0$. This observation motivates us to redefine the mean and standard deviation of the interarrival time of customer i, *who is served by the crowd*, to be μ_i^a/P and σ_i^a/P, respectively, since the mean interarrival time for a Poisson process is $1/\lambda$. Similar to the approaches in [16] and [17], we define the interarrival times uncertainty set for these customers to be

$$U_A(P) = \left\{ (A_1, \cdots, A_n) \geq 0 : \left| \frac{\sum_{i=k+1}^{j} (A_i - \mu_i^a/P)}{\sqrt{\sum_{i=k+1}^{j} (\sigma_i^a/P)^2}} \right| \leq \gamma^a, 0 \leq k < j \leq n \right\}, \tag{3.1}$$

which is motivated by the central limit theorem (CLT): $\lim_{j\to\infty} P\left(\frac{\sum_{i=k+1}^{j}\left(A_i - \mu_i^a/P\right)}{\sqrt{\sum_{i=k+1}^{j}\left(\sigma_i^a/P\right)^2}} \leq x\right) = \Phi(x)$, where Φ is the standard normal distribution. The parameter γ^a can be chosen based on the probability with which the firm wants the interarrival time inequalities to be satisfied, such as $\gamma^a \in \{2, 3\}$, since, if Z is a standard normal random variable, $P(|Z| \leq 2) \approx 0.954$

and $P(|Z| \leq 3) \approx 0.997$. If customers' delay sensitivity is high, then the firm should choose a higher γ^a value.

Our approach to modeling interarrival times captures trend and seasonality in customer arrivals, via proper selection of the μ_i^a and σ_i^a parameters, and is different from [16] and [17], whose approaches do not consider non-stationary arrivals. Our modeling of uncertainty sets with non-stationarity is motivated by [149] and [18].

Driver Route Durations. We assume there is a single depot in the service area and each crowd driver, with capacity Q, serves a set of q customers at a time, between visits to the depot, where $q \in \{1, \ldots, Q\}$; these sets are formed and served using a First-Come First-Served (FCFS) policy. We assume each driver takes an optimal traveling salesman problem (TSP) tour through the q customer locations and not FCFS within a set. This assumption is motivated by observations from industry: the Amazon Flex application gives the driver directions (presumably) for the optimal tour (logistics.amazon.com); similarly, Walmart also offers the optimal delivery tour to its drivers [146]. The sets themselves are served in the FCFS order.

We denote the TSP tour length that visits q city locations $x_1, \cdots, x_q \in \mathbb{R}^2$ as $L_q(x_1, \cdots, x_q)$; for simplicity we write $L_q = L_q(x_1, \cdots, x_q)$. We assume that the number of crowdsourced customers n is a multiple of q: $n = mq$, for some positive integer $m \geq 1$; if we consider n to be large enough, this approximation will not change our results materially. Therefore, the drivers collectively serve m sets of size q. This routing policy is inspired by a similar one in [32].

We assume customers are located in M ZIP codes, each with area $\mathbb{A}_i$ and population p_i, $i = 1, \ldots, M$, where the probability density function that each customer is in a given ZIP code is proportional to the ZIP code's population, and, given a ZIP code, the customer's location is conditionally uniformly distributed over the ZIP code's area. The probability density function of a customer's location $x \in \mathbb{R}^2$, which is potentially located in one of M ZIP codes, each with area $\mathbb{A}_i$ and population $p_i,\ i = 1, \dots, M$, is a generalization of a result in [50] and is given by

$$f(x) = \begin{cases} \dfrac{p_i}{\sum_{j=1}^{M} \mathbb{A}_j p_j}, & \text{if } x \text{ is in ZIP code } i,\ \ i = 1, \dots, M \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

There exists a probabilistic limit theorem for the optimal TSP tour length, which is effectively a law of large numbers: originally derived by [21] and further developed by [193], there exists a constant $\tilde{\beta}$ such that $\lim_{q\to\infty} \frac{L_q}{\sqrt{q}} = \tilde{\beta} \iint\limits_{\mathbb{R}^2} f(x)^{1/2} dx$, almost surely. From this asymptotic result and the probability density function $f(x)$, we can define

$$\lim_{q\to\infty} \frac{L_q}{\sqrt{q}} = \tilde{\beta} \sum_{i=1}^{M} \iint\limits_{x \text{ in ZIP code } i} \sqrt{\frac{p_i}{\sum_{j=1}^{M} \mathbb{A}_j p_j}} dx = \tilde{\beta} \sum_{i=1}^{M} \sqrt{\frac{p_i}{\sum_{j=1}^{M} \mathbb{A}_j p_j}} \mathbb{A}_i. \tag{3.3}$$

The above analysis has not accounted for the depot, where the driver picks up packages. For simplicity, we assume that the depot has the same distribution as the customers' locations. Therefore, a driver's tour starts from the depot, visits q customers, and returns to the depot, which has length L_{q+1}. Note that the driver might have to return to the depot for returning undeliverable packages and/or customer-returned items, which motivates us to use a tour, rather than a path. Let L^j_{q+1} denote the optimal length of the jth tour through the depot, which serves the q customers $(j-1)q+1, (j-1)q+2, \dots, jq$, and let v_j denote the average travel speed for the jth tour. Thus, L^j_{q+1}/v_j represents the optimal tour *duration* of the jth tour. Motivated by the [21] limit theorem, we define

$$U^\ell_L = \left\{ \frac{L^\ell_{q+1}}{v_\ell} : \beta_L \sum_{i=1}^{M} \sqrt{\frac{p_i}{\sum_{j=1}^{M} \mathbb{A}_j p_j}} \mathbb{A}_i \le \frac{L^\ell_{q+1}}{\sqrt{q+1}} \le \beta_U \sum_{i=1}^{M} \sqrt{\frac{p_i}{\sum_{j=1}^{M} \mathbb{A}_j p_j}} \mathbb{A}_i \right\} \tag{3.4}$$

as the uncertainty set for the ℓth tour duration. The β_L and β_U parameters are lower and upper bounds on $\tilde{\beta}$, which reflect the fact that the limit theorem does not hold exactly for a finite q. These parameters also control the level of conservatism in our model; as customers' delay sensitivity increases, the firm should choose a higher β_U value and a lower β_L value. A similar uncertainty set, motivated by the strong law of large numbers, was utilized in [206]. Note that, while the motivating TSP asymptotic results are based on the Euclidean distance metric, the robust uncertainty set does not require any specific distance metric; indeed, in our numerical experiments we utilize road distances measured on a real transportation network in the Seattle area. Similarly, [50], [42], and [178] utilize this limit theorem when in reality the distances are calculated using the Manhattan metric. In the sequel we show how we are modeling traffic patterns via the sets U_L^ℓ by appropriately setting the speeds v_ℓ.

Customer On-Site Service Times. The arrival of a customer entails a request for a package delivery. Once the server travels to the customer's location, there is an on-site service time (e.g., parking, finding the appropriate apartment, etc.). Let X_i denote the on-site service time of the ith customer with mean and standard deviation μ_i^s and σ_i^s, respectively. We adopt and modify the service time uncertainty set, for multiple servers in a non-routing queuing context, from [16]. More specifically, we adapt their uncertainty set to be defined in terms of the service times of *sets* of customers, rather than individual customer service times. Furthermore, we define the uncertainty set for non-stationary service times. We again assume there are $n = mq$ crowdsourced customers, resulting in m tours of size q that are served by N available crowd drivers; in the sequel, we discuss how N is determined endogenously by self-selecting drivers. Let $\tau^m \triangleq \frac{m}{N}$, assumed to be integer, denote the average number of tours taken by each driver. We next assign the m tours of customers, in order of their arrivals, into the following partitions: $J_1 = \{1, \dots, N\}$, $J_2 = \{N+1, \dots, 2N\}, \dots,$ $J_{\tau^m} = \{(\tau^m - 1)N + 1, \dots, m\}$. The uncertainty set U_X^N for service times in a multiple-driver scenario is defined as

$$U_X^N = \left\{ (X_1, \cdots, X_{mq}) \geq 0 : \left| \frac{\sum_{i \in I} \sum_{k=(j_i-1)q+1}^{j_i q} (X_k - \mu_k^s)}{\sqrt{\sum_{i \in I} \sum_{k=(j_i-1)q+1}^{j_i q} (\sigma_k^s)^2}} \right| \leq \gamma^s, \forall j_i \in J_i,\ \forall I \subseteq \{1, \ldots, \tau^m\} \right\}, \tag{3.5}$$

where γ^s plays the same role as γ^a and j_i is the index that refers to the members in set J_i. Simply stated, the uncertainty set U_X^N considers all different inter-set combinations of service times under all possible partitions for serving m sets by N available crowd drivers; we refer to [16] for a more in-depth discussion of this uncertainty set. Similar to the interarrival times uncertainty set, we can model non-stationarity in service times in the set U_X^N via the μ_i^s and σ_i^s parameters.

3.2.3 Our Crowdsourcing Model

Service Time Guarantees. We let S_i denote the *system time* of customer i in the crowd-sourced delivery system, defined as the time that passes between a customer placing an order until receiving the package, which entails wait time, transportation time and on-site time. The service levels of all customers are satisfied if $S_i \leq \alpha,\ i = 1, \cdots, mq$. If time e is needed for processing and preparation of delivery orders, we should replace α with $\alpha - e$; however, for simplicity in the analysis we consider $e = 0$ throughout. Regarding 3PL delivery, we assume that all customers are guaranteed to receive their packages on time; to realistically capture this assumption, we let c_T be decreasing in α, as supported by the FedEx delivery fees. Therefore, we only analyze the system times of customers in the crowdsourced delivery system.

Crowdsourced System Stability Condition. Implicit in our discussion above is a queuing system. Customers must wait for their sets (of size q) to form, and customer sets must wait for one of the N available crowd drivers to be free to serve them; in particular, a queue of sets will form while all available drivers are busy serving other sets. Therefore, we require a stability condition for this set queuing system. The set service time for this queuing system consists of tour travel time and on-site time. The expected on-site time for a set of q customers is $\frac{\sum_{i=1}^{mq}\mu_i^s}{m}$. From uncertainty set U_L^ℓ, defined in Equation (3.4), an upper bound on the average tour time for a set of q customers is $\frac{\sum_{i=1}^{m}\beta_U\left(\sum_{z=1}^{M}\sqrt{\frac{p_z}{\sum_{j=1}^{M}\mathbb{A}_j p_j}}\mathbb{A}_z\right)\sqrt{q+1}/v_i}{m}$; in the following analysis, for expository clarity, we replace $\beta_U\left(\sum_{z=1}^{M}\sqrt{\frac{p_z}{\sum_{j=1}^{M}\mathbb{A}_j p_j}}\mathbb{A}_z\right)$ with β. Therefore, $\frac{\sum_{i=1}^{mq}\mu_i^s}{m}+\frac{\sum_{i=1}^{m}\beta\sqrt{q+1}/v_i}{m}$ is an upper bound on the expected service time of each set. The expected interarrival time between sets of q customers, who are served via crowdsourced delivery, is $\frac{\sum_{i=1}^{mq}\mu_i^a/P}{m}$. As a result, $\rho \triangleq P\left(\frac{\sum_{i=1}^{mq}\mu_i^s}{m}+\frac{\sum_{i=1}^{m}\beta\sqrt{q+1}/v_i}{m}\right)/\left(N\frac{\sum_{i=1}^{mq}\mu_i^a}{m}\right)$ is an upper bound on the utilization of the queuing system serving sets of q customers. For queue stability, we require ρ to be strictly less than 1: $\rho<1$. For simplicity in the analysis, we write $\rho<1$ as $\rho\leq 1-\epsilon$, where ϵ is smaller than any other number.

Crowd Participation Constraint. We assume that there is a sufficiently large and heterogeneous supply of *potential* crowd drivers for the firm to tap into for its delivery work, with the population given by $\mathbb{N}$. The most important factor that motivates the participation of these crowd drivers is financial remuneration [8]. The manner in which crowd delivery platforms typically function is that the firm announces the delivery work that it needs, along with associated details, including the start date and time, and then each crowd driver who sees this announcement can choose to sign up for it and provide the requested delivery services, or not, depending on the compensation that the firm offers compared to their opportunity costs. Let K_i be crowd driver i's uncertain opportunity cost, with mean μ_i^k and standard deviation σ_i^k. Drivers have uncertainty in their opportunity costs, driven by different factors, such as the impact of weather or the different days of the week. Similar to the uncertainty set U_X^N, we define a CLT-motivated uncertainty set for the crowd's opportunity costs as

$$U_K = \left\{ (K_1, \cdots, K_{\mathbb{N}}) \geq 0 : \left| \frac{\sum_{i \in I} K_i - \mu_i^k}{\sqrt{\sum_{i \in I} \left(\sigma_i^k\right)^2}} \right| \leq \gamma^k, \forall I \subseteq \{1, \ldots, \mathbb{N}\} \right\}, \tag{3.6}$$

where γ^k controls the level of conservatism in crowd participation. If the μ_i^k and σ_i^k parameters are not known, the firm can use a single mean $\mu_i^k = \mu^k$ and standard deviation $\sigma_i^k = \sigma^k$, for all i.

During the time horizon that we are solving the problem for, let N denote the number of crowd drivers that the firm needs in order to deliver all packages to its customers on-time. The firm can make sure to have at least N willing-to-participate crowd drivers by offering them a sufficiently high compensation (but not higher than necessary). Each of these N crowd driver's expected hourly income under the utilization of ρ and hourly wage w is ρw or, equivalently, $wP\left(\frac{\sum_{i=1}^{mq} \mu_i^s}{m} + \frac{\sum_{i=1}^{m} \beta\sqrt{q+1}/v_i}{m}\right) / \left(N\frac{\sum_{i=1}^{mq} \mu_i^a}{m}\right)$. As crowd drivers are compensated only for their busy times, and not their idle times, the firm should consider the hourly income ρw of crowd drivers and not just their hourly wage w. If the utilization ρ of crowd drivers is small, then the drivers are idle most of the time which results in a small crowd hourly income and the crowd is not as willing to participate in on-demand delivery services; as a result the firm should offer a higher hourly wage to motivate the crowd to participate. However, if the utilization ρ of crowd drivers is high, then customers might not get their deliveries on-time, due to extended wait times. In the sequel, we analytically solve for the optimal hourly wage the firm should offer to the crowd to minimize its delivery cost while satisfying its customers' orders on-time.

Motivated by [200], if the expected hourly income of the crowd is at least its opportunity cost, then the crowd participates. We next analyze the opportunity costs of the crowd supply via the set U_K. Similar to [200], we assume crowd drivers with higher opportunity costs participate if and only if crowd drivers with lower opportunity costs participate. Thus, if N crowd drivers participate, these are the N cheapest drivers from the supply of available crowd drivers in set U_K. To guarantee the availability of N crowd drivers, we analyze the worst-case opportunity costs of the cheapest N crowd drivers, for a given level of conservatism γ^k. This can be mathematically written as $\min_{\substack{S\subseteq\{1,\ldots,\mathbb{N}\},\\ |S|=N}} \max_{(K_1,\ldots,K_{\mathbb{N}})\in U_K} \dfrac{\sum_{i\in S} K_i}{N}$. In the next lemma, we solve for these optimization problems. For simplicity in the analysis, we assume that μ_i^k depends on i but $\sigma_i^k = \sigma^k,\ \forall i$ does not (as in, for instance, linear discriminant analysis).

Lemma 3.1.

$$\min_{\substack{S\subseteq\{1,\ldots,\mathbb{N}\},\\ |S|=N}} \max_{(K_1,\ldots,K_{\mathbb{N}})\in U_K} \frac{\sum_{i\in S} K_i}{N} = \frac{\sum_{i=1}^{N} \mu_{(i)}^k + \gamma^k\sqrt{N}\sigma^k}{N},$$

where $\mu_{(1)}^k \le \mu_{(2)}^k \le \cdots \le \mu_{(\mathbb{N})}^k$ *are the order statistics of the mean parameters.*

The term $\frac{\sum_{i=1}^{N} \mu_{(i)}^k + \gamma^k\sqrt{N}\sigma^k}{N}$ is the worst-case average opportunity cost of the cheapest crowd drivers of size N, with the level of conservatism set at γ^k. Motivated by [200], the number of participating crowd drivers satisfies

$$wP\left(\frac{\sum_{i=1}^{mq}\mu_i^s}{m} + \frac{\sum_{i=1}^{m}\beta\sqrt{q+1}/v_i}{m}\right)\Big/\left(N\frac{\sum_{i=1}^{mq}\mu_i^a}{m}\right) \ge \frac{\sum_{i=1}^{N} \mu_{(i)}^k + \gamma^k\sqrt{N}\sigma^k}{N}.$$

This constraint guarantees that the participation constraint of the crowd drivers is satisfied with $P(|Z| \le \gamma^k)$ probability, where Z is a standard normal random variable (due to the CLT). The firm can influence the availability of crowd drivers through a) the hourly wage and b) the amount of on-demand delivery work it offers to the crowd. In Proposition 3.1, we show the above inequality is tight at optimality and we obtain the crowd supply function $w = \left(\sum_{i=1}^{N}\mu_{(i)}^k + \gamma^k\sqrt{N}\sigma^k\right) P\left(\frac{\sum_{i=1}^{mq}\mu_i^s}{m} + \frac{\sum_{i=1}^{m}\beta\sqrt{q+1}/v_i}{m}\right)\Big/\left(\frac{\sum_{i=1}^{mq}\mu_i^a}{m}\right)$ that *endogenously* determines the number of available crowd drivers. In this crowd supply function, the availability of the crowd, namely N, is an increasing function of the hourly wage offered by the firm, an increasing function of the amount of on-demand delivery work, and a decreasing function of the crowd opportunity costs. According to this supply function, as variability in the opportunity cost of the crowd increases, the firm should offer a higher hourly wage w to guarantee the availability of crowd drivers.

Our proposed model for crowd participation is different than the firm-employee model, where crowd drivers participate within the γ^k level of conservatism, whereas in the firm-employee model, all the employees participate with 100% certainty. Furthermore, unlike the the firm-employee model, our modeling of crowd participation captures variability in crowd opportunity costs. Additionally, in the firm-employee model, the number of employees is fixed and is not a function of the amount of available on-demand delivery work and the offered hourly wage, as is the case in crowdsourced delivery systems.

We next analyze the firm's objective function. The drivers' expected value-adding time for m sets of size q consists of traversing m tours, which takes $\sum_{j=1}^{m} L_{q+1}^{j}/v_j$ time, plus the on-site service time for mq customers, namely $\sum_{i=1}^{mq} X_i$. The sum of these expressions, divided by mq, is the value-adding time per package (e.g., hour/package). If we multiply this value-adding time per package by w, the driver's wage rate per time unit (e.g., \$/hour), we obtain the crowdsourced delivery cost rate per package (e.g., \$/package) as $w\frac{\sum_{j=1}^{m} L_{q+1}^{j}/v_j+\sum_{i=1}^{mq} X_i}{mq}$. As detailed previously, a P proportion of customers are served via crowdsourced delivery and the remaining $(1-P)$ proportion are served via a 3PL firm at cost c_T, which we consider to be the average of the 3PL delivery costs for all customers with different distances to the depot. Thus, the expected delivery cost rate per package is $Pw\frac{\sum_{j=1}^{m} L_{q+1}^{j}/v_j+\sum_{i=1}^{mq} X_i}{mq} + (1-P)c_T$; this expression will serve as our objective, to be minimized by appropriately selecting P and w; we subsequently optimize over q numerically as well. To obtain a conservative estimate of the impact of crowdsourcing last-mile deliveries, we solve the following robust optimization model:

$$
\begin{aligned}
\min_{P\in(0,1],w>0} \quad & \max_{\substack{(X_1,\dots,X_{mq})\in U_X^N \\ L_{q+1}^j/v_j\in U_L^j, j=1,\cdots,m}} \quad Pw\left(\frac{\sum_{j=1}^m L_{q+1}^j/v_j+\sum_{i=1}^{mq}X_i}{mq}\right)+(1-P)c_T \\
\text{s.t.} \quad & \max_{\substack{(A_1,\dots,A_{mq})\in U_A(P) \\ (X_1,\dots,X_{mq})\in U_X^N \\ L_{q+1}^j/v_j\in U_L^j, j=1,\dots,m}} \quad S_i\le\alpha,\ i=1,\dots,mq \\
& P\left(\frac{\sum_{i=1}^{mq}\mu_i^s}{m}+\frac{\sum_{i=1}^m\beta\sqrt{q+1}/v_i}{m}\right)\Big/\left(N\frac{\sum_{i=1}^{mq}\mu_i^a}{m}\right)\le 1-\epsilon \\
& wP\left(\frac{\sum_{i=1}^{mq}\mu_i^s}{m}+\frac{\sum_{i=1}^m\beta\sqrt{q+1}/v_i}{m}\right)\Big/\left(N\frac{\sum_{i=1}^{mq}\mu_i^a}{m}\right)\ge \min_{\substack{S\subseteq\{1,\dots,\mathbb{N}\}, \\ |S|=N}} \max_{(K_1,\dots,K_{\mathbb{N}})\in U_K} \frac{\sum_{i\in S}K_i}{N}.
\end{aligned}
\tag{3.7}
$$

The objective function is the maximum expected delivery cost rate per package, with respect to uncertainties that exist in the crowd delivery system. The left-hand side of the first constraint is the worst-case system time for customer i, with respect to uncertainties in the crowd delivery system; this constraint guarantees that the system time is at most the service level α; the index i refers to any customer who is served via crowdsourced delivery. The second constraint ensures the underlying queue is stable. The third constraint is the crowd participation constraint, that endogenously determines the number N of participating independent crowd drivers. In the proposed robust queueing system, the number of service providers is not exogenous and is endogenously determined by the crowd compensation. Similarly, [72] study queueing systems with a random number of servers for on-demand service providers using fluid and stochastic-fluid approximations, where the firm can use either full-time employees or flexible workers.

Problem (3.7) allows a firm to intelligently decide on crowd compensation and labor planning to minimize logistical costs, while making sure that there are enough willing-to-participate crowd drivers that are 1) not too busy to risk late deliveries or 2) not too idle, where the drivers' effective earning rate becomes too low for participation. In the next section, we simplify Problem (3.7) by solving the inner maximization problems over the uncertainty sets $U_A(P)$, U_X^N, U_L^j, and U_K.

3.2.4 Solution Methodology

In the following lemma, we analyze the objective function of Problem (3.7) and derive an attainable upper bound for it; in the remainder of our analysis of the robust model, we utilize this bound as our objective.

Lemma 3.2.

$$\max_{\substack{(X_1,\ldots,X_{mq})\in U_X^N \\ L_{q+1}^j/v_j\in U_L^j, j=1,\cdots,m}} Pw\left(\frac{\sum_{j=1}^m L_{q+1}^j/v_j+\sum_{i=1}^{mq} X_i}{mq}\right)+(1-P)c_T\leq$$

$$\frac{\sum_{j=1}^m \beta\sqrt{q+1}/v_j+\sum_{j=1}^N\left(\sum_{i=1}^{\tau^m}\sum_{k=(j_i-1)q+1}^{j_iq}\mu_k^s+\gamma^s\sqrt{\sum_{i=1}^{\tau^m}\sum_{k=(j_i-1)q+1}^{j_iq}(\sigma_k^s)^2}\right)}{mq}.$$

We next study the first constraint of Problem (3.7) and, for this purpose, we analyze the worst-case system times of customers. [16] study the worst-case system time of the last customer in a queue formed by customers. However, in contrast, we are studying the worst-case system times of *all* customers in a queue formed by *sets* of customers, whose arrival process is a superposition of customer arrival processes, interarrival times and service times are non-stationary, and there is an additional routing layer whose duration depends on traffic patterns.

In order to analyze the customers' system times, we first let $D_{(j-1)q+k}$, $k=1,\cdots,q$ denote the *departure time* of customer k in set $j=1,\ldots,m$. We provide set-specific upper bounds T_j for the customers' departure times D_k. The first N sets of customers start getting served once they are formed, as there are N idle drivers in the system at time zero. Therefore, we have the following upper bounds for all customer departure times in the first N sets:

$$\text{(Set 1)} \quad D_k \leq T_1 \triangleq \sum_{\ell=1}^{q} A_\ell + \sum_{\ell=1}^{q} X_\ell + L_{q+1}^1/v_1, \ k = 1, \cdots, q \tag{3.8}$$

$$\text{(Set 2)} \quad D_{q+k} \leq T_2 \triangleq \sum_{\ell=1}^{2q} A_\ell + \sum_{\ell=q+1}^{2q} X_\ell + L_{q+1}^2/v_2, \ k = 1, \cdots, q \tag{3.9}$$

$$\cdots$$

$$\text{(Set } N\text{)} \quad D_{(N-1)q+k} \leq T_N \triangleq \sum_{\ell=1}^{Nq} A_\ell + \sum_{\ell=(N-1)q+1}^{Nq} X_\ell + L_{q+1}^N/v_N, \ k = 1, \cdots, q. \tag{3.10}$$

For instance, the bound T_2 for Set 2 consists of three components: 1) $\sum_{\ell=1}^{2q} A_\ell$ is the time when Set 2 is formed and ready for service, 2) $\sum_{\ell=q+1}^{2q} X_\ell$ is the total on-site service time for all the customers in Set 2, and 3) L_{q+1}^2/v_2 is the optimal tour duration to visit all customers in Set 2.

A common difficulty in the analysis of multiserver queues is that overtaking can occur, in that set i arrives after set j, but is serviced and leaves the system first. This results in the order of arrivals being different than the order of departures, which creates technical difficulties in the analysis. We perform an analysis motivated by that in [16], which precludes overtaking from happening, and sets depart in the same order that they arrive to the crowdsourcing queue. We denote the policy that precludes overtaking as Θ, and the subsequent results are for this policy. Later, we show that the worst-case system time under policy Θ is equal to the worst-case system time under the FCFS policy.

Motivated by [128], which generalized the recursive equations of [142] to the multiserver $G/G/m$ queue, we define the following recursive set of upper bounds for the departure times of customers in the subsequent sets $N+1, N+2, \ldots, m$, that might require waiting for a driver to become free:

$$D^{\Theta}_{(j-1)q+k} \le T^{\Theta}_{j} \triangleq \max\left\{\sum_{\ell=1}^{jq} A_\ell, T^{\Theta}_{j-N}\right\} + \sum_{\ell=(j-1)q+1}^{jq} X_\ell + L^{j}_{q+1}/v_j,\ j = N+1, \cdots, m,\ k = 1, \cdots, q, \tag{3.11}$$

where, for $j = 1, \ldots, N$, $T^{\Theta}_{j} = T_j$ as defined in Equations (3.8) – (3.10). $\sum_{\ell=1}^{jq} A_\ell$ is the time when the jth set is formed and ready for service; $\max\left\{\sum_{\ell=1}^{jq} A_\ell, T^{\Theta}_{j-N}\right\}$ is the first time that set j begins service by a free driver, where T^{Θ}_{j-N} is the exact time that a driver becomes free from serving the $(j-N)$th departed set.

Customers' system times can be calculated from the departure times as follows:

$$S_{(j-1)q+k} = D_{(j-1)q+k} - \sum_{\ell=1}^{(j-1)q+k} A_\ell,\ k = 1, \ldots, q,\ j = 1, \ldots, m, \tag{3.12}$$

where $\sum_{\ell=1}^{(j-1)q+k} A_\ell$ is the arrival time of customer k in the jth set to the system. In the next lemma, we utilize the above bounds on departure times to derive upper bounds on the system times S_i for customers $i = 1, \ldots, mq$.

Lemma 3.3. *System times of customers k in set j in a multiple-driver queue under policy Θ have the following upper bounds:*

$$\begin{aligned}
S^{\Theta}_{(j-1)q+k} \le B^{\Theta}_{(j-1)q+k} &\triangleq \sum_{\ell=(j-1)q+k+1}^{jq} A_\ell + \sum_{\ell=(j-1)q+1}^{jq} X_\ell + L^{j}_{q+1}/v_j,\ k = 1, \ldots, q,\ j = 1, \ldots N \\
S^{\Theta}_{(j-1)q+k} \le B^{\Theta}_{(j-1)q+k} &\triangleq \max_{i=0,\ldots a^j}\left\{\sum_{\ell=1}^{(j-iN)q} A_\ell + \sum_{\ell=0}^{i}\left(\sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q} X_p + \frac{L^{j-\ell N}_{q+1}}{v_{j-\ell N}}\right)\right\} \\
&\quad - \sum_{\ell=1}^{(j-1)q+k} A_\ell,\ k = 1, \ldots, q,\ j = N+1, \ldots, m,
\end{aligned}$$

where $a^j = \lceil \frac{j}{N} \rceil - 1$.

Lemma 3.3 defines B_i^Θ as the upper bound on S_i^Θ, for any i. Considering the bounds for the customers within an arbitrary set $j \in \{1, \ldots, m\}$, we observe that $B_{(j-1)q+1}^\Theta \geq B_{(j-1)q+2}^\Theta \geq \cdots \geq B_{jq}^\Theta$; in other words, the upper bound for the ith customer assigned to the set is at least the upper bound of the $(i+1)$th customer, and the first customer assigned to the set (in FCFS) has the largest upper bound (due to the largest wait time for the set to form). Therefore, we have the following implications $B_{(j-1)q+1}^\Theta \leq \alpha \Rightarrow B_{(j-1)q+k}^\Theta \leq \alpha,\ k = 1, \ldots, q,\ j = 1, \ldots, m$, which further imply

$$
\begin{aligned}
& \max_{\substack{(A_1,\ldots,A_{mq})\in U_A(P) \\ (X_1,\ldots,X_{mq})\in U_X^N \\ L_{q+1}^j/v_j \in U_L^j, j=1,\ldots,m}} B_{(j-1)q+1}^\Theta \leq \alpha,\ j = 1, \ldots, m \\
\Rightarrow\ & \max_{\substack{(A_1,\ldots,A_{mq})\in U_A(P) \\ (X_1,\ldots,X_{mq})\in U_X^N \\ L_{q+1}^j/v_j \in U_L^j, j=1,\ldots,m}} B_{(j-1)q+k}^\Theta \leq \alpha,\ k = 1, \ldots, q,\ j = 1, \ldots, m \\
\Rightarrow\ & \max_{\substack{(A_1,\ldots,A_{mq})\in U_A(P) \\ (X_1,\ldots,X_{mq})\in U_X^N \\ L_{q+1}^j/v_j \in U_L^j, j=1,\ldots,m}} S_i^\Theta \leq \alpha,\ i = 1, \ldots, mq.
\end{aligned}
$$

Consequently, we replace the $n = mq$ service constraints of Problem (3.7) with the m constraints in the first inequality above. In the next set of results, we analyze this constraint, namely the maximum upper bound on the system time of the first customer in the jth set, $j \in \{1, \ldots, m\}$. Letting $\mathbf{L}/\mathbf{v} = (L_{q+1}^1/v_1, \ldots, L_{q+1}^m/v_m)$, $\mathbf{A} = (A_1, \ldots, A_{mq})$, and $\mathbf{X} = (X_1, \ldots, X_{mq})$, we solve the inner optimization problems, for $j = 1, \ldots, m$, of the the first constraint sequentially:

$$
\max_{\substack{\mathbf{A}\in U_A(P) \\ \mathbf{X}\in U_X^N \\ \mathbf{L}/\mathbf{v}\in \times_{j=1}^m U_L^j}} B_{(j-1)q+1}^\Theta(\mathbf{L}/\mathbf{v}, \mathbf{A}, \mathbf{X}) = \max_{\mathbf{L}/\mathbf{v}\in \times_{j=1}^m U_L^j} \left\{ \max_{\mathbf{A}\in U_A(P)} \left\{ \max_{\mathbf{X}\in U_X^N} B_{(j-1)q+1}^\Theta(\mathbf{L}/\mathbf{v}, \mathbf{A}, \mathbf{X}) \right\} \right\}. \tag{3.13}
$$

We first analyze the innermost optimization problem over U_X^N in the right-hand side of

Equation (3.13), whose solution is presented in the next lemma.

Lemma 3.4. *For $j = 1, \ldots, m$ we have that*

$$\max_{\mathbf{X} \in U_X^N} B^{\Theta}_{(j-1)q+1}(\frac{\mathbf{L}}{\mathbf{v}}, \mathbf{A}, \mathbf{X}) \leq$$

$$\begin{cases} \displaystyle\sum_{\ell=(j-1)q+2}^{jq} A_\ell + \sum_{\ell=(j-1)q+1}^{jq} \mu_\ell^s + \gamma^s \sqrt{\sum_{\ell=(j-1)q+1}^{jq} (\sigma_\ell^s)^2} + L_{q+1}^j / v_j, \;\; j = 1, \cdots, N \\ \displaystyle\max_{i=0,\ldots a^j} \Bigg\{ \sum_{\ell=1}^{(j-iN)q} A_\ell + \sum_{\ell=0}^{i} \sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q} \mu_p^s + \gamma^s \sqrt{\sum_{\ell=0}^{i} \sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q} \left(\sigma_p^s\right)^2} + \sum_{\ell=0}^{i} \frac{L_{q+1}^{j-\ell N}}{v_{j-\ell N}} - \\ \displaystyle\qquad \sum_{\ell=1}^{(j-1)q+1} A_\ell \Bigg\}, \;\; j = N+1, \ldots, m. \end{cases}$$

In the next lemma, we solve the second-tier optimization over $U_A(P)$ in the right-hand side of Equation (3.13).

Lemma 3.5. *For $j = 1, \ldots, m$ we have the following upper bounds:*

$$\max_{\mathbf{A} \in U_A(P)} \left\{ \max_{\mathbf{X} \in U_X^N} B^{\Theta}_{(j-1)q+1}(\mathbf{L}/\mathbf{v}, \mathbf{A}, \mathbf{X}) \right\} \leq \begin{cases} g^N, & j = 1, \cdots, N \\ \max\left\{ \displaystyle\max_{i=1,\ldots,a^j} f_i^N, g^N \right\}, & j = N+1, \cdots, m, \end{cases}$$

where $g^N = \sum_{\ell=(j-1)q+2}^{jq} \mu_\ell^a / P + \gamma^a \sqrt{\sum_{\ell=(j-1)q+2}^{jq} (\sigma_\ell^a / P)^2} + \sum_{\ell=(j-1)q+1}^{jq} \mu_\ell^s + \gamma^s \sqrt{\sum_{\ell=(j-1)q+1}^{jq} (\sigma_\ell^s)^2} + L_{q+1}^j / v_j$ *and* $f_i^N = -\sum_{\ell=(j-iN)q+1}^{(j-1)q+1} \mu_\ell^a / P + \gamma^a \sqrt{\sum_{\ell=(j-iN)q+1}^{(j-1)q+1} (\sigma_\ell^a / P)^2} + \sum_{\ell=0}^{i} \sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q} \mu_p^s + \gamma^s \sqrt{\sum_{\ell=0}^{i} \sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q} \left(\sigma_p^s\right)^2} + \sum_{\ell=0}^{i} L_{q+1}^{j-\ell N} / v_{j-\ell N}$.

In the next lemma, we solve the final outer optimization over $\bigtimes_{j=1}^{m} U_L^j$ in the right-hand side of Equation (3.13).

Lemma 3.6. *For $j = 1, \ldots, m$ we have the following upper bounds:*

$$\max_{\mathbf{L}/\mathbf{v} \in \times_{j=1}^{m} U_L^j} \left\{ \max_{\mathbf{A} \in U_A(P)} \left\{ \max_{\mathbf{X} \in U_X^N} B^{\Theta}_{(j-1)q+1}(\mathbf{L}/\mathbf{v}, \mathbf{A}, \mathbf{X}) \right\} \right\} \leq \begin{cases} \hat{g}^N, & j = 1, \cdots, N \\ \max\left\{ \max\limits_{i=1,\ldots,a^j} \hat{f}_i^N, \hat{g}^N \right\}, & j = N+1, \cdots, m, \end{cases}$$

where $\hat{g}^N = \sum_{\ell=(j-1)q+2}^{jq} \mu_\ell^a/P + \gamma^a \sqrt{\sum_{\ell=(j-1)q+2}^{jq} (\sigma_\ell^a/P)^2} + \sum_{\ell=(j-1)q+1}^{jq} \mu_\ell^s + \gamma^s \sqrt{\sum_{\ell=(j-1)q+1}^{jq} (\sigma_\ell^s)^2} + \beta\sqrt{q+1}/v_j$ *and* $\hat{f}_i^N = -\sum_{\ell=(j-iN)q+1}^{(j-1)q+1} \mu_\ell^a/P + \gamma^a \sqrt{\sum_{\ell=(j-iN)q+1}^{(j-1)q+1} (\sigma_\ell^a/P)^2} + \sum_{\ell=0}^{i} \sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q} \mu_p^s + \gamma^s \sqrt{\sum_{\ell=0}^{i} \sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q} \left(\sigma_p^s\right)^2} + \sum_{\ell=0}^{i} \beta\sqrt{q+1}/v_{j-\ell N}$.

The next result shows that the bounds B_i^{Θ} are also applicable to the FCFS policy.

Lemma 3.7. *The system time of customer i, $i = 1, \ldots, mq$, under the FCFS policy S_i^{FCFS} is at most the bound B_i^{Θ} defined in Lemma 3.3 for the no-overtaking policy Θ:*

$$S_i^{FCFS}(\mathbf{L}/\mathbf{v}, \mathbf{A}, \mathbf{X}) \leq B_i^{\Theta}(\mathbf{L}/\mathbf{v}, \mathbf{A}, \mathbf{X}),\ \forall \mathbf{X} \in U_X^N,\ \forall \mathbf{A} \in U_A(P),\ \forall \mathbf{L}/\mathbf{v} \in \bigtimes_{j=1}^{m} U_L^j.$$

As a result of Lemma 3.7, all of the analyses of the Θ policy are applicable, and equivalent, to the FCFS policy; we therefore replace, in the sequel, the Θ superscript with $FCFS$. Thus, Lemmas 3.1 – 3.6 allow us to define the following approximation to Problem (3.7):

$$
\begin{aligned}
z = \min_{P\in(0,1],w>0}\quad & Pw\left(\frac{\sum_{j=1}^{m}\beta\sqrt{q+1}/v_j+\sum_{j=1}^{N}\left(\sum_{i=1}^{\tau^m}\sum_{k=(j_i-1)q+1}^{j_iq}\mu_k^s+\gamma^s\sqrt{\sum_{i=1}^{\tau^m}\sum_{k=(j_i-1)q+1}^{j_iq}\left(\sigma_k^s\right)^2}\right)}{mq}\right)\\
& +(1-P)c_T\\
\text{s.t.}\quad & \max_{i=1,\dots,a^j}\left\{-\sum_{\ell=(j-iN)q+1}^{(j-1)q+1}\mu_\ell^a/P+\gamma^a\sqrt{\sum_{\ell=(j-iN)q+1}^{(j-1)q+1}\left(\sigma_\ell^a/P\right)^2}+\sum_{\ell=0}^{i}\sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q}\mu_p^s+\right.\\
&\qquad\left.\gamma^s\sqrt{\sum_{\ell=0}^{i}\sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q}\left(\sigma_p^s\right)^2}+\sum_{\ell=0}^{i}\frac{\beta\sqrt{q+1}}{v_{j-\ell N}}\right\}\le\alpha,\ j=N+1,\dots,m\\
& \sum_{\ell=(j-1)q+2}^{jq}\mu_\ell^a/P+\gamma^a\sqrt{\sum_{\ell=(j-1)q+2}^{jq}\left(\sigma_\ell^a/P\right)^2}+\sum_{\ell=(j-1)q+1}^{jq}\mu_\ell^s+\gamma^s\sqrt{\sum_{\ell=(j-1)q+1}^{jq}\left(\sigma_\ell^s\right)^2}+\\
& \frac{\beta\sqrt{q+1}}{v_j}\le\alpha, j=1,\dots,m\\
& P\left(\frac{\sum_{i=1}^{mq}\mu_i^s}{m}+\frac{\sum_{i=1}^{m}\beta\sqrt{q+1}/v_i}{m}\right)/\left(N\frac{\sum_{i=1}^{mq}\mu_i^a}{m}\right)\le 1-\epsilon\\
& wP\left(\frac{\sum_{i=1}^{mq}\mu_i^s}{m}+\frac{\sum_{i=1}^{m}\beta\sqrt{q+1}/v_i}{m}\right)/\left(N\frac{\sum_{i=1}^{mq}\mu_i^a}{m}\right)\ge\frac{\sum_{i=1}^{N}\mu_{(i)}^k+\gamma^k\sqrt{N}\sigma^k}{N}.
\end{aligned}
\tag{3.14}
$$

For each customer set j, $i=1,\dots,a^j$ indexes the customer sets formed before set j that are also served by the same crowd driver who serves set j. For analyzing the system times in Problem (3.14), let $E_i^j\triangleq\sum_{\ell=(j-iN)q+1}^{(j-1)q+1}\mu_\ell^a-\gamma^a\sqrt{\sum_{\ell=(j-iN)q+1}^{(j-1)q+1}\left(\sigma_\ell^a\right)^2}$ denote the best-case interarrival time between set j and the i-th preceding set, all served by the same driver. Let $F_i^j\triangleq\sum_{\ell=0}^{i}\sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q}\mu_p^s+\gamma^s\sqrt{\sum_{\ell=0}^{i}\sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q}\left(\sigma_p^s\right)^2}+\sum_{\ell=0}^{i}\frac{\beta\sqrt{q+1}}{v_{j-\ell N}}$ denote the cumulative worst-case system times (including the on-site service times and tour times) of set j and the i preceding sets, all of which served by the same driver. Finally, let $G_j\triangleq\sum_{\ell=(j-1)q+2}^{jq}\mu_\ell^a+\gamma^a\sqrt{\sum_{\ell=(j-1)q+2}^{jq}\left(\sigma_\ell^a\right)^2}$ denote the worst-case waiting time for the first customer in set j until her set forms. Using these terms, the closed-form solutions to Problem (3.14) are provided in Proposition 3.1.

Proposition 3.1. *If* $\alpha > \max_{j=1,\ldots,m}\left\{F_{i=0}^{j}\right\}$*, then Problem* (3.14) *is feasible if and only if* $\hat{P} \leq \min\left\{\overline{P}^{N}, \tilde{P}, 1\right\}$*; if feasible, the optimal solution is* $P^{*} = \min\left\{\overline{P}^{N}, \tilde{P}, 1\right\}$*, with the crowd supply function* $w^{*} = \frac{\frac{\sum_{i=1}^{N}\mu_{(i)}^{k}+\gamma^{k}\sqrt{N}\sigma^{k}}{N}\left(\frac{\sum_{i=1}^{mq}\mu_{i}^{a}}{m}\right)}{P^{*}\left(\frac{\sum_{i=1}^{mq}\mu_{i}^{s}}{m}+\frac{\sum_{i=1}^{m}\beta\sqrt{q+1}/v_{i}}{m}\right)}$*, and the minimized worst-case delivery cost rate* $z(N) = P^{*}\left(w^{*}\frac{\sum_{j=1}^{m}\beta\sqrt{q+1}/v_{j}+\sum_{j=1}^{N}\left(\sum_{i=1}^{\tau^{m}}\sum_{k=(j_i-1)q+1}^{j_i q}\mu_{k}^{s}+\gamma^{s}\sqrt{\sum_{i=1}^{\tau^{m}}\sum_{k=(j_i-1)q+1}^{j_i q}\left(\sigma_{k}^{s}\right)^{2}}\right)}{mq}-c_{T}\right)+c_{T}$*,*
where $\overline{P}^{N} = \min_{j=N+1,\ldots,m}\left\{\min_{i\in s_2^j}\left\{\frac{E_i^j}{F_i^j-\alpha}\right\}\right\}$, $\hat{P} = \max_{j=1,\ldots,m}\left\{\frac{G_j}{\alpha-F_{i=0}^{j}}\right\}$, $\tilde{P} = \frac{(1-\epsilon)N\frac{\sum_{i=1}^{mq}\mu_i^a}{m}}{\frac{\sum_{i=1}^{mq}\mu_i^s}{m}+\frac{\sum_{i=1}^{m}\beta\sqrt{q+1}/v_i}{m}}$,
$s_2^j = \left\{i = 1,\ldots,a^j : \alpha - F_i^j < 0\right\},\ j = N+1,\ldots,m.$

According to Proposition 3.1, the optimal hourly wage that the firm should offer to crowd drivers changes with the delivery time window and the different uncertainties that exist in the delivery system, including the uncertainty in customer purchasing patterns, delivery service times and crowd opportunity costs. In particular, the hourly wage that the firm should offer to crowd drivers is higher as the uncertainty in their opportunity costs (i.e., σ^k) increases; this is to ensure their participation. The precise nature of the dependence of the optimal crowd hourly wage on the other various uncertainties in the system is not straightforward (i.e., non-linear) due to the underlying queue; we study this dependence in Section 3.5.1 via a numerical analysis. Finally, if the firm wants to ensure crowd participation with an even higher probability (i.e., increasing γ^k), the hourly wage that the firm should offer should be even higher.

Our analysis shows that there exists an upper bound on P, denoted by $\overline{P}^{N}$, that, if exceeded, results in the on-time delivery of packages being at risk. $\tilde{P}$ is another upper bound on P, which ensures that the underlying queue in the crowd delivery system is stable. $\hat{P}$ is a lower bound on P that guarantees that sufficient delivery orders are assigned to the available crowd drivers such that customers do not wait too long for the formation of their delivery tours, and thus can receive their orders within α. Respecting these bounds means that the number of packages assigned to crowd drivers is large enough to help with the reduction of logistical cost, but at the same time small enough to not jeopardize the promise of on-time delivery. In other words, if firms want to use a crowdsourced system for their last-mile deliveries, they can not assign too little nor too much work to the crowd – it has to be a moderate workload.

3.3 Benchmark 1: A Stochastic Crowdsourcing Model

In this section, we propose and solve a stochastic counterpart to our robust model, where we replace the worst-case expressions by expected values; we aim to compare our worst-case analysis approach against an expected-value analysis approach. The comparison of these two models, in Section 3.5, will evaluate the benefit of using robust optimization over stochastic optimization. Furthermore, while trend and seasonality can be rather easily incorporated into the robust model, we found them to be intractable to incorporate into our stochastic queueing framework; consequently, we develop the stochastic model for stationary random variables, and compare it equitably with a robust model that also has stationary parameters (e.g., $\mu_i^a = \mu^a$ for some μ^a and all i).

We assume customers arrive according to a Poisson Process with rate λ and, similar to the robust crowdsourcing model, a proportion P of customers are randomly assigned to crowdsourced delivery and the remaining $(1 - P)$ proportion are served via a 3PL firm. The mean interarrival time between crowdsourced customers is $\frac{1}{P\lambda}$. We again form sets of q customers based on a FCFS policy and these sets are served by the first available crowd driver, who then visits the customers within a set using the optimal TSP tour. The time for a set to form has an Erlang distribution with mean $\frac{q}{P\lambda}$ and variance $\frac{q}{P^2\lambda^2}$. The on-site service times have a general distribution, with mean μ^s and variance σ_s^2. Customer locations are distributed according to the density $f(x)$, and L_{q+1} again denotes the shortest tour through

q customer locations and, according to [32], $E[L_{q+1}] = \hat{\beta}\sqrt{q+1}$, where, from Equation (3.3), we set $\hat{\beta} = \tilde{\beta} \sum_{i=1}^{M} \sqrt{\frac{p_i}{\sum_{j=1}^{M} \mathbb{A}_j p_j}} \mathbb{A}_i$. We denote $\hat{\beta}\sqrt{q+1}/v$ and $\sigma^2_{L_{q+1}}/v^2$ as the mean and variance of the tour duration L_{q+1}/v, respectively, where v is the average travel speed for all crowd delivery tours.

We can model the queue formed by the sets as a $G/G/m$ queue. Assuming N crowd drivers are willing to provide on-demand delivery services for the firm, the utilization of the crowd delivery system is $\frac{P\lambda}{qN}\left(\hat{\beta}\sqrt{q+1}/v + q\mu^s\right)$. We assume that the opportunity costs of available crowd drivers are distributed according to a general distribution with mean μ^k and standard deviation σ^k. Similar to the approach in the robust model, the number of participating crowd drivers satisfies $w\frac{P\lambda}{qN}\left(\hat{\beta}\sqrt{q+1}/v + q\mu^s\right) \geq \mu^k$; in Proposition 3.2, we show this inequality is tight and we obtain the supply function $w = N\mu^k \frac{q}{P\lambda\left(\hat{\beta}\sqrt{q+1}/v+q\mu^s\right)}$ that endogenously determines the number of participating crowd drivers, as a function of the crowd hourly wage and the delivery work load.

Using a similar analysis to that in [32], we calculate the expected system times of customers. The expected time a customer must wait until her set forms is $W_{\text{expected-time-to-form-a-set}} = \frac{1}{q}\left(0 + \frac{1}{\lambda} + \frac{2}{\lambda} + \ldots + \frac{q-1}{\lambda}\right) = \frac{(q-1)}{2\lambda}$. Using the heavy traffic limit introduced by [122], we write an approximation for the expected waiting time in queue (for a set of size q) as $W_{\text{expected-waiting-time-for-sets}} \approx \frac{\frac{P\lambda}{q}\left(\frac{q}{P^2\lambda^2} + \frac{1}{N^2}\left({\sigma_L}^2/v^2 + q{\sigma_s}^2\right)\right)}{2\left(1-\frac{P\lambda}{qN}\left(\hat{\beta}\sqrt{q+1}/v+q\mu^s\right)\right)}$; this approximation becomes more accurate as the system utilization increases towards 1. In our simulation study, covering the Seattle area, the utilization of crowd drivers under the stochastic policy is at least 0.8, which is high enough for this heavy traffic approximation [210]. Hence, we assume the quality of the [122] approximation for our stochastic analysis is practically reasonable.

Finally, the expected wait time for a customer to be served, once the driver starts serving the customer's set, is at most $W_{\text{expected-service-time-of-sets}} = \hat{\beta}\sqrt{q+1}/v + \frac{1}{q}\left(\mu^s + 2\mu^s + \ldots + q\mu^s\right) = \hat{\beta}\sqrt{q+1}/v + \frac{(q+1)\mu^s}{2}$. As a result, the expected system time of a customer in the stable crowdsourced delivery system (i.e., $\frac{P\lambda}{qN}\left(\hat{\beta}\sqrt{q+1}/v+q\mu^s\right) \leq 1-\epsilon$), is at most $\frac{(q-1)}{2\lambda} + \frac{\frac{P\lambda}{q}\left(\frac{q}{P^2\lambda^2}+\frac{1}{N^2}\left({\sigma_L}^2/v^2+q{\sigma_s}^2\right)\right)}{2\left(1-\frac{P\lambda}{qN}\left(\hat{\beta}\sqrt{q+1}/v+q\mu^s\right)\right)} + \hat{\beta}\sqrt{q+1}/v + \frac{(q+1)\mu^s}{2}$. As a result of the above analysis, a stochastic optimization variant of Model (3.7) can be written as the following model, with its solution provided in the subsequent proposition.

$$
\begin{aligned}
z_s = \min_{P\in(0,1],w>0} \quad & Pw\left(\frac{\hat{\beta}\sqrt{q+1}/v}{q}+\mu^s\right)+(1-P)c_T \\
\text{s.t.} \quad & \frac{(q-1)}{2\lambda} + \frac{\frac{P\lambda}{q}\left(\frac{q}{P^2\lambda^2}+\frac{1}{N^2}\left({\sigma_L}^2/v^2+q{\sigma_s}^2\right)\right)}{2\left(1-\frac{P\lambda}{qN}\left(\hat{\beta}\sqrt{q+1}/v+q\mu^s\right)\right)} + \hat{\beta}\sqrt{q+1}/v + \frac{(q+1)\mu^s}{2} \leq \alpha \\
& \frac{P\lambda}{qN}\left(\hat{\beta}\sqrt{q+1}/v+q\mu^s\right) \leq 1-\epsilon \\
& w\frac{P\lambda}{qN}\left(\hat{\beta}\sqrt{q+1}/v+q\mu^s\right) \geq \mu^k.
\end{aligned}
\tag{3.15}
$$

Proposition 3.2. *If* $\frac{(q-1)}{2\lambda} + \hat{\beta}\sqrt{q+1}/v + \frac{(q+1)\mu^s}{2} \leq \alpha$*, and* $\Delta \geq 0$*, then Problem* (3.15) *is feasible if and only if* $P_1 \leq \min\{P_2, P_3, 1\}$*; if feasible, the optimal solution is* $P^* = \min\{P_2, P_3, 1\}$ *with the crowd supply function* $w^* = \frac{\mu^k qN}{P^*\lambda\left(\hat{\beta}\sqrt{q+1}/v+q\mu^s\right)}$*, and the minimized delivery cost rate* $z_s(N) = \frac{\mu^k N}{\lambda} + (1-P^*)c_T$*, where* $U = \frac{\lambda}{qN^2}\left({\sigma_L}^2/v^2+q{\sigma_s}^2\right) + \frac{2\lambda}{qN}(\alpha - \frac{(q-1)}{2\lambda} - \hat{\beta}\sqrt{q+1}/v - \frac{(q+1)\mu^s}{2})(\hat{\beta}\sqrt{q+1}/v+q\mu^s)$*,* $\Delta = \left(\alpha - \frac{(q-1)}{2\lambda} - \hat{\beta}\sqrt{q+1}/v - \frac{(q+1)\mu^s}{2}\right)^2 - 4U/\lambda$*,* $P_1 = \frac{\alpha - \frac{(q-1)}{2\lambda} - \hat{\beta}\sqrt{q+1}/v - \frac{(q+1)\mu^s}{2} - \sqrt{\Delta}}{2U}$*,* $P_2 = \frac{\alpha - \frac{(q-1)}{2\lambda} - \hat{\beta}\sqrt{q+1}/v - \frac{(q+1)\mu^s}{2} + \sqrt{\Delta}}{2U}$*, and* $P_3 = \frac{(1-\epsilon)qN}{\lambda\left(\hat{\beta}\sqrt{q+1}/v+q\mu^s\right)}$*.*

Similar to the robust analysis in Proposition 3.1, P_1 is a lower bound on P that guarantees enough delivery orders are assigned to the available crowd such that customers do not wait too long for their crowd delivery set to form. P_2 is an upper bound on P that guarantees the number of assigned delivery orders to the available crowd drivers is small enough so that deliveries can be made on time, and P_3 is another upper bound on P to guarantee a stable crowd delivery queueing system. Similar to the robust approach, even though the number of crowd drivers N is determined endogenously, the supply function allows us to indirectly optimize over w by using N.

3.4 Benchmark 2: A Non-Randomized Heuristic Crowdsourcing Model

In this section, we propose and analyze a CVRP-based heuristic benchmark model that uses package characteristics (i.e., delivery location) to determine which packages should be crowdsourced and which should be assigned to the 3PL firm. In Section 3.5, we compare the quality of this dynamic and heuristic-based benchmark approach, which utilizes package characteristics, with that of the static and randomized allocation policy of Section 3.2. This benchmark utilizes and modifies the Savings Algorithm, originally proposed by [57], which is a heuristic method for solving the static deterministic CVRP, and it is used in many commercial vehicle routing packages [191].

In contrast to the static deterministic CVRP, where all customer locations are known a priori, if customers arrive (revealing their location and demand) dynamically over time, it is a dynamic CVRP. In order to solve the dynamic CVRP, we apply the Savings Algorithm in a rolling horizon framework. In particular, whenever q customers arrive – a batch of size q – we reapply the algorithm.

We next modify the algorithm to accommodate the case where each demand point can either be served by an available crowd driver or a 3PL firm. In our Modified Savings Algorithm, as the initial solution, we assign each customer to a 3PL firm at cost c_T^i, where the superscript i indicates customer i, whose distance to the depot influences this cost (i.e., if customer i is located farther from the depot, c_T^i increases); in our subsequent numerical analysis, we use FedEx delivery prices with volume discounts to parameterize c_T^i. The cost of this initial solution is $\sum_{i=1}^{n} c_T^i$.

We next apply the basic idea of the Savings Algorithm and calculate the savings for:

1) serving customer i in a single crowdsourced delivery tour, instead of the 3PL firm, and 2) combining customers i and j and serving them using a single crowdsourced delivery tour. The savings s_{ii} we obtain from reassigning customer i from the 3PL firm to a single crowdsourced delivery tour is $s_{ii} = c_T^i - 2wd_i$, where c_T^i is the delivery cost via the 3PL, d_i is the distance from customer i's location to the depot, and $2wd_i$ is the delivery cost via a crowd driver with hourly wage w. Note that, for simplicity, we consider d_i to be the travel time, between customer i and the depot, in hours. The savings we obtain from reassigning customers i and j from the 3PL firm to a crowdsourced delivery tour, that serves both customers i and j, is $s_{ij} = c_T^i + c_T^j - w(d_i + d_j + d_{ij})$, where $c_T^i + c_T^j$ is the delivery cost for customers i and j if they are both served via the 3PL, d_{ij} is the distance between customers i and j, and $w(d_i + d_j + d_{ij})$ is the delivery cost if customers i and j are served in a single tour via a crowd driver. Note that a crowd driver also experiences an on-site service time X_i when delivering a package to customer i (e.g., parking, finding the appropriate apartment, etc.); we include these on-site service times in the savings matrix as $s_{ii} = c_T^i - w(2d_i + X_i), s_{ij} = c_T^i + c_T^j - w(d_i + d_j + d_{ij} + X_i + X_j)$.

The Algorithm below outlines the steps of our proposed Modified Savings Algorithm, based on the Savings Algorithm in [132] and [191], for every batch arrival of size q. Loosely speaking, the proposed algorithm outsources deliveries, where either their location is not close to other deliveries or they require large on-site service times, to the 3PL. Note that, as guaranteed by step 4, the service level constraints of all customers are satisfied under this non-randomized allocation policy.

3.5 Numerical Analysis

In this section, we evaluate the performances of the optimal policies from the robust crowdsourcing model and the two benchmark models, via a real-world simulation analysis covering 17 ZIP codes in Seattle, Washington. These ZIP codes include the downtown area as well

Algorithm 1

Step 1. Assign each customer to a 3PL firm.
Step 2. Calculate the savings s_{ij} , $\forall i, j$.
Step 3. List the savings in descending order of magnitude.
Step 4. Choose the first feasible link (i, j) from the savings, where

1. i and j are not assigned to the same route,
2. i and j are traveled first or last in their routes,
3. there is an available crowd driver for serving the newly formed route from merging the routes of i and j,
4. merging the routes of i and j does not violate the service level constraints.

Add the new route to the solution, delete the merged routes consisting of i and j, and delete s_{ij} from the savings list.
Step 5. Repeat step 4 until there is no feasible link.

as sparsely populated areas, served by one active Amazon Flex depot [107]. Our simulation experiment is designed as follows. We randomly generated 1000 customer locations according to the probability density function $f(x)$ in Equation (3.2). Figure 3.1 shows the 17 ZIP codes, the 1000 randomly generated customers (green diamonds), and the Amazon Flex depot (black star). The boundaries of the ZIP codes are from the 2016 U.S. ZIP codes via ArcGIS.

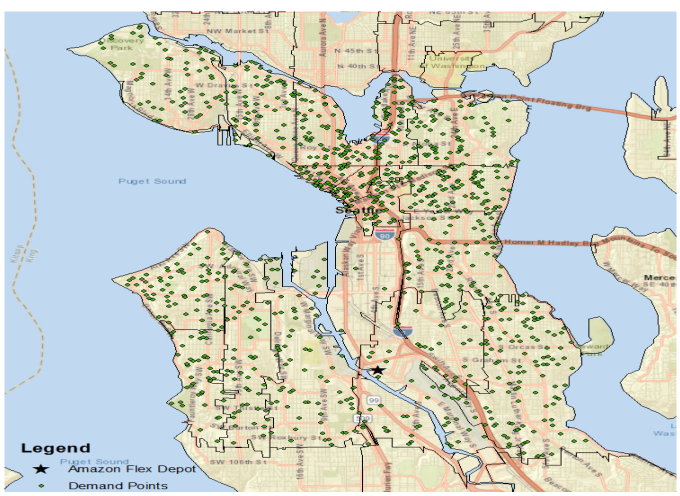

Figure 3.1: 1000 Randomly Generated Customers in 17 Seattle ZIP Codes.

We assume orders arrive from the 1000 customers in Figure 3.1 via a Poisson process, where customer on-site service times are exponentially distributed with a mean of 3 minutes.

We consider mean customer interarrival times that follow the pattern in the left-hand-side plot in Figure 3.2, where μ_L^a and μ_H^a are the mean interarrival times during busy and normal hours, respectively, which captures heavier buying patterns before/after normal business hours as well as the lunch hour. As a baseline, we consider $(\mu_H^a, \mu_L^a) = (10, 5)$ minutes. The plot in the middle of Figure 3.2 shows a heterogeneous supply of crowd drivers of size $\mathbb{N} = 50$, with two different expected opportunity costs, split evenly among the $\mathbb{N}$ drivers. According to [59], on average, the weekly earnings of Uber drivers from January 2015–March 2017 was \$376.38 for 17.06 hours covering 29.83 trips, where there was an average wait time of approximately 8 minutes between trips. This is equivalent to working for 21.04 hours and earning \$376.38 weekly, which results in earnings of \$17.9 per hour. In our numerical analysis, we assume that crowd drivers' opportunity costs are normally distributed, and the crowd drivers with lower opportunity costs have an expected opportunity cost of $\mu_L^k = \$18$ an hour, and the ones with higher opportunity costs have an expected opportunity cost of $\mu_H^k = \$22$ dollars an hour. The baseline standard deviation of the opportunity costs is $\sigma^k = \$5$ an hour. Finally, we consider a crowd driver's mean vehicle speed that follows the pattern in the right-hand-side plot in Figure 3.2, where v_L and v_H are delivery speeds during busy and normal hours, respectively, which captures rush hour traffic patterns. As a baseline, we consider $(v_H, v_L) = (30, 15)$ mph, which are motivated by the speed limits in the city of Seattle [166].

In order to calculate the distances between each customer and the depot as well as that between customers, we use *real distances* (i.e., on a street network) via the origin-destination (OD) cost matrix implemented in the Network Analyst tool in ArcGIS. We calculate the TSP tours of each instance using the nearest neighbor (NN) algorithm [115]; we also tested our results using the 2-opt local search algorithm [62] and our results remain unchanged.

We set the drivers' capacity $Q = 50$ as the baseline value; this number is supported

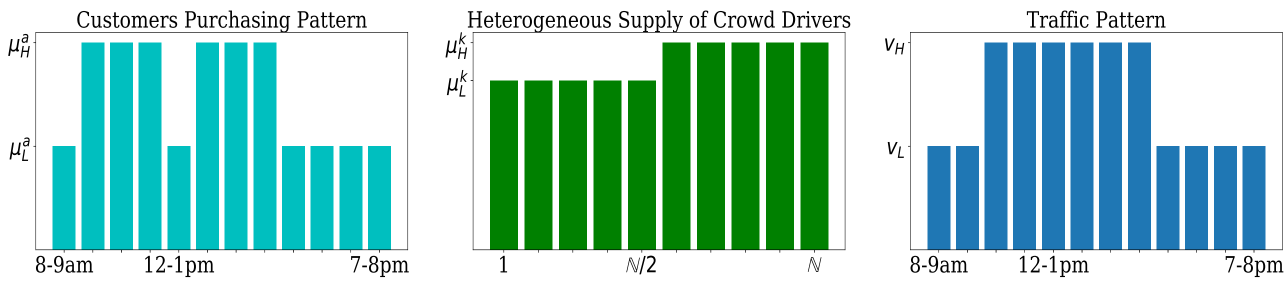

Figure 3.2: Patterns of Random Variables over a Day.

by Amazon Flex drivers' YouTube videos [110, 75]. Amazon's Prime Now program (primenow.amazon.com) offers 2-hour deliveries. Additionally, Amazon offers same-day delivery services: if customers order by noon, they can receive their orders by 9pm the same day. Therefore, we consider $\alpha \in \{2, 4, 9\}$ hours as the range of customer service levels.

We also need to have c_T^i estimates. The delivery fees of USPS, UPS, and FedEx vary significantly based on the distance between origin and destination, and the delivery time window. Therefore, c_T^i, the 3PL delivery cost for package i, should be a function of the distance between a customer's location and the depot, as well as the service level α. We use distance-dependent FedEx delivery fees for same-day, 4-hour, and 2-hour deliveries [66, 67, 65], for packages weighing between 0 and 50 pounds (87% of Amazon packages weigh less than 5 pounds [209]). According to [79], depending on a firm's annual shipping volume, the firm can get discounts of up to 30%. Thus, we consider the FedEx delivery prices, discounted by $\eta = 30\%$, as representative pricing for a 3PL firm.

We parameterize our experiment around the FedEx service, and depending on where a customer is located in the service region, different delivery rates are charged. Note that we calculate the real mile distance between customer $i = 1, \ldots, n$ and the Amazon Flex depot using the origin-destination (OD) cost matrix implemented in Network Analyst tool in ArcGIS and depending on the mile distance between the customer and the depot, the

Distance between origin and destination (miles)	$\alpha = 2$ hours	$\alpha = 4$ hours	$\alpha = 9$ hours
$0 - 5$	\$18	\$15	\$12
$6 - 10$	\$23	\$19	\$15
$11 - 15$	\$27	\$23	\$18
$16 - 25$	NA	\$35	\$28
$26 - 35$	NA	\$47	\$38
$36 - 45$	NA	\$62	\$46

Table 3.1: FedEx Delivery Fees for a 0-50 Pound Package, Before any Volume Discounts.

FedEx delivery cost is chosen from Table 3.1, and after applying the discount rate $\eta = 0.3$, the discounted cost is set as c_T^i.

In our robust and stochastic models, we use a single α-dependent value of c_T (the Modified Savings Algorithm uses all c_T^i values). We set $c_T = \sum_{i=1}^{n} c_T^i/n$, where c_T^i is the FedEx delivery cost for customer i who is located d_i miles from the depot. Our results are as follows: the average c_T values, after an $\eta = 30\%$ discount, for all $n = 1000$ customers in all 17 ZIP codes, are $c_T = \$9.55$ for $\alpha = 9$ hours, $c_T = \$12.04$ for $\alpha = 4$ hours, and $c_T = \$14.51$ for $\alpha = 2$ hours.

In the robust and stochastic models, we estimate upper bounds on tour lengths for each tour size $q \in \{1, \ldots, Q\}$, starting and ending at the Amazon Flex depot, by $L_{q+1} = \beta_U \sum_{i=1}^{M} \sqrt{\frac{p_i}{\sum_{j=1}^{M} \mathbb{A}_j p_j} \mathbb{A}_i} \sqrt{q+1}$. Using the population and area of each of the 17 Seattle ZIP codes in Figure 3.1, we calculated the multiplier $\sum_{i=1}^{M} \sqrt{\frac{p_i}{\sum_{j=1}^{M} \mathbb{A}_j p_j} \mathbb{A}_i} = 6.94$. Additionally, we consider $\beta_U = 2\tilde{\beta} \approx 1.4$, in which $\tilde{\beta}$ is the Bearwood-Halton-Hammersley constant for Euclidean distances on a plane [194], and the 2 multiplier reflects a finite q in the limit theorem. Additionally, in the stochastic model we set $\sigma_{L_{q+1}} = 0$, although we get qualitatively similar results for $\sigma_{L_{q+1}} > 0$.

We also perform an additional layer of numerical optimization for the robust model, to determine the optimal value of q: we insert the solutions from Proposition 3.1 into Problem

(3.14), which results in an objective function that appears to be convex in q (our conclusion is from extensive numerical analysis as we could not prove this analytically), which is easily solvable numerically.

3.5.1 Analysis of the Robust Solution

Figure 3.3 shows the worst-case cost of the robust optimal policies as a function of demand uncertainty σ^a for same-day deliveries, for different levels of conservatism in the robust model. We learn that, as there is more uncertainty in demand, the delivery cost increases according to a step function. Each of the sharp increases is due to having an extra crowd driver in the crowdsourced delivery system. An increase in the number of crowd drivers decreases the utilization of the system and, hence, it enables the firm to better respond to demand uncertainties. Although in each of the plateaus the number of crowd drivers does not change, crowd compensation and delivery assignments change as we move across the plateaus, as indicated for $\gamma^r = 3$ and 4 crowd drivers. In this example, on the left-hand side of the plateau, which corresponds to lower levels of demand uncertainty, our model suggests that the firm should assign a smaller number of packages ($q^* = 9$) to crowd drivers for the delivery tours. This allows the firm to create a stream of available delivery work for these independent crowd drivers, which helps with their participation, without having to offer them higher hourly wages. But as we move to the other end of the plateau, to the right-hand side, which is the side with higher demand uncertainty, our model suggests that the firm should assign a larger number of packages ($q^* = 15$) to the crowd drivers for the delivery tours. Assigning a larger number of packages to crowd drivers implies that the interarrival times between delivery tours increases. Consequently, the utilization of the system decreases, resulting in decreased income for the crowd drivers, which can adversely impact their participation. Therefore, the firm should increase the crowd hourly wage; in this example, on the left-hand side, the hourly wage is \$22 and on the right-hand side the hourly wage is \$26. This policy guarantees that the crowd drivers earn the same amount of income and, hence, do not get discouraged to participate. The changes of the optimal policy with respect to uncertainty in service times is qualitatively similar.

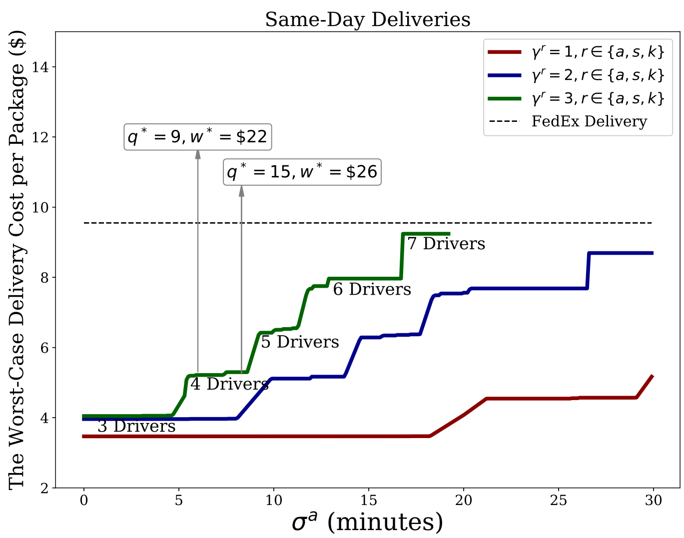

Figure 3.3: Robust Optimal Policy as a Function of Demand Uncertainty.

Figure 3.4 shows the robust optimal policies as a function of variability in crowd opportunity costs. As variability in crowd opportunity costs increases, the firm should not change the labor planning and crowd delivery assignment, but rather simply offer a higher hourly wage to guarantee the availability of independent crowd drivers.

Increasing the service guarantee level from 95% to 99% (i.e., increasing the γ^r parameters in the robust uncertainty sets from 2 to 3) is more expensive than increasing it from 68% to 95%, and this is due to a significant decrease in crowd utilization ρ. This results in the firm offering a higher hourly wage w for guaranteeing crowd participation. Thus, the firm, by offering satisfactory levels of an on-time delivery guarantee (e.g. 95%), can significantly re-

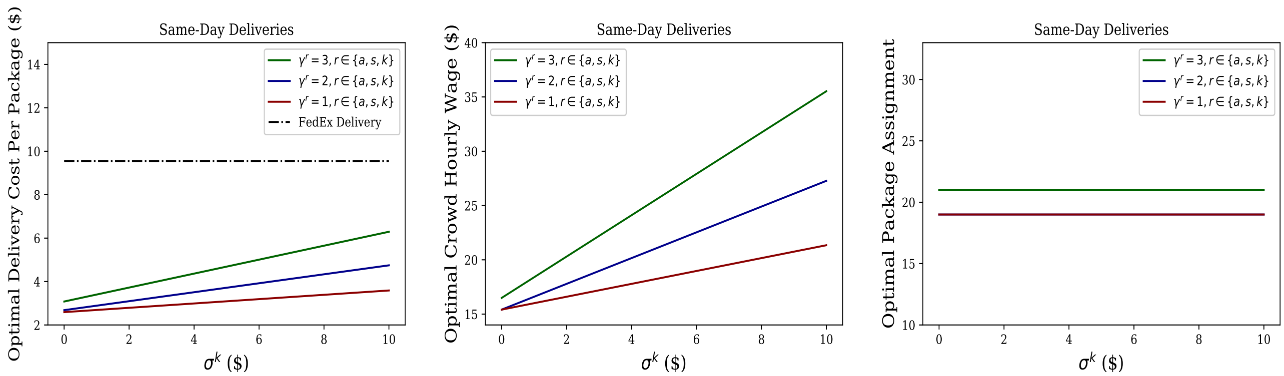

Figure 3.4: Robust Optimal Policy as a Function of Uncertainty in Crowd Opportunity Costs.

duce its delivery cost via crowdsourcing; we subsequently confirm this via the real simulation study over Seattle ZIP codes.

For lower levels of conservatism γ^r, the firm should assign a smaller number of deliveries to each crowd delivery tour to create a stream of available delivery work for the independent crowd drivers to make them willing to participate without having to offer them high hourly wages. However, for higher levels of conservatism γ^r, the firm wants to keep utilization low to guarantee on-time deliveries. Thus, the firm should assign larger set sizes q to crowd drivers to increase the interarrival times between delivery tours and, as a result, decrease crowd utilization, and instead compensate them via higher hourly wages. Currently Amazon [82] and Postmates [173] offer crowd hourly wages of \$18-\$25, which is consistent with our findings for service level guarantees with 68%-99% probabilities for same-day deliveries.

3.5.2 Comparison of the Robust and Benchmark Solutions

For our simulation studies over the 17 Seattle ZIP codes, for each parameter set, we generate 1000 simulation trials. We simulate the customer arrivals to each of the ZIP codes during 12-hour time intervals, from 8am-8pm, as a function of the daily pattern of mean interarrival times.

3.5.2.1 Comparison of the Robust and Stochastic Solutions.

The plots in Figure 3.5 summarize the comparisons between the robust and stochastic optimal solutions for different levels of conservatism in the robust model, as a function of $\gamma^a = \gamma^s = \gamma^k = \gamma$. The goal of this section is to see how our robust analysis compares against an expected value analysis approach. Note that, since our stochastic model can not handle non-stationary random variables, we consider stationary arrivals, service times, delivery speeds, and crowd opportunity costs for the comparison. We set $v_H = v_L = 15$ mph and $\mu_H^k = \mu_L^k = \$18$, as supported by the average speed and average hourly wage of Uber drivers [59], and interarrival mean $\mu_H^a = \mu_L^a = 5$ minutes, where interarrival times of customer orders are exponentially distributed. The left-hand-side plot provides the percentage of on-time deliveries and the right-hand-side plot shows the cost savings percentage per day with respect to the discounted delivery costs that FedEx offers to retailers. The left-hand-side

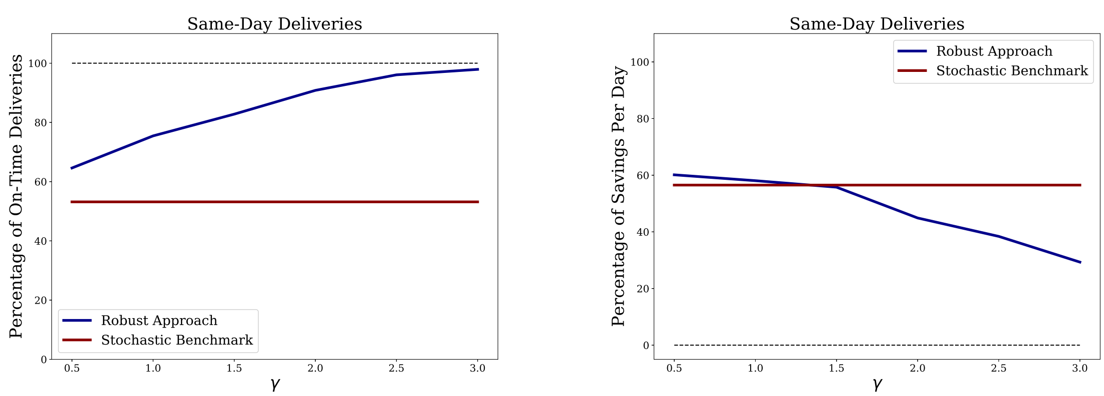

Figure 3.5: Comparison of the Performances of the Robust and Stochastic Optimal Solutions.

plot in Figure 3.5 shows that in the robust model, for $\gamma = 3$, almost all customers receive their same-day delivery orders on-time. Capturing 3 standard deviations, as motivated by the CLT, means capturing approximately 99% of variability, and that results in the on-time delivery of almost all orders. In contrast, under the stochastic optimal policy, approximately half of the customer orders were not delivered on-time. The right-hand-side plot shows that for $\gamma \leq 1$, the cost savings under the robust optimal policy are higher than that of the stochastic policy. However, as γ increases further, the robust model becomes conservative, which results in lower cost savings but higher on-time deliveries; in other words, the stochastic model can reduce costs more than the robust model, but at the expense of late deliveries. Another advantage of the robust model is that it gives the firm flexibility in addressing the trade-off between cost savings and responsiveness. Finally, the robust model is easily able to handle non-stationary uncertainties, characteristics that we found intractable to analyze in the stochastic model.

3.5.2.2 Comparison of the Robust Solution and the Heuristic Benchmark Algorithm.

The proposed robust model is a static model that uses a randomized allocation policy whereas the heuristic dynamic benchmark algorithm, a Modified Savings Algorithm, uses package characteristics (e.g., customer locations) to assign packages to the crowd or the 3PL provider. In this section, we evaluate how our robust model performs against the non-randomized heuristic benchmark. For this comparison, we use the baseline parameter values, where we assume customer orders arrive according to a Poisson process with rate λ. Table 3.2 shows the cost savings of the robust model and the non-randomized heuristic benchmark, respectively, as a function of α and λ. The baseline for these cost saving calculations is the discounted FedEx delivery cost. The results show that, despite being a simpler and faster strategy, the robust model results in higher cost savings for the firm.

According to Table 3.2, as the customer order arrival rate λ decreases, the percentage

α	$\lambda = 24$	$\lambda = 12$	$\lambda = 6$	$\lambda = 3$
Same-Day	(69.2,14.4)%	(64.7,13.7)%	(51.0,14.1)%	(0,0)%
4-Hour	(49.7,20.4)%	(35.9,11.4)%	(16.5,0.6)%	(0,0)%
2-Hour	(9.7,3.7)%	(0,0)%	(0,0)%	(0,0)%

Table 3.2: Percentage of Cost Savings Under the Robust Optimal Policy and the Non-Randomized Heuristic

of cost savings decreases until it becomes zero. The reason is that, as λ decreases, the interarrival time between orders increases, and, consequently, the utilization of crowd drivers decreases. This decreased utilization translates to decreased income for crowd drivers, which adversely impacts their participation. Thus, the firm should offer a higher hourly wage to crowd drivers to ensure their participation. As λ decreases further, utilizing crowd drivers becomes more expensive than outsourcing to 3PL companies. At that point, the firm should outsource all package deliveries to a 3PL firm and the cost savings from crowdsourcing becomes zero.

3.6 Conclusion

In this chapter, we analytically studied crowdsourcing last-mile deliveries, with guaranteed delivery time windows, under non-stationary uncertainties. We developed our optimization model by combining crowdsourcing, robust queuing, and robust routing theory. We analytically solved the proposed robust crowdsouring model and provided closed-form solutions, which allowed us to derive managerial insights on how the operations of crowdsourced last-mile delivery systems for on-demand orders should be designed. Our results show that crowdsourcing helps firms significantly decrease their delivery costs, while keeping the promise of on-time delivery to their customers for their on-demand orders. We validated the performance of the robust policy via a realistic simulation study over the 17 Seattle ZIP codes, that modeled a real transportation network and varying purchasing and traffic patterns. Finally, the robust model outperformed the two proposed benchmark models in terms of the firm's delivery cost savings as well as the percentage of on-time deliveries.

Chapter 4

CAPACITY FLEXIBILITY VIA ON-DEMAND WAREHOUSING

4.1 Introduction

Big e-commerce retailers such as Amazon are changing customer expectations [189]: They offer a variety of products and deliver goods as fast as same-day. Other retailers are forced to adapt themselves to provide similarly high service levels; however, not all of them have the resources to build a wide network of warehouses necessary to do so. Additionally, they may not have enough demand volumes to justify the network. While retailers can utilize 3PL warehouses, they do not give much flexibility and require commitments to a fixed structure contract over 1-3 years.

We study capacity decisions in the presence of on-demand warehousing platforms. This business model is also called the 'Airbnb for warehousing' [192], as no long-term lease or upfront fees are required to access on-demand capacity, and the retailer can obtain capacity as needed, enabling them to offer high service levels.

To the best of our knowledge, we are the first to study capacity flexibility via on-demand warehousing. This chapter is related to three streams of research, the sharing economy and on-demand platforms, capacity investments under uncertainty, and resource flexibility in operations management

4.2 Model

As in [199, 204, 91], we assume the unit price is determined via the (inverse) linear demand function $p(Q) = A - Q$, where A represents total market size or willingness to pay and Q is the production quantity. To capture uncertainty in the demand curve, we assume A is a non-negative continuous random variable with probability density distribution f_A, mean μ_A, and standard deviation σ_A. For simplicity and tractability, we assume the firm's production quantity Q is equal to its capacity investment.

We first analyze a benchmark model without on-demand warehousing in Section 4.2.1. Then we analyze the capacity investment decision with on-demand warehousing platforms in Section 4.2.2.

4.2.1 Benchmark

We first analyze the benchmark setting without on-demand warehousing platforms, which is similar to the analysis in [199]. The firm may invest in traditional warehousing capacity K_w for the unit cost of c_w. The firm's expected profit in the benchmark is given by

$$\begin{aligned} \pi_b &= \max_{K_w^b \geq 0} E_A\left(\left(p(Q) - c_w\right) K_w^b\right) \\ &= \max_{K_w^b \geq 0} E_A\left(\left(A - K_w^b - c_w\right) K_w^b\right). \end{aligned} \tag{4.1}$$

Theorem 4.1. *If $c_w \leq \mu_A$, then the firm should utilize traditional capacity ${K_w^b}^* = \dfrac{\mu_A - c_w}{2}$, which results in the optimal profit $\pi_b = \dfrac{(\mu_A - c_w)^2}{4}$. Otherwise, ${K_w^b}^* = 0$ and the optimal profit is $\pi_b = 0$.*

In the absence of on-demand warehousing, the firm's choice of traditional capacity is determined by the expected demand; variability in demand does not come into play. If the

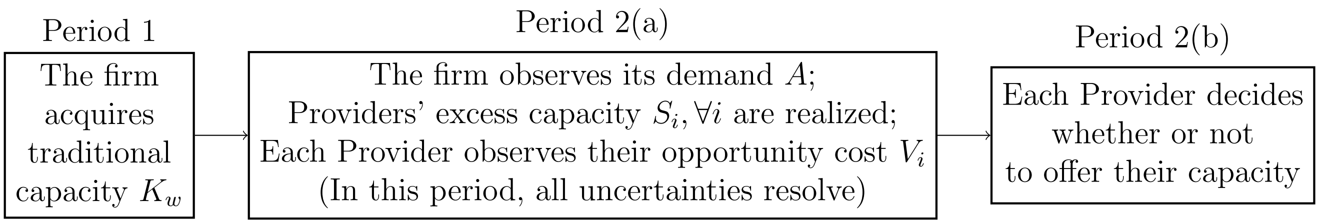

Figure 4.1: Problem Definition

expected demand is low or the traditional warehousing cost is high, then the firm should not invest in traditional warehousing capacity.

4.2.2 On-Demand Capacity Model

We now analyze the firm's capacity investment decision in the presence of on-demand warehousing. Figure 4.1 depicts the sequence of events. Starting from Period 1, the firm is committed to traditional capacity K_w. In Period 2(a), the firm observes its realized demand A and demands K_f on-demand capacity. We assume K_f is visible to providers and providers use this demand information to decide whether or not to offer their excess capacity. This assumption is similar to the current practice on Flexe.com where the platform informs providers about the firm's demand [106].

There is a pool of potential providers of size M, who can potentially participate and provide their excess capacity to the firm in the form of on-demand capacity in Period 2(b). In Period 1, provider i's excess capacity S_i is uncertain and is distributed according to probability density function f_S^i with mean μ_S^i and standard deviation σ_S^i. Provider i's excess capacity S_i is realized in Period 2(a) and the realized capacity is visible to both the firm and other providers, as motivated by the current practice of Flowspace (www.flow.space).

Opportunity costs of providers for unit excess capacity are drawn from the probability density function g and cumulative probability density function G. Provider i observes their opportunity cost V_i in Period 2(a). However, the firm does not observe V_i and only knows they are distributed according to the probability density function g. Additionally, provider i knows their own V_i but not $V_j,\ j \neq i$. Although, g is a common knowledge among providers.

In Period 2(a), after observing demand A, the firm may need K_f on-demand capacity, and for this K_f decision making, the firm should consider the on-demand capacity that it can get from participant providers in Period 2(b). Let the firm's expectation in Period 2(a) for the on-demand capacity provided by participant providers in Period 2(b) by $\bar{S}$. Next, we calculate $\bar{S}$:

In Period 2(a), for provider i with S_i excess capacity, the expected revenue from her participation and providing her excess capacity is given by $c_f \frac{S_i K_f}{\bar{S}}$, where c_f is the price for unit on-demand capacity. Note that the allocation $S_i/\bar{S}$ is similar to going on "allocation" in supply chain which is commonly used in practice [46]. Additionally note that here provider i has rational expectations and calculates her expected revenue based on the correct on-demand capacity that she expects to be provided by all participant providers, at equilibrium, in Period 2(b). A similar approach is used in [44] in studying service platforms with self-scheduling providers.

In Period 2(b), only those providers whose expected revenue from their unit excess capacity (which is given by $c_f \frac{K_f}{\bar{S}}$) is at least their opportunity cost participates; note that the ratio $\frac{K_f}{\bar{S}}$ is the utilization for each participant provider. Hence, in Period 2(a), the firm expects the on-demand capacity provided by participant providers (from the supply $\{1, \ldots, M\}$) in Period 2(b) to be

$$\bar{S} = \sum_{i=1}^{M} S_i G\left(c_f \frac{K_f}{\bar{S}}\right). \tag{4.2}$$

Equation (4.2) can be considered as the *on-demand capacity supply function.*

Note that Equation (4.2) is calculated in Period 2(a) where uncertainties in A as well as S_i are resolved. However, since provider i's opportunity cost V_i is private information of provider i, we need the c.d.f. G.

The demand to supply ratio of on-demand capacity is given by $\frac{K_f}{\bar{S}}$, which results in the following price for unit on-demand warehousing capacity:

$$c_f = b + \gamma \frac{K_f}{\bar{S}}, \tag{4.3}$$

where b can be interpreted as the minimum cost of providing one unit on-demand capacity. c_f captures surge pricing for on-demand warehousing, motivated by the findings of Flexe [189]; γ can be interpreted as the surge effect. Ride-sharing platforms such as Uber and Lyft similarly use surge pricing and [44] show that surge pricing is nearly optimal for service platforms with self-scheduling providers. The interpretation that utilization – represented by $\frac{K_f}{\bar{S}}$ – impacts the surge price is also observed in ride-sharing [201]. Using the Cournot competition framework [195], as $\bar{S} \to \infty$ the on-demand warehousing market becomes very competitive and price converges to marginal cost: $c_f \to b$. If the firm demands K_f, then the unit on-demand capacity is $c_f = b + \gamma \frac{K_f}{\bar{S}(K_f)}$; in the remainder of the analysis, we consider $b = 0$ for simplicity. If we input (4.3) in (4.2) we get

$$\bar{S} = \sum_{i=1}^{M} S_i G\left(\gamma \frac{K_f^2}{\bar{S}^2}\right). \tag{4.4}$$

Note that $c_f = \gamma$ is the maximum possible unit revenue that each provider can earn for every unit of her excess capacity. Hence, in our analysis we assume $G(\gamma) = 1$ for the supply of independent providers $\{1, \ldots, M\}$; because otherwise if $G(\gamma) < 1$ for a provider, then the provider would not participate even when all her excess capacity is used at the maximum possible unit price γ, and hence, this provider is not a potential provider as her participation constraint can never be satisfied and therefore we do not consider this provider in the supply $\{1, \ldots, M\}$.

4.2.2.1 Exogenous K_w.

In traditional warehousing, there is no flexibility for the firm and the firm is committed to its traditional capacity K_w. In this subsection, we show that even when the firm is constrained by a fixed warehousing capacity, it can still benefit from on-demand warehousing.

If the firm demands K_f on-demand capacity in Period 2(a), the firm can only acquire $\min\left\{K_f, \bar{S}(K_f)\right\}$ in Period 2(b), in which $\bar{S}(K_f)$ is the on-demand capacity provided by independent providers from the supply function (4.2). If the firm demands K_f, then the price of unit on-demand capacity is $c_f = \gamma\frac{K_f}{\bar{S}(K_f)}$. For the case where the firm's traditional capacity K_w is fixed and given, the firm's expected profit in the presence of on-demand warehousing is given by

$$\begin{aligned}\pi &= E_A\left[E_{(S_1,\ldots,S_M)}\left[\max_{K_f\geq 0}\left((p(Q)-c_w)K_w + (p(Q)-c_f(K_f))\min\left\{K_f,\bar{S}(K_f)\right\}\right)\right]\right]\\ &= E_A\left[E_{(S_1,\ldots,S_M)}\left[\max_{K_f\geq 0}\left(\left(A-K_w-\min\left\{K_f,\bar{S}(K_f)\right\}-c_w\right)K_w + \right.\right.\right.\\ &\qquad \left.\left.\left.\left(A-K_w-\min\left\{K_f,\bar{S}(K_f)\right\}-\gamma\frac{K_f}{\bar{S}(K_f)}\right)\min\left\{K_f,\bar{S}(K_f)\right\}\right]\right],\end{aligned} \tag{4.5}$$

where $\bar{S}(K_f)$ is given by (4.4).

Lemma 4.1. *Problem* (4.5) *can be simplified to*

$$\begin{aligned}\pi &= E_A\left[E_{(S_1,\ldots,S_M)}\left[\max_{0\leq K_f\leq\sum_{i=1}^M S_i}\left(\left(A-K_w-K_f-c_w\right)K_w + \right.\right.\right.\\ &\qquad \left.\left.\left.\left(A-K_w-K_f-\gamma\frac{K_f}{\bar{S}(K_f)}\right)K_f\right)\right]\right].\end{aligned} \tag{4.6}$$

Lemma 4.1 shows that if the platform uses surge pricing then during high demand realizations providers provide all their excess capacity to the firm due to higher prices from the surge.

Theorem 4.2. • *If $A > 2K_w$, then the firm should acquire on-demand capacity. The optimal on-demand warehouse capacity is given by*

$$K_f^* = \left\{ 0 \leq K_f \leq \sum_{i=1}^{M} S_i : A - 2K_w = 2K_f + \gamma \frac{K_f}{\bar{S}} \left(1 + \frac{1}{1 + \gamma \frac{2K_f^2}{\bar{S}^3} \sum_{i=1}^{M} S_i^2 g\left(\gamma S_i \frac{K_f^2}{\bar{S}^2}\right)} \right) \right\},$$

which is unique for $\dfrac{d^2 \bar{S}}{dK_f^2} \leq 0$ *in which* $\dfrac{d\bar{S}}{dK_f} = \dfrac{\gamma \frac{2K_f}{\bar{S}^2} \sum_{i=1}^{M} S_i g\left(\gamma \frac{K_f^2}{\bar{S}^2}\right)}{1 + \gamma \frac{2K_f^2}{\bar{S}^3} \sum_{i=1}^{M} S_i g\left(\gamma \frac{K_f^2}{\bar{S}^2}\right)}.$

- *Otherwise* $K_f^* = 0$.

Next, we analyze the results in Theorem 4.2 for the case where independent providers' opportunity costs are uniformly distributed: $V_i \sim \text{Uniform}(0, U)$. This is similar to the analysis by [118], for matching platforms, and [45], for pricing capacity, where they model individual valuations using uniform distributions. From equation (4.2), in which $G(c_f \frac{K_f}{\bar{S}}) = \frac{c_f \frac{K_f}{\bar{S}}}{U}$ for $c_f \frac{K_f}{\bar{S}} \leq U$, and, $G(c_f \frac{K_f}{\bar{S}}) = 1$ for $c_f \frac{K_f}{\bar{S}} > U$, we conclude

$$\bar{S} = \begin{cases} \dfrac{\sum_{i=1}^{M} S_i c_f \dfrac{K_f}{\bar{S}}}{U}, & \text{if } c_f \dfrac{K_f}{\bar{S}} \leq U \\ \sum_{i=1}^{M} S_i, & \text{Otherwise.} \end{cases} \tag{4.7}$$

Inputting c_f from (4.3) in (4.7) and solving for $\bar{S}(K_f)$, we get the supply function

$$\bar{S}(K_f) = \begin{cases} \left(\dfrac{\gamma K_f^2}{U} \sum_{i=1}^{M} S_i \right)^{1/3}, & \text{if } K_f \leq \sqrt{\dfrac{U}{\gamma} \sum_{i=1}^{M} S_i} \\ \sum_{i=1}^{M} S_i, & \text{Otherwise.} \end{cases} \tag{4.8}$$

Theorem 4.3. *The optimal on-demand capacity* K_f^* *is given by*

- *if* $0 < A \leq 2K_w$, *then* $K_f^* = 0$;
- *if* $2K_w < A \leq \hat{A}(K_w)$, *then* $K_f^* = \tilde{K}_f(K_w)$;
- *if* $\hat{A}(K_w) < A \leq 2K_w + 2(1 + \frac{\gamma}{\sum_{i=1}^M S_i})\sqrt{\frac{U}{\gamma}\sum_{i=1}^M S_i}$, *then* $K_f^* = \sqrt{\frac{U}{\gamma}\sum_{i=1}^M S_i}$;
- *if* $2K_w + 2(1 + \frac{\gamma}{\sum_{i=1}^M S_i})\sqrt{\frac{U}{\gamma}\sum_{i=1}^M S_i} < A \leq 2K_w + 2(1 + \frac{\gamma}{\sum_{i=1}^M S_i})\sum_{i=1}^M S_i$, *then* $K_f^* = \dfrac{A - 2K_w}{2(1 + \frac{\gamma}{\sum_{i=1}^M S_i})}$;
- *otherwise if* $A > 2K_w + 2(1 + \frac{\gamma}{\sum_{i=1}^M S_i})\sum_{i=1}^M S_i$, *then* $K_f^* = \sum_{i=1}^M S_i$;

where $\tilde{K}_f(K_w) = \Big(\big(\frac{A-2K_w}{4} + \sqrt{\frac{(A-2K_w)^2}{16} + \frac{8\left(\frac{\gamma^2 U}{\sum_{i=1}^M S_i}\right)}{27^2}}\big)^{1/3} + \big(\frac{A-2K_w}{4} - \sqrt{\frac{(A-2K_w)^2}{16} + \frac{8\left(\frac{\gamma^2 U}{\sum_{i=1}^M S_i}\right)}{27^2}}\big)^{1/3}\Big)^3$, *and* $\hat{A}$ *is the unique solution to the equation* $\tilde{K}_f(K_w) = \sqrt{\frac{U}{\gamma}\sum_{i=1}^M S_i}$.

Theorem 4.3 shows that only if the realization of demand A is high, does the firm choose to acquire on-demand warehousing to complement its primary capacity K_w. Therefore, a hybrid strategy where the firm operates both traditional and on-demand warehousing capacity may be optimal. This finding is consistent with practice where established big firms utilize on-demand capacity as a "filler" for managing variability [189]. Our analysis can help firms decide what capacity mix should be chosen in order to benefit from the flexibility provided via on-demand warehousing.

Figure 4.2(a) shows the firm's profit under the benchmark and on-demand warehousing option, as a function of the firm's traditional capacity K_w. The results show that firms with either small or large levels of traditional capacity investment K_w can benefit from on-demand warehousing, although the benefit is higher for firms with moderate levels of traditional capacity. Figure 4.2(b) shows that firms with lower K_w capacity should acquire more on-demand capacity to be able to not only absorb variability but to also expand their capacity.

The results in Figure 4.2(b) also show that in the presence of on-demand warehousing firms with relatively higher values of K_w end up with higher capacity investments as they still need to invest in on-demand warehousing to absorb fluctuations in their demand.

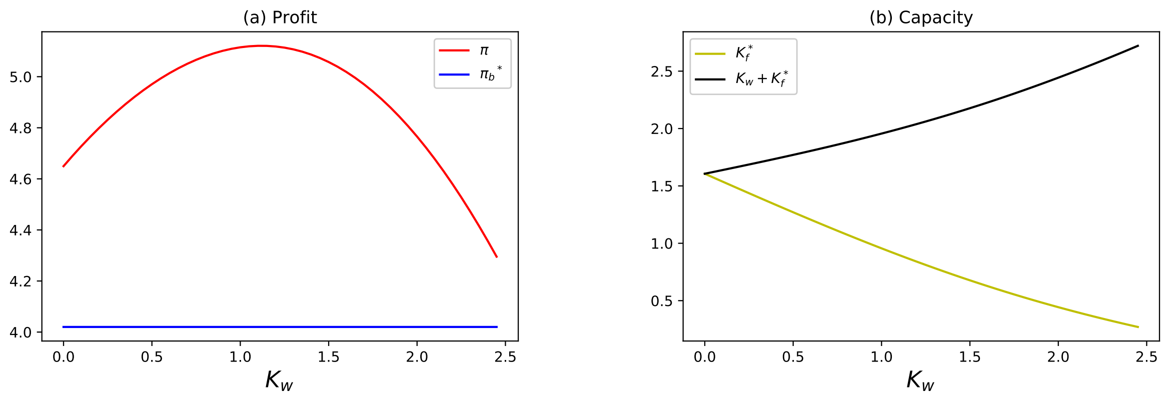

Figure 4.2: Comparison of the performance under the benchmark and general on-demand models for $A \sim \text{Normal}(6.5, 6.5/3)$ and $V_i \sim \text{Uniform}(0, 1)$, $S_i \sim \text{Normal}(0.5, 0.15)$, $i = 1, \ldots, M$, $\gamma = 10$, $M = 20$, and $c_w = 2.49$.

4.2.2.2 Endogenous K_w.

In this subsection, we analyze a case where the firm is able to choose traditional warehousing capacity K_w in Period 1.

$$\begin{aligned}\pi \quad = \quad \max_{K_w \geq 0} E_A \Bigg[E_{(S_1,\ldots,S_M)} \Bigg[\max_{0 \leq K_f \leq \sum_{i=1}^M S_i} \Bigg(& (A - K_w - K_f - c_w) K_w + \\ & \left(A - K_w - K_f - \gamma \frac{K_f}{\bar{S}(K_f)} \right) K_f \Bigg) \Bigg] \Bigg]. \end{aligned} \tag{4.9}$$

All the analysis in this subsection are presented for the case where S_i, $i = 1, \ldots, M$ are distributed according to probability density function f_S with mean μ_S and standard deviation σ_S, and, independent providers' opportunity costs are uniformly distributed: $V_i \sim \text{Uniform}(0, U),\ i = 1, \ldots, M$.

Theorem 4.4. *The optimal traditional warehouse capacity K_w^*, which is chosen in Period 1, is given by:*

- *if* $c_w < \mu_A - 2\int_{S=0}^{\infty}\left(\int_{A=0}^{\hat{A}(K_w=0)} \tilde{K}_f(K_w = 0) f_A dA + M\sqrt{\frac{U}{\gamma}S}\int_{\hat{A}(K_w=0)}^{2(1+\frac{\gamma}{MS})\sqrt{\frac{U}{\gamma}}MS} f_A dA + \int_{A=2(1+\frac{\gamma}{MS})\sqrt{\frac{U}{\gamma}}MS}^{2(1+\frac{\gamma}{MS})MS} \frac{A}{2(1+\frac{\gamma}{MS})} f_A dA + MS\int_{A=2(1+\frac{\gamma}{MS})MS}^{\infty} f_A dA\right) f_S dS$, *then the firm should utilize traditional capacity $K_w^* > 0$, which is the unique solution to the equation*

$$\frac{\mu_A - c_w}{2} - \int_{S=0}^{\infty}\Big(\int_{A=2K_w}^{\hat{A}} \tilde{K}_f f_A dA + M\sqrt{\frac{U}{\gamma}S}\int_{\hat{A}}^{2K_w+2(1+\frac{\gamma}{MS})\sqrt{\frac{U}{\gamma}}MS} f_A dA$$
$$+ \int_{A=2K_w+2(1+\frac{\gamma}{MS})\sqrt{\frac{U}{\gamma}}MS}^{2K_w+2(1+\frac{\gamma}{MS})MS} \frac{A - 2K_w}{2(1+\frac{\gamma}{\sum_{i=1}^{M} S_i})} f_A dA + MS\int_{A=2K_w+2(1+\frac{\gamma}{MS})MS}^{\infty} f_A dA\Big) f_S dS = K_w.$$

- *Otherwise,* $K_w^* = 0$.

Theorem 4.4(i) shows that if the unit cost of traditional capacity c_w is low, then the firm should acquire traditional capacity. Comparing with Theorem 4.1, it is easy to see that this threshold is smaller than under the benchmark. This indicates that even firms with cheaper traditional capacity cost c_w can benefit from on-demand capacity.

The following proposition compares the capacity choices in the presence and absence of on-demand warehousing.

Proposition 4.1.

1. *Traditional warehouse capacity in the presence of on-demand warehousing is at most that under the benchmark. In particular,*

- *If* $c_w < \mu_A$*, then* $0 \leq K_w^* \leq {K_w^b}^*$*.*
- *Otherwise,* $K_w^* = {K_w^b}^* = 0$*.*

2. *Total warehouse capacity in the presence of on-demand warehousing depends on the realization of* A*. In particular, there exists* $\hat{A} \geq 0$ *such that*

- $K_f^* + K_w^* < {K_w^b}^*$ *for* $A < \hat{A}$*,*
- *and* $K_f^* + K_w^* \geq {K_w^b}^*$ *for* $A \geq \hat{A}$*.*

Recall that the firm commits to traditional capacity before the selling season. Thus, the firm invests in lower traditional capacity, interestingly before demand A and on-demand capacity supply $\bar{S}$ are realized. However, depending on demand realization A, the total capacity investment can be smaller or larger in the presence of on-demand warehousing. This is the capacity flexibility effect and is illustrated in Figure 4.3, where the dotted line represents $\hat{A}$. For demand realizations smaller than $\hat{A}$, total capacity $K_w^* + K_f^*$ acquired in the presence of on-demand warehousing is smaller than ${K_w^b}^*$ under the benchmark, and for demand realizations greater than $\hat{A}$, total capacity $K_w^* + K_f^*$ is higher than ${K_w^b}^*$. However, interestingly, the following corollary shows that the expected values of ${K_w^b}^*$ and $K_w^* + K_f^*$ are equal.

Corollary 4.1. *In the presence of on-demand warehousing, if the firm invests in traditional capacity (i.e.* $K_w^* \neq 0$*), then at optimality the inequality* $K_w^* + E_A\left(K_f^*\right) = {K_w^b}^*$ *always holds; indicating that the firm should rely on-demand warehousing to only absorb variability in demand.*

Corollary 4.1 shows on-demand warehousing only provides flexibility to the firm to cope with its demand uncertainty. Figure 4.4(b) depicts this as a function of demand variability.

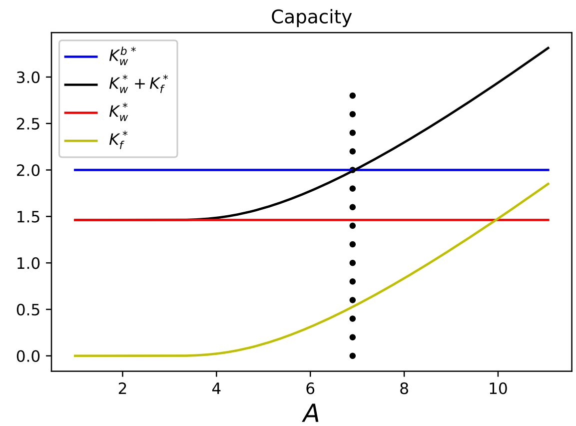

Figure 4.3: Comparison of the total capacity under the benchmark and the on-demand model for $A \sim \text{Normal}(\mu_A = 6.49, \mu_A/3)$, $V_i \sim \text{Uniform}(0, 2)$, $S_i = 0.5$, $i = 1, \ldots, M$, $M = 20$, $\gamma = 10$, and $c_w = 2.49$, over 200 simulation trials.

Corollary 4.2. *The firm's optimal profit in the presence of on-demand warehousing is at least that under the benchmark:* $\pi_b \leq \pi$.

According to Corollary 4.2, the firm always benefits from the presence of on-demand warehousing, if it is utilized optimally. We next conduct a numerical study.

Figure 4.4(a) shows as demand variability increases, the firm's benefit from on-demand model increases. As σ_A increases, the probability of large demand realizations increases, and as K_f^* can adapt to demand variability and attain profits from higher demand realizations, this results in higher profits for the firm in the on-demand warehousing model; this finding is similar to the results in [199]. As σ_A increases, the probability of small demand realizations increases as well but according to Theorem 4.3, the firm optimally chooses not to utilize on-demand capacity. Effectively, the firm's benefit from adjusting its on-demand capacity based on the realized demand is amplified when variability is larger.

Figure 4.4(b) shows the capacity decisions of the firm, as a function of demand variability. As demand variability increases, the firm utilizes less traditional capacity K_w^* to better

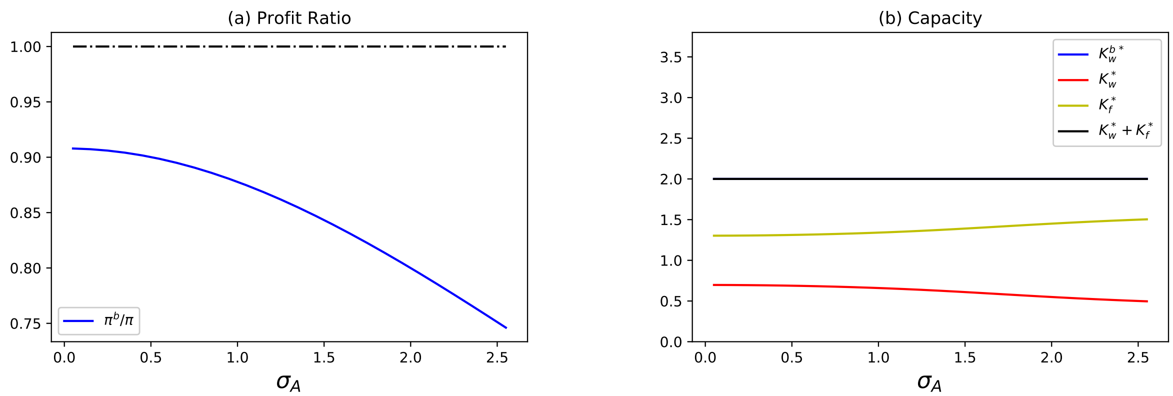

Figure 4.4: Comparison of the performance under the benchmark and the on-demand model for $A \sim \text{Normal}(\mu_A = 6.49, \sigma_A)$, $V_i \sim \text{Uniform}(0, 1)$, $S_i = 0.5$, $i = 1, \ldots, M$, $\gamma = 10$, $M = 20$, and $c_w = 2.49$.

cope with demand uncertainty. The probability of larger demand realizations increases with demand variability and this results in an increase in on-demand capacity K_f^*. In contrast, under the benchmark, the firm can not respond to variability and it is committed to early capacity investments based on its demand expectations. As Figure 4.4(b) shows, the expected capacity in the presence of on-demand warehousing is equal to that under the benchmark, and the presence of on-demand capacity allows the firm to shift its capacity from earlier in the season to later in the selling season when uncertainty in demand is resolved; this signifies that the benefit of on-demand warehousing is due to its flexibility.

Figure 4.5 shows the firm's optimal capacity decisions as a function of expected demand μ_A. The results show that firms, with either small or large demand expectations, can benefit significantly from the flexibility of on-demand warehousing. Interestingly, the firm's optimal on-demand capacity does not change with the expected demand, but according to Figure 4.4(b), it does change with σ_A. This is interesting as it shows that on-demand warehousing should be acquired based on variabilities in demand.

Figure 4.6(a) shows that the firm's profit decreases as U increases. This is because

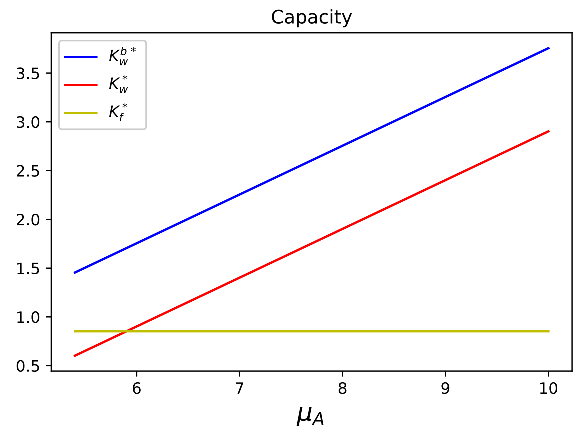

Figure 4.5: Comparison of the performance under the benchmark and the on-demand model for $A \sim \text{Normal}(\mu_A, 2)$, $V_i \sim \text{Uniform}(0,1)$, $S_i = 0.5$, $i = 1, \ldots, M$, $\gamma = 10$, $M = 20$, and $c_w = 2.49$.

any increase in U means that acquiring on-demand capacity becomes more expensive, and hence, according to Figure 4.6(b), the firm should invest more in traditional capacity K_w and utilize less on-demand capacity. This results in the firm not being able to respond to demand uncertainties and not being able to benefit from high demand realizations and therefore lower profits for the firm.

The plots in Figure 4.7 show that as uncertainty in providers' excess capacity increases, the firm should invest more in traditional capacity K_w and rely less on on-demand capacity K_f. This results in the firm's not being able to adapt to demand uncertainty and therefore decrease in the firm's profits.

4.3 Conclusion

In this chapter, we analyze how the presence of on-demand warehousing can impact firms' capacity investment decisions. We find that firms can benefit significantly from capacity flexibility via on-demand warehousing, and more so when they experience higher demand

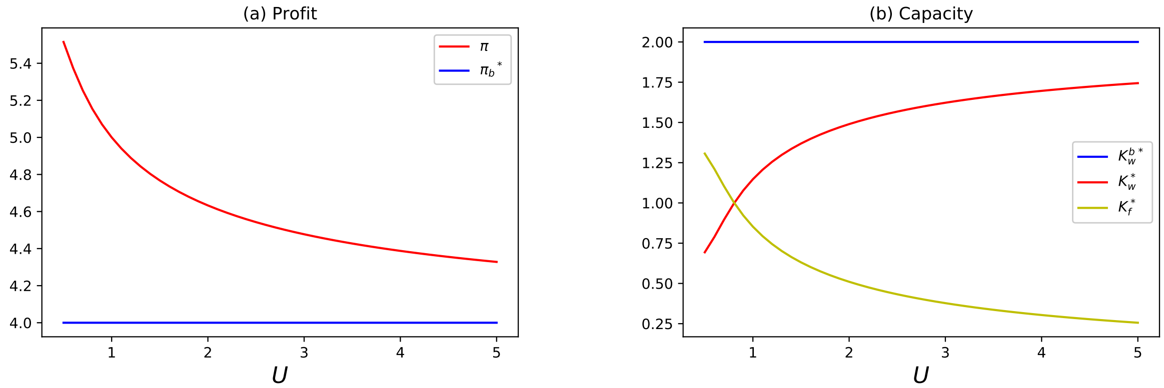

Figure 4.6: Comparison of the performance under the benchmark and the on-demand model for $A \sim \text{Normal}(6.49, 2)$, $V_i \sim \text{Uniform}(0, U)$, $S_i = 0.5,\ i = 1, \ldots, M$, $\gamma = 10$, $M = 20$, and $c_w = 2.49$.

variability. Our results indicate that established firms can utilize on-demand warehousing to complement their traditional capacity, which is consistent with practice; Walmart currently uses on-demand capacity along with its warehousing networks [98].

Our results demonstrate that start-ups with low traditional capacity investments can also benefit from on-demand warehousing and rely on on-demand warehousing to expand their capacity after observing their demand. On-demand warehousing allows start-ups to absorb their demand variability and utilize capacity based on their realized demand, rather than committing to traditional capacity based on demand forecasts. This may explain why Cargo, an e-commerce start-up, uses on-demand warehousing capacity exclusively [98].

On-demand warehousing gives structural flexibility to firms and allows them to cope with their demand variability better and compete with big e-commerce retailers such as Amazon.

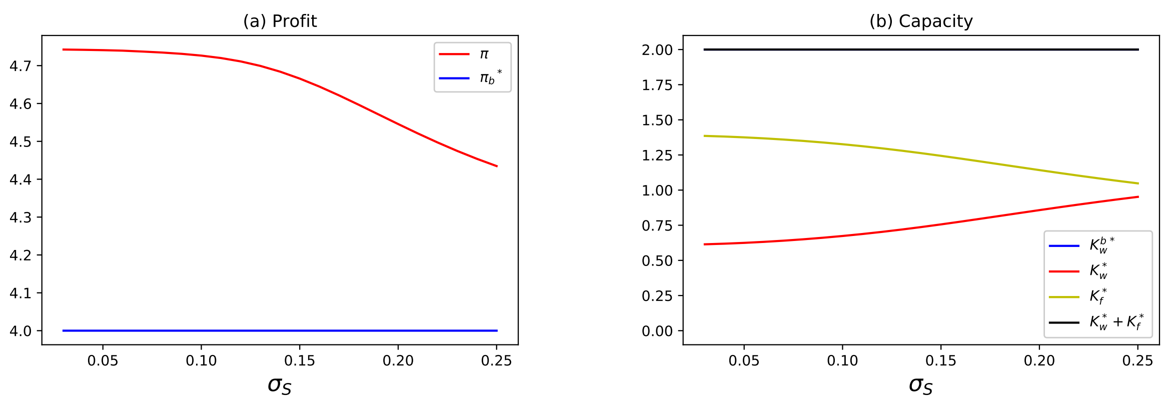

Figure 4.7: Comparison of the performance under the benchmark and the on-demand model for $S \sim \text{Normal}(0.5, \sigma_S)$, $A \sim \text{Normal}(6.49, 2)$, $V_i \sim \text{Uniform}(0, 1)$, $i = 1, \ldots, M$, $\gamma = 10$, $M = 20$, and $c_w = 2.49$.

Chapter 5

CONCLUSION AND FUTURE WORK

In this chapter, we provide a summary of this thesis. We also suggest some extensions for the research conducted in this thesis.

In Chapter 1, we provide an overview of the sharing economy and on-demand service platforms and present a literature review of the papers relevant to this research stream.

In Chapter 2, we analyze a new model of crowdfunding recently introduced by Bolstr, Localstake, and Startwise. A platform acts as a matchmaker between a firm needing funds and a crowd of investors willing to provide capital. Once the firm is funded, it pays back the investors using revenue sharing contracts, with a pre-specified investment multiple (investors will receive $M \geq 1$ dollars for every dollar invested) and a revenue-sharing proportion, over an investment horizon of uncertain duration. We analyze the revenue-sharing contract approach to crowdfunding, and we assist the firm to determine its optimal contract parameters to maximize its expected net present value, subject to investor participation constraints and platform fees. A natural multi-period formulation for the firm's problem results in an intractable stochastic optimization model, which we approximate using a deterministic model. In the approximation model, we use a cash buffer for dealing with cash-flow uncertainties; we are able to solve the approximation model analytically. Parametrized on real data from Bolstr campaigns, our approximation solutions give a NPV, in the stochastic problem, that is within 0.2% of the simulation-based optimal NPV, for all levels of cash-flow uncertainty. We compare revenue-sharing contracts with equity crowdfunding and observe that the former result in higher NPVs and comparable bankruptcy probabilities. We also compare revenue-sharing contracts with fixed-rate loans, and find that, for most cases considered, revenue-sharing contracts provide a higher NPV and a lower probability of bankruptcy than a fixed-rate loan. We also show that these benefits are more significant for firms with higher levels of cash-flow uncertainty. Revenue-sharing contracts are a novel approach to crowdfunding, which we show are superior to other financing models.

In Chapter 3, we propose a novel robust crowdsourcing optimization model to study labor planning and pricing for crowdsourced last-mile delivery systems that are utilized for satisfying on-demand orders with guaranteed delivery time windows. We develop our model by combining crowdsourcing, robust queueing, and robust routing theories. We show the value of the robust optimization approach by analytically studying how to provide fast and guaranteed delivery services utilizing independent crowd drivers under uncertainties in customer demands, crowd availability, service times, and traffic patterns; we also allow for trend and seasonality in these uncertainties. For a given delivery time window and an on-time delivery guarantee level, our model allows us to analytically derive the optimal delivery assignments to available independent crowd drivers and their optimal hourly wage. We evaluate the performance of our robust model against a stochastic counterpart and a heuristic benchmark algorithm, via a realistic simulation study based on the Seattle transportation network that allows for non-stationary customer purchasing patterns, heterogeneous crowd drivers, as well as time-varying traffic patterns. We show that our robust solution performs significantly better than the two benchmarks, both in terms of the percentage of on-time deliveries and cost savings. Our results show that crowdsourcing can help firms decrease their delivery costs significantly, while keeping the promise of on-time delivery to their customers.

In Chapter 4, we study the impact of capacity flexibility, facilitated by on-demand warehousing, on supply chain management. In particular, we study firms' capacity investment decisions in the presence of on-demand warehousing. Our findings show that as firms' demand variability increases, the firms' benefit from on-demand warehousing increases. Our results indicate that firms should mainly rely on on-demand warehousing to absorb their uncertainties. We further show that as uncertainty in providers' excess capacity decreases, firms' benefit from on-demand warehousing increases.

There are a number of directions that the research in this thesis can be extended. In particular,

- In Chapter 2, a comparison between crowdfunding revenue-sharing contracts and crowdfunding profit-sharing contracts, and, comparing firms' expected NPV and crowd investors' rate of return under the two contracts would be interesting.

- An interesting research question in the area of the sharing economy and on-demand services is studying the impact of these platforms on customers' demand. For example, retailers can use the crowd delivery model studied in Chapter 3 to offer on-demand delivery services to their customers. These fast delivery services can allow firms to learn demands for customers who would not order in the absence of on-demand delivery. This results in firms learning their real demand and make more optimal pricing, inventory, assortment, and logistical decisions.

- In Chapter 4, an interesting extension is studying the impact of on-demand warehousing on logistical costs as well as delivery service levels that retailers can offer to their customers. This allows retailers to transform their supply chain into a decentralized supply chain that could enable retailers to offer fast and free delivery services to their customers which in turn could increase retailers' competitiveness with big e-commerce retailers such as Amazon.

- Another interesting research question to explore as an extension of Chapter 4 is studying the role of on-demand warehousing on supply chain disruption created by natural disasters or pandemics (e.g., COVID-19 pandemic) and studying how this business model could allow firms to cope with sudden disruptions in their supply chain.

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Appendix

Proof. **Proposition 2.1.** Using $\gamma = \frac{MY}{\sum_{t=1}^{T} R_t}$, we first substitute out γ. We then solve the problem in stages. We first fix Y and T, and solve for the optimal $M^*(Y,T)$; Problem (2.11) simplifies to

$$
\begin{aligned}
\hat{z}_F(Y,T) = \quad \max_{M} \quad & \left(1-(\beta+1)\frac{MY}{\sum_{t=1}^{T} R_t}\right)\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t} + \sum_{t=T+1}^{\infty}\frac{R_t}{(1+r_t)^t} - \sum_{t=1}^{\infty}\frac{C_t}{(1+r_t)^t} \\
& +(1-\alpha)Y \\
\text{s.t.} \quad & \left(\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta+(1-\alpha)Y}{\sum_{t=1}^{\min\{\tau,T\}} R_t}\right)\left(\frac{\sum_{t=1}^{T} R_t}{(\beta+1)Y}\right) \geq M, \quad \tau \geq 1 \\
& M \geq (\hat{A}+1)\frac{\sum_{t=1}^{T} R_t}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}};
\end{aligned} \tag{1}
$$

note that $(\hat{A}+1)\frac{\sum_{t=1}^{T} R_t}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} > 1$, which implies $M \geq 1$ is redundant. Problem (1), if feasible, has the solution $M^*(Y,T) = (\hat{A}+1)\frac{\sum_{t=1}^{T} R_t}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}}$. We next solve for the optimal $Y^*(T)$, where the objective is obtained by inserting the optimal $M^*(Y,T)$ into the objective of Problem (1); the constraints are derived by 1) making sure the lower bound on M is smaller than the upper bound on M, making Problem (1) feasible, and 2) inserting the optimal $M^*(Y,T)$ into the resulting constraints.

Using Equation (2.13), the constraints become $Z_\tau(T)Y \leq \sum_{t=1}^{\tau}(R_t-C_t)-\theta$, for $\tau \geq 1$. If $\tau \in X$ (i.e., $\sum_{t=1}^{\tau}(R_t-C_t) < \theta$), then, since $Y \geq 0$, for the feasibility, we require $Z_\tau(T) < 0$. If $\tau \notin X$ (i.e., $\sum_{t=1}^{\tau}(R_t-C_t) \geq \theta$), then we consider two cases: 1) if $Z_\tau(T) \leq 0$, the constraint $Z_\tau(T)Y \leq \sum_{t=1}^{\tau}(R_t-C_t)-\theta$ is trivially true; 2) if $Z_\tau(T) > 0$, then $Y \leq \frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}$. Thus, the problem can be written as

$$\begin{aligned}
\hat{z}_F(T) = \quad & \max_{Y\geq 0} \quad \sum_{t=1}^{\infty}\frac{R_t - C_t}{(1+r_t)^t} - \left((\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha)\right)Y \\
& \text{s.t.} \quad Y \geq \max_{\tau\in X}\left\{\frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)}\right\} \\
& \qquad\; Y \leq \min_{\substack{\tau\notin X \\ Z_\tau(T)>0}}\left\{\frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)}\right\}.
\end{aligned} \tag{2}$$

Problem (2) is feasible if and only if $Z_\tau(T) < 0$ for all $\tau \in X$ and $\max_{\tau\in X}\left\{\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}\right\} \leq \min_{\substack{\tau\notin X \\ Z_\tau(T)>0}}\left\{\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}\right\}$; If $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) \geq 0$, then the objective function is a decreasing function of Y and if feasible, the solution is $Y^*(T) = \max_{\tau\in X}\left\{\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}\right\}$ if $X \neq \emptyset$ and $Y^*(T) = 0$ if $X = \emptyset$. Otherwise if $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) < 0$, then the objective function is an increasing function of Y and if feasible, the solution is $Y^*(T) = \min_{\substack{\tau\notin X \\ Z_\tau(T)>0}}\left\{\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}\right\}$. $\square$

Proof. **Lemma 2.1.** For $T > \max\{\tau : \tau \in X\}$, $Z_\tau(T)$ equals $(\hat{A}+1)(\beta+1)\frac{\sum_{t=1}^{\tau}R_t}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha)$, which is strictly decreasing in T. The lemma's condition $(\hat{A}+1)(\beta+1)\frac{\sum_{t=1}^{\tau}R_t}{R^\infty} - (1-\alpha) < 0$ implies that there exists T_X where, for $T > T_X$, $Z_\tau(T) < 0$ for all $\tau \in X$. $\square$

Proof. **Lemma 2.2.** We first show that $T > \tau$, for all $\tau \in X$. For a contradiction, suppose that $\exists \tau_0 \in X$ such that $T \leq \tau_0$. The cash-flow constraint of Problem (2.11) for $\tau = \tau_0$ is $\sum_{t=1}^{\tau_0}(R_t - C_t) + (1-\alpha)Y - (\beta+1)\gamma\sum_{t=1}^{T}R_t \geq \theta$. Using the contractual obligation constraint of Problem (2.11), $\gamma\sum_{t=1}^{T}R_t = MY$, this inequality implies that $MY \leq \frac{\sum_{t=1}^{\tau_0}(R_t-C_t)-\theta+(1-\alpha)Y}{(\beta+1)} \underbrace{<}_{\text{since } \tau_0\in X} \frac{(1-\alpha)}{(\beta+1)}Y \underbrace{\leq}_{\text{since } \alpha,\beta\in[0,1]} Y$, which contradicts the constraint $M \geq 1$. Thus, $T > \tau$ for all $\tau \in X$.

Noting that, for $\tau \in X$, $\sum_{t=1}^{\tau}(R_t - C_t) - \theta < 0$ and $Z_\tau(T) < 0$, we expand the expression

$\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}$ using positive expressions:

$$\begin{aligned}\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)} &= \frac{\sum_{t=1}^{\tau}(C_t-R_t)+\theta}{(1-\alpha)-(\hat{A}+1)(\beta+1)\frac{\sum_{t=1}^{\min\{\tau,T\}}R_t}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}}} \\ &= \frac{\sum_{t=1}^{\tau}(C_t-R_t)+\theta}{(1-\alpha)-(\hat{A}+1)(\beta+1)\frac{\sum_{t=1}^{\tau}R_t}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}}}, \tau \in X,\end{aligned}$$

where the last equality is due to $T > \tau$, for all $\tau \in X$. Consequently, as T is increased, $\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}$ strictly decreases since $R_t > 0$. Since all its arguments are strictly decreasing in T, so is the maximization problem $\max\limits_{\tau\in X}\left\{\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}\right\}$. □

Proof. **Lemma 2.3.** For any $\tau \notin X$ and $T \geq \tau$, the minimization argument $\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)} = \frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{(\hat{A}+1)(\beta+1)\frac{\sum_{t=1}^{\tau}R_t}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}}-(1-\alpha)}$ is strictly increasing in T. Note that if $Z_\tau(T)$ drops to or below zero, that argument is eliminated from consideration; this is only possible if $(\hat{A}+1)(\beta+1)\frac{\sum_{t=1}^{\tau}R_t}{R^\infty} < (1-\alpha)$ (this condition, in Lemma 2.1, is for $\tau \in X$). Thus, for large enough T, each minimization argument is strictly increasing in T, and we may conclude that $\min\limits_{\substack{\tau\notin X\\ Z_\tau(T)>0}}\left\{\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}\right\}$ is also strictly increasing in T. □

Proof. **Lemma 2.4.** First we prove the lemma for $r_t = \delta_t, \forall t$: According to Proposition 2.1, the objective function of Problem 2.14 is simplified to

$$\hat{z}_F(T) = \max_{Y\geq 0}\sum_{t=1}^{\infty}\frac{R_t-C_t}{(1+r_t)^t} - \Big((\beta+1)(\hat{A}+1)-(1-\alpha)\Big)\max_{\tau\in X}\left\{\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}\right\}.$$

As proved in Lemma 2.2, $\max_{\tau\in X}\left\{\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}\right\}$ is decreasing in T and so the objective function is increasing in T, which completes the proof for $r_t = \delta_t, \forall t$.

We next prove the lemma for $r_t = r \geq \delta = \delta_t, \forall t$ for the cases where 1) $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta)^t}} - (1-\alpha) \geq 0$ and 2) $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta)^t}} - (1-\alpha) < 0$ separately. First consider case 1: as shown in Proposition 2.1, the objective function under this case is simplified to $\max_{Y\geq 0}\sum_{t=1}^{\infty}\frac{R_t-C_t}{(1+r)^t}-\left((\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta)^t}}-(1-\alpha)\right)\max_{\tau\in X}\left\{\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}\right\}$. The derivative of the objective function with respect to T is

$$\begin{aligned}\frac{d\hat{z}_F(T)}{dT} = & -\left((\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta)^t}}-(1-\alpha)\right)\frac{d\left(\max_{\tau\in X}\left\{\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}\right\}\right)}{dT} \\ & -\frac{d\left((\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta)^t}}-(1-\alpha)\right)}{dT}\max_{\tau\in X}\left\{\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}\right\}. \quad (3)\end{aligned}$$

If we show $\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta)^t}}$ is non-increasing in T, then since $\max_{\tau\in X}\left\{\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}\right\}$ is positive and decreasing in T and $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta)^t}}-(1-\alpha)\geq 0$, we conclude the objective function is increasing in T.

To show $\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta)^t}}$ is non-increasing in T, we calculate differences $\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta)^t}}-\frac{\sum_{t=1}^{T-1}\frac{R_t}{(1+r)^t}}{\sum_{t=1}^{T-1}\frac{R_t}{(1+\delta)^t}}=\frac{\frac{R_T}{(1+r)^T}\sum_{t=1}^{T-1}\frac{R_t}{(1+\delta)^t}\left(1-\frac{\sum_{t=1}^{T-1}R_t(1+r)^{T-t}}{\sum_{t=1}^{T-1}R_t(1+\delta)^{T-t}}\right)}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta)^t}\sum_{t=1}^{T-1}\frac{R_t}{(1+\delta)^t}}\leq 0$ and the last inequality holds due to $r\geq\delta$, which completes the proof for case 1. Now consider case 2: as shown in Proposition 2.1, the objective function under this case is simplified to

$$\max_{Y\geq 0}\sum_{t=1}^{\infty}\frac{R_t-C_t}{(1+r)^t}-\left((\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta)^t}}-(1-\alpha)\right)\min_{\substack{\tau\notin X\\ Z_\tau(T)>0}}\left\{\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}\right\}.$$

The derivative of the objective function with respect to T is

$$\begin{aligned}\frac{d\hat{z}_F(T)}{dT} = & -\left((\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta)^t}}-(1-\alpha)\right)\frac{d\left(\min_{\substack{\tau\notin X\\ Z_\tau(T)>0}}\left\{\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}\right\}\right)}{dT} \\ & -\frac{d\left((\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta)^t}}-(1-\alpha)\right)}{dT}\min_{\substack{\tau\notin X\\ Z_\tau(T)>0}}\left\{\frac{\sum_{t=1}^{\tau}(R_t-C_t)-\theta}{Z_\tau(T)}\right\}. \quad (4)\end{aligned}$$

As $\min_{\substack{\tau \notin X \\ Z_\tau(T)>0}} \left\{ \frac{\sum_{t=1}^{\tau}(R_t - C_t) - \theta}{Z_\tau(T)} \right\}$ is positive and increasing in T and $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T} \frac{R_t}{(1+r)^t}}{\sum_{t=1}^{T} \frac{R_t}{(1+\delta)^t}} - (1-\alpha)$ is negative and non-increasing in T, as proved above, then we conclude the objective function is increasing in T, which completes the proof for case 2. □

Proof. **Lemma 2.5.** If i) $b < d$ or ii) $b = d$ and $a < c$, then $\sum_{t=1}^{\tau}(R_t - C_t) - \theta = (a-c)\tau + (b-d)\frac{\tau(\tau+1)}{2} - \theta$ is unbounded from below, implying that, in either case, $\exists \tau_0$ such that $\tau \in X$ for all $\tau > \tau_0$. For any finite $T \in \mathbb{N}$, pick a $\tau \in X$ with $\tau > T$; this implies $Z_\tau(T) = (\hat{A}+1)(\beta+1)\frac{\sum_{t=1}^{T}(a+bt)}{\sum_{t=1}^{T} \frac{a+bt}{(1+\delta_t)^t}} - (1-\alpha)$. Since T was chosen arbitrarily, we can select it so that $Z_\tau(T) \geq 0$, since $\sum_{t=1}^{T}(a+bt)$ is quadratic in T and $\sum_{t=1}^{T} \frac{a+bt}{(1+\delta_t)^t}$ grows sub-quadratically. This proves Problem (2.14) is infeasible, since the first constraint is violated, which implies that Problem (2.11) is also infeasible.

For iii) $b = d$ and $c \leq a < c + \theta$, $f(\tau)$ can be simplified to $f(\tau) = \dfrac{(a-c)\tau - \theta}{\eta(T)\left(a\tau + b\frac{\tau(\tau+1)}{2}\right) - (1-\alpha)}$. Note that $\lim_{\tau\to\infty} f(\tau) = 0$. Therefore $\min_{\substack{\tau \notin X \\ Z_\tau(T)>0}} \{f(\tau)\} \leq 0$ and as a result from the second constraint in Problem (2.14), $0 \leq \max_{\tau \in X}\{f(\tau)\} \leq \min_{\substack{\tau \notin X \\ Z_\tau(T)>0}} \{f(\tau)\} \leq 0$, we conclude $Y^* = 0$. However $Y^* = 0$ is infeasible since for $b = d$ and $c \leq a < c + \theta$, the firm is cash-flow below θ in the first month: $R_1 - C_1 = (a+b) - (c+d) = a - c \leq \theta$. Therefore the problem is infeasible for $b = d$ and $c \leq a < c + \theta$. □

Proof. **Lemma 2.6.** We first consider the case where $b > d$, $(b-d) \geq (c-a) + \theta$, and $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T} \frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T} \frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) \geq 0$. Condition $b > d$ implies $\sum_{t=1}^{\tau}(R_t - C_t) - \theta = (a-c)\tau + (b-d)\frac{\tau(\tau+1)}{2} - \theta \triangleq F(\tau)$ is quadratic and convex in τ, and is minimized when $\tau^* = \frac{2(c-a)-(b-d)}{2(b-d)}$ with value $F(\tau^*) = -\frac{(2(c-a)-(b-d))^2}{8(b-d)} - \theta < 0$. If $\tau^* < 1$ and $F(1) \geq 0$, then X is empty; these conditions are equivalent to $(b-d) > \frac{2}{3}(c-a)$ and $(b-d) \geq (c-a)+\theta$ and the latter constraint dominates (for both $c \geq a$ and $c < a$, constraint $(b-d) > \frac{2}{3}(c-a)$ is trivially true). Since set X is empty for $b > d$ and $(b-d) \geq (c-a)+\theta$, according to Proposition 2.1, we conclude $Y^*(T) = 0$ for $b > d$, $(b-d) \geq (c-a)+\theta$, and $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T} \frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T} \frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) \geq 0$.

Next consider the case where $b = d$, $a \geq c+\theta$, and $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) \geq 0$. Under conditions $b = d$ and $a \geq c + \theta$, $\sum_{t=1}^{\tau}(R_t - C_t) - \theta = (a-c)\tau - \theta$ is linear, and X is empty, which according to Proposition 2.1, we conclude $Y^*(T) = 0$ for $b = d$, $a \geq c + \theta$, and $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) \geq 0$.

Finally, consider the case where $b = d$, $a \geq c + \theta$, and $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) < 0$. According to Proposition 2.1, under the condition $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) < 0$ we have $Y^*(T) = \min_{\substack{\tau \notin X \\ Z_\tau(T)>0}} f(\tau)$. As shown earlier, under conditions $b = d$ and $a \geq c + \theta$, set X is empty which results in $Y^*(T) = \min_{\substack{\tau \in \mathbb{N} \\ Z_\tau(T)>0}} f(\tau)$. Note that $Z_\tau(T) = \eta(T)\left(a\tau + b\frac{\tau(\tau+1)}{2}\right) - (1-\alpha)$, where $\eta(T) = \frac{(\hat{A}+1)(\beta+1)}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} > 0$, is a convex function of τ; the roots of this quadratic are $\frac{-(a+\frac{b}{2})\eta(T) \pm \sqrt{(a+\frac{b}{2})^2\eta(T)^2 + 2b\eta(T)(1-\alpha)}}{b\eta(T)}$. By inspection, the smaller root is negative and the larger root is positive. This implies that $\{\tau \in \mathbb{N} : Z_\tau(T) > 0\} = \{\tau \in \mathbb{N} : \tau > \frac{-(a+\frac{b}{2})\eta(T) + \sqrt{(a+\frac{b}{2})^2\eta(T)^2 + 2b\eta(T)(1-\alpha)}}{b\eta(T)}\}$. Thus, $Y^*(T) = \min_{\substack{\tau \in \mathbb{N} \\ Z_\tau(T)>0}} f(\tau) = \min_{\substack{\tau \in \mathbb{N} \\ Z_\tau(T)>0}} \{\frac{(a-c)\tau - \theta}{\eta(T)\left(a\tau + b\frac{\tau(\tau+1)}{2}\right) - (1-\alpha)}\} = \lim_{\tau\to\infty} f(\tau) = 0$ for $b = d$, $a \geq c+\theta$, and $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) < 0$. □

For simplicity and tractability in the analysis, we solve the problem for $\theta \leq \theta_L(T)$ where $\theta_L(T)$ is given by $\theta_L(T) \triangleq \frac{2b(1-\alpha)(b-d)/\eta(T) + (cb-da)(2a+b) - \sqrt{((cb-da)(2a+b))^2 + 8(1-\alpha)b(cb-da)^2/\eta(T)}}{2b^2}$. Note that θ should be chosen such that the approximation solutions from Problem (2.11) are feasible for Problem (2.6), and also give high quality approximations for Problem (2.6). In all the experiments with real data from Bolstr, $\theta \leq \theta_L(T)$ suffices for high quality approximations. In extensive numerical experiments θ_L is positive. In the next lemma we show $\theta_L(T)$ is non-negative for $b > d$ and $b - d < c - a$ which indicate the firm that has cash-flow shortages.

Lemma .1. *$\theta_L(T)$ is non-negative for $b > d$ and $b - d < c - a$ $\forall$ $T \geq 1$.*

Proof. **Lemma .1.** Define

$$\Delta(\theta) \triangleq ((1-\alpha)(b-d) - \eta(T)\theta b)^2 - \eta(T)(cb - da)\left((2(c-a) - (b-d))(1-\alpha) + \eta(T)\theta(2a+b)\right).$$

It can be verified that $\Delta(\theta)$ is a quadratic convex function of θ in which $\theta_L(T)$ is its smaller root and its larger root is given by

$$\theta_H(T) \triangleq \frac{2b(1-\alpha)(b-d)/\eta(T) + (cb-da)(2a+b) + \sqrt{((cb-da)(2a+b))^2 + 8(1-\alpha)b(cb-da)^2/\eta(T)}}{2b^2}.$$

As $b > d$ at feasibility, it can be shown by inspection that $\theta_H(T) > 0$. If we show $\Delta(\theta = 0) \geq 0$, then since $\Delta(\theta)$ is convex with the larger root being positive, we conclude the smaller root, $\theta_L(T)$, is non-negative. Therefore we next show $\Delta(0) = (1-\alpha)^2(b-d)^2 - \eta(T)(1-\alpha)(cb - da)(2(c-a) - (b-d)) = (1-\alpha)(b-d)^2 \left((1-\alpha) - \frac{\eta(T)(cb-da)(2(c-a)-(b-d))}{(b-d)^2}\right)$ is non-negative. From $F(\overline{\tau}) = 0$, we know: $F(\overline{\tau}) = 0 \Longleftrightarrow \overline{\tau}(\overline{\tau}+1)/2 = \frac{\theta + (c-a)\overline{\tau}}{(b-d)}$. Additionally from $Z_\tau(T) < 0 \ \forall \tau \in X$, we conclude $Z_{\overline{\tau}}(T) \leq 0$ which results in $Z_{\overline{\tau}}(T) = \eta(T)\left(a\overline{\tau} + b\overline{\tau}(\overline{\tau}+1)/2\right) \leq (1-\alpha)$. Next we replace $\overline{\tau}(\overline{\tau}+1)/2 = \frac{\theta + (c-a)\overline{\tau}}{(b-d)}$ in the above inequality and this results in $\eta(T)\left(\frac{cb-da}{b-d}\frac{\left(c-a-\frac{b-d}{2}\right) + \sqrt{\left(c-a-\frac{b-d}{2}\right)^2 + 2\theta(b-d)}}{b-d} + \frac{b\theta}{(b-d)}\right) \leq (1-\alpha)$. Assuming the problem is feasible for $\theta \geq 0$, we next replace $\theta = 0$ in the above inequality; since $b - d < c - a$, or equivalently $c - a - (b-d) > 0$, and $b - d > 0$, we can conclude $c - a - (b-d)/2 > 0$. From the inequality $c - a - (b-d)/2 > 0$, we can simplify the left-hand side of the above inequality and write it as $\eta(T)\left(\frac{(cb-da)2\left(c-a-\frac{b-d}{2}\right)}{(b-d)^2}\right) \leq (1-\alpha) \Longleftrightarrow \Delta(0) \geq 0$ and this completes the proof. □

Proof. **Lemma 2.7.** According to Proposition 2.1, for $b > d$, $b - d \geq (c-a) + \theta$, and $(\beta+1)(\hat{A}+1)\frac{\sum_{t=1}^{T}\frac{R_t}{(1+r_t)^t}}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} - (1-\alpha) < 0$, we have $Y^*(T) = \min_{\substack{\tau \notin X \\ Z_\tau(T) > 0}} f(\tau)$. As shown in the proof of Lemma 2.6, for $b > d$ and $b - d \geq (c-a) + \theta$, set X is empty. Thus, we obtain $Y^*(T) = \min_{\substack{\tau \in \mathbb{N} \\ Z_\tau(T) > 0}} f(\tau)$. The derivative of $f(\tau)$ is

$$\frac{df(\tau)}{d\tau} = 2\frac{(cb-da)\eta(T)\tau^2 - 2\left((1-\alpha)(b-d) - \eta(T)\theta b\right)\tau + (2(c-a) - (b-d))(1-\alpha) + \eta(T)\theta(2a+b)}{(\eta(T)b\tau^2 + (2a+b)\eta(T)\tau - 2(1-\alpha))^2}.$$

We next prove the lemma for the following two cases: 1) $cb - da \geq 0$ and 2) $cb - da < 0$.

First consider case 1: $cb - da \geq 0$. The numerator $n(\tau)$ of $\frac{df(\tau)}{d\tau}$ is a convex quadratic due to $cb > da$. Note that, for $b > d$ and $b - d \geq (c-a) + \theta$ set X is empty, the numerator of $f(\tau)$ is increasing and positive for $\tau \in [1, \infty)$. Additionally as shown in the proof of Lemma 2.6, the denominator of $f(\tau)$, $Z_\tau(T) = \eta(T)\left(a\tau + b\frac{\tau(\tau+1)}{2}\right) - (1-\alpha)$ is a convex function of τ and the roots of this quadratic are $\frac{-(a+\frac{b}{2})\eta(T) \pm \sqrt{(a+\frac{b}{2})^2\eta(T)^2 + 2b\eta(T)(1-\alpha)}}{b\eta(T)}$. By inspection, the smaller root is negative and the larger root is positive. Thus for $\tau \in \left[1, \frac{-(a+\frac{b}{2})\eta(T) + \sqrt{(a+\frac{b}{2})^2\eta(T)^2 + 2b\eta(T)(1-\alpha)}}{b\eta(T)}\right)$, the denominator of $f(\tau)$ is negative and increasing and approaches zero from below at $\tau \nearrow \frac{-(a+\frac{b}{2})\eta(T) + \sqrt{(a+\frac{b}{2})^2\eta(T)^2 + 2b\eta(T)(1-\alpha)}}{b\eta(T)}$. Therefore as the numerator of $f(\tau)$ is positive and increasing and the denominator is negative and increasing, $f(\tau)$ is strictly decreasing on this interval, or equivalently $n(\tau) < 0$ for $\tau \in \left[1, \frac{-(a+\frac{b}{2})\eta(T) + \sqrt{(a+\frac{b}{2})^2\eta(T)^2 + 2b\eta(T)(1-\alpha)}}{b\eta(T)}\right)$.

Alternatively, $\lim_{\tau \to \infty} n(\tau) = \infty$. Since the numerator is convex, these observations imply that $f(\tau)$ is unimodal on $[1, \infty)$, with a minimum at the larger root of $n(\tau)$, or $\tau^{**} = \frac{(1-\alpha)(b-d) - \eta(T)\theta b + \sqrt{\Delta(\theta)}}{(cb-da)\eta(T)}$. Note that τ^{**} is real for $\theta \leq \theta_L(T)$, our domain of interest. We finally prove that $\tau^{**} > \frac{-(a+\frac{b}{2})\eta(T) + \sqrt{(a+\frac{b}{2})^2\eta(T)^2 + 2b\eta(T)(1-\alpha)}}{b\eta(T)}$, which implies that $\tau^{**} \in \{\tau \in \mathbb{N} : Z_\tau(T) > 0\}$ and $\min_{\substack{\tau \in \mathbb{N} \\ Z_\tau(T) > 0}} \{f(\tau)\} = \min\{f(\lfloor \tau^{**} \rfloor), f(\lceil \tau^{**} \rceil)\}$. If $\tau^{**} \leq \frac{-(a+\frac{b}{2})\eta(T) + \sqrt{(a+\frac{b}{2})^2\eta(T)^2 + 2b\eta(T)(1-\alpha)}}{b\eta(T)}$, then $f(\tau)$ is unimodal with a minimizer in $\Big[1, \frac{-(a+\frac{b}{2})\eta(T)}{b\eta(T)} +$

$\frac{\sqrt{(a+\frac{b}{2})^2\eta(T)^2+2b\eta(T)(1-\alpha)}}{b\eta(T)}\Big]$. In particular, this implies $f(\tau)$ is increasing as $\tau \nearrow \Big(\frac{-(a+\frac{b}{2})\eta(T)}{b\eta(T)} + \frac{+\sqrt{(a+\frac{b}{2})^2\eta(T)^2+2b\eta(T)(1-\alpha)}}{b\eta(T)}\Big)$; however, this contradicts $\lim_{\tau \nearrow \frac{-(a+\frac{b}{2})\eta(T)+\sqrt{(a+\frac{b}{2})^2\eta(T)^2+2b\eta(T)(1-\alpha)}}{b\eta(T)}} f(\tau) = -\infty$, since, on this domain, the denominator of $f(\tau)$, $Z_\tau(T)$, is negative and approaches zero from below, and the numerator is positive, by the definition of X.

Next consider case 2) $cb - da < 0$: The numerator $n(\tau)$ of $\frac{df(\tau)}{d\tau}$ is a concave quadratic due to $cb \leq da$. Consider the smaller and larger roots of $n(\tau)$, $\tau^{**} = \frac{-(1-\alpha)(b-d)+\eta(T)\theta b-\sqrt{\Delta(\theta)}}{(da-cb)\eta(T)}$ and $\tau^* = \frac{-(1-\alpha)(b-d)+\eta(T)\theta b+\sqrt{\Delta(\theta)}}{(da-cb)\eta(T)}$, respectively. As mentioned earlier, we solve the problem for $\theta \leq \theta_L(T)$ which guarantees the discriminant of $n(\tau)$ is non-negative. Next we show τ^{**} and τ^* are negative $\forall\ \theta \leq \theta_L(T)$. First we show $-(1-\alpha)(b-d) + \eta(T)\theta b < 0,\ \forall\theta \leq \theta_L(T)$: $-(1-\alpha)(b-d) + \eta(T)\theta b < 0,\ \forall\theta \leq \theta_L(T) \iff -\eta(T)(da-cb)(2a+b) - \sqrt{(\eta(T)(cb-da)(2a+b))^2 + 8\eta(T)(1-\alpha)b(cb-da)^2} < 0$ and this inequality holds due to $(da-cb) \geq 0$. For $-(1-\alpha)(b-d) + \eta(T)\theta b < 0,\ \forall\theta \leq \theta_L(T)$, it can be easily verified that τ^{**} is negative. We next prove τ^* is negative. If we rearrange the inequality $-(1-\alpha)(b-d) + \eta(T)\theta b < 0,\ \forall\theta \leq \theta_L(T)$, we get $\theta\eta(T) < (1-\alpha)(b-d)/b,\ \forall\theta \leq \theta_L(T)$. Next we show for $\theta\eta(T) < (1-\alpha)(b-d)/b,\ \forall\theta \leq \theta_L(T)$, the inequality $-(1-\alpha)(a-c) + \eta(T)\theta a < 0,\ \forall\theta \leq \theta_L(T)$ holds by replacing $\theta\eta(T)$ with its upper bound $(1-\alpha)(b-d)/b$ into $-(1-\alpha)(a-c)+\eta(T)\theta a < 0$: $-(1-\alpha)(a-c)+\eta(T)\theta a < 0,\ \forall\theta \leq \theta_L(T) \iff -(1-\alpha)(a-c) + [(1-\alpha)(b-d)/b]\,a < 0 \iff -(1-\alpha)(ad-cb)/b < 0$ and this inequality holds due to $(da-cb) \geq 0$. If $-(1-\alpha)(b-d)+\eta(T)\theta b < 0$ and $-(1-\alpha)(a-c)+\eta(T)\theta a < 0,\ \forall\theta \leq \theta_L(T)$, then we conclude $((2(c-a)-(b-d))(1-\alpha)+\eta(T)\theta(2a+b)) < 0,\ \forall\theta \leq \theta_L(T)$. From this result, we show that τ^* is negative:

$$\tau^* \leq 0 \Longleftrightarrow$$
$$\sqrt{((1-\alpha)(b-d)-\eta(T)\theta b)^2+\eta(T)(da-cb)\left((2(c-a)-(b-d))(1-\alpha)+\eta(T)\theta(2a+b)\right)}$$
$$\leq (1-\alpha)(b-d)-\eta(T)\theta b$$

and the above inequality holds due to $(1-\alpha)(b-d)-\eta(T)\theta b > 0$, $(da-cb) \geq 0$, and $((2(c-a)-(b-d))(1-\alpha)+\eta(T)\theta(2a+b)) < 0$. As a result we conclude $\tau^* < 0$. As $n(\tau)$ is concave with negative roots, we conclude $f(\tau)$ is strictly decreasing in $[0,\infty)$ and as a result $\min\limits_{\substack{\tau\in\mathbb{N}\\ Z_\tau(T)>0}} \{f(\tau)\} = \lim\limits_{\tau\to\infty} f(\tau) = (b-d)/\eta(T)b.$ □

Proof. **Lemma 2.8.** If $b > d$ and $b-d < (c-a)+\theta$, then $\sum_{t=1}^{\tau}(R_t - C_t) - \theta = (a-c)\tau + (b-d)\frac{\tau(\tau+1)}{2} - \theta \triangleq F(\tau)$ is a convex quadratic function of τ, and as $F(0) = -\theta$, the smaller root is negative and the larger root is positive and is equal to $\overline{\tau} \triangleq \frac{\left(c-a-\frac{b-d}{2}\right)+\sqrt{\left(c-a-\frac{b-d}{2}\right)^2+2\theta(b-d)}}{b-d}$. Note that as the discriminant is positive, the roots are real. Thus, $F(\tau) < 0$ for $1 \leq \tau < \overline{\tau}$. □

Proof. **Lemma 2.9.** Recall that $Z_\tau(T) = \eta(T)\left(a\tau + b\frac{\tau(\tau+1)}{2}\right) - (1-\alpha)$, where $\eta(T) = \frac{(\hat{A}+1)(\beta+1)}{\sum_{t=1}^{T}\frac{R_t}{(1+\delta_t)^t}} > 0$, is a convex function of τ; the roots of this quadratic are

$$\frac{-(a+\frac{b}{2})\eta(T) \pm \sqrt{(a+\frac{b}{2})^2\eta(T)^2 + 2b\eta(T)(1-\alpha)}}{b\eta(T)}.$$

By inspection, the smaller root is negative and the larger root is positive. This implies that $\{\tau \in \mathbb{N} : Z_\tau(T) < 0\} = \left\{\tau \in \mathbb{N} : \tau < \frac{-(a+\frac{b}{2})\eta(T)+\sqrt{(a+\frac{b}{2})^2\eta(T)^2+2b\eta(T)(1-\alpha)}}{b\eta(T)}\right\}$ and $\{\tau \in \mathbb{N} : Z_\tau(T) > 0\} = \left\{\tau \in \mathbb{N} : \tau > \frac{-(a+\frac{b}{2})\eta(T)+\sqrt{(a+\frac{b}{2})^2\eta(T)^2+2b\eta(T)(1-\alpha)}}{b\eta(T)}\right\}$. Therefore the condition $Z_\tau(T) < 0$ for all $\tau \in X$, in which $X = \{\tau \in \mathbb{N} : 0 \leq \tau < \overline{\tau}\}$, is equivalent to

$\overline{\tau} < \frac{-(a+\frac{b}{2})\eta(T)+\sqrt{(a+\frac{b}{2})^2\eta(T)^2+2b\eta(T)(1-\alpha)}}{b\eta(T)}$ or equivalently $\sqrt{(c-a-(b-d)/2)^2+2\theta(b-d)} < \frac{(ad-bc)+(b-d)\sqrt{(a+\frac{b}{2})^2+2b(1-\alpha)/\eta(T))}}{b}$. This inequality holds if its right-hand-side is non-negative or equivalently $(ad-bc)+(b-d)\sqrt{(a+b/2)^2+2(1-\alpha)b/\eta(T)} \geq 0$. Under this condition, the inequality can be rearranged as $\theta \leq \hat{\theta}(T)$ where

$$\begin{aligned}\hat{\theta}(T) &= \frac{(b-d)^2(a+\frac{b}{2})^2+\frac{2(1-\alpha)b(b-d)^2}{\eta(T)}+(ad-bc)^2+2(b-d)(ad-bc)\sqrt{(a+\frac{b}{2})^2+\frac{2(1-\alpha)b}{\eta(T)}}}{2b^2(b-d)} \\ &\quad -\frac{b^2(c-a-\frac{(b-d)}{2})^2}{2b^2(b-d)}.\end{aligned}$$

□

According to Lemma 2.9, the problem is feasible for $\theta \leq \hat{\theta}(T)$. As a result we consider $\theta \leq \min\left\{\hat{\theta}(T), \theta_L(T)\right\} \triangleq \tilde{\theta}(T)$ throughout the next set of analysis.

Proof. **Lemma 2.10.** The numerator $n(\tau)$ of $\frac{df(\tau)}{d\tau}$ is a convex quadratic due to $cb > da$. We consider the following two cases separately:

Case 1. If $n(0) = (2(c-a)-(b-d))(1-\alpha)+\eta(T)\theta(2a+b) \geq 0$, or equivalently $\theta \geq \frac{((b-d)-2(c-a))(1-\alpha)}{\eta(T)(2a+b)}$, then $f(\tau)$ is increasing at $\tau = 0$. In contrast $f(\tau)$ is strictly decreasing in $\left[\overline{\tau}, \frac{-(a+\frac{b}{2})\eta(T)+\sqrt{(a+\frac{b}{2})^2\eta(T)^2+2b\eta(T)(1-\alpha)}}{b\eta(T)}\right)$; the proof is similar to that of Lemma 2.7. Since $n(\tau)$ is continuous in $[0, \overline{\tau}]$, these observations imply that there is one point τ^* in this interval where $\frac{df(\tau^*)}{d\tau} = 0$. Furthermore, τ^* must be the smaller root of the convex numerator $n(\tau)$, or $\tau^* = \frac{(1-\alpha)(b-d)-\eta(T)\theta b-\sqrt{\Delta(\theta)}}{(cb-da)\eta(T)}$. As shown in Lemma 1, the $\theta_L(T)$ is the smaller root of the convex quadratic function $\Delta(\theta)$. Therefore for $\theta \leq \tilde{\theta}(T) \leq \theta_L(T)$ which we are solving the problem for, τ^* is real. Consequently, $\frac{df(\tau)}{d\tau}$ is positive for $\tau \in [0, \tau^*)$ and negative for $\tau \in (\tau^*, \overline{\tau}]$, which implies $f(\tau)$ is unimodal on $[0, \overline{\tau}]$, with a maximum at τ^*. Therefore, since X is a set of integers, $\max_{\tau \in X} \{f(\tau)\} = \max\{f(\lfloor \tau^* \rfloor), f(\lceil \tau^* \rceil)\}$.

Case 2. If $n(0) = (2(c-a)-(b-d))(1-\alpha)+\eta(T)\theta(2a+b) < 0$, or equivalently $\theta <$

$\frac{((b-d)-2(c-a))(1-\alpha)}{\eta(T)(2a+b)}$, then $f(\tau)$ is strictly decreasing at $\tau=0$. As shown above, $f(\tau)$ is strictly decreasing on $[\overline{\tau}, \frac{-(a+\frac{b}{2})\eta(T)+\sqrt{(a+\frac{b}{2})^2\eta(T)^2+2b\eta(T)(1-\alpha)}}{b\eta(T)})$. Since $n(\tau)$ is a convex quadratic, these observations implies $f(\tau)$ is strictly decreasing on $[0,\overline{\tau}]$ with a maximizer at $\tau=0$: $\max\limits_{\tau\in X}\{f(\tau)\} = f(1) = \dfrac{a+b-c-d-\theta}{\eta(T)(a+b)-(1-\alpha)}$. □

Proof. **Lemma 2.11.** From Lemma 2.9 we conclude that $\{\tau \in \mathbb{N} : \tau \notin X,\ Z_\tau(T) > 0\} = \{\tau : \tau > \frac{-(a+\frac{b}{2})\eta(T)+\sqrt{(a+\frac{b}{2})^2\eta(T)^2+2b\eta(T)(1-\alpha)}}{b\eta(T)}\}$. Next, building upon the proof of Lemma 2.10, we recall that the numerator of $\frac{df(\tau)}{d\tau}$, evaluated at $\tau=\overline{\tau}$, is non-positive: $n(\overline{\tau}) \le 0$ or equivalently $f(\tau)$ is decreasing at $\tau=\overline{\tau}$. Alternatively, $\lim_{\tau\to\infty} n(\tau)=\infty$. Since the numerator is convex, these observations imply that $f(\tau)$ is unimodal on $[\overline{\tau},\infty)$, with a minimum at the larger root of $n(\tau)$, or $\tau^{**} = \frac{(1-\alpha)(b-d)-\eta(T)\theta b+\sqrt{\Delta(\theta)}}{(cb-da)\eta(T)}$. Note that τ^{**} is real for $\theta \le \tilde{\theta}(T) \le \theta_L(T)$, our domain of interest. We finally prove that $\tau^{**} > \frac{-(a+\frac{b}{2})\eta(T)+\sqrt{(a+\frac{b}{2})^2\eta(T)^2+2b\eta(T)(1-\alpha)}}{b\eta(T)}$, which implies that $\tau^{**} \in \{\tau \in \mathbb{N} : \tau \notin X,\ Z_\tau(T) > 0\}$ and $\min\limits_{\substack{\tau\notin X\\ Z_\tau(T)>0}}\{f(\tau)\} = \min\{f(\lfloor\tau^{**}\rfloor), f(\lceil\tau^{**}\rceil)\}$. If $\tau^{**} \le \frac{-(a+\frac{b}{2})\eta(T)+\sqrt{(a+\frac{b}{2})^2\eta(T)^2+2b\eta(T)(1-\alpha)}}{b\eta(T)}$, then $f(\tau)$ is unimodal with a minimizer in $[\overline{\tau}, \frac{-(a+\frac{b}{2})\eta(T)+\sqrt{(a+\frac{b}{2})^2\eta(T)^2+2b\eta(T)(1-\alpha)}}{b\eta(T)}]$. In particular, this implies $f(\tau)$ is increasing as $\tau \nearrow \frac{-(a+\frac{b}{2})\eta(T)+\sqrt{(a+\frac{b}{2})^2\eta(T)^2+2b\eta(T)(1-\alpha)}}{b\eta(T)}$; however, this contradicts $\lim\limits_{\tau\nearrow\frac{-(a+\frac{b}{2})\eta(T)+\sqrt{(a+\frac{b}{2})^2\eta(T)^2+2b\eta(T)(1-\alpha)}}{b\eta(T)}} f(\tau) = -\infty$, since, on this domain, the denominator of $f(\tau)$, $Z_\tau(T)$, is negative and approaches zero from below, and the numerator is positive, by the definition of X. □

Proof. **Lemma 2.12.** We prove this lemma under the conditions of Lemmas 2.10 and 2.11. We first show that the left-hand-side of $\max\limits_{\tau\in X}\{f(\tau)\} - \min\limits_{\substack{\tau\notin X\\ Z_\tau(T)>0}}\{f(\tau)\} \le 0$ is increasing in θ and the inequality holds at $\theta=0$. Then we conclude $\exists\ \bar{\theta}(T)$ where $\max\limits_{\tau\in X}\{f(\tau)\} \le \min\limits_{\substack{\tau\notin X\\ Z_\tau(T)>0}}\{f(\tau)\}$ holds for $\theta \le \bar{\theta}(T)$.

First consider Case 1 in which $n(0) = (2(c-a)-(b-d))(1-\alpha)+\eta(T)\theta(2a+b) \ge 0$, which according to Lemmas 2.10–2.11 we have $\max_{\tau\in X}\{f(\tau)\} = \max\{f(\lfloor\tau^*\rfloor), f(\lceil\tau^*\rceil)\}$ and $\min_{\substack{\tau\notin X \\ Z_\tau(T)>0}}\{f(\tau)\} = \min\{f(\lfloor\tau^{**}\rfloor), f(\lceil\tau^{**}\rceil)\}$. For ease of exposition, we assume that τ^* and τ^{**} are integers, so that we need to prove $f(\tau^*) \le f(\tau^{**})$. Expanding the definition of $f(\tau)$, we observe

$$f(\tau^{**}) - f(\tau^*) = \frac{\left((a-c)\tau^{**} + (b-d)\frac{\tau^{**}(\tau^{**}+1)}{2} - \theta\right) Z_{\tau^*}(T)}{Z_{\tau^*}(T)Z_{\tau^{**}}(T)} - \frac{\left((a-c)\tau^{*} + (b-d)\frac{\tau^{*}(\tau^{*}+1)}{2} - \theta\right) Z_{\tau^{**}}(T)}{Z_{\tau^*}(T)Z_{\tau^{**}}(T)}. \tag{5}$$

From Lemma 2.10, we know that $Z_{\tau^*}(T) < 0$; similarly, from Lemma 2.11, we know that $Z_{\tau^{**}}(T) > 0$. Thus, the denominator of Equation (5) is negative. To analyze the numerator of Equation (5), we first consider the expression $\left((a-c)t + (b-d)\frac{t(t+1)}{2} - \theta\right) Z_\tau(T)$, which can be manipulated as $\left((a-c)t + (b-d)\frac{t(t+1)}{2} - \theta\right) Z_\tau(T) = \frac{\eta(T)b(b-d)}{4}t^2\tau^2 + \frac{\eta(T)(2a+b)(b-d)}{4}t^2\tau - \frac{(b-d)(1-\alpha)}{2}t^2 - \eta(T)b\left(\frac{2(c-a)-(b-d)}{4}\right)t\tau^2 - \eta(T)(2a+b)\left(\frac{2(c-a)-(b-d)}{4}\right)t\tau + (1-\alpha)\left(\frac{2(c-a)-(b-d)}{2}\right)t - \theta\left(\eta(T)\left(a\tau + b\frac{\tau(\tau+1)}{2}\right) - (1-\alpha)\right)$. Since $t^2\tau^2 = \tau^2t^2$ and $t\tau = \tau t$, we can write

$$\begin{aligned} &\left((a-c)t + (b-d)\tfrac{t(t+1)}{2} - \theta\right) Z_\tau(T) - \left((a-c)\tau + (b-d)\tfrac{\tau(\tau+1)}{2} - \theta\right) Z_t(T) \\ &= (t-\tau)\left(\left(\tfrac{2\eta(T)(bc-ad)}{4}\right)t\tau - \tfrac{(b-d)(1-\alpha)}{2}(t+\tau) + (1-\alpha)\left(\tfrac{2(c-a)-(b-d)}{2}\right)\right) + \theta\left(Z_t(T) - Z_\tau(T)\right) \end{aligned} \tag{6}$$

Letting $t = \tau^{**}$ and $\tau = \tau^*$, two roots of a quadratic equation, we have that $t + \tau = 2\frac{(1-\alpha)(b-d)-\eta(T)\theta b}{(cb-da)\eta(T)}$, $t\tau = \frac{(2(c-a)-(b-d))(1-\alpha)+\eta(T)\theta(2a+b)}{(cb-da)\eta(T)}$, and $(t-\tau) = \frac{2\sqrt{\Delta(\theta)}}{\eta(T)(cb-da)}$. Plugging these expressions into Expression (6), along with $t = \tau^{**}$ and $\tau = \tau^*$, we obtain $\frac{2\sqrt{\Delta(\theta)}}{\eta(T)(cb-da)}\left((2(c-a)-(b-d))(1-\alpha) + \eta(T)\theta(a+b/2) - (b-d)(1-\alpha)\frac{(b-d)(1-\alpha)-\eta(T)\theta b}{(cb-da)\eta(T)}\right) + \theta\left(Z_{\tau^{**}}(T) - Z_{\tau^*}(T)\right)$. As the denominator of Equation (5) is negative, $f(\tau^{**}) - f(\tau^*) \ge 0$ holds if the numerator of Equation (5) is non-positive or equivalently $\frac{2\sqrt{\Delta(\theta)}}{\eta(T)(cb-da)}\Big((2(c-a)-$

$(b-d))(1-\alpha)+\eta(T)\theta(a+b/2)-(b-d)(1-\alpha)\frac{(b-d)(1-\alpha)-\eta(T)\theta b}{(cb-da)\eta(T)}\Big)+\theta(Z_{\tau^{**}}(T)-Z_{\tau^{*}}(T))\leq 0$. Let $g(\theta)=\frac{2\sqrt{\Delta(\theta)}}{\eta(T)(cb-da)}\left((2(c-a)-(b-d))(1-\alpha)+\eta(T)\theta(a+b/2)-(b-d)(1-\alpha)\frac{(b-d)(1-\alpha)-\eta(T)\theta b}{(cb-da)\eta(T)}\right)$, and $h(\theta)=\theta\eta(T)(cb-da)\left(Z_{\tau^{**}}(T)-Z_{\tau^{*}}(T)\right)$. The above inequality is then written as $g(\theta)+h(\theta)\leq 0$. We first show $g(\theta)<0,\ \forall\theta\geq 0$. The $g(\theta)$ can be rearranged and written as $g(\theta)=\frac{2\sqrt{\Delta(\theta)}}{\eta^2(T)(cb-da)^2}\left(-\left(\sqrt{\Delta(\theta)}\right)^2-\eta(T)\theta b\left((1-\alpha)(b-d)-\eta(T)\theta b\right)\right)\leq 0$. This inequality holds due to $(1-\alpha)(b-d)-\eta(T)\theta b\geq 0$: As proved in Lemma 2.10, for Case 1 in which $n(0)=(2(c-a)-(b-d))(1-\alpha)+\eta(T)\theta(2a+b)\geq 0$, we have $\tau^*\in[0,\overline{\tau})$. Therefore from the definition of τ^*, we conclude $(1-\alpha)(b-d)-\eta(T)\theta b\geq 0$ must hold to have $\tau^*\geq 0$. We can easily show that $g(\theta)+h(\theta)<0$ at $\theta=0$. Note that $\theta=0$, if feasible, belongs to this case. The reason is that from the feasibility condition $b-d<c-a+\theta$ at $\theta=0$ we conclude $b-d<c-a$ and as a result $n(0)\geq 0$, which is the condition of Case 1.

Recall that at feasibility we have $(Z_{\tau^{**}}(T)-Z_{\tau^{*}}(T))>0$ and as a result $h(\theta)\geq 0$. If $h(\theta)$ is decreasing in θ, then $g(\theta)+h(\theta)<0,\ \forall\theta\geq 0$. Otherwise there exists some $\overline{\theta}(T)\geq 0$ where $g(\theta)+h(\theta)<0$ holds for $\theta\leq\overline{\theta}(T)\leq\tilde{\theta}(T)$ and this completes the proof for Case 1.

Next we prove the left-hand-side of $\max\limits_{\tau\in X}\{f(\tau)\}-\min\limits_{\substack{\tau\notin X\\ Z_\tau(T)>0}}\{f(\tau)\}\leq 0$ is increasing in θ for Case 2, in which $n(0)=(2(c-a)-(b-d))(1-\alpha)+\eta(T)\theta(2a+b)<0$. Then we claim, if Case 2 is feasible, there exists some $\overline{\theta}(T)\geq 0$ where $\max\limits_{\tau\in X}\{f(\tau)\}-\min\limits_{\substack{\tau\notin X\\ Z_\tau(T)>0}}\{f(\tau)\}\leq 0$ holds for $\theta\leq\overline{\theta}(T)\leq\tilde{\theta}(T)$ and this completes the proof for Case 2. According to Lemmas 2.10–2.11, under the conditions of Case 2, we have $\max\limits_{\tau\in X}\{f(\tau)\}=\dfrac{a+b-c-d-\theta}{\eta(T)(a+b)-(1-\alpha)}$ and $\min\limits_{\substack{\tau\notin X\\ Z_\tau(T)>0}}\{f(\tau)\}=\min\{f(\lfloor\tau^{**}\rfloor),f(\lceil\tau^{**}\rceil)\}$. For ease of exposition, we assume that τ^{**} is integer, so $\max\limits_{\tau\in X}\{f(\tau)\}\leq\min\limits_{\substack{\tau\notin X\\ Z_\tau(T)>0}}\{f(\tau)\}$ is equivalent to $\theta-((1-\alpha)-\eta(T)(a+b))\,f(\tau^{**})-(a+b)+(c+d)\leq 0$. For calculating the derivative of the left-hand-side of the above inequality with respect to θ, first we calculate $\frac{d(f(\tau^{**}(\theta),\theta))}{d\theta}$:

$$\begin{aligned}\frac{d\left(f(\tau^{**}(\theta),\theta)\right)}{d\theta} &= -\frac{\eta(T)\left(a\tau^{**}(\theta)+b\tau^{**}(\theta)(\tau^{**}(\theta)+1)/2\right)-(1-\alpha)}{Z^2_{\tau^{**}(\theta)}(T)} + \frac{\frac{d\tau^{**}(\theta)}{d\theta}}{Z^2_{\tau^{**}(\theta)}(T)} \times \\ &\quad \Big(\eta(T)\tau^{**2}(\theta)(cb-da)/2+(1-\alpha)\left(2(c-a)-(b-d)\right)/2 \\ &\quad -(1-\alpha)(b-d)\tau^{**}(\theta)+\theta\eta(T)(2a+b)/2+\theta\eta(T)b\tau^{**}(\theta)\Big).\end{aligned}$$

We know $\tau^{**}(\theta)$ is the root of $n(\tau)$ and therefore $n(\tau^{**}(\theta))=0$ which implies $(cb-da)\eta(T)\tau^{**2}(\theta)/2 = \left((1-\alpha)(b-d)-\eta(T)\theta b\right)\tau^{**}(\theta)-(2(c-a)-(b-d))(1-\alpha)/2-\eta(T)\theta(2a+b)/2$. If we replace the left-hand side of the above equation with its right-hand side into $\frac{d(f(\tau^{**}(\theta)(\theta),\theta))}{d\theta}$, the second term in $\frac{d(f(\tau^{**}(\theta)(\theta),\theta))}{d\theta}$ becomes zero and as a result we get $\frac{d(f(\tau^{**}(\theta),\theta))}{d\theta} = -\frac{1}{Z_{\tau^{**}(\theta)}(T)}$. Therefore, the derivative of the left-hand side of the inequality $\theta-\left((1-\alpha)-\eta(T)(a+b)\right)f(\tau^{**})-(a+b)+(c+d)\leq 0$, with respect to θ, is $1+\left((1-\alpha)-\eta(T)(a+b)\right)\frac{1}{Z_{\tau^{**}(\theta)}(T)}>0$, where the above inequality is due to $Z_{\tau^{**}(\theta)}(T)\geq 0$, and $\left((1-\alpha)-\eta(T)(a+b)\right)>0$ or equivalently $Z_{\tau=1}(T)<0$, and this completes the proof for Case 2. □

Proof. **Lemma 2.13.** Note that for $cb\leq da$, the condition

$$(ad-bc)+(b-d)\sqrt{(a+b/2)^2+2(1-\alpha)b/\eta(T)}\geq 0$$

is always satisfied. The numerator $n(\tau)$ of $\frac{df(\tau)}{d\tau}$ is a concave quadratic due to $cb\leq da$. As proved in Lemma 2.7, τ^{**} and τ^{*} are negative $\forall\ \theta\leq\theta_L(T)$ and $cb\leq da$. As $n(\tau)$ is concave with negative roots, we conclude $f(\tau)$ is strictly decreasing in $[0,\infty)$ and as a result $\max_{\tau\in X}\{f(\tau)\}=f(1)=\dfrac{a+b-c-d-\theta}{\eta(T)(a+b)-(1-\alpha)}$. □

Proof. **Lemma 2.14.** Building upon the proof of Lemma 2.13, $f(\tau)$ is decreasing in $[0,\infty)$. These observations imply that $\min_{\substack{\tau\notin X\\ Z_\tau(T)>0}}\{f(\tau)\}=\lim_{\tau\to\infty}f(\tau)=(b-d)/\eta(T)b$. □

Proof. **Lemma 2.15.** Building upon the proofs of Lemmas 2.13 and 2.14, the inequality $\max_{\tau \in X} \{f(\tau)\} \leq \min_{\substack{\tau \notin X \\ Z_\tau(T)>0}} \{f(\tau)\}$ is equivalent to $\theta \leq \frac{(1-\alpha)(b-d)+\eta(T)(da-cb)}{b\eta(T)}$. Therefore the inequality $\max_{\tau \in X} \{f(\tau)\} \leq \min_{\substack{\tau \notin X \\ Z_\tau(T)>0}} \{f(\tau)\}$ holds for $\theta \leq \min\left\{\tilde{\theta}(T), \frac{(1-\alpha)(b-d)+\eta(T)(da-cb)}{b\eta(T)}\right\}$, $b > d$, $b-d < (c-a)+\theta$, and $cb \leq da$, and this completes the proof. $\square$

Proof. **Lemma 3.1**. From set U_K for $\sigma_i^k = \sigma^k,\ \forall i$, we can conclude $\max_{(K_1,\ldots,K_{\mathbb{N}})\in U_K} \sum_{i\in S} K_i \leq \sum_{i\in S} \mu_i^k + \gamma^k\sigma^k\sqrt{|S|},\ \forall S \subseteq \{1,\ldots,\mathbb{N}\}$. The upper bound is achieved for opportunity cost vector $\hat{\mathbf{K}} = (\hat{K}_1, \cdots, \hat{K}_{\mathbb{N}})$ defined as

$$\hat{K}_\ell = \begin{cases} \mu_\ell^k + \gamma^k\sigma^k\left(\sqrt{|S|-t+1} - \sqrt{|S|-t}\right), & \ell \in S \text{ and } t\text{th element of } S \\ \mu_\ell^k, & \ell \notin S \end{cases}$$

It can be easily proved that $\sum_{\ell\in S} \hat{K}_\ell = \sum_{\ell\in S} \mu_\ell^k + \gamma^k\sigma^k\sqrt{|S|}$, and thus $\hat{\mathbf{K}} \in U_K$. Consequently, we conclude that $\max_{(K_1,\ldots,K_{\mathbb{N}})\in U_K} \sum_{i\in S} K_i = \sum_{i\in S} \hat{K}_i = \sum_{i\in S} \mu_i^k + \gamma^k\sigma^k\sqrt{N},\ \forall S \subseteq \{1,\ldots,\mathbb{N}\}, |S| = N$. Next, we solve the outer minimization problem, which is

$$\min_{\substack{S\subset\{1,\ldots,\mathbb{N}\}, \\ |S|=N}} \left\{\sum_{i\in S} \mu_i^k + \gamma^k\sigma^k\sqrt{N}\right\} = \sum_{i=1}^{N} \mu_{(i)}^k + \gamma^k\sigma^k\sqrt{N},$$

where $\mu_{(1)}^k \leq \mu_{(2)}^k \leq \cdots \leq \mu_{(\mathbb{N})}^k$ are the order statistics of the mean parameters, and this completes the proof. $\square$

Proof. **Lemma 3.2**. We can decompose the objective function of Problem (3.7) as

$$\begin{aligned}
&\max_{\substack{(X_1,\ldots,X_{mq})\in U_X^N\\ L^j_{q+1}/v_j\in U_L^j, j=1,\cdots,m}} Pw\left(\frac{\sum_{j=1}^{m} L^j_{q+1}/v_j + \sum_{i=1}^{mq} X_i}{mq}\right) + (1-P)c_T\\
= \quad & Pw\left(\frac{\max\limits_{\substack{L^j_{q+1}/v_j\in U_L^j\\ j=1,\cdots,m}} \sum_{j=1}^{m} L^j_{q+1}/v_j + \max\limits_{(X_1,\ldots,X_{mq})\in U_X^N} \sum_{i=1}^{mq} X_i}{mq}\right) + (1-P)c_T.
\end{aligned}$$

We first consider the optimization problem $\max\limits_{L^i_{q+1}/v_i\in U_L^i, i=1,\cdots,m} \sum_{i=1}^{m} L^i_{q+1}/v_i$. From set U_L^i we have $L^i_{q+1}/v_i \le \beta\sqrt{q+1}/v_i, i = 1,\cdots,m$. As a result, we can conclude

$$\max_{L^i_{q+1}/v_j\in U_L^i, i=1,\cdots,m} \sum_{i=1}^{m} L^i_{q+1}/v_i \le \sum_{i=1}^{m} \beta\sqrt{q+1}/v_i.$$

We now consider optimization problem $\max_{(X_1,\ldots,X_{mq})\in U_X^N} \sum_{i=1}^{mq} X_i$, whose objective can be decomposed as $\sum_{i=1}^{mq} X_i = \sum_{i=1}^{\tau^m} \sum_{j_i\in J_i} Y_{j_i}$ where $Y_{j_i} = \sum\limits_{k=(j_i-1)q+1}^{j_i q} X_k$ is the service time of a set of q customers; this decomposition allows us to focus on the service times Y_{j_i} of sets of customers. Exchanging the order of the summations, we obtain $\sum_{i=1}^{\tau^m} \sum_{j_i\in J_i} Y_{j_i} = \sum_{j=1}^{N} \sum_{i=1}^{\tau^m} Y_{j_i}$ where j_i is defined to be the jth element of the set J_i. From uncertainty set U_X^N with $I = \{1,\ldots,\tau^m\}$, we can write the upper bound $\sum_{j=1}^{N} \sum_{i=1}^{\tau^m} Y_{j_i} = \sum_{j=1}^{N} \sum_{i=1}^{\tau^m} \sum_{k=(j_i-1)q+1}^{j_i q} X_k \le \sum_{j=1}^{N} \left(\sum_{i=1}^{\tau^m} \sum_{k=(j_i-1)q+1}^{j_i q} \mu_k^s + \gamma^s \sqrt{\sum_{i=1}^{\tau^m} \sum_{k=(j_i-1)q+1}^{j_i q} (\sigma_k^s)^2} \right)$, where τ^m is the average number of tours each driver makes. As a result, we conclude that

$$\max_{(X_1,\ldots,X_{mq})\in U_X^N} \sum_{i=1}^{mq} X_i \le \sum_{j=1}^{N} \left(\sum_{i=1}^{\tau^m} \sum_{k=(j_i-1)q+1}^{j_i q} \mu_k^s + \gamma^s \sqrt{\sum_{i=1}^{\tau^m} \sum_{k=(j_i-1)q+1}^{j_i q} (\sigma_k^s)^2} \right).$$

Combining the above analyses, we have that $\dfrac{\max\limits_{L^j_{q+1}/v_j \in U^j_L, j=1,\cdots,m} \sum\limits_{j=1}^{m} L^j_{q+1}/v_j + \max\limits_{(X_1,\ldots,X_{mq}) \in U^N_X} \sum\limits_{i=1}^{mq} X_i}{mq} \leq$
$\dfrac{\sum_{j=1}^{m} \beta\sqrt{q+1}/v_j + \sum_{j=1}^{N}\left(\sum_{i=1}^{\tau^m}\sum_{k=(j_i-1)q+1}^{j_i q} \mu^s_k + \gamma^s\sqrt{\sum_{i=1}^{\tau^m}\sum_{k=(j_i-1)q+1}^{j_i q} \left(\sigma^s_k\right)^2}\right)}{mq}$.

□

Proof. **Lemma 3.3.** The definition of customer system times in Equation (3.12), combined with the departure-time bounds in Equations (3.8) - (3.10), result in the following upper bounds for customers' systems times, for customers $k = 1, \ldots, q$ in sets $j = 1, \ldots, N$ (which encounter an empty system):

$$\begin{aligned} S^{\Theta}_{(j-1)q+k} &= D_{(j-1)q+k} - \sum_{\ell=1}^{(j-1)q+k} A_\ell \\ &\leq T_j - \sum_{\ell=1}^{(j-1)q+k} A_\ell = \sum_{\ell=1}^{jq} A_\ell + \sum_{\ell=(j-1)q+1}^{jq} X_\ell + L^j_{q+1}/v_j - \sum_{\ell=1}^{(j-1)q+k} A_\ell \triangleq B^{\Theta}_{(j-1)q+k}. \end{aligned}$$

For $j > N$, we first solve for the T^{Θ}_j bounds in closed-form, which are presented recursively in Equation (3.11) for $j = N+1, \cdots, m$ and $k = 1, \cdots, q$, replicated here with the substitutions $Y_j = \sum_{\ell=(j-1)q+1}^{jq} X_\ell$ and $Z_j = \sum_{\ell=1}^{jq} A_\ell$: $T^{\Theta}_j = \max\left\{Z_j, T^{\Theta}_{j-N}\right\} + Y_j + L^j_{q+1}/v_j$. Suppose that $j = a^j N + b^j$, where $a^j \in \mathbb{N}^+$ and $b^j \in \{1, \ldots, N\}$ are integers; with this assignment, j is the b^jth member of partition J_{a^j+1}. Using this decomposition, we can expand the recursion to

$j = N+1, \ldots, m$, with the base cases $T_j = Y_j + Z_j + L^j_{q+1}/v_j$ for $j = 1, \ldots, N$:

$$
\begin{aligned}
T_j^\Theta &= \max\left\{Z_j, T^\Theta_{j-N}\right\} + Y_j + L^j_{q+1}/v_j \\
&= \max\left\{Z_j, \max\left\{Z_{j-N}, T^\Theta_{j-2N}\right\} + Y_{j-N} + L^{j-N}_{q+1}/v_{j-N}\right\} + Y_j + L^j_{q+1}/v_j \\
&= \max\left\{Z_j, \max\left\{Z_{j-N}, \max\left\{Z_{j-2N}, T^\Theta_{j-3N}\right\} + Y_{j-2N} + L^{j-2N}_{q+1}/v_{j-2N}\right\} + Y_{j-N} + L^{j-N}_{q+1}/v_{j-N}\right\} \\
&\quad + Y_j + L^j_{q+1}/v_j \\
&\quad \ldots \\
&= \max_{\ell=0,\ldots a^j}\left\{Z_{j-\ell N} + \sum_{k=0}^{\ell}\left(Y_{j-kN} + L^{j-kN}_{q+1}/v_{j-kN}\right)\right\}.
\end{aligned}
$$

Using Equation (3.12), we obtain upper bounds on the system times of all customers $k = 1, \ldots, q$ in sets $j = N+1, \ldots, m$:

$$
\begin{aligned}
S^\Theta_{(j-1)q+k} &= D^\Theta_{(j-1)q+k} - \sum_{\ell=1}^{(j-1)q+k} A_\ell \le \max_{i=0,\ldots a}\left\{Z_{j-iN} + \sum_{\ell=0}^{i}\left(Y_{j-\ell N} + L^{j-\ell N}_{q+1}/v_{j-\ell N}\right)\right\} - \sum_{\ell=1}^{(j-1)q+k} A_\ell \\
&= \max_{i=0,\ldots a^j}\left\{\sum_{\ell=1}^{(j-iN)q} A_\ell + \sum_{\ell=0}^{i}\left(\sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q} X_p + L^{j-\ell N}_{q+1}/v_{j-\ell N}\right)\right\} - \sum_{\ell=1}^{(j-1)q+k} A_\ell \\
&\triangleq B^\Theta_{(j-1)q+k}.
\end{aligned}
$$

Recall that customers are not served FCFS in a tour, but rather in the optimal sequence specified by the optimal TSP tour; if customer $j \in \{1, \ldots, q\}$ is served last in her set, then the upper bound on her system time S_j, B^Θ_j, minus the time it takes the driver to go back to the depot, is tight. $\square$

Proof. **Lemma 3.4.** We first perform the analysis for sets $j = N+1, \cdots, m$, and using

Lemma 3.3, we have that

$$\max_{\mathbf{X}\in U_X^N} B^{\Theta}_{(j-1)q+1}(\mathbf{L},\mathbf{A},\mathbf{X}) =$$
$$\max_{\mathbf{X}\in U_X^N}\left\{\max_{i=0,\ldots a^j}\left\{\sum_{\ell=1}^{(j-iN)q} A_\ell + \sum_{\ell=0}^{i}\left(\sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q} X_p + L_{q+1}^{j-\ell N}/v_{j-\ell N}\right)\right\} - \sum_{\ell=1}^{(j-1)q+1} A_\ell\right\}.$$

Exchanging the order of the maximizations, which results in $a+1$ maximizations over the uncertainty set U_X^N, provides the following inequality:

$$\max_{\mathbf{X}\in U_X^N} B^{\Theta}_{(j-1)q+1}(\mathbf{L},\mathbf{A},\mathbf{X}) \leq$$
$$\max_{i=0,\ldots a^j}\left\{\max_{\mathbf{X}\in U_X^N}\left\{\sum_{\ell=1}^{(j-iN)q} A_\ell + \sum_{\ell=0}^{i}\left(\sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q} X_p + L_{q+1}^{j-\ell N}/v_{j-\ell N}\right)\right\} - \sum_{\ell=1}^{(j-1)q+1} A_\ell\right\} \quad (7)$$

We analyze the above inequality. From uncertainty set U_X^N, for $I=\{j-\ell N : \ell = 0,\ldots,i\}$, we conclude that $\sum_{\ell=0}^{i}\sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q} X_p \leq \sum_{\ell=0}^{i}\sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q}\mu_p^s + \gamma^s\sqrt{\sum_{\ell=0}^{i}\sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q}\left(\sigma_p^s\right)^2}$. Consequently, we conclude that $\max_{\mathbf{X}\in U_X^N}\sum_{\ell=0}^{i}\sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q} X_p \leq \sum_{\ell=0}^{i}\sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q}\mu_p^s + \gamma^s\sqrt{\sum_{\ell=0}^{i}\sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q}\left(\sigma_p^s\right)^2}$, $i=0,\ldots,a^j$. We conclude the proof by analyzing the case of sets $j=1,\ldots,N$. Utilizing Lemma 3.3, consider the worst-case upper bounds on the system times of the first customers in sets $j=1,\ldots,N$:

$$\max_{\mathbf{X}\in U_X^N} B^{\Theta}_{(j-1)q+1}(\mathbf{L},\mathbf{A},\mathbf{X}) = \max_{\mathbf{X}\in U_X^N}\left\{\sum_{\ell=(j-1)q+2}^{jq} A_\ell + \sum_{\ell=(j-1)q+1}^{jq} X_\ell + L_{q+1}^{j}/v_j\right\}.$$

From uncertainty set U_X^N with $I=\{j\}$, and using a similar analysis, we know that

$$\max_{\mathbf{X}\in U_X^N}\sum_{\ell=(j-1)q+1}^{jq} X_\ell \leq \sum_{\ell=(j-1)q+1}^{jq}\mu_\ell^s + \gamma^s\sqrt{\sum_{\ell=(j-1)q+1}^{jq}\left(\sigma_\ell^s\right)^2},$$

and this completes the proof. $\square$

Proof. **Lemma 3.5.** We want to solve the following optimization problems

$$
\max_{\mathbf{A}\in U_A(P)}\left\{\max_{\mathbf{X}\in U_X^N} B^{\Theta}_{(j-1)q+1}(\mathbf{L},\mathbf{A},\mathbf{X})\right\}\le
\begin{cases}
\displaystyle\sum_{\ell=(j-1)q+2}^{jq} A_\ell+\sum_{\ell=(j-1)q+1}^{jq}\mu_\ell^s+\gamma^s\sqrt{\sum_{\ell=(j-1)q+1}^{jq}\left(\sigma_\ell^s\right)^2}+L_{q+1}^j/v_j,\ j=1,\cdots,N\\
\displaystyle\max_{i=0,\ldots a^j}\left\{\sum_{\ell=1}^{(j-iN)q} A_\ell+\sum_{\ell=0}^{i}\sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q}\mu_p^s+\gamma^s\sqrt{\sum_{\ell=0}^{i}\sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q}\left(\sigma_p^s\right)^2}\right.\\
\displaystyle\qquad\left.+\sum_{\ell=0}^{i}\frac{L_{q+1}^{j-\ell N}}{v_{j-\ell N}}-\sum_{\ell=1}^{(j-1)q+1}A_\ell\right\},\ j=N+1,\ldots,m.
\end{cases}
$$

where the inequality is due to Lemma 3.4. For the $j=N+1,\ldots,m$ case, exchanging the order of the maximizations, results in the following inequality for $j=N+1,\ldots,m$:

$$
\max_{\mathbf{A}\in U_A(P)}\left\{\max_{\mathbf{X}\in U_X^N} B^{\Theta}_{(j-1)q+1}(\mathbf{L},\mathbf{A},\mathbf{X})\right\}\le\max_{i=0,\ldots a^j}\left\{\max_{\mathbf{A}\in U_A(P)}\left\{\sum_{\ell=1}^{(j-iN)q}A_\ell-\sum_{\ell=1}^{(j-1)q+1}A_\ell\right\}\right.
$$
$$
\left.+\sum_{\ell=0}^{i}\sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q}\mu_p^s+\gamma^s\sqrt{\sum_{\ell=0}^{i}\sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q}\left(\sigma_p^s\right)^2}+\sum_{\ell=0}^{i}\frac{L_{q+1}^{j-\ell N}}{v_{j-\ell N}}\right\}.
$$

For expository simplicity, we analyze the outer maximization over i separately for $i=0$ and

$i = 1, \ldots, a$. Therefore, the above inequality can be written as

$$\begin{aligned}
&\max_{\mathbf{A}\in U_A(P)} \left\{ \max_{\mathbf{X}\in U_X^N} B^{\Theta}_{(j-1)q+1}(\mathbf{L},\mathbf{A},\mathbf{X}) \right\} \leq \\
&\max\left\{ \max_{i=1,\ldots a^j} \left\{ -\min_{\mathbf{A}\in U_A(P)} \sum_{\ell=(j-iN)q+1}^{(j-1)q+1} A_\ell + \sum_{\ell=0}^{i} \sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q} \mu_p^s + \gamma^s \sqrt{\sum_{\ell=0}^{i} \sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q} \left(\sigma_p^s\right)^2} \right.\right. \\
&\qquad \left. + \sum_{\ell=0}^{i} \frac{L_{q+1}^{j-\ell N}}{v_{j-\ell N}} \right\}, \max_{\mathbf{A}\in U_A(P)} \sum_{\ell=(j-1)q+2}^{jq} A_\ell + \sum_{p=(j-1)q+1}^{jq} \mu_p^s + \gamma^s \sqrt{\sum_{p=(j-1)q+1}^{jq} \left(\sigma_p^s\right)^2 + L_{q+1}^j / v_j} \Bigg\}, \\
&j = N+1, \cdots, m.
\end{aligned}$$

We first analyze $\min_{\mathbf{A}\in U_A(P)} \sum_{\ell=(j-iN)q+1}^{(j-1)q+1} A_\ell$. From uncertainty set $U_A(P)$, we know $\sum_{\ell=(j-iN)q+1}^{(j-1)q+1} A_\ell \geq \sum_{\ell=(j-iN)q+1}^{(j-1)q+1} \mu_\ell^a/P - \gamma^a \sqrt{\sum_{\ell=(j-iN)q+1}^{(j-1)q+1} \left(\sigma_\ell^a/P\right)^2}$. Therefore, we conclude that $\min_{\mathbf{A}\in U_A(P)} \sum_{\ell=(j-iN)q+1}^{(j-1)q+1} A_\ell \geq \sum_{\ell=(j-iN)q+1}^{(j-1)q+1} \mu_\ell^a/P - \gamma^a \sqrt{\sum_{\ell=(j-iN)q+1}^{(j-1)q+1} \left(\sigma_\ell^a/P\right)^2}$. Similarly, from $U_A(P)$, we have the upper bound $\sum_{\ell=(j-1)q+2}^{jq} A_\ell \leq \sum_{\ell=(j-1)q+2}^{jq} \mu_\ell^a/P + \gamma^a \sqrt{\sum_{\ell=(j-1)q+2}^{jq} \left(\sigma_\ell^a/P\right)^2}$. This shows $\max_{\mathbf{A}\in U_A(P)} \sum_{\ell=(j-1)q+2}^{jq} A_\ell \leq \sum_{\ell=(j-1)q+2}^{jq} \mu_\ell^a/P + \gamma^a \sqrt{\sum_{\ell=(j-1)q+2}^{jq} \left(\sigma_\ell^a/P\right)^2}$; note that this expression completes the proof of the

$j = 1, \ldots, N$ case. As a result, we have that

$$\max_{\mathbf{A} \in U_A(P)} \left\{ \max_{\mathbf{X} \in U_X^N} B^{\Theta}_{(j-1)q+1}(\mathbf{L}, \mathbf{A}, \mathbf{X}) \right\} \leq$$

$$\max \Bigg\{ \max_{i=1,\ldots a^j} \Bigg\{ -\sum_{\ell=(j-iN)q+1}^{(j-1)q+1} \frac{\mu_\ell^a}{P} + \gamma^a \sqrt{\sum_{\ell=(j-iN)q+1}^{(j-1)q+1} \left(\frac{\sigma_\ell^a}{P}\right)^2} + \sum_{\ell=0}^{i} \sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q} \mu_p^s +$$

$$\gamma^s \sqrt{\sum_{\ell=0}^{i} \sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q} \left(\sigma_p^s\right)^2} + \sum_{\ell=0}^{i} \frac{L_{q+1}^{j-\ell N}}{v_{j-\ell N}} \Bigg\}, \sum_{\ell=(j-1)q+2}^{jq} \mu_\ell^a/P +$$

$$\gamma^a \sqrt{\sum_{\ell=(j-1)q+2}^{jq} \left(\sigma_\ell^a/P\right)^2} + \sum_{\ell=(j-1)q+1}^{jq} \mu_\ell^s + \gamma^s \sqrt{\sum_{\ell=(j-1)q+1}^{jq} \left(\sigma_\ell^s\right)^2} + L_{q+1}^j/v_j \Bigg\},$$

$j = N+1, \cdots, m.$

□

Proof. **Lemma 3.6.** The proof is similar to those of Lemmas 3.4 - 3.5: we have that

$$\max_{\mathbf{L}/\mathbf{v} \in \times_{j=1}^m U_L^j} \left\{ \max_{\mathbf{A} \in U_A(P)} \left\{ \max_{\mathbf{X} \in U_X^N} B_{(j-1)q+1}(\mathbf{L}, \mathbf{A}, \mathbf{X}) \right\} \right\} \leq \max_{\mathbf{L}/\mathbf{v} \in \times_{j=1}^m U_L^j} \left\{ \max \left\{ \max_{1 \leq k \leq j-1} \{f_k^N\}, g^N \right\} \right\}$$

$$\leq \max \left\{ \max_{1 \leq k \leq j-1} \left\{ \max_{\mathbf{L}/\mathbf{v} \in \times_{j=1}^m U_L^j} \left\{ f_k^N \right\} \right\}, \max_{\mathbf{L}/\mathbf{v} \in \times_{j=1}^m U_L^j} \left\{ g^N \right\} \right\} \leq \max \left\{ \max_{1 \leq k \leq j-1} \left\{ \hat{f}_k^N \right\}, \hat{g}^N \right\},$$

where the first inequality is due to Lemma 3.5, the second inequality is due to exchanging the order of the optimizations (resulting in j optimizations over $\times_{j=1}^m U_L^j$ rather than one), and the final inequality is due to $\max_{L_{q+1}^i/v_i \in U_L^i} L_{q+1}^i/v_i \leq \beta\sqrt{q+1}/v_i$ for any i. □

Proof. **Lemma 3.7.** We use mathematical induction for the proof, which is inspired by a similar proof in [16]. We show that the set systems times for the FCFS policy are at most the upper bounds B_i^{Θ}, for all i.

Base case: For the first N sets, as the system is empty, the sets start getting service upon their arrival. Similarly to the Θ analysis, the system times of customers in set $j = 1, \ldots, N$

under a FCFS policy are defined as $S^{\text{FCFS}}_{(j-1)q+k}$, and upper bounds on these system times are calculated as

$$S^{\text{FCFS}}_{(j-1)q+k} \leq B^{\text{FCFS}}_{(j-1)q+k} \triangleq \sum_{\ell=(j-1)q+k+1}^{jq} A_\ell + \sum_{\ell=(j-1)q+1}^{jq} X_\ell + L^j_{q+1}/v_j,\ j=1,\cdots,N,\ k=1,\cdots,q.$$

Thus, we conclude $B^{\text{FCFS}}_{(j-1)q+k} = B^{\Theta}_{(j-1)q+k},\ j=1,\cdots,N,\ k=1,\cdots,q$ and this completes the proof for the base case.

Inductive step: We next assume the inductive hypothesis holds up to an arbitrary set $n-1$, and we then show the hypothesis holds for n. Similarly to Inequality (3.11), the departure times of customers in set j under the FCFS policy have the following upper bounds for $j=N+1,\cdots,m,\ k=1,\cdots,q$

$$D^{\text{FCFS}}_{(j-1)q+k} \leq T^{\text{FCFS}}_j \triangleq \max\left\{T^{\text{FCFS}}_{[j-N]}, \sum_{\ell=1}^{jq} A_\ell\right\} + \sum_{\ell=(j-1)q+1}^{jq} X_\ell + L^j_{q+1}/v_j,$$

where $T^{\text{FCFS}}_{[j-N]}$ refers to the departure time of the $(j-N)$th departed set; since overtaking might happen, the $(j-N)$th departed set might differ from the $(j-N)$th arrived set. As a result, the upper bounds on the system times of customers in set j under the FCFS policy are calculated as

$$S^{\text{FCFS}}_{(j-1)q+k} \leq B^{\text{FCFS}}_{(j-1)q+k} \ \triangleq \ T^{\text{FCFS}}_j - \sum_{\ell=1}^{(j-1)q+k} A_\ell,\ j=N+1,\cdots,m,\ k=1,\cdots,q. \quad (8)$$

Let an arbitrary set s be the last set, prior to set n, being served by the *same* driver who is now serving set n. The system time of the kth customer, $k=1,\cdots,q$, in set n, is written as

$$\begin{aligned}
S^{\text{FCFS}}_{(n-1)q+k} \leq B^{\text{FCFS}}_{(n-1)q+k} &\triangleq \max\left\{T^{\text{FCFS}}_s, \sum_{\ell=1}^{nq} A_\ell\right\} + \sum_{\ell=(n-1)q+1}^{nq} X_\ell + L^n_{q+1}/v_n - \sum_{\ell=1}^{(n-1)q+k} A_\ell \\
&= \max\left\{B^{\text{FCFS}}_{(s-1)q+1} + \sum_{\ell=1}^{(s-1)q+1} A_\ell, \sum_{\ell=1}^{nq} A_\ell\right\} + \sum_{\ell=(n-1)q+1}^{nq} X_\ell + L^n_{q+1}/v_n \\
&\quad - \sum_{\ell=1}^{(n-1)q+k} A_\ell,
\end{aligned}$$

where the second equality comes from the definition in Equation (8). According to the inductive hypothesis, there is no overtaking up to set $n-1$, which includes set s, which implies that $B^{\text{FCFS}}_{(s-1)q+1} = B^{\Theta}_{(s-1)q+1}$. We next replace $B^{\text{FCFS}}_{(s-1)q+1}$ with $B^{\Theta}_{(s-1)q+1}$ in the above expression, where $s = aN + b$ for some integers a and $b \leq N$ such that $a = \lceil \frac{s}{N} \rceil - 1$, and we apply Lemma 3.3:

$$\begin{aligned}
B^{\text{FCFS}}_{(n-1)q+k} &= \max\left\{\max_{i=0,\dots a}\left\{\sum_{\ell=1}^{(s-iN)q} A_\ell + \sum_{\ell=0}^{i}\left(\sum_{p=(s-\ell N-1)q+1}^{(s-\ell N)q} X_p + L^{s-\ell N}_{q+1}/v_{s-\ell N}\right)\right\}, \sum_{\ell=1}^{nq} A_\ell\right\} \\
&\quad + \sum_{\ell=(n-1)q+1}^{nq} X_\ell + \frac{L^n_{q+1}}{v_n} - \sum_{\ell=1}^{(n-1)q+k} A_\ell \\
&= \max\left\{\max_{i=0,\dots a}\left\{\sum_{\ell=1}^{(s-iN)q} A_\ell + \sum_{\ell=0}^{i}\left(\sum_{p=(s-\ell N-1)q+1}^{(s-\ell N)q} X_p + L^{s-\ell N}_{q+1}/v_{s-\ell N}\right)\right\}\right. \\
&\qquad + \sum_{\ell=(n-1)q+1}^{nq} X_\ell + L^n_{q+1}/v_n - \sum_{\ell=1}^{(n-1)q+k} A_\ell, \sum_{\ell=1}^{nq} A_\ell + \sum_{\ell=(n-1)q+1}^{nq} X_\ell \\
&\qquad \left. + L^n_{q+1}/v_n - \sum_{\ell=1}^{(n-1)q+k} A_\ell\right\}. \qquad (9)
\end{aligned}$$

Since sets s and n are served by the same driver, this implies that the other $N-1$ drivers are serving the sets arrived after s and before n. Due to the inductive hypothesis of no overtaking for sets $1, \ldots, n-1$, we conclude that $n = s + N$. We replace $s = n - N$ in Equation (9) and then replace $N + \eta N$ with $(1+\eta)N$, for $\eta \in \{i, \ell\}$ and obtain:

$$
\begin{aligned}
&B^{\text{FCFS}}_{(n-1)q+k} = \\
&\max\left\{ \max_{i=0,\ldots a} \left\{ \sum_{\ell=1}^{(n-(i+1)N)q} A_\ell + \sum_{\ell=0}^{i} \left(\sum_{p=(n-(\ell+1)N-1)q+1}^{(n-(\ell+1)N)q} X_p + \frac{L_{q+1}^{n-(\ell+1)N}}{v}_{n-(\ell+1)N} \right) \right\} + \right. \\
&\left. \sum_{\ell=(n-1)q+1}^{nq} X_\ell + L_{q+1}^{n}/v_n - \sum_{\ell=1}^{(n-1)q+k} A_\ell, \sum_{\ell=1}^{nq} A_\ell + \sum_{\ell=(n-1)q+1}^{nq} X_\ell + L_{q+1}^{n}/v_n - \sum_{\ell=1}^{(n-1)q+k} A_\ell \right\},
\end{aligned}
$$

Next, note that $\displaystyle\sum_{\ell=0}^{i} \left(\sum_{p=(n-(\ell+1)N-1)q+1}^{(n-(\ell+1)N)q} X_p + \frac{L_{q+1}^{n-(\ell+1)N}}{v_{n-(\ell+1)N}} \right) = \sum_{\ell=1}^{i+1} \left(\sum_{p=(n-\ell N-1)q+1}^{(n-\ell N)q} X_p + \frac{L_{q+1}^{n-\ell N}}{v_{n-\ell N}} \right)$. If we replace the left-hand side of this equation with its right-hand side in our expression for $B^{\text{FCFS}}_{(n-1)q+k}$, and then replace $i+1,\ i = 0, \cdots, a$ with $i,\ i = 1, \cdots, a+1$ and perform the maximization over $i = 1, \cdots, a+1$, and then incorporate $\displaystyle\sum_{\ell=(n-1)q+1}^{nq} X_\ell + L_{q+1}^{n}/v_n$ inside the inner maximization, we obtain

$$
\begin{aligned}
B^{\text{FCFS}}_{(n-1)q+k} \;=\; \max&\left\{ \max_{i=1,\ldots a+1} \left\{ \sum_{\ell=1}^{(n-iN)q} A_\ell + \sum_{\ell=0}^{i} \left(\sum_{p=(n-\ell N-1)q+1}^{(n-\ell N)q} X_p + \frac{L_{q+1}^{n-\ell N}}{v_{n-\ell N}} \right) \right\} - \sum_{\ell=1}^{(n-1)q+k} A_\ell, \right. \\
&\left. \sum_{\ell=1}^{nq} A_\ell + \sum_{\ell=(n-1)q+1}^{nq} X_\ell + L_{q+1}^{n}/v_n - \sum_{\ell=1}^{(n-1)q+k} A_\ell \right\}.
\end{aligned}
$$

Finally, we fold the second expression of the outer maximization into the inner maximization, which results in the simplified expression

$$
B^{\text{FCFS}}_{(n-1)q+k} = \max_{i=0,\ldots a+1} \left\{ \sum_{\ell=1}^{(n-iN)q} A_\ell + \sum_{\ell=0}^{i} \left(\sum_{p=(n-\ell N-1)q+1}^{(n-\ell N)q} X_p + \frac{L_{q+1}^{n-\ell N}}{v_{n-\ell N}} \right) \right\} - \sum_{\ell=1}^{(n-1)q+k} A_\ell = B^{\Theta}_{(n-1)q+k},
$$

since $n = s + N = (a+1)N + b$; c.f., Lemma 3.3. Thus, we can conclude that $B^{\text{FCFS}}_{(j-1)q+k} =$

$B^{\Theta}_{(j-1)q+k},\ j \geq N+1,\ k = 1, \cdots, q$; in other words, the upper bound on a customer's system time under FCFS has the same bound as under policy Θ. [16] show that the no-overtaking policy is equivalent to the FCFS policy for non-decreasing service times. $\square$

Proof. **Proposition 3.1.** We want to solve Problem (3.14):

$$
\begin{aligned}
z = \min_{P \in (0,1], w>0} \quad & P\Bigg(w \frac{\sum_{j=1}^{m} \beta\sqrt{q+1}/v_j + \sum_{j=1}^{N} \Big(\sum_{i=1}^{\tau^m} \sum_{k=(j_i-1)q+1}^{j_i q} \mu_k^s}{mq} + \\
& \frac{\gamma^s \sqrt{\sum_{i=1}^{\tau^m} \sum_{k=(j_i-1)q+1}^{j_i q} (\sigma_k^s)^2} \Big)}{mq} - c_T \Bigg) + c_T \\
\text{s.t.} \quad & \max_{i=1,\ldots,a} \Bigg\{ - \sum_{\ell=(j-iN)q+1}^{(j-1)q+1} \mu_\ell^a/P + \gamma^a \sqrt{\sum_{\ell=(j-iN)q+1}^{(j-1)q+1} (\sigma_\ell^a/P)^2} + \sum_{\ell=0}^{i} \sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q} \mu_p^s \\
& \quad + \gamma^s \sqrt{\sum_{\ell=0}^{i} \sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q} \left(\sigma_p^s\right)^2} + \sum_{\ell=0}^{i} \frac{\beta\sqrt{q+1}}{v_{j-\ell N}} \Bigg\} \leq \alpha,\ j = N+1, \ldots, m \\
& \sum_{\ell=(j-1)q+2}^{jq} \frac{\mu_\ell^a}{P} + \gamma^a \sqrt{\sum_{\ell=(j-1)q+2}^{jq} \left(\frac{\sigma_\ell^a}{P}\right)^2} + \sum_{\ell=(j-1)q+1}^{jq} \mu_\ell^s + \gamma^s \sqrt{\sum_{\ell=(j-1)q+1}^{jq} \left(\sigma_\ell^s\right)^2} \\
& + \frac{\beta\sqrt{q+1}}{v_j} \leq \alpha, j = 1, \ldots, m \\
& P\left(\frac{\sum_{i=1}^{mq} \mu_i^s}{m} + \frac{\sum_{i=1}^{m} \beta\sqrt{q+1}/v_i}{m} \right) / \left(N \frac{\sum_{i=1}^{mq} \mu_i^a}{m} \right) \leq 1 - \epsilon \\
& wP\left(\frac{\sum_{i=1}^{mq} \mu_i^s}{m} + \frac{\sum_{i=1}^{m} \beta\sqrt{q+1}/v_i}{m} \right) / \left(N \frac{\sum_{i=1}^{mq} \mu_i^a}{m} \right) \geq \frac{\sum_{i=1}^{N} \mu_{(i)}^k + \gamma^k \sqrt{N} \sigma^k}{N}.
\end{aligned}
\tag{10}
$$

First, we solve Problem (10) for w as a function of P. The objective function is an increasing function of w and according to the fourth constraint, for feasibility we require $w \geq \frac{\frac{\sum_{i=1}^{N} \mu_{(i)}^k + \gamma^k \sqrt{N}\sigma^k}{N} \left(\frac{\sum_{i=1}^{mq} \mu_i^a}{m}\right)}{P\left(\frac{\sum_{i=1}^{mq} \mu_i^s}{m} + \frac{\sum_{i=1}^{m} \beta\sqrt{q+1}/v_i}{m}\right)}$. Thus, we conclude $w^*(P) = \frac{\frac{\sum_{i=1}^{N} \mu_{(i)}^k + \gamma^k \sqrt{N}\sigma^k}{N} \left(\frac{\sum_{i=1}^{mq} \mu_i^a}{m}\right)}{P\left(\frac{\sum_{i=1}^{mq} \mu_i^s}{m} + \frac{\sum_{i=1}^{m} \beta\sqrt{q+1}/v_i}{m}\right)}$, which we interpret as the crowd supply function. We next replace w in the model with the crowd supply function and calculate the optimal crowd hourly wage indirectly via optimizing N. Problem (10), for fixed w, or equivalently fixed N from the crowd supply function, is written as

$$
\begin{aligned}
z(N) = \min_{P\in(0,1]} \quad & \frac{\frac{\sum_{i=1}^{N}\mu_{(i)}^{k}+\gamma^{k}\sqrt{N}\sigma^{k}}{N}\left(\frac{\sum_{i=1}^{mq}\mu_i^a}{m}\right)}{\left(\frac{\sum_{i=1}^{mq}\mu_i^s}{m}+\frac{\sum_{i=1}^{m}\beta\sqrt{q+1}/v_i}{m}\right)}\times \\
& \frac{\sum_{j=1}^{m}\beta\sqrt{q+1}/v_j+\sum_{j=1}^{N}\left(\sum_{i=1}^{\tau^m}\sum_{k=(j_i-1)q+1}^{j_iq}\mu_k^s+\gamma^s\sqrt{\sum_{i=1}^{\tau^m}\sum_{k=(j_i-1)q+1}^{j_iq}\left(\sigma_k^s\right)^2}\right)}{mq} \\
& +(1-P)c_T \\
\text{s.t.} \quad & \max_{i=1,\ldots,a}\Bigg\{-\sum_{\ell=(j-iN)q+1}^{(j-1)q+1}\mu_\ell^a/P+\gamma^a\sqrt{\sum_{\ell=(j-iN)q+1}^{(j-1)q+1}\left(\sigma_\ell^a/P\right)^2}+\sum_{\ell=0}^{i}\sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q}\mu_p^s+ \\
& \qquad \gamma^s\sqrt{\sum_{\ell=0}^{i}\sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q}\left(\sigma_p^s\right)^2}+\sum_{\ell=0}^{i}\frac{\beta\sqrt{q+1}}{v_{j-\ell N}}\Bigg\}\le\alpha,\ j=N+1,\ldots,m \\
& \sum_{\ell=(j-1)q+2}^{jq}\frac{\mu_\ell^a}{P}+\gamma^a\sqrt{\sum_{\ell=(j-1)q+2}^{jq}\left(\sigma_\ell^a/P\right)^2}+\sum_{\ell=(j-1)q+1}^{jq}\mu_\ell^s+\gamma^s\sqrt{\sum_{\ell=(j-1)q+1}^{jq}\left(\sigma_\ell^s\right)^2} \\
& +\frac{\beta\sqrt{q+1}}{v_j}\le\alpha, j=1,\ldots,m \\
& P\left(\frac{\sum_{i=1}^{mq}\mu_i^s}{m}+\frac{\sum_{i=1}^{m}\beta\sqrt{q+1}/v_i}{m}\right)/\left(N\frac{\sum_{i=1}^{mq}\mu_i^a}{m}\right)\le 1-\epsilon.
\end{aligned}
\tag{11}
$$

First, we analyze the first set of constraints in Problem (11) for $j = N+1,\ldots,m$: Let $F_i^j = \sum_{\ell=0}^{i}\sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q}\mu_p^s+\gamma^s\sqrt{\sum_{\ell=0}^{i}\sum_{p=(j-\ell N-1)q+1}^{(j-\ell N)q}\left(\sigma_p^s\right)^2}+\sum_{\ell=0}^{i}\frac{\beta\sqrt{q+1}}{v_{j-\ell N}}$, $E_i^j = \sum_{\ell=(j-iN)q+1}^{(j-1)q+1}\mu_\ell^a-\gamma^a\sqrt{\sum_{\ell=(j-iN)q+1}^{(j-1)q+1}(\sigma_\ell^a)^2}$, $j = N+1,\ldots,m$. As a result, we can write the constraints as $\max_{i=1,\ldots,a}\left\{-E_i^j/P+F_i^j\right\}\le\alpha,\ j=N+1,\ldots,m$. We consider the constraints separately for each $j = N+1,\ldots,m$ and $i = 1,\ldots,a$: $-E_i^j \le P\left(\alpha-F_i^j\right)$. For analyzing these constraints, we next define the following sets, which are defined based on the sign of the multiplier P: $s_1^j = \left\{i=1,\ldots,a^j : \alpha-F_i^j>0\right\}$, $s_2^j = \left\{i=1,\ldots,a^j : \alpha-F_i^j<0\right\}$, $s_3^j = \left\{i=1,\ldots,a^j : \alpha-F_i^j=0\right\}$ for $j = N+1,\ldots,m$. For the $i\in s_1^j$ or $i\in s_3^j$, the constraint $-E_i^j\le P\left(\alpha-F_i^j\right)$ is always satisfied as $E_i^j = \min_{\mathbf{A}\in U_A(P)}\sum_{\ell=(j-iN)q+1}^{(j-1)q+1}A_\ell\ge 0$. Thus, we only analyze set s_2^j. We can write the first set of constraints in Problem (11) as

$P \leq \min_{i\in s_2^j}\left\{\frac{E_i^j}{F_i^j-\alpha}\right\},\ j = N+1,\ldots,m$. We can combine each of the above constraints over j and write it as $P \leq \min\limits_{j=N+1,\ldots,m}\left\{\min\limits_{i\in s_2^j}\left\{\dfrac{E_i^j}{F_i^j-\alpha}\right\}\right\}$. Next, we analyze the second set of constraints in Problem (11). We define $G_j = \sum_{\ell=(j-1)q+2}^{jq}\mu_\ell^a + \gamma^a\sqrt{\sum_{\ell=(j-1)q+2}^{jq}(\sigma_\ell^a)^2},\ j = 1,\ldots,m$ and write the second set of constraints as $\frac{G_j}{P} \leq \alpha - F_{i=0}^j,\ j = 1,\ldots,m$. As the left-hand-side is non-negative, for feasibility we require $\alpha > F_{i=0}^j,\ j = 1,\ldots,m$. Under these conditions, the second constraint can be written as $\frac{G_j}{\alpha-F_{i=0}^j} \leq P,\ j = 1,\ldots,m$, or equivalently $\max\limits_{j=1,\ldots,m}\left\{\dfrac{G_j}{\alpha-F_{i=0}^j}\right\} \leq P$ for $\alpha > \max_{j=1,\ldots,m}\left\{F_{i=0}^j\right\}$. Finally, the third constraint can be written as $P \leq \dfrac{(1-\epsilon)N\frac{\sum_{i=1}^{mq}\mu_i^a}{m}}{\frac{\sum_{i=1}^{mq}\mu_i^s}{m}+\frac{\sum_{i=1}^{m}\beta\sqrt{q+1}/v_i}{m}}$. Let $\overline{P}^N = \min\limits_{j=N+1,\ldots,m}\left\{\min\limits_{i\in s_2^j}\left\{\dfrac{E_i^j}{F_i^j-\alpha}\right\}\right\}$, $\hat{P} = \max\limits_{j=1,\ldots,m}\left\{\dfrac{G_j}{\alpha-F_{i=0}^j}\right\}$, and $\tilde{P} = \dfrac{(1-\epsilon)N\frac{\sum_{i=1}^{mq}\mu_i^a}{m}}{\frac{\sum_{i=1}^{mq}\mu_i^s}{m}+\frac{\sum_{i=1}^{m}\beta\sqrt{q+1}/v_i}{m}}$. Thus, Problem (11) for $\alpha > \max_{j=1,\ldots,m}\left\{F_{i=0}^j\right\}$ can be written as

$$
\begin{aligned}
z(N) = \min_P \quad & \frac{\frac{\sum_{i=1}^{N}\mu_{(i)}^k+\gamma^k\sqrt{N}\sigma^k}{N}\left(\frac{\sum_{i=1}^{mq}\mu_i^a}{m}\right)}{\left(\frac{\sum_{i=1}^{mq}\mu_i^s}{m}+\frac{\sum_{i=1}^{m}\beta\sqrt{q+1}/v_i}{m}\right)}\times \\
& \frac{\sum_{j=1}^{m}\beta\sqrt{q+1}/v_j+\sum_{j=1}^{N}\left(\sum_{i=1}^{\tau^m}\sum_{k=(j_i-1)q+1}^{j_iq}\mu_k^s+\gamma^s\sqrt{\sum_{i=1}^{\tau^m}\sum_{k=(j_i-1)q+1}^{j_iq}(\sigma_k^s)^2}\right)}{mq} \\
& +(1-P)c_T \\
\text{s.t.} \quad & \hat{P} \leq P \leq \min\left\{\overline{P}^N, \tilde{P}, 1\right\}.
\end{aligned}
\tag{12}
$$

Problem (12) is feasible if the smallest upper bound on P is at least its largest lower bound: $\hat{P} \leq \min\left\{\overline{P}^N, \tilde{P}, 1\right\}$. Finally, the optimal proportion $P^* = \min\left\{\overline{P}^N, \tilde{P}, 1\right\}$, since the objective is decreasing in P. □

Proof. **Proposition 3.2.** First we solve Problem (3.15) for w as a function of P: The first two constraints in the problem are independent of w so assuming the first two constraints hold at feasibility we have:

$$\begin{aligned} z_s(P) = \min_{w>0} \quad & Pw\left(\frac{\hat{\beta}\sqrt{q+1}/v}{q} + \mu^s\right) + (1-P)c_T \\ \text{s.t.} \quad & w\frac{P\lambda}{qN}\left(\hat{\beta}\sqrt{q+1}/v + q\mu^s\right) \geq \mu^k. \end{aligned} \tag{13}$$

As the objective function of Problem (13) is increasing in w, the optimal solution is $w^*(P) = \frac{\mu^k qN}{P\lambda\left(\hat{\beta}\sqrt{q+1}/v+q\mu^s\right)}$ which we interpret as the crowd supply function. We next replace w in the model with the crowd supply function and calculate the optimal crowd hourly wage indirectly via optimizing N. Problem (3.15), for fixed w, or equivalently fixed N from the crowd supply function, is

$$\begin{aligned} z_s(N) = \min_{P\in(0,1]} \quad & \frac{\mu^k N}{\lambda} + (1-P)c_T \\ \text{s.t.} \quad & \frac{(q-1)}{2\lambda} + \frac{\frac{P\lambda}{q}\left(\frac{q}{P^2\lambda^2} + \frac{1}{N^2}\left({\sigma_L}^2/v^2 + q{\sigma_s}^2\right)\right)}{2\left(1 - \frac{P\lambda}{qN}\left(\hat{\beta}\sqrt{q+1}/v + q\mu^s\right)\right)} + \hat{\beta}\sqrt{q+1}/v + \frac{(q+1)\mu^s}{2} \leq \alpha \\ & \frac{P\lambda}{qN}\left(\hat{\beta}\sqrt{q+1}/v + q\mu^s\right) \leq 1 - \epsilon. \end{aligned} \tag{14}$$

According to the first two constraints, for feasibility we require $\frac{(q-1)}{2\lambda} + \hat{\beta}\sqrt{q+1}/v + \frac{(q+1)\mu^s}{2} \leq \alpha$. The first constraint can be written as $P_1 \leq P \leq P_2$ where $P_1 = \frac{\alpha - \frac{(q-1)}{2\lambda} - \hat{\beta}\sqrt{q+1}/v - \frac{(q+1)\mu^s}{2} - \sqrt{\Delta}}{2U}$, $P_2 = \frac{\alpha - \frac{(q-1)}{2\lambda} - \hat{\beta}\sqrt{q+1}/v - \frac{(q+1)\mu^s}{2} + \sqrt{\Delta}}{2U}$, $\Delta = \left(\alpha - \frac{(q-1)}{2\lambda} - \hat{\beta}\sqrt{q+1}/v - \frac{(q+1)\mu^s}{2}\right)^2 - 4U/\lambda$, and $U = \frac{\lambda}{qN^2}\left({\sigma_L}^2/v^2 + q{\sigma_s}^2\right) + \frac{2\lambda}{qN}(\alpha - \frac{(q-1)}{2\lambda} - \hat{\beta}\sqrt{q+1}/v - \frac{(q+1)\mu^s}{2})(\hat{\beta}\sqrt{q+1}/v + q\mu^s)$. Note that $P_1, P_2 \geq 0$ due to $\frac{(q-1)}{2\lambda} + \hat{\beta}\sqrt{q+1}/v + \frac{(q+1)\mu^s}{2} \leq \alpha$, and for feasibility we require $\Delta \geq 0$. The second constraint can be written as $P \leq P_3$ where $P_3 = \dfrac{(1-\epsilon)qN}{\lambda\left(\hat{\beta}\sqrt{q+1}/v + q\mu^s\right)}$. As a result, we can write the problem as the following

$$\begin{aligned} z_s(N) = \min_{P} \quad & \frac{\mu^k N}{\lambda} + (1-P)c_T \\ \text{s.t.} \quad & P_1 \leq P \leq \min\{P_2, P_3, 1\}. \end{aligned} \tag{15}$$

The problem is feasible for $P_1 \leq \min\{P_2, P_3, 1\}$, and as the objective function is decreasing in P, the optimal solution is $P^* = \min\{P_2, P_3, 1\}$. $\square$

Proof. **Theorem 4.1.** The objective function π_b is concave in K_w^b and can be solved via the first-order derivative. The optimal solution is equal to $K_w^{b\,*} = \max\left\{\frac{\mu_A - c_w}{2}, 0\right\}$. If we input $K_w^{b\,*}$ into π_b we get $\pi_b = \frac{(\mu_A - c_w)^2}{4}$ for $\mu_A - c_w \geq 0$ and $\pi_b = 0$ for $\mu_A - c_w < 0$. $\square$

Proof. **Lemma 4.1.** If $K_f > \bar{S}$, then from (4.3) we conclude $c_f > \gamma$. Additionally, from the probability of participation for provider i, which is $G\left(c_f \frac{K_f}{\bar{S}}\right)$, we conclude that $G\left(c_f \frac{K_f}{\bar{S}}\right) = 1$ due to having $c_f \frac{K_f}{\bar{S}} > c_f > \gamma$, as this means that provider i is able to earn the maximum unit revenue γ for every unit of her excess capacity S_i. Note that if $G\left(c_f \frac{K_f}{\bar{S}}\right) < 1$ for $c_f \frac{K_f}{\bar{S}} > \gamma$ then provider i's participation contraint can never be satisfied even when all her excess capacity is used at the maximum possible price γ and therefore provider i would not be in the supply $\{1, \ldots, M\}$ in the first place. If $G\left(c_f \frac{K_f}{\bar{S}}\right) = 1,\ i = 1, \ldots, M$, then from (4.2) we conclude $\bar{S} = \sum_{i=1}^{M} S_i$, and this means that $K_f > \bar{S}$ is only feasible when $\bar{S} \geq \sum_{i=1}^{M} S_i$. Below we show that the case where $K_f > \sum_{i=1}^{M} S_i$ is sub-optimal to the case where $K_f = \sum_{i=1}^{M} S_i$ and we then conclude that at optimality the ineqaulity $K_f \leq \sum_{i=1}^{M} S_i$ holds.

Let $K_f^{(1)}$ be such that $K_f^{(1)} = \sum_{i=1}^{M} S_i$ and $K_f^{(2)}$ such that $K_f^{(2)} > \sum_{i=1}^{M} S_i$. We compare the firm's profit under $K_f^{(1)}$ and $K_f^{(2)}$:

For $K_f^{(1)}$, from (4.5), the firm's profit is

$$\pi(K_f^{(1)}) =$$
$$E_A\bigg[\max_{K_f\geq 0} E_{(S_1,\ldots,S_M)}\bigg[\bigg(A-K_w-\sum_{i=1}^{M}S_i-c_w\bigg)K_w+\bigg(A-K_w-\sum_{i=1}^{M}S_i-\gamma\bigg)\sum_{i=1}^{M}S_i\bigg]\bigg].$$

For $K_f^{(2)}$, from (4.5), the firm's profit is

$$\pi(K_f^{(2)}) =$$
$$E_A\bigg[\max_{K_f\geq 0} E_{(S_1,\ldots,S_M)}\bigg[\bigg(A-K_w-\sum_{i=1}^{M}S_i-c_w\bigg)K_w+\bigg(A-K_w-\sum_{i=1}^{M}S_i-\gamma\frac{K_f^{(2)}}{\sum_{i=1}^{M}S_i}\bigg)\sum_{i=1}^{M}S_i\bigg]\bigg].$$

By comparing $\pi(K_f^{(1)})$ and $\pi(K_f^{(2)})$, we conclude $\pi(K_f^{(1)}) > \pi(K_f^{(2)})$ and hence $K_f = K_f^{(2)}$ is sub-optimal to $K_f = K_f^{(1)}$. As a result of the above analysis, we conclude that at optimality we have $\min\{K_f, \bar{S}(K_f)\} = K_f$ and $K_f \leq \sum_{i=1}^{M} S_i$, and this completes the proof. $\square$

Proof. **Theorem 4.2.** The first-order derivative of the objective function in Problem (4.6) with respect to K_f is given by $A - 2K_w - 2K_f - \gamma\frac{K_f}{\bar{S}}\left(2 - \frac{K_f}{\bar{S}}\frac{d\bar{S}}{dK_f}\right)$, where $\frac{d\bar{S}}{dK_f} = \frac{\gamma\frac{2K_f}{\bar{S}^2}\sum_{i=1}^{M}S_i g\left(\gamma\frac{K_f^2}{\bar{S}^2}\right)}{1+\gamma\frac{2K_f^2}{\bar{S}^3}\sum_{i=1}^{M}S_i g\left(\gamma\frac{K_f^2}{\bar{S}^2}\right)}$ from (4.4). The second-order derivative of the objective function is given by $-2 - \gamma\frac{2}{\bar{S}}\left(\frac{K_f}{\bar{S}}\frac{d\bar{S}}{dK_f} - 1\right)^2 + \gamma\frac{K_f^2}{\bar{S}^2}\frac{d^2\bar{S}}{dK_f^2}$. As we are solving the problem for $\frac{d^2\bar{S}}{dK_f^2} \leq 0$, we conclude the objective function is concave and the optimal on-demand capacity K_f^*, if feasible, is the root of the first-order derivative:

$$A - 2K_w = 2K_f + \gamma\left(\frac{2K_f}{\bar{S}} - \frac{K_f^2}{\bar{S}^2}\frac{d\bar{S}}{dK_f}\right) = 2K_f + \gamma\frac{K_f}{\bar{S}}\left(2 - \frac{\gamma\frac{2K_f^2}{\bar{S}^3}\sum_{i=1}^{M}S_i g\left(\gamma\frac{K_f^2}{\bar{S}^2}\right)}{1+\gamma\frac{2K_f^2}{\bar{S}^3}\sum_{i=1}^{M}S_i g\left(\gamma\frac{K_f^2}{\bar{S}^2}\right)}\right)$$

The right-hand-side of the above equation is positive and hence for feasibility we require the left-hand-side to be positive or equivalently $A > K_w$. The right-hand-side of the equation is zero at $K_f = 0$, and as $A > K_w$, for feasibility the right-hand-side should be increasing in K_f. Hence, if the derivative of the right-hand-side is positive then the above equation has a unique root. The derivative of the right-hand-side of the equation is given by

$$\frac{d\left(2K_f + \gamma\left(\frac{2K_f}{\bar{S}} - \frac{K_f^2}{\bar{S}^2}\frac{d\bar{S}}{dK_f}\right)\right)}{dK_f} = 2 + \gamma\frac{2}{\bar{S}}\left(\frac{K_f}{\bar{S}}\frac{d\bar{S}}{dK_f} - 1\right)^2 - \gamma\frac{K_f^2}{\bar{S}^2}\frac{d^2\bar{S}}{dK_f^2}.$$

Hence, if $2 + \gamma\frac{2}{\bar{S}}\left(\frac{K_f}{\bar{S}}\frac{d\bar{S}}{dK_f} - 1\right)^2 - \gamma\frac{K_f^2}{\bar{S}^2}\frac{d^2\bar{S}}{dK_f^2} > 0$, then we conclude K_f^* is unique and is given by

$$K_f^* = \left\{0 \leq K_f \leq \sum_{i=1}^{M} S_i : A - 2K_w = 2K_f + \gamma\left(\frac{2K_f}{\bar{S}} - \frac{K_f^2}{\bar{S}^2}\frac{d\bar{S}}{dK_f}\right)\right\}.$$

□

Note for Theorem 4.3. For the supply of independent providers, the condition $U \leq \gamma$ always holds due to the condition $G(\gamma) = 1$.

Proof. **Theorem 4.3.** We do the proof by considering the two cases in the supply function (4.2):

- From the supply function (4.8), for the first case we have $\bar{S}(K_f) = \left(\frac{\gamma K_f^2}{U}\sum_{i=1}^{M} S_i\right)^{1/3}$ for which the objective function is given by

$$\pi(K_f) = \left[(A - K_w - K_f - c_w)\, K_w + \left(A - K_w - K_f - \gamma\frac{K_f}{\bar{S}}\right) K_f\right]$$

  and is feasible for $K_f \leq \sqrt{\frac{U}{\gamma}\sum_{i=1}^{M} S_i}$. The first-order derivative of the objective function with respect to K_f is given by

$$\frac{d\pi(K_f)}{dK_f} = -2K_w - 2K_f + A - \frac{4}{3}\left(\frac{\gamma^2 U K_f}{\sum_{i=1}^M S_i}\right)^{1/3}. \qquad (16)$$

The second-order derivative of the objective function is given by

$$\frac{d^2\pi(K_f)}{dK_f^2} = -2 - \frac{4}{9}\left(\frac{\gamma^2 U}{K_f^2 \sum_{i=1}^M S_i}\right)^{1/3}.$$

The root of the first-order derivative, if feasible, is the optimal K_f^* and is the root of the equation $2K_f + \frac{4}{3}\left(\frac{\gamma^2 U}{\sum_{i=1}^M S_i}\right)^{1/3} K_f^{1/3} + (2K_w - A) = 0$. Using the Cardano's result for cubic equations, we conclude that the above equation has one real root which is given by
$$\tilde{K}_f = \left(\left(\frac{A-2K_w}{4} + \sqrt{\frac{(A-2K_w)^2}{16} + \frac{8\left(\frac{\gamma^2 U}{\sum_{i=1}^M S_i}\right)}{27^2}}\right)^{1/3} + \left(\frac{A-2K_w}{4} - \sqrt{\frac{(A-2K_w)^2}{16} + \frac{8\left(\frac{\gamma^2 U}{\sum_{i=1}^M S_i}\right)}{27^2}}\right)^{1/3}\right)^3.$$
It can be easily shown that if $A \geq 2K_w$ then $\tilde{K}_f \geq 0$, otherwise $\tilde{K}_f < 0$. The root of the first-order derivative is feasible if it satisfies the constraint $\tilde{K}_f \leq \sqrt{\frac{U}{\gamma}} \sum_{i=1}^M S_i$; note that as explained above, $U \leq \gamma$ holds, and hence the capacity constraint is satisfied: $\tilde{K}_f \leq \sqrt{\frac{U}{\gamma}} \sum_{i=1}^M S_i \leq \sum_{i=1}^M S_i$. The constraint $\tilde{K}_f \leq \sqrt{\frac{U}{\gamma}} \sum_{i=1}^M S_i$ is equivalent to $A \leq \hat{A}$, where $\hat{A}$ is the unique solution to the equation $\tilde{K}_f = \sqrt{\frac{U}{\gamma}} \sum_{i=1}^M S_i$; note that the root is unique as the right-hand-side in $\tilde{K}_f(K_w)$ can be easily shown to be increasing in A.

As a result, we conclude

- if $A < 2K_w$ then $K_f^* = 0$;
- otherwise if $2K_w \leq A \leq \hat{A}$ then the optimal on-demand capacity is $K_f^* = \tilde{K}_f$.
- otherwise if $A > \hat{A}$, or equivalently $\tilde{K}_f \geq \sqrt{\frac{U}{\gamma}} \sum_{i=1}^M S_i$, then since the objective function is concave we conclude the optimal on-demand capacity is $K_f^* =$

$\sqrt{\frac{U}{\gamma}} \sum_{i=1}^{M} S_i$.

- According to the supply function (4.8), for $\bar{S} = \sum_{i=1}^{M} S_i$, the problem is feasible for $K_f \geq \sqrt{\frac{U}{\gamma}} \sum_{i=1}^{M} S_i$ and the objective function is given by

$$\pi(K_f) = \left[(A - K_w - K_f - c_w) K_w + \left(A - K_w - K_f - \gamma \frac{K_f}{\sum_{i=1}^{M} S_i} \right) K_f \right]. \quad (17)$$

The first-order derivative of the objective function with respect to K_f is given by

$$\frac{d\pi(K_f)}{dK_f} = -2K_w - 2K_f + A - \frac{2\gamma K_f}{\sum_{i=1}^{M} S_i}. \quad (18)$$

The second-order derivative of the objective function is given by $\dfrac{d^2\pi(K_f)}{dK_f^2} = -2 - \dfrac{2\gamma}{\sum_{i=1}^{M} S_i} \leq 0$. The root of the first-order derivative, if feasible, is the optimal K_f^* and is given by $\hat{K}_f(K_w) = \dfrac{A - 2K_w}{2(1 + \frac{\gamma}{\sum_{i=1}^{M} S_i})}$. The root of the first-order derivative is feasible if it satisfies the inequality $\hat{K}_f(K_w) \geq \sqrt{\dfrac{U}{\gamma}} \displaystyle\sum_{i=1}^{M} S_i$ and the capacity constraint $\hat{K}_f(K_w) \leq \sum_{i=1}^{M} S_i$. These contraints are equivalent to $2K_w + 2(1 + \frac{\gamma}{\sum_{i=1}^{M} S_i})\sqrt{\frac{U}{\gamma}} \sum_{i=1}^{M} s_i \leq A \leq 2K_w + 2(1 + \frac{\gamma}{\sum_{i=1}^{M} S_i}) \sum_{i=1}^{M} S_i$; note that here the upper bound is always greater than the lower bound due to the condition $U \leq \gamma$.

From the analysis above we conclude that if $2K_w + 2(1 + \frac{\gamma}{\sum_{i=1}^{M} S_i})\sqrt{\frac{U}{\gamma}} \sum_{i=1}^{M} S_i \leq A \leq 2K_w + 2(1 + \frac{\gamma}{\sum_{i=1}^{M} S_i}) \sum_{i=1}^{M} S_i$, then the optimal on-demand capacity is given by $K_f^*(K_w) = \dfrac{A - 2K_w}{2(1 + \frac{\gamma}{\sum_{i=1}^{M} S_i})}$. Otherwise if $A > 2K_w + 2(1 + \frac{\gamma}{\sum_{i=1}^{M} S_i}) \sum_{i=1}^{M} S_i$ or equivalently $\hat{K}_f(K_w) \geq \sum_{i=1}^{M} S_i$, then since the objective function is concave, the optimal on-demand capacity is $K_f^*(K_w) = \sum_{i=1}^{M} S_i$. Otherwise if $A < 2K_w + 2(1 + \frac{\gamma}{\sum_{i=1}^{M} S_i})\sqrt{\frac{U}{\gamma}} \sum_{i=1}^{M} S_i$, or equivalently $\hat{K}_f(K_w) \leq \sqrt{\frac{U}{\gamma}} \sum_{i=1}^{M} S_i$, since the objective function is concave, the optimal on-demand capacity is given by $K_f^*(K_w) = \sqrt{\frac{U}{\gamma}\sum_{i=1}^M S_i}$.

As a result, we conclude

- if $A < 2K_w + 2(1 + \frac{\gamma}{\sum_{i=1}^M S_i})\sqrt{\frac{U}{\gamma}\sum_{i=1}^M S_i}$, or equivalently $\hat{K}_f(K_w) \leq \sqrt{\frac{U}{\gamma}\sum_{i=1}^M S_i}$, then since the objective function is concave, we conclude the optimal on-demand capacity is $K_f^* = \sqrt{\frac{U}{\gamma}\sum_{i=1}^M S_i}$;
- otherwise if $2K_w + 2(1 + \frac{\gamma}{\sum_{i=1}^M S_i})\sqrt{\frac{U}{\gamma}\sum_{i=1}^M S_i} \leq A \leq 2K_w + 2(1 + \frac{\gamma}{\sum_{i=1}^M S_i})\sum_{i=1}^M S_i$, then the optimal on-demand capacity is $K_f^* = \dfrac{A - 2K_w}{2(1 + \frac{\gamma}{\sum_{i=1}^M S_i})}$.
- otherwise if $A > 2K_w + 2(1 + \frac{\gamma}{\sum_{i=1}^M S_i})\sum_{i=1}^M S_i$, or equivalently $\hat{K}_f(K_w) \geq \sum_{i=1}^M S_i$, then since the objective function is concave, we conclude the optimal on-demand capacity is $K_f^* = \sum_{i=1}^M S_i$.

From the above analysis we conclude the following:

- If $A \leq 2K_w$ then $K^* = 0$.
- Otherwise if $2K_w < A \leq \hat{A}$ then $K_f^* = \tilde{K}_f$.
- Otherwise if $\hat{A} < A \leq 2K_w + 2(1 + \frac{\gamma}{\sum_{i=1}^M S_i})\sqrt{\frac{U}{\gamma}\sum_{i=1}^M S_i}$ then $K_f^* = \sqrt{\frac{U}{\gamma}\sum_{i=1}^M S_i}$.
- Otherwise if $2K_w + 2(1 + \frac{\gamma}{\sum_{i=1}^M S_i})\sqrt{\frac{U}{\gamma}\sum_{i=1}^M S_i} < A \leq 2K_w + 2(1 + \frac{\gamma}{\sum_{i=1}^M S_i})\sum_{i=1}^M S_i$ then $K_f^* = \dfrac{A - 2K_w}{2(1 + \frac{\gamma}{\sum_{i=1}^M S_i})}$.
- Otherwise if $A > 2K_w + 2(1 + \frac{\gamma}{\sum_{i=1}^M S_i})\sum_{i=1}^M S_i$ then $K_f^* = \sum_{i=1}^M S_i$.

$\square$

Proof. **Theorem 4.4.** If we input K_f^* from Theorem 4.3 into the objective function in Problem (4.9), we get the following optimal profit function, where, we have exchanged the order of expectations between A and S. Note that as $S_i,\ i = 1, \ldots, M$ are distributed according to the same probability distribution function f_S with mean σ_S and standard deviation σ_S, for the expressions inside the integral, we let $S_i = S,\ i = 1, \ldots, M$.

$$\begin{aligned}
\pi \ &= \ \int_{S=0}^{\infty}\left(\int_{A=0}^{\infty}(A-K_w-c_w)\,K_w+\int_{A=2K_w}^{\hat{A}}\left(A-2K_w-\tilde{K}_f-(\frac{\gamma^2\tilde{K}_fU}{MS})^{1/3}\right)\tilde{K}_ff_AdA\right.\\
&\qquad +\int_{\hat{A}}^{2K_w+2(1+\frac{\gamma}{MS})\sqrt{\frac{U}{\gamma}}MS}\left(A-2K_w-\sqrt{\frac{U}{\gamma}}MS-(\sqrt{\gamma U}\right)\sqrt{\frac{U}{\gamma}}MSf_AdA\\
&\qquad +\int_{A=2K_w+2(1+\frac{\gamma}{MS})\sqrt{\frac{U}{\gamma}}MS}^{2K_w+2(1+\frac{\gamma}{MS})MS}\left(A-2K_w-\hat{K}_f-\frac{\gamma\hat{K}_f}{MS}\right)\hat{K}_ff_AdA\\
&\qquad \left.+\int_{A=2K_w+2(1+\frac{\gamma}{MS})MS}^{\infty}(A-2K_w-MS-\gamma)\,MSf_AdA\right)f_SdS\\
&= \ \mu_AK_w-K_w^2-c_wK_w+\int_{S=0}^{\infty}\int_{A=2K_w}^{\hat{A}}\left(A-2K_w-\tilde{K}_f-(\frac{\gamma^2\tilde{K}_fU}{MS})^{1/3}\right)\tilde{K}_ff_AdAf_SdS\\
&\quad +\int_{S=0}^{\infty}\int_{\hat{A}}^{2K_w+2(1+\frac{\gamma}{MS})\sqrt{\frac{U}{\gamma}}MS}\left(A-2K_w-\sqrt{\frac{U}{\gamma}}MS-\sqrt{\gamma U}\right)\sqrt{\frac{U}{\gamma}}MSf_AdAf_SdS\\
&\quad +\int_{S=0}^{\infty}\int_{A=2K_w+2(1+\frac{\gamma}{MS})\sqrt{\frac{U}{\gamma}}MS}^{2K_w+2(1+\frac{\gamma}{MS})MS}\left(A-2K_w-\hat{K}_f-\frac{\gamma\hat{K}_f}{MS}\right)\hat{K}_ff_AdAf_SdS\\
&\quad +\int_{S=0}^{\infty}\int_{A=2K_w+2(1+\frac{\gamma}{MS})MS}^{\infty}(A-2K_w-MS-\gamma)\,MSf_AdAf_SdS.
\end{aligned}$$

To solve for K_w^* we need Lemma .2.

Lemma .2. $\pi(K_w)$ *is strictly concave.*

Proof. **Lemma .2.**

The first-order derivative of $\pi(K_w)$ with respect to K_w is given by

$$\begin{aligned}
\frac{d\pi(K_w)}{dK_w} &= \mu_A - 2K_w - c_w \\
&+ \int_{S=0}^{\infty}\int_{A=2K_w}^{\hat{A}} \left(\left(A - 2\tilde{K}_f - 2K_w - \frac{4}{3}(\frac{\gamma^2\tilde{K}_f U}{MS})^{1/3}\right)\frac{d\tilde{K}_f}{dK_w} - 2\tilde{K}_f\right) f_A dA f_S dS \\
&-2M\sqrt{\frac{U}{\gamma}}\int_{S=0}^{\infty} S \int_{\hat{A}}^{2K_w+2(1+\frac{\gamma}{MS})\sqrt{\frac{U}{\gamma}}MS} f_A dA f_S dS \\
&+ \int_{S=0}^{\infty}\int_{A=2K_w+2(1+\frac{\gamma}{MS})\sqrt{\frac{U}{\gamma}}MS}^{2K_w+2(1+\frac{\gamma}{MS})MS} \left(\left(A - 2\hat{K}_f - 2K_w - 2\frac{\gamma\hat{K}_f U}{MS}\right)\frac{d\hat{K}_f}{dK_w} - 2\hat{K}_f\right) f_A dA f_S dS \\
&-2M\int_{S=0}^{\infty} S \int_{A=2K_w+2(1+\frac{\gamma}{MS})MS}^{\infty} f_A dA f_S dS.
\end{aligned}$$

From eqaution (16), we know $\left(A - 2\tilde{K}_f - 2K_w - \frac{4}{3}(\frac{\gamma^2\tilde{K}_f U}{MS})^{1/3}\right) = 0$. Similarly, from equation (18) we know $\left(-2K_w - 2K_f + A - \frac{2\gamma K_f}{MS}\right) = 0$. Hence, we simplify $\frac{d\pi(K_w)}{dK_w}$ to

$$\begin{aligned}
\frac{d\pi(K_w)}{dK_w} &= (\mu_A - c_w) - 2K_w \\
&-2\int_{S=0}^{\infty}\Bigg(\int_{A=2K_w}^{\hat{A}} \tilde{K}_f f_A dA + M\sqrt{\frac{U}{\gamma}} S \int_{\hat{A}}^{2K_w+2(1+\frac{\gamma}{MS})\sqrt{\frac{U}{\gamma}}MS} f_A dA + \\
&\qquad \int_{A=2K_w+2(1+\frac{\gamma}{MS})\sqrt{\frac{U}{\gamma}}MS}^{2K_w+2(1+\frac{\gamma}{MS})MS} \hat{K}_f f_A dA + MS\int_{A=2K_w+2(1+\frac{\gamma}{MS})MS}^{\infty} f_A dA\Bigg) f_S dS.
\end{aligned}$$

Thus, the second order derivative of $\pi(K_w)$ with respect to K_w can be calculated as

$$\begin{aligned}
\frac{d^2\pi(K_w)}{dK_w^2} &= -2 - 2 \times \\
&\int_{S=0}^{\infty}\left(\int_{A=2K_w}^{\hat{A}} \frac{1}{1+\frac{2}{9}\left(\frac{\gamma^2 U}{MS}\right)^{1/3}\tilde{K}_f^{-2/3}} f_A dA - \frac{1}{(1+\frac{\gamma}{MS})}\int_{A=2K_w+2(1+\frac{\gamma}{MS})\sqrt{\frac{U}{\gamma}}MS}^{2K_w+2(1+\frac{\gamma}{MS})MS} f_A dA\right) f_S dS.
\end{aligned}$$

Hence, we conclude $\dfrac{d^2\pi(K_w)}{dK_w^2} \leq 0$. Hence, $\pi(K_w)$ is strictly concave and this completes the proof. □

From Lemma .2, we know

$$\begin{aligned}
\frac{d\pi(K_w)}{dK_w} &= (\mu_A - c_w) - 2K_w - 2 \times \\
& \int_{S=0}^{\infty} \Bigg(\int_{A=2K_w}^{\hat{A}} \tilde{K}_f f_A dA + M\sqrt{\frac{U}{\gamma}} S \int_{\hat{A}}^{2K_w+2(1+\frac{\gamma}{MS})\sqrt{\frac{U}{\gamma}}MS} f_A dA \\
& + \int_{A=2K_w+2(1+\frac{\gamma}{MS})\sqrt{\frac{U}{\gamma}}MS}^{2K_w+2(1+\frac{\gamma}{MS})MS} \hat{K}_f f_A dA + MS \int_{A=2K_w+2(1+\frac{\gamma}{MS})MS}^{\infty} f_A dA \Bigg) f_S dS.
\end{aligned}$$

For $K_w \to \infty$, it can be easily shown that $\dfrac{d\pi(K_w)}{dK_w} \to -\infty$. If

$$\begin{aligned}
\frac{d\pi(K_w)}{dK_w}|_{K_w=0} &= (\mu_A - c_w) - 2 \times \\
& \int_{S=0}^{\infty} \Bigg(\int_{A=0}^{\hat{A}(K_w=0)} \tilde{K}_f(K_w=0) f_A dA + M\sqrt{\frac{U}{\gamma}} S \int_{\hat{A}(K_w=0)}^{2(1+\frac{\gamma}{MS})\sqrt{\frac{U}{\gamma}}MS} f_A dA \\
& + \int_{A=2(1+\frac{\gamma}{MS})\sqrt{\frac{U}{\gamma}}MS}^{2(1+\frac{\gamma}{MS})MS} \frac{A}{2(1+\frac{\gamma}{MS})} f_A dA + MS \int_{A=2(1+\frac{\gamma}{MS})MS}^{\infty} f_A dA \Bigg) f_S dS
\end{aligned}$$

is non-negative then as $\pi(K_w)$ is strictly concave (c.f. Lemma .2), $\dfrac{d\pi(K_w)}{dK_w}$ has a positive unique root. Thus, $K_w^* > 0$ exists and is unique. Otherwise, as $\pi(K_w)$ is strictly concave (c.f. Lemma .2), $\dfrac{d\pi(K_w)}{dK_w}$ does not have a root in $K_w \in (0, \infty)$ and we conclude $K_w^* = 0$. □

Proof. **Proposition 4.1.** We first prove that in the presence of on-demand warehousing, the firm utilizes traditional capacity no more than the benchmark case: From Theorem 4.1, we know ${K_w^b}^* = \max\left\{\frac{\mu_A - c_w}{2}, 0\right\}$. For comparing ${K_w^b}^*$ and K_w^*, we input ${K_w^b}^*$ into the first-order derivative $\frac{d\pi(K_w)}{dK_w}$ and show it is non-positive. Then as $\pi(K_w)$ is strictly concave with a unique maximizer, if exists, (c.f. Lemmas .2), we conclude $K_w^* \leq {K_w^b}^*$. From Lemma .2, we know

$$\begin{aligned}\frac{d\pi(K_w)}{dK_w} &= (\mu_A - c_w) - 2K_w - 2\times \\ &\int_{S=0}^{\infty}\left(\int_{A=2K_w}^{\hat{A}} 2\tilde{K}_f f_A dA + M\sqrt{\frac{U}{\gamma}}S\int_{\hat{A}}^{2K_w+2(1+\frac{\gamma}{MS})\sqrt{\frac{U}{\gamma}}MS} f_A dA + \right. \\ &\left. \int_{A=2K_w+2(1+\frac{\gamma}{MS})\sqrt{\frac{U}{\gamma}}MS}^{2K_w+2(1+\frac{\gamma}{MS})MS} \hat{K}_f f_A dA + MS\int_{A=2K_w+2(1+\frac{\gamma}{MS})MS}^{\infty} f_A dA\right) f_S dS.\end{aligned}$$

We do the proof for the following two cases separately:

- If $\mu_A - c_w > 0$, from Theorem 4.1 we know ${K_w^b}^* = \frac{\mu_A - c_w}{2}$, and

$$\begin{aligned}\frac{d\pi(K_w)}{dK_w}\Big|_{{K_w^b}^*=\frac{\mu_A-c_w}{2}} &= -2\int_{S=0}^{\infty}\left(\int_{A=\mu_A-c_w}^{\hat{A}(K_w=\frac{\mu_A-c_w}{2})} \tilde{K}_f(K_w = \frac{\mu_A - c_w}{2}) f_A dA\right. \\ &+M\sqrt{\frac{U}{\gamma}}S\int_{\hat{A}(K_w=\frac{\mu_A-c_w}{2})}^{\mu_A-c_w+2(1+\frac{\gamma}{MS})\sqrt{\frac{U}{\gamma}}MS} f_A dA \\ &+\int_{A=\mu_A-c_w+2(1+\frac{\gamma}{MS})\sqrt{\frac{U}{\gamma}}MS}^{\mu_A-c_w+2(1+\frac{\gamma}{MS})MS} \hat{K}_f(K_w = \frac{\mu_A - c_w}{2}) f_A dA \\ &\left.+MS\int_{A=\mu_A-c_w+2(1+\frac{\gamma}{MS})MS}^{\infty} f_A dA\right) f_S dS \le 0.\end{aligned}$$

  Thus, we conclude $0 \le K_w^* \le K_w^b$.

- If $\mu_A - c_w \le 0$, from Theorem 4.1 we know ${K_w^b}^* = 0$. Furthermore from Theorem 4.4, we know $K_w^* = 0$. Thus, we conclude $K_w^* = K_w^b = 0$.

We next show there exists a threshold $\hat{A} \ge 0$ such that $K_f^* + K_w^* \le {K_w^b}^*$ for $A \le \hat{A}$ and $K_f^* + K_w^* > {K_w^b}^*$ for $A > \hat{A}$. According to the analysis above, we know$K_w^* \le {K_w^b}^*$. Additionally, we know ${K_w^b}^*$ and K_w^* are independent of the realization A. Hence, if we show K_f^* is non-decreasing in A the proof is complete. Fom Theorem 4.3, we know $\tilde{K}_f \le \sqrt{\frac{U}{\gamma}}MS$

and $\sqrt{\frac{U}{\gamma}}MS \leq \hat{K}_f \leq MS$. Thus, if we show $\tilde{K}_f$ and $\hat{K}_f$ are non-decreasing in A the proof is complete. Below we show $\tilde{K}_f(K_w^*)$ is increasing in A:

$$\frac{d\tilde{K}_f(K_w^*)}{dA} =$$
$$3\left(\left(\frac{A-2K_w^*}{4}+\sqrt{\frac{(A-2K_w^*)^2}{16}+\frac{8\left(\frac{U}{MS}\right)}{27^2}}\right)^{\frac{1}{3}}+\left(\frac{A-2K_w^*}{4}-\sqrt{\frac{(A-2K_w^*)^2}{16}+\frac{8\left(\frac{U}{MS}\right)}{27^2}}\right)^{\frac{1}{3}}\right)^2$$
$$\left(\left(\frac{1}{4}+\frac{(A-2K_w^*)}{16}\left(\frac{(A-2K_w^*)^2}{16}+\frac{8\left(\frac{U}{MS}\right)}{27^2}\right)^{\frac{-1}{2}}\right)\left(\frac{A-2K_w^*}{4}+\sqrt{\frac{(A-2K_w^*)^2}{16}+\frac{8\left(\frac{U}{MS}\right)}{27^2}}\right)^{\frac{-2}{3}}\right.$$
$$\left.+\left(\frac{1}{4}+\frac{(A-2K_w^*)}{16}\left(\frac{(A-2K_w^*)^2}{16}+\frac{8\left(\frac{U}{MS}\right)}{27^2}\right)^{\frac{-1}{2}}\right)\left(\frac{A-2K_w^*}{4}-\sqrt{\frac{(A-2K_w^*)^2}{16}+\frac{8\left(\frac{U}{MS}\right)}{27^2}}\right)^{\frac{-2}{3}}\right),$$

which is non-negative. Similarly, we show $\hat{K}_f$ in increasing in A: $\frac{d\hat{K}_f(K_w^*)}{dA} = \frac{1}{2\left(1+\frac{\gamma}{MS}\right)} \geq 0$. $\square$

Proof. **Corollary 4.1.** From Theorem 4.4, we know if

$$c_w < \mu_A - 2\int_{S=0}^{\infty}\left(\int_{A=0}^{\hat{A}}\tilde{K}_f(K_w=0)f_A dA + M\sqrt{\frac{U}{\gamma}}S\int_{\hat{A}(K_w=0)}^{2(1+\frac{\gamma}{MS})\sqrt{\frac{U}{\gamma}}MS} f_A dA\right.$$
$$\left.+\int_{A=2(1+\frac{\gamma}{MS})\sqrt{\frac{U}{\gamma}}MS}^{2(1+\frac{\gamma}{MS})MS}\hat{K}_f(K_w=0)f_A dA + MS\int_{A=2(1+\frac{\gamma}{MS})MS}^{\infty} f_A dA\right)f_S dS,$$

then $K_w^* \neq 0$ and is given by

$$\begin{aligned}
K_w^* &= {K_w^b}^* - \int_{S=0}^{\infty}\Big(\int_{A=0}^{2K_w^*}0 f_A dA + \int_{A=2K_w^*}^{\hat{A}(K_w^*)}\tilde{K}_f(K_w^*)f_A dA + M\sqrt{\frac{U}{\gamma}}S\int_{\hat{A}(K_w^*)}^{2K_w^*+2(1+\frac{\gamma}{MS})\sqrt{\frac{U}{\gamma}}MS} f_A dA \\
&\quad + \int_{A=2K_w^*+2(1+\frac{\gamma}{MS})\sqrt{\frac{U}{\gamma}}MS}^{2K_w^*+2(1+\frac{\gamma}{MS})MS}\hat{K}_f(K_w^*)f_A dA + MS\int_{A=2K_w^*+2(1+\frac{\gamma}{MS})MS}^{\infty} f_A dA\Big)f_S dS \\
&= {K_w^b}^* + E\left(K_f^*\right),
\end{aligned}$$

and this completes the proof. $\square$

Proof. **Corollary 4.2.** We know the following inequality holds for $K_w \geq 0$ and $0 \leq K_f \leq \sum_{i=0}^{M} S_i$:

$$h(K_w, K_f) \triangleq E_A \left(E_{(S_1,\ldots,S_M)} \left(\left[\left(A - K_w - K_f - c_w \right) K_w + \left(A - K_w - K_f - c_f(K_f) \right) K_f \right] \right) \right) \leq$$
$$\max_{K_w \geq 0} E_A \left(E_{(S_1,\ldots,S_M)} \left(\max_{0 \leq K_f \leq \sum_{i=0}^{M} S_i} \left[\left(A - K_w - K_f - c_w \right) K_w + \left(A - K_w - K_f - c_f(K_f) \right) K_f \right] \right) \right)$$
$$= \pi.$$

For $c_w \leq \mu_A$, for which ${K_w^b}^* = \frac{(\mu_A - c_w)}{2}$. If we input $(K_w, K_f) = (\frac{\mu_A - c_w}{2}, 0)$ into $h(K_w, K_f)$, we get $h(\frac{\mu_A - c_w}{2}, 0) = \pi_b \leq \pi$. Similarly for $c_w > \mu_A$, for which ${K_w^b}^* = 0$, if we input $(K_w, K_f) = (0, 0)$ into $h(K_w, K_f)$, we get $h(0, 0) = \pi_b \leq \pi$, and this completes the proof. $\square$